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Intersections Of Hyperconics And Configurations In Classical Planes

James Michael McQuillan

INTERSECTIONS OF HYPERCONICS AND CONFIGURATIONS IN CLASSICAL PLANES

by

James M. McQuillan

Department of Mathematics

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Abstract.

Let $\pi = PG(2, F)$, where F is a field of characteristic 2 and of order greater than 2. Given a conic, its tangents all pass through a common point, the nucleus. A conic, together with its nucleus, is called a hyperconic. All conics considered are non-degenerate.

First, a relationship is established between hyperconics and certain symmetric unipotent Latin squares for all finite projective planes.

Intersection properties of hyperconics in $PG(2, F)$, Fano configurations containing points of a hyperconic, as well as certain subplanes of $PG(2, F)$ are studied. An open question in $\pi = PG(2, q)$, q even, is: what is the size and structure of a set of maximum size of hyperovals (or hyperconics) pairwise intersecting in exactly 2 points? In $PG(2, 4)$, such a set is shown to have size 16 and to have one of 2 'dual' structures: 16 hyperconics missing a fixed line, or 16 hyperconics through a fixed point.

The former is a $2-(16, 6, 2)$-design of grid type which can be obtained from the $5-(24, 8, 1)$ Mathieu design, and which can be related to singular points of a Kummer surface in $PG(2, q)$ for q odd (see [Bruen 2]).

The latter is shown to be an affine plane in 2 ways:

i) taking the hyperconics which all contain the fixed point, as well as the lines through that fixed point (in the original plane) to be the lines of an $AG(2, 4)$; and

ii) taking the hyperconics in the original plane to be the points, and the points (except the fixed point in all 16 hyperconics) in the original plane to be the lines of an $AG(2, 4)$.

In $PG(2, F)$ let the field F contain a subfield of order 4. Then, in $PG(2, F)$ we describe certain sets of 6 points no 3 collinear called hexagons. It is then shown how the much studied even intersection property in $PG(2, 4)$ can be lifted (extended) to certain sets of hyperconics in $PG(2, F)$.

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Chapter 1. Introduction.

A **projective plane** π consists of objects called points and subsets of points called lines such that

1) given two points, there exists a unique line containing both;

2) given two lines, they meet in a unique point;

3) there exists a quadrangle. (See [Hartshorne 1].)

Let π be a classical or non-classical projective plane.

If π is a finite projective plane with $n+1$ points on l_∞, then each point of π has $n+1$ lines through it; each line of π has $n+1$ points on it; the total number of points is n^2+n+1; and the total number of lines is n^2+n+1.

Denote π by $PG(2.n)$.

A k**-arc** in a projective plane is a set of k points, no 3 collinear. It follows that k is at most $n+1$ if n is odd, and k is at most $n+2$ if n is even. A **hyperoval** in $PG(2,n)$ is an $(n+2)$-arc. Thus, hyperovals exist only if n is even. It follows that, for a fixed line and a fixed hyperoval, the line is either disjoint from the hyperoval, or the line intersects the hyperoval in exactly 2 points.

If n is a prime power, there exists a field of order n, and therefore there exists a projective plane of order n. The question remains (prime power conjecture) as to whether every finite projective plane must have prime power order.

Theorem 1.1. *(Bruck-Ryser)*
If $n \equiv 1$ or $2 \pmod 4$ then unless $n = a^2 + b^2$, for some integers a and b, there is no projective plane of order n.

The smallest integer which is not a prime power and which can't be ruled out by the Bruck-Ryser Theorem is $n = 10$. It was recently proved there is no $PG(2,10)$. An important part of the proof of the non-existence of a $PG(2,10)$ is the non-existence of hyperovals in a $PG(2,10)$ via a computer search. The number of hyperovals was connected to an incidence matrix of a $PG(2,10)$ via weight enumerators and the MacWilliams identities for algebraic codes.

Notation. *$PG(2,F)$ denotes the projective plane over the finite or infinite field F. Using non-homogenous coordinates, its points are*
$\{(a,b)|a,b \in F\} \cup \{(m=a)|a \in F \cup \{\infty\}\}$

and its lines are

$Y = mX + b$, $X = c$, *and* l_∞ *where* $m, b, c \in F$ *and* $l_\infty = \{(m = d) | d \in F \cup \infty\}$.

Using homogeneous coordinates its points are

$\{(a_0, a_1, a_2) \mid a_i \in F,\ a_i$ *not all zero*$\}$, *with* (a_0, a_1, a_2), *and* (b_0, b_1, b_2) *representing the same point iff* $a_i = cb_i \ \forall i$, *for some non-zero* $c \in F$; *using homogenous coordinates its lines are*

$$[a_0, a_1, a_2] := \{\underbrace{(x_0, x_1, x_2)}_{\textit{a point}} | a_0x_0 + a_1x_1 + a_2x_2 = 0, \quad a_i \in F \text{ not all zero}\}$$

with $[a_0, a_1, a_2]$, *and* $[b_0, b_1, b_2]$ *representing the same line iff* $a_i = cb_i \quad \forall i$, *for some* $c \in F \setminus \{0\}$.

A **classical projective plane**, *denoted* $PG(2, q)$, *is the projective plane over the field* $\mathbf{F}_q$.

A **conic** C over a field F is a set of points satisfying a quadratic equation $aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, where $a, b, \ldots, f \in$ F. This conic is **non-degenerate** if its equation does not factor. If charF = 2, a conic is non-degenerate if $(f, e, d) \notin C$. All conics considered here are non-degenerate.

Any 5-arc is on a *unique* conic. The celebrated theorem of B. Segre asserts that in $PG(2, \text{F})$, if charF $\neq 2$, every $q + 1$-arc is a conic, and conversely (see [Hirschfeld 1], p.168). Also, if charF $\neq 2$, the tangents to a conic form a conic in the dual plane.

If charF = 2, Segre's Theorem is false. Also, if char F = 2, the tangents to a conic form a degenerate conic in the dual plane. The tangents to a conic form a pencil, i.e., they all go through a single point, the **nucleus**. The nucleus of the non-degenerate conic $aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, where $a, b, \ldots, f \in$ F, is (f, e, d). When F = $\mathbf{F}_q$, a conic has $q + 1$ points, no 3 collinear. Further, we can adjoin the nucleus to the conic to give a **hyperconic**. If F = $\mathbf{F}_q$, a hyperconic is a hyperoval; however, not every hyperoval is a hyperconic. For example, $Y = X^4 \cup \{(m = 0), (m = \infty)\}$ is hyperoval but is not a hyperconic in $PG(2, 32)$ (see [Segre 1]).

All fields considered here have characteristic 2.

All conics considered here are non-degenerate.

We will denote by $H = C \cup \{N\}$ a hyperconic where C is a conic with nucleus N. A hyperoval is a **affine** if it is skew to (disjoint from) l_∞, and is **projective** otherwise.

In Chapter 2, we investigate a connection between hyperovals and certain symmetric Latin squares. Every hyperoval in a projective plane, classical or non-classical, gives rise to a symmetric unipotent Latin square.

In Chapter 3, we investigate hyperovals in $PG(2,4)$. We consider sets of hyperovals that pairwise intersect one another in an even number of points, and the nature of their intersections. The main result in that chapter is such a set has size less than or equal to 16, with equality if the set has one of 2 'dual' structures. These structures yield designs.

Of course, $PG(2,4)$ is a well studied object with connections to group theory.

"Among projective planes, $PG(2,4)$ is particularly remarkable. A 2-design, it is three times extendable to a $5-(24,8,1)$ design admitting the 5-transitive Mathieu group M_{24} ... by far the most elementary construction [of this] emerges from a study of the binary code of the projective plane. "[1]

The finite projective plane $PG(2,4)$ is interesting because of its relationship to the Golay code, the Conway group, the Mathieu group, and the Leech lattice (see [Cameron 1] and [van Lint 1]).

"$5-(12,6,1)$ and $5-(24,8,1)$ designs ... are intimately related to their automorphism groups, the 5-fold transitive groups M_{12} and M_{24} discovered by Mathieu (1861), (1873). ... The easiest way to construct (and prove uniqueness of) these designs is via coding theory, using the ternary and binary Golay codes associated with them. ... Luneburg's construction of the $5-(24,8,1)$ design is based on ... combinatorial properties of the unique projective plane of order 4 ...".[2]

In $PG(2,4)$, one of the sets of maximum size of hyperovals pairwise intersecting in exactly 2 points can be obtained from the $5-(24,8,1)$ Mathieu design.

"The [Mathieu] group M_{24} is one of the most remarkable of all finite groups. Many properties of the larger sporadic groups reduce on examination to properties of M_{24}. This centenarian group can still startle us with its youthful acrobatics. The automorphism group of the Leech lattice, modulo a centre of order 2, is the Conway group

[1][Lander 1], p. 53

[2][Cameron 1], p. 22

C_{o_1}, and by stabilizing sublattices of dimensions 1 and 2 we obtain the other Conway groups C_{o_2}, C_{o_3}, the McLaughlin group McL, and the Higman-Sims group HS. The sporadic Suzuki group Suz, and the Hall-Janko group $HJ = J_2$, can also be obtained from the Leech lattice by enlarging the ring of definition. The Leech lattice is a 24-dimensional Euclidean lattice which is easily defined in terms of the Mathieu group M_{24}."[3]

In Chapter 4, we investigate intersection properties of hyperconics in projective planes over fields. Of particular interest are 6-arcs where the conic through every 5 of the points has as nucleus the remaining point. These 6-arcs only occur in projective planes over fields that have subfields of order 4. Of these 6-arcs, it is shown that those contained in a fixed hyperconic form a 3-design. (See theorem 4.37.)

Orbits of 5-tuples on the complex projective line have been studied by L. Renner (see [Renner 1]). Here, we consider orbits of certain sets of 5 points on a projective line which result from some special 6-arcs in a projective plane. The images of a set of these 5 points under the Mobius group yield a 3-design (see theorem 4.41).

In Chapter 5, we show how the famous 'even intersection property' can be lifted (extended) from the 168 hyperconics in a subplane of order 4, to the (6)(168) hyperconics in the larger plane containing them. This works in projective planes over fields that do not have a subfield of order 8.

The Appendix contains some facts on hyperconics in $PG(2,q)$, $q = 2^t$ when $q = 4$ and when $q \geq 16$.

[3] [Conway 1], p. viii

Chapter 2. Hyperovals and Latin Squares.

Section 2.1. A relationship between hyperovals and certain symmetric unipotent Latin squares.

Theorem 2.1. *Let π be a classical or non-classical projective plane of order n even. Then every hyperoval in π skew to (disjoint from) a fixed line l gives rise to a unique (up to the reordering of the rows and columns) symmetric unipotent Latin square of size $(n+2) \times (n+2)$.*

Also, every hyperoval in π intersecting l gives rise to a unique (up to a reordering of the rows and columns) symmetric unipotent Latin square of size $n \times n$.

Proof: Let $H = \{P_1, P_2, \ldots, P_{n+2}\}$ be a hyperoval in π. Let l be a line missing H. Let $Q_1, Q_2, \ldots, Q_{n+1}$ be the points of l. Define a matrix

$$A = [a_{ij}]$$

where

$$a_{ij} = \begin{cases} t & \text{if the line } P_iP_j \text{ intersects } l \text{ in } Q_t \\ * & \text{if } i = j. \end{cases}$$

Clearly A is symmetric, and A is a Latin square with entries $\{*, 1, 2, \ldots, n+1\}$ appearing exactly once in each row and column. Thinking of the symmetric Latin square as the table of a quasigroup (see [Denes 1]) with identity $*$, it is unipotent, ie., $a_{ii} = * \quad \forall i$ (the square of any element is the identity).
Suppose now that l is a line intersecting H. Let $\{P_1, P_2, \ldots, P_n\}$ be the points of H off l, and let $Q_1, Q_2, \ldots, Q_{n-1}$ be the points of l not on H. Define A as above. Then A is a symmetric unipotent Latin square. □

Corollary 2.2. *Suppose $\pi = PG(2,n)$ is a classical or non-classical projective plane, where n is even. Let H be a hyperoval in π. Let $\{P_1, P_2, \ldots, P_t\}$ be the points of H not on l_∞, where $t = n+2$ if H is an affine hyperoval and $t = n$ if H is a projective hyperoval. Let $Q_1, Q_2, \ldots, Q_{t-1}$ be the points of l off H. Then $\{P_2, P_3, \ldots, P_t\}$ can* *be relabelled to give two standard forms for the resulting Latin square as follows.*

1) Relabelling so that

$$P_1P_2 \cap l_\infty = Q_1,$$
$$P_1P_3 \cap l_\infty = Q_2,$$
$$\vdots$$
$$P_1P_t \cap l_\infty = Q_{t-1}$$

yields a Latin square with the form

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_t \end{array}
\begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & \cdots & P_t \end{array} \\
\left(\begin{array}{ccccc} * & 1 & 2 & \cdots & t-1 \\ 1 & * & & & \\ 2 & & * & & \\ \vdots & & & \ddots & \\ t-1 & & & & * \end{array}\right) \end{array}.$$

2) Relabelling $P_2, P_3, \ldots$ so that

$$P_1P_2 \cap l_\infty = Q_1$$
$$P_3P_4 \cap l_\infty = Q_1$$
$$\vdots$$
$$P_{t-1}P_t \cap l_\infty = Q_1$$

yields a Latin square of the form

$$A = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_{t-1} \\ P_t \end{array}
\begin{array}{c} \begin{array}{ccccccc} P_1 & P_2 & P_3 & P_4 & \cdots & P_{t-1} & P_t \end{array} \\
\left(\begin{array}{ccccccc} * & 1 & & & & & \\ 1 & * & & & & & \\ & & * & 1 & & & \\ & & 1 & * & & & \\ & & & & \ddots & & \\ & & & & & * & 1 \\ & & & & & 1 & * \end{array}\right) \end{array}.$$

□

Lemma 2.3. *1) There exist symmetric unipotent Latin squares which cannot be constructed from a hyperoval in the way described above.*
2) Different hyperovals may give rise to the same symmetric unipotent Latin square.

Proof: 1) For example, in $PG(2,8)$, there is no hyperoval that gives rise to the symmetric unipotent Latin square

$$A = \begin{array}{c} \\ P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{array} \begin{array}{c} \begin{array}{cccccccccc} P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \end{array} \\ \begin{pmatrix} * & 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & 0 & \infty \\ 1 & * & \alpha^4 & \alpha^3 & \alpha^2 & \alpha & \alpha^6 & \alpha^5 & \infty & 0 \\ \alpha & \alpha^4 & * & 1 & \alpha^5 & \alpha^6 & 0 & \infty & \alpha^2 & \alpha^3 \\ \alpha^2 & \alpha^3 & 1 & * & \infty & 0 & \alpha & \alpha^4 & \alpha^6 & \alpha^5 \\ \alpha^3 & \alpha^2 & \alpha^5 & \infty & * & 1 & \alpha^4 & 0 & \alpha & \alpha^6 \\ \alpha^4 & \alpha & \alpha^6 & 0 & 1 & * & \infty & \alpha^3 & \alpha^5 & \alpha^2 \\ \alpha^5 & \alpha^6 & 0 & \alpha & \alpha^4 & \infty & * & \alpha^2 & \alpha^3 & 1 \\ \alpha^6 & \alpha^5 & \infty & \alpha^4 & 0 & \alpha^3 & \alpha^2 & * & 1 & \alpha \\ 0 & \infty & \alpha^2 & \alpha^6 & \alpha & \alpha^5 & \alpha^3 & 1 & * & \alpha^4 \\ \infty & 0 & \alpha^3 & \alpha^5 & \alpha^6 & \alpha^2 & 1 & \alpha & \alpha^4 & * \end{pmatrix} \end{array}$$

where $\mathbf{F}_8 \setminus \{0\} =< \alpha >, \alpha^3 = 1 + \alpha^2$.
If we choose $P_0 = (0,0)$ and $P_1 = (1,1)$, then $P_2 = (a, \alpha a)$, $P_3 = (b, \alpha^2 b)$, $P_4 = (c, \alpha^3 c)$, $P_5 = (d, \alpha^4 d)$, $P_6 = (e, \alpha^5 e)$, $P_7 = (f, \alpha^6 f)$ $P_8 = (g, 0)$ and $P_9 = (0, h)$, for some $a, b, \dots, h \in \mathbf{F}_8$. Write $A = [A_{i,j}]$ where $i, j \in \{0, \dots, 9\}$. Now $g = 1$ since $A_{1,8} = \infty$, $h = 1$ since $A_{1,9} = 0$, $a = \alpha^6 e$ since $A_{2,6} = 0$, $b = c$ since $A_{3,4} = \infty$, $b = \alpha^2 d$ since $A_{3,5} = 8$, $c = \alpha^3 f$ since $A_{4,7} = 0$ and $d = e$ $A_{5,6} = \infty$. This yields a contradiction. Thus A does not arise from any hyperoval in $PG(2,8)$.
2) For example, in $PG(2,4)$, the 48 hyperovals skew to a line give rise to the same symmetric unipotent Latin square; the 120 hyperovals intersecting a line give rise to the same symmetric unipotent Latin square. ☐

Note that in the proof of 1) in lemma 2.3, we needed to use a projective plane of order at least 8.

In $PG(2,2)$, the unique symmetric unipotent Latin square with first row $*, 1, 0, \infty$ is

$$\begin{array}{c} \\ P_0 \\ P_1 \\ P_2 \\ P_3 \end{array}\begin{array}{c} \begin{array}{cccc} P_0 & P_1 & P_2 & P_3 \end{array} \\ \begin{pmatrix} * & 1 & 0 & \infty \\ 1 & * & \infty & 0 \\ 0 & \infty & * & 1 \\ \infty & 0 & 1 & * \end{pmatrix} \end{array}.$$

Given a hyperoval H in $PG(2,2)$ skew to line l, choose coordinates so H contains $(0,0)$ and $(1,1)$ and so that $l = l_\infty$. Then $H = \{(0,0),(1,1),(1,0),(0,1)\}$. Each of the 4 choices for P_0 amongst the points of H yields the Latin square above (if you insist on having the standard form where the first row is $*, 1, 0, \infty$).

Now consider $PG(2,4)$ where $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$. There are 6 symmetric unipotent Latin squares with the first row $*, 1, \omega, \omega^2, 0, \infty$. Consider

$$A = \begin{array}{c} \\ P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array}\begin{array}{c} \begin{array}{cccccc} P_0 & P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{pmatrix} * & 1 & \omega & \omega^2 & 0 & \infty \\ 1 & * & c & & & \\ \omega & & * & a & & \\ \omega^2 & & a & * & & \\ 0 & & & & * & b \\ \infty & & & & b & * \end{pmatrix} \end{array}.$$

If $a = 1$ and A is a symmetric unipotent Latin square, then $b = 1$; moreover, c is 0 or ∞, and for each of these choices of c, there is exactly one such Latin square.

If $a \neq 1$, then $a = 4$ or 5 and $b = 2$ or 3. For each of these 4 possibilities, there is a unique symmetric unipotent Latin square. Thus A extends in exactly 6 ways to a symmetric unipotent Latin square.

Given a hyperoval H skew to l, choose $l = l_\infty$. In theorem 2.7, there is an example of a symmetric unipotent Latin square that H gives rise to. There are 6 choices for labelling the points of H $P_0, \ldots, P_5$ so that they give rise to a symmetric unipotent Latin square with first row $*, 1, \omega, \omega^2, 0, \infty$ as there are 6 choices for P_0. □

Section 2.2. Translations of Hyperovals.

Consider $\pi = PG(2, \mathrm{F})$, where F is a field. Let H be a hyperoval in π, where $\mathrm{F} = \mathbf{F}_q$, $q = 2^t$.

Define $H + (a, b) := \{(x + a, y + b) \mid (x, y) \in H \setminus l_\infty\} \cup (H \cap l_\infty)$, where $a, b \in \mathrm{F}$. I.e., $H + (a, b)$ is an affine translation of H.

Lemma 2.4. *Suppose $\pi = PG(2, F)$, where F is a field. If H is a hyperconic in π, then $H + (k, l)$ is a hyperconic in π, where $k, l \in F$. Moreover, H and $H + (k, l)$ give rise to the same symmetric unipotent Latin square. If H is a hyperoval in π with $F = \mathbf{F}_q$, $q = 2^t$, then $H + (k, l)$ is a hyperoval in π. Moreover, H and $H + (k, l)$ give rise to the same symmetric unipotent Latin square.*

Proof: Let $H = C \cup \{N\}$ where

$$C : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$$

and where $N = (f, e, d)$. Consider the translation C' of C where

$$C' : a(X + kZ)^2 + b(Y + lZ)^2 + cZ^2 + d(X + kZ)(Y + lZ) + e(X + kZ)Z + f(Y + lZ)Z = 0.$$

Now

$$C' : aX^2 + bY^2 + (c + ak^2 + bl^2 + ek + fl)Z^2 + dXY + (e + dl)XZ + (f + dk)YZ = 0$$

which has nucleus $N' = (f + dk, e + dl, d)$. Let $H' = H + (k, l)$, i.e., H' is a translation of H.

Now N is on l_∞ iff N' is on l_∞; moreover, if N and N' are on l_∞, they are equal. Also, N is affine iff N' is affine; moreover, if N and N' are affine, then $N' = N + (k, l)$. Setting $Z = 0$ in the equations of C and C' gives the same equation, i.e., these 2 conics have the same points on l_∞. For affine points of C and C', $(X, Y) \in C$ iff $(X, Y) + (k, l) \in C'$.

Thus H' is a hyperconic in π.

Next, for an arbitrary hyperoval H in $PG(2, q)$, $q = 2^t$, consider $H + (k, l)$. Let P, Q, and R be 3 points of H. Since $R + (k, l)$ is not on l_∞ if R is not on l_∞, it follows that if P and Q are on l_∞, then P, Q and $R+(k,l)$ are not collinear. If P is on l_∞, but Q and R are not on l_∞, then P, $Q+(k,l)$ and $R+(k,l)$ are not collinear as the line joining $Q+(k,l)$ and $R+(k,l)$ intersects l_∞ in the same point as the line QR does. If P, Q and R are not on l_∞, then $P+(k,l)$, $Q+(k,l)$ and $R+(k,l)$ are not collinear since the line joining $P+(k,l)$ and $Q+(k,l)$ intersects l_∞ in the same point as the line PQ does, and the line joining $P+(k,l)$ and $R+(k,l)$ intersects l_∞ in the same point as the line PR does. Thus $H+(k,l)$ is a hyperoval.

Moreover, if H is a hyperconic or hyperoval, and if P and Q are affine points of H, then the line joining $P+(k,l)$ and $Q+(k,l)$ meets l_∞ in the same point as the line PQ does. Therefore, H and $H+(k,l)$ give rise to the same symmetric unipotent Latin square. □

Define a relation $\sim$ amongst the hyperovals in $PG(2,4)$ as follows.
Given 2 hyperovals, H_1 and H_2, define $H_1 \sim H_2$ if $|H_1 \cap H_2|$ is even.
It is well known this is an *equivalence relation* with 3 equivalence classes of size $168/3 = 56$ (see [Lander 1]).

Theorem 2.5. *Let $\pi = PG(2,4)$. If H is an affine hyperoval, then $\{H+(a,b) | a,b \in F_4\}$ is a set of 16 distinct hyperovals, any 2 of which have exactly 2 common points.*

Proof: Let $H = \{P_1, \ldots, P_6\}$ where $P_i = (a_i, b_i)$.
First we need to establish the following claim.

Claim 1: $|(H+(a,b)) \cap (H+(c,d))|$ is even.
If P is in both $H+(a,b)$ and $H+(c,d)$, then $\exists P_1 \in H$ such that $P = P_1+(a,b)$. Also $P = P_1+(a,b) \in H+(c,d)$. Therefore $P_1+(a,b)+(c,d) \in H$. Therefore $P_1+(c,d) \in H+(a,b)$. Therefore $P+(a,b)+(c,d) \in H+(a,b)$. Similarly $P+(a,b)+(c,d) \in H+(c,d)$. This establishes Claim 1.
Next, we prove Claim 2.

Claim 2: The 15 differences $P_i - P_j = (a_i,b_i)-(a_j,b_j)$, where $P_i \neq P_j$, are distinct.
If $P_i + P_j = P_i + P_k$, then $P_j = P_k$.
Suppose by way of contradiction that $P_1+P_2 = P_3+P_4$. Let $Q_1 = P_1P_2 \cap l_\infty$. Let $Q_2 = P_1P_3 \cap l_\infty$. Let $Q_3 = P_1P_4 \cap l_\infty$. Thus, hyperoval H gives rise to the symmetric

unipotent Latin square

$$\begin{pmatrix} * & 1 & 2 & 3 & 4 & 5 \\ 1 & * & 3 & 2 & & \\ 2 & 3 & * & 1 & & \\ 3 & 2 & 1 & * & & \\ 4 & & & & & \\ 5 & & & & & \end{pmatrix}.$$

However, this can't be completed to a symmetric unipotent Latin square, yielding a contradiction. This proves Claim 2.

Next, we establish Claim 3.

Claim 3: $(H + (a,b)) \cap (H + (c,d)) \neq \emptyset$.

We have $(a,b) + (c,d) = P_1 + P_2$, say, by claim 2. Consider $P_1 + (a,b)$. $P_1 + (a,b) = P_2 + (c,d)$. But P_1 and P_2 are in H. Therefore, $(H + (a,b)) \cap (H + (c,d)) \neq \emptyset$. This establishes Claim 3.

We now prove Claim 4.

Claim 4: $H + (a,b) \neq H + (c,d)$ unless $(a,b) = (c,d)$.

If $H + (a,b) = H + (c,d)$, then $H + (a,b) + (c,d) = H$. Therefore, suppose by way of contradiction that $H + (a,b) = H$ and $(a,b) \neq (0,0)$. Therefore $P_1 + (a,b) \in H$. Say $P_1 + (a,b) = P_2$. Say $P_3 + (a,b) = P_4$. Say $P_5 + (a,b) = P_6$. Therefore $H = \{P_1, P_1 + (a,b), P_3, P_3 + (a,b), P_5, P_5 + (a,b)\}$. The line joining P_1 to $P_1 + (a.b)$, the line joining P_2 to $P_2 + (a,b)$, and the line joining P_3 to $P_3 + (a,b)$ each meet l_∞ in Q_1, say. The line P_1P_3 and the line joining $P_1 + (a,b)$ to $P_3 + (a,b)$ both meet l_∞ in Q_2, say. The line P_1P_5 and the line joining $P_1 + (a,b)$ to $P_5 + (a,b)$ both meet l_∞ in Q_3, say. But, then H gives rise to a symmetric unipotent Latin square that looks like

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{array} \begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \\ \begin{pmatrix} * & 1 & 2 & 3 & 4 & 5 \\ 1 & * & & 2 & & 4 \\ 2 & & * & 1 & & \\ 3 & 2 & 1 & * & ? & ? \\ 4 & & & & * & 1 \\ 5 & 4 & & & 1 & * \end{pmatrix} \end{array}.$$

This cannot be extended to a symmetric unipotent Latin square, thus yielding a contradiction. This establishes Claim 4. □

Theorem 2.6. *Let $\pi = PG(2,4)$. If H is a projective hyperoval, then $\{H + (a,b) | a,b \in \mathbb{F}_4\}$ contains exactly 4 different hyperovals. Any 2 hyperovals in this set which are different meet in P, Q and no other points, where P and Q are the points of intersection of H with l_∞. Also, these 4 different hyperovals must belong to the same equivalence class under the even intersection equivence relation.*

Proof: Consider $(H + (a,b)) \cap (H + (c,d))$. If P is an affine point on both $H + (a,b)$ and $H + (c,d)$, then so is $P + (a,b) + (c,d)$. Therefore $|(H + (a,b)) \cap (H + (c,d))|$ is even. Since 4 points determine a unique hyperoval, if $H + (a,b)$ and $H + (c,d)$ have a common affine point, then they are equal. Let (a,b) be an affine point not in H. Let (c,d) be an affine point not in H or $H + (a,b)$. Let (e,f) be an affine point not in H, $H + (a,b)$ or $H + (c,d)$. Each affine point is on exactly one of the 4 hyperovals H, $H + (a,b)$, $H + (c,d)$, and $H + (e,f)$. □

Section 2.3. Translations of Hyperovals in $PG(2,4)$.

Consider the classical plane $PG(2,4)$. An easy counting argument (see [Lander 1]) shows there are 168 hyperovals in $PG(2,4)$. Also there are 12 hyperovals containing a given pair of points. l_∞ has 10 pairs of points. Therefore, there are exactly 120 projective hyperovals and $168 - 120 = 48$ affine hyperovals. Also, every quadrangle is contained in a unique hyperoval.

Under the equivalence relation for hyperovals defined in section 2.2,

$$H_1 \sim H_2 \quad \text{if } |H_1 \cap H_2| \text{ is even,}$$

the 168 hyperovals fall into 3 equivalence classes of 56 hyperovals (see [Lander 1]).

Theorem 2.7. *Let $\pi = PG(2,4)$. The relation $H_1 \sim H_2$ if $|H_1 \cap H_2|$ is even, where H_1 and H_2 are affine hyperovals in π, is an equivalence relation amongst the 48 affine hyperovals of π. There are 3 equivalence classes, each of which contains 16 affine hyperovals. Moreover, the 16 affine hyperovals in a fixed equivalence class are translations of each other.*

Also, all 48 affine hyperovals give rise to the same symmetric unipotent Latin square.

Proof: We use $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, $\omega^2 = 1 + \omega$.

Let

$$H_1 = \{(0,0), (1,1), (1+\omega, 1), (1, 1+\omega), (1+\omega, 0), (0, 1+\omega)\},$$

$$H_2 = \{(0,0), (\omega,\omega), (1,\omega), (\omega, 1), (1,0), (0,1)\},$$

and

$$H_3 = \{(0,0), (1+\omega, 1+\omega), (\omega, 1+\omega), (1+\omega, \omega), (\omega, 0), (0,\omega)\}.$$

Note that

$$H_1 \cap H_2 = \{(0,0)\}, \quad H_2 \cap H_3 = \{(0,0)\}, \quad \text{and } H_1 \cap H_3 = \{(0,0)\}.$$

Thus, H_1, H_2, and H_3 are pairwise non-equivalent. Therefore, by theorem 2.5,

$$\{H_1 + (a,b) | a, b \in \mathbb{F}_4\} \cup \{H_2 + (a,b) | a, b \in \mathbb{F}_4\} \cup \{H_3 + (a,b) | a, b \in \mathbb{F}_4\}$$

is a set of 48 distinct hyperovals; there are 16 hyperovals in each of the 3 equivalence classes of π. Moreover, the 16 hyperovals in a fixed equivalence class are translations of each other.

Also, all 3 of H_1, H_2 and H_3 give rise to the symmetric unipotent Latin square

$$\begin{array}{c} \\ P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{cccccc} P_0 & P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{pmatrix} * & 1 & 2 & 3 & 4 & 5 \\ 1 & * & 4 & 5 & 3 & 2 \\ 2 & 4 & * & 1 & 5 & 3 \\ 3 & 5 & 1 & * & 2 & 4 \\ 4 & 3 & 5 & 2 & * & 1 \\ 5 & 2 & 3 & 4 & 1 & * \end{pmatrix} \end{array}.$$

Therefore, by lemma 2.4, all 48 affine hyperovals give rise to the same symmetric unipotent Latin square. □

Section 2.4. Incidence Matrix of $PG(2,4)$.

We now digress to consider the 2-rank of the incidence matrix of $PG(2,4)$. Using a hyperoval, the 2-rank is shown to be at most 10. It is known that the 2-rank is exactly 10 (see [Assmus 1]). Suppose $\pi = PG(2,n)$ has lines $l_1, \ldots, l_v$ and points $P_1, \ldots, P_v$, where $v = n^2 + n + 1$. Define an incidence matrix

$$N = [n_{ij}], \qquad \text{where } n_{ij} = \begin{cases} 1 & \text{if } P_j \text{ on } l_i \\ 0 & \text{otherwise.} \end{cases}$$

Consider the rank of N over $\mathbf{F}_p$, denoted $rank_p N$. (If N' is a different incidence matrix for π it differs from N by a relabelling of the P's and l's, i.e., by permutation matrices with $det = \pm 1$. Therefore, the rank is independent of the incidence matrix.) It is known for $\pi = PG(2,4)$ that $rank_2 N = 10$ (see [Assmus 1]). Here we now provide a geometrical proof of this result.

Theorem 2.8. *Let $\pi = PG(2,4)$. Let N be the incidence matrix for π. Then the rank of N over $\mathbf{F}_2$ is at least 10 (see [Assmus 1]).*

Proof: Note that if the lines of $\pi = PG(2,4)$ are reordered so that l_j contains a point not on any of $l_1, \ldots, l_{j-1}$, then the corresponding rows in N are linearly independent and $rank_2 N \geq j$.

$$\begin{array}{c} \\ l_1 \\ \vdots \\ \\ \\ l_j \\ l_{j+1} \\ \vdots \\ l_v \end{array} \begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & \cdots & & P_v \end{array} \\ \left(\begin{array}{ccccc} 1 & & 0 & 0 & \\ & & \vdots & \vdots & \\ & & 0 & 0 & \\ & & 1 & 0 & \\ & & & 1 & \\ & & & & \\ & & & & \\ & & & & \end{array} \right) \end{array}$$

The 10 lines in the geometric basis for $PG(2,4)$ can be chosen from the 15 lines hitting a hyperoval H. These 10 lines can be chosen with the above property as follows.
Choose a hyperoval $H = \{P_1, \ldots, P_6\}$ in $PG(2,4)$.
Let $l_1 = P_1P_2$, $l_2 = P_2P_3$, $l_3 = P_1P_3$, $l_4 = P_4P_5$, $l_5 = P_5P_6$,
$l_6 = P_4P_1$, $l_7 = P_6P_3$, $l_8 = P_5P_2$, $l_9 = P_5P_3$, and $l_{10} = P_4P_3$.
We now establish the following claim.

Claim: l_i contains at least one point not on $l_1, \ldots, l_{i-1}$, $i = 2, 3, \ldots, 10$.

l_2 contains P_3 which is not on l_1 since H is a hyperoval.

l_3 contains P_1 and P_3; but, l_3 contains no further points of l_1 or l_2.

l_4 contains P_4, which is not on l_1, l_2 or l_3.

l_5 contains P_6, which is not on $l_1, \ldots, l_4$.

l_6 contains P_1 and P_4. l_6 also contains $l_2 \cap l_5$ since it must intersect l_4. l_6 contains no further points of $l_1, \ldots, l_5$.

l_7 contains P_3 and P_6. l_7 contains $l_1 \cap l_4$ since it must intersect l_4. l_7 contains none of $P_4 = l_6 \cap l_4$, $P_1 = l_6 \cap l_1$ or $l_6 \cap l_2$. Thus, l_7 contains 1 of the 2 points of l_6 not on any of $l_1, \ldots, l_5$.

l_8 contains P_2, P_5 and the unique point of l_3 that is not on any of $l_2, l_4, l_5, \ldots, l_7$. l_8 either contains $l_6 \cap l_7$ or both the point of l_6 not on $l_1, \ldots, l_5, l_7$ and the point of l_7 not on $l_1, \ldots, l_6$. Upon examining l_9, we will find l_8 contains $l_6 \cap l_7$. Therefore, l_8 contains exactly one point that is not on any of $l_1, \ldots, l_7$.

$l_9 = P_5P_3$ does not contain $l_6 \cap l_4$, $l_6 \cap l_2$, $l_6 \cap l_1$ or $l_6 \cap l_7$. Therefore, l_9 contains the point of l_6 not on $l_1, \ldots, l_5, l_7$. Also, l_9 intersects l_8 in P_5. Therefore, l_8 does not contain $l_6 \cap l_8$. Thus l_8 must contain $l_6 \cap l_7$. Therefore l_8 contains a point not on any of $l_1, \ldots, l_7$. Also, l_9 contains the point of l_1 not on $l_2, \ldots, l_8$. Thus, there is one point of l_9 that is not on any of $l_1, \ldots, l_8$.

$l_{10} = P_4P_3$. l_{10} meets l_1 in $l_1 \cap l_5$. l_{10} meets l_8 in the point of l_8 not on $l_1, \ldots, l_7$. Thus l_{10} contains one point not on any of $l_1, \ldots, l_9$. $\square$

Section 2.5. Hyperconics, hexagons, and symmetric unipotent Latin squares.

We introduce hexagons in Chapter 4. By considering the hexagons contained in a hyperconic, we can reorder the points of a hyperconic so that the hyperconic gives rise to a symmetric unipotent Latin square with identical square blocks of size 4 along the main diagonal.

Corollary 2.9. *Let $\pi = PG(2, F)$ where F is a field of order greater than 4 which contains a subfield of order 4. Let $H = C \cup \{N\}$ be a hyperconic (where C is a conic with nucleus N). Let l_∞ be a line through the nucleus N and a point P on C. Order the affine points $P_1, P_2, \ldots$ of H so that N and P along with quadrangles $P_1P_2P_3P_4$, $P_5P_6P_7P_8, \ldots$, are all hexagons. Then, by rearranging the order of the points within each of these quadrangles, the symmetric unipotent Latin square resulting from H will have identical 4×4 blocks along the diagonal.*

Proof: Hexagons are defined in chapter 4, just before theorem 4.5. The Fano configurations through the quadrangles $P_1 \cdots P_4$, $P_5 \cdots P_8, \ldots$ all intersect l_∞ in the same 3 points by theorem 4.51. Thus, if the lines P_1P_2, $P_3P_4, \ldots$ intersect l_∞ in Q_1, say; and the lines P_1P_3, P_2P_4, P_5P_7, $P_6P_8, \ldots$ intersect l_∞ in Q_2, say; and the lines P_1P_4, P_2P_3, P_5P_8, $P_6P_7, \ldots$ intersect l_∞ in Q_3, say, then the symmetric unipotent Latin square resulting from H is

$$A = \begin{pmatrix} B & & & \\ & B & & \\ & & B & \\ & & & \ddots \end{pmatrix}$$

where

$$B = \begin{array}{c} \\ P_{4i+1} \\ P_{4i+2} \\ P_{4i+3} \\ P_{4i+4} \end{array} \begin{array}{c} \begin{array}{cccc} P_{4i+1} & P_{4i+2} & P_{4i+3} & P_{4i+4} \end{array} \\ \begin{pmatrix} * & 1 & 2 & 3 \\ 1 & * & 3 & 2 \\ 2 & 3 & * & 1 \\ 3 & 2 & 1 & * \end{pmatrix} \end{array}. \quad \square$$

Chapter 3. $PG(2,4)$ hyperovals pairwise intersecting in 2 points

An open problem in $\pi = PG(2,q)$ q even is: what is the size and structure of a set of maximum size of hyperovals (or hyperconics) pairwise intersecting in exactly 2 points?

It is known that the size is at least q^2 (see [Bruen 3]). An example of q^2 hyperconics containing a common point that pairwise intersect in exactly 2 points is as follows.

Example 3.1. Consider the q^2 conics $C_{a,b} : bX^2 + Y^2 + aZ^2 + XZ = 0$ where $a, b \in \mathbb{F}_q$. These conics all have nucleus $(m = \infty)$. We will now show that each pair of these conics intersect in a unique point.

Let $H_{a,b} = C_{a,b} \cup \{(m = \infty)\}$. Now $C_{a,b} \cap l_\infty = \{(m = \sqrt{b})\}$.

Consider $C_{a,b}$ and $C_{c,d}$ where $a, b, c, d \in \mathbb{F}_q$ and $(a, b) \neq (c, d)$. We have $b = d$ iff $C_{a,b}$ and $C_{c,d}$ intersect in a point on l_∞ since when $Z = 0$, $C_{a,b} : bX^2 + Y^2 = 0$ and $C_{c,d} : dX^2 + Y^2 = 0$.

Consider the affine points of $C_{a,b}$ and $C_{c,d}$. Let $Z = 1$.

There is no point (X, Y) satisfying both $bX^2 + Y^2 = a + X$ and $bX^2 + Y^2 = c + X$ unless $a = c$. Th s, if $b = d$, the only common points of $C_{a,b}$ and $C_{c,d}$ are on l_∞.

Suppse next that $b \neq d$ and consider

$$X_0 = \left(\frac{a+c}{b+d}\right)^{\frac{1}{2}}$$

$$Y_0 = \left(X_0 + \frac{b(a+c) + a(b+d)}{b+d}\right)^{\frac{1}{2}}.$$

The point (X_0, Y_0) is on both $C_{a,b}$ and $C_{c,d}$. Conversely, this is the only solution to both $a + bX^2 + Y^2 = X$ and $c + dX^2 + Y^2 = X$.

Thus $H_{a,b}$ and $H_{c,d}$ have exactly 2 points in common. □

The main result of this chapter is to show that in $PG(2,4)$, the maximum size of a set of hyperovals, pairwise intersecting in 2 points, is 16; and moreover, to show that there are only 2 possibilities for the structure (which are actually 'dual' structures) for such a set with maximum size.

In this chapter, we will only consider hyperovals in $PG(2,4)$.

It follows from the definition that every hyperoval in $PG(2,4)$ is a hyperconic. Recall the well known equivalence relation amongst the hyperovals of $PG(2,4)$ discussed in section 2.2 and in section 2.3.

Two hyperovals are said to be equivalent if they intersect in an even number of points. This equivalence relation has 3 equivalence classes. Fix on an equivalence class. Then all hyperovals considered for the remainder of this chapter will be contained in this equivalence class, which we refer to as **Type I**.

Recall that in $PG(2,4)$, every quadrangle is contained in a unique hyperoval. Also recall that, given a hyperoval, a given line is either disjoint from that hyperoval, or it intersects that hyperoval in exactly 2 points.

Also recall that there are 56 hyperovals per equivalence class of which 40 intersect and 16 are skew to (disjoint from) a given line.

We have 56 = (40 projective + 16 affine) hyperovals of Type I.

Let S_P be the set of all hyperovals of Type I through the point P. There are 16 such hyperovals. The set S_P is called a **Point-16**.

Let S^l be the set of all hyperovals of Type I skew to the line l. There are 16 such hyperovals. The set S^l is called an **Affine-16**.

A **2-intersecting family** S is a set of hyperovals of Type I such that $H_1, H_2 \in S \Rightarrow |H_1 \cap H_2| = 2$.

A point-16 and an affine-16 are examples of 2-intersecting families. Therefore, a 2-intersecting family of maximum size must have size at least 16.

We now consider 2-intersecting families of maximum size.

The main theorem in this chapter is theorem 3.2.

Theorem 3.2. *Let $\pi = PG(2,4)$. Let S be a 2-intersecting family of maximum size. Then either there exists a point common to all the hyperovals of S, i.e., $S = S_P$ for some P, or all hyperovals of S are skew to a unique line, i.e., $S = S^l$ for some l. In both cases $|S| = 16$.*

Thinking of the 6 lines skew to a hyperoval as the points of a 'dual' hyperoval, these two structures are 'dual' to each other.

After we prove this theorem, we will show that the latter case is a $2-(16,6,2)$-design of grid type, which can be related to singular points of a Kummer surface in $PG(2,q)$ for q odd (see [Bruen 2]). This design can also be obtained from the Mathieu $5-(24,8,1)$ design M by taking only blocks of M meeting a fixed block B in exactly

2 points (see [Kantor 1] and [Hughes 1]).

We will also see that the first case of theorem 3.2 has each of the 21 points contained in a hyperoval of S and yields an affine plane $AG(2,4)$ in two different ways:

1) Let P be the point on all 16 hyperovals and l_∞ be a line through P. Define the structure π_1 to have points the affine points of π. A line through two points P_1 and P_2 in π_1 is defined to be the hyperoval in π through P_1, P_2, and P if there is one; or the line P_1P_2 in π if there is no hyperoval through P_1, P_2, and P in π. We will show that π_1 is an affine plane.

2) A dual affine plane results by taking the hyperovals in π as points and the points in π as lines.

We will see later that it can be proven (see [Cameron 2]) that a 2-intersecting family has size at most 16 via Hoffman's inequality for strongly regular graphs. (See section 3.8.)

To prove theorem 3.2 it is enough to prove theorem 3.3.

Theorem 3.3. *Let $\pi = PG(2,4)$. Suppose S is a 2-intersecting family of maximum size. Either*

1) there are at least 3 hyperovals in S through some pair of points; or

2) there are at most 2 hyperovals in S through each pair of points.

If 1) holds, then all hyperovals in S pass through a unique point (one of the 2 points on the 3 hyperovals), and S is a point-16.

If 2) holds, then there exists a line which is skew to every hyperoval in S and S is an affine-16.

Example 3.4. If $H = \{(a_1,b_1),\dots,(a_6,b_6)\}$,
then $H + (a,b) = \{(a_1+a, b_0+b),\dots,(a_6+a,b_6+b)\}$.
We use $\mathbf{F}_4 = \{0,1,\omega,\omega^2\}$, $\omega^2 = 1+\omega$.
Let $H = \{(0,0),(1,1),(1+\omega,1),(1,1+\omega),(1+\omega,0),(0,1+\omega)\}$.
H is an affine hyperoval in $PG(2,4)$. Therefore $\{H+(a,b)|a,b\in\mathbf{F}_4\}$ is a set of 16 affine hyperovals pairwise meeting in exactly 2 points by theorem 2.5, i.e., this set is an affine-16. □

Example 3.5. Let $P = (m=\infty)$. The following 16 hyperovals are a 2-intersecting family that all pass through P, i.e., these hyperovals are a point-16.

$$H_0 = \{(0,0),(1,1),(\omega,\omega^2),(\omega^2,\omega),(m=0),(m=\infty)\}$$
$$H_0+(0,1) = \{(0,1),(1,0),(\omega,\omega),(\omega^2,\omega^2),(m=0),(m=\infty)\}$$
$$H_0+(0,\omega) = \{(0,\omega),(1,\omega^2),(\omega,1),(\omega^2,0),(m=0),(m=\infty)\}$$
$$H_0+(0,\omega^2) = \{(0,\omega^2),(1,\omega),(\omega,0),(\omega^2,1),(m=0),(m=\infty)\}$$
$$H_1 = \{(0,0),(m=1),(\omega^2,1),(\omega,1),(1,0),(m=\infty)\}$$
$$H_1+(0,1) = \{(0,1),(m=1),(\omega^2,0),(\omega,0),(1,1),(m=\infty)\}$$
$$H_1+(0,\omega) = \{(0,\omega),(m=1),(\omega^2,\omega^2),(\omega,\omega^2),(1,\omega),(m=\infty)\}$$
$$H_1+(0,\omega^2) = \{(0,\omega^2),(m=1),(\omega^2,\omega),(\omega,\omega),(1,\omega^2),(m=\infty)\}$$
$$H_\omega = \{(0,0),(\omega^2,\omega^2),(m=\omega),(1,\omega^2),(\omega^2,0),(m=\infty)\}$$
$$H_\omega+(0,1) = \{(0,1),(\omega^2,\omega),(m=\omega),(1,\omega),(\omega^2,1),(m=\infty)\}$$
$$H_\omega+(0,\omega) = \{(0,\omega),(\omega^2,1),(m=\omega),(1,1),(\omega^2,\omega),(m=\infty)\}$$
$$H_\omega+(0,\omega^2) = \{(0,\omega^2),(\omega^2,0),(m=\omega),(1,0),(\omega^2,\omega^2),(m=\infty)\}$$
$$H_{\omega^2} = \{(0,0),(\omega,\omega),(1,\omega),(m=\omega^2),(\omega,0),(m=\infty)\}$$
$$H_{\omega^2}+(0,1) = \{(0,1),(\omega,\omega^2),(1,\omega^2),(m=\omega^2),(\omega,1),(m=\infty)\}$$
$$H_{\omega^2}+(0,\omega) = \{(0,\omega),(\omega,0),(1,0),(m=\omega^2),(\omega,\omega),(m=\infty)\}$$
$$H_{\omega^2}+(0,\omega^2) = \{(0,\omega^2),(\omega,1),(1,1),(m=\omega^2),(\omega,\omega^2),(m=\infty)\}$$

Notice that the hyperovals H_0, $H_0+(0,1)$, $H_0+(0,\omega)$, $H_0+(0,\omega^2)$ all pass through both $(m=\infty)$ and $(m=0)$, ..., the hyperovals H_{ω^2}, $H_{\omega^2}+(0,1)$, $H_{\omega^2}+(0,\omega)$, $H_{\omega^2}+(0,\omega^2)$ all pass through $(m=\infty)$ and $(m=\omega^2)$. $\square$

To prove theorem 3.3 we need some lemmas describing how hyperovals in a 2-intersecting family intersect each other.

Section 3.1. Hyperovals through a pair of points.

In order to prove the main theorem of this chapter, theorem 3.2, we will investigate the intersection of certain hyperovals. A result we will use frequently in considering how various hyperovals intersect is the following.

Through every pair of points, P, Q, there are exactly 4 hyperovals of Type I; moreover, each point not on the line PQ is on exactly one of these 4 hyperovals. This is theorem 3.7.

Example 3.6. Notice that in example 3.5, we had 16 hyperovals through $(m = \infty)$. These consisted of the 4 hyperovals through both $(m = \infty)$ and $(m = 0)$, the 4 hyperovals through both $(m = \infty)$ and $(m = 1)$, the 4 hyperovals through both $(m = \infty)$ and $(m = \omega)$, and the 4 hyperovals through both $(m = \infty)$ and $(m = \omega^2)$. This is also true for the points on any line through $(m = \infty)$. □

Theorem 3.7. *Consider $\pi = PG(2,4)$. Let P_1 and P_2 be points of π. Then there are exactly 4 hyperovals of Type I through both P_1 and P_2. Moreover, every point not on the line P_1P_2 is on exactly one of these hyperovals.*

Proof: Recall that in $\pi = PG(2,4)$, each quadrangle is contained in a unique hyperoval, and given a hyperoval, each line is either skew to that hyperoval, or intersects that hyperoval in exactly 2 points.

First, we establish the following claim.

Claim: There are exactly 4 hyperovals of Type I through both P_1 and P_2.

There are 56 hyperovals of Type I, $\frac{(21)(20)}{2}$ pairs of points, and 6 points/hyperoval. Now

$$(15 \text{ pairs/hyperoval})(56 \text{ hyperovals}) = \left(\binom{21}{2} \text{ pairs}\right)(\#\text{hyperovals/pair}).$$

This establishes the claim.

Each of these hyperovals contains the points P_1 and P_2 and 4 points not on the line P_1P_2. Thus, each of the $16 = (4)(4)$ points not on the line P_1P_2 is on exactly one of these 4 hyperovals. □

Corollary 3.8. *Let $\pi = PG(2,4)$. Consider two points, P_1 and P_2 not on the line l_0. Then l_0 intersects exactly 2 of the 4 hyperovals of Type I that contain both P_1 and P_2.*

Proof: Let $l = P_1P_2$. Let $H_1, \ldots, H_4$ be the 4 hyperovals through both P_1 and P_2 of Type I. Therefore, of the 16 points off l, each is on exactly one of $H_1, \ldots, H_4$ by theorem 3.7. Therefore, l_0, which has 4 points off l, must intersect H_1 (say) in 2 points. Therefore l_0 must intersect H_2, say. Therefore, l_0 cannot intersect H_3 or H_4. □

We now consider the 'affine-16 case' of theorem 3.3.

Section 3.2. A 2-intersecting family of maximum size of hyperovals of that are skew to a fixed line.

Theorem 3.9. *Let $\pi = PG(2,4)$. If S is a 2-intersecting family of maximum size with the property that each pair of points of π is on at most 2 hyperovals of S, then S is an affine-16.*

Proof: Let $H \in S$, and $H = \{P_1, \ldots, P_6\}$. There are 15 pairs P_i, P_j, $1 \le i \ne j \le 6$. Now, by assumption, there are at most 2 hyperovals in S through both P_i and P_j, including H. Therefore, there are at most $1+15 = 16$ hyperovals of Type I intersecting H. Therefore $|S| \le 16$. But, by corollary 3.8, an affine-16 satisfies the hypothesis of this theorem. Therefore, $|S| = 16$. Thus, for any $H \in S$ and for any pair $P_i, P_j \in H$, there exists a unique hyperoval, besides H, in S through both P_i and P_j. There are 5 pairs P_i, P_j for fixed i. Therefore, there are exactly 6 hyperovals in S which contain P_i. This must be true for any point on any hyperoval in S.
Let $x = \#$ points of π contained in some hyperoval of S. Therefore,

$$\begin{aligned} 16 \cdot 6 &= (\#\text{hyperovals})(\#\text{points/hyperoval}) \\ &= \#\text{incidences} \qquad = \sum_P \#\text{hyperovals of } S \text{ containing } P \\ &= (\#\text{points contained in some hyperoval of } S)(\#\text{hyperovals/point}) \\ &= (x)(6). \end{aligned}$$

Therefore, $x = 16$. Therefore $21 - 16 = 5$ points are not contained in any of the hyperovals of S.
We establish the following claim.

Claim: These 5 points are collinear.

Let $P_1, \ldots, P_5$ be these 5 points. Let $l = P_1P_2$. Suppose, by way of contradiction, that Q is a point on l which is not one of $P_1, \ldots, P_5$. Then there are exactly 6 hyperovals in S which contain Q; moreover, each of these 6 hyperovals intersects l in another point. However, there are at most 2 hyperovals in S through a pair of points on l. Therefore, there must be at least 4 points of l which are contained in some hyperoval in S. This is a contradiction.
This establishes the claim. □

Section 3.3. Intersection properties of hyperovals that all intersect with a fixed line.

We now prove a series of lemmas which will be used in proving the 'point-16 case' of theorem 3.3. The following lemmas describe how hyperovals that all intersect a fixed line intersect each other.

Lemma 3.10. *Consider $\pi = PG(2,4)$. Let P_1, P_2 be points of π, let $l = P_1P_2$ and let the remaining points of l be P_3, P_4 and P_5. Denote by H_1, H_2 2 of the 4 hyperovals of Type I through both P_1 and P_2. Then among the (3)(4) hyperovals which intersect l in 2 of P_3, P_4 or P_5, exactly 2 also intersect both H_1 and H_2; moreover, these 2 hyperovals intersect l in the same 2 points.*

Proof: Let $H_1 = \{P_1, P_2, Q_1, Q_2, Q_3, Q_4\}$ and $H_2 = \{P_1, P_2, R_1, R_2, R_3, R_4\}$. By corollary 3.8, exactly 2 of the 4 hyperovals of Type I through both R_1 and Q_1 intersect l. Consider one of these 2 hyperovals. If it contains P_1 or P_2, then it contains no more points in $\{Q_1, \ldots, Q_4, R_1, \ldots, R_4\}$. If it contains neither P_1 nor P_2, then it must contain exactly 2 of P_3, P_4, and P_5 and also one more point in $\{Q_1, \ldots, Q_4\}$, and one more point in $\{R_1, \ldots, R_4\}$. Similarly for the other hyperoval through R_1 and Q_1 which intersects l.

There are 16 pairs R_i, Q_j. For each pair, as with the pair R_1, Q_1, we have exactly 2 hyperovals of Type I containing them which intersect l.

There are (4)(3) hyperovals of Type I through P_1 which do not contain P_2. Each of these must contain exactly one pair Q_i, R_j $(i, j \in \{1, \ldots, 4\})$. Similarly, there are (4)(3) hyperovals of Type I through P_2 which do not contain P_1. Each of these hyperovals must contain exactly one pair Q_i, R_j $(i, j \in \{1, \ldots, 4\})$. The remaining hyperovals that contain a pair Q_i, R_j contain exactly 2 of $Q_1, \ldots, Q_4$ and exactly 2 of $R_1, \ldots, R_4$, i.e., they contain 4 pairs of the form Q_i, R_j. Thus, there are exactly $\frac{(16)(2)-((4)(3))(2)}{4} = 2$ hyperovals of Type I which intersect l and which intersect both H_1 and H_2 in points other that P_1 or P_2.

We establish the following claim.

Claim: The 2 hyperovals of Type I which intersect l and also intersect H_1 and H_2 in points off l, must intersect l in the same 2 points.

Write $l_\infty = l$. Now $H_2 = H_1 + (a, b)$, for some $a, b \in \mathbf{F}_4$ by theorem 2.6. Thus, if $P \in H_1$, then $P + (a, b) \in H_2$, and if $Q \in H_2$, then $Q + (a, b) \in H_1$. Therefore, if H

is one of the hyperovals which intersects $l \setminus \{P_1, P_2\}$ and H_1 and H_2, then $H + (a, b)$ must the the other, i.e., the 2 hyperovals intersecting $l \setminus \{P_1, P_2\}$, and also H_1 and H_2 must intersect l in the same 2 points.

This establishes the claim. □

Lemma 3.11. *Let $\pi = PG(2,4)$ and let $P_1, \ldots, P_5$ be the points of a line l. Suppose H_1, H_2, and H_3 are 3 of the 4 hyperovals of Type I containing both P_1 and P_2. Then there are no hyperovals of Type I through both P_i and P_j that intersect all of H_1, H_2 and H_3, where $i, j \in \{3, 4, 5\}$; moreover, all hyperovals of Type I intersecting each of H_1, H_2, H_3 and l must intersect P_1 and/or P_2.*

Proof: Suppose by way of contradiction that G is a hyperoval of Type I in S that intersects H_1, H_2 and H_3 and also contains both P_3 and P_4. Therefore G must intersect each of H_1, H_2, and H_3 in points not on l. But then it must contain 2 points of H_1, 2 points of H_2, and 2 points of H_3 with only its 4 points that are not on l, yielding a contradiction. □

Corollary 3.12. *Let S be a 2-intersecting family in $\pi = PG(2,4)$. Let $P_1, \ldots, P_5$ be the points of a line l. Suppose that at least 3 of the 4 hyperovals through both P_1 and P_2 are in S. Suppose further that at least 3 of the 4 hyperovals through both P_1 and P_i are in S, where $i \in \{3, 4, 5\}$. Then there is no hyperoval in S through P_2 that does not contain either P_1 or P_i; all hyperovals in S intersecting l either contain P_1, or contain both P_2 and P_i.*

Proof: This is a corollary of lemma 3.11. □

Section 3.4. Intersection properties of the 4 hyperovals through a pair of points on a line l with the 16 hyperovals skew to l.

We are still working towards proving the 'point-16 case' of theorem 3.3. The following lemmas describe how the 16 hyperovals skew to a line intersect with the 4 hyperovals through a pair of points on that line.

Lemma 3.13. *Let $\pi = PG(2,4)$. Let P_1 and P_2 be points in π and let $l = P_1P_2$. Suppose $H_1, \ldots, H_4$ are the 4 hyperovals of Type I through both P_1 and P_2. Consider a hyperoval H of Type I skew to l. Then exactly 3 of $H_1, \ldots, H_4$ intersect H.*

Proof: H has 6 points and each H_i meets H in 0 or 2 points. Since each point off l is on a unique H_i, exactly 3 of $H_1, \ldots, H_4$ meets H. □

Lemma 3.14. *Let S be a 2-intersecting family in $PG(2,4)$. Let l be a line through both P_1 and P_2, where P_1 and P_2 are points of π. Denote by $H_1, \ldots, H_4$ the 4 hyperovals of Type I containing both P_1 and P_2.*

a) Exactly 12 of the 16 hyperovals of Type I skew to l intersect H_i for a fixed i; if there is a hyperoval through both P_1 and P_2 in S, then there are at most 12 hyperovals skew to l in S.

b) Exactly 8 of the 16 hyperovals of Type I skew to l intersect both H_i and H_j for fixed i and j; if there are at least 2 hyperovals through both P_1 and P_2 in S, then there are at most 8 hyperovals skew to l in S;

c) Exactly 4 of the 16 hyperovals of Type I skew to l intersect each of H_i, H_j and H_k for fixed i, j and k $(0 \leq i \neq j \neq k \leq 4)$; if there are at least 3 hyperovals through both P_1 and P_2 in S, then there are at most 4 hyperovals skew to l in S;

d) None of the hyperovals of Type I skew to l of intersects all of $H_1, \ldots, H_4$; if there are 4 hyperovals through both P_1 and P_2, then there is no hyperoval in S skew to l.

Proof: There are 16 hyperovals skew to l (i.e., S^l). Therefore there are at most 16 hyperovals of Type I skew to l in S.

d) This is a corollary of lemma 3.13.

a) There are 4 points on H_4 that are not on l, i.e., there are 6 pairs of points on H_4 that are not in l. Now there are 2 hyperovals/(pair of points off l) of Type I that are skew to l by corollary 3.8. Therefore, there are exactly 12 hyperovals of Type I skew to l that intersect H_4. Therefore, there are exactly $16 - 12 = 4$ hyperovals of Type I skew to both l and H_4.

b) There are exactly 4 hyperovals of Type I skew to l which are also skew to H_3. There are exactly 4 hyperovals skew to l which are also skew to H_4. These 8 hyperovals are distinct, i.e., 8 of the 16 hyperovals skew to l are also skew to H_3 or H_4. These 8 hyperovals are distinct because a hyperoval skew to l is also skew to exactly one of $H_1, \ldots, H_4$. Therefore if $H_3, H_4 \in S$, then these 8 hyperovals missing l are not in S.

c) Exactly 4 of the hyperovals of Type I skew to l are also skew to H_1, and therefore intersect each of H_2, H_3, H_4. □

We can combine lemma 3.11 and lemma 3.14 to give corollary 3.15.

Corollary 3.15. *Let $\pi = PG(2,4)$. Let S be a 2-intersecting family containing H_1, H_2, and H_3, where H_1, H_2, and H_3 are hyperovals of Type I which contain both P_1 and P_2. Let l be a line through $P_1, \ldots, P_5$. Then by lemma 3.11, there are no hyperovals in S through both P_i and P_j, $3 \leq i \neq j \leq 5$, and by lemma 3.14, there are at most 4 hyperovals in S skew to l.*

□

Section 3.5. Some more intersection properties of hyperovals.

These are the remaining lemmas we will use in proving the 'point-16 case' of theorem 3.3.

Lemma 3.16. *Let $\pi = PG(2,4)$. Suppose $P_1, \ldots, P_5$ are the points of l_∞. Denote by H_1, H_2 and H_3 3 of the 4 hyperovals of Type I through both P_1 and P_2, and let G_1 be a hyperoval of Type I through both P_1 and P_3. Then there are exactly 3 affine hyperovals of Type I intersecting each of H_1, H_2, H_3, G_1.*

Further, if S is a 2-intersecting family containing H_1, H_2, H_3, and G_1, then S contains at most 3 affine hyperovals.

Proof: Let $H_1, \ldots, H_4$ be the hyperovals of Type I through both P_1 and P_2. Let $G_1, \ldots, G_4$ be the hyperovals of Type I through both P_1 and P_3. Now exactly 12 of the 16 affine hyperovals of Type I intersect H_4; each of these 12 hyperovals is skew to exactly one of H_1, H_2, or H_3.
We establish the following claim.

Claim: Exactly nine of these 12 hyperovals intersect G_1.
Let Q_i be the affine point where H_4 and G_i intersect $(i = 1,2,3,4)$. There are exactly 6 affine hyperovals containing Q_i and thus intersecting both H_4 and G_i. Recall that there are exactly 12 affine hyperovals of Type I intersecting H_4, of which there are exactly 2 through each pair Q_i, Q_j. Consider the 6 affine hyperovals of Type I which either contain both Q_2 and Q_3, or both Q_2 and Q_4, or both Q_3 and Q_4. The 2 through both Q_2 and Q_3 must intersect G_2 and G_3 again, which means one of these hyperovals intersects G_1 but not G_4, the other intersects G_4 but not G_1. Therefore, of the 12 affine hyperovals of Type I intersecting H_4, exactly 9 intersect G_1 (6 through Q_1 and 3 others).
This establishes the claim.

Therefore, there are exactly 3 affine hyperovals of Type I intersecting H_4 but which are skew to G_1; however, there are a total of 4 affine hyperovals of Type I missing G_1. Therefore, exactly one affine hyperoval of Type I is skew to both G_1 and H_4. An affine hyperoval of Type I skew to H_4 is an affine hyperoval hitting H_1, H_2, H_3. Thus there is a unique affine hyperoval of Type I intersecting each of H_1, H_2, H_3 but which is skew to G_4. I.e., 3 of the 4 affine hyperovals of Type I intersecting H_1, H_2, H_3 also intersects G_4. □

Lemma 3.17. *Let $P_1, \ldots, P_5$ be the points of a line l in $\pi = PG(2,4)$. Let S be a 2-intersecting family containing H_1, H_2, G_1, G_2 where H_1, H_2 are hyperovals of Type I through both P_1 and P_3; and G_1, G_2 are hyperovals of Type I through both P_2 and P_4, say. Then there is no hyperoval through P_5 in S.*

Proof: This is a corollary of lemma 3.10. □

Lemma 3.18. *Let $\pi = PG(2,4)$. Suppose H is a hyperoval of Type I through both P_1 and P_2. Consider 2 points, P_3, P_4, on the line P_1P_2 with $P_1, \ldots, P_4$ distinct. Then exactly 2 of the 4 hyperovals of Type I through both P_3 and P_4 intersect H. If S is a 2-intersecting family containing H, then it contains at most 2 of these 4 hyperovals of Type I through both P_3 and P_4.*

Proof: Each point off l is on a unique hyperoval of Type I through both P_3 and P_4, there are 4 points on H that are not on the line l. □

We are now ready to prove theorem 3.3.

Section 3.6. Sets of maximum size of hyperovals pairwise meeting in 2 points.

Theorem 3.3. *Let $\pi = PG(2,4)$. Suppose S is a 2-intersecting family of maximum size. Either*

1) there are at least 3 hyperovals in S through some pair of points; or

2) there are at most 2 hyperovals in S through every pair of points.

If 1) holds, then all hyperovals in S pass through a unique point (one of the 2 points on the 3 hyperovals), i.e., S is a point-16.

If 2) holds, then there exists a line which is skew to every hyperoval in S, i.e., S is an affine-16.

Proof: If each pair of points is on at most 2 hyperovals of S, then S is an affine-16 and $|S| = 16$ by theorem 3.9.

Suppose now that H_1, H_2, and H_3 are in S and each contains both P_1 and P_2. Let the points of the line $l := P_1P_2$ be $P_1, \ldots, P_5$, say. The point-16 through P_1 and also the point-16 through P_2 satisfy the hypothesis of this theorem. Therefore $|S| \geq 16$. We establish the following claim.

Claim 1: $|S| \leq (4)(8)$.

By lemma 3.14, there are at most 4 hyperovals in S which are skew to l. By lemma 3.11 there are no hyperovals in S through both P_i and P_j, where $3 \leq i \neq j \leq 5$. By theorem 3.7 there are exactly 4 hyperovals of Type I that contain both P_i and P_j. Therefore,

possibilities for # hyperovals in S through the pair...

$$\begin{pmatrix} P_1P_2 & P_1P_3 & P_1P_4 & P_1P_5 & P_2P_3 & P_2P_4 & P_2P_5 & \#\text{ skew to } l \\ 3,4 & 0,\ldots,4 & 0,\ldots,4 & 0,\ldots,4 & 0,\ldots,4 & 0,\ldots,4 & 0,\ldots,4 & 0,\ldots,4 \end{pmatrix}$$

are the only possibilities for hyperovals in S.

This establishes the claim.

We also know that $|S| \geq 16$. Since H_1, H_2, and H_3 are in S, there must be another hyperoval in S, denoted by H_4 through both P_1 and P_3, say, which intersects l.

Therefore, by lemma 3.16, the number of hyperovals in S skew to l is at most 3. Also, by lemma 3.11 (all hyperovals intersecting l in S must contain P_2 and/or P_i if there are 3 hyperovals in S through both these points), the number of hyperovals in S through both P_2 and P_i, $i = 4, 5$, is at most 2.

We now prove the following claim.

Claim 2: There are at most 2 hyperovals in S through both P_2 and P_3.

For otherwise suppose there are 3 through both P_2 and P_3. Then, by Lemma 3.11 there are no hyperovals in S through both P_1 and P_4, nor through both P_1 and P_5. Also if there is a hyperoval through both P_2 and P_4 in S, then there are at most 2 hyperovals in S through both P_1 and P_3. Therefore $|S| \leq 15$.

This proves claim 2.

Thus

possibilities for # hyperovals in S through the pair...

$$\begin{pmatrix} P_1P_2 & P_1P_3 & P_1P_4 & P_1P_5 & P_2P_3 & P_2P_4 & P_2P_5 & \#\text{ skew to } l \\ 3,4 & 1,\ldots,4 & 0,\ldots,4 & 0,\ldots,4 & 0,1,2 & 0,1,2 & 0,1,2 & 0,\ldots,3 \end{pmatrix}.$$

By way of contradiction, suppose that S contains $H_1, \ldots, H_4$ but S is not the point-16 through P_1.

We establish the following claim.

Claim 3: there exists a hyperoval through P_2 (and not P_1) in S.

Suppose, by way of contradiction, that the only hyperovals in S containing P_2 also contain P_1. Therefore, there is a hyperoval in S skew to l, and thus by lemma 3.14 there are at most 12 hyperovals in S which intersect l, and by lemma 3.16 there are at most 3 hyperovals in S which are skew to l.

possibilities for # hyperovals in S through the pair...

$$\begin{pmatrix} P_1P_2 & P_1P_3 & P_1P_4 & P_1P_5 & P_2P_3 & P_2P_4 & P_2P_5 & \#\text{ skew to } l \\ 3 & 1,2,3 & 0,\ldots,3 & 0,\ldots,3 & 0 & 0 & 0 & 1,2,3 \end{pmatrix}.$$

Therefore, $|S| \leq 15$ and S is not a maximum 2-intersecting family, a contradiction. This establishes the claim.

Therefore, let G_1 be a hyperoval in S through both P_2 and P_i, for some $i \neq 1, 2$. Say $\{P_1, \ldots, P_5\} = \{P_1, P_2, P_i, P_j, P_k\}$. Now by lemma 3.18, there are at most 2 hyperovals in S through both P_1 and P_j, and at most 2 hyperovals in S through both P_1 and P_k.

Therefore,

possibilities for # hyperovals in S through the pair...

$$\left(\begin{array}{cccccccc} P_1P_2 & P_1P_i & P_1P_j & P_1P_k & P_2P_i & P_2P_j & P_2P_k & \#\text{ skew to } l \\ 3,4 & 0,\ldots,4 & 0,1,2 & 0,1,2 & 1,2 & 0,1,2 & 0,1,2 & 0,\ldots,3 \end{array}\right).$$

Now, by lemma 3.14, if there are 4 hyperovals in S through both P_1 and P_2 and/or 4 hyperovals in S through both P_1 and P_i, then there are no hyperovals in S skew to l. Therefore, there exists a hyperoval in S through P_2 but which does not contain P_1 or P_i. Say G_2 is a hyperoval through both P_2, and P_j in S.

Therefore, by lemma 3.18, the number of hyperovals in S through both P_1 and P_i is at most 2. Therefore,

possibilities for # hyperovals in S through the pair...

$$\left(\begin{array}{cccccccc} P_1P_2 & P_1P_i & P_1P_j & P_1P_k & P_2P_i & P_2P_j & P_2P_k & \#\text{ skew to } l \\ 3,4 & 0,1,2 & 0,1,2 & 0,1,2 & 1,2 & 1,2 & 0,1,2 & 0,\ldots,3 \end{array}\right).$$

Recall that if there are 4 hyperovals in S through both P_1 and P_2, then there are no hyperovals in S skew to l; therefore, there must be 2 hyperovals in S through both P_1 and P_l; and also 2 through both P_2 and P_m, for some $3 \leq l \neq m \leq 5$. Let $\{P_3, P_4, P_5\} = \{P_l, P_m, P_w\}$.

Therefore, by lemma 3.17, there are no hyperovals in S through both P_1 and P_w, and none through both P_2 and P_w. Therefore,

possibilities for # hyperovals in S through the pair ...

$$\left(\begin{array}{cccccccc} P_1P_2 & P_1P_l & P_1P_m & P_1P_w & P_2P_l & P_2P_m & P_2P_w & \#\text{ skew to } l \\ 3,4 & 2 & 0,1,2 & 0 & 0,1,2 & 2 & 0 & 0,\ldots,3 \end{array}\right).$$

Therefore $|S| \leq 15$, yielding a contradiction. Thus S is the point-16 through P_1. □

We have therefore proved the following main theorem:

Theorem 3.2. *Let $\pi = PG(2,4)$. Let S be a 2-intersecting family of maximum size in π. Then either all the hyperovals of S contain a unique point, or all hyperovals of S are skew to a unique line. In both cases $|S| = 16$.*

□

In the next section we will see that both cases yield designs.

Section 3.7. Some designs resulting from the 16 hyperovals through a point and from the 16 hyperovals skew to a line.

In the previous sections we proved theorem 3.2.

Theorem 3.2. *Let $\pi = PG(2,4)$. Let S be a 2-intersecting family of maximum size in π. Then either all the hyperovals of S contain a unique point, or all hyperovals of S are skew to a unique line. In both cases $|S| = 16$.*

□

We will now show that both cases in theorem 3.2 yield designs.
The latter case is a $2-(16,6,2)$-design of grid type which can be related to singular points of a Kummer surface in $PG(2,q)$ for q odd (see [Bruen 2]). This design can be obtained from the Mathieu $5-(24,8,1)$ design M by taking only blocks of M that intersect a fixed block B in exactly 2 points.

Theorem 3.19. *The 2-intersecting family consisting of 16 hyperovals skew to a fixed line is a $2-(16,6,2)$-design of grid type which can be obtained from the Mathieu $5-(24,8,1)$ design M by taking only the blocks of M that intersect a fixed block B in exactly 2 points.*

Proof: Consider

$$A = \begin{pmatrix} (\omega^2,0) & (1,\omega) & (\omega,0) & (0,\omega) \\ (\omega,1) & (0,\omega^2) & (\omega^2,1) & (1,\omega^2) \\ (1,1) & (\omega^2,\omega^2) & (0,1) & (\omega,\omega^2) \\ (0,0) & (\omega,\omega) & (1,0) & (\omega^2,\omega) \end{pmatrix}.$$

The $2-(16,6,2)$-design of grid type obtained from this, with each point defining a block through it to be the other elements in its row and column, has blocks the 16 affine hyperovals which are the affine translations of

$$H = \{(\omega^2,0),(\omega,1),(1,1),(\omega,\omega),(1,0),(\omega^2,\omega)\}.$$

These 16 affine hyperovals pairwise meet in 2 points.
Now consider the Mathieu $5-(24,8,1)$ design M. Recall that of the 77 blocks of M

through 2 fixed points P_1 and P_2, $P_1 \neq P_2$, the 21 through a point P_3 $(P_3 \neq P_1, P_2)$ are isomorphic to a projective plane $\mathbf{P} = PG(2,4)$; moreover, the remaining 56 blocks are the hyperovals of one equivalence class of $\mathbf{P}$. Therefore, consider all the blocks of M through two fixed points P_1 and P_2 $(P_1 \neq P_2)$ in a fixed block B_0 of M. Let P_3 be another point of B_0 $(P_3 \neq P_1, P_2)$. Then the blocks through P_1, P_2 and P_3 form a projective plane $\mathbf{P} = PG(2,4)$ and the remaining blocks (through P_1 and P_2 but not containing P_3) are the hyperovals of one equivalence class of $\mathbf{P}$. Thus B_0 is a line in $\mathbf{P}$ (actually the points of B_0, except P_1 and P_2, form a line) and the blocks through P_1 and P_2 and no other points of B_0 are the hyperovals of one equivalence class of $\mathbf{P}$ that miss that line. □

The former case of theorem 3.2 has each of the 21 points of π contained in a hyperoval of S, and yields an affine plane $AG(2,4)$ in two different ways.

1) Let P be the point on all 16 hyperovals, and let l_∞ be a line through P. Consider the structure π_1 whose points are the affine points of π. The line in π_1 through the points P_1 and P_2 is defined to be the hyperoval of Type I in π containing each of P_1, P_2, and P, if there is one, and is defined to be the line P_1P_2 in π if there is no hyperoval of Type I in π containing the 3 points P_1, P_2 and P. Then π_1 is an affine plane.

2) A dual affine plane results by taking the hyperovals of Type I through a fixed point P in π as points, and the points in π as lines.

Consider the structure 1) first:

Theorem 3.20. *Consider $\pi = PG(2,4)$. Let P be the point on 16 hyperovals of one equivalence class and let l_∞ be a line through P. Define the structure π_1 to have as points the affine points of π. A line through two affine points P_1 and P_2 in π_1 is defined to contain the points of the hyperoval of Type I in π through P_1, P_2 and P if there is one, and is defined to contain the points of the line P_1P_2 in π if there is no hyperoval of Type I through P_1, P_2 and P in π. Then π_1 is an affine plane.*

Proof: Through any pair Q_1, Q_2 of points of π, there are exactly 4 hyperovals of Type I. Any line l not containing Q_1 or Q_2 intersects exactly 2 of these 4 hyperovals; each of the 4 points of l not on the line Q_1Q_2 are on one of these 2 hyperovals. Thus, given affine points P_1 and P_2, either exactly one of the 4 hyperovals of Type I through both P_1 and P_2 contains P, or else P, P_1 and P_2 are collinear.

Thus, let the points of π_1 be the affine points of π. A line through two points, P_1 and P_2 in π_1, is defined to be the hyperoval of Type I in π through P_1, P_2 and P if there is one, and the line P_1P_2 in π if there is no hyperoval of Type I containing all of P_1, P_2 and P in π. Thus π_1 has 16 points, 20 lines, 1 line/pair of points, 4 points/line and 5 lines/point. Therefore π_1 is a $2-(16,4,1)$ design, an affine plane of order 4. □

Now, consider the second structure, 2):

Theorem 3.21. *Let $\pi = PG(2,4)$. Let P be a point on 16 hyperovals of one equivalence class. A dual affine plane π_2 results by taking these 16 hyperovals in π as points and the points in π as lines.*

Proof: Define the structure π_2 to have as points the hyperovals of Type I through P in π, and as blocks the points of π, except for point P. There are 16 points and 20 blocks in π_2. There is one block through each pair of points in π_2 as there are 2 points, one of which is P, on each pair of hyperovals through P of Type I in π. There are 4 points on a block in π_2 since in π there are 4 hyperovals of Type I through P and any other fixed point of π. There are 5 blocks on a point in π_2 since each hyperoval of Type I through P in π has 6 points, of which one is P. Thus π_2 is a $2-(16,4,1)$-design, i.e., an affine plane of order 4. □

Section 3.8. A strongly regular graph.

It is known that the 56 hyperovals in $PG(2,4)$ from one equivalence class form a strongly regular graph as follows. Here we prove this well known fact.

Theorem 3.22. *Let Γ be the graph with vertices the hyperovals of one fixed equivalence class of $PG(2,4)$. Define 2 hyperovals to be adjacent if they are skew and distinct. Then $v = 56$, $d = 10$, $\mu = 2$, $\nu = 0$, where v is the number of vertices, d is the number of vertices adjacent to a given vertex, ν is the number of vertices adjacent to 2 adjacent vertices, and μ is the number of vertices adjacent to 2 non-adjacent vertices.*

Proof: $v = 56$ as there are 56 hyperovals.

We now establish the following claim.

Claim 1: $d = 10$.

There are 4 hyperovals through each pair of points. For each pair of points on a fixed hyperoval H, there are 3 hyperovals through that pair that are distinct from H. Therefore there are $3\binom{6}{2} + 1 = 46$ hyperovals hitting a fixed hyperoval. Therefore, there are $56 - 46 = 10$ hyperovals skew to a given hyperoval.

This establishes claim 1. Next, we prove claim 2.

Claim 2: $\nu = 0$.

It is well known that there do not exist 3 pairwise skew hyperovals; for check the 5 lines through one point of one hyperoval.

This proves claim 2. Finally, we establish claim 3.

Claim 3: $\mu = 2$.

Given 2 hyperovals meeting in 2 points, let $H_1 \cap H_2 = \{P_1, P_2\}$. Let $l_\infty = P_1P_2$. There are no affine hyperovals missing both H_1 and H_2 by lemma 3.13.

Consider the hyperovals hitting l_∞. Let H_3, H_4 be the other 2 hyperovals through both P_1 and P_2. Let P_3, P_4, and P_5 be the other points on l_∞. Exactly 2 of the hyperovals through both P_i and P_j, $i \neq j \in \{3,4,5\}$, intersect both H_3 and H_4 by lemma 3.10; say both P_3 and P_4. Thus the other 2 hyperovals through P_3 and P_4 are the only hyperovals to miss both H_1 and H_2.

This establishes claim 3.

Thus Γ is a strongly regular graph. □

In a graph Γ, an **independent set** is a set of vertices of Γ with the property that no 2 of the vertices are adjacent. The **independence number** α of a graph Γ is the size of an independent set of maximum size. The **adjacency matrix** A of Γ has i,j entry 1 if the i^{th} vertex is adjacent to the j^{th} vertex, and i,j entry 0 otherwise (see [Bondy 1]).

Theorem 3.23. *(Hoffman)*
Suppose Γ is a regular graph with v vertices and independence number α. Denote by A the adjacency matrix for Γ. Let $\lambda_1 \geq \cdots \geq \lambda_v$ be the eigenvalues of A. Then

$$\alpha \leq \frac{v|\lambda_v|}{|\lambda_1| + |\lambda_v|}.$$

Proof: See [Tonchev 1]. □

This gives us an alternative proof that the size of a maximum set of hyperovals in $PG(2,4)$ pairwise intersecting in exactly 2 points is 16; however, it tells nothing of the structure of such a set.

Corollary 3.24. *([Cameron 2])*
A 2-intersecting family of maximum size in $PG(2,4)$ has size 16.

Proof: We use the notation of theorem 3.22 and theorem 3.23. Let Γ be the strongly regular graph of theorem 3.22. Let S be a 2-intersecting family of maximum size of hyperovals. Then S is an independent set in Γ, as it is a set of vertices, pairwise non-adjacent. Let α be the independence number of Γ, the size of a maximum independence set. Let A be the adjacency matrix for the graph Γ. Let $\lambda_1 \geq \cdots \geq \lambda_v$ be the eigenvalues of A. Here $\lambda_1 = 10$, the degree of Γ, and $\lambda_{56} = -4$ (see [Tonchev 1]). Then by Hoffman's inequality,

$$\alpha \leq \frac{v|\lambda_v|}{|\lambda_1| + |\lambda_v|} = \frac{(56)(4)}{10+4} = 16.$$

However, a point-16 has size 16. □

Section 3.9. Hoffman's Inequality.

In this section we digress to prove an inequality using Hoffman's inequality and some results on strongly regular graphs. We now prove an inequality for a strongly regular graph which involves the independence number of the graph but which does not involve the eigenvalues of an adjacency matrix of the graph.

Theorem 3.25. *Let Γ be a strongly regular graph with v vertices and independence number α. Denote by ν the number of vertices adjacent to 2 adjacent vertices, and by μ the number of vertices adjacent to 2 non-adjacent vertices. Then*

$$\frac{(v-\alpha)}{d}(d+(\alpha-1)\mu) \geq d\alpha.$$

Proof: Let $\lambda_1 \geq \cdots \geq \lambda_v$ be the eigenvalues of the adjacency matrix A for Γ. Now λ_1 is the degree of Γ (see [Tonchev 1]). Therefore, Hoffman's inequality

$$\alpha \leq \frac{v|\lambda_v|}{|\lambda_1| + |\lambda_v|}$$

can be rewritten as

$$\alpha \leq \frac{v|\lambda_v|}{d + |\lambda_v|}$$

or

$$|\lambda_v| \geq \frac{\alpha d}{v-\alpha}.$$

We also have

$$|\lambda_v|^2 + (\nu - \mu)|\lambda_v| + (\mu - d) = 0 \qquad \text{(see [Tonchev 1])}.$$

This gives

$$\left(\frac{\alpha d}{v-\alpha}\right)^2 + (\nu-\mu)\left(\frac{\alpha d}{v-\alpha}\right) + (\mu - d) \leq 0$$

which can be rewritten as

$$(\alpha d)^2 + (v-\alpha)(\nu-\mu)(\alpha d) + (v-\alpha)^2(\mu-d) \leq 0$$

or as

$$(d^2 - \nu d + \mu d - d)\alpha^2 + (\nu v d - \mu v d - 2\mu\nu + 2dv)\alpha + (\mu v^2 - dv^2) \leq 0.$$

However, $d^2 + d\mu - d\nu + \mu - d = \mu v$ since $d(d - \nu - 1) = \mu(v - 1 - d)$ (see [Tonchev 1]). Therefore

$$\mu v\alpha^2 + (\nu v d - \mu v d - 2\mu\nu + 2dv)\alpha + (\mu v^2 - dv^2) \leq 0$$

which can be rewritten as

$$\mu\alpha^2 - \big(d(\mu - \nu) - 2(d - \mu)\big)\alpha - (d - \mu)v \leq 0.$$

However, $d(\mu - \nu - 1) = v\mu - \mu - d^2$ since $d(d - \nu - 1) = (v - 1 - d)\mu$ (see [Tonchev 1]). Therefore

$$\mu\alpha^2 - (v\mu + \mu - d^2 - d)\alpha - (d - \mu)v \leq 0.$$

Rearranging this gives $vd + (\alpha - 1)v\mu - \alpha d - \alpha(\alpha - 1)\mu \geq d^2\alpha$. Factoring this yields $(v - \alpha)\big(d + (\alpha - 1)\mu\big) \geq d^2\alpha$. Therefore

$$\left(\frac{v - \alpha}{d}\right)(d + (\alpha - 1)\mu) \geq d\alpha. \quad \square$$

Lemma 3.26. *With the notation of theorem 3.25, if $\{v_1, \ldots, v_\alpha\}$ are the vertices of an independent set of maximum size, and $\{w_1 \ldots, w_d\}$ are the vertices adjacent to v_1, then there are $d + (\alpha - 1)\mu$ edges between the set $\{v_1 \ldots, v_\alpha\}$ and the set $\{w_1, \ldots, w_d\}$.*

Proof: v_1 is adjacent to each of $w_1, \ldots, w_d$ giving d edges. Each of $v_2, \ldots, v_\alpha$ is adjacent to exactly μ of the vertices $w_1, \ldots, w_d$ giving $(\alpha - 1)\mu$ edges. $\square$

Chapter 4. Hexagons.

Section 4.1. Subplanes of order 4 of a projective plane.

Lemma 4.1. *Let $\pi = PG(2,F)$, where F is a field of order greater than 2. Let $H = C \cup \{N\}$ be a hyperconic in π. Then given any hyperconic $H' = C' \cup \{N'\}$, there exists ϕ in $PGL(3,F)$ such that $\phi N' = N$, $\phi C' = C$ and $\phi H' = H$, where $PGL(3,F)$ is the projective general linear group of π.*

Proof: See [Hirschfeld 1], p.179. □

Given any quadrangle $P_1P_2P_3N$, in $\pi = PG(2,F)$, it is known also that there is a unique conic through P_1, P_2 and P_3 which has nucleus N (see [Hirschfeld 1]). Therefore, using lemma 4.1, if we consider an arbitrary conic, we may choose its coordinates by specifying the coordinates of any 3 points on the conic as well as the coordinates of its nucleus.

Two choices we will use frequently are conics of the form given in the next two examples.

Example 4.2. *Let $\pi = PG(2,F)$ where F is a field. A conic through $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ has equation $a(X^2+XZ) = Y^2+YZ$ where $a \in F$. This conic C_a : $a(X^2+XZ) = Y^2+YZ$ also contains the point $(m = a^{\frac{1}{2}})$ and has as nucleus $N_a = (m = a)$. If F contains a subfield $\{0,1,\omega,\omega^2\}$ of order 4, then two such conics are $C_\omega : \omega(X^2+XZ) = Y^2+YZ$ and $C_{\omega^2} : \omega^2(X^2+XZ) = Y^2+YZ$.* □

Example 4.3. *Let $\pi = PG(2,F)$ where F is a field. The conic $\{(\gamma^2,\gamma) | \gamma \in F\} \cup \{(m = 0)\}$ has equation $Y^2 = XZ$. This conic contains the points $(0,0)$, $(1,1)$, $(m = 0)$ and has as nucleus $(m = \infty)$. If F contains a subfield $\{0,1,\omega,\omega^2\}$ of order 4, then this conic also contains the points (ω,ω^2) and (ω^2,ω).* □

Proposition 4.4. *Let $\pi = PG(2,F)$ where F is any field. If F contains a subfield $\mathbf{F}_4$ of order 4, then every quadrangle in π is contained in a unique $PG(2,4)$-subplane of π. Conversely, if π contains a $PG(2,4)$-subplane then F contains a subfield of* *order 4. In particular, for $F = \mathbf{F}_q$, $q = 2^t$, a quadrangle is contained in a unique $PG(2,4)$-subplane of π if t even.*

Proof: Let $\pi = PG(2, F)$, where F is a field containing a subfield $\mathbf{F}_4$ of order 4. Given a quadrangle $Q = \{(0,0),(1,0),(0,1),(1,1)\}$ in π, Q is contained in the $PG(2,4)$-subplane π_0 with points

$$\{(a,b)|a,b \in \mathbf{F}_4\} \cup \{(m=a)|a \in \mathbf{F}_4 \cup \{\infty\}\}.$$

There is at most one subfield of order 4 in F. Thus, if Q is contained in a $PG(2,4)$-subplane of π containing the point $(b,0)$, $b \neq 0,1$, then $b^2 = 1 + b$ and so $b = \omega$ or ω^2. Therefore Q is contained in a unique $PG(2,4)$-subplane of π.

Conversely, if π_0 is a $PG(2,4)$-subplane of π, then let $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ be in π_0. Therefore

$$(m=\infty) = (0,0)(0,1) \cap (1,0)(1,1),$$
$$(m=0) = (0,0)(1,0) \cap (0,1)(1,1),$$
$$\text{and } (m=1) = (0,0)(1,1) \cap (0,1)(1,0)$$

are on $l_\infty \cap \pi_0$.

We establish the following claim.

Claim: If $(a,0)$ and $(b,0)$ are points in π_0, then $(a+b,0)$ is a point in π_0 (see [Hartshorne 1]).

$(0,0)$, $(m=1)$, $(b,0)$ and $(m=\infty)$ are all points in π_0. Therefore

$$(b,b) = (0,0)(m=1) \cap (b,0)(m=\infty)$$

is a point in π_0. Thus, $(m=0)$, (b,b), $(0,0)$ and $(m=\infty)$ are all points in π_0. Therefore

$$(0,b) = (m=0)(b,b) \cap (0,0)(m=\infty)$$

is a point in π_0. Now we know that $(m=0)$, $(m=1)$, $(a,0)$ and $(0,b)$ are all points in π_0. Therefore

$$(m=b/a) = (m=0)(m=1) \cap (a,0)(0,b)$$

is a point in π_0. Thus we know that $(m = b/a)$, (b, b), $(0,0)$ and $(1,0)$ are all points in π_0. Therefore

$$(a + b, 0) = (m = b/a)(b, b) \cap (0,0)(1,0)$$

is also a point in π_0.

This establishes the claim.

Let $(a, 0)$ be a point in π_0 on $(0,0)(m = 0)$, where $a \neq 0, 1$.

Now $(0,1)$, $(a,0)$, $(m = 0)$ and $(m = 1)$ are all points in π_0. Therefore

$$(m = \frac{1}{a}) = (0,1)(a,0) \cap (m = 0)(m = 1)$$

is a point in π_0. Also, in proving the above claim we saw that if $(a, 0)$ is a point in π_0, then $(0, a)$ is a point in π_0. Thus, $(m = \frac{1}{a})$, $(0, a)$, $(0,0)$ and $(1,0)$ are all points in π_0. Therefore

$$(a^2, 0) = (m = \frac{1}{a})(0, a) \cap (0,0)(1,0)$$

is a point in π_0. Now we have $(a, 0)$ and $(a^2, 0)$ are both points in π_0. Therefore, by the above claim, $(a^2 + a, 0)$ must be a point of π_0. Therefore $a^2 + a + 1 = 0$. Therefore F contains a subfield of order 4. □

Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let A be a 5-arc in π. Let C be the unique conic containing A. Let N be the nucleus of C. If the conic through N and some 4 points of A has as nucleus the remaining point of A, then $A \cup \{N\}$ is called a **hexagon.**

Theorem 4.5. *Let $\pi = PG(2, F)$ where F is a field. If $\{0, 1, \omega, \omega^2\}$ is a subfield of order 4 of F, then $\{(0,0), (1,0), (0,1), (1,1), (m = \omega), (m = \omega^2)\}$ is a hexagon in π.*

Proof: Suppose $\{0, 1, \omega, \omega^2\}$ is a subfield of order 4 of F. Consider

$$\{(0,0), (1,0), (0,1), (1,1), (m = \omega), (m = \omega^2)\}.$$

The conic through any 5 of these points has as nucleus the sixth point. □

Theorem 4.6. *Let $\pi = PG(2, F)$ where F is a field. Let A be a 5-arc and let C be the unique conic through A. Let N be the nucleus of C. Suppose $A \cup \{N\}$ is a hexagon, i.e., suppose for some 4 points of A, the conic through those points as well* *as N has nucleus as the remaining point of A.*

Then 1) F must contain a subfield of order 4;

2) The conic through N and each quadrangle of A has as nucleus the remaining point of A, i.e., the conic through 5 points of $A \cup \{N\}$ has as nucleus the remaining point of $A \cup \{N\}$; and

3) $A \cup \{N\}$ is a hyperconic in a $PG(2,4)$-subplane of π.

Conversely, if F has a subplane of order 4, then any hyperoval (hyperconic) in a $PG(2,4)$-subplane is a hexagon in π.

Proof: Let A be a 5-arc, $A = \{P_1, \dots, P_5\}$, say. Let C be the conic through A. Let N be the nucleus of C. Suppose the conic C' through $N, P_1, \dots, P_4$ has nucleus P_5. Since the projective linear group $PGL(3, F)$ of π is transitive on the quadrangles of π, we may assume $\{P_1, \dots, P_4\} = \{(0,0),(1,0),(0,1),(1,1)\}$. Thus, the conics through $P_1, \dots, P_4$ are $a(X^2 + XZ) = Y^2 + YZ$, where $a \in F$. Therefore $\exists a \in F$ such that C is the conic $a(X^2 + XZ) = Y^2 + YZ$. Thus $N = (m = a)$. Also the conic C' through $N, P_1, \dots, P_4$ is $b(X^2 + XZ) = Y^2 + YZ$, for some $b \in F$. N is a point on the conic C'. Therefore $b = a^2$. The nucleus of the conic $C' : b(X^2 + XZ) = Y^2 + YZ$ is P_5. Therefore $a = b^2$. Thus $a^3 = b^3 = 1$, $a^2 = 1 + a$, $b^2 = 1 + b$, $b^2 = a$. Thus $\{0, 1, a, a^2\}$ is a subfield of order 4 of F. We have $A = \{(0,0),(1,0),(0,1),(1,1),(m = a^2)\}$ and $N = (m = a)$. The conic through N and any 4 of the points of A has as nucleus the fifth point of A. Also,

$$A \cup \{N\} = \{(0,0),(1,0),(0,1),(1,1),(m = a^2),(m = a)\}$$

is a hyperconic in a $PG(2,4)$-subplane of π. □

Theorem 4.7. *Let $\pi = PG(2, F)$ where F is a field.*

1) An equivalent definition of a hexagon is a hyperconic in a $PG(2,4)$-subplane of π.

2) If F contains a subfield of order 4, then every quadrangle is contained in a unique hexagon.

Proof: 1) A hyperconic in a $PG(2,4)$-subplane is a 6-arc such that the conic through every 5 points of these points has as nucleus the sixth. Thus, a hyperconic in a $PG(2,4)$-subplane is a hexagon.

Conversely, by theorem 4.6, a hexagon is a hyperconic in a $PG(2,4)$-subplane.

2) If F contains a subfield of order 4, then every quadrangle is contained in a unique

$PG(2,4)$-subplane; this subplane contains a unique $PG(2,4)$-hyperconic through that quadrangle. Thus every quadrangle is contained in a unique hexagon. □

Proposition 4.8. *Let $\pi = PG(2,F)$ where F is a field. Suppose H is a hexagon in π. $H = \{P_1,\ldots,P_6\}$, say. Since $PGL(3,F)$ is transitive on the quadrangles of π, we may take $P_1,\ldots,P_4$ to be $(0.0),(1,0),(0,1),(1,1)$. The unique hexagon through these 4 points is*

$$\{(0,0),(1,0),(0,1),(1,1),(m=\omega),(m=\omega^2)\},$$

where $\{0,1,\omega,\omega^2\}$ is the subfield of order 4 of F.
Alternatively, we could have chosen $P_1,\ldots,P_4$ to be $(m=\infty),(m=0),(0,0),(1,1)$. The unique hexagon through these 4 points is

$$\{(m=\infty),(m=0),(0,0),(1,1),(\omega,\omega^2),(\omega^2,\omega)\}.$$

□

Example 4.9. *Suppose $\pi = PG(2,F)$ where F is a field containing the subfield $\{0,1,\omega,\omega^2\}$. The hexagon*

$$\{(0,0),(1,0),(0,1),(1,1),(m=\omega),(m=\omega^2)\}$$

is contained in the 6 hyperconics

$$\begin{aligned}
X^2+Y^2+Z^2+XY=0 &\quad \cup \quad \{(0,0)\}\\
X^2+Y^2+XY+YZ=0 &\quad \cup \quad \{(1,0)\}\\
X^2+Y^2+XY+YZ=0 &\quad \cup \quad \{(0,1)\}\\
X^2+Y^2+XY+XZ+YZ=0 &\quad \cup \quad \{(1,1)\}\\
\omega X^2+Y^2+\omega XZ+YZ=0 &\quad \cup \quad \{(m=\omega)\}\\
\omega^2X^2+Y^2+\omega^2XZ+YZ=0 &\quad \cup \quad \{(m=\omega^2)\}.
\end{aligned}$$

□

Example 4.10. *Suppose $\pi = PG(2, F)$ where F is a field containing the subfield $\{0, 1, \omega, \omega^2\}$. The hexagon*

$$\{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$$

is contained in the 6 hyperconics

$$\begin{aligned} Y^2 &= XZ \cup (m = \infty) \\ X^2 &= YZ \cup (m = 0) \\ Z^2 &= XY \cup (0,0) \\ XY + \omega XZ + \omega^2 YZ &= 0 \cup (\omega^2, \omega) \\ XY + \omega^2 XZ + \omega YZ &= 0 \cup (\omega, \omega^2) \\ XY + XZ + YZ &= 0 \cup (1,1). \quad \square \end{aligned}$$

We will call the hexagons

$$\begin{aligned} &\{(0,0), (1,0), (0,1), (1,1), (m = \omega), (m = \omega^2)\} \\ &\quad = \{(0,0,1), (1,0,1), (0,1,1), (1,1,1), (1,\omega,0), (1,\omega^2,0)\} \\ &\{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\} \\ &\quad = \{(0,1,0), (1,0,0), (0,0,1), (1,1,1), (\omega, \omega^2, 1), (\omega^2, \omega, 1)\} \end{aligned}$$

fundamental hexagons.

Section 4.2. Most 5-arcs are not contained in hexagons.

Example 4.11. In $\pi = PG(2, F)$ where F is a field of order greater than 4 which does not contain a subfield of order 4, there are no hexagons by theorem 4.6. Thus no 5-arc in π is contained in a hexagon. □

Example 4.12. In $PG(2, F)$ where F is a field of order greater than 4 that contains a subfield $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$ of order 4, none of the 5-arcs

$$\{(0,0),(1,0),(0,1),(1,1),(m = a)\},$$

is contained in a hexagon, where $a \in F \setminus \mathbf{F}_4$. □

Let $\pi = PG(2, F)$ where F is a field. Let $P_1, \ldots, P_4$ be a quadrangle. The **Fano plane** or **Fano configuration** containing the points $P_1, \ldots, P_4$ is the projective plane of order 2 that contains $P_1, \ldots, P_4$. This plane contains the 7 points $P_1, \ldots, P_4$, $P_1P_2 \cap P_3P_4$, $P_1P_3 \cap P_2P_4$ and $P_1P_4 \cap P_2P_3$. The 3 points $P_1P_2 \cap P_3P_4$, $P_1P_3 \cap P_2P_4$ and $P_1P_4 \cap P_2P_3$ are the points of one of the lines in this plane. The **Fano line** of the quadrangle $P_1 \cdots P_4$ is the unique line of the Fano plane through $P_1, \ldots, P_4$ that contains none of $P_1, \ldots, P_4$, i.e., the Fano line of $P_1 \cdots P_4$ is the line joining the points $(P_1P_2 \cap P_3P_4)$ and $(P_1P_3 \cap P_2P_4)$. *This line also contains the point* $P_1P_4 \cap P_2P_3$.

Example 4.13. *The Fano line of the quadrangle* $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ *is* l_∞*; the Fano configuration through this quadrangle has points*

$$(0,0),(1,0),(0,1),(1,1),(m = 0),(m = 1),(m = \infty). \quad \square$$

Lemma 4.14. *Let* $\pi = PG(2, F)$ *where* F *is a field of order greater than 2. Then given any 4 points on a conic, the Fano line of those 4 points contains the nucleus of that conic. Thus, the Fano lines of the quadrangles of a conic form a pencil of lines through the nucleus of that conic.*

Proof: Let $(0,0),(1,0),(0,1),(1,1)$ be 4 points of a conic. Thus the nucleus of this conic must be $(m = a)$, for some $a \in F$. Thus the nucleus is on l_∞ which is the

Fano line of $P_1, \ldots, P_4$.

Alternatively, consider Pascal's theorem, which states that given any 6 points on a conic, $R_1, \ldots, R_6$, the points $R_1R_5 \cap R_2R_4$, $R_1R_6 \cap R_3R_4$ and $R_2R_6 \cap R_3R_5$ are collinear (see [Samuel 1]).

Consider $P_1, \ldots, P_6$ where $P_3 = P_6$ and $P_2 = P_5$. Let $Q_1 := P_1P_3 \cap P_2P_4$ and $Q_2 := P_1P_2 \cap P_3P_4$. Q_1, Q_2 and Q_3 are on the Fano line (and in the Fano configuration) of $P_1, \ldots, P_4$. Now, by Pascal's theorem, $P_1P_2 \cap P_3P_4$, $P_1P_6 \cap P_4P_5$ and $P_3P_6 \cap P_2P_5$ are collinear. I.e., $P_1P_2 \cap P_3P_4$, $P_1P_3 \cap P_2P_4$ and $P_3P_3 \cap P_2P_2$ are collinear. I.e., Q_1, Q_2 and N are collinear. □

Lemma 4.15. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Given a 5-arc $P_1, \ldots, P_5$, the quadrangles in this 5-arc have distinct Fano lines.*

Proof: Consider the quadrangles $P_1 \cdots P_4$ and $P_1P_2P_3P_5$. Suppose by way of contradiction that l is the Fano line of both. Since $P_1 \cdots P_4$ has Fano line l, the point $P_2P_3 \cap P_1P_4$ is on l. Since $P_1P_2P_3P_5$ has Fano line l, the point $P_2P_3 \cap P_1P_5$ is on l. Thus $P_1P_4 = P_1P_5$. Thus $P_4 = P_5$, a contradiction. □

Theorem 4.16. *Let $\pi = PG(2, F)$, where F is a field of order greater than 2. Suppose A is a 5-arc. Let N be the nucleus of the conic through A. There is a unique conic through any 5 of the 6 points in the 6-arc $A \cup \{N\}$. The set of nuclei of these conics is a 6-arc, B say. Moreover, this 6-arc is $A \cup \{N\}$ iff $A \cup \{N\}$ is a hexagon. If $A \cup \{N\}$ is not a hexagon, then the 6-arc B does not necessarily lie on a conic.*

Proof: Write $A = \{P_2, \ldots, P_6\}$. Let $P_1 = N$ be the nucleus of the conic through A. $P_1, \ldots, P_6$ is a hexagon iff the conic through any 5 of these points has as nucleus the sixth point. Let P_6' be the nucleus of the conic through $P_1, \ldots, P_5$; let P_5' be the nucleus of the conic through P_1, P_2, P_3, P_4, P_6; etc. Thus $P_1' = P_1$ is the nucleus of the conic through $P_2, \ldots, P_6$. Suppose by way of contradiction that three of these nuclei are collinear. Say P_1', P_2' and P_3' are collinear. Let l be this line. Now the nucleus of the conic through a quadrangle lies on the Fano line of that quadrangle. Thus both P_1' and P_3' are on the Fano line of $P_2P_4P_5P_6$; and both P_1' and P_2' are on the Fano line of $P_3P_4P_5P_6$. But as $P_2, \ldots, P_6$ is a 5-arc, $P_3P_4P_5P_6$ and $P_2P_4P_5P_6$ must have different Fano lines (by lemma 4.15), yielding a contradiction.

Suppose $A \cup \{N\}$ is not a hexagon. Then $B = \{P'_1, \ldots, P'_6\}$ does not necessarily lie on a conic as the next example shows. □

Example 4.17. Let $\pi = PG(2,16)$, where $\mathbf{F}_{16} \setminus \{0\} =< \alpha >$, and $\alpha^4 = 1 + \alpha$. Let $A = \{P_2, \ldots, P_6\}$ where $P_2 = (1,0)$, $P_3 = (0,1)$, $P_4 = (1,1)$, $P_5 = (\alpha^5, \alpha^7)$, $P_6 = (\alpha^7, \alpha^5)$. The conic through this 5-arc is $X^2 + Y^2 + Z^2 + XY = 0$; this conic has nucleus $P_1 = (0,0)$.
The conic through $P_1, \ldots, P_5$ is $\alpha X^2 + Y^2 + \alpha XZ + YZ = 0$; the nucleus of this conic is $P'_6 = (m = \alpha)$.
The conic through $P_1P_2P_3P_4P_6$ is $\alpha^{14}X^2 + Y^2 + \alpha^{14}XZ + YZ = 0$; the nucleus of this conic is $P'_5 = (m = \alpha^{14})$.
The conic through $P_1P_2P_3P_5P_6$ is $\alpha^8X^2 + \alpha^8Y^2 + XY + \alpha^8XZ + \alpha^8YZ = 0$; the nucleus of this conic is $P'_4 = (\alpha^8, \alpha^8)$.
The conic through $P_1P_2P_4P_5P_6$ is $X^2 + \alpha^6Y^2 + XY + XZ + \alpha^9YZ = 0$; the nucleus of this conic is $P'_3 = (\alpha^9, 1)$. The conic through $P_1P_3P_4P_5P_6$ is $\alpha^7X^2 + Y^2 + XZ + \alpha^9XZ + YZ = 0$; the nucleus of this conic is $P'_2 = (1, \alpha^9)$.
There is no conic through $P_1P'_2, \ldots, P'_6$ since the conic through $P'_2, \ldots, P'_6$ is $\alpha^8X^2 + \alpha^8Y^2 + \alpha Z^2 + XY + \alpha^{12}XZ + \alpha^{12}YZ = 0$ which does not contain $(0,0)$. □

The proof given for the following theorem depends on theorem 4.20. Theorem 4.18 is not used in the future, and thus can be omitted. However, this theorem fits nicely in this section.

Theorem 4.18. *Let $\pi = PG(2,F)$ where F is a field of order greater than 2. Let C be the conic through the 5-arc $A = \{P_2, \ldots, P_6\}$. Let P_1 be the nucleus of C. Let P'_i be the nucleus of the conic through the 5 points $P_1 \cdots P_6 \setminus \{P_i\}$, $i = 2, \ldots, 6$. If one of the P'_i's is in C, say $P'_6 \in C$, then replacing P_6 with P'_6 in A yields a hexagon $P_1 \cdots P_5P'_6$.*

Proof: Consider the 6-arc $P_1, \ldots, P_5, P'_6$. P_1 is the nucleus of the conic through the 5-arc $P_2, \ldots, P_5, P'_6$. P'_6 is the nucleus of the conic through $P_1, \ldots, P_5$. Thus $P_1, \ldots, P_5, P'_6$ is a 6-arc contained in at least 2 hyperconics. We will see in the next section that an equivalent definition of a hexagon is a 6-arc contained in 2 hyperconics. □

Section 4.3. An equivalent definition of hexagon in terms of hyperconics through a 6-arc.

Recall example 4.9.

Example 4.19. *In $PG(2,F)$ where F contains the subfield $\{0,1,\omega,\omega^2\}$, the fundamental hexagon*

$$\{(0,0),(1,0),(0,1),(1,1),(m=\omega),(m=\omega^2)\}$$

is contained in the 6 hyperconics

$$\begin{aligned}
X^2+Y^2+Z^2+XY=0 &\quad \cup \quad \{(0,0)\}\\
X^2+Y^2+XY+YZ=0 &\quad \cup \quad \{(1,0)\}\\
X^2+Y^2+XY+YZ=0 &\quad \cup \quad \{(0,1)\}\\
X^2+Y^2+XY+XZ+YZ=0 &\quad \cup \quad \{(1,1)\}\\
\omega X^2+Y^2+\omega XZ+YZ=0 &\quad \cup \quad \{(m=\omega)\}\\
\omega^2X^2+Y^2+\omega^2XZ+YZ=0 &\quad \cup \quad \{(m=\omega^2)\}.
\end{aligned}$$

The set of nuclei of these hyperconics is a fundamental hexagon; this hexagon is contained in each of these hyperconics. □

Theorem 4.20. *Let $\pi = PG(2,F)$ where F is a field of order greater than 4 and containing a subfield of order 4. Then an equivalent definition of a hexagon is a 6-arc contained in 2 hyperconics. Moreover, given a hexagon, there are exactly 6 hyperconics pairwise intersecting in that hexagon. The hexagon is the set of nuclei of these 6 hyperconics.*

Proof: The fundamental hexagon in example 4.19 is contained in 6 hyperconics; the set of nuclei of these hyperconics is this fundamental hexagon.
Suppose $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ are 2 hyperconics and $H_1 \cap H_2 = \{P_1,\dots,P_6\}$. Say $P_1 = N_1$ and $P_2 = N_2$. Suppose

$$\{P_3,\dots,P_6\} = \{(0,0),(1,0),(0,1),(1,1)\}.$$

Therefore $N_1 = (m = a)$, $N_2 = (m = b)$, for some $a, b \in F$. Now $a^2 = b$ since $N_1 \in C_2$; $b^2 = a$ since $N_2 \in C_1$. Therefore $P_1, \ldots, P_6$ is the fundamental hexagon in example 4.19. □

Corollary 4.21. *Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Through every quadrangle there are exactly 2 conics whose corresponding hyperconics contain the hexagon through the given quadrangle.*

Proof: Given quadrangle $P_1 \cdots P_4$, it is contained in a unique hexagon by theorem 4.7. Let P_5 and P_6 be the other points of this hexagon. There are exactly 6 hyperconics through $P_1, \ldots, P_6$ of which exactly 2 have nuclei in $\{P_5, P_6\}$. □

Section 4.4. Fano configurations and quadrangles in a hyperconic.

Lemma 4.22. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H = C \cup \{N\}$. If F contains a subfield of order 4, then given any 3 points P_1, P_2 and P_3 in C, there are exactly 2 points of C on the Fano line of $NP_1P_2P_3$. If F does not contain a subfield of order 4, then given any 3 points P_1, P_2, P_3 in C, there is no point of C on the Fano line of $NP_1P_2P_3$.*

Proof: Let $N = (0,0)$, $P_1 = (1,0)$, $P_2 = (0,1)$, $P_3 = (1,1)$. Therefore $C : X^2 + Y^2 + Z^2 + XY = 0$. All points on $C \cap l_\infty$ satisfy $X^2 + Y^2 + XY = 0$. Choose $X = 1$. Therefore $Y^2 = Y + 1$. Therefore there exists points on $C \cap l_\infty$ iff $\exists Y \in F$ such that $Y^2 = Y + 1$, i.e., iff F contains a subfield of order 4. $\square$

Let $\pi = PG(2, F)$ where F is a field. Given affine points $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$ of π, where $a_i, b_i \in F$, we denote $P_1 + P_2 := (a_1 + a_2, b_1 + b_2)$.

Lemma 4.23. *Let $\pi = PG(2, F)$ where F is a field. Suppose $P_1 \cdots P_4$ is an affine quadrangle (quadrangle of affine points) of points of C. Write $P_i = (a_i, b_i)$, where $a_i, b_i \in F$, say. Then $P_1 + \cdots + P_4 = (0,0)$ iff $P_1 \cdots P_4$ has Fano line l_∞.*

Proof: Let $m(P_i, P_j) := P_iP_j \cap l_\infty$.
Suppose $P_1 + \cdots + P_4 = (0,0)$. Therefore, $P_1 + P_2 = P_3 + P_4$ and thus $m(P_1, P_2) = m(P_3, P_4)$. Also, $P_1 + P_3 = P_2 + P_4$, so that $m(P_1, P_3) = m(P_2, P_4)$. Also, $P_1 + P_4 = P_2 + P_3$, so that $m(P_1, P_4) = m(P_2, P_3)$. Therefore $P_1 \cdots P_4$ has Fano line l_∞.
Conversely, let $P_1 \cdots P_4$ be an affine quadrangle such that $m(P_1, P_2) = m(P_3, P_4)$, $m(P_1, P_3) = m(P_2, P_4)$ and $m(P_1, P_4) = m(P_2, P_3)$. Write $P_i = (a_i, b_i)$ where $a_i, b_i \in F$.
Suppose first that $a_1 = a_2$. Therefore $m(P_1, P_2) = (m = \infty)$ and thus $a_3 = a_4$. Therefore $a_1 + \cdots + a_4 = 0$. Since $m(P_1, P_3) = m(P_2, P_4)$, we have

$$\frac{b_1 + b_3}{a_1 + a_3} = \frac{b_2 + b_4}{a_2 + a_4} = \frac{b_2 + b_4}{a_1 + a_3}.$$

It follows that $b_1 + b_3 = b_2 + b_4$. Thus $P_1 + \cdots + P_4 = (0,0)$.
Now suppose that $b_1 = b_2$. Therefore, $m(P_1, P_2) = (m = 0)$ and so $b_3 = b_4$. Therefore

$b_1 + \cdots + b_4 = 0$. Since $m(P_1, P_3) = m(P_2, P_4)$, we have $a_1 + a_3 = a_2 + a_4$. Therefore, without loss of generality, assume that all the a_i's are distinct and all the b_i's are distinct.

Now P_1P_2 is the line $Y = kX + c$, and P_3P_4 is the line $Y = kX + d$, for some $c, d, k \in F$ where $d \neq c$. Therefore, $P_1 = (a_1, ka_1 + c)$, $P_2 = (a_2, ka_2 + c)$, $P_3 = (a_3, ka_3 + d)$, and $P_4 = (a_4, ka_4 + d)$. Now $m(P_1, P_3) = m(P_2, P_4)$. Therefore

$$\frac{k(a_1 + a_3) + (c + d)}{a_1 + a_3} = \frac{k(a_2 + a_4) + (c + d)}{a_2 + a_4}.$$

Therefore $(c + d)(a_2 + a_4) = (c + d)(a_1 + a_3)$. Therefore $a_1 + \cdots + a_4 = 0$. Therefore $b_1 + \cdots + b_4 = 0$. □

Suppose we have a hyperconic H in $PG(2, F)$, with nucleus N on the line l. The following theorem shows that given any two points P_1 and P_2 of H that are not on l, the remaining points of $H \setminus l$ can be partioned into pairs yielding quadrangles each of which contain both P_1 and P_2, and each of which has Fano line l. Moreover, no other quadrangle of points containing both P_1 and P_2 has Fano line l.

Theorem 4.24. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H = C \cup \{N\}$ be any hyperconic with its nucleus N on l_∞. Then given any pair P_1, P_2 of points in $C \setminus l_\infty$, the remaining points $P_3, P_4, \ldots$ of $C \setminus l_\infty$ can be reordered such that*

$$P_1P_2P_3P_4,\ P_1P_2P_5P_6,\ \ldots$$

all have Fano line l_∞; moreover, no other quadrangle $P_1P_2P_iP_j$ has Fano line l_∞.

Proof: Suppose the point $N = (m = k)$ is on l_∞ and also suppose C contains $(0,0)$, $(0,1)$, and $(1,0)$. Therefore $C : k(X^2 + XZ) = Y^2 + YZ$. Given (a_1, b_1), $(a_2, b_2) \in C \setminus l_\infty$, then for any point $(a_3, b_3) \in C \setminus l_\infty$,

$$(a_1, b_1) + (a_2, b_2) + (a_3, b_3) = (a_1 + a_2 + a_3, b_1 + b_2 + b_3)$$

which is a point on C since

$$\begin{aligned}
&k((a_1 + a_2 + a_3)^2 + (a_1 + a_2 + a_3) \\
&\quad = k(a_1^2 + a_2^2 + a_3^2 + a_1 + a_2 + a_3) \\
&\quad = k(a_1^2 + a_1) + k(a_2^2 + a_2) + k(a_3^2 + a_3) \\
&\quad = (b_1^2 + b_1) + (b_2^2 + b_2) + (b_3^2 + b_3) \\
&\quad = (b_1 + b_2 + b_3)^2 + (b_1 + b_2 + b_3).
\end{aligned}$$

Therefore, by lemma 4.23, the quadrangle through (a_1, b_1), (a_2, b_2), (a_3, b_3) and (a_4, b_4) has Fano line l_∞. Moreover, if the quadrangle through (a_1, b_2), (a_2, b_2), (a_3, b_3), and (a_4, b_4) has Fano line l_∞, then by lemma 4.23,

$$(a_1, b_1) + (a_2, b_2) + (a_3, b_3) + (a_4, b_4) = (0, 0).$$

Therefore $a_4 = a_1 + a_2 + a_3$ and $b_4 = b_1 + b_2 + b_3$. Therefore $(a_4, b_4) = (a_1 + a_2 + a_3, b_1 + b_2 + b_3)$. □

Corollary 4.25. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H = C \cup \{N\}$ be any hyperconic with its nucleus N on l_∞. Given $P_1, P_2, P_3 \in C \setminus l_\infty$, there exists a unique affine point P of C such that the quadrangle $P_1P_2P_3P$ has Fano line l_∞; this is the unique point P of $C \setminus l_\infty$ such that $P_1 + P_2 + P_3 + P = (0, 0)$ and $P = P_1 + P_2 + P_3$.*
□

Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Suppose H is a hyperconic in π with its nucleus N not on the line l. Suppose further that either F does not contain a subfield of order 4 and l is skew to H, or F does contain a subfield of order 4 and l intersects H. The following theorem shows that the points of $C \setminus l$ can be partitioned into triples yielding quadrangles through N in H that all have Fano line l. Moreover, no other quadrangles in H through N have Fano line l.

Theorem 4.26. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H = C \cup \{N\}$ be a hyperconic with its nucleus N not on the line l_∞. If F does not contain a subfield of order 4 and also H is affine, or if F does contain a subfield of order 4 and also H is projective, then the affine points $P_1, P_2, \ldots$ of C can be reordered such that the quadrangles*

$$NP_1P_2P_3,\ NP_4P_5P_6,\ \ldots$$

all have Fano line l_∞; moreover, no other quadrangles $NP_iP_jP_k$ have Fano line l_∞.

We will see in theorem 4.31 that if F contains a subfield of order 4, then each of these quadrangles together with the two points of H on l_∞ are hexagons in H through those 2 points on l_∞.

Proof: Let $N = (0,0)$. Let $P_1 = (1,0)$, $P_2 = (0,1)$ and $P_3 = (1,1)$ be 3 points in C. Therefore $N + P_1 + P_2 + P_3 = (0,0)$. C is the conic $X^2 + Y^2 + Z^2 + XY = 0$. Therefore, given (a,b) in C, the points $(b, a+b)$ and $(a+b, a)$ are also in C. Now $(0,0) + (a,b) + (b, a+b) + (a+b, a) = (0,0)$. Therefore, given $N = (0,0)$, $P_1 = (1,0)$, $P_2 = (0,1)$ and $P_3 = (1,1)$, pick any affine point $P_4 = (a_4, b_4)$ from the remaining elements of $C \setminus l_\infty$. Let $P_5 = (b_4, a_4 + b_4)$, $P_6 = (a_4 + b_4, a_4)$. Pick any affine point P_7 from the remaining elements of $C \setminus l_\infty$. Etc. Therefore

$$N + P_1 + P_2 + P_3 = (0,0)$$
$$N + P_4 + P_5 + P_6 = (0,0)$$
$$\vdots \quad .$$

Therefore, by lemma 4.23, the quadrangles $NP_1P_2P_3$, $NP_4P_5P_6, \ldots$ all have Fano line l_∞.

Suppose, by way of contradiction, that NP_1QR and $NP_1Q'R'$ are 2 quadrangles through both N and P_1 with Fano line l_∞, where Q, R, Q', and R' are affine points in C. Then

$$N + P_i + Q + R = (0,0)$$
$$\text{and } N + P_i + Q' + R' = (0,0).$$

Therefore

$$Q + R + Q' + R' = (0,0).$$

Therefore

$$Q, R, Q', R' \text{ is a quadrangle with Fano line } l_\infty.$$

But then, by lemma 4.14, N must be on the Fano line of $QRQ'R'$, yielding a contradiction. □

Let $\pi = PG(2,F)$ where F is a field. Suppose H is a hyperconic in π with nucleus N a point not on the line l. Suppose further that either F does not contain a subfield of order 4 and l intersects H, or F does contain a subfield of order 4 and l is skew to H. Then the following theorem shows that no quadrangle in H, that contains N, has Fano line l.

Theorem 4.27. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H = C \cup \{N\}$ be a hyperconic with its nucleus N not on l_∞. If F does not contain a subfield of order 4 and H is projective, or if F contains a subfield of order 4 and H is affine, then there is no quadrangle of points of H that has Fano line l_∞.*

Proof: This is a corollary of lemma 4.14 and lemma 4.22. □

Section 4.5. Hyperconics intersecting in exactly 5 points.

Theorem 4.28. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π with $|H_1 \cap H_2| = 5$. Then $|C_1 \cap C_2| = 4$.*

Proof: We have $|C_1 \cap C_2| = 3$, 4 or 5 since $|H_1 \cap H_2| = 5$.

Now $N_1 \neq N_2$ since there is a unique conic through 3 points with a given point as nucleus.

Also, $|C_1 \cap C_2| \neq 5$ since $H_1 \neq H_2$.

Suppose, by way of contradiction, that $|C_1 \cap C_2| = 3$.

Therefore, $N_1 \in C_2$ and $N_2 \in C_1$. Let $l_\infty = N_1N_2$ and consider $C_1 \cap C_2 = \{P_1, P_2, P_3\}$. P_1, P_2, and P_3 must be affine points. Therefore $P_1 + P_2 + P_3$ is a point in both C_1 and C_2 by corollary 4.25. Therefore $|C_1 \cap C_2| \geq 4$ — a contradiction.

Thus $|C_1 \cap C_2| = 4$. □

Note that if F contains a subfield of order 4, then of the conics through the quadrangle $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, only the two containing a hexagon through this quadrangle contain any more points of the $PG(2,4)$ subplane through that quadrangle. This is true since in $PG(2,4)$, 4 points determine a unique hyperoval through them. See also section 4.10.

Section 4.6. Hyperconics and quadrangles.

Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Let $H = C \cup \{N\}$ be a hyperconic in π. Then, the hexagon through N and 3 points of C is contained in H. This important result is the following lemma.

Lemma 4.29. *Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Let $H = C \cup \{N\}$ be a hyperconic in F. Every hexagon through N and 3 points of C is contained in H.*

Proof: Let P_2, P_3, P_4 be 3 points of C. Let $P_1 = N$. There is a unique hexagon through the quadrangle $P_1, \ldots, P_4$ by theorem 4.7. Let P_5 and P_6 be the other 2 points of this hexagon. There are 6 hyperconics through $P_1, \ldots, P_6$; moreover, $P_1, \ldots, P_6$ is the set of their nuclei by theorem 4.20. Therefore, the unique conic through $P_2, \ldots, P_6$ with nucleus P_1 must be C. Thus, any hexagon through N and 3 points of C is contained in H. □

Let $\pi = PG(2, F)$ where F is a field of order greater than 4 which contains a subfield of order 4. We now show that if two hyperconics intersect in exactly 4 points, then the nucleus of a conic is not on the other conic.

Theorem 4.30. *Let $\pi = PG(2, F)$ where F is a field of order more than 4 and containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$. Then $|H_1 \cap H_2| = 4$ iff $|C_1 \cap C_2| = 4$ and $N_1 \notin C_2$, $N_2 \notin C_1$.*

Proof: Suppose first that H_1 and H_2 have the same nucleus. Then $H_1 = H_2$ as there is a unique conic through 3 points with a given point as nucleus.

Next suppose, by way of contradiction, that $N_1 \in C_2$ and $N_2 \in C_1$ and $N_1 \neq N_2$. Therefore $H_1 \cap H_2 = \{N_1, N_2, P_3, P_4\}$ where $P_3, P_4 \in C_1 \cap C_2$. There is a unique hexagon through this quadrangle by theorem 4.7. Moreover, it is contained in both H_1 and H_2 by lemma 4.29, i.e., $|H_1 \cap H_2| = 6$ — a contradiction.

Now, suppose by way of contradiction that $N_1 \in C_2$ but $N_2 \notin C_1$.

$$H_1 \cap H_2 = \{N_1, P_2, P_3, P_4\}, \quad \text{where } P_2, P_3, P_4 \in C_1 \cap C_2.$$

Let $l_\infty = N_1 N_2$. Thus P_2, P_3, and P_4 are affine points. Now there exists a unique point P in H_1 such that $P_2 + P_3 + P_4 + P = (0,0)$ by corollary 4.25. Also, there exists a unique point Q in H_2 such that $P_2 + P_3 + P_4 + Q = (0,0)$. Therefore $P = Q$. Therefore $P \in H_1 \cap H_2$ — a contradiction. □

Thus two hyperconics meet in exactly 4 points iff their conics meet in exactly 4 points and the nucleus of each is not on the other conic.

Theorem 4.31. *Let $\pi = PG(2,F)$ where F is a field containing a subfield of order 4. Let $H = C \cup \{N\}$ be a hyperconic and suppose the points Q and R of C are on l_∞. Then, adjoining Q and R to each of the quadrangles $NP_1P_2P_3$, $NP_4P_5P_6, \ldots$ of points of H that have Fano line l_∞, gives all the hexagons in H that contain Q and R.*

Proof: By theorem 4.26, the affine points $P_1, P_2, \ldots$ of C can be reordered such that the quadrangles $NP_1P_2P_3$, $NP_4P_5P_6, \ldots$ all have Fano line l_∞.
Consider the quadrangle $NP_1P_2P_3$ which has Fano line l_∞. There is a unique hexagon containing $NP_1P_2P_3$ by theorem 4.7. Moreover, this hexagon is contained in H by lemma 4.29. Let $N = (0,0)$, $P_1 = (1,0)$, $P_2 = (0,1)$ and $P_3 = (1,1)$. Then the unique hexagon containing $NP_1P_2P_3$ has its remaining 2 points on l_∞. Thus Q and R must be on the hexagon containing the quadrangle $NP_1P_2P_3$.
Since there is a unique hexagon containing N and any 3 points of C, the hexagons containing Q and R and one of the quadrangles $NP_1P_2P_3$, $NP_4P_5P_6, \ldots$ are all the hexagons in H that contain Q and R. □

Section 4.7. Canonical forms for pairs of hyperconics.

Theorem 4.32. *Let $\pi = PG(2, F)$ where F is a field of order greater than 4 which contains a subfield of order 4. Then*

1) A hexagon contained in a hyperconic must contain the nucleus of that hyperconic;

2) Every hexagon in π can be extended in exactly 6 ways to a hyperconic in π that contains it, i.e., every hyperconic in a $PG(2,4)$-subplane of a projective plane π is contained in (can be 'lifted to') exactly 6 hyperconics in the projective plane π.

Proof: 1) This is a corollary of theorem 4.20.

2) For each point P in a hexagon, there is a unique hyperconic with nucleus P containing that hexagon by theorem 4.20. □

Thus, to summarize, in $\pi = PG(2, F)$ where F is a field of order greater that 4, suppose $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ are hyperconics. Then $|C_1 \cap C_2| \leq 4$. $|H_1 \cap H_2| = 5$ iff the nucleus of exactly one of the conics C_1, C_2 is on the other conic. If F contains no subfield of order 4, then $|H_1 \cap H_2| \leq 5$. If $|H_1 \cap H_2| = 6$, then F contains a subfield of order 4 and the hexagon $H_1 \cap H_2$ is contained in 6 hyperconics (including H_1 and H_2) as the set of their nuclei; the hexagon $H_1 \cap H_2$ is a hyperconic in a unique $PG(2,4)$-subplane of π. A hexagon contained in a hyperconic contains the nucleus of that hyperconic. Every hexagon can be extended to exactly 6 hyperconics containing that hexagon.

Theorem 4.33. *Suppose $\pi = PG(2, F)$ where F is a field containing a subfield $\{0, 1, \omega, \omega^2\}$ of order 4. Let $H_1 = C_1 \cup \{N_1\}$, and $H_2 = C_1 \cup \{N_2\}$ be hyperconics intersecting in a hexagon. Then there exists ϕ in $PGL(3, F)$ (where $PGL(3, F)$ is the projective general linear group on π) such that*

$$\phi C_1 : \omega(X^2 + XZ) = Y^2 + YZ$$

and

$$\phi C_2 : \omega^2(X^2 + XZ) = Y^2 + YZ.$$

These hyperconics,

$$\omega(X^2 + XZ) = Y^2 + YZ \cup (m = \omega)$$
$$\omega^2(X^2 + XZ) = Y^2 + YZ \cup (m = \omega^2),$$

meet in the fundamental hexagon

$$\{(0,0), (1,0), (0,1), (1,1), (m = \omega), (m = \omega^2)\}.$$

Thus, if we are given two hyperconics meeting in a hexagon, we may change coordinates so these are the two hyperconics.

Proof: Let H_1 and H_2 be 2 hyperconics intersecting in the hexagon $\{P_1, \ldots, P_6\}$, where P_1 is the nucleus of H_1 and P_2 is the nucleus of H_2. Now there exists ϕ in $PGL(3, F)$ such that

$$\phi\{P_3, \ldots, P_6\} = \{(0,0), (1,0), (0,1), (1,1)\}.$$

Thus $\phi\{P_1, P_2\} = \{(m = \omega), (m = \omega^2)\}$. □

The hyperconics

$$\omega(X^2 + XZ) = Y^2 + YZ \cup (m = \omega)$$
$$\omega^2(X^2 + XZ) = Y^2 + YZ \cup (m = \omega^2)$$

will be used as a *canonical form for two hyperconics that meet in a hexagon.*

The following proposition is used to determine common points of the conic $Y^2 = XZ$ and another conic. It will be used extensively in chapter 5.

Proposition 4.34. *Let $\pi = PG(2, F)$ where F is a field of order greater that 2. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π where*

$$C_1 : Y^2 = XZ$$
$$C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0, \quad a, b, \ldots, f \in F.$$

Consider any common points on H_1 and H_2. $N_1 \in C_2$ iff $b = 0$. $N_2 \in C_1$ iff $e^2 = df$. There is at most one common point, $(m = 0)$, on l_∞. $(m = 0)$ is a common point of

C_1 and C_2 *iff* $a = 0$. *An affine point* (X, Y) *is on both* C_1 *and* C_2 *iff* $X = Y^2$ *and* Y *is a root of the polynomial*

$$p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c.$$

Proof: $N_1 \in C_2$ iff $b = 0$. $N_2 \in C_1$ iff $e^2 = df$. If $C_1 \cap C_2$ has a common point on l_∞, it must be $(m = 0)$. $(m = 0) \in C_1 \cap C_2$ iff $a = 0$. If $C_1 \cap C_2$ has common points off l_∞, say $(X, Y) \in C_1 \cap C_2$, then $Y^2 = X$ and $aX^2 + bY^2 + cZ^2 + dXY + eX + fY = c$. Thus $aY^4 + bY^2 + dY^3 + eY^2 + fY + c = 0$. Thus Y is a root of the polynomial $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$.
Conversely, if Y is a root of $p(t)$, then $aY^4 + dY^3 + (b+e)Y^2 + fY + c = 0$; so $aY^2Y^2 + dY^2Y + bY^2 + eY^2 + fY + a = 0$. Then (Y^2, Y) is a point on $Y^2 = XZ$ and $aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$. □

Theorem 4.35. *Let* $\pi = PG(2, F)$ *where* F *is a field of order greater than 4.* F *may or may not contain a subfield of order 4. Let* $H_1 = C_1 \cup \{N_1\}$ *and* $H_2 = C_2 \cup \{N_2\}$ *be hyperconics intersecting in exactly 5 points. Say* $N_1 \in C_2$ *but* $N_2 \notin C_1$.
Then, up to a collineation, $C_1 : Y^2 = XZ$ *and* $C_2 : XY + kXZ + (1+k)YZ = 0$, *where* $k \in F \setminus \{0, 1\}$ *and* k *is not contained in a subfield of order 4, and*

$$H_1 \cap H_2 = \{(m = \infty), (m = 0), (0,0), (1,1), (1 + k^2, 1 + k)\}.$$

Alternatively, *up to a collineation* $C_1 : a(X^2 + XZ) = Y^2 + YZ$ *and* $C_2 : a^2(X^2 + XZ) = Y^2 + YZ$, *where* $a \in F \setminus \{0, 1\}$ *and* a *is not contained in a subfield of order 4 of* F, *and* $H_1 \cap H_2 = \{(0,0), (1,0), (0,1), (1,1), (m = a)\}$.

Proof: $|H_1 \cap H_2| = 5$. Let $N_1 = (m = \infty)$. Let $C_1 \cap C_2 = \{P_2, \ldots, P_5\}$. Therefore $H_1 \cap H_2 = \{N_1, P_2, \ldots, P_5\}$. Let $P_2 = (m = 0)$, $P_3 = (0,0)$, $P_4 = (1,1)$. Therefore $C_1 : Y^2 = XZ$ and $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ for some $a, b, \ldots, f \in F$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. Then $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$ by proposition 4.34. $b = 0$ since $(m = \infty) \in C_2$. $a = 0$ since $(m = 0) \in C_2$. $c = 0$ since $(0,0) \in C_2$. $d + e + f = 0$ since $(1,1) \in C_2$. $d = 1$ since $N_2 \notin l_\infty$. Thus $C_2 : XY + eXZ + (1+e)YZ = 0$ and

$$\begin{aligned} p(t) &= t^3 + et^2 + (1+e)t \\ &= t(t^2 + et + (1+e)) \\ &= t(t+1)(t+1+e). \end{aligned}$$

Thus $(1+e^2, 1+e) \in C_1 \cap C_2$. The nucleus of $XY + eZ^2 + (1+e)YZ = 0$ is $N_2 = (1+e, e)$. We know $N_2 \notin C_1$ and C_1 contains $(0,0)$ and $(1,1)$. Thus $e \neq 0, 1$. If F contains a subfield $\{0, 1, \omega, \omega^2\}$ of order 4, then (ω, ω^2) and (ω^2, ω) are also in C_1. Therefore $e \neq \omega, \omega^2$. Therefore $e \neq 0, 1$ and e is not contained in a subfield of order 4 of F (if there is one). Thus

$$H_1 \cap H_2 = \{(m=\infty), (m=0), (0,0), (1,1), (1+e^2, 1+e)\} \text{ where } e \in F.$$

Alternatively, if $N_1 \in C_2$ but $N_2 \notin C_1$, then let $C_1 \cap C_2 = \{(0,0), (1,0), (0,1), (1,1)\}$. Therefore $C_1 : a(X^2 + XZ) = Y^2 + YZ$ and $C_2 : b(X^2 + XZ) = Y^2 + YZ$, for some $a, b \in F$. Thus $b = a^2$ since $N_1 \in C_2$; but, $a \neq b^2$ since $N_2 \notin C_1$. Thus $a^3 \neq 1$, $b^3 \neq 1$ and

$$H_1 \cap H_2 = \{(0,0), (1,0), (0,1), (1,1), (m=a)\}$$

where $a \in F \setminus \{0,1\}$ but a is not in a subfield of order 4 of F. □

Theorem 4.36. *Let $\pi = PG(2, F)$ where F is a field of order greater than 4 and containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics meeting in 4 exactly points. Then, up to a collineation, $C_1 : a(X^2 + XZ) = Y^2 + YZ$ and $C_2 : b(X^2 + XZ) = Y^2 + YZ$ where $a, b \in F \setminus \{0,1\}$ and where $a^2 \neq b$, and $b^2 \neq a$.*

Proof: By theorem 4.30, $|C_1 \cap C_2| = 4$ and $N_1 \notin C_2$, $N_2 \notin C_1$.
Let $C_1 \cap C_2 = \{(0,0), (1,0), (0,1), (1,1)\}$. Therefore $C_1 : a(X^2 + XZ) = Y^2 + YZ$ and $C_2 : b(X^2 + XZ) = Y^2 + YZ$ for some $a, b \in F \setminus \{0,1\}$. Now $N_1 = (m=1) \notin C_2$. Therefore $b^2 \neq a$. Also $N_2 = (m=b) \notin C_1$. Therefore $a^2 \neq b$.
Conversely, if $C_1 : a(X^2 + XZ) = Y^2 + YZ$ and $C_2 : b(X^2 + XZ) = Y^2 + YZ$ where $a, b \in F \setminus \{0,1\}$ and where $a^2 \neq b$ and $b^2 \neq a$, then $|H_1 \cap H_2| = |C_1 \cap C_2| = 4$. □

Section 4.8. Hexagons contained in a hyperconic.

Let $H = C \cup \{N\}$ be a hyperconic in $PG(2, F)$ where F is a field containing a subfield of order 4. Consider the hexagons contained in H.

Theorem 4.37. *Let $\pi = PG(2, F)$ where F is a field with a subfield of order 4. Let $H = C \cup \{N\}$ be a hyperconic in π. Consider the structure D where the points of D are the points of C, and the blocks of D are those 5-arcs in C that are contained in hexagons of H. A given point of D is on a given block of D if the point is on the hexagon containing that block. If $F = \mathbf{F}_q$, then D is $3-(q+1, 5, 1)$-design.*

Proof: Let $F = \mathbf{F}_q$. This is a well defined structure since every 5-arc of C is contained in at most one hexagon of H.

There are $q+1$ points on every conic in π. Thus, the number of points in D is $q+1$. Each hexagon in H consists of 5 points of C plus the nucleus N of C. Thus, each block in D contains 5 points.

Let P_1, P_2, P_3 be any 3 points of D, i.e., P_1, P_2, P_3 are any 3 points of C. There is a unique hexagon through N, P_1, P_2, P_3 since by theorem 4.7 every quadrangle in π is contained in a unique hexagon in π. Moreover, this hexagon through N, P_1, P_2, P_3 is contained in H by lemma 4.29. Let $N, P_1, \ldots, P_5$ be this unique hexagon. $P_1, \ldots, P_5$ must all be in C. $\{P_1, \ldots, P_5\}$ is the block through P_1, P_2, P_3. Further, since every 3 points of $P_1, \ldots, P_5$ are contained in the unique hexagon $N, P_1, \ldots, P_5$, those 3 points are also on the block $\{P_1, \ldots, P_5\}$. Thus, there is exactly one block through each set of 3 points of C. Thus D is a $3-(q+1, 5, 1)$-design. □

Theorem 4.38. *With the notation of the previous theorem, the design D has $\frac{(q+1)q(q-1)}{60}$ blocks and each point is in exactly $\frac{q(q-1)}{12}$ blocks.*

Proof: Every three points of D are contained in a unique block; each triple of the 5 points of a block are contained in that block only. Thus, the number of hexagons contained in a hyperconic

$$= \# \text{ blocks of } D$$

$$= \frac{\binom{q+1}{3}}{\binom{5}{3}}.$$

Given a point P of D, the number of blocks of D through P is the number of triples through P of points of D divided by the number of triples through P of points on a block through P. Thus, the number of blocks through a point is

$$= \frac{\binom{q}{2}}{\binom{4}{2}}. \quad \square$$

Proposition 4.39. *Let $\pi = PG(2,q)$ where $q = 2^t$. Let C be the conic $Y^2 = XZ$. Denote by $PGL(3,q)$ the projective linear group of π and by $PGO(3,q)$ the subgroup of $PGL(3,q)$ fixing C. Then*

$$PGO(3,q) = \left\{ \begin{pmatrix} a^2 & ab & b^2 \\ 0 & ad+bc & 0 \\ c^2 & cd & d^2 \end{pmatrix} : a,b,c,d \in \mathbf{F}_q,\ ad \neq bc \right\}.$$

Proof: See [Hirschfeld 1]. $\square$

Let $\pi = PG(2,q)$ where $q = 2^t$ and t is even. *Consider the structure J whose points consist of the points of $PG(1,q)$, and whose blocks consist of the images of $PG(1,4)$ under the Mobius group of appropriate dimension.* It is known that this is a $3-(q+1,5,1)$-design. (See [Hughes 1].)

Theorem 4.41. *Let $\pi = PG(2,q)$ where $q = 2^t$ and t is even. Let D be the $3-(q+1,5,1)$-design described in theorem 4.37 and theorem 4.38. Then D is isomorphic to J where J is the design described above.*

Proof: Let $\mathbf{F}_4 = \{0,1,\omega,\omega^2\}$ be the subfield of order 4 of $\mathbf{F}_q$. Let $H = C \cup \{N\}$ be a hyperconic. Consider the hexagons ir H. Without loss of generality we can take C to be the conic $Y^2 = XZ$ with nucleus $(0,1,0)$ (in homogenous coordinates). Consider the map

$$\phi : \text{points of } C \mapsto \text{ points of } PG(1,q)$$
$$\text{given by } \phi(t^2,t,1) = (t,1)$$
$$\text{and } \phi(1,0,0) = (1,0).$$

Let G_0 be the block

$$G_0 = \{(1,0,0),(0,0,1),(1,1,1),(\omega^2,\omega,1),(\omega,\omega^2,1)\}$$

in D. Therefore

$$\phi G_0 = \{(1,0),(0,1),(1,1),(\omega,1),(\omega^2,1)\}$$

which is $PG(1,4)$. Let $PGO(3,q)$ be the subgroup of $PGL(3,q)$ fixing C (and thus N and H). Consider

$$T : PGO(3,q) \mapsto PGL(2,q)$$

defined by $T\begin{pmatrix} a^2 & ab & b^2 \\ 0 & ad+bc & 0 \\ c^2 & cd & d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbf{F}_q$ and $ad \neq bc$.

T is an isomorphism (see [Hirschfeld 1]). Given block G in D,

$$G = \begin{pmatrix} a^2 & ab & b^2 \\ 0 & ad+bc & 0 \\ c^2 & cd & d^2 \end{pmatrix} G_0$$

for some $a, b, c, d \in \mathbf{F}_q$ with $ad \neq bc$ by proposition 4.39. Thus

$$\begin{aligned} \phi G &= \phi \begin{pmatrix} a^2 & ab & b^2 \\ 0 & ad+bc & 0 \\ c^2 & cd & d^2 \end{pmatrix} G_0 \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi G_0 \end{aligned}$$

which is the Mobius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

applied to $\phi G_0 = PG(1,4)$. □

Section 4.9 Fano configurations and hexagons.

We now determine the number of quadrangles in a conic which have the Fano line l, where l is a line containing the nucleus of that conic.

Lemma 4.42. *Let $\pi = PG(2,q)$. Suppose $H = C \cup \{N\}$ is a hyperconic in π. Let l_∞ be the line through N and one point, P_{q+1} say, of C. Then there are*

$$\frac{\binom{q+1}{4} \text{ quadrangles in } C}{q+1 \text{ lines through } N}$$

quadrangles in C which have Fano line l_∞.

Proof: If the point P_{q+1} of C is on l_∞, we may reorder the remaining points, $P_1, \ldots, P_q$, of C such that P_1P_2, P_3P_4, ..., $P_{q-1}P_q$ are the lines ($\neq l_\infty$) through P intersecting C. Thus $P_iP_{i+1}P_jP_{j+1}$, $i \neq j \in \{1, 3, \ldots, q-1\}$ are the quadrangles in C with Fano configurations containing P and Fano line l_∞. There are

$$\frac{q-2}{2} + \left(\frac{q-2}{2} - 1\right) + \cdots + 1 = \frac{q(q-2)}{8}$$

such quadrangles ($\frac{q-2}{2}$ through each pair P_i, P_{i+1}). Thus there are

$$\begin{aligned}\left(\frac{q(q-2)}{8}\right)\left(\frac{q-1}{3}\right) &= \frac{q(q-1)(q-2)}{(4)(3)(2)} \\ &= \frac{\frac{(q+1)q(q-1)(q-2)}{(4)(3)(2)}}{q+1} \\ &= \frac{\binom{q+1}{4} \text{ quadrangles in } C}{q+1 \text{ lines through } N}\end{aligned}$$

quadrangles in C which have Fano line l_∞ (as expected). □

The following theorem gives an equivalent definition of a hexagon.

Theorem 4.43. *Let $\pi = PG(2,F)$ where F is any field containing a subfield of order 4. An equivalent definition of hexagon is a 6-arc such that the Fano line of each quadrangle of the 6-arc contains the remaining 2 points of the 6-arc.*

Proof: Suppose $P_1, \ldots, P_6$ is a 6-arc such that each quadrangle of these points has Fano line through the other two points. Let $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (0,1)$, and $P_4 = (1,1)$. $P_1 \cdots P_4$ has Fano line l_∞. Therefore $P_5 = (m = a)$, $P_6 = (m = b)$, for some $a, b \in F \setminus \{0,1\}$. Also, $P_5P_1P_2P_3$ has Fano line $Y = a^2X + a$ which must contain P_4 and P_6. Therefore $(1,1)$ is on $Y = a^2X + a$. Therefore, $a^2 = 1 + a$, i.e., a is contained in a subfield $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$ of F of order 4. Also, $(m = b)$ is on $Y = a^2X + a$. Therefore, $a^2 = b$. Therefore $P_1, \ldots, P_6$ is a fundamental hexagon. Conversely, given a hexagon, we may choose coordinates so this is a fundamental hexagon. This fundamental hexagon has the property that the Fano line ofeach quadrangle of the hexagon contains the remaining 2 points of the hexagon. □

Example 4.44. Consider $\mathbf{F}_{16}$, where $\mathbf{F}_{16} \setminus \{0\} = < \alpha >$, and $\alpha^4 = 1 + \alpha$. Let

$$Q_1 = \{(0,0), (1,0), (0,1), (1,1)\}$$
$$Q_2 = \{(\alpha^8, \alpha^4), (\alpha^2, \alpha^4), (\alpha^8, \alpha), (\alpha^2, \alpha)\}$$
$$Q_3 = \{(\alpha^5, \alpha^8), (\alpha^{10}, \alpha^8), (\alpha^5, \alpha^2), (\alpha^{10}, \alpha^2)\}$$
$$Q_4 = \{(\alpha^4, \alpha^5), (\alpha, \alpha^5), (\alpha^4, \alpha^{10}), (\alpha, \alpha^{10})\}.$$

Then $\{(m = \omega)\} \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ is the conic $\omega^2(X^2 + XZ) = Y^2 + YZ$; this conic has nucleus $(m = \omega^2)$. Note that $\{(m = \omega^2), (m = \omega)\} \cup Q_i$ is a hexagon, $i = 1, \ldots, 4$, and

$$Q_2 = Q_1 + (\alpha^8, \alpha^4)$$
$$Q_3 = Q_1 + (\alpha^5, \alpha^8)$$
$$Q_4 = Q_1 + (\alpha^4, \alpha^5). \quad \square$$

Example 4.45. Consider $\mathbf{F}_{16}$, where $\mathbf{F}_{16} \setminus \{0\} = < \alpha >$, and $\alpha^4 = 1 + \alpha$. Let $\omega = \alpha^5$. Let

$$Q_1 = \{(0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$$
$$Q_2 = \{(\alpha^2, \alpha), (\alpha^8, \alpha^4), (\alpha, \alpha^8), (\alpha^4, \alpha^2)\}$$
$$Q_3 = \{(\alpha^6, \alpha^3), (\alpha^{13}, \alpha^{14}), (\alpha^9, \alpha^{12}), (\alpha^7, \alpha^{11})\}$$
$$Q_4 = \{(\alpha^{12}, \alpha^6), (\alpha^{11}, \alpha^{13}), (\alpha^{14}, \alpha^7), (\alpha^3, \alpha^9)\}.$$

Then $\{(m = 0)\} \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ is the conic $Y^2 = XZ$; this conic has nucleus $(m = \infty)$. □

Lemma 4.46. *Consider $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $P_1, \ldots, P_5$ be a 5-arc. Let l be the Fano line of $P_1 \cdots P_4$ and m be the Fano line of $P_1P_2P_3P_5$. Then $l \cap m$ is the nucleus of the conic through $P_1, \ldots, P_5$.*

Proof: Given a quadrangle, the nucleus of each conic through that quadrangle lies on the Fano line of the quadrangle by lemma 4.14. □

Theorem 4.47. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Given a 5-arc, we can geometrically construct the points of the conic through that 5-arc using Fano configurations. Also, given a quadrangle, we can geometrically construct the points of the conic through any 3 of the points with nucleus the fourth.*

Proof: A 5-arc determines a conic C. We can find the nucleus N by lemma 4.46. Using N with three of the original points, say P_1, P_2, P_3, we can construct any point on the conic.

Given N, P_1, P_2, P_3, let C be the conic through P_1, P_2, P_3 with nucleus N. Let $l_4, l_5, \ldots$ be the lines through N different from NP_1, NP_2, NP_3. Let

$$P_i = ((P_1P_2 \cap l_i)P_3) \cap ((P_2P_3 \cap l_i) \cdot P_1), \qquad i = 4, 5, \ldots ;$$

i.e., let P_i be the unique point such that $P_1P_2P_3P_i$ has Fano line l_i. Thus

$$N, P_1, P_2, P_3, P_i \text{ is a 5-arc, } i = 4, 5, \ldots.$$

We now establish the following claim.

Claim: P_i is on C, $i = 4, 5, \ldots$.

There is a unique point P on C such that $P_1P_2P_3P$ has Fano line l_i by corollary 4.25. This establishes the claim.

$P_4, P_5, \ldots$ are distinct since $l_4, l_5, \ldots$ are distinct. Therefore $C = \{P_1, P_2, \ldots\}$. □

Theorem 4.48. *Let $\pi = PG(2, F)$ where F is a field of order greater than 4 and containing a subfield of order 4. Given a triangle P_1, P_2, N, consider the conics through P_1 and P_2 with nucleus N. There is a partition of the points that are not on the triangle through P_1, P_2, N into 3's yielding distinct hexagons through NP_1P_2. Thus, if $F = \mathbb{F}_q$, there are $q - 1$ conics through P_1 and P_2 with nucleus N; moreover,* *they partition the $(q-1)^2$ points that are not on the triangle through P_1, P_2, N into 3's yielding distinct hexagons through NP_1P_2.*

Proof: Pick a point P not on the triangle through N, P_1, P_2. There is a unique conic C containing P, P_1, and P_2 with nucleus N. Write $C = \{P_1, P_2, P_3, \dots\}$ say. There is a partition of the points of $C \setminus \{P_1, P_2\}$ into 3's by theorem 4.26. Reorder P_3, P_4, ... if necessary so that the quadrangles $NP_3P_4P_5$, $NP_6P_7P_8, \dots$ all have Fano line l. These quadrangles are in distinct hexagons all of which contain P_1 and P_2 by theorem 4.31. Pick a point P' not on C, such that P' is not on the triangle through N, P_1, P_2. There is a unique conic C' through P', P_1, P_2 with nucleus N. As with C, there is a similar partition of the points of $C' \setminus \{P_1, P_2\}$, into 3's. Moreover $C \cap C' = \{P_1, P_2\}$ since there is a unique conic through 3 points with nucleus N. This can be repeated for all points of π not on the triangle through N, P_1, P_2. Thus we have a partition of the points of π not on the triangle through N, P_1, P_2 into triples. □

We return to this discussion in section 4.17 to obtain yet another consequence of theorem 4.47.

Section 4.10. Hexagons and the corresponding $PG(2,4)$-subplanes of π.

Recall some results about hexagons.

Let $\pi = PG(2, F)$ where F is a field of order greater than 4 which contains a subfield of order 4.

1) A hexagon contained in a hyperconic contains the nucleus of that hyperconic.

2) There is a unique hexagon through three points of a conic that is contained in the hyperconic through that conic.

3) Every hexagon can be extended in exactly 6 ways to a hyperconic in π.

4) Every hexagon is contained in a unique $PG(2,4)$-subplane of π.

Theorem 4.49. *Let $\pi = PG(2, F)$ where F is a field of order greater than 4 which contains a subfield of order 4. Let $H = C \cup \{N\}$ be a hyperconic in π. Given 2 distinct hexagons in H, the corresponding $PG(2,4)$-subplanes containing them are distinct. Thus, each of the hexagons in a hyperconic gives rise to a distinct $PG(2,4)$-subplane containing the point N.*

Proof: Let G_1, and G_2 be hexagons in H. Therefore $N \in G_1 \cap G_2$. In a projective plane of order 4, a maximum set of points, no 3 collinear is a hyperconic containing 6 points. Thus G_1, G_2 are in distinct $PG(2,4)$-subplanes. □

Lemma 4.50. *Let $\pi = PG(2, F)$ where F is a field of order greater than 4 which contains a subfield of order 4. Let $H = C \cup \{N\}$ be a hyperconic in π. Let l_∞ be a line containing the nucleus N of C and also a point P_0 of C. Let $NP_0P_1 \cdots P_4$ be a hexagon contained in H. Then given any affine point P in C, the 6-arc containing N, P_0, and also the quadrangle $P_1 + P, \dots, P_4 + P$ is also a hexagon in H.*

Proof: Let $N = (m = \omega^2)$, $P_0 = (m = \omega)$, $P_1 = (0,0)$ and $P_2 = (1,0)$. Therefore $P_3 = (0,1)$, $P_4 = (1,1)$ and $C : \omega^2(X^2 + XZ) = Y^2 + YZ$. Let $P = (a, b)$ be an affine point of C. Consider $P_1 + P, \dots, P_4 + P$. These are all affine points of C. Moreover, N, P_0 and $P_1 + P, \dots, P_4 + P$ are all points on the hyperconic $\omega(X^2 + XZ) = Y^2 + YZ$ $\cup \{(m = \omega)\}$. Therefore $NP_0P_1 + P \cdots P_4 + P$ is a hexagon. □

Theorem 4.51. *Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Suppose $H = C \cup \{N\}$ is a hyperconic in π. Let l_∞ be the line through N and one point, P_0 say, of C. Then the remaining points $P_1, P_2, \dots$ of C can be rearranged so* *that*

$$NP_0P_1P_2P_3P_4, NP_0P_5P_6P_7P_8, \ldots$$

are hexagons through N and P_0 with the property that the Fano configurations through

$$P_1P_2P_3P_4, P_5P_6P_7P_8, \ldots$$

all meet l_∞ in the same 3 points. Thus the hexagons of C that contain both N and P_0 partition the points of $l_\infty \setminus \{N, P_0\}$ into 3's.

Proof: Let $l_\infty = NP_0$. Given any point Q on the line NP, which is not N or P, let l_1 be any line through Q that meets l in 2 points. Let P_1 and P_2 be these 2 points. There is a unique hexagon through NPP_1P_2. Let P_3 and P_4 be the other points on this hexagon. By lemma 4.50, the remaining points of C can be partitioned giving hexagons $NPP_1 \cdots P_4$, $NPP_5 \cdots P_8$, ... such that the Fano configurations through $P_1 \cdots P_4$, $P_5 \cdots P_8$, ... all meet the line NP in Q and 2 other fixed points. □

Corollary 4.52. *With the notation of the previous theorem, consider the hexagons in H through N and P. Each of these hexagons is contained in a $PG(2,4)$-subplane of π. Moreover, given any 2 of these subplanes, their lines through N and P either have only points N and P in common, or else they are equal with 5 points in common.*

Proof: Let the 2 hexagons through N and P be $NPP_1 \cdots P_4$ and $NPP_1' \cdots P_4'$. The Fano configurations through $P_1 \cdots P_4$ and $P_1' \cdots P_4'$ either meet the line NP in the same 3 points, or in different 3 points by theorem 4.31. □

Corollary 4.33. *With the notation of theorem 4.31, consider the hexagons in H through N and P. Each of these hexagons is contained in a $PG(2,4)$-subplane of π. For each hexagon H through N and P in $PG(2,q)$, there are exactly $\frac{q}{4}$ of these hexagons, including H, such that the corresponding $PG(2,4)$-subplanes all have the same line NP.*

Proof: Each hexagon through N and P contains 4 of the q points of $H \setminus \{N, P\}$. Thus, there are $\frac{q}{4}$ such subplanes. □

Section 4.11. Involutions on a line resulting from the conics through a quadrangle.

Let $\pi = PG(2, F)$ where F is a field. Choose a quadrangle Q in π. Let l be any line skew to Q. Consider the conics through Q. A conic through Q intersects l in 0, 1 or 2 points. Define the mapping ϕ on the points of l by $\phi : P \mapsto P'$ if there is a conic through Q and both P and P'; and by $\phi : P \mapsto P$ if there is a conic through Q tangent to P.

We now examine Desargues' involution theorem. We will show the following:

1) If l is the Fano line of Q, then ϕ fixes all points of l, i.e., all conics through Q intersecting l are tangent to l.

2) If l passes through exactly one of the three points where the Fano configuration through Q meets the Fano line of Q, then this is the unique fixed point of l. I.e., none of the non-degenerate conics through Q are tangent to l.

3) If l misses all 3 of these points, then there is a unique point on l which is fixed by ϕ. I.e., there is a unique (non-degenerate) conic through Q which is tangent to l since there is a unique point on l which is the nucleus of a conic through Q.

Lemma 4.54. *Let $\pi = PG(2, F)$ where F is a field. Let Q be a quadrangle in π with Fano line l. Then for every point N on l which is not a point in the Fano configuration through Q, there exists a conic through Q with nucleus N.*

Proof: Let l be the Fano line of the quadrangle Q. Let N be a point on l which is not a point in the Fano configuration of Q. Suppose $Q = \{P_1, \dots, P_4\}$. There is a unique conic C through P_1, P_2, and P_3 with nucleus N. This conic intersects l in a point P Consider the conic C_2 through $P, P_1, \dots, P_4$. C_2 has nucleus on l since l is the Fano line of $P_1 \cdots P_4$. Thus C_1 and C_2 have nuclei on l. C_1 and C_2 also have nuclei on the Fano line of P, P_1, P_2, P_3. Therefore they have the same nuclei. Therefore $C_1 = C_2$. □

Theorem 4.55. *Let $\pi = PG(2, F)$ where F is a field. Choose a quadrangle Q in π. Let Q_1, Q_2, Q_3 be the points where the Fano configuration through Q meets the Fano line of Q. Let l be a line skew to Q and let l_∞ be the Fano line of Q. Then* *the conics (including both degenerate and non-degenerate conics) through Q define an involution on the points of l which either fixes every point of l, or fixes a unique point of l.*

Proof: Define the mapping ϕ on points of l by $\phi : P \mapsto P'$ if there is a conic through Q and both P and P'; and by $\phi : P \mapsto P$ if there is a conic through Q tangent to P. Suppose $Q = \{(0,0),(1,0),(0,1),(1,1)\}$. l_∞ is the Fano line of Q. $Q_1 = (m = \infty)$, $Q_2 = (m = 0)$ and $Q_3 = (m = 1)$ are the points on l_∞ where the Fano configuration through Q meets the Fano line of Q. *The nucleus of each conic through Q is on l_∞.*
1) Suppose $l = l_\infty$. Then for every conic through Q intersecting l, the nucleus must be on l. Thus those conics are tangent to l, and ϕ fixes every point of l.
2) Suppose l intersects Q_1, say, but does not contain the points Q_2 or Q_3. Then of the conics through Q intersecting l, only the one with nucleus Q_1 (a degenerate conic through Q_1) has nucleus on l. Thus Q_1 is the only point fixed by ϕ; none of the conics intersecting l are not tangent to l, except the degenerate one through Q_1.
3) Suppose l does not contain Q_1, Q_2, or Q_3. Let $R = l \cap l_\infty$. There is a unique conic through Q with nucleus R. This conic is tangent to l, and intersects l in R', say. All the other conics through Q have nucleus on $l_\infty \setminus l$ and are therefore not tangent to l. Thus R' is the unique fixed point of ϕ. □

Section 4.12. Orbits of the conics that contain a fixed quadrangle.

To summarize some results in $PG(2,F)$, where F is a field of order more than 4, if 2 conics intersect in exactly 4 points, then either

1) their hyperconics intersect in a hexagon, i.e., their hyperconics intersect in 6 points; or

2) their hyperconics intersect in 5 points and the nucleus of one conic is on the second conic, but the nucleus of that second conic is not contained in the first conic; or

3) their hyperconics meet in 4 points and the nucleus of each conic is not on the other conic.

Let us consider the conics through a fixed quadrangle. Given quadrangle Q and one conic C_0 through Q let us define a new conic C_1 to be the conic containing both the quadrangle Q and the nucleus of C_0. Now we can look at the conic C_2 defined to be the conic containing both the quadrangle Q and the nucleus of C_1. Etc. This gives us an 'orbit' of a conic through a quadrangle. We will see that in $PG(2,F)$, when F has a subfield of order 4, there is a unique orbit of length 2 (i.e., only 2 distinct conics in that orbit) corresponding to the 2 conics through Q whose hyperconics meet in the hexagon through Q. The other orbits will have length s where $\mathbf{F}_{2^s}$ is a subfield of the given field.

Theorem 4.56. *Let $\pi = PG(2,F)$, where F is a field of order greater than four. Let Q be a quadrangle in π. Given a conic C containing Q, define ϕC to be the conic containing the 4 points of the quadrangle Q as well as point that is the nucleus of C. Let the* **orbit** *of C be $\{C, \phi C, \phi^2 C, \dots\}$. Then there is an orbit of conics through Q of length s iff F contains a subfield of order 2^s. When it exists, it is unique. Moreover, if F contains a subfield of order 4, then the orbit of length 2 contains the 2 conics through Q whose hyperconics contain the hexagon through Q.*

Proof. Let $Q = \{(0,0),(1,0),(0,1),(1,1)\}$. Then the conics through Q are C_a : $a(X^2+XZ) = Y^2+YZ$ where $a \in F$. Consider the map $\phi : a \mapsto a^2$ or $\phi : C_a \mapsto C_{a^2}$. I.e., ϕC is the conic containing the quadrangle Q as well as the nucleus of C. Note that if $a \in \mathbf{F}_s$ where $\mathbf{F}_s$ is a subfield of F, then $a^{2^s} = a$. Conversely, suppose $\mathbf{F}_{2^s}$ is the smallest subfield of F containing a. Then C_a is in an orbit of length s. Note that C_a

is in an orbit of length 2 iff $(0,0),(1,0),(0,1),(1,1),(m=a),(m=a^2)$ is the unique hexagon through $(0,0),(1,0),(0,1),(1,1)$. Therefore there is an orbit of length s if F contains a subfield of order 2^s. In particular, if F contains a subfield of order 4, there is a unique orbit of length 2 corresponding to the 2 hyperconics containing the hexagon through Q with nuclei on the Fano line of Q. In $F = \mathbf{F}_q$, where $q = 2^t$, there is an orbit of length t. □

Section 4.13. Intersection of a hyperconic with certain lines through a fixed point.

Consider a hexagon $H = C \cup \{N\}$ and a line l_∞ through the nucleus of this hyperconic. Let P_0 be the point of C that is on l_∞. Now the hexagons in H containing N and P_0 partition the points of $l_\infty \setminus \{N, P_0\}$ into 3's (see theorem 4.51). Consider one of these triples. Recall that this triple along with P_0 and N are in some $PG(2,4)$-subplanes that contain some of the hexagons in H. Consider the lines through a given point not on l_∞ that hit this triple. Then, either exactly one, or exactly 3 of these lines intersects H. This is the main theorem of this section.

Theorem 4.57. *Let $\pi = PG(2, F)$ where F contains a subfield of order 4. Suppose $H = C \cup \{N\}$ is a hyperconic containing a hexagon G. Let π_0 be the $PG(2,4)$-subplane of π containing G. Choose l_∞ to be a line through N. Write $G = \{P_1, \dots, P_6\}$, where $P_1 := N$ and $l_\infty = P_1P_2$. Let the other points of l_∞ in π_0 be Q_1, Q_2, Q_3. Let P be a point of π. Then, of the lines PQ_1, PQ_2, PQ_3, either exactly 1 or exactly 3 intersects H.*

In theorem 4.57 $P_1 \cdots P_4 N P_0$ must be a hexagon as lemma 4.58 shows.

Lemma 4.58. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H = C \cup \{N\}$. Let l_∞ be a line containing N and a point P_0, say, of C. Suppose $P_1, \dots, P_4$ are points of C such that the quadrangle $P_1 \cdots P_4$ has Fano line l_∞, but the 6-arc $P_1 \cdots P_4 P_0 N$ is not a hexagon. Let Q_1, Q_2, Q_3 be the points on l_∞ in the Fano plane through $P_1 \cdots P_4$. I.e.,*

$$Q_1 = P_1P_2 \cap P_3P_4, \quad Q_2 = P_1P_3 \cap P_2P_4, \quad Q_3 = P_1P_4 \cap P_2P_3, \text{ say.}$$

Then there may exist a point $P \in \pi$ such that PQ_1, PQ_2, and PQ_3 all are skew to H.

Proof: Consider, for example, $C : \alpha^3X^2 + Y^2 + \alpha^3XZ + YZ = 0$ in $\pi = PG(2,16)$ where $\mathbf{F}_{16} \setminus \{0\} = <\alpha>$, $\alpha^4 = 1 + \alpha$. Let $H = C \cup \{N\}$ where $N = (m = \alpha^3)$. Now $P_0 := C \cap l_\infty = (m = \sqrt{\alpha^3})$. Consider the quadrangle

$$P_1 = (0,0), \; P_2 = (1,0), \; P_3 = (0,1), \; P_4 = (1,1)$$

of points in C. The 6-arc $P_1 \cdots P_4 P_0 N$ is not a hexagon. $Q_1 = (m = 0)$, $Q_2 = (m = \infty)$, and $Q_3 = (m = 1)$ are the points of l_∞ in the Fano plane containing $P_1 \cdots P_4$. Let P be the point (α^2, α^9). Therefore PQ_1 is the line $Y = \alpha^9$, PQ_2 is the line $X = \alpha^2$, and PQ_3 is the line $Y = X + \alpha^{11}$. All three of these lines are skew to H. □

To prove the main theorem of this section, theorem 4.57, we need some other results first.

Proposition 4.59. *Let F be a field. Let $S = \{c^2 + c | c \in F\}$. If $F \cap S \neq F$, then let $T_k = \{c^2 + c + k \mid c \in F\}$ where $k \in F \setminus S$. Then*

1) $1 \in S$ iff F contains a subfield of order 4. Thus T_1 is non-empty if F contains a subfield of order 4.

2) $a, b \in S \Rightarrow a + b \in S$.

Suppose T_k is non-empty. Then

3) $a, b \in T_k \Rightarrow a + b \in S$.

4) $a \in S, b \in T_k \Rightarrow a + b \in T_k$.

5) $S \cap T_k = \emptyset$.

6) $F = S \cup T_k$.

Proof: 1) F contains a subfield of order 4 iff $\exists \omega \in F$ such that $\omega^2 + \omega + 1 = 0$, i.e., iff $1 \in S$.

2) If $a, b \in S$ then $a = a_1^2 + a_1$, $b = b_1^2 + b_1$, for some $a_1, b_1 \in F$. Therefore $a + b = (a_1 + b_1)^2 + (a_1 + b_1) \in S$.

3) If $a, b \in T_k$ then $a = a_1^2 + a_1 + k$, $b = b_1^2 + b_1 + k$, for some $a_1, b_1 \in F$. Therefore $a + b = (a_1 + b_1)^2 + (a_1 + b_1) \in S$.

4) If $a \in S, b \in T_k$ then $\exists a_1, b_1 \in F$ such that $a = a_1^2 + a_1$, $b = b_1^2 + b_1 + k$. Therefore $a + b = (a_1 + b_1)^2 + (a_1 + b_1) + k \in T_k$.

5) If $S \cap T_k \neq \emptyset$, $\exists a \in F$ such that $a = b^2 + b = c^2 + c + k$ where $b, c \in F$. I.e., $0 = (b+c)^2 + (b+c) + k$. I.e., $(b+c)^2 + (b+c) = k$. But $k \in F \setminus S$. This is a contradiction.

6) Given $a \in F$, consider a, and $a + k$. Now $a + (a + k) = k \in T_k$. Thus by 2), 3), 4), and 5), exactly one of a, and $a + k$ is in S; the other is in T_k. Therefore $\mathbf{F}_q = S \cup T_k$ and $S \cap T_k = \emptyset$. □

We now prove the main theorem of this section.

Theorem 4.60. *Suppose $\pi = PG(2, F)$, where F is a field containing the subfield $\{0, 1, \omega, \omega^2\}$ of order 4. Let $H = C \cup \{N\}$. Let l_∞ be a line through N and P_0, where P_0 is a point in C. Suppose $P_1, \ldots, P_4$ are points in C such that $P_1 \cdots P_4 P_0 N$ is a hexagon. Let Q_1, Q_2, Q_3 be the points on l_∞ in the Fano plane of the quadrangle $P_1 \cdots P_4$. I.e., N, P_0, Q_1, Q_2, Q_3 are the points of l_∞ in the $PG(2,4)$-subplane containing the hexagon $P_1, \ldots, P_4, N, P_0$. Then, for any point P, exactly 1 or 3 of PQ_1, PQ_2, PQ_3 intersects H.*

Proof: Let $N = (m = \omega^2)$, $P_0 = (m = \omega^2)$, $P_1 = (0,0)$, and $P_2 = (1,0)$. Therefore $P_3 = (0,1)$, $P_4 = (1,1)$ and $C : \omega(X^2 + XZ) = Y^2 + YZ$. Therefore $Q_1 = (m = 0)$, $Q_2 = (m = \infty)$, and $Q_3 = (m = 1)$.
If $P \in H$ or $P \in l_\infty$, the result is immediate.
Suppose $P \notin H$, and $P \notin l_\infty$. Suppose $P = (a, b)$. Therefore PQ_1 is the line $Y = b$, PQ_2 is the line $X = a$, and PQ_3 is the line $Y = X + a + b$. Let $S = \{c^2 + c \mid c \in F\}$, $T_1 = \{c^2 + c + 1 \mid c \in F\}$.

$$
\begin{aligned}
Y = b \text{ misses } C &\text{ iff there is no solution to } \omega(X^2 + X) = b^2 + b \\
&\text{ iff there is no solution to } X^2 + X = \omega^2(b^2 + b) \\
&\text{ iff } \omega^2(b^2 + b) \notin S \\
X = a \text{ misses } C &\text{ iff there is no solution to } \omega(a^2 + a) = Y^2 + Y \\
&\text{ iff } \omega(a^2 + a) \notin S
\end{aligned}
$$

$$
\begin{aligned}
&Y = X + a + b \text{ misses } C \\
&\quad\text{iff there is no solution to } \omega(X^2 + X) = (X + a + b)^2 + (X + a + b) \\
&\quad\text{iff there is no solution to } \omega^2(X^2 + X) = ((a + b)^2 + (a + b)) \\
&\quad\text{iff } \omega((a + b)^2 + (a + b)) \notin S.
\end{aligned}
$$

We now establish the following claim.

Claim: $Y = b$, $X = a$ and $Y = X + a + b$ cannot all be skew to H.
Let $d = \omega^2(b^2 + b)$, $e = \omega(a^2 + a)$, and $f = \omega((a^2 + a) + b^2 + b) = \omega(a^2 + a) + \omega(b^2 + b)$.
If all of $Y = b$, $X = a$ and $Y = X + a + b$ are skew to C, then the sum of 2 of d, e

and f is in S; thus, the sum $d + e + f$ is not in S. However, the sum of these is

$$\begin{aligned} d + e + f &= \omega^2(b^2 + b) + \omega(a^2 + a) + (\omega(a^2 + a) + \omega(b^2 + b)) \\ &= (\omega^2 + \omega)(b^2 + b) \\ &= (b^2 + b) \text{ which is in } S. \end{aligned}$$

This establishes the claim.

Moreover, as $d + e = (b^2 + b) + f$, $d + f = (b^2 + b) + e$, and $e + f = (b^2 + b) + d$, then if 2 of the lines $Y = b$, $X = a$, and $Y = X + a + b$ intersect H it follows that the third line does too, since S is closed under addition.

□

Section 4.14. Hyperconics pairwise meeting in distinct hexagons.

Example 4.61. Let $\pi = PG(2,16)$, where $\mathbf{F}_{16} \setminus \{0\} = <\alpha>$, $\alpha^4 = 1 + \alpha$. Let $\omega = \alpha^5$. Consider the hyperconics

$$H_1 : \big(\omega^2(X^2 + XZ) = Y^2 + YZ\big) \cup \{(m = \omega^2)\}$$

and

$$H_2 : \big(\omega(X^2 + XZ) = Y^2 + YZ\big) \cup \{(m = \omega)\}.$$

$$\begin{aligned} H_1 : \big(\omega^2(X^2 + XZ) = Y^2{+}YZ\big) &\cup \{(m = \omega^2)\} \\ &= \{(m=\omega^2),(m=\omega),(0,0),(1,0),(0,1),(1,1), \\ &\quad (\alpha^8,\alpha^4),(\alpha^4,\alpha^5),(\alpha^5,\alpha^8), \\ &\quad (\alpha^8,\alpha),(\alpha,\alpha^{10}),(\alpha^{10},\alpha^8), \\ &\quad (\alpha^2,\alpha^4),(\alpha^4,\alpha^{10}),(\alpha^{10},\alpha^2), \\ &\quad (\alpha^2,\alpha),(\alpha,\alpha^5),(\alpha^5,\alpha^2)\} \end{aligned}$$

$$\begin{aligned} H_2 : \big(\omega^2(X^2 + XZ) = Y^2 + YZ\big) &\cup \{(m = \omega^2)\} \\ &= \{(m=\omega),(m=\omega^2),(0,0),(0,1),(1,0),(1,1), \\ &\quad (\alpha^4,\alpha^8),(\alpha^5,\alpha^4),(\alpha^8,\alpha^5), \\ &\quad (\alpha,\alpha^8),(\alpha^{10},\alpha),(\alpha^8,\alpha^{10}), \\ &\quad (\alpha^4,\alpha^2),(\alpha^{10},\alpha^4),(\alpha^2,\alpha^{\text{[illegible]}}), \\ &\quad (\alpha,\alpha^2),(\alpha^5,\alpha),(\alpha^2,\alpha^5)\}. \end{aligned}$$

H_1 and H_2 intersect in the hexagon $\{(m = \omega^2),(m = \omega),(0,0),(1,0),(0,1),(1,1)\}$. Each of the 4 hyperconics

$$H_{a,b} : X^2 + Y^2 + (a^2 + b^2 + ab)Z^2 + XY = 0 \cup (0,0),$$

where $(a,b) \in \{(\alpha^8,\alpha^4),(\alpha^8,\alpha),(\alpha^2,\alpha^4),(\alpha^2,\alpha)\}$, intersects

H_1 and H_2 in different hexagons.

$$\begin{aligned}
H_1 \cap H_{\alpha^8,\alpha^4} &= \{(m=\omega^2),(m=\omega),(0,0),(\alpha^8,\alpha^4),(\alpha^4,\alpha^5),(\alpha^5,\alpha^8)\}\\
H_2 \cap H_{\alpha^8,\alpha^4} &= \{(m=\omega),(m=\omega^2),(0,0),(\alpha^4,\alpha^8),(\alpha^5,\alpha^4),(\alpha^8,\alpha^5)\}\\
H_1 \cap H_{\alpha^8,\alpha} &= \{(m=\omega^2),(m=\omega),(0,0),(\alpha^8,\alpha),(\alpha,\alpha^{10}),(\alpha^{10},\alpha^8)\}\\
H_2 \cap H_{\alpha^8,\alpha} &= \{(m=\omega),(m=\omega^2),(0,0),(\alpha,\alpha^8),(\alpha^{10},\alpha),(\alpha^8,\alpha^{10})\}\\
H_1 \cap H_{\alpha^2,\alpha^4} &= \{(m=\omega^2),(m=\omega),(0,0),(\alpha^2,\alpha^4),(\alpha^4,\alpha^{10}),(\alpha^{10},\alpha^2)\}\\
H_2 \cap H_{\alpha^2,\alpha^4} &= \{(m=\omega),(m=\omega^2),(0,0),(\alpha^4,\alpha^2),(\alpha^{10},\alpha^4),(\alpha^2,\alpha^{10})\}\\
H_1 \cap H_{\alpha^2,\alpha} &= \{(m=\omega^2),(m=\omega),(0,0),(\alpha^2,\alpha),(\alpha,\alpha^5),(\alpha^5,\alpha^2)\}\\
H_2 \cap H_{\alpha^2,\alpha} &= \{(m=\omega),(m=\omega^2),(0,0),(\alpha,\alpha^2),(\alpha^5,\alpha),(\alpha^2,\alpha^5)\}. \quad \Box
\end{aligned}$$

Theorem 4.62. *Let $\pi = PG(2,F)$, where F is a field of order greater than. Given 2 hyperconics, $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ that intersect in a hexagon, let N_3 be any point of $H_1 \cap H_2$ other than N_1 or N_2. Then, for any point P of H_1 different from N_1, N_2 and N_3, the hyperconic containing N_1, N_2 and P that has nucleus N_3 intersects H_1 in a hexagon and intersects H_2 is a (possibly different) hexagon. These 2 hexagons are different if $P \notin H_1 \cap H_2$. This gives a partition of H_1 and H_2 into triples. If $F = \mathbf{F}_q$, there are exactly $\frac{q-4}{3}$ hyperconics through N_1 and N_2 with nuclei N_3 that intersect H_1 in a hexagon, and H_2 in a different hexagon.*

Proof: There are exactly 6 hyperconics containing the hexagon $H_1 \cap H_2$ by theorem 4.20.

Recall that a hexagon contained in a hyperconic must contain the nucleus of that hyperconic by theorem 4.32. Thus, if H_3 is a hypercnoic that intersects H_1 in a hexagon, H_3 must contain the nucleus of H_2, and H_1 and H_2 must both contain the nucleus of H_3. Since $H_1 \cap H_2$ is a hexagon, H_1 contains the nucleus of H_2 and H_2 contains the nucleus of H_1. Therefore the nuclei of H_1, H_2 and H_3 are all contained in $H_1 \cap H_2 \cap H_3$. There can be no more points in $H_1 \cap H_2 \cap H_3$ as a quadrangle in contained in a unique hexagon.

We are given that N_1 is the nucleus of H_1, N_2 is the nucleus of H_2, and N_3 is one other point of $H_1 \cap H_2$. Now, every quadrangle is contained in a unique hexagon by theorem 4.7, and the hexagon through N_1 and 3 more points of H_1 is contained in H_1

by lemma 4.29. Thus, the points of H_1, other that N_1, N_2 and N_3, can be partitioned into triples each of which, together with N_1, N_2 and N_3 is a hexagon in H.
Let $N_1 = (m = \omega)$, $N_2 = (m = \omega^2)$, and $N_3 = (0,0)$. Suppose $(1,0) \in H_1 \cap H_2$. Therefore

$$H_1 \cap H_2 = \{(m = \omega^2), (m = \omega), (0,0), (1,0), (0,1), (1,1)\}.$$

Thus $H_1 : \omega^2(X^2 + XZ) = Y^2 + YZ$ and $H_2 : \omega(X^2 + XZ) = Y^2 + YZ$. Let (a, b) be a point of H_1 that is not on the hexagon $H_1 \cap H_2$. Define $H_3 : X^2 + Y^2 + (a^2 + b^2 + ab)Z^2 + XY = 0$. Now

$$(m = \omega^2), (m = \omega), (0,0), (a, b), (b, a + b), (a + b, a) \in H_1 \cap H_3.$$

Therefore $H_1 \cap H_3$ is a hexagon in H_1 and H_3. Also

$$(m = \omega^2), (m = \omega), (0,0), (b, a), (a + b, b), (a, a + b) \in H_2 \cap H_3$$

Therefore $H_2 \cap H_3$ is a hexagon in H_2 and H_3. Thus every point (a, b) of H_1 not on the hexagon $H_1 \cap H_2$ gives rise to a hyperconic intersecting H_1 and H_2 in different hexagons; (a, b), $(b, a + b)$, $(a + b, a)$ along with N_1, N_2, and $(0,0)$ are the points of a hexagon in H_1.
In $\mathbb{F}_q$, there are $q + 2 - 6$ points of $H_1 \setminus (H_1 \cap H_2)$. Thus there are $\frac{q-4}{3}$ hyperconics with nuclei $(0,0)$ that intersect H_1 and H_2 in distinct hexagons. □

Section 4.15. Maximum sets of hyperconics pairwise meeting in six points.

Lemma 4.63. *Let $\pi = PG(2,q)$, $q = 2^t$, t even, and suppose $q \geq 16$. Let S be a set of hyperconics any 2 of which pairwise meet in 6 points. If $|S| \geq 4$ then they have a common hexagon.*

Proof: Suppose $H_1, \ldots, H_4 \in S$, where $H_i = C_i \cup \{N_i\}$. Suppose $|H_i \cap H_j| = 6$. Therefore $H_i \cap H_j$ is a hexagon and $N_j \in C_i$, $N_i \in C_j$ by theorem 4.32, $i \neq j \in \{1, \ldots, 4\}$. Therefore $N_2, N_3, N_4 \in C_1$. Therefore the hexagon through $N_1, \ldots, N_4$ is contained in H_1 by lemma 4.29. Similarly, the hexagon through $N_1, \ldots, N_4$ is contained in H_2, H_3, and H_4. Therefore $H_i \cap H_j$ is the hexagon through $N_1N_2N_3N_4$, $i \neq j \in \{1, \ldots, 4\}$. □

Corollary 4.64. *Let $\pi = PG(2,q)$, $q = 2^t$, t even, and suppose $q \geq 16$. If S is a maximum set of hyperconics pairwise meeting in 6 points, then $|S| = 6$ and all hyperconics in S contain the hexagon which is the set of their nuclei.*

□

Section 4.16. Two conics with no common hexagon that have the nucleus of each conic on the other conic.

Theorem 4.65. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Suppose we have two conics with the nucleus of each conic on the other conic, but with no common hexagons. Then the two conics can have at most 2 points in common.*

Proof: Suppose $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ are hyperconics with $N_2 \in C_1$ and $N_1 \in C_2$.

If $|C_1 \cap C_2| = 4$, then $|H_1 \cap H_2| = 6$, i.e., $H_1 \cap H_2$ is a hexagon. Thus $|C_1 \cap C_2| \neq 4$.

If $|C_1 \cap C_2| = 3$, then $|H_1 \cap H_2| = 5$. But then by theorem 4.28 we can't have both $N_1 \in C_2$ and $N_2 \in C_1$. Thus $|C_1 \cap C_2| \neq 3$.

Therefore $|C_1 \cap C_2| \leq 2$.

It is possible here that $|C_1 \cap C_2| = 2$, for consider $PG(2, F)$ where F is a field of order more than 2 but not containing a subfield of order 4. Consider $C_1 : Y^2 = XZ$ and $C_2 : X^2 = YZ$. Then $|C_1 \cap C_2| = 2$.

If F contains a subfield of order 4, then $|C_1 \cap C_2| \leq 1$ since N_1, N_2 and 2 points of $C_1 \cap C_2$ would be contained in a hexagon which is in both H_1 and H_2. □

Section 4.17. Conics through 2 fixed points with a common nucleus.

Theorem 4.66. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. The conics through 2 fixed points with a fixed point as nucleus contain no further common points and thus partition the points off the triangle through P, Q, N. If $F = \mathbf{F}_q$, there are exactly $q-1$ conics through two fixed points that have a common nucleus. No two of these conics have a further point in common. These conics through points P and Q with nucleus N partition the points of π that are off the triangle PQN into $(q-1)$-tuples.*

Proof: There are

$$\begin{aligned}(q^2 + q + 1) &- ((q+1) + q + (q-1)) \\ &= q^2 - 2q + 1 \\ &= (q-1)^2\end{aligned}$$

points off the triangle through PQN. Thus, there are $(q-1)^2$ choices for a point R off this triangle which gives a conic through P, Q, R with nucleus N. $(q-1)$ of these choices yield the same hvperconic. Thus there are $q-1$ distinct conics through P and Q with nucleus N.

No two of these conics can have a further point in common since 3 points together with a nucleus uniquely determines a conic. □

Section 4.18. Intersection of an arbitrary hyperconic with an arbitrary $PG(2,4)$-subplane.

Theorem 4.67. *Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Consider the intersection of an arbitrary subplane of order 4 of π with an arbitrary hyperconic in π. If there are 6 common points, then the common points are a hexagon and one of these points must be the nucleus of H.*

Proof: A 6-arc in π_0 is a hexagon. A hexagon in H contains the nucleus. □

Theorem 4.68. *Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Consider the intersection of an arbitrary subplane of order 4 of π with an arbitrary hyperconic in π. There cannot be exactly 5 common points.*

Proof: Suppose the hyperconic $H = C \cup \{N\}$ meets the subplane π_0 in 5 points. These 5 points extend uniquely to a hyperconic in the subplane, i.e., to a hexagon. If the nucleus of the hyperconic is in the subplane, then this hexagon must be contained in the hyperconic, yielding a contradiction. Otherwise, the 5 points common points of the hyperconic H and the subplane must be on C and thus the nucleus is in the subplane. □

Theorem 4.69. *Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Consider the intersection of an arbitrary subplane of order 4 of π with an arbitrary hyperconic in π. If there are exactly 4 common points, then the nucleus of the hyperconic is not contained in the subplane; moreover, the 4 common points are not contained in any hexagon with the nucleus.*

Proof: Suppose by way of contradiction that hyperconic H and a $PG(2,4)$-subplane π_0 have exactly 4 common points. If the nucleus of the hyperconic were one of these points, then the hexagon through those points would be contained in the hyperconic — a contradiction. If $H \cap \pi_0$ is contained in a hexagon through N, then the hexagon is in H — a contradiction. □

Theorem 4.70. *Let $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Consider the intersection of an arbitrary subplane π_0 of π of order 4 with an arbitrary hyperconic H in π. If there are exactly 4 common points, these points extend uniquely* *to a hexagon G in the subplane. This hexagon is not in the hyperconic H. This hexagon can be extended in 6 ways to distinct hyperconics in π (six choices for a nucleus). Of these 6 hyperconics, 2 meet H in exactly 4 points and 4 meet H in exaclty 5 points.*

Proof: Let $G_1, \ldots G_4$ be the hyperconics with nuclei in $H \cap \pi_0$. Let G_5, G_6 be the hyperconics with nuclei in $G \setminus (H \cap \pi_0)$. Note that the nuclei of G_5, G_6 are not in H; the nucleus of H is not on G_5, G_6. Therefore $|H \cap G_5| < 5$ and $|H \cap G_6| < 5$. Therefore

$$H \cap G_5 = H \cap G_6 = H \cap \pi_0.$$

Note that the nuclei of $G_1 \ldots, G_6$ are in H. Also $|G_i \cap H| \geq 4, \; i = 1, \ldots, 4$, since $\pi_0 \cap H \subset G_i$. Thus, as two hyperconics meeting in exactly 4 points can't have the nucleus of one conic on the other conic, we have $|G_i \cap H| = 5, \; i = 1, \ldots, 4$. □

Chapter 5. Hyperconics containing hexagons from the same subplane of order 4.

The main result of this chapter (see section 5.3) is a generalization of the famous even-intersection property of hyperconics in $PG(2,4)$. In $PG(2,4)$, even intersection amongst hyperconics is an equivalence relation.

Let $\pi = PG(2,F)$ where F is a field containing a subfield of order 4. Two hexagons are **coplanar** if they are in the same subplane of order 4 of π.

Theorem 5.1. *Let $\pi = PG(2,F)$ where F is a field containing a subfield of order 4 but not containing a subfield of order 8. Suppose H_1 and H_2 are hyperconics in π and G_1 and G_2 are coplanar hexagons with G_i a hexagon in H_i, $i = 1,2$. If G_1 and G_2 meet in an even number of points, then so do H_1 and H_2; if G_1 and G_2 meet in an odd number of points, then so do H_1 and H_2. I.e., we have a 'lifting' of the even intersection property of a plane of order 4.*
If F contains a subfield of order 8, then the above lifting fails in three cases. If $|G_1 \cap G_2| = 2$ and $G_1 \cap G_2$ contains exactly one of the nuclei of H_1 and H_2, then $|H_1 \cap H_2| = 5$ iff F contains a subfield of order 8. If $|G_1 \cap G_2| = 1$ and the nuclei of H_1 and H_2 along with the point $G_1 \cap G_2$ are 3 distinct points not on a line, then $|H_1 \cap H_2| = 4$ iff F contains a subfield of order 8. If $|G_1 \cap G_2| = 3$ and the nuclei of H_1 and H_2 are 2 of the 3 points of $G_1 \cap G_2$, then $H_1 \cap H_2$ is a hexagon iff F contains a subfield of order 8.

In section 5.1 we will prove this for the case where $G_1 \cap G_2$ is even. In section 5.2 we will prove this for the case where $G_1 \cap G_2$ is odd. In section 5.3 we will obtain a generalization of the even intersection property for hyperconics in $PG(2,4)$.

To prove this theorem, we will consider the different possibilities of how two hyperconics in a $PG(2,4)$ plane intersect. We will then regard the $PG(2,4)$ hyperconics as hexagons in a $PG(2,4)$- subplane of a projective plane. We will consider the 6 hyperconics through each of the hexagons to see how pairs of these hyperconics meet.
A result that will be used frequently in this chapter is proposition 4.34. Recall this proposition.

Proposition 4.34. *Let $\pi = PG(2, F)$ where F is a field of order greater than 2. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π where*

$$C_1 : Y^2 = XZ$$
$$C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0, \quad a, b, \dots, f \in F.$$

Consider any common points on H_1 and H_2. $N_1 \in C_2$ iff $b = 0$. $N_2 \in C_1$ iff $e^2 = df$. C_1 and C_2 have at most one common point, $(m = 0)$, on l_∞. $(m = 0)$ is a common point of C_1 and C_2 iff $a = 0$. An affine point (X, Y) is on both C_1 and C_2 iff $X = Y^2$ and Y is a root of the polynomial

$$p(t) = at^4 + dt^3 + (b + e)t^2 + ft + c.$$

Section 5.1. 'Lifting' hyperconics in $PG(2,4)$ that meet in an even number of points.

Theorem 5.3. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. If $|G_1 \cap G_2|$ is even and F does not contain a subfield of order 8, then $|H_1 \cap H_2|$ is even.*

We will prove this via 3 separate theorems, theorems 5.4 through 5.6, depending on whether $|G_1 \cap G_2|$ is 0, 2 or 6.

Theorem 5.4. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $G_1 = G_2$. Then $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: $|H_1 \cap H_2| = 6$. □

Theorem 5.5. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 2$.*

1) If $N_1 = N_2$ then $H_1 \cap H_2 = G_1 \cap G_2$.

2) Suppose N_1, $N_2 \in G_1 \cap G_2$ but $N_1 \neq N_2$. If F contains a subfield of order 16, then $H_1 \cap H_2$ is a hexagon; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

3) If N_1, $N_2 \notin G_1 \cap G_2$ then either some three of the four points $\{N_1, N_2\} \cup (G_1 \cap G_2)$ are collinear and $H_1 \cap H_2 = G_1 \cap G_2$, or $\{N_1, N_2\} \cup (G_1 \cap G_2)$ is a quadrangle and $|H_1 \cap H_2| = 4$.

4) Suppose exactly one of N_1 and N_2 is on both G_1 and G_2. If F contains a subfield of order 8 then $|H_1 \cap H_2| = 5$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

Proof: Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of F of order 4.

1) Let $N_1 = N_2 = (m = \infty)$. Let $(m = 0) \in G_1 \cap G_2$. Let $(0,0)$ and $(1,1) \in G_1$. Therefore

$$G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$$

and $C_1 : Y^2 = XZ$. By theorem 2.6, $G_2 = G_1 + (a,b)$ for some $a, b \in \mathbb{F}_4$. Therefore

$$G_2 = \{(m = \infty), (m = 0), (a,b), (a+1,b), (a,b+1), (a+1,b+1)\}$$

and $C_2 : Y^2 + (a + b^2)Z^2 + XZ = 0$. Therefore $H_1 \cap H_2 = G_1 \cap G_2$.

2) Let $N_1 = (m = \infty)$ and $N_2 = (m = 0)$. Pick $P \in G_2 \setminus \{(m = \infty)\}$. Let $(0,0)$ and N_1 be the points of G_1 on the line N_1P. Let $(1,1)$ and N_2 be the points of G_1 on the line N_2P. Thus $P = (0,1)$ and

$$G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}.$$

There are exactly 4 hyperconics in the $PG(2,4)$-subplane of π through G_1 that meets G_1 only in the points $(m = \infty)$ and $(m = 0)$, and only one of these contains P by theorem 3.7. Thus

$$G_2 = \{(m = \infty), (m = 0), (0,1), (1,0), (\omega, \omega), (\omega^2, \omega^2)\}.$$

Thus $C_1 : Y^2 = XZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, for some $a, b, \dots, f \in \mathbf{F}_4$. Let $p(t) = at^4 + dt^3 + (b + e)t^2 + ft + c$. By proposition 4.34 $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Now $f = 1$, $d = 0$, $e = 0$ since $N_2 = (m = 0)$, $b = 0$ since $(m = \infty) \in G_2$, $b + c + f = 0$ since $(0,1) \in G_2$, and $a + c + e = 0$ since $(1,0) \in G_2$. Thus $c = 1$ and $a = 1$. Therefore $C_2 : X^2 + Z^2 + YZ = 0$ and $p(t) = t^4 + t + 1$. Thus $p(t)$ has 4 roots if F contains a subfield of order 16; $p(t)$ has no roots otherwise. Therefore, by proposition 4.34, $|H_1 \cap H_2| = 6$ if F contains a subfield of order 16; otherwise $H_1 \cap H_2 = G_1 \cap G_2$.

3) In this case N_1, $N_2 \notin G_1 \cap G_2$. Let $G_1 \cap G_2 = \{P_1, P_2\}$.

Suppose first that $N_1N_2P_1P_2$ is a quadrangle.

Let $N_1 = (m = \infty)$, $P_1 = (m = 0)$, and $P_2 = (0,0)$. Let N_1 and $(1,1)$ be the points of G_1 on N_1N_2 so that N_2 is on $X = 1$. Thus

$$G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}.$$

Therefore $C_1 : Y^2 = XZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, for some $a, b, \dots, f \in \mathbf{F}_4$. Let $p(t) = at^4 + dt^3 + (b + e)t^2 + ft + c$. By proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff Y is a root of $p(t)$. Now $a = 0$ since $(m = 0) \in G_2$, $c = 0$ since $(0,0) \in G_2$, and $d = f = 1$ since N_2 is on $X = 1$. Thus $C_2 : bY^2 + XY + eXZ + YZ = 0$, and $p(t) = t^3 + (b+e)t^2 + t = t(t^2 + (b+e)t + 1$. Let $p_1(t) = t^2 + (b+e)t + 1$. Notice that $p_1(0) = 1 \neq 0$, $p_1(1) = b + e \neq 0$ since $(1,1) \notin C_2$, $p_1(\omega) = \omega^2 + (b+e)\omega + 1 \neq 0$ since $(\omega, \omega^2) \notin C_2$, and $p_1(\omega^2) = \omega + (b+e)\omega^2 + 1 \neq 0$ since $(\omega^2, \omega) \notin C_2$. Thus $p(t)$

has one root in $\mathbf{F}_4$. $p(t)$ has exactly 3 roots in F if $\mathbf{F}_{16}$ is a subfield of F and exactly one root otherwise. Therefore, by proposition 4.34, if F contains a subfield of order 16 then $|H_1 \cap H_2| = 4$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

Now suppose N_1, N_2 and P_1 are collinear.

Let $N_1 = (m = \infty)$, $P_1 = (m = 0)$ and $P_2 = (0,0)$. Let $(1,1)$ be on G_1 also. Therefore $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega,\omega^2), (\omega^2,\omega)\}$. Now N_1, N_2 and P_1 are collinear. Therefore N_2 is on l_∞. Therefore $C_1 : Y^2 = XZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ for some $a, b, \ldots, f \in \mathbf{F}_4$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. By proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff Y is a root of $p(t)$. Now $a = 0$ since $(m = 0) \in G_2$, $c = 0$ since $(0,0) \in G_2$, and $d = 0$, $f = 1$ since N_2 is on l_∞. Thus $C_2 : bY^2 + eXZ + YZ = 0$, and $p(t) = (b+e)t^2 + t = t((b+e)t+1)$. Now $b+e+1 \neq 0$ since $(0,0) \notin C_2$, $(b+e)\omega \neq \omega^2$ since $(\omega,\omega^2) \notin C_2$, and $(b+e)\omega^2 \neq \omega$ since $(\omega^2,\omega) \notin C_2$. Therefore $b = e$. Thus $p(t) = t$. Therefore, by proposition 4.34, $H_1 \cap H_2 = G_1 \cap G_2$.

4) Suppose $N_1 \in G_1 \cap G_2$, but $N_2 \notin G_1 \cap G_2$. Let $N_1 = (m = \infty)$. Let $P = (m = 0)$ be the other point of $G_1 \cap G_2$. N_1N_2 meets G_1 in N_1 and one other point, $(0,0)$, say. PN_2 meets G_1 in P and one other point, $(1,1)$, say. Thus $N_2 = (0,1)$ and $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega,\omega^2), (\omega^2,\omega)\}$. Therefore $C_1 : Y^2 = XZ$. There are 4 hyperconics in the $PG(2,4)$-subplane through G_1 that meet G_1 only in $(m = \infty)$ and $(m = 0)$ by theorem 3.7. The one that also passes through $(0,1)$ is $G_2 = \{(m = \infty), (m = 0), (0,1), (1,0), (\omega,\omega), (\omega^2,\omega^2)\}$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, for some $a, b, \ldots, f \in F$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. By proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Now $f = 0$, $e = 1$, and $d = 1$ since $N_2 = (0,1)$, $b = 0$ since $(m = \infty) \in G_2$, $a = 0$ since $(m = 0) \in G_2$, and $c = 1$ since $(1,0) \in G_1$. Thus $C_2 : Z^2 + XY + XZ = 0$ and $p(t) = t^3 + t^2 + 1$. Thus $p(t)$ has 3 roots if F contains a subfield of order 8; otherwise, $p(t)$ has no roots. Therefore, by proposition 4.34, if F contains a subfield of order 8 then $|H_1 \cap H_2| = 5$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$. □

Theorem 5.6. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $G_1 \cap G_2 = \emptyset$. If F contains a subfield of order 256, then $|H_1 \cap H_2| = 4$; otherwise, $H_1 \cap H_2 = \emptyset$.*

Proof: Let $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of F of order 4.

Given hyperconics G_1, G_2, D_1 and D_2 in $PG(2,4)$ such that $G_1 \cap G_2 = \emptyset$ and $D_1 \cap D_2 = \emptyset$, and given points $P_1 \in G_1$, $P_2 \in G_2$, $Q_1 \in D_1$ and $Q_2 \in D_2$, there is a unique element ϕ of $PG(3,4)$ such that $\phi(P_1) = Q_1$, $\phi(P_2) = Q_2$ and also $\phi(G_1) = D_1$ and $\phi(G_2) = D_2$ (see [Brouwer 1]).

Therefore, without loss of generality we can choose

$$G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$$
$$G_2 = \{(m = 1), (m = \omega^2), (0,1), (\omega, \omega), (\omega, 1), (0, \omega)\}$$

and also $N_1 = (m = \infty)$ and $N_2 = (m = 1)$. Therefore $C_1 : Y^2 = XZ$ and $C_2 : \omega^2 X^2 + \omega Y^2 + \omega^2 Z^2 + XZ = 0$. Let $p(t) = \omega^2 t^4 + \omega^2 t^2 + t + \omega^2$. Therefore, by proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Let $p_1(t) = t^4 + t^2 + \omega t + 1$. Therefore $p(t) = \omega^2 p_1(t)$.

We now establish the following claim.

Claim: $p_1(t)$ is irreducible over $\mathbf{F}_4$.

Differentiating $p_1(t)$ yields ω, which is not 0. Thus $p_1(t)$ has no multiple roots. Also, none of 0, 1, ω, or ω^2 is a root of $p_1(t)$. Suppose, by way of contradiction, that $p_1(t)$ factors as a product of quadratics.

$$\begin{aligned} t^4 + t^2 + \omega t + 1 &= (t^2 + \gamma_1 t + \gamma_2)(t^2 + \gamma_3 t + \gamma_4), \text{ for some } \gamma_1, \ldots, \gamma_4 \in \{0, 1, \omega, \omega^2\} \\ &= t^4 + (\gamma_1 + \gamma_3)t^3 + (\gamma_2 + \gamma_4 + \gamma_1\gamma_3)t^2 + (\gamma_1\gamma_4 + \gamma_2\gamma_3)t + \gamma_2\gamma_4. \end{aligned}$$

Thus $\gamma_1 = \gamma_3$. Therefore $\gamma_2 + \gamma_4 = 1 + \gamma_1^2$, $\gamma_1(\gamma_2 + \gamma_4) = k$ and $\gamma_2\gamma_4 = 1$. Now γ_2, $\gamma_4 \neq 0, 1$ since $\gamma_1(\gamma_2 + \gamma_4) = k$ and $\gamma_2\gamma_4 = 1$. Therefore $\gamma_2, \gamma_4 \in \{\omega, \omega^2\}$. We also know $\gamma_2\gamma_4 = 1$. Thus $\gamma_4 = \gamma_2^2$. Therefore $\gamma_2 + \gamma_4 = 1$. Therefore $\gamma_1 = \omega$. Therefore $1 = \gamma_2 + \gamma_4 = 1 + \omega^2 = \omega$, yielding a contradiction.

This establishes the claim.

Thus $p(t)$ is irreducible over $\{0, 1, \omega, \omega^2\}$. Therefore, by proposition 4.34, if F contains a subfield of order 256, then $|H_1 \cap H_2| = 4$; otherwise $H_1 \cap H_2 = \emptyset$. □

The previous 3 theorems combine to give this main result.

Theorem 5.3. *Consider $\pi = PG(2, F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and*

G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. If $|G_1 \cap G_2|$ is even and F does not contain a subfield of order 8, then $|H_1 \cap H_2|$ is even.

□

Section 5.2. 'Lifting' hyperconics in $PG(2,4)$ that meet in an odd number of points.

Theorem 5.7. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. If $|G_1 \cap G_2|$ is odd and F does not contain a subfield of order 8, then $|H_1 \cap H_2|$ is odd.*

We will prove this via 2 separate theorems, theorems 5.8 and 5.9, depending on whether $|G_1 \cap G_2|$ is 1 or 3.

Theorem 5.8. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 1$.*

1) Suppose $N_1 = N_2$, i.e., suppose C_1 and C_2 have common tangents. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 3$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

2) Suppose exactly one of N_1 and N_2 is on both G_1 and G_2. If F contains a subfield of order 256, then $|H_1 \cap H_2| = 5$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$.

3) Suppose N_1 and N_2 are not the common point of G_1 and G_2, and further that this common point is on a common tangent to both C_1 and C_2, i.e., the line N_1N_2 passes through $G_1 \cap G_2$. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 3$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

4) Suppose N_1 and N_2 are not the common point of G_1 and G_2, and further that this common point is not on a tangent to both C_1 and C_2, i.e., the line N_1N_2 does not pass through $G_1 \cap G_2$. If F contains a subfield of order 8, then $|H_1 \cap H_2| = 4$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

We will prove this via 4 separate lemmas, lemmas 5.11 through 5.14. Also, we need to look at coplanar hexagons that meet in exactly 3 points.

Theorem 5.9. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 3$.*

1) If $N_1 = N_2$ then $H_1 \cap H_2 = G_1 \cap G_2$.

2) Suppose $G_1 \cap G_2 = \{N_1, N_2, P\}$. If F contains a subfield of order 8, then $|H_1 \cap H_2| =$

6: otherwise $H_1 \cap H_2 = G_1 \cap G_2$.
3) Suppose exactly one of N_1 and N_2 is on both G_1 and G_2. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 5$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.
4) Suppose N_1 and $N_2 \notin G_1 \cap G_2$. Exactly one of the points of $G_1 \cap G_2$ must be on the common tangent to C_1 and C_2. Thus $H_1 \cap H_2 = G_1 \cap G_2$.

We will also prove this via 4 separate lemmas, lemmas 5.15 through 5.18.

First, we need to know what two hyperconics in $PG(2,4)$ look like when they meet in a single point.

In $PG(2,4)$, recall that two hyperconics are equivalent if they meet in an even number of points. This equivalence relation has 3 equivalence classes. Thus, given a fixed hyperconic G in $PG(2,4)$, there are 6 hyperconics from each of the 2 equivalence classes that do not contain G that intersect G only in the point P (see [Lander 1]). If we fix the hyperconic G and a point P on G, and pick an equivalence class not containing G, then we can consider the 6 hyperconics $D_1, \ldots, D_6$ such that $|G \cap D_i| = 1$, $i = 1, \ldots, 6$.

Proposition 5.10. *Let $\pi = PG(2,4)$. Let*

$$G = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\},$$

let

$$D_1 = \{(m = \infty), (m = 1), (0,1), (1,\omega^2), (\omega, 1), (\omega^2, \omega^2)\}$$
$$D_2 = \{(m = \infty), (m = 1), (0,\omega), (1,0), (\omega, \omega), (\omega^2, 0)\}$$
$$D_3 = \{(m = \infty), (m = \omega), (0,1), (1,0), (\omega, 0), (\omega^2, 1)\}$$
$$D_4 = \{(m = \infty), (m = \omega), (0,\omega^2), (1,\omega), (\omega, \omega), (\omega^2, \omega^2)\}$$
$$D_5 = \{(m = \infty), (m = \omega^2), (0,\omega^2), (1,\omega^2), (\omega, 0), (\omega^2, 0)\}$$
$$D_6 = \{(m = \infty), (m = \omega^2), (0,\omega), (1,\omega), (\omega, 1), (\omega^2, 1)\}$$

and let

$$E_1 = \{(m=\infty),(m=1),(0,1),(1,\omega),(\omega,\omega),(\omega^2,1)\}$$
$$E_2 = \{(m=\infty),(m=1),(0,\omega^2),(1,0),(\omega,0),(\omega^2,\omega^2)\}$$
$$E_3 = \{(m=\infty),(m=\omega),(0,\omega),(1,\omega),(\omega,0),(\omega^2,0)\}$$
$$E_4 = \{(m=\infty),(m=\omega),(0,\omega^2),(1,\omega^2),(\omega,1),(\omega^2,1)\}$$
$$E_5 = \{(m=\infty),(m=\omega^2),(0,\omega),(1,\omega^2),(\omega,\omega),(\omega^2,\omega^2)\}$$
$$E_6 = \{(m=\infty),(m=\omega^2),(0,1),(1,0),(\omega,1),(\omega^2,0)\}.$$

$D_1,\dots,D_6$ and $E_1,\dots,E_6$ are hyperconics in π that meet the hyperconic G only in the point $(m=\infty)$. $D_1,\dots,D_6$ are in the same equivalence class, $E_1,\dots,E_6$ are in a different equivalence class, and G is in the remaining equivalence class. Moreover, $D_1,\dots,D_6$ and $E_1,\dots,E_6$ are the only hyperconics intersecting G only in $(m=\infty)$.

Proof: Given a hyperconic, and a fixed point on that hyperconic, there are exactly 6 hyperconics in each of the other 2 equivalence classes that meet the given hyperconic only in the one fixed point. □

We now look at four cases where coplanar hexagon intersect in exactly one point.

Lemma 5.11. *Consider $\pi = PG(2,F)$, where F is a field containing a subfield $\{0,1,\omega,\omega^2\}$ of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose G_1 and G_2 have only one point in common, and that this point is the nucleus of both of the conics C_1 and C_2. If F does not contain a subfield of order 16, then $H_1 \cap H_2 = G_1 \cap G_2$; otherwise, F contains a subfield of order 16, and $|H_1 \cap H_2| = 3$.*

Proof: Let $\mathbf{F}_4 = \{0,1,\omega,\omega^2\}$ be the subfield of order 4 of F. Let $N_1 = (m=\infty)$. Let $G_1 = \{(m=\infty),(m=0),(0,0),(1,1),(\omega,\omega^2),(\omega^2,\omega)\}$. Thus $C_1 : Y^2 = XZ$. Then there are exactly 12 choices for G_2. Either G_2 must be one of $D_1,\dots,D_6$, or G_2 must be one of $E_1,\dots,E_6$ where $D_1,\dots,D_6$ and $E_1,\dots,E_6$ were defined in proposition 5.10. We will only consider $D_1,\dots,D_6$. $E_1,\dots,E_6$ can similarly be considered.
Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ for some $a,b,\dots,f \in \mathbf{F}_4$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. By proposition 4.34, $(Y^2,Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Now $d = f = 0$, and $e = 1$ since $N_2 = (m=\infty)$. Thus

$$p(t) = at^4 + (b+1)t^2 + c = (a^{\frac{1}{2}}t^2 + (b+1)^{\frac{1}{2}}t + c^{\frac{1}{2}})^2.$$

Case 1): $G_2 = D_1$.

Now $a = b$ since $(m = 0) \in G_2$. $b = c$ since $(0,1) \in G_2$, and $a + \omega b + c + 1 = 0$ since $(1, \omega^2) \in G_2$. Thus $a = b = c = \omega^2$, $d = f = 0$, and $e = 1$. Let $p_1(t) = t^2 + \omega t + 1$. Thus $p(t) = (\omega p_1(t))^2$. $p_1(t)$ has no roots in $\mathbf{F}_4$ and exactly 2 roots iff F contains a subfield of order 16.

Case 2): $G_2 = D_2$.

Now $a = b$ since $(m = 0) \in G_2$, $a + c = 1$ since $(1,0) \in G_2$, and $b\omega^2 = c$ since $(0, \omega) \in G_2$. Thus $a = b = \omega^2$, $c = \omega$, $d = f = 0$, and $e = 1$. Let $p_2(t) = t^2 + \omega t + \omega$. Thus $p(t) = (\omega p_2(t))^2$. $p_2(t)$ has no roots in $\mathbf{F}_4$ and exactly 2 roots iff F contains a subfield of order 16.

Case 3): $G_2 = D_3$.

Now $a = \omega^2 b$ since $(m = \omega) \in G_2$, $b = c$ since $(0,1) \in G_2$, and $a + c = 1$ since $(1,0) \in G_2$. Thus $a = \omega$, $b = c = \omega^2$, $d = f = 0$, and $e = 1$. Let $p_3(t) = t^2 + t + \omega^2$. Then $p(t) = (\omega^2 p_3(t))^2$. $p_3(t)$ has no roots in $\mathbf{F}_4$ and exactly 2 roots iff F contains a subfield of order 16.

Case 4): $G_2 = D_4$.

$a = \omega^2 b$ since $(m = \omega) \in G_2$, $\omega b = c$ since $(0, \omega^2) \in G_2$, and $a + \omega^2 b + c = 1$ since $(1, \omega) \in G_2$. Thus $a = \omega$, $b = \omega^2$, $c = 1$, $d = f = 0$, and $e = 1$. Let $p_4(t) = t^2 + t + \omega$. Then $p(t) = (\omega^2 p_4(t))^2$. $p_4(t)$ contains no roots in $\mathbf{F}_4$ and exactly 2 roots iff F contains a subfield of order 16.

Case 5): $G_2 = D_5$.

$a = \omega b$ since $(m = \omega^2) \in G_2$, $\omega b = c$ since $(0, \omega^2) \in G_2$, and $a + \omega b + c = 1$ since $(1, \omega^2) \in G_2$. Thus $a = \omega$, $b = \omega^2$, $c = 1$, $d = f = 0$, and $e = 1$. Let $p_5(t) = t^2 + t + \omega$. Then $p(t) = (\omega^2 p_5(t))^2$. $p_5(t)$ has no roots in $\mathbf{F}_4$ and exactly 2 roots iff F contains a subfield of order 16.

Case 6): $G_2 = D_6$.

$a = \omega b$ since $(m = \omega^2) \in G_2$, $\omega^2 b + c = 0$ since $(0, \omega^2) \in G_2$, and $a + \omega^2 b + c = 1$ since $(1, \omega) \in G_2$. Let $p_6(t) = t^2 + \omega^2 t + 1$. Then $p(t) = (p_6(t))^2$. $p_6(t)$ has no roots in $\mathbf{F}_4$ and exactly 2 roots iff F contains a subfield of order 16.

Therefore, by proposition 4.34, if F contains a subfield of order 16, then $|H_1 \cap H_2| = 3$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$. □

Lemma 5.12. *Consider $\pi = PG(2, F)$, where F is a field containing a subfield of* *order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $G_1 \cap G_2 = \{N_1\}$ but $N_2 \notin G_1$. If F contains a subfield of order 256, then $|H_1 \cap H_2| = 5$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of F of order 4. Let $N_1 = (m = \infty)$. Let $(m = 0)$ be the other point of G_1 on the line N_1N_2. Let $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$. Therefore $C_1 : Y^2 = XZ$. Then there are exactly 12 choices for G_2. G_2 must be one of $D_1, \ldots, D_6$ or one of $E_1, \ldots, E_6$ defined in proposition 5.10. We will consider $D_1, \ldots, D_6$; $E_1, \ldots, E_6$ can similarly be considered. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ for some $a, b, \ldots, f \in \mathbb{F}_4$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. Therefore, by proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. $b = 0$ since $(m = \infty) \in C_2$, and $d = 0$ and $f = 1$ since the nucleus of C_2 is on l_∞. Thus $C_2 : aX^2 + cZ^2 + eXZ + YZ = 0$ and $p(t) = at^4 + et^2 + t + c$.

Case 1) $G_2 = D_1$.

$e = 1$ since $N_2 = (m = 1)$, $c = 1$ since $(0,1) \in G_2$, and $a + c + e + \omega^2 = 0$ since $(1, \omega^2) \in G_2$. Thus $a = \omega^2$, $b = 0$, $c = 1$, $d = 0$, $e = 1$, and $f = 1$. Let $p_1(t) = t^4 + \omega t^2 + \omega t + \omega$. Thus $p(t) = \omega^2 p_1(t)$.

Case 2) $G_2 = D_2$.

$e = 1$ since $N_2 = (m = 1)$, $a + c + e = 0$ since $(1,0) \in G_2$, and $c = \omega$ since $(0, \omega) \in G_2$. Thus $a = \omega^2$, $b = 0$, $c = \omega$ $d = 0$, $e = 1$, and $f = 1$. Let $p_2(t) = t^4 + \omega t^2 + \omega t + \omega^2$. Thus $p(t) = \omega^2 p_2(t)$.

Case 3) $G_2 = D_3$.

$e = \omega$ since $N_2 = (m = \omega)$, $c = 1$ since $(0,1) \in G_2$, and $a + c + e = 0$ since $(1,0) \in G_2$. Thus $a = \omega^2$, $b = 0$, $c = 1$, $d = 0$, $e = \omega$, and $f = 1$. Let $p_3(t) = t^4 + \omega^2 t^2 + \omega t + 1$. Thus $p(t) = \omega^2 p_3(t)$.

Case 4) $G_2 = D_4$.

$e = \omega$ since $N_2 = (m = \omega)$, $c = \omega^2$ since $(0, \omega^2) \in G_2$, and $a + c + e + \omega = 0$ since $(1, \omega) \in G_2$. Thus $a = \omega^2$, $b = 0$, $c = \omega^2$, $d = 0$, $e = \omega$, and $f = 1$. Let $p_4(t) = t^4 + \omega^2 t^2 + \omega t + 1$. Thus $p(t) = \omega^2 p_4(t)$.

Case 5) $G_2 = D_5$.

$e = \omega^2$ since $N_2 = (m = \omega^2)$, $c = \omega^2$ since $(0, \omega^2) \in G_2$, and $a + c + e + \omega^2 = 0$

since $(1,\omega^2) \in G_2$. Thus $a = \omega^2$, $b = 0$, $c = \omega^2$, $d = 0$, $e = \omega^2$, and $f = 1$. Let $p_5(t) = t^4 + t^2 + \omega t + 1$. Thus $p(t) = \omega^2 p_5(t)$.

Case 6) $G_2 = D_6$.

$e = \omega^2$ since $N_2 = (m = \omega^2)$, $c = \omega^2$ since $(0,\omega) \in G_2$, and $a + c + e + \omega = 0$ since $(1,\omega) \in G_2$. Thus $a = \omega^2$, $b = 0$, $c = \omega$, $d = 0$, $e = \omega^2$, and $f = 1$. Let $p_6(t) = t^4 + t^2 + \omega t + \omega^2$. Thus $p(t) = \omega^2 p_6(t)$.

We now establish the following claim.

Claim 1: Either all or none of $p_1(t), \ldots, p_6(t)$ is irreducible over $\{0, 1, \omega, \omega^2\}$.

γ is a root of $p_1(t)$ iff $\gamma + 1$ is a root of $p_2(t)$. γ is a root of $p_3(t)$ iff $\gamma + \omega$ is a root of $p_4(t)$. γ is a root of $p_5(t)$ iff $\gamma + 1$ is a root of $p_6(t)$. γ is a root of $p_4(t)$ iff $\omega\gamma$ is a root of $p_1(t)$. γ is a root of $p_6(t)$ iff $\omega^2\gamma$ is a root of $p_1(t)$.

This establishes claim 1.

We now establish claim 2.

Claim 2: $p_1(t)$ is irreducible over $\{0, 1, \omega, \omega^2\}$.

There are no multiple roots (differentiating $p_1(t)$ yields ω).

Suppose

$$\begin{aligned} p_1(t) &= (t^2 + \gamma_1 t + \gamma_2)(t^2 + \gamma_3 t + \gamma_4) \\ &= t^4 + (\gamma_1 + \gamma_3)t^3 + (\gamma_4 + \gamma_2 + \gamma_1\gamma_3)t^2 + (\gamma_1\gamma_4 + \gamma_2\gamma_3)t + \gamma_2\gamma_4. \end{aligned}$$

Thus $\gamma_1 = \gamma_3$. Thus $\gamma_1 + \gamma_4 = \omega$, $\gamma_2\gamma_4 = \omega$. These equations have no solutions in $\{0, 1, \omega, \omega^2\}$.

This establishes claim 2.

Thus $p(t)$ is irreducible in all 6 cases. Therefore, by proposition 4.34, if F contains a subfield of order 256, then $|H_1 \cap H_2| = 5$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$. □

Lemma 5.13. *Consider $\pi = PG(2, F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 1$ but N_1 and N_2 are not the common point of G_1 and G_2. Suppose further that there is a common tangent to both C_1 and C_2 through this point, i.e., the line N_1N_2 passes through $G_1 \cap G_2$. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 3$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: Let $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of order 4 of F. Let $G_1 \cap G_2 = \{(m = \infty)\}$. Let $N_1 = (m = 0)$. Let $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$. Thus $C_1 : Y^2 = XZ$. There are exactly 12 choices for G_2. G_2 must be one of $D_1, \ldots, D_6$ or one of $E_1, \ldots, E_6$ defined in proposition 5.10. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ for some $a, b, \ldots, f \in \mathbf{F}_4$. Let $q(t) = bt^4 + dt^3 + (a + f)t^2 + et + c$. By a slight modification of proposition 4.34, $(X, X^2) \in C_1 \cap C_2 \setminus l_\infty$ iff $q(X) = 0$. Now $b = 0$ since $(m = \infty) \in C_2$, and $d = 0$, $f = 1$ since the nucleus of C_2 is on l_∞. Thus $C_2 : aX^2 + cZ^2 + eXZ + YZ = 0$ is identical to the conic in lemma 5.12 for corresponding cases. However, C_1 is different here. $C_1 : X^2 = YZ$. Thus we are using the polynomial

$$q(t) = bt^4 + dt^3 + (a + f)t^2 + et + c = (a + 1)t^2 + et + c$$

instead of $p(t)$.

Case 1) $G = D_1$. Thus, by lemma 5.12, $q(t) = \omega t^2 + t + 1$.

Case 2) $G = D_2$. $q(t) = \omega t^2 + t + \omega$.

Case 3) $G = D_3$. $q(t) = \omega t^2 + \omega t + 1$.

Case 4) $G = D_4$. $q(t) = \omega t^2 + \omega t + \omega^2$.

Case 5) $G = D_5$. $q(t) = \omega t^2 + \omega^2 t + \omega^2$.

Case 6) $G = D_6$. $q(t) = \omega t^2 + \omega^2 t + \omega$.

In each of these cases, $|H_1 \cap H_2| = 3$ iff F contains a subfield of order 16; and $H_1 \cap H_2 = G_1 \cap G_2$ iff F does not contain a subfield of order 16. □

Lemma 5.14. *Suppose $\pi = PG(2, F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 1$ but N_1 and N_2 are not the common point of G_1 and G_2. Suppose further that this common point is not on a tangent to both C_1 and C_2, i.e., the line N_1N_2 does not pass through $G_1 \cap G_2$. If F contains a subfield of order 8, then $|H_1 \cap H_2| = 4$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: We can choose coordinates so there are only 2 possibilities. We will choose coordinates for the following quadrangle in G_1: the nucleus N_1 of C_1; the common point P of G_1 and G_2; the point of $G_1 \setminus \{N_1\}$ on the line N_1N_2; and the point of $G_1 \setminus G_2$ on the line through N_2 and P. In this way, we know the coordinates of C_1, the nucleus N_2 of C_2 and $P \in C_2$. Thus, there will only be 2 choices for C_2.

Let $N_1 = (m = 0)$. Let $(m = \infty) = G_1 \cap G_2$. Let $\{(0,0), N_1\} = G_1 \cap (N_1N_2)$. Let $\{(1,1), (m = \infty)\} = G_1 \cap (m = \infty)N_2$. Therefore $N_2 = (X = 1) \cap (Y = 0) = (1,0)$. Thus $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$. Therefore $C_1 : X^2 = YZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, for some $a, b, \ldots, f \in \mathbf{F}_4$. Let $q(t) = bt^4 + dt^3 + (a+f)t^2 + et + c$. By a slight modification of proposition 4.34, $(X, X^2) \in C_1 \cap C_2 \setminus l_\infty$ iff $q(X) = 0$. Now $d = f = 1$, $e = 0$ since $N_2 = (1,0)$, and $b = 0$ since $(m = \infty)$ is on C_2. Thus $C_2 : aX^2 + cZ^2 + XY + YZ = 0$ and $q(t) = t^3 + (a+1)t^2 + c$. Since $N_2 = (1,0)$, only 2 of the 6 choices for G_2 amongst $D_1, \ldots, D_6$ defined in proposition 5.10 are possible ($E_1, \ldots, E_6$ can be similarly considered). G_2 must be either D_2 or D_3.

If $G_2 = D_2$ then $a = 1$ since $(m = 1) \in C_2$, and $c = \omega$ since $(0, \omega) \in C_2$. Thus $q(t) = t^3 + \omega$.

If $G_2 = D_3$ then $a = \omega$ since $(m = \omega) \in C_2$, and $c = 1$ since $(0,1) \in C_2$. Thus $q(t) = t^3 + \omega^2 t^2 + 1$.

In both cases, $|H_1 \cap H_2| = 4$ iff F contains a subfield of order 8; and $H_1 \cap H_2 = G_1 \cap G_2$ iff F does not contain a subfield of order 8. □

The previous 4 lemmas prove the following theorem.

Theorem 5.8. *Consider $\pi = PG(2, F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 1$.*

1) Suppose $N_1 = N_2$. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 3$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

2) Suppose exactly one of N_1 and N_2 is on both G_1 and G_2. If F contains a subfield of order 256, then $|H_1 \cap H_2| = 5$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$.

3) Suppose N_1 and N_2 are not the common point of G_1 and G_2 and further that this common point is on a common tangent to both C_1 and C_2, i.e., the line N_1N_2 passes through $G_1 \cap G_2$. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 3$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

4) Suppose N_1 and N_2 are not the common point of G_1 and G_2 and further that this common point is not a tangent to both C_1 and C_2, i.e., the line N_1N_2 does not pass through $G_1 \cap G_2$. If F contains a subfield of order 8, then $|H_1 \cap H_2| = 4$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

□

We now consider the 4 cases where 2 coplanar hexagons intersect in exactly 3 points.

Lemma 5.15. *Consider $\pi = PG(2,F)$ where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$, $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 3$ and $N_1 = N_2 \in G_1 \cap G_2$. Then $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of order 4 of F. Let $N_1 = N_2 = (m = \infty)$. Let $(m = 0)$ and $(0,0)$ be the other 2 points of $G_1 \cap G_2$. Let $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$. Thus $C_1 : Y^2 = XZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ for some $a, b, \ldots, f \in F$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. By proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Now $d = f = 0$ and $e = 1$ since $N_2 = (m = \infty)$, $a = 0$ since $(m = 0) \in C_2$, and $c = 0$ since $(0,0) \in C_2$. Thus $C_2 : bY^2 + XZ = 0$ and $p(t) = (b+1)t^2$. Therefore, by proposition 4.34, $H_1 \cap H_2 = G_1 \cap G_2$. □

Lemma 5.16. *Consider $\pi = PG(2,F)$ where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$, $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 3$ and $N_1, N_2 \in G_1 \cap G_2$. If F contains a subfield of order 8, then $|H_1 \cap H_2| = 6$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of order 4 of F. Let $N_1 = (m = \infty)$ and $N_2 = (m = 0)$. Let the remaining point of $G_1 \cap G_2$ be $(0,0)$. Suppose $(1,1) \in G_1$. Therefore $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$. Thus $C_1 : Y^2 = XZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ for some $a, b, \ldots, f \in \mathbb{F}_4$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. By proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Now $d = e = 0$, and $f = 1$ since $N_2 = (m = 0)$, $b = 0$ since $(m = \infty) \in C_2$, and $c = 0$ since $(0,0) \in C_2$. Thus $C_2 : aX^2 + YZ = 0$, $p(t) = at^4 + t = t(at^3 + 1)$. Also $a \neq 0, 1$ since C_2 is non-degenerate and $(1,1)$ is not in C_2. Therefore $at^3 + 1$ contains no roots in the subfield $\{0, 1, \omega, \omega^2\}$ of F. Thus $at^3 + 1$ has 3 solutions in F if F contains a subfield of order 8 and no solutions otherwise. Thus, by proposition 4.34, $|H_1 \cap H_2| = 6$ if F contains a subfield of order 8; otherwise $H_1 \cap H_2 = G_1 \cap G_2$. □

Lemma 5.17. *Consider $\pi = PG(2,F)$ where F is a field containing a subfield of* *order 4. Let $H_1 = C_1 \cup \{N_1\}$, $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 3$ and $N_1 \in G_1 \cap G_2$ but $N_2 \notin G_1 \cap G_2$. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 5$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: Let $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of order 4 of F. Let $N_1 = (m = \infty)$. Let the other 2 points of $G_1 \cap G_2$ be $(m = 0)$ and $(0,0)$. Let $G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}$. Thus $C_1 : Y^2 = XZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, for some $a, b, \ldots, f \in \mathbf{F}_4$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. By proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Now $b = 0$ since $(m = \infty) \in C_2$, $a = 0$ since $(m = 0) \in C_2$, $c = 0$ since $(0,0) \in C_2$, and $d = 1$ since $N_2 \notin l_\infty$. Thus $C_2 : XY + eXZ + fYZ = 0$ and $p(t) = t^3 + et^2 + ft = t(t^2 + et + f)$. Now $(m = \infty)$ and $(0,0)$ are on $X = 0$ and C_2. Thus $(0,1) \notin C_2$. Thus $f \neq 0$. Also $(m = 0)$ and $(0,0)$ are on $Y = 0$ and C_2. Thus $(1,0) \notin C_2$. Therefore $e \neq 0$. Also, $1 + e\omega + f\omega^2 \neq 0$ since $(\omega^2, \omega) \in C_1 \setminus C_2$, and $1 + e\omega^2 + f\omega \neq 0$ since $(\omega, \omega^2) \in C_1 \setminus C_2$. Therefore $t^2 + et + f$ has 2 solutions in F iff F contains a subfield of order 16; otherwise, $t^2 + et + f$ has no solutions in F. Therefore, by proposition 4.34, if F contains a subfield of order 16, then $|H_1 \cap H_2| = 5$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$. □

Lemma 5.18. *Consider $\pi = PG(2, F)$ where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$, $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. If $|G_1 \cap G_2| = 3$ and $N_1, N_2 \notin G_1 \cap G_2$ then $H_1 \cap H_2 = G_1 \cap G_2$.*

Proof: Let $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$ be the subfield of order 4 of F. Let $N_1 = (m = \infty)$. Let $\{(m = 0), (0,0), (1,1)\} = G_1 \cap G_2$. Thus

$$G_1 = \{(m = \infty), (m = 0), (0,0), (1,1), (\omega, \omega^2), (\omega^2, \omega)\}.$$

Therefore $C_1 : Y^2 = XZ$. Now $C_2 : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$, for some $a, b, \ldots, f \in \mathbf{F}_4$. Let $p(t) = at^4 + dt^3 + (b+e)t^2 + ft + c$. By proposition 4.34, $(Y^2, Y) \in C_1 \cap C_2 \setminus l_\infty$ iff $p(Y) = 0$. Now $a = 0$ since $(m = 0) \in C_2$, $c = 0$ since $(0,0) \in C_2$, and $a + b + \cdots + f = 0$ since $(1,1) \in C_2$. Therefore $C_2 : bY^2 + dXY + eXZ + fYZ = 0$ and $p(t) = dt^3 + (b+e)t^2 + ft = t(dt^2 + (b+e)t + f)$ and $b + d + e + f = 0$. Note that the line N_1N_2 meets exactly one of the 3 points on both G_1 and G_2.

Suppose first that $(m = 0)$ is on N_1N_2. Thus N_2 is on l_∞. Therefore $f = 1$ and $d = 0$. Therefore $C_2 : bY^2 + (1+b)XZ + YZ = 0$ and $p(t) = t(t+1)$. Thus $C_1 \cap C_2 = G_1 \cap G_2$.

Suppose next that $(0,0)$ is on N_1N_2. Therefore N_2 is on $X = 0$. Therefore $f = 0$ and $d = 1$ and thus $0 = b+d+e+f = b+e+1$. Therefore $C_2 : bY^2 + XY + eXZ = 0$ and $p(t) = t^2 + (b+e)t^2 = t^2(t + (b+e))$ and $b + e = 1$. Therefore $C_1 \cap C_2 = G_1 \cap G_2$.

Lastly, suppose that $(1,1)$ is on N_1N_2. Therefore N_2 is on $X = 1$. Therefore $f = 1$ and $d = 1$ and thus $0 = b+d+e+f = b+e$. Therefore $C_2 : bY^2 + XY + eXZ + YZ = 0$ and $p(t) = t^3 + t = t(t+1)^2$. Therefore $C_1 \cap C_2 = G_1 \cap G_2$.

Therefore, by proposition 4.34, $H_1 \cap H_2 = G_1 \cap G_2$. □

The previous 4 lemmas prove the following theorem:

Theorem 5.9. *Consider $\pi = PG(2, F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. Suppose $|G_1 \cap G_2| = 3$.*

1) If $N_1 = N_2$, then $H_1 \cap H_2 = G_1 \cap G_2$.

2) Suppose $G_1 \cap G_2 = \{N_1, N_2, P\}$. If F contains a subfield of order 8, then $|H_1 \cap H_2| = 6$; otherwise $H_1 \cap H_2 = G_1 \cap G_2$.

3) Suppose exactly one of N_1 and N_2 is on both G_1 and G_2. If F contains a subfield of order 16, then $|H_1 \cap H_2| = 5$; otherwise, $H_1 \cap H_2 = G_1 \cap G_2$.

4) Suppose N_1 and $N_2 \notin G_1 \cap G_2$. Exactly one of the points of $G_1 \cap G_2$ must be on the common tangent to C_1 and C_2. Then $H_1 \cap H_2 = G_1 \cap G_2$.

□

Theorems 5.8 and 5.9 combine to give us our main result:

Theorem 5.7. *Consider $\pi = PG(2, F)$, where F is a field containing a subfield of order 4. Let $H_1 = C_1 \cup \{N_1\}$ and $H_2 = C_2 \cup \{N_2\}$ be hyperconics in π. Let G_1 and G_2 be coplanar hexagons satisfying $G_1 \subset H_1$, $G_2 \subset H_2$. If $|G_1 \cap G_2|$ is odd and F does not contain a subfield of order 8, then $|H_1 \cap H_2|$ is odd.*

□

Section 5.3. A generalization of the 'even intersection' property of a projective plane of order 4.

The main results of the previous 2 sections, theorems 5.3 and 5.7, combine to give us the following results regarding a lifting of the even intersection property of hyperconics in $PG(2,4)$.

Theorem 5.19. *Let $\pi = PG(2,F)$, where F is a field containing a subfield of order 4 but containing no subfield of order 8. Given any 2 hyperconics, if there is a $PG(2,4)$-subplane intersecting each of the hyperconics in 6 points, i.e., a $PG(2,4)$-subplane containing a hexagon from each hyperconic, then either both the 2 hyperconics intersect evenly and the 2 hexagons intersect evenly; or, both hyperconics intersect in an odd number of points, and both hexagons intersect in an odd number of points.*

□

The following theorem is a generalization of the even intersection property of hyperconics in $PG(2,4)$.

Theorem 5.20. *Let $\pi = PG(2,F)$, where F is a field containing a subfield of order 4 but containing no subfield of order 8. Given any subplane of π of order 4, each of the 168 hyperconics in the subplane are contained in exactly 6 hyperconics in π. These $168 \cdot 6$ hyperconics in π are distinct. Define a relation $\sim$ on these hyperconics by two hyperconics are related if they meet in an even number of points. Then $\sim$ is an equivalence relation, i.e., we have an extension of the 'even intersection' property of the hyperconics in $PG(2,4)$.*

□

Appendix

In $\pi = PG(2,q)$, $q = 2^t$, t even, where $q \geq 16$, we have the following.

$$\#\text{ points in } \pi = q^2 + q + 1 = \#\text{ lines in } \pi$$

$$\#\text{points/line} = q + 1 = \#\text{lines/point}$$

$$\#\text{hyperconics} = q^2(q^3 - 1)$$

$$\#\text{4-arcs} = \frac{(q^2+q+1)(q^2+q)q^2(q^2-2q+1)}{(4)(3)(2)}$$

$$\#\text{hexagons} = \frac{\#\text{4-arcs}}{\binom{6}{4}}$$

$$\#\text{triples/conic} = \frac{(q+1)q(q-1)}{(3)(2)}$$

$$\#\text{hexagons/hyperconic} = \frac{\#\text{triples/conic}}{\binom{5}{3}}$$

$$\#\text{hyperconics/hexagon} = 6$$

$$\#PG(2,4)\text{-subplanes} = \frac{\#\text{hexagons}}{168}.$$

In $PG(2,4)$,

$$\#\text{hyperconics} = 168$$

$$\text{and } \#\text{hyperconics/equivalence class} = 56.$$

In a fixed equivalence class of the hyperconics in $PG(2,4)$,

$$\#\text{hyperconics/pair of points} = 4$$

$$\#\text{hyperconics/point} = 16$$

$$\#\text{hyperconics skew to a line} = 16$$

$$\#\text{hyperconics intersecting a line} = 56 - 16 = 40$$

$$\#\text{hyperconics/3-arc} = 1$$

$$\#\text{hyperconics intersecting a fixed hyperconic} = 1 + 3\binom{6}{2} = 46$$

$$\#\text{hyperconics skew to a given hyperconic} = 56 - 46 = 10.$$

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