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Essays On Information Economics

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by

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ABSTRACT

This thesis consists of three essays on information economics.

The first essay is "Dynamic Insurance between Two Risk Averse Agents with Bilateral Asymmetric Information." There are two infinitely lived agents in our model, both risk averse, and each has an i.i.d. random endowment stream which is unobservable to the other. Dynamic incentive compatibility in the Nash sense is studied. Feasible and incentive compatible coinsurance contracts are characterized. We give sufficient and necessary conditions for the existence of a constrained efficient contract. We show that a constrained efficient contract can be characterized in a Bellman equation. Algorithms for numerical solution to the Bellman equation are discussed and an example with exponential utility is computed. Our computational results show that, among other things, the wealth position of each agent follows a random walk with reflecting barriers.

The Second essay, "Adverse Selection in Credit Markets with Costly Screening," is a joint work with Steve Williamson. In this essay, we develop a credit market model with adverse selection where risk-neutral borrowers self select because lenders make use of a costly screening technology. The model has some features which are similar to the Rothschild-Stiglitz adverse selection model. If an equilibrium exists it is a separating equilibrium, and there exist parameter values for which an equilibrium does not exist. Equilibrium contracts are debt contracts, and it is robust to randomization, in contrast to results for the costly state verification model. This framework can be extended to permit financial intermediary structures, and it potentially has many applications.

The third essay, "Incentives, CEO Compensation, and Shareholder Wealth in A Dynamic Agency Model, " uses a simple dynamic agency model to address a

CEO compensation issue raised by Jensen and Murphy (1990). Jensen and Murphy argue that the observed pay-performance sensitivity, though positive, is too low to be consistent with formal agency theory. Two observations are made from computational results. First, in levels, CEO compensation and shareholder wealth are nonpositively correlated. Second, the first differences in CEO compensation and shareholder wealth can be positively or negatively correlated, depending on the degree of risk-sharing achieved with the optimal contract. Furthermore, for a wide variety of plausible parameter values, our model is capable of generating data where the pay-performance sensitivity can be significantly positive but very small, as in Jensen and Murphy's data. We therefore conclude that Jensen and Murphy's empirical finding is consistent with dynamic agency theory.

In memory of my grandparents

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Dynamic Insurance Between Two Risk Averse Agents With Bilateral Asymmetric Information

1.1 Introduction

This paper studies a dynamic insurance problem that arises between two risk averse agents with bilateral asymmetric information. The two agents, both infinitely lived in a pure exchange economy, face idiosyncratic risks in the endowments they receive. Specifically, at each date, they each draw independently a stochastic, privately observed endowment. The endowment is perishable, and there exist no opportunities for the two agents to borrow and lend with outside parties. Self-imposed punishments are also infeasible, due to a problem of ex post inefficiency. The two agents hence are constrained to consume the aggregate endowment they receive at each date. Being risk averse, they would wish to pool their endowments together. But this is impeded by the private information about the endowments they receive. The problem that the two agents face, therefore, is to design a feasible trading mechanism which achieves Pareto efficiency subject to the constraint of incentive compatibility: they must both be given the incentives to truthfully reveal their endowments.

The problem we study here is closely related to the following two types of dynamic insurance models studied in the literature. Townsend (1982), Spear and Srivastava (1987), and Thomas and Worrall (1990) model relationships between a risk neutral principal and a single risk averse agent. Green (1987), Phelan and Townsend (1990), and Atkeson and Lucas (1992) examine relationships between a risk neutral principal and a continuum of risk averse agents. In these models, as in standard principal-agent models such as Holmstrom (1979, 1982), Allen (1985), and Radner (1985), a key feature is that the risk neutral principal, who typically has access to credit markets, plays a crucial role in attaining constrained efficiency. Essentially, what the principal does is to permit violation of the aggregate budget-balancing constraint by serving as a residual claimant.

Our model is different from those studied in the dynamic insurance literature mainly in that such a role played by the principal is completely discarded. In our model, the two agents are constrained to consume the entire aggregate endowment at each date. In addition, unlike in the single-agent dynamic insurance models where there is a clear distinction between the insurer - the principal - and the insuree - the agent, here the two agents are identical in every respect, and each plays the role of an insurer and an insuree. In contrast to the continuum-of-agent models where the principal-social planner-can treat each individual agent on an one-to-one basis, and the agents do not interact directly, the two agents here have to deal with each other directly.

One way of looking at the differences between our modelling strategy and that adopted by the existing dynamic insurance literature is to view the present paper as modelling a closed long-term insurance partnership, whereas earlier writers have aimed at modelling other types of long-term insurance relationships. Spear and Srivastava (1987), which is a simple dynamic version of Holmstrom (1979), can be viewed as modelling capitalistic firms which feature separation between owners and labor.

The problem that Townsend (1982), and Thomas and Worrall (1990) study can be viewed as one faced by an income insurance company. And of course, the problems that Green (1987), Phelan and Townsend (1991), and Atkeson and Lucas (1992) consider can be interpreted as a relationship between the members of a society and its government or the social planner.

One chief advantage of the modelling strategy we adopt in this paper is that it creates a situation with bilateral asymmetric information and hence allows us to model dynamic incentive compatibility in the Nash sense. [1] This, too, makes a substantial difference between the model here and the ones in the dynamic insurance literature. Suppose here the two agents have a principal with whom the two agents can conduct lending and borrowing, as is the case in the earlier models, then our problem simply breaks down into a problem which is exactly the same as what Townsend (1982) and Thomas and Worrall (1990) studied. There, private information comes in a simple one-sided fashion, and there is an incentive problem only with respect to the risk averse agent. This same feature is also embodied in the continuum-of-agent dynamic insurance models. In Green (1987), for example, the relationship between the principal and the agents can be viewed as being composed of a continuum of one-to-one relationships between the principal and each individual agent. From the perspective of each single agent then, the incentive problem he faces is exactly the same as he would face if he was the only agent the principal is dealing with, just as in Thomas and Worrall. When there are two agents, both with private information, and having to deal with each other directly without going though a principal, as is the case here, the situation becomes quite different. In our model, for each agent, in deciding on whether to tell the truth or not, he has to take into consideration whether

[1] A static Nash incentive problem is studied in Mookherjee (1984) where there is a risk-neutral principal contracting with many risk-averse agents whose actions are unobservable.

the other agent is to tell the truth or not. This implies that, in general, incentive compatibility can only be achieved by making each agent's consumption a function of not only his own endowment history, but also that of the other's. This is in contrast to the case with the existing models in the literature where each agent's consumption depends exclusively on his own reported history, and not on the report of anyone else.

Our technical approach builds on the work of not only the authors in the dynamic insurance literature, but also authors in the repeated game literature, especially Abreu, Pearce, and Stacchetti (1986,1990). In particular, following Green (1987), the concept of temporary incentive compatibility is developed to decompose the problem of super incentive compatibility into a sequence of temporary incentive problems. Our innovation however is that, unlike Green (1987) (and many others including Thomas and Worrall (1990), and Atkeson and Lucas (1992),) here incentive compatibility requires that both agents truthfully revealing their endowments at each date constitute a subgame perfect Nash equilibrium. This is solved by conditioning each agent's current consumption not only on his own reported history, but also on that of the other's. In terms of formulating, characterizing, and solving for the efficient contract, a central task here is to characterize and compute the set of expected utilities that are achievable by a feasible and incentive compatible contract. This is where we borrow from Abreu, Pearce, and Stacchetti (1986,1990) the concept of self-generation as our major mathematical instrument. We show that the set of feasible and incentive compatible expected utilities is compact, and this solves the problem of existence. We also show that this set is a fixed point of a one-step mapping. This in turn leads to a Bellman equation which characterizes the efficient contract, where the expected utility of one agent is defined as the state variable to summarize the histories of reported endowments. Based on these, two related algorithms, both in the spirit of self-generation, are formulated for numerical computation of an efficient contract.

In Section 1.2, the model is presented. We define a coinsurance contract as a sequence of functions that map from histories of endowments into current net transfers from agent 1 to agent 2.

In Section 1.3, feasible and incentive compatible coinsurance contracts are discussed. A contract is said to be feasible if it never requires any agent to transfer to the other agent more than what he claims to have received. A contract is said to be perfectly (Nash) incentive compatible if, at each date and conditional on any history up to that date, the continuation of this contract is such that both agents truthfully reporting their endowments from tomorrow on constitutes a Nash equilibrium. A contract is said to be temporarily incentive compatible if, at each date and conditional on any history, given that both agents are to report truthfully from tomorrow on, then they both reporting truthfully today constitutes a Nash equilibrium. We show that a temporarily incentive compatible contract is perfectly incentive compatible. We also show as an intuitive property for a temporarily incentive compatible contract is that, in any circumstances, each agent should receive less transfer from the other agent, and be entitled a higher expected utility from tomorrow on, if he reports a higher endowment today.

Section 1.4 is the key part of the paper. We demonstrate that a constrained efficient contract that delivers *ex ante* expected utility V to agent 2 exists if and only if V is in some compact set we denote as Φ_V. We then show that a constrained efficient contract can be characterized in a Bellman equation. Our Bellman equation is quite different from those studied by earlier writers in the dynamic insurance literature. Among other things, a unique feature of our Bemman equation is that the value function enters both the objective and the incentive constraints.

A common important result in Green (1987), Atkeson and Lucas (1992), and Thomas and Worrall (1990) is, for efficient risk sharing, the expected utility of each agent converges to negative infinity with probability one. Due to the different modelling strategy we adopt, this however is not the case here. In Section 1.5, we show that the expected utility of each agent converges to every level in the set of possible expected utilities with probability zero. We also show in this section that the constrained efficient contract is nontrivial and strictly dominates autarky.

As our model is not amenable to analytic solutions, two algorithms for numerical computation of an efficient contract are discussed in Section 1.6. Then in Section 1.7, an example, where utility is exponential and endowment takes on two values, is computed for numerical solution. Among other things, a central finding from our computational results is that for each agent, his consumption path forms a Markov chain with stationary transitions and his expected utility follows a random walk with reflecting barriers. Hence in the long-run, the distributions of consumption and expected utilities each converge to an ergodic distribution.

Section 1.8 concludes the paper with several short remarks.

1.2 The Model

Consider the following economy. Time is discrete and lasts forever: $t = 1, 2, \ldots$. There are two infinitely lived agents, indexed by $a = 1, 2$. Both agents are risk averse and maximize their *ex ante* expected life-time utilities, and discount the future by the common discount factor $\beta \in (0,1)$. There is one perishable good which the agents consume. The instaneous utility function $u : R \rightarrow R$, [2] shared by both agents, is assumed to satisfy the following conditions: $u'(c) > 0$, $u''(c) < 0$, for all $c \geq 0$. At each date, each agent has a random endowment e_t^a drawn from a finite set $\{\theta_1, \theta_2, \ldots, \theta_n\}$, where $0 < \theta_1 < \theta_2 < \ldots < \theta_n$. Index this set by $\Theta = \{1, 2, \ldots, n\}$. We

[2] Note that here, as in Green (1987), Thomas and Worrall (1990), and the exponential utility case of Atkeson and Lucas (1992), utility is assumed to be well defined on both positive and negative consumptions. As we will see later, this facilitates formulating the problem of incentive compatibility.

assume that e_t^1 and e_t^2 are identically and independently distributed and $\text{Prob}\{e_t^a = \theta_i\} = \pi_i > 0$, for all $t \geq 1$, $i \in \Theta$ and $a = 1, 2$.

There exists no opportunities for the two agents to borrow or lend with outside parties. Self-imposed punishments are also infeasible, due to a problem of enforcement: *ex post* it is not in the interest of either of the two agents to waste some of the endowments. [3] Since endowments are perishable, the two agents hence are constrained to consume the entire aggregate endowment at each date. Being risk averse, they would wish to pool their endowments together. Suppose the endowment received by each agent at every date is publicly observed. Then the Pareto problem in this repeated setting can be reduced to a static optimization problem, and the stationary efficient allocation rule would require that the marginal rates of substitution between the two agents be identical across the n^2 states of nature. This, however, is not the case we study here. To complete the basic framework of our model, private information is introduced in the following way. At each date, the history of realized endowments of each agent is his private information. No monitoring technology is available to either of the two agents or any third party.

Clearly, the problem here is to design a long-term contract which will achieve the best risk sharing possible through borrowing and lending between the two agents. To facilitate analysis, we assume as in Green (1987) and others in the dynamic insurance literature that the two agents can bind themselves to long-term participation in a contract that may dictate an *ex post* lower welfare level than autarky in some circumstances. In other words, we assume that a contract, once agreed upon by the two agents, is legally enforcable.

[3] See Holmstrom (1982) for a brief discussion on this issue.

For all $t \geq 1$, denote the history of realized endowments up to date t by

$$h^t = (h^{1t}, h^{2t}) = (e_1, ..., e_t) \in (\Theta \times \Theta)^t = H^t,$$

where $h^{at} = (e_1^a, ..., e_t^a)$, $a = 1, 2$, and $e_t = (e_t^1, e_t^2)$. H^t is the set of all possible histories up to and including date t. Note that here we have slightly abused the notation to let e_t^a and its index be used interchangeably. Of course any possible trades in our model are going to be based solely upon what is reported by the two agents. For all $t \geq 1$, we denote by $g^{at} = (r_1^a, ..., r_t^a)$ agent a's reported history of endowments up to date t, where r_t^a is agent a's reported endowment at date t. Next, let the overall history of reported endowments up to date t be denoted by

$$g^t = (g^{1t}, g^{2t}) = (r_1, ..., r_t) \in (\Theta \times \Theta)^t = H^t,$$

where $r_t = (r_t^1, r_t^2)$. At each date, based on what has happened in the past and what the agent has received today, each agent will have to make a decision on what to report today: $r_t^1 = r_t^1(h^{1t}, g^{2(t-1)})$, $r_t^2 = r_t^2(g^{1(t-1)}, h^{2t})$. Note that here by convention we let $h^0 = g^0 = (\emptyset, \emptyset)$ denote the null history and reported history prior to any realization of endowments. Call $r_t^1 : \Theta^t \times \Theta^{t-1} \rightarrow \Theta$ and $r_t^2 : \Theta^{t-1} \times \Theta^t \rightarrow \Theta$ the date t reporting strategies for the two agents, respectively. Denote by $r^a = (r_t^a)_{t=1}^{\infty}$ the overall reporting strategy profile for agent a, $a = 1, 2$. Accordingly, agent a's truthful reporting strategy r^{a*} is such that $r_t^{a*} = e_t^a$, for all t.

Definition 1 A coinsurance contract σ is a sequence of functions $\{\sigma_t\}_{t=1}^{\infty}$ where $\sigma_t : H^t \rightarrow R$. Call $\sigma_t(g^t)$ the amount of good transfered from agent 1 to agent 2 at date t, conditional on reported history g^t up to date t.

Therefore under contract σ, at date t, agent 1 will consume $-\sigma(g^t) + e_t^1$, and agent 2 $\sigma(g^t) + e_t^2$. Note that our way of defining a contract here automatically ensures that the two agents will consume exactly the entire aggregate endowment at each date. Also note that the case where $\sigma_t = 0$ independet of date and history corresponds to the autarky.

Once a contract σ is agreed upon by the two agents, they are essentially engaged in an infinitely repeated reporting game whose payoff function is determined by the form of the contract. The two players, being assumed to be *ex ante* expected life-time utility maximizers, each choose a reporting strategy to maximize their expected payoffs. Let $U(\sigma; r^1, r^2)$ be the *ex ante* expected life-time utility that contract σ will deliver to agent 1 if he chooses reporting strategy r^1 and agent 2 chooses r^2. Define $V(\sigma; r^1, r^2)$ analoguously. We have

$$U(\sigma; r^1, r^2) = E_0 \sum_{t=1}^{\infty} \beta^{t-1} u(-\sigma_t(g^t | r^1, r^2) + e_t^1)$$

and

$$V(\sigma; r^1, r^2) = E_0 \sum_{t=1}^{\infty} \beta^{t-1} u(\sigma_t(g^t | r^1, r^2) + e_t^2).$$

Note that here $(g^t | r^1, r^2)$ is used to denote the date t reported history given that the reporting strategies r^1 and r^2 are adopted by the two agents respectively. E_0 denotes the *ex ante* exnectation.

1.3 Formulating Constrained Efficiency

We begin this section by defining feasibility for a coinsurance contract. Let σ be a coinsurance contract and let $\sigma_t(g^{t-1}, (i,j))$ be the date t net transfer of endowment from agent 1 to agent 2 if reported history up to date $t-1$ has been g^{t-1} and date t current reports by the two agents are θ_i and θ_j respectively.

Definition 2 A coinsurance contract σ is feasible if for all $t \geq 1$ and $g^{t-1} \in H^{t-1}$,

$$-\theta_j \leq \sigma_t(g^{t-1}, (i,j)) \leq \theta_i, \ \ \forall (i,j) \in \Theta^2. \tag{1.1}$$

Condition (1.1) simply requires that, at any date, the contract will not take from any agent more than what he claims to have received. Therefore, suppose the two agents both report truthfully about their endowments at each date, as they will under the conditions of incentive compatibility to be given shortly, then both agents will consume a non-negative amount of the consumption good, and of course together they will consume the entire aggregate endowment.

We proceed now to tackle the issue of incentive compatibility which is central to the problem of constrained efficiency we seek to formulate. As claimed in Section 1, one chief novelty of this paper is that our modelling strategy permits us to model incentive compatibility in the Nash sense. Here, a contract is said to be incentive compatible if it is in each agent's interest to adopt the truthful reporting strategy, given that the other does so. Precisely, *ex ante*, given a contract σ, both agents will choose a reporting strategy, r^1 and r^2, respectively. We say that σ is incentive compatible if (r^{1*}, r^{2*}) constitutes a Nash equilibrium. That is, r^{1*} is agent 1's best reporting strategy if $r^2 = r^{2*}$, and r^{2*} is agent 2's best reporting strategy if $r^1 = r^{1*}$.

Now that an infinitely repeated reporting game is under consideration, agents are allowed to reassess their reporting strategies at any stage of the game. Therefore, as is conventional in the repeated game literature, (e.g., Abreu, Pearce, and Stacchetti (1986, 1990),) it is natural that we require subgame perfection in our definition of Nash incentive compatibility. Basically, a contract σ is said to be perfectly incentive compatible if. at any date $t \geq 1$, conditional on any history, the continuation profile of σ is such that both agents keeping reporting truthfully from $t+1$ on constitutes a Nash equilibrium. To formulate this, for all $t \geq 0$, [4] let $\sigma|g^t$ be the continuation profile of σ conditional on the reported history g^t. By convention then, $\sigma|g^0 = \sigma$. Let $U(\sigma|g^t; r^1, r^2)$ denote the date t expected future utility (discounted to date $t+1$)

[4] Note that here and after, $t = 0$ is always used to denote the state of the *ex ante*.

that $\sigma|g^t$ will deliver to agent 1 if the reporting strategies r^1 and r^2 are adopted by the two agents respectively from date $t+1$ on. Note that here we abuse the notation slightly to let r^a in the expression $U(\sigma|g^1; r^1, r^2)$ stand for agent a's reporting strategy for the subgame starting from $t+1$ and conditional on reported history g^t. Analoguously define $V(\sigma|g^t; r^1, r^2)$. Also note that $U(\sigma|g^0; r^1, r^2) = U(\sigma; r^1, r^2)$ and $V(\sigma|g^0; r^1, r^2) = V(\sigma; r^1, r^2)$.

Definition 3 A coinsurance contract σ is perfectly (Nash) incentive compatible if

$$U(\sigma|g^t; r^{1*}, r^{2*}) \geq U(\sigma|g^t; r^1, r^{2*}), \ \ \forall r^1, \tag{1.2}$$

$$V(\sigma|g^t; r^{1*}, r^{2*}) \geq V(\sigma|g^t; r^{1*}, r^2), \ \ \forall r^2, \tag{1.3}$$

for all $t \geq 0$ and $g^t \in H^t$.

Following Green (1987), we now want to develop the concept of temporary incentive compatibility for this environment to help reformulate the incentive compatibility above defined in a recursive manner. The main idea here is to decompose the super incentive problem that each agent faces at the beginning of each date into a sequence of one-step incentive problems, each associated with a single coming date. Some more notation is needed here. Denote by $U(\sigma|g^{t-1}, (i,j))$ the date t expected utility (discounted to date $t+1$) that the continuation profile (from date $t+1$ on) of σ will deliver to agent 1, conditional on reported history $(g^{t-1}, (i,j))$ up to date t and that both will report truthfully from date $t+1$ on. Define $V(\sigma|g^{t-1}, (i,j))$ analoguously.

Definition 4 A contract σ is (Nash) temporarily incentive compatible at node (g^{t-1}, t) if, for all $(i,j) \in \Theta^2$ and $k \in \Theta$,

$$\begin{aligned} &u(-\sigma_t(g^{t-1}, (i,j)) + \theta_i) + \beta U(\sigma \mid g^{t-1}, (i,j)) \geq \\ &\quad u(-\sigma_t(g^{t-1}, (k,j)) + \theta_i) + \beta U(\sigma \mid g^{t-1}, (k,j)), \end{aligned} \tag{1.4}$$

and, for all $(i,j) \in \Theta^2$ and $l \in \Theta$,

$$u(\sigma_t(g^{t-1},(i,j)) + \theta_j) + \beta V(\sigma \mid g^{t-1},(i,j)) \geq$$
$$u(\sigma_t(g^{t-1},(i,l)) + \theta_j) + \beta V(\sigma \mid g^{t-1},(i,l)). \tag{1.5}$$

A contract is said to be temporarily incentive compatible if it is temporaril, incentive compatible at every possible node.

In constraint (1.4), given that θ_i and θ_j are the endowments the two agents receive at date t, and reported history up to date $t-1$ has been g^{t-1}, on the left hand of the inequality, $-\sigma_t(g^{t-1},(i,j)) + \theta_i$ is agent 1's current consumption and $U(\sigma|g^{t-1},(i,j))$ his future utility if he and agent 2 both report truthfully, today and from tomorrow on. On the right hand of the inequality, $-\sigma_t(g^{t-1},(k,j)) + \theta_i$ is agent 1's current consumption and $U(\sigma|g^{t-1},(k,j))$ his future utility, if agent 1 cheats by reporting θ_k rather than θ_i, given that agent 2 reports honestly θ_j and that they both will report truthfully from tomorrow on. Hence by Definition 4, temporary incentive compatibility means that, at any date, given any reported history, if one agent chooses to adopt the truthful reporting strategy from that date on, then the other can not benefit from any one-period misrepresentation at that date. Note that here we do not impose the restriction that reports must be made simultaneously at every date. Also note that although feasibility will guarantee that the "truth reporting" current consumptions $-\sigma_t(g^{t-1},(i,j)) + \theta_i$ and $\sigma_t(g^{t-1},(i,j)) + \theta_j$ are non-negative, it may still be the case that the "deviating" –off equilibrium path–current consumptions $-\sigma_t(g^{t-1},(k,j)) + \theta_i$ and $\sigma_t(g^{t-1},(i,l)) + \theta_j$ take on negative values. This is why we required for mathematical convenience in Section 1.2 that the utility function u be defined for negative consumption.

The following two lemmas present two important properties of a temporarily incentive compatible contract. Lemma 1 characterizes the impact of current endowment realizations on the quantities of current tradings, and on the continuation values of the contract to the two agents. At each date, given any history, no matter what agent 1 reports, agent 2 should receive a smaller transfer of current endowment from agent 1, and be entitled to a higher expected utility from tomorrow on, if he reports a higher endowment. Similarly, if he reports a lower endowment, then he should receive a larger transfer of current endowment from agent 1 and be entitled to a lower expected utility from tomorrow on. Then in Lemma 2, we show that if a contract is temporarily incentive compatible, it is perfectly incentive compatible.

Lemma 1 Let σ be temporarily (Nash) incentive compatible. Let $i, j, k, l \in \Theta$ be such that $k \leq i$, $l \leq j$. Then for all $t \geq 1$ and $g^{t-1} \in H^{t-1}$,

$$\sigma_t(g^{t-1}, (k,j)) \leq \sigma_t(g^{t-1}, (i,j)) \leq \sigma_t(g^{t-1}, (i,l)),$$

$$U(\sigma|g^{t-1}, (k,j)) \leq U(\sigma|g^{t-1}, (i,j)) \leq U(\sigma|g^{t-1}, (i,l)),$$

$$V(\sigma|g^{t-1}, (i,l)) \leq V(\sigma|g^{t-1}, (i,j)) \leq V(\sigma|g^{t-1}, (k,j)).$$

A corollary of Lemma 1 is that for each agent, at each date, conditional on his current endowment, temporary incentive compatibility implies that his current consumption increases with the current aggregate endowment, but his expected utility decreases with the current aggregate endowment.

Lemma 2 If contract σ is temporarily (Nash) incentive compatible, then it is perfectly (Nash) incentive compatible.

The proofs of the two lemmas are in the appendix. The proof of Lemma 1 involves simply manipulating conditions (1.4) and (1.5), and using the concavity of the utility function. The proof of Lemma 2 is in the spirit of Green (1987) and Atkeson and Lucas (1991). The idea is, given that the truthful reporting strategy is adopted by one agent, if the other agent can not benefit from any one-shot deviation from telling the truth, then he can not benefit from misrepresentation at all. Readers familar with the repeated game literature should note that temporary incentive compatibility and perfect incentive compatibility here are analogues of unimprovability and subgame perfection in the repeated game literature.

In the rest of the paper, we will focus on studying contracts that are temporarily incentive compatible (hence perfectly incentive compatible by Lemma 2). It is thus always legitimate for us to use h^t and g^t as interchangeable, as long as the temporary incentive compatibility conditions are being imposed. Also, for simplicity, we will denote $U(\sigma; r^{1*}, r^{2*})$ and $V(\sigma; r^{1*}, r^{2*})$ by $U(\sigma)$ and $V(\sigma)$, respectively.

We are now in a position to define constrained efficiency. We say that a feasible and perfectly incentive compatible contract is constrained efficient if it maximizes the expected utility of agent 1 subject to deliving a given expected utility to agent 2. Formally,

Definition 5 A coinsurance contract σ is constrained efficient at V if it maximizes $U(\sigma)$ subject to constraints (1.1), (1.4), (1.5), and

$$V(\sigma) = V. \tag{1.6}$$

To close this section, we note that a problem that arises immediately with Definition 5 is that the so defined efficient contracts do not exist for all values of V. In fact, If we let $\underline{\alpha}$ and $\overline{\alpha}$ denote the expected values of consuming nothing and everything (i.e., $e_t^1 + e_t^2$) at all the dates, respectively, then for any $V \leq \underline{\alpha}$ or $V \geq \overline{\alpha}$, there are simply no feasible and incentive compatible contracts that can deliver expected utility V to either of the two agents. The question is: on what domain of V is the Pareto problem in Definition 5 well specified? [5] i.e., for what values of V does a constrained

[5]This domain obviously contains the closed interval $[\underline{V}, \overline{V}]$, here $\underline{V}$ (respectively $\overline{V}$) is the ex ante expected life-time utility that one agent can receive if the contract is such that a fixed amount $-\theta_1$

efficient contract exist? One aim of the next section to address this question.

1.4 The Efficient Contract

In this section, we first solve the problem of existence by showing that an efficient contract exists if and only if V is in some compact set we call Φ_V. We then show that to solve for an efficient contract, it is sufficient to solve for a stationary allocation rule which in turn can be obtained by solving a Bellman equation.

We begin by defining the set of feasible and incentive compatible expected utilities $\Phi \subseteq R^2$ as follows:

$$\Phi \equiv \{(U,V)(\sigma) \in R^2 | \sigma \; s.t. \; (1.1), (1.4), and \; (1.5)\},$$

where $(U,V)(\sigma) = (U(\sigma), V(\sigma))$. Set Φ is nonempty because the autarkic contract is always feasible and incentive compatible. Φ is also bounded. A primary task of this section is to to characterize this set. We shall show that Φ is compact and is a fixed point of a one-step operator. To this end, we borrow from Abreu, Pearce and Stacchetti (1986,1990) the concept "self-generation" as our major mathematical instrument in this section. [6]

Let Ψ be any nonempty and bounded set in R^2. Define a set of current controls C as follows: $C \equiv [\sigma(i,j)]_{(i,j)\in\Theta^2}$. Define a function $\mathcal{U}:\Theta^2 \to R^2$ to be a continuation

(respectively θ_1) is transfered to him from the other agent, i.e.,

$$\underline{V} = \frac{1}{1-\beta}\sum \pi_i u(\theta_i - \theta_1), \quad \overline{V} = \frac{1}{1-\beta}\sum \pi_i u(\theta_i + \theta_1).$$

[6] Although self-generation is developed in its inventors' studies of sequential equilibrium outcomes in a repeated cournot oligopoly game (1986) and a generalized repeated stage game with imperfect monitoring (1990), the principle behind it is quite general and has the potential of being applied to many other issues that have a repeated nature. For example, Atkeson (1991) has applied this principle to his study of international lending with moral hazard and risk of repudiation; Others have applied it to the simple case of perfect monitoring in repeated games, see Sabourian (1989) and Croushaw and Luenberger (1990). For a highly readable discussion of this technique, see Fudenberg and Tirole (1991).

value function with respect to Ψ if $\mathcal{U}(i,j) = (U(i,j), V(i,j)) \in \Psi$, for all $(i,j) \in \Theta^2$.

Definition 6 Given Ψ, a pair $(\mathcal{C},\mathcal{U})$ is said to be admissible to Ψ if $\mathcal{U}$ is a continuation value function with respect to Ψ, and the following conditions are satisfied:

$$-\theta_j \leq \sigma(i,j) \leq \theta_i, \quad \forall (i,j) \in \Theta^2; \tag{1.7}$$

$\forall (i,j) \in \Theta^2$ and $\forall k \in \Theta$,

$$u(-\sigma(i,j) + \theta_i) + \beta U(i,j) \geq u(-\sigma(k,j) + \theta_i) + \beta U(k,j); \tag{1.8}$$

and, $\forall (i,j) \in \Theta^2$ and $\forall l \in \Theta$,

$$u(\sigma(i,j) + \theta_j) + \beta V(i,j) \geq u(\sigma(i,l) + \theta_j) + \beta V(i,l). \tag{1.9}$$

Note that the constraints in the above definition are the one-step analogues of constraints (1.1), (1.4), and (1.5) in the previous section. Specifically, (1.7) is feasibility, and (1.8) and (1.9) are temporary incentive compatibility. Now for any given $(\mathcal{C},\mathcal{U})$, which is admissible to Ψ, define the one-step expected utilities generated by this pair by

$$E(\mathcal{C},\mathcal{U}) = \{\sum \pi_i \pi_j [u(-\sigma(i,j) + \theta_i) + \beta U(i,j)], \sum \pi_i \pi_j [u(\sigma(i,j) + \theta_j) + \beta V(i,j)]\}.$$

If we let $L^\infty(\Theta^2; R^2)$ denote the space of bounded functions mapping from Θ^2 to R^2, then function $E : R^{\Theta^2} \times L^\infty(\Theta^2; R^2) \rightarrow R^2$ which is defined above is continuous when the product space $R^{\Theta^2} \times L^\infty(\Theta^2; R^2)$ is endowed with the proper product topology. Further, define operator B stepwise in the following way:

$$B(\Psi) \equiv \{E(\mathcal{C},\mathcal{U}) \mid (\mathcal{C},\mathcal{U}) \textit{ admissible to } \Psi\}.$$

Note that the operator B which maps from the collection of all nonempty and bounded sets in R^2 to R^2, is nonempty, bounded valued, and is monotone increasing in the sense that $\Psi_1 \subseteq \Psi_2$ implies $B(\Psi_1) \subseteq B(\Psi_2)$.

<u>Definition 7</u> Ψ is self-generating if $\Psi \subseteq B(\Psi)$.

That is, Ψ is self-generating if its image under operator B contains Ψ itself. In the remainder of this section, we will follow Abreu, Pearce, and Stacchetti (1990) closely in using the concept of self-generation to serve our purpose of characterizing the set Φ.

<u>Lemma 3</u> If Ψ is self-generating, then $B(\Psi) \subseteq \Phi$.

<u>Lemma 4</u> Φ is self-generating.

<u>Theorem 1</u> Φ is compact and $\Phi = B(\Phi)$.

<u>Corollary 1</u> $\Phi_V = \{V \in R \mid \text{there exists } U \text{ such that } (U, V) \in \Phi\}$ is compact.

<u>Corollary 2</u> $\Phi(V) = \{U \in R \mid (U, V) \in \Phi\}$ is compact, for all $V \in \Phi_V$.

The proofs of Lemma 3, Lemma 4, and Theorem 1 are in the appendix. The proofs of Corollary 1 and Corollary 2 are straightforward exercises using the compactness of Φ.

What Corollary 1 and Corollary 2 tell us is that a constrained efficient contract exists for any V in a compact set Φ_V. Given that there is not a feasible and incentive compatible contract for any V outside the set Φ_V, we can then conclude that a constrained efficient contract exists if and only if $V \in \Phi_V$.

Having solved the problem of existence, we are now in a position to look more deeply into the problem by characterizing the constrained efficient contract in a Bellman equation, as is a standard exercise in the literature. The idea here is to let agent 2's expected utility at each date serve as a state variable which summarizes the history of reported endowments up to that date.[7] Then, let the value function $U^* : \Phi_V \to R$ be defined in the following way:

$$U^*(V) \equiv \max_{U \in \Phi(V)} U, \;\; \forall\, V \in \Phi_V.$$

That is, given that agent 2 receives an expected utility V, $U^*(V)$ is the maximum expected utility that can be delivered to agent 1 by a feasible and incentive compatible contract. Note that by our eailier discussion in this section, function U^* is well defined. Our aim now is to show that U^* satisfies a Bellman functional equation that will be specified shortly.

Let $C(V) = [\sigma(i,j)(V), V(i,j)(V)]_{(i,j)\in\Theta^2}$ denote a set of current controls at V, for all $V \in \Phi_V$. Let $\bar{U} : \Phi_V \to R$ be any bounded function. Given $\bar{U}$, for all $V \in \Phi_V$, let

$$E(C(V), \bar{U}) = \sum \pi_i \pi_j [u(-\sigma(i,j) + \theta_i) + \beta \bar{U}(V(i,j))],$$

for all $C(V)$ such that the following constraints are satisfied:
$\forall (i,j) \in \Theta^2$,

$$-\theta_j \leq \sigma(i,j)(V) \leq \theta_i, \;\; V(i,j)(V) \in \Phi_V; \tag{1.10}$$

$\forall (i,j) \in \Theta^2$ and $\forall k \in \Theta$,

$$\begin{aligned} &u(-\sigma(i,j)(V) + \theta_i) + \beta \bar{U}(V(i,j)(V)) \geq \\ &\quad u(-\sigma(k,j)(V) + \theta_i) + \beta \bar{U}(V(k,j)(V)); \end{aligned} \tag{1.11}$$

$\forall (i,j) \in \Theta^2$ and $\forall l \in \Theta$,

$$u(\sigma(i,j)(V) + \theta_j) + \beta V(i,j)(V) \geq u(\sigma(i,l)(V) + \theta_j) + \beta V(i,l)(V); \tag{1.12}$$

[7] This idea is due to Spear and Srivastava (1987). In their repeated principal-agent model, the expected utility of the agent is defined as the state variable of the Bellman equation.

and

$$\sum_{(i,j)\in\Theta^2} \pi_i\pi_j[u(\sigma(i,j)(V)+\theta_j)+\beta V(i,j)(V)] = V. \tag{1.13}$$

Condition (1.10) requires that $C(V)$ be feasible. Constraints (1.11) and (1.12) require that $C(V)$ be incentive compatible, given $\bar{U}$. Constraint (1.13) is the one-step analogue of (1.6) which promises that expected utility V will be delivered to agent 2.

Now define an operator T, which maps from bounded functions to bounded functions, as follows. Given function $\bar{U}$, let

$$T(\bar{U})(V) \equiv \sup_{C(V)} E(C(V),\bar{U}), \ \ \forall V \in \Phi_V,$$

where $C(V)$ satisfies constraints (1.10) through (1.13). The following theorem provides us with a Bellman equation for the efficient contract by saying that U^* is a fixed point of the operator T.

Theorem 3 $T(U^*)(V) = U^*(V)$, for all $V \in \Phi_V$.

Proof. Fix V. Let $C(V)$ be such that $(C(V),U^*)$ meets constraints (1.10), (1.12), (1.13), and

$$u(-\sigma(i,j)(V)+\theta_i)+\beta U^*(V(i,j)(V)) \geq u(-\sigma(i,j)(V)+\theta_j)+\beta U^*(V(i,j)(V)).$$

To show $T(U^*)(V) \leq U^*(V)$, we need only show that there exists a contract σ which is feasible and incentive compatible and is such that $V(\sigma)=V$, and $U(\sigma) = T(U^*)(V)$. Now for each $(i,j)\in\Theta^2$, since $(U^*(V(i,j)(V)), V(i,j)(V)) \in \Phi$, there exists a feasible and incentive compatible contract σ_{ij} such that

$$U(\sigma_{ij}) = U^*(V(i,j)(V)), \ \ V(\sigma_{ij}) = V(i,j)(V), \ \ \forall (i,j)\in\Theta^2.$$

We can then let the contract $\sigma = \{\sigma_t(h^t)\}$ be constructed in the following way:

$$\sigma_1(h^0,(i,j)) = \sigma(i,j)(V), \quad \sigma|h^0,(i,j) = \sigma_{ij}, \quad \forall (i,j)\in\Theta^2.$$

We now proceed to show that $U^*(V) \leq T(U^*)(V)$. For all $U(\sigma) \in \Phi(V)$, we have

$$U(\sigma) = \sum \pi_i \pi_j [u(-\sigma_1(h^0,(i,j)) + \theta_i) + \beta U(\sigma \mid h^0,(i,j))].$$

and

$$V = \sum \pi_i \pi_j [u(\sigma_1(h^0,(i,j)) + \theta_j) + \beta V(\sigma | h^0,(i,j))],$$

where $\{\sigma_1(h^0,(i,j)), U(\sigma|h^0,(i,j)), V(\sigma|h^0,(i,j))\}_{(i,j)\in\Theta^2}$ satisfies (1.1), (1.4), and (1.5). But by definition of $U^*(V)$,

$$U(\sigma \mid h^0,(i,j)) \leq U^*(V(\sigma \mid h^0,(i,j))).$$

Therefore for all $U(\sigma) \in \Phi(V)$,

$$\begin{aligned} U(\sigma) &\leq \sum \pi_i \pi_j [u(-\sigma_1(h^0,(i,j)) + \theta_i) + \beta U^*(V(\sigma | h^0,(i,j)))] \\ &\leq \sup_{C(V)} E(C(V), U^*) \\ &= T(U^*)(V). \end{aligned}$$

Thus taking the maximum across $U(\sigma)$ yields

$$U^*(V) = \max_{U(\sigma)\in\Phi(V)} U(\sigma) \leq T(U^*)(V).$$

Q.E.D.

The following lemma states that the "sup" in the Bellman equation is actually attained.

Lemma 5 $T(U^*)(V) = max_{C(V)} E(C(V), U^*)$, for all $V \in \Phi_V$.

proof. Fix V. Since $\Phi(V)$ is compact, there exists a contract σ^* such that $U(\sigma^*) = U^*(V)$. Then $[\sigma_1^*(h^0,(i,j)), V(\sigma^*|h^0,(i,j))]_{(i,j)\in\Theta^2} \in argmax_{C(V)} E(C(V), U^*)$. Q.E.D.

Now let $C^{*\prime}V) = [\sigma^*(i,j)(V), V^*(i,j)(V)]_{(i,j)\in\Theta^2}$. Let $C^* = \{C^*(V) : C^*(V) \in argmax_{C(V)} E(C(V), U^*),\ V \in \Phi_V\}$ be called an efficient allocation rule, [8] where $\{\sigma^*(i,j)(V) : (i,j) \in \Theta^2,\ V \in \Phi_V\}$ is the efficient trading scheme and $\{V^*(i,j)(V) : (i,j) \in \Theta^2,\ V \in \Phi_V\}$ is the optimal law of motion of the state variable. We say that an efficient allocation rule C^* can generate a contract σ, (or σ can be generated by C^*,) if for all $t \geq 1$, $h^{t-1} \in H^{t-1}$, and $(i,j) \in \Theta^2$,

$$\begin{aligned} \sigma_t(h^{t-1},(i,j)) &= \sigma^*(i,j)(V(\sigma|h^{t-1})), \\ V(\sigma|h^{t-1},(i,j)) &= V^*(i,j)(V(\sigma|h^{t-1})), \\ U(\sigma|h^{t-1},(i,j)) &= U^*(V^*(i,j)(V(\sigma|h^{t-1}))). \end{aligned}$$

The following lemma establishes in some loose sense an equivalence relationship between efficient allocation rules and efficient contracts.

Lemma 6 (i) Let σ be an efficient contract. Then there exists an efficient allocation rule C^* that generates σ. (ii) Let C^* be an efficient allocation rule. Then for all $V \in \Phi_V$, an efficient contract σ can be generated by C^* such that $V(\sigma) = V$.

A formal proof of Lemma 6 is left for the reader. Due to (ii) of Lemma 6 then, to solve for an efficient contract, it is sufficient to solve for an efficient allocation rule which in turn amounts to solving the Bellman equation. For illustrative purposes, we now describe briefly how the contract σ that is generated by C^* in (ii) of Lemma 6 works. Suppose, at date 1, θ_i and θ_j are reported respectively by the two agents. Then the contract says that $\sigma^*(i,j)(V)$ amount of the consumption good is to be transfered from agent 1 to agent 2. In the mean time, the contract also determines that, from date 2 on, agent 2 is entitled to an expected utility $V_1 = V(i,j)^*(V)$. Now as the two agents move to date 2, suppose $\theta_{i'}$ and $\theta_{j'}$ are reported, then $\sigma^*(i',j')(V_1)$

[8] This terminology is borrowed from Atkeson and Lucas (1992) but used here in a slightly different sense.

will be transfered from agent 1 to agent 2, and $V_2 = V^*(i',j')(V_1)$ will be promised to agent 2 as his expected utility from date 3 on. In this way the contract rolls forward date by date. Notice that here the expected utility of agent 2, V_t, acts as a state variable that summarizes all the history up to date t. At the beginning of each date t, nothing but V_{t-1} matters, for today and for the future.

Of course, an efficient allocation rule C^* will have the following properties. For all $V \in \Phi_V$, if i, j, k, l in Θ are such that $k \leq i, l \leq j$, then

$$\sigma^*(k,j)(V) \leq \sigma^*(i,j)(V) \leq \sigma^*(i,l)(V),$$

$$V^*(i,l)(V) \leq V^*(i,j)(V) \leq V^*(k,j)(V),$$

$$U^*(V^*(k,j)(V)) \leq U^*(V^*(i,j)(V)) \leq U^*(V^*(i,l)(V)).$$

To fully characterize an efficient allocation rule, it is desirable that the Bellman equation can be solved analytically. Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992) have shown that, for some special forms of the utility function, (exponential utility functions in particular,) it is possible to derive closed form solutions to their Bellman equations. This, however, is infeasible here, even for the simpliest case where the endowment takes on only two values . There are two reasons for this, each has to do with the modelling strategy we adopt in this paper, and each can be viewed as a unique feature of our Bellman equation, compared to those in related models. First, in the Bellman equation here, there are explicit upper and lower boundaries for the net transfers, $\sigma(i,j)(V)$, whereas in the Bellman equations of Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992) for the exponential utility case, there is not a boundary on an individual agent's consumption. Second, in our Bellman equation, the value function $U^*(\cdot)$ enters not only the objective but also the constraints.

1.5 Further Characterization of the Efficient Contract: The Case of Two Endowment Values

In this section, we present two theorems to further characterize the constrained efficient contract without solving for it analytically. For tractability, we consider the simplest case where the endowment takes on only two values, i.e., $n = 2$ and $\Theta = \{1, 2\}$. First, we show in Theorem 4 that a contract where agent 1 transfers a constant amount of the endowment to agent 2 can not be constrained efficient. A corollary of Theorem 4 hence is that the autarkic contract is domimanted by an efficient contract.

<u>Theorem 4</u> The contract σ^a where $\sigma_t^a = a$ for all t and $a \in [-\theta_1, \theta_1]$ is not constrained efficient.

Proof. We prove the theorem by constructing a feasible and perfectly incentive compatible contract which strictly improves upon contract σ^a in the Pareto sense. Without losing generality assume $a \geq 0$. Let $\delta \in [0, \theta_2 - a]$ and let $\Delta \in [0, a]$. Construct a contract called $\sigma(\delta, \Delta)$ in the following way.

For $t = 1$, let

$$\begin{aligned}\sigma_1(\delta,\Delta)(g^0,(2,1)) &= a+\delta,\\ \sigma_1(\delta,\Delta)(g^0,(i,j)) &= a, \quad \forall (i,j) \neq (2,1).\end{aligned}$$

For $t = 2$, let

$$\begin{aligned}\sigma_2(\delta,\Delta)(g^2) &= a-\Delta, \;\; if\; g^1 = (g^0,(2,1)),\\ \sigma_2(\delta,\Delta)(g^2) &= a, \quad\; if\; g^1 \neq (g^0,(2,1)).\end{aligned}$$

Finally, for $t \geq 3$, let

$$\sigma_t(\delta,\Delta)(g^t) = a, \quad \forall g^t \in H^t.$$

Obviously then, $\sigma(0,0)$ is just σ^a. Note that for all δ and Δ the contract $\sigma(\delta,\Delta)$ thus constructed is certainly feasible, and it is also temporarily incentive compatible at all the dates $t \geq 2$. We now proceed to show that by choosing the magnitudes of δ and Δ properly, $\sigma(\delta,\Delta)$ can be made temporarily incentive compatible at date 1 as well and satisfy the desired Pareto dominance requirement.

Define functions $F_i, G_i, i = 1,2$, all mapping from $[0,\theta_1]^2$ to R as follows:

$$F_1(\delta,\Delta) = u(-a-\delta+\theta_1) + \beta[\pi_1 u(-a+\Delta+\theta_1) + \pi_2 u(-a+\Delta+\theta_2)] \\ -u(-a+\theta_1) - \beta[\pi_1 u(-a+\theta_1) + \pi_2 u(-a+\theta_2)],$$

$$F_2(\delta,\Delta) = u(-a-\delta+\theta_2) + \beta[\pi_1 u(-a+\Delta+\theta_1) + \pi_2 u(-a+\Delta+\theta_2)] \\ -u(-a+\theta_2) - \beta[\pi_1 u(-a+\theta_1) + \pi_2 u(-a+\theta_2)],$$

$$G_1(\delta,\Delta) = u(a+\delta+\theta_1) + \beta[\pi_1 u(a-\Delta+\theta_1) + \pi_2 u(a-\Delta+\theta_2)] \\ -u(a+\theta_1) - \beta[\pi_1 u(a+\theta_1) + \pi_2 u(a+\theta_2)],$$

$$G_2(\delta,\Delta) = u(a+\delta+\theta_2) + \beta[\pi_1 u(a-\Delta+\theta_1) + \pi_2 u(a-\Delta+\theta_2)] \\ -u(a+\theta_2) - \beta[\pi_1 u(a+\theta_1) + \pi_2 u(a+\theta_2)].$$

Then it is easy to see that $\sigma(\delta,\Delta)$ is temporarily incentive compatible at date 1 if and only if the following inequalities hold:

$$F_1(\delta,\Delta) \leq 0, \tag{1.14}$$

$$F_2(\delta,\Delta) \geq 0, \tag{1.15}$$

$$G_1(\delta,\Delta) \geq 0, \tag{1.16}$$

$$G_2(\delta,\Delta) \leq 0. \tag{1.17}$$

And, $\sigma(\delta,\Delta)$ strictly dominates autarky if and only if either (1.15) or (1.16) holds in strict inequality. Let

$$\Omega = \{(\delta,\Delta) \in [0,\theta_1]^2 \mid (\delta,\Delta)\ s.t.\ (1.14),(1.15),(1.16),(1.17)\}.$$

Notice that since $F_i(0.0) = G_i(0,0) = 0,\ i = 1,2$, we have $(0,0) \in \Omega$. In the following we will show that Ω contains a point at which either (1.15) or (1.16) holds with strict inequality. To this end, we find that Ω is characterized by the following facts:

$$\frac{d\Delta}{d\delta}\,|_{F_i(\delta,\Delta)=0} > 0, \qquad \frac{d\Delta}{d\delta}\,|_{G_i(\delta,\Delta)=0} > 0, \qquad i = 1,2;$$

$$\frac{d^2\Delta}{d\delta^2}\,|_{F_i(\delta,\Delta)=0} > 0, \qquad \frac{d^2\Delta}{d\delta^2}\,|_{G_i(\delta,\Delta)=0} < 0, \qquad i = 1,2;$$

$$\frac{d\Delta}{d\delta}\,|_{F_1(0,0)=0} = \frac{d\Delta}{d\delta}\,|_{G_1(0,0)=0} = K_1 > K_2 = \frac{d\Delta}{d\delta}\,|_{F_2(0,0)=0} = \frac{d\Delta}{a\delta}\,|_{G_2(0,0)=0}\ .$$

[9] With these facts in hand, the set Ω is depicted graphically in Figure (1.1). [10]

[9]To show these facts, define functions α and γ that map from R to R as follows:

$$\alpha(x) = (1/\beta)[\pi_1 u'(x - a + \theta_1) + \pi_2 u'(x - a + \theta_2)]^{-1},$$
$$\gamma(x) = (1/\beta)[\pi_1 u'(x + a + \theta_1) + \pi_2 u'(x + a + \theta_2)]^{-1}.$$

Then for $i = 1,2$, we have:

$$\frac{d\Delta}{d\delta}\,|_{F_i(\delta,\Delta)=0} = \alpha(\Delta)u'(a - \delta + \theta_i) > 0,$$

$$\frac{d\Delta}{d\delta}\,|_{G_i(\delta,\Delta)=0} = \gamma(-\Delta)u'(a + \delta + \theta_i) > 0,$$

$$\frac{d^2\Delta}{d\delta^2}\,|_{F_i(\delta,\Delta)=0} = -\alpha(\Delta)u''(-a - \delta + \theta_i)$$
$$-\alpha(\Delta)^3\beta u'(-a - \delta + \theta_i)^2[\pi_1 u''(-a + \Delta + \theta_1) + \pi_2 u''(-a + \Delta + \theta_2)] > 0,$$

$$\frac{d^2\Delta}{d\delta^2}\,|_{G_i(\delta,\Delta)=0} = \gamma(-\Delta)u''(a + \delta + \theta_i)$$
$$+\gamma(-\Delta)^3\beta u'(a + \delta + \theta_i)^2[\pi_1 u''(a - \Delta + \theta_1) + \pi_2 u''(a - \Delta + \theta_2)] < 0,$$

$$\frac{d\Delta}{d\delta}\,|_{F_i(0,0)=0} = \alpha(0)u'(-a\theta_i),$$

$$\frac{d\Delta}{\delta}\,|_{G_i(0,0)=0} = \gamma(0)u'(a + \theta_i).$$

It is straightforward to verify that $\alpha(0)u'(-a + \theta_i) > \gamma(0)u'(a + \theta_i)$.

[10]Note that the arrows in the figure point to the directions that are consistent with the inequalities from (1.14) to (1.17).

Obviously then, for all $(\delta, \Delta) \in \Omega - \{A, E\}$, $\bar{\sigma}(\delta, \Delta)$ strictly dominates autarky. Q.E.D.

The rest of this section is devoted to looking at the long-run behavior of the two agents' expected utilities. To motivate our result, note that a central proposition in Green (1987) is that the long-run distribution of expected utilities across agents is degenerate: for each individual agent in the population, his expected utility converges to negative infinity with probability one. Green assumes that his agents have an exponential utility function. Thomas and Worrall (1990) in their single-agent model show that for a family of utility functions including the exponential one, the agent's expected utility also converges to the negative infinity with probability one. In the examples that Atkeson and Lucas (1992) solve, the same result is replicated. This, however, can obviously not be the case here, due to the different modelling strategy we adopt. First, here by assuming that a feasible contract never takes away from any agent more than what he receives, the two agents in our model will never consume a negative amount of the endowment. This implies that their expected utilities are essentially bounded from below, although the two agents here may have the same unbounded utility function as the agents in for example Green (1987) have. This certainly rules out possibilities for the expected utilities of the two agents to converge to negative infinity. Further, given that in our model the two agents are identical and they are constrained to consume the entire aggregate endowment each date, it is also unlikely that their expected utilities will converge to the minimum in the expected utility possibilities with probability one. Our aim in the rest of the section is to show that the expected utility of each agent actually converges to every expected utility, including the minimum, in Φ_V with probability zero. This will guarantee that the long-run distributions of expected utilities of the two agents are not degenerate.

Let $V_0 \in \Phi_V$ be any arbitrary *ex ante* expected utility that the constrained efficient contract would promise to agent 2. Let V_t ($t \geq 1$) be the random variable representing the expected utility to which agent 2 is entitled at the end of date t.

Theorem 5 Prob$\{\lim_{t\to\infty} V_t = V\} = 0$, for all $V \in \Phi_V$.

Let $\{v_t\}_{t=1}^{\infty}$ [11] be any time series (or path) of agent 2's expected utilities that has the following property

$$\lim_{t\to\infty} v_t = V. \tag{1.18}$$

For each v_t, let $\{\sigma^*(i,j)(v_t), V^*(i,j)(v_t)\}_{(i,j)\in\Theta^2}$ be the one-step profile of the constrained efficient contract at the state $V = v_t$. We show for the first step of the proof that either

$$\lim_{t\to\infty} V^*(1,n)(v_t) \neq V, \tag{1.19}$$

or

$$\lim_{t\to\infty} V^*(n,1)(v_t) \neq V. \tag{1.20}$$

must be true. Suppose not and $\lim_{t\to\infty} V^*(1,n)(v_t) = \lim_{t\to\infty} V^*(n,1)(v_t) = V$. Then since

$$V^*(n,1)(v_t) \leq V^*(i,j)(v_t) \leq V^*(1,n)(v_t), \;\; \forall (i,j) \in \Theta^2,$$

for all v_t, it is immediate that

$$\lim_{t\to\infty} V^*(i,j)(v_t) = V, \;\; \forall (i,j) \in \Theta^2. \tag{1.21}$$

Apply this to the incentive constraints for agent 2 in the Bellman equation to yield

$$\lim_{t\to\infty} [\sigma^*(i,j)(v_t) - \sigma^*(i',j')(v_t)] = 0, \;\; \forall (i,j),(i',j') \in \Theta^2.$$

[11]Note that as is standard we use the upper case for the random variable and the lower case for its realization.

Since for each $(i,j) \in \Theta^2$, the sequence $\{\sigma^*(i,j)(v_t)\}_{t=1}^{\infty}$ is bounded and hence contains a convergent subsequence. For convenience we assume that this subsequence is the sequence itself. We therefore can write:

$$\lim_{t\to\infty} \sigma^*(i,j)(v_t) = a, \quad \forall (i,j) \in \Theta^2, \tag{1.22}$$

where a is some constant in $[-\theta_1, \theta_1]$. Now notice that for each v_t,

$$v_t = \sum_{(i,j)\in\Theta^2} \pi_i \pi_j [u(\sigma^*(i,j)(v_t) + \theta_j) + \beta V^*(i,j)(v_t)].$$

Let $t \to \infty$ and due to (1.21) and (1.22), the above will yield:

$$V = \frac{1}{1-\beta} \sum_{j\in\Theta} \pi_j u(a + \theta_j).$$

This implies that the contract σ^a where $\sigma_t^a = a$ for all t is efficient, contradicting Theorem 4. Therefore, either (1.19) or (1.20) must be true.

We now proceed with the second step of the proof. Suppose (1.19) is true. Define two subsequences $\{x_q\}$ and $\{y_q\}$ of $\{v_t\}$ be such that

$$x_q = V^*(1,n)(y_q), \ \forall q. \tag{1.23}$$

We show that $\{x_q\}$ can not contain infinitely many elements. Suppose the contrary, then due to (1.18),

$$\lim_{q\to\infty} x_q = \lim_{q\to\infty} y_q = V.$$

However, $\lim_{q\to\infty} y_q = V$ and (1.23) together would imply $\lim_{q\to\infty} x_q \neq V$, due to (1.19).[12] This is a contradiction. Therefore $\{x_q\}$ can have at most finitely many elements and hence the path $\{v_t\}$ allows only finitely many $(1,n)$ to occur. Such paths have a measure zero. Supposing (1.20) is true will lead us to the same conclusion. Q.E.D.

[12] Remember that in (1.19) $\{v_t\}$ is any arbitrary sequence converging to V_{min}.

1.6 Computing the Efficient Contract: Algorithm

As our discussion in Section 1.4 indicates, the complex nature of our Bellman equation makes it infeasible for us to solve analytically for an efficient contract. In order to obtain more quantitative insight into the structure of an efficient contract, in this and the next sections, we turn to pursue a computational approach. As a first step, we explore in this section two related algorithms, both in the spirit of self-generation, for numerical computation of an efficient contract.

Following our analysis in Section 1.4, the key to solving for an efficient contract is to solve for Φ, the set of admissible expected utility pairs. Once Φ is obtained, then the set of admissible states, i.e., the set of admissible expected utilities of agent 2, Φ_V, and the value function of the Bellman equation, $U^*(V), V \in \Phi_V$, are readily computed. Finally, solving the Bellman equation given Φ_V and $U^*(V)$ will yield the efficient trading scheme $\{\sigma^*(i,j)(V)\}$ and the optimal law of motion of the state variable $\{V^*(i,j)(V)\}$.

The following Lemma, which is in the spirit of Abreu-Pearce-Stacchetti (1986,1990), provides an algorithm of solving for Φ through an iterative procedure. Basically, starting with a set $W_0 \subseteq R^2$ which is large enough, and operating on it iteratively using the operator B, we will then obtain a monotone sequence of sets converging to Φ.

<u>Lemma 8</u> Let W_0 be the space on which (U,V), the pair of expected utilities of the two agents, are allowed to take values, and assume $B(W_0) \subseteq W_0$. Let $W_{t+1} = B(W_t)$, $\forall t \geq 0$. Then $\{W_t\}$ is monotone decreasing and $\lim_{t\to\infty} W_t = W_\infty = \Phi$.

Proof. $\{W_t\}$ is monotone decreasing because B is monotonic and $B(W_0) \subseteq W_0$. To prove the limiting result, first, we show that the sequence $\{W_t\}$ converges. It is obvious that $B(W_0) \subseteq W_0$. Now operate B repeatedly on both sides of this expression to yield: $W_{t+1} = B(W_t) \subseteq W_t$, for all $t \geq 0$, as the operator B is monotone increasing. Therefore $\{W_t\}$ is a bounded and monotone decreasing set sequence. It converges and in fact $W_\infty = \lim_{t\to\infty} W_t = \bigcap_{t=0}^{\infty} W_t$. Second, we show that $\Phi \subseteq W_\infty$. Obviously, $\Phi \subseteq W_0$. Monotonicity of B implies $B(\Phi) \subseteq B(W_0)$. But $B(\Phi) = \Phi$ by Theorem 1 and $B(W_0) = W_1$ by construction, we thus have: $\Phi \subseteq W_1$. Iterate the above procedure to obtain: $\Phi \subseteq W_t$, for all $t \geq 0$. Therefore $\Phi \subseteq W_\infty$. Third, we show that $W_\infty \subseteq \Phi$. By the construction and convergence of $\{W_t\}$, $B(W_\infty) = W_\infty$. Therefore W_∞ is self-generating. By Lemma 1 then, $W_\infty = B(W_\infty) \subseteq \Phi$. Q.E.D.

Here a natural candicate for W_0 is $[\underline{\alpha}, \overline{\alpha}] \times [\underline{\alpha}, \overline{\alpha}]$, where $\underline{\alpha} = -1/(1-\beta)$, and $\overline{\alpha} = \sum \pi_i \pi_j u(\theta_i + \theta_j)/(1-\beta)$. That is, $\underline{\alpha}$ is the expected life-time utility of the agent if he consumes zero units of the consumption good at every date, and $\overline{\alpha}$ is the expected life-time utility of the agent if he consumes the sum of the endowments of the two agents at every date. Of course any set in R^2 that contains $[\underline{\alpha}, \overline{\alpha}] \times [\underline{\alpha}, \overline{\alpha}]$ will also do the job.

To numerically implement the above algorithm, W_0 is not allowed to take on continuous values. We can assume that W_0 contains N^2 grid points uniformly distributed over the space $[\underline{\alpha}, \overline{\alpha}] \times [\underline{\alpha}, \overline{\alpha}]$. That is, $W_0 = \{(U_p, V_q), p, q = 1, 2, ..., N\}$, where $U_p = V_p = \underline{\alpha} + \frac{p-1}{N-1}(\overline{\alpha} - \underline{\alpha})$, $p = 1, 2, ..., N$. To obtain $W_1 = B(W_0)$, we are essentially searching over W_0 for all the (U_p, V_q)s where there exist $[(U(i,j), V(i,j)) \in W_0]_{(i,j)\in\Theta^2}$, and $[\sigma(i,j) \in R]_{(i,j)\in\Theta^2}$, such that they satisfy conditions (1.7) through (1.9) and

$$U_p = \sum \pi_i \pi_j [u(-\sigma(i,j) + \theta_i) + \beta U(i,j)], V_q = \sum \pi_i \pi_j [u(\sigma(i,j) + \theta_j) + \beta V(i,j)]$$

Since now we are dealing with a finite space of possible expected utilities, convergence of the sequence $\{W_t\}$ will occur after a finite number of iterations and a solution to the Bellman equation is guaranteed.

A deficiency of the above algorithm however is that the amount of computation that it requires to reach the solution can be large. At any $(t+1)$th iteration, given the nature of bilateral incentive compatibility, and that the space W_t over which we search for admissible expected utilities is two-dimensional, a large number of nonliear programming problems must be solved. To reduce computation, we now proceed to develop an alternative algorithm for determining the efficient contract. The idea here is to compute the value function of the Bellman equation directly, without having to keep track of the whole set of admissible expected utilities Φ, as we do in Lemma 8.

Instead of gridding the whole space on which the pair of expected utilities (U, V) takes on values, we now assume that only V is restricted to take values on a discrete space containing N grid points $\{V(K): K = 1, 2, ..., N\}$, which are distributed over the interval $[\underline{\alpha}, \overline{\alpha}]$. But for each V, U is allowed to take continuous values on the closed interval $[\underline{\alpha}, \overline{\alpha}]$. The following lemma lays out an alg thm for computing an efficient contract under these assumptions.

<u>Lemma 9</u> Let $\Phi_V^0 = \{V(K) : K = 1, 2, ..., N\}$. Let $U_{min}^0(V) = \underline{\alpha}$, and $U_{max}^0(V) = \overline{\alpha}$, $\forall V \in \Phi_V^0$. Let $\Phi_0 = \{(U, V) | U \in [U_{min}^0(V), U_{max}^0(V)],\ \ V \in \Phi_V^0\}$. For $t \geq 0$ and $V \in \Phi_V^t$, let

$$S_{t+1}(V) = \{\sum \pi_i \pi_j [u(-\sigma(i,j) + \theta_i) + \beta U(i,j)]\},$$

where $[\sigma(i,j), U(i,j)]_{(i,j)\in\Theta^2}$ is such that there exists $[V(i,j)]_{(i,j)\in\Theta^2}$ so that $[\sigma(i,j), U(i,j), V(i,j)]_{(i,j)\in\Theta^2}$ is admissible to Φ_t and satisfies equation (1.13). Let

$$\Phi_{t+1} = \bar{B}(\Phi_t) = \{(U, V) \in \Phi_t | V \in \Phi_V^t,\ S_{t+1}(V) \neq \emptyset,\ U \in [U_{min}^{t+1}(V), U_{max}^{t+1}(V)]\},$$

where $U_{max}^{t+1}(V)$ and $U_{min}^{t+1}(V)$ respectively are the maximum and minimum values in the set $S_{t+1}(V)$. Then, let

$$\Phi_V^{t+1} = \{V \in \Phi_V^t :\ \exists\, U \ s.t.\ (U, V) \in \Phi_{t+1}\}.$$

Let $\Phi_V^\infty = \lim_{t\to\infty} \Phi_V^t$, and $U_{max}^\infty(V) = \lim_{t\to\infty} U_{max}^t(V), \forall V \in \Phi_V^\infty$. If $T(U_{max}^\infty)(V) = U_{max}^\infty(V)$, $\forall V \in \Phi_V^\infty$, then $\Phi_V = \Phi_V^\infty$, and $U^*(V) = U_{max}^\infty(V), \forall V \in \Phi_V$.

Proof. We begin the proof by noticing several simple facts. First, $\bar{B}$ is monotonic and $\bar{B}(\Phi_0) \subseteq \Phi_0$. Second, $B(\Psi) \subseteq \bar{B}(\Psi)$, $\forall \Psi \in R^2$. Third, the two set sequences $\{\Phi_t\}$ and $\{\Phi_V^t\}$ are monotone decreasing. Fourth, for all $V \in \Phi_V^\infty$, $\{U_{max}^t(V)\}$ is monotone decreasing but $\{U_{min}^t(V)\}$ is monotone increasing.

Given the conditions, we have: $\{(U_{max}^\infty(V), V) : V \in \Phi_V^\infty\} \subseteq \Phi$, and it then follows immediately that $\Phi_V^\infty \subseteq \Phi_V$, and $U_{max}^\infty(V) \leq U^*(V), \forall V \in \Phi_V^\infty$.

Now let $\{W_t\}$ be the monotone sequence of sets generated by operating B (rather than $\bar{B}$) iteratively on Φ_0. Then

$$\Phi \subseteq B(\Phi) \subseteq B(\Phi_0) = W_1 \subseteq \bar{B}(\Phi_0) = \Phi_1.$$

By induction it can be shown that in general,

$$\Phi \subseteq W_t \subseteq \Phi_t, \forall t \geq 0.$$

But this implies that $\Phi_V \subseteq \Phi_V^\infty$ and $U^*(V) \leq U_{max}^\infty(V), \forall V \in \Phi_V$. Q.E.D.

Notice that unlike in Lemma 8 where the whole set of admissible expected utilities Φ is computed, Lemma 9 computes the value function $U^*(V), V \in \Phi_V$, or the efficient expected utility pairs directly. Lemma 9 therefore is able to reduce considerably the amount of computation that is required to obtain an efficient contract. Here, allowing U to take continuous values leads to a large deduction in the amount of search over the grid points. At any $(t+1)$th iteration, we only search for admissible expected utilities of agent 2 along a single dimensional space Φ_V^t, whereas in Lemma 8, we were searching over a two-dimensional space W_t for admissible expected utilitiy pairs.

As is commonplace in dynamic programming problems where the mapping that defines the Bellman equation is not a contraction and uniqueness of a fixed point of this mapping is not guaranteed, Lemma 9 as an algorithm only works when a certain requirement is satisfied. Here the key requirement is that the function U_{max}^{∞} must in fact be a fixed point of operator T. Obviously, a sufficient condition for this to hold is that the operator B preserves convexity, in the sense that for all t and V, $(U_1, V) \in W_t$ and $(U_2, V) \in W_t$ together will imply $(\alpha U_1 + (1-\alpha)U_2, V) \in W_t$, $\forall \alpha \in (0,1)$. In this case, B and $\bar{B}$ will essentially be equivalent. Finally, we note that although the condition that U_{max}^{∞} is a fixed point of T may be hard to verify analytically, it is straightforward to check computationally after the convergence occurs.

1.7 Computing the Efficient Contract: An Example

This section is devoted to looking at a parameterized example of our model that we solve numerically using the algorithm provided by Lemma 9. We will examine quantitatively the structure of the efficient contract. We will also look at the consumption and wealth time series that the efficient contract generates.

In the example, utility is exponential, i.e., $u(c) = -exp(-c)$. Exponential utility not only has the property of being well-defined on the whole real line, but also is computationally easy to handle. Assume $\beta = 0.96$. Assume the endowment can be either low or high: $\theta_1 = 0.2$ and $\theta_2 = 0.4$. The low and high endowments are received by each agent with equal probabilities: $\pi_1 = \pi_2 = 0.5$. Assume that the expected utility of agent 2 can only take values on a finite set that contains one hundred grid points, $\{V(1), ..., V(100)\}$, which are uniformly distributed over the interval $[\underline{\alpha}, \overline{\alpha}]$, where $\underline{\alpha}$ and $\overline{\alpha}$ are calculated accordingly using the formulas given in the previous section.

We find, for the efficient contract, $\Phi_V = \{V(22), ..., V(84)\}$. That is, any expected utility which is below $V(22)$ or above $V(84)$ is not achievable. The value function $U^*(V)$, $V \in \Phi_V$ is depicted in Figure 1.2. Notice that $U^*(V)$ is concave and monotone decreasing. Figure 1.3 is a graph of the efficient trading scheme, where $a(i,j)(K)$ stands for $\sigma^*(i,j)(V(K))$. Notice that $a(1,2)(K) < a(i,i)(K) < a(2,1)(K), i = 1,2$. Remember that this is the property we prove theoretically in Lemma 1, which describes the impact of current endowments on current trades. Also notice that $a(i,j)(K)$ are monotone increasing in K, which means that $\sigma^*(i,j)(V)$, and hence $\sigma^*(i,j)(V) + \theta_j$, which is the current consumption of agent2, tends to increase as V increases. That is, agent 2 will receive more transfer of the consumption good from agent 1 and hence consume more currently, as his wealth accumulates. Similarly, agent 1's current consumption $-\sigma(i,j)(V) + \theta_i$ decreases as V increases. Note that this is how history affects current consumption.

The optimal law of motion of the state variable is as follows:

$$\begin{aligned} V^*(1,1)(V(K)) &= V(K), K = 22, ..., 84. \\ V^*(1,2)(V(K)) &= V(K+1), K = 22, ..., 83; \; V^*(1,2)(V(84)) = V(84). \\ V^*(2,1)(V(K)) &= V(K-1), K = 23, ..., 84; \; V^*(2,1)(V(22)) = V(22). \\ V^*(2,2)(V(K)) &= V(K), K = 22, ..., 84. \end{aligned}$$

Notice that for all (i,j), $V^*(i,j)(V(K))$ is monotone increasing in K. Note that this is how history affects future wealth: for given current endowment realizations, the agent will be in a better wealth position tomorrow if he is in a better wealth position today.

More important, the above law of motion of the state variable indicates that the time series of the expected utilities of each agent form a random walk with reflecting barriers. For agent 2, the barriers here are the lowest state $V(22)$ and the highest state $V(84)$, respectively. Between these barriers, if the two agents receive the same endowment, then the state does ot change. If agent 2 receives less than agent 1 does, then the state moves upward, i.e., the expected utility of agent 2 falls but that of agent 1 rises. Similarly, if agent 2 receives more endowment than agent 1 does, then the state moves downward, or the expected utility of agent 2 rises and that of agent 1 falls. The barriers are reflecting because $V^*(1,2)(V(22)) = V(23)$, and $V^*(2,1)(V(84)) = V(83)$. That is, if the state hits either of the barriers, then at each date the state will bounce back with positive probabilities.

This has two implications. First, the expected utility process of each agent will have a unique invariant distribution on the ergodic set Φ_V. Second, the consumption process of each agent will form a Markov chain with stationary transitions, and it also has an unique invariant distribution on an ergodic set. For agent 1, this ergodic set is $\{-\sigma^*(i,j)(V)+\theta_i : (i,j) \in \Theta^2, V \in \Phi_V\}$, and for agent 2, $\{\sigma^*(i,j)(V)+\theta_j : (i,j) \in \Theta^2, V \in \Phi_V\}$. We emphasize that these results are in contrast to the ones obtained by Green (1987), Phelan and Townsend (1991), and Atkeson and Lucas (1992), where inequality in consumption and wealth across agents grows over time without bound.

To illustrate graphically the processes of consumptions and expected utilities, Figure 1.4 plots an example of the expected utility paths of the two agents who start with almost the same *ex ante* expected life-time utilities over a period of 400 dates. Notice that although the two agents have ergodic long-run distributions in expected utilities, their wealth positions may still fan out temporarily. Finally, Figure 1.5 and Figure 1.6 plot respectively the associated consumption paths of the two agents. Notice the persistence in consumption that shows up in these two figures.

1.8 Concluding Remarks

This paper studies a simple model of dynamic insurance under private information in a pure exchange economy. There are two infinitely-lived agents in our model, both risk-averse, and each having an i.i.d. stochastic endowment stream which is unobservable to the other. Incentive compatibility in the Nash sense is studied. We give sufficient and necessary conditions for the existence of a constrained efficient contract. We show that a constrained efficient contract can be characterized in a Bellman equation. Algorithms for numerical computation of an efficient contract are discussed and an example with exponential utility is computed. Among other things, our computational results show that the wealth position of each agent follows a random walk with reflecting barriers.

Our model here is simple and restricted. For example, there are only two agents in our model. One natural extension of our model is to allow for multiple agents, and it is clear that the technical approach developed here is able to be modified to confront this situation. Specifically, in the case of N agents, an efficient contract can be defined as one which maximizes the expected utilty of the Nth agent, subject to delivering a given vector of expected utilities to the rest $N-1$ agents. For the Bellman equation, $N-1$ state variables, each corresponding to the expected utilities of the $N-1$ agents, will need to be defined.

Other extensions of the model are also possible. For example, the only consumption good here is perishable, it will be interesting to see how savings can be determined in a bilateral trading context by allowing for storage in our model. Of course it is also important to understand to what extent the efficient allocations in our model can be achieved in a decentralized environment with price-taking traders.

Finally, we note that although the model studied in this paper is very restricted, it does capture some of the key features that exist in certain real world economic relationships. For instance, labor contracts often are long-term contracts and it is likely that there is bilateral asymmetric information between the manager and the labor. The two agents in our model may also be viewed as two countries in an international trading partnership where each country has private information about its own aggregate shocks. It is also our speculation that the approach developed here is applicable to dynamic macro or money models with bilateral tradings and asymmetric information.

Appendix

Proof of Lemma 1 Fix t and g^{t-1}. Fix $(i,j) \in \Theta^2$. Let $l < j$. Manipulate agent 2's incentive compatibility constraints to get

$$u(\sigma_t(g^{t-1},(i,j)) + \theta_j) - u(\sigma_t(g^{t-1},(i,l)) + \theta_j) \geq \beta[V(\sigma|g^{t-1},(i,l)) - V(\sigma|g^{t-1},(i,j))],$$

$$u(\sigma_t(g^{t-1},(i,l)) + \theta_l) - u(\sigma_t(g^{t-1},(i,j)) + \theta_l) \geq \beta[V(\sigma|g^{t-1},(i,j)) - V(\sigma|g^{t-1},(i,l))].$$

Adding these two inequalities yields:

$$u(\sigma_t(g^{t-1},(i,j)) + \theta_j) - u(\sigma_t(g^{t-1},(i,l)) + \theta_j)$$
$$\geq u(\sigma_t(g^{t-1},(i,j)) + \theta_l) - u(\sigma_t(g^{t-1},(i,l)) + \theta_l).$$

Define fuuction $f : R_+ \rightarrow R$ as follows:

$$f(\theta) = u(\sigma_t(g^{t-1},(i,j)) + \theta) - u(\sigma_t(g^{t-1},(i,l)) + \theta).$$

Then we have: $f(\theta_j) \geq f(\theta_l)$. Suppose, by way of contradiction, that $\sigma_t(g^{t-1},(i,j)) > \sigma_t(g^{t-1},(i,l))$, then

$$f'(\theta) = u'(\sigma_t(g^{t-1},(i,j)) + \theta) - u'(\sigma_t(g^{t-1},(i,l) + \theta) < 0.$$

Since $\theta_l < \theta_j$, $f(\theta_l) > f(\theta_j)$, we have a contradiction. Therefore $\sigma_t(g^{t-1},(i,j)) \leq \sigma_t(g^{t-1},(i,l))$ must be the case. Applying this result to the first inequality in this proof, it is immediate that $V(\sigma|g^{t-1},(i,l)) \leq V(\sigma|g^{t-1},(i,j))$. In almost the same way the remaining parts of the lemma can be shown to be true. Q.E.D.

Proof of Lemma 2 We prove by way of contradiction. Let σ be temporarily incentive compatible at all possible nodes. Suppose, for some $\tau \geq 0$ and $g^\tau \in H^t$, there exists for agent 1 a reporting strategy r^1 covering dates from $\tau + 1$ and after such that

$$U(\sigma|g^\tau; r^1, r^{2*}) > U(\sigma|g^\tau; r^{1*}, r^{2*}).$$

Define a sequence of reporting strategies $(r^{1n})_{n=\tau+1}^{\infty}$ as follows. $r_t^{1n} = e_t^1$, for all $t > n$; and $r_t^{1n} = r_t^1$, for all $t \leq n$. Since utilities are bounded, it is immediate that

$$\lim_{n\to\infty} U(\sigma|g^\tau; r^{1n}, r^{2*}) = U(\sigma|g^\tau; r^1, r^{2*}).$$

Therefore, there exists N large enough such that

$$U(\sigma|g^\tau; r^{1N}, r^{2*}) > U(\sigma|g^\tau; r^{1*}, r^{2*}).$$

For convenience, let N be the minimum among such integers. N is finite. Also, date N is the last date at which agent 1 would misrepresent if the reporting strategy r^{1N} is adopted. Now consider a deviation from the reporting strategy r^{1N}, denoted by $\bar{r}^{1N}$, such that $\bar{r}_t^{1N} = e_t^1$, if $t = N$; and $\bar{r}_t^{1N} = r_t^{1N}$, if $t \neq N$. But σ is temporarily incentive compatible at date $(t-1)$, we have

$$U(\sigma|g^\tau; \bar{r}^{1N}, r^{2*}) \geq U(\sigma|g^\tau; r^{1N}, r^{2*}).$$

This is a contradiction since $\bar{r}^{1N}$ involves strictly less misrepresentations than r^{1N} does. Therefore, the assumed reporting strategy r^1 can not exist and hence (1.1) must hold. In a similar way, (1.2) can also be shown to hold. Q.E.D.

Proof of Lemma 3 Let Ψ be self-generating and let $\psi(h^0) = (\psi^1(h^0),\ \psi^2(h^0)) \in B(\Psi)$. We need to show that $\psi(h^0) \in \Phi$, i.e., there exists a feasible and incentive compatible coinsurance contract $\sigma(\psi(h^0))$ such that $(U,V)(\sigma(\psi(h^0))) = \psi(h^0)$.

We start by constructing the contract $\sigma(\psi(h^0))$. By the definition of $B(\Psi)$, there exists a pair $(\mathcal{C}\ (\psi(h^0)), \mathcal{U}(\psi(h^0)))$, where $\mathcal{C}(\psi(h^0)) = [\sigma(\psi(h^0))(i,j)]_{(i,j)\in\Theta^2}$, admissible with respect to Ψ such that

$$E(\mathcal{C}(\psi(h^0)), \mathcal{U}(\psi(h^0))) = \psi(h^0).$$

Define, for all $(i,j) \in \Theta^2$, that

$$\sigma_1(h^0,(i,j)) = \sigma(\psi(h^0))(i,j).$$

Then for any date 1 reported realization of endowments, say (i,j), let

$$\psi(h^1) = \psi(h^0,(i,j)) = \mathcal{U}(\psi(h^0))(i,j) \in \Psi \subseteq B(\Psi).$$

Where the " $\in$ " is due to the fact that $\mathcal{U}(\psi(h^0))$ is a selection from Ψ and the " $\subseteq$ " is due to the fact that Ψ is self-generating.

Now for $\psi(h^1) \in B(\Psi)$ instead of $\psi(h^0) \in B(\Psi)$, follow the above procedure to obtain $\sigma_2(h^2)$ and $\psi(h^2)$. Repeat this for all t to obtain:

$$\begin{aligned} \sigma(\psi(h^0)) &= \{\sigma_1(h^1), \sigma_2(h^2), ..., \sigma_t(h^t), ...\}, \\ S(\psi(h^0)) &= \{\psi(h^0), \psi(h^1), ..., \psi(h^t), ...\}. \end{aligned}$$

We now demonstrate that $S(\psi(h^0))$ is the sequence of expected utility vectors that the contract $\sigma(\psi(h^0))$ will generate for the two agents. Precisely,

$$(U,V)(\sigma(\psi(h^0)) \mid h^t) = \psi(h^t), \quad t = 0,1,2,.... \tag{1.24}$$

Note that if the above equation is indeed true, then it is easy to perceive that the coinsurance contract $\sigma(\psi(h^0))$ is feasible, incentive compatible and gives the two agents the expected utility vector $\psi(h^0)$, as is desired. To show that (1.24) is true, observe that a simple fact from the above recursive construction of $\sigma(\psi(h^0))$ is:

$$\sigma(\psi(h^0)) \mid h^t = \sigma(\psi(h^t)), \quad t = 0,1,2....$$

Use this relationship to write:

$$\begin{aligned} U(\sigma(\psi(h^0))) &= \sum \pi_i\pi_j[u(-\sigma_1(h^0,(i,j)) + \theta_i) + \beta U(\sigma(\psi(h^0)) \mid h^0,(i,j))] \\ &= \sum \pi_i\pi_j[u(-\sigma_1(h^0,(i,j)) + \theta_i) + \beta U(\sigma(\psi(h^0,(i,j))))]. \end{aligned}$$

On the other hand, by the construction of $\psi(h^0)$, we have:

$$\begin{aligned} \psi^1(h^0) &= \sum \pi_i\pi_j[u(-\sigma(\psi(h^0))(i,j) + \theta_i) + \beta U(\psi(h^0))(i,j))] \\ &= \sum \pi_i\pi_j[u(-\sigma_1(h^0,(i,j)) + \theta_i) + \beta\psi^1(h^0,(i,j))]. \end{aligned}$$

Therefore,

$$\begin{aligned} \mid \psi^1(h^0) - U(\sigma(\psi(h^0)) \mid &\leq \beta \sum \pi_i \pi_j \mid \psi^1(h^0,(i,j)) - U(\sigma(\psi(h^0,(i,j)))) \mid \\ &\leq \beta \sup_{(i,j)\in\Theta^2} \mid \psi^1(h^0,(i,j)) - U(\sigma(\psi(h^0,(i,j)))) \mid \\ &\dots \\ &\leq \beta^t \sup_{(i,j)\in\Theta^2} \mid \psi^1(h^{t-1},(i,j)) - U(\sigma(\psi(h^{t-1},(i,j)))) \mid . \end{aligned}$$

Note that the above is true for all $t \geq 1$ and all $h^{t-1} \in H^{t-1}$. Now let $t \to \infty$. Since $0 < \beta < 1$ and utilities are bounded, it is immediate that $\psi^t(h^0) = U(\sigma(\psi)(h^t))$, $\forall t \geq 0$. Therefore half of (1.24) is proven. In the same way we can show the other half to be true. Q.E.D.

Proof of Lemma 4 Let $\phi = (\phi^1, \phi^2) \in \Phi$. We need to show that $\phi \in B(\Phi)$. By definition of Φ, there exists a coinsurance contract $\sigma(\phi)$ such that $(U,V)(\sigma(\phi)) = \phi$, or equally,

$$\begin{aligned} \phi^1 &= \sum \pi_i \pi_j [u(-\sigma_1(\phi)(h^0,(i,j) + \theta_i) + \beta U(\sigma(\phi) \mid h^0,(i,j))], \\ \phi^2 &= \sum \pi_i \pi_j [u(\sigma_1(\phi)(h^0,(i,j) + \theta_j) + \beta V(\sigma(\phi) \mid h^0,(i,j))]. \end{aligned}$$

Define a pair $(\mathcal{C},\mathcal{U})(\phi)$ such that for all $(i,j) \in \Theta^2$,

$$\begin{aligned} \sigma(\phi)(i,j) &= \sigma_1(\phi)(h^0,(i,j)), \\ U(\phi)(i,j) &= U(\sigma(\phi)|h^0,(i,j)), \\ V(\phi)(i,j) &= V(\sigma(\phi)|h^0,(i,j)). \end{aligned}$$

It is then obvious that $(\mathcal{C},\mathcal{U})(\phi)$ thus constructed is feasible and incentive compatible in the one-step sense defined by constraints (1.7), (1.8) and (1.9), and also gives the vector of expected utilities (ϕ^1, ϕ^2) to the two agents. Q.E.D.

Proof of Theorem 1 $\Phi = B(\Phi)$ is an immediate consequence of Lemma 3 and Lemma 4. The proof for the compactness of Φ takes two steps. We first show that if Ψ is closed, then $B(\Psi)$ is closed, too. In other words, B preserves closedness. Let Ψ be closed and let sequence $\{\psi_n\} \subseteq B(\Psi)$ be such that $\psi_n \rightarrow \psi$, as $n \rightarrow \infty$. By the definition of $B(\Psi)$, there exists a sequence $\{(\mathcal{C}_n, \mathcal{U}_n)\}$ with each element admissible with respect to Ψ such that $E(\mathcal{C}_n, \mathcal{U}_n) = \psi_n$, $\forall n$. Since the space of all admissible pairs with respect to Ψ is bounded, $\{(\mathcal{C}_n, \mathcal{U}_n)\}$ has a convergent subsequence $(\mathcal{C}_{n_q}, \mathcal{U}_{n_q}) \rightarrow (\mathcal{C}, \mathcal{U})$, *as* $q \rightarrow \infty$. But $E(\mathcal{C}, \mathcal{U})$ is continuous in $(\mathcal{C}, \mathcal{U})$, we have $E(\mathcal{C}, \mathcal{U}) = \lim_{q \rightarrow \infty} E(\mathcal{C}_{n_q}, \mathcal{U}_{n_q}) = \lim_{n \rightarrow \infty} \psi_n = \psi$. Left to be shown are: (a) $\mathcal{U}$ is a continuation value function with respect to Ψ; and (b) $(\mathcal{C}, \mathcal{U})$ satisfies equations (7) through (9). To show (a), simply notice that since $\mathcal{U}_{n_q}(i,j) \in \Psi$, $\forall (i,j) \in \Theta^2$, and Ψ is closed, we have $\mathcal{U}(i,j) = \lim_{q \rightarrow \infty} \mathcal{U}_{n_q}(i,j)$, $\forall (i,j) \in \Theta^2$. (b) is obvious. Therefore we have shown that $\psi \in B(\Psi)$, and hence $B(\Psi)$ is closed.

Now we can proceed with the second step of the proof. We need only show that Φ is closed since it is certainly bounded. Let $\overline{\Phi}$ be the closure of Φ. By definition, $\Phi \subseteq \overline{\Phi}$. Since the operator B is monotone increasing, we thus have $B(\Phi) \subseteq B(\overline{\Phi})$. But $B(\Phi) = \Phi$, therefore $\Phi \subseteq B(\overline{\Phi})$. Now since $\overline{\Phi}$ is closed, by the result of the first step of the proof then, $B(\overline{\Phi})$ is also closed. However, since $\overline{\Phi}$ is the smallest closed set containing Φ, it must be the case that $\overline{\Phi} \subseteq B(\overline{\Phi})$, that is, $\overline{\Phi}$ is self-generating. Therefore by Lemma 3, $B(\overline{\Phi}) \subseteq \Phi$, implying $\overline{\Phi} \subseteq \Phi$. Hence we have shown that $\overline{\Phi} = \Phi$, or Φ is closed. Q.E.D.

Chapter 2

Adverse Selection In Credit Markets With Costly Screening

2.1 Introduction

In this paper, we develop a model where the costly screening of borrowers in a credit market with adverse selection yields debt contracts as an optimal financial arrangement. The model has at least two advantages over a widely used and tractable alternative model of debt contracts, the costly state verification model (though it would be difficult to argue that our model strictly dominates). First, debt contracts survive randomization here and, second, the frictions in our environment which imply the optimality of debt are closer to the frictions that play an important role in real-world credit markets. The model can be extended in straightforward ways to yield optimal intermediary structures, and it potentially has other useful applications.

There is now a large literature on optimal financial arrangements under private information. A primary aim of this literature has been to show how private information frictions give rise to the simple financial contracts and intermediary structures observed in practice. A widely used financial contracting setup is the costly state verification (CSV) model, first developed by Townsend (1979). In the CSV model, if attention is restricted to deterministic verification strategies, debt contracts arise as an optimal means for economizing on verification costs, and these costs can then be interpreted as costs of bankruptcy. The CSV model has been extended to study investment (Gale and Hellwig 1985), credit rationing (Williamson 1987a), financial intermediation (Williamson 1986), and macroeconomic issues (Bernanke and Gertler 1989, Williamson 1987b).

There are at least two problems with the CSV model as an explanation for debt contracts and financial intermedation. First, as Townsend (1979) and Mookherjee and Png (1989) have shown, debt contracts are not optimal if stochastic verification is permitted, and this result holds even if attention is restricted to environments where all agents are risk neutral (Border and Sobel 1987, Boyd and Smith 1993). Thus, the model loses some of its appeal if very restrictive assumptions are required to obtain the simple contracts observed in practice.[1] Second, as a vehicle for studying financial intermediary structures, the CSV model relies on delegated monitoring results (Diamond 1984), whereby intermediation economizes on ex post verification costs; an intermediary's depositors delegate verification to the intermediary. However, the costs that appear to be most important for real world financial intermediaries are not ex post verification costs (i.e. auditing costs) but ex ante costs of information acquisition. For banks, these costs are primarily associated with the screening of loan applicants.

In the model we construct, there are two periods and three types of agents, lenders, good borrowers, and bad borrowers. Each lender is endowed with an investment good in the first period; borrowers receive no endowment, but each has access to an indivisible investment project which takes the investment good as input in period one, and yields a random return in the second period. The investment projects of

[1] However, Boyd and Smith (1993) argue that the suboptimality arising from the use of deterministic auditing where random auditing would be optimal actually implies a very small welfare loss in practice. To the extent that we accept this argument, it dampens the force of our objection.

good and bad borrowers differ according to a property that is somewhat stronger than first-order stochastic dominance. Type is private information. In contrast to standard adverse selection models, such as Rothschild and Stiglitz (1977), where self selection is achieved with risk averse agents who have different preferences (typically, some version of the single-crossing property is necessary for self selection), all agents are risk neutral here. To obtain self selection, lenders must make use of a costly screening technology, which reveals a borrower's type at a cost.

We use an equilibrium concept similar to that studied by Rothschild and Stiglitz (1977). That is, in equilibrium there exists no contract that, given the equilibrium contracts, makes the agents who choose it better off while earning nonnegative expected profits. A contract is a payment schedule/screening probability pair. That is, if a borrower accepts a particular contract, she agrees to make contingent payments in period two as specified by the contract, and submits to random screening in period one. Potentially, there might be a pooling equilibrium, where each borrower is offered the same contract and there is no screening, or a separating equilibrium where payment schedules and screening probabilities are different for different types of borrowers.

We show that, if the equilibrium exists, it must be a separating equilibrium. Pooling equilibria do not exist here for much the same reason they do not exist in the Rothschild-Stiglitz model. That is, any pooling equilibrium involves an implicit subsidy of bad borrowers by good borrowers. A contract can always be offered which makes the good type better off while not attracting any bad types and earning nonnegative profits. The separating equilibrium has the property that good borrowers are screened with positive probability, while bad borrowers are not screened. Incentive constraints are binding for bad borrowers and do not bind for good borrowers. A central result is that the unique equilibrium separating contract for good borrowers is a debt contract. This contract is relatively unattractive for bad borrowers, and therefore permits self selection while minimizing the screening probability for good borrowers. Debt contracts are also optimal for bad borrowers, but there exists a continuum of contracts which do just as well.

The separating equilibrium exists provided that the cost of screening is sufficiently low and the fraction of bad borrowers in the population is sufficiently large. A feature of interest here is that, for *any* positive screening cost, if there are few enough good borrowers relative to bad borrowers in the population, then a separating equilibrium exists. Thus, if the adverse selection problem is sufficiently severe, the screening technology will be used no matter how large the screening cost. Some comparative statics results show that the screening probability decreases and loan interest rates increase as the cost of screening increases. That is, as screening becomes more costly, the screening technology is used less intensively, and the higher costs of lending result in an increase in market interest rates. Also, an increase in the risk-free interest rate faced by lenders leads to an increase in loan interest rates and in the screening probability. The screening probability increases in this case as higher interest rates tend to aggravate the adverse selection problem.

The model can be extended in a straightforward way to obtain financial intermediation with "delegated screening," analogous to the delegated monitoring results obtained with the CSV model (e.g. Williamson 1986). That is, if the investment projects of borrowers are large in scale relative to the endowment of an individual lender, then in general there will be replication of screening by the individual lenders who fund a particular borrower's project. This replication can be circumvented by a financial intermediary which exploits the law of large numbers, holding a perfectly diversified portfolio of loans and making noncontingent payments to its depositors.

Our model thus yields many results which are reminiscent of those obtained with costly state verification, in particular the optimality of debt contracts and intermediary structures. However, this environment is more appealing in that debt contracts survive the introduction of random screening strategies, and because ex ante screening costs appear to be a much more important component of intermediation costs than are ex post auditing costs. Also, the model permits applications to problems where adverse selection plays an important role in credit markets and financial intermediation. It would be difficult to argue, though, that our model dominates the CSV model in all respects. For example, we rely extensively on risk neutrality for our results. Also, to obtain debt contracts as an equilibrium, we require a monotonicity restriction on payment schedules (see also Innes 1990). This latter assumption can be justified, but only by appealing to features of the environment which are not completely worked out.

Related work that deals with the optimality of debt contracts in an environment different from that of the CSV model is Lacker (1992). Lacker appeals to differences in valuation of collateral by borrowers and lenders to obtain optimal debt contracts. A model with costly screening, which focuses on multiple equilibrium issues is in Gale (1992), and an adverse selection model with costly screening is studied in De Meza and Webb (1988). De Meza and Webb do not deal with the optimal contracting issues which are central to this paper.

The remainder of the paper is organized as follows. In Section 2.2 we construct the model, and then define an equilibrium in Section 2.3. We then characterize the separating equilibrium and pooling equilibrium in Sections 2.4 and 2.5, respectively. In Section 2.6, we study the existence of equilibrium and describe some comparative statics results. We show how the model can be extended to yield financial intermediary structures in Section 2.7, and derive conclusions in Section 2.8.

2.2 The Model

There are two periods, denoted 1 and 2, where investment takes place in period 1 and agents consume in period 2. There are three types of agents: lenders, type g borrowers, and type b borrowers. There is a continuum of borrowers and lenders, with the measure of borrowers being strictly less than the measure of lenders. Among the group of borrowers, a fraction α is type g, and the remaining fraction, 1 - α, is type b.

Each lender has one unit of an investment good in period 1, which can either go to a borrower in exchange for some promise to pay consumption in period 2, or it can earn a certain return of r units of consumption in period 2 for each unit invested in period 1, through a risk-free investment technology. Lenders maximize the expected value of $u(c, e) = c - e$, where c is consumption in period 2, and e is effort in screening borrowers in period 1.

Borrowers have no endowment in period 1 and maximize the expected value of period 2 consumption. Each borrower has access to an investment project which requires 1 unit of the investment good in period 1 to operate, and which yields a random quantity of the consumption good as output in period 2, if funded. If a borrower of type i funds her project, the return in period 2 is distributed according to the probability distribution function $F_i(\cdot)$, with the corresponding probability density function $f_i(\cdot)$. We assume that $f_i(x) > 0$ for $x \in [0,1]$, $f_i(x) = 0$ otherwise, that $f_i(x)$ is continuous on $[0,1]$, and that $\mu_i > r$ for $i = g, b$, where μ_i is the mean investment return faced by borrower i. We also assume

$$\frac{f_g(x)}{f_b(x)} < \frac{f_g(y)}{f_b(y)}; x, y \in [0,1]; x < y. \tag{2.1}$$

Condition (2.1) is essentially identical to the monotone likelihood ratio property often assumed in principal agent problems with moral hazard. In the appendix, we show that condition (2.1) implies that $F_g(x) < F_b(x)$ for all $x \in (0,1)$, that is $F_g(\cdot)$ dominates $F_b(\cdot)$ in terms of first-order stochastic dominance. Therefore, condition (2.1) is stronger than first-order stochastic dominance of $F_b(\cdot)$ by $F_g(\cdot)$. However, there are some probability distributions for which (2.1) is equivalent to first-order stochastic dominance. For example, if $F_g(x) = x^{\alpha}$ and $F_b(x) = x^{\beta}$, with $\alpha > \beta$, then $F_g(x) < F_b(x)$ for all $x \in (0,1)$ and condition (2.1) holds. Another example is the exponential case, where $F_g(x) = \frac{1-e^{-\alpha x}}{1-e^{-\alpha}}$ and $F_b(x) = \frac{1-e^{-\beta x}}{1-e^{-\beta}}$, with $\beta > \alpha > 0$.

If borrowers are not screened, then type is private information. However, each lender has access to a technology which allows the lender to observe a borrower's type by incurring a fixed cost γ, in units of effort, where $\gamma \geq 0$. We assume that a given borrower can contact at most one lender in period 1, but that a lender may contact as many borrowers as she likes. This is somewhat similar to the costly state verification technology studied by Townsend (1979), in that there exists a costly technology for revealing information which would otherwise be private. This environment differs, however, in that there is adverse selection rather than moral hazard (as in the costly state verification approach), and because here private information is revealed by the technology before investment occurs rather than after. Also, the return on a borrower's investment project is publicly observable here.

2.3 Equilibrium

Our equilibrium concept is similar to that in Rothschild and Stiglitz (1977). Equilibrium contracts written in period 1 between lenders and borrowers consist of payment schedule/screening probability pairs $[R_i(x), \pi_i], i = g, b, x \in [0,1]$, where $R_i(x)$ denotes the payment made by the borrower to the lender in the event that the return on the borrower's investment is x, and π_i denotes the probability that a lender uses the screening technology to reveal the type of a borrower who claims to be type i. We consider two types of contracts, separating contracts and pooling contracts. A separating contract is offered by a lender to a particular agent type, while a pooling contract is offered to all agent types. There are potentially two types of equilibria: separating equilibria where all contracts offered by lenders are separating contracts, and pooling equilibria, where all contracts are pooling contracts.

2.3.1 Separating Equilibrium

A separating equilibrium is a pair of contracts $[R_i(x), \pi_i], i = g, b$, which must satisfy the following conditions, in addition to some other conditions which will be discussed later.

$$0 \leq R_i(x) \leq x, x \in [0,1], i = g, b. \tag{2.2}$$

$$x \leq y \Rightarrow R_i(x) \leq R_i(y); x, y \in [0,1]; i = g, b. \tag{2.3}$$

$$\int_0^1 R_i(x)dF_i(x) \leq (1 - \pi_j) \int_0^1 R_j(x)dF_i(x) + \pi_j\mu_i; i, j = g, b. \tag{2.4}$$

$$\int_0^1 R_i(x)dF_i(x) \geq r + \pi_i\gamma, i = g, b. \tag{2.5}$$

Condition (2.2) states that the payment schedule must be feasible for each type. Condition (2.3) imposes monotonicity on the payment schedules. This monotonicity condition is also used by Innes (1990), and he provides several justifications for it that rely on an appeal to implicit features of the environment. Among these justifications are the following. First, suppose that a loan market opens in period 2 after borrowers observe their output realization, but before lenders do. Loans are repaid at the end of the period at a zero interest rate, and these transactions are unobserved by lenders. Then, if the payment schedule for a given type is decreasing over some range, it is in the interest of that agent type to borrow temporarily in period 2 in some states of the world in order to fake a higher quantity of output. Thus, the payment schedule in this case is effectively monotonic, so we can simply impose condition (2.3). A problem with this justification for monotonicity is that borrowers' participation on the loan market must be limited to borrowing, otherwise they can fake lower levels of output as well.[2] A second implicit assumption that will yield monotonicity is to suppose that a lender can unilaterally destroy any or all of the borrower's output in period 2, if he/she wishes. The lender would destroy output only if the payment schedule were decreasing over some range, but destroying output is clearly suboptimal, so payment schedules must be monotonic. This second justification may appear implausible when taken at face value, but it suggests that there may be well-specified (and plausible) moral hazard problems in lending (effort by the lender affects the borrower's return) which would guarantee monotonicity and not change the flavor of our results.

The conditions (2.4) are incentive compatibility constraints. Here, a borrower of type i has probability π_i of being screened. The left side of (2.4) is the loss in expected utility for the borrower if she reports her true type. The right side of (2.4) is the loss in expected utility if the borrower falsifies her type, reporting type j. In this case, with probability $1-\pi_j$ the agent is not screened and cheating is not detected. In this case, the loss in expected utility is the expected cost of making payments as for a type j borrower. Alternatively, with probability $\pi_j$the borrower is screened, in which case cheating is discovered and the borrower is denied a loan. If a loan is denied, then the borrower cannot fund her project (as she can contact only one borrower) and consumes zero, so that the loss in expected utility is μ_i. Condition (2.5) states that the expected return to a lender from each separating contract is at least as great as

[2]If there is unconstrained lending and borrowing, and if the participants in the period 2 loan market include agents outside the credit market, then the credit market would shut down entirely. If borrowers can lend and borrow only among themselves in period 2, this would imply that payment schedules must be linear.

from the alternative risk-free investment opportunity. As in the Rothschild-Stiglitz (1977) insurance model, each contract earns nonnegative expected profits; there is no cross-subsidization.

2.3.2 Pooling Equilibrium

Pooling contracts clearly do not involve the use of the screening technology, so that these contracts can be characterized by the payment schedule, denoted by $R(x)$. A pooling equilibrium is then an equilibrium payment schedule $R(x)$ which satisfies the following three properties, in addition to some other conditions to be discussed later.

$$0 \leq R(x) \leq x, x \in [0,1]. \tag{2.6}$$

$$x \leq y \Rightarrow R(x) \leq R(y); x, y \in [0,1]. \tag{2.7}$$

$$\alpha \int_0^1 R(x) dF_g(x) + (1-\alpha) \int_0^1 R(x) dF_b(x) \geq r. \tag{2.8}$$

Conditions (2.6) and (2.7) require, respectively, feasibility and monotonicity of the equilibrium payment schedule. These conditions are the counterparts of (2.2) and (2.3), respectively. Condition (2.8) states that the expected return from the equilibrium contract for a lender must be no less than the return on the alternative risk-free investment. Note here that, when a lender offers the contract $R(x)$, the probability of lending to a good borrower is α. Condition (2.8) is the counterpart of (2.5) in the separating equilibrium.

2.3.3 Remaining Equilibrium Conditions

In either a separating equilibrium or a pooling equilibrium, it must be the case that, given the equilibrium contracts, no contract could be offered which is (i) weakly preferred by a lender to the safe alternative investment, given the borrowers who would accept it, and (ii) strictly preferred by some borrower to the equilibrium contract she would otherwise accept. Therefore, in a separating equilibrium, it must be the case that, for $i = g, b$, there exists no separating contract $[R_i^*(x), \pi_i^*]$, such that

$$0 \leq R_i^*(x) \leq x, x \in [0,1]. \tag{2.9}$$

$$x \leq y \Rightarrow R_i^*(x) \leq R_i^*(y); x, y \in [0,1]. \tag{2.10}$$

$$\int_0^1 R_i^*(x)dF_i(x) \leq (1 - \pi_j) \int_0^1 R_j(x)dF_i(x) + \pi_j\mu_i; j \neq i. \tag{2.11}$$

$$\int_0^1 R_j(x)dF_j(x) \leq (1 - \pi_i^*) \int_0^1 R_i^*(x)dF_j(x) + \pi_i^*\mu_j; j \neq i. \tag{2.12}$$

$$\int_0^1 R_i^*(x)dF_i(x) \geq r + \pi_i^*\gamma \tag{2.13}$$

$$\int_0^1 R_i^*(x)dF_i(x) < \int_0^1 R_i(x)dF_i(x) \tag{2.14}$$

Conditions (2.9)-(2.12) state that the proposed separating contract must be feasible, monotonic, and incentive compatible for each type, respectively. Condition (2.13) requires that the proposed separating contract be weakly preferred by a lender to the risk-free alternative asset, while (2.14) states that the proposed separating contract be strictly preferred by type i to the equilibrium contract for type i.

Similarly, in a separating equilibrium there can exist no pooling contract $R^*(x)$ with the following properties.

$$0 \leq R^*(x) \leq x, x \in [0,1]. \tag{2.15}$$

$$x \leq y \Rightarrow R^*(x) \leq R^*(y); x, y \in [0,1]. \tag{2.16}$$

$$\alpha \int_0^1 R^*(x)dF_g(x) + (1 - \alpha) \int_0^1 R^*(x)dF_b(x) \geq r. \tag{2.17}$$

$$\int_0^1 R^*(x)dF_i(x) < \int_0^1 R_i(x)dF_i(x); i = g, b. \tag{2.18}$$

Here, note that both types must strictly prefer the proposed pooling contract to the equilibrium contract for their type.

In a pooling equilibrium, there can exist no separating contract $[R_i^*(x), \pi_i^*]$ for which the following hold.

$$0 \leq R_i^*(x) \leq x, x \in [0,1]. \tag{2.19}$$

$$x \leq y \Rightarrow R_i^*(x) \leq R_i^*(y); x, y \in [0,1]. \tag{2.20}$$

$$\int_0^1 R(x)dF_j(x) \leq (1-\pi_i^*)\int_0^1 R_i^*(x)dF_j(x) + \pi_i^*\mu_j; j \neq i. \tag{2.21}$$

$$\int_0^1 R_i^*(x)dF_i(x) \geq r + \pi_i^*\gamma \tag{2.22}$$

$$\int_0^1 R_i^*(x)dF_i(x) < \int_0^1 R(x)dF_i(x) \tag{2.23}$$

Also, in a pooling equilibrium there can exist no pooling contract $R^*(x)$ with the following properties.

$$0 \leq R^*(x) \leq x, x \in [0,1]. \tag{2.24}$$

$$x \leq y \Rightarrow R^*(x) \leq R^*(y); x, y \in [0,1]. \tag{2.25}$$

$$\alpha\int_0^1 R^*(x)dF_g(x) + (1-\alpha)\int_0^1 R^*(x)dF_b(x) \geq r. \tag{2.26}$$

$$\int_0^1 R^*(x)dF_i(x) < \int_0^1 R(x)dF_i(x); i = g, b. \tag{2.27}$$

2.4 Characterization of the Separating Equilibrium

Now that separating and pooling equilibria have been defined, we can proceed to establish some basic properties of these equilibria, starting with the separating case. We first need to show that conditions (2.5), which play the role of nonnegative expected profit conditions, hold with equality.

Lemma 1 In a separating equilibrium, $\pi_b = 0$.

Proof: Suppose not, and suppose that condition (2.4) is a strict inequality for $i = g$. Then, there exists an alternative separating contract for type b, $[R_b^*(x), \pi_b^*]$, with $\pi_b^* = 0 < \pi_b$, $R_b^*(x) = \delta R_b(x)$, $0 < \delta < 1$, and conditions (2.9)-(2.14) hold. Alternatively, suppose that $\pi_b > 0$ and condition (2.4) is an equality for $i = g$. Consider the pooling contract $R^*(x) = R_b(x)$. Since this contract is monotonic, and since $F_g(x) < F_b(x)$, we have $\int_0^1 R^*(x)dF_g(x) > \int_0^1 R^*(x)dF_b(x) > r$, where the first inequality is due to the monotonicity of $R^*(x)$ and the first-order stochastic dominance of $F_b(x)$ by $F_g(x)$, and the second inequality follows from (2.5). Therefore, condition (2.17) holds with strict inequality. In addition, this contract is strictly preferred by type g, as $\int_0^1 R_g(x)dF_g(x) > \int_0^1 R^*(x)dF_g(x)$. Finally, $R^*(x)$ is weakly preferred by type b. Thus, (2.15)-(2.18) hold for $R^*(x)$. Q.E.D.

The above lemma states that type b borrowers will never be screened in a separating equilibrium. If type b borrowers were screened, then we have one of two cases. First, it might be the case that a separating contract could be offered to type b borrowers with a lower screening probability and a preferable payment schedule. This contract could earn nonnegative expected profits as the lower expected screening costs make up for the reduction in expected payments to the lender. Second, the type b contract could be offered to both types as a pooling contract. This contract is preferred by both types, and earns positive expected profits since it involves no screening and will yield a higher expected profit from a type g borrower than from a type b.

Lemma 2 In a separating equilibrium, if condition (2.5) is a strict inequality for given i, then condition (2.4) is an equality for $j \neq i$.

Proof: Suppose not. Then, it is possible to offer an alternative separating contract for type i, $[R_i^*(x), \pi_i^*]$, with $\pi_i^* = \pi_i$, and with $R_i^*(x) \leq R_i(x), x \in [0,1]$ with strict inequality for some $x \in S$ with positive measure, such that (2.9)-(2.14) are satisfied. Q.E.D.

Thus, if there were a separating equilibrium where the contract for agent type i earned positive expected profits, then it must be the case that a type j agent, $j \neq i$, must be indifferent between the type i contract and the type j contract. Otherwise, some lender could offer an alternative contract which type i strictly prefers and which is incentive compatible for type j.

Proposition 1 In a separating equilibrium, (2.5) holds with equality for $i = g, b$.

Proof: Suppose that (2.5) is a strict inequality for $i = b$. Then, from the previous lemma, (2.4) holds with equality for $i = g$. Consider the alternative pooling contract $R^{**}(x) = \delta R_b(x)$, where $0 < \delta < 1$, with δ sufficiently close to 1. This contract is strictly preferred by both types of borrowers and, we have ,

$$\alpha \int_0^1 R_b(x) dF_g(x) + (1-\alpha) \int_0^1 R_b(x) dF_b(x) > r.$$

Therefore, for δ sufficiently close to 1, $R^*(x)$ satisfies (2.15)-(2.18), a contradiction.

Now, suppose (2.5) is a strict inequality for $i = g$. Since (2.5) holds with equality for $i = b$, and since $\pi_b = 0$, we have $\int_0^1 R_b(x) dF_b(x) = r$. Therefore, since $\mu_i > r$, the above lemma and (2.4) for $i = b$ imply that $\pi_g < 1$. Now, consider an alternative separating contract for type g, $[R_g^*(x), \pi_g^*]$, with $R_g^*(x) = \delta R_g(x)$ for all x, and $\pi_g^* > \pi_g$. For properly chosen π_g^* and δ, this alternative contract can be constructed so that (2.9)-(2.14) hold for $i = g$, a contradiction. Q.E.D.

The above proposition states that expected profits on each separating contract must be zero in equilibrium. Given separating contracts which are incentive compatible but which earn positive expected profits for a lender, an alternative contract exists which, if offered, earns nonnegative expected profits and .. incentive compatible.

Lemma 3 In a separating equilibrium, $\pi_g > 0$.

Proof: Suppose not. Then, given Lemma 1, Proposition 3, and (2.4), we have $\int_0^1 R_g(x)dF_g(x) \leq \int_0^1 R_g(x)dF_b(x)$, a contradiction given (2.1) and the monotonicity of $R_g(x)$. Q.E.D.

Given Lemma 1 and Lemma 3, any separating equilibrium must involve screening for type g borrowers, with no screening for type b borrowers. Without screening of type g borrowers, it is not possible to achieve self selection while satisfying the zero expected profit conditions. However, screening of type b borrowers simply wastes resources.[3]

Lemma 4 Condition (2.4) holds with equality for $i = b$.

Proof: Suppose not. Then, given Proposition 3, Lemma 1, and Lemma 4, there exists an alternative separating contract for type g, $[R_g^*(x), \pi_g^*]$, with $R_g^*(x) \leq R_g(x), x \in [0,1]$, and $\pi_g^* < \pi_g$, such that (2.9)-(2.14) hold. Q.E.D.

The above lemma states that the incentive constraint must be binding for a type b borrower. Type g borrowers are screened with positive probability for the purpose of preventing type b agents from falsely reporting their type to be g. Since the type g borrowers effectively incur the expected screening costs, given that each contract earns zero expected profits in equilibrium, type g agents can always be made better off with a lower screening probability and preferable payment schedule if the incentive constraint for type b borrowers is not binding.

Now, since $\pi_b = 0$ in a separating equilibrium and (2.5) holds with equality for $i = b$, the expected utility of type b borrowers is effectively fixed in a separating equilibrium at $\mu_b - r$. As type g borrowers bear the expected screening costs for

[3]De Meza and Webb (1988) also obtain the result that good types are screened and bad types are not, in a model with costly screening and adverse selection.

their type, to find separating contracts which are optimal (that is, immune from alternative separating contracts which earn nonnegative expected profits while making some agents better off) we need to minimize the verification probability π_g subject to the binding incentive compatibility constraint for type b borrowers and the zero expected profit constraint for type g borrowers. This will serve to relax the incentive constraint for type g borrowers as much as possible. We will ignore (2.4) for $i = g$ in solving for the optimal contract, and then show that, given the optimal contract, this constraint is not binding. Thus, the separating contract for type $i = g$ is determined by choosing $R_g(x)$ and π_g to solve:

$$\max(-\pi_g)$$

subject to

$$\int_0^1 R_g(x)dF_g(x) = r + \pi_g\gamma, \tag{2.28}$$

$$r = (1 - \pi_g)\int_0^1 R_g(x)dF_b(x) + \pi_g\mu_b, \tag{2.29}$$

(2.2) and (2.3). From the previous analysis, the separating contracts which solve this problem then only need to be checked to insure that there is not a feasible, monotonic pooling contract which makes each type of borrower better off while earning nonnegative expected profits. We are now ready to prove the main proposition of this section.

Proposition 2 If a separating equilibrium exists, the unique equilibrium separating contract for type $i = g$ is a standard debt contract. That is, $R_g(x) = \bar{R}_g, x \in [\bar{R}_g, 1]; R_g(x) = x, x \in [0, \bar{R}_g]$ for some $\bar{R}_g \in (0, 1)$.

Proof: The Lagrangian for the above optimization problem is

$$\begin{aligned}\mathcal{L} = -\pi_g + \lambda_1[r + \pi_g\gamma - \textstyle\int_0^1 R_g(x)f_g(x)dx] \\ +\lambda_2[-r + \pi_g\mu_b + (1 - \pi_g)\textstyle\int_0^1 R_g(x)f_b(x)dx],\end{aligned}$$

and we want to choose $\lambda_i \neq 0, i = 1,2$, $\pi_g \in [0,1]$, and $R_g(x)$ to maximize $\mathcal{L}$ subject to (2.2) and (2.3). We can rewrite $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L} = -\pi_g + \textstyle\int_0^1[-\frac{f_g(x)}{f_b(x)} + \frac{\lambda_2}{\lambda_1}(1-\pi_g)]\lambda_1 f_b(x)R_g(x)dx \\ +\lambda_1(r+\pi_g\gamma) + \lambda_2(-r+\pi_g\mu_b).\end{aligned} \tag{2.30}$$

If $\lambda_1 > 0$ and $\lambda_2 < 0$ or if $\lambda_1 < 0$ and $\lambda_2 > 0$, then maximizing (2.30), we get $R_g(x) = 0$, which violates (2.28). If $\lambda_1 < 0$ and $\lambda_2 < 0$, then maximizing (2.30), we get $\pi_g = 0$, which violates Lemma 4. We therefore have $\lambda_i > 0, i = 1,2$. Now, given optimal $\lambda_i, i = 1,2$, and π_g, we must have $-\frac{f_g(x)}{f_b(x)} + \frac{\lambda_2}{\lambda_1}(1-\pi_g) < 0$ for some $x \in (0,1)$ otherwise $R_g(x) = x$ is optimal for all $x \in [0,1]$. and (2.29) is violated. Similarly, we must have $-\frac{f_g(x)}{f_b(x)} + \frac{\lambda_2}{\lambda_1}(1-\pi_g) > 0$ for some $x \in (0,1)$, otherwise $R_g(x) = 0$ is optimal for all $x \in (0,1)$, and (2.28) is violated. Therefore, since $-\frac{f_g(x)}{f_b(x)} + \frac{\lambda_2}{\lambda_1}(1-\pi_g)$ is continuous and monotonic in x, there exists some $\bar{x} \in (0,1)$, such that $-\frac{f_g(x)}{f_b(x)} + \frac{\lambda_2}{\lambda_1}(1-\pi_g) > 0$ for all $x \in (0,\bar{x})$, $-\frac{f_g(x)}{f_b(x)} + \frac{\lambda_2}{\lambda_1}(1-\pi_g) = 0$ for $x = \bar{x}$, and $-\frac{f_g(x)}{f_b(x)} + \frac{\lambda_2}{\lambda_1}(1-\pi_g) < 0$ for $x \in [\bar{x}, 1]$. Now, given the monotonicity of $R_g(\cdot)$, it is clear that for optimality we must have $R_g(x) = R_g(\bar{x}), x \in [\bar{x}, 1]$. For $x \in [0, \bar{x}]$, given constraints (2.2) and (2.3), the optimal payment schedule is $R_g(x) = x$, $x \in [0, R_g(\bar{x})]$ and $R_g(x) = R_g(\bar{x}), x \in [R_g(\bar{x}), \bar{x}]$. The proof is concluded by letting $\bar{R}_g = R_g(\bar{x})$.□

Given the above proposition, the separating contract for a type g borrower is characterized by $(\bar{R}_g, \pi_g)$, where $\bar{R}_g$ is interpreted as the gross loan interest rate. From (2.28) and (2.29), $\bar{R}_g$ and π_g are determined by the following two equations.

$$\int_0^{\bar{R}_g} x dF_g(x) + \bar{R}_g[1 - F_g(\bar{R}_g)] = r + \pi_g\gamma$$

$$r = (1-\pi_g)\{\int_0^{\bar{R}_g} x dF_b(x) + [1 - F_b(\bar{R}_g)]\} + \pi_g\mu_b$$

Alternatively, using integration by parts to simplify the above two equations, we get

$$\bar{R}_g - \int_0^{\bar{R}_g} F_g(x)dx = r + \pi_g\gamma \tag{2.31}$$

$$r = (1 - \pi_g)[\bar{R}_g - \int_0^{\bar{R}_g} F_b(x)dx] + \pi_g \mu_b \tag{2.32}$$

In the appendix, we show that there exists a unique $\bar{R}_g \in (0, 1)$ and a unique $\pi_g \in (0, 1)$ that solve (2.31) and (2.32).

Proposition 3 In a separating equilibrium, given an optimal separating contract for type $i = g$ satisfying (2.31) and (2.32), an optimal separating contract for type $i = b$ is a debt contract characterized by $\bar{R}_b > \bar{R}_g$.

Proof: We need only show that a debt contract for type $i = b$ satisfies the zero expected profit condition for $i = b$, and the incentive constraint for $i = g$. First, since $\mu_b > r$, there exists a unique $\bar{R}_b \in (0, 1)$ satisfying $\bar{R}_b - \int_0^{\bar{R}_b} F_b(x)dx = r$, so that (2.5) for $i = b$ is satisfied with equality. Second, since $\pi_g > 0$ in a separating equilibrium, and $\mu_b > r$, from (2.32) we must have $\bar{R}_g - \int_0^{\bar{R}_g} F_b(x)dx < r$. Now, given that $R - \int_0^R F_b(x)dx$ is an increasing function of R, we therefore have $\bar{R}_g < \bar{R}_b$, which in turn implies that (2.4) is satisfied for $i = g$. Q.E.D.

From Propositions 6 and 7, debt contracts are optimal for both types in a separating equilibrium. A debt contract for type $i = g$ is the least attractive monotonic and feasible contract for type $i = b$, as it provides the largest rewards for a type g borrower in the upper tail of the probability distribution of returns, where type g has relatively more probability mass than type b. Thus, a debt contract for type g induces as much self selection as is possible, which minimizes the screening probability for type g, and maximizes the welfare of type g borrowers. For type g borrowers, a debt contract is the unique equilibrium contract. However, a debt contract is only one in a continuum of equilibrium contracts for type b borrowers. Any contract which, given the optimal type g contract, satisfies the zero expected profit constraint for $i = b$ and the incentive constraint for $i = g$ is an equilibrium contract for type b. In fact, another equilibrium contract for type b is $R_b(x) = 0, x \in [0, x^*); R_b(x) = x, x \in [x^*, 1]$, where x^* solves $\int_{x^*}^{1} x dF_b(x) = r$. This contract is feasible, monotonic, and incentive compatible and it is therefore an equilibrium contract. However, it is as far removed as possible from a debt contract.

A separating equilibrium, if it exists, has some attractive features. First, as we showed above, debt contracts are equilibrium contracts for both types, which is consistent with the widespread use of debt contracts in practice. Second, the optimality of the debt contract for type $i = g$ arises because of the costs of screening borrowers. In credit markets, screening costs appear to be a large component of the costs incurred by financial intermediaries. This second feature is an advantage of this approach in explaining debt contracts, relative to the costly state verification approach. In costly state verification models, for example Townsend (1979), Gale and Hellwig (1985), or Williamson (1986, 1987), the existence of debt contracts is explained by costs to a lender of verifying the ex post return of a borrower. In equilibrium, these costs can be interpreted as bankruptcy costs, i.e. auditing costs. These costs would seem to represent a small fraction of the costs of advancing credit, relative to the fraction accounted for by screening costs.[4] Third, since screening costs are incurred for the low interest rate contracts, these are the contracts which would be intermediated (with some further assumptions about the scale of investment projects), as we will show later. Thus, low quality borrowers pay high interest rates outside the intermediation sector, while high quality borrowers are served by financial intermediaries in equilibrium.

[4]Some casual evidence is the following. Our conversations with commercial loan officers suggest that, in a commercial loan operation, the ratio of those employed screening loans to those employed in "working out" loans (essentially getting all the bank can before a borrower fails) is in the range of 5 to 8.

2.5 Characterization of the Pooling Equilibrium

Analysis of the pooling equilibrium is more straightforward than for the separating case, as there is no screening here; that is $\pi_i = 0$ for all i. The (potential) equilibrium contract is then completely described by the payment schedule, $R(x)$, which both types of borrowers face.

The constraint (2.8) is satisfied with equality in a pooling equilibrium, since otherwise there exists another pooling contract $R^*(x)$ satisfying (2.24)-(2.27). That is, given a pooling contract earning positive expected profits, both agents can be made better off with another contract which is feasible, monotonic, and earns nonnegative expected profits. Thus, we can restrict attention to the set of pooling contracts which earn zero expected profits. If a contract within this set is an equilibrium, there can be no other pooling contract earning zero expected profits which makes both types better off, and no incentive compatible separating contract for type i that makes type i better off.

<u>lemma 6</u> An equilibrium pooling contract $R(x)$ solves

$$\min \int_0^1 R(x)dF_g(x)$$

subject to

$$\alpha \int_0^1 R(x)dF_g(x) + (1-\alpha)\int_0^1 R(x)dF_b(x) = r, \tag{2.33}$$

(2.6), and (2.7).

Proof: We have already shown that an equilibrium pooling contract must satisfy (2.33). Suppose that the equilibrium pooling contract does not solve the above optimization problem. Then, we can have one of two cases. First, there may exist a contract $R^*(x)$ satisfying (2.6) and (2.7) which both types strictly prefer to $R(x)$, and which satisfies

$$\alpha \int_0^1 R^*(x)dF_g(x) + (1-\alpha)\int_0^1 R^*(x)dF_b(x) = r.$$

Otherwise, there exists a separating contract $[R_g^*(x), 0]$ satisfying (2.19) and (2.20) which type g strictly prefers, while type b weakly prefers contract $R(x)$. This contract is constructed so as to satisfy

$$\alpha \int_0^1 R_g^*(x)dF_g(x) + (1-\alpha)\int_0^1 R_g^*(x)dF_b(x) = r.$$

Now, since $F_g(x) < F_b(x)$ for all $x \in (0,1)$, and since $R_g^*(x)$ is monotonic, we have $\int_0^1 R_g^*(x)dF_g(x) > r$. Therefore, the contract $[R_g^*(x), 0]$ satisfies (2.19)-(2.23). Q.E.D.

The above lemma states that an equilibrium pooling contract is the pooling contract which maximizes the welfare of type $i = g$ borrowers subject to the zero expected profit constraint. If this were not the case, then, either a pooling contract earning zero expected profits could be found which both borrower types strictly prefer, or there exists an incentive compatible separating contract for type g borrowers which earns positive expected profits while making type g borrowers better off.

Proposition 4 If a pooling equilibrium exists, the pooling contract, $R(x)$, is a standard debt contract with $R(x) = \bar{R}, x \in [\bar{R}, 1]; R(x) = x, x \in [0, \bar{R}]$; for some $\bar{R} \in (0,1)$.

Proof: From the optimization problem in Lemma 8, form the Lagrangian

$$\mathcal{L} = -\int_0^1 R(x)dF_g(x) + \lambda[\alpha \int_0^1 R(x)dF_g(x) + (1-\alpha)\int_0^1 R(x)dF_b(x) - r].$$

Then, the optimization problem is one of choosing $R(x)$ and $\lambda > 0$ to maximize $\mathcal{L}$ subject to (2.6) and (2.7). Now, if $\lambda\alpha - 1 \geq 0$, then $R(x) = x$ would be optimal, but then (2.33) is not satisfied, so $\lambda\alpha - 1 < 0$. We can then rewrite $\mathcal{L}$ as

$$\mathcal{L} = \int_0^1 [-\frac{f_g(x)}{f_b(x)} + \frac{\lambda(1-\alpha)}{(1-\lambda\alpha)}]R(x)(1-\lambda\alpha)f_b(x)dx - \lambda r.$$

If $-\frac{f_g(x)}{f_b(x)}+\frac{\lambda(1-\alpha)}{(1-\lambda\alpha)}<0$ for all $x\in[0,1]$, then $R(x)=0$ for all $x\in[0,1]$ and (2.33) does not hold. Alternatively, if $-\frac{f_g(x)}{f_b(x)}+\frac{\lambda(1-\alpha)}{(1-\lambda\alpha)}>0$ for all $x\in[0,1]$, then $R(x)=x$ for all $x\in[0,1]$, and (2.33) does not hold. Therefore, since $\frac{f_g(x)}{f_b(x)}$ is continuous and strictly increasing in x, there is some $x^*\in(0,1)$ such that $-\frac{f_g(x)}{f_b(x)}+\frac{\lambda(1-\alpha)}{(1-\lambda\alpha)}>0, x\in[0,x^*); -\frac{f_g(x)}{f_b(x)}+\frac{\lambda(1-\alpha)}{(1-\lambda\alpha)}<0, x\in(x^*,1]; -\frac{f_g(x)}{f_b(x)}+\frac{\lambda(1-\alpha)}{(1-\lambda\alpha)}=0, x=x^*$. Therefore, given the feasibility and monotonicity constraints on $R(x)$, the optimal contract is $R(x)=x, x\in[0,\bar{R}]; R(x)=\bar{R}, x\in[\bar{R},1]$, for some $\bar{R}\in(0,1)$. Q.E.D.

The optimal gross loan interest rate, $\bar{R}$, is determined by the zero expected profit constraint. Substituting in (2.33) and using integration by parts to simplify, we get

$$\bar{R}-\int_0^{\bar{R}}[\alpha F_g(x)+(1-\alpha)F_b(x)]dx=r. \tag{2.34}$$

The left side of (2.34) is continuous and strictly increasing in $\bar{R}$. In addition, the left side is less than r for $\bar{R}=0$ and greater than r for $\bar{R}=1$. Therefore, (2.34) has a unique solution in $(0,1)$. From (2.34), a useful result is that

$$\frac{d\bar{R}}{d\alpha}=\frac{\int_0^{\bar{R}}[F_g(x)-F_b(x)]dx}{1-F_b(\bar{R})-\alpha[F_g(\bar{R})-F_b(\bar{R})]}<0. \tag{2.35}$$

As for an equilibrium separating contract, the optimal pooling contract is a standard debt contract. In contrast to the separating case, where debt was not the unique optimal contract for type $i=b$, here both borrower types face a debt contract which is a unique equilibrium. A debt contract is optimal in a pooling equilibrium, as it makes type g borrowers as well off as possible subject to the constraint that the pooling contract earns zero expected profits. That is, the debt contract minimizes the implicit subsidy that type g borrowers pay to type b borrowers.

2.6 Existence of Equilibrium and Comparative Statics

In this section we show that, if an equilibrium exists, it is a separating equilibrium. We then explore how the parameters of the model, α, r, and γ, determine whether or not an equilibrium exists, and study how changes in the parameters affect equilibrium interest rates and screening probabilities.

In deriving the equilibrium separating and pooling contracts in the previous two sections, we omitted some conditions which are required for the existence of each type of equilibrium. For the separating equilibrium, we need to check that there exists no pooling contract satisfying (2.15)-(2.18), and for the pooling equilibrium, we need to check that there exists no separating contract satisfying (2.19)-(2.23).

Proposition 5 A pooling equilibrium does not exist.

Proof: Suppose a pooling equilibrium exists, which is characterized by the gross loan interest rate $\bar{R}$ which solves (2.34). Now, consider a separating contract for type $i = g$ which is a debt contract $[\bar{R}_g^*, \pi_g^*]$. This contract is constructed to satisfy the zero expected profit constraint

$$\bar{R}_g^* - \int_0^{\bar{R}_g^*} F_g(x)dx = r + \pi_g^*\gamma, \tag{2.36}$$

and the incentive constraint for type $i = b$,

$$\hat{r} = (1 - \pi_g^*)[\bar{R}_g^* - \int_0^{\bar{R}_g^*} F_b(x)dx] + \pi_g^*\mu_b, \tag{2.37}$$

where $\hat{r} = \bar{R} - \int_0^{\bar{R}} F_b(x)dx < r$. Now, equations (2.36) and (2.37) yield a unique solution for $\bar{R}_g^*$ and π_g^*, where, substituting for π_g^* in (2.37) using (2.36), we get an equation which solves for $\bar{R}_g^*$.

$$\bar{R}_g^* - \int_0^{\bar{R}_g^*} F_g(x)dx = r + \gamma\left[\frac{\hat{r} - \bar{R}_g^* + \int_0^{\bar{R}_g^*} F_b(x)dx}{\mu_b - \bar{R}_g^* + \int_0^{\bar{R}_g^*} F_b(x)dx}\right]. \tag{2.38}$$

Note that the left side of (2.38) is increasing and continuous in $\bar{R}_g^*$ on [0,1], while the right side is decreasing and continuous in $\bar{R}_g^*$ on [0,1]. Therefore, given the values of the left and right sides of (2.38) at the endpoints of [0,1], there exists a unique solution for $\bar{R}_g^*$. Now, note that, for $\bar{R}_g^* = \bar{R}$, the right side of (2.38) equals r, while the left side equals $\bar{R} - \int_0^{\bar{R}} F_g(x)dx >$ · Therefore, we have $\bar{R}_g^* < \bar{R}$. Thus, the alternative separating contract satisfies (2.19)-(2.23), a contradiction. Q.E.D.

Therefore, given any potential pooling equilibrium, there always exists a separating contract which type $i = g$ borrowers strictly prefer, and which attracts only type $i = g$ borrowers while earning nonnegative expected profits. We can thus confine attention to the separating equilibrium.

For the separating equilibrium, which can be characterized by the loan interest rates faced by each type of borrower, $\bar{R}_i, i = g, b$, and the screening probability π_g, solving (2.31), (2.32), and

$$\bar{R}_b - \int_0^{\bar{R}_b} F_b(x)dx = r, \tag{2.39}$$

we need only check that there is no pooling contract which is feasible, monotonic, earns nonnegative expected profits, and makes both types better off. Consider the pooling contract which is a debt contract characterized by $\bar{R}$ solving (2.34). This contract clearly makes type b borrowers better off than with the separating contract, since $\bar{R}_b - \int_0^{\bar{R}_b} F_b(x)dx = r > \bar{R} - \int_0^{\bar{R}} F_b(x)dx$. The pooling contract also maximizes the expected utility of type g borrowers in the set of pooling contracts which earns nonnegative expected profits. Therefore, if type g agents do not strictly prefer the pooling contract $\bar{R}$ to the separating contract $\bar{R}_g$, then the separating contracts characterized by $\bar{R}_i, i = g, b$, and π_g, constitute an equilibrium. That is, an equilibrium exists if and only if $\bar{R} \geq \bar{R}_g$.

We will now show how the parameters α, γ, and r determine existence of equilibrium, and how changes in these parameters affect equilibrium interest rates and the screening probability. First, to study the conditions under which an equilibrium exists, use (2.32) to substitute for π_g in (2.31) to obtain an equation which solves implicitly for $\bar{R}_g$.

$$\bar{R}_g - \int_0^{\bar{R}_g} F_g(x)dx = r + \gamma \left[1 - \frac{\mu_b - r}{\mu_b - \bar{R}_g + \int_0^{\bar{R}_g} F_b(x)dx} \right] \tag{2.40}$$

In (2.40), the left side is increasing and continuous in $\bar{R}_g$ on [0,1], while the right side is decreasing and continuous in $\bar{R}_g$ on [0,1]. Now note, from (2.40), that $\bar{R}_g$ is independent of α, while from (2.34), $\bar{R}$ is decreasing in α. Substituting $\alpha = 1$ in (2.34), we obtain $\bar{R} = \bar{R}_1$ as the solution to

$$\bar{R}_1 - \int_0^{\bar{R}_1} F_g(x)dx = r. \tag{2.41}$$

Substituting $\bar{R}_g = \bar{R}_1$ into (2.40), the left side of (2.40) is equal to r, while the right side is greater than r. Therefore, we have $\bar{R}_g > \bar{R}_1$. Similarly, substituting $\alpha = 0$ in (2.34), we obtain $\bar{R} = \bar{R}_0$ as the solution to

$$\bar{R}_0 - \int_0^{\bar{R}_0} F_b(x)dx = r. \tag{2.42}$$

Now, substituting $\bar{R}_g = \bar{R}_0$ into (2.40), the left side of (2.40) is greater than r, while the right side is equal to r. Therefore, we have $\bar{R}_g < \bar{R}_0$. Thus, by continuity and (2.35), given any r and γ, there exists $\alpha^* \in (0,1)$ such that an equilibrium exists for $\alpha \in (0,\alpha^*]$, and an equilibrium does not exist for $\alpha \in (\alpha^*,1)$. Note here that, for any screening cost, no matter how large, if the fraction of type b borrowers in the population is sufficiently large, then an equilibrium exists where screening is worthwhile.

Next, note in (2.34) that $\bar{R}$ is independent of γ, while in (2.40) $\bar{R}_g$ is increasing in γ. If $\gamma = 0$, from (2.40) and (2.34) we have $\bar{R}_g < \bar{R}$, so that an equilibrium exists. Alternatively, as $\gamma \to \infty$, from (2.40) we get $r - \bar{R}_g + \int_0^{\bar{R}_g} F_b(x)dx \to 0$, or $\bar{R}_g \to \bar{R}_b > \bar{R}$. Therefore, an equilibrium does not exist as $\gamma \to \infty$. Thus, from continuity and monotonicity, for given r and α, there exists $\gamma^* > 0$ such that an equilibrium exists for $\gamma \in [0, \gamma^*]$, and an equilibrium does not exist for $\gamma > \gamma^*$. Thus, for any r and α, an equilibrium exists if the screening cost is sufficiently small.

We do not get clear results for how the interest rate r affects the existence of equilibrium. Here, from (2.34) and (2.40), changes in r affect both $\bar{R}$ and $\bar{R}_g$, and the relative effects depend on the other parameters.

From (2.31), (2.32), and (2.39), it is clear that loan interest rates $\bar{R}_i, i = g, b$, and the screening probability π_g are independent of α in equilibrium. Thus, the relative fractions of borrower types in the population are irrelevant for equilibrium interest rates and the screening intensity. From (2.40), an increase in γ results in an increase in $\bar{R}_g$. Then, from (2.32), π_g must decrease when γ increases. Therefore, an increase in screening costs causes the screening technology to be used less intensively (the screening probability falls) and the loan interest rate faced by type g borrowers rises. Note that there is no change in the interest rate faced by type b borrowers, from (2.39).

To determine the effects of a change in r on $\bar{R}_i, i = g, b$, and on π_g, comparative statics using (2.31), (2.32), and (2.39) gives

$$\frac{d\bar{R}_b}{dr} = \frac{1}{1 - F_b(\bar{R}_b)} > 0$$

$$\frac{d\bar{R}_g}{dr} = \frac{\mu_b - \bar{R}_g + \int_0^{\bar{R}_g} F_b(x)dx + \gamma}{[1 - F_g(\bar{R}_g)][\mu_b - \bar{R}_g + \int_0^{\bar{R}_g} F_b(x)dx] + \gamma(1 - \pi_g)[1 - F_b(\bar{R}_g)]} > 0$$

$$\frac{d\pi_g}{dr} = \frac{F_b(\bar{R}_g) - F_g(\bar{R}_g) + \pi_g[1 - F_b(\bar{R}_g)]}{[1 - F_g(\bar{R}_g)][\mu_b - \bar{R}_g + \int_0^{\bar{R}_g} F_b(x)dx] + \gamma(1 - \pi_g)[1 - F_b(\bar{R}_g)]} > 0$$

Thus, when the interest rate faced by lenders increases, interest rates on loans increase for both types of borrowers, and the screening probability increases for type g borrowers. The increase in loan interest rates makes the loan contract for type g borrowers look relatively more attractive to type b borrowers than the type b loan contract. Therefore, the verification probability must rise in order to induce self selection.

2.7 Financial Intermediation

Costly state verification models of debt contracts (Townsend 1979, Gale and Hellwig 1985, Williamson 1987a) can be extended in such a way that financial intermediation arises endogenously as an efficient means to economize on verification costs (Diamond 1984, Williamson 1986). The costly verification of returns on investment projects is delegated to a financial intermediary by its depositors, and the intermediary is able to guarantee certain returns to its depositors by diversifying its loan portfolio.

The model constructed here has some advantages over the costly state verification framework as a model of debt contracts, particularly in that debt contracts survive randomization, and because screening costs seem more important for credit market activity than do ex post verification costs (essentially auditing costs). Our model can also be extended to generate financial intermediation endogenously in a straightforward way. In fact, intermediaries here will play a "delegated screening" role which is analogous to the "delegated monitoring" role of a financial intermediary in a costly state verification environment.

We extend the model by changing the investment technology available to each borrower so that it requires $K > 1$ rather than one unit of the investment good to fund an investment project in period 1. This implies that more than one lender is needed to fund any one investment project. We retain the assumption that one borrower can meet at most one lender in period 1. Also, we assume that a given lender can observe the results of his/her screening only.

Suppose first that all lending is done directly. However, since one borrower can meet one lender at most, some coordination among lenders is required. Thus, let lenders who wish to contract with borrowers form coalitions of K members each. Coalition members agree in advance on nothing, except that they will offer their contracts, jointly meet one borrower, and then the borrower will report his/her type to the coalition. A Nash equilibrium has similar properties to the equilibrium for the model with $K = 1$, in that, if an equilibrium exists it is a separating equilibrium, only type g borrowers are screened, and debt contracts are equilibrium contracts. An equilibrium consists of promised payments R_i^D, $i = g, b$, and a screening probability for a type g agent, π_g^D, satisfying

$$rK = (1-\pi_g^D)^K \left[R_g^D - \int_0^{R_g^D} F_b(x)dx\right] + \left[1-(1-\pi_g^D)^K\right]\mu_b \tag{2.43}$$

and

$$\frac{1}{K}\left[R_g^D - \int_0^{R_g^D} F_g(x)dx\right] = \pi_g^D\gamma + r. \tag{2.44}$$

Note, in the incentive constraint (2.43), that each lender in a coalition screens type g borrowers with probability π_g^D, and that a borrower who is denied a loan from any member of the coalition cannot fund his/her project.

Now, suppose that some lenders act as financial intermediaries. That is, they take deposits of the investment good in period 1, make loans to borrowers, and offer depositors returns contingent on the returns on their loan portfolios. Now, if a financial intermediary diversifies across a large number of borrowers, and borrows from a large number of depositors, by the law of large numbers it can guarantee each depositor a certain return in period 2 of r per unit deposited in period 1. The optimal contract results derived in the previous sections apply here, so that debt contracts are equilibrium contracts for both types of borrowers (and the unique equilibrium contract for type g borrowers). An equilibrium consists of promised payments R_i^I, $i = g, b$, and a screening probability for a type g agent, π_g^I, satisfying

$$rK = (1 - \pi_g^I)\left[R_g^I - \int_0^{R_g^I} F_b(x)dx\right] + \pi_g^I \mu_b \tag{2.45}$$

and

$$R_g^I - \int_0^{R_g^I} F_g(x)dx = \pi_g^I \gamma + rK. \tag{2.46}$$

Now, suppose that financial intermediaries offer a contract to type g agents with $R_g^I = R_g^D$, and $\pi_g^I = \min(K\pi_g^D, 1)$. In the case where $K\pi_g^D \geq 1$, from (2.44), (2.45), and (2.46), this contract earns nonnegative expected profits, and the incentive constraint for a type b agent is satisfied as a strict inequality. Therefore, there is some alternative contract with $R_g^I < R_g^D$ and $\pi_g^I < 1$ that earns nonnegative expected profits and is incentive compatible. Therefore, direct lending is dominated in this case by intermediated lending. Alternatively, suppose that $K\pi_g^D < 1$. Then, the proposed contract earns zero expected profits, from (2.44) and (2.46), and from (2.43) and (2.45), it satisfies the incentive constraint for a type b borrower as a strict inequality if and only if

$$1 - \pi_g^D K < (1 - \pi_g^D)^K. \tag{2.47}$$

But (2.47) holds by induction, for any integer $K \geq 2$. Therefore, financial intermediation dominates in this case as well.

In moving from a direct lending arrangement to intermediated lending there are two effects. First, with intermediated lending, screening is delegated to the intermediary, so that, for a given screening probability, expected screening costs are reduced as screening is not replicated by several lenders. Second, since a loan can be denied if any one lender discovers cheating in the direct lending case, each individual lender can screen with a lower probability than with financial intermediation, and this will tend to make expected screening costs lower in the direct lending case than with financial intermediation. Inequality (2.47) implies that the first effect outweighs the second, so that financial intermediation is more efficient than direct lending.

Note here that the only contracts which need be intermediated are those to type g borrowers, as these are the contracts for which costly screening takes place. Therefore, this framework yields a financial structure where some lending is intermediated while some lending is not. Again, this is an advantage over related costly state verification environments (e.g Williamson 1986), in that those models typically have all lending intermediated.

2.8 Conclusion

We have developed an adverse selection model of a credit market, where self selection is achieved with risk neutral borrowers through the use by lenders of a costly screening technology. In some ways, the model has features that are similar to the adverse selection model of Rothschild and Stiglitz (1977), where self selection occurs with risk averse agents who have different preferences. For example, if an equilibrium exists in our model, it is a separating equilibrium where good borrowers submit to screening with positive probability. Also, for some parameter values an equilibrium does not exist.

In this model, the unique equilibrium contract for a good borrower is a debt contract. Given the characteristics of the probability distributions of investment returns faced by borrowers, a debt contract is relatively unattractive for bad borrowers, so that a debt contract for good borrowers achieves self selection while minimizing the screening probability for good borrowers, and therefore minimizes expected screening costs. Debt contracts are also equilibrium contracts for bad borrowers if an equilibrium exists, but then there exists a continuum of equilibrium contracts for bad borrowers which are not debt contracts. We conjecture that, if the model were extended to include many types of borrowers whose investment projects are ordered in the same way as they are here, then debt contracts would be optimal for each type except the lowest ranked.

Comparative statics results show that equilibrium loan interest rates and the screening probability increase as the risk free interest rate faced by lenders increases. Also, an increase in the screening cost results in a decrease in the screening probability and an increase in loan interest rates. An equilibrium exists for sufficiently low screening costs and given a sufficiently large fraction of bad borrowers in the population. Somewhat surprisingly, for any positive screening cost, no matter how large, an equilibrium exists provided the fraction of bad borrowers in the population is large enough.

This model has at least two advantages over the alternative costly state verification model, which also can produce debt contracting as an optimal financial arrangement. First, debt contracts survive here even when random screening is permitted; costly state verification models do not yield debt contracts with random verification strategies. Second, ex ante screening costs would appear to be much more important for the functioning of credit markets than ex post auditing costs. In addition to having these advantages, our model is capable of generating intermediary structures, just as in costly state verification setups (e.g. Williamson 1986). Here, if borrowers' investment projects are large in scale relative to a lender's endowment, it is optimal for perfectly diversified financial intermediaries to be delegated the role of screening borrowers, and this is also an equilibrium financial arrangement.

Given that this model can generate debt contracting and intermediary structures, it potentially has many applications. For example, the model could be extended to study the effects of deposit insurance on the portfolio behavior and screening behavior of banks. It is often asserted that deposit insurance causes banks to take on too much risk and to expend too little effort in screening borrowers. An extension of the model with aggregate risk could be used to evaluate this assertion. Another possible extension would be to embed the model in a dynamic framework to study cyclical behavior. Here, we could make the relative numbers of good and bad borrowers endogenous by giving these agents some alternative to operating their own projects. The endogenous mix of borrower types in the population could then produce an interesting propagation mechanism, and allow us to study "credit crunches."

Appendix

Proposition 1 Condition (2.1) implies that $F_g(x) < F_b(x), x \in (0,1)$.

Proof: First, note that

$$\int_0^1 \left[\frac{f_g(y)}{f_b(y)} - 1\right] f_b(y)dy = 0. \tag{2.48}$$

Now, (2.1) and (2.48) imply that $\frac{f_g(0)}{f_b(0)} - 1 < 0$ and $\frac{f_g(1)}{f_b(1)} - 1 > 0$. Therefore, there exists $\bar{x} \in (0,1)$ such that $\frac{f_g(x)}{f_b(x)} - 1 < 0, x \in [0, \bar{x}]$ and $\frac{f_g(x)}{f_b(x)} - 1 > 0, x \in [x, 1]$. Therefore, we have

$$F_g(x) - F_b(x) = \int_0^x \left[\frac{f_g(y)}{f_b(y)} - 1\right] f_b(y)dy < 0. \tag{2.49}$$

Q.E.D.

Proposition 2 Given $\mu_b > r$ and $\gamma > 0$, there exists a unique $\bar{R}_g \in (0,1)$ and a unique $\pi_g \in (0,1)$ that solve (2.31) and (2.32).

Proof: First, solving (2.32) for π_g in terms of $\bar{R}_g$, we get

$$\pi_g = 1 - \frac{\mu_b - r}{\mu_b - \bar{R}_g + \int_0^{\bar{R}_g} F_b(x)dx}. \tag{2.50}$$

Thus, from (2.50), given $\bar{R}_g$ we get a unique solution for π_g. Next, substitute for π_g in (2.31) using (2.50) to get

$$\bar{R}_g - \int_0^{\bar{R}_g} F_g(x)dx = r + \gamma\left[1 - \frac{\mu_b - r}{\mu_b - \bar{R}_g + \int_0^{\bar{R}_g} F_b(x)dx}\right]. \tag{2.51}$$

Now, note that the left side of (2.51) is increasing and continuous in $\bar{R}_g$ on $[0,1]$, and that right side of (2.51) is decreasing and continuous in $\bar{R}_g$ on [0,1). For $\bar{R}_g = 0$, the left side of (2.51) equals 0, while the right side equals $r(1 + 1/\mu_b) > 0$. For $\bar{R}_g = 1$, the left side of (2.51) equals $\mu_g > r$, while the right side goes to $-\infty$ as $\bar{R}_g \to 1$. Therefore, there exists a unique solution to (2.51) for $\bar{R}_g$ in the interval (0,1). Q.E.D.

Chapter 3

Incentives, CEO Compensation And Shareholder Wealth In A Dynamic Agency Model

3.1 Introduction

A standard example of an agency problem is the conflict of interest between the owners of a corporation (shareholders) and its chief executive officer (CEO). Managerial actions taken by the CEO are crucial to the performance of the firm, yet these actions are often unobservable to the firm's shareholders. To implement desired actions, a compensation policy must be designed to give the CEO proper incentives. In general, agency theory predicts that the CEO should be paid on the basis of firm performance, leaving aside the issue of how firm performance should be measured. In a recent paper, Jensen and Murphy (1990) raise the following issue: is real world practice concerning CEO compensation and firm performance consistent with the predictions of formal agency models? The answer which they provide is negative. Specifically, using the first difference in shareholder wealth as a measure of firm performance, their estimation of the pay-performance relation indicates that CEO compensation changes too little in response to a change in shareholder wealth. The estimated pay-performance sensitivity, defined as the dollar change in the CEO's compensation associated with a dollar change in shareholder wealth, though significantly positive, is as low as 0.00325. This, along with other related observations, leads Jensen and Murphy to reject the agency model. They hypothesize that either CEOs are not important agents of the shareholders, or CEO incentives are unimportant because their actions depend only on innate ability or competence, or public and private political forces have imposed constraints on large payoffs for exceptional performance and hence reduce the pay-performance sensitivity.

Is the Jensen and Murphy evidence really a problem for agency models? The purpose of this paper is to consider this question in the context of a dynamic principal-agent relationship. Our model is a simple dynamic version of Holmstrom (1979), where the agent is the CEO and the principal is the shareholders. We solve the model numerically, and then, using model generated data on CEO compensation and shareholder wealth, we estimate pay-performance sensitivities in the same way as Jensen and Murphy did using real world data. We find that, first, there are circumstances where changes in CEO compensation and changes in shareholder wealth are positively correlated, and there are circumstances where they are negatively correlated, depending on the degree of risk sharing that is achieved with the optimal contract. We therefore claim that the hypothesis that changes in CEO compensation and in shareholder wealth should be positively and highly correlated, is at odds with the simple dynamic agency model we consider here. Second, our model is capable of generating data where the pay-performance sensitivity can be significantly positive but very small, just as in Jensen and Murphy's data. Our test for robustness shows that the above results are true for a wide variety of plausible parameter values. We therefore conclude that Jensen and Murph's empirical finding is in fact consistent with dynamic agency theory.

Our approach in this paper builds on the dynamic insurance analyses carried out by Spear and Srivastava (1987), Green (1987), Taub (1990), Phelan and Townsend (1990), Thomas and Worrall (1990), and Atkeson and Lucas (1992). These earlier papers show that, in environments where people are able to commit to long-term relationships, dynamic contracts can act both as efficient incentive devices to tackle information frictions and as risk sharing instrumrent to achieve optimal consumption smoothing. Our contribution to the literature is made in two ways. First, this paper directly tackles a quantitative issue whereas most previous work in the literature is theoretically focused. Second, this paper is new in the computational technique it uses to solve a dynamic insurance contract. [1] Our work here is compared with Phelan and Townsend (1990) who pioneered the computational line of research on dynamic contracting. In Phelan and Townsend, there is a social planner contracting with a continuum of agents, and they use lotteries to convexify spaces. In this paper, we are concerned with dynamic contracting between a risk neutral principal and a single risk averse agent, and we do not allow lotteries to play a role. Consequently, our innovation here is that a computational program based on Abreu, Pearce, and Stacchetti (1990)'s concept of self-generation is developed to compute the set of admissible states, before applying the contraction mapping theorem to the Bellman equation. Potentially, our program can be modified to solve other dynamic contracting problems. [2]

This paper, though, is not the first which tries to provide an answer to Jensen and Murphy's puzzle. Recently, Haubrich (1994) finds that, under certain assumptions, a simple static agency model can yield quantitative solutions in line with the empirical

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[1]Note that the models in the literature have somewhat different environments, but technically they all share a common method of dealing with the dynamic incentive problem. Namely, a state variable of some kind is defined to capture history dependence and, based on this, the problem of solving for the efficient super dynamic contract is reduced to one of solving a Bellman equation. In general, however, analytic solutions to Bellman equations are not easily derived. In the absence of a closed form solution, computational methods become an important vehicle to understanding the dynamics of the optimal contract and to confronting the model with data.

[2]For example, our program can be easily modified to solve a dynamic contracting problem with hidden endowment as in Thomas and Worrall (1990).

observations of Jensen and Murphy. In particular, he finds that for many parameter values, CEO compensation need increase only by about 10 dollars for every 1000 dollar additional shareholder wealth. Although valuable examination into the issue, Haubrich's evidence is less than totally satisfactory in that, first, for very plausible values of the coefficient of constant absolute risk aversion, greater than 0.5 specifically, the agency model solved predicts zero pay-performance sensitivities, indicating that compensation pay is state independent and the CEO always takes managerial actions that require the least effort. Second, although small but non-zero pay-performance sensitivities (ranging from 0.01 to 0.03) are predicted for lower values of the coefficient of absolute risk aversion, Haubrich relies on negative compensation in bad states, which is counterfactual, to generate sufficient incentives for the CEO to exert higher managerial effort than the minimum.

Compared to Haubrich, this paper takes a different approach. We explore implications of a dynamic, rather than static, agency model with Jensen and Murphy's hypotheses in mind. We argue that a dynamic agency model would provide a better theory benchmark, especially for the issue at hand. A dynamic agency model where CEO compensation, firm profit, and shareholder wealth are laid out explicitly will allow us to examine the issue under consideration in a more systematic way.[3] More

[3]Note that in Jensen and Murphy, longitudinal data are used in the estimation. These data are collected by following a large set of CEOs listed in the Executive Compensation Surveys for a period ranging from 1974 to 1986. When the CEOs are hired repeatedly, a sequence of one-period short contracts is dominated by a dynamic long-term contract. Second, in a static agency model, it is informationally meaningless to treat shareholder wealth (or firm value) and total net profit as two different variables. This is reflected in Haubrich where total net profit and shareholder wealth are used interchangably, even though in practice they stand respectively for two quite different entities both of a great deal of importance to the CEO and the shareholders. This is also reflected in Jensen and Murphy who have to argue that shareholder wealth as a measure of firm performance is superior to accounting profit, although empirically the latter can explain CEO compensation better than the former does. (Table 6, Jensen and Murphy (1990).) In our model, there is a natural distinction between the roles that these two variables play. Specifically, in our model, the expected discounted net profit of the firm from tomorrow on is readily interpreted as shareholder wealth, while current net profit is what the shareholders earn today.

important, a dynamic agency model will do better in confronting Jensen and Murphy's empirical observation. In Haubrich, the reason that the CEO is exerting the least managerial effort, even though he is not highly risk averse, is that the principal is unable to provide him with sufficient incentives. There the principal is assumed to have no access to other incentive tools other than current cash compensation. This problem is solved in our model which allows both current compensation and expected future utility to be used as incentive devices by the principal. In a dynamic environment, if the agent performs well today, he is rewarded both today and tomorrow. Similarly, if the agent performs badly today, then he is punished both today and tomorrow. Consequently, in our model, sufficient incentives can almost always be generated for the CEO to make the best managerial effort, and this is robust for a large set of plausible values of the coefficient of absolute risk aversion. For the same reason, unlike Haubrich, our model does not rely on negative compensation to achieve non-zero pay-performance sensitivities.

This paper is also related to an empirical study by Margiotta and Miller (1991), who also adopts a dynamic approach in dealing with a CEO compensation issue. However, their modeling strategy is quite different from ours. In Margiotta and Miller, it is assumed, among other things, that a complete set of markets exist, and that the optimal long-term contract can be implemented via a sequence of one-period short-term contracts. There, an intertemporal incentive scheme is not necessary at all, and the principal is able to reward good performance and punish bad performance immediately, rather than postponing it. Consequently, in Margiotta and Miller, the dynamics concerns only the savings behavior of the agent. This is in sharp contrast to the modeling strategy we adopt in this paper. Here, we assume the consumption good is perishable, and our model relies critically on an intertemporal compensation scheme built into the contract to deliver proper incentives to the agent.

The rest of the paper is organized as follows. In Section 3.2 we present the model and discuss an algorithm for solving it. In Section 3.3 we take a parameterized example of the model to the computer and characterize the optimal contract numerically. In Section 3.4, we use numerical results from Section 3.3 to address the issue concerning CEO compensation and shareholder wealth. In Section 3.5, various perturbations to the example solved in Section 3.3 are discussed to test for the robustness of the model's predictions. Section 3.6 concludes the paper.

3.2 A Dynamic Agency Model

In this section, we first formulate a dynamic version of the standard principal-agent model d·e to Holmstrom (1979), and then discuss an algorithm to solve it numerically. Our mode · essentially that of Spear and Srivastava (1987), except that there the principal has an unlimited access to a credit market where he can borrow at the discount rate, whereas here a closed economy is considered and compensation can not exceed current output. [4] In addition, our formulation of the model will be somewhat different from theirs and lead naturally to a two-step algorithm for computing the optimal contract.

Consider an infinite horizon world where time is discrete and denoted by $t = 0, 1, 2...$, where $t = 0$ is the ex ante contracting period. There are two agents in the model: the principal and the agent. They both are ex ante expected discounted utility maximizers and discount the future by the common discount factor $\beta \in (0, 1)$. There is a single perishable good in the model which the principal and agent consume. The principal is risk neutral but the agent is risk averse. The instantaneous utility

[4]Note that Spear and Srivastava are primarily interested in qualitative properties of the optimal contract which they characterize under the assumption that the solution to the Bellman equation is continuously differentiable. In this paper, we are more concerned with quantitative aspects of the optimal contract and a computational approach will be adopted to solve the optimal contract.

function of the agent, denoted by $v(c,a)$, is bounded and strictly concave in c. Here c is the amount of the consumption good consumed at the end of each date, and a is the unobservable action taken by the agent at the beginning of the date. Let A be the set of actions available. Assume by the choice of a, the agent then effectively chooses a probability distribution over $y = y(a) \in Y$, where Y is the set of possible outcomes. Assume Y has finitely many elements. Denote the probability distribution by $f(y;a)$. Assume $f(y;a) > 0$, $\forall y \in Y$ and $a \in A$. Also assume that $v(c_1, a) > v(c_2, a)$, for $c_1 > c_2$ and $\forall a$, and $v(c, a_1) < v(c, a_2)$, for $a_1 > a_2$ and $\forall c$.

The nature of this paper makes it important for us to focus on the relationship between the static model and the dynamic model. It is hence useful to write down the static contracting problem as a benchmark, before the dynamic one is described in detail. Here the static contracting problem is to choose an action $a \in A$ and a compensation scheme $w(y) \in [0, y], \forall y \in Y$ to maximize the expected income of the principal subject to the constraints that a is incentive compatible and that the contract delivers certain expected utility to the agent. Mathematically, the problem here is to

$$maximize \int_Y [y - w(y)] f(y;a) dy$$

subject to

$$\int_Y v(w(y), a) f(y;a) dy \geq \int_Y v(w(y), a') f(y;a') dy, \ \forall a' \in A,$$

and

$$\int_Y v(w(y), a) f(y;a) dy = v,$$

where the first constraint is incentive compatibility which ensures that it is not in the agent's interest to deviate from the action recommened. The second constraint promises that a given expected utility v will be delivered to the agent.

In a dynamic setting, the strategy of formulating the contracting problem is essentially the same as that used in formulating the static problem, except now history comes into the decision variables. Specifically, the problem here is to construct a history-dependent compensation scheme $\{w_t(h^t)\}$ where $h^t = \{y_1, y_2, \ldots, y_t\} \in Y^t$ is the history up to date t and y_t is the date-t realization of outcome, along with an action recommendation plan $\{a_t(h^{t-1})\}$, which maximizes the ex ante expected discounted income of the principal, subject to the constraint of incentive compatibility, and to delivering a given ex ante expected discounted utility V_0 to the agent. A central issue here is how to model history dependence in a tractable way. Spear and Srivastava (1987) use the expected discounted utility of the agent as the state variable, similar to the capital stock in a growth model, to summarize history. Note that this approach is also taken by Green, Phelan and Townsend, Thomas and Worrall, and Atkeson and Lucas in related models. At any date $t \geq 1$, given state V_{t-1} that is determined by forgone history h^{t-1}, the expected discounted utility of the agent from tomorrow on, V_t, along with the action to take today, a_t, and current compensation to the agent, w_t, is considered as a choice variable today. Here a_t is a function of V_{t-1}: $a_t = a_t(V_{t-1})$. V_t and w_t are both functions of V_{t-1} and output realization today: $V_t = V_t(y_t, V_{t-1})$, $w_t = w_t(y_t, V_{t-1})$, and V_t is to serve as the state tomorrow. In this way, the problem of solving the history-dependent supercontract can be decomposed into solving a sequence of static problems which can then be described in a Bellman equation. We now go on to formalize this.

Let $\sigma = \{a_t(h^{t-1}), w_t(h^t)\}_{t=1}^{\infty}$ denote a contract. Call σ feasible if

$$a_t(h^{t-1}) \in A, \quad \forall t > 0, h^{t-1} \in Y^{t-1}; \tag{3.1}$$

$$0 \leq w_t(h^t) \leq y_t, \quad \forall t > 0, h^t \in Y^t. \tag{3.2}$$

Here (3.2) is the compensation constraint which says that the principal can not borrow from outside parties and compensation must be bounded above by total current output.[5] Note that (3.2) also requires that the agent can not consume a negative amount of the consumption good.

At any date t, given history h^t, the continuation profile of σ from date $t+1$ on is denoted by $\sigma|h^t$. Note that $\sigma = \sigma|h^0$, where h^0 represents the null history. Given that the agent follows the action recommendation scheme in the continuation profile $\sigma|h^t$ from $t+1$ on, the contract dictates a continuation value $V(\sigma|h^t)$ for the agent, and $U(\sigma|h^t)$ for the principal, from date $t+1$ on. We can now define σ to be incentive compatible if, for all $t > 0$ and all $h^{t-1} \in Y^{t-1}$,

$$\int [v(w_t(h^t), a_t(h^{t-1})) + \beta V(\sigma|h^t)] f(y_t; a_t(h^{t-1})) dy_t \geq \int [v(w_t(h^t), a_t^{'}) + \beta V(\sigma|h^t)] f(y_t; a_t^{'}) dy_t, \quad \forall a_t^{'} \in A. \tag{3.3}$$

The left side of (3.3) is the agent's expected discounted utility if he chooses, at date t, to take the recommended action $a_t(h^{t-1})$, and to follow the action recommendation scheme in $\sigma|h^t$ from $t+1$ on. The right side of (3.3) is the agent's expected discounted utility if he takes an arbitrary action $a_t^{'}$ at date t, and then follows the action recommendation scheme in $\sigma|h^t$ from $t+1$ on. Therefore (3.3) guarantees that the agent would not benefit from any one-step deviation from the recommended action plan. Here it is important to note that, given $0 < \beta < 1$ and that the agent has bounded utilities, the unimprovablity principle applies and (3.3) essentially implies that the agent can not benefit from deviating from the action recommendation plan in σ at all. In other words, (3.3) guarantees that $\{a_t(h^{t-1})\}$, the whole action recommendation plan, will be actually adopted by the agent. Furthermore, this is subgame perfect, in the sense that at the beginning of any date t, given any history h^{t-1}, the agent

[5]This assumption will be relaxed in Section 5 where the principal is allowed to have limited access to a credit market. Also note here that, implicitly, the agent can not borrow either.

will have no incentive to deviate in future dates from the continuation of that plan conditional on h^{t-1}, i.e., $\{a_\tau(h^{\tau-1})|h^{t-1}\}_{\tau=t}^{\infty}$.

Let $\mathcal{V}$ be the set of all expected discounted utilities of the agent that can be generated by a feasible and incentive compatible contract. i.e.,

$$\mathcal{V} \equiv \{V \in \mathcal{W}|\ \exists\ \sigma\ s.t.\ (3.1),\ (3.2)\ , (3.3)\ , V(\sigma|h^0) = V\}$$

where $\mathcal{W}$ is the space on which V is allowed to take values. Assume $\mathcal{W}$ is nonempty, compact, and is such that $\mathcal{V}$ is nonempty, too. Now for each $V \in \mathcal{V}$, the optimization problem that the principal faces can be specified as follows:

$$\max_{\sigma} U(\sigma|h^0)\ s.t.\ (3.1), (3.2), (3.3),\ V(\sigma|h^0) = V.$$

Call a solution to the above problem an optimal contract delivering expected discounted utility V to the agent. Now for each $V \in \mathcal{V}$, let

$$\mathcal{U}(V) \equiv \{U(\sigma|h^0)|\ \sigma\ s.t.\ (3.1),\ (3.2),\ (3.3),\ V(\sigma|h^0) = V\}. \tag{3.4}$$

$\mathcal{U}(V)$ is the set of feasible and incentive compatible expected discounted incomes of the principal, given that V must be delivered to the agent. The following Lemma guarantees that an optimal contract exists, for every V in $\mathcal{V}$.

Lemma 1 $\mathcal{U}(V)$ is compact, $\forall V \in \mathcal{V}$.

Having specified the optimization problem and shown that optimal contracts exist, we are now in a position to characterize them in a Bellman equation. First, for all $V \in \mathcal{V}$, given Lemma 1, we can define

$$U^*(V) \equiv \max\{U(\sigma|h^0) \in \mathcal{U}(V)\}.$$

That is, $U^*(V)$ is the best expected discounted income achievable by a feasible and incentive compatible contract which delivers expected discounted utility V to the agent. Note that in what follows, function U^* will serve as the value function in the Bellman equation. Next, define an operator Γ, which maps from the space of bounded continuous functions $U: \mathcal{V} \to R$ with the sup norm into itself, as follows:

$$\Gamma(U)(V) \equiv \max_{a(V), w(y,V), \bar{V}(y,V)} \int [y - w(y,V) + \beta U(\bar{V}(y,V))] f(y; a(V)) dy$$

subject to

$$\int [v(w(y,V), a(V)) + \beta \bar{V}(y,V)] f(y; a(V)) dy \geq$$
$$\int [v(w(y,V), a') + \beta \bar{V}(y,V)] f(y; a') dy, \quad \forall a' \in A \tag{3.5}$$

$$\int [v(w(y,V), a(V)) + \beta \bar{V}(y,V)] f(y; a(V)) dy = V \tag{3.6}$$

$$a(V) \in A \tag{3.7}$$

$$0 \leq w(y,V) \leq y, \quad \forall y \in Y \tag{3.8}$$

$$\bar{V}(y,V) \in \mathcal{V}, \quad \forall y \in Y \tag{3.9}$$

Note that here the three decision variables $a(V)$, $w(y,V)$, and $\bar{V}(y,V)$ are the one-step counterparts of $a_t(h^{t-1})$, $w_t(h^t)$, and $V(\sigma|h^t)$ respectively. Conditions (3.7), (3.8), (3.5), and (3.6) are the one-step counterparts of the four constraints in the principal's optimization problem respectively. Finally, (3.9) reflects the idea that, at any date, looking forward, the contractors face exactly the same set of feasible and incentive compatible expected discounted utilities, $\mathcal{V}$, on which the contract's continuation value to the agent can take values. The following Proposition shows U^* is a fixed point of Γ.

<u>Proposition 1</u> $U^*(V) = \Gamma(U^*)(V)$, $\forall V \in \mathcal{V}$.

Note that all the proofs in this paper are relegated to Appendix I. Now with the Bellman equation described above, the optimal contract evolves as follows. At the beginning of an arbitrary date t, given state V_{t-1} which is determined yesterday, $a_t = a(V_{t-1})$ is recommended by the principal to the agent as the optimal action to take at date t. By the end of date t, suppose output y_t is observed, then $w_t = w(y_t, V_{t-1})$ is paid to the agent as current compensation, and $V_t = \bar{V}(y_t, V_{t-1})$ is promised to the agent as his expected discounted utility tomorrow. Correspondingly, for the principal, he gets $y_t - w(y_t, V_{t-1})$ for current consumption and $U_t = U^*(V_t)$ as expected discounted income from tomorrow on. Now as the date moves from t to $t+1$, every variable is to be determined in the same way except now the state is V_t rather than V_{t-1}. The contract thus rolls forward date by date, and at the beginning of each date, only the state counts. Note it is now clear that $\mathcal{V}$ is the set of all admissible states.

As predicted earlier, the problem of solving the optimal history-dependent supercontract is now reduced to one of solving the Bellman equation in Proposition 1. Suppose $\mathcal{V}$ is known, then it is easily verified that operator Γ satisfies Blackwell's sufficient conditions, i.e., monotonicity and discounting, for a contraction. It thus follows, by the contraction mapping theorem, that the Bellman equation in Proposition 1 has a unique solution which can be obtained through an iterative procedure, starting with an arbitrary guess for the value function.

The rest of the section is devoted to finding a way of solving for the set $\mathcal{V}$, before applying the contraction mapping theorem to the Bellman equation. To this end, we borrow from Abreu, Pearce and Stacchetti (1990) the concept of self-generation, and show that $\mathcal{V}$ must be the largest set that can generate itself. This, in turn, leads to an algorithm to compute $\mathcal{V}$.

Define an operator B in the following way. For an arbitrary set $\Phi \subset R$,

$$B(\Phi) = \{V \in \mathcal{W} | \exists\, a(V),\ w(y,V),\ \tilde{V}(y,V)\ s.t.\ (3.5), (3.6), (3.7), (3.8), (3.10)\}$$

where (3.10) is the following:

$$\tilde{V}(y,V) \in \Phi. \tag{3.10}$$

It is useful to note that B is monotone increasing in the sense that $\Phi_1 \subseteq \Phi_2$ implies $B(\Phi_1) \subseteq B(\Phi_2)$. Then, we call Φ self-generating if

$$\Phi \subseteq B(\Phi),$$

i.e., Φ is self-generating if its image under B contains Φ itself.

In Appendix I it is shown in Lemma 2 that, first, $\mathcal{V}$ is self-generating. Second, if a set Φ is self-generating, then $B(\Phi) \subseteq \mathcal{V}$. These are important properties of the mapping B which lead us to the following proposition.

Proposition 2 (i) $\mathcal{V} = B(\mathcal{V})$; (ii) Let $W_0 = \mathcal{W}$, and let $W_{n+1} = B(W_n)$, for $n = 0, 1, 2....$ Then $\lim_{n \to \infty} W_n = \mathcal{V}$.

What this proposition provides us with is essentially an algorithm for computing the set $\mathcal{V}$. Starting with the set $\mathcal{W}$, and operating on it iteratively using the operator B, we will be led to $\mathcal{V}$ which, due to (i), is a fixed point of B. In the following section where we take a parameterized example of the model to the computer, A and $\mathcal{W}$ are assumed to be finite discrete spaces. Therefore $\mathcal{V}$ is also finite and discrete, and convergence will occur after a finite number of iterations.

To conclude this section, we developed a two-step algorithm for solving (or computing) the optimal contract. The first step is to solve $\mathcal{V}$ through an iterative procedure formulated in Proposition 2. Then, with $\mathcal{V}$ solved, the second step is to find the solution to the Bellman equation in Proposition 1, and hence the optimal contract, through another iterative procedure which is indicated by the contraction mapping theorem.

3.3 Computing the Optimal Contract: An Example

In this section, the dynamic agency model described in the previous section is parameterized and taken to the computer for numerical solution. Our objective here is to determine quantitatively the structure of the optimal contract for particular parameter values and functional forms. For computational convenience the model is parameterized in a rather simple way, though our program is easily adapted to more sophisticated parameterizations. Specifically, the instantaneous utility function of the agent is assumed to be exponential, i.e.,

$$v(c,a) = -e^{\gamma(a-\alpha c)},$$

where γ is the coefficient of absolute risk aversion, and α measures how the agent would value leisure against consumption. In the example we compute in this section, we set as a benchmark $\gamma = \alpha = 1$. Note that an exponential utility function is not only computationally easy to handle, but also provides a simple interpretation of effort as negative consumption. Here, to the agent, his cost of making one unit of effort is equivalent to α units of the consumption good. [6] Now for the production technology, there are available to the agent two effort levels,

$$A = \{a_L = 0.1, a_H = 0.2\}.$$

That is, the agent can choose either to shirk or to work diligently, and $a_H > a_L$ indicates that diligence is more distasteful than shirking. Next, assume output can

[6] In general, α can be a function of c. We assume α to be constant for simplicity.

either be low or high, i.e.,

$$Y = \{y_L = 0.4, y_H = 0.8\},$$

and the probability distribution that links action to output is as follows:

$$Prob(y_L|a_L) = Prob(y_H|a_H) = 2/3;$$

$$Prob(y_L|a_H) = Prob(y_H|a_L) = 1/3.$$

These probabilities are designated to capture the idea that the more diligently the agent works, the higher the probability of a high output. Lastly, for the discount factor, we set $\beta = 0.96$.

Note that at this stage, the parameter values are given quite arbitrarily. In Section 3.5 we will examine quantitatively how changing the magnitudes of the parameters would affect the structure of the optimal contract and hence its implications for the issue at hand. We will show that, for a wide variety of plausible parameter values, our model's predictions concerning CEO compensation and shareholder wealth are quite robust.

Now for the space of the state variable, a hundred grid points are designated uniformly over $[\underline{V}, \overline{V}]$, where

$$\underline{V} = -exp(a_H)/(1-\beta), \quad \overline{V} = -[\frac{1}{3}exp(a_L - y_L) + \frac{2}{3}exp(a_L - y_H)]/(1-\beta).$$

Here $\underline{V}$ is the expected discounted utility of the agent if he works the most and consumes zero at every date, and $\overline{V}$ is his expected discounted utility if he works the least but enjoys consuming everything that could be produced if only he works the most at every date. Obviously then, anything below $\underline{V}$ or above $\overline{V}$ is not admissible. Each grid point corresponds to a potential expected discounted utility level, denoted with $V(K)$, i.e., $\mathcal{W} = \{V(K), K = 1, 2, ..., 100\}$, where $V(1) = \underline{V}$ and $V(100) = \overline{V}$.

The program takes two steps to reach the solution, as was indicated in the previous section. First, following Proposition 2, the operator B is applied iteratively upon set W_0 to reach the set of admissible states $\mathcal{V}$. It is found that convergence occurs after a finite number of iterations, and $\mathcal{V} = \{V(K), K = 19, ..., 90\}$. In the second step, starting with an all-zero initial guess for the value function in the Bellman equation, an iterative procedure is then applied to reach the true value function and the optimal decision rules.

The solution to the Bellman equation is composed of eight vectors: $K, U(K), V(K), A(K), w(L,K), w(H,K), V(L,K), V(H,K)$, $(K = 19, ..., 90)$. Here K is the index of the state, i.e., the expected discounted utility of the agent. $U(K)$ and $V(K)$ respectively are the values of the expected discounted utilities of the principal and the agent when the state is K. $w(I,K)(I = L,H)$ is current compensation to the agent given . ate K and the realization of current output y_I. And finally, $V(I,K)(I = L,H)$ is the index of tomorrow's state given today's state K and output realization y_I. These vectors fully describe how the optimal contract works. Suppose for instance that the contract starts with an initial state K_0. that is, the agent is promised an ex ante expected discounted utility $V(K_0)$. Looking into the K_0th line of the Bellman equation, the principal will find $U(K_0)$ as his ex ante expected discounted income, and $A(K_0)$ the optimal action to recommmend to the agent for the first date. Then at the end of the first date, if a low (high) outcome $y_L(y_H)$ is realized, $w(L,K)(w(H,K))$ is paid to the agent as current compensation. Meanwhile, the agent is entitled to an expected discounted utility at the level of $K_1 = V(L,K)(V(H,K))$ from date 2 on. As the two agents move from date 1 to date 2, K_1 then serves as the new state. In this way the contract rolls forward infinitely.

As a benchmark, for given parameters, the static model in Section 3.2 is also solved numerically, at the reservation expected utilities corresponding to the admissible expected discounted utilities in the dynamic model. Specifically, these expected utilities are $\{V_s(K)\}, K = 19, ..., 90$, where $V_s(K) = (1-\beta)V(K)$. For these reservation expected utilities, the solution to the static model was found to exist. Without confusion in notation, for each K, the optimal compensation rule is also denoted by $w(I, K), I = L, H$, and recommended action $A(K)$.

We now go on to examine the structure of the optimal contract computed. First, allowing history dependence and using expected discounted utility as an additional incentive device in the dynamic model induces better effort. It turns out that for the dynamic contract, the optimal actions are: $A(19) = a_L$, and $A(K) = a_H, K = 20, ..., 90$. That is, the high effort action is implemented in the optimal contract except for the case where the initial expected discounted utility is the lowest. This is compared with the static case where the corresponding optimal actions are: $A(K) = a_L, K = 19, ..., 33$, and $A(K) = a_H, K = 34, ..., 90$. Better effort in turn brings higher utilities and this is reflected in Figure 3.1 which plots the value function of the dynamic model in contrast with that of the static model. The value function of the dynamic model exceeds that of the static model uniformly, and particularly for Ks ranging from 20 to 33, where high actions can be implemented in the dynamic model but not in the static model. Note that, as $V(K)$ approaches its highest value, the two value functions tend to merge together. In other words, the dynamic model does not improve upon the staic model significantly (and not at all for $K = 85, ..., 90$), if a very high expected discounted utility is reserved for the agent. An explanation for this will be given later when the properties of the decision rules are examined. Figure 1 also shows that the value function of the dynamic contract is concave. [7] Finally, as in Phelan and Townsend, for the dynamic model, the value function has an upward-sloping portion. This is because the only way to make the lowest expected

[7] A formal proof of the concavity is in Spear and Srivastava (1987).

discounted utility incentive compatible is to let the agent take the low action and consume the minimum.

Figure 3.2 is the graph for the decision rules concerning tomorrow's state, i.e., $V(L, K)$ and $V(H, K)$. It tells us how the state variable that summarizes history evolves. First observe that these decision rules are both monotone in K. The implication therefore is: given the same output realization today, the agent who did well in the past and hence is in a better welfare position today will continue to be in a better welfare position tomorrow. Note that this is how history affects future welfare of the agent. Also observe that $V(L, K) \leq K \leq V(H, K)$ (for most Ks, the inequalities are strict). This is how current performance affects future welfare. What this tells us is that the agent will be in a better welfare position tomorrow (than today) if output today takes on the high value, and a lower welfare position tomorrow if output today takes on the low value. In other words, if the agent does well today, he is promised more in the future, and vice versa. Here we see how expected discounted utility is used as an incentive device to reward good outcomes and punish bad ones and hence induce good effort today. Perhaps more interesting is the observation that, on the two tails of the graph, the two functions overlay on each other. Specifically, $V(L, 19) = V(H, 19) = 19$, and $V(L, K) = V(H, K) = K, K = 85, ..., 90$. Obviously, either $V(L, 19) < 19$ or $V(H, 19) > 19$ would mean that $V(19)$ is not the minimal feasible expected discounted utility. The explanation for what happens on the other tail has to do with the concavity and monotonicity of the value function and is to be discussed shortly.

Figure 3.3.a is the graph for the compensation policies $w(L, K)$ and $w(H, K)$. Parallel to the case concerning expected discounted utilities tomorrow, here we can also make three observations. First, these compensation policies are monotonic in K.

[8] This implies that given the same output realization, the agent who did relatively well in the past and hence holds a better welfare position today gets to consume a larger share of total current output than a agent who holds a lower welfare position. This is where history affects current consumption. Next, observe that compensation is (weakly) monotone in output: $w(H, K) \geq w(L, K)$, i.e., higher output means higher current consumption, and low output lower current consumption. Note that this is how current performance affects current consumption. In this way, current compensation is used as another incentive device to reward good outcomes and punish bad ones. Third, observe that $w(L, K) = w(H, K)$, for $K = 28, ..., 62$, i.e., in these states, current compensation does not play a role as an incentive instrument. In other words, in these states, the optimal contract relies solely on expected discounted utility to achieve incentive compatibility. Compare this with what occurs in states 85 through 90, where it is current compensation, rather than expected discounted utility, which is used as the dominant incentive instrument.

The fact that the two incentive instruments, current compensation and expected discounted utility, may dominate each other in the role they play as incentive devices can be explained in the following way. The key lies on the shape of the value function. As the expected discounted utility of the agent approaches its highest possible value, i.e., as K gets close to 90, since the value function is concave and decreasing, the cost to the principal (measured in terms of his expected discounted income) of using expected discounted utility as an incentive device to reward a good outcome increases. Eventually, expected discounted utility is too costly to be used at all and hence current compensation becomes the dominant incentive tool. Note that now the optimal dynamic contract essentially degenerates to the one-period static one and

[8]Except for $w(H, K)$ at $K = 85$, where the obvious reason goes to the nonconvexity of the state space.

this explains the overlay of the two value functions for very high expected utilities in Figure 1. On the other hand, as K goes down from 90, the opposite occurs and it becomes less and less costly to use expected discounted utility to reward good outcomes, and for $K = 65, ..., 84$, both expected discounted utility and current compensation are used as incentive tools. As K goes down further, the expected discounted utility as an incentive tool becomes so efficient that it precludes any role as an efficient incentive device for current compensation. This is why for Ks ranging from 28 to 62, current compensation is independent of current output. However, finally as K gets close to the lowest level, the value function is now concave but increasing, expected discounted utility now gets increasingly costly to be used as an centive device to punish bad outcomes, and this is why we see current consumption return to play an incentive role together with expected discounted utility.

Finally, Figure 3.3.a is to be compared with Figure 3.3.b, which plots the compensation rules for the static model, to see how our dynamic contract functions as a better risk-sharing arrangement than the static contract. Clearly, allowing history dependence and letting expected discounted utility play an incentive role greatly narrows the distance between $w(L, K)$ and $w(H, K)$ in the static model through pushing $w(L, K)$ upward and $w(H, K)$ downward. This is optimal because the risk-averse agent's compensation is now less closely tied to the current output realization, which in turn means smaller fluctuations in consumption. This is achieviable in the dynamic model because some of the fluctuations in consumption generated by the randomness of output realizations are now shifted to that in expected discounted utility. In other words, here expected discounted utility is helping share the risk which in a static model falls entirely on current compensation. Figure 3.4 shows the agent's consumption is greatly smoothed by entering the optimal dynamic contract. Figure 3.5 shows how the agent's expected discounted utility fluctuates over time, compared to the static case where the agent has a constant expected utility.

3.4 CEO Compensation and Shareholder Wealth

The aim of this section is to use the optimal contract solved numerically in the previous section to tackle the CEO compensation issue which motivated this paper. Remember Jensen and Murphy have claimed that real world practice concerning CEO compensation and shareholder wealth is not consistent with the prediction of formal agency models. Their key argument is that the observed pay-performance sensitivity, defined as the dollar change in the CEO's compensation associated with a dollar change in shareholder wealth, though significantly positive, is too low to provide the CEO with proper incentives. With static agency models in mind, they seem to believe that CEO compensation and shareholder wealth should be positively correlated and that the pay-performance sensitivity should be high. In this section we want to ask the following two related questions: First, does our model imply a positive and reasonably high correlation between CEO compensation and shareholder wealth, measured either in levels or in first differences? Second, is our model capable of producing, using model-generated data, pay-performance sensitivities that are significantly positive, but very small, similar to what Jensen and Murphy observed with real world data?

Our first step is to interpret the model to fit the CEO-shareholders environment we are dealing with. The principal in our model is interpreted to be shareholders and the agent is the CEO. The outcome at each date is interpreted as current profit. The expected discounted income of the principal, i.e., U, is readily interpreted as shareholder wealth in Jensen and Murphy's language, while w is just the CEO's current compensation. Note that for simplicity of expression, in the rest of this section, we will ignore the tails of the vectors which constitute the Bellman equation.

In other words, we will not be interested in cases where the CEO has extremely low or extremely high levels of expected discounted utility.

We now proceed to examine the correlation between the levels of CEO compensation and shareholder wealth. Remember y_t, w_t, U_t are date-t profit, CEO compensation, and shareholder wealth respectively, and V_{t-1} is the time-t state, which is determined by the history to date t-1. Therefore, according to the way our dynamic contract works, $w_t = w(y_t, V_{t-1})$ and $U_t = U^*(\bar{V}(y_t, V_{t-1}))$. That is, w_t and U_t are correlated because they are both determined by y_t and V_{t-1}. To determine the direction of the correlation, first we hold y_t fixed. Then w_t changes positively with V_{t-1} for the compensation rule $w(I, K)$ is monotone increasing in K. However, U_t changes negatively with V_{t-1} for the value function is decreasing but $V(I, K)$ is increasing in K. Therefore, for a fixed realization of current profit, there is nonpositive correlation between w_t and U_t. Next, we hold V_{t-1} fixed. Observe now that w_t decreases as y_t increases, and U_t increases as y_t decreases. This is due to the facts observed in the previous section that for fixed K, $w(H, K) > w(L, K)$, and $V(H, K) > V(L, K)$. Note that the second inequality implies $U(H, K) < U(L, K)$. Thus. holding V_{t-1} fixed, there is again a nonpositive correlation between w_t and U_t. Intuitively, this is easy to explain. Notice that in the optimal contract, expected discounted utility and current compensation are used as incentive and consumption-smoothing devices. Suppose the CEO receives a good output shock. Then, for incentive purposes, he should be allowed to consume more (than he would if a bad output shock were received) today. However. for the purpose of consumption smoothing, the agent should not consume the whole fruit of the surplus today. Instead, part of the fruit is saved for future consumption, in the form of a greater expected discounted utility. But this in turn means a lower expected discounted income for the shareholders. Conversely, suppose a bad output shock is received. Then, the CEO, in general, should not bear this bad luck entirely today. Instead, he should consume more (than what he would) by borrowing from tomorrow which in turn means a lower expected discounted utility for him but a higher expected discounted income for the shareholders. Now to summarize, we find there is a nonpositive correlation between w_t and U_t. Of course, holding output realizations y_t and y_{t-1} fixed, w_t is also negatively correlated with U_{t-1} for the latter decreases when V_{t-1} increases, and increases when V_{t-1} decreases Further, w_t is also negatively correlated with any U_{t-s} $(s > 1)$, for the simple reason that, holding output realizations fixed, U_t increases when U_{t-s} increases. and decreases when U_{t-s} decreases.

To summarize then, our first observation here about the relationship between CEO compensation and shareholder wealth is that, in levels, they are nonpositively correlated. Although this is a straightforward implication of the model, it is important for the issue at hand. While it is natural for Jensen and Murphy to draw from a static agency model the notion that CEO compensation should be positively correlated with shareholder wealth, what a simple dynamic agency model like ours tells us is that there is no reason to expected this.

However, in Jensen and Murphy's estimation of what they call pay-performance sensitivity, the regression variables are defined in first differences. In general, nonpositive correlation between two variables does not necessarily imply nonpositive correlation between the first differences of those variables. One then wonders what our model says about the correlation between the first difference in CEO compensation and that in shareholder wealth? To analyze this, we fix the initial state at some K_0. Let y_1 and y_2 be the realizations of profit at date 1 and date 2. Correspondingly, CEO compensations are w_1 and w_2, values of shareholder wealth are U_1 and U_2, and next period's states, K_1 and K_2. Let $\Delta y = y_2 - y_1$, $\Delta w = w_2 - w_1$, $\Delta U = U_2 - U_1$. We want to examine how Δw and ΔU are correlated locally around state K_0. Here we have four different cases to deal with.

$$(1) : \Delta y = 0 \ (y_1 = y_H, y_2 = y_H), \Delta w > 0, \Delta U < 0;$$

$$(2) : \Delta y < 0 \ (y_1 = y_H, y_2 = y_L), \Delta w =?, \Delta U > 0;$$

$$(3) : \Delta y > 0 \ (y_1 = y_L, y_2 = y_H), \Delta w =?, \Delta U < 0;$$

$$(4) : \Delta y = 0 \ (y_1 = y_L, y_2 = y_L), \Delta w < 0, \Delta U > 0;$$

where $\Delta w =?$ means that Δw can be either positive or negative. We now explain these cases in detail. In case (1), $y_2 = y_H$ implies $K_2 > K_1$, for $V(H, K) > K$, which in turn means $U_2 < U_1$, or $\Delta U < 0$, since the value function is decreasing. To determine the sign of Δw, notice that $y_1 = y_H$ implies $K_1 > K_0$, therefore, $\Delta w = w(H, K_1) - w(H, K_0) > 0$, as $w(H, K)$ is increasing in K. Case (4) is the opposite of case (1) and can be analyzed in a parallel way. Therefore, in both cases (1) and (4), CEO compensation and shareholder wealth move in opposite directions. The mechanism at work here, as the above analysis has revealed, is one which provides proper incentives to the CEO.

Cases (2) and (3) are a little more interesting. In Case (2), $y_2 = y_L$ implies $K_2 < K_1$ and hence $\Delta U > 0$. For Δw, we can write

$$\Delta w = [w(L, K_1) - w(L, K_0)] + [w(L, K_0) - w(H, K_0)] \tag{3.11}$$

Now that $y_1 = y_H$, we have $K_1 > K_0$ for $V(H, K) > K$. But $w(L, K)$ is increasing in K, we then know the first term on the right side of equation (3.11) is positive. But the second term is nonnegative, obviously. This explains why the sign of Δw is undetermined. Here if V_0 is a state where the distance between $w(L, K_0)$ and $w(H, K_0)$ is less than that between $w(L, K_1)$ and $w(L, K_0)$, then $\Delta w > 0$, otherwise, $\Delta w < 0$. Obviously then, at least for K_0s where $w(L, K_0) = w(L', K_0)$, w and U will move in the same direction. Case (3) can be analysed in the same way. Here $y_2 = y_H$

implies $\Delta U < 0$ and

$$\Delta w = [w(H, K_1) - w(H, K_0)] + [w(H, K_0) - w(L, K_0)] \tag{3.12}$$

where $K_1 < K_0$, $w(H, K_1) - w(H, K_0) < 0$, and $w(H, K_0) - w(L, K_0) \geq 0$. Here we have with certainty $\Delta w < 0$, and hence w and U move together, for those initial states where $w(L, K_0) = w(H, K_0)$.

Here, it is important to note that, in a partial sense, $|w(L, K_0) - w(H, K_0)|$ stands as a measure for the extent of risk-sharing or consumption smoothing that is achieved with the optimal contract around state K_0. In the static case where there is relatively poor risk-sharing, $|w(L, K_0) - w(H, K_0)|$ is large, as plotted in Figure 3.3.b. In contrast, a necessary condition for full insurance or perfect consumption smoothing will be $|w(L, K_0) - w(H, K_0)| = 0$. In the situation under current consideration, smaller $|w(L, K_0) - w(H, K_0)|$ means less heavy use of current compensation as an incentive device because there are other more efficient ones available. This in turn means more consumption smoothing can be achieved. Here of course a more obvious fact is that a smaller $|w(L, K_0) - w(H, K_0)|$ implies less uncertainty in the CEO's current consumption. In particular, $|w(L, K_0) - w(H, K_0)| = 0$ implies no risks at all in current consumption. With the above notion in mind then, we can now claim that better rish-sharing achieved in the optimal contract gives more chances for Δw and ΔU to move in the same direction. We will shortly demonstrate that this, in turn, is crucial to the explanation of a positive correlation between Δw and ΔU.

Before we go on to summarize the implications of the four cases concerning the relation between Δw and ΔU, it is interesting enough to observe that CEO compensation and current profits may move in opposite directions. In particular, $\Delta w > 0$ in case (2) indicates that a CEO may get a raise in compensation even when the firm is experiencing a decrease in profit. Similarly, in case (3), $\Delta w < 0$ implies that the CEO may face a compensation cut even when the firm is experiencing a gain in profit. The explaination for these again has to do with the better risk sharing which is achieved by allowing history dependence in our model. Particularly, in case (2), $y_1 = y_H$ indicates that the CEO is in a better welfare position at date two than at date one. Now that current compensation is independent of profit realization and is monotone increasing in the state, the CEO will get an increase in compensation no matter what date-2 profit is. Here, the increase in compensation is certainly not a reward for the bad profit realization today, it is a postponed reward for the CEO's good performance yesterday. A similar story can be told to explain why $\Delta w < 0$ can be the case in case (3) where the decrease in compensation is a postponed punishment for bad performance yesterday. Finally, we should note that in circumstances where good and bad outputs alternate and current compensation is used heavily as an incentive instrument, i.e., when current compensation depends heavily on current profit, w and y will move together ($\Delta w < 0$ in case (2) and $\Delta w > 0$ in case (3).)

The implications of the four cases in determining the correlation between Δw and ΔU can be best understood by locating graphically the pairs $(\Delta w, \Delta U)$ associated with different cases on a coordinate plane. For case (1), $(\Delta w, \Delta U)$ is a point in the second quadrant, and for case (4), the forth quadrant. For cases (2) and (3), the situation is a little more complicated. Suppose first there is insufficient risk-sharing with the optimal contract, in the sense that $|w(L, K_0) - w(H, K_0)|$ is not close enough to zero, and we have $\Delta w > 0$ in case (2) and $\Delta w < 0$ in case (3). Then $(\Delta w, \Delta U)$ is in the second quadrant for case (2) and the fourth quadrant for case (3). This situation is depicted in Figure 3.6.a for a specific initial state $K_0 = 70$. It appears that Δw and ΔU in Figure 3.6.a are negatively correlated. Next, suppose there is sufficient risk-sharing, in the sense that $|w(L, K_0) - w(H, K_0)|$ is close enough to zero, and hence $\Delta w > 0$ in case (2) and $\Delta w < 0$ in case (3). Then, $(\Delta w, \Delta U)$ is in the first quadrant for case (2) and the third quadrant for case (3). This situation is depicted in Figure 3.6.b for a specific initial state $K_0 = 30$. Now we can see clearly how better risk-sharing makes a difference in terms of determining the sign of the correlation between Δw and ΔU. In Figure 3.6.b, the clear negative correlation between Δw and ΔU as observed in Figure 3.6.a is gone. Essentially, what better risk-sharing does is to put $(\Delta w, \Delta U)$ corresponding to case (2) in the first quadrant, and $(\Delta w, \Delta U)$ corresponding to case (3) in the third quadrant. With less risk-sharing, these two points are in the second and the fourth quadrants respectively, as shown by Figure 3.6.a. The consequence of this, as is clear from the figures, is an increase in the chance for Δw and ΔU to be positively correlated with each other.

Two preliminary conclusions can now be drawn. First, our model does not predict with certainty a positive correlation between Δw and ΔU. Particularly, in circumstances where there is insufficient risk-sharing achieved with the optimal contract, as is the case depicted in Figure 3.6.a, the correlation between Δw and ΔU may well be negative. Second, our model does not rule out positive but small pay-performance sensitivities, as found by Jensen and Murphy with real world data. With CEOs moving in states associated with good risk-sharing achievable with the optimal contract, observations of $(\Delta w, \Delta U)$ will be scattered over all the four quadrants in the coordinate plane. Conceivably, the conflicting forces in the formation of the average correlation between Δw and ΔU may offset each other and the outcome may just be a slight dominance of the positive over the negative, which, in turn, means a positive but small pay-performance sensitivity. To obtain sharper insightment into this, we next will conduct a simple experiment aimed at obtaining more specific quantitative information about the pay-performance sensitivities our model may deliver.

In the following experiment, we estimate, using model-generated data, pay-performance sensitivities across the spectrum of initial expected discounted utilities. To be consistent with Jensen and Murphy's work, for each estimation of the pay-performance sensitivity, a large number of CEOs are involved who each is recorded for 13 consecutive time periods. Specifically, the experiment is conducted in the following way. First, for each of the initial expected discounted utility levels from $V(25)$ to $V(85)$, 2000 sequences, representing 2000 CEOs starting with the same initial expected discounted utility, each containing 100 randomly generated values of profit are created. The random number generator is designed in accordance with the action recommendation scheme in the optimal contract and the action-output probability distribution that is specified at the beginning of Section 3.3. Then, for each of these sequences, two more sequences are generated through the Bellman equation solved. The first is a sequence of CEO compensation, and the second is a sequence of shareholder wealth. Now for each of the expected discounted utility levels, using the (panel) data set which includes the first 13 observations of these sequences, regress the first difference in CEO compensation linearly on that in shareholder wealth. For each of the expected discounted utilities, the estimated coefficients, i.e., pay-performance sensitivities, and their standard errors are displayed in Table 3.1.a. Next, use the last 13 observations in the sequences to run the same regressions, with the estimated pay-performance sensitivities and the corresponding standard errors reported in Table 3.1.b.

Tables 3.1.a and 3.1.b show that, across different initial states, our estimates using model generated data of the pay-performance sensitivity can either be significantly positive, significantly negative, or insignificantly different from zero. In other words, our model does not necessarily imply even a positive pay-performance sensitivity. This, again, contradicts Jensen and Murph's hypothesis that, to provide proper incentives, the pay-performance sensitivity should be positive and reasonably high. However, observe that in general, for initial states that are relatively low, the estimated pay-performance sensitivities are positive and small, similar to those found by

Jensen and Murphy with real world data. Now, suppose that we have a large panel of CEOs identified by different initial levels of expected discounted utilities distributed heavily on the range where the estimated sensitivities are positive but small, and we then estimate the pay-performance sensitivity by using the panel data generated by these CEOs. It is possible to obtain a significantly positive but small estimate for the pay-performance sensitivity. We now implement this conjecture through the following experiment.

Let there be 2000 CEOs initially having expected discounted utilities distributed over levels from 25 to 80. The distribution function is Poisson with mean 7. [9] Again, for each of the CEOs, two sequences, one for CEO compensation, one for shareholder wealth, both containing 100 observations, are generated. The regression that uses the panel data set which includes all the first 13 observations of the 2000 sequences is reported as follows:

$$\Delta w_t = 0.00034 + 0.00432\Delta U_t, \quad R^2 = 0.13313 \tag{3.13}$$

where the t-statistics for the intercept and the estimated pay-performance sensitivity are 3.29126 and 5.97161, respectively. The regression that uses the last 13 observations produces the following result:

$$\Delta w_t = 0.00018 + 0.00161\Delta U_t, \quad R^2 = 0.0346541 \tag{3.14}$$

here the t-statistics for the intercept and the estimated pay-performance sensitivity are 2.21724 and 3.16441, respectively. To summarize, in both regressions, the pay-performance sensitivities estimated are significantly positive, but very small, similar to those observed by Jensen and Murphy with real world data. Finally, observe

[9]The distribution is $Prob(K) = e^{-\theta}\theta^{(K-25)}/(K-25)!$, where $\theta = 7$ and $K = 25, 26, \ldots$. Note that if a number greater than 80 is generated, (the probability of this event is rather low,) for convenience we will simply set it to be 80.

that the R-squares in both regressions are very small, indicating that the extent to which changes in shareholder wealth can contribute to the explanation of that in CEO compensation is rather low. This, also, is consistent with the corresponding observations made by Jensen and Murphy.

To conclude this section, we claim two major findings. First, our model does not predict a positive correlation between CEO compensation and shareholder wealth, measured either in levels or in first differences. This is in contrast to the assertion that a formal agency model would imply a positive correlation between CEO compensation and shareholder wealth, and a high pay-performance sensitivity. Second, the fact that our model is capable of generating pay-performance sensitivities that are very close to what is observed by Jensen and Murphy suggests that the real world data is consistent with the simple dynamic agency model considered here.

3.5 Perturbations to the Example

In this section, we examine various perturbations to the numerical example solved in Section 3.3 and used in Section 3.4 to address the CEO compensation issue. Our purpose here is to see how robust our results in the previous two sections are. First, we will allow the principal to have access to a credit market where he can borrow up to a limited amount. Second, we will examine the effect of a change in the magnitude of the discount factor. Third, we will discuss how changes in other parameter values, including those in the agent's utility function and in the production technology, will affect our model's predictions concerning CEO compensation and shareholder wealth.

3.5.1 Principal Can Borrow A Limited Amount

In the basic model we presented in Section 3.2 and solved in a numerical example in Section 3.3, the economy is a closed one: there is no borrowing from any outside parties and the agent is at no date paid more than total current output. This assumption can be very restrictive in terms of the extent to which the risk-averse agent can be insured by the principal, especially when the agent is at relatively high states where the compensation constraint is binding if low output is realized. In fact, it was shown in Figures 3.2 and 3.3.a that the optimal dynamic contract tends to degenerate to the optimal static one as the expected discounted utility of the agent gets sufficiently close to its highest level. In this section, the principal is assumed to have access to a credit market where he can borrow up to a limited amount. In particular, we assume that if the firm makes a low profit, the principal can then borrow from a risk neutral insurance company a maximum amount of $(y_H - y_L)$ units of the consumption good. That is, whether the firm is making a high or low profit, the principal is now always able to pay the CEO compensation up to the amount y_H. Finally, for convenience, we follow Green (1987) and other authors to assume that credit is available at the rate $(1-\beta)/\beta$.

Here again we grid the space of potentially admissible expected discounted utilities $[\underline{V}, \overline{V}]$ uniformly with 100 points, where $\underline{V}$ is the same as defined in Section 3.3, but $\overline{V}$ is now the expected discounted utility of the agent if he is allowed to consume a constant amount y_H and works the least, i.e, $\overline{V} = -exp(a_L - y_H)/(1-\beta)$. The program discussed in Section 3.2 is readily modified for application to the current situation, except now the compensation constraint is $0 \leq w(I, K) \leq y_H, I = L, H$. Here computational results show that the set of admissible states is $\{V(K), K = 17, ..., 100\}$. The solution to the Bellman equation is characterized in Figures 3.7, 3.8.a and 3.8.b.

First, as expected, allowing borrowing from an outside party improves welfare. This is shown in Figure 3.7 where the value function of the optimal contract with borrowing is plotted against that without borrowing. Welfare is improved because borrowing allows for more risk-sharing or consumption smoothing through expanding the choice domains of current compensation and expected discounted utility. Obviously, this expansion is most effective (in the wealfare-improving sense) in states where expected discounted utilities of the agent are relatively high. For these states, except for the highest one, the compensation constraint is no longer binding at the optimum, regardless of output realization. This explains why the value function with borrowing significantly exceeds that without borrowing only in the high states. The fact that expanding the choice domains allows for more risk sharing or consumption smoothing is most clearly reflected in Figure 3.8.a which plots the compensation rules for the economy with borrowing and can be contrasted to Figure 3.3.a. Most notably, with borrowing, we have $w(L, K) = w(H, K)$, for every admissible K. Here, current compensation, which is independent of current output, is not used for incentive purposes at all. Rather, it is only arranged to meet the single target of consumption smoothing. Remember the argument we made in Section 3.4 that the magnitude of $|w(H, K) - w(L, K)|$ in some sense serves as a partial measure for the degree of risk-sharing that is achieved around state K with the optimal contract.

What does borrowing say about the relationship between CEO compensation and shareholder wealth? Due to more risk-sharing achieved in the optimal contract, specifically $w(L, K) = w(H, K)$ for all Ks, there are now better chances for the first differences of CEO compensation and shareholder wealth to be positively related. Here, following the discussion in Section 4, it is now certain that $\Delta w > 0$ in case (2) and $\Delta w < 0$ in case (3). Tables 3.2.a and 3.2.b present pay-performance sensitivities estimated from model-generated data in the same way were those in Tables 3.1.a and 3.1.b respectively. Tables 3.2.a and 3.2.b show that almost all the coefficients estimated are either significantly positive but small, or insignificantly different from zero. Clearly then, allowing the principal to borrow greatly enhances the model's ability to generate positive but small pay-performance sensitivities. Here we can also do the same sort of experiments associated with regressions 3.13 and 3.14 in the previous section. We find, for almost any arbitrary distribution of initial expected discounted utilities entitled to the CEOs, and for any arbitrary number of observations, the estimated pay-performance sensitivity is positive and small. For example, if the distribution of initial states is Poisson with mean 10, over Ks equal to or greater than 20, the estimated pay-performance sensitivity is 0.003719, with a t-statistic of 6.2917, if the first 13 observations are used in the estimation. If the last 13 observations are used, the estimated pay-performance sensitivity is 0.0023705, with a t-statistic of 5.2918.

To conclude this subsection, we note that theoretically the principal can be allowed to borrow up to any arbitrary amount, and our program is readily modified to meet the change. However, as can be easily seen, allowing the principal to be able to borrow more only enhances our model's ability to generate pay-performance sensitivities that are positive and small, as desired.

3.5.2 Effect of A Change in the Discount Factor

One other important parameter in the model is the discount factor β. The discount factor is important because it affects how efficiently expected discounted utility can be used as an incentive device in the optimal contract. A larger discount factor means tomorrow is more important to the agent and hence allows for more efficient use of expected discounted utility in rewarding good outcomes and punishing bad ones. This, in turn, leaves more room for risk-sharing or consumption smoothing. In terms of the pay-performance sensitivity, given our discussion in Section 3.4, better risk-sharing associated with a higher discount factor makes a positive but small pay-performance sensitivity more likely to occur. Conversely, a lower discount factor will make a positive but small pay-performance sensitivity less likely to occur. In the numerical examples studied so far, the discount factor was set to 0.96. In this subsection, in order to examine quantitatively the effect of a change in the discount factor, we solve the optimal contract for $\beta = 0.9$ and $\beta = 0.8$, respectively.

First, a lower discount factor makes it more costly to induce good effort. Remember for $\beta = 0.96$, the ' .gh effort action is implemented in every state except for the lowest one. Here for $\beta = 0.9$, the high effort action is implemented for $K \geq 22$. For $\beta = 0.8$, the high action is only implemented for $K \geq 23$.

Second, a lower discount factor allows for less risk-sharing to be achieved with the optimal contract. This is reflected in Figures 3.9.a through 3.10.b which should be compared to Figures 3.2 and 3.3.a. Observe that, for $\beta = 0.90$ and $\beta = o.80$, the magnitudes of $|w(H,K) - w(L,K)|$ and $|V(H,K) - V(L,K)|$ are significantly greater, compared to that for $\beta = 0.96$. A lower discount factor therefore implies more volatility in consumption.

Third, due to less risk-sharing achievable , a lower discount factor makes it less likely for the first difference in CEO compensation to be positively correlated with that in shareholder wealth. In particular, we find that, for $\beta = 0.9$ and $\beta = 0.8$, the pay-performance sensitivities estimated using model-generated data are uniformly negative across the initial states. [10]

3.5.3 Effects of Changes in Other Parameters

Changing the value of γ in the agent's utility function changes his absolute risk aversion. To test for the sensitivity of our model's predictions concerning CEO compensation and shareholder wealth to γ, we experiment with various γs ranging from 0.1 to 10, holding all the other parameters fixed and assuming the principal can not borrow. We find, first, our results in the previous section are quite robust with re-

[10] These results are not reported for brevity but available on request.

spect to different values of γ in [0.1, 10]. This is compared with the results in Haubrich where for $\gamma > 0.5$, the agent simply makes the least managerial effort and the pay-performance sensitivities are constant at zero. Second, for larger γ, our model is able to deliver more positive but small pay-performance sensitivities across initial expected discounted utilities. The intuition here is that when the agent becomes more risk averse, the marginal value of smoothing his current consumption increases, relative to the marginal value of smoothing his future consumptions, due to discounting. Consequently, it is optimal to have more states where current compensation is profit independent. But this in turn provides more chances for the changes in current compensation and shareholder wealth to be positively correlated.

There is another parameter in the agent's utility function, α, which determines the cost of exerting effort relative to the benefits from consumption. So far, we have set $\alpha = 1$, i.e., to the agent, one unit of effort is equivalent to one unit of the consumption good. Therefore, for the CEO, the cost of taking a_H rather than a_L is equivalent to 0.1 units of the consumption good, which is 1/4 of the gross profit in a bad year, or 1/8 of the gross profit in a good year. For the shareholders, this cost is even bigger, taking into consideration the cost of overcoming moral harzard. In reality however, for an average CEO and an average firm, this incentive cost is probably much smaller, given the fact that CEO compensation is usually a small fraction of total profit. [11] To make a good effort less costly, α must be larger. Notice that the optimal contract with utility function $-e^{\gamma(a-\alpha c)}$ and production set $A = \{0.4, 0.8\}$ is equivalent to that with utility function $-e^{\gamma(a-c)}$ and production set $A = \{0.4\alpha, 0.8\alpha\}$. Further notice that for $\alpha = 1$ and $A = \{0.4, 0.8\}$, for states associated with positive but small

[11]In Haubrich's base case, if $a_H = 0.2$ is taken by the agent, the expected firm value is $0.525 \times 700 + 0.475 \times 300 = 510$, which can be translated into an expected total profit in the amount of $(1 - 0.96) \times 510 = 20.4$, indicating that the cost of making a good rather than bad effort is about 1/200 of expected total profit.

pay-performance sensitivities, the feasibility constraint is not binding. Therefore for $\alpha > 1$, the model will do at least as well as it does with $\alpha = 1$, in terms of generating pay-performance sensitivities that are positive but small, as we desire.

We now go on to examine the effects of changes in parameters associated more with the production technology, particularly the ratio y_H/y_L and the probability $\pi = Prob(y_H|a_H) = Prob(y_L|a_L)$, which determine the impact that the CEO's effort has on the firm's expected gross profit, i.e., how powerful the CEO is. In the examples we have been studying so far, we have set $y_H/y_L = 2$ and $\pi = 2/3$, implying that the CEO can increase the firm's expected gross profit by 25 percent if he takes action $a_H(= 0.2)$ rather than $a_L(= 0.1)$. Our CEO seems too powerful, compared to those in Jensen and Murphy and in Haubrich. In Haubrich, by taking a_H rather than a_L, the CEO can only increase gross profit by 2 percent.

To make the CEO less powerful is to give smaller values to y_H/y_L and π. In the experiments we conduct in this respect, we first hold y_H fixed at 0.8 and π at 2/3, and let y_L to increase from 0.4 to 0.78. Next, we hold y_H and y_L fixed and lower the value of π from 0.6667 to 0.51. We found that smaller values of y_H/y_L and π only work to enhance our model's ability to generate pay-performance sensitivities that are positive but small, as we desire. Intuitively, this is easy to explain. A smaller y_H/y_L means a relatively narrower gap between y_H and y_L, which in turn would make "sufficient consumption smoothing" an easier goal to achieve and hence, according to our analysis in Section 4, expands chances of having positive but small pay-performance sensitivities. On the other hand, other things equal, a smaller π means a greater randomness in profit realization, which in turn implies a grater marginal value of smoothing consumption across different levels of profit realization. Consequently, more consumption smoothing is optimal and this enhances our model's ability to generate positive and small pay-performance sensitivities, again due to our analysis in Section 4.

3.6 Conclusion

In this paper we have used a simple dynamic agency model to address the CEO compensation issue raised by Jensen and Murphy in an empirical study which reveals that the observed pay-performance sensitivity, defined as the dollar change in the CEO compensation associated with a dollar change in shareholder wealth, is positive but rather small. Jensen and Murphy then propose to reject formal agency models, as the low pay-performance sensitivity seems to indicate insufficient incentives for the CEO. In this paper, a simple dynamic version of Holmstrom (1979)'s standard agency model is presented, and a computation algorithm is developed to solve numerical solutions to it. We made two observations based on computational results. First, our model predicts nonpositive correlation between levels of CEO compensation and shareholder wealth. Second, changes in CEO compensation and that in shareholder wealth can be positively or negatively correlated, depending on the extent of risk-sharing that is achieved with the optimal contract. Obviously, what these observations tell us is that a positive and reasonably high correlation between CEO compensation and shareholder wealth, either in levels or in first differences, is not a prediction of a dynamic agency model like ours. Further, for a wide variety of plausible parameter values, our model is robust in being able to generate pay-performance sensitivities that are significantly positive, but very small, similar to what is observed by Jensen and Murphy. It therefore is our conclusion that the small pay-performance sensitivity observed by Jensen and Murphy is in fact consistent with formal agency theory for which our model serves as a simple dynamic version.

Appendix 1

Proof of Lemma 1 Fix V. We need only show that $\mathcal{U}(V)$ is closed, since it is bounded obviously. Let $\{U_n\} \subseteq \mathcal{U}(V)$, and $U_n \to U_\infty$, as $n \to \infty$. We want to show $U_\infty \in \mathcal{U}(V)$. That is, there exists a contract σ_∞ which satisfies (3.1), (3.2), (3.3), $U(\sigma_\infty|h^0) = U_\infty$, and $V(\sigma_\infty|h^0) = V$. We construct σ_∞ in the following way. By definition of $\mathcal{U}(V)$ there exists a sequence of contracts $\{\sigma_n\} = \{a_t^n(h^{t-1}), w_t^n(h^t)\}$, each satisfying (3.1), (3.2), (3.3) and $V(\sigma_n|h^0) = V$, such that

$$U_\infty = \lim_{n\to\infty} \sum_{t=1}^{\infty} \beta^{t-1} \int [y_t - w_t^n(h^t)] f(y_t; a_t^n(h^{t-1})) dh^t.$$

For $t = 1$, notice that $\{a_1^n(h^0), w_1^n(h^1)\}$ is a finite collection of bounded sequences, therefore there exists a collection of subsequences $\{a_1^{n_q}(h^0), w_1^{n_q}(h^1)\}$ such that

$$\lim_{n_q\to\infty} a_1^{n_q}(h^0) = a_1^\infty(h^0), \lim_{n_q\to\infty} w_1^{n_q}(h^1) = w_1^\infty(h^1).$$

Now move to $t = 2$, and notice that $\{a_2^{n_q}(h^1), w_2^{n_q}(h^2)\}$ is a finite collection of bounded sequences. We can then define $\{a_2^\infty(h^1), w_2^\infty(h^2)\}$, in the same way through which $\{a_1^\infty(h^0), w_1^\infty(h^1)\}$ is defined. Iterate this procedure for $t = 3, 4, ...$, and let $\sigma_\infty \equiv \{a_t^\infty(h^{t-1}), w_t^\infty(h^t)\}$. It is straightforward to verify that σ_∞ thus constructed is what is desired. Q.E.D.

Proof of Proposition 1. Fix V. We first show that $\Gamma(U^*)(V) \leq U^*(V)$. Notice that this must be true if there exists a feasible and incentive compatible contract σ such that $V(\sigma|h^0) = V$, and $U(\sigma|h^0) = \Gamma(U^*)(V)$. Such a σ can be constructed in the following way. Let $a(V), w(y, V), \tilde{V}(y, V)$ denote the solution to the optimization problem associated with the definition of $\Gamma(U^*)(V)$. Let

$$a_1(h^0) = a(V);\ w_1(h^1) = w(y_1, V), \forall h^1.$$

Next, for given outcome realization y_1 of date 1, there exists a feasible and incentive compatible contract σ_{y_1} which delivers expected discounted utility $\bar{V}(y_1, V)$ to the agent, and $U^*(\bar{V}(y_1, V))$ to the principal. Let

$$\sigma|h^1 = \sigma_{y_1}, \ \forall h^1.$$

It is straight forward to verify that σ thus defined is what is desired. This completes the first half of the proof. Next, we show $U^*(V) \leq \Gamma(U^*)(V)$. Let σ^* be an optimal contract that delivers an expected discounted utility V to the agent, then

$$\begin{aligned} U^*(V) &= U(\sigma^*|h^0) \\ &= \int [y_1 - w_1^*(y_1) + \beta U^*(\sigma^*|h^1)] f(y_1; a_1^*(h^0)) dy_1 \\ &\leq \Gamma(U^*)(V), \end{aligned}$$

where the last inequality is derived by letting $a(V) = a_1^*(h^0)$, $w(y, V) = w_1^*(y)$, and $\bar{V}(y, V) = V(\sigma^*|y)$. These decision rules satisfy (3.5) through (3.9). Q.E.D.

<u>Lemma 2</u> (i) $\mathcal{V}$ is self-generating. (ii) If Φ is self-generating, then $B(\Phi) \subseteq \mathcal{V}$.

Proof. To prove (i), let $V \in \mathcal{V}$. There exists $\sigma = \{a_t(h^{t-1}), w_t(h^t)\}$ which satisfies (3.1),(3.2),(3.3), and $V(\sigma|h^0) = V$. Now let

$$a(V) = a_1(h^0); \quad w(y, V) = w_1(y), \forall y \ ; \bar{V}(y, V) = V(\sigma|y), \forall y.$$

It is clear that the triple $\{a(V), w(y, V), \bar{V}(y, V)\}$ thus defined satisfies (3.5) through (3.9). Therefore we have shown that $V \in B(\mathcal{V})$, and (i) is proven.

We now proceed to show (ii). Let Φ be self-generating and let $V_{h^0} \in B(\Phi)$. We need only construct a contract $\sigma = \{a_t(h^{t-1}), w_t(h^t)\}$ subject to (1), (3.2), (3.3), and $V(\sigma|h^0) = V_{h^0}$. This is done in the following recursive way. First, there exists a triple $\{a(V_{h^0}), w(y, V_{h^0}), \bar{V}(y, V_{h^0})\}$ which satisfies (3.5), (3.7), (3.8), (3.11), and

$$\int [v(w(y, V_{h^0}), a(V_{h^0})) + \beta \bar{V}(y; a(V_{h^0}))] f(y, a(V_{h^0})) dy = V_{h^0}.$$

Let, for $t = 1$, $a_1(h^0) = a(V_{h^0})$, and $w_1(h^1) = w(y_1, V_{h^0})$, $\forall h^1$. Also let $V_{h^1} = \bar{V}(y_1, V_{h^0}), \forall h^1$. Now notice that $V_{h^1} \in \Phi \in B(\Phi)$, there exists a triple $\{a(V_{h^1}), w(y, V_{h^1}),$ $\bar{V}(y, V_{h^1})\}$ which meets the conditions (3.5), (3.7), (3.8), (3.11), and

$$\int [v(w(y, V_{h^1}), a(V_{h^1})) + \beta \bar{V}(y; a(V_{h^1}))] f(y; a(V_{h^1}))dy = V_{h^1}.$$

For $t = 2$ then, we can let $a_2(h^1) = a(V_{h^1})$, $w_2(h^2) = w(y_2, V_{h^1})$, and $V_{h^2} = \bar{V}(y_2, V_{h^1})$. Iterate the above procedure for $t = 3, 4, \ldots$ to construct the whole profile of σ. Now observe that, by construction, for any given $t \geq 0$ and h^t,

$$V(\sigma|h^t) - V_{h^t} = \beta \int [V(\sigma|h^{t+1}) - V_{h^{t+1}}] f(y_{t+1}; a(V_{h^t}))dy_{t+1}.$$

Given $0 < \beta < 1$ and that the utilities are bounded, the above equality implies immediately that

$$V(\sigma|h^t) = V_{h^t}, \;\; \forall t \geq 0, \; \forall h^t.$$

Therefore the contract σ we constructed is truly what is desired. Q.E.D.

Proof of Proposition 2 (i) is obvious. To show (ii), first we show that the sequence $\{W_n\}$ converges. It is obvious that $B(W_0) \subseteq W_0$. Now operate B repeatedly on both sides of this expression to yield: $W_{n+1} = B(W_n) \subseteq W_n$, for all $n \geq 0$, as the operator B is monotone increasing. Therefore $\{W_n\}$ is a bounded and monotone decreasing set sequence. It converges and in fact $W_\infty = \lim_{n\to\infty} W_n = \bigcap_{n=0}^{\infty} W_n$. Second, we show that $\mathcal{V} \subseteq W_\infty$. Since $\mathcal{V} \subseteq W_0$, monotonicity of B implies $B(\mathcal{V}) \subseteq B(W_0)$. But $B(\mathcal{V}) = \mathcal{V}$ by (i), and $B(W_0) = W_1$ by construction, we thus have: $\mathcal{V} \subseteq W_1$. Iterate the above procedure to obtain: $\mathcal{V} \subseteq W_n$, for all $n \geq 0$. Therefore $\mathcal{V} \subseteq W_\infty$. Third, we show that $W_\infty \subseteq \Phi$. By the construction and convergence of $\{W_n\}$, $B(W_\infty) = W_\infty$. Therefore W_∞ is self-generating, and $W_\infty = B(W_\infty) \subseteq \mathcal{V}$. This concludes the proof of the Theorem.

Appendix 2

This is a note on the computation program we used to solve the Bellman equation. The program is a two-step one. The aim of the first step is to get the set of admissible expected discounted utilities, i.e., $\mathcal{V}$, through an iteration process. We start with $\mathcal{V}_0 = \mathcal{W} = \{V(K), K = 1, ..., N\}$, here $N = 100$ and

$$V(K) = V(1) + \frac{K-1}{N-1}[V(N) - V(1)], \ \ \forall K \in I_0$$

where $I_0 = \{1, 2, ..., N\}$. Let P_{ij} be the probability of having outcome i after making an effort j, $i, j = H, L$. In the first iteration, for all $K \in I_0$, and for all $P, Q \in I_0$, let $F_H(I_0; K, P, Q) = 1$ if the following set of constraints, (3.15) through (3.18), has a feasible point; $F_H(I_0; K, P, Q) = 0$, otherwise.

$$0 \le w_H \le y_H, \tag{3.15}$$

$$0 \le w_L \le y_L, \tag{3.16}$$

$$p_{HH}[v(w_H, a_H) + \beta V(P)] + p_{LH}[v(w_L, a_H) + \beta V(Q)] \ge$$

$$p_{HL}[v(w_H, a_L) + \beta V(Q)] + p_{LH}[v(w_L, a_L) + \beta V(Q)], \tag{3.17}$$

$$p_{HH}[v(w_H, a_H) + \beta V(P)] + p_{LH}[v(w_L, a_H) + \beta V(Q)] = V(K). \tag{3.18}$$

Parallelly, $F_L(I_0; K, P, Q) = 1$ if there exists (w_H, w_L) which satisfies (3.15),(3.16), and the following two additional constraints:

$$p_{HL}[v(w_H, a_L) + \beta V(P)] + p_{LL}[v(w_L, a_L) + \beta V(Q)] \ge$$

$$p_{HH}[v(w_H, a_H) + \beta V(P)] + p_{LH}[v(w_L, a_H) + \beta V(Q)], \tag{3.19}$$

$$p_{HL}[v(w_H, a_L) + \beta V(P)] + p_{LL}[v(w_L, a_L) + \beta V(Q)] = V(K). \tag{3.20}$$

Otherwise, $F_L(I_0; K, P, Q) = 0$. Now define

$$I_1 = \{K \in I_0 : \sum_{i,P,Q} F_i(I_0; K, P, Q) > 0, i = H, K; P, Q \in I_0\}$$

and this concludes the first iteration. Generally, suppose we are at the tth iteration with I_{t-1} given, then

$$I_t = \{K \in I_{t-1} : \sum_{i,P,Q} F_i(I_{t-1}; K, P, Q) > 0, i = H, L; P, Q \in I_{t-1}\}$$

The sequence $\{I_t\}$ by the Lemma is guaranteed to converge to some I, and we then have $\mathcal{V} = \{V(K), K \in I\}$.

Note that constraints (3.17) through (3.20) are nonlinear. In cases where utility is separable, however, they can be easily transformed into linear ones. In this paper where utility is exponential, constraints (3.15) through (3.20) are transformed into the following linear ones.

$$e^{-y_H} \leq x_H \leq 1 \tag{3.21}$$

$$e^{-y_L} \leq x_L \leq 1 \tag{3.22}$$

$$\begin{aligned}(p_{HH}e^{a_H} - p_{HL}e^{a_L})x_H + (p_{LH}e^{a_H} - p_{LL}e^{a_L})x_L \leq \\ (p_{HH} - p_{HL})\beta V(P) + (p_{LH} - p_{LL})\beta V(Q)\end{aligned} \tag{3.23}$$

$$p_{HH}e^{a_H}x_H + p_{LH}e^{a_H}x_L = p_{HH}\beta V(P) + p_{LH}\beta V(Q) - V(K), \tag{3.24}$$

$$\begin{aligned}(p_{HL}e^{a_L} - p_{HH}e^{a_H})x_H + (p_{LL}e^{a_L} - p_{LH}e^{a_L})x_L \leq \\ (p_{HL} - p_{LH})\beta V(P) + (p_{LL} - p_{LL})\beta V(Q)\end{aligned} \tag{3.25}$$

$$p_{HL}e^{a_L}x_H + p_{LL}e^{a_L})x_L = p_{HL}\beta V(P) + p_{LL}\beta V(Q) - V(K) \tag{3.26}$$

where $x_H = e^{-w_H}$ and $x_L = e^{-w_L}$. The above constraints correspond to (3.15) through (3.20) respectively. Since these are linear constraints, a linear programming program can then be applied to determine the values of $F_i(I_{t-1}; K, P, Q)$, for all i, K, P, Q, at any iteration $t \geq 1$.

Given $\mathcal{V}$ determined in Step 1, the aim of the second step is to reach the solution to the Bellman equation. We start with an all-zero guess for the value function:

$U_0(K) = 0, \forall K \in I$. Suppose now we are at the tth iteration, $t \geq 1$,

$$U_t(K) = max\{U_t^i(K; P, Q) : i = H, L; P, Q \in I\}, \forall K \in I$$

where

$$U_t^H(K; P, Q) = \max_{w_H, w_L} p_{HH}[y_H - w_H + \beta U_{t-1}(P)] + p_{LH}[y_L - w_L + \beta U_{t-1}(Q)] \tag{3.27}$$

subject to (3.20),(3.21),(3.22),(3.23). And,

$$U_t^L(K; P, Q) = \max_{w_H, w_L} p_{HL}[y_H - w_H + \beta U_{t-1}(P)] + p_{LL}[y_L - w_L + \beta U_{t-1}(Q)] \tag{3.28}$$

subject to (3.21),(3.22),(3.25),(3.26). Again, with exponential utility, the nonlinear programing problems (3.27) and (3.28) can be transformed into programming problems with linear constraints but nonlinear objectives. Specifically, (3.15) through (3.20) are replaced by (3.21) through (3.26) respectively; $-w_H$ and $-w_L$ in the objective functions of (3.27) and (3.28) are replaced by $log x_H$ and $log x_L$ respectively. Once this is done, (3.27) and (3.28) are to be solved by a modified simplex method.

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TABLE 3.1.a: Estimates of Pay-Performance Sensitivities Across Initial Expected Discounted Utilities. First 13 Observations used.

K	Sensitivity (S.E.)	K	Sensitivity (S.E.)
21	-0.012948 (0.000546)	51	0.000216 (0.000249)
22	-0.015643 (0.000620)	52	0.000292 (0.000248)
23	-0.013950 (0.000611)	53	-0.000798 (0.000248)
24	-0.008616 (0.000715)	54	-0.000039 (0.000248)
25	-0.005753 (0.000723)	55	0.000099 (0.000248)
26	-0.002382 (0.000804)	56	-0.000791 (0.000249)
27	0.005070 (0.000851)	57	-0.001161 (0.000250)
28	0.006857 (0.000938)	58	-0.002140 (0.000253)
29	0.006039 (0.000849)	59	-0.004602 (0.000265)
30	0.008190 (0.000803)	60	-0.007551 (0.000274)
31	0.003702 (0.000582)	61	-0.013607 (0.000306)
32	0.002236 (0.000525)	62	-0.019405 (0.000328)
33	0.002038 (0.000465)	63	-0.029349 (0.000354)
34	0.002283 (0.000410)	64	-0.040075 (0.000395)
35	0.000698 (0.000360)	65	-0.053234 (0.000491)
36	0.001144 (0.000337)	66	-0.070296 (0.000505)
37	0.000229 (0.000306)	67	-0.088188 (0.000571)
38	0.000281 (0.000276)	68	-0.105353 (0.000647)
39	-0.000057 (0.000272)	69	-0.121981 (0.000720)
40	0.000567 (0.000260)	70	-0.141273 (0.000816)
41	0.000236 (0.000260)	71	-0.157500 (0.000903)
42	0.000263 (0.000254)	72	-0.174900 (0.001000)
43	-0.000563 (0.000251)	73	-0.193018 (0.001094)
44	-0.000565 (0.000248)	74	-0.208354 (0.001201)
45	0.000114 (0.000249)	75	-0.221913 (0.001297)
46	-0.000183 (0.000249)	76	-0.242980 (0.001504)
47	-0.000172 (0.000249)	77	-0.256160 (0.001628)
48	0.000275 (0.000247)	78	-0.272295 (0.001787)
49	-0.000276 (0.000249)	79	-0.280994 (0.002090)
50	-0.000053 (0.000248)	80	-0.290370 (0.002414)

TABLE 3.1.b: Estimates of Pay-Performance Sensitivities Across Initial Expected Discounted Utilities. Last 13 Observations used.

K	Sensitivity (S.E.)	K	Sensitivity (S.E.)
21	0.001647 (0.000511)	51	-0.033203 (0.000807)
22	0.002075 (0.000524)	52	-0.038730 (0.000873)
23	0.000681 (0.000513)	53	-0.040613 (0.000937)
24	0.000293 (0.000510)	54	-0.045080 (0.001025)
25	0.001343 (0.000493)	55	-0.044611 (0.001024)
26	0.000620 (0.000495)	56	-0.057241 (0.001243)
27	0.001434 (0.000504)	57	-0.055169 (0.001217)
28	0.000433 (0.000504)	58	-0.056911 (0.001241)
29	0.001359 (0.000503)	59	-0.066303 (0.001406)
30	0.001276 (0.000500)	60	-0.070464 (0.001529)
31	0.000312 (0.000492)	61	-0.076561 (0.001583)
32	0.000982 (0.000487)	62	-0.083877 (0.001632)
33	-0.000573 (0.000481)	63	-0.082990 (0.001883)
34	-0.001456 (0.000470)	64	-0.092507 (0.002038)
35	-0.001109 (0.000469)	65	-0.096709 (0.002094)
36	-0.002222 (0.000473)	66	-0.098467 (0.002101)
37	-0.004557 (0.000509)	67	-0.103221 (0.002346)
38	-0.004003 (0.000498)	68	-0.102081 (0.002549)
39	-0.003856 (0.000521)	69	-0.107492 (0.002582)
40	-0.005277 (0.000470)	70	-0.111344 (0.002986)
41	-0.008444 (0.000504)	71	-0.114145 (0.003168)
42	-0.009716 (0.000538)	72	-0.124573 (0.003438)
43	-0.015505 (0.000612)	73	-0.121308 (0.003590)
44	-0.011238 (0.000599)	74	-0.121843 (0.003946)
45	-0.014796 (0.000566)	75	-0.133787 (0.004153)
46	-0.021313 (0.000598)	76	-0.141727 (0.004607)
47	-0.026570 (0.000688)	77	-0.133675 (0.005045)
48	-0.022811 (0.000741)	78	-0.137024 (0.005894)
49	-0.024871 (0.000778)	79	-0.136234 (0.006225)
50	-0.032730 (0.000784)	80	-0.140867 (0.007059)

TABLE 3.2.a: Estimates of Pay-Performance Sensitivities Across Initial Expected Discounted Utilities: Principal Can Borrow. First 13 Observations Used.

K	Sensitivity (S.E.)	K	Sensitivity (S.E.)
21	0.001731 (0.000644)	56	-0.000014 (0.000250)
22	0.003532 (0.000724)	57	0.000497 (0.000249)
23	0.004023 (0.000736)	58	-0.000523 (0.000249)
24	0.006727 (0.000828)	59	0.000132 (0.000249)
25	0.007541 (0.000865)	60	0.000355 (0.000249)
26	0.007766 (0.000953)	61	-0.000486 (0.000248)
27	0.009582 (0.000858)	62	0.000256 (0.000248)
28	0.009407 (0.000786)	63	0.000365 (0.000249)
29	0.003921 (0.000599)	64	0.000351 (0.000248)
30	0.003758 (0.000538)	65	0.000385 (0.000248)
31	0.002297 (0.000466)	66	0.000091 (0.000248)
32	0.001141 (0.000405)	67	-0.000467 (0.000248)
33	0.000995 (0.000364)	68	-0.000173 (0.000247)
34	0.000668 (0.000320)	69	-0.000198 (0.000247)
35	0.000227 (0.000305)	70	0.000176 (0.000248)
36	0.000390 (0.000277)	71	-0.000475 (0.000248)
37	-0.000078 (0.000270)	72	0.000630 (0.000247)
38	-0.000118 (0.000259)	73	0.000009 (0.000248)
39	0.000134 (0.000254)	74	0.000732 (0.000246)
40	-0.000442 (0.000257)	75	0.000013 (0.000246)
41	0.000097 (0.000249)	76	0.000115 (0.000247)
42	-0.000121 (0.000250)	77	0.000105 (0.000244)
43	-0.000345 (0.000250)	78	0.000394 (0.000245)
44	-0.000317 (0.000248)	79	0.000283 (0.000246)
45	0.000686 (0.000249)	80	0.000330 (0.000246)
46	-0.000182 (0.000245)	81	0.000289 (0.000244)
47	-0.000019 (0.000249)	82	0.000110 (0.000244)
48	-0.000724 (0.000251)	83	0.000116 (0.000250)
49	-0.000145 (0.000249)	85	0.000005 (0.000254)
50	-0.000564 (0.000249)	85	0.000462 (0.000266)
51	-0.000291 (0.000249)	86	0.000577 (0.000289)
52	-0.000470 (0.000250)	87	0.000259 (0.000295)
53	-0.000087 (0.000249)	88	0.001545 (0.000325)
54	-0.000751 (0.000249)	89	0.000791 (0.000344)
55	-0.000150 (0.000249)	90	0.001924 (0.000377)

TABLE 3.2.b: Estimates of Pay-Performance Sensitivities Across Initial Expected Discounted Utilities: Principal Can Borrow. Last 13 Observations Used.

K	Sensitivity (S.E)	K	Sensitivity (S.E.)
21	0.001370 (0.000524)	56	-0.000216 (0.000262)
22	0.001771 (0.000511)	57	-0.000052 (0.000264)
23	0.003593 (0.000517)	58	0.000095 (0.000259)
24	0.002090 (0.000491)	59	0.000173 (0.000266)
25	0.002679 (0.000497)	60	0.000230 (0.000266)
26	0.001987 (0.000506)	61	0.000711 (0.000256)
27	0.002242 (0.000487)	62	0.000046 (0.000267)
28	0.003176 (0.000509)	63	0.000149 (0.000262)
29	0.003438 (0.000494)	64	0.000171 (0.000256)
30	0.002493 (0.000468)	65	0.000324 (0.000256)
31	0.002255 (0.000476)	66	-0.000449 (0.000261)
32	0.001237 (0.000460)	67	-0.000046 (0.000256)
33	0.001398 (0.000442)	68	-0.000003 (0.000266)
34	0.001194 (0.000442)	69	0.000101 (0.000264)
35	0.000827 (0.000434)	70	-0.000589 (0.000273)
36	0.001043 (0.000415)	71	0.000401 (0.000264)
37	0.001030 (0.000405)	72	0.000025 (0.000272)
38	0.000532 (0.000400)	73	-0.000247 (0.000267)
39	0.000671 (0.000384)	74	0.000190 (0.000266)
40	0.000904 (0.000359)	75	0.000837 (0.000269)
41	0.000757 (0.000357)	76	0.000477 (0.000279)
42	0.000559 (0.000347)	77	0.000672 (0.000284)
43	0.001114 (0.000344)	78	-0.000142 (0.000286)
44	0.001211 (0.000336)	79	0.000153 (0.000278)
45	0.000532 (0.000329)	80	-0.000323 (0.000271)
46	0.000425 (0.000325)	81	0.001250 (0.000293)
47	0.000264 (0.000301)	82	-0.000139 (0.000279)
48	0.000235 (0.000304)	83	0.000857 (0.000280)
49	0.000063 (0.000293)	84	0.000670 (0.000296)
50	0.000266 (0.000282)	85	0.000968 (0.000284)
51	0.000052 (0.000290)	86	0.000492 (0.000276)
52	0.000075 (0.000281)	87	-0.000771 (0.000293)
53	0.000192 (0.000268)	88	0.000014 (0.000291)
54	0.001002 (0.000271)	89	0.000432 (0.000288)
55	-0.000324 (0.000266)	90	0.000042 (0.000292)

FIGURE 1.1
An Efficient Contract Dominates Autarky

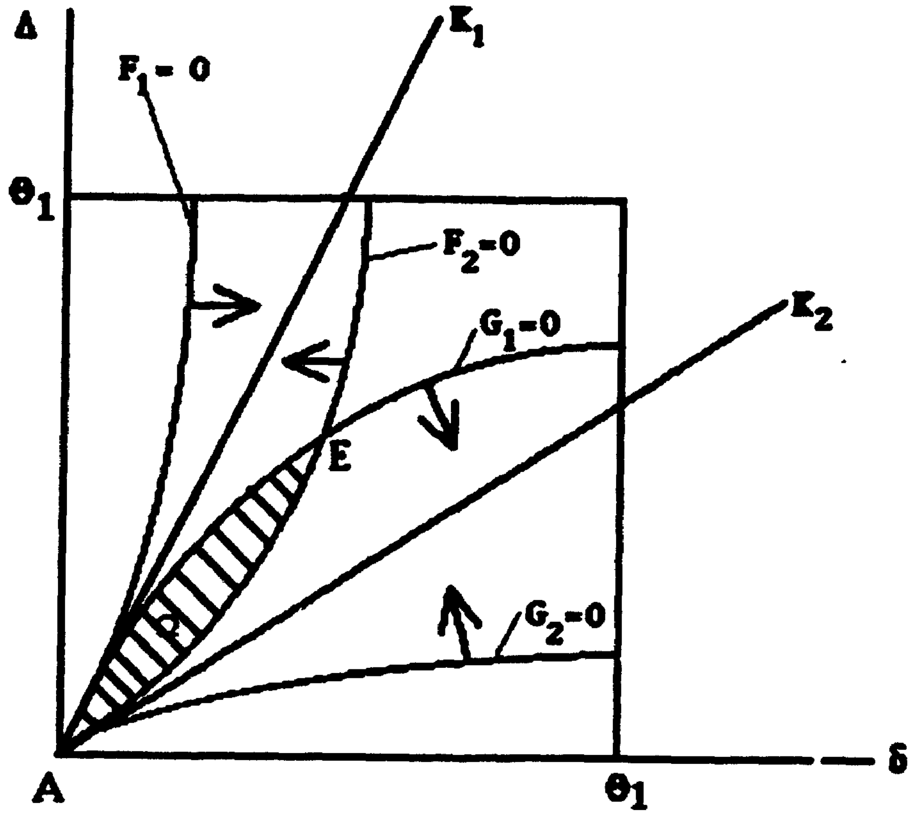


Figure 1.2
The Value Function

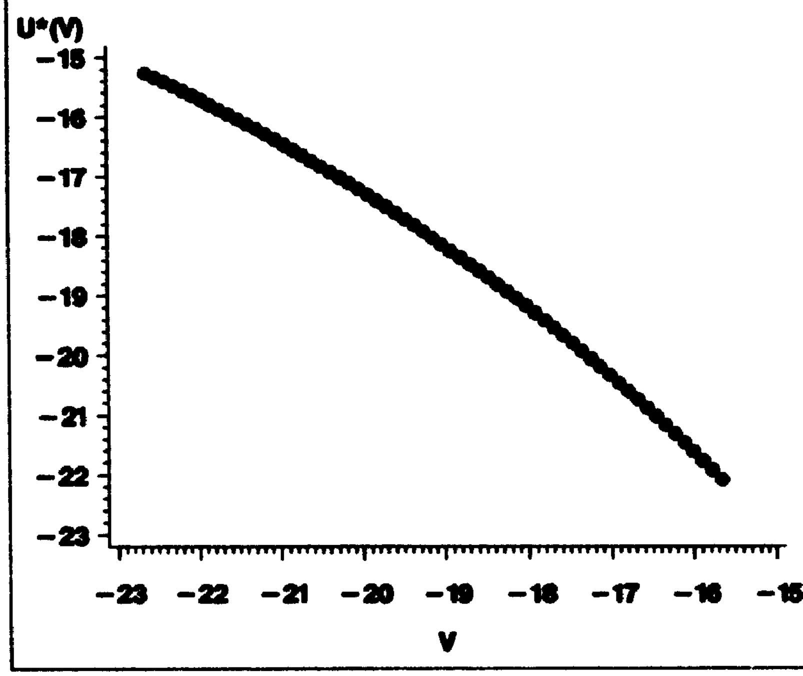


Figure 1.3

Net Transfer From Agent 1 To Agent 2

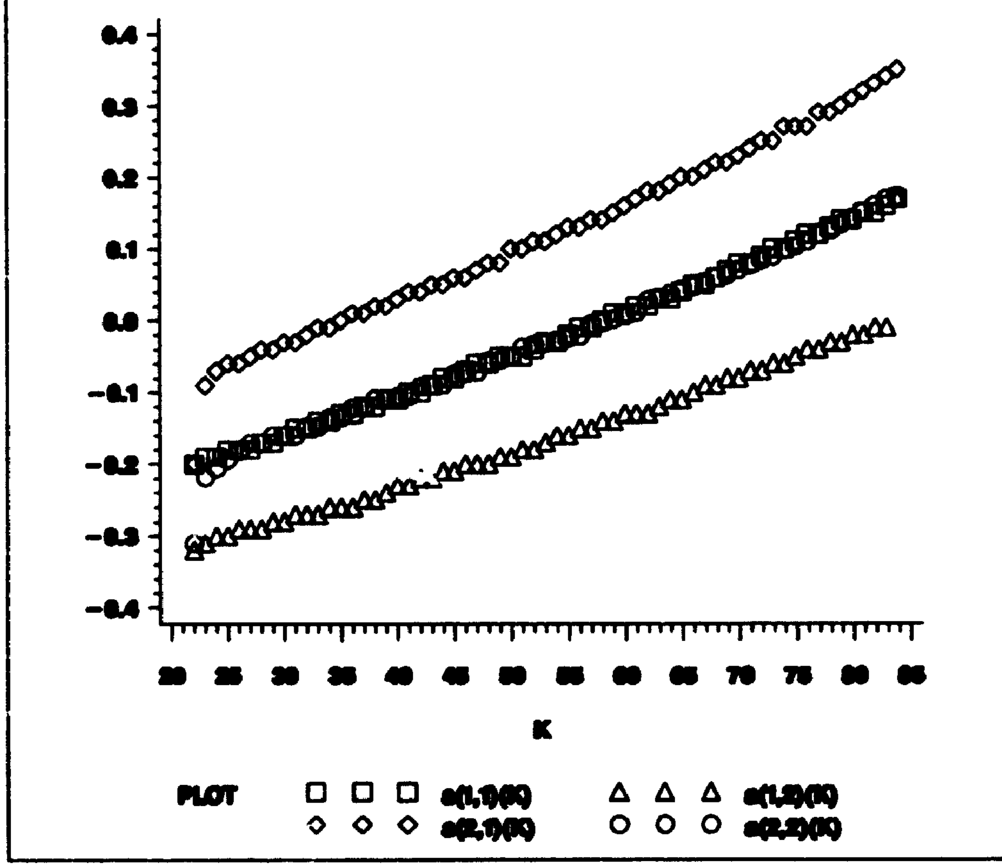
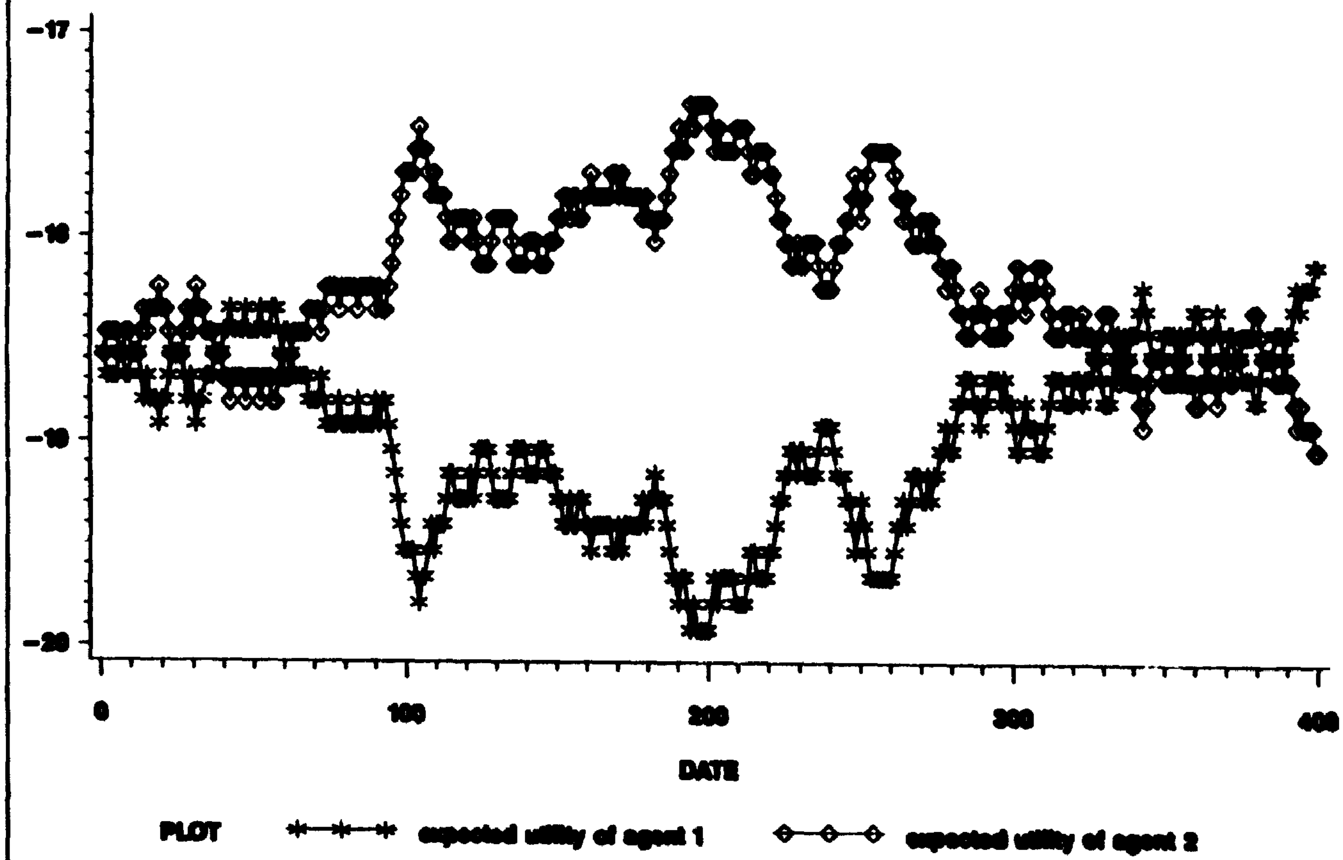


Figure 1.4

Expected Utility Paths of the Two Agents



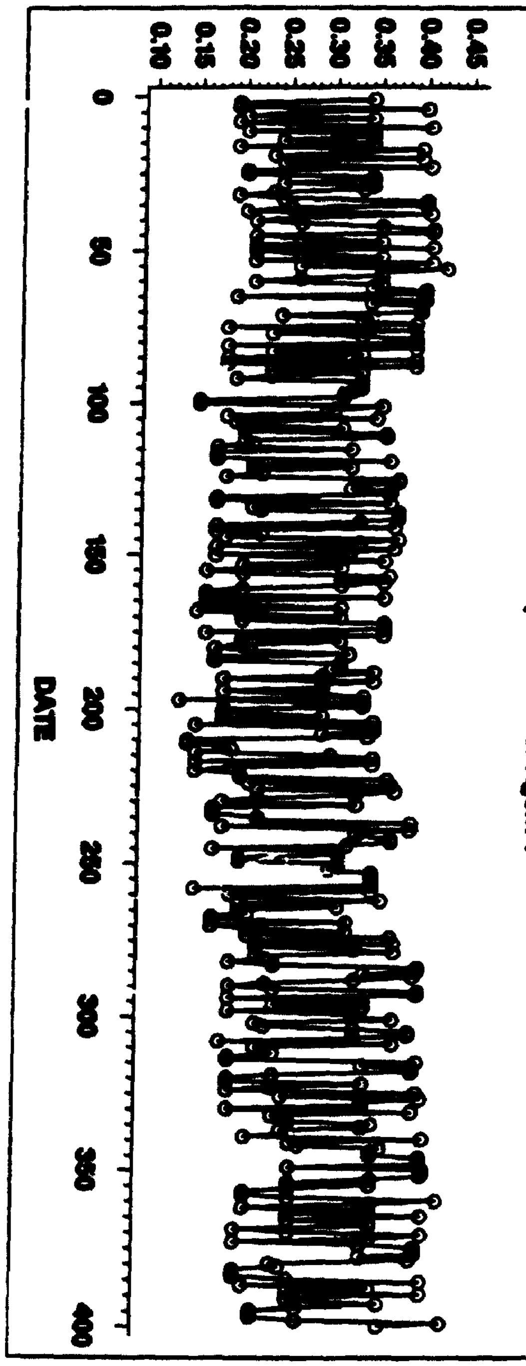


Figure 1.5
Consumption Path of Agent 1

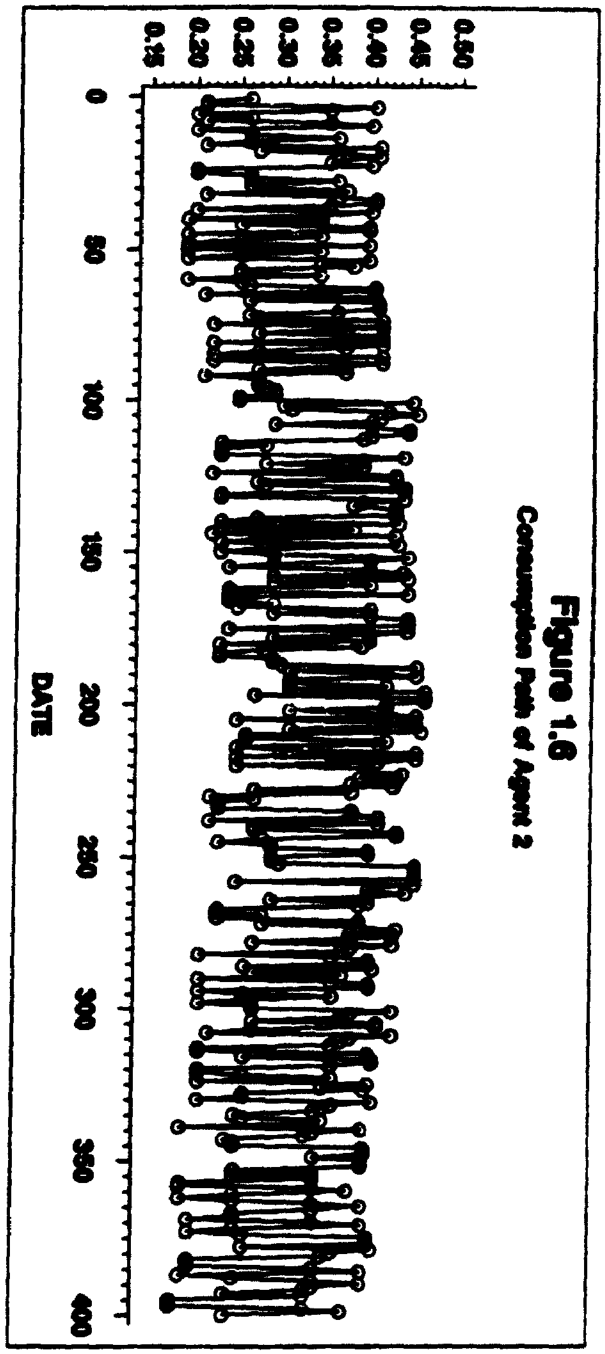


Figure 1.6
Consumption Path of Agent 2

Figure 3.1

Value Functions: Dynamic Model vs Static Model

Discount Factor=0.96

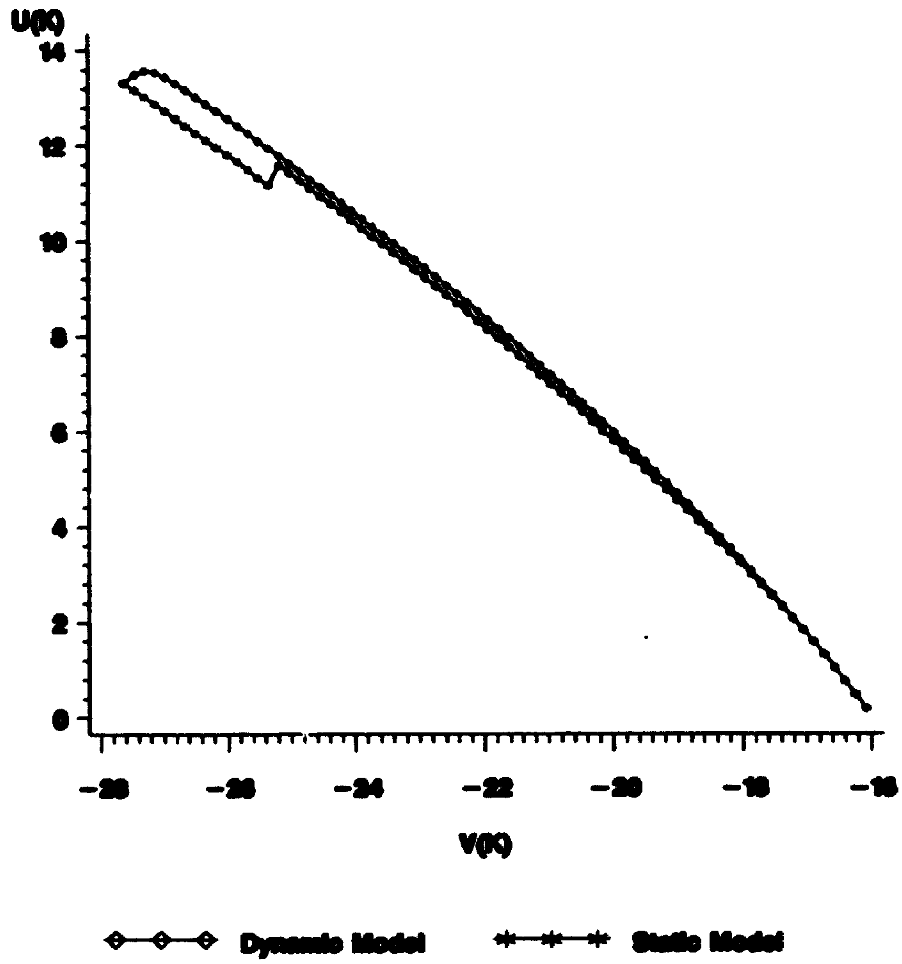


Figure 3.2
Expected Discounted Utilities Tomorrow
Discount Factor=0.96

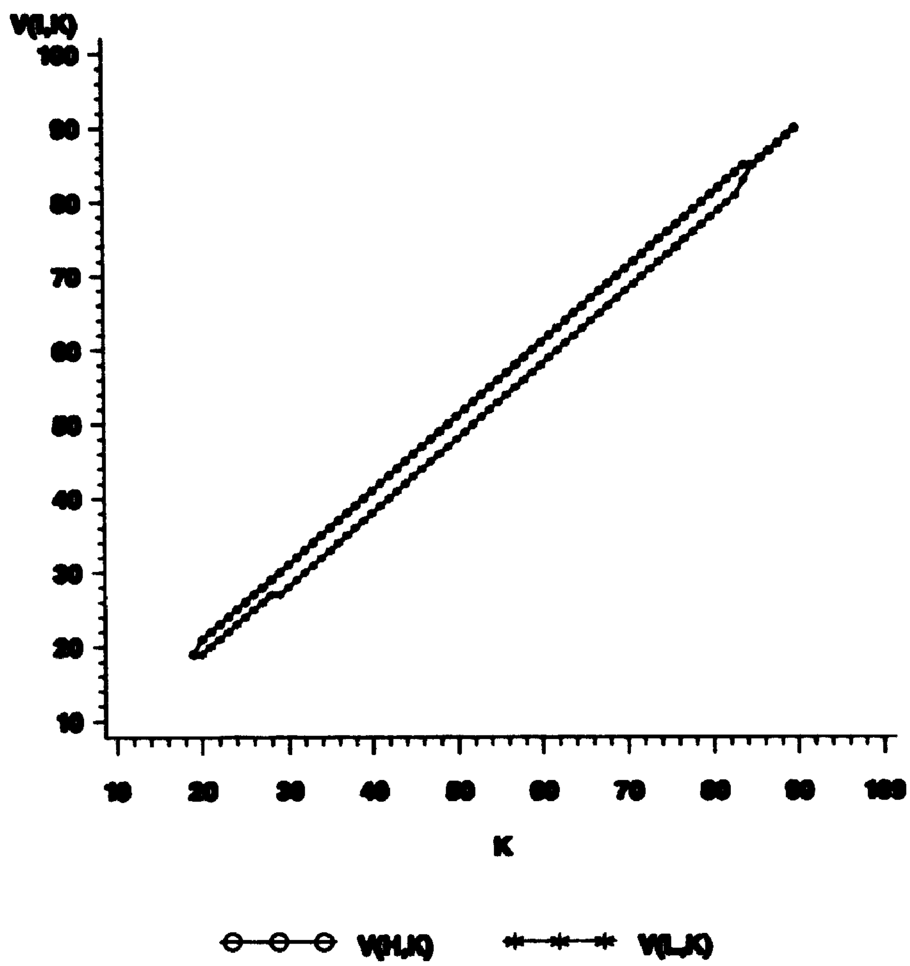


Figure 3.3.a

Current Compensations of the Agent: Dynamic Model
Discount Factor=0.96

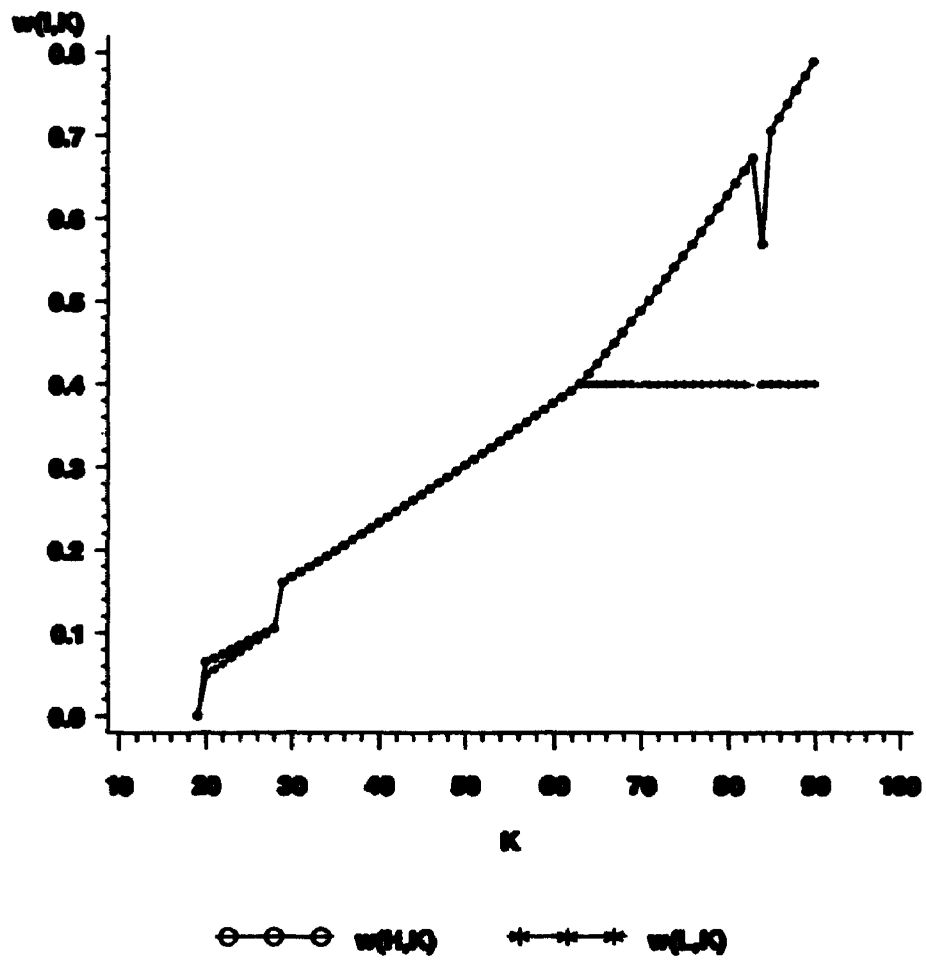


Figure 3.3.b

Current Compensations of the Agent: Static Model

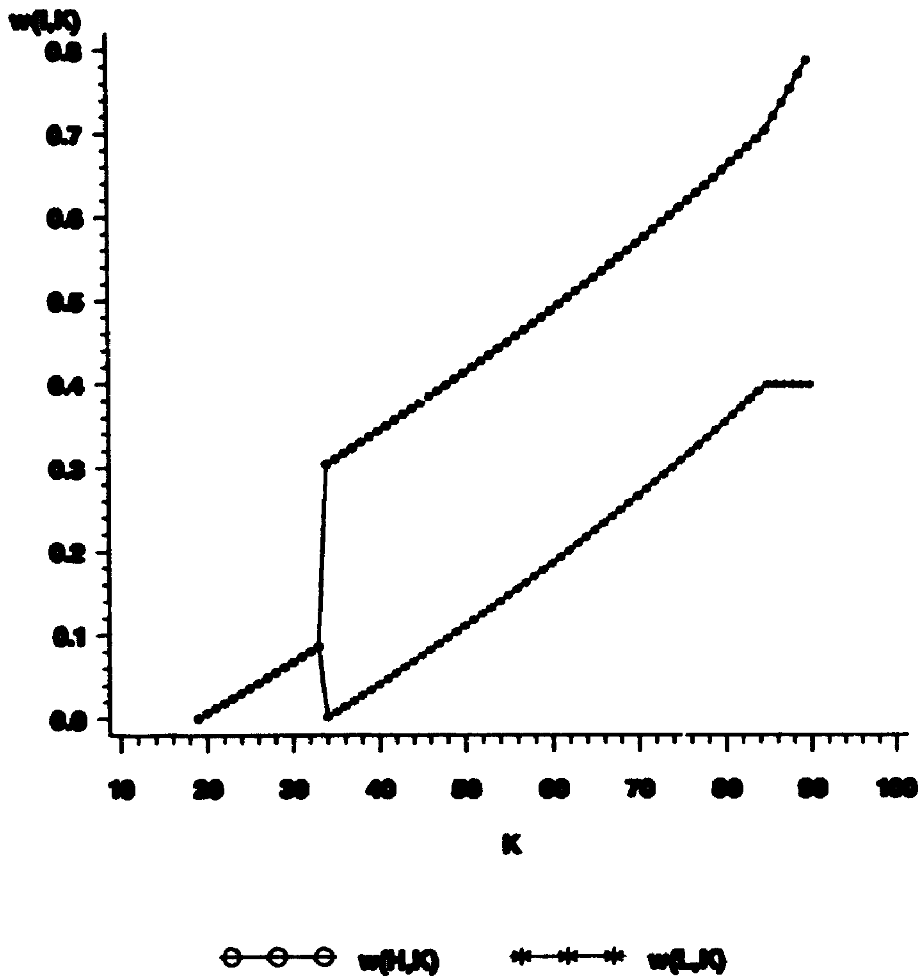


Figure 3.4

Consumption Paths of the Agent: Dynamic vs Static Model

Initial Expected Utility=V(40)

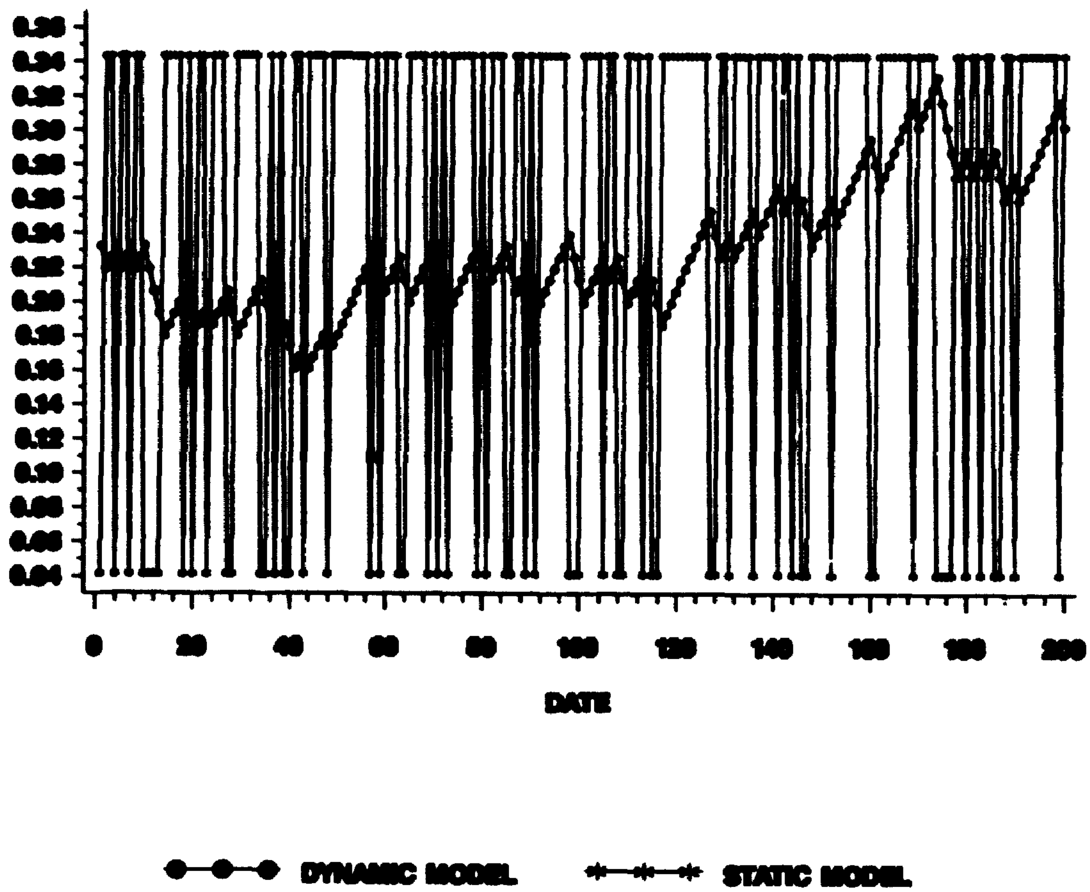


Figure 3.5
Expected Utilities of the Agent: Dynamic vs Static Model
Initial Expected Utility = V(40)

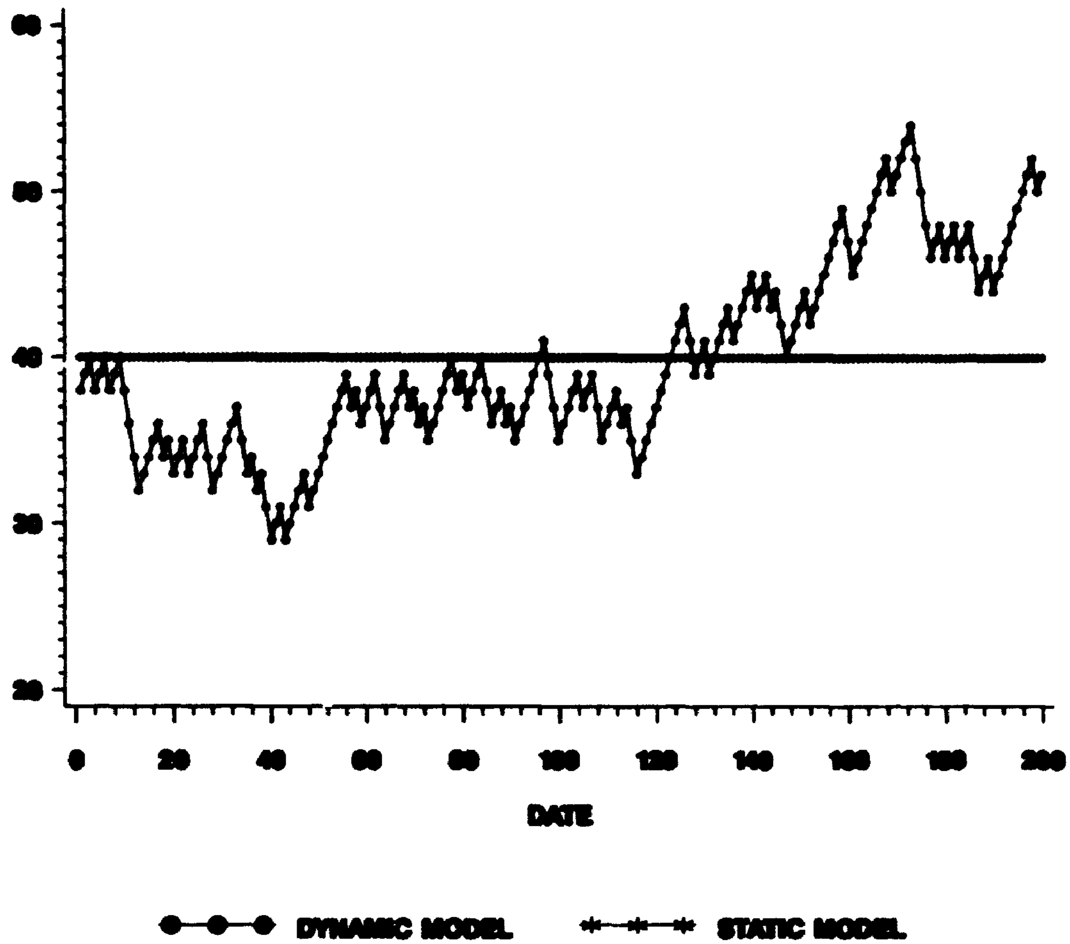


FIGURE 3.6.a

First Differences in CEO Compensation and ShareholderWealth
Insufficient Risk Sharing

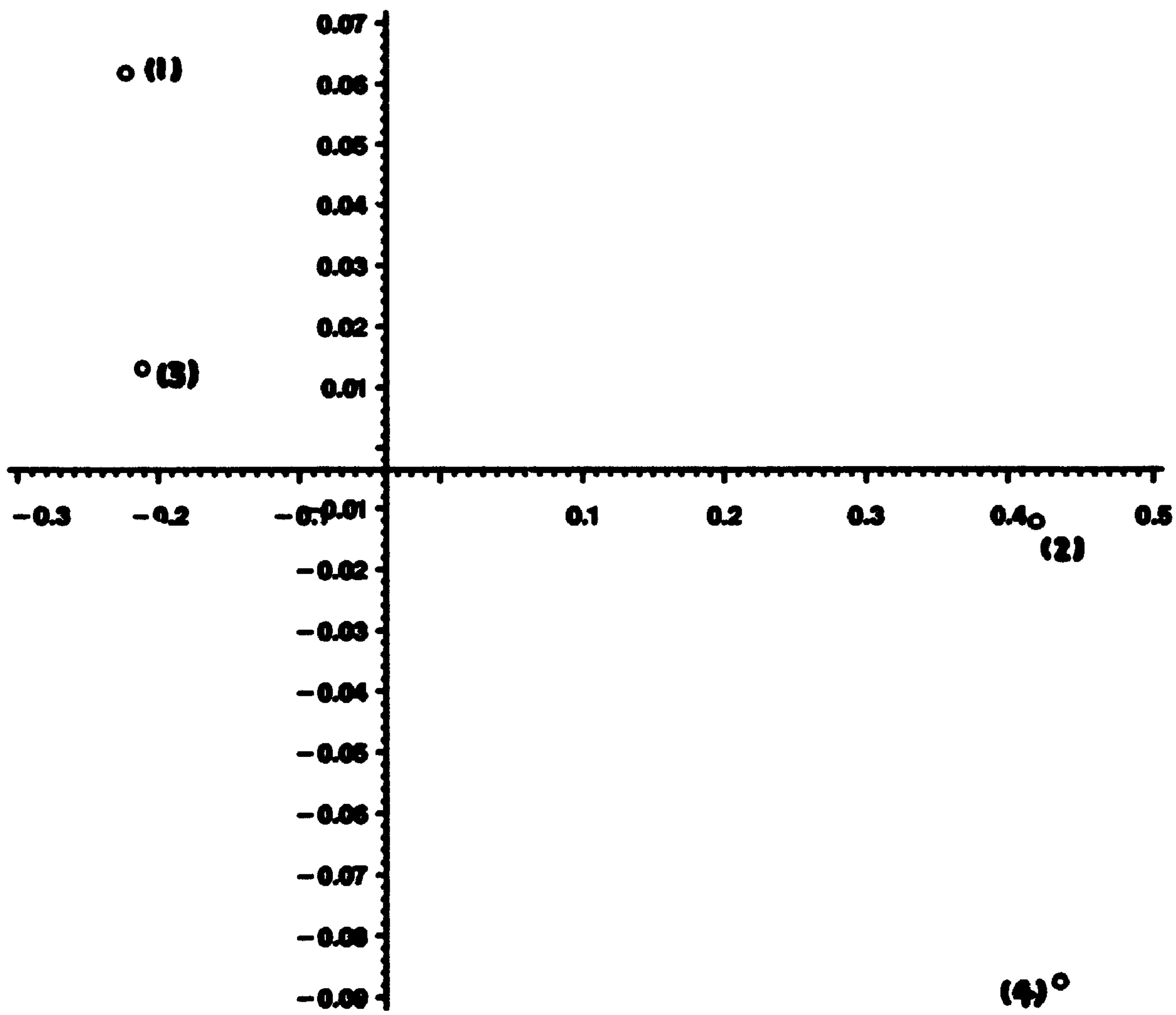


FIGURE 3.6.b

First Differences in CEO Compensation and ShareholderWealth

Sufficient Risk Sharing

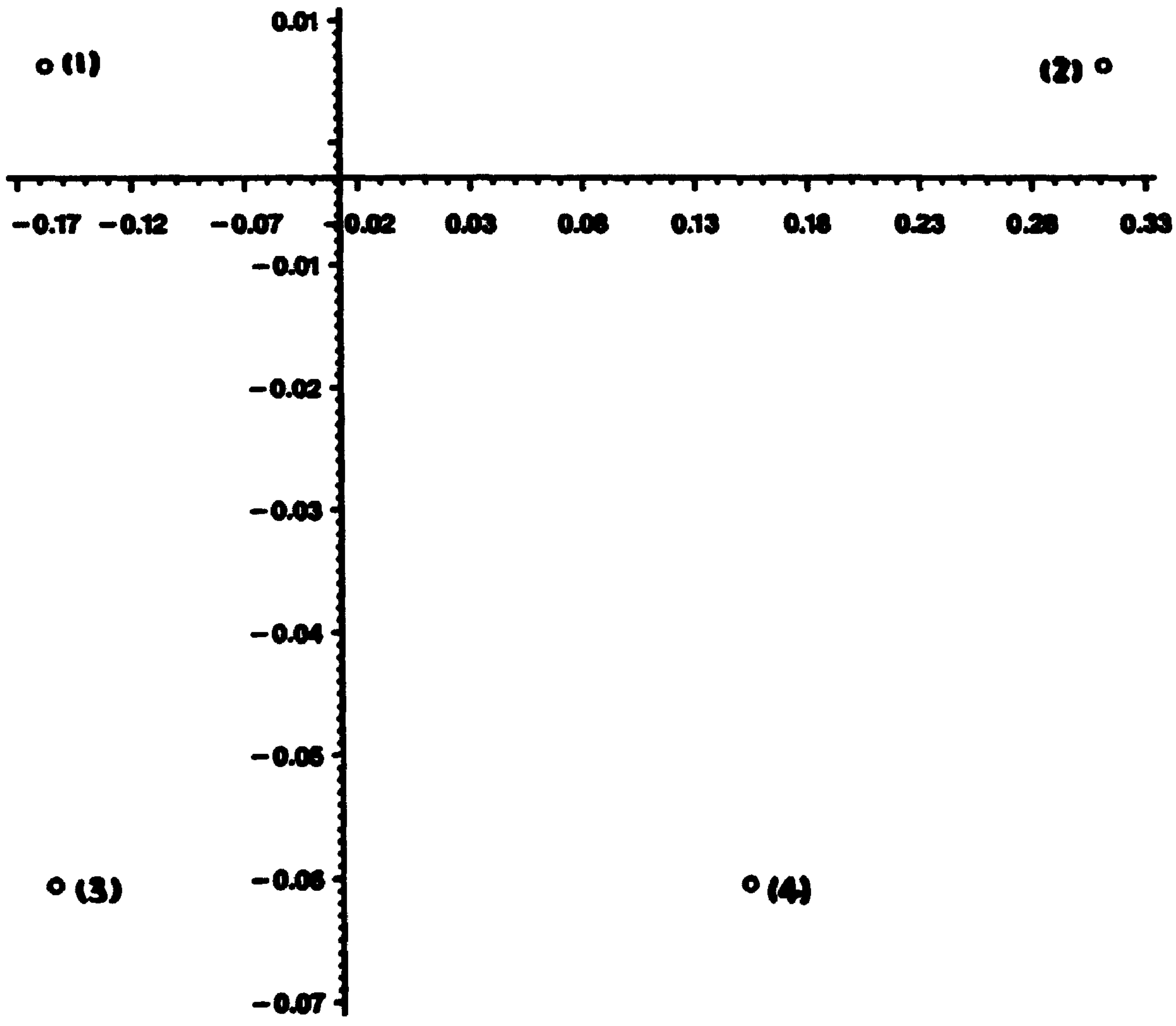


Figure 3.7

Value Functions: Closed Economy vs Economy with Borrowing

Discount Factor=0.96

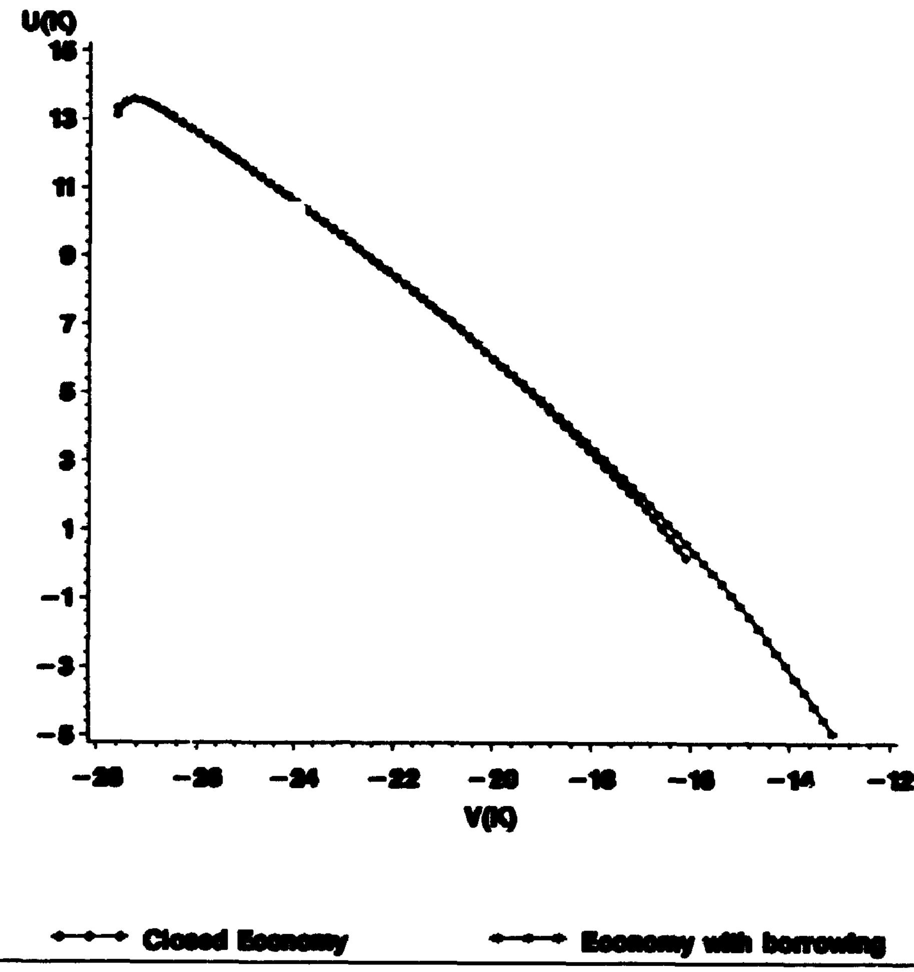


Figure 3.8.a

Expected Discounted Utilities Tomorrow: Principal CanBorrow

Discount Factor=0.96

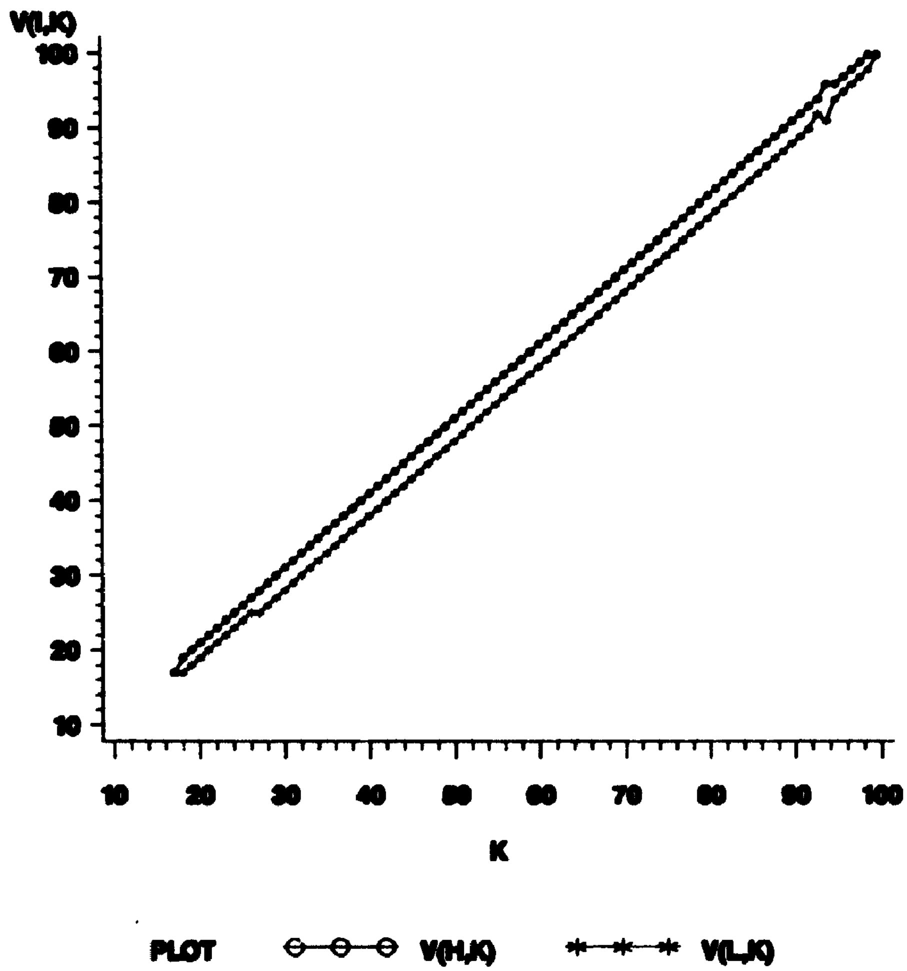


Figure 3.8.b

Current Compensations: Principal Can Borrow

Discount Factor=0.96

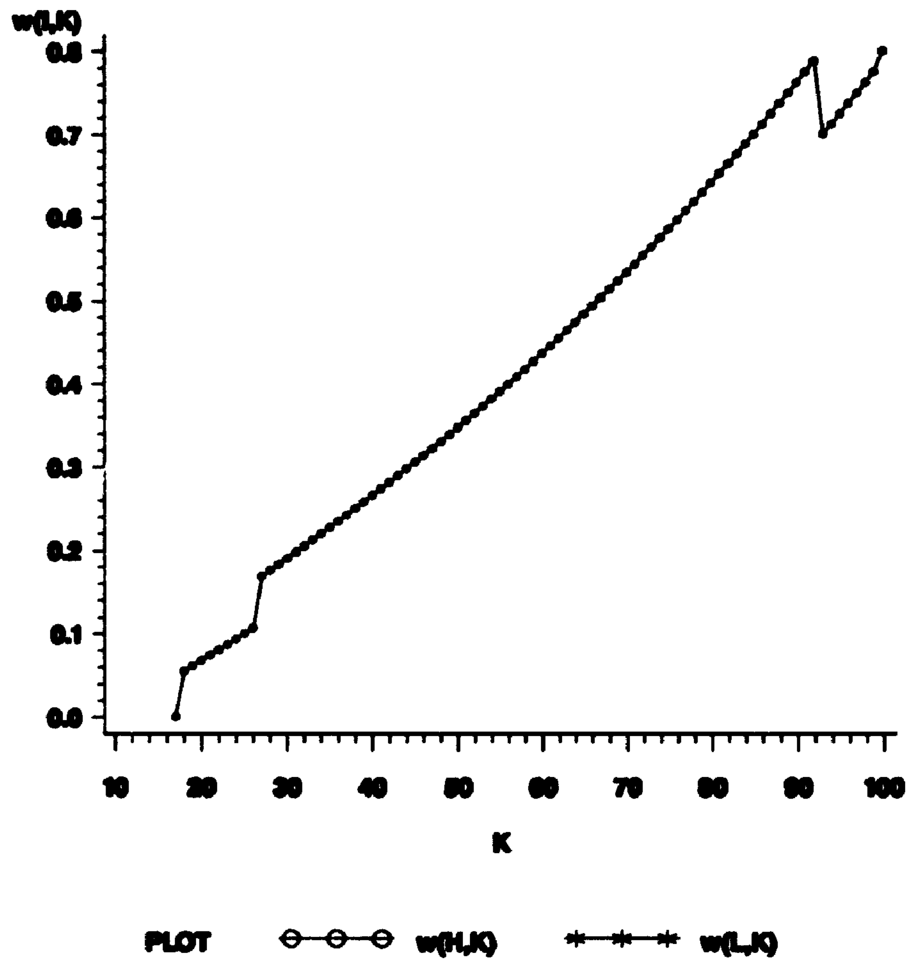


Figure 3.9.a

Expected Discounted Utilities Tomorrow

Discount Factor=0.90

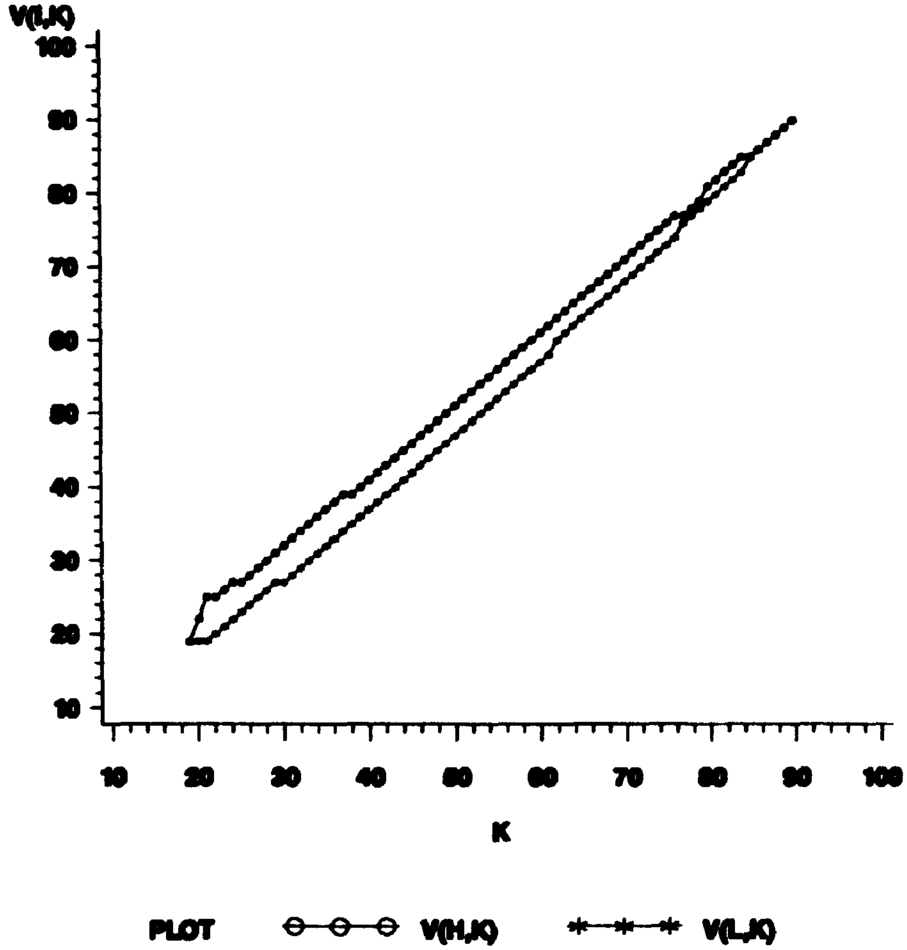


Figure 3.9.b

Current Compensations: Dynamic Model

Discount Factor=0.90

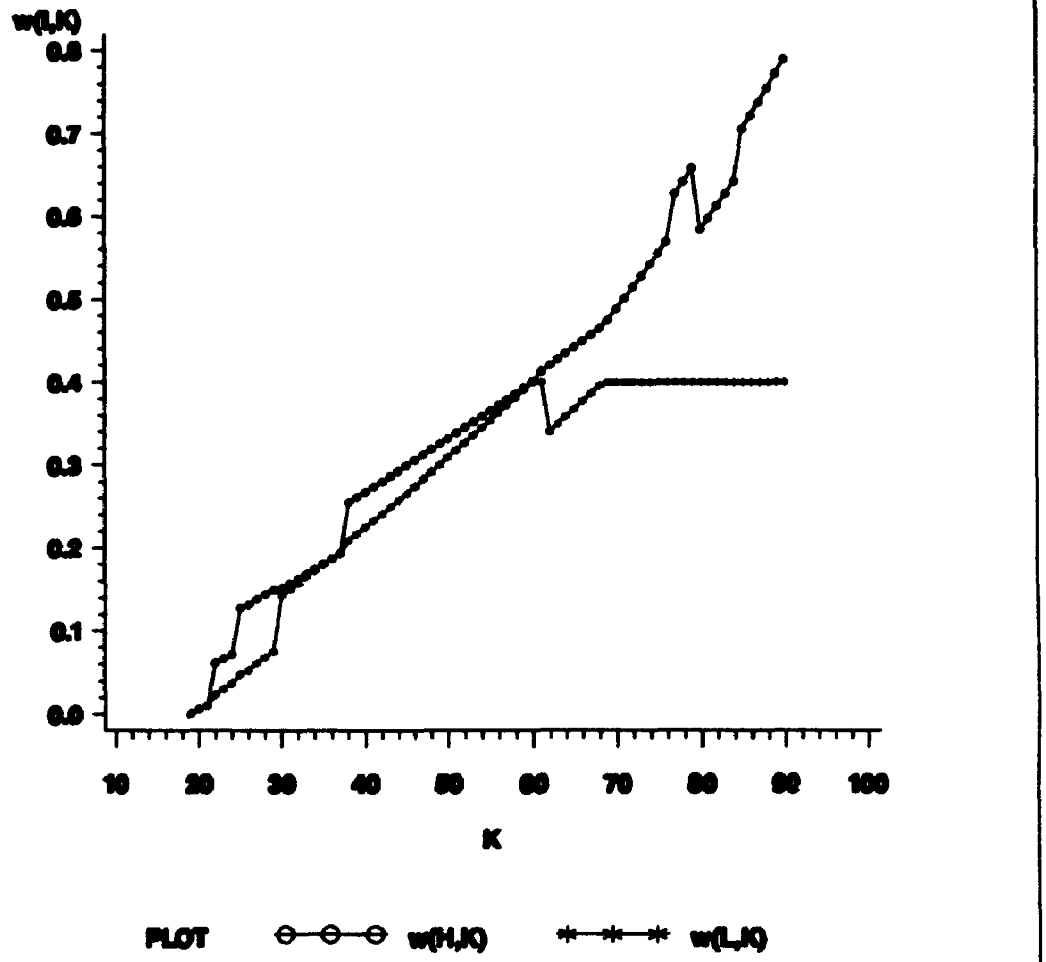


Figure 3.10.a

Expected Discounted Utilities Tomorrow

Discount Factor=0.80

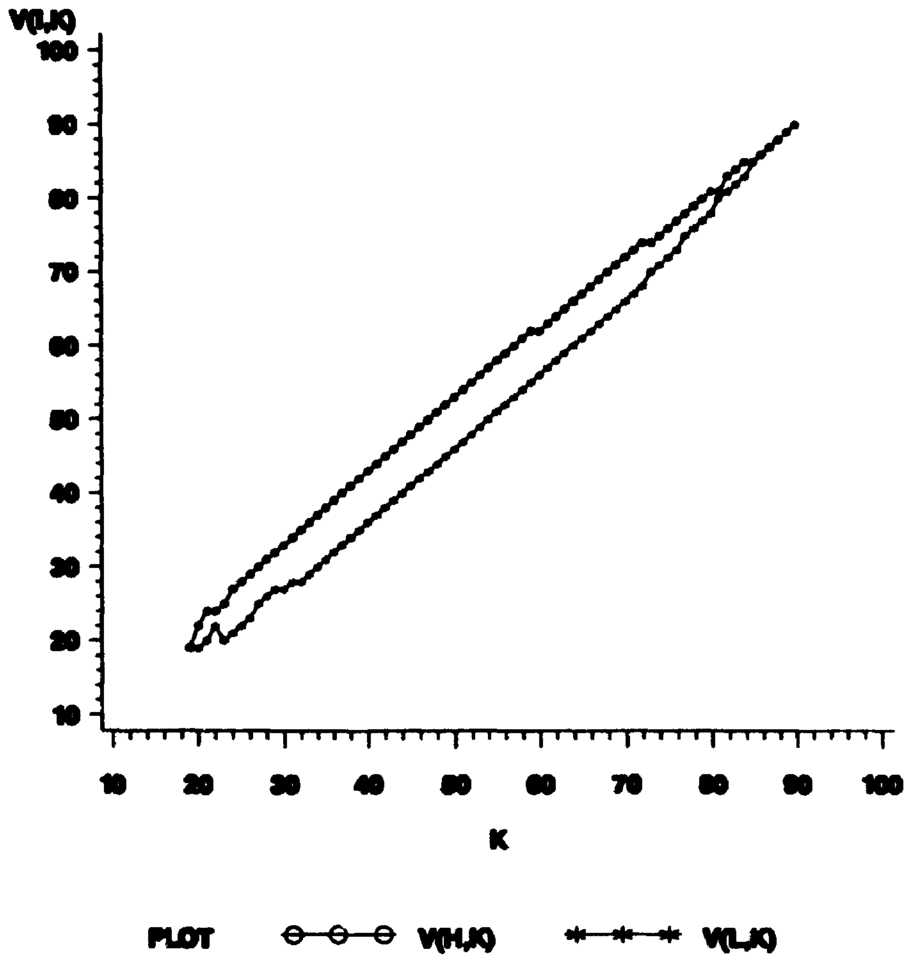


Figure 3.10.b

Current Compensations: Dynamic Model

Discount Factor=0.80