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by

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ABSTRACT

The problem of studying the behaviour of a fluid moving past a body constitutes a classical area of research in fluid dynamics. In this work, the unsteady and steady flow of a viscous, incompressible fluid past a flat plate, situated normal to the flow and started impulsively from rest, is considered. The Navier-Stokes equations and the continuity equation are formulated in terms of the streamfunction and the vorticity.

For the case of flow in an unbounded region, transformational and perturbational techniques are used to obtain exact solutions for initial and small time values. These solutions are valid for large values of the Reynolds numbers.

Finite-differencing techniques are employed in solving the problem of channel-contained steady flow of a viscous, incompressible fluid past an impulsively-started flat plate. These solutions were found to be in close agreement with experimental work done on the same problem. Calculations were carried out for small values of the Reynolds number, ranging from 5 to 20.

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CHAPTER 1

FORMULATION OF THE GENERAL PROBLEM

1.1 INTRODUCTION

The problem of fluid flow past a body constitutes a classical area of research in fluid dynamics. Since the formulation in the last century of the Navier-Stokes equations, which mathematically describe fluid flow, investigators have responded to the intractability of these highly non-linear differential equations by fixing certain flow parameters and specifying the flow geometry. Indeed, the resulting patchwork of successively better approximations to simplified flow equations has marked the development of the subject of fluid dynamics. The problem to be considered in this study is that of viscous flow of an incompressible fluid past an impulsively-started flat plate normal to the direction of flow.

In ideal fluid theory, the viscosity is assumed to be identically zero, and it is predicted that the fluid will slip along a boundary. However, this theory is in disagreement with experimental observations in that the fluid immediately adjacent to the boundary is in fact at rest. One finds that for fluids with extremely small viscosity, the viscous forces are usually negligible compared to the inertial forces, except in limited regions of flow where they are locally very important. One such region is the thin layer adjacent to the body -- termed the "boundary layer" (by L. Prandtl) in 1904.

Prandtl noted that the order of the inviscid flow equations is less than the order of the viscous flow equations. It follows that a solution of the inviscid equations will be incapable of simultaneously satisfying all the boundary conditions. To resolve this difficulty, Prandtl postulated a two-region theory. The first region is that of a thin boundary layer immediately adjacent to the solid/fluid interface. In this region, the tangential velocity changes from being zero on the boundary to attaining a freestream value at the outer edge. The second region, external to the boundary layer, is then considered to be inviscid in nature.

A further advantage offered by the two-region theory is that an order analysis of the terms in the full Navier Stokes equations, based on the thinness of the boundary layer, reduces these equations significantly to the simpler Prandtl boundary-layer equations. These equations are solved in the boundary layer subject to the conditions of no-slip on the solid/fluid interface and potential flow on the edge of the boundary layer.

In 1908, Blasius used the boundary layer equations in considering the problem of fluid flow past an impulsively started cylinder. He calculated the initial flow in terms of the first two terms of a series in powers of time. Goldstein & Rosenhead (1936) later extended these results by obtaining the third term in the power series expansion.

Subsequent to these classical works, various investigators have further expanded our collective understanding of the theoretical approach to flow past impulsively started bodies by focusing on particular cross-sections of the bodies. In 1967, Wang considered symmetrical flow past a circular cylinder and asymmetrical flow past an elliptic cylinder. Collins and Dennis (1971) studied the initial flow past a circular cylinder. Viscous flow past a parallel flat plate was tackled by Dennis and Dunwoody in 1966.

Due to the non-linearity of the Navier-Stokes equations, rigorous theoretical treatment of the general problem is still out of reach, while treatment of specific flow configurations embodies a fascinating, but limited, body of work. While numerical approaches seem to be most practical in attempting to obtain solutions, one serious drawback in these approaches is their failure to accurately describe the initial flow profile. Such errors may then propagate to yield inaccuracies in the description of the flow at subsequent times.

One notes that the problem of flow past an impulsively started body involves a singularity in the vorticity at time $t = 0$. Indeed, from boundary layer theory, it is known that the vorticity on the surface of the cylinder is proportional to $\left(\frac{R}{t}\right)^{1/2}$ while the boundary layer thickness is proportional to $\left(\frac{t}{R}\right)^{1/2}$. Thus at $t = 0$, an infinitesimally thin ring of infinite vorticity is generated at the cylinder surface. It is difficult to incorporate this circumstance into numerical procedures.

Groundbreaking work by Collins and Dennis (1971) and Dennis and Staniforth (1971) employed boundary layer transformations to the full Navier Stokes equations which virtually eliminated the singularity inherent in the equations at $t = 0$.

One motivation of the present work is to apply the careful and rigorous techniques of Collins, Dennis and Staniforth to the yet unsolved problem of flow past an impulsively-started flat plate normal to the direction of flow. Indeed, a theoretical analysis has been carried out to understand the nature of the initial flow of a viscous, incompressible fluid past an impulsively started normal flat plate and an impulsively started thin ellipse. These results are valid for very large values of the Reynolds number.

As well, a numerical study of the steady state solution of flow past an impulsively started normal flat plate has been undertaken, and the results have been compared with existing numerical studies and experimental work. In particular, Hudson and Dennis (1985) had published work on the steady state flow past such a plate in an open-field. Theirs was a strictly numerical study, based on flow equations given in terms of the primary variables. By contrast, this study employs a streamfunction-vorticity formulation of the governing equations, and we consider flow in a channel. Experimental work published by Coutanceau and Launay (1993) offers data on channel flow with which our results are compared. This segment of the work was carried out for small values of the Reynolds number, $5 \leq R \leq 20$.

1.2 BASIC EQUATIONS GOVERNING THE FLOW

For a viscous, incompressible fluid, the Navier-Stokes equations can be written in the usual dimensionless form

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla p + \frac{2}{R} \nabla^2 \vec{v} \tag{1.2.1}$$

where $\vec{v}$ and p are, respectively, the dimensionless velocity vector and the pressure, t is the dimensionless time and R is the Reynolds number based on the reference length and reference velocity in the flow field.

Definitions of these quantities in terms of dimensional quantities are given in Appendix I. Equation (1.2.1) together with the equation of continuity

$$\text{div } \vec{v} = 0, \tag{1.2.2}$$

and the appropriate boundary conditions are assumed to govern the flow.

Eliminating the pressure term from the linear momentum equations of (1.2.1), and introducing the vorticity vector function, $\vec{\omega}$, we have the flow being governed by the following system of equations (see Appendix I):

$$\vec{\omega} = \text{curl } \vec{v} \tag{1.2.3}$$

$$\frac{\partial \vec{\omega}}{\partial t} - \text{curl}(\vec{v} \times \vec{\omega}) = -\frac{2}{R} \text{curl}(\text{curl } \vec{\omega}) \tag{1.2.4}$$

$$\text{div } \vec{v} = 0 \qquad (1.2.2)$$

The above equations together with appropriate boundary conditions on $\vec{v}$ and $\vec{\omega}$ are taken to govern the flow.

For two-dimensional motion, $\vec{v} = (u,v)$ where u and v are the components in the Cartesian-defined (x,y) directions, respectively. It is then possible to introduce the stream function $\psi(x,y,t)$ from the conservation of mass equation (1.2.2):

$$u(x,y,t) = \frac{\partial \psi}{\partial y} \qquad v(x,y,t) = -\frac{\partial \psi}{\partial x} \tag{1.2.5}$$

We further observe that the vorticity vector function, defined as the curl of the velocity $\vec{v}$, will have only one non-zero component:

$$\vec{\omega}(x,y,t) = \left(0,0,\zeta(x,y,t)\right) \tag{1.2.6}$$

Employing equations (1.2.5) and (1.2.6) in equations (1.2.3) and (1.2.4), we obtain the following governing equations, in terms of the streamfunction, $\psi(x,y,t)$, and the scalar vorticity function, $\zeta(x,y,t)$:

$$\frac{\partial \zeta}{\partial t} = \frac{2}{R} \nabla^2 \zeta + \left(\frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x}\right) \tag{1.2.7}$$

$$\nabla^2 \psi + \zeta = 0 \tag{1.2.8}$$

We shall refer to this formulation of the problem in terms of $\psi(x,y,t)$ and $\zeta(x,y,t)$ as System (I), and shall now seek the boundary conditions to be prescribed on these dependent variables.

We consider flow past a normal flat plate which is started impulsively from rest at $t = 0$ and which continues to move with dimensionless velocity 1 along the direction of the x-axis. The fluid at large enough distances from the plate is assumed to remain undisturbed. In actuality, we adopt the following equivalent formulation of the problem. For $t < 0$, the plate and the fluid are moving at velocity 1 in the direction of the positive x-axis. At $t = 0$, the plate is immediately brought to rest. The fluid at large enough distances from the plate is assumed to be moving with uniform velocity, $\vec{v} = 1\,\vec{i}$ for all time t.

In defining the physical domain of the problem, we can consider two separate cases: flow through a channel and flow in an open-field. By defining the parameter λ (blockage ratio) to be the ratio of the plate length to the channel width, the case of flow in an open-field may be thought of merely as a limiting case of flow through a channel ($\lambda \rightarrow 0$). Furthermore, taking the position at which the flat plate is brought to rest to be coincident with the y-axis with the centre point on the plate being y = 0, we find that our problem is symmetric with respect to the x-axis, and hence we need only consider one-half the domain. This problem is illustrated in Fig. I.1a. The problem domain for the case of flow in an open-field is illustrated in Figure I.1b.

We consider, for the present, flow in an open-field. The boundary conditions for the dimensionless vorticity and stream functions re:

$t < 0$: $\psi = y$ throughout the flow field.

$t \geq 0$: (i) $\psi = \dfrac{\partial \psi}{\partial x} = 0$ along the plate (no-slip condition),

(ii) $\dfrac{\partial \psi}{\partial y} \rightarrow 1$, $\dfrac{\partial \psi}{\partial x} \rightarrow 0$ far from the plate,

(iii) $\zeta \rightarrow 0$ far from the plate,

and (iv) $\psi = 0$, $\zeta = 0$ along the x-axis

(anti-symmetric condition). **(1.2.9)**

System (I) must be solved subject to conditions (1.2.9).

To facilitate a more feasible computational domain and also to attempt management of the vorticity singularity at the top edge of the flat plate, it is advantageous to consider the alternate curvilinear system of elliptical cylindrical coordinates (see Appendix II). The relationship between the elliptical cylindrical coordinate system, and the cartesian system is given by the conformal transformation equations:

$$x = \sinh\xi \cos\eta \quad \text{and} \quad y = \cosh\xi \sin\eta. \tag{1.2.10}$$

The use of such transformations is well established in the literature. Furthermore, under the transformation (1.2.10), the domain of our problem becomes a semi-infinite strip spanning a width of π in the η-direction. The transformed domain is illustrated in Figure I.2.

Under the transformation (1.2.10), equations (1.2.7) and (1.2.8) of System (I) become respectively

$$M^2 \frac{\partial\zeta}{\partial t} = \frac{2}{R}\left[\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2}\right] + \left[\frac{\partial\zeta}{\partial\eta}\frac{\partial\psi}{\partial\xi} - \frac{\partial\zeta}{\partial\xi}\frac{\partial\psi}{\partial\eta}\right] \tag{1.2.11}$$

and

$$M^2 \zeta = -\left[\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}\right] \tag{1.2.12}$$

where

$$M^2 = \frac{1}{2}[\cos 2\eta + \cosh 2\xi]$$

Likewise, the boundary conditions given by (1.2.9) are transformed to the following set of conditions:

$t < 0$: $\psi = \cosh \xi \sin \eta$ $\forall (\xi, \eta) \in D'$.

$t \geq 0$: (i) $\psi = \dfrac{\partial \psi}{\partial \xi} = 0$ along the plate.

(ii) $\psi \to \cosh \xi \sin \eta$ as $\xi \to \infty$.

(iii) $\zeta \to 0$ as $\xi \to \infty$.

and (iv) $\left.\begin{array}{l} \psi = 0 \\ \zeta = 0 \end{array}\right\}$ along $\eta = 0$ and $\eta = \pi$, $\forall \xi$. **(1.2.13)**

We shall refer to the formulation of our flow problem involving equations (1.2.11) and (1.2.12), together with the boundary and initial conditions given in (1.2.13), as System (II).

CHAPTER II

SINE SERIES EXPANSIONS OF THE VORTICITY AND STREAMFUNCTION

2.1 THE METHOD OF SERIES TRUNCATION: OBTAINING A GLOBAL CONDITION

It is assumed that the streamfunction $\psi(\xi, \eta, t)$ and the vorticity $\zeta(x, \eta, t)$ can be represented by the following sine series expansions.

$$\psi(\xi,\eta,t) = \sum_{n=1}^{\infty} f_n(\xi,t)\sin n\eta \tag{2.1.1}$$

$$\zeta(\xi,\eta,t) = \sum_{n=1}^{\infty} g_n(\xi,t)\sin n\eta \tag{2.1.2}$$

Employing the above expansions in governing equations (1.2.11) and (1.2.12), we obtain the differential equations for the functions $f_K(\xi, t)$, $g_K(\xi, t)$ given by

$$\frac{\partial^2 f_K}{\partial \xi^2} - K^2 f_K = -\frac{1}{4}\{2 \cosh 2\xi \; g_K + g_{K-2} + g_{K+2} - g_{2-K}\} \tag{2.1.3}$$

and

$$2 \cosh 2\xi \frac{\partial g_K}{\partial t} + \frac{\partial g_{K-2}}{\partial t} + \frac{\partial g_{K+2}}{\partial t} - \frac{\partial g_{2-K}}{\partial t}$$

$$= \frac{8}{R}\left\{\frac{\partial^2 g_K}{\partial \xi^2} - K^2 g_K\right\} + S_K(\xi,t) \tag{2.1.4}$$

where

$$S_K(\xi,t) = 2\sum_{m=1}^{\infty} m\left\{g_m\left[\frac{\partial f_{K-m}}{\partial \xi} + \frac{\partial f_{K+m}}{\partial \xi} - \frac{\partial f_{m-K}}{\partial \xi}\right] - f_m\left[\frac{\partial g_{K-m}}{\partial \xi} + \frac{\partial g_{K+m}}{\partial \xi} - \frac{\partial g_{m-K}}{\partial \xi}\right]\right\}$$

In the above, functions with negative subscripts are taken to be zero. The boundary conditions for (2.1.3) and (2.1.4) follow from those given in (1.2.13). They are given by

i. $f_K(0, t) = \frac{\partial f_K}{\partial \xi}(0, t) = 0 \qquad \forall\, k \in \mathbf{N},$

ii. As $\xi \to \infty$, $g_K \to 0 \qquad \forall\, k \in \mathbf{N},$

and iii. As $\xi \to \infty$,

$$\begin{cases} 2e^{-\xi} f_K \to \delta_{K,1} \\ 2e^{-\xi} \dfrac{\partial f_K}{\partial \xi} \to \delta_{K,1} \end{cases}$$

where

$$\delta_{K,1} = \begin{cases} 1 & ; \quad K = 1 \\ 0 & ; \quad K \neq 1 \end{cases} \qquad (2.1.5)$$

In considering the boundary conditions on $f_K(\xi, t)$ and $g_K(\xi, t)$, we note that while there is only one condition imposed on g_K, an excess of conditions are imposed on f_K. We attempt to find an integral or global condition on $g_K(\xi, t)$ by employing the excess conditions on $f_K(\xi, t)$. Dennis and Quartapelle (1989) have documented the theory behind, and the manner in which, global conditions are sought.

Multiplying equation (2.1.3) by $e^{-K\xi}$, and integrating the resulting equation wrt ξ from ξ

= 0 to $\xi \to \infty$, we deduce that

$$-\frac{1}{4}\int_0^\infty e^{-K\xi}\{2\cosh 2\xi\, g_K + g_{K-2} + g_{K+2} - g_{2-K}\}\, d\xi$$

$$= \left\{ e^{-K\xi}\frac{\partial f_K}{\partial \xi} + Ke^{-K\xi} f_K \right\}_{\xi=0}^{\xi\to\infty}$$

$$= \frac{1}{2}(1+K)\delta_{K,1}$$

That is, the desired integral condition is

$$\int_0^\infty e^{-K\xi}\{2\cosh 2\xi\, g_K + g_{K-2} + g_{K+2} - g_{2-K}\}\, d\xi = -4\delta_{K,1} \qquad (2.1.6)$$

Equations (2.1.3) and (2.1.4), together with boundary conditions (2.1.5i,ii) and integral condition (2.1.6) shall be referred to as System (III).

2.2 BOUNDARY-LAYER TRANSFORMATIONS

It is known from boundary-layer theory that in the initial boundary layer after an impulsive start, the boundary-layer thickness is proportional to $(t/R)^{1/2}$. Furthermore, the vorticity ζ and streamfunction ψ are proportional to $(t/R)^{-1/2}$ and $(t/R)^{1/2}$ respectively. Hence, to study the motion of fluid in the region near the flat plate, the boundary-layer transformations together with the appropriate scaling of the dependent functions take the form:

$$\xi = \lambda x \ , \ F_n(x,t) = \frac{1}{\lambda} f_n(\xi,t),$$

$$G_n(x,t) = \lambda g_n(\xi,t) \tag{2.2.1}$$

where

$$\lambda = 2\left(\frac{2t}{R}\right)^{1/2}.$$

Under the transformation (2.2.1), the equations (2.1.3) and (2.1.4) take the forms

$$\frac{\partial^2 F_K}{\partial x^2} - K^2\lambda^2 F_K = -\frac{1}{4}\{2 \cosh(2\lambda x) G_K + G_{K-2} + G_{K+2} - G_{2-K}\} \tag{2.2.2}$$

and

$$\left\{2\left[1+\frac{2^2x^2}{2!}\left(\frac{8t}{R}\right)+\frac{2^4x^4}{4!}\left(\frac{8t}{R}\right)^2+\frac{2^6x^6}{6!}\left(\frac{8t}{R}\right)^3+\ldots\right]\right.$$

$$\cdot\left[2t\frac{\partial G_K}{\partial t}-x\frac{\partial G_K}{\partial x}-G_K\right]$$

$$+\left[2t\frac{\partial G_{K-2}}{\partial t}-x\frac{\partial G_{K-2}}{\partial x}-G_{K-2}\right]+\left[2t\frac{\partial G_{K+2}}{\partial t}-x\frac{\partial G_{K+2}}{\partial x}-G_{K+2}\right]$$

$$\left.-\left[2t\frac{\partial G_{2-K}}{\partial t}-x\frac{\partial G_{2-K}}{\partial x}-G_{2-K}\right]\right\}$$

$$=2\left[\frac{\partial^2 G_K}{\partial x^2}-k^2\frac{8t}{R}G_K\right]$$

$$+4t\sum_{m=1}^{\infty} m\left\{G_m\left[\frac{\partial F_{K-m}}{\partial x}+\frac{\partial F_{K+m}}{\partial x}-\frac{\partial F_{m-K}}{\partial x}\right]\right.$$

$$\left.-F_m\left[\frac{\partial G_{K-m}}{\partial x}+\frac{\partial G_{K+m}}{\partial x}-\frac{\partial G_{m-K}}{\partial x}\right]\right\}. \qquad (2.2.3)$$

Equations (2.2.2) and (2.2.3) govern the scaled coefficient functions, $F_K(x, t)$ and $G_K(x, t)$, of the sine series expansions of the streamfunction and scalar vorticity, respectively. The boundary and initial conditions to be imposed on F_K and G_K are obtained by applying transformations (2.2.2) on conditions (2.1.5i), (2.1.5ii) and (2.1.6). We have

i. $F_K(0, t) = \frac{\partial F_K}{\partial x}(0, t) = 0,$

ii. $G_K(x, t) \to 0$ as $x \to \infty,$

and iii. $$\int_0^{\infty} e^{-t\lambda x}\{2\cosh 2\lambda x\; G_K + G_{K-2} + G_{K+2} - G_{2-K}\}\,dx = -4\delta_{K,1} \qquad (2.2.4)$$

where $K \varepsilon N$.

Hence, the flow inside the boundary layer may be determined by solving equations (2.2.2) and (2.2.3), subject to conditions (2.2.4). We shall refer to this formulation as System (IV).

Taking $t = 0$ in equations (2.2.2), (2.2.3) and in the conditions (2.2.4), we deduce the formulation (System (V)) which governs the initial flow.

$$\frac{\partial^2 F_K}{\partial x^2} = -\frac{1}{2} G_K - \frac{1}{4}\left[G_{K-2} + G_{K+2} - G_{2-K}\right] \qquad (2.2.5)$$

and

$$2\left[\frac{\partial^2 G_K}{\partial x^2} + x\frac{\partial G_K}{\partial x} + G_K\right] = -\left\{\left[x\frac{\partial G_{K-2}}{\partial x} + G_{K-2}\right] + \left[x\frac{\partial G_{K+2}}{\partial x} + G_{K+2}\right] - \left[x\frac{\partial G_{2-K}}{\partial x} + G_{2-K}\right]\right\} \qquad (2.2.6)$$

govern the initial flow, subject to:

i. $F_K(0, 0) = \dfrac{\partial F_K}{\partial x}(0, 0) = 0,$

ii. $G_K(x, 0) \to 0$ as $x \to \infty$, and **iii.** $$\int_0^\infty \{2\,G_K + G_{K-2} + G_{K+2} - G_{2-K}\}\,dx = -4\,\delta_{K,1} \qquad (2.2.7)$$

for all k ε **N**.

CHAPTER III

ANALYTICAL CONSIDERATIONS FOR INITIAL AND SMALL-TIME FLOW OF A VISCOUS, INCOMPRESSIBLE FLUID PAST AN IMPULSIVELY STARTED FLAT PLATE

3.1 FORMULATION OF THE PROBLEM IN THE BOUNDARY-LAYER REGION

The equations for unsteady, viscous, incompressible flow, expressed in terms of the vorticity function, $\zeta(\xi,\eta,t)$ and the streamfunction, $\psi(\xi,\eta,t)$ are

$$\frac{1}{2}[\cos 2\eta + \cosh 2\xi]\,\frac{\partial\zeta}{\partial t} = \frac{2}{R}\left[\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2}\right] + \left[\frac{\partial\zeta}{\partial\eta}\frac{\partial\psi}{\partial\xi} - \frac{\partial\zeta}{\partial\xi}\frac{\partial\psi}{\partial\eta}\right] \qquad (3.1.1)$$

and

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = -\frac{1}{2}[\cos 2\eta + \cosh 2\xi]\zeta, \qquad (3.1.2)$$

where the independent variables ξ and η form the elliptical cylindrical coordinat. .iet. As noted earlier, the boundary conditions on ζ and ψ are as follows:

i Along the flat plate ($\xi = 0$):

$$\psi = \frac{\partial\psi}{\partial\xi} = 0\ ,$$

and

ii Far from the plate ($\xi \to \infty$):

$$\psi \to \cosh\xi \sin\eta \ ,$$
$$\frac{\partial\psi}{\partial\xi} \to \sinh\xi \sin\eta \ ,$$

and $$\zeta \to 0 \ .$$

From boundary-layer theory it is known that after an impulsive start of a body, in the initial boundary-layer:

$$\lambda \ \alpha \left(\frac{t}{R}\right)^{1/2} ,$$

$$\zeta \ \alpha \left(\frac{t}{R}\right)^{-1/2} ,$$

and $$\psi \ \alpha \left(\frac{t}{R}\right)^{1/2} ,$$

where λ is the boundary-layer thickness.

Accordingly, we apply the following boundary-layer transformations wherein the vorticity and streamfunction have been scaled appropriately:

$$\left.\begin{aligned} &\xi = \lambda x, \quad \psi(\xi,\eta,t) = \lambda\Psi(x,\eta,t) \\ &\zeta(\xi,\eta,t) = \frac{1}{\lambda}\,\Omega(x,\eta,t) \\ &\text{where} \\ &\lambda = 2\left(\frac{2t}{R}\right)^{1/2}. \end{aligned}\right\} \qquad (3.1.3)$$

Under the transformation (3.1.3) the equations (3.1.1) and (3.1.2) take the forms

$$\left(\frac{\partial^2\Omega}{\partial x^2} + \lambda^2\frac{\partial^2\Omega}{\partial\eta^2}\right) + [\cos 2\eta + \cosh 2\lambda x]\left(x\frac{\partial\Omega}{\partial x} + \Omega\right)$$

$$= 2t\left\{[\cos 2\eta + \cosh 2\lambda x]\frac{\partial\Omega}{\partial t} - 2\left[\frac{\partial\Omega}{\partial\eta}\frac{\partial\Psi}{\partial x} - \frac{\partial\Omega}{\partial x}\frac{\partial\Psi}{\partial\eta}\right]\right\} \tag{3.1.4}$$

and

$$\frac{\partial^2\Psi}{\partial x^2} + \lambda^2\frac{\partial^2\Psi}{\partial\eta^2} = -\frac{1}{2}[\cos 2\eta + \cosh 2\lambda x]\Omega. \tag{3.1.5}$$

Along the flat plate, the transfor ed no-slip boundary condition takes the form

$$\Psi = \frac{\partial\Psi}{\partial x} = 0 \qquad \text{at} \qquad \xi = 0. \tag{3.1.6a}$$

Far from the plate, the condition on the vorticity function is:

$$\Omega \to 0 \qquad \text{as} \qquad \xi \to \infty. \tag{3.1.6b}$$

The behaviour of the streamfunction far from the normal flat plate is considered.

i.

$$\lambda\Psi \to \cosh \lambda x \sin\eta \qquad \text{as} \qquad x \to \infty$$

i.e.

$$\lambda\Psi \to \left[\frac{e^{\lambda x} + e^{-\lambda x}}{2}\right]\sin\eta \qquad \text{as} \qquad x \to \infty$$

As $x \to \infty$, $e^{-\lambda x} \to 0$

Thus,

$$e^{-\lambda x}\,\lambda\,\Psi \rightarrow \frac{1}{2}\sin\eta \qquad \text{as} \qquad x \rightarrow \infty. \tag{3.1.6c}$$

ii. Since $\dfrac{\partial\Psi}{\partial x} \rightarrow \sinh\lambda x\,\sin\eta \qquad \text{as} \qquad x \rightarrow \infty,$

therefore:

$$e^{-\lambda x}\,\frac{\partial\Psi}{\partial x} \rightarrow \frac{1}{2}\sin\eta \qquad \text{as} \qquad x \rightarrow \infty. \tag{3.1.6d}$$

Considering the boundary conditions on the vorticity and streamfunction given by (3.1.6), it is observed that there is an excess of conditions on $\Psi(x, \eta, t)$. A global integral condition on the vorticity function is sought in order to make compensation for this fact. Taking $t = 0$, and hence $\lambda = 0$, in equation (3.1.5), one has:

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{1}{2}\left[\cos 2\eta + 1\right]\Omega.$$

Both sides of this equation are multiplied by $e^{-\lambda x}$ to obtain

$$e^{-\lambda x}\,\frac{\partial^2\Psi}{\partial x^2} = -\frac{1}{2}\,e^{-\lambda x}\left[\cos 2\eta + 1\right]\Omega.$$

Since

$$e^{-\lambda x}\frac{\partial^2\Psi}{\partial x^2} = \lim_{\lambda\to 0}\left\{e^{-\lambda x}\frac{\partial^2\Psi}{\partial x^2} - \lambda e^{-\lambda x}\frac{\partial\Psi}{\partial x} + \lambda e^{-\lambda x}\frac{\partial\Psi}{\partial x} - \lambda^2 e^{-\lambda x}\Psi\right\}$$

$$= \lim_{\lambda\to 0}\left\{\frac{\partial}{\partial x}\left[e^{-\lambda x}\frac{\partial\Psi}{\partial x} + \lambda e^{-\lambda x}\Psi\right]\right\},$$

substitution of the above into the equation governing $\Psi(x, \eta, t)$ yields

$$\lim_{\lambda\to 0}\left\{\frac{\partial}{\partial x}\left[e^{-\lambda x}\frac{\partial\Psi}{\partial x} + \lambda e^{-\lambda x}\Psi\right]\right\},$$

$$= \lim_{\lambda\to 0}\left\{-\frac{1}{2}[\cos 2\eta + 1]\Omega\right\}.$$

Integration of this equation wrt x, from $x = 0$ to $x \to \infty$ gives

$$\int_0^\infty (\cos 2\eta + 1)\Omega(x,\eta,t{=}0)\,dx = -2\sin\eta \;. \qquad (3.1.7)$$

This is the desired integral condition on the vorticity function, for time $t = 0$. This condition is compatible with the condition (2.4.7) on $G_K(x, \eta, t{=}0)$ found earlier. Indeed, multiplication of both sides of (3.1.7) by $\sin k\eta$, and subsequent integration wrt η from $n = 0$ to $\eta = \pi$ yields the following equivalent integral condition:

$$\int_0^\pi\int_0^\infty (\cos 2\eta + 1)\Omega(x,\eta,t{=}0)\sin k\eta \;dx\,d\eta = -\pi\delta_{k,1}, \qquad (3.1.8)$$

where

$$\delta_{k,1} = \begin{cases} 1 & , \; k = 1 \\ 0 & , \; k \neq 1 \end{cases}.$$

3.2 THE INITIAL SOLUTION

At initial time t = 0, the flow of a viscous, incompressible fluid past an impulsively started flat plate, normal to the direction of the flow - in terms of the boundary-layer transformed vorticity function $\Omega(x, \eta, t)$ and streamfunction $\Psi(x, \eta, t)$ - is governed by the system of equations:

$$\frac{\partial^2\Omega}{\partial x^2} + (\cos 2\eta + 1)\left(x\frac{\partial\Omega}{\partial x} + \Omega\right) = 0 \tag{3.2.1}$$

and

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{1}{2}(\cos 2\eta + 1)\Omega. \tag{3.2.2}$$

The boundary and global conditions given in (3.1.6a), (3.1.6b) and (3.1.7) constitute a sufficient set of conditions with which to solve equations (3.2.1) and (3.2.2). Applying the transformation

$$x = f(\eta)z \tag{3.2.3}$$

to equation (3.2.1) yields

$$\frac{\partial^2\Omega}{\partial z^2} + f^2(\eta)\ (1 + \cos 2\eta)\left[z\frac{\partial\Omega}{\partial z} + \Omega\right] = 0.$$

By selecting

$$f(\eta) = (1 + \cos 2\eta)^{-1/2} \tag{3.2.4}$$

the above equation reduces to

$$\frac{\partial^2\Omega}{\partial z^2} + z\frac{\partial\Omega}{\partial z} + \Omega = 0. \tag{3.2.5}$$

By inspection,

$$\Omega_1(z,\eta,t{=}0) = g(\eta)\, e^{-1/2z^2}$$

is a solution of (3.2.5).

The second linearly independent solution of (3.2.5) is obtained through knowledge of the properties of Wronskians:

$$\Omega_2(z,\eta,t{=}0) = h(\eta)e^{-1/2z^2}\int_0^z e^{+1/2u^2du}.$$

Both Ω_1 and Ω_2 satisfy the condition $\Omega \to 0$ as $z \to \infty$, and hence a linear combination of these functions would also satisfy this far-field condition. However, employing $\Omega_2(z, \eta, t{=}0)$ in the integral condition (3.1.7) yields a divergent integral. Hence, $h(\eta)$ must be taken to be zero, and thus the solution to (3.2.5) is

$$\Omega(x,\eta,t{=}0) = g(\eta)e^{-\frac{(1+\cos 2\eta)}{2}x^2}$$

where $g(\eta)$ may be determined through the integral condition (3.1.7).

$$\int_0^\infty (\cos 2\eta + 1)g(\eta) \cdot e^{-1/2(1+\cos 2\eta)x^2}\, dx = -2\sin\eta\ .$$

Hence:

$$g(\eta) = \frac{-2\sin\eta}{(\cos 2\eta + 1)\left[\int_0^\infty e^{-1/2(1+\cos 2\eta)x^2}dx\right]}.$$

Since

$$\int_0^\infty e^{-ku^2} = \frac{1}{2}\sqrt{\frac{\pi}{k}},$$

therefore

$$g(\eta) = \frac{-2^{3/2}\sin\eta}{\sqrt{\pi}\,(1+\cos 2\eta)^{1/2}},$$

and hence

$$\Omega(x,\eta,t{=}0) = \frac{-2^{3/2}\sin\eta}{\sqrt{\pi}\,(1+\cos 2\eta)^{1/2}}\, e^{-1/2(1+\cos 2\eta)x^2}. \tag{3.2.6}$$

A singularity in the vorticity exists at $\eta = \pi/2$, i.e. at the top edge of the normal flat plate. Employing (3.2.6) in the governing equation for $\Psi(x, \eta, t{=}0)$, yields

$$\frac{\partial^2\Psi}{\partial x^2} = \sqrt{\frac{2}{\pi}}\,(1+\cos 2\eta)^{1/2}\,\sin\eta\; e^{-1/2(1+\cos 2\eta)x^2}$$

Integration of this equation wrt x, from x = 0 to x = x, and employing the no-slip condition $\left(\frac{\partial\Psi}{\partial x}\right)_{x=0} = 0$, gives

$$\frac{\partial\Psi}{\partial x} = (\sin\eta)\ \mathrm{erf}\left(\sqrt{\frac{1+\cos 2\eta}{2}}\ x\right)$$

where

$$\mathrm{erf}\,(\sqrt{k}\ x) = 2\sqrt{\frac{k}{\pi}}\int_0^x e^{-ku^2}\,du\ .$$

Integrating again wrt x from x = 0 to x = x yields

$$\Psi(x,\eta,t=0) = \sin\eta\left\{x\ \mathrm{erf}\left(\sqrt{\frac{1+\cos 2\eta}{2}}\,x\right)\right.$$

$$\left. - \sqrt{\frac{2}{\pi}}\ (1+\cos 2\eta)^{-1/2}\left(1 - e^{-\frac{(1+\cos 2\eta)}{2}x^2}\right)\right\}. \tag{3.2.7}$$

3.3 SERIES EXPANSIONS IN POWERS OF λ

Equations (3.1.4) through (3.1.7) provide the mathematical formulation for the problem of unsteady flow of a viscous, incompressible fluid past an impulsively started normal flat plate. Furthermore, the expressions given in (3.2.6) and (3.2.7) describe the initial-time behaviour of the fluid. We now embark on a study of the behaviour of the fluid for small time t. This is facilitated by first assuming the following series expansions of the vorticity and streamfunction in powers of λ:

$$\Omega(x,\eta,t) = \Omega_0(x,\eta,t) + \lambda\Omega_1(x,\eta,t) + \lambda^2\Omega_2(x,\eta,t) + \ldots \tag{3.3.1}$$

and

$$\Psi(x,\eta,t) = \Psi_0(x,\eta,t) + \lambda\Psi_1(x,\eta,t) + \lambda^2\Psi_2(x,\eta,t) + \dots \qquad (3.3.2)$$

We note that the dependence of the problem on the Reynolds number R can be considered to be contained in the variable $\lambda = \left(\frac{8t}{R}\right)^{1/2}$. Thus if we consider flows of low viscosity at small times, λ will be small, and expansions (3.3.1) and (3.3.2) are viable. Additionally, we replace the term cosh $2\lambda x$ in equations (3.1.4) and (3.1.5) by its power series expansion, so that

$$\begin{aligned} M^2 &= \frac{1}{2}(\cos 2\eta + \cosh 2\lambda x) \\ &= \frac{1}{2}\left[(1+\cos 2\eta) + \lambda^2 \cdot \frac{(2x)^2}{2!} + \lambda^4 \cdot \frac{(2x)^4}{4!} + \dots\right]. \end{aligned}$$

Employing the above expansion along with expansions (3.3.1), (3.3.2) in equations (3.1.4) and (3.1.5), and equating the coefficients of like powers of λ to zero, we have the following equations governing the behaviour of coefficient functions $\Psi_K(x, \eta, t)$ and $\Omega_K(x, \eta, t)$:

$$\begin{aligned} \frac{\partial^2\Psi_K}{\partial x^2} + \frac{\partial^2\Psi_{K-2}}{\partial\eta^2} = -\frac{1}{2}\Big\{(1+\cos 2\eta)\Omega_k \\ + \frac{(2x)^2}{2!}\Omega_{k-2} + \frac{(2x)^4}{4!}\Omega_{k-4} + \dots + \frac{(2x)^m}{m!}\Omega_{K-m}\Big\} \end{aligned} \qquad (3.3.3)$$

and

$$
\begin{aligned}
&\frac{\partial^2\Omega_K}{\partial x^2} + (1+\cos 2\eta)\left[x\,\frac{\partial\Omega_K}{\partial x} + (1-k)\Omega_k\right] \\
&= (1+\cos 2\eta)\cdot 2t\cdot\frac{\partial\Omega_K}{\partial t} - \frac{\partial^2\Omega_{K-2}}{\partial\eta^2} \\
&+ \frac{(2x)^2}{2!}\left[2t\frac{\partial\Omega_{K-2}}{\partial t} - x\frac{\partial\Omega_{K-2}}{\partial x} + (k-3)\Omega_{K-2}\right] \\
&+ \frac{(2x)^4}{4!}\left[2t\frac{\partial\Omega_{K-4}}{\partial x} - x\frac{\partial\Omega_{K-4}}{\partial x} + (k-5)\Omega_{K-4}\right] \\
&+ \dots + \frac{(2x)^m}{m!}\left[2t\frac{\partial\Omega_{K-m}}{\partial t} - x\frac{\partial\Omega_{K-m}}{\partial x} + (k-m-1)\Omega_{K-m}\right] \\
&-4t\left[\sum_{p=0}^{K}\left(\frac{\partial\Omega_p}{\partial\eta}\frac{\partial\Psi_{K-p}}{\partial x} - \frac{\partial\Psi_p}{\partial\eta}\frac{\partial\Omega_{K-p}}{\partial x}\right)\right], \qquad (3.3.4)
\end{aligned}
$$

where

$$
m = \begin{cases} K & ; \quad K \text{ is even} \\ K-1 & ; \quad K \text{ is odd} \end{cases} .
$$

The boundary conditions for $\Psi_K(x, \eta, t)$ and $\Omega_K(x, \eta, t)$ are

$$
\Psi_K = \frac{\partial\Psi_K}{\partial x} = 0 \quad \text{on} \quad x = 0 , \qquad (3.3.5)
$$

and

$$
\Omega_K \to 0 \quad \text{as} \quad x \to \infty. \qquad (3.3.6)
$$

The integral conditions on the vorticity coefficient functions, $\Omega_K(x, \eta, t)$, are obtained by expressing the coefficients of the sine series expansion of the vorticity in the following manner:

$$G_n(x,t) = \frac{2}{\pi}\int_0^{\pi} \Omega(x,\eta,t)\sin n\eta \; d\eta ,$$

and substituting in the integral conditions given in (1.4.4).

With the resulting expression,

$$\int_0^{\pi}\int_0^{\infty} e^{-k\lambda x}\{2\cosh 2\lambda x \sin k\eta + \sin(k-2)\eta$$

$$+ \sin(k+2)\eta - \sin(2-k)\eta\}\Omega(x,\eta,t)\,dx\, d\eta$$

$$= -2\pi\delta_{k,1},$$

we expand the functions of λ contained in the equations in powers of λ to obtain:

$$\int_0^{\pi}\int_0^{\infty}\{[2\sin k\eta + \sin(k-2)\eta + \sin(k+2)\eta - \sin(2-k)\eta]$$

$$+ \lambda\left[(-kx)(2\sin k\eta + \sin(k-2)\eta + \sin(k+2)\eta - \sin(2-k)\eta)\right]$$

$$+ \lambda^2\frac{x^2}{2!}\left[((2-k)^2 + (-2-k)^2)\sin k\eta\right.$$

$$\left.+ k^2(\sin(k-2)\eta + \sin(k+2)\eta - \sin(2-k)\eta)\right]$$

$$+ \lambda^3\frac{x^3}{3!}\left[((2-k)^3 + (-2-k)^3)\sin k\eta\right.$$

$$\left.-k^3(\sin(k-2)\eta + \sin(k+2)\eta - \sin(2-k)\eta)\right]$$

$$+ \ldots\}\left\{\Omega_0 + \lambda\Omega_1 + \lambda^2\Omega_2 + \ldots\right\} dx\, d\eta$$

$$= -2\pi\delta_{k,1}.$$

Now, equating coefficients of like powers of λ in this equation, we generate the following desired set of integral conditions on the coefficient vorticity functions:

i
$$\int_0^{\pi}\int_0^{\infty}\left\{\left[2\sin k\eta+\sin(k-2)\eta+\sin(k+2)\eta-\sin(2-k)\eta\right]\Omega_0\right\}dx\,d\eta = -2\pi\,\delta_{k,1}, \tag{3.3.7i}$$

ii
$$\int_0^{\pi}\int_0^{\infty}\left\{\left[2\sin k\eta+\sin(k-2)\eta+\sin(k+2)\eta-\sin(2-k)\eta\right]\left[\Omega_1-kx\Omega_0\right]\right\}dx\,d\eta = 0, \tag{3.3.7ii}$$

iii
$$\int_0^{\pi}\int_0^{\infty}\left\{\left[2\sin k\eta+\sin(k-2)\eta+\sin(k+2)\eta-\sin(2-k)\eta\right]\left[\Omega_2-kx\Omega_1\right] + \frac{x^2}{2!}\left[\left((2-k)^2+(-2-k)^2\right)\sin k\eta+k^2\left(\sin(k-2)\eta + \sin(k+2)\eta-\sin(2-k)\eta\right)\right]\Omega_0\right\}dx\,d\eta = 0, \tag{3.3.7iii}$$

3.4 SERIES EXPANSIONS IN POWERS OF TIME IN THE BOUNDARY LAYER

In the boundary-layer case ($\lambda = 0$), it is known that for small time t, the functions $\Psi_0(x, \eta, t)$ and $\Omega_0(x, \eta, t)$ can be expressed in series of powers of time, with functional coefficients depending on x and η. That is,

$$\Psi_0(x,\eta,t) = \Psi_{00}(x,\eta) + t\,\Psi_{01}(x,\eta) + t^2\Psi_{02}(x,\eta) + \dots \tag{3.4.1}$$

and

$$\Omega_0(x,\eta,t) = \Omega_{00}(x,\eta) + t\,\Omega_{01}(x,\eta) + t^2\Omega_{02}(x,\eta) + \dots\ . \tag{3.4.2}$$

These expansions, when employed in the governing equations, facilitate the isolation of time-dependence. Taking k = 0 in equations (3.3.3) and (3.3.4), we have the equations which govern the boundary-layer coefficient functions, $\Psi_0(x, \eta, t)$ and $\Omega_0(x, \eta, t)$:

$$\frac{\partial^2\Psi_0}{\partial x^2} = -\frac{1}{2}(1+\cos 2\eta)\Omega_0 \tag{3.4.3}$$

and

$$\frac{\partial^2\Omega_0}{\partial x^2} + (1+\cos 2\eta)\left[x\frac{\partial\Omega_0}{\partial x} + \Omega_0\right] = 2t\left\{(1+\cos 2\eta)\frac{\partial\Omega_0}{\partial t} - 2\left(\frac{\partial\Omega_0}{\partial\eta}\frac{\partial\Psi_0}{\partial x} - \frac{\partial\Omega_0}{\partial x}\frac{\partial\Psi_0}{\partial\eta}\right)\right\}. \tag{3.4.4}$$

We substitute expansions (3.4.1) and (3.4.2) into the above equations.

From (3.4.3):

$$\left[\frac{\partial^2\Psi_{00}}{\partial x^2} + t\frac{\partial^2\Psi_{01}}{\partial x^2} + t^2\frac{\partial^2\Psi_{02}}{\partial x^2} + \ldots\right]$$

$$= -\frac{1}{2}(1+\cos 2\eta)\left[\Omega_{00} + t\Omega_{01} + t^2\Omega_{02} + \ldots\right]$$

Equating coefficients of like powers of t on either side of this equation, we find that

$$\frac{\partial^2\Psi_{0n}}{\partial x^2} = -\frac{1}{2}(1+\cos 2\eta)\Omega_{0n}, \qquad (3.4.5)$$

for n = 0,1,2,3,...

From (3.4.4):

$$\text{LHS of (3.4.4)} = \sum_{n=0}^{\infty} t^n \left\{\frac{\partial^2\Omega_{0n}}{\partial x^2} + (1+\cos 2\eta)\left[x\frac{\partial\Omega_{0n}}{\partial x} + \Omega\right]\right\}.$$

Also:

$$2t\frac{\delta\Omega_0}{\delta t} = 2t\left[\Omega_{01} + 2t\Omega_{02} + 3t^2\Omega_{03} + \ldots\right]$$

$$= 2\sum_{n=0}^{\infty} nt^n\Omega_{0n}.$$

$$2t\frac{\partial\Omega_0}{\partial t} = \left[\Omega_{01} + 2t\Omega_{02} + 3t^2\Omega_{03} + \dots\right]\cdot 2t$$

$$= 2\sum_{n=0}^{\infty} n t^n \Omega_{0n},$$

$$4t\frac{\partial\Omega_0}{\partial\eta}\frac{\partial\Psi_0}{\partial x} = \left\{\left[\frac{\partial\Omega_{00}}{\partial\eta} + t\frac{\partial\Omega_{01}}{\partial\eta} + t^2\frac{\Omega_{02}}{\partial\eta} + \dots\right]\right.$$

$$\left.\cdot\left[\frac{\partial\Psi_{00}}{\partial x} + t\frac{\partial\Psi_{01}}{\partial x} + t^2\frac{\partial\Psi_{02}}{\partial x} + \dots\right]\right\}\cdot 4t$$

$$= 4\left\{t\left[\frac{\partial\Omega_{00}}{\partial\eta}\frac{\partial\Psi_{00}}{\partial x}\right] + t^2\left[\frac{\partial\Omega_{00}}{\partial\eta}\frac{\partial\Psi_{01}}{\partial x} + \frac{\partial\Omega_{01}}{\partial\eta}\frac{\partial\Psi_{00}}{\partial x}\right]\right.$$

$$+ t^3\left[\frac{\partial\Omega_{00}}{\partial\eta}\frac{\partial\Psi_{02}}{\partial x} + \frac{\partial\Omega_{01}}{\partial\eta}\frac{\partial\Psi_{01}}{\partial x} + \frac{\partial\Omega_{02}}{\partial\eta}\frac{\partial\Psi_{00}}{\partial x}\right]$$

$$\left. + \dots\right\}$$

$$= 4\sum_{n=0}^{\infty}\sum_{m=0}^{n-1} t^n\left(\frac{\partial\Omega_{0m}}{\partial\eta}\frac{\partial\Psi_{0,n-m-1}}{\partial x}\right),$$

and

RHS of (3.4.4) =

$$\sum_{n=0}^{\infty} t^n\left\{2n\,(1+\cos 2\eta)\Omega_{on} + 4\sum_{m=0}^{n-1}\left(\frac{\partial\Omega_{0m}}{\partial x}\frac{\partial\Psi_{0,n-m-1}}{\partial\eta} - \frac{\partial\Omega_{0m}}{\partial\eta}\frac{\partial\Psi_{0,n-m-1}}{\partial x}\right)\right\}.$$

Hence, equating coefficients of like powers of t on the LHS and RHS of equation (3.4.4), we have that for n = 0,1,2,3,...:

$$\frac{\partial^2\Omega_{0n}}{\partial x^2}+(1+\cos 2\eta)\left[x\frac{\partial\Omega_{0n}}{\partial x}+\Omega_{0n}\right]$$

$$= 2n(1+\cos 2\eta)\Omega_{0n} + 4\sum_{m=0}^{n-1}\left(\frac{\partial\Omega_{0m}}{\partial x}\frac{\partial\Psi_{0,n-m-1}}{\partial\eta}-\frac{\partial\Omega_{0m}}{\partial\eta}\frac{\partial\Psi_{0,n-m-1}}{\partial x}\right).$$

That is,

$$\frac{\partial^2\Omega_{0n}}{\partial x^2}+(1+\cos 2\eta)\left[x\frac{\partial\Omega_{0n}}{\partial x}+(1-2n)\Omega_{0n}\right]$$

$$= 4\sum_{m=0}^{n-1}\left(\frac{\partial\Omega_{0m}}{\partial x}\frac{\partial\Psi_{0,n-m-1}}{\partial\eta}-\frac{\partial\Omega_{0m}}{\partial\eta}\frac{\partial\Psi_{0,n-m-1}}{\partial x}\right). \qquad (3.4.6)$$

The associated boundary conditions obtained from (3.3.5) and (3.3.6) are that:

$$\Psi_{0n} = \frac{\partial\Psi_{0n}}{\partial x} = 0 \quad \text{when} \quad x=0, \qquad (3.4.7)$$

and

$$\Omega_{0n} \to 0 \quad \text{as} \quad x \to \infty. \qquad (3.4.8)$$

The integral conditions on the coefficient functions $\Omega_{0n}(x, \eta)$ are obtained by substitution of the expansion (3.4.2) into condition (3.1.7). We get:

$$\int_0^{\infty} (1 + \cos 2\eta)\Omega_{00}(x,\eta)\, dx = -2\sin\eta, \tag{3.4.9}$$

and

$$\int_0^{\infty} (1 + \cos 2\eta)\Omega_{0n}(x,\eta)\, dx = 0, \quad \forall n \varepsilon \mathbf{N} \tag{3.4.10}$$

Multiplying equations (3.4.9) by sin p η and integrating wrt η from $\eta = 0$ to $\eta = \pi$, we may rewrite this integral condition in the following equivalent form:

$$\int_0^{\pi}\int_0^{\infty} (1 + \cos 2\eta)\sin p\eta \; \Omega_{00}\, dx\, d\eta = -\pi\, \delta_{p,1}, \tag{3.4.11}$$

where

$$\delta_{p,1} = \begin{cases} 1 & ; \quad p = 1 \\ 0 & ; \quad p \neq 1 \end{cases}$$

Likewise, we integrate equation (3.4.10) wrt η from $\eta = 0$ to $\eta = \pi$ to obtain the following form of that integral condition

$$\int_0^{\pi}\int_0^{\infty} (1 + \cos 2\eta)\; \Omega_{0n}\, dx\, d\eta = 0\,. \tag{3.4.12}$$

From the preceding analysis, it is clear that the zeroth order coefficient functions $\Psi_{00}(x, \eta)$ and $\Omega_{00}(x, \eta)$ are governed by the equations

$$\frac{\partial^2 \Psi_{00}}{\partial x^2} = -\frac{1}{2}(1+\cos 2\eta)\Omega_{00},$$

and

$$\frac{\partial^2 \Omega_{00}}{\partial x^2} + (1+\cos 2\eta)\left[x\frac{\partial \Omega_{00}}{\partial x} + \Omega_{00}\right] = 0,$$

subject to the conditions

$$\Psi_{00} = \frac{\partial \Psi_{00}}{\partial x} = 0 \quad \text{when} \quad x = 0,$$

$$\Omega_{00} \to 0 \quad \text{as} \quad x \to \infty,$$

&

$$\int_0^{\pi}\int_0^{\infty} (1+\cos 2\eta)\sin p\eta \; \Omega_{00} \, dx \, d\eta = -\pi \, \delta_{p,1}.$$

The exact solutions to this system were obtained earlier ((3.2.6) and (3.2.7)), and are given here again. We have

$$\Omega_{00}(x,\eta) = -\frac{2^{3/2}\sin\eta}{\sqrt{\pi}(1+\cos 2\eta)^{1/2}} e^{-\frac{1}{2}(1+\cos 2\eta)x^2} \tag{3.4.13}$$

and

$$\Psi_{00}(x,\eta) = \left\{x \operatorname{erf}\left(\sqrt{\frac{1+\cos 2\eta}{2}}\, x\right) - \sqrt{\frac{2}{\pi}}(1+\cos 2\eta)^{-\frac{1}{2}} \left(1 - e^{\frac{-(1+\cos 2\eta)}{2}x^2}\right)\right\} \sin \eta. \tag{3.4.14}$$

The first-order coefficient equations are governed by the equations

$$\begin{cases} \dfrac{\partial^2 \Psi_{01}}{\partial x^2} = -\dfrac{1}{2}(1+\cos 2\eta)\Omega_{01} \\ \dfrac{\partial^2 \Omega_{01}}{\partial x^2} + (1+\cos 2\eta)\left[x\dfrac{\partial \Omega_{01}}{\partial x} - \Omega_{01}\right] \\ = 4\left[\dfrac{\partial \Omega_{00}}{\partial x}\dfrac{\partial \Psi_{00}}{\partial \eta} - \dfrac{\partial \Omega_{00}}{\partial \eta}\dfrac{\partial \Psi_{00}}{\partial x}\right] \end{cases}$$

subject to the conditions

$$\Psi_{01} = \frac{\partial \Psi_{01}}{\partial x} = 0 \quad \text{when} \quad x = 0,$$

$$\Omega_{01} \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty,$$

and

$$\int_0^\pi \int_0^\infty (1+\cos 2\eta)\Omega_{01}\, dx\, d\eta = 0.$$

We apply the transformation obtained earlier, i.e. $z = (1 + \cos 2\eta)^{1/2} x$, to the above.

Since

$$\Omega_{00}(z,\eta) = -\frac{2^{3/2}\sin \eta}{\sqrt{\pi}(1+\cos 2\eta)^{1/2}} e^{-1/2z^2}$$

and

$$\Psi_{00}(z,\eta) = \frac{\sin\eta}{(1+\cos 2\eta)^{1/2}} \left\{ z \operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right) - \sqrt{\frac{2}{\pi}} \left(1 - e^{-1/2z^2}\right) \right\},$$

therefore:

i.

$$\frac{\partial \Omega_{00}}{\partial \eta} = \left(-\frac{2^{3/2}}{\sqrt{\pi}} e^{-\frac{1}{2} z^2}\right)\{\sin\eta \ \sin 2\eta \ (1+\cos 2\eta)^{-3/2} + \cos\eta(1+\cos 2\eta)^{-1/2}\} ,$$

ii.

$$\frac{\partial \Omega_{00}}{\partial z} = \frac{2^{3/2}}{\sqrt{\pi}} \sin\eta \ (1+\cos 2\eta)^{-1/2} \ z \ e^{-1/2 \ z^2} ,$$

iii.

$$\frac{\partial \Psi_{00}}{\partial \eta} = \left\{z \ \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}\left(1 - e^{-1/2 \ z^2}\right)\right\}$$

$$\{\sin\eta \ \sin 2\eta \ (1+\cos 2\eta)^{-3/2} + \cos\eta(1+\cos 2\eta)^{-1/2}\},$$

and

iv.

$$\frac{\partial \Psi_{00}}{\partial z} = \frac{\sin\eta}{(1+\cos 2\eta)^{1/2}} \ \mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right).$$

Hence, the transformed system of equations governing the behaviour of $\Psi_{01}(z, \eta)$ and $\Omega_{01}(z, \eta)$ takes the form

$$\frac{\partial^2 \Psi_{01}}{\partial z^2} = -\frac{1}{2} \Omega_{01} , \qquad (3.4.15)$$

and

$$\frac{\partial^2\Omega_{01}}{\partial z^2} + z\frac{\partial\Omega_{01}}{\partial z} - \Omega_{01} = 8\sqrt{\frac{2}{\pi}}\ \sin\eta$$

$$\left[\sin\eta\ \sin 2\eta\ (1+\cos 2\eta)^{-5/2} + \cos\eta(1+\cos 2\eta)^{-3/2}\right]$$

$$\cdot\, e^{-\frac{1}{2}z^2} \cdot \left\{(z^2+1)\ \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}\ z\ (1 - e^{-1/2z^2})\right\}. \tag{3.4.16}$$

The transformed boundary conditions take the form

$$\Psi_{01} = \frac{\partial\Psi_{01}}{\partial z} = 0 \quad \text{when} \quad z = 0\,, \tag{3.4.17}$$

$$\Omega_{01} \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty, \tag{3.4.18}$$

and

$$\int_0^{\pi}\int_0^{\infty} (1+\cos 2\eta)^{1/2}\,\Omega_{01}\ dz\ d\eta = 0. \tag{3.4.19}$$

In order to facilitate a fuller understanding of the solution of the system of governing equations and boundary/integral conditions describing the behaviour of $\Psi_{01}(z, \eta)$ and $\Omega_{01}(z, \eta)$, we consider again the system of equations describing the general functions $\Psi_{0n}(x, \eta)$ and $\Omega_{0n}(x, \eta)$, given by equations (3.4.5) to (3.4.10). Applying the transformation $z = (1 + \cos 2\,\eta)^{1/2}\,x$ to the governing equations (3.4.5) and (3.4.6), we have

$$\frac{\partial^2\Psi_{0n}}{\partial z^2} = -\frac{1}{2}\Omega_{0n}, \tag{3.4.20}$$

and

$$\frac{\partial^2 \Omega_{0n}}{\partial z^2} + z\,\frac{\partial \Omega_{0n}}{\partial z} + (1-2n)\Omega_{0n} = S_n(z,\eta), \qquad (3.4.21)$$

where

$$S_n(z,\eta) = 4\cdot(1+\cos 2\eta)^{-\frac{1}{2}} \sum_{m=0}^{n-1}\left[\frac{\partial \Omega_{0m}}{\partial z}\frac{\partial \Psi_{0,n-m-1}}{\partial \eta} - \frac{\partial \Omega_{0m}}{\partial \eta}\frac{\partial \Psi_{0,n-m-1}}{\partial z}\right].$$

The corresponding boundary and integral conditions are transformed likewise:

$$\Psi_{0n} = \frac{\partial \Psi_{0n}}{\partial z} = 0 \quad \text{at} \quad z = 0, \qquad (3.4.22)$$

$$\Omega_{0n} \to 0 \quad \text{as} \quad z \to \infty, \qquad (3.4.23)$$

and

$$\int_0^{\pi}\int_0^{\infty} (1+\cos 2\eta)^{1/2}\,\Omega_{0n}\, dz\, d\eta = 0. \qquad (3.4.24)$$

In constructing exact solutions for $\Psi_{0n}(z,\eta)$ and $\Omega_{0n}(z,\eta)$, we first consider the homogeneous differential equation corresponding to (3.4.21):

$$\frac{\partial^2 (\Omega_{0n})_h}{\partial z^2} + Z\,\frac{\partial (\Omega_{0n})_h}{\partial z} + (1-2n)(\Omega_{0n})_h = 0. \qquad (3.4.25)$$

Assuming a solution of the form

$$(\Omega_{0n})_h\,(z,\eta) = e^{az^2}\, F(z,\eta),$$

we have

$$\frac{\partial^2 F}{\partial z^2} + (4a+1)z\frac{\partial F}{\partial z} + (4a^2z^2+2az^2+2a+1-2n)F = 0.$$

Taking a = -1/4, we have

$$\frac{\partial^2 F}{\partial z^2} - \left(\frac{1}{4}z^2 + 2n - \frac{1}{2}\right)F = 0. \tag{3.4.26}$$

Equation (3.4.26) is a differential equation of the standard form:

$$y'' + (ax^2 + bx + c)y = 0,$$

which is well-known to have solutions which are parabolic cylindrical functions (see Appendix II).

Even and odd power solutions of (3.4.26) are given by

$$f_1(z) = e^{-1/4\,z^2}\left\{1 + (2n)\frac{z^2}{2!} + (2n)(2n+2)\frac{z^4}{4!} + \dots\right\}$$

$$= e^{1/4\,z^2}\left\{1 + (2n-1)\frac{z^2}{2!} + (2n-1)(2n-3)\frac{z^4}{4!} + \dots\right\}$$

and

$$f_2(z) = e^{-1/4\,z^2}\left\{z + (2n+1)\frac{z^3}{3!} + (2n+1)(2n+3)\frac{z^5}{5!} + \dots\right\}$$

$$= e^{1/4\,z^2}\left\{z + (2n-2)\frac{z^3}{3!} + (2n-2)(2n-4)\frac{z^5}{5!} + \dots\right\}.$$

and standard solutions of the differential equation are constructed from these series solutions. Two linearly independent solutions of (3.4.26) are:

$$U\left(2n-\frac{1}{2},z\right) = \cos(n\pi)\cdot F_1$$

and

$$V\left(2n-\frac{1}{2},z\right) = \frac{1}{\Gamma(1-2n)}\cos(n\pi)\cdot F_2,$$

where

$$F_1 = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}-n\right)}{2^n} f_1$$

and

$$F_2 = \frac{1}{\sqrt{\pi}}\frac{\Gamma(1-n)}{2^{n-\frac{1}{2}}} f_2 .$$

Hence the solution of the homogeneous differential equation governing the behaviour of $(\Omega_{0_n})_h(z,\eta)$, and given by (3.4.25), may be written as

$$\left[\Omega_{0_n}(z,\eta)\right]_h = e^{-\frac{1}{4}z^2}\left[A_n(\eta)\ U(2n-\frac{1}{2},z) + B_n(\eta)\ V(2n-\frac{1}{2},z)\right],$$

where $A_n(\eta)$ and $B_n(\eta)$ are functions of η to be determined through use of the far-field condition on Ω_{0_n}, (3.4.23) and the global integral condition, (3.4.24). Using the method of undetermined coefficients, a particular solution can be found for the non-homogeneous differential equation in (3.4.21), and a complete solution for $\Omega_{0n}(z, \eta)$ is then

$$\Omega_{0n} = e^{-\frac{1}{4}z^2}\left[A_n(\eta)\ U(2n-\frac{1}{2},z) + B_n(\eta)\ V(2n-\frac{1}{2},z)\right] + (\Omega_{0n})_p\ .$$

Returning our consideration to equations (3.4.15) and (3.4.16) which govern the behaviour of the functions $\Psi_{01}(z, \eta)$ and $\Omega_{01}(z, \eta)$, we mimic the construction of the general solutions and write

$$(\Omega_{01})_h = e^{-\frac{1}{4}z^2}\ K(\eta)\ w(z).$$

Then, w(z) satisfies the simpler ordinary differential equation

$$w''(z) - \left(\frac{1}{4}z^2+\frac{3}{2}\right)w(z) = 0$$

Employing the theory of parabolic cylinder functions, we find that

$$w(z) = \frac{c_1}{\sqrt{2\pi}}\ e^{1/4z^2}z + c_2\left[\sqrt{2\pi}\ z\ e^{1/4z^2}\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right) + 2e^{-1/4z^2}\right],$$

where c_1, c_2 $\varepsilon\mathbb{R}$. Since $w(z) \to 0$ as $z \to \infty$, therefore $c_1 = -2\pi\ c_2$.

The homogeneous solution

$$(\Omega_{01})_h = K(\eta)\left\{z\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right) + \sqrt{\frac{2}{\pi}}\ e^{-\frac{1}{2}z^2} - z\right\}$$

will satisfy the necessary anti-symmetric conditions. The function $K(\eta)$ shall be determined later through use of the integral condition (3.4.19) on the complete solution.

A particular solution of the non-homogeneous differential equation (3.4.16) is easily determined and is given here:

$$(\Omega_{01})_p = M(\eta)\left\{\left[-\frac{1}{4}z^2+\frac{1}{4}\right]\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)e^{-\frac{1}{2}z^2} + \left[\frac{1}{3}\sqrt{\frac{2}{\pi}}\,z-\frac{1}{2}\right]e^{-\frac{1}{2}z^2} - \frac{1}{4}\sqrt{\frac{2}{\pi}}\,z\,e^{-z^2} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\,z\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right) + \frac{1}{2}\sqrt{\frac{\pi}{2}}\,z\left(\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)\right)^2\right\}$$

where

$$M(\eta) = 8\sqrt{\frac{2}{\pi}}\,(1+\cos 2\eta)^{-5/2}\left\{\sin\eta\,\cos\eta\,(1+\cos 2\eta)+\sin^2\eta\,\sin 2\eta\right\}.$$

A complete solution for $\Omega_{01}(z,\eta)$ is then

$$\Omega_{01} = K(\eta)\cdot\left\{z\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)\cdot e^{-1/4z^2} + \sqrt{\frac{2}{\pi}}\,e^{-3/4z^2} - z\,e^{-1/4z^2}\right\}$$

$$+ M(\eta)\left\{\left[-\frac{1}{4}z^2+\frac{1}{4}\right]\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)e^{-1/2z^2} + \left[\frac{1}{3}\sqrt{\frac{2}{\pi}}\,z-\frac{1}{2}\right]\right.$$

$$e^{-1/2z^2} - \frac{1}{4}\sqrt{\frac{2}{\pi}}\,z\,e^{-z^2} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\,z\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)$$

$$\left.+\frac{1}{2}\sqrt{\frac{\pi}{2}}\,z\left(\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)\right)^2\right\}.$$

The integral condition (3.4.19) shall be satisfied if $K(\eta)$ is taken to be $M(\eta)$ in the above. Hence,

$$\Omega_{01}(z,\eta) = M(\eta)\cdot\left\{\frac{1}{2}\sqrt{\frac{\pi}{2}}\,z\left(\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)\right)^2 - \frac{1}{2}\sqrt{\frac{\pi}{2}}\,z\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)\right.$$

$$+ z\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)e^{-1/4z^2} + \left[-\frac{1}{4}z^2+\frac{1}{4}\right]\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)e^{-1/2z^2}$$

$$+\left[\frac{1}{3}\sqrt{\frac{2}{\pi}}\,z-\frac{1}{2}\right]e^{-1/2z^2} + \sqrt{\frac{2}{\pi}}\,e^{-3/4z^2} - \frac{1}{4}\sqrt{\frac{2}{\pi}}\,z\,e^{-z^2}$$

$$\left.- z\,e^{-1/4z^2}\right\}. \qquad (3.4.27)$$

The solution for Ψ_{01} is obtained by integrating (3.4.27) twice with respect to z, subject to conditions (3.4.17). We have

$$\frac{\partial \Psi_{01}}{\partial z} = M(\eta) \cdot \left\{ \frac{1}{4}\sqrt{\frac{\pi}{2}}\ (z^2-1)\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}}\ z\right)\right]^2 \right.$$

$$+ \frac{1}{2} z\ e^{-1/2\ z^2}\ \mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right) + \frac{1}{4}\sqrt{\frac{2}{\pi}}\left(e^{-z^2}-1\right)$$

$$-\frac{1}{4}\sqrt{\frac{\pi}{2}}\ z^2\ \mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right) - \frac{1}{4} z\ e^{-1/2\ z^2}$$

$$+ \frac{1}{4}\sqrt{\frac{\pi}{2}}\ \mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right) - 2\ \mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right) e^{-1/4\ z^2}$$

$$+ 2\sqrt{\frac{2}{3}}\ \mathrm{erf}\left(\frac{\sqrt{3}}{2}\ z\right)$$

$$- \frac{1}{8}\sqrt{\frac{\pi}{2}}\ \left[\mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right)\right]^2 + \frac{1}{4}\ z\ e^{-1/2\ z^2}\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}\ z\right)$$

$$+ \frac{1}{8}\sqrt{\frac{2}{\pi}}\left(e^{-z^2}-1\right) + \frac{1}{8}\sqrt{\frac{\pi}{2}}\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right)\right]^2$$

$$+ \frac{1}{3}\sqrt{\frac{2}{\pi}}\left(1-e^{-1/2\ z^2}\right) - \frac{1}{2}\sqrt{\frac{\pi}{2}}\ \mathrm{erf}\left(\frac{1}{\sqrt{2}} z\right)$$

$$+ \sqrt{\frac{2}{3}}\ \mathrm{erf}\left(\frac{\sqrt{3}}{2}\ z\right) + \frac{1}{8}\sqrt{\frac{2}{\pi}}\left(e^{-z^2}-1\right)$$

$$\left. + 2\left(e^{-1/4\ z^2}-1\right)\right\},$$

<u>i.e.</u>

$$\frac{\partial \Psi_{01}}{\partial z} = M(\eta)\left\{\frac{1}{4}\sqrt{\frac{\pi}{2}}\,(z^2-1)\left[\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right)\right]^2\right.$$

$$-\frac{1}{4}\sqrt{\frac{\pi}{2}}\,(z^2+1)\left[\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right)\right]$$

$$+\frac{3}{4}\,z\,e^{-1/2\,z^2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right) + e^{-1/2\,z^2}\left[-\frac{1}{4}\,z - \frac{1}{3}\sqrt{\frac{2}{\pi}}\right]$$

$$+ e^{-z^2}\frac{1}{2}\sqrt{\frac{2}{\pi}} + 2e^{-1/4\,z^2} - 2\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right)e^{-1/4\,z^2}$$

$$\left. + \sqrt{6}\operatorname{erf}\left(\frac{\sqrt{3}}{2}\,z\right) - \frac{1}{6}\sqrt{\frac{2}{\pi}} - 2\right\}$$

That is,

$$\frac{\partial \Psi_{01}}{\partial z} = M(\eta)\left\{\frac{1}{4}\sqrt{\frac{\pi}{2}}\left[(z^2-1)\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right)\right)^2 - (z^2+1)\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right)\right]\right.$$

$$+ e^{-1/2\,z^2}\left[\frac{3}{4}z\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right) + \left(-\frac{1}{4}z - \frac{1}{3}\sqrt{\frac{2}{\pi}}\right)\right]$$

$$+ e^{-1/4\,z^2}\left[-2\operatorname{erf}\left(\frac{1}{\sqrt{2}}z\right) + 2\right]$$

$$\left. + \frac{1}{2}\sqrt{\frac{2}{\pi}}\,e^{-z^2} + \sqrt{6}\operatorname{erf}\left(\frac{\sqrt{3}}{2}\,z\right) - \frac{1}{6}\sqrt{\frac{2}{\pi}} - 2\right\}$$

Integrating again wrt z from z = 0 to z = z, we obtain the expression for $\Psi_{01}(z, \eta)$

$$\Psi_{01}(z,\eta) = M(\eta)\cdot\left\{\frac{1}{4}\sqrt{\frac{\pi}{2}}\left[\frac{1}{3}z^3\left(\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)\right)^2 + \frac{2}{3\pi}z\,e^{-z^2}\right.\right.$$

$$+\frac{2}{3}\sqrt{\frac{2}{\pi}}(z^2+2)\,e^{-1/2\,z^2}\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right) - \frac{5}{3\sqrt{\pi}}\mathrm{erf}(z)$$

$$-z\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)\right]^2 - 2\sqrt{\frac{2}{\pi}}\,e^{-1/2\,z^2}\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)$$

$$+\frac{2}{\sqrt{\pi}}\mathrm{erf}(z) - \frac{1}{3}z^3\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)$$

$$-\frac{1}{3}\sqrt{\frac{2}{\pi}}(z^2+2)\,e^{-1/2\,z^2} - \frac{2}{3}\sqrt{\frac{2}{\pi}}$$

$$\left.-z\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right) + \sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{\pi}}\,e^{-1/2\,z^2}\right]$$

$$+\frac{3}{4\sqrt{2}}\mathrm{erf}(z) - \frac{3}{4}e^{-1/2\,z^2}\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)$$

$$+\frac{1}{4}e^{-1/2\,z^2} - \frac{1}{4} - \frac{1}{3}\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right)$$

$$+2\sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2}z\right) + \frac{1}{4}\sqrt{2}\,\mathrm{erf}(z)$$

$$\left.-\left(\frac{1}{6}\sqrt{\frac{2}{\pi}} + 2\right)z + \sqrt{6}\,z\,\mathrm{erf}\left(\frac{\sqrt{3}}{2}z\right) + 2\sqrt{\frac{2}{\pi}}\left(e^{-3/4\,z^2} - 1\right)\right\} \qquad (3.4.28)$$

The process can be repeated to obtain higher order solutions in the boundary layer region. The homogeneous solution, $(\Omega_{on})_h$, of (3.4.21) will simply be a parabolic cylindrical function, and the particular solution, $(\Omega_{on})_p$, is obtained using the method of undetermined coefficients. Integrating the vorticity twice, as per equation (3.4.20), gives us the streamfunction Ψ_{on}.

3.5 THE FIRST-ORDER CORRECTION TO BOUNDARY-LAYER THEORY

Putting K = 1 in equations (3.3.3) and 93.3.4), we obtain equations for Ψ_1 and Ω_1 given by

$$\frac{\partial^2\Psi_1}{\partial x^2} = -\frac{1}{2}(1+\cos 2\eta)\Omega_1 \ , \tag{3.5.1}$$

$$\frac{\partial^2\Omega_1}{\partial x^2} + (1+\cos 2\eta)\left(x\frac{\partial\Omega_1}{\partial x}\right) = 2t\left\{(1+\cos 2\eta)\frac{\partial\Omega_1}{\partial t}\right.$$
$$\left. + 2\sum_{m=0}^{1}\left(\frac{\partial\Omega_m}{\partial x}\frac{\partial\Psi_{1-m}}{\partial\eta} - \frac{\partial\Omega_m}{\partial\eta}\frac{\partial\Psi_{1-m}}{\partial x}\right)\right\} . \tag{3.5.2}$$

Series for Ψ_1 and Ω_1 in power of time may be taken. We write these expansions as

$$\Psi_1(x,\eta,t) = \Psi_{10}(x,\eta) + t\,\Psi_{11}(x,\eta) + t^2\Psi_{12}(x,\eta) + \ldots , \tag{3.5.3}$$

$$\Omega_1(x,\eta,t) = \Omega_{10}(x,\eta) + t\,\Omega_{11}(x,\eta) + t^2\Omega_{12}(x,\eta) + \ldots . \tag{3.5.4}$$

$$\frac{\partial^2 \Psi_{1n}}{\partial x^2} = -\frac{1}{2}(1 + \cos 2\eta)\Omega_{1n}, \tag{3.5.3}$$

$$\frac{\partial^2 \Omega_{1n}}{\partial x^2} + x(1 + \cos 2\eta)\frac{\partial \Omega_{1n}}{\partial x} - 2n\Omega_{1n} = r_{1n} ; \tag{3.5.5}$$

where

$$r_{1n} = 4 \sum_{m=0}^{1} \sum_{p=0}^{n-1} \left(\frac{\partial \Omega_{m,p}}{\partial x} \frac{\partial \Psi_{1-m,n-1-p}}{\partial \eta} - \frac{\partial \Omega_{m,p}}{\partial \eta} \frac{\partial \Psi_{1-m,n-1-p}}{\partial x} \right) \quad ; n \in N$$

and

$$r_{10} = 0.$$

Here, as in the case of the boundary-layer expansion, the functions r_{1n} for a given $n > 0$ can be expressed entirely in terms of functions Ω_{ij}, Ψ_{ij} for $j < n$. Hence, the equations (3.5.5), (3.5.6) can be solved successively.

The functions Ψ_{1n} must satisfy

$$\Psi_{1n} = \frac{\partial \Psi_{1n}}{\partial x} = 0 \quad \text{when} \quad x = 0,$$

and the farstream vorticity condition is expressed thus:

$$\Omega_{1n} \to 0 \qquad \text{as} \qquad x \to \infty.$$

Employing expansions (3.5.3), (3.5.4) in integral condition (3.3.7ii), we obtain the following global condition on $\Psi_{1n}(x, \eta)$:

$$\int_0^{\pi}\int_0^{\infty}\left[2\ \sin k\eta + \sin(k-2)\eta + \sin(k+2)\eta - \sin(2-k)\eta\right]\Omega_{1n}\ dx\ deta$$

$$= \iint kx\left[2\ \sin k\eta + \sin(k-2)\eta + \sin(k+2)\eta - \sin(2-k)\eta\right]\ \Omega_{on}\ dx\,d\eta\ .$$

Now, taking n = 0 in the above, we have the following equations for Ψ_{10} and Ω_{10}:

$$\frac{\partial^2\Psi_{10}}{\partial x^2} = -\frac{1}{2}(1 + \cos 2\eta)\Omega_{10},$$

$$\frac{\partial^2\Omega_{10}}{\partial x^2} + x(1 + \cos 2\eta)\frac{\partial\Omega_{10}}{\partial x} = 0.$$

Applying the transformation

$$x = (1 + \cos 2\eta)^{-1/2}z$$

to this system of equations, we have

$$\frac{\partial^2\Psi_{10}}{\partial z^2} = -\frac{1}{2}\Omega_{10},$$

$$\frac{\partial^2\Omega_{10}}{\partial z^2} + z\frac{\partial\Omega_{10}}{\partial z} = 0.$$

the latter equation may be written as

$$\frac{\partial}{\partial z}\left[e^{1/2z^2}\frac{\partial\Omega_{10}}{\partial z}\right] = 0$$

Integrating twice with respect to z yields

$$\Omega_{10}(z,\eta) = C_1(\eta)\cdot\sqrt{\frac{\pi}{2}}\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}\,z\right) + C_2(\eta).$$

Since

$$\Omega_{10}(z,\eta) \to 0 \quad \text{as} \quad z \to \infty\ ,$$

therefore we must take

$$C_1(\eta) = -C_2(\eta)$$

Hence,

$$\Omega_{10}(z,\eta) = C_1(\eta)\left\{\sqrt{\frac{\pi}{2}}\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}\,z\right) - 1\right\},$$

where $C_1(\eta$ is an arbitrary function of η which may be determined from the integral condition on $\Omega_{10}(z, \eta)$.

Integrating twice with respect to z yields the expression for $\Psi_{10}(z, \eta)$.

$$\Psi_{10}(z,\eta) = -\frac{1}{4}\,C_1(\eta)\left\{\sqrt{\frac{\pi}{2}}\,(z^2+1)\mathrm{erf}\left(\frac{1}{\sqrt{2}}z\right) + z\,e^{-1/2z^2} - z^2\right\}.$$

That is,

$$\Omega_{10}(x,\eta) = C_1(\eta)\left\{\sqrt{\frac{\pi}{2}}\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}(1 + \cos 2\eta)^{1/2}x\right) - 1\right\}$$

and

$$\Psi_{10}(x,\eta) = -\frac{1}{4}C_1(\eta)\left\{\left((1 + \cos 2\eta)x^2 + 1\right) \text{erf}\left(\sqrt{\frac{1+\cos 2\eta}{2}}\, x\right)\right.$$

$$\left. + \sqrt{1+\cos 2\eta}\, x\, e^{-1/2(1+\cos 2\eta)x^2} - (1 + \cos 2\eta)\, x^2 \right\}.$$

For $n > 0$, we have:

$$\frac{\partial^2 \Omega_{1n}}{\partial x^2} + (1 + \cos 2\eta)\left[x \frac{\partial \Omega_{1n}}{\partial x} - 2n\Omega_{1n}\right] = r_{1n}\, .$$

Employing the transformation

$$x = (1 + \cos 2\eta)^{-1/2}\, z$$

in the above equation, we have

$$\frac{\partial^2 \Omega_{1n}}{\partial z^2} + z\frac{\partial \Omega_{1n}}{\partial z} - 2n\Omega_{1n} = (1 + \cos 2\eta)^{-1}\, R_{1n}(z,\eta)$$

where

$$R_{1n}(z,\eta) = 4(1 + \cos 2\eta)^{1/2} \sum_{m=0}^{1} \sum_{p=0}^{n-1} \left(\frac{\partial \Omega_{m,p}}{\partial x}\frac{\partial \Psi_{1-m,n-1-p}}{\partial \eta} - \frac{\partial \Omega_{m,p}}{\partial \eta}\frac{\partial \Psi_{1-m,n-1-p}}{\partial x}\right)$$

Hence, for $n > 0$, the equation governing $\Omega_{1n}(z, \eta)$ is

$$\frac{\partial^2 \Omega_{1n}}{\partial z^2} + z\frac{\partial \Omega_{1n}}{\partial z} - 2n\Omega_{1n}$$

$$= 4(1 + \cos 2\eta)^{-1/2} \sum_{m=0}^{1} \sum_{p=0}^{n-1} \left(\frac{\partial \Omega_{m,p}}{\partial x}\frac{\partial \Psi_{1-m,n-1-p}}{\partial \eta} - \frac{\partial \Omega_{m,p}}{\partial \eta}\frac{\partial \Psi_{1-m,n-1-p}}{\partial x}\right).$$

We consider the homogeneous differential equation corresponding to this equation:

$$\frac{\partial^2(\Omega_{1n})_h}{\partial z^2} + z\,\frac{\partial(\Omega_{1n})_h}{\partial z} - 2n(\Omega_{1n})_h = 0\ .$$

Assuming a solution of the form

$$(\Omega_{1n})_h\ (z,\eta) = e^{az^2}\ F(z,\eta)\ ,$$

we have:

$$\frac{\partial^2 F}{\partial z^2} + (4a+1)z\frac{\partial F}{\partial z} + (4a^2z^2+2az^2+2a-2n)F = 0.$$

Taking $a = -\frac{1}{4}$ yields

$$\frac{\partial^2 F}{\partial z^2} - \left(\frac{1}{4}z^2 + \frac{1}{2} + 2n\right)F = 0\ . \qquad (3.5.7)$$

Equation (3.5.7) is a differential equation of the standard form

$$y'' + (ax^2+bx+c)y = 0.$$

The solutions of such equations are parabolic cylindrical functions. Hence, we will have the homogeneous part of the vorticity coefficient functions being various indices of parabolic cylindrical functions. The particular solutions may be sought using the methods of undetermined coefficients.

CHAPTER IV

FLOW PAST AN ELLIPTIC CYLINDER

4.1 EQUATIONS GOVERNING SYMMETRICAL FLOW PAST AN IMPULSIVELY-STARTED ELLIPTIC CYLINDER

In this chapter, we apply the techniques employed in Chapter 3 for initial flow past an impulsively-started normal flat plate, to the case of initial flow past an elliptic cylinder which is started impulsively from rest. The elliptic cylinder is given, in Cartesian coordinates, by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where b > a.

The transformation

$$\xi + i\eta = \sinh^{-1}(x+y) - \xi^*, \qquad (4.1.1)$$

where $\tanh \xi^* = a/b$, is employed to transform the physical domain of the problem from the upper-half of the xy-plane to a semi-infinite strip of width π. The inverse transformation equations of (4.1.1) are given by

$$x = \sinh(\xi + \xi^*)\cos\eta,$$

and

$$y = \cosh(\xi + \xi^*)\sin\eta. \qquad (4.1.2)$$

This transformation successfully maps the surface of the elliptic cylinder to $\xi = 0$, with η varying from 0 to π and also ensures the periodicity of the physical properties of the fluid.

Application of transformation equations (4.1.2) to the equations governing flow of viscous, incompressible fluids given by (3.1.1) and (3.1.2) yields the following set of equations. Here, the streamfunction and vorticity, both functions of spatial variables ξ and η, and time t, determine the flow past an elliptic cylinder. We have:

$$\frac{\partial \zeta}{\partial t} = H^2 \left[\frac{2}{R} \left(\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} \right) + \left(\frac{\partial \psi}{\partial \xi} \frac{\partial \zeta}{\partial \eta} - \frac{\partial \zeta}{\partial \xi} \frac{\partial \psi}{\partial \eta} \right) \right] \tag{4.1.3}$$

and

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = \frac{1}{H^2} \zeta , \tag{4.1.4}$$

where

$$H^2 = \left(\frac{\partial \xi}{\partial x} \right)^2 + \left(\frac{\partial \xi}{\partial y} \right)^2$$

$$= 2 \left[\cosh 2(\xi + \xi^*) + \cos 2\eta \right]^{-1}. \tag{4.1.5}$$

It has been noted by Staniforth (1971) that for the transformation performed, as $\xi \rightarrow \infty$, the following relationships exist between the Cartesian and elliptic cylindrical coordinates:

$$x \sim \frac{1}{2} e^{\xi} \cos\eta ,$$

and

$$y \sim \frac{1}{2} e^{\xi} \sin\eta . \tag{4.1.6}$$

In specifying the boundary conditions to be satisfied by the vorticity and streamfunction, in the transformed coordinate system, we employ the original boundary conditions given by (1.2.9) together with relations (4.1.6) and

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\xi},$$
$$\frac{\partial\psi}{\partial\eta} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\eta}. \qquad \textbf{(4.1.7)}$$

The transformed formulation of the problem may then be summarized as follows.

For $t < 0$: $\zeta = 0$ throughout the flow field.

For $t \geq 0$;

$$\frac{\partial\zeta}{\partial t} = H^2\left\{\frac{2}{R}\left(\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2}\right) + \frac{\partial(\psi,\zeta)}{\partial(\xi,\eta)}\right\}, \qquad \textbf{(4.1.3)}$$

and

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = \frac{1}{H^2}\,\zeta, \qquad \textbf{(4.1.4)}$$

subject to the conditions

$$\psi = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{on} \quad \xi = 0, \qquad \textbf{(4.1.8)}$$

$$\psi = \zeta = 0 \quad \text{on} \quad \eta = 0,\pi\,, \qquad \textbf{(4.1.9)}$$

$$e^{-\xi}\frac{\partial\psi}{\partial\xi} \to \frac{1}{2}\,e^{\xi'}\sin\eta$$

and

$$e^{-\xi}\frac{\partial\psi}{\partial\eta} \to \frac{1}{2}e^{\xi^*}\cos\eta \quad \text{as} \quad \xi \to \infty, \tag{4.1.10}$$

and

$$\zeta \to 0 \quad \text{as} \quad \xi \to \infty. \tag{4.1.11}$$

We observe that apart from the anti-symmetric condition expressed in (4.1.9) and the far-field condition of (4.1.11), all the boundary conditions of the problem can be considered as being imposed solely on the streamfunction ψ. However, conditions on the vorticity ζ are contained implicitly in the non-linear coupling of the two equations (4.1.3) and (4.1.4). While the uncoupling of these equations is difficult, we can extract from the equations the desired information in the form of global integral conditions on the vorticity function.

We assume sine series expansions for the streamfunction and the vorticity of the following form:

$$\psi(\xi,\eta,t) = \sum_{n=1}^{\infty} f_n(\xi,t)\ \sin n\eta, \tag{4.1.12}$$

and

$$\zeta(\xi,\eta,t) = \sum_{n=1}^{\infty} g_n(\xi,t)\ \sin n\eta. \tag{4.1.13}$$

Substitution of the above series in equations (4.1.3) and (4.1.4) yields the following equations in $f_n(\xi,t)$ and $g_n(\xi, t)$.

$$\frac{\partial g_n}{\partial t} = H^2 \left\{ \frac{2}{R} \left(\frac{\partial^2 g_n}{\partial \xi^2} - n^2 g_n \right) \right.$$

$$\left. + \sum_{m=1}^{\infty} \left(m g_m \frac{\partial f_n}{\partial \xi} - m f_m \frac{\partial g_n}{\partial \xi} \right) \cos m\eta \right\}. \tag{4.1.14}$$

and

$$\frac{\partial^2 f_n}{\partial \xi^2} - n^2 f_n = \frac{1}{H^2} g_n . \tag{4.1.15}$$

Employing (4.1.12) and (4.1.13) in boundary conditions (4.1.8), (4.1.10) and (4.1.11), we obtain the following boundary conditions on the coefficient functions $f_n(\xi, t)$ and $g_n(\xi, t)$:

$$f_n = \frac{\partial f_n}{\partial \xi} = 0 \quad \text{on} \quad \xi = 0, \tag{4.1.16}$$

$$e^{-\xi} \frac{\partial f_n}{\partial \xi} \rightarrow \frac{1}{2} e^{\xi^*} \delta_{n,1} \quad \text{and}$$

$$e^{-\xi} f_n \rightarrow \frac{1}{2} e^{\xi^*} \delta_{n,1} \quad \text{as} \quad \xi \rightarrow \infty, \tag{4.1.17}$$

and

$$g_n \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty, \tag{4.1.18}$$

where

$$\delta_{n,1} = \begin{cases} 1 & ; \quad n = 1 \\ 0 & ; \quad n \neq 1 \end{cases}$$

We consider equation (4.1.15), and in particular, we observe that the LHS of this equation can be manipulated as follows:

$$\frac{\partial^2 f_n}{\partial \xi^2} - n^2 f_n = e^{n\xi} \left\{ e^{-n\xi} \frac{\partial^2 f_n}{\partial \xi^2} - n\, e^{-n\xi} \frac{\partial f_n}{\partial \xi} + n\, e^{-n\xi} \frac{\partial f_n}{\partial \xi} - n^2 e^{-n\xi} f_n \right\}$$

$$= e^{n\xi} \left\{ \frac{\partial}{\partial \xi} \left[e^{-n\xi} \frac{\partial f_n}{\partial \xi} + n\, e^{-n\xi} f_n \right] \right\}.$$

Then, equation (4.1.15) may be expressed as

$$\frac{\partial}{\partial \xi} \left[e^{-n\xi} \frac{\partial f_n}{\partial \xi} + n\, e^{-n\xi} f_n \right] = \frac{1}{H^2} e^{-n\xi} g_n,$$

which is integrated wrt ξ, from $\xi = 0$ to $\xi \to \infty$, to yield the integral condition:

$$\int_0^\infty \left[\cosh 2(\xi + \xi^*) + \cos 2\eta\right] e^{-n\xi} g_n(\xi,t)\, d\xi = 2e^{\xi^*} \delta_{n,1}. \qquad (4.1.19)$$

Since

$$\zeta(\xi,\eta,t) = \sum_{n=1}^{\infty} g_n(\xi,t) \sin n\eta,$$

therefore

$$\int_0^\pi \zeta \sin k\eta\, d\eta = \sum_{n=1}^{\infty} \left[g_n(\xi,t) \cdot \int_0^\pi \sin n\eta \sin k\eta\, d\eta \right],$$

and we have:

$$g_k(\xi,t) = \frac{2}{\pi}\int_0^{\pi} \zeta(\xi,\eta,t)\sin k\eta \, d\eta .$$

Employing the above in the integral condition given in (4.1.19), we have the following equivalent double integral expression of that global condition:

$$\int_0^{\pi}\int_0^{\infty} \left[\cosh 2(\xi+\xi^*) + \cos 2\eta\right] \sin n\eta \; e^{-n\xi}\, \zeta(\xi,\eta,t)\, d\xi\, d\eta$$

$$= \pi\, e^{\xi^*}\, \delta_{n,1} . \tag{4.1.20}$$

4.2 THE BOUNDARY-LAYER TYPE TRANSFORMATION

Application of boundary-layer transformations through introduction of the scaled variable

$$z = \frac{1}{\lambda}\,\xi$$

allows scrutiny of the details of the flow for small time t, and close to the surface of the elliptic cylinder. Here, λ is the boundary-layer thickness, given by $\lambda = 2(2t/R)^{1/2}$. It is known from the boundary-layer theory of this problem that the streamfunction is proportional to λ, while the vorticity is inversely proportional to λ. Thus, these functions are also scaled accordingly. The following transformations are used:

$$\xi = \lambda z, \quad \psi(\xi,\eta,t) = \lambda\Psi(z,\eta,t), \quad \zeta(\xi,\eta,t) = \frac{1}{\lambda}\,\Omega(z,\eta,t), \tag{4.2.1}$$

where

$$\lambda = 2\left(\frac{2t}{R}\right)^{1/2}. \tag{4.2.2}$$

The coefficient functions are scaled in a like manner:

$$f_n(\xi,t) = \lambda\ F_n(z,t) \quad , \quad g_n(\xi,t) = \frac{1}{\lambda}\ G_n(z,t). \tag{4.2.3}$$

Employing transformation equations (4.2.3) in equations (4.1.15) and (4.1.14), we obtain the following equations:

$$\frac{\partial^2 F_n}{\partial z^2} - n^2\lambda^2 F_n = \frac{1}{H^2}\ G_n \tag{4.2.4}$$

and

$$\frac{1}{2} H^2 \frac{\partial^2 G_n}{\partial z^2} + z\frac{\partial G_n}{\partial z} + (1 - \frac{1}{2} H^2 n^2 \lambda^2)\ G_n$$

$$= 2t\frac{\partial G_n}{\partial t} - 2t\ H^2 \sum_{m=1}^{\infty}\left(m\ G_m \frac{\partial F_n}{\partial z} - m\ F_m \frac{\partial G_n}{\partial z}\right)\cos m\eta. \tag{4.2.5}$$

The boundary conditions given in (4.1.16) to (4.1.18) are easily scaled according to the boundary-layer variables to obtain conditions on $F_n(z, t)$ and $G_n(z, t)$:

$$F_n = \frac{\partial F_n}{\partial z} = 0 \quad \text{along} \quad z = 0, \tag{4.2.6}$$

$$e^{-\lambda z}\,\frac{\partial F_n}{\partial z} \rightarrow \frac{1}{2}\,e^{\xi^*}\,\delta_{n,1},$$

$$e^{-\lambda z}\cdot \lambda\, F_n \rightarrow \frac{1}{2}\,e^{\xi^*}\,\delta_{n,1} \quad \text{as} \quad z \rightarrow \infty, \tag{4.2.7}$$

and

$$G_n \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty. \tag{4.2.8}$$

The integral conditions on the coefficient functions $G_n(z, t)$ are obtained through similar alterations of (4.1.19):

$$\int_0^\infty \left[\cosh\, 2(\lambda z + \xi^*) + \cos\, 2\eta\right] e^{-n\lambda z}\, G_n(z,t)\, dz = 2e^{\xi^*}\,\delta_{n,1}. \tag{4.2.9}$$

Scaling of (4.1.20) yields a global integral condition on the vorticity function, $\Omega(z, \eta, t)$:

$$\int_0^\pi \int_0^\infty \left[\cosh\, 2(\lambda z + \xi^*) + \cos\, 2\eta\right] \sin\, n\eta\; e^{-n\lambda z}\Omega(z,\eta,t)\, dz\, d\eta$$

$$= \pi\, e^{\xi^*}\,\delta_{n,1}. \tag{4.2.10}$$

4.3 THE INITIAL SOLUTION

While the introduction of sine series expansions for the vorticity and streamfunctions facilitated the development of the global integral condition on the vorticity function, it is useful to again consider the equations governing $\zeta(\xi, \eta, t)$ and $\psi(\xi, \eta, t)$. Applying now the boundary layer transformations to equations (3.1.3) and (3.1.4), we obtain

$$\frac{\partial^2\Omega}{\partial z^2} + \frac{2}{H^2} z \frac{\partial\Omega}{\partial z} + \frac{2}{H^2}\Omega = 4t \left\{ \frac{1}{H^2}\frac{\partial\Omega}{\partial t} + \frac{\partial(\Psi,\Omega)}{\partial(\eta,z)} \right\} - \lambda^2 \frac{\partial^2\Omega}{\partial\eta^2} \quad (4.3.1)$$

and

$$\frac{\partial^2\Psi}{\partial z^2} + \lambda^2 \frac{\partial^2\Psi}{\partial\eta^2} = \frac{1}{H^2}\Omega. \quad (4.3.2)$$

Taking $t = 0$, and hence $\lambda = 0$, in the above equations, we arrive at equations governing the initial vorticity and streamfunction in the boundary-layer.

$$\frac{\partial^2\Omega}{\partial z^2} + \frac{2}{H^2} z \frac{\partial\Omega}{\partial z} + \frac{2}{H^2}\Omega = 0 \quad (4.3.3)$$

and

$$\frac{\partial^2\Psi}{\partial z^2} = \frac{1}{H^2}\Omega. \quad (4.3.4)$$

The boundary conditions to be satisfied by $\Psi(z, \eta, t = 0)$ and $\Omega(z, \eta, t = 0)$ are easily derived from conditions (4.1.8) and (4.1.11):

$$\Psi = \frac{\partial\Psi}{\partial z} = 0 \quad \text{along} \quad z = 0 \quad (4.3.5)$$

and

$$\Omega \to 0 \quad \text{as} \quad z \to \infty. \quad (4.3.6)$$

Taking $\lambda = 0$ in the integral equation expressed in (4.2.10) yields the integral condition to be satisfied by the initial vorticity function.

$$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^* + \cos 2\eta)\sin n\eta\ \Omega(z,\eta,t=0)\ dz\ d\eta = \pi\ e^{\xi^*}\ \delta_{n,1} \qquad (4.3.7)$$

Using the expression for the metric H^2 given in (4.1.5) in equations (4.3.3) and (4.3.4), we rewrite these equations as

$$\frac{\partial^2\Omega}{\partial z^2} + \left[\cosh 2\xi^* + \cos 2\eta\right]\left\{z\frac{\partial\Omega}{\partial z}+\Omega\right\} = 0 \qquad (4.3.8)$$

and

$$\frac{\partial^2\Psi}{\partial z^2} = \left[\cosh 2\xi^* + \cos 2\eta\right]\Omega. \qquad (4.3.9)$$

Applying the transformation

$$z = f(\eta)u \qquad (4.3.10)$$

to equation (4.3.8), we get

$$\frac{\partial^2\Omega}{\partial u^2} + f^2(\eta)\left[\cosh 2\xi^* + \cos 2\eta\right]\left\{u\frac{\partial\Omega}{\partial u}+\Omega\right\} = 0.$$

Taking

$$f^2(\eta) = \left[\cosh 2\xi^* + \cos 2\eta\right]^{-1}, \qquad (4.3.11)$$

i.e. $\quad z = (\cosh 2\xi^* + \cos 2\eta)^{-1/2}\ u,$

we have

$$\frac{\partial^2\Omega}{\partial u^2} + u\frac{\partial\Omega}{\partial u} + \Omega = 0. \tag{4.3.12}$$

Likewise, equation (4.3.9) is transformed to yield

$$\frac{\partial^2\Psi}{\partial u^2} = \frac{1}{2}\,\Omega\,. \tag{4.3.13}$$

Two linearly independent solutions of equation (4.3.12) which satisfy $\Omega \to 0$ as $u \to \infty$ are given by

$$\Omega_1(u,\eta,t{=}0) = g(\eta)\; e^{-1/2\, u^2}$$

and

$$\Omega_2(u,\eta,t{=}0) = h(\eta)\; e^{-1/2\, u^2} \int_0^{\frac{u}{2}} e^{v^2}\, dv.$$

Employing Ω_2 in the integral condition (4.3.7) yields a divergent integral. Thus, $h(\eta)$ must be taken to be zero, and hence, the solution to (4.3.12) is

$$\Omega(u,\eta,t{=}0) = g(\eta)\; e^{-1/2\, u^2}. \tag{4.3.14}$$

That is,

$$\Omega(z,\eta,t{=}0) = g(\eta)\ e^{-1/2(\cosh 2\xi^* + \cos 2\eta)z^2}. \tag{4.3.15}$$

In the above, $g(\eta)$ may be determined through the integral condition (4.3.7). That is,

$$\int_0^\pi \int_0^\infty (\cosh 2\xi^* + \cos 2\eta) \sin n\eta \cdot g(\eta)\ e^{-1/2(\cosh 2\xi^* + \cos 2\eta)z^2}\ dz\ d\eta = \pi\ e^{\xi^*}\ \delta_{n,1}.$$

We have:

$$\int_0^\pi (\cosh 2\xi^* + \cos 2\eta) \sin n\eta\ g(\eta) \cdot \int_0^\infty e^{-1/2(\cosh 2\xi^* + \cos 2\eta)z^2}\ dz\ d\eta$$

$$= \pi\ e^{\xi^*}\ \delta_{n,1}. \tag{4.3.16}$$

We consider the integral:

$$I = \int_0^\infty e^{-1/2(\cosh 2\xi^* + \cos 2\eta)z^2}\ dz.$$

Letting $w = \left[\dfrac{\cosh 2\xi^* + \cos 2\eta}{2}\right]^{1/2} z, \qquad dw = \left[\dfrac{\cosh 2\xi^* + \cos 2\eta}{2}\right]^{1/2} dz$

and $dz = \left[\dfrac{2}{\cosh 2\xi^* + \cos 2\eta}\right]^{1/2} dw.$

Therefore:

$$I = \int_0^\infty e^{-w^2} \cdot \left(\frac{2}{\cosh 2\xi^* + \cos 2\eta}\right)^{1/2} dw$$

$$= \left[\frac{2}{\cosh 2\xi^* + \cos 2\eta}\right]^{1/2} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\frac{\pi}{2}}\ (\cosh 2\xi^* + \cos 2\eta)^{-1/2}.$$

Employing the above in (4.3.16), we have

$$\sqrt{\frac{\pi}{2}} \int_0^\pi (\cosh 2\xi^* + \cos 2\eta)^{1/2} \sin n\eta\ g(\eta)\ d\eta = \pi\ e^{\xi^*}\ \delta_{n,1} \tag{4.3.17}$$

For n = 1,

$$\int_0^\pi (\cosh 2\xi^* + \cos 2\eta)^{1/2} \sin\eta\ g(\eta)\ d\eta = \sqrt{2\pi}\ e^{\xi^*}. \tag{4.3.18}$$

Observing that

$$g(\eta) = 2\sqrt{\frac{2}{\pi}}\ (\cosh 2\xi^* + \cos 2\eta)^{-1/2}\ e^{\xi^*} \sin\eta \tag{4.3.19}$$

satisfies equation (4.3.18), we have the following expression for the initial vorticity:

$$\Omega(z,\eta,t{=}0) = 2\sqrt{\frac{2}{\pi}}\; e^{\xi^*} (\cosh 2\xi^* + \cos 2\eta)^{-1/2} \sin\eta \cdot e^{-1/2\,(\cosh 2\xi^* + \cos 2\eta) z^2}. \qquad (4.3.20)$$

Taking $n \neq 1$ in (4.3.17), we have

$$\int_0^\pi (\cosh 2\xi^* + \cos 2\eta)^{1/2} \sin n\eta\; g(\eta)\; d\eta = 0$$

Employing (4.3.19) in the above, the integral on the LHS takes the form

$$I = 2\sqrt{\frac{2}{\pi}}\; e^{\xi^*} \int_0^\pi \sin n\eta\; \sin\eta\; d\eta = 0.$$

Hence, the integral condition expressed in equation (4.3.17) is satisfied by the expression for $g(\eta)$ given by (4.3.19). Furthermore, equation (4.3.8) is satisfied by the expression for the initial vorticity function given in (4.3.20). As well, this expression satisfies the far-field condition of zero vorticity as $z \to \infty$. It is valuable to plot the expression for initial vorticity for decreasing values of ξ^*, corresponding to slimmer and slimmer ellipses.

Employing (4.3.20) in equation (4.3.13) yields

$$\frac{\partial^2 \Psi}{\partial u^2} = \sqrt{\frac{2}{\pi}}\; (\cosh 2\xi^* + \cos 2\eta)^{-1/2}\; e^{\xi^*} \sin\eta\; e^{-1/2\, u^2}.$$

Integrating this equation twice wrt u, we can obtain an expression for the initial streamfunction. Taking $u = 0$ and $u = u$ as the limits of integration, we have first

$$\frac{\partial \Psi}{\partial u} = \sqrt{\frac{2}{\pi}}\ (\cosh 2\xi^* + \cos 2\eta)^{-1/2}\, e^{\xi^*} \sin\eta \int_0^u e^{-1/2\, t^2}\, dt.$$

That is,

$$\frac{\partial \Psi}{\partial u} = (\cosh 2\xi^* + \cos 2\eta)^{-1/2}\, e^{\xi^*} \sin\eta \ \mathrm{erf}\left(\frac{1}{\sqrt{2}}\, u\right).$$

Integrating again wrt u from u = 0 to u = u yields:

$$\Psi = (\cosh 2\xi^* + \cos 2\eta)^{-1/2}\, e^{\xi^*} \sin\eta \int_0^u \mathrm{erf}\left(\frac{1}{\sqrt{2}}t\right) dt,$$

i.e.

$$\Psi = \frac{e^{\xi^*} \sin\eta}{(\cosh 2\xi^* + \cos 2\eta)^{1/2}} \left\{ u\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}\, u\right) + \sqrt{\frac{2}{\pi}} \left[e^{-1/2\, u^2} - 1\right] \right\}.$$

Hence,

$$\Psi(z,\eta,t{=}0) = e^{\xi^*} \sin\eta \left\{ z\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}(\cosh 2\xi^* + \cos 2\eta)^{1/2}\, z\right) \right.$$

$$\left. + \sqrt{\frac{2}{\pi}}\ (\cosh 2\xi^* + \cos 2\eta)^{-1/2} \left[e^{-1/2(\cosh 2\xi^* + \cos 2\eta)\, z^2} - 1 \right] \right\}. \qquad (4.3.21)$$

4.4 SERIES EXPANSIONS IN POWERS OF λ

Equations (4.3.1) and (4.3.2) are the governing equations - in terms of the boundary-layer-transformed streamfunction, $\Psi(z, \eta, t)$ and vorticity, $\Omega(z, \eta, t)$ - for the unsteady, two-dimensional flow of a viscous, incompressible fluid past an impulsively-started elliptic cylinder. The expressions given in (4.3.20) and (4.3.21) describe the initial behaviour of the fluid. As in the previous chapter, it is possible to undertake a study of the behaviour of the fluid for small values of the time. Expansions of the streamfunction and vorticity in powers of λ are assumed:

$$\Psi(z,\eta,t) = \Psi_0(z,\eta,t) + \lambda\Psi_1(z,\eta,t) + \lambda^2\Psi_2(z,\eta,t) + \ldots \qquad (4.4.1)$$

and

$$\Omega(z,\eta,t) = \Omega_0(z,\eta,t) + \lambda\Omega_1(z,\eta,t) + \lambda^2\Omega_2(z,\eta,t) + \ldots \qquad (4.4.2)$$

These expansions are employed in the governing equations (4.3.1) and (4.3.2), and the coefficients of successive powers of λ are equated to zero. We first observe that

$$\begin{aligned} H^{-2} &= \frac{1}{2}\left[\cosh 2(\lambda x + \xi^*) + \cos 2\eta\right] \\ &= \frac{1}{2}\left[\cosh 2\lambda z \cosh 2\xi^* + \sin 2\lambda z \sinh 2\xi^* + \cos 2\eta\right]. \end{aligned}$$

Since

$$\cosh 2\lambda z = 1 + \frac{1}{2!}(2\lambda z)^2 + \frac{1}{4!}(2\lambda z)^4 + \ldots$$

and

$$\sinh 2\lambda z = (2\lambda z) + \frac{1}{3!}(2\lambda z)^3 + \frac{1}{5!}(2\lambda z)^5 + \ldots ,$$

hence

$$H^{-2} = \frac{1}{2}\Big\{[\cosh 2\xi^* + \cos 2\eta] + \lambda[2z\sinh 2\xi^*]$$

$$+ \lambda^2\left[\frac{1}{2!}(2z)^2 \cosh 2\xi^*\right] + \lambda^3\left[\frac{1}{3!}(2z)^3 \sinh 2\xi^*\right]$$

$$+ \dots \Big\} .$$

Then, equation (4.3.2) yields:

$$\frac{\partial^2\Psi_0}{\partial z^2} + \lambda\frac{\partial^2\Psi_1}{\partial z^2} + \lambda^2\left(\frac{\partial^2\Psi_2}{\partial z^2} + \frac{\partial^2\Psi_0}{\partial\eta^2}\right) + \lambda^3\left(\frac{\partial^2\Psi_3}{\partial z^2} + \frac{\partial^2\Psi_1}{\partial\eta^2}\right) + \dots$$

$$= \frac{1}{2}\Big\{[\cosh 2\xi^* + \cos 2\eta] + \lambda[(2z)\sinh 2\xi^*]$$

$$+ \lambda^2\left[\frac{1}{2!}(2z)^2 \cosh 2\xi^*\right] + \dots\Big\}\left\{\Omega_0 + \lambda\Omega_1 + \lambda^2\Omega_2 + \dots\right\} ,$$

$$= \frac{1}{2}\Big\{[\cosh 2\xi^* + \cos 2\eta]\Omega_0 + \lambda\left[(\cosh 2\xi^* + \cos 2\eta)\Omega_1 + (2z \sinh 2\xi^*)\Omega_0\right]$$

$$+ \lambda^2\left[(\cosh 2\xi^* + \cos 2\eta)\Omega_2 + (2z \sinh 2\xi^*)\Omega_1 + \left(\frac{1}{2!}\, 2^2 z^2 \cosh 2\xi^*\right)\Omega_0\right]$$

$$+ \lambda^3\Big[(\cosh 2\xi^* + \cos 2\eta)\Omega_3 + (2z \sinh 2\xi^*)\Omega_2$$

$$+ \left(\frac{1}{2!}\, 2^2 z^2 \cosh 2\xi^*\right)\Omega_1 + \left(\frac{1}{3!}\, 2^3 z^3 \sinh 2\xi^*\right)\Omega_0\Big] + \dots .$$

Hence, we can generate the following equations:

$$\frac{\partial^2\Psi_0}{\partial z^2} = \frac{1}{2}[\cosh 2\xi^* + \cos 2\eta]\Omega_0 ,$$

$$\frac{\partial^2\Psi_1}{\partial z^2} = \frac{1}{2}\left\{[\cosh 2\xi^* + \cos 2\eta]\Omega_1 + [2z \sinh 2\xi^*]\Omega_0\right\} .$$

$$\frac{\partial^2\Psi_2}{\partial z^2} + \frac{\partial^2\Psi_0}{\partial \eta^2} = \frac{1}{2}\Big\{ \left[\cosh 2\xi^* + \cos 2\eta\right]\Omega_2 + \left[2z \sinh 2\xi^*\right]\Omega_1 + \left[\frac{1}{2!}\, 2^2\, z^2 \cosh 2\xi^*\right]\Omega_0 \Big\}\,.$$

$$\frac{\partial^2\Psi_3}{\partial z^2} + \frac{\partial^2\Psi_1}{\partial \eta^2} = \frac{1}{2}\Big\{ \left[\cosh 2\xi^* + \cos 2\eta\right]\Omega_3 + \left[2z \sinh 2\xi^*\right]\Omega_2 + \left[\frac{1}{2!}\, 2^2\, z^2 \cosh 2\xi^*\right]\Omega_1 + \left[\frac{1}{3!}\, 2^3\, z^3 \sinh 2\xi^*\right]\Omega_0 \Big\}\,.$$

In general, we have the following governing equations for $\Psi_K(z, \eta, t)$, where $K \varepsilon \mathbf{N}_0$:

$$\frac{\partial^2\Psi_K}{\partial z^2} + \frac{\partial^2\Psi_{K-2}}{\partial \eta^2} = \frac{1}{2}\Big\{ \left[\cosh 2\xi^* + \cos 2\eta\right]\Omega_K + \left[2z \sinh 2\xi^*\right]\Omega_{K-1} + \left[\frac{1}{2!}\, 2^2\, z^2 \cosh 2\xi^*\right]\Omega_{K-2} + \dots + \left[\frac{1}{(K-1)!}\, 2^{K-1}\, z^{K-1}\, cs(K-1)\right]\Omega_1 + \left[\frac{1}{(K)!}\, 2^K\, z^K\, cs(K)\right]\Omega_0 \Big\}\,. \tag{4.4.3}$$

where

$$cs(K) = \begin{cases} \cosh(2\xi^*) & ; \quad K \text{ is even} \\ \sinh(2\xi^*) & ; \quad K \text{ is odd.} \end{cases}$$

Likewise, equation (4.3.1) implies:

$$\left[\frac{\partial^2\Omega_0}{\partial z^2} + \lambda\frac{\partial^2\Omega_1}{\partial z^2} + \lambda^2\frac{\partial^2\Omega_2}{\partial z^2} + \ldots\right]$$

$$+ \left\{[\cosh 2\xi^* + \cos 2\eta] + \lambda[2\ z \sinh 2\xi^*] + \lambda^2\left[\frac{1}{2!}\ 2^2\ z^2 \cosh 2\xi^*\right] + \ldots\right\}$$

$$\cdot \left\{\left[z\frac{\partial\Omega_0}{\partial z} + \Omega_0\right] + \lambda\left[z\frac{\partial\Omega_1}{\partial z} + \Omega_1\right] + \lambda^2\left[z\frac{\partial\Omega_2}{\partial z} + \Omega_2\right] + \ldots\right\}$$

$$= \left\{[\cosh 2\xi^* + \cos 2\eta] + \lambda[2\ z \sinh 2\xi^*] + \lambda^2\left[\frac{1}{2!}\ 2^2\ z^2 \cosh 2\xi^*\right] + \ldots\right\}$$

$$\cdot \left[\left\{\frac{\partial\Omega_0}{\partial t} + \lambda\frac{\partial\Omega_1}{\partial t} + \lambda^2\frac{\partial\Omega_2}{\partial t} + \ldots\right\} \cdot 2t + \left\{\lambda\Omega_1 + 2\lambda^2\Omega_2 + 3\lambda^3\Omega_3 + \ldots\right\}\right]$$

$$- \left[\lambda^2\frac{\partial^2\Omega_0}{\partial\eta^2} + \lambda^3\frac{\partial^2\Omega_1}{\partial\eta^2} + \lambda^4\frac{\partial^2\Omega_2}{\partial\eta^2} + \ldots\right]$$

$$+ 4t \sum_{n=0}^{\infty} \sum_{p=0}^{n} \lambda^n \left[\frac{\partial\Psi_p}{\partial\eta}\frac{\partial\Omega_{n-p}}{\partial z} - \frac{\partial\Psi_p}{\partial z}\frac{\partial\Omega_{n-p}}{\partial\eta}\right] .$$

That is,

$$\left[\frac{\partial^2 \Omega_0}{\partial z^2} + (\cosh 2\xi^* + \cos 2\eta)\left(z\frac{\partial \Omega_0}{\partial z} + \Omega_0\right)\right]$$

$$+ \lambda\left[\frac{\partial^2 \Omega_1}{\partial z^2} + (\cosh 2\xi^* + \cosh 2\eta)\left(z\frac{\partial \Omega_1}{\partial z} + \Omega_1\right) + (2\ z \sinh 2\xi^*)\left(z\frac{\partial \Omega_0}{\partial z} + \Omega_0\right)\right]$$

$$+ \lambda^2\left[\frac{\partial^2 \Omega_2}{\partial z^2} + \frac{\partial^2 \Omega_0}{\partial \eta^2} + (\cosh 2\xi^* + \cos 2\eta)\left(z\frac{\partial \Omega_2}{\partial z} + \Omega_2\right) + (2\ z \sinh 2\xi^*)\left(z\frac{\partial \Omega_1}{\partial z} + \Omega_1\right)\right.$$

$$\left. + \left(\frac{1}{2!}\ 2^2\ z^2 \cosh 2\xi^*\right)\left(z\frac{\partial \Omega_0}{\partial z} + \Omega_0\right)\right]$$

$$+ \ldots + \lambda^K\left[\frac{\partial^2 \Omega_K}{\partial z^2} + \frac{\partial^2 \Omega_{K-2}}{\partial \eta^2} + (\cosh 2\xi^* + \cos 2\eta)\left(z\frac{\partial \Omega_K}{\partial z} + \Omega_K\right)\right.$$

$$+ (2\ z \sinh 2\xi^*)\left(z\frac{\partial \Omega_{K-1}}{\partial z} + \Omega_{K-1}\right)$$

$$\left. + \ldots + \left(\frac{1}{K!}\ 2^K\ z^K \cdot cs(K)\right)\left(z\frac{\partial \Omega_0}{\partial z} + \Omega_0\right)\right]$$

$$+ \ldots$$

$$= 2t \left\{ \left[(\cosh 2\xi^* + \cos 2\eta) \frac{\partial \Omega_0}{\partial t} \right] \right.$$

$$+ \lambda \left[(\cosh 2\xi^* + \cos 2\eta) \frac{\partial \Omega_1}{\partial t} + (2\ z \sinh 2\xi^*) \frac{\partial \Omega_0}{\partial t} \right]$$

$$+ \lambda^2 \left[(\cosh 2\xi^* + \cos 2\eta) \frac{\partial \Omega_2}{\partial t} + (2\ z \sinh 2\xi^*) \frac{\partial \Omega_1}{\partial t} \right.$$

$$\left. + \left(\frac{1}{2!}\ 2^2\ z^2 \cosh 2\xi^* \right) \frac{\partial \Omega_0}{\partial t} \right]$$

$$+ \ldots + \lambda^K \left[(\cosh 2\xi^* + \cos 2\eta) \frac{\partial \Omega_K}{\partial t} + (2\ z \sinh 2\xi^*) \frac{\partial \Omega_{K-1}}{\partial t} \right.$$

$$\left. \left. + \ldots + \left(\frac{1}{K!}\ 2^K\ z^K\ cs(K) \right) \frac{\partial \Omega_0}{\partial t} \right] + \ldots \right\}$$

$$+ 4t \left\{ \left[\frac{\partial \Psi_0}{\partial \eta} \frac{\partial \Omega_0}{\partial z} - \frac{\partial \Psi_0}{\partial z} \frac{\partial \Omega_0}{\partial \eta} \right] + \lambda \left[\frac{\partial \Psi_0}{\partial \eta} \frac{\partial \Omega_1}{\partial z} + \frac{\partial \Psi_1}{\partial \eta} \frac{\partial \Omega_0}{\partial z} \right. \right.$$

$$\left. \left. - \frac{\partial \Psi_0}{\partial z} \frac{\partial \Omega_1}{\partial \eta} - \frac{\partial \Psi_1}{\partial z} \frac{\partial \Omega_0}{\partial \eta} \right] + \ldots + \lambda^K \sum_{p=0}^{K} \left(\frac{\partial \Psi_p}{\partial \eta} \frac{\partial \Omega_{K-p}}{\partial z} - \frac{\partial \Psi_p}{\partial z} \frac{\partial \Omega_{K-p}}{\partial \eta} \right) \right\}$$

$$+ \Big\{\lambda\Big[(\cosh 2\xi^* + \cos 2\eta)\Omega_1\Big] + \lambda^2\Big[(\cosh 2\xi^* + \cos 2\eta) \cdot 2\Omega_2$$

$$+ (2\ z \sinh 2\xi^*)\Omega_1\Big] + \lambda^3\Big[(\cosh 2\xi^* + \cos 2\eta) \cdot 3\Omega_3$$

$$+ (2\ z \sinh 2\xi^*) \cdot (2\Omega_2) + \left(\frac{1}{2!}\ 2^2\ z^2 \cosh 2\xi^*\right)\Omega_1\Big]$$

$$+ \ldots + \lambda^K\Big[(\cosh 2\xi^* + \cos 2\eta)(k\Omega_K) + (2\ z \sinh 2\xi^*)((K-1)\Omega_{K-1})$$

$$+ \ldots + \left(\frac{1}{(K-1)!}\ 2^{K-1}\ z^{K-1}\ cs(K-1)\right)\Omega_1\Big] + \ldots\Big\}.$$

Equating the coefficients of the like powers of λ to zero, we have the following equation governing the behaviour of $\Omega_K(z, \eta, t)$:

$$\frac{\partial^2\Omega_K}{\partial z^2} + \frac{\partial^2\Omega_{K-2}}{\partial\eta^2} + (\cosh 2\xi^* + \cos 2\eta)\left(z\frac{\partial\Omega_K}{\partial z} + \Omega_K\right)$$

$$+ (2z \sinh 2\xi^*)\left(z\frac{\partial\Omega_{K-1}}{\partial z} + \Omega_{K-1}\right)$$

$$+ \ldots + \left(\frac{1}{K!}\ 2^K\ z^K\ cs(k)\right)\left(z\frac{\partial\Omega_0}{\partial z} + \Omega_0\right)$$

$$= 2t \left\{ \left[(\cosh 2\xi^* + \cos 2\eta) \frac{\partial \Omega_K}{\partial t} + (2z \sinh 2\xi^*) \frac{\partial \Omega_{K-1}}{\partial t} + \dots + \left(\frac{1}{K!} 2^K z^K cs(k) \right) \frac{\partial \Omega_0}{\partial t} \right] + 2 \sum_{p=0}^{K} \left(\frac{\partial \Psi_p}{\partial \eta} \frac{\partial \Omega_{K-p}}{\partial z} - \frac{\partial \Psi_p}{\partial z} \frac{\partial \Omega_{K-p}}{\partial \eta} \right) \right\}$$

$$+ \left[(\cosh 2\xi^* + \cos 2\eta)(k\Omega_K) + (2z \sinh 2\xi^*)((K-1)\Omega_{K-1}) + \dots + \left(\frac{1}{(K-1)!} 2^{K-1} z^{K-1} cs(K-1)\Omega_1 \right) \right].$$

That is,

$$\frac{\partial^2 \Omega_K}{\partial z^2} + \frac{\partial^2 \Omega_{K-2}}{\partial \eta^2} + (\cosh 2\xi^* + \cos 2\eta) \left[-2t \frac{\partial \Omega_K}{\partial t} + z \frac{\partial \Omega_K}{\partial z} + (1-k)\Omega_k \right]$$

$$+ (2z \sinh 2\xi^*) \left[-2t \frac{\partial \Omega_{K-1}}{\partial t} + z \frac{\partial \Omega_{K-1}}{\partial z} + (2-K)\Omega_{K-1} \right]$$

$$+ \left(\frac{1}{2!} 2^2 z^2 \cosh 2\xi^* \right) \left[-2t \frac{\partial \Omega_{K-2}}{\partial t} + z \frac{\partial \Omega_{K-2}}{\partial z} + (3-K)\Omega_{K-2} \right]$$

$$+ \dots + \left(\frac{1}{(K-1)!} 2^{K-1} z^{K-1} cs(K-1) \right) \left[-2t \frac{\partial \Omega_1}{\partial t} + z \frac{\partial \Omega_1}{\partial z} \right]$$

$$\left(\frac{1}{K!} 2^K z^K cs(K) \right) \left[-2t \frac{\partial \Omega_0}{\partial t} + z \frac{\partial \Omega_0}{\partial z} + \Omega_0 \right]$$

$$= 4t \sum_{p=0}^{K} \left(\frac{\partial \Psi_p}{\partial \eta} \frac{\partial \Omega_{K-p}}{\partial z} - \frac{\partial \Psi_p}{\partial z} \frac{\partial \Omega_{K-p}}{\partial \eta} \right). \tag{4.4.4}$$

The boundary conditions for $\Psi_K(z, \eta, t)$ and $\Omega_K(z, \eta, t)$ are easily derived by employing expansions (4.1.1) and (4.2.2) in conditions (4.3.5) and (4.3.6). We have:

$$\Psi_K = \frac{\partial \Psi_K}{\partial z} = 0 \quad \text{on} \quad z = 0 \tag{4.4.5}$$

and

$$\Omega_K \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty \,. \tag{4.4.6}$$

In a like manner, we employ the power series expansion for $\Omega(z, \eta, t)$ in the integral condition (4.2.10) to generate global conditions on $\Omega_K(z, \eta, t)$. Since

$$\begin{aligned}
&\left[\cosh 2(\lambda z + \xi^*) + \cos 2\eta\right] e^{-n\lambda z} \\
&= \left\{\left[\cosh 2\xi^* + \cos 2\eta\right] + \lambda(2z \sinh 2\xi^*) + \lambda^2\left(\frac{1}{2!}\, 2^2\, z^2 \cosh 2\xi^*\right) + \ldots\right\} \\
&\quad \cdot \left\{1 - n\lambda z + \frac{1}{2!}\, n^2\, \lambda^2\, z^3 + \ldots\right\} \\
&= \left[\cosh 2\xi^* + \cos 2\eta\right] + \lambda\left[(\cosh 2\xi^* + \cos 2\eta)(-nz) + (2z \sinh 2\xi^*)\right] \\
&\quad + \lambda^2\left[(\cosh 2\xi^* + \cos 2\eta)\left(\frac{1}{2!} n^2 z^2\right) + (2z \sinh 2\xi^*)(-nz) + \left(\frac{1}{2!} 2^2 z^2 \cosh 2\xi^*\right)\right] \\
&\quad + \ldots + \lambda^K\left[(\cosh 2\xi^* + \cos 2\eta)\left((-1)^K \frac{1}{K!} n^K z^K\right) + (2z \sinh 2\xi^*)\left((-1)^{K-1} \frac{1}{(K-1)!} n^{K-1} z^{K-1}\right)\right. \\
&\quad \left. + \ldots + \left(\frac{1}{K!}\, 2^K\, z^K\, cs(K)\right)\right],
\end{aligned}$$

therefore

$$\left[\cosh 2(\lambda z+\xi^{*})+\cos 2\eta\right]e^{-n\lambda z}\,\Omega(z,\eta,t)$$

$$=\left[\cosh 2\xi^{*}+\cos 2\eta\right]\Omega_0$$

$$+\lambda\left\{\left[\cosh 2\xi^{*}+\cos 2\eta\right]\Omega_1+\left[(\cosh 2\xi^{*}+\cos 2\eta)(-nz)+(2z\sinh 2\xi^{*})\right]\Omega_0\right\}$$

$$+\lambda^2\left\{\left[\cosh 2\xi^{*}+\cos 2\eta\right]\Omega_2+\left[(\cosh 2\xi^{*}+\cos 2\eta)(-nz)+(2z\sinh \xi^{*})\right]\Omega_1\right.$$

$$\left.+\left[(\cosh 2\xi^{*}+\cos 2\eta)\left(\frac{1}{2!}n^2z^2\right)+(2z\sinh 2\xi^{*})(-nz)+\left(\frac{1}{2!}2^2z^2\cosh 2\xi^{*}\right)\right]\Omega_0\right\}$$

$$+\ldots+\lambda^K\left\{\left[\cosh 2\xi^{*}+\cos 2\eta\right]\Omega_K+\left[(\cosh 2\xi^{*}+\cos 2\eta)(-nz)+(2z\sinh 2\xi^{*})\right]\right.$$

$$\Omega_{K-1}+\ldots+\left[(\cosh 2\xi^{*}+\cos 2\eta)\left((-1)^K\frac{1}{K!}n^Kz^K\right)+(2z\sinh 2\xi^{*})\left((-1)^{K-1}\right.\right.$$

$$\left.\left.\left.\frac{1}{(K-1)!}n^{K-1}z^{K-1}\right)+\ldots+\left(\frac{1}{K!}2^Kz^K\,cs(k)\right)\right]\Omega_0\right\}+\ldots\quad.$$

We thus generate the following set of integral conditions

(i) $$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^{*}+\cos 2\eta)\sin n\eta\;\Omega_0(z,\eta,t)\,dz\,d\eta=\pi e^{\xi^{*}}\delta_{n,1},$$

(ii) $$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^{*}+\cos 2\eta)\sin n\eta\;\Omega_1(z,\eta,t)\,dz\,d\eta$$

$$=-\int_0^{\pi}\int_0^{\infty}\left[(\cosh 2\xi^{*}+\cos 2\eta)(-nz)+(2z\sinh 2\xi^{*})\right]\sin n\eta\;\Omega_0\,dz\,d\eta$$

$$+\pi e^{\xi^{*}}\delta_{n,1}\,,$$

(iii)

$$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^* + \cos 2\eta)\sin n\eta\ \Omega_2(z,\eta,t)\,dz\,d\eta$$

$$= -\int_0^{\pi}\int_0^{\infty}\Big\{\Big[(\cosh 2\xi^* + \cos 2\eta)(-nz) + (2z\sinh 2\xi^*)\Big]\Omega_1$$

$$+ \Big[(\cosh 2\xi^* + \cos 2\eta)\Big(\frac{1}{2!}n^2 z^2\Big) + (2z\sinh 2\xi^*)(-nz)$$

$$+ \Big(\frac{1}{2!}n^2 z^2 \cosh 2\xi^*\Big)\Big]\Omega_0\Big\}$$

$$\sin n\eta\,dz\,d\eta + \pi e^{\xi^*}\delta_{n,1}\ .$$

et cetera.

In general, we have the following integral condition on the $\Omega_K(z, \eta, t)$:

$$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^* + \cos 2\eta)\sin n\eta\ \Omega_K(z,\eta,t)\,dz\,d\eta$$

$$= -\int_0^{\pi}\int_0^{\infty}\Big\{\Big[(\cosh 2\xi^* + \cos 2\eta)(-nz) + (2z\sinh 2\xi^*)\Big]\Omega_{K-1}$$

$$+ \dots + \Big[(\cosh 2\xi^* + \cos 2\eta)\Big((-1)^K\frac{1}{K!}n^K z^K\Big)$$

$$+ (2z\sinh 2\xi^*)\Big((-1)^{K-1}\frac{1}{(K-1)!}n^{K-1} z^{K-1}\Big)$$

$$+ \dots + \Big(\frac{1}{K!}2^K z^K \mathrm{cs}(K)\Big)\Big]\Omega_0\Big\}\sin n\eta\,dz\,d\eta$$

$$+ \pi e^{\xi^*}\delta_{n,1}\ . \qquad (4.4.7)$$

4.5 SERIES EXPANSIONS IN POWERS OF TIME IN THE BOUNDARY LAYER

We consider the boundary-layer case of $\lambda = 0$. In order to isolate the t-dependence we make further expansions in powers of time as follows:

$$\Psi_0(z,\eta,t) = \Psi_{00}(z,\eta) + t\Psi_{01}(z,\eta) + t^2\Psi_{02}(z,\eta) + \ldots \tag{4.5.1}$$

and

$$\Omega_0(z,\eta,t) = \Omega_{00}(z,\eta) + t\Omega_{01}(z,\eta) + t^2\Omega_{02}(z,\eta) + \ldots \; . \tag{4.5.2}$$

We substitute these expansions, which are valid for small values of t, in equations (4.4.3) and (4.4.4) after taking K = 0 in those equations.

Equation (4.4.3) implies:

$$\frac{\partial^2\Psi_{0p}}{\partial z^2} = \frac{1}{2}\left[\cosh\, 2\xi^* + \cos\, 2\eta\right]\Omega_{0p} \, , \tag{4.5.3}$$

for p=0,1,2,3,....

Since

$$\frac{\partial\Omega_0}{\partial t} = \Omega_{01} + 2t\,\Omega_{02} + 3t^2\,\Omega_{03} + \ldots \, ,$$

therefore

$$2t\left(\cosh\, 2\xi^* + \cos\, 2\eta\right)\frac{\partial\Omega_0}{\partial t}$$

$$= 2\left(\cosh\, 2\xi^* + \cos\, 2\eta\right)\left[\Omega_{01}t + 2t^2\Omega_{02} + 3t^3\Omega_{03} + \ldots\right],$$

and so equation (4.4.4) yields:

$$\left(\frac{\partial\Omega_{00}}{\partial z^2} + t\frac{\partial\Omega_{01}}{\partial z^2} + t^2\frac{\partial\Omega_{02}}{\partial z^2} + \dots\right)$$

$$+ (\cosh 2\xi^* + \cos 2\eta)\left[\left(z\frac{\partial\Omega_{00}}{\partial z} + \Omega_{00}\right) + t\left(z\frac{\partial\Omega_{01}}{\partial z} + \Omega_{01}\right)\right.$$

$$\left. + t^2\left(z\frac{\partial\Omega_{02}}{\partial z} + \Omega_{02}\right) + \dots\right]$$

$$= 4t\left\{\left(\frac{\partial\Psi_{00}}{\partial\eta}\frac{\partial\Omega_{00}}{\partial z} - \frac{\partial\Psi_{00}}{\partial z}\frac{\partial\Omega_{00}}{\partial\eta}\right)\right.$$

$$+ t\left[\sum_{q=0}^{1}\left(\frac{\partial\Psi_{0q}}{\partial\eta}\frac{\partial\Omega_{0,1-q}}{\partial z} - \frac{\partial\Psi_{0q}}{\partial z}\frac{\partial\Omega_{0,1-q}}{\partial\eta}\right)\right]$$

$$+ t^2\left[\sum_{q=0}^{2}\left(\frac{\partial\Psi_{0q}}{\partial\eta}\frac{\partial\Omega_{0,2-q}}{\partial z} - \frac{\partial\Psi_{0q}}{\partial z}\frac{\partial\Omega_{0,2-q}}{\partial\eta}\right)\right]$$

$$+ \dots + t^p\left[\sum_{q=0}^{p}\left(\frac{\partial\Psi_{0q}}{\partial\eta}\frac{\partial\Omega_{0,p-q}}{\partial z} - \frac{\partial\Psi_{0q}}{\partial z}\frac{\partial\Omega_{0,p-q}}{\partial\eta}\right)\right]$$

$$\left. + \dots \right\}$$

$$+ 2t(\cosh 2\xi^* + \cos 2\eta)\left\{\Omega_{01} + 2t\Omega_{02} \quad 3t^2\Omega_{03} + \dots + (p+1)t^p\Omega_{0,p+1} + \dots\right\}.$$

Equating coefficients of like powers of t, we have, for p=0,1,2,3,...:

$$\frac{\partial^2\Omega_{0p}}{\partial z^2} + (\cosh 2\xi^* + \cos 2\eta)\left[z\frac{\partial\Omega_{0p}}{\partial z} + (1-2p)\Omega_{0p}\right]$$

$$= 4\sum_{q=0}^{p-1}\left(\frac{\partial\Psi_{0q}}{\partial\eta}\frac{\partial\Omega_{0,p-q}}{\partial z} - \frac{\partial\Psi_{0q}}{\partial z}\frac{\partial\Omega_{0,p-q}}{\partial\eta}\right) \tag{4.5.4}$$

Thus, equations (4.5.3) and (4.5.4) govern the behaviour of the coefficient functions $\Psi_{0p}(z, \eta)$ and $\Omega_{0p}(z, \eta)$. The associated boundary conditions obtained from (4.4.5) and (4.4.6) are that, for p=0,1,2,3,...:

$$\Psi_{0p} = \frac{\partial\Psi_{0p}}{\partial z} = 0 \quad \text{when} \quad z = 0\ , \tag{4.5.5}$$

and

$$\Omega_{0p} \to 0 \quad \text{as} \quad z \to \infty\ . \tag{4.5.6}$$

The integral conditions on the coefficient functions $\Omega_{0p}(z, \eta)$ are obtained by substitution of the expansion (4.5.2) into condition (4.4.7). We obtain:

$$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^* + \cos 2\eta)\ \sin n\eta\ \Omega_{00}(z,\eta)\ dz\,d\eta = \pi\ e^{\xi^*}\ \delta_{n,1} \tag{4.5.7}$$

and for p=1,2,3,...:

$$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^* + \cos 2\eta)\ \sin n\eta\ \Omega_{0p}(z,\eta)\ dz\,d\eta = 0 \tag{4.5.8}$$

Focusing our attention on the zeroth order coefficient functions, $\Psi_{00}(z, \eta)$ and $\Omega_{00}(z, \eta)$, it is clear from the preceding analysis that the mathematical formulation for this sub-problem can be summarized as follows:

$$\frac{\partial^2 \Psi_{00}}{\partial z^2} = \frac{1}{2}\left[\cosh 2\xi^* + \cos 2\eta\right]\Omega_{00}$$

and

$$\frac{\partial^2 \Omega_{00}}{\partial z^2} + \left(\cosh 2\xi^* + \cos 2\eta\right)\left[z\frac{\partial \Omega_{00}}{\partial z} + \Omega_{00}\right] = 0$$

subject to the conditions

$$\Psi_{00} = \frac{\partial \Psi_{00}}{\partial z} = 0 \quad \text{on} \quad z = 0 ,$$

$$\Omega_{00} \to 0 \quad \text{as} \quad z \to \infty ,$$

and

$$\int_0^{\pi}\int_0^{\infty} \left(\cosh 2\xi^* + \cos 2\eta\right) \sin n\eta \, \Omega_{00}(z,\eta) \, dz d\eta = \pi \, e^{\xi^*} \delta_{n,1} .$$

In fact, this formulation duplicates the system of governing equations and conditions for the initial solution of flow past an impulsively-started elliptic cylinder. The solutions, obtained earlier and summarized in equations (4.3.20) and (4.3.21), are given here again:

$$\Omega_{00}(z,\eta) = 2\sqrt{\frac{2}{\pi}} \; e^{\xi^*} \left(\cosh 2\xi^* + \cos 2\eta\right)^{-1/2} \sin\eta \cdot e^{-1/2(\cosh 2\xi^* + \cos 2\eta)z^2} \qquad (4.5.9)$$

and

$$\Psi_{00} = e^{\xi^*} \sin\eta \left\{ z \operatorname{erf}\left(\frac{1}{\sqrt{2}} (\cosh 2\xi^* + \cos 2\eta)^{1/2} z \right) \right.$$

$$\left. + \sqrt{\frac{2}{\pi}} (\cosh 2\xi^* + \cos 2\eta)^{-1/2} \left[e^{-1/2 z^2 (\cosh 2\xi^* + \cos 2\eta)} - 1 \right] \right\}. \qquad (4.5.10)$$

The first-order coefficient equations are governed by the equations:

$$\frac{\partial^2 \Psi_{01}}{\partial z^2} = \frac{1}{2} (\cosh 2\xi^* + \cos 2\eta) \Omega_{01},$$

and

$$\frac{\partial^2 \Omega_{01}}{\partial z^2} + (\cosh 2\xi^* + \cos 2\eta) \left(z \frac{\partial \Omega_{01}}{\partial z} - \Omega_{01} \right)$$

$$= 4 \left[\frac{\partial \Psi_{00}}{\partial \eta} \frac{\partial \Omega_{00}}{\partial z} - \frac{\partial \Psi_{00}}{\partial z} \frac{\partial \Omega_{00}}{\partial \eta} \right],$$

subject to the conditions

$$\Psi_{01} = \frac{\partial \Psi_{01}}{\partial z} = 0 \quad \text{on} \quad z = 0,$$

$$\Omega_{01} \to 0 \quad \text{as} \quad z \to \infty$$

and

$$\int_0^{\pi} \int_0^{\infty} (\cosh 2\xi^* + \cos 2\eta) \sin n\eta \; \Omega_{01} \, dx \, d\eta = 0 .$$

We employ the transformation $u = (\cosh 2\xi^* + \cos 2\eta)^{1/2} z$ to the preceding formulation.

Since

$$\Omega_{00}(u,\eta) = 2\sqrt{\frac{2}{\pi}}\; e^{\xi^*} \sin\eta(\cosh 2\xi^* + \cos 2\eta)^{-1/2}\, e^{-1/2\, u^2}$$

and

$$\Psi_{00}(u,\eta) = e^{\xi^*} \sin\eta(\cosh 2\xi^* + \cos 2\eta)^{-1/2} \left\{ u \,\mathrm{erf}\left(\frac{1}{\sqrt{2}}\, u\right) + \sqrt{\frac{2}{\pi}}\left(e^{-1/2\, u^2} - 1\right)\right\},$$

therefore:

i.

$$\frac{\partial \Omega_{00}}{\partial u} = -2\sqrt{\frac{2}{\pi}}\; e^{\xi^*} \sin\eta(\cosh 2\xi^* + \cos 2\eta)^{-1/2}\, u\, e^{-1/2\, u^2},$$

ii.

$$\frac{\partial \Omega_{00}}{\partial \eta} = 2\sqrt{\frac{2}{\pi}}\; e^{\xi^*}\, e^{-1/2 u^2}\Big\{\cos\eta(\cosh 2\xi^* + \cos 2\eta)^{-1/2}$$
$$+ \sin\eta\, \sin 2\eta(\cosh 2\xi^* + \cos 2\eta)^{-3/2}\Big\},$$

iii.

$$\frac{\partial \Psi_{00}}{\partial u} = e^{\xi^*} \sin\eta(\cosh 2\xi^* + \cos 2\eta)^{-1/2} \left\{ \mathrm{erf}\left(\frac{1}{\sqrt{2}}\, u\right)\right\},$$

and

iv.

$$\frac{\partial \Psi_{00}}{\partial \eta} = e^{\xi^*}\left\{u \operatorname{erf}\left(\frac{1}{\sqrt{2}}u\right)+\sqrt{\frac{2}{\pi}}\left(e^{-1/2u^2}-1\right)\right\}$$

$$\cdot\left\{\cos\eta(\cosh 2\xi^*+\cos 2\eta)^{-1/2}+\sin\eta \sin 2\eta(\cosh 2\xi^*+\cos 2\eta)^{-3/2}\right\}.$$

Hence, the transformed system of equations governing the behaviour of $\Psi_{01}(u,\eta)$ and $\Omega_{01}(u,\eta)$ takes the form

$$\frac{\partial^2 \Psi_{01}}{\partial u^2} = \frac{1}{2}\Omega_{01} . \qquad \textbf{(4.5.11)}$$

and

$$\frac{\partial^2 \Omega_{01}}{\partial z^2} + z\frac{\partial \Omega_{01}}{\partial z} - \Omega_{01}$$

$$= 4(\cosh 2\xi^*+\cos 2\eta)^{-1/2}\left[\frac{\partial \Psi_{00}}{\partial \eta}\frac{\partial \Omega_{00}}{\partial u} - \frac{\partial \Psi_{00}}{\partial u}\frac{\partial \Omega_{00}}{\partial z}\right]. \qquad \textbf{(4.5.12)}$$

The transformed boundary conditions take the form:

$$\Psi_{01} = \frac{\partial \Psi_{01}}{\partial u} = 0 \quad \text{at} \quad u = 0, \qquad \textbf{(4.5.13)}$$

$$\Omega_{01} \to 0 \quad \text{as} \quad u \to \infty, \qquad \textbf{(4.5.14)}$$

and

$$\int_0^{\pi}\int_0^{\infty}(\cosh 2\xi^*+\cos 2\eta)^{1/2} \sin n\eta \; \Omega_{01} \, du \, d\eta = 0. \qquad \textbf{(4.5.15)}$$

The functions $\Psi_{01}(\xi, \eta)$ and $\Omega_{01}(\xi, \eta)$ may be obtained from this system of governing equations and boundary conditions. These solutions may then be used to compute the right hand sides of equations (4.5.3) and (4.5.4), hence defining the equations governing the functions Ψ_{02} and Ω_{02}. It is possible to proceed in this manner, and thus exact solutions are in principle obtainable for any power of t. Clearly, however, such solutions are too cumbersome to obtain and would be more easily obtained numerically.

CHAPTER V

NUMERICAL INTEGRATION OF THE EQUATIONS FOR STEADY FLOW THROUGH A CHANNEL

5.1. BASIC THEORETICAL EQUATIONS AND GENERAL METHOD OF SOLUTION

In this chapter, we consider the steady two-dimensional viscous flow of an incompressible fluid normal to an infinite flat plate of finite breadth in a fluid bounded by channel walls (Figure I.1a). The corresponding problem of flow in an unbounded region was studied by Hudson and Dennis (1985), wherein numerical computations were carried out on a primary variable formulation of the governing equations. By contrast, this study involves calculations based on a solution procedure in terms of the vorticity and streamfunction. Singularities in the vorticity at the edges of the plate are driven to the perimeter of the flow domain by the introduction of the elliptical cylindrical coordinates ξ and η (Figure V.1). Furthermore, a strategy of numerical handling by L.C. Woods (1954) allows for complete avoidance of the effects of the vorticity singularities.

A comparable experimental study of this problem was carried out by Coutanceau & Launay (1993). In particular, this work examined the influence of the ratio of the plate length to the channel width - termed the blockage ratio, λ - on the nature of the flow. Corresponding to this study, numerical calculations have been carried out for $\lambda = 0.05$, $\lambda = 0.1$ and $\lambda = 0.2$. Solutions have been obtained in the Reynolds number range $5 \leq R \leq 20$.

The governing equations for the flow are:

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = -\frac{1}{2}(\cosh 2\xi + \cos 2\eta)\zeta \qquad \textbf{(5.1.1)}$$

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2} = \frac{1}{2}R\left[\frac{\partial\psi}{\partial\eta}\frac{\partial\zeta}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\eta}\right]. \qquad \textbf{(5.1.2)}$$

The solution of these equations is required, subject to the conditions

$$\psi = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{when} \quad \xi = 0; \qquad \textbf{(5.1.3a)}$$

$$\psi = \zeta \quad \text{when} \quad \eta = 0, \pi; \qquad \textbf{(5.1.3b)}$$

$$\frac{1}{\sinh\xi}\frac{\partial\psi}{\partial\xi} \to \sin\eta\ , \quad \frac{1}{\cosh\xi}\frac{\partial\psi}{\partial\eta} \to \cos\eta \quad \text{as } \xi \to \infty; \qquad \textbf{(5.1.4a)}$$

$$\frac{\partial\psi}{\partial\xi} = \sinh\xi \ \sin\eta, \quad \frac{\partial\psi}{\partial\eta} = \cosh\xi \ \cos\eta \quad \text{when} \quad \cosh\xi\ \sin\eta = \frac{1}{\lambda}. \qquad \textbf{(5.1.4b)}$$

A viable computational domain is created by imposing the conditions of (5.1.4a) along $\xi = \xi_{max}$, where ξ_{max} is taken to be sufficiently far form the flat plate that its influence is negligible. As well, the introduction of a modified streamfunction:

$$\Psi(\xi,\eta) = \psi(\xi,\eta) - \cosh\xi \ \sin\eta \qquad \textbf{(5.1.5)}$$

preserves the governing equation(5.1.1) while simplifying the far stream conditions and the conditions along the channel wall. Our modified streamfunction equation is:

$$\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2} = -\frac{1}{2}(\cosh 2\xi + \cos 2\eta)\zeta\ , \qquad \textbf{(5.1.1)}$$

and this must be solved subject to the conditions:

$$\Psi = -\sin\eta\ ,\ \frac{\partial\Psi}{\partial\xi} = 0 \quad \text{when} \quad \xi = 0; \tag{5.1.3a}$$

$$\Psi = 0 \quad \text{when} \quad \eta = 0,\pi; \tag{5.1.3b}$$

$$\frac{\partial\Psi}{\partial\xi} = 0\ ,\ \frac{\partial\Psi}{\partial\eta} = 0 \quad \text{when} \quad \cosh\xi\ \sin\eta = \frac{1}{\lambda} \quad \text{and when} \quad \xi = \xi_{max}\ . \tag{5.1.4}$$

A solution to our mathematically-defined problem is sought through finite-differencing techniques. A grid is imposed over the computational domain as a series of lines parallel to the ξ and η coordinate axes, spaced equally a distance of 'h' units apart. In terms of numerical handling, the region may be regarded as being composed of three subregions: (i) the set of three grid points in the direct neighbourhood of the vorticity singularity at $\xi = 0$, $\eta = \pi/2$, (ii) the set of grid points adjacent to the channel wall, and (iii) the remaining grid points which constitute the bulk of the computational domain. The general strategy of solution involves approximating the derivatives of equations (5.1.1) and (5.1.2) with differences (appropriate to the particular subregion), and hence the differential equations with difference equations. A "two-diagram" approach is undertaken, wherein the difference equation for the streamfunction is solved at all of the grid points in the computational domain. Then, the difference equation for the vorticity is solved at all of the grid points. This sequence is repeated until the difference between two successive iterates, summed over the entire flow field, falls below a prescribed tolerance.

5.2 NUMERICAL TECHNIQUES USED IN THE INTEGRATIONS

The numerical strategies involved in integrating equations (5.1.1) and (5.1.2), subject to conditions (5.1.3a,b) and (5.1.4) are summarized in Figure V.2.

1. Dealing with the Channel Wall

Along the channel wall, we have that:

$$\frac{\partial \Psi}{\partial \xi} = \frac{\partial \Psi}{\partial \eta} = 0.$$

Using this knowledge, it is possible to derive simplified expressions of the Taylor series expansions of $\Psi(\xi,\eta)$, in both the ξ and η directions, about appropriate points on the channel wall. We have:

$$\Psi_a = 0.5\ (1+\alpha)^2\ h^2 \left(\frac{\partial^2 \Psi}{\partial \xi^2}\right)_w + \ldots$$

$$\Psi_b = 0.5 * (\alpha h)^2 \left(\frac{\partial^2 \Psi}{\partial \xi^2}\right)_w + \ldots$$

$$\Psi_c = 0.5\ (1-\alpha)^2\ h^2 \left(\frac{\partial^2 \Psi}{\partial \xi^2}\right)_w + \ldots$$

Here, the subscripts b and a express respectively the grid points which are left-adjacent and right-adjacent to the channel wall, while the subscript c indicates the grid point to the left of b. The subscript w indicates the point on the channel wall itself, and α expresses the fraction of one grid spacing which is on the left side of the channel wall.

From the above, we can easily obtain formulae relating the value of Ψ at internal grid points directly adjacent to the channel wall to the grid point further inside the flow field, and relating the value of Ψ at the external grid point directly adjacent to the channel wall to the internal grid point one removed from the channel wall. For example, for fixed η, we have:

$$\begin{cases} \Psi(\eta^*, b) = \dfrac{(1-\alpha)^2}{(1+\alpha)^2}\,\Psi(\eta^*, a) \\ \Psi(\eta^*, c) = \dfrac{\alpha^2}{(1+\alpha)^2}\,\Psi(\eta^*, a) \end{cases}$$

2. Streamfunction Computations

At the internal grid points, the modified streamfunction Ψ is computed using difference approximations developed by Hudson and Dennis (1989). The differential equation (5.1.1) is replaced by a nine-point difference approximation. In accordance with Southwell's notation (1946), illustrated in Figure V.2, this fourth-order accurate formula applied at a typical grid point (ξ_0, η_0) is

$$\begin{aligned} \Psi_0 = &\left\{8(\Psi_1+\Psi_2+\Psi_3+\Psi_4)+2(\Psi_5+\Psi_6+\Psi_7+\Psi_8)\right\} \\ &-\frac{1}{2}\,h^2\left\{20(M_0^2)\zeta_0+M_5^2\zeta_5+M_6^2\zeta_6+M_7^2\zeta_7+M_8^2\zeta_8\right\}\,, \end{aligned} \qquad (5.2.1)$$

where h is the grid size and $M^2 = \frac{1}{2}(\cosh 2\xi + \cos 2\eta)$.

3. Vorticity Computations

The vorticity function is computed at internal grid points of the computational domain using the h^4-accuracy finite differencing schemes developed by Hudson and Dennis (1989). Equation (5.1.2) is replaced by the approximating difference equation

$$\begin{aligned}(1 - \tfrac{1}{2}hf_0 + \alpha h^2 f_0^2)\zeta_1 + (1 - \tfrac{1}{2}hg_0 + \alpha h^2 g_0^2)\zeta_2 \\ + (1 + \tfrac{1}{2}hf_0 + \alpha h^2 f_0^2)\zeta_3 + (1 + \tfrac{1}{2}hg_0 + \alpha h^2 g_0^2)\zeta_4 \\ - (4 + 2\alpha h^2 [f_0^2 + g_0^2])\zeta_0 = 0 \, , \end{aligned} \tag{5.2.2}$$

where $f = \frac{1}{2}R\frac{\partial\psi}{\partial\eta}$, $g = -\frac{1}{2}R\frac{\partial\psi}{\partial\xi}$ and α is a parameter to be defined. If $\alpha = 0$, the approximation given above reduces to the standard central difference scheme. The point however is to take non-zero values of α ($\alpha \geq 1/16$) which will cause the associated matrices to be diagonally dominant. This guarantees the convergence of a successive over-relaxation iterative method of solution. In the computations documented here, the value of α was taken to be 1/12.

To compute the vorticity at the three grid points directly in the neighbourhood of the point of singularity, strategies which avoided the singularity were employed. In particular, the technique of "cross-differencing" was employed whereby the differences involving the point of singularity were replaced by equivalent differences obtained by considering the coordinate system arrived at by rotating in the counterclockwise direction the x- and y-axes 45 degrees. Employing this strategy, the standard h^2-accurate central differencing approximation for the Laplacian of a function:

$$\frac{\partial^2 f_0}{\partial x^2} + \frac{\partial^2 f_0}{\partial y^2} = \frac{1}{h^2}(f_1 + f_2 + f_3 + f_4 - 4f_0) + O(h^3)$$

would be replaced by:

$$\frac{\partial^2 f_0}{\partial x^2} + \frac{\partial^2 f_0}{\partial y^2} = \frac{1}{h^2}(f_5 + f_6 + f_7 + f_8 - 4f_0) + O(h^3).$$

Using the alternative difference formula allows for the avoidance of problematic (singular) points.

At the flat plate itself, the vorticity was computed using a simple condition formulated by L.C. Woods (1954)

$$\zeta_{wall} = \frac{3}{h^3} \cdot \frac{1}{(\cosh 2\xi + \cos 2\eta)} \Psi_{adjacent} - \frac{1}{2} \zeta_{adjacent} . \qquad (5.2.3)$$

5.3 RESULTS

Calculations were carried out over the range of Reynolds numbers $5 \leq R \leq 20$, and for blockage ratios $\lambda = 0.05, 0.1, 0.2$. An iterative procedure of solution was employed in which relaxation parameters speeded the rate of convergence. Initializing to zero the values of the streamfunction and vorticity in the flow domain, equation (5.2.1) was solved at all of the interior grid points. The surface vorticity was then computed using (5.2.3). This was followed by a complete sweep of the equations (5.2.2). Incorporating as well the special considerations made near the channel wall and the point of singularity, the entire sequence was repeated until convergence, determined by the criteria that the sum of all the differences between successive iterates would fall below 0.001.

In their experimental work, Coutanceau and Launay studied the effect of the blockage ratio on the nature of the flow. In particular, they examined the variation of the length of the closed wake as the blockage ratio increased (Fig. V.4), the distance of the vortex centres from the plate (fig. V.5) and the distance between the two vortex centres (Fig. V.6). Experimental results by Taneda (1968) are also given in the figures, as are numerical results obtained by Dennis & Qiang (1993) and Hudson & Dennis (1985), for the case of flow in an open field.

CHAPTER VI

SUMMARY AND CONCLUSION

The objective of this thesis is to consider symmetrical viscous fluid flow past a normal flat plate started impulsively from rest. We have obtained the initial profile of the flow, as well as analytical solutions for small values of time. In addition, a theoretical basis has been developed from which a numerical forward march in time can be carried out to understand the nature of the flow at subsequent stages. The soundness of the exact solutions obtained seems to be confirmed through comparison with the spatial-singularity-free problem of flow past a thin ellipse. It appears that the normal flat plate may be regarded as a limiting case of increasingly thinner ellipses.

For small values of the Reynolds number, we have obtained numerical solutions for the steady state achieved by fluid passing a normal flat plate whilst confined by channel walls. The solutions obtained are in close agreement with existing solutions derived experimentally.

APPENDIX I: DERIVATION OF GOVERNING EQUATIONS

Obtaining the Dimensionless Form

For a viscous, incompressible fluid, the Navier-Stokes equations, expressed in terms of dimensional quantities, take the familiar form:

$$\frac{\partial \vec{v}'}{\partial t'} + (\vec{v}' \cdot \nabla')\vec{v}' = -\frac{1}{\rho}\,\text{grad}\,p' + \frac{\mu}{\rho}\nabla'^2\,\vec{v}\ . \qquad \text{(I.1)}$$

In the above formulation, $\vec{v}'$ and p' are, respectively, the dimensional velocity vector field and the dimensional pressure function. The operator ∇' denotes the dimensional del operator based on the dimensional coordinates, x' and y'. Also, t' is the dimensional (or actual) time.

We assume a flow field in which 'd' is the representative length and 'U' is the representative velocity. We may then define the following dimensionless quantities:

$$x = \frac{x'}{d}\ ,\quad y = \frac{y'}{d}\ ,\quad \vec{v} = \frac{\vec{v}'}{U}\ ,\quad p = \frac{p'}{\rho U^2}\ ,\quad t = \frac{Ut'}{d}\ . \qquad \text{(I.2)}$$

Employing the above transformations in equation (I.1) yields

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\text{grad}\,p + \frac{2}{R}\nabla^2\,\vec{v}\ . \qquad \text{(I.3)}$$

Here, the Reynolds number R is defined as $R = \dfrac{2Ud}{\nu}$, where ν is the coefficient of kinematic viscosity. Equation (I.3) along with the equation of continuity:

$$\text{div}\,\vec{v} = 0\,, \tag{I.4}$$

subject to the appropriate boundary conditions, forms the required system of non-dimensional governing equations in terms of the basic variables, $\vec{v}$ and p.

Introduction of the Vorticity Function

We employ the following well-known vector identities in equation (I.3):

$$(\vec{v}\cdot\nabla)\vec{v} = \nabla\left(\frac{1}{2}v^2\right) - \vec{v}\times(\text{curl}\;\vec{v})$$

$$\nabla^2\,\vec{v} = \nabla(\text{div}\,\vec{v}) - \text{curl}(\text{curl}\;\vec{v})$$

and obtain

$$\frac{\partial\vec{v}}{\partial t} + \nabla\left(\frac{1}{2}v^2\right) - \vec{v}\times(\text{curl}\;\vec{v}) = -\text{grad}\,p - \frac{2}{R}\,\text{curl}(\text{curl}\,\vec{v})\,.$$

Taking the curl of this equation, and further defining the vorticity function as

$$\vec{\omega} = \text{curl}\;\vec{v}\,, \tag{I.5}$$

we have

$$\frac{\partial\vec{\omega}}{\partial t} - \text{curl}(\vec{v}\times\vec{\omega}) = -\frac{2}{R}\,\text{curl}(\text{curl}\,\vec{\omega})\,. \tag{I.6}$$

Hence, equations (I.5), (I.6) and (I.4) form a system of three equations in the three unknowns, $\vec{\omega} = (0, 0, \zeta(x,y,t))$ and $\vec{v} = (u(x,y,t)\,,\, v(x,y,t), 0)$.

Introduction of the Streamfunction

The continuity equation (I.4), for the case of two-dimensional flow, implies the existence of the streamfunction $\psi(x,y,t)$, such that

$$u = \psi_y \quad , \quad v = -\psi_x \, . \tag{I.7}$$

Employing (I.7) in (I.5) and (I.6), we obtain the vorticity-streamfunction formulation of the flow problem. That is,

$$\frac{\partial\zeta}{\partial t} = \frac{2}{R}\,\nabla^2\zeta + \left(\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}\right) \tag{I.8}$$

and

$$\nabla^2\psi + \zeta = 0 \, . \tag{I.9}$$

Elliptical Cylindrical Coordinates: Transformation of the Independent Variables

Introduction of the elliptic cylindrical coordinates (ξ,η), related to the rectangular coordinates (x,y) by the equations

$$\begin{cases} x = \sinh\xi\,\cos\eta \\ y = \cosh\xi\,\sin\eta \end{cases} , \tag{I.10}$$

transforms the upper-half of the Cartesian plane to a semi-infinite strip of width π. From equations (I.10), it is clear that

$$dx = \cosh\xi\,\cos\eta\,d\xi - \sinh\xi\,\sin\eta\,d\eta$$

and

$$dy = \sinh\xi \sin\eta \, d\xi + \cosh\xi \cos\eta \, d\eta.$$

Thus,

$$\begin{aligned}(ds)^2 &= (dx)^2 + (dy)^2 \\ &= [\cosh^2\xi \cos^2\eta + \sinh^2\xi \sin^2\eta][(d\xi)^2 + (d\eta)^2] \\ &= M^2(\xi,\eta)[(d\xi)^2 + (d\eta)^2],\end{aligned}$$

where

$$\begin{aligned}M^2 &= [\cosh^2\xi \cos^2\eta + \sinh^2\xi \sin^2\eta] \\ &= \frac{1}{2}(\cos 2\eta + \cosh 2\xi).\end{aligned} \tag{I.11}$$

From differential geometry, it is well known that if

$$(ds)^2 = g_1^2(\xi,\eta)(d\xi)^2 + g_2^2(\xi,\eta)(d\eta)^2,$$

Then:

$$\text{(i)} \qquad \operatorname{div} \underset{\sim}{F} = \frac{1}{g_1 g_2} \left\{ (g_2 F_1)_\xi + (g_1 F_2)_\eta \right\},$$

$$\text{(ii)} \qquad \operatorname{grad} \phi = \left(\frac{1}{g_1} \phi_\xi , \frac{1}{g_2} \phi_\eta \right),$$

$$\text{(iii)} \qquad \operatorname{curl} \underset{\sim}{F} = \frac{1}{g_1 g_2} \begin{vmatrix} g_1 \underset{\sim}{e}_\xi & g_2 \underset{\sim}{e}_\eta & \underset{\sim}{k} \\ \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial z} = 0 \\ g_1 F_1 & g_2 F_2 & 0 \end{vmatrix},$$

$$\text{and} \quad \text{(iv)} \qquad \nabla^2 \phi = \frac{1}{g_1 g_2} \left\{ \frac{\partial}{\partial \xi} \left(\frac{g_2}{g_1} \phi_\xi \right) + \frac{\partial}{\partial \eta} \left(\frac{g_1}{g_2} \phi_\eta \right) \right\}. \qquad \text{(I.12)}$$

In our case, $g_1(\xi, \eta) = g_2(\xi, \eta) = M(\xi, \eta)$, where M^2 is given in (I.11).

Letting velocity vector, $\bar{v} = u\bar{e}_\xi + v\bar{e}_\eta$ and vorticity vector $\bar{\omega} = 0\bar{e}_\xi + 0\bar{e}_\eta + \zeta\bar{k}$, we employ the formulae in (I.12) to compute the following:

$$\text{(i)} \qquad \operatorname{div} \underset{\sim}{v} = \frac{1}{M^2} \left\{ (Mu)_\xi + (Mv)_\eta \right\}$$

Since

$$\operatorname{div} \underset{\sim}{v} = 0,$$

$\therefore$

$$(Mu)_\xi + (Mv)_\eta = 0.$$

This equation implies that ∃ a function $\psi = \psi(\xi, \eta, t)$ such that

$$\frac{\partial\psi}{\partial\eta} = Mu$$

and

$$\frac{\partial\psi}{\partial\xi} = -Mv.$$

(ii) $\underline{\omega} = \text{curl } \bar{v}$

$$= \frac{1}{M^2}\begin{vmatrix} M\underline{e}_\xi & M\underline{e}_\eta & \underline{k} \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial z}\equiv 0 \\ Mu & Mv & 0 \end{vmatrix}$$

$$= \frac{1}{M^2}\begin{vmatrix} M\underline{e}_\xi & M\underline{e}_\eta & \underline{k} \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial z}\equiv 0 \\ \psi_\eta & -\psi_\xi & 0 \end{vmatrix}$$

$$= \frac{1}{M_2}\left[M\underline{e}_\xi\{0\} + M\underline{e}_\eta\{0\} + \underline{k}\,\{-\psi_{\xi\xi} - \psi_{\eta\eta}\} \right]$$

$$= -\frac{1}{M^2}(\psi_{\xi\xi} + \psi_{\eta\eta})\,\underline{k}$$

(iii)

$$\underset{\sim}{v}\times\underset{\sim}{w} = \begin{vmatrix} \underset{\sim}{e}_\xi & \underset{\sim}{e}_\eta & \underset{\sim}{k} \\ u & v & 0 \\ 0 & 0 & -\frac{1}{M^2}(\psi_{\xi\xi}+\psi_{\eta\eta}) \end{vmatrix}$$

$$= -\frac{v}{M^2}(\psi_{\xi\xi}+\psi_{\eta\eta})\,\underset{\sim}{e}_\xi + \frac{u}{M^2}(\psi_{\xi\xi}+\psi_{\eta\eta})\underset{\sim}{e}_\eta$$

$$= \frac{1}{M^3}(\psi_{\xi\xi}+\psi_{\eta\eta})\left[\psi_\xi\,\underset{\sim}{e}_\xi + \psi_\eta\,\underset{\sim}{e}_\eta\right]$$

(iv)

$$\text{curl}\,\underset{\sim}{\omega} = \frac{1}{M^2}\begin{vmatrix} M\underset{\sim}{e}_\xi & M\underset{\sim}{e}_\eta & \underset{\sim}{k} \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial z}\equiv 0 \\ 0 & 0 & \zeta \end{vmatrix}$$

$$= \frac{1}{M^2}\left[M\,\zeta_\eta\,\underset{\sim}{e}_\xi - M\,\zeta_\xi\,\underset{\sim}{e}_\eta\right]$$

$$= \frac{1}{M}\left[\zeta_\eta\,\underset{\sim}{e}_\xi - \zeta_\xi\,\underset{\sim}{e}_\eta\right]$$

(v)

$$\operatorname{curl}(\operatorname{curl}\,\underset{\sim}{\omega}) = \frac{1}{M^2}\begin{vmatrix} M\underset{\sim}{e}_\xi & M\underset{\sim}{e}_\eta & \underset{\sim}{k} \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial z}\equiv 0 \\ \zeta_\eta & -\zeta_\xi & 0 \end{vmatrix}$$

$$= \frac{1}{M^2}\left[-\frac{\partial^2\zeta}{\partial\xi^2} - \frac{\partial^2\zeta}{\partial\eta^2}\right]\underset{\sim}{k}$$

$$= -\frac{1}{M^2}\left[\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2}\right]\underset{\sim}{k}$$

(vi)

$$\operatorname{curl}(\underset{\sim}{v}\times\underset{\sim}{\omega}) = \frac{1}{M^2}\begin{vmatrix} M\underset{\sim}{e}_\xi & M\underset{\sim}{e}_\eta & \underset{\sim}{k} \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial z}\equiv 0 \\ -\zeta\psi_\xi & -\zeta\psi_\eta & 0 \end{vmatrix}$$

$$= \frac{\mathbf{k}}{M^2}\left[\frac{\partial}{\partial\xi}[-\zeta\psi_\xi] + \frac{\partial}{\partial\eta}[\zeta\psi_\xi]\right]$$

$$= \frac{\mathbf{k}}{M^2}[\zeta_\eta\psi_\xi - \zeta_\xi\psi_\eta]$$

Employing the above results in equations (I.6) and (I.5), we obtain the following equations governing the behaviour of $\psi(\xi,\eta,t)$ and $\zeta(\xi,\eta,t)$:

$$\frac{RM^2}{2}\frac{\partial\zeta}{\partial t} = \left(\frac{\partial^2\zeta}{\partial\xi^2}+\frac{\partial^2\zeta}{\partial\eta^2}\right) + \frac{R}{2}\left(\frac{\partial\zeta}{\partial\eta}\frac{\partial\psi}{\partial\xi}-\frac{\partial\zeta}{\partial\xi}\frac{\partial\psi}{\partial\eta}\right) \tag{I.12}$$

and

$$\zeta = -\frac{1}{M^2}\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}\right) \tag{I.13}$$

APPENDIX II: PARABOLIC CYLINDRICAL FUNCTIONS

The parabolic cylindrical functions are solutions of the differential equation

$$\frac{d^2y}{dx^2} + (ax^2 + bx + c)y = 0 . \tag{II.1}$$

Two real and distinct forms of equation (II.1) are

$$\frac{d^2y}{dx^2} - \left(\frac{1}{4}x^2 + a\right)y = 0 \tag{II.2}$$

and

$$\frac{d^2y}{dx^2} + \left(\frac{1}{4}x^2 + a\right)y = 0 . \tag{II.3}$$

In fact, taking $a = -\left(n + \frac{1}{2}\right)$ in (II.2), we have the differential equation whose solution

is the parabolic cylindrical function of index n — denoted $D_n(x)$. That is,

$$\frac{d^2\,D_n(x)}{dx^2} - \left(\frac{1}{4}x^2 - n - \frac{1}{2}\right)D_n(x) = 0 . \tag{II.4}$$

Even and odd power solutions of (II.4) are given by

$$y_1 = e^{-\frac{1}{4}x^2}\left\{1 + (-n)\frac{x^2}{2!} + (-n)(2-n)\frac{x^4}{4!} + \ldots\right\} \tag{II.5}$$

$$y_1 = e^{\frac{1}{4}x^2}\left\{1 + (-n-1)\frac{x^2}{2!} + (-n-1)(-n-3)\frac{x^4}{4!} + \ldots\right\} \tag{II.6}$$

and

$$y_2 = e^{-\frac{1}{4}x^2}\left\{x + (1-n)\frac{x^3}{3!} + (1-n)(3-n)\frac{x^5}{5!} + \ldots\right\} \tag{II.7}$$

$$= e^{\frac{1}{4}x^2}\left\{x + (-n-2)\frac{x^3}{3!} + (-n-2)(-n-4)\frac{x^5}{5!} + \ldots\right\}. \tag{II.8}$$

Standard solutions of the differential equation (II.4) are constructed from these series solutions. Two linearly independent solutions of (II.4) are

$$U\left(-n-\frac{1}{2}\ ;\ x\right) = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\cos\left(\frac{n\pi}{2}\right)\Gamma\left(\frac{1+n}{2}\right)y_1 \tag{II.9}$$

$$V\left(-n-\frac{1}{2}\ ;\ x\right) = \frac{2^{\frac{n+1}{2}}}{\sqrt{\pi}}\cos\left(\frac{n\pi}{2}\right)\frac{\Gamma\left(\frac{2+n}{2}\right)}{\Gamma(1+n)}y_2\ . \tag{II.10}$$

and the Wronskian of these two functions is known to satisfy

$$W(U,V) = \sqrt{\frac{2}{\pi}}\ . \tag{II.11}$$

Taking n = -2 in (II.4), we have the differential equation which governs a component of the solution for $(\Omega_{01})_h$:

$$\frac{d^2y}{dx^2} - \left(\frac{1}{4}x^2 + \frac{3}{2}\right)y = 0 . \qquad \text{(II.12)}$$

From (II.10), we have that

$$V\left(\frac{3}{2} ; x\right) = \frac{1}{\sqrt{2\pi}} x e^{\frac{1}{4}x^2} .$$

Employing this expression for $V\left(\frac{3}{2} ; x\right)$ in (II.11), $U\left(\frac{3}{2} ; x\right)$ is found to be governed

by the equation

$$\frac{dU}{dx} - \left(\frac{1}{2}x + \frac{1}{x}\right)U = -\frac{2}{x} e^{-\frac{1}{4}x^2} .$$

Hence,

$$U\left(\frac{3}{2} ; x\right) = \sqrt{2\pi}\; x\, e^{\frac{1}{4}x^2} \operatorname{erf}\left(\frac{1}{\sqrt{2}}x\right) + 2e^{-\frac{1}{4}x^2} ,$$

and the general solution of (II.12) is

$$y = c_1 \cdot \frac{1}{\sqrt{2\pi}} x\, e^{\frac{1}{4}x^2} + c_2 \left\{ \sqrt{2\pi}\; x\, e^{\frac{1}{4}x^2} \operatorname{erf}\left(\frac{1}{\sqrt{2}}x\right) + 2e^{-\frac{1}{4}x^2} \right\} ,$$

where c_1 and c_2 can be determined through the application of appropriate boundary conditions.

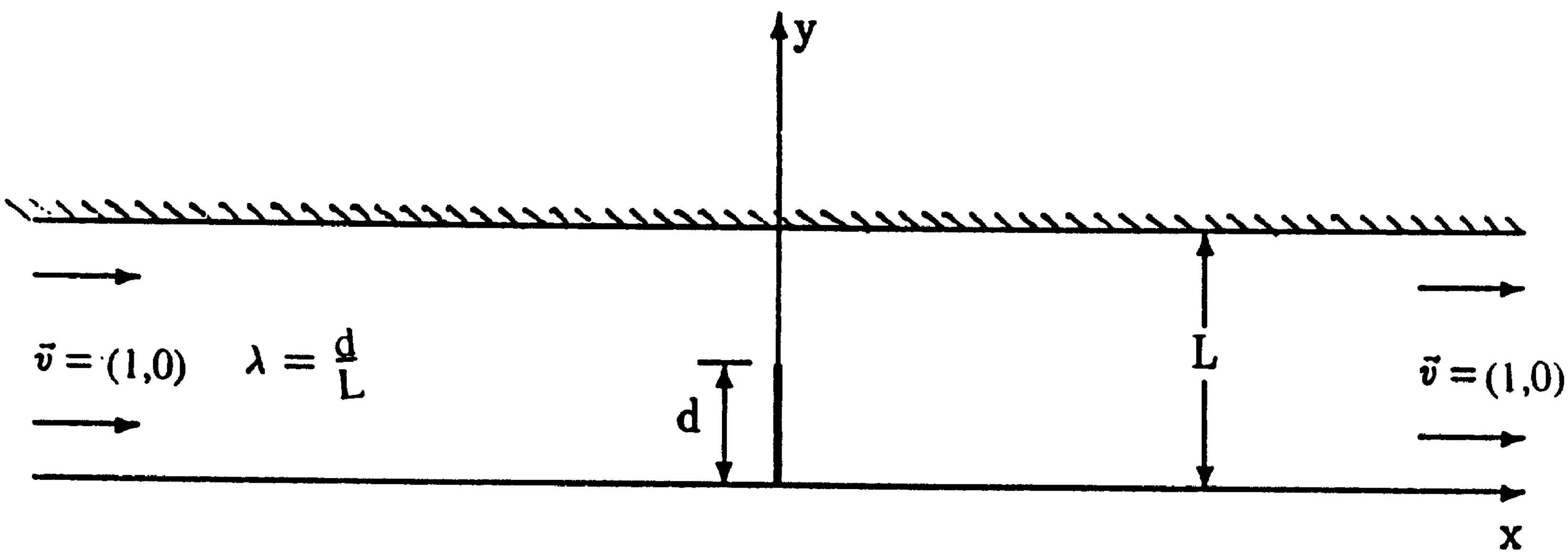

Figure I.1a : Flow through a Channel

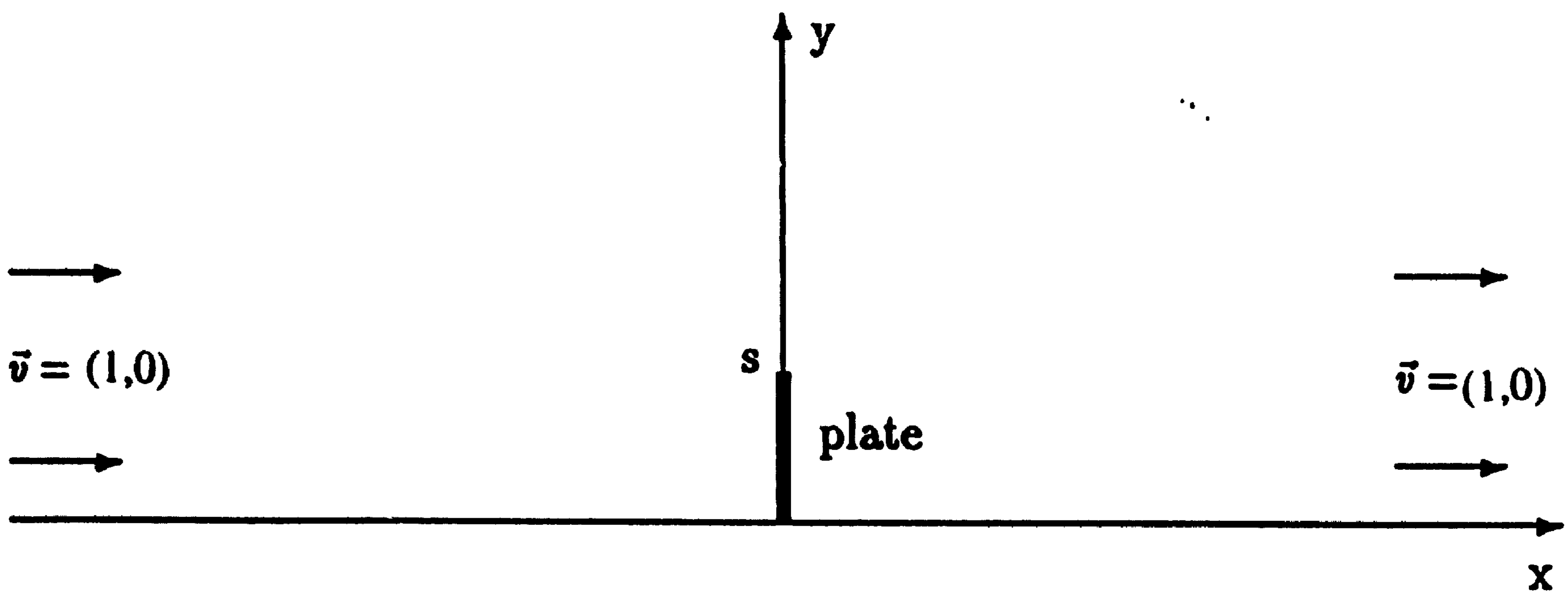

Figure I.1b : Flow in an Open-Field

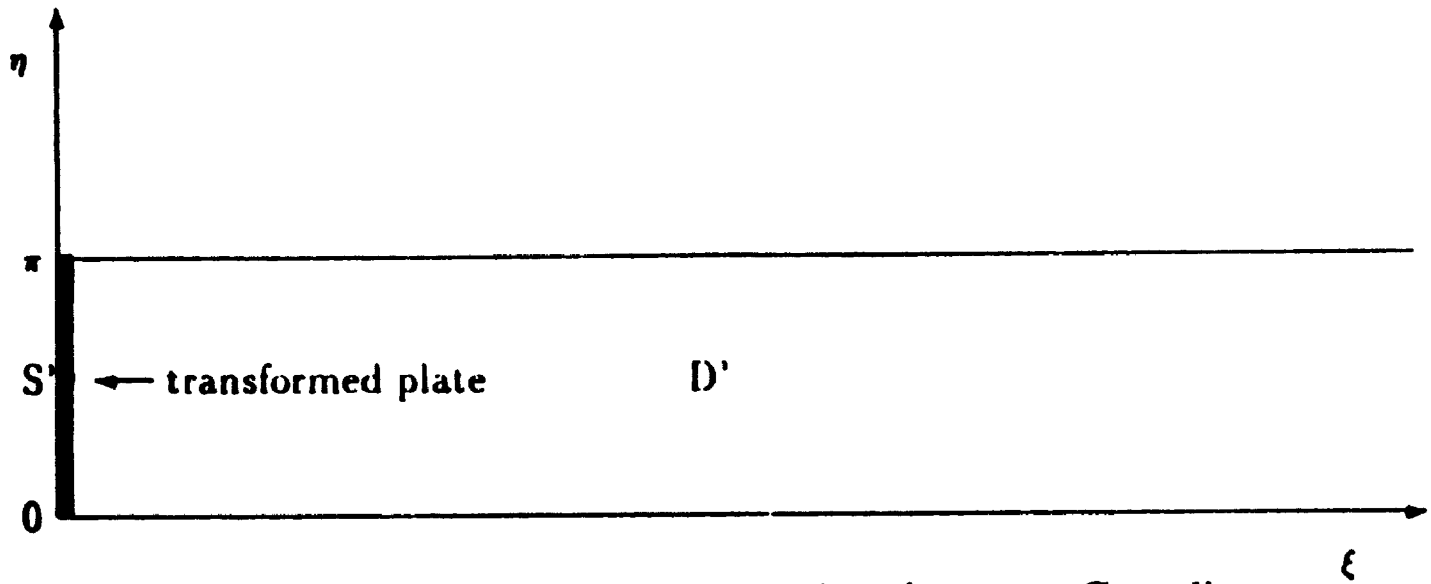

Figure I.2 : Domain of problem in (ξ, η). Coordinates

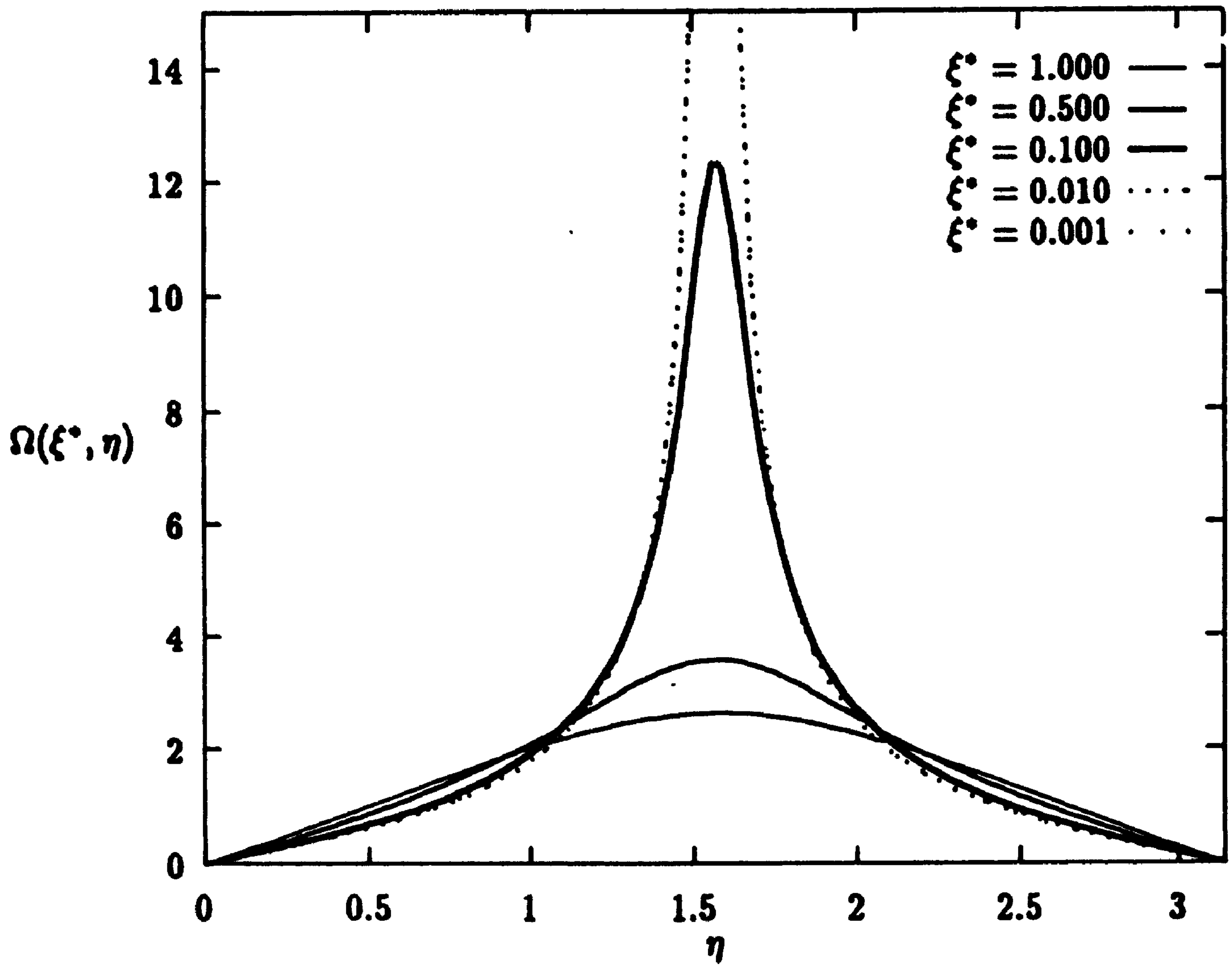

Figure IV.1 : Vorticity values along the surface of the ellipse

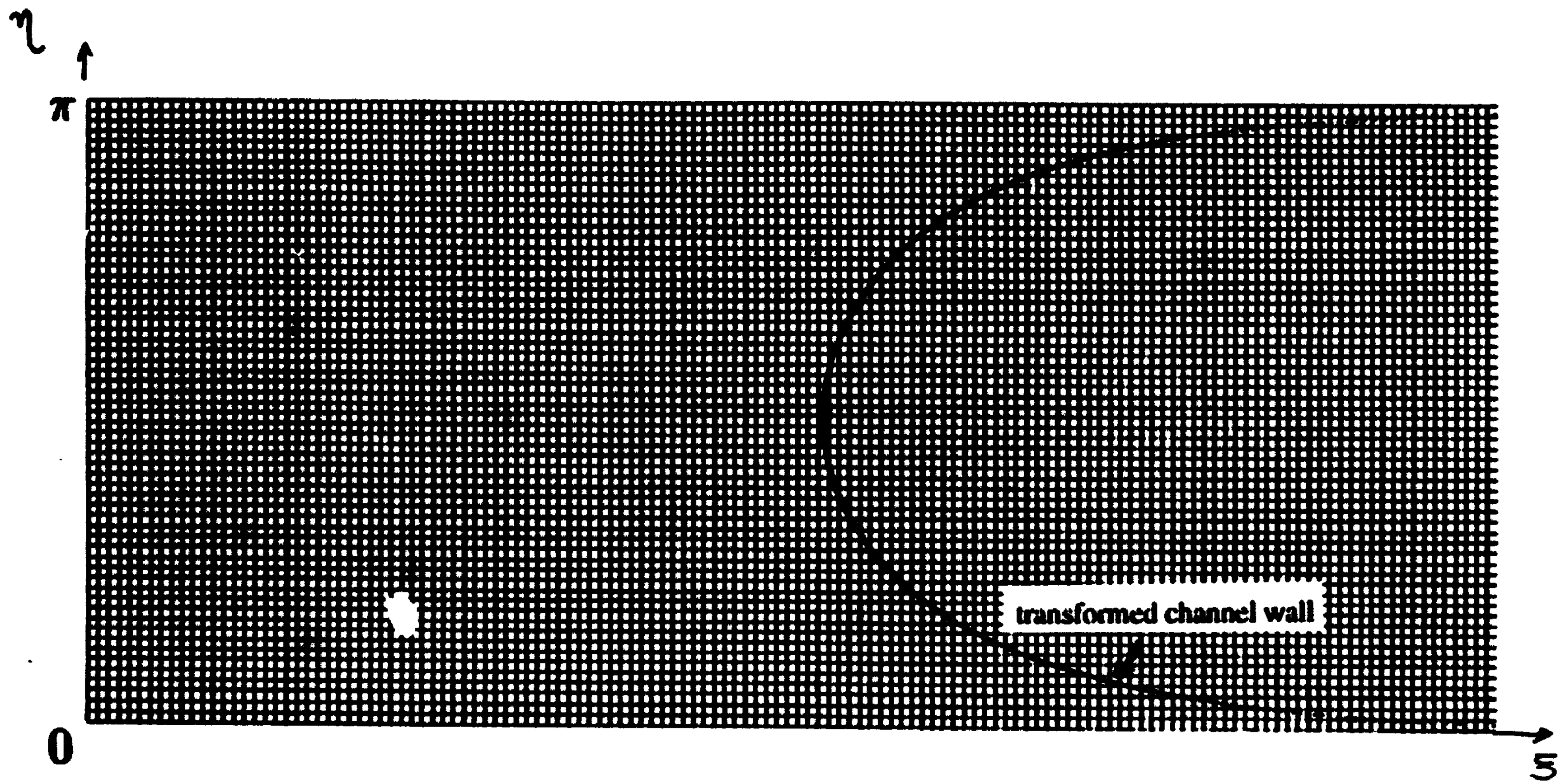

Figure V.1 : Transformed Domain for flow through a Channel

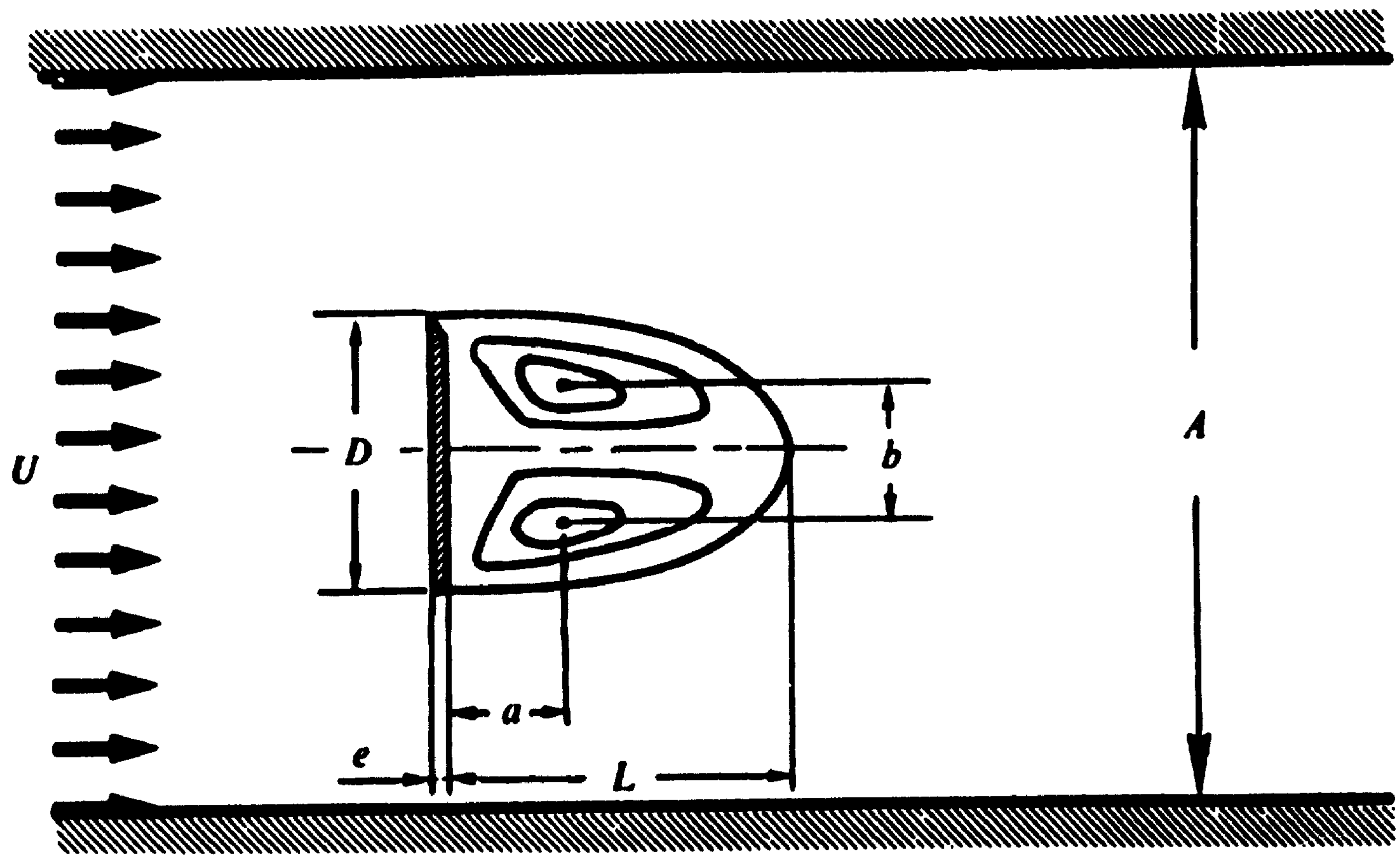

Figure V.3 : Geometrical parameters of the closed wake

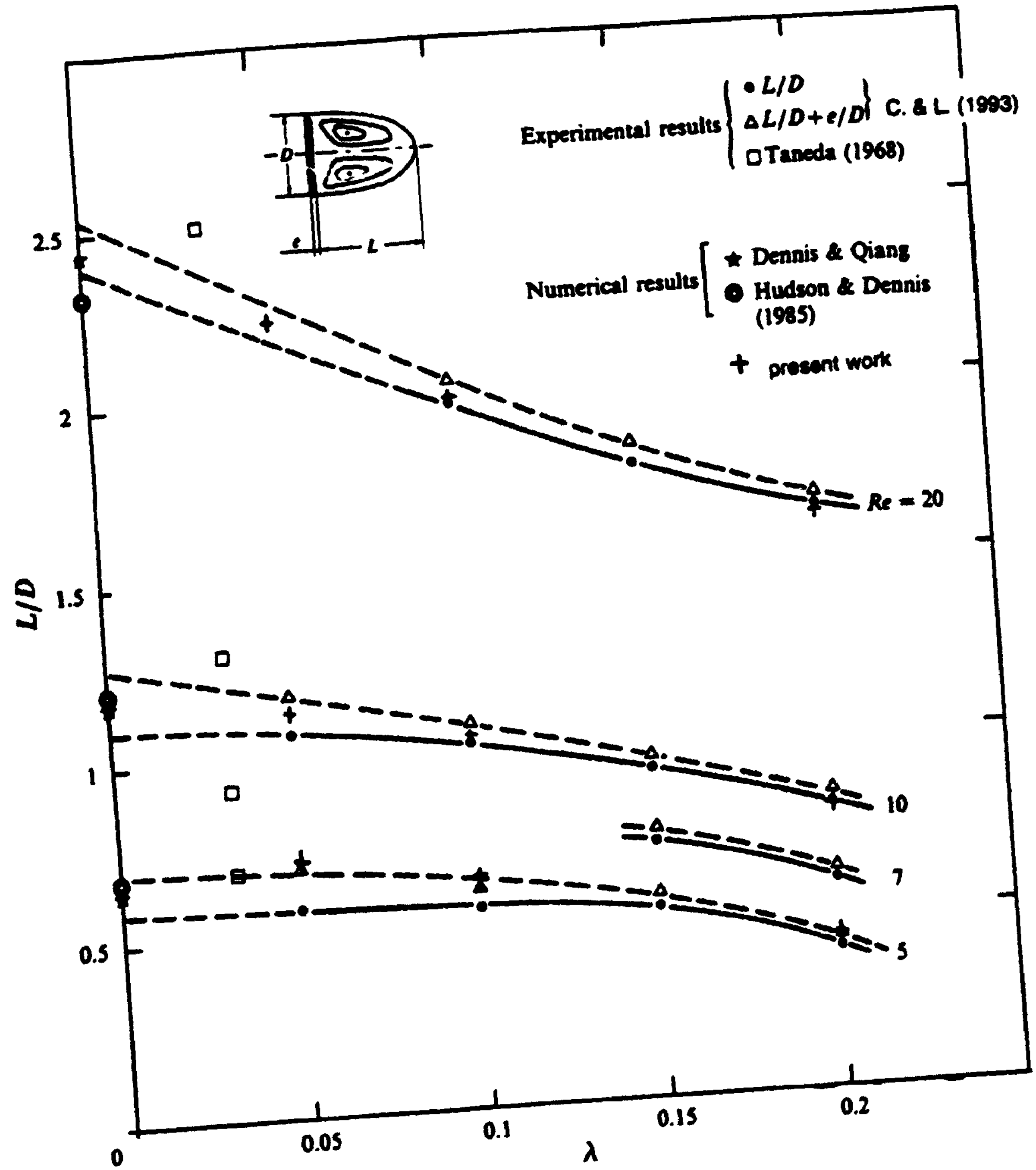

Figure V.4 : Relationship between wake-length and blockage ratio

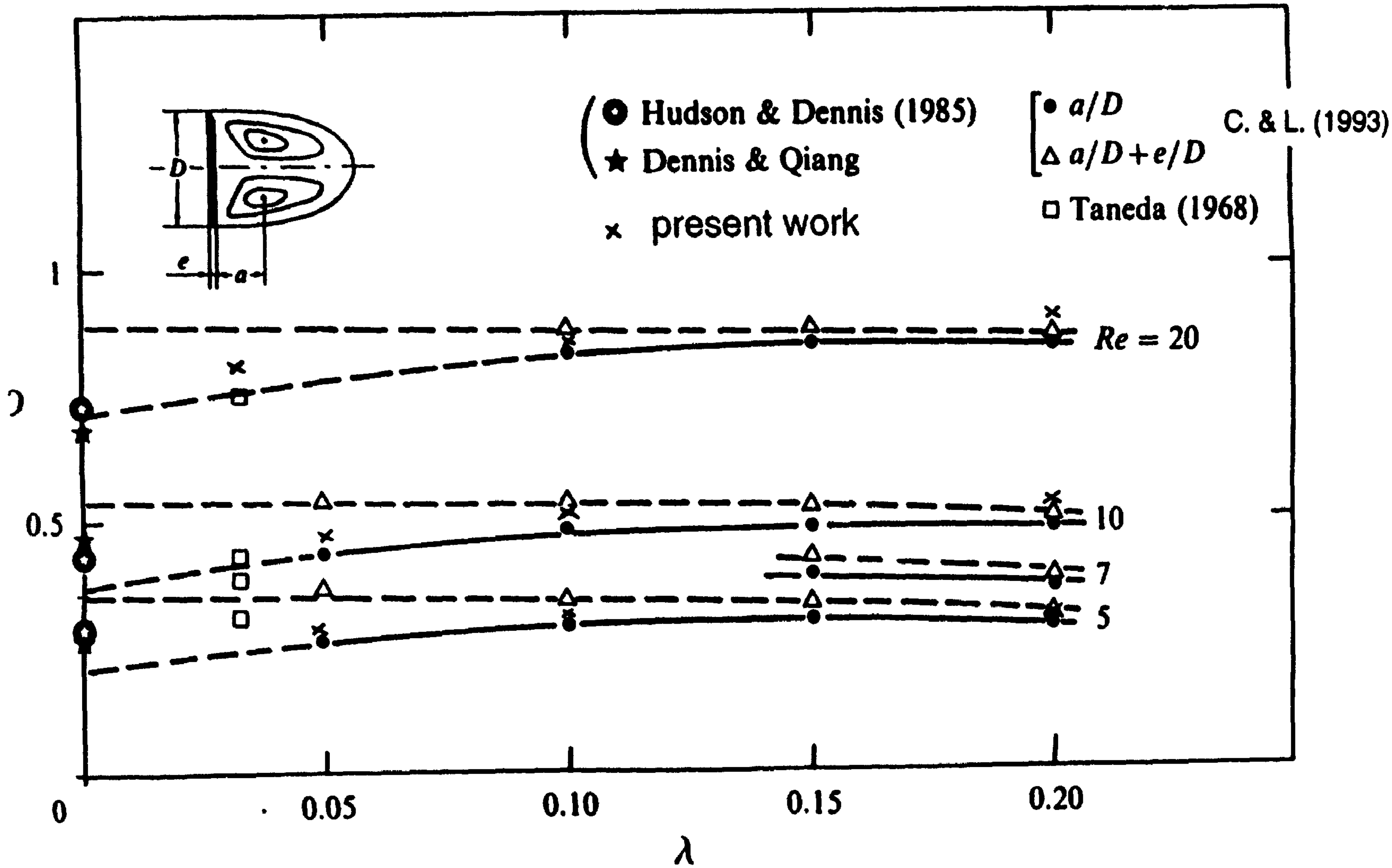

Figure V.5 : The distance between the plate and the vortex centre plotted against the blockage ratio

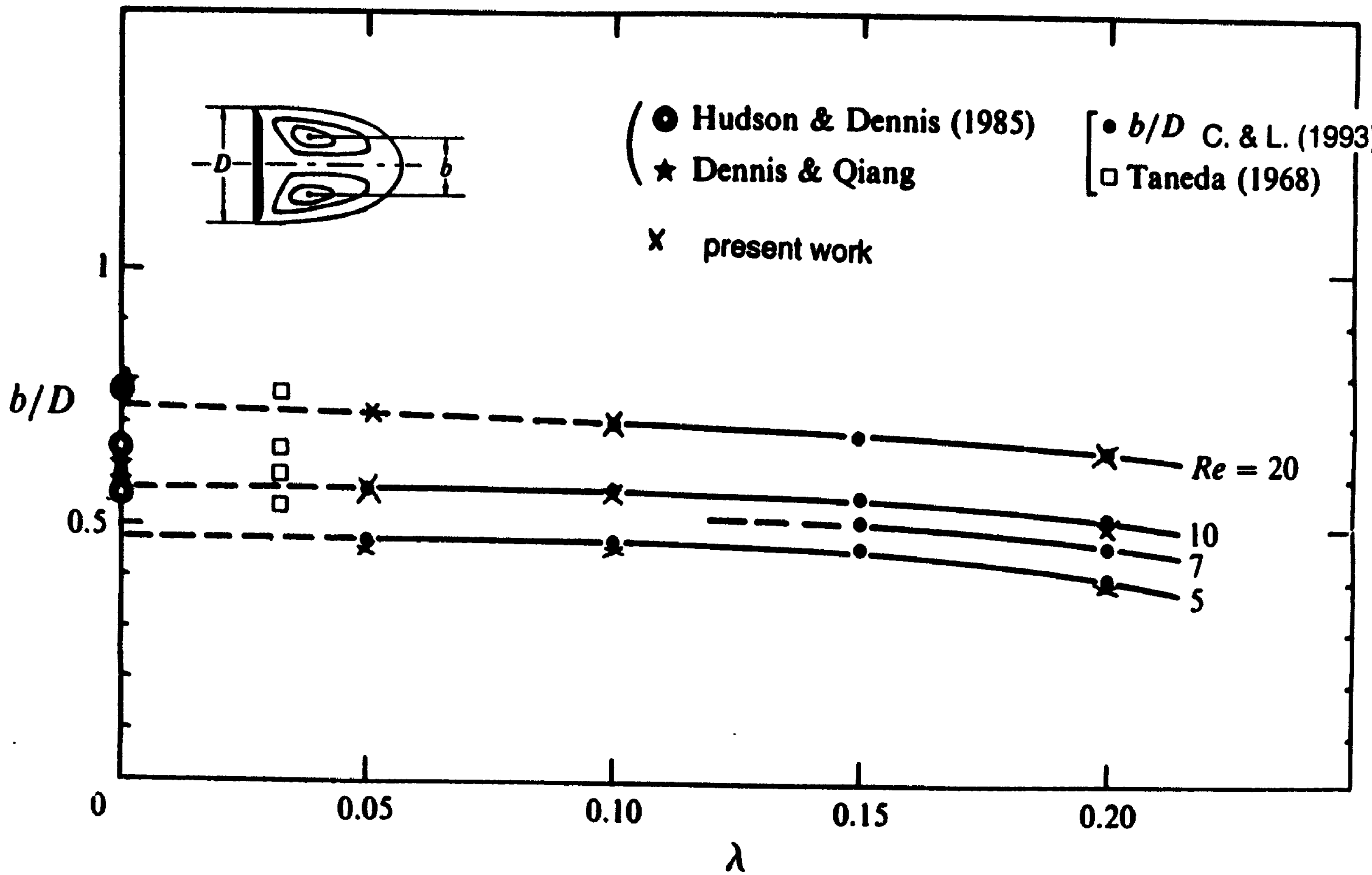

Figure V.6 : The distance between the two vortex centres plotted against the blockage ratio

START

↓

Initialize:
$\psi = \zeta = 0$;
$\forall$ grid points.

↓

Compute modified streamfunction:
$\Psi = \psi - \cosh\xi \sin\eta$

↓

⊛ ⟶ Set error in
ψ, Z1 = 0.0

↓

1. DEALING WITH THE CHANNEL WALL

(a) Using Taylor series expansions in the ξ-direction and knowledge of the streamfunction at the channel wall, compute $\Psi(M,b)$ & $\Psi(M,c)$ from $\Psi(M,a)$.

(b) Using Taylor series expansions in the η-direction and knowledge of the streamfunction at the channel wall, compute $\Psi(b,N)$ & $\Psi(c,N)$ from $\Psi(a,N)$.

(c) From the streamfunction equation, compute ζ at the points adjacent to the channel wall, i.e. compute $\zeta(M,b)$ & $\zeta(b,N)$.

↓

2. STREAMFUNCTION COMPUTATIONS

(a) Compute Ψ at all the remaining internal grid points, using h^4-technique.

(b) Determine differences between present and previous iterates.

$$\text{DIFF} = \text{ABS}\left(\sum_{i,j}\left[\Psi_{i,j}^{(k)} - \Psi_{ij}^{(k-1)}\right]\right)$$

$$Z1 = Z1 + \text{DIFF}$$

(c) Compute $\psi = \psi + \cosh\xi \sin\eta$

↓

3. VORTICITY COMPUTATIONS

(a) Set ZZ = 0.0 (error in vorticity)

(b) Use "avoidance strategies" (cross-differencing, etc.) to compute vorticity at (M2, 2), (M3, 2) & (M5, 2).

(c) Use h^4-technique to compute vorticity at remaining grid points. Determine differences between present and previous iterates.

$$DIFF = ABS\left(\sum_{i,j}\left(\zeta_{ij}^{(k)} - \zeta_{ij}^{(k-1)}\right)\right)$$

$$ZZ = ZZ + DIFF$$

(d) Use Woods' condition to compute the vorticity along ξ=0.

↓

Is (Z1+ZZ)<TOL — YES → STOP

↓ NO

*

Figure V.7: Program Flow Chart

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