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Various Distributional Results For Ratios Of Quadratic Forms With Applications To Serial Correlation

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VARIOUS DISTRIBUTIONAL RESULTS FOR RATIOS OF QUADRATIC FORMS WITH APPLICATIONS TO SERIAL CORRELATION

by

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ABSTRACT

Quadratic forms in normal vectors are central building blocks in statistics, and ratios of quadratic forms arise in a variety of contexts. They are connected, for example, to regression and analysis of variance problems associated with linear models and to correlation analysis.

The matrices of the quadratic forms may be for example positive definite, positive semidefinite or indefinite; the normal vectors may be central or noncentral, nonsingular or singular. All these cases are considered for a single quadratic form as well as for the ratio of two quadratic forms. The distributional results also apply to bilinear forms, sums of quadratic and bilinear forms and the ratios thereof.

The inverse Mellin transform technique is used to obtain a representation of the density function of a nonnegative definite quadratic form. The density function of an indefinite quadratic form is then obtained in terms of Whittaker's function. Approximate density functions based on Patnaik and Pearson's approximations are also proposed. Computable expressions are then derived for the corresponding distribution functions. Three techniques are proposed for determining the distribution of ratios of quadratic forms: (1) the problem is transformed so that the results obtained for indefinite quadratic forms may be used; (2) the inverse Mellin transform technique is applied directly to the ratio in order to obtain the exact density function in terms of generalized hypergeometric functions; these techniques apply to ratios of quadratic forms which are not necessarily independently distributed; the moments which are used to approximate the distribution are also given; (3) the density function is obtained by differentiation; this approach requires the independence of the quadratic forms. These theoretical results are illustrated via several

examples, and then corroborated by simulations. For the lag- k serial correlation coefficient, the exact distribution function and an approximation are obtained. The cumulants of the serial covariance are derived in two ways: by introducing a special operator and by solving a system of second–order difference equations.

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CHAPTER 1

PRELIMINARY RESULTS

Introduction

Most of the distributional results derived in this thesis are based on the representation of the density function of a linear combination of independent chi-square random variables given at the end of Section 1.2. This representation is derived using the inverse Mellin transform technique as well as some identities involving a generalized hypergeometric function called the H-function. This technique is also applied directly to ratios of quadratic forms in Chapter 3; the representations of the density functions so obtained are expressed in terms of H-functions. In Section 1.1 we define the H-function and some of its properties. The density function of a linear combination of chi-square variables is derived in Section 1.2. A series representation of the H-function is given in Section 1.3. A discussion of the Pearson curves which are used to obtain approximate distributions is included in Section 1.4. Quadratic forms are defined in Section 1.5 where some of their properties are described. It is shown in particular that they are distributed as linear combinations of chi-square variables. Section 1.6 outlines the original contributions of the thesis and proposes some topics for future research.

Computable representations of the exact and approximate distribution and density functions of indefinite quadratic forms in noncentral singular or nonsingular random vectors are derived in Chapter 2 and used in the following chapter which contains several distributional results - exact and approximate - for ratios of quadratic forms. Several examples illustrate the theoretical results which are corroborated by simulations.

Some of the results are used in Chapter 4 in order to obtain the approximate and exact distribution function of the sample serial correlation coefficient. Some derivations are included in Appendix A in order to make Chapter 4 more readable. The moments of the lag- k serial correlation coefficient for a Gaussian white noise process are derived from the cumulants of the serial covariance. The cumulants are derived in two ways: by introducing a special operator and by solving a system of second-order difference equations. Since the distributional results obtained in this thesis are directly or indirectly (from the representation of the probability density function (p.d.f.) of a nonnegative definite quadratic form given in (1.56)) derived by means of the inverse Mellin transform, several other statistical applications of the inverse Mellin transform technique are listed in Appendix B. Finally some key computer programs which were used in the various calculations are included in Appendix C.

1.1 The H-Function as an Inverse Mellin Transform

The H-function is applicable in a number of problems arising in physical sciences, engineering and statistics. The importance of this function lies in the fact that nearly all the special functions occurring in applied mathematics and in statistics are its special cases.

The H-function has been studied by Fox (1971), Braaksma (1964), Nair (1973), Buschman (1974), Oliver and Kalla (1971) and Mathai and Saxena (1978) among others.

A representation of the density of a linear combination of chi-square variates is provided in Section 1.2. Such linear combinations are connected to various problems in many areas. For instance, for their connection to random division of intervals and distribution of spacings see Dwass (1961), to content of a frustrum of a simplex see Ali (1973), to storage capacities and queues see Prabhu (1965). Linear combinations of chi-square are also related to test statistics and traces of Wishart matrices as can be seen in Mathai (1980) and in Mathai and Pillai (1982). Their connection to time series problems can be seen from MacNeill (1974). They also appear in the study of probability content of offset ellipsoids in Gaussian hyperspace and of distribution of quadratic forms, see for example, Ruben (1962) and Sheil and O'Muircheartaigh (1979).

The basic tools for deriving distributions of sums, differences, products, ratios, powers, and more generally, algebraic functions of continuous random variables are the integral transforms. The most commonly used integral transforms are the Laplace transform, the Fourier transform and the Mellin transform. The aforementioned transforms, each corresponding to a function $f(x)$, are now defined, together with their inverse transforms.

If $f(x)$ is a real piecewise smooth function which is defined and single valued almost everywhere for $x \geq 0$ and which is such that the integral

$$\int_0^\infty |f(x)|e^{-kx}dx \tag{1.1}$$

converges for some real value k, then

$$L_f(r) = \int_0^\infty e^{-rx} f(x)dx \tag{1.2}$$

is the Laplace transform of $f(x)$, where r is a complex variable and the inverse

Laplace transform is

$$f(x) = (2\pi i)^{-1} \int_{c+i\infty}^{c-i\infty} e^{rx} L_f(r) dr \tag{1.3}$$

for values of x where $f(x)$ is continuous. Equation (1.3) determines $f(x)$ uniquely, if $L_f(r)$ is analytic in a strip consisting of the portion of the plane to the right of (and including) the Bromwich path $(c - i\infty, c + i\infty)$, the latter denoting the straight line given by

$$\lim_{a \to \infty} (c - ia, c + ia) \tag{1.4}$$

where $i = (-1)^{1/2}$ and c is any value greater than k in (1.1).

The Laplace transform provides the means for deriving and analyzing the distribution of sums of nonnegative random variables. On the other hand, if the random variables may take on both positive and negative values, the Fourier transform as defined in (1.6) is an appropriate tool for deriving the probability density function of their sums and their differences.

If $f(x)$ is a real piecewise smooth function which is defined and single valued almost everywhere for $-\infty < x < \infty$, and which is such that

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \tag{1.5}$$

where x is a real parameter, then

$$F_f(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx \tag{1.6}$$

is the Fourier transform of $f(x)$ and, where $f(x)$ is continuous, the corresponding inverse Fourier transform is

$$f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} F_f(t) dt. \tag{1.7}$$

The expression $F_f(t)$ is also called the characteristic function of $f(x)$ when $f(x)$ is a density function and e^{itx} is called the kernel.

The Mellin transform as defined in (1.9) constitutes the counterpart of the Laplace transform in deriving the distribution of products and ratios of nonnegative random variables.

If $f(x)$ is a real piecewise smooth function which is defined and single valued almost everywhere for $x > 0$ and which is such that

$$\int_0^\infty x^{k-1}|f(x)|dx \tag{1.8}$$

converges for some real value k, then

$$M_f(s) = \int_0^\infty x^{s-1} f(x)dx \tag{1.9}$$

where s is a complex number, is the Mellin transform of $f(x)$ and the inverse Mellin transform is

$$f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} M_f(s)ds \tag{1.10}$$

for values of x where $f(x)$ is continuous. Equation (1.10) determines $f(x)$ uniquely, if the Mellin transform is an analytic function of the complex variable s for

$$c_1 \leq Re(s) = c \leq c_2$$

where c_1 and c_2 are real. This condition is sufficient because the analyticity of the transform ensures that the integrand of the inversion integral is expressible as a Laurent expansion, which is always unique.

At this point we give some important properties of these integral transforms. It is assumed that each transform pair exists within the region of convergence.

1. **Linearity property**

$$\text{Laplace}: \; L_{c_1 f_1 + c_2 f_2}(r) = c_1 L_{f_1}(r) + c_2 L_{f_2}(r)$$

$$\text{Fourier}: \; F_{c_1 f_1 + c_2 f_2}(t) = c_1 F_{f_1}(t) + c_2 F_{f_2}(t)$$

$$\text{Mellin}: \; M_{c_1 f_1 + c_2 f_2}(s) = c_1 M_{f_1}(s) + c_2 M_{f_2}(s) \qquad (1.11)$$

2. **Shifting property**

$$\text{Laplace}: \; L_{e^{ax} f}(r) = L_f(r - a)$$

$$\text{Fourier}: \; F_{e^{ax} f}(t) = F_f(t - ia)$$

$$\text{Mellin}: \; M_{x^{-a} f}(s) = M_f(s - a) \qquad (1.12)$$

3. **Scaling**

$$\text{Laplace}: L_{f(ax)}(r) = (1/a) L_{f(x)}(r/a)$$

$$\text{Fourier}: F_{f(ax)}(t) = (1/a) F_{f(x)}(t/a)$$

$$\text{Mellin}: M_{f(ax)}(s) = (1/a)^s M_{f(x)}(s) \qquad (1.13)$$

4. **Exponentiation**

$$\text{Mellin}: \; M_{f(x^a)}(s) = (1/a) M_{f(x)}(s/a) \; . \qquad (1.14)$$

Since the great majority of cases, both theoretical and applied, involving the use of Mellin transform in connection with products, ratios and powers of independent random variables, are concerned with Mellin transforms of real variables, we henceforth restrict our discussion to Mellin transforms of real variables only.

Let

$$M_f(s) = \frac{\left[\prod_{j=1}^{m} \Gamma(b_j + B_j s)\right] \left[\prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)\right]}{\left[\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s)\right] \left[\prod_{j=n+1}^{p} \Gamma(a_j + A_j s)\right]} = h(s) , \qquad \text{say}, \tag{1.15}$$

where $M_f(s)$ is defined in (1.9); m, n, p, q are nonnegative integers such that $0 \le n \le p$, $0 \le m \le q$; A_j, $(j = 1, \ldots, p)$, B_j, $(j = 1, \ldots, q)$ are positve numbers and a_j, $(j = 1, \ldots, p)$, b_j, $(j = 1, \ldots, q)$ are complex numbers such that

$$-A_j(b_k + \nu) \neq B_k(1 - a_j + \lambda) \tag{1.16}$$

for $\nu, \lambda = 0, 1, 2, \ldots$; $k = 1, \ldots, m$; $j = 1, \ldots, n$. Then the H-function may be defined in terms of the inverse Mellin transform of $M_f(s)$ as follows:

$$\begin{aligned} f(x) &= H_{p,q}^{m,n}(x) \\ &= H_{p,q}^{m,n}\left\{x \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix}\right\} \\ &= H_{p,q}^{m,n}\left\{x \middle| \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix}\right\} \\ &= (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} h(s) x^{-s} ds \end{aligned} \tag{1.17}$$

where the Bromwich path $(c - i\infty, c + i\infty)$ defined in (1.4), separates the points

$$s = -(b_j + \nu)/B_j, \quad j = 1, \ldots, m; \quad \nu = 0, 1, 2, \ldots, \tag{1.18}$$

which are the poles of $\Gamma(b_j + B_j s)$, $j = 1, \ldots, m$, from the points

$$s = (1 - a_j + \lambda)/A_j, \quad j = 1, \ldots, n; \quad \lambda = 0, 1, 2, \ldots \ . \tag{1.19}$$

which are the poles of $\Gamma(1 - a_j - A_j s)$, $j = 1, \ldots, n$.

Hence, one must have that

$$\max_{1 \leq j \leq m} \mathrm{Re}\{-b_j/B_j\} < c < \min_{1 \leq j \leq n} \mathrm{Re}\{(1-a)/A_j\}. \tag{1.20}$$

The H -function is an analytic function of x and it is defined if the following existence conditions are satisfied.

$$\text{Case i. For all } x > 0, \quad \text{when } \mu > 0 \tag{1.21}$$

$$\text{Case ii. For } 0 < x < \beta^{-1}, \quad \text{when } \mu = 0 \tag{1.22}$$

where

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \tag{1.23}$$

and

$$\beta = \left\{ \prod_{j=1}^{p} A_j^{A_j} \right\} \left\{ \prod_{j=1}^{q} B_j^{-B_j} \right\}. \tag{1.24}$$

When

$$A_j = B_h = 1 \ , \ \ j = 1, \ldots, p; \ \ h = 1, \ldots, q \ \ ,$$

the H -function reduces to a Meijer's G -function. Hence

$$\begin{aligned} &H_{p,q}^{m,n}\left\{x \,\middle|\, \begin{matrix} (a_1, 1), \ldots, (a_p, 1) \\ (b_1, 1), \ldots, (b_q, 1) \end{matrix} \right\} \\ &= G_{p,q}^{m,n}\left\{x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right\} \end{aligned} \tag{1.25}$$

exists for all $x > 0$ when $q > p$ and for $0 < x < 1$ when $q = p$. A detailed account of Meijer's G -function can be found in Mathai and Saxena (1973).

The behavior of the H -function for small and large values of the argument has been discussed by Braaksma (1964). The two main results are

$$H_{p,q}^{m,n}(x) = O(|x|^c) \tag{1.26}$$

for small x, where $\mu \geq 0$ and $c = \min(b_j/B_j)$, $j = 1, \ldots, m$; and

$$H_{p,q}^{m,n}(x) = O(|x|^d) \tag{1.27}$$

for large x, where $\mu \geq 0$ and $d = \max((a_j - 1)/A_j)$, $j = 1, \ldots, n$.

Here we give a useful property of the H-function which follows readily from its definition.

$$H_{p,q}^{m,n}\left\{x\Big|\begin{matrix}(a_j, A_j)\\(b_j, B_j)\end{matrix}\right\} = H_{q,p}^{n,m}\left\{\frac{1}{x}\Big|\begin{matrix}(1-b_j, B_j)\\(1-a_j, A_j)\end{matrix}\right\} \tag{1.28}$$

When the H-function in (1.17) is nonnegative and when it is normalized by a normalizing constant so that $\int_0^\infty f(x)dx = 1$, we will refer to $f(x)$ as the probability density function of a random variable belonging to the class of H-functions or simply as an H-function random variable. Because this class includes so many basic distributions as well as the distribution of many test statistics in multivariate analysis (see for instance Mathai (1970), (1971), (1972), (1972a), Mathai and Saxena (1969), (1971), (1978), and Mathai and Rathie (1971), it is important to represent the H-function in computable form.

In fact, $H_{p,q}^{m,n}(x)$ is available as the sum of the residues of $h(s)x^{-s}$ in the points (1.18). The proof of the applicability of the residue theorem as well as a discussion about the contours for evaluating the integrals may be found in Springer (1979). In the next section, we will give some computable representations of the H-function for the cases of interest. Furthermore a computer program has been written by Eldred (1978) for the evaluation of the H-function. This program evaluates the probability density function (p.d.f) and the cumulative distribution function (c.d.f.) of an H-function random variables at any value of the random variable and also plots the p.d.f. and the c.d.f. The computer program compiled on an MNF compiler and run on a CDC 6600, is very efficient and poses no precision problem. Should precision problems arise when the program is run on smaller computers, the problem may be solved by compiling the program under IBM's Extended H -compiler. Furthemore it will be possible to evaluate the H -function using the forthcoming version of MAPLE.

Let $f(x)$ denote a p.d.f. as defined in (1.10) and $F(x)$, the corresponding distribution function. Then an analogous result for $F(x)$ to that of $f(x)$ in (1.10) is given by

$$F(x) = 1 - (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} s^{-1} M_f(s+1) ds.$$

Writing $F(x)$ in the form of an H -function inversion integral, we have

$$F(x) = 1 - (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Big\{ \Big[\prod_{j=1}^{m} \Gamma(b_j + B_j(s+1)) \Big] \Big[\prod_{j=1}^{n} \Gamma(1 - a_j - A_j(s+1)) \Big] \Big\}$$
$$\times \Big\{ s \Big[\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j(s+1)) \Big] \Big[\prod_{j=n+1}^{p} \Gamma(a_j + A_j(s+1)) \Big] \Big\}^{-1} x^{-s} ds. \quad (1.29)$$

The evaluation of this integral is discussed in detail in Springer (1979).

The definition of the H -function is slightly modified in (1.17) to present it as an inverse Mellin transform. This modification does not affect the results given by Braaksma (1964) on its properties and the conditions for its existence, nor the representation in computable forms found in Mathai and Saxena (1978), p.71, where the H -function which will be denoted by $\bar{\text{H}}$ is defined as follows:

$$\text{when} \quad \max_{1\le j\le n} \text{Re}(a_j - 1)/A_j < c' < \min_{1\le j\le m} \text{Re}(b_j/B_j), \quad (1.30)$$

$$\bar{H}_{p,q}^{m,n}\left\{x \Big| \begin{matrix}(a_j, A_j)\\(b_j, B_j)\end{matrix}\right\} = (2\pi i)^{-1} \int_{c'-i\infty}^{c'+i\infty} h(-s) x^{s} ds \quad (1.31)$$

where, according to (1.15), $h(-s)$ is equal to

$$\left\{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1-a_j+A_j s)\right\}$$

$$\times\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)\right\}^{-1}.$$

The identity

$$\bar{H}^{m,n}_{p,q}\left\{x\Big|{(a_j,A_j)\atop (b_j,B_j)}\right\} \equiv H^{m,n}_{p,q}\left\{x\Big|{(a_j,A_j)\atop (b_j,B_j)}\right\} \tag{1.32}$$

follows by changing direction; we simply have to replace s by $-s$ in (1.17).

1.2. Linear Combinations of Chi-square Variables

We will first derive an expression for the $h-th$ moment of the reciprocal of a linear combination of independent chi-square variables. This result which is stated in the next theorem will be used to determine the exact density of a linear combination of independent chi-square variates.

Theorem 1.1 Let $G = m_1X_1 + \cdots + m_nX_n$ where $m_j > 0,\ j = 1,\ldots,n,$ and $X_1,\ldots,X_n$ are independent chi-square variates with $r_1,\ldots,r_n$ degrees of freedom respectively; let $\rho = \frac{r_1}{2} + \cdots + \frac{r_n}{2}$, then

$$E(G^{-h}) = \left\{\prod_{j=1}^{n}(\mu_j)^{-\frac{r_j}{2}}\right\}\left\{\Gamma(q)/\Gamma(\rho)\right\}F_D(q;\frac{r_1}{2},\ldots,\frac{r_n}{2};\rho;\gamma_1,\ldots,\gamma_n) \tag{1.33}$$

where $q = \rho - h$, $0 < Re(h) < \rho$ and $|\gamma_j| = |(\mu_j - 1)/\mu_j| < 1$, that is $\mu_j = 2mj > \frac{1}{2},\ j = 1,\ldots,n,.$

Proof

Clearly the $h-th$ moment of 1/G is given by

$$E(G^{-h}) = \int_0^\infty \cdots \int_0^\infty G^{-h}\prod_{j=1}^{n}\left\{\frac{x_j^{\frac{r_j}{2}-1}e^{\frac{-x_j}{2}}}{2^{\frac{r_j}{2}}\Gamma(\frac{r_j}{2})}dx_j\right\}. \tag{1.34}$$

But since

$$G^{-h} = (\Gamma(h))^{-1} \int_0^\infty t^{h-1} e^{-Gt} dt \ , \tag{1.35}$$

for $\mathrm{Re}(h) > 0,\ G > 0$, where $Re(\cdot)$ denotes the real part of $(\cdot)$, we have

$$E(G^{-h}) = (\Gamma(h))^{-1} \int_0^\infty \cdots \int_0^\infty \int_0^\infty t^{h-1} e^{-t(m_1 x_1 + \ldots + m_n x_n)}$$
$$\times \prod_{j=1}^n \Big\{ \frac{x_j^{\frac{r_j}{2}-1} e^{\frac{-x_j}{2}}}{2^{\frac{r_j}{2}} \Gamma(\frac{r_j}{2})} dx_j \Big\} dt$$
$$= (\Gamma(h))^{-1} \int_0^\infty t^{h-1} \prod_{j=1}^n \Big\{ \int_0^\infty \frac{e^{-x_j(m_j t + \frac{1}{2})} x_j^{\frac{r_j}{2}-1}}{2^{\frac{r_j}{2}} \Gamma(\frac{r_j}{2})} dx_j \Big\} dt. \tag{1.36}$$

Noticing that for $j = 1, 2, \ldots, n$,

$$\int_0^\infty (\Gamma(\frac{r_j}{2}))^{-1} e^{-x_j(m_j t + \frac{1}{2})} x_j^{\frac{r_j}{2}-1} dx_j = (m_j t + \frac{1}{2})^{-\frac{r_j}{2}} \ , \tag{1.37}$$

(1.36) becomes

$$E(G^{-h}) = (\Gamma(h))^{-1} \int_0^\infty t^{h-1} \prod_{j=1}^n \Big\{ (m_j t + \frac{1}{2}) 2 \Big\}^{-\frac{r_j}{2}} dt$$
$$= (\Gamma(h))^{-1} \int_0^\infty t^{h-1} \prod_{j=1}^n (\mu_j t + 1)^{-\frac{r_j}{2}} dt \ , \tag{1.38}$$

where

$$\mu_j = 2mj \ , \quad j = 1, \ldots, n. \tag{1.39}$$

Letting $u = 1/(1+t)$, that is, $t = (1-u)/u$, one has

$$|\frac{dt}{du}| = \frac{1}{u^2}$$

and (1.38) becomes

$$E(G^{-h}) = (\Gamma(h))^{-1} \int_0^1 (\frac{1-u}{u})^{h-1} \prod_{j=1}^n \Big\{ \frac{\mu_j}{u} (1 - u + \frac{u}{\mu_j}) \Big\}^{-\frac{r_j}{2}} u^{-2} du \tag{1.40}$$
$$= (\Gamma(h))^{-1} \prod_{j=1}^n (\mu_j^{-\frac{r_j}{2}}) \int_0^1 u^{\rho-h-1} (1-u)^{h-1} \prod_{j=1}^n \Big\{ 1 - \frac{u(\mu_j - 1)}{\mu_j} \Big\}^{-\frac{r_j}{2}} du$$

where

$$\rho = \frac{r_1}{2} + \frac{r_2}{2} + \cdots + \frac{r_n}{2} \, . \tag{1.41}$$

Let

$$\rho - h = q \tag{1.42}$$

$$E(G^{-h}) = \left\{(\prod_{j=1}^{n} \mu_j^{-\frac{r_j}{2}})\Gamma(q)/\Gamma(\rho)\right\}[\left\{\Gamma(\rho)/(\Gamma(q)\Gamma(\rho - q))\right\}$$

$$\times \int_0^1 u^{q-1}(1-u)^{\rho-q-1} \prod_{j=1}^{n}(1-\gamma_j u)^{-\frac{r_j}{2}} du] \, . \tag{1.43}$$

At this point, we give a multiple series representation as well as a single integral representation of a type–D Lauricella's hypergeometric function of n variables denoted by F_D ( ).

$F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)$

$$= \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_n=j} \left\{(a)j/(c)j\right\}\left\{(b_1)_{j_1} \cdots (b_n)_{j_n}(x_1^{j_1} \ldots x_n^{j_n})\right\}/\left\{(j_1!)\cdots(j_n!)\right\} \tag{1.44}$$

where $|x_i| < 1$, $i = 1, \ldots, n$, and for instance, $(a)_j = \Gamma(a+j)/\Gamma(a)$, and

$F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)$

$$= \left\{\Gamma(c)/(\Gamma(a)\Gamma(c-a))\right\} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux_1)^{-b_1} \cdots$$

$$\times (1-ux_n)^{-b_n} du \, , \tag{1.45}$$

where

$$Re(a) > 0, \; Re(c-a) > 0.$$

These representations are given in Mathai and Saxena (1978), p. 162, where other types of Lauricella's hypergeometric functions are also defined. We may now rewrite

(1.43) as follows:

$$E(G^{-h}) = \Big\{\prod_{j=1}^{n}\mu_j^{-\frac{r_j}{2}}\Big\}\Big\{\Gamma(q)/\Gamma(\rho)\Big\}F_D(q;\frac{r_1}{2},\ldots,\frac{r_n}{2};\rho;\gamma_1,\ldots,\gamma_n), \qquad (1.46)$$

provided $\mathrm{Re}(q) = \mathrm{Re}(\rho - h) > 0$, $\mathrm{Re}(\rho - q) = Re(h) > 0$ and

$$|\gamma_j| = |(\mu_j - 1)/\mu_j| < 1 \ , \qquad (1.47)$$

that is,

$$\mu_j > \frac{1}{2} \text{ for } j = 1,\ldots,n \ .$$

Condition (1.47) allows us to express the $h-th$ moment of 1/G as a multiple series. Hence

$$E(G^{-h}) = \Big\{\prod_{j=1}^{n}\mu_j^{-\frac{r_j}{2}}\Big\}\Big\{\Gamma(q)/\Gamma(\rho)\Big\}\sum_{\nu=0}^{\infty}\sum_{\nu_1+\ldots+\nu_n=\nu}\Big\{(q)_\nu/(\rho)_\nu\Big\}$$

$$\times\Big\{(\frac{r_1}{2})_{\nu_1}\cdots(\frac{r_n}{2})_{\nu_n}\Big\}(\gamma_1^{\nu_1}\cdots\gamma_n^{\nu_n})/(\nu_1!\cdots\nu_n!) = \gamma_h \ \text{ say}, \qquad (1.48)$$

for $|\gamma_j| < 1, \quad j = 1,\ldots,n.$

If the condition $\mu_j > \frac{1}{2}$ is not satisfied for $j = 1,\ldots,n$, then we use the following technique to satisfy the conditions in the integral (1.43) as well as in the multiple series (1.48). We multiply G by δ/δ where δ is a non zero constant, this allows us to express in terms of multiple series the $h-th$ moment of $1/G$, that is,

$$E(G^{-h}) = E((G'/\delta)^{-h}) = \delta^h E(G'^{-h}),$$

where $G' = \delta G$. If δ is a such that $\mu'_j = \delta\mu_j = \delta 2m_j > \frac{1}{2}$ for all j, condition (1.47) is satisfied. We obtain the density of G' by taking the inverse Mellin transform of $E(G'^{-h})$ and thereby we can easily get the density of G by making an appropriate change of variables.

Noting that

$$\Gamma(q)\ (q)_\nu = \Gamma(q)\Gamma(q+\nu)/\Gamma(q) = \Gamma(q+\nu) = \Gamma(\rho+\nu-h) \tag{1.49}$$

and that

$$\Gamma(\rho)\ (\rho)_\nu = \Gamma(\rho+\nu)\ ,$$

(1.48) may be rewritten as

$$E(G^{-h}) = \sum_{\nu=0}^{\infty} K_\nu \Gamma(\rho+\nu-h) \tag{1.50}$$

where

$$\begin{aligned}
K_\nu &= \sum_{\nu_1+\ldots+\nu_n=\nu} k_{\nu_1\ldots\nu_n} \\
k_{\nu_1,\ldots,\nu_n} &= \Big(\prod_{j=1}^{n} \mu_j^{-\frac{r_j}{2}}\Big)\Big\{(\frac{r_1}{2})_{\nu_1}\cdots(\frac{r_n}{2})_{\nu_n}\Big\}\frac{(\gamma_1^{\nu_1}\cdots\gamma_n^{\nu_n})}{\Gamma(\rho+\nu)(\nu_1!\ldots\nu_n!)}\ , \\
\gamma_j &= |(\mu_j-1)/\mu_j| < 1,\ j=1,\ldots,n\ , \\
\mu_j &= 2m_j\ ,\ j=1,\ldots,n\ , \\
\rho &= \frac{r_1}{2}+\ldots+\frac{r_n}{2}
\end{aligned} \tag{1.51}$$

and

$$0 < Re(h) < \rho\ .$$

We obtain the density of $J = 1/G$ denoted $f(J)$ as the inverse Mellin transform of $E(J^h) = E(G^{-h})$.

Hence

$$f(J) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} E(J^h)J^{-(h+1)}dh \tag{1.52}$$

where

$$i = (-1)^{\frac{1}{2}} \quad \text{and} \quad c < \min_{\nu=0,1\ldots}(\rho+\nu) = \rho\ .$$

Since the infinite series in (1.50) is uniformly convergent within its region of convergence, $\mu_j > \frac{1}{2}$, $j+1,\ldots,n$, the density of J may be written as follows

$$f(J) = \sum_{\nu=0}^{\infty} K_\nu J^{-1}\Big\{(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\Gamma(\rho+\nu-h)J^{-h}dh\Big\}$$
$$= \sum_{\nu=0}^{\infty} K_\nu J^{-1} H_{1,0}^{0,1}\Big\{J\Big|{(1-\rho-\nu,\ 1) \atop }\Big\}$$

for $c < \rho$. Then according to (1.28), we have

$$f(J) = \sum_{\nu=0}^{\infty} K_\nu J^{-1} H_{0,1}^{1,0}\Big\{\frac{1}{J}\Big|{ \atop (\rho+\nu,\ 1)}\Big\}\ . \tag{1.53}$$

Now since

$$H_{0.1}^{1,0}\Big\{x\Big|{ \atop (b,B)}\Big\} = B^{-1}x^{\frac{b}{B}}\ \exp(-x^{\frac{1}{B}})$$

(equation 1.7.2 in Mathai and Saxena (1978)), we can express the density of J as follows for $J > 0$.

$$f(J) = \sum_{\nu=0}^{\infty} K_\nu J^{-(\rho+\nu+1)}\ \exp(-\frac{1}{J})\ .$$

Let $g_2(G)$ denote the probability density function of G. Since

$$J = \frac{1}{G}\ .$$

we have

$$\left|\frac{dJ}{dG}\right| = \frac{1}{G^2}$$

and

$$g_2(G) = \sum_{\nu=0}^{\infty} K_\nu G^{\rho+\nu-1} e^{-G} \qquad (1.55)$$

$$= \sum_{\nu=0}^{\infty} k_\nu G^{\rho+\nu-1} e^{-G} / \Gamma(\rho+\nu) \, .$$

where $k_\nu = K_\nu \Gamma(\rho + \nu)$. Therefore the density of G is available as a linear combination of densities of gamma variates with parameters $(\rho + \nu, 1)$.

To accelerate the convergence of the series representing the density of linear combination of chi-square variables we divide it by a constant β , a positive real number which is some average of the $\beta_j = 2l_j$, $j = 1, \ldots, n$ (the geometric mean or the harmonic mean for example), where

$$G = Y/\beta = \left(\sum_{j=1}^{n} l_j X_j \right) / \beta, \quad l_j > 0$$

and X_j are independent chi-square variables.

Then the coefficients μ_i in (1.51) are replaced with β_j / β and the simple change of variables, $Y = \beta G$, yields the density of Y . This result is stated in the following corollary.

Corollary 1. The exact density $f(y)$ of Y where

$$Y \sim \sum_{j=1}^{n} l_j \chi^2_{r_j}$$

is according to (1.55)

$$f(y) = \sum_{\nu=0}^{\infty} k_\nu y^{\rho+\nu-1} e^{-y/\beta} / \Gamma(\rho+\nu) \qquad (1.56)$$

for $y > 0$, where

$$\rho = \frac{r_1}{2} + \cdots + \frac{r_n}{2} ,$$

$$k_\nu = \sum_{\nu_1+\ldots+\nu_n=\nu} \{\prod_{j=1}^{n}(\beta_j^{-r_j/2}\}\{(r_1/2)_{\nu_1} \ldots (r_n/2)_{\nu_k}\} \times \{\gamma_1^{\nu_1} \ldots \gamma_n^{\nu_n}\}/\{(\nu_1! \ldots \nu_n!)\}$$

and

$$|\gamma_j| = \left|\frac{1}{\beta} - \frac{1}{\beta_j}\right| < 1, \; j = 1, \ldots, n .$$

The coefficients k_ν can be obtained using the recursive relationship

$$\begin{cases} k_0 = \prod_{j=1}^{n} \beta_j^{-\frac{r_j}{2}} , \\ k_i = \frac{1}{i} \sum_{k=0}^{i-1} g_{i-k} k_k , \\ g_m = \sum_{j=1}^{n} \frac{r_j}{2} \left(\frac{1}{\beta} - \frac{1}{\beta_j}\right)^m \end{cases}$$

which can be established by induction.

1.3. Series Representation of the H-Function

In this section, we give a series representation of the H-function which will enable us to express certain density functions in computable forms. The following representation of $\bar{H}_{p,q}^{m,n}(x)$ which are valid for $H_{p,q}^{m,n}$ in view of (1.32), can be deduced from the results of Mathai and Saxena (1978). A computable representation of (1.32) is obtained by evaluating (1.32) as the sum of the residues at the poles of the gamma product after indentifying the poles.

Here we use the following form of the H-function

$$\bar{H}_{p,q}^{m,n}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h(-s) x^s ds , \tag{1.57}$$

where

$$h(-s)=\frac{\left\{\prod_{j=1}^{m}\Gamma(b_j-B_js)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j+A_js)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j-A_js)\right\}}\ . \tag{1.58}$$

Case I When the poles of $\prod_{j=1}^{m}\Gamma(b_j+B_js)$ are assumed to be simple, that is, when

$$B_k(b_j+\lambda)\neq B_j(b_k+\nu)$$

for $j\neq k$; $j,\ k=1,\ldots,m$; $\lambda,\ \nu=0,1,2,\ldots$ we have the following expansion for $\bar{H}_{p,q}^{m,n}(x)$:

$$\bar{H}_{p,q}^{m,n}\left\{x\middle|\begin{matrix}(a_1,A_1), & \ldots, & (a_p,A_p)\\ (b_1,B_1), & \ldots, & (b_q,B_q)\end{matrix}\right\}=\sum_{k=1}^{m}\sum_{v=0}^{\infty}\frac{\prod_{j=1,j\neq k}^{m}\Gamma(b_j-B_j(b_k+v)/B_k)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j(b_k+v)/B_k)}\ .$$

$$\times\frac{\prod_{j=1}^{n}\Gamma(1-a_j+A_j(b_k+v)/B_k)}{\prod_{j=n+1}^{p}\Gamma(a_j-A_j(b_k+v)/B_k)}\frac{(-1)^v x^{(b_k+v)/B_k}}{(v!)B_k}\ , \tag{1.59}$$

which exists for all $x>0$ if $\mu>0$ and for $0<x<\beta^{-1}$ if $\mu=0$ where μ and β are defined in (1.23) and (1.24), respectively.

Case II When the poles of $\prod_{j=1}^{n}\Gamma(1-a_j+A_js)$ are assumed to be simple that is, when

$$A_k(1-a_j+v)\neq A_j(1-a_j+\lambda)\ ,$$

$j\neq k$; $j,\ k=1,2,\ldots,n$; $\lambda,\ \nu=0,1,\ldots,$ we have the following expansion for $\bar{H}_{p,q}^{m,n}(x)$:

$$\bar{H}_{p,q}^{m,n}\left\{x\middle|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\ (b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right\}=\sum_{k=1}^{n}\sum_{v=0}^{\infty}\frac{\prod_{j=1,j\neq k}^{n}\Gamma(1-a_j-A_j(1-a_k+v)/A_k)}{\prod_{j=n+1}^{p}\Gamma(a_j+A_j(1-a_k+v)/A_k)}\ .$$

$$\times \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j(1 - a_k + v)/A_k)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j(1 - a_k + v)/A_k)} \frac{(-1)^v x^{-(1-a_k+v)/A_k}}{(v!)A_k} , \tag{1.60}$$

which exists for all $x > 0$ if $\mu > 0$ and for $x > \beta^{-1}$ if $\mu = 0$ where μ and β are defined in (1.23) and (1.24), respectively.

Formula (1.60) can be obtained from (1.59) by making the following changes in it

$$\begin{aligned} &m \leftrightarrow n,\ q \leftrightarrow p\ , \\ &b_j \leftrightarrow 1 - a_j\ , \\ &B_j \leftrightarrow A_j\ , \text{ and} \\ &x \leftrightarrow 1/x\ , \end{aligned} \tag{1.61}$$

where $a \leftrightarrow b$ means that a and b are interchanged. The case of multiple poles is not discussed as it is not required for the numerical calculations performed in connection with the results derived in this thesis.

1.4. Approximation of the Distributions through the Moments

Since a description of the Pearson curves system is readily available from many sources we will not go into the details. Only the basic ideas will be given.

Pearson curves are probability densities

$$y = Q(x)$$

which are solution of the differential equation

$$\frac{dy}{dx} = \frac{(x - a)y}{(b_0 + b_1 x + b_2 x^2)} . \tag{1.62}$$

Given the mean value μ and the central moments μ_2, μ_3 and μ_4 of the distribution to be approximated, the selection of a particular Pearson curve is based on the following moment ratios

$$r_1 = \mu_3^2/\mu_2^3 \ , \tag{1.63}$$

$$r_2 = \mu_4/\mu_2^2 \tag{1.64}$$

and

$$\kappa = r_1(r_2+3)^2/\{4(2r_2-3r_1-6)(4r_2-3r_1)\} \ . \tag{1.65}$$

There are twelve types of curves and the set of rules for determining which curve best approximates a given probability distribution has been developed by Karl Pearson in the late 1880's. A complete development of the curves and the associated rules can be found in many books, see for example Elderton and Johnson (1969). Tables of standardized percentage points are included in Pearson and Hartley (1972) together with examples of their use. The steps used in fitting a Pearson curve to approximate a theoretical probability density function are given in Solomon and Stephens (1978).

From the first four moments about the origin, namely μ, μ_2', μ_3' and μ_4' from which one can compute μ_2, μ_3 and μ_4 , a Pearson curve may be fitted to approximate the theoretical p.d.f. of R.

It is worth mentioning that Carter (1970) has written a programm in FORTRAN language named STOFAN (stochastic function analyzer) which includes procedures to find the moments of the probability density function of an algebraic function of H-function independent random variables and to approximate the probability density function and the cumulative distibution function from the moments.

1.5. Quadratic Forms

Quadratic forms in singular and nonsingular normal variables are defined in this section. Various representations are also given.

Definition 1.1. Let $\boldsymbol{X} = (X_1, \ldots, X_n)'$ denote a random vector with mean $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$ and covariance matrix Σ. The quadratic form in the random variables $X_1, \ldots, X_n$ associated with an $n \times n$ symmetric matrix $A = (a_{ij})$ is defined as

$$Q(\boldsymbol{X}) = Q(X_1, \ldots, X_n) = \boldsymbol{X}'A\boldsymbol{X} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}X_iX_j \ . \tag{1.66}$$

Definition 1.2. A central quadratic form is a quadratic form in random variables whose means are all equal to zero; otherwise the quadratic form is said to be noncentral.

Definition 1.3. A real quadratic form $\boldsymbol{X}'A\boldsymbol{X}$ is said to be positive definite if $\boldsymbol{X}'A\boldsymbol{X} > 0$ for all $\boldsymbol{X} \neq \boldsymbol{0}$; positive semidefinite if $\boldsymbol{X}'A\boldsymbol{X} \geq 0$ for all $\boldsymbol{X}$, negative definite if $\boldsymbol{X}'A\boldsymbol{X} < 0$ for all $\boldsymbol{X} \neq \boldsymbol{0}$ and negative semidefinite if $\boldsymbol{X}'A\boldsymbol{X} \leq 0$ for all $\boldsymbol{X}$.

Definition 1.4. A quadratic form is indefinite if it does not belong to any of the categories described in Definition 1.3.

The matrix of an indefinite quadratic form has both positive and negative eigenvalues.

When $\boldsymbol{X}$ is distributed as a $N_p(\boldsymbol{\mu}, \Sigma)$ random vector (this is denoted by $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$), $\Sigma > 0$, the quadratic form $Q = \boldsymbol{X}'A\boldsymbol{X}$, $A = A'$ has the following representation

$$Q = \boldsymbol{X}'A\boldsymbol{X} = \sum_{j=1}^{p} \lambda_j (U_j + b_j)^2 \quad \text{in noncentral case} \tag{1.67}$$

Some representations of the density of linear combination of chi-square variables are available in Mathai and Saxena (1978).

2.2 The Exact Distribution Function

As explained in Section 2.1, indefinite quadratic forms in normal vectors are distributed as the difference of two linear combinations of chi-square variables. Let

$$Z = U - V = \sum_{j=1}^{t} l_j Y_j - \sum_{j=t+1}^{t+w} l_j Y_j \tag{2.23}$$

where the Y_j 's are independent noncentral chi-square variables with noncentrality parameter d_j^2 and α_i degrees of freedom and the l_j 's are positive real numbers $j = 1, \ldots, t+w$.

For $z \le 0$, the distribution function, $F(z)$, can be evaluated by integrating from $-\infty$ to z the representation of the density of Z given in (2.15) . Considering only the terms involving z and letting $\lambda = -(\alpha + k + s - \alpha' - \nu - \eta)/2$ and $\mu = (1 - \alpha' - \nu - \eta - \alpha - k - s)/2$, we have

$$\int_{-\infty}^{z} (-s)^{-\mu-\frac{1}{2}} e^{s/\beta'} \Bigg[(-bs)^{\mu+\frac{1}{2}} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \sum_{i=0}^{\infty} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{(-bs)^i}{i!}$$
$$+ (-bs)^{-\mu+\frac{1}{2}} \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \sum_{i=0}^{\infty} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{(-bs)^i}{i!} \Bigg] ds$$
$$= \sum_{i=0}^{\infty} \Bigg[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} (\frac{1}{i!}) \int_{-\infty}^{z} b^{\mu+\frac{1}{2}+i} e^{\frac{s}{\beta'}} (-s)^i ds$$
$$+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} (\frac{1}{i!}) \int_{-\infty}^{z} b^{-\mu+\frac{1}{2}+i} e^{\frac{s}{\beta'}} (-s)^{-2\mu+i} ds \Bigg] .$$

Now letting

$$I_{1i} = \Gamma(i+1, -\frac{z}{\beta'}) \beta'^{(i+1)} \quad \text{and} \quad I_{2i} = \Gamma(-2\mu+i+1, -\frac{z}{\beta'}) \beta'^{(-2\mu+i+1)} , \tag{2.24}$$

where

$$\Gamma(a, x) = \int_{x}^{\infty} e^{-t} t^{a-1} dt ,$$

$Q \sim Q_1 - Q_2$, where $Q_1 = \sum_{j=1}^{k} \lambda_j \chi^2_{r_j}(\delta_j^2)$ and $Q_2 = \sum_{j=k+1}^{m}(-\lambda_j)\chi^2_{r_j}(\delta_j^2)$, Q_1 and Q_2 being independently distributed with density representations given in Section 1.2. Then, the density of Q can be obtained by using the densities of Q_1 and Q_2 , see Khatri (1980) and Mathai (1983).

Gurland (1956) and Shah (1963) considered respectively central and noncentral indefinite quadratic forms but as pointed out by Shah (1963), the expansions obtained are not practical. Press (1966) obtained more useful representations in terms of confluent hypergeometric functions. An extension of Mathai's results to the case of arbitrary λ_j 's is given in Mathai (1983).

Box (1954a) considered a linear combination $Q = \sum_{j=1}^{m} \lambda_j \chi^2_{2\nu_j}$ of independent central chi-squares of even degrees of freedom $2\nu_j$, $j = 1, \ldots, m$ and obtained a finite series representations of the density of Q in central chi-square densities.

A modified form of Box's (1954a) result for arbitrary λ_j 's has been obtained by Imhof (1961) upon inverting the characteristic function of Q . A brief outline of the technique follows. For the general case assume that some of the λ_j 's are positive and some are negative. Let

$$Y = Q(X) = \lambda_1 \chi^2_{2\nu_1} + \cdots + \lambda_s \chi^2_{2\nu_s} - \lambda_{s+1}\chi^2_{2\nu_{s+1}} - \cdots - \lambda_m \chi^2_{2\nu_m} \tag{1.70}$$

where $\lambda_j > 0,\ j = 1, 2, \ldots, s,\ s+1, \ldots, m$, are assumed to be distinct within sets for $j = 1, \ldots, s$ and $j = s+1, \ldots, m$. The characteristic function of Y is then

$$\phi(t) = E\left(e^{itY}\right) = \left\{\prod_{j=1}^{s}(1 - 2i\lambda_j t)^{-\nu_j}\right\}\left\{\prod_{j=s+1}^{m}(1 + 2i\lambda_j t)^{-\nu_j}\right\}, \tag{1.71}$$

where $\nu_j,\ j = 1, \ldots, m$ are positive integers. Imhof (1961) inverted (1.71) to get the density and distribution function of Y . Numerical methods were suggested by Rice (1980) and Davis (1973) for the evaluation of the distribution function of indefinite quadratic forms.

Additional materials may be found in Baksalary and Kala (1983), Bhat (1962), Bhattacharyya (1945), Brown (1986), Charnet and Rathie (1985), Chattopadhyay (1982), Dick and Gurst (1985), Farebrother (1984, 1990), Grad and Solomon (1955), Gurland (1955), Harville (1971), Hotelling (1948), Jensen (1982), Kanno (1977), Khatri (1977b), Krishnaiah and Waiker (1973), Krishnan (1976), Mäkaläinen (1966), Moschopoulos and Canada (1984), Pachares (1952), Ponomarenko (1985), Provost (1988a), Robbins (1948), Rohde, Urquhart and Searle (1966), Rotar (1975), Ruben (1963, 1978), Shanbhag (1970), Slater (1960), Solomon (1960a, 1960b), Tziritas (1987) and Zorin and Iyubimov (1988).

1.6. Original Contributions and Future Research

The main original results contained in this thesis are enumerated below:

(1) an exact representation of the density function of an indefinite quadratic form in noncentral normal and possibly singular normal vectors is given in Section 2.1, Theorem 2.1;

(2) closed forms representations of the corresponding distribution functions are given in Section 2.2, Equations (2.25) and (2.27);

(3) we apply known approximations (Patnaik and Pearson) for positive definite quadratic forms to indefinite quadratic forms and express the results in terms of Whittaker's function (Sections 2.3.1 and 2.3.2);

(4) in connection with the distribution of ratios of noncentral quadratic forms, the exact distribution function is obtained by transforming the problem and applying the results obtained in Chapter 2 (Section 3.1.1, Theorem 3.1);

(5) the moments of the ratio of two noncentral quadratic forms are given explicitly (Section 3.2.2, Equation (3.28));

(6) the inverse Mellin transform technique is applied directly to the ratio of two noncentral quadratic forms (not necessarily independently distributed) in order to determine their exact density function in terms of generalized hypergeometric functions (Section 3.3.2, Equation (3.41);

(7) the density function of the ratio of two independently distributed quadratic forms is obtained by differentiation (Section 3.3.2, Equation (3.77));

(8) two new methods are proposed for the derivation of the cumulants of the lag–k sample serial correlation coefficient allowing us to obtain an explicit representation of the fifth moment which is not available in the literature;

(9) we give a closed form representation of the distribution function of the lag–k sample serial correlation coefficient for a series of length T and any lag k, $k \leq T-1$.

Possible directions for future research are

(1) obtaining the corresponding distributional results for quadratic forms in

 (a) random matrices,

 (b) elliptically contoured random vectors,

 (c) non–normal vectors;

(2) determining the eigenvalues of the matrix $A_k V$ in connection with the distribution of the lag- k sample serial correlation coefficient;

(3) creating a computer program that would give the percentage points of the lag- k sample serial correlation coefficient for series of any length T and any $k \leq T-1$;

(4) considering other time series models involving ratios of quadratic forms.

CHAPTER 2

THE DISTRIBUTION FUNCTION OF INDEFINITE QUADRATIC FORMS IN NONCENTRAL NORMAL VARIABLES

The distribution of quadratic forms in normal variables has been studied by several authors. Various representations of the distribution function of quadratic forms have been derived, and several different procedures have been proposed for computing percentage points and preparing tables. Gurland (1948, 1953, 1956), Pachares (1955), Ruben (1960, 1962), Shah and Khatri (1961) and Kotz *et al.*(1967) among others, gave representations of the distribution function of quadratic forms in terms of MacLaurin series, Laguerre polynomials and chi-squares distribution functions. Gurland (1956) and Shah (1963) considered respectively central and noncentral indefinite quadratic forms but as pointed by Shah (1963), the expansions obtained are not practical. Numerical methods were suggested by Rice (1980) and Davis (1973) for the evaluation of the distribution function of indefinite quadratic forms. Provost (1989a) obtained a representation of the density function of an indefinite quadratic form in central normal vectors.

This chapter contains three main results: (1) an exact representation of the density function of an indefinite quadratic form in noncentral normal and possibly singular normal vectors is given in Section 2.1, Theorem 2.1; (2) closed forms representations of the corresponding distribution functions are given in Section 2.2, Equations (2.25) and (2.27); (3) we apply known approximations (Patnaik and Pearson) for positive definite quadratic forms to indefinite quadratic forms and express the results in terms of Whittaker's function (Sections 2.3.1 and 2.3.2).

2.1 The Exact Density Function

We give in this section a representation of the density function of a linear combination of noncentral chi-square variables involving positive and negative coefficients. This representation applies to indefinite quadratic forms.

Let $U = \boldsymbol{X}'A\boldsymbol{X}$ be a positive definite quadratic form in noncentral normal variables where $\boldsymbol{X} \sim N_r(\boldsymbol{m}, V)$ and $V > 0$; let $V = LL'$, $P'L'ALP = \text{diag}(\lambda_1, \ldots, \lambda_r)$, $\lambda_i > 0$, $i = 1, \ldots, r$, $PP' = I$. Then, as shown in Section 1.5, U can be expressed in the following form

$$U = \sum_{i=1}^{r} \lambda_i (X_i + \delta_i)^2 \tag{2.1}$$

where the X_i 's are independent standard normal variables, the λ_i 's are the eigenvalues of $L'AL$ (or equivalently the eigenvalues of VA) and the δ_i 's are such that $(\delta_1, \ldots, \delta_r) = (PL^{-1}\boldsymbol{m})'$.

Note that some of the λ_i 's may be equal in which case we will have a sum of noncentral chi-squares some of which having more than one degree of freedom. Let l_j , $j = 1 \ldots, t$, denote the t distinct positive eigenvalues among the λ_i 's and let α_j denote the multiplicity of l_j (that is, the numbers of λ_i 's in U defined in (2.1) which are equal to l_j). Then it is seen that

$$U = \sum_{j=1}^{t} l_j \sum_{i=1}^{\alpha_j} (X_{j_i} + \delta_{j_i})^2$$

is distributed as a linear combination of chi-square variables each having α_j degrees of freedom and noncentrality parameter

$$d_j^2 = \sum_{i=1}^{\alpha_j} \delta_{j_i}^2 \ . \tag{2.2}$$

The moment generating function of U is given by

$$\begin{aligned}
M_U(\tau) =& E(\exp\{\sum_{j=1}^{t} \tau l_j \sum_{i=1}^{\alpha_j} (X_{j_i} + \delta_{j_i})^2\}) = \prod_{j=1}^{t} \prod_{i=1}^{\alpha_j} E(\exp\{\tau l_j (X_{j_i} + \delta_{j_i})^2\}) \\
&= \prod_{j=1}^{t} \prod_{i=1}^{\alpha_j} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}x^2 + \tau l_j (x + \delta_{j_i})^2\} dx \\
&= \prod_{j=1}^{t} \prod_{i=1}^{\alpha_j} (1 - 2l_j\tau)^{-\frac{1}{2}} \exp\{\tau \delta_{j_i}^2 l_j (1 - 2l_j\tau)^{-1}\} \\
&= \prod_{j=1}^{t} (1 - 2l_j\tau)^{-\frac{\alpha_j}{2}} e^{-\frac{d_j^2}{2}} e^{\frac{d_j^2}{2(1-2l_j\tau)}} \\
&= e^{-\frac{d^2}{2}} \prod_{j=1}^{t} \sum_{s_j=0}^{\infty} (\frac{d_j^2}{2})^{s_j} [(1 - 2l_j\tau)^{s_j + \frac{\alpha_j}{2}} s_j!]^{-1}
\end{aligned}$$

where $d_j^2 = \delta_{j_1}^2 + \cdots + \delta_{j_{\alpha_j}}^2$ and $d^2 = \sum_{j=1}^{t} d_j^2$.

Let $\beta_j = 2l_j$, then

$$M_U(\tau) = e^{-\frac{d^2}{2}} \sum_{s=0}^{\infty} \sum_{s_1+\cdots+s_t=s} \prod_{j=1}^{t} \left(\frac{d_j^2}{2}\right)^{s_j} (s_j!)^{-1} \prod_{j=1}^{t} (1 - \beta_j\tau)^{-(s_j + \frac{\alpha_j}{2})} \ .$$

We expand $\prod_{j=1}^{t}(1-\beta_j\tau)^{-(s_j+\frac{\alpha_j}{2})}$ as an infinite series in $(1-\beta\tau)^{-1}$ where β can be taken as a some average of the β_j 's, $j = ,\ldots ,t$ such as the geometric mean or the harmonic mean which were used by R.ben (1962) in a similar context.

Writing

$$(1 - \beta_j\tau) = \beta^{-1}\beta_j(1 - \beta\tau)[1 - c_j(1 - \beta\tau)^{-1}]$$

where

$$c_j = 1 - \frac{\beta}{\beta_j}$$

and assuming that $|c_j/(1-\beta\tau)| < 1$ or equivalently that $\tau < 1/\beta_j, \quad 1 \le j \le t$, one has the following identity

$$(1-\beta_j\tau)^{-(s_j+\frac{\alpha_j}{2})} = \left(\frac{\beta}{\beta_j}\right)^{s_j+\frac{\alpha_j}{2}} \sum_{k_j=0}^{\infty} \frac{(s_j+\frac{\alpha_j}{2})_{k_j}}{k_j!} c_j^{k_j} (1-\beta\tau)^{-(s_j+\frac{\alpha_j}{2})-k_j}$$

where, for example, $(s_1+\frac{\alpha_1}{2})_{k_1} = \Gamma(s_1+\frac{\alpha_1}{2}+k_1)/\Gamma(s_1+\frac{\alpha_1}{2})$. Note that it suffices to require that

$$|c_j| = |1-\beta/\beta_j| < 1 \ , \ j = 1,\ldots,t \ , \tag{2.3}$$

as it is then always possible to select τ small enough so that $|1-\beta\tau| > |c_j|$, $j = 1,\ldots,t$.

Then

$$M_U(\tau) = e^{-\frac{d^2}{2}} \sum_{s=0}^{\infty} \sum_{s_1+\cdots+s_t=s} \prod_{j=1}^{t} \left(\frac{d_j^2}{2}\right)^{s_j} (s_j!)^{-1} \left(\frac{\beta}{\beta_j}\right)^{s_j+\frac{\alpha_j}{2}} \tag{2.4}$$
$$\times \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_t=k} \frac{(s_1+\frac{\alpha_1}{2})_{k_1}\cdots(s_t+\frac{\alpha_t}{2})_{k_t}}{k_1!\cdots k_t!} c_1^{k_1}\cdots c_t^{k_t} (1-\beta\tau)^{-\alpha-k-s}$$

where $\alpha = (\alpha_1+\cdots+\alpha_t)/2$.

Noting that $(1-\beta\tau)^{-\alpha-k-s}$ is the moment generating function of gamma variates with parameters $\alpha+k+s$ and β, the inversion of $M_U(\tau)$ yields the following expression for the density of U

$$g_U(u) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \theta_{s,k} u^{\alpha+k+s-1} e^{-\frac{u}{\beta}}/\Gamma(\alpha+k+s) \ , \quad u > 0 \tag{2.5}$$

where

$$\theta_{s,k} = e^{-\frac{d^2}{2}} \sum_{s_1+\cdots+s_t=s} \prod_{j=1}^{t} \left(\frac{d_j^2}{2}\right)^{s_j} (s_j!)^{-1} \left(\frac{\beta}{\beta_j}\right)^{s_j+\frac{\alpha_j}{2}} \sum_{k_1+\cdots+k_t=k} \prod_{j=1}^{t} \frac{(s_j+\frac{\alpha_j}{2})_{k_j} c_j^{k_j}}{k_j!\beta^{\alpha+k+s}}$$
$$= e^{-\frac{d^2}{2}} \sum_{s_1+\cdots+s_t=s} \sum_{k_1+\cdots+k_t=k} \prod_{j=1}^{t} \left[\frac{d_j^{2s_j}(s_j+\frac{\alpha_j}{2})_{k_j}}{\beta_j^{s_j+\frac{\alpha_j}{2}} s_j! k_j! 2^{s_j}} (\beta^{-1}-\beta_j^{-1})^{k_j}\right] \tag{2.6}$$

and $d^2 = \sum_{j=1}^{t} d_j^2$. Note that the representation (2.5) can also be obtained using the inverse Mellin transform technique as shown in Section 1.2.

We now consider a difference of two linear combinations of independent noncentral chi-square random variables with α_j degrees of freedom and noncentality parameter d_j^2 , $j = 1, \ldots, t+w$. Let

$$Z = U - V \tag{2.7}$$

where

$$U = \sum_{j=1}^{t} l_j Y_j \ , \quad Y_j \sim \chi^2_{\alpha_j}(d_j^2) \tag{2.8}$$

$$V = \sum_{j=t+1}^{t+w} l_j Y_j \ , \quad Y_j \sim \chi^2_{\alpha_j}(d_j^2) \tag{2.9}$$

and $l_j > 0$, $j = 1, \ldots, t+w$. Then U and V respectively have the probability density functions

$$g_U(u) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \theta_{s,k} u^{\alpha+k+s-1} e^{-\frac{u}{\beta}} / \Gamma(\alpha+k+s) \ , \quad u > 0$$

where $\theta_{s,k}$ is given in (2.6), and

$$g_V(v) = \sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \theta'_{\nu,\eta} v^{\alpha'+\nu+\eta-1} e^{-\frac{v}{\beta'}} / \Gamma(\alpha'+\nu+\eta) \ , \quad v > 0$$

where

$$\theta'_{\nu,\eta} = e^{-\frac{\gamma^2}{2}} \sum_{\nu_{t+1}+\cdots+\nu_{t+w}=\nu} \ \sum_{\eta_{t+1}+\cdots+\eta_{t+w}=\eta} \ \prod_{j=t+1}^{t+w} \left[\frac{\gamma_j^{2\nu_j}(\nu_j + \frac{\alpha_j}{2})_{\eta_j}}{(\beta_j)^{\nu_j + \frac{\alpha_j}{2}} \nu_j! \eta_j! 2^{\nu_j}} \left(\frac{1}{\beta'} - \frac{1}{\beta_j} \right)^{\eta_j} \right] \tag{2.10}$$

with $\gamma^2 = \sum_{i=t+1}^{t+w} d_j^2$ and $\alpha' = (\alpha_{t+1} + \cdots + \alpha_{t+w})/2$, $\beta_j = 2l_j$, and β' is such that $|1 - \beta'/\beta_j| < 1$, $j = t+1, \ldots, t+w$. Since U and V are independently distributed, the joint density of $Y = U$ and $Z = U - V$ is given by

$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta} y^{\alpha+k+s-1} e^{-y/\beta} (y-z)^{\alpha'+\nu+\eta-1} e^{-(y-z)/\beta'}}{\Gamma(\alpha+k+s)\Gamma(\alpha'+\nu+\eta)}$$

for $z > 0$ and $y > z$ or for $z \leq 0$ and $y > 0$.

The marginal density of Z is therefore

$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}e^{z/\beta'}}{\Gamma(\alpha+k+s)\Gamma(\alpha'+\nu+\eta)} \int_z^{\infty} y^{\alpha+k+s-1}(y-z)^{\alpha'+\nu+\eta-1}e^{-by}dy \tag{2.11}$$

for $z > 0$ and

$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}e^{z/\beta'}}{\Gamma(\alpha+k+s)\Gamma(\alpha'+\nu+\eta)} \int_0^{\infty} y^{\alpha+k+s-1}(y-z)^{\alpha'+\nu+\eta-1}e^{-by}dy \tag{2.12}$$

for $z \leq 0$ where $b = \beta^{-1} + (\beta')^{-1}$.

Using Eq. 4, p. 319 of Gradshteyn and Ryzhik (1980) and noting that $\alpha' + v + \eta > 0$ and $bz > 0$ in (2.11), one can express the integral in (2.11) as follows

$$b^{-(\alpha+k+s+\alpha'+\nu+\eta)/2} z^{(\alpha+k+s+\alpha'+\nu+\eta-2)/2}\Gamma(\alpha'+\nu+\eta)e^{-bz/2}$$
$$\times W_{\frac{(\alpha+k+s-\alpha'-\nu-\eta)}{2},\frac{(1-\alpha-k-s-\alpha'-\nu-\eta)}{2}}(bz)$$

for $z > 0$ where W($\cdot$) denotes Whittaker's function.

Now letting $\rho = y + \xi$ where $\xi = -z$ the integral in (2.12) becomes

$$\int_{\xi}^{\infty} (\rho-\xi)^{\alpha+k+s-1}\rho^{\alpha'+\nu+\eta-1}e^{-b(\rho-\xi)}d\rho$$
$$= e^{b\xi}\int_{\xi}^{\infty} (\rho-\xi)^{\alpha+k+s-1}\rho^{\alpha'+\nu+\eta-1}e^{-b\rho}d\rho$$
$$= e^{-bz}b^{-(\alpha+k+s+\alpha'+\nu+\eta)/2}(-z)^{(\alpha+k+s+\alpha'+\nu+\eta-2)/2}\Gamma(\alpha+k+s)e^{bz/2}$$
$$\times W_{\frac{(-\alpha-k-s+\alpha'+\nu+\eta)}{2},\frac{(1-\alpha-k-s-\alpha'-\nu-\eta)}{2}}(-bz)$$

for $z \leq 0$.

When $z = 0$, the integral in (2.12) is equal to

$$b^{-(\alpha+k+s+\alpha'+\nu+\eta-1)}\Gamma(\alpha+k+s+\alpha'+\nu+\eta-1) .$$

Making use of some identities of Section 9.220, 9.210.1 in Gradshteyn and Ryzhik (1980), one can express Whittaker's function as follows:

$$W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} \; {}_1F_1(\mu-\lambda+\frac{1}{2}, 2\mu+1; z)$$
$$+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} z^{-\mu+\frac{1}{2}} e^{-\frac{z}{2}} \; {}_1F_1(-\mu-\lambda+\frac{1}{2}, -2\mu+1; z) \quad (2.13)$$

whenever the various quantities are defined, where for example

$${}_1F_1(\delta,\gamma,z) = \sum_{i=0}^{\infty} (\delta)_i z^i [(\gamma)_i i!]^{-1} \; .$$

The exact density of Z is given in terms of Whittaker's function in the following theorem.

Theorem 2.1

Let

$$Z = U - V = \sum_{j=1}^{t} l_j Y_j - \sum_{j=t+1}^{t+w} l_j Y_j$$

where the Y_j 's are independent noncentral chi-square variables with noncentrality parameter d_j^2 and α_i degrees of freedom and the l_j 's are positive real numbers $j = 1,\ldots,t+w$. Let $\alpha = (\alpha_1 + \cdots + \alpha_t)/2$, $\alpha' = (\alpha_{t+1} + \cdots + \alpha_{t+w})/2$, $\beta_j = 2l_j$, $j = 1,\ldots,t+w$, and $b = \beta^{-1} + \beta'^{(-1)}$, where β and β' are such that $|1-\beta/\beta_j| < 1$, $j = 1,\ldots,t$, and $|1-\beta'/\beta_j| < 1$, $j = t+1,\ldots,t+w$. Then the density of Z is given by

$$f(z) = \begin{cases} \displaystyle\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha+s+k)} b^{\frac{-(\alpha+k+\alpha'+\nu+s+\eta)}{2}} z^{\frac{(\alpha+k+\alpha'+\nu+s+\eta-2)}{2}} & (2.14) \\ \quad \times e^{z(\beta'^{-1}-\beta^{-1})/2} W_{\frac{\alpha+k+s-\alpha'-\nu-\eta}{2}, \frac{1-\alpha'-\upsilon-\eta-\alpha-k-s}{2}}(bz) \quad \text{for } z > 0, & \\ \displaystyle\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+\nu+\eta)} b^{\frac{-(\alpha+k+\alpha'+\nu+s+\eta)}{2}} (-z)^{\frac{(\alpha+k+\alpha'+\nu+s+\eta-2)}{2}} & (2.15) \\ \quad \times e^{z(\beta'^{-1}-\beta^{-1})/2} W_{\frac{\alpha'+\nu+\eta-\alpha-k-s}{2}, \frac{1-\alpha-k-s-\alpha'-\nu-\eta}{2}}(-bz) \quad \text{for } z \le 0, & \end{cases}$$

where $W_{\lambda,\mu}(\cdot)$ denotes Whittaker's function defined in (2.13), and $\theta_{s,k}$ and $\theta'_{\nu,\eta}$ are given in (2.6) and (2.10). When $z = 0$, the ordinate of the density of Z obtained from (2.15) is equal to

$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\theta_{s,k}\theta'_{\nu,\eta}b^{-(\alpha+k+s+\alpha'+v+\eta-1)}}{\Gamma(\alpha+k+s)\Gamma(\alpha'+\nu+\eta)}\Gamma(\alpha+k+s+\alpha'+v+\eta-1)\,. \tag{2.16}$$

<u>Central Case</u>

For the central case, $d_j^2 = 0$, $e^{(d_j^2/2(1-2l_j\tau))} = 1$, $j = 1\ldots,t+w$, and the sums over s and ν disappear. Hence the density of Z in the central case is

$$f(z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\theta_k\theta'_\eta}{\Gamma(\alpha+k)}b^{\frac{-(\alpha+k+\alpha'+\eta)}{2}}z^{\frac{(\alpha+k+\alpha'+\eta-2)}{2}} & (2.17)\\ \qquad\times e^{z(\beta'^{-1}-\beta^{-1})/2}W_{\frac{\alpha+k-\alpha'-\eta}{2},\frac{1-\alpha'-\eta-\alpha-k}{2}}(bz) \qquad \text{for } z>0, & \\ \displaystyle\sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\theta_k\theta'_\eta}{\Gamma(\alpha'+\eta)}b^{\frac{-(\alpha+k+\alpha'+\eta)}{2}}(-z)^{\frac{(\alpha+k+\alpha'+\eta-2)}{2}} & (2.18)\\ \qquad\times e^{z(\beta'^{-1}-\beta^{-1})/2}W_{\frac{\alpha'+\eta-\alpha-k}{2},\frac{1-\alpha-k-\alpha'-\eta}{2}}(-bz) \qquad \text{for } z\le 0, & \end{cases}$$

where

$$\theta_k = \sum_{k_1+\cdots+k_t=k}\left[\prod_{i=1}^{t}\left(\beta_i^{\frac{-\alpha_i}{2}}(\frac{\alpha_i}{2})_{k_i}(\beta^{-1}-\beta_i^{-1})^{k_i}/(k_i)!\right)\right], \tag{2.19}$$

$$\theta'_\eta = \sum_{\eta_{t+1}+\cdots+\eta_{t+w}=\eta}\left[\prod_{i=t+1}^{t+w}\left(\beta_i^{\frac{-\alpha_i}{2}}(\frac{\alpha_i}{2})_{\eta_i}(\beta'^{-1}-\beta_i^{-1})^{\eta_i}/(\eta_i)!\right)\right], \tag{2.20}$$

$$|1-\beta/\beta_i|<1,\ i=1,\ldots,t,$$

and (2.21)

$$|1-\beta'/\beta_i|<1,\ i=t+1,\ldots,t+w.$$

When $z = 0$, the ordinate of the density of Z obtained from (2.18) is equal to

$$\sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\theta_k\theta'_\eta b^{-(\alpha+k+\alpha'+\eta-1)}\Gamma(\alpha+k+\alpha'+\eta-1)}{\Gamma(\alpha+k)\Gamma(\alpha'+\eta)}\,. \tag{2.22}$$

Some representations of the density of linear combination of chi-square variables are available in Mathai and Saxena (1978).

2.2 The Exact Distribution Function

As explained in Section 2.1, indefinite quadratic forms in normal vectors are distributed as the difference of two linear combinations of chi-square variables. Let

$$Z = U - V = \sum_{j=1}^{t} l_j Y_j - \sum_{j=t+1}^{t+w} l_j Y_j \tag{2.23}$$

where the Y_j 's are independent noncentral chi-square variables with noncentrality parameter d_j^2 and α_i degrees of freedom and the l_j 's are positive real numbers $j = 1, \ldots, t+w$.

For $z \le 0$, the distribution function, $F(z)$, can be evaluated by integrating from $-\infty$ to z the representation of the density of Z given in (2.15). Considering only the terms involving z and letting $\lambda = -(\alpha + k + s - \alpha' - \nu - \eta)/2$ and $\mu = (1 - \alpha' - \nu - \eta - \alpha - k - s)/2$, we have

$$\int_{-\infty}^{z} (-s)^{-\mu-\frac{1}{2}} e^{s/\beta'} \left[(-bs)^{\mu+\frac{1}{2}} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \sum_{i=0}^{\infty} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{(-bs)^i}{i!} \right.$$
$$\left. + (-bs)^{-\mu+\frac{1}{2}} \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \sum_{i=0}^{\infty} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{(-bs)^i}{i!} \right] ds$$
$$= \sum_{i=0}^{\infty} \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} (\frac{1}{i!}) \int_{-\infty}^{z} b^{\mu+\frac{1}{2}+i} e^{\frac{s}{\beta'}} (-s)^i ds \right.$$
$$\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} (\frac{1}{i!}) \int_{-\infty}^{z} b^{-\mu+\frac{1}{2}+i} e^{\frac{s}{\beta'}} (-s)^{-2\mu+i} ds \right] .$$

Now letting

$$I_{1i} = \Gamma(i+1, -\frac{z}{\beta'}) \beta'^{(i+1)} \quad \text{and} \quad I_{2i} = \Gamma(-2\mu+i+1, -\frac{z}{\beta'}) \beta'^{(-2\mu+i+1)} , \tag{2.24}$$

where

$$\Gamma(a, x) = \int_{x}^{\infty} e^{-t} t^{a-1} dt ,$$

or

$$\Gamma(a,x) = \Gamma(a) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{a+n}}{n!(a+n)} ,$$

the distribution function of Z can be represented as follows:

$$F(z) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)}\, b^{\mu-\frac{1}{2}} \times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)}\frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i}\frac{b^{\mu+\frac{1}{2}+i}}{i!}I_{1i} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)}\frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i}\frac{b^{-\mu+\frac{1}{2}+i}}{i!}I_{2i}\right]. \tag{2.25}$$

For $z > 0$ the distribution function, $F(z)$, can be evaluated by integrating from $-\infty$ to z the representations of the density of Z given in (2.14) for $z > 0$ and in (2.15) for $z \le 0$ which is equivalent to integrating the representation (2.14) from 0 to z and adding $F(0)$. Considering only the terms involving z and letting $\lambda_2 = (\alpha + k + s - \alpha' - \nu - \eta)/2$ and $\mu = (1 - \alpha' - \nu - \eta - \alpha - k - s)/2$, we have

$$\int_0^z (s)^{-\mu-\frac{1}{2}} e^{-s/\beta}\left[(bs)^{\mu+\frac{1}{2}}\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)}\sum_{i=0}^{\infty}\frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i}\frac{(bs)^i}{i!} + (bs)^{-\mu+\frac{1}{2}}\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)}\sum_{i=0}^{\infty}\frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i}\frac{(bs)^i}{i!}\right] ds$$

$$= \sum_{i=0}^{\infty}\left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)}\frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i}(\frac{1}{i!})\int_0^z b^{\mu+\frac{1}{2}+i}e^{-\frac{s}{\beta}}(s)^i ds + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)}\frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i}(\frac{1}{i!})\int_0^z b^{-\mu+\frac{1}{2}+i}e^{-\frac{s}{\beta}}(s)^{-2\mu+i}ds\right] .$$

Now letting

$$I_{3i} = \gamma(i+1,\frac{z}{\beta})\beta^{(i+1)} \quad \text{and} \quad I_{4i} = \gamma(-2\mu+i+1,\frac{z}{\beta})\beta^{(-2\mu+i+1)} , \tag{2.26}$$

where the incomplete gamma function,

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt$$

or

$$\gamma(a,x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{a+n}}{n!(a+n)} ,$$

the distribution function of Z can be represented as follows:

$$F(z) = F(0) + \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha+s+k)}\, b^{\mu-\frac{1}{2}} \times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)} \frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{\mu+\frac{1}{2}+i}}{i!} I_{3i} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)} \frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} I_{4i} \right] , \quad (2.27)$$

where

$$F(0) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)}\, b^{\mu-\frac{1}{2}} \times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{\mu+\frac{1}{2}+i}}{i!} \Gamma(i+1)\beta'^{(i+1)} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} \Gamma(-2\mu+i+1)\beta'^{(-2\mu+i+1)} \right] .$$

The distribution function of Z for quadratic forms in central normal variables is given below: it is a special case of the distributions given in (2.25) and (2.26).

$$F(z) = \sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_k\theta'_\eta}{\Gamma(\alpha'+\eta)}\, b^{\mu-\frac{1}{2}} \times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{\mu+\frac{1}{2}+i}}{i!} I_{1i} + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} I_{2i} \right] , \quad \text{for } z \le 0 \quad (2.28)$$

where I_{1i} and I_{2i} are given in (2.20) and

$$F(z) = F(0) + \sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_k\theta'_\eta}{\Gamma(\alpha+k)}\, b^{\mu-\frac{1}{2}}$$

$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)} \frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{\mu+\frac{1}{2}+i}}{i!} I_{3i} \right. \tag{2.29}$$

$$\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)} \frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} I_{4i} \right] , \quad \text{for } z > 0$$

where $\lambda = -(\alpha+k-\alpha'-\eta)/2$, $\lambda_2 = (\alpha+k-\alpha'-\eta)/2$ and $\mu = (1-\alpha'-\eta-\alpha-k)/2$, θ_k and θ'_η are given in (2.19) and (2.20) respectively, I_{3i} and I_{4i} are given in (2.26), and

$$F(0) = \sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_k \theta'_\eta}{\Gamma(\alpha'+\eta)} b^{\mu-\frac{1}{2}}$$

$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{\mu+\frac{1}{2}+i}}{i!} \Gamma(i+1)\beta'^{(i+1)}) \right.$$

$$\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} \Gamma(-2\mu+i+1)\beta'^{(-2\mu+i+1)} \right] .$$

When α and α' are both nonnegative integers plus 1/2 then, $\mu-\lambda+1/2$ is a negative integer plus 1/2, $2\mu+1$ is a negative integer, and the representation (2.25) and (2.27) will both diverge since $(\mu-\lambda+\frac{1}{2})_i/(2\mu+1)_i$ is infinite for any $i \geq -2\mu$ in that case.

In order to find the distribution function in this case, we modify the equation (2.25) by isolating one of the $l_j Y_j$'s (denoted below by Y) having an odd number degrees of freedom $2a$ (where a is a multiple of $1/2$) i.e., let

$$Z = U - V = Y + U_1 - V = Y + Z^* \tag{2.30}$$

where Y is a gamma variable with density

$$g(y) = \frac{1}{\Gamma(a)} \frac{y^{a-1} e^{-y/B}}{B^a} . \tag{2.31}$$

The linear combination Z^* has the density given in (2.14) and (2.15) with $\alpha^* = \alpha - a$ where α^* is equal to a nonnegative integer. Using definition (2.13) of the

Whittaker's function, the joint density of $W = Y$ and $Z = Y + Z^*$ is

$$
f(z,w) = \begin{cases}
\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{1}{\Gamma(a)} \frac{w^{a-1}e^{-w/B}}{B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)} \quad (2.32) \\
\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} (w-z)^i e^{(z-w)/\beta'} \right. \\
\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} (w-z)^{-2\mu+i} e^{(z-w)/\beta'} \right], \\
\qquad \text{for } z-w \le 0 \\
\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{1}{\Gamma(a)} \frac{w^{a-1}e^{-w/B}}{B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha+k+s)} \quad (2.33) \\
\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)} \frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} (z-w)^i e^{(w-z)/\beta} \right. \\
\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)} \frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} (z-w)^{-2\mu+i} e^{(w-z)/\beta} \right]. \\
\qquad \text{for } z-w > 0,
\end{cases}
$$

where $\lambda = -(\alpha - a + k + s - \alpha' - \nu - \eta)/2$, $\lambda_2 = (\alpha - a + k + s - \alpha' - \nu - \eta)/2$ and $\mu = (1 - \alpha' - \nu - \eta - \alpha + a - k - s)/2$.

For $z \le 0$ the marginal density of Z has the form

$$
\begin{aligned}
f_1(z) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} & \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)} \qquad (2.34) \\
\times & \left[\frac{(-1)^i\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \int_0^\infty (z-w)^i w^{a-1} e^{(z-w)/\beta'} e^{-w/B} dw \right. \\
+ & \left. \frac{(-1)^{-2\mu+i}\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} \int_0^\infty (z-w)^{-2\mu+i} w^{a-1} e^{(z-w)/\beta'} e^{-w/B} dw \right]
\end{aligned}
$$

and, for $z > 0$, it has a following form:

$$
\begin{aligned}
f_2(z) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} & \left\{ \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)} \right. \\
\times & \left[\frac{(-1)^i\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \int_z^\infty (z-w)^i w^{a-1} e^{(z-w)/\beta'} e^{-w/B} dw \right.
\end{aligned}
$$

$$+ \frac{(-1)^{-2\mu+i}\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} \int_z^{\infty} (z-w)^{-2\mu+i} w^{a-1} e^{(z-w)/\beta'} e^{-w/B} dw \Bigg]$$

$$+ \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha+k+s)} \tag{2.35}$$

$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)} \frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \int_0^z (z-w)^i w^{a-1} e^{-(z-w)/\beta} e^{-w/B} dw \right.$$

$$\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)} \frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} \int_0^z (z-w)^{-2\mu+i} w^{a-1} e^{-(z-w)/\beta} e^{-w/B} dw \right] \Bigg\} .$$

To obtain a series representation for $f_1(z)$ when $z \leq 0$, we use binomial expansions in the integrands of (2.34) and then we perform the integration. This yields

$$f_1(z) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{\eta=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)} \tag{2.36}$$

$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \right.$$

$$\times \sum_{n=0}^{i} \binom{i}{n} \left(\frac{1}{\beta'}+\frac{1}{B}\right)^{-(i-n+a)} \Gamma(i-n+a)(-z)^n e^{\frac{z}{\beta'}}$$

$$+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!}$$

$$\left. \sum_{n=0}^{-2\mu+i} \binom{-2\mu+i}{n} \left(\frac{1}{\beta'}+\frac{1}{B}\right)^{-(-2\mu+i-n+a)} \Gamma(-2\mu+i-n+a)(-z)^n e^{\frac{z}{\beta'}} \right] .$$

On integrating this expression from $-\infty$ to z, we have the distribution function for $z \leq 0$:

$$F(z) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{\eta=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)} \tag{2.37}$$

$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \right.$$

$$\times \sum_{n=0}^{i} \binom{i}{n} \left(\frac{1}{\beta'}+\frac{1}{B}\right)^{-(i-n+a)} \Gamma(i-n+a)(\beta')^{n+1}\Gamma(n+1,\frac{z}{\beta'})$$

$$+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!}$$

$$\times \sum_{n=0}^{-2\mu+i} \binom{i-2\mu}{n} \left(\frac{1}{\beta'} + \frac{1}{B}\right)^{(2\mu-i+n-a)} \Gamma(i-2\mu-n+a)(\beta')^{n+1}\Gamma(n+1, \frac{z}{\beta'})\Bigg] .$$

Using Eq. 4.p. 319 of Gradshteyn and Ryzhik (1980), we can express the first two integrals in (2.35) in terms of Whittaker's function and the remaining two in terms of hypergeometric function and obtain a series representation for $f_2(z)$ for the case $z > 0$:

$$\begin{aligned} f_2(z) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \Bigg\{ & \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)} \\ & \times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} I_1 \right. \\ & \left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} I_2 \right] \\ & + \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha-a+k+s)} \\ & \times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)} \frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} I_3 \right. \\ & \left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)} \frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} I_4 \right] \Bigg\} , \end{aligned} \tag{2.38}$$

where

$$\begin{aligned} I_1 = \sum_{p=0}^{\infty} \Bigg\{ & \frac{\Gamma(i+a)}{p!} \frac{(1-a)_p}{(1-a-i)_p} \left(\frac{1}{\beta'} + \frac{1}{B}\right)^{p-i-a} z^p e^{-\frac{z}{B}} \\ & + \frac{\Gamma(i+1)}{p!} \frac{\Gamma(-i-a)}{\Gamma(1-a)} \frac{(i+1)_p}{(1+a+i)_p} \left(\frac{1}{\beta'} + \frac{1}{B}\right)^{p-i-a} z^{i+a+p} e^{-\frac{z}{B}} \Bigg\} \end{aligned}$$

and

$$I_2 = \sum_{p=0}^{\infty} \left\{ \frac{\Gamma(-2\mu+i+a)}{p!} \frac{(1-a)_p}{(1-a+2\mu-i)_p} \left(\frac{1}{\beta'}+\frac{1}{B}\right)^{2\mu+p-i-a} z^p e^{-\frac{z}{B}} \right.$$
$$+ \frac{\Gamma(-2\mu+i+1)}{p!} \frac{\Gamma(2\mu-i-a)}{\Gamma(1-a)} \frac{(-2\mu+i+1)_p}{(-2\mu+1+a+i)_p}$$
$$\left. \times \left(\frac{1}{\beta'}+\frac{1}{B}\right)^{2\mu+p-i-a} z^{-2\mu+i+a+p} e^{-\frac{z}{B}} \right\}$$
$$I_3 = \sum_{p=0}^{\infty} \frac{(a)_p}{(a+i+1)_p} \frac{B(i+1,a)}{p!} \left(\frac{1}{\beta}-\frac{1}{B}\right)^p z^{i+a+p} e^{-\frac{z}{\beta}}$$
$$I_4 = \sum_{p=0}^{\infty} \frac{(a)_p}{(-2\mu+a+i+1)_p} \frac{B(-2\mu+i+1,a)}{p!} \left(\frac{1}{\beta}-\frac{1}{B}\right)^p z^{-2\mu+i+a+p} e^{-\frac{z}{\beta}} .$$

The distribution function of Z for $z > 0$, is obtained by integrating the representation (2.38) from 0 to z and adding $F(0)$:

$$F(z) = F(0) + \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \left\{ \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)} \right.$$
$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \int_0^z I_1 \right.$$
$$\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} \int_0^z I_2 \right]$$
$$+ \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha-a+k+s)} \tag{2.39}$$
$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)} \frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \int_0^z I_3 \right.$$
$$\left.\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)} \frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!} \int_0^z I_4 \right] \right\},$$

where

$$F(0) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{1}{\Gamma(a)B^a} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v+\eta)}$$

$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \right.$$

$$\times \sum_{n=0}^{i} \binom{i}{n} \left(\frac{1}{\beta'} + \frac{1}{B} \right)^{-(i-n+a)} \Gamma(i-n+a)(\beta')^{n+1}\Gamma(n+1)$$

$$+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^i}{i!}$$

$$\left. \times \sum_{n=0}^{-2\mu+i} \binom{i-2\mu}{n} \left(\frac{1}{\beta'} + \frac{1}{B} \right)^{(2\mu-i+n-a)} \Gamma(i-2\mu-n+a)(\beta')^{n+1}\Gamma(n+1) \right],$$

$$\int_0^z I_1 = \sum_{p=0}^{\infty} \left\{ \frac{\Gamma(i+a)}{p!} \frac{(1-a)_p}{(1-a-i)_p} \left(\frac{1}{\beta'} + \frac{1}{B} \right)^{p-i-a} B^{p+1} \gamma(p+1, \frac{z}{B}) \right.$$

$$+ \frac{\Gamma(i+1)}{p!} \frac{\Gamma(-i-a)}{\Gamma(1-a)} \frac{(i+1)_p}{(1+a+i)_p} \left(\frac{1}{\beta'} + \frac{1}{B} \right)^{p-i-a} B^{i+a+p} \gamma(i+a+p, \frac{z}{B}),$$

$$\int_0^z I_2 = \sum_{p=0}^{\infty} \left\{ \frac{\Gamma(-2\mu+i+a)}{p!} \frac{(1-a)_p}{(1-a+2\mu-i)_p} \left(\frac{1}{\beta'} + \frac{1}{B} \right)^{2\mu+p-i-a} \right.$$

$$\times B^{p+1}\gamma(p+1, \frac{z}{B}) + \frac{\Gamma(-2\mu+i+1)}{p!} \frac{\Gamma(2\mu-i-a)}{\Gamma(1-a)} \frac{(-2\mu+i+1)_p}{(-2\mu+1+a+i)_p}$$

$$\left. \times \left(\frac{1}{\beta'} + \frac{1}{B} \right)^{2\mu+p-i-a} B^{-2\mu+i+a+p+1} \gamma(-2\mu+i+a+p+1, \frac{z}{B}) \right\},$$

$$\int_0^z I_3 = \sum_{p=0}^{\infty} \frac{(a)_p}{(a+i+1)_p} \frac{B(i+1,a)}{p!} \left(\frac{1}{\beta} - \frac{1}{B} \right)^p$$

$$\times \beta^{i+a+p+1} \gamma(i+a+p+1, \frac{z}{\beta})$$

and

$$\int_0^z I_4 = \sum_{p=0}^{\infty} \frac{(a)_p}{(-2\mu+a+i+1)_p} \frac{B(-2\mu+i+1,a)}{p!} \left(\frac{1}{\beta} - \frac{1}{B} \right)^p$$

$$\times \beta^{-2\mu+i+a+p+1} \gamma(-2\mu+i+a+p+1, \frac{z}{\beta}).$$

2.3 Approximations

The various methods used for computing the exact distribution of quadratic forms in normal variables require in general extensive numerical computations. As for the existing tables, not only an interpolation is frequently required, but extensive tabulation for larger p (dimensionality of the random vector of the quadratic form) is prohibitive because of the number of param' 'ers involved. Furthermore, there is a need for tables of values for the noncentral case as well.

Numerical integrations on the other hand, though sufficiently accurate for solving the general problem, require a considerable amount of computer time.

Several approximation methods have been proposed to reduce those difficulties. Some of these approximations are valid for special classes of quadratic forms involved in certain applications. We adapt in the next two sections two known approximate distributions for positive definite quadratic forms to indefinite quadratic forms.

2.3.1 Patnaik's Chi-square Type Approximation for the Central Case

The simplest approximation proposed by Patnaik (1949) consists of replacing the distribution of $Q \sim \sum_{j=1}^{n} \lambda_j U_j^2$ where the U_j 's are independent standard normal variables, by that of $\theta\chi_\nu^2$, where θ and ν are chosen so that Q and $\theta\chi_\nu^2$ have the same first two moments, that is,

$$E(Q) = E(\theta\chi_\nu^2) \ , \ Var(Q) = Var(\theta\chi_\nu^2) \tag{2.40}$$

or

$$\sum_{j=1}^{n} \lambda_j = \theta\nu \ , \ \sum_{j=1}^{n} \lambda_j^2 = \theta^2 \nu \ . \tag{2.41}$$

From these relations one gets

$$\theta = \Big(\sum_{j=1}^{n} \lambda_j^2\Big) \Big/ \Big(\sum_{j=1}^{n} \lambda_j\Big) \quad \text{and} \quad \nu = \Big(\sum_{j=1}^{n} \lambda_j\Big)^2 \Big/ \Big(\sum_{j=1}^{n} \lambda_j^2\Big) \ .$$

As seen from the above expression for ν, it can happen that ν is fractional. Hence in the following discussion the notation χ^2_ν will mean a gamma variable with parameters $\alpha = \nu/2$ and $\beta = 2$. Then one may write

$$Pr(Q \le x) \simeq Pr(\chi^2_\nu \le x/\theta) \tag{2.42}$$

where the symbol $\simeq$ means "is approximately equal to".

The distribution of the χ^2_ν may be approximately normalized using the Wilson-Hilferty (1931) transformation. Johnson and Kotz (1970) comment that this method depends not so much on the accuracy of the approximation as on the smoothness of the error as a function of the λ_j 's.

Let

$$Z = U - V = \sum_{j=1}^{t} l_j Y_j - \sum_{j=t+1}^{t+w} l_j Y_j$$

where the Y_j 's are independent central chi-square variables with α_i degrees of freedom and the l_j 's are positive real numbers $j = 1, \ldots, t + w$.

Using Patnaik's approximation we replace the linear combinations U and V by $\theta_1 W_1$ and $\theta_2 W_2$ respectively, where $W_1 \sim \chi^2_m$ and $W_2 \sim \chi^2_n$. Then

$$Z^* \sim \theta_1 \chi^2_m - \theta_2 \chi^2_n \tag{2.43}$$

where Z^* approximate Z. The distribution of Z^* is found from the expressions (2.28) and (2.29) of the distribution function of the difference of two quadratic forms in central normal variables:

$$
F_{z^*}(z) = \begin{cases}
\left[(2\theta_1)^{\frac{m}{2}}(2\theta_2)^{\frac{n}{2}}\Gamma(\frac{n}{2})\right]^{-1} \sum_{i=0}^{\infty} \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \right. \\
\times (2\theta_2)^{i+1}\Gamma(i+1, -\frac{z}{2\theta_2}) \\
+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} \\
\left. \times\, (2\theta_2)^{-2\mu+i+1}\Gamma(-2\mu+i+1, -\frac{z}{2\theta_2})\right] , \text{ for } z \le 0 \\
\left[(2\theta_1)^{\frac{m}{2}}(2\theta_2)^{\frac{n}{2}}\Gamma(\frac{m}{2})\right]^{-1} \sum_{i=0}^{\infty} \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda_2)} \frac{(\mu-\lambda_2+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{2\mu+i}}{i!} \right. \\
\times (2\theta_2)^{i+1}\gamma(i+1, \frac{z}{2\theta_1}) \\
+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda_2)} \frac{(-\mu-\lambda_2+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} \\
\left. \times\, (2\theta_2)^{-2\mu+i+1}\gamma(-2\mu+i+1, \frac{z}{2\theta_1})\right] + F_{Z^*}(0) , \text{ for } z > 0
\end{cases} \tag{2.44}
$$

where $b = 1/2\theta_1 + 1/2\theta_2$, $\lambda = (n-m)/4$, $\lambda_2 = (m-n)/4$, $\mu = (2-m-n)/4$, $\gamma(\cdot,\cdot)$ is defined in (2.26) and

$$
\begin{aligned}
F_{Z^*}(0) = {} & \left[(2\theta_1)^{\frac{m}{2}}(2\theta_2)^{\frac{n}{2}}\Gamma(\frac{n}{2})\right]^{-1} \sum_{i=0}^{\infty} \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} b^{2\mu+i}\, (2\theta_2)^{i+1} \right. \\
& + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} \\
& \left. \times\, (2\theta_2)^{-2\mu+i+1}\Gamma(-2\mu+i+1)\right] .
\end{aligned} \tag{2.45}
$$

2.3.2 Pearson's Approach for the Noncentral Case

If $\theta\chi^2_\nu + c$ is used instead of $\theta\chi^2_\nu$, a further improvement might be expected analogous to Pearson's (1959) three-moment central chi-square approximation to the distribution of a noncentral chi-square.

Let $E(Q)$ and σ_Q be the mean and standard deviation of the noncentral quadratic form $Q = \sum_{j=1}^{n} \lambda_j (U_j + d_j)^2$. Following Pearson's approach, one can write

$$Q \approx Q^* \quad , \quad Q^* \sim \Big(\frac{\chi^2_\nu - \nu}{(2\nu)^{\frac{1}{2}}}\Big)\sigma_Q + E(Q) \tag{2.46}$$

where the symbol $\approx$ means "is approximately distributed as" and determine ν so that both Q and Q^* have equal third cumulants. The s-th cumulant of Q is $\kappa_s = 2^{s-1}(s-1)!\sum_{j=1}^{n}\lambda_j^s(1+sd_j^2)$. Since $E(\chi^2_\nu) = \nu, \mathrm{Var}(\chi^2_\nu) = 2\nu$ note that $E(Q^*) = E(Q)$ and $\mathrm{Var}\,(Q^*) = \sigma_Q^2$. Denoting the moment generating function of Q^* by $M_{Q^*}(t)$ one has

$$M_{Q^*}(t) = (1 - t\sigma_Q\sqrt{(2/\nu)})^{-\nu/2}\exp(t\{E(Q) - \sqrt{(\nu/2)}\}) \ , \tag{2.47}$$

$$\ln M_{Q^*}(t) = -\frac{\nu}{2}\ln(1 - t\sigma_Q\sqrt{(2/\nu)}) + t\{E(Q) - \sigma_Q\sqrt{(\nu/2)}\} \ , \tag{2.48}$$

and

$$\frac{\partial^3}{\partial t^3}\ln M_{Q^*}(t)|_{t=0} = \frac{2\sqrt{2}}{\sqrt{\nu}}\sigma_Q^3 \ . \tag{2.49}$$

Thus the third cumulant of Q^* is $2\sqrt{2}\sigma_Q^3/\sqrt{\nu}$ and the second and first cumulants of Q^* coincide with those of Q. Equating the third cumulants of Q^* and Q one gets

$$\nu = \frac{k_2^3}{k_3^2} \ , \tag{2.50}$$

where

$$k_s = \sum_{j=1}^{n}\lambda_j^s(1+sd_j^2) \ , \quad s = 1,2,3 \ .$$

Thus,

$$Q \approx \frac{k_3}{k_2}\chi^2_\nu - \frac{k_2^2}{k_3} + k_1 \ , \tag{2.51}$$

or $Q \approx \theta\chi^2_\nu + e$, where $\theta = k_3/k_2$, $e = -k_2^2/k_3 + k_1$. Hence

$$Pr(Q \le x) \simeq Pr(\theta\chi^2_\nu + e \le x) = Pr(\theta\chi^2_\nu \le y) \ , \tag{2.52}$$

where

$$y = x - e \, .$$

In both Patnaik's and Pearson's approximations, ν is usually fractional so that an interpolation is needed if standard chi-square tables are used. Then, the Wilson-Hilferty approximation can be used to evaluate the right-hand side of (2.42). As can be seen in Imhof (1961), Pearson's approximation, which requires very little more work, gives a much better fit than is achieved with Patnaik's approximation, particularly in the upper tail.

Let

$$Z = U - V = \sum_{j=1}^{t} l_j Y_j - \sum_{j=t+1}^{t+w} l_j Y_j$$

where the Y_j 's are independent noncentral chi-square variables with noncentrality parameter d_j^2 as defined in (2.2) and α_i degrees of freedom and the l_j 's are positive real numbers $j = 1, \ldots, t + w$.

Using Pearson's approximation we replace the linear combinations U and V by $\theta_1 U^* + e_1$ and $\theta_2 V^* + e_2$ where $U^* \sim \chi_m^2$ and $V^* \sim \chi_n^2$, respectively. Then

$$Z^* \sim \theta_1 \chi_m^2 - \theta_2 \chi_n^2 + e_1 - e_2 \tag{2.53}$$

where Z^* approximates Z . Then

$$\begin{aligned} Pr(Z \leq z) \simeq & Pr(U^* - V^* + e_1 - e_2 \leq z) \\ & = Pr(U^* - V^* \leq y) = Pr(Z^* \leq y) \end{aligned}$$

where $y = z - e_1 - e_2$ The distribution of Z^* is found from the representations (2.44) and (2.45) of the distribution function of the difference of two central chi-square variables.

2.4 Noncentral Quadratic Forms in Singular Normal Vectors

We show in this section that the previous results can be used in the singular case. Let $Q(\boldsymbol{X}) = \boldsymbol{X}'A\boldsymbol{X}$ be a semi positive definite quadratic form in noncentral normal variables where $\boldsymbol{X}$ is $p \times 1$ vector with $E(\boldsymbol{X}) = \boldsymbol{\mu}$, Cov $\boldsymbol{X} = \Sigma \geq 0$, $\rho(\Sigma) = r \leq p$; let $\Sigma = BB'$, where B is $p \times r$, $B'AB \neq O$, $P'B'ABP = \text{diag}(\lambda_1, \ldots, \lambda_r)$, $\lambda_i \geq 0$, $i = 1, \ldots, r$, and $PP' = I$. Then, as shown in Section 1.5, $Q(\boldsymbol{X}) = \boldsymbol{X}'A\boldsymbol{X}$ can be expressed in the following form

$$Q(\boldsymbol{X}) = \boldsymbol{X}'A\boldsymbol{X} = \sum_{i=1}^{r} \lambda_i (U_i + \delta_i)^2 + c \tag{2.54}$$

where the U_i 's are independent standard normal variables, the λ_i 's are the eigenvalues of $P'B'ABP$,

$$\delta_i = \frac{b_j}{\lambda_j}\,, \quad c = a - \sum_{j=1}^{r} \frac{b_j^2}{\lambda_j}\,,$$

$\boldsymbol{b}' = (b_1, \ldots, b_r) = \boldsymbol{\mu}'ABP$, and $a = \boldsymbol{\mu}'A\boldsymbol{\mu}$.

Note that some of the λ_i 's may be equal in which case we will have a sum of noncentral chi-squares where some of the chi-squares have more than one degree of freedom. Let l_j, $j = 1 \ldots, t$, denote the t distinct positive eigenvalues among the λ_i 's and let α_j denote the multiplicity of l_j (that is, the numbers of λ_i 's in U which are equal to l_j). Then it is seen that

$$Q(\boldsymbol{X}) = \sum_{j=1}^{t} l_j \sum_{i=1}^{\alpha_j} (U_{ji} + \delta_{ji})^2 + c\,. \tag{2.55}$$

Let

$$Z = U - V = \sum_{j=1}^{t} l_j Y_j + c_1 - \sum_{j=t+1}^{t+w} l_j Y_j - c_2 = U^* - V^* + c_1 - c_2$$

represent the difference of two quadratic forms, where the Y_j 's are independent noncentral chi-square variables with noncentrality parameter d_j^2 and α_i degrees of freedom, the l_j 's are positive real numbers $j = 1, \ldots, t + w$ and let c_1 , c_2 be real constants as given in (2.54). Note that U^* and V^* are linear combinations of noncentral chi-square variables.

Then the distribution function of Z is

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) = \Pr(U^* - V^* + c_0 \leq z) \\ &= \Pr(U^* - V^* \leq z - c_0) = F_{Z^*}(z - c_0) \ , \end{aligned} \tag{2.56}$$

where $c_0 = c_1 - c_2$ and $Z^* = U^* - V^*$.

In view of the above relationship, the distribution function of Z can be evaluated from the representations given for the nonsingular case in (2.25) (when $(z - c_0) \leq 0$) and in (2.27) (when $(z - c_0) > 0$).

2.5 Sums of Quadratic and Bilinear Forms

It should be pointed out that the results derived in this chapter also apply to bilinear forms, i.e. expressions of the type $\boldsymbol{x}'A\boldsymbol{y}$, in view of the following identity:

$$\boldsymbol{x}'A\boldsymbol{y} = (\boldsymbol{x}', \boldsymbol{y}') \begin{pmatrix} O & \frac{1}{2}A' \\ \frac{1}{2}A & O \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \tag{2.57}$$

which shows that any bilinear form can be expressed as a quadratic form.

Furthemore, it is seen from the following identity that sums of quadratic forms may be expressed as a single quadratic form:

$$\begin{aligned} &(\boldsymbol{x}', \boldsymbol{y}') \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} + (\boldsymbol{y}', \boldsymbol{z}') \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} \\ &= (\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') \begin{pmatrix} A_{11} & A_{12} & O \\ A_{21} & (A_{22} + B_{11}) & B_{12} \\ O & B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} . \end{aligned} \tag{2.58}$$

In view of the above remarks the distributional results given in Chapter 3 are also applicable to ratios involving linear combinations of quadratic and/or bilinear forms either in their numerators or their denominators or both.

CHAPTER 3

THE DISTRIBUTION OF THE RATIO OF QUADRATIC FORMS IN NONCENTRAL NORMAL VARIABLES

Ratios of quadratic forms arise in a variety of contexts. They are connected, for example, to regression and analysis of variance problems associated with linear models. Koerts and Abrahamse (1969) have investigated the distribution of ratios of quadratic forms in the context of the general linear model. Von Neumann (1941), using a geometrical approach has derived an expression for the moments of the ratio of the mean square successive difference to the sample variance for n consecutive observations of a normal population with mean value zero and variance σ^2. White (1957) used the sample circular serial correlation coefficient, r, for a first order Gaussian auto-regressive process, X_t, as an estimator for the parameter ρ of the stochastic difference equation

$$X_t = \rho X_{t-1} + u_t$$

where the u_t's are independent standard normal variables and ρ is an unknown parameter. Shenton and Johnson (1965) derived the first few terms of the series expansions of the first two moments of this sample circular serial correlation coefficient. This type of correlation coefficient can be expressed as the ratio of two quadratic forms. Morin-Wahhab (1985) derived the integer moments of the ratio of two nonnegative quadratic forms expressed as linear combinations of chi-square variables. One of the first papers on the study of ratios of quadratic forms was written by Pitman and Robbins (1949). They expressed the distribution function of the ratio of two independent positive definite central quadratic forms in normal variables in terms of a double series of beta distribution functions. Box (1954a) also obtained the distribution function of the above ratio with some restrictions. We consider the more general case where the quadratic forms, Q_1 and Q_2, in the ratio, denoted by R, are not necessarily independent.

This chapter contains four original results in connection with the distribution of ratios of quadratic forms: (1) for ratios of noncentral quadratic forms, the exact distribution function is obtained by transforming the problem and applying the results obtained in Chapter 2 (see Section 3.1.1, Theorem 3.1); (2) the moments of the ratio of two noncentral quadratic forms are given explicitly (Section 3.2.2, Equation (3.28)); (3) the inverse Mellin transform technique is applied directly to the ratio of two noncentral quadratic forms (not necessarily independently distributed) in order to determine their exact density function in terms of generalized hypergeometric functions (Section 3.3.2, Equation (3.23); (4) the density function of the ratio of two independently distributed quadratic forms is obtained by differentiation (Section 3.3.2, Equation (3.77)).

3.1 Exact Distribution Function (Case of Dependence)

3.1.1 Representations in Closed Forms

Let

$$R = \frac{Q_1}{Q_2} \tag{3.1}$$

where $Q_1 = (X', Y')A(X', Y')'$, $Q_2 = (Y', Z')B(Y', Z')'$, X is $m \times 1$, Y is $t \times 1$ Z is $n \times 1$, $A = A' = (A_{ij})$ $i, j = 1, 2$, A_{11} is $m \times m$, A_{12} is $m \times t$, $A_{21} = A'_{12}$, A_{22} is $t \times t$, $B = B' = (B_{ij})$ $i, j = 1, 2$, B_{11} is $t \times t$, $B_{12} = B'_{21}$ is $t \times n$, B_{22} is $n \times n$, $(X', Y', Z')' \sim N_p(\mu, \Sigma)$, $\Sigma > 0$.

Now letting $F_R(c)$ denote the distribution function of R at the point c, $W = (X', Y', Z')'$, $\Sigma = LL'$ with L of dimension $p \times p$, $W = LV$ where $V \sim N_p(L^{-1}\mu, I)$, and

$$S_c = \begin{pmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} - cB_{11} & -cB_{12} \\ O & -cB_{21} & -cB_{22} \end{pmatrix},$$

one has that

$$\begin{aligned} F_R(c) = \Pr(R \leq c) &= \Pr(Q_1 - cQ_2 \leq 0) \\ &= \Pr(W'S_cW \leq 0) \\ &= \Pr((LV)'S_c(LV) \leq 0) \\ &= \Pr(V'M_cV \leq 0) \\ &= \Pr((Z + L^{-1}\mu)'M_c(Z + L^{-1}\mu) \leq 0) \end{aligned}$$

where $M_c = M'_c = L'S_cL$ and $Z \sim N_p(0, I)$.

Let P be a $p \times p$ orthogonal matrix which diagonalizes M_c (see for example 1c.3(i) in Rao (1965)). That is

$$P'M_cP = \text{diag}(\lambda_1, \ldots, \lambda_p), \quad PP' = I$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of M_c. Letting $\boldsymbol{U} = P'\boldsymbol{Z}$, one has that $\boldsymbol{Z} = P\boldsymbol{U}$, $E(\boldsymbol{U}) = \mathbf{0}$ and $\mathrm{Cov}(\boldsymbol{U}) = I$.

$$F_R(c) = \Pr((\boldsymbol{U} + \boldsymbol{\delta})'P'M_cP(\boldsymbol{U} + \boldsymbol{\delta}) \le 0)$$
$$= \Pr((\boldsymbol{U} + \boldsymbol{\delta})'\mathrm{diag}(\lambda_1, \ldots, \lambda_p)(\boldsymbol{U} + \boldsymbol{\delta}) \le 0)$$

where $\boldsymbol{U}' = (U_1, \ldots, U_p)$ and $\boldsymbol{\delta} = (P'L^{-1}\boldsymbol{\mu})' = (\delta_1, \ldots, \delta_p)$ and the U_j's are mutually independent standard normal variables. Hence

$$F_R(c) = \Pr\left(\sum_{j=1}^{p} \lambda_j(U_j + \delta_j)^2 \le 0\right) \quad \text{for } E(\boldsymbol{W}) = \boldsymbol{\mu},$$
$$= \Pr\left(\sum_{j=1}^{p} \lambda_j U_j^2 \le 0\right) \quad \text{for } E(\boldsymbol{W}) = \mathbf{0}, \tag{3.2}$$

that is, $F_R(c)$ can be obtained from the distribution of a general linear combination (in the sense that the coefficients in the linear combination may also be negative) of independent central chi-square variables when $\boldsymbol{\mu} = \mathbf{0}$ and independent noncentral chi-square variables when $\boldsymbol{\mu} \neq \mathbf{0}$. Equivalently, one may use the distribution function of the difference of two linear combinations of chi-square variables:

$$F_R(c) = \Pr(Z \le 0)$$

where

$$Z = U - V, \tag{3.3}$$

$U = \sum_{i=1}^{r} \lambda_i S_i$ and $V = \sum_{i=r+1}^{r+q} \lambda_i S_i$, $r + q \le p$, the λ_i's, $i = 1, \ldots, r$, being equal to the r positive λ_i's and the λ_i's, $i = r + 1, \ldots, r + q$, being equal to the absolute value of the q negative λ_i's (assuming that the λ_i's in (3.2) are already ordered so that $\lambda_i > 0$, $i = 1, \ldots, r$, $\lambda_i < 0$, $i = r + 1 \ldots, r + q$ and $\lambda_i = 0$, $i = r + q, \ldots, p$) and the S_i's being mutually independent noncentral chi-square variables with one degree of freedom and noncentrality parameter δ_i^2 corresponding to $(U_i+\delta_i)^2$ in (3.2).

We now derive computable representations for the distribution function of

$$R=\frac{Q_1}{Q_2}\ ,$$

the ratio of two quadratic forms defined in (3.1).

In view of (3.2) and (3.3), the distribution function of R is

$$F_R(r)=\Pr(R\le r)=\Pr(Z\le 0) \tag{3.4}$$

where $Z=U-V$.

Note that in view of the discussion leading to (2.2), if the chi-square variables having the same coefficients in U or in V as defined in (3.3) are combined, then Z can be expressed as in (2.7) where the l_j's, denote the t distinct eigenvalues among the r positive λ_i's in (3.3) and α_j represents the multiplicity of l_j, $j=1\ldots,t$, whereas the l_j's denote the w distinct eigenvalues among the q negative λ_i's in (3.3) and α_j represents the multiplicity of l_j, $j=t+1,\ldots,t+w$. The noncentrality parameters d_j^2 are defined in (2.2) for $j=1,\ldots,t$ and the noncentrality parameters d_j^2 for $j=t+1,\ldots,t+w$ are defined similarly, that is, $d_j^2=\sum_{i=1}^{\alpha_j}d_{j_i}^2$ where α_j is defined above and $d_{j_i}^2$, $i=1,\ldots,\alpha_j$, are the noncentrality parameters of the random variables S_i in V as defined in (3.3) which have the same coefficient l_j.

The distribution function, $F_R(c)$, can be found by evaluating the expression (2.25) or (2.28) at zero for the noncentral or central cases respectively.

Now letting

$$\delta_{1i}=\Gamma(i+1)\beta'^{(i+1)} \quad \text{and} \quad \delta_{2i}=\Gamma(-2\mu+i+1)\beta'^{(-2\mu+i+1)}\ , \tag{3.5}$$

the distribution function of R in the noncentral case can be represented as follows:

$$F_R(c) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha'+v)}\, b^{\mu-\frac{1}{2}}$$
$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{\mu+\frac{1}{2}+i}}{i!}\delta_{1i} \right.$$
$$\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!}\delta_{2i} \right]. \tag{3.6}$$

This main result is stated in the next theorem.

Theorem 3.1

Let R be defined as in (3.1) and U and V be related to R according to (3.4) where $U = \sum_{i=1}^{t} l_i Y_i$, $V = \sum_{i=t+1}^{t+w} l_i Y_i$, the l_i's , $i = 1,\dots,t+w$, are real positive numbers, and the Y_i 's are independently distributed noncentral chi-square variables with noncentrality parameter d_i^2 and α_i degrees of freedom; let $\alpha = (\alpha_1 + \cdots + \alpha_t)/2$, $\alpha' = (\alpha_{t+1} + \cdots + \alpha_{t+w})/2$, $\beta_i = 2l_i$ for $i = 1,\dots,t+w$, and $b = \beta^{-1} + \beta'^{(-1)}$ where β and β' are such that the inequalities in (2.21) are satisfied. Then the distribution function of R is given by (3.6) where $\theta_{s,k}$ and $\theta'_{\nu,\eta}$ are defined in (2.6) and (2.10), respectively, $\lambda = -(\alpha + k + s - \alpha' - \nu - \eta)/2$, $\mu = (1 - \alpha' - \eta - \nu - \alpha - k - s)/2$, and δ_{1i} and δ_{2i} are defined in (3.5).

<u>Note</u> A simplified representation of the distribution function in (3.6) can be obtained when $\mu - \lambda + \frac{1}{2}$ and $2\mu + 1$ are integers, that is, when both α and α' are integers. The following identities are used. When x and y are negative integers with $-x < -y$, then

$$\frac{(x)_i}{(y)_i} = \begin{cases} \frac{\Gamma(-y-i+1)}{\Gamma(-x-i+1)}\frac{\Gamma(-x+1)}{\Gamma(-y+1)} & \text{when } 0 \le i \le -x, \\ 0 & \text{when } -x+1 < i < -y, \\ (-1)^{x-y}\frac{\Gamma(x+i)}{\Gamma(y+i)}\frac{\Gamma(-x+1)}{\Gamma(-y+1)} & \text{when } i > -y, \end{cases} \tag{3.7}$$

and when $x - y$ is a positive integer, we define $\Gamma(x)/\Gamma(y)$ as follows

$$\frac{\Gamma(x)}{\Gamma(y)} = (-1)^{x-y}\frac{\Gamma(-y+1)}{\Gamma(-x+1)}. \tag{3.8}$$

In view of (3.7) and (3.8), (3.6) becomes

$$F_R(c)=\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\theta_k\theta'_v}{\Gamma(\alpha'+v)}b^{\mu-\frac{1}{2}}$$

$$\times\left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)}\left(\sum_{i=0}^{\alpha'+v+\eta-1}\delta_{1i}\frac{\Gamma(-2\mu-i)}{\Gamma(-\mu+\lambda+\frac{1}{2}-i)}\frac{\Gamma(-\mu+\lambda+\frac{1}{2})}{\Gamma(-2\mu)}\frac{b^{\mu+\frac{1}{2}+i}}{i!}\right.\right. \tag{3.9}$$

$$\left.+\sum_{i=-2\mu}^{\infty}\delta_{1i}\frac{\Gamma(\mu-\lambda+\frac{1}{2}+i)}{\Gamma(2\mu+1+i)}\frac{\Gamma(-\mu+\lambda+\frac{1}{2})}{\Gamma(-2\mu)}\frac{b^{\mu+\frac{1}{2}+i}}{i!}(-1)^{-\mu-\lambda-\frac{1}{2}}\right)$$

$$\left.+(-1)^{\mu+\lambda-\frac{1}{2}}\frac{\Gamma(-\mu-\lambda+\frac{1}{2})}{\Gamma(-2\mu+1)}\sum_{i=0}^{\infty}\delta_{2i}\frac{\Gamma(-\mu-\lambda+\frac{1}{2}+i)}{\Gamma(-\mu-\lambda+\frac{1}{2})}\frac{\Gamma(-2\mu+1)}{\Gamma(-2\mu+1+i)}\frac{b^{-\mu+\frac{1}{2}+i}}{i!}\right].$$

Now noting that $-\mu+\lambda+\frac{1}{2}=\alpha'+v+\eta$ and $\frac{1}{2}-\mu-\lambda=\alpha+k+s$, we have

$$F_R(c)=\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha+k)}$$

$$\times\left(\sum_{i=0}^{\alpha'+\nu+\eta-1}\delta_{1i}\frac{\Gamma(-2\mu-i)}{\Gamma(-\mu+\lambda+\frac{1}{2}-i)}\frac{b^{2\mu+i}}{i!}\right.$$

$$+\sum_{i=-2\mu}^{\infty}\delta_{1i}\frac{\Gamma(\mu-\lambda+\frac{1}{2}+i)}{\Gamma(2\mu+1+i)}\frac{b^{2\mu+i}}{i}(-1)^{-\mu-\lambda-\frac{1}{2}}$$

$$\left.+(-1)^{\mu+\lambda-\frac{1}{2}}\sum_{i=0}^{\infty}\delta_{2i}\frac{\Gamma(-\mu-\lambda+\frac{1}{2}+i)}{\Gamma(-2\mu+1+i)}\frac{b^{i}}{i!}\right)$$

$$=\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\theta_k\theta'_\nu}{\Gamma(\alpha+k)}$$

$$\times\left(\sum_{i=0}^{\alpha'+\nu+\eta-1}\beta'^{(i+1)}\frac{\Gamma(-2\mu-i)}{\Gamma(-\mu+\lambda+\frac{1}{2}-i)}b^{2\mu+i}\right.$$

$$+(-1)^{\alpha+k+s-1}\sum_{i=-2\mu}^{\infty}\beta'^{(i+1)}\frac{\Gamma(\mu-\lambda+\frac{1}{2}+i)}{\Gamma(2\mu+1+i)}b^{2\mu+i}$$

$$\left.+(-1)^{\alpha+k+s}\sum_{i=0}^{\infty}\beta'^{(i-2\mu+1)}\frac{\Gamma(-\mu-\lambda+\frac{1}{2}+i)}{\Gamma(1+i)}b^{i}\right). \tag{3.10}$$

The second and the third sums in the round brackets cancel out and the resulting simplified expression for the distribution function of R is given in the next theorem.

Theorem 3.2

Let R be defined as in (3.1) and U and V be related to R according to (3.4) where $U = \sum_{i=1}^{t} l_i Y_i$, $V = \sum_{i=t+1}^{t+w} l_i Y_i$, the l_i's, $i = 1, \ldots, t+w$ are real positive numbers, and the Y_i's are independently distributed noncentral chi-square variables with noncentrality parameter d_i^2 and α_i degrees of freedom; let both $\alpha = (\alpha_1 + \cdots + \alpha_t)/2$ and $\alpha' = (\alpha_{t+1} + \cdots + \alpha_{t+w})/2$ be integers, $\beta_i = 2l_i$ for $i = 1, \ldots, t+w$, and $b = \beta^{-1} + \beta'^{(-1)}$ where β and β' are such that the inequalities in (2.21) are satisfied. Then the distribution function of R has a following form:

$$F_R(c) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(\alpha+k+s)} \times \left(\sum_{i=0}^{\alpha'+\nu+\eta-1} \beta'^{(i+1)} \frac{\Gamma(-2\mu-i)}{\Gamma(-\mu+\lambda+\frac{1}{2}-i)} b^{2\mu+i} \right) \tag{3.11}$$

where $\theta_{s,k}$ and $\theta'_{\nu,\eta}$ are given in (2.6) and (2.10), respectively, $\lambda = -(\alpha+k+s-\alpha'-\nu-\eta)/2$ and $\mu = (1-\alpha'-\eta-\nu-\alpha-k-s)/2$.

Central Case

The distribution function of R is obtained similarly in the case of quadratic forms in central normal variables. Let

$$Z = U - V = (l_1 X_1 + \cdots + l_t X_t) - (l_{t+1} X_{t+1} + \cdots + l_{t+w} X_{t+w}) \tag{3.12}$$

where the l_i's are all real positive numbers and the X_i's are independent central chi-square variables with α_i degrees of freedom, $i = 1, \ldots, t+w$. Let $\alpha = (\alpha_1 + \cdots + \alpha_t)/2$, $\alpha' = (\alpha_{t+1} + \cdots + \alpha_{t+w})/2$, $\beta_i = 2l_i$ for $i = 1, \ldots, t+w$ and $b = \beta^{-1} + \beta'^{(-1)}$, where β and β' are such that the inequalities in (2.21) are satisfied, and let $W_{\lambda,\mu}(w)$ denotes Whittaker's function defined in (2.13) where $\lambda = -(\alpha+k-\alpha'-\eta)/2$ and $\mu = (1-\alpha'-\eta-\alpha-k)/2$. Then the distribution function of R is

$$F_R(c) = \sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty}\sum_{i=0}^{\infty} \frac{\theta_k \theta'_\eta}{\Gamma(\alpha'+\eta)} b^{\mu-\frac{1}{2}}$$

$$\times \left[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \frac{(\mu-\lambda+\frac{1}{2})_i}{(2\mu+1)_i} \frac{b^{\mu+\frac{1}{2}+i}}{i!} \delta_{1i} \right.$$

$$\left. + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} \frac{(-\mu-\lambda+\frac{1}{2})_i}{(-2\mu+1)_i} \frac{b^{-\mu+\frac{1}{2}+i}}{i!} \delta_{2i} \right], \qquad (3.13)$$

where δ_{1i} , δ_{2i} are defined in (3.5), and θ_k and θ'_η are respectively given in (2.19) and (2.20).

When $\mu - \lambda + \frac{1}{2}$ and $2\mu + 1$ are integers, that is, when both α and α' are integers the expression given in (3.13) for the distribution function of R can be expressed in the following form in view of Theorem 3.2:

$$F_R(c) = \sum_{k=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_k \theta'_\eta}{\Gamma(\alpha+k)}$$

$$\times \left(\sum_{i=0}^{\alpha'+\eta-1} \beta'^{(i+1)} \frac{\Gamma(-2\mu-i)}{\Gamma(-\mu+\lambda+\frac{1}{2}-i)} b^{2\mu+i} \right). \qquad (3.14)$$

where θ_k and θ'_η are defined in (2.19) and (2.20) respectively.

The representations (3.11) for the noncentral case and (3.14) for the central case apply

(i) when both α and α' are positive integers, in which case

$$\mu - \lambda + 1/2 = 1 - \alpha' - v - \eta$$

and

$$2\mu + 1 = 2 - \alpha' - v - \eta - \alpha - k - s$$

are negative integers;

(ii) when α is a positive integer plus $1/2$ and α' is a positive integer, in which case

$$\mu - \lambda + 1/2 \text{ is a negative integer,}$$

$$2\mu + 1 \text{ is a negative integer plus } 1/2\,,$$

$$\frac{\Gamma(2\mu)}{\Gamma(\mu - \lambda + \frac{1}{2})} = 0$$

[For $\beta = 0, 1, 2, \ldots$, we define $1/\Gamma(-\beta)$ as 0]

and

$$\frac{(\mu - \lambda + \frac{1}{2})_i}{(2\mu + 1)_i} = 0 \quad \text{for } i = -\mu + \lambda + 1/2, \ldots, \infty \;;$$

in view of these last two equalities, one can easily obtain the expresssions respectively given in (3.11) and (3.14) from the representations (3.6) and (3.13);

(iii) when α is a positive integer and α' is a positive integer plus $1/2$, in which case one must first make use of the following relationship:

$$Pr(U - V \leq 0) = 1 - Pr(V - U \leq 0)$$

which leads to case (ii).

Note that when α and α' are both positive integers plus 1/2 then, $\mu - \lambda + 1/2$ is an integer plus $1/2$ and $2\mu + 1$ is a negative integer, and the representations (3.6) and (3.11) will both diverge since $(\mu - \lambda + \frac{1}{2})_i/(2\mu + 1)_i$ is infinite for any $i \geq -2\mu$ in that case. In this case we use the representations (2.37) and (2.39) for $c \leq 0$ and for $c \geq 0$, respectively.

The representations (3.6) and (3.11) also apply to ratios involving chi-square variables whose degrees of freedom may be fractional; such a situation may occur for example if Patnaik's (1949) approximation is used on each of $L_1, \ldots, L_\rho$ or a subset thereof where the L_i's are linear combinations of chi-square variables such that $\sum_{i=1}^{\rho} L_i = \sum_{j=1}^{p} \lambda_j (U_j + \delta_j)^2$ as given in (3.2).

3.1.2 Examples

Three examples involving the evaluation of the series representation of the distribution function of the ratio R are given below.

Example 3.1

In the first example, the numerator and the denominator of R are independently distributed and the quadratic forms are already expressed in terms of central chi-squares. Let

$$R = \frac{\frac{1}{2}\chi_4^2 + \chi_2^2}{\chi_6^2} . \tag{3.15}$$

In this case, it is possible though tedious to obtain the distribution function of R using the transformation of variables technique. It is given by

$$F_R(c) = \frac{16c^7 + 80c^6 + 168c^5 + 140c^4 + 40c^3}{(2c+1)^4(c+1)^3} , \quad c \geq 0 . \tag{3.16}$$

We now show that the representation of the distribution function given in (3.14) yields the same expression.

Note that we can use the simplified expression since the number of degrees of freedom of every chi-square is an even integer. In this case

$$a_1 = 2, \quad a_2 = 1, \quad \alpha = 3. \quad a_3 = 3, \quad \alpha' = 3. \quad \beta_1 = 1, \quad \beta_2 = 2, \quad \beta_3 = 2c;$$

we set $\beta' = 2c$ and we choose β to be the harmonic mean of β_1 and β_2, that is, $\frac{1}{\beta} = \frac{1}{2}(\frac{1}{\beta_1} + \frac{1}{\beta_2}) = \frac{3}{4}$. Then

$$b = \frac{1}{\beta} + \frac{1}{\beta'} = \frac{3}{4} + \frac{1}{2c} = \frac{3c+2}{4c}$$

$$\theta_k = \sum_{i=0}^{k} \frac{i+1}{2} \left(\frac{1}{4}\right)^k (-1)^i,$$

$$\theta'_\eta = \theta'_0 = \left(\frac{1}{2c}\right)^3,$$

$$\lambda = -\frac{k}{2}, \qquad \mu = \frac{-5-k}{2}$$

and

$$F_R(c) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(3+k)} \left[\sum_{i=0}^{k} \frac{i+1}{2} \left(\frac{1}{4}\right)^k (-1)^i\right] \left(\frac{1}{2c}\right)^3 \left[2c\frac{\Gamma(5+k)}{2}\left(\frac{3c+2}{4c}\right)^{-5-k}\right.$$

$$\left. + (2c)^2 \frac{\Gamma(4+k)}{1}\left(\frac{3c+2}{4c}\right)^{-4-k} + (2c)^3\frac{\Gamma(3+k)}{1}\left(\frac{3c+2}{4c}\right)^{-3-k}\right]$$

$$= \left[\sum_{k=0}^{\infty}\left(\sum_{i=0}^{n}(i+1)(-1)^i\right)\left(\frac{3c+2}{c}\right)^{-3-k}\left(\frac{1}{4}\right)^{-3}\right]$$

$$\times \left(\frac{9c^2+30c+6kc+2k^2+18k+40}{2(3c+2)^2}\right).$$

Noting that

$$\sum_{i=0}^{k}(i+1)(-1)^i = \begin{cases} \frac{k}{2}+1, & \text{if } k=2n; \\ \frac{-k-1}{2}, & \text{if } k=2n+1, \quad n=0,1,2,\ldots, \end{cases}$$

we have

$$F_R(c) = \sum_{n=0}^{\infty}(n+1)\left(\frac{3c+2}{c}\right)^{-3-2n}\left(\frac{1}{4}\right)^{-3}\left[\frac{9c^2+30c+12nc+8n^2+36n+40}{2(3c+2)^2}\right.$$

$$\left. - \left(\frac{3c+2}{c}\right)^{-1}\frac{9c^2+30c+12nc+6c+8n^2+44n+60}{2(3c+2)^2}\right].$$

Now using the following equality

$$\sum_{n=0}^{\infty} n^k x^{2n} = \frac{x}{2}\frac{d}{dx}\left(\sum_{n=0}^{\infty} n^{k-1}x^{2n}\right),$$

$F_R(c)$ takes the form

$$F_R(c) = \left(\frac{c}{3c+2}\right)^3\left(\frac{1}{3c+2}\right)\left(\frac{1296c^8+9936c^7+34344c^6+66444c^5}{(c+1)^3(2c+1)^4}\right.$$

$$\left. + \frac{77704c^4+56288c^3+24768c^2+6080c+640}{(c+1)^3(2c+1)^4}\right)$$

$$= \frac{c^3(16c^4+80c^3+168c^2+140c+40)}{(c+1)^3(2c+1)^4} \tag{3.17}$$

as required. The computer program used for evaluating (3.14) and (3.16) is included in Appendix C.1. The exact probability density function of R can be readily obtained by differentiating $F_R(c)$ with respect to c.

Example 3.2

In the second example which follows, we consider two central quadratic forms which are not independently distributed. Let

$$R = \frac{(X_1, X_2, Y)A(X_1, X_2, Y)'}{(Y, Z)B(Y, Z)'}$$

where

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & (2\sqrt{2})^{-1} \\ (2\sqrt{2})^{-1} & 1 \end{pmatrix},$$

and $(X_1, X_2, Y, Z) \sim N_4(0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The covariance matrix Σ can be expressed as the product of a triangular matrix by its transpose as follows:

$$\Sigma = LL' = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to find the distribution function of the ratio R at the point c, we evaluate the matrix $M_c = M_c' = L'S_cL$ (defined in (3.2)) and its eigenvalues λ_i. Then the distribution function of the ratio R can be expressed in terms of the distribution function of $\sum_{i=1}^{4} \lambda_i U_i^2$ evaluated at zero where, in view of (3.2), the U_i's are independent standard normal variables and the λ_i's are the eigenvalues of M_c.

The values of the distribution function obtained from the representation (3.13) for $c = 2.0$ and $c = 4.0$ are respectively $F_R(2.0) = 0.1188$ and $F_R(4.0) = 0.3414$. They are very close to the simulated values obtained with the IMSL package which are respectively $F_R(2.0) = 0.1219$ and $F_R(4.0) = 0.3398$. The computer program used for evaluating the distribution function at c=4.0 is included in Appendix C.2.

Example 3.3

In the third example, we consider two quadratic forms in noncentral normal variables which can be expressed as the following ratio:

$$R = \frac{Q_1}{Q_2} = \frac{1.5(Y_1 + 0.4)^2 + 1.2(Y_2 + 0.5)^2}{1.2(Y_3 + 0.5)^2 + 1.8(Y_4 + 0.6)^2}$$

Here $Z = U - V$ is equal to

$$1.5(Y_1 + 0.4)^2 + 1.2(Y_2 + 0.5)^2 - r[1.2(Y_3 + 0.5)^2 + 1.8(Y_4 + 0.6)^2]$$

and in the notation of Theorem 3.2, $l_1 = 1.5$, $l_2 = 1.2$, $d_1 = 0.4$, $d_2 = 0.5$,

$l_3 = 1.2\ r$, $l_4 = 1.8\ r$, $d_3 = 0.5$, $d_4 = 0.6$,

$\alpha = 1$, $\alpha' = 1$,

$\mu = (-\nu - \eta - s - k - 1)/2$, $\lambda = (-s - k + \nu + \eta)/2$,

$-2\mu - i = \nu + \eta + s + k - i$, $-\mu + \lambda + 1/2 - i = \nu + \eta + 1 - i$,

$$\theta_{s,k} = e^{-\frac{\delta^2}{2}} \sum_{i=0}^{s} \sum_{j=0}^{k} \left[\frac{d_1^{2i} d_2^{2(s-i)}}{\beta_1^{i+\frac{1}{2}} \beta_2^{s-i+\frac{1}{2}} i!(s-i)! 2^s} \frac{\Gamma(i + j + \frac{1}{2})\Gamma(s - i + k - j + \frac{1}{2})}{\Gamma(i + \frac{1}{2})\Gamma(s - i + \frac{1}{2}) j!(k-j)!} \right.$$
$$\left. \times (\beta^{-1} - \beta_1^{-1})^j (\beta^{-1} - \beta_2^{-1})^{k-j} \right] ,$$

$$\theta'_{\nu,\eta} = e^{-\frac{\gamma^2}{2}} \sum_{i=0}^{\nu} \sum_{j=0}^{\eta} \left[\frac{d_3^{2i} d_4^{2(\nu-i)}\ 2^{-\nu}}{(\beta_3)^{i+\frac{1}{2}} (\beta_4)^{\nu-i+\frac{1}{2}} \nu!(\nu-i)!} \frac{\Gamma(i + j + \frac{1}{2})\Gamma(\nu - i + \eta - j + \frac{1}{2})}{\Gamma(i + \frac{1}{2})\Gamma(\nu - i + \frac{1}{2}) j!(\eta - j)!} \right.$$
$$\left. \times ((\beta')^{-1} - (\beta_3)^{-1})^j ((\beta')^{-1} - (\beta_4)^{-1})^{\eta-j} \right] .$$

Hence

$$F_R(c) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{\theta_{s,k}\theta'_{\nu,\eta}}{\Gamma(s+k+1)} \times \left(\sum_{i=0}^{\alpha'+\eta-1} \beta'^{(i+1)} \frac{\Gamma(\nu+\eta+s+k-i+1)}{\Gamma(\nu+\eta+1-i)} b^{2\mu+i} \right) \tag{3.18}$$

where $b = \beta^{-1} + (\beta')^{-1}$. Some values of the distribution function of R obtained by evaluating (3.18) are compared with the corresponding values obtained by simulation in the following table.

r	F(r) exact	F(r) simulated
0.05	0.0566	0.0572
0.10	0.1073	0.1075
0.20	0.1941	0.1942
0.30	0.2658	0.2659
0.40	0.3259	0.3260
0.60	0.4209	0.4209
1.00	0.5486	0.5485
1.20	0.5935	0.5935
2.00	0.7093	0.7097
2.80	0.7739	0.7743
4.00	0.8303	0.8300
6.00	0.8803	0.8801
10.0	0.9245	0.9243
15.0	0.9463	0.9485

Table 3.3

The computer program used for evaluating (3.18) is included in Appendix C 3.

3.2 The Exact Density Obtained as an Inverse Mellin Transform

A computable expression is derived for the moments about the origin of the random variable $Z = N/D$, where $N = \sum_1^n m_i X_i + \sum_{n+1}^s m_i X_i$, $D = \sum_{n+1}^s l_i X_i + \sum_{s+1}^r n_i X_i$, and the X_i's are independently distributed noncentral chi-square variables. The first four moments of Z are required for approximating its distribution by means of Pearson curves. The exact density of Z is obtained in terms of sums of generalized hypergeometric functions for the case $n = 1$, $s = 2$ and $r = 3$ by taking the inverse Mellin transform of its h-th moment where h is a complex number. Based on the same approach, a method for determining the exact density of any random variable having the structure of Z is also described.

3.2.1 Introduction

We are considering the random variable

$$Z = \frac{\sum_1^n m_i X_i + \sum_{n+1}^s m_i X_i}{\sum_{n+1}^s l_i X_i + \sum_{s+1}^r n_i X_i} \tag{3.19}$$

where the m_i's are real numbers, $l_i > 0$, $i = n+1, \ldots, s$, $n_i > 0$, $i = s+1, \ldots, r$,

$$X_i \overset{\text{ind}}{\sim} \chi^2_{r_i}(d_i^2) \ , \quad i = 1, \ldots, r \ ,$$

that is, the X_i's are independently distributed noncentral chi-square variables having respectively r_i degrees of freedom and noncentrality parameter d_i^2.

Such a structure arises in a variety of contexts. Chaubey and Nur Enayet Talukder (1983, eq. 2.1) obtained the moments of the quantity Q_1/Q_2 for the case where $Q_1 = \sum a_i X_i + \sum c_i Z_i$, $Q_2 = \sum b_i Y_i + \sum d_i Z_i$, where X_i, Y_i, Z_i are mutually independent chi-square random variables; a representation of the moments about the origin of the ratio Q_1/Q_2 was obtained in closed form by Morin–Wahhab (1985). Statistics having the structure of Z also appear in Lauer and Han (1972, p.255) where probabilities of certain ratios of chi-square variables are examined;

in von Neumann (1941, p.369) where the ratio of the mean square successive difference to the variance is studied; in Toyoda and Ohtani (1986, eq. 8) where a statistic involved in a two-stage test is considered; and in Provost (1986, p. 291) where a statistic is derived in connection with tests on the structural coefficients of a multivariate linear functional relationship model.

One may also use the random variable Z in the case of statistics expressed as ratios of quadratic forms in noncentral normal variables where some variables may be common to the numerator and the denominator (by diagonalizing the matrices of the quadratic forms).

The technique of the inverse Mellin transform (see for instance, Springer (1979) or Mathai and Saxena (1978)) is used in Section 3 to obtain in closed form a representation of the exact density of Z . Many researchers have utilized this technique in order to solve various distributional problems. For example Mathai and Tan (1977) obtained the distribution of the likelihood ratio criterion for testing the hypothesis that the covariance matrix in a multivariate distribution is diagonal; Pederzoli and Rathie (1983) derived the exact distribution of Bartlett's criterion for testing the equality of covariance matrices; Bagai (1972) obtained the distribution of a statistic used to test the hypotheses of equality of two dispersion matrices, equality of the multidimensional mean vectors, and the independence between a p set and a q set of variates. This technique was also used by Gupta and Rathie (1982) who considered the distribution of the likelihood ratio criterion for testing the hypothesis of equality of variances in k-normal populations; Nagarsenker and Nagarsenker (1982) who gave the distribution of the likelihood criterion for testing the equality of p two-parameter exponential distributions; and Kulp and Nagarsenker (1984) who obtained the asymptotic distribution of Wilks' criterion.

A representation of the moments of Z about the origin is obtained in Section

3.2.2. The exact density of Z is derived for a particular case in Section 3.2.3; it is also shown that the approach used applies in the general case.

3.2.2 Approximate Distribution

One needs the first four moments of Z in order to approximate the distribution of Z by means of Pearson curves described in Section 1.4. We derive in this section the h-th moment of Z for any positive integer h.

Let $Z = N/D$ and h be positive integer; then $Z^h = N^h/D^h$ where

$$N^h = (\sum_{i=1}^{s} m_i X_i)^h = \sum h!(\prod_{i=1}^{s}(m_i X_i)^{h_i}/h_i!) , \tag{3.20}$$

the unindexed summation sign denoting a sum over the nonnegative integers $h_1, \ldots, h_s$ such that $h_1 + \cdots + h_s = h$, and

$$\begin{aligned} D^{-h} &= (\sum_{i=n+1}^{s} l_i X_i + \sum_{i=s+1}^{r} n_i X_i)^{-h} \\ &= \Gamma(h)^{-1} \int_0^\infty t^{h-1}\, e^{-t(\sum_{i=n+1}^{s} l_i X_i + \sum_{i=s+1}^{r} n_i X_i)} dt\ . \end{aligned} \tag{3.21}$$

The h-th moment of Z is

$$\begin{aligned} E(Z^h) &= \int_0^\infty \cdots \int_0^\infty N^h D^{-h} \prod_{i=1}^{r} f_i(x_i) dx_i, \quad \text{(where } f_i(x_i) \text{ denotes the pdf of } X_i) \\ &= \int_0^\infty \cdots \int_0^\infty \sum h! \left(\prod_{i=1}^{s} \frac{(m_i x_i)^{h_i}}{h_i!} \right) \Gamma(h)^{-1} \int_0^\infty t^{h-1}\, e^{-t(\sum_1^s l_i x_i + \sum_{s+1}^r n_i x_i)} dt \\ &\quad \times \prod_{i=1}^{r} \left[e^{-d_i} \sum_{j_i=0}^{\infty} \frac{d_i^{j_i}}{j_i!} \frac{x_i^{\frac{r_i}{2}+j_i-1} e^{-x_i/2}}{2^{\frac{r_i}{2}+j_i} \Gamma(\frac{r_i}{2}+j_i)} dx_i \right] . \end{aligned}$$

The last product is equal to

$$\sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_r=j} e^{-d} \prod_{i=1}^{r} \frac{d_i^{j_i}}{j_i!} \frac{x_i^{\frac{r_i}{2}+j_i-1} e^{-x_i/2}}{2^{\frac{r_i}{2}+j_i} \Gamma(\frac{r_i}{2}+j_i)}$$

where $d = d_1 + \cdots + d_r$. Then

$$E(Z^h) = \sum_{h_1+\cdots+h_s=h} \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_r=j} \mu \prod_{i=1}^{r} d_i^{j_i}(j_i!\, 2^{\frac{r_i}{2}+j_i}\, \Gamma(\frac{r_i}{2}+j_i))^{-1}$$
$$\times \int_0^\infty t^{h-1} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} m_i^{h_i} x_i^{h_i+\frac{r_i}{2}+j_i-1}\, e^{-x_i/2}$$
$$\times \left(\prod_{i=n+1}^{s} m_i^{h_i} x_i^{h_i+\frac{r_i}{2}+j_i-1} e^{-x_i(l_it+\frac{1}{2})}\right)\left(\prod_{i=s+1}^{r} x_i^{\frac{r_i}{2}+j_i-1}\, e^{-x_i(n_it+\frac{1}{2})}\right) dx_1 \ldots dx_r dt$$

where

$$\mu = e^{-d} h! (\prod_{i=1}^{s} h_i!)^{-1} (\Gamma(h))^{-1}.$$

Noting that

$$\int_0^\infty \frac{e^{-x_i(n_it+\frac{1}{2})} x_i^{\frac{r_i}{2}+j_i-1}}{\Gamma(\frac{r_i}{2}+j_i)} dx_i = (n_it + 1/2)^{-\frac{r_i}{2}-j_i} \quad \text{for } i = s+1, \ldots, r\ ,$$

$$\int_0^\infty \frac{e^{-\frac{x_i}{2}} x_i^{h_i+\frac{r_i}{2}+j_i-1}}{\Gamma(h_i+\frac{r_i}{2}+j_i)} dx_i = (\frac{1}{2})^{-(h_i+\frac{r_i}{2}+j_i)} \quad \text{for } i = 1, \ldots, n\ ,$$

and that

$$\int_0^\infty \frac{e^{-x_i(l_it+\frac{1}{2})} x_i^{h_i+\frac{r_i}{2}+j_i-1}}{\Gamma(h_i+\frac{r_i}{2}+j_i)} dx_i = (l_it + \frac{1}{2})^{-(h_i+\frac{r_i}{2}+j_i)} \quad \text{for } i = n+1, \ldots, s\ ,$$

one has

$$E(Z^h) = \sum_{h_1+\cdots+h_s=h} \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_r=j} \rho \int_0^\infty t^{h-1} \prod_{i=n+1}^{s} (l_it + \frac{1}{2})^{-(h_i+\frac{r_i}{2}+j_i)}$$
$$\times \prod_{i=s+1}^{r} (n_it + \frac{1}{2})^{-\frac{r_i}{2}-j_i}\, dt \qquad (3.22)$$

where

$$\rho = \mu \left(\prod_{i=1}^{n} m_i^{h_i} \frac{\Gamma(h_i+\frac{r_i}{2}+j_i)}{2^{-h_i}\Gamma(\frac{r_i}{2}+j_i)}\right)\left(\prod_{i=n+1}^{s} m_i^{h_i} \frac{\Gamma(h_i+\frac{r_i}{2}+j_i)}{2^{\frac{r_i}{2}-j_i}\Gamma(\frac{r_i}{2}+j_i)}\right) \prod_{i=s+1}^{r} 2^{-\frac{r_i}{2}-j_i}\ .$$

Writing $\prod_{i=n+1}^{s}(l_i t+\frac{1}{2})^{-(h_i+r_i/2+j_i)}$ in the right-hand side of (3.22) as $\theta\prod_{i=n+1}^{s}(t+1/(2l_i))^{-(h_i+r_i/2+j_i)}$ where

$$\theta=\prod_{i=n+1}^{s} l_i^{-(h_i+\frac{r_i}{2}+j_i)}\ , \tag{3.23}$$

and $\prod_{i=s+1}^{r}(n_i t+\frac{1}{2})^{-(\frac{r_i}{2}+j_i)}$ in the right-hand side of (3.22) as $\eta\prod_{i=s+1}^{r}(t+1/(2n_i))^{-(\frac{r_i}{2}+j_i)}$ where

$$\eta=\prod_{i=s+1}^{r} n_i^{-(\frac{r_i}{2}+j_i)}\ , \tag{3.24}$$

and then letting $v=(1+t)^{-1}$, that is, $t=(1-v)/v$ so that $|dt/dv|=v^{-2}$, the integral in (3.24) becomes

$$\theta\eta\int_0^1 v^{\alpha-h-1}(1-v)^{h-1}\left(\prod_{i=n+1}^{s}(1-\gamma_i v)^{-(h_i+\frac{r_i}{2}+j_i)}\right)\left(\prod_{i=s+1}^{r}(1-\gamma_i v)^{-\frac{r_i}{2}-j_i}\right)dv \tag{3.25}$$

where $\gamma_i=1-1/(2l_i)$, $i=n+1,\ldots,s$, $\gamma_i=1-1/(2n_i)$, $i=s+1,\ldots,r$,

$$\alpha=\sum_{n+1}^{s}(h_i+\frac{r_i}{2}+j_i)+\sum_{s+1}^{r}(\frac{r_i}{2}+j_i)\ .$$

Then letting $\alpha-h=q$, one may write

$$E(Z^h)=\sum_{h_1+\cdots+h_s=h}\ \sum_{j=0}^{\infty}\ \sum_{j_1+\cdots+j_r=j}\theta\eta\rho\left[\frac{\Gamma(q)\Gamma(\alpha-q)}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{\Gamma(q)\Gamma(\alpha-q))}I\right]$$

$$=\sum_{h_1+\cdots+h_s=h}\ \sum_{j=0}^{\infty}\ \sum_{j_1+\cdots+j_r=j}\theta\eta\rho B(q,\alpha-q) \tag{3.26}$$

$$\times F_D(q;h_{n+1}+\frac{r_{n+1}}{2}+j_{n+1},..h_s+\frac{r_s}{2}+j_s,\frac{r_{s+1}}{2}+j_{s+1},..\frac{r_r}{2}+j_r;\alpha;\gamma_{n+1},..\gamma_r),$$

provided $q>0$, that is, $\alpha>h$, $\alpha-q=h>0$, and $|\gamma_i|<1$, that is,

$$2l_i>\frac{1}{2},\quad i=n+1,\ldots,s,\quad \text{and}\quad 2n_i>\frac{1}{2},\ i=s+1,\ldots,r, \tag{3.27}$$

where $B(a,b)=(\Gamma(a)\Gamma(b))/\Gamma(a+b)$ and $F_D(\cdot)$ denotes Lauricella's type D hypergeometric function (see Mathai and Saxena (1978), p. 162 for a series representation and p.163 for an integral representation). The h-th moment given in (3.26) may be expressed in terms of infinite series as follows:

$$E(Z^h) = \sum_{h_1+\cdots+h_s=h} \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_r=j} \theta\, \eta\, \rho\, B(q,\alpha-q) \sum_{\nu=0}^{\infty} \sum_{\nu} [(q)_\nu/(\alpha)_\nu]$$
$$\times \left(\prod_{i=n+1}^{s} \gamma_i^{\nu_i} \frac{(h_i + \frac{r_i}{2} + j_i)_{\nu_i}}{\nu_i!} \right) \left(\prod_{i=s+1}^{r} \gamma_i^{\nu_i} \frac{(\frac{r_i}{2} + j_i)_{\nu_i}}{\nu_i!} \right) \quad ,(3.28)$$

where the last sum Σ_ν is over the nonnegative ν_j's such that $\sum_{i=n+1}^{r} \nu_i = \nu$, $(\epsilon)_\nu = \Gamma(\epsilon+\nu)/\Gamma(\epsilon)$, and ρ, η and θ are given in (3.22), (3.23) and (3.24), respectively.

If the conditions specified in (3.27) are not satisfied, both the numerator and the denominator of Z^h may be multiplied by a scalar quantity b^h chosen so that $2bl_i > \frac{1}{2}$ for $i = 1,\ldots,s$, and $2bn_i > \frac{1}{2}$ for $i = s+1,\ldots,r$. Then the h-th moment of Z can be evaluated using (3.28).

After evaluating the first four moments of Z, one can select the Pearson curve which best approximates the exact density of Z. For a complete development of the curves and the associated rules, see for example Elderton and Johnson (1969).

It is worth noting that the representation (3.28) still holds whether the coefficients m_i are positive or negative since the expansion of the numerator of Z given in (3.20) applies to sums of positive and negative numbers. However, in view of (3.21), none of the terms in the denominator of Z may be negative.

3.2.3 The Exact Density of the Ratio

First, an expression for the h-th moment of Z, where h is a complex number, is derived for the case $n = 1$, $s = 2$ and $r = 3$. A representation of the exact density of Z is then obtained as the inverse Mellin transform of the h-th moment for this particular case. It is finally shown that the same technique also applies to the general case.

Let

$$Z = \frac{m_1 X_1 + m_2 X_2}{l X_2 + n X_3} \tag{3.29}$$

where

$$X_i \overset{ind}{\sim} \chi^2_{r_i}(d_i) , \qquad i = 1,2,3 \quad .$$

Then the h-th moment of Z where h is a complex number, is

$$E(Z^h) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{(m_1 x_1 + m_2 x_2)^h}{(l x_2 + n x_3)^h} \prod_{i=1}^{3} \left(e^{-d_i} \sum_{j_i=0}^{\infty} \frac{d_i^{j_i}}{j_i!} \frac{x_i^{\frac{r_i}{2}+j_i-1}}{2^{\frac{r_i}{2}+j_i}} \frac{e^{-\frac{x_i}{2}}}{\Gamma(\frac{r_i}{2}+j_i)} dx_i \right) . \tag{3.30}$$

Letting

$$(l x_2 + n x_3)^{-h} = \int_0^\infty \frac{t^{h-1} e^{-(l x_2 + n x_3)t}}{\Gamma(h)} dt , \tag{3.31}$$

one can integrate out x_3 as follows:

$$\begin{aligned} &\int_0^\infty \sum_{j_3=0}^{\infty} \frac{d_3^{j_3}}{j_3!} \frac{x_3^{\frac{r_3}{2}+j_3-1}}{2^{\frac{r_3}{2}+j_3}} \frac{e^{-x_3(nt+\frac{1}{2})}}{\Gamma(\frac{r_3}{2}+j_3)} dx_3 \\ &\quad = \sum_{j_3=0}^{\infty} \frac{d_3^{j_3}}{j_3!} \int_0^\infty \frac{x_3^{\frac{r_3}{2}+j_3-1}}{2^{\frac{r_3}{2}+j_3}} \frac{e^{-x_3(nt+\frac{1}{2})}}{\Gamma(\frac{r_3}{2}+j_3)} dx_3 \\ &\quad = \sum_{j_3=0}^{\infty} \frac{d_3^{j_3}}{j_3!} (2nt+1)^{-(\frac{r_3}{2}+j_3)} \\ &\quad = (2nt+1)^{-(\frac{r_3}{2}+j_3)} e^{(\frac{d_3}{2nt+1})} . \end{aligned} \tag{3.32}$$

Then, letting $\psi = m_1 x_1 + m_2 x_2$ and $x_2 = x_2$, one has

$$\begin{aligned} &\int_0^\infty \sum_{j_2=0}^{\infty} \frac{d_2^{j_2}}{j_2!} \frac{x_2^{\frac{r_2}{2}+j_2-1}}{2^{\frac{r_2}{2}+j_2}} \frac{e^{-x_2(lt+\frac{1}{2})}}{\Gamma(\frac{r_2}{2}+j_2)} \int_0^\infty (m_1 x_1 + m_2 x_2)^h \sum_{j_1=0}^{\infty} \frac{d_1^{j_1}}{j_1!} \frac{x_1^{\frac{r_1}{2}+j_1-1}}{2^{\frac{r_1}{2}+j_1}} \frac{e^{-\frac{x_1}{2}} dx_1 dx_2}{\Gamma(\frac{r_1}{2}+j_1)} \\ &= \sum_{j_2=0}^{\infty} \sum_{j_1=0}^{\infty} \frac{d_2^{j_2}}{j_2!} \frac{d_1^{j_1}}{j_1!} \frac{1}{2^{\frac{r_1}{2}+j_1} 2^{\frac{r_2}{2}+j_2} \Gamma(\frac{r_1}{2}+j_1) \Gamma(\frac{r_2}{2}+j_2)} \\ &\quad \times \int_0^\infty x_2^{\frac{r_2}{2}+j_2-1} e^{-x_2(lt+\frac{1}{2})} \int_0^\infty (m_1 x_1 + m_2 x_2)^h x_1^{\frac{r_1}{2}+j_1-1} e^{-\frac{x_1}{2}} dx_1 dx_2 \end{aligned} \tag{3.33}$$

where the double integral in (3.33), denoted by $I_1(t)$, is equal to

$$\int_0^\infty x_2^{\frac{r_2}{2}+j_2-1} e^{-x_2(nt+\frac{1}{2})} \int_{m_2x_2}^\infty \frac{1}{m_1}\psi^h \left(\frac{1}{m_1}(\psi - m_2x_2)\right)^{\frac{r_1}{2}+j_1-1} e^{-\frac{\psi-m_2x_2}{2m_1}} d\psi dx_2 .$$

Note that all the integrands in this paper are measurable functions. We may therefore change the order of integration by invoking Fubini's theorem . In order to express

$$E(Z^h) = \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \sum_{j_3=0}^\infty e^{-d} \frac{d_1^{j_1}}{j_1!}\frac{d_2^{j_2}}{j_2!}\frac{d_3^{j_3}}{j_3!} \int_0^\infty \frac{t^{h-1}}{\Gamma(h)} \frac{(1+2nt)^{\frac{-r_3}{2}+j_3} I_1(t)\, dt}{2^{\frac{r_1}{2}+j_1} 2^{\frac{r_2}{2}+j_2} \Gamma(\frac{r_1}{2}+j_1)\Gamma(\frac{r_2}{2}+j_2)}$$

in terms of series, one needs the following lemmas.

Lemma 3.1

$$\begin{aligned} &\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{-\beta x}dx \\ &= \sum_{r=0}^\infty \frac{(1-\nu)_r}{(2-\mu-\nu)_r}\frac{1}{r!}\,\Gamma(\mu+\nu-1)\,\beta^{r+1-\mu-\nu}u^r e^{-\beta u} \\ &+ \sum_{r=0}^\infty \frac{(\mu)_r}{(\mu+\nu)_r}\frac{1}{r!}\,\frac{\Gamma(\mu)\Gamma(1-\mu-\nu)}{\Gamma(1-\nu)}\,\beta^r u^{\mu+\nu+r-1}e^{-\beta u} \end{aligned} \tag{3.34}$$

where $1-\mu-\nu$ and $1-\nu$ are not integers, and $(a)_b = \Gamma(a+b)/\Gamma(a)$.

Proof. (The numbers appearing on the right of the following equalities refer to Gradshteyn and Ryzhik (1980)) The left-hand side of (3.34) is equal to

$$\beta^{-\frac{\mu+\nu}{2}} u^{\frac{\mu+\nu-2}{2}} \Gamma(\mu) e^{-\frac{\beta u}{2}} W_{\frac{\nu-\mu}{2},\frac{1-\mu-\nu}{2}}(\beta u) \qquad \text{by } 3.203, 4.$$

for $R(\mu)>0$ and $R(\beta u)>0$ where $R(\cdot)$ denotes the real part of $(\cdot)$, with

$$\begin{aligned} W_{\frac{\nu-\mu}{2},\frac{1-\mu-\nu}{2}}(\beta u) &= \frac{\Gamma(\mu+\nu-1)}{\Gamma(\mu)} M_{\frac{\nu-\mu}{2},\frac{1-\mu-\nu}{2}}(\beta u) \\ &\quad + \frac{\Gamma(1-\mu-\nu)}{\Gamma(1-\nu)} M_{\frac{\nu-\mu}{2},-\frac{(1-\mu-\nu)}{2}}(\beta u) \qquad \text{by } 9.220,4. \\ &= \frac{\Gamma(\mu+\nu-1)}{\Gamma(\mu)} (\beta u)^{1-\frac{(\mu+\nu)}{2}} e^{-\frac{\beta u}{2}} \phi(1-\nu, 2-\mu-\nu; \beta u) \\ &\quad + \frac{\Gamma(1-\mu-\nu)}{\Gamma(1-\nu)} (\beta u)^{\frac{\mu+\nu}{2}} e^{\frac{-\beta u}{2}} \phi(\mu, \mu+\nu; \beta u) \quad \text{by } 9.220,2. \text{ and } 3., \end{aligned}$$

$$\phi(1-\nu, 2-\mu-\nu; \beta u) = \sum_{r=0}^{\infty} \frac{(1-\nu)_r}{(2-\mu-\nu)_r} \frac{1}{r!} (\beta u)^r ,$$

and

$$\phi(\mu,\mu+\nu;\beta u) = \sum_{r=0}^{\infty} \frac{(\mu)_r}{(\mu+\nu)_r} \frac{1}{r!} (\beta u)^r \quad \text{by } 9.210,1 .$$

Taking h such that $-\frac{r_1}{2}-j_1-h$ and h are not integers and using Lemma 1, $I_1(t)$ becomes

$$\begin{aligned} &\int_0^\infty x_2^{\frac{r_2}{2}+j_2-1} e^{-(lt+\frac{1}{2})x_2} \left(\frac{1}{m_1}\right)^{\frac{r_1}{2}+j_1} \Gamma(\frac{r_1}{2}+j_1+h)(2m_1)^{\frac{r_1}{2}+j_1+h} \\ &\quad \times \left(\sum_{r=0}^{\infty} \frac{(-h)_r}{(1-\frac{r_1}{2}-j_1-h)_r} \frac{1}{r!} \left(\frac{m_2 x_2}{2m_1}\right)^r \right) dx_2 \\ &+ \int_0^\infty x_2^{\frac{r_1}{2}+\frac{r_2}{2}+j_1+j_2+h-1} e^{-(lt+\frac{1}{2})x_2} \left(\frac{1}{m_1}\right)^{\frac{r_1}{2}+j_1} \frac{\Gamma(\frac{r_1}{2}+j_1)\Gamma(-\frac{r_1}{2}-j_1-h)}{\Gamma(-h)} \\ &\quad \times \left(\sum_{r=0}^{\infty} \frac{(\frac{r_1}{2}+j_1)_r}{(\frac{r_1}{2}+j_1+h+1)_r} \frac{1}{r!} \left(\frac{m_2 x_2}{2m_1}\right)^r \right) dx_2 \\ &= \sum_{r=0}^{\infty} \frac{(-h)_r}{(1-\frac{r_1}{2}-j_1-h)_r} \frac{1}{r!} \Gamma(\frac{r_1}{2}+j_1+h)(2m_1)^{\frac{r_1}{2}+j_1+h-r} \left(\frac{1}{m_1}\right)^{\frac{r_1}{2}+j_1} m_2^r \\ &\quad \times \int_0^\infty x_2^{\frac{r_2}{2}+j_2+r-1} e^{-(lt+\frac{1}{2})x_2} dx_2 \\ &+ \sum_{r=0}^{\infty} \frac{(\frac{r_1}{2}+j_1)_r}{(\frac{r_1}{2}+j_1+h+1)_r} \frac{1}{r!} \frac{\Gamma(\frac{r_1}{2}+j_1)\Gamma(-\frac{r_1}{2}-j_1-h)}{\Gamma(-h)} (2m_1)^{-r} \left(\frac{1}{m_1}\right)^{\frac{r_1}{2}+j_1} \\ &\quad \times m_2^{\frac{r_1}{2}+j_1+h+r} \int_0^\infty x_2^{\frac{r_1}{2}+\frac{r_2}{2}+j_1+j_2+h+r-1} e^{-(lt+\frac{1}{2})x_2} dx_2 \end{aligned}$$

$$= \left\{ \sum_{r=0}^{\infty} \frac{(-h)_r}{(1-\frac{r_1}{2}-j_1-h)_r} \frac{1}{r!} \Gamma(\frac{r_1}{2}+j_1+h)(2m_1)^{\frac{r_1}{2}+j_1+h-r}(m_2)^r \left(\frac{1}{m_1}\right)^{\frac{r_1}{2}+j_1} \right.$$
$$\left. \times\Gamma(\frac{r_2}{2}+j_2+r)(lt+\frac{1}{2})^{-(\frac{r_2}{2}+j_2+r)} \right\}$$
$$+ \left\{ \sum_{r=0}^{\infty} \frac{(\frac{r_1}{2}+j_1)_r}{(\frac{r_1}{2}+j_1+h+1)_r} \frac{1}{r!} \frac{\Gamma(\frac{r_1}{2}+j_1)\Gamma(-\frac{r_1}{2}-j_1-h)}{\Gamma(-h)}(2m_1)^{-r} m_1^{-\frac{r_1}{2}-j_1} \right.$$
$$\left. \times m_2^{\frac{r_1}{2}+j_1+h+r}\Gamma(\frac{r_1}{2}+\frac{r_2}{2}+j_1+j_2+h+r)(lt+\frac{1}{2})^{-(\frac{r_1}{2}+\frac{r_2}{2}+j_1+j_2+h+r)} \right\}$$
$$= P(t)+Q(t) \ \ \text{(say)}\ . \tag{3.35}$$

Then

$$E(Z^h) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} e^{-d}\frac{d_1^{j_1}}{j_1!}\frac{d_2^{j_2}}{j_2!}\frac{d_3^{j_3}}{j_3!}$$
$$\times \int_0^{\infty} \frac{t^{h-1}}{\Gamma(h)} \frac{(1+2nt)^{\frac{-r_3}{2}+j_3}}{2^{\frac{r_1}{2}+j_1}2^{\frac{r_2}{2}+j_2}\Gamma(\frac{r_1}{2}+j_1)\Gamma(\frac{r_2}{2}+j_2)}[P(t)+Q(t)]dt$$
$$= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} \{ e^{-d}\frac{d_1^{j_1}}{j_1!}\frac{d_2^{j_2}}{j_2!}\frac{d_3^{j_3}}{j_3!}$$
$$\times \left[\sum_{r=0}^{\infty} \frac{(-h)_r}{(1-\frac{r_1}{2}-j_1-h)_r} \frac{\Gamma(\frac{r_1}{2}+j_1+h)(2m_1)^{\frac{r_1}{2}+j_1+h-r}\, m_2^r\, \Gamma(\frac{r_2}{2}+j_2+r)}{r!\, 2^{\frac{r_2}{2}+j_2+\frac{r_1}{2}+j_1}\, m_1^{\frac{r_1}{2}+j_1}\, \Gamma(\frac{r_1}{2}+j_1)\,\Gamma(\frac{r_2}{2}+j_2)\,\Gamma(h)} \right.$$
$$\times \int_0^{\infty} t^{h-1}(1+2nt)^{-\frac{r_3}{2}-j_3}\,(lt+\frac{1}{2})^{-(\frac{r_2}{2}+j_2+r)}dt \tag{3.36}$$
$$+ \sum_{r=0}^{\infty} \frac{\Gamma(\frac{r_1}{2}+j_1)\Gamma(-(\frac{r_1}{2}+j_1+h))}{r!\, m_1^{\frac{r_1}{2}+j_1}(2m_1)^r\Gamma(-h)\Gamma(h)} \frac{m_2^{\frac{r_1}{2}+j_1+h+r}\ \Gamma(\frac{r_1}{2}+\frac{r_2}{2}+j_1+j_2+h+r)}{(\frac{r_1}{2}+j_1+h+1)_r\ 2^{\frac{r_1}{2}+\frac{r_2}{2}+j_1+j_2}\Gamma(\frac{r_2}{2}+j_2)}$$
$$\times \frac{(\frac{r_1}{2}+j_1)_r}{\Gamma(\frac{r_1}{2}+j_1)} \int_0^{\infty} t^{h-1}\ (1+2nt)^{-\frac{r_3}{2}+j_3}\ (lt+\frac{1}{2})^{-(\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+h+r)}dt.$$

Lemma 3.2

Let $\eta = -\sigma + \sum_r^q k_i + \sum_{q+1}^p a_i$, then

$$\int_0^{\infty} t^{\sigma-1}\left(\prod_{i=q+1}^{p}(1+c_it)^{-a_i}\right)\left(\prod_{i=r}^{q}(\theta_it+1/b_i)^{-k_i}\right)dt$$

$$= \left(\prod_{i=r}^{q} \theta_i^{-k_i}\right) \left(\prod_{i=q+1}^{p} c_i^{-a_i}\right) [\Gamma(\eta)\Gamma(\sigma)(\Gamma(\eta+\sigma))^{-1}] \sum_{j=0}^{\infty} \sum_{j_r+\cdots+j_p=j} \frac{(\eta)_j}{(\eta+\sigma)_j \prod_{i=r}^{p} j_i!}$$

$$\times \left(\prod_{i=q+1}^{p} (a_i)_{j_i}\right) \left(\prod_{i=r}^{q} (k_i)_{j_i}\right) \left(\prod_{i=q+1}^{p} (1-\frac{1}{c_i})^{j_i}\right) \left(\prod_{i=r}^{q} (1-\frac{1}{\theta_i b_i})^{j_i}\right) \tag{3.37}$$

provided

$$|1 - 1/(\theta_i b_i)| < 1 \text{ for } 1 = r, \ldots, q \;;\; |1 - 1/c_i| < 1 \text{ for } i = q+1, \ldots, p \;;$$

$$R(\eta) > 0 \text{ and } R(\sigma) > 0 \,. \tag{3.38}$$

Proof. The right hand side of (3.37) can be expressed as

$$C \int_0^{\infty} t^{\sigma-1} \left(\prod_{i=q+1}^{p} (t + 1/c_i)^{-a_i}\right) \left(\prod_{i=r}^{q} (t + 1/(\theta_i b_i))^{-k_i}\right) dt \;=\; I_2 \;\; \text{(say)} \,,$$

where $C = \prod_{i=r}^{q} \theta_i^{-k_i} \prod_{i=q+1}^{p} c_i^{-a_i}$.

Letting $u = 1/(1+t)$, i.e., $t = (1-u)/u$ so that $|dt/du| = u^{-2}$,

$$I_2 = C \int_0^1 \left(\frac{1-u}{u}\right)^{\sigma-1} \left\{\prod_{i=q+1}^{p} \left(\frac{(1-u)}{u} + \frac{1}{c_i}\right)^{-a_i}\right\} \prod_{i=r}^{q} \left(\frac{(1-u)}{u} + \frac{1}{\theta_i b_i}\right)^{-k_i} u^{-2} dt$$

$$= C \int_0^1 u^{\eta-1}(1-u)^{\sigma-1} \left\{\prod_{i=q+1}^{p} (1 - u(1-1/c_i))^{-a_i}\right\} \prod_{i=r}^{q} (1 - u(1 - 1/(\theta_i b_i)))^{-k_i} dt$$

where $\eta = -\sigma + \sum_{i=q+1}^{p} a_i + \sum_{i=r}^{q} k_i$. Therefore

$$I_2 = C \,\frac{\Gamma(\eta)\Gamma(\sigma)}{\Gamma(\eta+\sigma)} F_D(\eta; a_{q+1}, \ldots, a_p, k_r, \ldots, k_q; \eta+\sigma; (1 - 1/c_{q+1}),$$

$$\ldots, (1 - 1/c_p), (1 - 1/(\theta_r b_r)), \ldots, (1 - 1/(\theta_q b_q)))$$

provided $R(\eta) > 0$ and $R(\sigma) > 0$ where $F_D(\cdot)$ is Lauricella's type D hypergeometric function (discussed in Section 2) whose series representation leads to (3.37) provided the conditions specified in (3.38) are satisfied.

In order to apply Lemma 2 to the integrals appearing on the right hand side of (3.36), the following substitutions are made: for the first integral, $\sigma = h$, $r = q = 1$, $p = 2$, $c_2 = 2n$, $\theta_1 = l$, $b_1 = 2$, $a_2 = r_3/2 + j_3$ and $k_1 = r_2/2 + j_2 + r$, and for the second integral, $\sigma = h$, $r = q = 1$, $p = 2$, $c_2 = 2n$, $\theta_1 = l$, $b_1 = 2$, $a_2 = r_3/2 + j_3$ and $k_1 = r_1/2 + r_2/2 + j_1 + j_2 + h + r$. Then,

$$
\begin{aligned}
E(Z^h) = & \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} e^{-d}\frac{d_1^{j_1}}{j_1!}\frac{d_2^{j_2}}{j_2!}\frac{d_3^{j_3}}{j_3!}\Big\{\Big[\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\lambda+u=j}\frac{(-h)_r}{(1-\frac{r_1}{2}-j_1-h)_r} \\
& \times \frac{\Gamma(\frac{r_1}{2}+j_1+h)(2m_1)^{\frac{r_1}{2}+j_1+h-r}m_2^r\Gamma(\frac{r_2}{2}+j_2+r)}{r!2^{\frac{r_1}{2}+j_1}2^{\frac{r_2}{2}+j_2}m_1^{\frac{r_1}{2}+j_1}\Gamma(\frac{r_1}{2}+j_1)\Gamma(\frac{r_2}{2}+j_2)\Gamma(h)} \\
& \times \frac{\Gamma(-h+\frac{r_2}{2}+j_2+\frac{r_3}{2}+j_3)\Gamma(h)}{l^{\frac{r_2}{2}+j_2+r}2n^{\frac{r_3}{2}+j_3}\Gamma(\frac{r_2}{2}+j_2+r+\frac{r_3}{2}+j_3)}\cdot\frac{(-h+\frac{r_2}{2}+j_2+\frac{r_3}{2}+j_3)_j}{(\frac{r_2}{2}+j_2+\frac{r_3}{2}+j_3)_j\ \lambda!u!} \\
& \times (\frac{r_3}{2}+j_3)_u\ (\frac{r_2}{2}+j_2+r)_\lambda\ (1-1/(2l))^\lambda(1-1/(2n))^u\Big] \\
& + \Big[\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\lambda+u=j}\frac{\Gamma(\frac{r_1}{2}+j_1)\Gamma(-\frac{r_1}{2}-j_1-h)}{\Gamma(-h)\Gamma(h)} \\
& \times \frac{(\frac{r_1}{2}+j_1)_r\ (2m_1)^{-r}m_2^{\frac{r_1}{2}+j_1+h+r}\Gamma((\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+h+r)}{(\frac{r_1}{2}+j_1+h+1)_r\ r!\ 2^{(\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2}m_1^{\frac{r_1}{2}+j_1}\Gamma(\frac{r_1}{2}+j_1)\Gamma(\frac{r_2}{2}+j_2)} \\
& \times \frac{\Gamma(\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+\frac{r_3}{2}+j_3+r)\Gamma(h)(2n)^{-\frac{r_3}{2}-j_3}}{l^{\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+h+r}\Gamma(\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+h+r+\frac{r_3}{2}+j_3)} \\
& \times \frac{(\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+\frac{r_3}{2}+j_3+r)_j}{(\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+h+r+\frac{r_3}{2}+j_3)_j\ \lambda!u!} \\
& \times (\frac{r_3}{2}+j_3)_u\ (\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+h+r)_\lambda\ (1-1/(2n))^u(1-1/(2l))^\lambda\Big]\Big\} \\
= & \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} e^{-d}\frac{d_1^{j_1}}{j_1!}\frac{d_2^{j_2}}{j_2!}\frac{d_3^{j_3}}{j_3!}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\lambda+u=j}\Big\{K_1\ \Gamma(r-h) \\
& \times \frac{\Gamma(1-\frac{r_1}{2}-j_1-h)\Gamma(\frac{r_1}{2}+j_1+h)\Gamma(j+r_2/2+r_3/2+j_2+j_3+r-h)}{(2m_1)^{-h}\ \Gamma(-h)\Gamma(r+1-\frac{r_1}{2}-j_1-h)} \\
& + \frac{\Gamma(-\frac{r_1}{2}-j_1-h)\Gamma(\frac{r_1}{2}+j_1+1+h)}{\Gamma(-h)\Gamma(r+\frac{r_1}{2}+j_1+1+h)l^h}\ \frac{m_2^h\Gamma(\lambda+\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+r+h)K_2}{\Gamma(j+\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+\frac{r_3}{2}+j_3+r+h)}\Big\}
\end{aligned}
\tag{3.39}
$$

where

$$K_1 = \frac{\Gamma(\frac{r_3}{2} + j_3 + u)\Gamma(\frac{r_2}{2} + j_2 + r + \lambda) m_2^r \, (2m_1)^{-\frac{r_1}{2} - j_1 + r}}{r! \, 2^{\frac{r_1}{2} + j_1 + \frac{r_2}{2} + j_2} \, \Gamma(\frac{r_1}{2} + j_1)\Gamma(\frac{r_2}{2} + j_2)\Gamma(\frac{r_3}{2} + j_3) \, m_1^{\frac{r_1}{2} + j_1} \, l^{\frac{r_2}{2} + j_2 + r}}$$

$$\times \frac{(1 - 1/(2n))^u (1 - 1/(2l))^\lambda}{\Gamma(\frac{r_2}{2} + j_2 + \frac{r_3}{2} + j_3 + r + j) \, \lambda! u!} (2n)^{\frac{r_3}{2} + j_3} \quad ,$$

and

$$K_2 = \frac{\Gamma(\frac{r_1}{2} + j_1 + r)\Gamma(\frac{r_3}{2} + j_3 + u)\Gamma(\frac{r_1}{2} + j_1 + \frac{r_2}{2} + j_2 + \frac{r_3}{2} + j_3 + r + j)}{r! \, 2^{\frac{r_1}{2} + j_1 + \frac{r_2}{2} + j_2} \, (2m_1)^r \, m_1^{\frac{r_1}{2} + j_1} \, \Gamma(\frac{r_1}{2} + j_1)\Gamma(\frac{r_2}{2} + j_2)\Gamma(\frac{r_3}{2} + j_3)}$$

$$\times \frac{(1 - 1/(2n))^u (1 - 1/(2l))^\lambda m_2^{\frac{r_1}{2} + j_1 + r}}{l^{\frac{r_1}{2} + j_1 + \frac{r_2}{2} + j_2 + r} \, (2n)^{\frac{r_3}{2} + j_3} \, \lambda! u!} \quad .$$

Note that K_1 and K_2 do not involve h.

The inverse Mellin transform of the moment expression in (3.39) yields the density of Z :

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_C E(Z^h) z^{-(h+1)} dh \\
&= \frac{1}{2\pi i} z^{-1} \int_C E(Z^{-s}) z^s ds \qquad \text{(letting } s = -h\text{)} \\
&= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} e^{-d} \frac{d_1^{j_1}}{j_1!} \frac{d_2^{j_2}}{j_2!} \frac{d_3^{j_3}}{j_3!} \\
&\times \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\lambda+u=j} \Bigg\{ z^{-1} \Bigg[\frac{K_1}{2\pi i} \int_C \Gamma(r+s)\Gamma(1 - \tfrac{r_1}{2} - j_1 + s) \\
&\times \frac{\Gamma(j + \frac{r_2}{2} + j_2 + \frac{r_3}{2} + j_3 + r + s)\Gamma(\frac{r_1}{2} + j_1 - s)(z/2m_1)^s}{\Gamma(s)\Gamma(r + 1 - \frac{r_1}{2} - j_1 + s)} ds \qquad (3.40) \\
&+ \frac{K_2}{2\pi i} \int_C \frac{\Gamma(-\frac{r_1}{2} - j_1 + s)}{\Gamma(r + \frac{r_1}{2} + j_1 + 1 - s)} \\
&\times \frac{\Gamma(\frac{r_1}{2} + j_1 + 1 - s)\Gamma(\lambda + \frac{r_1}{2} + j_1 + \frac{r_2}{2} + j_2 + r - s)(lz/m_2)^s}{\Gamma(j + \frac{r_1}{2} + j_1 + \frac{r_2}{2} + j_2 + \frac{r_3}{2} + j_3 + r - s)\Gamma(s)} ds \Bigg] \Bigg\}
\end{aligned}$$

where C is a suitable contour in the complex plane (for a definition of the Mellin transform and its inverse, see for instance Springer (1979) or Mathai (1992a)).

In view of (1.30), for a fixed integer j_1, it is seen that the poles of the first integrand in (3.40) are seperated for any number $c(j_1)$ satisfying

$$r_1/2 + j_1 - 1 < c(j_1) < r_1/2 + j_1$$

and the poles of the second integrand are seperated for any number $c_1(j_1)$ satisfying

$$r_1/2 + j_1 < c_1(j_1) < r_1/2 + j_1 + 1 \ .$$

Taking C as the left Bromwich contour corresponding to the path $(c_1(0) - i\infty, c_1(0) + \infty)$, the right-hand side of (3.40) may be expressed in terms of H-functions as follows:

$$f(z) = z^{-1} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} e^{-d} \frac{d_1^{j_1}}{j_1!} \frac{d_2^{j_2}}{j_2!} \frac{d_3^{j_3}}{j_3!} \tag{3.41}$$

$$\times \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\lambda+u=j} \Bigg\{ K_1 \Bigg(H_{3\ 3}^{1\ 3} \left[\frac{z}{2m_1} \middle| \begin{matrix} (1-r,1), (\frac{r_1}{2}+j_1,1), \\ (\frac{r_1}{2}+j_1,1), (1,1), \end{matrix} \right.$$

$$\left. \begin{matrix} (1-j-(\frac{r_2}{2}-j_2-\frac{r_3}{2}-j_3-r,1) \\ (\frac{r_1}{2}+j_1-r,1) \end{matrix} \right] + \mathrm{Res}_1\left(\frac{r_1}{2}\right) - \sum_{k=0}^{j_1-1} \mathrm{Res}_1\left(\frac{r_1}{2}+k\right) \Bigg) + K_2 \Bigg(H_{3\ 3}^{2\ 1}$$

$$\left[\frac{lz}{m_2} \middle| \begin{matrix} (1+\frac{r_1}{2}+j_1,1), (r+\frac{r_1}{2}+j_1+1,1), (j+(\frac{r_1+r_2+r_3}{2}+j_1+j_2+j_3+r,1) \\ (\frac{r_1}{2}+j_1+1,1), (\lambda+\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+r,1), (1,1) \end{matrix} \right]$$

$$- \sum_{k=1}^{j_1} \mathrm{Res}_2\left(\frac{r_1}{2}+k\right) \Bigg) \Bigg\}$$

where $\mathrm{Res}_l\ (\frac{r_1}{2}+k)$ is the residue of the first integrand in (3.40) at the pole $(\frac{r_1}{2}+k)$, $l = 1, 2$. The residue of a function $g(\cdot)$ at the simple pole x_0 is defined as follows:

$$\mathrm{Res}_g(x_0) = \lim_{x \to x_0} (x - x_0) g(x) \ .$$

Note that in view of condition (1.20), the first integral in (1.40) cannot be expressed as an H-function when the Bromwich path, $(c_1(0) - i\infty, c_1(0) + \infty)$, is used. Hence, for a given positive integer j_1, one must use the Bromwich path, $(c(j_1) - i\infty, c(j_1) + \infty)$, in order to obtain a representation of this first integral in terms of an H-function - which is equal to the sum of the residues of the integrand at the poles located to the left of the Bromwich path, $(c(j_1) - i\infty, c(j_1) + \infty)$ - and then subtract the additional residues evaluated at the poles located between the Bromwich paths, $(c_1(0) - i\infty, c_1(0) + \infty)$ and $(c(j_1) - i\infty, c(j_1) + \infty)$. Note that when $j_1 = 0$, the $\mathrm{Res}_1(\frac{r_1}{2}$ has to be added since the additional pole is to the left of the path $(c_1(0) - i\infty, c_1(0) + \infty)$. Similarly residues have to be subtracted for the evaluation of the second integrand.

The two H-functions in (3.41) exist for $0 < |\frac{z}{2m_1}| < 1$ and $0 < |\frac{lz}{m_2}| < 1$, respectively; the relationship (1.28) allows us to compute them when the arguments are greater than one in absolute value. For a definition of the H-function, a description of the conditions for its existence, and series representations, the reader is referred to Mathai and Saxena (1978). The H-function is defined as an inverse Mellin transform of $M_f(x)$ in (1.17).

The right-hand side of (3.41) may also be expressed in terms of G-functions:

$$f(z) = z^{-1} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} e^{-d} \frac{d_1^{j_1}}{j_1!} \frac{d_2^{j_2}}{j_2!} \frac{d_3^{j_3}}{j_3!} \tag{3.42}$$

$$\times \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\lambda+u=j} \Bigg\{ K_1 \Bigg(G_{3\,3}^{1\,3} \left[\frac{z}{2m_1} \middle| \begin{matrix} 1-r,\ \frac{r_1}{2}+j_1,\ 1-j-\frac{r_2}{2}-j_2-\frac{r_3}{2}-j_3-r \\ \frac{r_1}{2}+j_1,\ 1,\ \frac{r_1}{2}+j_1-r \end{matrix} \right]$$

$$+ \mathrm{Res}_1(\frac{r_1}{2}) - \sum_{k=0}^{j_1-1} \mathrm{Res}_1(\frac{r_1}{2}+k) \Bigg) + K_2 \Bigg(G_{3\,3}^{2\,1}$$

$$\left[\frac{lz}{m_2} \middle| \begin{matrix} 1+\frac{r_1}{2}+j_1,\ r+\frac{r_1}{2}+j_1+1,\ j+\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+\frac{r_3}{2}+j_3+r \\ \frac{r_1}{2}+j_1+1,\ \lambda+\frac{r_1}{2}+j_1+\frac{r_2}{2}+j_2+r,\ 1 \end{matrix} \right] - \sum_{k=1}^{j_1} \mathrm{Res}_2(\frac{r}{2}+k) \Bigg) \Bigg\}.$$

For a definition of the G-function and series representations, the reader is referred to Mathai (1971,1992a).

For the central case, $d_j^2 = 0$, $e^{-d} = 1$ and the sums over j_1, j_2, j_3 disappear. Hence the density of Z in the central case is

$$f(z) = z^{-1} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\lambda+u=j} \Bigg(K_1 \left(G_{3\,3}^{1\,3} \left[\frac{z}{2m_1} \middle| \begin{matrix} 1-r,\ \frac{r_1}{2},\ 1-j-\frac{r_2}{2}-\frac{r_3}{2}-r \\ \frac{r_1}{2},\ 1,\ \frac{r_1}{2}-r \end{matrix} \right] \right.$$
$$\left. - \operatorname{Res}\left(-\frac{r_1}{2}\right)\right) \tag{3.43}$$
$$+ K_2 G_{3\,3}^{2\,1} \left[\frac{lz}{m_2} \middle| \begin{matrix} 1+\frac{r_1}{2},\ r+\frac{r_1}{2}+1,\ j+\frac{r_1}{2}+\frac{r_2}{2}+\frac{r_3}{2}+r \\ \frac{r_1}{2}+1,\ \lambda+\frac{r_1}{2}+\frac{r_2}{2}+r,\ 1 \end{matrix} \right] \Bigg) .$$

The technique described above may also be used for determining the exact density of Z in its general form:

$$Z = \frac{\sum_1^n m_i X_i + \sum_{n+1}^{s} m_i X_i}{\sum_{n+1}^{s} l_i X_i + \sum_{s+1}^{r} n_i X_i} .$$

The h-th moment of Z is

$$E(Z^h) = \int_0^{\infty} \cdots \int_0^{\infty} z^h \prod_{i=1}^{r} f(x_i) dx_i .$$

Making use of the following identities

$$\left(\sum_{n+1}^{s} l_i x_i + \sum_{s+1}^{r} n_i x_i \right)^{-h} = \int_0^{\infty} \frac{t^{h-1}}{\Gamma(h)} e^{-(\sum_{n+1}^{s} l_i x_i + \sum_{s+1}^{r} n_i x_i) t} dt \tag{3.44}$$

and

$$\prod_{i=s+1}^{r} \int_0^{\infty} \sum_{j_i=0}^{\infty} \frac{d_i^{j_i}}{j_i!} \frac{x_i^{\frac{r_i}{2}+j_i-1}}{2^{\frac{r_i}{2}+j_i}} \frac{e^{-x_i(n_i t + \frac{1}{2})}}{\Gamma(\frac{r_i}{2}+j_i)} dx_i = \prod_{i=s+1}^{r} (2n_i t + 1)^{-\frac{r_i}{2}+j_i} , \tag{3.45}$$

$$\int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=n+1}^{s} x_i^{\frac{r_i}{2}+j_i-1} e^{-(l_i t + \frac{1}{2}) x_i} \int_0^{\infty} \cdots \int_0^{\infty} \left(\sum_1^{s} m_i x_i \right)^h \prod_{i=1}^{n} (x_i^{\frac{r_i}{2}+j_i-1} e^{-\frac{x_i}{2}})$$
$$dx_1 \ldots dx_s$$

can be written as

$$\int_0^{\infty} \psi_s^{\frac{r_s}{2}+j_s-1} e^{-\psi_s(b_s-b_{s-1})} \int_{\psi_s}^{\infty} \cdots \int_{\psi_3}^{\infty} \prod_{i=2}^{s-1} ((\psi_i - \psi_{i+1})^{\frac{r_i}{2}+j_i-1} e^{-\psi_i(b_i-b_{i-1})} .$$
$$\times \int_{\psi_2}^{\infty} \psi_1^h (\psi_1 - \psi_2)^{r_1/2-1} e^{-\psi_1 b_1} d\psi_1 d\psi_2 \ldots d\psi_s , \tag{3.46}$$

with $\psi_i = m_i x_i + \cdots + m_s x_s$, $i = 1, \ldots, s$, i.e., $m_i x_i = \psi_i - \psi_{i+1}$, $i = 1, \ldots, s-1$ and $m_s x_s = \psi_s$, and $b_i = (1/2 + \xi_i\ t)$ where $\xi_i = l_i$ for $i = n+1, \ldots, s$ and $\xi_i = 0$ for $i = 1, \ldots, n$.

Each integral in (3.46) can be evaluated successively using Lemma 1 except the last one (over ψ_s) whose integrand is seen to be proportional to a gamma probability density function, as was the case in (3.35).

The sums thus obtained will be of the type of those appearing in (3.36) except that the integral over t will be of the general type given in Lemma 2. With the appropriate substitutions, the resulting expression will involve multiple sums of the type appearing in (3.39). Hence, the only functions involving h will be gamma functions and constants raised to the power h. The exact density of Z can therefore be expressed in terms of sums of G-functions by taking the inverse Mellin transform of $E(Z^h)$ as was done in (3.40) and (3.41). This density can then be evaluated using the series representations of the G-functions.

Sums of gamma random variables rather than linear combinations of chi-square variables could have been considered. The latter was chosen as the results are then readily applicable to some ratios of quadratic forms. Furthermore the case of linear combinations of gamma variables can be obtained from the results derived in this paper by assigning positive real values to the degrees of freedom of the chi-square variables and by selecting the coefficients of the linear combinations appropriately. Various distributional results on linear functions of certain noncentral versions of independent gamma variables are given in Mathai (1992).

3.2.4. Examples.

First, a theoretical example shows that the inverse Mellin transform technique yields exact density of a ratio whose density can be obtained by means of the transformation of variables technique. In the second and third examples, the exact distribution functions are evaluated at various points covering the range of the variables and then compared with values obtained by simulation and, for the third example, with values obtained from the approximate distribution function.

Example 3.4

In the first example, the following ratio of independently distributed chi-square variables is considered

$$Z \sim \frac{\frac{1}{2}\chi_4^2 + \chi_2^2}{\chi_6^2} . \quad (3.47)$$

Using the approach described in Section 3.2.2, the h-th moment of R is found to be equal to

$$E(Z^h) = 2\Gamma(-h+3)\Gamma(h+1) - \Gamma(h+2)\Gamma(-h+3)(\frac{1}{2})^h - 2\Gamma(h+1)\Gamma(-h+3)(\frac{1}{2})^h.$$

The inverse Mellin transform of the moment expression gives the following representation of the density of Z :

$$f(z) = 2\, G_1^1\, {}_1^1[z| {}^{-3}_{0}] - G_1^1\, {}_1^1[2z| {}^{-3}_{1}] - 2\, G_1^1\, {}_1^1[2z| {}^{-3}_{0}]. \quad (3.48)$$

The first G-function can be expressed as the following series

$$G_1^1\, {}_1^1[z| {}^{-3}_{0}] = \sum_{k=0}^{\infty} \Gamma(4+k)\frac{(-1)^k}{k!} z^k = \frac{6}{(1+z)^4} \quad ,$$

for $0 < z < 1$, by considering the residues at the poles $0, -1, -2, \ldots$, and

$$G_1^1\, {}_1^1[z| {}^{-3}_{0}] = \sum_{k=0}^{\infty} \Gamma(4+k)\frac{(-i)^k}{k!} z^{-k-4} = \frac{6}{(1+z)^4}$$

for $z > 1$, by considering the residues at the poles at $4, 5, 6, \ldots$. The infinite series are simplified by applying the general binomial theorem. The following identities are obtained similarly:

$$G_{1\,1}^{1\,1}\left[2z\Big|\begin{matrix}-3\\1\end{matrix}\right] = \frac{48z}{(1+2z)^5} \quad , \qquad \text{for } z > 0 \tag{3.49}$$

$$G_{1\,1}^{1\,1}\left[2z\Big|\begin{matrix}-3\\0\end{matrix}\right] = \frac{6}{(1+2z)^4} \quad , \qquad \text{for } z > 0 \ .$$

This yields

$$\begin{aligned} f(z) &= 2\frac{6}{(1+z)^4} - \frac{48z}{(1+2z)^5} - 2\frac{6}{(1+2z)^4} \\ &= \frac{312z^5 + 660z^4 + 480z^3 + 120z^2}{(2z+1)^5(z+1)^4} \quad \text{for } z > 0 \ , \end{aligned} \tag{3.50}$$

which is the expression obtained with the transformation of variables technique.

Example 3.5

The next example involves a ratio of linear combinations of chi-square variables whose numerator and denominator are not independently distributed. Let

$$Z = \frac{X_1 + \frac{1}{2}X_2}{\frac{1}{2}X_2 + X_3} \tag{3.51}$$

where $X_i \overset{\text{ind}}{\sim} \chi^2_{r_i}$, $r_1 = 4,\ r_2 = 1$ and $r_3 = 6$. We multiply both the numerator and denominator by the parameter θ in order to accelerate the convergence of the series expression representing the density function. The h-th moment of Z calculated from (3.21) is

$$\begin{aligned} E(Z^h) &= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{u=0}^{j} K_1 (2\theta)^h \frac{\Gamma(r-h)\Gamma(-1-h)\Gamma(2+h)\Gamma(3.5+j+r-h)}{\Gamma(-h)\Gamma(r-1-h)} \\ &+ \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{u=0}^{j} K_2 \frac{\Gamma(-2-h)\Gamma(3+h)\Gamma(2.5+j-u+r+h)}{\Gamma(-h)\Gamma(r+3+h)\Gamma(5.5+j+r+h)} \end{aligned}$$

where

$$K_1 = \frac{(u+1)(u+2)\Gamma(0.5+r+j-u)(1-\frac{1}{2\theta})^u(1-\frac{1}{\theta})^{j-u}}{r!\,(2\theta)^{r+3}\ 2\ \theta^{0.5}\ \Gamma(0.5)\Gamma(3.5+r+j)\ (j-u)!}$$

and

$$K_2 = \frac{(1+r)(u+1)(u+2)\Gamma(5.5+r+j)(1-\frac{1}{2\theta})^u(1-\frac{1}{\theta})^{j-u}}{(2\theta)^{r+5}\;2\;\theta^{0.5}\;\Gamma(0.5)(j-u)!} \quad .$$

The density function of Z obtained from (3.25) is

$$f(z) = z^{-1}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{u=0}^{j}\left\{K_1 G_{3\,3}^{1\,3}\left[\frac{z}{2\theta}\middle|\begin{matrix}1-r,\ 2,\ -2.5-j-r\\ 2,\ 1,\ 2-r\end{matrix}\right]\right.$$
$$\left. - \operatorname{Res}(-2) + K_2 G_{3\,3}^{2\,1}\left[z\middle|\begin{matrix}3,\ r+3,\ 5.5+j+r\\ 3,\ 2.5+j-u+r,\ 1\end{matrix}\right]\right\} . \quad (3.52)$$

By applying the method of residues (see Springer (1979)), the G-functions can be expressed in terms of series as follow :

$$G_{3\,3}^{1\,3}\left[\frac{z}{2\theta}\middle|\begin{matrix}1-r,\ 2,\ -2.5-j-r\\ 2,\ 1,\ 2-r\end{matrix}\right] = \sum_{v=2}^{\infty}(r+v-1)\Gamma(3.5+v+j+r)\frac{(-1)^v}{(v-1)!}\left(\frac{z}{2\theta}\right)^v$$

for $z/2\theta < 1$,

$$G_{3\,3}^{1\,3}\left[\frac{z}{2\theta}\middle|\begin{matrix}1-r,\ 2,\ -2.5-j-r\\ 2,\ 1,\ 2-r\end{matrix}\right] = \sum_{v=0}^{\infty}(4.5+v+j)\Gamma(4.5+v+j+r)$$
$$\times\frac{(-1)^v}{v!}\left(\frac{z}{2\theta}\right)^{-(3.5+r+j+v)}$$

for $\frac{z}{2\theta} > 1$,

$$G_{3\,3}^{2\,1}\left[z\middle|\begin{matrix}3,\ r+3,\ 5.5+j+r\\ 3,\ 2.5+j-u+r,\ 1\end{matrix}\right] = \sum_{v=3}^{r+2}\frac{\Gamma(2.5-u+j+r-v)}{\Gamma(5.5+j+r-v)\Gamma(r-v+3)}\frac{(-1)^{v-1}}{(v-1)!}z^v$$
$$+\sum_{v=0}^{u+2}\frac{\Gamma(-2.5-r-j+u-v)}{\Gamma(0.5-j+u-v)\Gamma(u-v+3)}\frac{(-1)^v}{v!}z^{v+j+r-u+2.5}$$

for $z < 1$.

$$G_{3\ 3}^{2\ 1}\left[z\middle|\begin{matrix}3, & r+3, & 5.5+j+r\\ 3, & 2.5+j-u+r, & 1\end{matrix}\right] = \frac{\Gamma(0.5+j-u+r)}{\Gamma(1+r)\Gamma(3.5+j+r)}z^2 - \frac{\Gamma(1.5+j-u+r)}{\Gamma(2+r)\Gamma(4.5+j+r)}z$$

for $z > 1$,

and

$$\begin{aligned}\text{Res}(-2) &= \lim_{h\to -2}(h+2)\left(\frac{2\theta}{z}\right)^h \frac{\Gamma(r-h)\Gamma(-1-h)\Gamma(2+h)\Gamma(3.5+j+r-h)}{\Gamma(-h)\Gamma(r-1-h)}\\ &= (r+1)\Gamma(5.5+j+r)\left(\frac{z}{2\theta}\right)^2 .\end{aligned}$$

Upon substituting in (3.52) the series expansions of the G-functions for the intervals (0,1), $(1, 2\theta)$ and $(2\theta, \infty)$, one can evaluate the probability density function of Z . For comparison purposes the density function of Z was obtained from its simulated distribution function by numerical differentiation using a sample size of 1,000,000; the simulated values are denoted by $f^*(z)$. The numerical results are given below.

z	$f(z)$ exact	$f^*(z)$ simulated
0.1	0.501	0.502
0.2	0.843	0.842
0.4	0.944	0.944
0.6	0.778	0.771
0.8	0.570	0.577
1.0	0.412	0.421
1.4	0.221	0.226
1.6	0.168	0.170
2.0	0.091	0.098
4.2	0.009	0.010

Table 3.5

Note that this accuracy has been achieved by summing only the first 25 terms in the series representations of the G-functions. The computer program used for evaluating (3.52) is included in Appendix C.4.

Example 3.6

In this third and last example, the following ratio is considered:

$$Z = \frac{X_1 + \frac{1}{2}X_2 + \frac{1}{2}X_3}{\frac{1}{2}X_2 + X_4} \tag{3.53}$$

where $X_i \overset{\text{ind}}{\sim} \chi^2_{r_i}$, $r_1 = 4$, $r_2 = 4$, $r_3 = 2$ and $r_4 = 6$.

The method described at the end of Section 3.2.2 for the general case is used to obtain the following representation of the h-th moment of Z and its density.

$$E(Z^h) = \frac{1}{2^6\theta^8}\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\left(\frac{1}{2}\right)^r\left(\frac{1}{\theta}\right)^i(l+1)(l+2)\left(1-\frac{1}{\theta}\right)^{j-l}\left(1-\frac{1}{2\theta}\right)^l$$
$$\times\left\{\frac{4\,\theta^3\,\Gamma(2+i+j-l)}{\Gamma(5+i+j)(j-l)!\,i!}I_1(h) - \frac{4\,\theta^{r-2}\,\Gamma(3+r+i+j-l)}{\Gamma(2+r+i)\Gamma(6+r+i+j)(j-l)!}I_2(h)\right.$$
$$\left. + \frac{\theta^3\,(r+1)\,\Gamma(2+i+j-l)}{\Gamma(5+i+j)\,(j-l)1\,i!}I_3(h) - \frac{\theta^{-r}\,(r+1)\,\Gamma(8+r+i+j)}{(j-l)!}I_4(h)\right\}$$

where

$$\begin{aligned} I_1(h) &= \frac{\Gamma(r-h)\Gamma(-1-h)\Gamma(2+h)\Gamma(5+i+j-h)}{\Gamma(-h)\Gamma(r-1-h)\,(2\theta)^h}\,. \\ I_2(h) &= \frac{\Gamma(r-h)\Gamma(-1-h)\Gamma(2+h)\Gamma(6+r+i+j-h)}{\Gamma(-h)\Gamma(r-1-h)\,(2\theta)^h}\,, \\ I_3(h) &= \frac{\Gamma(3+h)\Gamma(-2-h)\Gamma(5+i+j-h)}{\Gamma(-h)\,\theta^h}\,, \\ I_4(h) &= \frac{\Gamma(3+h)\Gamma(-2-h)\Gamma(5+r+i+j-l+h)}{\Gamma(-h)\Gamma(4+r+i+h)\Gamma(8+r+i+j+h)}\,. \end{aligned} \tag{3.54}$$

The Bromwich paths, $(c_i - i\infty, c_i + i\infty)$, $i = 1,2,3,4$ suitable for the functions $I_1(h)$, $I_2(h)$, $I_3(h)$ and $I_4(h)$ are respectively $c_1 \in (-2,-1)$, $c_2 \in (-2,-1)$, $c_3 \in (-3,-2)$ and $c_4 \in (-3,-2)$. On taking the inverse Mellin transform of

$E(Z^h)$ with respect to the Bromwich path, $(c_3 - i\infty, c_3 + i\infty)$, we obtain the following density function for Z :

$$f(z) = \frac{1}{2^6\theta^8}\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\left(\frac{1}{2}\right)^r\left(\frac{1}{\theta}\right)^i(l+1)(l+2)\left(1-\frac{1}{\theta}\right)^{j-l}\left(1-\frac{1}{2\theta}\right)^l$$
$$\times\left\{\frac{4\,\theta^3\,\Gamma(2+i+j-l)}{\Gamma(5+i+j)(j-l)!\,i!}\left(G_{3\,3}^{1\,3}\left[\frac{z}{2\theta}\middle|\begin{matrix}1-r, & 2, & -4-i-j\\ & 2, & 1, & 2-r\end{matrix}\right]-\mathrm{Res}_{I_1}(-2)\right)\right.$$
$$-\frac{4\,\theta^{r-2}\,\Gamma(3+r+i+j-l)}{\Gamma(2+r+i)\Gamma(6+r+i+j)(j-l)!}\left(G_{3\,3}^{1\,3}\left[\frac{z}{2\theta}\middle|\begin{matrix}1-r, & 2, & -5-r-i-j\\ & 2, & 1, & 2-r\end{matrix}\right]\right.$$
$$\left.-\mathrm{Res}_{I_2}(-2)\right)+\frac{\theta^3\,(r+1)\,\Gamma(2+i+j-l)}{\Gamma(5+i+j)\,(j-l)1\,i!}G_{2\,2}^{1\,2}\left[\frac{z}{\theta}\middle|\begin{matrix}3, & -4-i-j\\ 3, & 1\end{matrix}\right] \tag{3.55}$$
$$\left.-\frac{\theta^{-r}\,(r+1)\,\Gamma(8+r+i+j)}{(j-l)!}\,G_{3\,3}^{2\,1}\left[z\middle|\begin{matrix}3, & 4+r+i, & 8+r+i+j\\ 3, & 5+r+i+j-l, & 1\end{matrix}\right]\right\}.$$

3.3 The Exact Density Function of the Ratio of Two Independent Quadratic Forms

In this section, we derive the density function of the ratio of two independent quadratic forms (or equivalently of independent linear combinations of chi-square variates) by differentiating the distribution function of such a ratio. In Section 3.3.1, we give a representation of the exact distribution function of the ratio of two independent quadratic forms in normal variables. In Section 3.3.2, we differentiate this representation to obtain the density function. Two examples are given in Section 3.3.3, the first one involving positive definite quadratic forms and the second one involving an indefinite quadratic form.

3.3.1. The Exact Distribution Function

A closed form representation of the distribution function of the ratio of two quadratic forms is derived in this section. It is shown below that a quadratic form in central normal random variables can be expressed as a linear combination of chi-square variables.

Let $X \sim N_p(0, \Sigma), \Sigma > 0$ and $Q = X'AX, A = A'$, be a quadratic form. Taking $Y = \Sigma^{-\frac{1}{2}}X$, we have $E(Y) = 0$, and

$$\mathrm{Cov}(Y) = \Sigma^{-\frac{1}{2}}\mathrm{Cov}(X)\Sigma^{-\frac{1}{2}} = \Sigma^{-\frac{1}{2}}\Sigma\Sigma^{\frac{1}{2}} = I$$

where $\Sigma^{\frac{1}{2}}$ is the symmetric square root of Σ, i.e. $(\Sigma^{\frac{1}{2}})^2 = \Sigma$. Let P be $p \times p$ orthogonal matrix which diagonalizes $\Sigma^{-\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ and $U = P'Y$; we have $E(U) = 0$ and $\mathrm{Cov}(U) = I$. Then

$$Q(X) = Y'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}Y = U'\mathrm{diag}(\lambda_1, \ldots, \lambda_p)U = \sum_{j=1}^{p}\lambda_j U_j^2, \tag{3.56}$$

$U' = (U_1, \ldots, U_p)$, the U_i 's are mutually independent standard normal variables and the U_j^2 's are chi-square variates with one degree of freedom each.

Provost (1989a) obtained the distribution function of ratios of independent linear combinations of gamma variables. Clearly the result holds for linear combinations of chi-square variables. The main steps of the derivation are given below. Let

$$R = S_1/S_2 \tag{3.57}$$

where

$$S_1 = \sum_{i=1}^{t}\alpha_i X_i \quad \text{and} \quad S_2 = \sum_{i=t+1}^{t+u}\alpha_i X_i \tag{3.58}$$

where α_i, $i = 1, \ldots, t+u$ are real constants and ν of the X_i 's be common to both S_1 and S_2. These X_i 's and the remaining X_i 's in S_1 and S_2 are assumed to be mutually independently distributed chi-square variables.

Let c be a real constant and $S = S_1 - cS_2$, then

$$\Pr\{R \le c\} = \Pr\{S \le 0\}. \tag{3.59}$$

Note that when Y is a chi-square variable aY, $a > 0$ is a gamma variable. Hence without loss of generality we can write

$$S = Y^+ - Z^-$$

where

$$Y^+ = \sum_{i=1}^{r} Y_i \ , \ Z^- = \sum_{i=r+1}^{p} Y_i \tag{3.60}$$

and $Y_1, \ldots, Y_p$ are mutually independent gamma variables with parameters $a_i > 0$, $b_i > 0$, $i = 1, \ldots, p$.

Let

$$F_R(c) = \Pr\{R \le c\} \ .$$

Now, noting that

$$\Pr\{R \le c\} = \Pr\{Y^+ \le Z^-\}$$

one may divide Y^+ and Z^- by the same constant θ (for example, the geometric mean of the b_i's, $i = 1, \ldots, r, \ldots, p$) to make the series representations of their densities converge faster. So letting $Y = Y^+/\theta$ and $Z = Z^-/\theta$, we have that

$$\begin{aligned} F_R(c) &= \Pr\{Y \le Z\} \\ &= \int_0^\infty h_2(z) \int_0^z h_1(y) dy \, dz \end{aligned} \tag{3.61}$$

where

$$h_1(y) = \sum_{\nu=0}^{\infty} K_\nu y^{\rho_1+\nu-1} e^{-y} \ , \ h_2(z) = \sum_{w=0}^{\infty} K'_w z^{\rho_2+w-1} e^{-z} \ , \tag{3.62}$$

$$\begin{aligned} K_\nu &= \sum_{\nu_1+\cdots+\nu_r=\nu} \prod_{j=1}^{r} \mu_j^{-a_j} (a_j)_{\nu_j} \gamma_j^{\nu_j} / (\nu_j! \Gamma(\rho_1 + \nu)) \ , \\ K'_w &= \sum_{w_{r+1}+\cdots+w_p=w} \prod_{j=r+1}^{p} \mu_j^{-a_j} (a_j)_{w_j} \gamma_j^{w_j} / (w_j! \Gamma(\rho_2 + w)) \ , \end{aligned} \tag{3.63}$$

$$\gamma_j = (\mu_j - 1)/\mu_j \ , \qquad j = 1, \ldots, r, \ldots, p,$$

$$\mu_j = b_j/\theta > \tfrac{1}{2}, \qquad j = 1, \ldots, r, \ldots, p, \tag{3.64}$$

$$\rho_1 = a_1 + \cdots + a_r \ , \qquad \rho_2 = a_{r+1} + \cdots + a_p$$

and for example $\sum_{\nu_1+\cdots+\nu_r=\nu}$ represents a sum over the set $\{\nu_1, \ldots, \nu_r\}$ such that $\nu_1 + \cdots + \nu_r = \nu$. The series representations for $h_i(\cdot)$ were derived by Provost (1989b) using the technique of the inverse Mellin transform.

Using the identity

$$\int_0^z y^{\rho_1+\nu-1} e^{-y} = e^{-z} \sum_{j=0}^{\infty} z^{\rho_1+\nu+j}/(\rho_1+\nu)_{j+1} \tag{3.65}$$

(see Gradshteyn and Ryzhik (1980, eq 3.201.2)) and evaluating the remaining integral using the density of a gamma variable, Provost (1989b) obtained the following representation of the distribution function of R:

$$F_R(c) = \sum_{\nu=0}^{\infty} \sum_{w=0}^{\infty} \sum_{j=0}^{\infty} K_\nu K'_w \frac{(\Gamma(\rho_1+\rho_2+\nu+w+j) 2^{-(\rho_1+\rho_2+\nu+w+j)})}{(\rho_1+\nu)_{j+1}} \tag{3.66}$$

where K_ν and K'_w are given in (3.63).

3.3.2. The Density Function Obtained by Differentiation

The exact density function of R is obtained by differentiating its distribution function in this section.

Consider

$$R = \frac{Q_1}{Q_2} \sim \frac{S_1}{S_2}$$

where Q_1 and Q_2 are independent quadratic forms in normal variables.

Let Y_i, $i = 1, \ldots, p$, (defined in (3.60)) denote independently distributed gamma (a_i, b_i) variates and

$$S = S_1 - cS_2 = Y - Z$$

where

$$Y = \sum_{i=1}^{r} Y_i/\theta \ , \ Y_i/\theta \sim \text{Gamma}(a_i, \mu_i), \ \mu_i = b_i/\theta, \ i = 1, \ldots, r \ ;$$
$$b_i = c\beta_i, \ i = 1, \ldots, s; \ b_i = \beta_i, \ i = s+1, \ldots, r; \ \theta > 0, \tag{3.67}$$

and

$$Z = \sum_{i=r+1}^{p} Y_i/\theta, \ Y_i/\theta \sim \text{Gamma}(a_i, \mu_i), \ \mu_i = b_i/\theta, \ i = r+1, \ldots, p \ ;$$
$$b_i = c\beta_i, \ i = r+1, \ldots, q \ ; \ b_i = \beta_i, \ i = q+1, \ldots, p \ ; \theta > 0 \ . \tag{3.68}$$

Note that

$$|\gamma_i| = \left|\frac{c\beta_i - \theta}{c\beta_i}\right| < 1 \Rightarrow c > \theta/2\beta_i \ , \ i = 1, \ldots, s \ ; \ i = r+1, \ldots, q \ ;$$
$$|\gamma_i| = \left|\frac{\beta_i - \theta}{\beta_i}\right| < 1 \Rightarrow \theta < 2\beta_i, \ i = s+1, \ldots, r; \ i = q+1, \ldots, p \ . \tag{3.69}$$

The coefficients K_ν and K'_w appearing on the right-hand side of (3.66) can be expressed as follows

$$K_\nu = \sum_{\nu_1+\cdots+\nu_r=\nu} (\Gamma(\rho_1+\nu))^{-1} \prod_{i=1}^{s} \left(\frac{c\beta_i}{\theta}\right)^{-a_i} \left(\frac{c\beta_i - \theta}{c\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!}$$
$$\times \prod_{i=s+1}^{r} \left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i - \theta}{\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!} \ , \tag{3.70}$$

and

$$K'_w = \sum_{w_{r+1}+\cdots+w_p=w} (\Gamma(\rho_2+w))^{-1} \prod_{i=r+1}^{q} \left(\frac{c\beta_i}{\theta}\right)^{-a_i} \left(\frac{c\beta_i - \theta}{c\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!}$$
$$\times \prod_{i=q+1}^{p} \left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i - \theta}{\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!} \ . \tag{3.71}$$

Since only K_ν and K'_w depend on c, we differentiate these two quantities first:

$$\frac{\partial K_\nu}{\partial c} = (\Gamma(\rho_1+\nu))^{-1} \sum_{\nu_1+\cdots+\nu_r=\nu} \prod_{i=1}^{s} \left(\frac{c\beta_i}{\theta}\right)^{-a_i} \left(\frac{c\beta_i - \theta}{c\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!}$$
$$\times \prod_{i=s+1}^{r} \left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i - \theta}{\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!} c^{-1} \left(-A_s - V_s + \sum_{i=1}^{s} \frac{\nu_i \beta_i c}{c\beta_i - \theta}\right), \tag{3.72}$$

where

$$A_s = a_1 + \cdots + a_s, \qquad V_s = \nu_1 + \cdots + \nu_s, \tag{3.73}$$

and

$$\frac{\partial K_w}{\partial c} = (\Gamma(\rho_2 + w))^{-1} \sum_{w_{r+1}+\cdots+w_p=w} \prod_{i=r+1}^{q} \left(\frac{c\beta_i}{\theta}\right)^{-a_i} \left(\frac{c\beta_i - \theta}{c\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!}$$
$$\times \prod_{i=q+1}^{p} \left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i - \theta}{\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!} c^{-1} \Big(-A_q - W_q + \sum_{i=r+1}^{q} \frac{w_i \beta_i c}{c\beta_i - \theta}\Big), \tag{3.74}$$

where

$$A_q = a_{r+1} + \cdots + a_q, \qquad W_s = w_{r+1} + \cdots + w_s. \tag{3.75}$$

Then

$$\frac{\partial}{\partial c}(K_\nu K'_w) = K'_w \frac{\partial K_\nu}{\partial c} + K_\nu \frac{\partial K'_w}{\partial c} = \frac{1}{\Gamma(\rho_1 + \nu)\Gamma(\rho_2 + w)} \sum_{\nu_1+\cdots+\nu_r=\nu} \sum_{w_{r+1}+\cdots+w_p=w}$$
$$\times \prod_{i=s+1}^{r} \Big[\left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i - \theta}{\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!}\Big]$$
$$\times \prod_{i=q+1}^{p} \Big[\left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i - \theta}{\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!}\Big]$$
$$\times \prod_{i=1}^{s} \Big[\left(\frac{c\beta_i}{\theta}\right)^{-a_i} \left(\frac{c\beta_i - \theta}{c\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!}\Big] \tag{3.76}$$
$$\times \prod_{i=r+1}^{q} \Big[\left(\frac{c\beta_i}{\theta}\right)^{-a_i} \left(\frac{c\beta_i - \theta}{c\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!}\Big]$$
$$\times c^{-1} \Big(-A_s - A'_q - V_s - W_q + \sum_{i=1}^{s} \frac{\nu_i \beta_i c}{c\beta_i - \theta} + \sum_{i=r+1}^{q} \frac{w_i \beta_i c}{c\beta_i - \theta}\Big)$$
$$= K_\nu K'_w c^{-1} \Big(-A_s - A'_q - V_s - W_q + \sum_{i=1}^{s} \frac{\nu_i \beta_i c}{c\beta_i - \theta} + \sum_{i=r+1}^{q} \frac{w_i \beta_i c}{c\beta_i - \theta}\Big) .$$

and the density function of R can be expressed as follows:

$$f_R(c) = \frac{\partial F_R(c)}{\partial c} = \sum_{\nu=0}^{\infty}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\rho_1+\rho_2+\nu+w+j)2^{-(\rho_1+\rho_2+\nu+w+j)}}{(\rho_1+\nu)_{j+1}} \times \left[K'_w\frac{\partial K_\nu}{\partial c} + K_\nu\frac{\partial K'_w}{\partial c}\right], \tag{3.77}$$

where the expression in square brackets is given in (3.76).

Note that in view of (3.69), the expression obtained in (3.77) for the density will converge for

$$c \in (\theta/2\beta_i, \infty),\ i = 1,\ldots,s\ ;\ i = r+1,\ldots,q$$

and

$$\theta \in (0, 2\beta_i),\ i = s+1,\ldots,r\ ;\ i = q+1,\ldots,p\ . \tag{3.78}$$

In order to obtain the density for $0 < c < \theta/2\beta_i$, the following identities are used

$$\begin{aligned} F_R(c) = \Pr(R \le c) &= \Pr(S_1/S_2 \le c) \\ &= \Pr(S_2/S_1 \ge c^{-1}) \\ &= 1 - \Pr(S_2/S_1 \le c^{-1}) \\ &= 1 - \Pr\Big(\frac{S_2 - c^{-1}S_1}{\theta} \le 0\Big)\ . \end{aligned}$$

Letting

$$S = \frac{S_2 - c^{-1}S_1}{\theta} = G^+ - G^-\ ,$$

where

$$G^+ = \sum_{i=r+1}^{p} Y_i\ ,\ Y_i \sim \text{Gamma}(a_i, \mu_i),$$
$$\mu_i = \beta_i/c\theta,\ i = q+1,\ldots,p\ ;\ \mu_i = \beta_i/\theta,\ i = r+1,\ldots,q;\ \theta > 0, \tag{3.79}$$

and

$$G^- = \sum_{i=1}^{s} Y_i\ ,\ Y_i \sim \text{Gamma}(a_i, \mu_i),$$
$$\mu_i = \beta_i/c\theta,\ i = s+1,\ldots,r\ ;\ \mu_i = \beta_i/\theta,\ i = 1,\ldots,s;\ \theta > 0\ , \tag{3.80}$$

we have

$$F_R(c) = 1 - \sum_{\nu=0}^{\infty}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty} K_{\nu}^{*} K_{w}^{*'} \frac{(\Gamma(\rho_1+\rho_2+\nu+w+j)2^{-(\rho_1+\rho_2+\nu+w+j)})}{(\rho_2+w)_{j+1}} \tag{3.81}$$

where

$$K_{\nu}^{*} = \sum_{\nu_1+\cdots+\nu_r=\nu} (\Gamma(\rho_1+\nu))^{-1} \prod_{i=1}^{s} \left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i-\theta}{\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!}$$
$$\times \prod_{i=s+1}^{r} \left(\frac{\beta_i}{c\theta}\right)^{-a_i} \left(\frac{\beta_i-c\theta}{\beta_i}\right)^{\nu_i} \frac{(a_i)_{\nu_i}}{\nu_i!}\,, \tag{3.82}$$

and

$$K_{w}^{*'} = \sum_{w_{r+1}+\cdots+w_p=w} (\Gamma(\rho_2+w))^{-1} \prod_{i=r+1}^{q} \left(\frac{\beta_i}{\theta}\right)^{-a_i} \left(\frac{\beta_i-\theta}{\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!}$$
$$\times \prod_{i=q+1}^{p} \left(\frac{\beta_i}{c\theta}\right)^{-a_i} \left(\frac{\beta_i-c\theta}{\beta_i}\right)^{w_i} \frac{(a_i)_{w_i}}{w_i!}\,.$$

On differentiating (3.81) with respect to c we obtain the density of R for

$$c \in (0, 2\beta_i/\theta),\ i = s+1,\ldots,r;\ i = q+1,\ldots,p$$

and

$$\theta \in (0, 2\beta_i),\ i = 1,\ldots,s\ ;\ i = r+1,\ldots,q \tag{3.83}$$

in the following form

$$f_R(c) = \frac{\partial F_R(c)}{\partial c} = -\sum_{\nu=0}^{\infty}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty} \frac{\Gamma(\rho_1+\rho_2+\nu+w+j)2^{-(\rho_1+\rho_2+\nu+w+j)}}{(\rho_2+w)_{j+1}}$$
$$\times \left(K_{w}^{*'}\frac{\partial K_{\nu}^{*}}{\partial c} + K_{\nu}^{*}\frac{\partial K_{w}^{*'}}{\partial c}\right), \tag{3.84}$$

where

$$\frac{\partial}{\partial c}(K_{\nu}^{*}K_{w}^{*'}) = K_{w}^{*'}\frac{\partial K_{\nu}^{*}}{\partial c} + K_{\nu}^{*}\frac{\partial K_{w}^{*'}}{\partial c}$$
$$= K_{\nu}^{*}K_{w}^{*'}c^{-1}\left(A_s + A_q - \sum_{i=s+1}^{r}\frac{\nu_i\theta c}{\beta_i - c\theta} - \sum_{i=q+1}^{p}\frac{w_i\theta c}{\beta_i - c\theta}\right)\,. \tag{3.85}$$

In view of (3.78) and (3.83), we have that $\theta \in (0, 2\beta_i)$, $i = 1, \ldots, p$, that is

$$\theta \in (0, 2\beta^*)$$

where

$$\beta^* = \min \beta_i \ , \ i = 1, \ldots, p \ .$$

Then, noting that

$$\frac{\theta}{2\beta_i} \leq \frac{\theta}{2\beta^*} < \frac{2\beta^*}{\theta} \leq \frac{2\beta_i}{\theta} \ , \tag{3.86}$$

we see that the intersection of the intervals $(0, 2\beta_i/\theta)$, $i = s+1, \ldots, r$; $i = q+1, \ldots, p$, and $(\theta/2\beta_i, \infty)$, $i = 1, \ldots, s$; $i = r+1, \ldots, q$ is not empty. Therefore we have closed form representations of the density of R for every positive value of c .

For the negative values of c, let

$$R^* = -S_1/S_2 \tag{3.87}$$

and consider the following equalities:

$$\begin{aligned} \Pr(S_1/S_2 \leq c) &= \Pr(-S_1/S_2 \geq -c) \\ &= \Pr(R^* \geq -c) \\ &= 1 - \Pr(R^* < -c) \ . \end{aligned}$$

The density is then given by

$$f_R(c) = -\frac{\partial F_{R^*}(-c)}{\partial c} = \frac{\partial F_{R^*}(-c)}{\partial(-c)} = f_{R^*}(-c) \ . \tag{3.88}$$

Hence the density of R is available for every real value of c .

When ρ_1 is a positive integer which occurs for example when Y^+ defined in (3.60) is a linear combination of chi-square variates whose degrees of freedom add up to an even integer, successive integrations by parts of the integral on the right-hand side of (3.65) yield the following identity

$$\int_0^z y^{\rho_1+\nu-1}e^{-y}dy = (\rho_1+\nu-1)! - e^{-z}\sum_{j=0}^{\rho_1+\nu-1}\frac{(\rho_1+\nu-1)!}{j!}z^j \tag{3.89}$$

(see also Gradshteyn and Ryzhik (1980, Eq. 3.261.1)). Using (3.61) and (3.62), we have

$$\begin{aligned}
F_R(c) &= \int_0^\infty h_2(z)\int_0^z h_1(y)dydz = \sum_{\nu=0}^{\infty}K_\nu\int_0^\infty\sum_{w=0}^{\infty}K'_w z^{(\rho_2+w-1)}e^{-z}\\
&\quad\times\left(\Gamma(\rho_1+\nu)-e^{-z}\sum_{j=0}^{\rho_1+\nu-1}\frac{(\rho_1+\nu-1)!}{j!}z^j\right)dz\\
&=\sum_{\nu=0}^{\infty}\sum_{w=0}^{\infty}K_\nu K'_w\Big\{\int_0^\infty z^{(\rho_2+w-1)}e^{-z}\Gamma(\rho_1+\nu)dz\\
&\qquad-\sum_{j=0}^{\rho_1+\nu-1}\int_0^\infty z^{(\rho_2+w+j-1)}e^{-2z}\frac{(\rho_1+\nu-j)!}{j!}\Big\}\\
&=\sum_{\nu=0}^{\infty}\sum_{w=0}^{\infty}K_\nu K'_w\Big\{\Gamma(\rho_1+\nu)\Gamma(\rho_2+w)-\sum_{j=0}^{\rho_1+\nu-1}\frac{(\rho_1+\nu-1)!}{j!}\frac{\Gamma(\rho_2+w+j)}{2^{(\rho_2+w+j)}}\Big\}
\end{aligned} \tag{3.90}$$

where the symbols are defined in (3.63) and (3.64). The density function is obtained similarly in this case since only K_ν and K'_w depend on c .

3.3.3. Examples

For comparison purposes, we consider in this section two ratios of quadratic forms whose exact densities may also be obtained by means of the transformation of variables technique.

Example 3.7

In the first example, we let

$$R = \frac{Q_1}{Q_2} \sim \frac{2\chi_4^2 + 3\chi_2^2}{\chi_6^2}\ ; \tag{3.91}$$

then, in our notation,

$$a_1 = 2,\ a_2 = 1,\ \beta_1 = 4,\ \beta_2 = 6,\ a_3 = 3,\ \beta_3 = 2,\ \rho_1 = 3,\ \text{and}\ \rho_2 = 3.$$

Using the expressions (3.72), (3.74) and (3.77), we have

$$\begin{aligned} f_R(c) = {} & \frac{\theta^6}{c^4 2^9 3} \sum_{\nu=0}^{\infty} \sum_{w=0}^{\infty} \frac{1}{\Gamma(w+1)} \left(\frac{2c-\theta}{2c}\right)^w \\ & \times \left[\Gamma(w+3) - \sum_{j=0}^{\nu+2} \frac{\Gamma(6+\nu+w+j)}{\Gamma(3+\nu-j)2^{(6+\nu+w+j)}}\right] \left(\frac{w\theta}{2c-\theta} - 3\right) \\ & \times \left[\sum_{i=0}^{\nu} (i+1) \left(1-\frac{\theta}{4}\right)^i \left(1-\frac{\theta}{6}\right)^{\nu-i}\right], \end{aligned} \tag{3.92}$$

for $c \in (\theta/2\beta_i, \infty) = (\theta/4, \infty)$, $\theta \in (0, 2\beta_i) = (0, 8)$; then using the expressions (3.84) and (3.85), we have

$$\begin{aligned} f_R(c) = {} & -\frac{c^2\theta^6}{2^9 3} \sum_{\nu=0}^{\infty} \sum_{w=0}^{\infty} \frac{1}{\Gamma(\nu+3)} \left(\frac{2-\theta}{2}\right)^w (w+1)(w+2) \\ & \times \sum_{j=0}^{\infty} \frac{\Gamma(6+\nu+w+j)}{\Gamma(4+w+j)2^{(6+\nu+w+j)}} \sum_{i=0}^{\nu} (i+1) \left(1-\frac{c\vartheta}{4}\right)^i \left(1-\frac{c\theta}{6}\right)^{\nu-i} \\ & \times \left[3 - \frac{ci\theta}{2-c\theta} - \frac{c(\nu-i)\theta}{6-c\theta}\right] \end{aligned} \tag{3.93}$$

for $c \in (0, 2\beta_i/\theta) = (0, 8/\theta)$, $\theta \in (0, 2\beta_i) = (0, 4)$. The density function of R which in this case can be found by standard statistical methods is

$$f_R^*(c) = \frac{9}{16}\left(\frac{c}{6}+\frac{1}{2}\right)^{-4} - \frac{9}{16}\left(\frac{c}{4}+\frac{1}{2}\right)^{-4} - \frac{3c}{16}\left(\frac{c}{4}+\frac{1}{2}\right)^{-5}. \tag{3.94}$$

Certain values of the density function of R calculated from (3.92) and (3.93) are compared below with the corresponding values obtained from (3.94).

c	$f_R(c)$	$f_R^*(c)$
0.4	0.1477	0.1577
0.6	0.2159	0.2195
0.8	0.2602	0.2609
1.0	0.2796	0.2797
1.2	0.2828	0.2828
1.8	0.2429	0.2465
2.0	0.2256	0.2289
2.5	0.1847	0.1853
3.0	0.1410	0.1477

Table 3.7

Example 3.8

In the second example, we consider the following ratio

$$R = \frac{Q_1}{Q_2} \sim \frac{p_1\chi_4^2 - p_2\chi_2^2}{\chi_6^2}. \tag{3.95}$$

Clearly, R can take on positive and negative values and therefore we have to consider four overlapping intervals in order to obtain the density of R for any point $c \in \Re$.

From (3.88), (3.77), and (3.76),

$$K'_w \frac{\partial K_\nu}{\partial c} + K_\nu \frac{\partial K'_w}{\partial c} = \frac{\theta^6}{b_1^2 b_2 c^4 16} \frac{1}{\Gamma(5+w)} (i+1)(i+2)(w-i+1)$$
$$\times \left(\frac{b_2-\theta}{b_2}\right)^\nu \left(\frac{2c+\theta}{2c}\right)^i \left(\frac{b_1-\theta}{b_1}\right)^{w-i} \left(-\frac{i\theta}{2c+\theta} - 3\right) \tag{3.96}$$

in the interval $(-\infty, -\theta/2b_3)$;

from (3.84), (3.85), and (3.88),

$$K'_w \frac{\partial K_\nu}{\partial c} + K_\nu \frac{\partial K'_w}{\partial c} = \frac{c^2\theta^6}{b_1^2 b_2 16} \frac{1}{\Gamma(1+\nu)}(i+1)(i+2)(w-i+1)$$
$$\times \left(\frac{b_2+c\theta}{b_2}\right)^\nu \left(\frac{2-\theta}{2}\right)^i \left(\frac{b_1+c\theta}{b_1}\right)^{w-i} \left(3 + \frac{\nu\theta c}{b_2+c\theta} + \frac{(w-i)\theta c}{b_1+c\theta}\right) \quad (3.97)$$

in the interval $(-2b_1/\theta, 0)$;

from (3.84) and (3.85),

$$K'_w \frac{\partial K_\nu}{\partial c} + K_\nu \frac{\partial K'_w}{\partial c} = \frac{c^2\theta^6}{b_1^2 b_2 16} \frac{1}{\Gamma(1+\nu)}(i+1)(i+2)$$
$$\times \left(\frac{b_1-c\theta}{b_1}\right)^\nu \left(\frac{2-\theta}{2}\right)^i \left(\frac{b_2-c\theta}{b_2}\right)^{w-i} \left(3 - \frac{\nu\theta c}{b_1-c\theta} - \frac{(w-i)\theta c}{b_2-c\theta}\right) \quad (3.98)$$

in the interval $(0, 2b_1/\theta)$;

and from (3.76) and (3.77),

$$K'_w \frac{\partial K_\nu}{\partial c} + K_\nu \frac{\partial K'_w}{\partial c} = \frac{\theta^6}{b_1^2 b_2 c^4 16} \frac{1}{\Gamma(4+w)}(i+1)(i+2)$$
$$\times \left(\frac{b_1-\theta}{b_1}\right)^\nu \left(\frac{2c-\theta}{2c}\right)^i \left(\frac{b_2-\theta}{b_2}\right)^{w-i} \left(\frac{i\theta}{2c-\theta} - 3\right) \quad (3.99)$$

in the interval $(\theta/2b_3, \infty)$.

The density of R found by standard statistical methods is

$$f^*(c) = \begin{cases} \frac{3p_2}{16(p_2+p_1)^2}\left(\frac{c}{2p_1}+\frac{1}{2}\right)^{-4} + \frac{3c}{8(p_2+p_1)p_1}\left(\frac{c}{2p_1}+\frac{1}{2}\right)^{-5}, & \text{if } c \geq 0\,; \\ \frac{3p_2}{16(p_2+p_1)^2}\left(-\frac{c}{2p_2}+\frac{1}{2}\right)^{-4}, & \text{if } c \leq 0\,. \end{cases} \quad (3.100)$$

Letting $p_1 = 2.0$ and $p_2 = 3.0$, the density of R is evaluated from the expressions given in (3.96), (3.97), (3.98), and (3.99) in the respective intervals using (3.77), (3.81) and (3.88), and compared with the corresponding values of the density calculated from the right-hand side of (3.100) at various points:

c	$f_R(c)$	$f_R^*(c)$
-3.0	0.0202	0.0225
-2.5	0.0298	0.0318
-2.0	0.0458	0.0466
-1.7	0.0594	0.0597
-1.2	0.0936	0.0937
-1.0	0.1138	0.1139
-0.9	0.1257	0.1260
0.4	0.3612	0.3665
0.6	0.3196	0.3199
0.8	0.2721	0.2722
1.0	0.2291	0.2291
1.7	0.1248	0.1248
2.0	0.0911	0.0974
2.5	0.0596	0.0660
3.0	0.0398	0.0460

Table 3.8

Again we see that the values obtained for $f_R(\cdot)$ and $f_R^*(\cdot)$ are in close agreement. This accuracy was achieved by using the following ranges of indices: $\nu = 0{:}28$, $w = 0{:}28$, $j = 0{:}25$ in our computer program; accuracy can be further increased by including more terms in the summations. The computer program used for evaluating the density function in all cases is included in Appendix C.5.

It should be pointed out that even for such simple examples, the transformation of variables technique is very tedious to apply. However the computable representations of the density of the ratio of two independent quadratic forms derived in this paper can be readily used in every case.

CHAPTER 4

THE SAMPLING DISTRIBUTION OF THE SERIAL CORRELATION COEFFICIENT

As shown in the first section of this chapter, the serial correlation coefficient is distributed as the ratio of two quadratic forms; hence its distribution can be obtained from the new results derived in Chapter 3. Two methods are proposed for the derivation of cumulants of the lag- k serial covariance for the case of a Gaussian white-noise process: the first one requires a special operator; the other one requires the solving of a system of second-order difference equations and the representation so obtained is expressed in terms of trigonometric functions. Explicit representations of the first moments of the serial correlation coefficient - including the fifth moment for which no explicit representation is available in the literature - are then given. The first four moments are used to approximate the density of the serial correlation by means of Pearson curves. A new computable representation of the exact distribution function is also given. The exact, the approximate and the simulated distributions are compared in two numerical examples.

4.1. Introduction

Given a series of observations $y_1, \ldots, y_T$ having joint normal distribution with mean vector $\mu = 0$ and positive definite covariance matrix $\Sigma = \sigma^2 I$, one can define the serial covariance at lag k as

$$P_k(\bar{y}) = \frac{1}{T}\sum_{i=1}^{T-k}(y_i - \bar{y})(y_{i+k} - \bar{y}) \tag{4.1}$$

for $k = 0, 1, \ldots, T-1$, where $\bar{y} = \sum_{i=1}^{T} y_i/T$. Throughout this chapter, it is assumed that the length of the series under observation is equal to T. Note that a general process $\boldsymbol{x}' = (x_1, \ldots, x_T)$ with mean $\boldsymbol{\mu}$ and positive definite covariance matrix Σ can be transformed into a white-noise process having mean 0 by defining $\boldsymbol{y} = (\boldsymbol{x} - \boldsymbol{\mu})\Sigma^{-1/2}$ where $\Sigma^{-1/2}$ is the symmetric square root of Σ^{-1}.

In matrix notation,

$$P_k(\bar{y}) = \frac{1}{T}(\boldsymbol{y}' B_k \boldsymbol{y}) \tag{4.2}$$

where

$$\boldsymbol{y}' = (y_1, \ldots, y_T)$$

$$B_k = V A_k V\ , \tag{4.3}$$

$$V = (I - \frac{1}{T}\boldsymbol{\epsilon}\boldsymbol{\epsilon}')\ , \tag{4.4}$$

$$\boldsymbol{\epsilon} = (1, 1, \ldots, 1)'\ ,$$

$$A_k = \frac{1}{2} M_k\ , \tag{4.5}$$

$$M_k = L_k + L_k'\ , \tag{4.6}$$

and L_k is a null matrix with the zeros in its k-subdiagonal replaced by ones.

The lag-k sample mean corrected serial correlation coefficient is then given by

$$r_k(\bar{y}) = \frac{P_k(\bar{y})}{P_o(\bar{y})} = \frac{\boldsymbol{y}' B_k \boldsymbol{y}}{\boldsymbol{y}' V \boldsymbol{y}}\ . \tag{4.7}$$

Note that $B_o = VA_oV = VIV = V$, V being indempotent. This definition of the serial correlation coefficient was used by O.D. Anderson (1990) and T.W. Anderson

((1971), p.302) where a more general set-up is proposed. The case $k = 1$ is treated in Hannan ((1970), p.342). Moran (1948) calculated values of the lower moments of this serial correlation coefficient.

The above representation may be simplified by defining the noncentered lag- k serial covariance as follows

$$P_k(0) = \sum_{i=1}^{T-k} y_i y_{i+k}/T = \mathbf{y}' A_k \mathbf{y}/T \; ; \tag{4.8}$$

the corresponding noncentered lag- k serial correlation is then

$$r_k(0) = \frac{P_k(0)}{P_o(0)} \; . \tag{4.9}$$

Merikoski and Pukkila (1983) used this simpler representation in a moment problem. Clearly $P_k(\bar{y})$ can be obtained from $P_k(0)$ by replacing y_i by $y_i - \bar{y}$ in the latter.

4.2. The Moments of the Serial Correlation Coefficient for a Gaussian White Noise Process

We propose in this section a methodology for the derivation of the trace of the powers of $A_k V$ needed to evaluate the cumulants of the lag- k serial covariance defined in (4.2) for the case of a Gaussian white-noise process, that is, assuming that $\Sigma = \sigma^2 I$. We then give explicit representations of the first moments of the serial correlation coefficient. Since the distribution of the serial correlation coefficient does not involve σ^2, it is assumed that $\sigma^2 = 1$ in the sequel.

Since B_k is symmetric and V is symmetric and idempotent, a theorem due to Pitman which is stated in Hannan (1970, p.343) applies and hence the following equality holds

$$E(P_k(\bar{y})/P_o(\bar{y}))^m = E(P_k(\bar{y}))^m / E(P_o(\bar{y}))^m \; . \tag{4.10}$$

In order to derive the cumulants of the serial covariance, we use the following result (see Mathai and Provost ((1992), Theorem 3.3.2)): the s-th cumulant of $P_k(\bar{y})$ given in (4.2) is

$$K_s = 2^{s-1}(s-1)!tr(B_k/T)^s \tag{4.11}$$

where

$$\begin{aligned} tr(B_k/T)^s &= (\frac{1}{T})^s tr(VA_kV)^s \\ &= (\frac{1}{T})^s tr(A_kV)^s \ . \end{aligned}$$

It is shown in Appendix A.1 that the first five cumulants of $P_k(\bar{y})$ can be represented as follows

$$K_1 = -\frac{(T-k)}{T^2} \ , \tag{4.12}$$

$$K_2 = \frac{(T-k)(T^2-2k)-2T<T-2k>}{T^4} \ , \tag{4.13}$$

$$\begin{aligned} K_3 = (\frac{2}{T^6})\{&-(T-k)(T^2-2kT+4k^2)-6kT<T-2K> \\ &-3T^2<T-3k>\} \ , \end{aligned} \tag{4.14}$$

$$\begin{aligned} K_4 = (\frac{6}{T^8})\{&(T-k)(T^4-4T^2k+8Tk^2-8k^3) \\ &+<T-2k>(2T^4-4T^3-16Tk^2) \\ &-<T-3k>8T^2k-<T-4k>4T^3\} \ , \end{aligned} \tag{4.15}$$

$$\begin{aligned} K_5 = (\frac{24}{T^{10}})\{&(T-K)(-T^4+4T^3K-16T^2k^2+24Tk^3-16k^4) \\ &-<T-2k>10Tk(T^2+4k^2)-<T-3k>10T^2(T^2+Tk+2k^2) \\ &-<T-4k>10T^3k-<T-5k>5T^4\} \end{aligned} \tag{4.16}$$

where

$$<a>=\begin{cases} 0 & \text{if } a<0 \\ a & \text{if } a\geq 0 \ . \end{cases} \tag{4.17}$$

This explicit representation for K_5 is new. Anderson (1990) obtained K_2 and K_3 using a different approach. The expression for K_4 which can be derived from the product–cumulant formula given in Anderson (1991) agrees with (4.15).

From the relationship between the moments and the cumulants given in Rao ((1965), p.105), we obtain the moments about the origin of $P_k(\bar{y})$.

The moments about the origin of $(\boldsymbol{y}'V\boldsymbol{y})/T$ are

$$E(P_0(\bar{y}))^m = (T-1)(T+1)(T+3)\cdots(T+2m-3)/T^m \ , \qquad (4.18)$$

see Anderson ((1990), equation (9)). Then in view of (4.10), we have the following representation of the moments about the origin of $P_k(\bar{y})$, the lag– k serial correlation coefficient:

$$\mu = -\frac{T-k}{T(T-1)} \qquad (4.19)$$

$$\mu_2' = \frac{(T-k)(T^2+T-3k) - 2T < T-2k >}{T^2(T-1)(T+1)} \qquad (4.20)$$

$$\mu_3' = \frac{1}{T^3(T-1)(T+1)(T+3)}\{-3(T-k)[(T-k)(T^2+T-3k)+2k^2] + 6T\,(T-3k) < T-2k > -6T^2 < T-3k >\} \qquad (4.21)$$

$$\mu_4' = \frac{1}{T^4(T-1)(T+1)(T+3)(T+5)}\{(T-k)\,(3T^5+12T^4+9T^3 - 24T^3k - 3T^4k - 63T^2k + 135Tk^2 + 18T^2k^2 - 105k^2) + < T-2k > (-24T^3 - 72T^2k + 12T^3k - 180Tk^2) + < T-3k > (24T^3 - 72T^2k) - < T-4k > 24T^3\} \ . \qquad (4.22)$$

$$\mu_5' = \{-15(T-k)(T^6 - 2T^5k + T^4k^2 + 4T^5 + 3T^4 - 12T^4k - 24T^3k + 78T^2k^2 - 10T^2k^3 + 18T^3k^2 - 112Tk^3 + 63k^4)$$

$$+ < T - 2k > 60(2T^4 - 3T^4k + 3T^3k^2 - 6T^3k + 15T^kk^2 - 35Tk^3)$$

$$- < T - 3k > 60(T^5 + 3T^4 + 15T^2k^2 - T^4k)$$

$$+ < T - 4k > 120(T^4 - 3T^3k) - < T - 5k > 120T^4 \}$$

$$\times \left\{T^5(T-1)(T+1)(T+3)(T+5)(T+7)\right\}^{-1} \quad . \tag{4.23}$$

These representations of the moments hold for any positive values of T and k.

We need the mean and the central moments, μ_2, μ_3 and μ_4 in order to determine the appropriate Pearson curve. These are obtained from the relationships given in Elderton and Johnson ((1969), p.51) between the moments about the origin and the central moments. They are

$$\mu = -\frac{(T-k)}{T(T-1)} \tag{4.24}$$

$$\mu_2 = \frac{(T-k)(T-2)(T^2+T-2k) - 2T(T-1) < T - 2k >}{T^2(T-1)^2(T+1)} \tag{4.25}$$

$$\mu_3 = 2\left\{(T-k)\left[T^2(T+1)(5T-9) - 2Tk(2T^2+5T-15)\right.\right.$$

$$\left. - 4k^2(T-2)(T-3)\right] - < T - 2k > 6T(T-1)(2T+kT-3k)$$

$$\left. - < T - 3k > 3T^2(T-1)^2\right\}\left\{T^3(T-1)^3(T+1)(T+3)\right\}^{-1} \tag{4.26}$$

$$\mu_4 = \left\{(T-k)(T-1)^3\left(3T^5 + 12T^4 + 9T^3 - 24T^3k - 3T^4k\right.\right.$$

$$\left. - 63T^4k + 135Tk^2 + 18T^2k^2 - 105k^2\right)$$

$$- 12(T-1)^2(T+5)(T-k)^2[(T-k)(T^2+T-3k) + 2k^2]$$

$$+ (T-1)(T+3)(T+5)6(T-k)^3(T^2+T-3k)$$

$$- 3(T+1)(T+3)(T+5)(T-k)^4$$

$$+ < T - 2k > \left[(T-1)^3 12T(-2T^2 + 6Tk + T^2k - 15k^2)\right.$$

$$+ (T-1)^2(T+5)24T(T-4)(T-3k)$$

$$\left. - (T-1)(T+3)(T+5)12T(T-4)^2\right]$$

$$+ < T - 3k > 48T^2(-Tk - 3T + 4k)(T-1)^2 \tag{4.27}$$

$$- < T - 4k > (T-1)^3 24T^3\} \{T^4(T-1)^4(T+1)(T+3)(T+5)\}^{-1} .$$

4.3. Alternate Representation of the Cumulants in terms of Eigenvalues and Eigenvectors

First the eigenvalues and the eigenvectors of A_k defined in (4.5) are derived and expressed in terms of trigonometric functions. Then an explicit representation of the noncentered lag- k serial covariance, $P_k(0)$ defined in (4.8) is obtained. From the standard relationship between the eigenvalues λ and the eigenvectors x i.e.,

$$A_k x = \lambda x \ , \tag{4.28}$$

we obtain T linear equations which can be solved as a system of second-order difference equations. The details of the derivation are presented in Appendix A.2. For any nonnegative integers T and k, the eigenvalues are

$$\lambda_{t,T} = \lambda_{ks-a,kn+b} = \cos \frac{k\pi t}{T+k-b} = \cos \alpha_{t,T} \quad \text{if } a \geq b \tag{4.29}$$

$$\lambda_{t,T} = \lambda_{ks-a,kn+b} = \cos \frac{k\pi t}{T+(k-b)+k} = \cos \alpha_{t,T} \quad \text{if } a < b \tag{4.30}$$

where $t = ks - a$, $s \leq \frac{T+a}{k}$, $a = 0, 1, \ldots, k-1$ and $T = kn + b$, $b = 0, 1 \ldots, k-1$, $n \leq \frac{T-b}{k}$. The eigenvalues also appear in Table I.

The eigenvector corresponding to the eigenvalue $\lambda_{t,T} = \lambda_{ks-a,kn+b}$ is denoted by $x_{t,T}$ and its components are

$$x_i = \begin{cases} \sin(u\alpha_{t,T}), & i = uk - a, \ u = 1, \ldots, \ [\frac{T+a}{k}] \\ 0 & \text{otherwise} . \end{cases}$$

The row vectors $x'_{t,T}$, $t = 1, \ldots, T$, form an orthogonal matrix. We will denote the corresponding orthonormal matrix by P and its row vectors by p_t, $t = 1, \ldots, T$.

Note that the noncentered lag- k serial covariance can be expressed as

$$\begin{aligned} P_k(0) &= \mathbf{y}' A_k \mathbf{y}/T \\ &= \mathbf{y}' P' \Lambda P \mathbf{y}/T \\ &= \sum_{i=1}^{T} \lambda_i Z_i^2/T \end{aligned} \tag{4.31}$$

where Λ is the diagonal matrix whose diagonal elements are λ_t, the eigenvalues of A_k, P is an orthonormal matrix satisfying $A_k = P'\Lambda P$, and the Z_i's are independently distributed as $N(0, \sigma^2)$. Using Theorem 3.2b.2 given in Mathai and Provost ((1992), p.53), the first four moments of $T \times P_k(0)$ are found to be:

$$\begin{aligned} E[T \times P_k(0)] &= trA_k \ , \\ E[T \times P_k(0)]^2 &= 2trA_k^2 + (trA_k)^2 \ , \\ E[T \times P_k(0)]^3 &= 8trA_k^3 + 6trA_k^2 \ trA_k + (trA_k)^3 \ , \end{aligned} \tag{4.32}$$

and

$$E[T \times P_k(0)]^4 = 48trA_k^4 + 32trA_k^3 trA_k + 12trA_k^2(trA_k)^2 + 12(trA_k^2)^2 + (trA_k)^4 \ .$$

From Equation (10) in Johnson and Kotz ((1970), p.168), the moments of $T \times P_0(0)$ are found to be

$$\begin{aligned} E(T \times P_0(0)) &= T \ . \\ E(T \times P_0(0))^2 &= T(T+2) \ . \\ E(T \times P_0(0))^3 &= T(T+2)(T+4) \ . \end{aligned} \tag{4.33}$$

and

$$E(T \times P_0(0))^4 = T(T+2)(T+4)(T+6) \ .$$

Applying the relationship (4.10), we obtain the first four moments of the serial correlation coefficient:

$$\mu = \frac{trA_k}{T} \ ,$$

$$\mu_2' = \frac{2trA_k^2 + (trA_k)^2}{T(T+2)} ,$$

$$\mu_3' = \frac{8trA_k^3 + 6trA_k^2 trA_k + (trA_k)^3}{T(T+2)(T+4)} , \tag{4.34}$$

and

$$\mu_4' = \{48trA_k^4 + 32trA_k^3 trA_k + 12trA_k^2(trA_k)^2 + 12(trA_k^2)^2 + (trA_k)^4\}$$
$$\times \{T(T+2)(T+4)(T+6)\}^{-1} .$$

Note that $tr(A_k^\alpha) = \sum_{i=1}^{T} \lambda_i^\alpha$ where the λ_i's are the eigenvalues of A_k which are given in (4.29) and (4.30).

In order to obtain a representation of the moments of lag- k serial covariance $P_k(\bar{y})$ defined in (4.2), we have to evaluate expressions of the type $tr[A_k^{s-\ell}(A_k\epsilon\epsilon')^\ell]$ as seen from Appendix A.1.

For $\ell \neq 0$

$$tr[A_k^{s-\ell}(A_k\epsilon\epsilon')^\ell] = tr[(P'\Lambda P)^{s-\ell}(P'\Lambda P\epsilon\epsilon')^\ell]$$

$$= tr[(P'\Lambda P)^{s-\ell}(P'\Lambda P\epsilon\epsilon')(P'\Lambda P\epsilon\epsilon')^{\ell-1}] ;$$

noting that $\epsilon'(P\Lambda P\epsilon\epsilon')^{\ell-1} = (\epsilon'P'\Lambda P\epsilon)(\epsilon'P'\Lambda P\epsilon)\cdots(\epsilon'P'\Lambda P\epsilon)\epsilon' = (\sum_{i=1}^{T} \lambda_i p_i^2)^{\ell-1}\epsilon'$,

$$tr[A_k^{s-\ell}(A_k\epsilon\epsilon')^\ell] = (\sum_{\ell=1}^{T} \lambda_i p_i^2)^{\ell-1} tr[P'\Lambda^{s-\ell}PP'\Lambda P\epsilon\epsilon']$$
$$= (\sum_{i=1}^{T} \lambda_i p_i^2)^{\ell-1} tr[\Lambda^{s-\ell+1}(P\epsilon)(\epsilon'P')]$$
$$= (\sum_{i=1}^{T} \lambda_i p_i^2)^{\ell-1} (\sum_{\ell-1}^{T} \lambda_i^{s-\ell+1} p_i^2), \quad \ell = 1,2,3,\ldots \tag{4.35}$$

where $p_i = \sum_{j=1}^{T} p_{ij}$, is the sum of the components of the eigenvector corresponding to λ_i, $i = 1,2,\ldots,T$.

For $\ell = 0$,

$$tr(A_k^{s-\ell}(A_k\epsilon\epsilon')^\ell) = tr(A_k^s) = tr[(P'\Lambda P)^s] = tr(P'\Lambda^s P) = tr(\Lambda^s) = \sum_{i=1}^{T} \lambda_i^s \,. \tag{4.36}$$

The following alternate representations of $tr(A_k V)^s$ are obtained using some identities appearing in Appendix A.1

$$tr(A_k V) = -\frac{(T-k)}{T} \tag{4.37}$$

$$tr[(A_k V)^2] = \sum_{i=1}^{T} \lambda_i^2 - \frac{2}{T}\sum_{i=1}^{T} \lambda_i^2 p_i^2 + \frac{1}{T^2}(T-k)^2 \tag{4.38}$$

$$tr[(A_k V)^3] = -\frac{3}{T}(\sum_{i=1}^{T} \lambda_i^3 p_i^2) + \frac{3}{T^2}(T-k)(\sum_{i=1}^{T} \lambda_i^2 p_i^2) - \frac{1}{T^3}(T-k)^3 \tag{4.39}$$

$$tr[(A_k V)^4] = \sum_{i=1}^{T} \lambda_i^4 - \frac{4}{T}(\sum_{i=1}^{T} \lambda_i^4 p_i^2) + \frac{4}{T^2}(T-k)(\sum_{i=1}^{T} \lambda_i^3 p_i^2) + \frac{2}{T^2}(\sum_{i=1}^{T} \lambda_i^2 p_i^2)^2 - \frac{4}{T^3}(T-k)^2(\sum_{i=1}^{T} \lambda_i^2 p_i^2) + \frac{1}{T^4}(T-k)^4 \,. \tag{4.40}$$

The following identities were also used to obtain (4.40):

$$tr[(A_k\epsilon\epsilon' A_k)^2] = tr\{P'\Lambda P\epsilon\epsilon' P'\Lambda PP'\Lambda P\epsilon\epsilon' P'\Lambda P\} = (\sum_{i=1}^{T} \lambda_i^2 p_i^2)^2 \,.$$

Thus in view of (4.11) and the relationships between the moments about origin and the central moments, we have:

$$\mu = -\frac{(T-k)}{T(T-1)}\,, \tag{4.41}$$

$$\mu_2 = \frac{2T^2(T-1)\sum_{i=1}^{T} \lambda_i^2 - 4T(T-1)\sum_{i=1}^{T} \lambda_i^2 p_i^2 + 2(T-2)(T-k)^2}{T^2(T-1)^2(T+1)} \tag{4.42}$$

$$\mu_3 = \{6T(T^2 - 2T + 1)\Big(-4T\sum_{i=1}^{T} \lambda_i^3 p_i^2 + 6(T-k)\sum_{i=1}^{T} \lambda_i^2 p_i^2$$

$$- T(T-k)\sum_{i=1}^{T}\lambda_i^2\Big) - 8(T-k)^3(T^2-5T+6)$$

$$+ 6T(T-k)(T^2+2T-3)(T\sum_{i=1}^{T}\lambda_i^2 - 2\sum_{i=1}^{T}\lambda_i^2 p_i^2))\}$$

$$\times \{T^3(T-1)^3(T+1)(T+3)\}^{-1} , \tag{4.43}$$

$$\mu_4 = \{48T^3(T-1)^3(T\sum_{i=1}^{T}\lambda_i^4 - 4\sum_{i=1}^{T}\lambda_i^4 p_i^2)$$

$$+ 96T^2(T-k)(T-1)^2(2T-8)\sum_{i=1}^{T}\lambda_i^2 p_i^2$$

$$+ 144T^2(T-1)^3(\sum_{i=1}^{T}\lambda_i^2 p_i^2)$$

$$- 48T(T-k)^2(T-1)(5T^2-23T+30)\sum_{i=1}^{T}\lambda_i^2 p_i^2$$

$$- 12T^3(T-1)^3(\sum_{i=1}^{T}\lambda_i^2)(4\sum_{i=1}^{T}\lambda_i^2 p_i^2 - T\sum_{i=1}^{T}\lambda_i^2)$$

$$+ 24T^2(T-k)^2(T-1)(T^2-3T+14)\sum_{i=1}^{T}\lambda_i^2$$

$$+ 12(T-k)^4(5T^3-33T^2+76T-60)\}$$

$$\times \{T^4(T-1)^4(T+1)(T+3)(T+5)\}^{-1} . \tag{4.44}$$

4.4 The Distribution Function of the Serial Correlation Coefficient

We give in this section an explicit representation of the distribution function of the lag- k serial correlation for a series of length T. This representation can be used for any positive T and any $k = 1, \ldots, T-1$. It is believed that no such general representation is available in the literature. Pan Jie-Jiam (1968) obtained an integral representation for the distribution function of the lag-1 serial correlation coefficient.

In order to derive the exact distribution function of $r_k(\bar{y})$ given in (4.7) for a Gaussian white-noise process, we will use the expression (1.55) given in Section 1.2 for the density of linear combination of chi-square variables. Let $D = \sum_{j=1}^{k} m_j X_j$ where the X_j 's are independently distributed chi-square variables with r degrees of freedom and the m_j 's are constants, then the density of D is

$$f(d) = \sum_{v=0}^{\infty} K_v e^{-d} (d)^{v+\frac{rk}{2}-1} \tag{4.45}$$

where

$$K_v = \sum_{v_1+\cdots+v_k=v} \left(\prod_{j=1}^{k} \mu_j^{-\frac{r}{2}}\right)\left(\frac{r}{2}\right)_{v_1} \cdots \left(\frac{r}{2}\right)_{v_k} \frac{c_1^{v_1}\cdots c_k^{v_k}}{v_1!\cdots v_k!\,\Gamma(v+\frac{rk}{2})} \ , \tag{4.46}$$

$$c_j = \frac{\mu_j - 1}{\mu_j} \ , \ \mu_j = 2m_j \ , \ j = 1,\ldots,k \ ,$$

and for example $(\alpha)_\beta = \Gamma(\alpha+\beta)/\Gamma(\alpha)$.

The lag- k serial correlation coefficient given in (4.7) can be expressed as a ratio of two linear combinations of independent chi-square variables. Because $B_k = VA_kV$ and V commute, they can be simultaneously diagonalized with the same orthonormal matrix. Hence, noting that the rank of V is $T-1$ and that all its nonzero eigenvalues are equal to one (since $V = V^2$), we see that the lag- k serial correlation coefficient has the following distribution

$$r_k(\bar{y}) \sim \frac{\sum_{i=1}^{T-1} l_i X_i}{\sum_{i=1}^{T-1} X_i} \tag{4.47}$$

where the l_i 's, $i = 1,\cdots,T-1$, are the eigenvalues of VA_kV and the X_i 's are independent chi-square variables with 1 degree of freedom. It is worth noting that eigenvalues of $A_k V$ are the same as those of VA_kV ; the proof is included in Appendix A.3.

Let $S_1 = \sum_{i=1}^{T-1} l_i X_i$, $S_2 = \sum_{i=1}^{T-1} X_i$, c be a real constant and $S = S_1 - cS_2$, then

$$Pr(r_k(\bar{y}) \leq c) = Pr(S \leq 0) .$$

Now writing

$$S = Y^+ - Y^- \tag{4.48}$$

where

$$Y^+ = \sum_{i:l_i - c>0} (l_i - c)X_i \ ,$$
$$Y^- = \sum_{i:l_i - c<0} (c - l_i)X_i \ ,$$

we have

$$F_r(c) = Pr(r_k(\bar{y}) \leq c) = Pr(Y^+ \leq Y^-). \tag{4.49}$$

Letting $Y = Y^+/\theta$ and $Z = Y^-/\theta$ where θ is a positive constant which can be chosen to make the representations of the densities of Y and Z converge faster, we have

$$F_r(c) = Pr(Y \leq Z) = \int_0^\infty h_1(z) \int_0^z h_2(y) dy dz \tag{4.50}$$

where $h_1(z)$ and $h_2(y)$ denote the density functions of Z and Y. From the representation (4.45),

$$h_1(y) = \sum_{v=0}^{\infty} K_v e^{-y} (y)^{\frac{\rho_1}{2}+v-1} \ , \tag{4.51}$$

and

$$h_2(z) = \sum_{w=0}^{\infty} K_w e^{-z} (z)^{\frac{\rho_2}{2}+w-1}, \tag{4.52}$$

where

ρ_1 is the number of terms in Y^+,

ρ_2 is the number of terms in Y^-,

$$K_v = \sum_{v_1+\cdots+v_{\rho_1}=v} \Big(\prod_{j:\lambda_j - c>0} \mu_j^{-\frac{1}{2}} \Big) \frac{(\frac{1}{2})_{v_1} \cdots (\frac{1}{2})_{v_{\rho_1}} c_1^{v_1} \ldots c_{\rho_1}^{v_{\rho_1}}}{v_1! \cdots v_{\rho_1}! \Gamma(v + \frac{\rho_1}{2})}$$

with

$$c_j = \frac{\mu_j - 1}{\mu_j} \ , \ \mu_j = \frac{2(l_i - c)}{\theta} \tag{4.53}$$

and

$$K_w = \sum_{w_1+\cdots+w_{\rho_2}=w} \Big(\prod_{j:\lambda_j - c<0} \mu_j^{-\frac{1}{2}} \Big) \frac{(\frac{1}{2})_{w_1} \cdots (\frac{1}{2})_{w_{\rho_2}} d_1^{w_1} \cdots d_{\rho_2}^{w_{\rho_2}}}{w_1! \ldots w_{\rho_2}! \Gamma(w + \frac{\rho_2}{2})}$$

with

$$d_j = \frac{\mu_j - 1}{\mu_j} \ , \ \mu_j = \frac{2(c - l_i)}{\theta} \ . \tag{4.54}$$

Using the identity

$$\int_0^z y^{v+\frac{\rho_1}{2}-1} e^{-y} dy = e^{-z} \sum_{j=0}^{\infty} z^{\frac{\rho_1}{2}+v+j} / (\frac{\rho_1}{2} + v)_{j+1}$$

(see Gradshteyn and Ryzhik ([4], Eq. 3.381.2)), we obtain the following representation of the distribution function of $r_k(y)$

$$F_r(c) = \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \sum_{j=0}^{\infty} K_v K_w \frac{1}{(\frac{\rho_1}{2} + v)_{j+1}} \int_0^{\infty} e^{2z} z^{\frac{\rho_1+\rho_2}{2}+v+w+j-1} dz \tag{4.55}$$

$$= \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \sum_{j=0}^{\infty} K_v K_w \frac{\Gamma(\frac{\rho_1+\rho_2}{2} + v + w + j)}{(\frac{\rho_1}{2} + v)_{j+1} 2^{\frac{\rho_1+\rho_2}{2}+w+v+j}} \ .$$

4.5. Two Numerical Examples

The main theoretical results developed in this chapter are corroborated by the numerical examples presented in this section. For each case, we calculate the moments of the serial correlation coefficient with the formulas derived in Section 4.2 and 4.3, and we evaluate the approximate distribution function based on the appropriate Pearson curves (which is a function of the first four moments). Then we determine the exact distribution function of the lag- k serial correlation coefficient using (4.55). We also simulate series of observations from T independent standard normal variables for comparison purposes.

Example 4.1

In the first example, we let $T = 5$ (independent standard normal variables) and $k = 1$. Using the expressions (4.22)–(4.25), the moments are found to be

$$\mu = -0.2$$

$$\mu_2 = 0.09$$

$$\mu_3 = 0.002$$

$$\mu_4 = 0.0183 \ .$$

The same results were obtain^ed using the expressions (4.38)–(4.41).

From $\beta_1 = \mu_3^2/\mu_2^3 = 0.0054869$ and $\beta_2 = \mu_4/\mu_2^2 = 2.25926$, we determine that for the case $T = 5$ and $k = 1$, the appropriate Pearson frequency curve is of Type I, i.e.,

$$y = y_0(1 + \frac{x}{a_1})^{m_1} \left(\frac{x}{a_2}\right)^{m_2} \quad , \quad -a_1 < x < a_2 \ , \tag{4.56}$$

where $m_1 = 1.378119$, $m_2 = 1.6438605$, $a_1 = 0.6723952$, $a_2 = 08020532$, and $y_o = 0.86376$; see Elderton and Johnson ((1969), p.45). We evaluate by numerical integration the approximate distribution function of the serial correlation coefficient for some values at x. The eigenvalues of VA_1V (or equivalently those of A_1V, see Appendix A.3) are found to be

$$l_1 = 0.5, \ l_2 = 0.0583, \ l_3 = -0.5, \ l_4 = -0.8583,$$

and the distribution function is calculated using expression (4.55). In order to verify the validity of our theoretical results, certain values of the distribution function of the lag–1 serial correlation coefficient obtained from the moments (Pearson curve), formula (4.55) and by simulation (with the IMSL subroutine) are given in the following table assuming a series of observations of length 5.

c	Pearson Curve	Exact Distribution	Simulation
−0.4	0.28068	0.28576	0.28631
−0.3	0.39111	0.39541	0.39472
−0.2	0.50642	0.50487	0.50355
−0.1	0.62010	0.61433	0.61293
0.0	0.72604	0.72333	0.72428
0.1	0.81885	0.82932	0.82639
0.2	0.89474	0.89530	0.89607
0.3	0.94937	0.94628	0.94632
0.4	0.98335	0.98191	0.98197

Table 4.1

All these values are seen to be in close agreement.

Example 4.2

For the second example, we consider a series of observations of length 7 and find the distribution function of the lag–2 serial correlation coefficient. From the formulas (4.22)–(4.25), the moments are found to be

$$\mu = -0.1190476$$

$$\mu_2 = 0.0742629$$

$$\mu_3 = 0.00368346$$

$$\mu_4 = 0.0146199\ .$$

The corresponding coefficients of skewness and kurtosis are respectively $\beta_1 = 0.0331281$ and $\beta_2 = 2.650939$. For these values, the appropriate Pearson curve is of Type I with frequency curve given by (4.56), with $m_1 = 4.021127$, $m_2 = 6.150408$, $a_1 = 0.794222$, $a_2 = 1.2147897$, and $y_o = 0.7214296$. The eigenvalues of VA_2V or of A_2V are found to be $l_1 = 0.7515$, $l_2 = 0.3090$, $l_3 = 0$, $l_4 = -0.2753$, $l_5 = -0.6905$, and $l_6 = -0.8090$. Certain values of the distribution function of the lag-2 serial correlation coefficient obtained from the moments (Pearson curve), formula (4.55) and by simulation are given in the table below assuming a series of observations of length 7.

c	Pearson	Exact	Simulation
−0.5	0.07849	0.0817	0.0836
−0.4	0.15951	0.1619	0.1636
−0.3	0.27026	0.2642	0.2678
−0.2	0.40144	0.3984	0.3982
−0.1	0.53944	0.5372	0.5358
0.0	0.67013	0.5861	0.5869
0.1	0.78216	0.7835	0.7833
0.2	0.86903	0.8703	0.8711
0.3	0.92955	0.9297	0.9306
0.4	0.96691	0.9655	0.9650

Table 4.2

All the values are in close agreement except the Pearson curve approximation of the distribution function evaluated at $c = 0$. The computer program used for evaluating (4.55) is included in Appendix C.6.

4.6. A More General Case

We give in this section an expression for the moments of the noncentered serial correlation for a process which is not necessarily Gaussian.

Let $y_1, \ldots, y_T$ be a series of observations from a distribution with known mean vector μ and known positive definite covariance matrix Σ. Using the representation (1.67) of a quadratic form , the serial covariance can be expressed as

$$P_k(0) = \frac{1}{T}(\boldsymbol{y}'A_k\boldsymbol{y}) \sim \sum_{j=1}^{T} \lambda_j(U_j + b_j)^2 \tag{4.57}$$

where $\lambda_1, \ldots, \lambda_T$ are the eigenvalues of $\Sigma^{\frac{1}{2}} A_k \Sigma^{\frac{1}{2}}/T$, $\Sigma^{\frac{1}{2}}$ is the symmetric square root of Σ, A_k is defined in (4.5),

$$E(\boldsymbol{U}) = \boldsymbol{0} \quad , \mathrm{Cov}(\boldsymbol{U}) = I \quad , \boldsymbol{U}' = (U_1, \ldots, U_T)$$

$\boldsymbol{b}' = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'$ and P be a $T \times T$ orthonormal matrix which diagonalizes $\Sigma^{\frac{1}{2}} A_k \Sigma^{\frac{1}{2}}/T$.

In view of this representation, the moments about origin can be evaluated as follows using Theorem 3.26.1 given in Mathai and Provost ((1992), p.49):

$$\begin{aligned} E[(P_k(0))^r] &= E[\sum_{j=1}^{T} \lambda_j(U_j + b_j)^2]^r \\ &= \sum_{r_1+r_2+\cdots+r_T=r} \frac{r!\,\lambda_1^{r_1} \cdots \lambda_T^{r_T}}{r_1! \cdots r_T!} E(V_1^{r_1} \cdots V_T^{r_T}) \end{aligned} \tag{4.58}$$

where $V_j = (U_j + b_j)^2$. Note that this result applies to other models for which $P_k(0) = \boldsymbol{y}'C\boldsymbol{y}$ where C is a other type of symmetric matrix. The moments of the denominator of serial correlation coefficient can be obtained by setting $k = 0$ in (4.58).

Furthermore, it is seen from the representation (4.57) that, for a Gaussian process, $P_k(0)$ is distributed as a linear combination of noncentral chi-square variables when $\boldsymbol{\mu} \neq \boldsymbol{0}$; hence the distribution function can be obtained from the results derived in Chapter 2.

λ_t	$t=kn$ as $\frac{T}{k}$	$t=kn-1$ as $\frac{T+1}{k}$	$t=kn-2$ as $\frac{T+2}{k}$		$t=kn-(k-2)$ as $\frac{T+k-2}{k}$	$t=kn-(k-1)$ as $\frac{T+k-1}{k}$
$T=kn$	$\cos\frac{k\pi t}{T+k}$	$\cos\frac{k\pi t}{T+k}$				$\cos\frac{k\pi t}{T+k}$
$T=kn+1$	$\cos\frac{k\pi t}{T+(k-1)}$	–	–	$\cos\frac{k\pi t}{T+k-1}$	$\cos\frac{k\pi t}{T+k-1}$	$\cos\frac{k\pi t}{T+(h-1)+k}$
$T=kn+2$	$\cos\frac{k\pi t}{T+(k-2)}$	–	–	$\cos\frac{k\pi t}{T+(k-2)}$	$\cos\frac{k\pi t}{T+(k-2)+k}$	$\cos\frac{k\pi t}{T+(k-2)+k}$
$T=kn+3$	$\cos\frac{k\pi t}{T+(k-3)}$	–	–	$\cos\frac{k\pi t}{T+(k-3)+k}$	$\cos\frac{k\pi t}{T+(k-3)+k}$	$\cos\frac{k\pi t}{T+(k-3)+k}$
⋮	⋮			⋮	⋮	⋮
$T=kn+(k-2)$	$\cos\frac{k\pi t}{T+2}$	$\cos\frac{k\pi t}{T+2}$	$\cos\frac{k\pi t}{T+2+k}$ –	–	–	$\cos\frac{k\pi t}{T+2+k}$
$T=kn+(k-1)$	$\cos\frac{k\pi t}{T+1}$	$\cos\frac{k\pi t}{T+1+k}$	–	–	$\cos\frac{k\pi t}{T+1+k}$	$\cot\frac{k\pi t}{T+1+k}$

TABLE I. Eigenvalues of the matrix A_k

APPENDIX A

Some Intermediate Results Used in the Derivation of the Distribution of the Serial Correlation

Appendix A.1

One can see from (4.11) that the values of $tr(A_kV)^s$ are needed in order to find the s-th cumulant of $P_k(\bar{y})$. Let $A_k^* = 2A_k$. We introduce the operator $\zeta_{k,T}(\boldsymbol{v}) = A_k^*\boldsymbol{v}$. For example, $\zeta_{k,T}$ applied to the vector $\boldsymbol{w} = (w_1, \ldots, w_T)'$ can be expressed as follows:

$$\zeta_{k,T}(\boldsymbol{w}) = \begin{pmatrix} w_{k+1} \\ w_{k+2} \\ \vdots \\ w_T \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ w_1 \\ w_2 \\ \vdots \\ w_{T-k} \end{pmatrix} .$$

This operator is used for evaluating expressions of the type $tr[A_k^{s-\ell}(A_k\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')^\ell]$.

For $\ell = 0$, $tr[A_k^s] = 0$ for $s = 2i-1$, $i = 1, 2, \ldots$.

For $\ell = s$,

$$\begin{aligned} tr(A_k\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')^s &= tr[A_k\boldsymbol{\varepsilon}(\boldsymbol{\varepsilon}'A_k\boldsymbol{\varepsilon})(\boldsymbol{\varepsilon}'A_k\boldsymbol{\varepsilon})\cdots(\boldsymbol{\varepsilon}'A_k\boldsymbol{\varepsilon})\boldsymbol{\varepsilon}'] \\ &= tr[A_k\boldsymbol{\varepsilon}(T-k)^{s-1}\boldsymbol{\varepsilon}'] = (T-k)^{s-1}tr(\boldsymbol{\varepsilon}'A_k\boldsymbol{\varepsilon}) \\ &= (T-k)^s ; \end{aligned}$$

for $\ell = s-1$,

$$\begin{aligned} tr[A_k(A_k\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')^{s-1}] &= tr[A_kA_k\boldsymbol{\varepsilon}(\boldsymbol{\varepsilon}'A_k\boldsymbol{\varepsilon})\cdots(\boldsymbol{\varepsilon}A_k\boldsymbol{\varepsilon})\boldsymbol{\varepsilon}'] \\ &= (T-k)^{s-2}(\frac{1}{2})^2tr(\boldsymbol{\varepsilon}'A_k^*A_k^*\boldsymbol{\varepsilon}) = (T-k)^{s-2}\boldsymbol{v}'\boldsymbol{v} \end{aligned}$$

where $v = A_k^{*}\epsilon$; for other values of ℓ,

$$
\begin{aligned}
tr[A_k^{s-\ell}(A_k\epsilon\epsilon')^{\ell}] &= tr[A_k^{s-\ell}A_k\epsilon(\epsilon' A_k\epsilon)\cdots(\epsilon' A_k\epsilon)\epsilon] \\
&= (\frac{1}{2})^{s-\ell+1}(T-k)^{\ell-1}tr[\epsilon' A_k^{*s-\ell}A_k^{*}\epsilon] \\
&= (\frac{1}{2})^{s-\ell+1}(T-k)^{\ell-1}tr[\epsilon' A_k^{*}A_k^{*s-\ell-1}A_k^{*}\epsilon] \\
&= (\frac{1}{2})^{s-\ell+1}(T-k)^{\ell-1}v'\zeta_{k,T}^{s-\ell-1}(v)
\end{aligned}
$$

The traces of powers of A_kV are then obtained as follows

$$
\begin{aligned}
tr(A_kV) &= tr[A_k(I-\frac{1}{T}\epsilon\epsilon')] = tr A_k - \frac{1}{T}tr(\varepsilon' A_k\varepsilon) \\
&= -\frac{(T-k)}{T} \; ; \\
tr(A_kV)^2 &= tr(A_k^2) - \frac{2}{T}tr(A_k^2\epsilon\epsilon') + \frac{1}{T^2}tr(A_k\epsilon\epsilon')^2 \\
&= \frac{1}{2}(T-k) - \frac{2}{T}(\frac{1}{2})^2 v'v + \frac{1}{T^2}(T-k)^2 \\
&= \frac{(T-k)(T^2-2k) - 2T < T-2k >}{2T^2}
\end{aligned}
$$

where

$$
v'v = \begin{cases} 2(T-k) & \text{for } T-2k<0 \\ 4T-6k & \text{for } T-2k>0 \end{cases}
$$

and $tr(A_k^2) = \frac{1}{2}(T-k)$;

$$
\begin{aligned}
tr[(A_kV)^3] &= tr(A_k^3) - \frac{3}{T}tr(A_k^2A_k\epsilon\epsilon') \\
&\quad + \frac{3}{T^2}tr[A_k(A_k\epsilon\epsilon')^2] - \frac{1}{T^3}tr[(A_k\epsilon\epsilon')^3] \\
&= \frac{-(T-k)(T^2-2kT+4k^2) - 6kT < T-2k > -3T^2 < T-3k >}{4T^3}
\end{aligned}
$$

since

$$
v'\zeta_{k,T}(v) = \begin{cases} 2(T-k) & \text{for } T-2k<0 \\ 6T-10k & \text{for } T-2k>0,\ T-3k<0 \\ 8T-16k & \text{for } T-3k>0\ ; \end{cases}
$$

$$\begin{aligned} tr[(A_kV)^4] = {} & tr(A_k^4) - \frac{4}{T} tr(A_k^3 A_k ee') \\ & + \frac{4}{T^3} tr[A_k^2 (A_k ee')^2] + \frac{2}{T^2} tr[(A_k ee' A_k)^2] \\ & - \frac{4}{T^3} tr[A_k (A_k ee')^3] + \frac{1}{T^4} tr[(A_k ee')^4] \\ = {} & \frac{1}{8T^4} \{(T-k)(T^4 - 4T^2k + 8Tk^2 - 8k^3) \\ & + < T-2k > (2T^4 - 4T^3 - 16Tk^2) \\ & - < T-3k > 8T^2k - < T-4k > 4T^3 \} \end{aligned}$$

since

$$v'\zeta_{k,T}^2(v) = \begin{cases} 2(T-k) & \text{for } T-2k<0 \\ 10T-18k & \text{for } T-3k<0,\ T-2k>0 \\ 14T-30k & \text{for } T-4k<0,\ T-3k>0 \\ 16T-38k & \text{for } T-4k>0 \end{cases}$$

and $tr(A_k)^4 = \frac{1}{8}(T-k) + \frac{1}{4} < T-2k >$;

$$\begin{aligned} tr[(A_kV)^5] = {} & tr(A_k^5) - \frac{1}{T^5} tr[(A_k ee')^5] - \frac{5}{T} tr(A_k^4 A_k ee') \\ & + \frac{5}{T^2} tr[A_k^3 (A_k ee')^2] + \frac{5}{T^2} tr[A_k^2 A_k ee' A_k A_k ee'] \\ & - \frac{5}{T^3} tr[(A_k)^2 (A_k ee')^3] - \frac{5}{T^3} tr[A_k (A_k ee')^2 A_k A_k ee'] \\ & + \frac{5}{T^4} tr[A_k (A_k ee')^4] \\ = {} & \frac{1}{16T^5} \{(T-k)(-T^4 + 4T^3k - 16T^2k^2 + 24Tk^3 - 16k^4) \\ & - < T-2k > 10Tk(T^2 + 4k^2) \\ & - < T-3k > 10T^2(T^2 + Tk + 2k^2) \\ & - < T-4k > 10T^3k - < T-5k > 5T^4 \} \end{aligned}$$

since

$$v'\zeta_{k,T}^3(v) = \begin{cases} 2(T-k) & \text{for } T-2k<0 \\ 14T-26k & \text{for } T-3k<0,\ T-2k>0 \\ 26T-62k & \text{for } T-4k<0,\ T-3k>0 \\ 30T-78k & \text{for } T-5k<0,\ T-4k>0 \\ 32T-88k & \text{for } T-5k>0\ . \end{cases}$$

Appendix A.2

In this appendix, we derive explicit expressions for the eigenvalues and eigenvectors of the matrix A_k defined in (4.5) for the case $k = 3$.

From the relationship (4.28), we obtain T linear equations which can be solved as a system of second-order difference equations. The following relationships must hold

$$x_4 = 2\lambda x_1 \tag{1}$$

$$x_5 = 2\lambda x_2 \tag{2}$$

$$x_6 = 2\lambda x_3 \tag{3}$$

$$x_{t-3} + x_{t+3} = 2\lambda x_t \tag{4}$$

$$x_{T-5} = 2\lambda x_{T-2} \tag{5}$$

$$x_{T-4} = 2\lambda x_{T-1} \tag{6}$$

$$x_{T-3} = 2\lambda x_T \ . \tag{7}$$

When $t = 3s$, $s = 1, 2, 3, \ldots,$ we solve the following system of equations in order to obtain the eigenvalues:

$$x_6 = 2\lambda x_3$$

$$x_{3s-3} + x_{3s+3} = 2\lambda x_{3s}$$

$$x_{3n-3} = 2\lambda x_{3n} \quad \text{if } T = 3n \qquad \text{(from (7))}$$

$$x_{3n-3} = 2\lambda x_{3n} \quad \text{if } T = 3n+1 \qquad \text{(from (6))}$$

$$x_{3n-3} = 2\lambda x_{3n} \quad \text{if } T = 3n+2\ . \qquad \text{(from (5))}$$

Letting $x_{3s} = y_s$, we have

$$y_2 = 2\lambda y_1 \tag{8}$$

$$y_{s-1} + y_{s+1} = 2\lambda y_s \tag{9}$$

$$y_{n-1} = 2\lambda y_n \ . \tag{10}$$

From equation (9), the characteristic equation is

$$\theta^2 - 2\lambda\theta + 1 = 0$$

and its root θ_1 , and θ_2 are such that $\theta_1 = \frac{1}{\theta_2}$ and $\theta_1 + \theta_2 = 2\lambda$.

The general solution of (9) is given by

$$y_s = c_1\theta^s + c_2\theta^{-s}$$

which applied to (8) yields the following equality

$$c_1\theta^2 + c_2\theta^{-2} = (\theta + \theta^{-1})(c_1\theta + c_2\theta^{-1}) \ .$$

Noting that $c_1 + c_2 = 0$ and letting $c_1 = 1$ and $c_2 = -1$, we have

$$y_s = \theta^s - \theta^{-s} \ .$$

Then in view of (10) we have

$$\theta^{n-1} - \theta^{-(n-1)} = (\theta + \theta^{-1})(\theta^n - \theta^{-n})$$

so that $\theta^{n+1} - \theta^{-(n+1)} = 0$ and $\theta^{2(n+1)} = 1$. The roots of this equation are the $2(n+1)$ roots of 1 which are

$$e^{i2\pi s/[2(n+1)]} \quad , \quad s = 0, 1, 2, \ldots$$

The s -th characteristic root is

$$\lambda = \frac{1}{2}(\theta + \theta^{-1}) = \cos\frac{\pi s}{n+1}$$

and the characteristic roots for $t = 3s$ where $s \leq T/3$ are

$$\lambda_{t,T} = \begin{cases} \cos\frac{3\pi t}{T+3}, & T = 3n \\ \cos\frac{3\pi t}{T+2}, & T = 3n+1 \\ \cos\frac{3\pi t}{T+1}, & T = 3n+2\ . \end{cases} \tag{11}$$

When $t = 3s - 1$, $s = 1, 2, \ldots$, the system of equations can be written as follows:

$$x_5 = 2\lambda x_2$$

$$x_{3s-4} + x_{3s+2} = 2\lambda x_{3s-1}$$

$$x_{3n-4} = 2\lambda x_{3n-1} \text{ if } T = 3n \qquad \text{(from (7))}$$

$$x_{3n-4} = 2\lambda x_{3n-1} \text{ if } T = 3n+1 \qquad \text{(from (6))}$$

$$x_{3n-1} = 2\lambda x_{3n+2} \text{ if } T = 3n+2\ . \qquad \text{(from (5))}$$

Letting $x_{3s-1} = y_s$, we have

$$y_2 = 2\lambda y_1 \tag{12}$$

$$y_{s-1} + y_{s+1} = 2\lambda y_s \tag{13}$$

$$y_{n-1} = 2\lambda y_n \tag{14}$$

$$y_{n-1} = 2\lambda y_n \tag{15}$$

$$y_n = 2\lambda y_{n+1}\ . \tag{16}$$

For $T = 3n$ and $T = 3n+1$, $\lambda = \cos\frac{\pi s}{n+1}$ as before and (16) yields $\theta^{2(n+2)} = 1$ and $\lambda = \cos\frac{\pi s}{n+2}$. The characteristic roots for the case $t = 3s - 1$ are

$$\lambda_{t,T} = \begin{cases} \cos\frac{3\pi t}{T+3} & \text{if } T = 3n \\ \cos\frac{3\pi t}{T+2} & \text{if } T = 3n+1 \\ \cos\frac{3\pi t}{T+4} & \text{if } T = 3n+2 \end{cases} \tag{17}$$

provided $s \le \frac{T+1}{3}$. When $t = 3s - 2$, the system of equation is

$$x_4 = 2\lambda x_1$$

$$x_{3s-5} + x_{3s+1} = 2\lambda x_{3s-2}$$

$$x_{3n-5} = 2\lambda x_{3n-2} \quad \text{if } T = 3n \qquad \text{(from (5))}$$

$$x_{3n-2} = 2\lambda x_{3n+1}$$

$$x_{3n-2} = 2\lambda x_{3n+1} \; .$$

Letting $x_{3s-2} = y_s$, we have

$$y_2 = 2\lambda y_1 \tag{18}$$

$$y_{s-1} + y_{s+1} = 2\lambda y_s \tag{19}$$

$$y_{n-1} = 2\lambda y_n \tag{20}$$

$$y_n = 2\lambda y_{n+1} \tag{21}$$

$$y_n = 2\lambda y_{n+1} \; . \tag{22}$$

The eigenvalues for $t = 3s - 2$ are

$$\lambda_{t,T} = \begin{cases} \cos\frac{3\pi t}{T+3} & \text{if } T = 3n \\ \cos\frac{3\pi t}{T+5} & \text{if } T = 3n+1 \\ \cos\frac{3\pi t}{T+4} & \text{if } T = 3n+2 \end{cases} \tag{23}$$

provided $s \le \frac{T+2}{3}$.

Denoting the argument of the cosine function representing the eigenvalue as $\alpha_{t,T}$, we can write the corresponding eigenvector as

$$x_{t,T} = (x_j)$$

where

$$x_j = \begin{cases} \sin u\alpha_{t,T} & t = 3s - a, \; j = a - 2 + 3m, \; m = 1, 2, \ldots, [\frac{T}{3}] \\ 0 & \text{otherwise.} \end{cases}$$

For example, when $T = 3n$,

$$
x_{t,T} = \begin{cases} (0,0,\ \sin\ \alpha_{t,T},\ 0,0,\ \sin 2\ \alpha_{t,T},\ 0,0,\ \ldots \sin\ [\frac{T}{3}]\ \alpha_{t,T}) & \text{for } t = 3s-2 \\ (0,\ \sin\ \alpha_{t,T},\ 0,0,\ \sin\ 2\ \alpha_{t,T},\ 0,0,\ \ldots, \sin\ [\frac{T}{3}]\ \alpha_{t,T},\ 0) & \text{for } t = 3s-1 \\ (\sin\ \alpha_{t,T},\ 0,0,\ \sin\ 2\ \alpha_{t,T},\ 0,0,\ \ldots, \sin\ [\frac{T}{3}]\ \alpha_{t,T}, 0, 0) & \text{for } t = 3s\ . \end{cases} \tag{24}
$$

The t-th nonzero component of the eigenvector for corresponding the eigenvalue $\lambda_{t,T} = \cos \alpha_{t,T}$ is taken as

$$
\frac{1}{2i}\ x_t = \frac{1}{2i}(e^{it\alpha_{t,T}} - e^{-it\alpha_{t,T}}) = \sin\ t\ \alpha_{t,T}\ . \tag{25}
$$

For other values of k, one can similarly obtain explicit expressions for the eigenvalues of A_k and the corresponding eigenvectors in terms of trigonometric functions. These general results are presented in Section 3.

Appendix A.3

We show that the VA_kV and A_kV have the same eigenvalues. Let λ be an eigenvalue of A_kV, then

$$
A_k V x = \lambda V x\ ,
$$

and on multiplying both sides by V, we have

$$
V A_k V x = \lambda V x\ ,
$$

i.e.,

$$
V A_k V (V x) = \lambda (V x)
$$

which shows that λ is also the eigenvalue of VA_kV.

Now let θ be an eigenvalue of VA_kV Then

$$
V A_k V v = \theta v\ ,
$$

and on multiplying both sides by the idempotent matrix V, we have

$$VVA_kV\boldsymbol{v} = \theta V\boldsymbol{v} \ ,$$

i.e.,

$$VA_k(V\boldsymbol{v}) = \theta(V\boldsymbol{v}) \ ,$$

which shows that θ is also eigenvalue of VA_k.

Hence it suffices to find the eigenvalues of A_kV when those of VA_kV are required.

APPENDIX B

A Review on the Statistical Applications of the Mellin Transform

Introduction

This review deals with various statistical applications of the Mellin Transform and its inverse. More particularly the exact distributions of various test statistics will be given and the results will be expressed in terms of the G-function or the H-function. The H-function is also used in a number of problems arising in physical sciences, engineering and statistics. The importance of this function lies in the fact that nearly all the special functions occurring in applied mathematics and statistics are its special cases.

The theory as well as many applications of of H-functions and Mellin transform and its inverse are available in Mathai and Saxena (1973, 1978). Many applications are given in Rathie (1989). The Mellin transform and the inverse Mellin transform are defined in Chapter 1 where the H-function is expressed as an inverse Mellin transform. If the existence conditions are satisfied and the parameter values are such that the H-function remains positive for the whole domain, then the probability density functions can be created by normalizing the H-function. Series representation of the H-function are also given in Chapter 1.

The following problems are discussed.

1. Testing the hypothesis that the covariance matrix is diagonal.
2. Testing equality of covariance matrices.
3. Testing compound symmetry.
4. Sphericity test.
5. The density function of L.R.T. for testing sphericity structure for a complex normal covariance matrix.
6. The distribution of the determinant of a random beta matrix.
7. Neyman and Pearson criterion.
8. The distribution of L.R.T. for testing equality of two parameter exponential distribution.
9. The distribution of L.R.T. for testing equality of covariance matrices under intraclass correlation model.
10. The exact density of the statistic $\ln(\tilde{X}/\bar{X})$ related to the shape parameter of a Gamma variate.
11. The exact density of the ratio of a linear combination of chi-square variables over the root of a product of chi-square variables.
12. The distribution of a test statistic connected with the multivariate linear hypothesis.
13. The distribution of Pearson and Wilk's likelihood test statistics
14. The density function of Wilks' criterion for testing the multivariate linear hypothesis.
15. Wilks' L_{vc} criterion.
16. Testing equality of diagonal elements.
17. $L_1(vc)$ of Votaw.

18. $\bar{L}_1(vc)$ of Votaw.

Our aim is to include the main applications; hence the previous list should not be construed as being exhaustive.

B.1 Testing the hypothesis that the covariance matrix is diagonal

This problem is discussed in Anderson (1984), Mathai (1972b), Mathai and Katiyar (1979) and Mathai and Tan (1977). Consider a p-variate normal distribution with the parameters μ and Σ. The likelihood ratio criterion λ for testing the hypothesis that Σ is a diagonal matrix is

$$U = \lambda^{2/N} = |S| / \prod_{i=1}^{p} s_{ii} \qquad (B.1.1)$$

where N is the samp'e size; $|S|$ the determinant of $S = (s_{ij})$, and S is the corrected sum of products matrix,

$$s_{ij} = \sum_k (x_{ik} - \bar{x}_i)(x_{jk} - \bar{x}_j); \quad \bar{x}_i = \sum_k x_{ik}/N , \qquad (B.1.2)$$

where $X_i \sim N_p(\mu, \Sigma)$, $X_i' = (x_{i1}, \ldots, x_{ip})$. The $h-th$ null moment of U is:

$$E(U^h|H_0) = \left\{ \prod_{i=1}^{p} \frac{\Gamma(h + \frac{1}{2}(n-i+1))}{\Gamma(\frac{1}{2}(n-i+1))} \right\} \left\{ \frac{\Gamma(\frac{1}{2}n)}{\Gamma(h+\frac{1}{2}n)} \right\}^p ; n \geq p \qquad (B.1.3)$$

The inverse Mellin transform of the moment expression in (B.1.3) yields $f(u)$ the density of U

$$f(u) = Cu^{-1} \frac{1}{2\pi i} \int_L u^{-h} \left\{ \prod_{i=1}^{p} \frac{\Gamma(h + \frac{1}{2}(n-i+1))}{\Gamma(h+\frac{1}{2}n)} \right\} dh \qquad (B.1.4)$$

where L is a suitable contour and

$$C = \Gamma^p(\frac{1}{2}n) / \prod_{k=1}^{p} \Gamma(\frac{1}{2}n - k + 1) .$$

In terms of G-function the density is

$$f(u) = Cu^{-1}G^{p,0}_{p,p}\left\{u\Big|\begin{matrix}\frac{1}{2}n, \frac{1}{2}n, \cdots, \frac{1}{2}n \\ \frac{1}{2}(n-i+1),\ i=1,\ldots,p\end{matrix}\right\} . \tag{B.1.5}$$

The h-th non-null moment of $W = \lambda^{2/n}$ obtained by Mathai and Tan (1977) is

$$E(W^h) = [\prod_{j=1}^{p}\Gamma(\frac{n-j}{2})]^{-1}\sum_{m=0}\frac{1}{m!}\sum_{a_1=0}^{m}\cdots\sum_{a_p=0}^{m}\sum_{j_1=0}^{a_1}\cdots\sum_{j_p=0}^{a_p} \tag{B.1.6}$$

$$\ell(a_1,\ldots,\ a_p;j_1,\ldots,j_p)\times b^p\cdot\frac{\Gamma(\frac{n-1}{2}+r+m)}{\Gamma(\frac{n-1}{2}+r)}\times\prod_{s=1}^{p}\frac{\Gamma(\frac{n-s}{2}+r)}{\Gamma(\frac{n-1}{2}+r+a_s-j_s)} ,$$

where $b = |\Sigma|/\prod_{i=1}^{p}a_{ii}$ and

$$\ell(a_1,\ldots,a_p;j_1,\ldots,j_p) = (-1)^{j_1+\cdots+j_p}\binom{a_1}{j_1}\cdots\binom{a_p}{j_p}A^{(m)}_{\underline{a}}\prod_{s=1}^{p}\Gamma(a_s+\frac{n-1}{2}-j_s)$$

and $A^{(m)}_{\underline{a}}$ is the coefficient of $x_1^{a_1}x_2^{a_2}\cdots x_p^{a_p}$ for the joint distribution

$$f(x_1,\ldots,x_p) = \sum_{m=0}^{\infty}\frac{\Gamma(\frac{p}{2}+m)}{\Gamma(\frac{p}{2})m!}\sum_{a_1=0}^{m}\cdots\sum_{a_p=0}^{m}A^{(m)}_{\underline{a}}f_a(x_1,\ldots,x_p;\frac{p}{2},\ldots,\frac{p}{2}) .$$

With the help of the inverse Mellin transformation the non-null density $f(y)$ is found to be

$$f(y) = A\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{a_1=0}^{m}\cdots\sum_{a_p=0}^{m}\sum_{j_1=0}^{a_1}\cdots\sum_{j_p=0}^{a_p}\ell(a_1,\ldots,a_p;j_1,\ldots,j_p)L(y:m,\underline{j},\underline{a}) ,$$

where

$$\begin{aligned} L(y;m,\underline{j},\underline{a}) &= \frac{1}{2\pi i}\int_L\frac{\Gamma(\frac{n-1}{2}+r+m)\prod_{n=1}^{p}\Gamma(\frac{n-p}{2}+r)}{\Gamma(\frac{n-1}{2}+r)\prod_{s=1}^{p}\Gamma(\frac{n-1}{2}+r+a_s-j_s)}y^{-r}dr ,\\ &= G^{p+1,0}_{p+1,p+1}\left\{y\Big|\begin{matrix}\frac{n-1}{2}+a_i-j_s,\ i=1,2,\ldots,p,\ \frac{n-1}{2}\\ \frac{n-1}{2}, i=1,\ldots,p,\ \frac{n-1}{2}+m\end{matrix}\right\}\end{aligned} \tag{B.1.7}$$

$i=\sqrt{-1}$, and L is a suitable contour.

B.2 Testing equality of covariance matrices

This problem is discussed in Pillai, Al-Ani and Jouris (1969), Mathai and Rathie (1980), Anderson (1984) and Pederzoli and Rathie (1983). Let $X_{\alpha}^{(i)}$, $(a = 1, \ldots, N;\ i = 1, \ldots, q)$ be an observation from the population $N_p(\mu^{(i)}, \Sigma_i)$. For testing the hypothesis

$$H_1 : \Sigma_1 = \cdots = \Sigma_q \tag{B.2.1}$$

the likelihood ratio criterion used by Pederzoli and Rathie (1983) can be written as

$$\lambda = \left\{N^{\frac{1}{2}pN} / \prod_{i=1}^{q} N_i^{\frac{1}{2}pN_i}\right\}\left\{\prod_{i=1}^{q} |A_i|^{N_i/2}/|A|^{N/2}\right\}, \tag{B.2.2}$$

where $N = \sum_{i=1}^{q} N_i$; $A_i = \sum_{a=1}^{N_i}(x_a^{(i)} - \bar{x}^{(i)})(x_a^{(i)} - \bar{x}^{(i)})'$ $i = 1, 2, \ldots, q$, and $A = \sum_{i=1}^{q} A_i$. Bartlett (1937) considered the following test statistic

$$V_1 = \prod_{i=1}^{q} |A_i|^{n_i/2}/|A|^{n'/2} \tag{B.2.3}$$

where $n_i = N_i - 1$, $n' = \sum_{i=1}^{q} n_i = N - q$. The $h-th$ moment of V_1 under H_1 is

$$E(V_1^h) = c\prod_{i=1}^{p} \frac{\Gamma((hn + n + 1 - i)/2)}{\Gamma((nq + nqh + 1 - i)/2)} \tag{B.2.4}$$

where

$$c = \prod_{i=1}^{p} \frac{\Gamma((nq + 1 - i)/2)}{\Gamma^q((n + 1 - i)/2)} \text{ and } 0 < V_1 < q^{-pqn/2} .$$

By using the Gauss Legendre multiplication formulae, i.e.

$$\prod_{i=1}^{n} \Gamma(z + (i - 1)/n) = (2\pi)^{\frac{1}{2}(n-1)} n^{\frac{1}{2} - nz}\Gamma(nz) , \tag{B.2.5}$$

the $h-th$ moment of $U = V_1^{2/n}$, is given by

$$E(U^h) = C_{p,q}\prod_{i=1}^{p} \frac{\Gamma^q[(h + (n + 1 - i)/2]q^{-qh}}{\prod_{j=0}^{q-1} \Gamma[h + \frac{n}{2} + (1 - i + 2j)/2q]},\ 0 < u < q^{-pq} , \tag{B.2.6}$$

where

$$C_{p,q} = 2\pi^{-p(1-q)/2} q^{-2pq/2+p(p+1)/4} \prod_{i=1}^{p} \frac{\Gamma[(nq+1-i)/2]}{\Gamma^{q}[(n+1-i)/2]} .$$

Using the inverse Mellin transform, the density function $f(u)$ of U in terms of Meijers' G-function is:

$$f(u) = C_{pq} u^{-1} G^{pq,0}_{pq,pq}\left\{ uq^{pq} \Big| \begin{matrix} \frac{n}{2} + (1-i+2j)/2q, & {}^{i=1,2,\ldots,p,}_{j=0,\ldots,q-1} \\ \theta_1, \theta_2, \ldots, \theta_q & \end{matrix} \right\} . \qquad (B.2.7)$$

where $\theta_k = n/2, (n-1)/2, \ldots, (n-p+1)/2$; $k = 1, \ldots, q$.

B.3 Testing compound symmetry

In studying psychological examinations, or other measuring devices, one may wish to test whether several forms of an examination may be used interchangeably not only with each other but also with regard to their relation to some outside criterion measure (e.g. the criterion might be skill in a given task). The hypothesis of interchangeability is equivalent to the hypothesis of equality of means, equality of variances, and equality of covariances among these forms and equality of covariances between forms and criterion. A normal distribution for which this hypothesis is true is said to have compound symmetry.

This problem was considered by Votaw (1948) and Consul (1969).

Let $X_1, X_2, \ldots, X_N$ be random sample of $(p \times q)$-component vectors drawn from a $(p \times q)$ variate normal distribution $N(\mu, \Sigma)$. If the covariance matrix Σ is partitioned into four matrices, then the hypothesis H that Σ is of the bipolar form

$$\begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{bmatrix} \qquad (B.3.1)$$

where Σ_1 is a $p \times p$ matrix with all diagonal elements equal to σ_{aa}, and other elements equal to $\sigma_{aa'}$, Σ_3 is a $q \times q$ matrix with all diagonal elements equal to

σ_{bb} and other elements equal to $\sigma_{bb'}$, and Σ_2 is a $p \times q$ matrix whose elements are equal to σ_{ab}, is a hypothesis about compound symmetry. Consul (1969) using the Gauss Legendre multiplication theorem (B.2.5) expressed the h-th moment of the likelihood ratio criterion L^* defined by Votaw (1948) as a

$$E(L^*)^h = k_1(N)\frac{\prod_{j=1}^{p+q-2}\Gamma(h+\frac{1}{2}(N-2)-\frac{1}{2}j)}{\prod_{j=1}^{p-1}\Gamma(h+\frac{1}{2}(N-1)+\frac{j-1}{p-1})\prod_{j=1}^{q-1}\Gamma(h+\frac{1}{2}(N-1)+\frac{j-1}{q-1})}, \tag{B.3.2}$$

where

$$k_1(N) = \frac{\prod_{j=1}^{p-1}\Gamma(\frac{1}{2}(N-1)+\frac{j-1}{p-1})\prod_{j=1}^{q-1}\Gamma(\frac{1}{2}(N-1)+\frac{j-1}{q-1})}{\prod_{j=1}^{p+q-2}\Gamma(\frac{1}{2}(N-2)-\frac{1}{2}j)}.$$

From this result Consul (1969) found the exact p.d.f. of the criterion L^* as an inverse Mellin transform:

$$f(L^*) = k_1(N)(L^*)^{\frac{1}{2}(N-p-q-2)}G_{\lambda,\lambda}^{\lambda,0}\left\{L^*\Big|_{0,\frac{1}{2},1,\ldots,\frac{1}{2}(p+q-3)}^{a_r,a_s}\right\}, \tag{B.3.3}$$

where

$$\lambda = p+q-2,$$
$$a_r = \frac{1}{2}(p+q-1),\ \frac{1}{2}(p+q-1)+\frac{1}{p-1},\ldots,\frac{1}{2}(p+q-1)+\frac{p-2}{p-1},$$
$$a_s = \frac{1}{2}(p+q-1),\frac{1}{2}(p+q-1)+\frac{1}{q-1},\ldots,\frac{1}{2}(p+q-1)+\frac{q-2}{q-1}.$$

B.4 Sphericity test

This problem was considered by Consul (1969), Bagai (1972), and Mathai and Saxena (1973, p.204).

Consider a random sample $X_1, X_2, \ldots, X_N$ of p-component vectors drawn from the multivariate normal distribution $N(\mu, \Sigma)$. The hypothesis that the covariance matrix Σ is of the form $\sigma^2 I$, where σ^2 is an unknown scalar, and I is an identity matrix, is known as the sphericity test. This test is a combination of the tests for testing the hypothesis that the covariance matrix is diagonal and testing equality of the diagonal elements. The hypothesis $H_0 : \Sigma = \sigma^2 I$ can be put in the form that the arithmetic mean of roots $\phi_1, \ldots, \phi_p$ of $|\Sigma - \phi I| = 0$, which are the characteristic roots of Σ, is equal to their geometric mean, i.e.

$$\prod_{i=1}^{p} \phi_i^{\frac{1}{p}} / \left\{ (\sum_{i=1}^{p} \phi_i)/p \right\} = |\Sigma|^{\frac{1}{p}} / \left\{ (tr\Sigma)/p \right\} = 1 . \qquad (B.4.1)$$

As the squares of the lengths of principal axes of ellipsoids determined by setting the density equal to a constants are proportional to the roots ϕ_i, the ellipsoids become spheres under H_0.

Consul (1969) used the criterion W for testing this sphericity in a p-variate normal distribution as

$$W = |S| / \{(trS)/p\}^p, \qquad (B.4.2)$$

where S is the maximum likelihood estimate matrix of Σ, and obtained the h-moment of the sphericity criterion:

$$E(W^h) = (p^p)^h \frac{\Gamma[\frac{1}{2}p(N-1)]}{\Gamma[\frac{1}{2}p(N-1)+ph]} \prod_{j=1}^{p} \frac{\Gamma[\frac{1}{2}(N-j)+h]}{\Gamma[\frac{1}{2}(N-j)]}. \qquad (B.4.3)$$

Using the Gauss Legendre multiplication formula (B.2.5), Consul expressed the $h-th$ moment in the form:

$$E(W^h) = K(N) \prod_{j=1}^{p} \frac{\Gamma[\frac{1}{2}(N-j)+h]}{\Gamma[\frac{1}{2}(N-1)+(j-1)/p+h]}, \qquad (B.4.4)$$

where

$$K(N) = (2\pi)^{\frac{1}{2}(p-1)} p^{\frac{1}{2}-\frac{1}{2}p(N-1)} \frac{\Gamma[\frac{1}{2}p(N-1)]}{\prod_{j=1}^{p} \Gamma[\frac{1}{2}(N-j)]} .$$

Upon inversion the exact p.d.f for the criterion is

$$f(w) = K(N) w^{(N-p-2)/2} G_{p,p}^{p,0} \left\{ W \middle| \begin{matrix} \frac{N-1}{2}, \frac{N-1}{2}+\frac{1}{p}, \ldots, \frac{N-1}{2}+\frac{p-1}{p} \\ \frac{N-1}{2}, \frac{N-1}{2}, \ldots, \frac{N-p}{2} \end{matrix} \right\} . \qquad (B.4.5)$$

B.5 The density function of L.R.T. for testing sphericity structure for a complex normal covariance matrix.

Nagarsenker and Nagarsenker (1981) among others considered this problem.

Let the joint density function of the complex random matrices $X : p \times N$ and $S : p \times p$ be

$$CN(X; \mu, \Sigma)\ CW(S; p,\ n, \Sigma), \tag{B.5.1}$$

where $CN(X; \mu, \Sigma)$ is the complex multivariate normal distribution whose probability density function is

$$CN(X; \mu, \Sigma) = (\pi)^{-pN} |\Sigma|^{-N} \exp[-tr\Sigma^{-1}(X-\mu)(\overline{X-\mu})'] \ , \tag{B.5.2}$$

$CW(S; p, n, \Sigma)$ is the complex Wishart distribution whose p.d.f. is

$$CW(S; p, n, \Sigma) = \{\tilde{\Gamma}_p(n)\}^{-1} |S|^{n-p} |\Sigma|^{-n} \exp(-tr\Sigma^{-1}S) \ , \tag{B.5.3}$$

where $\overline{X} = (\overline{x}_{ij})$ and $\overline{x}_{ij}$ is the complex conjugate of x_{ij} with $X = (x_{ij})$; $n = N-1$; $\tilde{\Gamma}_p(n) = \pi^{p(p-1)/2} \prod_{j=1}^{p} \Gamma(n-i+1)$. Σ and S are Hermitian positive definite matrices of order p and μ: $p \times N$ is a complex matrix. The paper considers the testing of a sphericity structure for the Hermitian covariance matrix Σ, i.e. the hypothesis that all off-diagonal elements of Σ are zero and the diagonal elements are equal in sets. Consider the null hypothesis:

$$H_0 : \Sigma = \begin{bmatrix} \sigma_1^2 I_{p_1} & 0 & \cdots & 0 \\ 0 & \sigma_2^2 I_{p_2} & & \vdots \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \sigma_q^2 I_{p_q} \end{bmatrix} ;\ \sigma_i^2 > 0 \text{ unknown} \tag{B.5.4}$$

against the alternative, where Ip_1 is the identity matrix of order p_i and $p_1 + \cdots + p_q = p$ Let

$$S = \begin{bmatrix} S_{11}, & S_{12} & \cdots & S_{1q} \\ S_{21}, & S_{22} & & S_{2q} \\ & \vdots & & \vdots \\ S_{q1}, & S_{q2} & \cdots & S_{qq} \end{bmatrix}$$

be partitioned similarly to (B.5.4). The likelihood ratio criterion for testing H_0 is based on the statistic:

$$\lambda = |S|^{\frac{N}{2}} / \left\{ \prod_{i=1}^{q} [(trS_{ii}/p_i)^{p_i}]^{N/2} \right\} .$$

The h-th moment of $W = \lambda^{2/N}$ is given by:

$$E(W^h) = \prod_{i=1}^{q} p_i^{p_i h} \frac{\Gamma(np_i)}{\Gamma[(n+h)p_i]} \prod_{a=1}^{p} \frac{\Gamma(n+1-a+h)}{\Gamma(n+1-a)} . \qquad (B.5.7)$$

Using the inverse Mellin transform the density of W is given

$$f(w) = \frac{\prod_{i=1}^{q} \Gamma(np_i)}{\prod_{a=1}^{p} \Gamma(N-a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} w^{-h-1} \frac{\prod_{a=1}^{p} \Gamma(N-a+h)}{\prod_{i=1}^{q} \Gamma(np_i+hp_i)} \prod_{i=1}^{q} p_i^{hp_i} dh . \quad (B.5.8)$$

Letting $N - \lambda + h = s$ in (B.5.8), we have

$$f(w) = \frac{\prod_{i=1}^{q} \Gamma(np_i) p_i^{-(N-\lambda)p_i}}{\prod_{a=i}^{p} \Gamma(N-a)} \frac{1}{2\prod i} \int_{c-i\infty}^{c+i\infty} \left(\frac{w}{\prod_{i=1}^{q} p_i} p_i \right)^{-s} \frac{\prod_{a=1}^{p} \Gamma(s+\lambda-a)}{\prod_{i=1}^{q} \Gamma[p_i(s+\lambda-1)]} ds . \qquad (B.5.9)$$

Using the definition of the H-function as an inverse Mellin transform, the density of W is

$$f(w) = KH_{q,p}^{p,0} \left\{ \frac{w}{\prod_{i=1}^{q} p_i^{p_i}} \middle| \begin{matrix} (p_i(\lambda-1), p_i);\ i = 1, 2, \ldots, q \\ (\lambda - a, 1),\ a = 1, \ldots, p \end{matrix} \right\} \qquad (B.5.10)$$

where

$$K = \frac{\prod_{i=1}^{q} \{\Gamma(np_i) p_i^{-(N-\lambda)p_i}\}}{\prod_{a=1}^{p} \Gamma(N-a)} .$$

B.6 The distribution of the determinant of a random beta matrix

This problem is studied by Mathai and Tracy (1986). Consider a random Hermitian positive definite $p \times p$ matrix $A = (a_{ij})$, where the a_{ij} 's are scalar complex random variables such that the eigenvalues of A are in the open interval (0,1). The matrix A is said to have a complex multivariate beta distribution i.e. the joint distribution of the a_{ij} 's is complex multivariate beta, if its density is given by

$$f(A) = \begin{cases} \frac{\tilde{\Gamma}_p(\alpha+\beta)}{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)}|A|^{\alpha-p}|I-A|^{\beta-p}; & 0 < A < I \\ 0 & \text{elsewhere} \end{cases} \tag{B.6.1}$$

for $\text{Re}(\alpha) > p$, $\text{Re}(\beta) > p$, where

$$\tilde{\Gamma}_p(\alpha) = \pi^{p(p-1)/2}\Gamma(\alpha)\Gamma(\alpha-1)\cdots\Gamma(\alpha-p+1), \text{ and } 0 < A < I$$

means that A as well as $I - A$ are positive definite. Let $\lambda_1, \ldots, \lambda_p$ denote the eigenvalues of A. The quantities $tr(A) = \lambda_1 + \cdots + \lambda_p$ and $|A| = \lambda_1 \cdots \lambda_p$ appear as test statistics in a number of hypothesis testing problems in multivariate statistical analysis. The exact density of $U = |A|$ is derived in the article. The method used is that of the Mellin and inverse Mellin transform.

$$\begin{aligned} M_U(s) = E(U^{s-1}) = E|A|^{s-1} &= \int_{0<A<1} |A|^{s-1} f(A)dA \\ &= \frac{\tilde{\Gamma}_p(\alpha+s-1)}{\tilde{\Gamma}_p(\alpha)} \frac{\tilde{\Gamma}_p(\alpha+\beta)}{\tilde{\Gamma}_p(\alpha+\beta+s-1)} = c\prod_{j=1}^{p} \frac{\Gamma(\alpha+s-j)}{\Gamma(\alpha+\beta+s-j)}, \end{aligned} \tag{B.6.2}$$

for $\text{Re}(\alpha + s - 1) > p$, $\text{Re}(\beta) > p$ and

$$c = \prod_{j=1}^{p} \frac{\Gamma(\alpha+\beta-j+1)}{\Gamma(\alpha-j+1)} .$$

The density $f(u)$ of U in (B.6.2) is available from the inverse Mellin transform:

$$f(u) = \begin{cases} \frac{c}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{p} \frac{\Gamma(\alpha+s-j)}{\Gamma(\alpha+\beta+s-j)} u^{-s} ds, & 0 < u < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{B.6.3}$$

where $c > p - \Re(\alpha)$ and, in a view of (1.15), (1.17) and (1.25).

$$f(u) = \begin{cases} c\ G^{p,0}_{p,p}\left\{u \middle| \begin{matrix} \alpha+\beta-j, & \\ \alpha-j, & j=1,\ldots,p \end{matrix}\right\}, & 0 < u < 1 \\ 0 & \text{elsewhere} \end{cases} \qquad (B.6.4)$$

In the p-variate complex Gaussian case the likelihood ratio for the general linear hypothesis is

$$\lambda = |A_1(A_1 + A_2)^{-1}| , \qquad (B.6.5)$$

where A_1, A_2 are independent $p \times p$ complex Wishart matrices $W_p(\Sigma, n)$, $W_p(\Sigma, q)$ respectively with n, q positive integers $\geq p$. A complex Wishart matrix B of order p with degrees of freedom ν and parameter matrix $\Sigma > 0$ is a positive definite random Hermitian matrix with density function:

$$g(B) = \begin{cases} \dfrac{|B|^{\nu-p} e^{-tr(\Sigma^{-1}B)}}{|\Sigma|^{\nu}\tilde{\Gamma}_p(\nu)}, & B > 0 \\ 0 & \text{elsewhere} \end{cases}, \qquad (B.6.6)$$

and we write $B \sim W_p(\Sigma, \nu)$. The h-th moment of λ is

$$E(\lambda^h) = C \prod_{j=1}^{p} \frac{\Gamma(n+1+h-j)}{\Gamma(n+1+q+h-j)} , \qquad (B.6.7)$$

where

$$C = \prod_{j=1}^{p} \frac{\Gamma(n+1+q-j)}{\Gamma(n+1-j)} .$$

The expression (B.6.7) is the Mellin transform (B.6.2) with $\alpha = n$, $\beta = q$, $h = s-1$. Again using the inverse Mellin transform we obtain the density of λ

$$f(\lambda) = \begin{cases} \dfrac{C}{2\pi i} \displaystyle\int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{p} \frac{\Gamma(n+1+h-j)}{\Gamma(n+1+q+h-j)} \lambda^{-h} dh , & 0 < \lambda < 1 \\ 0 & \text{elsewhere} , \end{cases} \qquad (B.6.8)$$

which can be expressed in terms of the G-function as follows:

$$f(\lambda) = \begin{cases} C\ G^{p,0}_{p,p}\left\{\lambda \middle| \begin{matrix} n+1+q-j , & j=1,\ldots,p \\ n+1-j, & j=1,\ldots,p \end{matrix}\right\}, & 0 < \lambda < 1 \\ 0 & \text{elsewhere} . \end{cases} \qquad (B.6.9)$$

B.7 Neyman and Pearson criterion

This problem was studied by Gupta and Rathie (1982), Chao and Glaser (1978), and Nandi (1980) and Nandi (1981).

Consider k independent random samples $X_{ij}, i = 1, \ldots, k,\ j = 1, \ldots, n$ of equal size n $(n \geq 2)$ be taken from k normal populations $N(\mu_i, \sigma_i^2)$.

Neyman and Pearson (1931, p.117–118) have shown that for testing the null hypothesis

$$H_0 : \sigma_1 = \cdots = \sigma_k \tag{B.7.1}$$

the likelihood ratio criterion is

$$\Lambda = \prod_{i=1}^{k} \left(\frac{s_i^2}{s_a^2} \right)^{n/2} \tag{B.7.2}$$

where

$$s_i^2 = \sum_{j=1}^{n} \sum_{j=1}^{n} (X_{ij} - X_i)^2/n, \quad \bar{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij} ,$$

$$s_a^2 = \frac{1}{k} \sum_{i=1}^{k} s_i^2 .$$

The h-th moment of Λ is

$$E(\Lambda^h) = \frac{k^{hkn/2} \Gamma[k(n-1)/2] \Gamma^k[hn/2 + (n-1)/2]}{\Gamma^k[(n-1)/2] \Gamma[hkn/2 + k(n-1)/2]} . \tag{B.7.3}$$

The exact density of Λ is given by Gupta and Rathie (1982) in terms of Meijer's G-function and in a form suitable for the computation of the percentage points:

$$f(\lambda) = c\lambda^{-1/n} G^{k-1,0}_{k-1,k-1} \left\{ \lambda^{2/n} \Big| \begin{matrix} \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k} \\ 0, 0, \ldots, 0 \end{matrix} \right\},\ 0 < \lambda < 1 , \tag{B.7.4}$$

where

$$c = \frac{2k^{k(1-n)/2+1/2} \Gamma[k(n-1)/2]}{n(2\pi)^{(1-k)/2} \Gamma^k[(n-1)/2]} .$$

They gave the density function for the special cases $k = 2$ and 3 respectively

$$f(\lambda) = \frac{2\Gamma(\frac{n}{2})n^{-1}}{\pi^{1/2}\Gamma[(n-1)/2]}\lambda^{-1/n}(1-\lambda^{2/n})^{-1/2}, \ 0 < \lambda < 1 \ , \text{ for } k = 2 \ , \qquad (B.7.5)$$

and, for $k = 3$,

$$f(\lambda) = \frac{2\Gamma[(n-1)/2+1/3]\Gamma[(n-1)/2+2/3]}{n\Gamma^2[(n-1)/2]}\lambda^{-\frac{1}{n}} \ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\lambda^{2/n}\right) \ ; \qquad (B.7.6)$$

using the following representation of ${}_2F_1(a_1 b; c; 1-z)$,

$${}_2F_1(a,b;c;1-z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-1)\Gamma(b)} G_{22}^{22}\left\{z\Big|\begin{matrix}1-a, 1-b\\ 0, c-a-b\end{matrix}\right\} , \qquad (B.7.7)$$

and the density function (B.17.6) has the following form:

$$f(\lambda) = \frac{2\Gamma(\frac{n-1}{2}+\frac{1}{3})\Gamma(\frac{n-1}{2}+\frac{2}{3})}{n\Gamma^2(\frac{n-1}{2})\Gamma^2(\frac{1}{3})\Gamma^2(\frac{2}{3})}\lambda^{-1/n} G_{22}^{22}\left\{\lambda^{2/n}\Big|\begin{matrix}\frac{2}{3}, \ \frac{1}{3}\\ 0, \ 0\end{matrix}\right\} . \qquad (B.7.8)$$

B.8 The distribution of L.R.T. for testing the equality of two-parameter exponential distributions

Suppose that p samples have been drawn from a two parameter exponential distribution with probability density function

$$f(X,\theta_i,A_i) = \begin{cases} \theta_i^{-1}e^{-(X-A_i)/\theta_i}, & x > A_i, \ \theta_i > c \ , \ i = 1,2,\ldots,p \\ 0 \ , & \text{otherwise} \end{cases} \qquad (B.8.1)$$

and the i-th sample contains n_i observations X_{ij} with mean $\bar{X}_i$ $(i = 1,2,\ldots,p, \ j = 1,2,\ldots,n_i)$. Testing the hypothesis H_0 that the p samples have been randomly drawn from the same population is equivalent to testing that the p exponential populations in (B.8.1) are identical, i.e.

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_p \ \text{ and } \ A_1 = A_2 = \cdots = A_p \ . \qquad (B.8.2)$$

The likelihood ratio criterion used by B. Nagarsenker and P.B. Nagarsenker (1982) for testing the hypothesis H_0 has the following form:

$$\lambda = [\prod_{i=1}^{p}(\bar{X}_i - X_{(1)i})^{n_i}]/(\bar{X}_0 - X_{(1)})^n , \tag{B.8.3}$$

where $n = n_1 + \cdots + n_p$, $X_{(1)i}$ denotes the lowest observation in the i-th sample while $X_{(1)}$ denotes the smallest of the p lowest observations $X_{(1)i}$, and $\bar{X}_0 = \sum_{i=1}^{p} \bar{X}_i/p$.

Let $W = \lambda^{1/n}$. Then the h-th moment of W is given by

$$E(W^h) = Kn^h\Big\{\prod_{i=1}^{p}\Gamma(n_i - 1 + \frac{hn_i}{n})/n_i^{hn_i/n}\Big\}/\Gamma(n + h - 1) , \tag{B.8.4}$$

where

$$K = \Gamma(n-1)/\prod_{i=1}^{p}\Gamma(n_i - 1) .$$

Upon taking the inverse Mellin transform, the density function of W is:

$$f(w) = K\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-h-1}\left[\frac{n^h}{\Gamma(n+h-1)}\prod_{i=1}^{p} n_i^{-hn_i/n}\Gamma(n_i - 1 + hn_i/n)\right] dh . \tag{B.8.5}$$

Letting $n + h = t$, then

$$\begin{aligned} f(w) &= Kn^{-n}(\prod_{i=1}^{p} n_i^{n_i})w^{n-1}\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} w^{-t}n^t \frac{\prod_{i=1}^{p}\Gamma(\frac{n_i t}{n} - 1)}{(\prod_{i=1}^{p} n_i^{n_i t/n})\Gamma(t-1)}dt \\ &= Kn^{-n}(\prod_{i=1}^{p} n_i^{n_i})w^{n-1}H_{1,p}^{p,0}\Big\{\frac{w}{n}\prod_{i=1}^{p} n_i^{n_i/n}\Big|_{(-1,\frac{n_i}{n});\ i=1,\ldots,p}^{(-1,1)}\Big\} . \end{aligned} \tag{B.8.6}$$

B.9 The distribution of LRT for testing the equality of covariance matrices under intraclass correlation model

Gupta and Nagar (1987) among others considered this problem. Let $\{X^{(1)}, S^{(1)}\}, \ldots, \{X^{(m)}, S^{(m)}\}$ be independently distributed where $X^{(i)} \sim N_p(\mu^{(i)}, \Sigma^{(i)})$ and $S^{(i)} \sim W_p(n_i, \Sigma^{(i)})$, $i = 1, 2, \ldots, n$. Also, let

$$\Sigma^{(i)} = \sigma_i^2[(1-\delta_i)I_p + \delta ee'],\ i = 1, \ldots, m , \tag{B.9.1}$$

where I_p is the identity matrix of order p, $e = (1,\ldots,1)$, $\sigma_i^2 > 0$ and δ_i, $i = 1,\ldots,n$ are unknown scalars. This structure of the covariance matrix is known as an interclass correlation model.

We consider testing the following hypothesis:

$$H : \Sigma^{(1)} = \cdots = \Sigma^{(m)} = \sigma^2[(1-\delta)I_p + \delta ee'] \; . \qquad (B.9.2)$$

Let Γ be an orthogonal matrix whose first row is $p^{-\frac{1}{2}}e'$ and let us use the transformation

$$Y^{(i)} = \Gamma X^{(i)}, \qquad V^{(i)} = \Gamma S^{(i)}\Gamma', \; i = 1,\ldots,m \; . \qquad (B.9.3)$$

Then

$$Y^{(i)} \sim N_p(v^{(i)}, \xi^{(i)}) \text{ and } V^{(i)} \sim W_p(n_i, \xi^{(i)}) \; ,$$

where

$$v^{(i)} = \Gamma\mu^{(i)}, \; \xi^{(i)} = \Gamma\Sigma^{(i)}\Gamma' = \operatorname{diag}(\alpha_i, \beta_i, \ldots, \beta_i) \; ,$$

$$\alpha_i = \sigma_i^2[1 + (p-1)\rho_i] \; \text{ and } \beta_i = \sigma_i^2(1-\rho_i), \; i = 1,2,\ldots,m \; .$$

Since the transformation from (δ_i, σ_i^2) to (α_i, β_i) is one-to-one the problem can be solved equally well in terms (α_i, β_i). The null hypothesis of testing equality of covariance matrices is then reduced to:

$$H_0 : \; \alpha_1 = \cdots = \alpha_m; \;\; \beta_1 = \cdots = \beta_m \; . \qquad (B.9.4)$$

The likelihood ratio criterion for testing H_0 is given by

$$\Lambda = \frac{n_0^{n_0/2}}{\prod_{i=1}^m n_i^{n_i/2}} \frac{\prod_{i=1}^m W_{1i}^{n_i/2}}{(\sum_{i=1}^m W_{1i})^{n_0/2}} \frac{[n_0(p-1)]^{n_0(p-1)/2}}{\prod_{i=1}^m [n_i(p-1)]^{n_i(p-1)/2}} \frac{\prod_{i=1}^m W_{2i}^{n_i(p-1)/2}}{(\sum_{i=1}^m W_{2i})^{n_0(p-1)/2}} \; , \qquad (B.9.5)$$

where

$$W_{1i} = W_{11}^{(i)} \; ; \qquad W_{2i} = trV_{22}^{(i)}, \qquad i = 1,\ldots,m;$$

$$n_0 = n_1 + \cdots + n_m \; ; \qquad V^{(i)} = \begin{bmatrix} V_{11}^{(i)} & V_{12}^{(i)} \\ V_{21}^{(i)} & V_{22}^{(i)} \end{bmatrix} \; i = 1, \ldots$$

and $V_{11}^{(i)}$ is a scalar.

The h-th moment of the test statistic λ is given by

$$\begin{aligned} E(\Lambda^h) = & \frac{n_0^{\frac{1}{2}n_0 p h}}{\prod_{i=1}^m n_i^{\frac{1}{2}n_i p h}} \frac{\Gamma(\frac{1}{2}n_0)\Gamma(\frac{1}{2}n_0(p-1))}{\Gamma[\frac{1}{2}n_0(1+h)]\Gamma[\frac{1}{2}n_0(p-1)(1+h)]} \\ & \times \prod_{i=1}^m \frac{\Gamma[\frac{1}{2}n_i(1+h)]}{\Gamma(\frac{1}{2}n_i)(\eta_1^{-1}\alpha_i)^{\frac{1}{2}n}} \frac{\Gamma[\frac{1}{2}n_i(p-1)(1+h)]}{\Gamma[\frac{1}{2}n_i(p-1)](\eta_2^{-1}\beta_i)^{\frac{1}{2}n_i^{(p-1)}}} \\ & \times F_D^{(m)}\Big(\frac{1}{2}n_0; \frac{1}{2}n_1(1+h), \ldots, \frac{1}{2}n_m(1+h); \; \frac{1}{2}n_0(1+h); \; 1-\eta_1\alpha_1^{-1}, \ldots, 1-\eta_1\alpha_m^{-1}\Big) \\ & \times F_D^{(m)}\Big(\frac{1}{2}n_0(p-1); \; \frac{1}{2}n_1(p-1)(1+h), \ldots, \frac{1}{2}n_m(p-1)(1+h); \; \frac{1}{2}n_0(p-1)(1+h); \\ & \quad 1-\eta_2\beta_1^{-1}, \ldots, 1-\eta_2\beta_m^{-1}\Big) , \end{aligned}$$

where

$$|1-\eta_1\alpha_i^{-1}| < 1, \qquad |1-\eta_2\beta_i^{-1}| < 1, \quad i = 1, 2, \ldots, m \; . \tag{B.9.6}$$

Expanding the Lauricella's hypergeometric functions F_D as follows

$$\begin{aligned} F_D(a; \; b_1, \ldots, b_p; c; x_1, \ldots, x_p) = & \frac{F(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \{u^{a-1}(1-u)^{c-a-1} \\ & \cdot (1-ux_1)^{-b_1} \cdots (1-ux_p)^{-b_p}\} \, du \\ = & \sum_{m_1=0}^{\infty} \cdots \sum_{m_p=0}^{\infty} \frac{(a)_{m_1+\cdots+m_p}(b_1)_{m_1} \cdots (b_p)_{m_p}}{(c)_{m_1+\cdots+m_p} m_1! \cdots m_p!} x_1^{m_1} \cdots x_p^{m_p} \end{aligned} \tag{B.9.7}$$

for $\mathrm{Re}(c) > 0$, $\mathrm{Re}(a) > 0$, $\mathrm{Re}(c-a) > 0$, $|x_i| < 1$, $b_i > 0$, $i = 1, 2, \ldots, p$ and

$$(a)_n = a(a+1) \cdots (a+n-1) \; ,$$

in (B.9.6) and using the inverse Mellin transform we obtain the non-null density of Λ as

$$f(\lambda)=\prod_{i=1}^{m}\left\{(\eta_1^{-1}\alpha_i)^{\frac{1}{2}n_i}\,(\eta_2\beta_i)^{\frac{1}{2}n_i(p-1)}\,\Gamma(\frac{1}{2}n_i)\,\Gamma[\frac{1}{2}n_i(p-1)]\right\}^{-1}$$

$$\sum_{r_1=0}\cdots\sum_{r_m=0}\left\{\prod_{i=1}^{m}[(1-\eta_1 a_i^{-1})^{r_i}/r_i!]\right\}\Gamma(\frac{1}{2}n_0+r_1+\cdots+r_m)$$

$$\sum_{s_1=0}\cdots\sum_{s_m=0}\left\{\prod_{i=1}^{m}[(1-\eta_2\beta_i^{-1})^{s_i}/s_i!]\right\}\Gamma(\frac{1}{2}n_0(p-1)+s_1+\cdots+s_m)\quad (B.9.8)$$

$$\lambda^{-1}H_{2,2m}^{2m,0}\left\{\lambda\frac{\sum_{i=1}^{\infty}n_i^{\frac{1}{2}n_ip}}{n_0^{\frac{1}{2}n_0p}}\Big|\begin{matrix}(\frac{1}{2}n_0+r_1+\cdots+r_m,\frac{1}{2}n_0),\\ \{(\frac{1}{2}n_i+r_i,\frac{1}{2}n_i),\\ (\frac{1}{2}n_0(p-1)+s_1+\cdots+s_{m1},\frac{1}{2}n_0(p-1))\\ (\frac{1}{2}n_i(p-1)+s_i,\frac{1}{2}n_i(p-1))\}\ ,\ i=1,2,\ldots,m\end{matrix}\right\}.$$

Letting $\alpha_1=\cdots=\alpha_m=\eta_1^{-1}$, $\beta_1=\cdots=\beta_m=\eta_2^{-1}$ we can express the null density as follows:

$$f(\lambda|H_0)=\prod_{i=1}^{m}\{\Gamma(\frac{1}{2}n_i)\Gamma[\frac{1}{2}n_i(p-1)]\}^{-1}\Gamma(\frac{1}{2}n_0)\Gamma[\frac{1}{2}n_0(p-1)]$$

$$\lambda^{-1}H_{2,2m}^{2m,0}\left\{\lambda\frac{\prod_{i=1}^{m}n_i^{\frac{1}{2}n_ip}}{n_0^{\frac{1}{2}n_0p}}\Big|\begin{matrix}(\frac{1}{2}n_0,\frac{1}{2}n_0),(\frac{1}{2}n_0(p-1),\frac{1}{2}n_0(p-1))\\ \{(\frac{1}{2}n_i,\frac{1}{2}n_i)(\frac{1}{2}n_i(p-1),\frac{1}{2}n_i(p-1))\}\end{matrix}\right\}$$

$$i=1,2,\ldots,m\ .$$

B.10 The exact density of the statistic $(\tilde{X}/\bar{X})$ related to the shape parameter of a Gamma variate

This problem was solved by Provost (1988b). Let $X_1,\ldots,X_n$ be a random sample of size n from a two-parameter gamma distribution with density

$$g(x;\alpha,\beta)=x^{\alpha-1}e^{-x/\beta}/\beta^{\alpha}\Gamma(\alpha),\quad x>0,\ \alpha,\beta>0\ ,\qquad (B.10.1)$$

and let $\bar{x}$ and $\tilde{x}$ denote, respectively, the arithmetic and the geometric mean of the random sample, i.e.

$$\bar{x}=\sum_{i=1}^{n}x_i/n\ \text{ and }\ \tilde{x}=\prod_{i=1}^{n}x_i^{1/n}\ .$$

The technique of the inverse Mellin transform was used to derive the exact density of the statistic $\ln(\bar{X}/\tilde{X})$. Let

$$Z = (\tilde{X}/\bar{X})^n \ .$$

Then the h-th moment of Z is

$$E(Z^h) = \delta\Gamma(\alpha + h)^{n-1} / \prod_{j=1}^{n-1} \Gamma(a_j + h) \ ,$$

where

$$\delta = 2\pi^{(n-1)/2}\Gamma(n\alpha)/\{\Gamma(\alpha)^n n^{n\alpha-1/2}\} \ ,$$

and

$$a_j = \alpha + j/n, \qquad j = 0, 1, \ldots, n-1 \ .$$

The inverse Mellin transform of the moment expression in (B.10.3) yields $f(z)$ the density of Z

$$f(z) = z^{-1}\delta G^{n-1,0}_{n-1,n-1}\left\{z \middle| \begin{matrix} a_1, a_2, \ldots, a_{n-1} \\ \alpha, \alpha, \ldots, \alpha \end{matrix} \right\}, \quad 0 < z < 1 \ . \qquad (B.10.4)$$

Now letting

$$y = \ln(\bar{x}/\tilde{x}) = -(\ln(z))/n, \qquad y > 0,$$

the density of y is

$$g(y) = ne^{-ny} f(e^{-ny}), \qquad \text{for } y > 0,$$

$$g(y) = ne^{-ny}e^{ny}\delta G^{n-1,0}_{n-1,n-1}\left\{e^{-ny} \middle| \begin{matrix} a_1, \ldots, a_{n-1} \\ \alpha, \ldots, \alpha \end{matrix} \right\} \qquad (B.10.5)$$

$$= n\delta G^{n-1,0}_{n-1,n-1}\left\{e^{-ny} \middle| \begin{matrix} a_1, \ldots, a_{n-1} \\ \alpha, \ldots, \alpha \end{matrix} \right\} .$$

B.11 The exact density of the ratio of a linear combination of chi-square variables over the root of a product of chi-square variables

This problem was studied by Provost (1986a). Consider a statistic of the type

$$R = P/Q, \qquad (B.11.1)$$

where

$$\begin{aligned} P &= m_1X_1 + \cdots + m_kX_k, \quad k \geq 1, \quad m_j > 0, \quad j = 1, \ldots, k \\ Q &= (Y_1Y_2 \cdots Y_k)^{1/k}, \end{aligned} \qquad (B.11.2)$$

$X_1, \ldots, X_k$ are independent chi-square variables having $r_1, \ldots, r_k$ degrees of freedom respectively, $Y_1, \ldots, Y_k$ are independent chi-square variables having $q_1, \ldots, q_k$ degrees of freedom, and P and Q are assumed to be independently distributed. First the h-th moment of $1/R^k$ is derived. Let $U = 1/R$. If the density of U exists, then the h-th moment of U about the origin is given by

$$\begin{aligned} E(U^h) = \{\prod_{j=1}^{k}(2m_j)^{-r_j/2}\Gamma(q)\Gamma(\varrho)\} \sum_{v=0} \sum_{v_1+\cdots+v_k=v} \{(q)_v/(\varrho)_v\} \\ \times \{(r_1/2)_{v_1} \cdots (r_k/2)_{v_k}\}(\gamma_1^{v_1} \cdots \gamma_k^{v_k})/(v_1! \cdots v_k!)\Big\{2^h \prod_{i=1}^{k} \frac{\Gamma(\frac{q_i}{2} + \frac{h}{k})}{\Gamma(q_{i/2})}\Big\}, \end{aligned} \qquad (B.11.3)$$

where

$$\rho = (r_1 + \cdots + r_k)/2, \qquad \rho - h = q, \quad \gamma_j = (2m_j - 1)/(2m_j)\ .$$

Let $T = U^k$. Then the h-th moment of $(1/R)^k = U^k$ is

$$\begin{aligned} E(T^h) = \sum_{v=0} \sum_{v_1+\cdots+v_k=v} C_{v_1,\cdots,v_k}\{\prod_{j=1}^{k}\Gamma(q_j/2 + h)\} \\ \times \prod_{\ell=1}^{k}\Gamma[(\rho + v + \ell - 1)/k - h](2/k)^{hk}\ , \end{aligned} \qquad (B.11.4)$$

where

$$C_{v_1,\ldots,v_k} = \frac{\{(2\pi)^{(1-k)/2}\Gamma(\frac{q_i}{2})\}\Gamma(q+v)(v_1!\cdots v_k!)}{\{\prod_{i=1}^{k}\{(2m_i)^{r_i/2}\Gamma(\frac{q_i}{2})\}\Gamma(q+v)(v_1!\cdots v_k!)]} \, . \tag{B.11.5}$$

The inverse Mellin transform gives the density of T:

$$g(T) = \sum_{v=0} \sum_{v_1+\cdots+v_k=v} T^{-1} C_{v_1,\ldots,v_k} G_{k,k}^{k,k}\left\{\theta T \Big|{}^{a_1,\ldots,a_k}_{b_1,\ldots,b_k}\right\} , \tag{B.11.6}$$

where

$$\theta = (k/2)^2, \; 1 - a_i = (\rho + v + L - 1)/k, \quad b_j = q_j/2,$$

and a_i and b_j are such that $a_i - 1 - \lambda \neq b_j + n$. Since

$|\delta T/\delta R| = kr^{-(k+1)}$, the density of R is therefore

$$\begin{aligned} \Phi(R) &= kR^{-(k+1)} g(1/R^k) \\ &= kR^{-(k+1)} \sum_{v=0}^{\infty} \sum_{v_1+\cdots+v_k=v} R^k C_{v_1,\ldots,v_k} G_{k,k}^{k,k}\left\{\frac{\theta}{R^k} \Big|{}^{a_1,\ldots,a_k}_{b_1,\ldots,b_k}\right\} . \end{aligned} \tag{B.11.7}$$

B.12 The distribution of the test statistic connected with the multivariate linear functional relationship model

This problem is discussed by Provost (1986b). This article pointed out the connection of the function $G_{k,k}^{k,k}$ with the density of products of independent F and beta type-2 variables.

A random variable T_j is said to have a beta type-2 density with parameters α_j and β_j if its probability density function is

$$f_j(t_j) = \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)\Gamma(\beta)i)} t_j^{\alpha_j - 1}(1 + t_j)^{-(\alpha_j+\beta_j)} , \tag{B.12.1}$$

for

$$0 < t_j < \infty, \quad \mathrm{Re}(\alpha_j) > 0, \quad \mathrm{Re}(\beta_j) > 0 ,$$

and $f_j(t_j) = 0$ elsewhere.

Let $W_k = T_1 T_2 \cdots T_k$ where $T_1, T_2, \ldots, T_k$ are k independent beta type-2 variables with the respective densities (B.12.1), then the h-th moment of W_k is

$$E(W_k^h) = \frac{\prod_{j=1}^{k} \Gamma(\alpha_j + h)\Gamma(\beta_j - h)}{\prod_{j=1}^{k} \Gamma(\alpha_j)\Gamma(\beta_j)} \ . \tag{B.12.2}$$

The density of W_k is obtained by using inverse Mellin transform:

$$f(x) = kH_{k,k}^{k,k}\{x|{}_{(a_i - 1, 1)}^{(-\beta_i, 1)}\, i = 1, 2, \ldots, k\} \ , \tag{B.12.3}$$

which can be expressed in terms of the G-function as follows

$$f(x) = kG_{k,k}^{k,k}\{x|{}_{a_i - 1}^{-\beta_i}\ i = 1, \ldots, k\} \ . \tag{B.12.4}$$

Now if $Z_k = \sum_{j=1}^{k} F_j$, where F_j's are k independent F-variables each having $2\alpha_j$, $2\beta_j$ degrees of freedom, $j = 1, \ldots, k$, then the h-th moment of Z_k is equal to the h-th moment of a constant times W_k, that is

$$\begin{aligned} E(Z_k^h) &= a^h E(W_k^h) = E((aW_k)^h), \text{ where} \\ a &= \prod_{j=1}^{k} (\beta_j/\alpha_j), \end{aligned} \tag{B.12.5}$$

and therefore the density of Z_k is

$$f(z) = KG_{k,k}^{k,k}\{\frac{z}{a}|{}_{\alpha_j - 1}^{-\beta_j}\quad j = 1, \ldots, k\} \ . \tag{B.12.6}$$

B.13 The distribution of Pearson and Wilks' likelihood test statistic

The problem was discussed by Rathie (1982). Let each of k samples of two variables x and y be drawn from some normal population. Let n_i be the population from which X_i has been drawn and let the means of x and y be a_i and b_i, the standard deviations, σ_{xt} and σ_{yt}, and the correlation coefficient, ρ_t $(t = 1, \ldots, k)$. Then consider the hypothesis H that the population π_i are identical, that is

$$H : \sigma_{xt} = \sigma_x,\ \sigma_{yt} = \sigma_y, \quad \rho_t = \rho,\ a_t = a,\ b_t = b, \quad \text{for } t = 1, \ldots, k\,. \qquad (B.13.1)$$

To test H the likelihood ratio test statistic is:

$$\lambda = \prod_{t=1}^{k} \left[\frac{|V_{itj}|}{|V_{ijo}|} \right]^{\frac{n_t}{2}}, \qquad (B.13.2)$$

where

$$|V_{ijt}| = S_{xt}^2 S_{yt}^2 (1 - r_t^2) = \begin{vmatrix} V_{11t} & V_{12t} \\ V_{21t} & V_{22t} \end{vmatrix} \qquad (B.13.3)$$

with

$$\begin{aligned} n_t S_{xt}^2 &= \sum_{a=1}^{n_t} (x_{ta} - \bar{x}_t)^2 = n_t V_{11t} \\ n_t S_{yt}^2 &= \sum_{a=1}^{n_t} (y_{ta} - \bar{y}_t)^2 = n_t v_{22t}, \\ n_t s_{xt} s_{yt} r_t &= \sum_{a=1}^{n_t} (x_{ta} - \bar{x})(y_{ta} - \bar{y}) = n_t v_{12t} \end{aligned} \qquad (B.13.4)$$

and

$$|V_{ijo}| = \begin{vmatrix} V_{110} & V_{120} \\ V_{210} & V_{220} \end{vmatrix} = S_{x_0}^2 S_{y_0}^2 (1 - r_0^2) \qquad (B.13.5)$$

with

$$NS_{x0}^2 = \sum_{t=1}^{k}\sum_{a=1}^{n_t}(x_{ta} - \bar{x}_0)^2 = NV_{110} \ ,$$

$$NS_{y0}^2 = \sum_{t=1}^{k}\sum_{a=1}^{n_t}(y_{ta} - \bar{y}_0)^2 = NV_{220} \ , \qquad (B.13.6)$$

$$NS_{X0}^2 S_{y0} \ r_0 = \sum_{t=1}^{k}\sum_{a=1}^{n_t}(x_{t_a} - \bar{x}_0)(y_{ta} - \bar{y}_0) = NV_{120} \ .$$

Also $\bar{x}_0, \bar{y}_0, s_{x0}, s_{y0}, r_0$ denote respectively the means, standard deviations and correlation coefficient obtained on combining the N pairs of observations from the k-samples, when $N = \sum_{t=1}^{k} n_t$. The h-th moment of λ is given by

$$E(\lambda^h) = \frac{\Gamma(nk-2)}{\Gamma^k(n-2)} k^h \frac{\Gamma^k(n-2+\frac{h}{k})}{\Gamma(nk-2+h)} \ . \qquad (B.13.7)$$

Now using the inverse Mellin transform, and letting $n - 2 + \frac{h}{k} = t$, we obtain the following expression for the density of λ

$$f(z) = g_k z^{(n-2)k-1} G_{k,k}^{k,0}\left\{z^k \middle| \begin{matrix} 3-\frac{3}{k}, 3-\frac{4}{k}, \ldots, 3-\frac{2+k}{k} \\ 0,\ 0,\ \ldots,\ 0 \end{matrix}\right\}, \quad 0 < z < 1,\ k \geq 2, \qquad (B.13.8)$$

where

$$g_k = \frac{\Gamma(nk-2)}{(2\pi)^{\frac{1}{2}(1-k)} k^{nk-7/2} \Gamma^k(n-2)} \ .$$

Integration : (B.21.8) gives the following distribution function

$$F(z) = g_k z^{(n-2)k} G_{k+1,k+1}^{k,1}\left\{z^k \middle| \begin{matrix} 3-n, 3-\frac{3}{k}, \ldots, 3-\frac{k+2}{k} \\ 0, \qquad ,0,\ 2-n \end{matrix}\right\}, \ 0 < z < 1,\ k \geq 2 \ . \qquad (B.13.9)$$

B.14 The density function of Wilks' criterion for testing the multivariate linear hypothesis

This problem was discussed by Kulp and Nagarsenker (1984). Let each column vector of a $p \times N$ matrix X; $(X_1, \ldots, X_N)$ be distributed independently according to a p-variate normal distribution with common covariance matrix Σ, where $\Sigma_{(p\times p)}$ is positive definite and unknown. The problem consists of testing the hypothesis:

$$H_0 : E(X_j) = 0 \qquad (B.14.1)$$

for $j = 1, 2, \ldots, b$ and $s+1, \ldots, N$ with $b \leq s$ against the alternative:

$$\begin{aligned} K : E(X_j) &\neq 0 \quad \text{for some } j \ (1 \leq j \leq b) \\ &= 0 \quad \text{for } j = s+1, \ldots, N \ . \end{aligned} \qquad (B.14.2)$$

The likelihood ratio statistic for testing this hypothesis is given by

$$\Lambda = W^{1/2} = (|S_e|/|S_e + S_n|)^{N/2} \ , \qquad (B.14.3)$$

where

$$S_e = \sum_{i=s+1}^{N} X_i X_i' \quad \text{and } S_h = \sum_{i=1}^{b} X_i X_i' \ .$$

The h-th null moment of $U = W^{1/s}$ is

$$E(U^h) = \frac{\Gamma_p(m_1 + \delta + b/2)}{\Gamma_p(m_1 + \sigma)} \frac{\Gamma_p(h/s + m_1 + \delta)}{\Gamma_p(h/s + b/2 + m_1 + \delta)} \qquad (B.14.4)$$

where

$$\delta = (p - b + 1)/4, \quad m = n - (p + b + 1)/2, \quad n = N - s + b, \quad m_1 = m/2$$

and

$$\Gamma_p(a) = \pi^{p(p-1)/4}\Gamma(a)\Gamma(a - \frac{1}{2}) \cdots \Gamma(a - \frac{1}{2}(p-1)) = \pi^{p(p-1)/4} \times \prod_{i=1}^{p} \Gamma[a - \frac{1}{2}(i-1)] \ .$$

Hence

$$E(U^h) = \frac{\Gamma_p(m_1 + \delta - \frac{b}{2})}{\Gamma_p(m_1 + \delta)} \frac{\prod_{i=1}^{p} \Gamma[\frac{h}{s} + m_1 + \delta - \frac{1}{2}(i-1)]}{\prod_{i=1}^{p} \Gamma[\frac{h}{s} + \frac{b}{2} + m_1 + \delta - \frac{1}{2}(i-1)]} . \qquad (B.14.5)$$

Using the inverse Mellin transform, the the following representation of the density of U is obtained

$$f(u) = \frac{\Gamma_p(m_1 + \delta - \frac{b}{2})}{\Gamma_p(m_1 + \delta)} u^{-1} H_{p,p}^{p,0}\left\{ u \middle| \begin{matrix} (\frac{b}{2} + m_1 + \delta - \frac{1}{2}(i-1), \frac{1}{s}) \\ m_1 + \delta - \frac{1}{2}(i-1), \frac{1}{s} \end{matrix} \; i = 1, 2, \ldots, p \right\} . \qquad (B.14.6)$$

B.15 Wilks' L_{VC} criterion

This problem was discussed by Wilks (1946). Consider a random sample $X_1, \ldots, X_k$ of p-component vectors drawn from the multivariate normal distribution $N(\mu_i, \Sigma_i)$, $i = 1, 2, \ldots, k$. Let $X_i' = (X_{i1}, \ldots, X_{ip})$, $i = 1, 2, \ldots, k$. We consider hypotnesis H_{VC} that variances are equal, and covariances are equal, irrespective of the values of the means. The test criterion L_{VC} is derived using the Neyman-Pearson method of likelihood ratios for testing H_{VC}. The criterion for testing H_{VC} depends on the following quantities:

$$\bar{X}_i = \frac{1}{p}\sum_{\alpha=1}^{p} X_{i\alpha}, \quad \bar{X} = \frac{1}{k}\sum_{i=1}^{k} \bar{X}_i ,$$
$$s_{ij} = \frac{1}{p}\sum_{\alpha=1}^{p}(X_{i\alpha} - \bar{X}_i)(X_{j\alpha} - \bar{X}_j) = \frac{1}{p}\sum_{\alpha=1}^{p} X_{i\alpha}X_{j\alpha} - \bar{X}_i\bar{X}_j \qquad (B.15.1)$$
$$s^2 = \frac{1}{k}\sum_{i=1}^{k} s_{ii}, \quad r = \frac{1}{k(k-1)} \sum_{i \neq j = 1}^{k} s_{ij} .$$

The sample criterion, based on the method of likelihood ratios for testing H_{VC} is

$$L_{VC} = \frac{|s_{ij}|}{(s^2)^k (1-r)^{k-1}(1 + (k-1)r)}$$

where $|s_{ij}|$ is the determinant of sample variances and covariances. If H_{VC} is true, there will be $k+2$ parameters , $\mu_1, \ldots, \mu_k$, σ^2 , and ρ. The maximum likelihood estimates of these parameters are, respectively: $\bar{X}_1, \ldots, \bar{X}_k$, s^2 and r.

The h-th moment of the exact sampling distribution of L_{VC} when H_{VC} is true is:

$$M_1(L_{VC}) = (k-1)^{h(k-1)} \prod_{i=2}^{k} \frac{\Gamma[\frac{1}{2}(n-i)+h]}{\Gamma[\frac{1}{2}(n-i)]} \frac{\Gamma[\frac{1}{2}(n-i)(k-1)]}{\Gamma[\frac{1}{2}(n-i)+h+(i-1)/(k-1)]} . \tag{B.15.4}$$

Applying the Gauss-Legendre multiplication theorem (B.2.5) to $\Gamma[\frac{1}{2}(n-i)+h+(i-1)/(k-1)]$ we obtain

$$M_1(L_{VC}) = K(n) \prod_{i=1}^{k-1} \frac{\Gamma[\frac{1}{2}(n-i)+h]}{\Gamma[\frac{1}{2}(n-i)+h+(i-1)/(k-1)]} , \tag{B.15.5}$$

where

$$K(n) = (2\pi)^{\frac{k-2}{2}} (k-1)^{\frac{1}{2}-\frac{(k-1)(n-1)}{2}} \frac{\Gamma[\frac{1}{2}(k-1)(n-1)]}{\prod_{i=1}^{k-1} \Gamma[\frac{1}{2}(n-i)]} .$$

Thus the exact p.d.f. for the criterion is

$$f(L_{VC}) = K(n) L_{VC}^{(n-k-1)/2} G_{k-1,k-1}^{k-1,0} \left\{ L_{VC} \middle| \begin{matrix} \frac{1}{2}(k-2)+\frac{k-2}{k-1}, \cdots, \frac{1}{2}(k-2)-\frac{1}{k-1}, \frac{k-2}{2} \\ 0, \frac{1}{2}, 1, \ldots, \frac{1}{2}(k-2) \end{matrix} \right\} . \tag{B.15.6}$$

B.16 Testing equality of diagonal elements

This problem was considered by Mathai (1979). Consider a random sample $X_{i\alpha}$, $i=1,\ldots,k$; $\alpha=1,\ldots,p$, of p-component vectors drawn from the multinormal distribution $N(\mu_i, \Sigma_i)$, $i=1,\ldots,k$. We wish to test the equality of the diagonal elements given that the covariance matrix is diagonal, that is $\Sigma = (\delta_{ij})$ $\delta_{ij}=0$ if $i \neq j$. The hypothesis is:

$$H_0 : \sigma_{ii} = \sigma^2, \ i=1,\ldots,p . \tag{B.16.1}$$

The likelihood ratio criterion for testing H_0 is

$$\lambda = \prod_{i=1}^{p} s_{ii}^{k/2}/[(s_{11} + \cdots + s_{pp})/p]^{kp/2} , \tag{B.16.2}$$

where

$$s_{ii} = \sum_{j=1}^{n}\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2; \quad \bar{X}_i = \sum_{j=1}^{n} x_{ij}/n .$$

Mathai derived the h-th moment of $V = \lambda^{2/N}$ in terms of the Lauricella's hypergeometric function F_D, which is defined in (B.9.7), in the following form:

$$\begin{aligned} E(V_h) &= \frac{p^{ph}\Gamma^p(\frac{n}{2}+h)}{\Gamma^p(\frac{n}{2})\Gamma(\frac{np}{2}+ph)}\Gamma(\frac{np}{2})\Big\{\sigma_{11}\cdots\sigma_{pp}\Big\}^{-\frac{n}{2}} \\ &\times F_D(\frac{np}{2}; \frac{n}{2}+h, \cdots, \frac{n}{2}+h; \frac{np}{2}+ph; 1-\sigma_{11}^{-1}, \cdots, 1-\sigma_{pp}^{-1}) . \end{aligned} \tag{B.16.3}$$

One can use the series representation of Lauricella's function and invert term by term to obtain the non-null density using the inverse Mellin transform technique. These techniques are discussed in Mathai and Saxena (1973). The exact h-th non-null moment of V is:

$$\begin{aligned} E(V^h) &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_p=0}^{\infty} \frac{\Gamma^p(\frac{n}{2}+h)(\frac{n}{2}+h)_{m_1}\cdots(\frac{n}{2}+h)_{m_p}\Gamma(\frac{np}{2})(\frac{np}{2})_{m_1+\cdots+m_p}}{\Gamma(\frac{np}{2}+ph)(\frac{np}{2}+ph)_{m_1+\cdots+m_p}m_1!\cdots m_p!} \\ &\times \frac{p^{ph}}{\Gamma^p(\frac{n}{2})}(1-\sigma_{11}^{-1})^{m_1}\cdots(1-\sigma_{pp}^{-1})^{m_p}\{\sigma_{11}\cdots\sigma_{pp}\}^{-\frac{n}{2}} \\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_p=0}^{\infty} \frac{\Gamma(\frac{np}{2}+m_1+\cdots+m_p)(1-\sigma_{11}^{-1})^{m_1}\cdots(1-\sigma_{pp}^{-1})^{m_p}}{\Gamma^p(\frac{n}{2})m_1!\cdots m_p!(\sigma_{11}\cdots\sigma_{pp})^{n/2}} \\ &\quad \frac{(\prod_{i=1}^{p}\Gamma(\frac{n}{2}+h+m_i))p^{ph}}{\Gamma(\frac{np}{2}+m_1+\cdots+m_p+ph)} . \end{aligned} \tag{B.16.4}$$

Then

$$f(V) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_p=0}^{\infty} C_{m_1\cdots m_p}V^{-1}H_{1,p}^{p,0}\Big\{\frac{V}{p^p}\Big|{}_{(\frac{n}{2}+m_1,1),\cdots,(\frac{n}{2}+m_p,1)}^{(\frac{np}{2}+m_1+\cdots+m_p,p)}\Big\}, \tag{B.16.5}$$

where

$$C_{m_1,\ldots,m_p} = \frac{\gamma(\frac{np}{2} + m_1 + \cdots + m_p)(1-\sigma_{11}^{-1})^{m_1}\cdots(1-\sigma_{pp}^{-1})^{m_p}}{\Gamma^p(\frac{n}{2})m_1!\cdots m_p!(\sigma_{11}\cdots\sigma_{pp})^{n/2}} .$$

The null moment of V is:

$$E(V^h|H_0) = p^{ph}\frac{\Gamma^p(\frac{n}{2}+h)\Gamma(\frac{np}{2})}{\Gamma^p(\frac{n}{2})\Gamma(\frac{np}{2}+ph)} . \qquad (B.16.6)$$

The h-th moment in (B.16.6) can be inverted using the inverse Mellin transform technique yielding the density of V in terms of an H-function:

$$f(V) = cV^{-1}H_{1-p}^{p,0}\left\{\frac{V}{p^p}\Big|_{(\frac{n}{2},1),\ (\frac{n}{2},1),\cdots,(\frac{n}{2},1)}^{(\frac{np}{2},p)}\right\}, \qquad (B.16.7)$$

where $c = \Gamma(\frac{np}{2})/\Gamma^p(\frac{n}{2})$. Mathai expanded $\Gamma(np/2+ph)$ using the Gauss-Legendre multiplication formula (B.2.5) and then obtained the following density function $f(v)$ in terms of the G-function:

$$f(v) = p^{\frac{1}{2}-\frac{np}{2}}(2\pi)^{\frac{1}{2}(p-1)}cG_{p,p}^{p,0}\left\{V\Big|_{\frac{n}{2},\cdots,\frac{n}{2}}^{\frac{n}{2},\frac{n}{2}+\frac{1}{p_1},\cdots,\frac{n}{2}+\frac{p-1}{p}}\right\} . \qquad (B.16.8)$$

B.17 $L_1(VC)$ of Votaw

This problem was discussed by Votaw (1948). Consider a normal t-variate population, X_i $(i = 1,\ldots,t)$, $(t \geq 3)$. Let the set of variates $X_1,\ldots,X_t$ be partitioned into q mutually exclusive subsets of which b subsets contain exactly one variate each and the remaining $q-b=h$ subsets $(h \geq 1)$ contain $n_1,\ldots,n_h$ variates $(n_a \geq 2;\ a = 1,\ldots,h;\ b+\sum_{a=1}^{h} n_a = t)$.

The hypothesis that within each subset of variates the variances are equal and the covariances are equal and that between any two distinct subsets of variates the covariances are equal is called the $H_1(VC)$ hypothesis.

As the sample criterion for $H_1(VC)$ which is based on a sample of size N, Votaw (1948) proposed

$$L_1(VC) = [\lambda_1(VC)]^{2/N} = |V_{ij}|/|\bar{V}_{ij}|;\ i,j = 1,\ldots,t \tag{B.17.1}$$

where $\lambda_1(vc)$ is the likelihood ratio criterion for $H_1(vc)$, and

$$\begin{aligned} v_{ij} &= \sum_{\alpha=1}^{N}(X_{i\alpha} - \bar{X}_i)(X_{j\alpha} - \bar{X}_j) \\ \bar{v}_{aa'} &= v_{aa'} \\ \bar{v}_{ai_a} &= \frac{1}{n_a}\sum_{j_a} v_{aj_a} \\ \bar{v}_{i_a i_a} &= \frac{1}{n_a}\sum_{j_a} v_{j_a j_a} \\ \bar{v}_{i_a j_a} &= \frac{1}{n_a(n_a-1)}\sum_{i'_a \neq j'_a} v_{i'_a j'_a} \\ v_{i_a j'_a} &= \frac{1}{n_a n'_a}\sum_{i'_a, j'_a} v_{i'_a j'_a}\ . \end{aligned} \tag{B.17.2}$$

Under $H_1(VC)$, the d-th moment of $L_1(VC)$ is given by

$$E[L_1(VC)]^d = \prod_{a=1}^{h}\prod_{s_a=1}^{n_a-1} \frac{(\frac{N-q-s_a-n_a+a-1}{2})_d}{(\frac{N-1}{2} + \frac{s_a-1}{n_a-1})_d}\ , \tag{B.17.3}$$

where $(T)_d = \Gamma(T+d)/\Gamma(T)$. So

$$E[L_1(VC)]^d = c_1 \prod_{a=1}^{h}\prod_{s_a=1}^{n_a-1} \frac{\Gamma[\frac{1}{2}(N-q-s_a-n_a+a-1)+d]}{\Gamma[\frac{1}{2}(n-1) + \frac{s_a-1}{n_a-1} + d]}\ , \tag{B.17.4}$$

where

$$c_1 = \prod_{a=1}^{h}\prod_{s_a=1}^{n_a-1} \frac{\Gamma(\frac{n-1}{2} + \frac{s_1-1}{n_a-1})}{\Gamma[\frac{1}{2}(N-q-s_a-n_a+a-1)]}\ .$$

Upon taking the inverse Mellin transform we obtain the following density function for $L_1(VC)$

$$f(L_1(VC)) = c_1 L_1^{-1} VC G^{n-h,0}_{n-h,n-h}\left\{L_1(VC)\Big|^{\frac{n-1}{2} + \frac{s_1-1}{n_a-1},\quad a=1,\ldots,h \atop s_a=1,\ldots,n_a-1}_{\frac{1}{2}(N-q-s_a-n_a+a-1)}\right\}\ , \tag{B.17.5}$$

where $n = n_1 + \cdots + n_k$.

B.18 $\bar{L}_1(VC)$ of Votaw.

This problem was studied by Votaw (1948). Assumptions from B.18 are kept in this case. The hypothesis that within each subset of variates, the variances are equal and the covariances are equal and that between any two distinct subsets of variates the diagonal covariances are equal and the off-diagonal covariances are equal is called the $\bar{H}_1(VC)$ hypothesis. As the sample criterion for testing $\bar{H}_1(VC)$ Votaw (1948) proposed:

$$L_1(VC) = [\bar{\lambda}_1(VC)]^{2/N} = |v_{ij}|/|\bar{v}_{ij}| \qquad i,j = 1,\ldots,t, \tag{B.18.1}$$

where $\bar{\lambda}_1(VC)$ is the likelihood ratio criterion for $\bar{H}_1(VC)$, and

$$\begin{aligned}
v_{ij} &= \sum_{\alpha=1}^{n}(X_{i\alpha} - \bar{X}_i)(X_{j\alpha} - \bar{X}_j) , \\
\bar{v}_{i_a i_a} &= \frac{1}{n}\sum_{j_a} v_{j_a j_a} , \\
\bar{v}_{i_a j_a} &= \frac{1}{n(n-1)}\sum_{i'_a \neq j'_a} v_{i'_a j'_a} , \\
\bar{v}_{i_a k'_a} &= \frac{1}{n}\sum_{j_a, k'_a} v_{j_a k_a}; \quad k'_a = j_a + n(a'-a), \quad a \neq a', \\
\bar{v}_{i_a h'_a} &= \frac{1}{n(n-1)}\sum_{i_a,\, h'_a} v_{j_a h'_a}; \quad h'_a \neq j_a + n(a'-a), \; a \neq a' .
\end{aligned} \tag{B.18.2}$$

Under $\bar{H}_1(vc)$ the d-th moment of $\bar{L}_1(vc)$ is given by

$$\begin{aligned}
E[L(vc)]^d &= \prod_{a=1}^{h}\prod_{s=1}^{n-1} \frac{\left(\frac{N-h-a-(n-1)(a-1)}{2}\right)_d}{\left(\frac{N-1}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1}\right)_d} \\
&= c_2 \prod_{a=1}^{h}\prod_{s=1}^{n-1} \frac{\Gamma[\frac{1}{2}(N-h-a-(n-1)(a-1)) + d]}{\Gamma[\frac{N-1}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1} + d]}
\end{aligned} \tag{B.18.3}$$

where $$c_2 = \prod_{a=1}^{h}\prod_{j=1}^{n-1} \frac{\Gamma[\frac{N-1}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1}]}{\Gamma[\frac{1}{2}(N-h-a-(n-1)(a-1))]} .$$

Using the inverse Mellin transform technique we obtain the following density function for $\bar{L}_1(vc)$

$$f(\bar{L}_1(vc)) = c_2 L_1^{-1}(vc) G^{h(n-1),0}_{h(n-1),h(n-1)} \left\{ \bar{L}_1(vc) \Big| \begin{matrix} \frac{N-1}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1};\ a = 1,\ldots,h \\ \frac{N-h-a-(n-1)(s-1)}{2};\ s = 1,\ldots,n-1 \end{matrix} \right\}. \tag{B.18.4}$$

APPENDIX C

Computer Programs

The programs were written in FORTRAN and some subroutines from the IMSL package were used.

```
c                        APPENDIX C

c                    COMPUTER PROGRAMS

c          Appendix C.1


c                    Example 3.1

c        |||| A ratio whose  numerator and denominator||||
c        ||||      are independently distributed        ||||

      program ratio1
      integer nout,n,k,v,i,j,p,u,w
      real sumk,sumv,sumi,sumtk,arg(1:9)

      double precision tk,tv,d1i,d2i,l,m,b,b1,b2
      double precision c,a1,g1,g2,ex1,ex2,ex3,x,x1
      double precision dgamma,m5,b3,b4,t,fc

      double precision N1,N2,n3,n4
      external dgamma,umach
      open(unit=2,file='ratio3',status='unknown')
      data (arg(p),p=1,9) /0.1,0.3,0.5,0.8,1.0,1.2,2.5,10.0,13.0/


      do 22 p=1,9
      c=arg(p)

       sumk=0.0
      bp=(1.0/(2.0*c))
      b3=3.0/4.0
      b=b3+bp
      do 20 k=0,25
      sumtk=0.0
      do 2 n=0,k
      sumtk=sumtk+(k-n+1)*((b3-0.5)**n)*((b3-1.0)**(k-n))

 2    continue
      tk=sumtk/2.0
      sumv=0.0
      v=0
      tv=1/(8*c**3)
      sumi=0.0
      do 16 i=0,2
      call umach(2,nout)
      l=(-k)/2.0
      m=(-5.0-k)/2.0
      g1=I+1.0
      b1=bp**g1
      n2=-2.0*m-i
      u=n2
      n4=-m+l+0.5-i
      w=n4
      m5=3.0+k
      m5=dgamma(m5)
      t=1.0/m5
```

```
      do 10 s=v,u-1

      t=t*s
10    continue

      ex1=(t*b**(2.0*m+I ))/b1
      sumi=sumi+ex1
16    continue

      sumv=sumv+tv*sumi

c     |||     The distribution function obtained          |||
c     |||        from the representation (3.10)           |||

      sumk=sumk+tk*sumv

20    continue

c     |||     The distribution function obtained |||
c     ||| by the transformation of variables technique |||

      ex2=16.0*c**7+80.0*c**6+168.0*c**5+140.0*c**4+40.0*c**3
      fc=ex2/((2.0*c+1.0)**4*(c+1.0)**3)
      write(2,*),sumk,c,fc
22    continue

      close (unit=2)
      stop
      end
```

```
c       Appendix C.2

c         ||||                Example 3.2                 ||||
c         |||| A ratio of dependent quadratic forms ||||

        program ratio3
        integer nout,n,k,v,i,j,p,u,w
        real sumk,sumj,sumtk,sumtv,sumv

        double precision tk,l1,l2,b,b1,b2,tv
        double precision dgamma,b3,be,bep,fc,b4
        double precision a1,a2,a3,a4,a5,a6
        double precision N1,N2,n3,n4,n5,m1,m2,m3,m4,m5,m6
        external dgamma,umach
        open(unit=2,file='ratio4',status='unknown')



        b1=36.0
        b2=12.42
        b3=5.44
        b4=18.98
        be=2.0*b1*b2/(b1+b2)
        bep=2.0*b3*b4/(b3+b4)
        b=1/bep+1/be
         sumk=0.0
        do 20 k=0,15
        sumtk=0.0
        do 2 n=0,k
        call umach(2,nout)
        m1=n+0.5
        m1=dgamma(m1)
        m2=k-n+0.5
        m2=dgamma(m2)
        m3=n+1
        m3=dgamma(m3)
        m4=k-n+1
        m4=dgamma(m4)
        m5=(1/be-1/b1)**n
        m6=(1/be-1/b2)**(k-n)
        sumtk=sumtk+(m1/m3)*(m2/m4)*m5*m6
2       continue
        tk=sumtk
         sumv=0.0
        do 30 v=0,15
        sumtv=0.0
        do 16 n=0,v
        call umach(2,nout)
        a1=n+0.5
        a1=dgamma(a1)
        a2=v-n+0.5
        a2=dgamma(a2)
        a3=n+1
        a3=dgamma(a3)
        a4=v-n+1
        a4=dgamma(a4)
        a5=(1/bep-1/b3)**n
        a6=(1/bep-1/b4)**(v-n)
```

```
      sumtv=sumtv+(a1/a3)*(a2/a4)*a5*a6
 16   continue
      tv=sumtv
      sumj=0.0

      do 18 j=0,v
      n1=1.0
      do 10 i=1,k
      n1=n1*(v-j+i)/i

 10   continue
      n3=(b**(j-k-v-1))*(bep**(j+1))
      sumj=sumj+(n1)*n3
 18   continue

      sumv=sumv+tv*sumj
 30   continue
      sumk=sumk+tk*sumv
 20   continue
      l1=(b1*b2*b3*b4)**0.5
      l2=0.5
      l2=dgamma(l2)
c     |||   The value of the distribution function    |||
c     ||| obtained from the expression (3.13) at c=4.0 |||
      fc=sumk/(l1*l2**4)
      write(2,*) ,fc
      close (unit=2)
      stop
      end
```

Appendix 3.3

```
c     ||||                 Example 3.3                     ||||
c     |||| The distribution function of a ratio of quadratic ||||
c     ||||      forms in noncentral normal variables        ||||

      program distrib ratio of noncentral
      integer nout,n,k,i,j,s,p,ip,jp,v,l
      double precision sums,sumi,sumk,sumj,arg(1:9),dens
      double precision sumv,sumip,sumn,sumjp,suml
      double precision l1,l2,d1,d2,d,b,b1,b2,be
      double precision l1p,l2p,d1p,d2p,dp,bp,b1p,b2p
      double precision g1,g2,g3,g4,g5,r
      double precision dgamma,i1,i2,i3,i4,i5,g6,g7,g8
      double precision j1,j2,j3,j4,j5,j6,j7,j8
      double precision g1p,g2p,g3p,g4p,g5p
      double precision i1p,i2p,i3p,i4p,i5p,g6p,g7p
      double precision j1p,j2p,j3p,j4p,j5p,j6p,j7p,j8p
      external dgamma,umach,dbeta
      open(unit=2,file='ratnonc',status='unknown')
      data (arg(p),p=1,9) /0.05,0.1,0.2,0.3,0.4,0.6,1.0,1.2,2.0/

      write(2,*)'The distribution function of the ratio of '
      write(2,*)'quadratic form in noncentral normal variables'
      write(2,*)'    r                    dist'
      write(2,*)'         '

      do 22 p=1,9
      r=arg(p)
      l1=1.5
      l2=1.2
      d1=0.4
      d2=0.5
      l1p=r*1.2
      l2p=r*1.8
      d1p=0.5
      d2p=0.6
      d=d1**2+d2**2
      dp=d1p**2+d2p**2
      b1=2.0*l1
      b2=2.0*l2
      b1p=2.0*l1p
      b2p=2.0*l2p
      b=(1.0/b1+1.0/b2)/2.0
      bp=(1.0/b1p+1.0/b2p)/2.0
      be=b+bp
      sums=0.0
      do 20 s=0,13
      sumk=0.0
      do 18 k=0,13
      g5=s+k+1

      sumi=0.0
      do 2 i=0,s
      i1=(d1**2/2.0)**i*(d2**2/2.0)**(s-i)
      i2=i+1.0
      i2=dgamma(i2)
      i3=s-i+1.0
```

```
      i3=dgamma(i3)
      i4=b1**(i+0.5)*b2**(s-i+0.5)
      i5=i1/(i4*i2*i3)

      sumj=0.0
      call umach(2,nout)
      do 4 j=0,k
      j1=i+j+0.5
      j2=s-i+k-j+0.5
      j3=i+0.5
      j4=s-i+0.5
      j5=j+1.0
      j3=dbeta(j3,j5)
      j6=k-j+1.0
      j6=dbeta(j4,j6)
      j7=((b-1.0/b2)**(k-j))*(b-1.0/b1)**j
      j8=(1.0/(j3*j1))*(1.0/(j6*j2))*j7

      sumj=sumj+j8

 4    continue
      sumi=sumi+sumj*i5
 2    continue

      sumv=0.0
      do 16 v=0,11
      sumn=0.0
      do 14 n=0,11
      sumip=0.0
      do 6 ip=0,v
      i1p=(d1p**2/2.0)**ip*(d2p**2/2.0)**(v-ip)
      i2p=ip+1.0
      i2p=dgamma(i2p)
      i3p=v-ip+1.0
      i3p=dgamma(i3p)
      i4p=b1p**(ip+0.5)*b2p**(v-ip+0.5)
      i5p=i1p/(i4p*i2p*i3p)

      sumjp=0.0
      call umach(2,nout)
      do 8 jp=0,n
      j1p=ip+jp+0.5
      j2p=v-ip+n-jp+0.5
      j3p=ip+0.5
      j4p=v-ip+0.5
      j5p=jp+1.0
      j5p=dbeta(j3p,j5p)
      j6p=n-jp+1.0
      j6p=dbeta(j4p,j6p)
      j7p=((bp-1.0/b2p)**(n-jp))*(bp-1.0/b1p)**jp
      j8p=(1.0/(j1p*j5p))*(1.0/(j6p*j2p))*j7p

      sumjp=sumjp+j8p

 8    continue
      sumip=sumip+sumjp*i5p
 6    continue

      suml=0.0
      do 12 l=0,v+n
```

```
      g1=bp**(l+1)
      g7=(v+n+s+k-l+1)
      g2=(be**(v+n))/be**(l-1)
      g3=v+n+l-1
      g3=dbeta(g3,g5)
      g4=1.0/(g3*(g1*g7)*g2)

      suml=suml+g4
 12   continue
      sumn=sumn+sumip*suml
 14   continue
      sumv=sumv+sumn
 16   continue
      g8=be**(s+k)
      sumk=sumk+sumv*sumi/g8
 18   continue

      sums=sums+sumk
 20   continue
      g6=exp((d+dp)/2)

c     ||| The distribution function evaluated from |||
c     |||       the expression (3.11)              |||
      dist=sums/g6

      write(2,*),r,dist
 22   continue

      close (unit=2)
      stop
      end
```

Appendix C.4

```
c      ||||                 Example 3.5                          ||||
c      ||||     A representation of the density function         ||||
c      |||| in terms of the H-function applied to the ratio      ||||
c      ||||          of dependent quadratic forms                ||||

       program gcent
       integer nout,r,j,u,v,y,k,s
       real exu1,d,z,exu2,x
       double precision sumr,sumj,sumu,sumv,sumv1,sumv2
       double precision g1,g2,g3,g4,arg(1:8),g5,theta(1:8)
       double precision u1,u2,u3,u4,u5,m1,m2,m3,m4,f3,f4
       double precision p1,p2,p3,p4,p5,p6,p7,l1,l2,l3,f1,f2

       external dgamma,umach

       open(unit=2,file='gfunction',status='unknown')
       data (arg(n),n=1,8) /0.1,0.2,0.4,0.6,0.8,0.9,1.0,2.0/

       d=1.8

       do 100 n=1,6
       z=arg(n)
        call umach(2,nout)
       sumr=0.0
        do 30 r=0,9
       f1=r+1
       f1=dgamma(f1)
       sumj=0.0
       do 20 j=0,11

       sumu=0.0
       do 10 u=0,j

       g1=j-u+1
       g1=dgamma(g1)
       g2=0.5+r+j-u
       g2=dgamma(g2)
       g4=((1.0-1.0/(2.0*d))**u)*((1.0-1.0/d)**(j-u))
       g5=(g2/g1)*g4
       l2=5.5+r+j
       l2=dgamma(l2)

       call umach(2,nout)
       sumv=0.0
       do 60 v=3,11
       u2=r+v-1
       g3=3.5+v
       g3=dgamma(g3)
       u1=3.5
       u1=dgamma(u1)
       u1=g3/u1
       do 12 k=0,j+r-1
       u1=u1*(v+k+3.5)/(3.5+k)
12     continue
       u4=v+1
       u4=dgamma(u4)
```

```
      u5=(-1.0)**(v)*(z/(2.0*d))**v
      sumv=sumv+(u2/f1)*(u1/u4)*u5
 60   continue
      sumv1=0.0
      do 70 v=3,r+2
      m1=-u+2.5
      m1=dgamma(m1)
      m2=5.5
      m2=dgamma(m2)
      m1=m1/m2
      do 14 k=0,j+r-v-1
      m1=m1*(-u+2.5+k)/(5.5+k)
 14   continue
      m3=v+1
      m3=dgamma(m3)
      m4=r-v+3
      m4=dgamma(m4)
      sumv1=sumv1+(-1)**(v-1)*z**v*(m1/m3)*(12/g1)/m4
 70   continue
      sumv2=0.0
      do 75 v=0,u+2
      p1=v-r-j-3.5
      p1=dgamma(p1)
      p3=v-j-1.5
      p3=dgamma(p3)
      p4=j+r-v+4.5
      p7=(-1)**(u-v)*z**(p4)
      p5=u-v+3
      p5=dgamma(p5)
      p6=v+1
      p6=dgamma(p6)
      sumv2=sumv2+(p1/(p6*p3))*(12/g1)*p7/(p5*p4)
 75   continue
      l1=r+1
      l3=((1.0-1.0/(2.0*d))**u)*((1.0-1.0/d)**(j-u))*(l1)
      sumu=sumu+((2.0*d)**(2)*g5*sumv+l3*(sumv1+sumv2))
     &          *(u+1.0)*(u+2.0)
 10   continue

      sumj=sumj+sumu

 20   continue

      f3=dgamma(0.5)
      f4=(2.0*d)**(r+5)*d**0.5
      sumr=sumr+sumj/(2.0*f3*f4)
 30   continue

c     ||| Application of the representation (3.34) |||

      write(2,*)'z                sumr '
      write(2,*),z,sumr
100   continue


      close(unit=2)
      stop
      end
```

Appendix C.5

```
c          ||||             Example 3.8                  ||||
c          |||| The density function of the ratio        ||||
c          ||||   of independent quadratic forms          ||||
c          |||| Representations (3.84) and (3.88)         ||||
      program dist7
      integer nout,n,u,y,w,v,i,j,s
      double precision p1,p2,c,b1,b2,t,ex1,g5,b3,x,r,sumi1
      double precision sumv,sumw,sumj,sumi,ex2,sumv1,sumw1
      double precision g1,g2,g3,g4,u1,u2,u3,arg(1:10),ex3
      real dens,fest,fex
      external dgamma,umach
      open(unit=2,file='dist7',status='unknown')
      data (arg(n),n=1,8) /0.1,0.2,0.4,0.5,0.6,0.8,1.0,1.7/
      p1=2.0
      p2=3.0
      p3=1
      t=3.8
      b1=2*p1
      b2=2*p2
      b3=2*p3
      do 20 n=1,10
      c=-arg(n)
      sumv=0.0
      sumv1=0.0
      do 10 v=0,28

      sumw=0.0
      sumw1=0.0
      do 8 w=0,28
      sumj=0.0
      do 6 j=0,25
      call umach(2,nout)
      u=6+v+w+j
      y=6+w+j
      g3=2.0**(6+v+w+j)
      x=1.0/(g3)
      do 12 s=y,u-1
      x=x*s
12    continue
      sumj=sumj+x
 6    continue
      sumi1=0.0
      sumi=0.0
      do 4 i=0,w

      ex1=((1.0-t/b3)**i)*((1.0+c*t/b1)**(w-i))
      sumi=sumi+(i+1)*(i+2)*ex1*(w-i+1)
      ex3=3.0+c*v*t/(b2+c*t)+(c*(w-i)*t)/(b1+c*t)
      sumi1=sumi1+ex1*ex3*(i+1.0)*(i+2.0)*(w-i+1)
 4    continue
      sumw=sumw+sumj*sumi
      sumw1=sumw1+sumj*sumi1
 8    continue
      call umach(2,nout)
      g5=(1.0+t*c/b2)
      g5=g5**v
```

```
      g4=1.0+v
      g4=dgamma(g4)

      sumv=sumv+g5*sumw/(g4*((b3**3)*(b1**2)*b2*2.0))
      sumv1=sumv1+g5*sumw1/(g4*((b3**3)*(b1**2)*b2*2.0))
   10 continue
      fest=-sumv*(c**3*t**6)
      dens=sumv1*(c**2*t**6)

c     |||        Evaluation of the expression obtained   |||
c     ||| by the transformation of variables technique |||

      u2=3*p2/(16.0*((p2+p1)**2)*((-c/b2+0.5)**4))
      fex=u2
      write(2,*), dens,fest,c,fex
   20 continue
      close (Unit=2)
      stop
      end
```

Appendix C.5

```
c          ||||           Example 3.8            ||||
c          ||||  The density function of the ratio ||||
c          ||||    of independent quadratic forms  ||||
c          ||||  Representations (3.87) and (3.88) ||||

       program dist6
       integer nout,n,u,y,w,v,i,j,s
       double precision p1,p2,c,b1,b2,t,ex1,g5,b3,x,r,sumi1
       double precision sumv,sumw,sumj,sumi,ex2,sumv1,sumw1
       double precision g1,g2,g3,g4,u1,u2,u3,arg(1:10)
       real dens,fest,fex
       external dgamma,umach
       open(unit=2,file='dist6',status='unknown')
       data (arg(n),n=1,8) /0.9,1.0,1.2,1.7,2.0,2.5,3.5,5.0/
       p1=2.0
       p2=3.0
       p3=1
       t=3.0
       b1=2*p1
       b2=2*p2
       b3=2*p3
       do 20 n=1,10
       c=-arg(n)
       sumv=0.0
       sumv1=0.0
       do 10 v=0,31
        sumw=0.0
        sumw1=0.0
       do 8 w=0,29
       sumj=0.0
       do 6 j=0,25
       call umach(2,nout)
       u=6+v+w+j
       y=2+v+j
       g3=2.0**(6+v+w+j)
       x=1.0/(g3)
       do 12 s=y,u-1
       x=x*s
 12    continue
       sumj=sumj+x
  6    continue
       sumi1=0.0
       sumi=0.0
       do 4 i=0,w
       ex2=((i*t)/(c*b3+t)+3.0)
       ex1=((1.0+t/(b3*c))**i)*((1.0-t/b1)**(w-i))
       sumi=sumi+(i+1)*ex1*(i+2)*(w-i+1)
       sumi1=sumi1-(i+1)*(i+2)*ex1*ex2*(w-i+1)
  4    continue

       call umach(2,nout)
       g4=w+5
       g4=dgamma(g4)
```

```
      sumw=sumw+sumi*sumj/g4
      sumw1=sumw1+sumi1*sumj/g4
    8 continue
      sumv=sumv+sumw*(1.0-t/b2)**v
      sumv1=sumv1+sumw1*(1.0-t/b2)**v
   10 continue
      fest=1.0+sumv*(t**6)/((b3**3)*(c**3)*(b1**2)*b2*2.0)
      dens=sumv1*(t**6)/((b3**3)*(c**4)*(b1**2)*b2*2.0)
c     |||      Evaluation of the expression obtained      |||
c     ||| by the transformation of variables technique    |||
      u2=3*p2/(16.0*((p2+p1)**2)*((-c/b2+0.5)**4))
      fex=u2
      write(2,*), dens,fest,c,fex
   20 continue
      close (Unit=2)
      stop
      end
```

Appendix C.5

```
c         ||||            Example 3.8             ||||
c         |||| The density function of the ratio  ||||
c         ||||   of independent quadratic forms   ||||
c         |||| Representations (3.77) and (3.99)  ||||

      program dist2
      integer nout,n,u,y,w,v,i,j,s
      double precision p1,p2,c,b1,b2,t,ex1,g5,b3,x,r,sumi1
      double precision sumv,sumw,sumj,sumi,ex2,sumv1,sumw1
      double precision g1,g2,g3,g4,u1,u2,u3,arg(1:10)
      real dens,fest,fex
      external dgamma,umach
      open(unit=2,file='dist5',status='unknown')
      data (arg(n),n=1,8) /0.6,0.9,1.0,1.2,1.7,2.0,2.5,5.0/
      p1=2.0
      p2=3.0
      p3=1
      t=1.6
      b1=2*p1
      b2=2*p2
      b3=2*p3
      do 20 n=1,10
      c=arg(n)
      sumv=0.0
      sumv1=0.0
      do 10 v=0,31
       sumw=0.0
       sumw1=0.0
      do 8 w=0,29
      sumj=0.0
      do 6 j=0,25
      call umach(2,nout)
      u=6+v+w+j
      y=3+v+j
      g3=2.0**(6+v+w+j)
      x=1.C/(g3)
      do 12 s=y,u-1
      x=x*s
 12   continue
      sumj=sumj+x
  6   continue
      sumi1=0.0
      sumi=0.0
      do 4 i=0,w
      ex2=((i*t)/(c*b3-t)-3.0)
      ex1=((1.0-t/(b3*c))**i)*((1.0-t/b2)**(w-i))
      sumi=sumi+(i+1)*ex1*(i+2)
      sumi1=sumi1+(i+1)*(i+2)*ex1*ex2
  4   continue

      call umach(2,nout)
      g4=w+4
      g4=dgamma(g4)

      sumw=sumw+sumi*sumj/g4
```

```
      sumw1=sumw1+sumi1*sumj/g4
    8 continue
      sumv=sumv+(v+1)*sumw*(1.0-t/b1)**v
      sumv1=sumv1+(v+1)*sumw1*(1.0-t/b1)**v
   10 continue
      fest=sumv*(t**6)/((b3**3)*(c**3)*(b1**2)*b2*2.0)
      dens=sumv1*(t**6)/((b3**3)*(c**4)*(b1**2)*b2*2.0)

c     |||     Evaluation of the expression obtained      |||
c     ||| by the transformation of variables technique |||

      u1=c*3.0/((((c/b1+0.5)**5)*8.0*(p2+p1)*p1)
      u2=3*p2/(16.0*((p2+p1)**2)*((c/b1+0.5)**4))
      fex=u2+u1
      write(2,*), dens,fest,c,fex
   20 continue
      close (Unit=2)
      stop
      end
```

Appendix C.5

```
c         ||||          Example 3.8             ||||
c         ||||  The density function of the ratio ||||
c         ||||   of independent quadratic forms   ||||
c         |||| Representations (3.84) and (3.98)  ||||

      program dist3
      integer nout,n,u,y,w,v,i,j,s
      double precision p1,p2,c,b1,b2,t,ex1,g5,b3,x,r,sumi1
      double precision sumv,sumw,sumj,sumi,ex2,sumv1,sumw1
      double precision g1,g2,g3,g4,u1,u2,u3,arg(1:10),ex3
      real dens,fest,fex
      external dgamma,umach
      open(unit=2,file='dist4',status='unknown')
      data (arg(n),n=1,8) /0.1,0.2,0.4,0.6,0.8,1.0,1.2,1.7/
      p1=2.0
      p2=3.0
      p3=1
      t=3.6
      b1=2*p1
      b2=2*p2
      b3=2*p3
      do 20 n=1,10
      c=arg(n)
      sumv=0.0
      sumv1=0.0
      do 10 v=0,28

      sumw=0.0
      sumw1=0.0
      do 8 w=0,28
      sumj=0.0
      do 6 j=0,25
      call umach(2,nout)
      u=6+v+w+j
      y=5+w+j
      g3=2.0**(6+v+w+j)
      x=1.0/(g3)
      do 12 s=y,u-1
      x=x*s
 12   continue
      sumj=sumj+x
  6   continue
      sumi1=0.0
      sumi=0.0
      do 4 i=0,w

      ex1=((1.0-t/b3)**i)*((1.0-c*t/b2)**(w-i))
      sumi=sumi+(i+1)*(i+2)*ex1
      ex3=3.0-c*v*t/(b1-c*t)-(c*(w-i)*t)/(b2-c*t)
      sumi1=sumi1+ex1*ex3*(i+1.0)*(i+2.0)
  4   continue
      sumw=sumw+sumj*sumi
      sumw1=sumw1+sumj*sumi1
  8   continue
      call umach(2,nout)
```

```
      g5=(1.0-t*c/b1)
      g5=g5**v
      g4=1.0+v
      g4=dgamma(g4)

      sumv=sumv+g5*sumw/(g4*((b3**3)*(b1**2)*b2*2.0))
      sumv1=sumv1+g5*sumw1/(g4*((b3**3)*(b1**2)*b2*2.0))
 10   continue
      fest=1.0-sumv*(c**3*t**6)
      dens=-sumv1*(c**2*t**6)

c      |||      Evaluation of the expression obtained     |||
c      ||| by the transformation of variables technique |||

      u1=c*3.0/((((c+p1)/b1)**5)*8.0*(p2+p1)*p1)
      u2=3*p2/(16.0*((p2+p1)**2)*((c/b1+0.5)**4))
      fex=u2+u1
      write(2,*), dens,fest,c,fex
 20   continue
      close (Unit=2)
      stop
      end
```

Appendix C.6

```
c       ||||                Example 4.2                  ||||
c       ||||  Lag 2 serial correlation coefficient       ||||
c       ||||  of a series of observation of length 7     ||||
c       ||||  Expression (4.55),   c<0                   ||||
        program distr K inverse
        integer nout,n,w,v,i,j,m,l
        double precision p1,p2,c,b1,b2,t,ex1,g5,b3,x,r,sumi1
        double precision d1,d2,d3,d4,d5,d6,f1,f2,u,y
        double precision e1,e2,e3,e4,e5,e6,e7,e8
        double precision sumv,sumw,sumj,sumi,ex2,sumv1,sumw1
        double precision g1,g2,g3,g4,u1,u2,u3,arg(1:10),ex3
        real dens,fest,fex,b4,b5,p4,p5,p3
        external dgamma,umach,dbeta
        open(unit=2,file='distki7',status='unknown')
        data (arg(n),n=1,10) /0.05,0.1,0.15,0.2,0.25,0.3,0.35,0.4,
     &     0.45,0.5/
        write(2,*),'   c','distribution n=7,k=7       fest'
        do 100 m=4,4
        c=-arg(m)
        p1=1.0-0.7515/c
        p2=1.0-0.3090/c
        p3=-1.0-0.8090/c
        p4=-1.0-0.6905/c
        p5=-1.0-0.2753/c
        p6=1.0
        t=1.2
        b1=2*p1
        b2=2*p2
        b3=2*p3
        b4=2*p4
        b5=2*p5
        b6=2*p6
        f1=dgamma(0.5)
        sumv=0.0
        do 10 v=0,40

        sumw=0.0
        do 8 w=0,40
        sumj=0.0
        do 6 j=0,40
        call umach(2,nout)
        u=3.0+v+w+j
        y=2.5+v+j
        e4=w+1.5
        e4=dbeta(e4,y)
        g3=2.0**(3+v+j)
        x=g3*e4*u
        sumj=sumj+1.0/x
    6   continue
        sumv1=0.0
        do 3 i=0,v
        sumv2=0.0
        do 5 n=0,v-i
        d1=f1
        do 41 l=1,i
        d1=(d1*(l-0.5))/l
```

```
 41   continue
      d2=f1
      do 42 l=1,n
      d2=(d2*(l-0.5))/l
 42   continue
      d3=f1
      do 43 l=1,v-i-n
      d3=(d3*(l-0.5))/l
 43   continue
      d6=(1.0-t/b1)**i*(1.0-t/b2)**n*(1.0-t/b6)**(v-i-n)
      sumv2=sumv2+d1*d2*d3*d6
  5   continue
      sumv1=sumv1+sumv2
  3   continue

      sumi=0.0
      do 4 i=0,w
      sumi1=0.0
      do 9 n=0,w-i
      e1=f1
      do 31 l=1,i
      e1=(e1*(l-0.5))/l
 31   continue
      e2=f1
      do 32 l=1,n
      e2=(e2*(l-0.5))/l
 32   continue
      e3=f1
      do 33 l=1,w-i-n
      e3=(e3*(l-0.5))/l
 33   continue
      e7=(1.0-t/b3)**i*(1.0-t/b4)**n*(1.0-t/b5)**(w-i-n)
      sumi1=sumi1+e1*e2*e3*e7
  9   continue
      sumi=sumi+sumi1
  4   continue
      sumw=sumw+sumi*sumj/(2.0**w)
  8   continue
      call umach(2,nout)
      sumv=sumv+sumw*sumv1
 10   continue
      f2=(b1*b2*b3*b4*b5*b6)**0.5*f1**6
      fest=(t**3.0/f2)*sumv

      write(2,*),c,fest
100   continue
      close (Unit=2)
      stop
      end
```

Appendix C.6

```
c          ||||            Example 4.2                       ||||
c          ||||  Lag 2 serial correlation coefficient        ||||
c          ||||  of a series of observation of length 7      ||||
c          ||||  Representation (4.55),  c>0                 ||||
      program distr7 of ind ratio
      integer nout,n,w,v,i,j,s,l,m,k,v1,v2
      double precision p1,p2,c,b1,b2,t,ex1,g5,b3,x,r,sumi1
      double precision d1,d2,d3,d4,d5,d6,f1,f2,u,y
      double precision e1,e2,e3,e4,e5,e6,e7,e8,sumv2
      double precision sumv,sumw,sumj,sumi,ex2,sumv1,sumw1
      double precision g1,g2,g3,g4,u1,u2,u3,arg(1:10),ex3
      real dens,fest,fex,b4,b5,p4,p5,p3,p6,b6
      external dgamma,umach,dbeta
      open(unit=2,file='distk7',status='unknown')
      data (arg(n),n=1,10) /0.1,0.12,0.15,0.2,0.25,0.3,0.35,0.4,
     &   0.45,0.48/
      do 100 m=1,10
      p1=0.7515-c
      p2=0.3090-c
      p3=0.8090+c
      p4=0.6905+c
      p5=0.2753+c
      p6=c
      t=0.2
      b1=2*p1
      b2=2*p2
      b3=2*p3
      b4=2*p4
      b5=2*p5
      b6=2*p6
      f1=dgamma(0.5)
      sumv=0.0
      do 10 v=0,25
      sumw=0.0
      do 8 w=0,25
      sumj=0.0
      do 6 j=0,25
      call umach(2,nout)
      u=3.0+v+w+j
      y=2.0+v+j
      e4=w+2.0
      e4=dbeta(e4,y)
      g3=2.0**(3.0+v+j)
      x=g3*e4*u
      sumj=sumj+1.0/x
    6 continue
      sumv1=0.0
      do 41 v1=0,v
      d1=f1
      do 36 l=1,v1
      d1=(d1*(l-0.5))/l
   36 continue
      d3=f1
      do 38 l=1,v-v1
      d3=(d3*(l-0.5))/l
   38 continue
```

```
      d6=(1.0-t/b1)**v1*(1.0-t/b2)**(v-v1)
      sumv1=sumv1+d1*d3*d6
 41   continue
      sumi=0.0
      do 4 i=0,w
      sumi1=0.0
      do 9 n=0,w-i
      sumv2=0.0
      do 11 v2=0,w-i-n
      e1=f1
      do 31 l=1,i
      e1=(e1*(l-0.5))/l
 31   continue
       e2=f1
      do 32 l=1,n
      e2=(e2*(l-0.5))/l
 32   continue
      e3=f1
      do 33 l=1,v2
      e3=(e3*(l-0.5))/l
 33   continue
       e4=f1
      do 37 l=1,w-i-n-v2
      e4=(e4*(l-0.5))/l
 37   continue
      e7=(1.0-t/b3)**i*(1.0-t/b4)**n*(1.0-t/b5)**v2
     &  *(1.0-t/b6)**(w-i-n-v2)
      sumv2=sumv2+e1*e2*e3*e4*e7
 11   continue
      sumi1=sumi1+sumv2
  9   continue
      sumi=sumi+sumi1/(2.0**w)
  4   continue

      sumw=sumw+sumj*sumi
  8   continue
      call umach(2,nout)
      sumv=sumv+sumw*sumv1
 10   continue
      f2=(b1*b2*b3*b4*b5*b6)**0.5*f1**6
      fest=(t**3.0/f2)*sumv

      write(2,*),c,fest
100   continue
      close (Unit=2)
      stop
      end
```

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