1992

Some New Results In Topological Field Theories

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SOME NEW RESULTS IN TOPOLOGICAL FIELD THEORIES

by

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Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

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The University of Western Ontario
London, Ontario
October 1992

ISBN 0-315-81267-2

ABSTRACT

We report two results in topological field theories. Using the Hamiltonian approach, we reinterpret the partition functions of cohomology models as various character-valued indices. Secondly, We also show that, in the presence of Gribov zero modes, the topological field theories without regulating terms permit spontaneous symmetry breaking. The relation to reducible configurations is discussed.

ACKNOWLEDGMENTS

I would like to thank my supervisor, Dr. H.C. Lee, for his generous support and guidance as well as countless stimulating discussions of various aspects in the thesis. I also thank Dr. G.D. McKeon and Dr. S. Wu for very helpful discussions about topological field theories and Dr. V.A. Miransky for conversations on the general features of spontaneous symmetry breaking.

Finally, I would like to express my sincere appreciations to my family, my friends and my colleagues for their constant encouragement and invaluable support during the course of my Ph.D. program.

This research was funded in part by the Natural Sciences and Engineering Research Council of Canada and by the Ontario Ministry of Colleges and Universities through its Graduate Scholarship Program.

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Chapter 1

Introduction

The history of the developments of quantum field theories is closely connected to their successful applications to many infinite-dimensional physical systems. For many years, it has been a hope that eventually all of the continuum physical systems may be described by quantum field theories. While such a hope seems reasonable so long as perturbative calculation applies, it faces extreme difficulties when such calculation fails: in such situations nonperturbative regions are encountered, and we do not even know whether a quantum field theory exists [1]. An example is the gravitational theories where, due to the high nonlinearity, a nonperturbative treatment is deemed necessary. However, so far it is not known how to handle these theories non-perturbatively (except for the 2-dimensional gravitation theory, which has recently been solved non-perturbatively using the matrix model method [2-4]). It has been proposed that these nonperturbative regions may well be associated with different phases of the physical system considered, and these phases may appear very different from the one described by the naive Lagrangian (for the perturbative region) [5]. However, for each of these regions there could be a field theory formulation, with a Lagrangian looking entirely distinct from the naive one, that gives an equivalent description for the appropriate phase in its own perturbative region. Here, giving an "equivalent description" means producing the same behaviour for all physical observables [1].

One can also look at these phases from the symmetry point of view. In general, a phase with a higher symmetry, which usually means having fewer degrees of freedom, is associated with a higher energy scale. When the energy decreases, the system presumably undergoes a phase transition that leads itself to a phase with a lower symmetry. While the exact mechanism of this phase transition is still far from being understood, it has been suggested [5,6] that the phase with the highest symmetry should be the topological phase – a phase with only excitations that are topological in nature and a finite number of degrees of freedom. The field theories which may be used to describe such phases are the topological field theories, and this motivates the study o. ιopological quantum field theories (TQFTs).

It should be made clear that the idea that a topological phase transition could take place, although tantalizing, is so far only a conjecture. It is not even clear in what sense the term topological phase transition exactly means. Therefore, further clarification of this picture, such as a proof of the existence of the spontaneous breaking of the topological symmetry, would be very useful for setting it on a firmer ground.

TQFTs are characterized by observables (correlation functions) which depend only on global features of the space in which these theories are defined. This means that general covariance is not broken, and the observables are independent of any metric used for the space. Generally, a theory depending only on the global structures of the space does not necessarily mean that it is independent of the metric. For example, it can be dependent on the moduli of the space [38]. However, the TQFTs used for topological transition seem to be theories independent of the metric, and we will restrict our consideration solely on this type of models. It is an amazing result that one can achieve general covariance in a quantum theory without necessarily integrating over the metric, as one does in quantum gravity. Besides their significance in physics, TQFTs are also of prime interest in mathematics. TQFTs are quite generally soluble, and can be computable by standard techniques used in quantum field theory. From a purely mathematical point of view, TQFTs provide novel representations of certain global invariants whose properties are often transparent in the path integral approach [7,8]. Although such derivations cannot be considered rigorous, they can be checked by other physical (Hamiltonian) and mathematical methods.

The origin of TQFT can be traced to the works of A. Schwarz and E. Witten. Schwarz [9] showed in 1978 that the Ray-Singer torsion - a particular topological invariant - could be represented as the partition function of a certain quantum field theory. Quite distinct from this observation was the work of Witten [10] in 1982, in which a framework was given for understanding Morse theory in terms of supersymmetric quantum mechanics. These two constructions represent the prototypes of all known TQFTs.

The significance of Witten's approach was realized by A. Floer [11] who applied similar techniques in an infinite dimensional setting to obtain new results concerning the topology of 3-manifolds. It was thought that this work should be related in some way to the findings of S. Donaldson [7] on the geometry of 4-manifolds. In an influential paper, M. Atiyah [8] conjectured that a quantum field theory might provide an understanding of these results, and he gave a nonrelativistic Hamiltonian in 3-dimensions whose ground states are the Floer groups. A 4-dimensional relativistic Lagrangian description of Donaldson's work was supplied by Witten [5] in 1987, which established the link between the earlier three and four dimensional results.

Quite apart from these developments, a new polynomial invariant of knots was constructed by V. Jones [12] in 1985. It is noteworthy that this work was strongly influenced by problems in 2-dimensional statistical mechanics. As knots are objects living intrinsically in 3-dimensions, a longstanding puzzle for knot theorists is to understand these invariants from a 3-dimensional point of view. In a classic paper, Witten again [13] provided the answer by constructing knot polynomials as correlations of Wilson line operators in a 3-dimensional quantum field theory defined by the Chern-Simons action. Moreover, this theory incorporates significant generalizations of the previously known invariants. While these mathematical advances are self-evident, Chern-Simons theory also provides a unifying 3-dimensional viewpoint for 2-dimensional conformal field theory, as well as new results on quantum gravity in three dimensions.

Other examples of TQFTs were also given by Witten. The topological sigma models [6] were used to construct invariants of complex manifolds and related to other works of Floer [14]. Also of importance are the 2-dimensional topological gravity models [15,16]. There are many intriguing conjectures relating 2-dimensional topological gravity with string theory. One of which is that noncritical string theory with a certain matter content is equivalent to topological gravity coupled to topological matter [16-18].

These developments have made it very clear that TQFTs, which have been under intensive study in many directions for the past few years, are becoming one of the most fruitful areas in both contemporary physics and mathematics. Many important results have been surprisingly linked together using TQFTs, which is an outcome not suspected a few years ago. Since the structure of TQFTs so far is still not completely understood, one may expect that, as a deeper understanding of these theories is achieved, more interesting results will be discovered that could significantly extend our understanding in many areas of mathematics and physics in ways we cannot now imagine.

A complete understanding of TQFTs faces many problems. From a mathematical point of view, perhaps the most important problem is to give a mathematically rigorous treatment to the infinite dimensional configuration space in TQFTs. For instance, a complete treatment should include considering all the configurations involved, particularly the reducible configurations which have currently been ignored due to lack of a proper method to treat them [5]. While such a treatment is not yet available, we hope that further discoveries of relationships between topological invariants and TQFTs may help us understand its mathematical structure more. Since the topological invariant associated with each cohomological theory is typically of the Donaldson's type, it will be interesting to see if there is another type of topological invariants which have been independently defined that can also be associated with the theory [19]. This will at least show the mathematical equivalence between these two types of invariants. As far as physics is concerned, the most important problem is to understand the phase transition mentioned earlier and, in particular, to answer the question of how topological symmetry breaks spontaneously. So far, there is no satisfactory mechanism available in this regard [24-26].

This thesis is primarily concerned with the problems posed above. Chapter 2 is a summary of known results in TQFT. Following a brief discussion of some general features of a typical TQFT, we study in particular the 1-dimensional supersymmetric quantum mechanics for the reason that it is the simplest TQFT; it will be extensively used as a tractable model for further studies of various properties of TQFT in the subsequent chapters. Chapter 3 is devoted to a further investigation of some new topological invariants associated with every TQFT [19]. We find that partition functions of the cohomology TQFT may be interpreted as character-valued index theorems. In this context, the 1-dimensional supersymmetric quantum mechanics is further elaborated. In Chapter 4 we examine the idea of topological phase transition and address the question of how topological symmetry can be spontaneously broken. Generalizing a conventional approach to symmetry breaking, we ascertain the possibility of topological symmetry breaking by examining the response of expectation values of some variables in response to a vanishingly small symmetry-breaking term added in the action [21]. As examples, we study this mechanism in detail in the 0-dimensional and the 1-dimensional topological models. The same mechanism should be applicable to other models, too. In particular, we relate the possibility of symmetry breaking to the existence of reducible configurations and argue that it is these configurations that are responsible for the phase transitions. The fact that these reducible configurations have in previous studies been ignored [5] should partly address the puzzling question of why, so far, no symmetry breaking has been found in the context of conventional perturbative calculations. We summarize in Chapter 5 with some comments on problems that remain to be solved and possible directions for future research.

Chapter 2

General Aspects Of Topological Quantum Field Theory

In this chapter, we first present some general definitions and properties common to all topological theories. Among these are the simple formal arguments which establish, with some exceptions, the topological nature of a given model. We then use the 1-dimensional supersymmetric quantum mechanics as an example to illustrate these features.

2.1 General Properties of Topological Field Theories

Let us begin by recalling the essential ingredients present in a conventional BRST approach to quantization [23]. In such a formulation, we denote the collective field content by Φ, which includes both the matter fields and the ghosts. Corresponding to the local gauge symmetry, the BRST operator Q can be defined with the nilpotence condition; i.e. $Q^2 = 0$. The variation of any functional F of the fields Φ is denoted by $\delta F = \{Q, F\}$, where the bracket notation stands for the graded commutator with the fermionic charge Q. The complete quantum action, denoted by I, which comprises the classical action S_c and the necessary gauge fixing and ghost terms, is by construction Q-invariant.

The physical Hilbert space is composed of states $\{|phy\rangle\}$ defined by the condition $Q|phy\rangle = 0$; a physical state of the form $|phy\rangle' = |phy\rangle + Q|\chi\rangle$ is regarded as equivalent to $|phy\rangle$, for any state $|\chi\rangle$. A state which is annihilated by Q is said to be Q-closed, while a state of the form $Q|\chi\rangle$ is called Q-exact. This equivalence relation thus defines the physical Hilbert space as composed of Q-cohomology classes, that is, states which are Q-closed modulo Q-exact states.

Now, from the BRST invariant vacuum, it follows immediately that the vacuum expectation value of $\{Q, F\}$, for any functional F, is zero, i.e.

$$\langle 0|\{Q, F\}|0\rangle \equiv \langle\{Q, F\}\rangle = 0. \tag{2.1}$$

Let us now assume that our theory is defined on some n-dimensional manifold M, with a metric $g_{\alpha\beta}$. The energy momentum tensor $T_{\alpha\beta}$ is defined by the change in the action under an infinitesimal deformation of the metric

$$\delta_g I = \frac{1}{2}\int_M d^n x\sqrt{g}\,\delta g^{\alpha\beta}T_{\alpha\beta} \tag{2.2}$$

Finally, we assume that the functional measure in the path-integral is both Q-invariant and metric independent. This means

$$\int [D\Phi] Q(anything) = 0. \tag{2.3}$$

We are now in a position to define what we mean by a topological field theory. A topological field theory consists of

(a) A collection of fields Φ (which are Grassmann graded) defined on a Riemannian manifold (M, g).

(b) A nilpotent operator Q, usually referred as BRST operator, which is odd with respect to the Grassmann grading.

(c) Physical states defined to be Q-cohomology classes.

(d) An energy-momentum tensor which is Q-exact, i.e.

$$T_{\alpha\beta} = \{Q, V_{\alpha\beta}\}, \tag{2.4}$$

for some functional $V_{\alpha\beta}$ of the fields and the metric.

We now consider the change in the partition function

$$Z = \int [D\Phi] e^{-I}, \tag{2.5}$$

under an infinitesimal change in the metric. We have

$$\begin{aligned}\delta_g Z &= \int [D\Phi] e^{-I}\left(-\frac{1}{2}\int_M d^n x\sqrt{g}\,\delta g^{\alpha\beta} T_{\alpha\beta}\right)\\ &= \int [D\Phi] e^{-I}\left(-\frac{1}{2}\int_M d^n x\sqrt{g}\,\delta g^{\alpha\beta}\{Q, V_{\alpha\beta}\}\right)\\ &= \int [D\Phi] e^{-I}\{Q,\chi\}\\ &=<\{Q,\chi\}>=0\end{aligned} \tag{2.6}$$

where $\chi = -\frac{1}{2}\int_M d^n x\sqrt{g}\,\delta g^{\alpha\beta} V_{\alpha\beta}$. We thus see that given the BRST invariance of the vacuum, the partition function is metric independent. That is, the partition function depends not on the local structure of the manifold, but only on global properties: Z is a topological invariant. At this point, however, we should perhaps clarify the use of the terminology 'topological'. In all cases, our theory is defined with respect to a 'base' manifold M. This could be, for example, a Riemannian manifold with metric g, or a more general situation. What we have shown above is that if the condition (a)-(d) are satisfied, then the partition function takes a constant value on the space of all metrics on M. It is to this metric independence that the term topological invariance is referred.

Since the partition function Z can be thought as the 1-point correlation function of the identity operator 1, we can ask the question as to whether there exist other metric independent correlation functions in the theory: Does a given theory have a richer set of topological invariants?

Consider the vacuum expectation value of an observable F, (also called the correlation function for F),

$$< F >= \int [D\Phi] e^{-I} F(\Phi). \tag{2.7}$$

We wish to determine sufficient conditions for this expectation value to be a topological invariant, i.e., for $\delta_g < F >$ to be zero. Proceeding as before, we find

$$\delta_g < F >= \int [D\Phi] e^{-I}(\delta_g F - (\delta_g I)F). \tag{2.8}$$

Assuming that F enjoys the properties,

$$\delta_g F = \{Q, R\}, \tag{2.9a}$$

$$\{Q, F\} = 0, \tag{2.9b}$$

for some R, we have that

$$\delta_g < F > = < \{Q, R + \setminus F\} > = 0. \tag{2.10}$$

Now, clearly if $F = \{Q, F'\}$, for some F', we automatically have $< F > = 0$. Hence, our real interest is in Q-cohomology classes of operators i.e., BRST invariant operators which are not Q-exact and satisfy $\delta_g F = \{Q, R\}$. In deriving the above relations, we should note that we made essential use of the (assumed) metric independence of the functional measure, i.e., equation (2.3). To show that this assumption is in fact realized, one needs to check for metric anomalies [5].

Another important feature for TQFTs is that the action of the theory is determined only up to a BRST cohomology class. This means that we have the freedom to add an arbitrary Q-exact term without changing the values of the correlation function for F. To see this, let I' be a new action which differs from the original action by an exact term $\{Q, W\}$ for some W. The correlation function for F under I' is

$$< F >_{I'} = \int [D\Phi] F e^{-I-\{Q,W\}} = < F e^{-\{Q,W\}} >_I . \tag{2.11}$$

$e^{-\{Q,W\}}$ in (2.11) can be expanded as

$$e^{-\{Q,W\}} = \sum_{n=0}^{\infty} \frac{1}{n!} \{Q, W\}^n = 1 + \{Q, Y\}, \tag{2.12}$$

where

$$Y = \sum_{n=1}^{\infty} W \frac{1}{n!} \{Q, W\}^{n-1} \tag{2.13}$$

We now substitute (2.12) into (2.11)

$$< F >_{I'} = < F e^{-\{Q,W\}} >_I$$

$$= < F >_I + < F\{Q,Y\} >_I$$

$$= < F >_I + < \{Q, FY\} >_I = < F >_I, \tag{2.14}$$

where we have again made use of the assumed measure invariance (2.3). Therefore, the TQFT described by I is equivalent to the TQFT described by I'.

Usually, a TQFT can be classified as being either of the cohomology type or the Chern-Simons type. In this report, we shall primarily deal with TQFT of cohomology type. In this case, the complete quantum action I, which comprises the classical action plus all the necessary gauge fixing and ghost terms can be written as a BRST commutator, i.e.

$$I = \{Q, V\} \tag{2.15}$$

for some functional $V(\Phi, g)$ of the fields. In additional to the freedom of varying V, as we considered above, there is also the freedom to add topological terms to the action (2.15); i.e., terms for which the Lagrangian is locally a total derivative. Such terms change neither the equations of motion nor the energy momentum tensor. Clearly, as a consequence of (2.15), we have

$$T_{\alpha\beta} = \{Q, \frac{2}{\sqrt{g}} \frac{\delta V}{\delta g^{\alpha\beta}}\}, \tag{2.16}$$

which ensures us of the topological nature of the model. However, the stronger condition (2.15) allows us to prove that the partition function Z and the above mentioned correlation functions $< F >$ are also exact at the semiclassical level. By introducing a dimensionless parameter t (equivalently, $\frac{1}{\hbar}$) and rescaling the action $I \to tI$, we can consider the variation of Z under a change in t:

$$\delta_t Z = -\int [D\Phi] e^{-tI} I \delta t$$

$$= -\int [D\Phi] e^{-tI} \{Q, V\} \delta t = 0. \tag{2.17}$$

This shows that Z is independent of t as long as t is nonzero (one cannot set t to zero, since one needs a damping factor in the path integral). One can then evaluate Z in the large t limit. Such a limit corresponds to the semiclassical approximation, in which the path integral is dominated by fluctuations around the classical minima. This approximation is exact for Witten type theories. A similar argument applied to (2.7) establishes the semiclassical exactness of the correlation functions $< F >$.

Typically, these classical minima give rise to the notion of moduli spaces for the classical equations. It turns out [5] that these moduli spaces play the major role in evaluating all the correlators. It can be shown that the quantization of a Witten type theory can be done entirely within the moduli space. In this sense, a Witten type theory can be said to be essentially a quantum mechanical theory, rather than a quantum field theory as it appears to be.

2.2 Supersymmetric Quantum Mechanics

In the previous section, we outlined the general features common to all the topological field theories that have been studied. As an example of a topological field theory we now consider a relatively simple and tractable model, namely supersymmetric quantum mechanics. Although this model is interesting in its own right, the rationale for studying it in detail, in the present context, is to illustrate the fundamental features of Witten type theories. Supersymmetric quantum mechanics was identified as a topological field theory in [20], and as such, the techniques generally used in a TQFT are also applicable in this model. This example allows us to introduce these techniques in a relatively simple setting.

The action of supersymmetric quantum mechanics that we shall use has the form

$$I = \int dt\,\left(ib_i(\dot{x}^i + g^{ij}\partial_j w(x)) + \psi_i \mathcal{D}^i_j \phi^j\right)$$

$$+\alpha(b_i b_j g^{ij} + \frac{1}{2}R^{ij}{}_{kl}\phi^k\phi^l\psi_i\psi_j), \tag{2.18}$$

$$\mathcal{D}^i_j = \delta^i_j \frac{d}{dt} + \dot{x}^k\Gamma^i_{jk} + D_j D^i w(x),$$

where D_i is the Riemannian covariant derivative. x^i are the coordinates of the Riemannian manifold M with metric g^{ij} and curvature R_{ijkl}; b_i are bosonic fields; ϕ^i and ψ_j are Grassmannian fields ; w is a function on M. Our conventions for the Riemannian connection and curvature are given by

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$$

$$R^i_{ljk} = \partial_j\Gamma^i_{lk} - \partial_k\Gamma^i_{lj} + \Gamma^i_{jm}\Gamma^m_{lk} - \Gamma^i_{km}\Gamma^m_{lj}. \tag{2.19}$$

The supersymmetry of the action is

$$\{Q, x^i\} = \phi^i \tag{2.20a}$$

$$\{Q, \phi^i\} = 0 \tag{2.20b}$$

$$\{Q, \psi_i\} = b_i - \psi_j\Gamma^j_{ik}\phi^k \tag{2.20c}$$

$$\{Q, b_i\} = b_j\Gamma^j{}_{ik}\phi^k - \frac{1}{2}\psi_j R^j{}_{ilk}\phi^l\phi^k, \tag{2.20d}$$

It is straightforward to check that the supersymmetry generator Q is nilpotent: $Q^2 = 0$. Q therefore is referred to as a BRST operator.

At first glance, due to the presence of the connection terms in the r.h.s. which do not transform as tensors, it seems that the equations (2.20c) and (2.20d) are not defined covariantly. This is however not true. In fact, from (2.20a) we see that Q is not commutative with general coordinate transformation. This means that $\{Q,\ \}$ acting on a tensor may also not be covariant. In a sense, Q behaves like the space time derivatives d. For instance, in (2.20c), the general coordinate transformation of $\{Q, \psi_i\}$ is

$$\{Q, (\partial_i f^j)\psi_j\} = (\partial_i f^j)\{Q, \psi_j\} + \phi^l(\partial_l \partial_i f^j)\psi_j. \tag{2.21}$$

This is noncovariant because of the second term in the r.h.s. The same noncovariant term is reproduced from the connection term in the r.h.s. of (2.20c) (under the same transformation). Therefore, the equations in (2.20) are covariant in the sense that the forms of the equations do not change under a general coordinate transformation. In general, it is the operator $\{Q, \ \}\delta^i_j - \phi^k \Gamma^i_{jk}$ that is a covariant operator.

The action (2.18) can be rewritten as a BRST commutator:

$$I = \{Q, V\} \tag{2.22}$$

where

$$V = i \int dt \psi_i(\dot{x}^i + g^{ij}\partial_j w(x) + \alpha b^i) \tag{2.23}$$

As we stated earlier, this is a common feature for all of the cohomology models. Note that V is well defined on M. According to the arguments given in section **2.1**, the theory is a topological theory. This means that the expectation values of BRST invariants are independent of the metric g^{ij}.

The simplest such invariant is the partition function $Z =< 1 >$. We calculate its value for two different V's, (the one with $\alpha = 0$ and the other with $w = 0$), each of which is related to the other via a continuous deformation of its parameters [20]. A general statement we make is that such a deformation does not change the value of a topological invariant. We will show by detailed calculation that indeed the values of the partition function for two V's are equal. This will at least partially verify the general statement (2.14).

First, let us set α to zero. Actually, a comparison with the usual gauge theory shows that α plays the role of the gauge parameter; Setting α to zero amounts to taking a Landau gauge in the theory. In this gauge, the action (2.18) reads

$$I = \int dt \left(ib_i(\dot{x}^i + g^{ij}\partial_j w(x)) + \psi_i \mathcal{D}^i_j \phi^j \right). \tag{2.24}$$

Here, the lack of a quadratic term in b makes the action more singular, which means that more care is needed in detailed calculations. This is why usually the Feynman gauge is more favored. However, in our model the Landau gauge has a great advantage: the action becomes very simple allowing us to calculate all of the correlation functions explicitly. Later we will see how this happens.

The partition function Z is as usual defined by

$$Z = \int [DbDxD\psi D\phi] e^{-I} \tag{2.25}$$

The linear dependence of b_i in the action (2.24) allows us to integrate over b_i first, yielding a δ-functional constraint with which the evaluation of the integral becomes much easier. Also the integration of the two fermions can be carried out to yield a determinant of $\mathcal{D}$,

$$Z = \int [Dx] \prod_t \delta(\dot{x}^i + g^{ij}\partial_j w) det(\mathcal{D}). \tag{2.26}$$

With $x^i_s(t)$ being solutions (parameterized by s) of the equation

$$\dot{x}^i + g^{ij}\partial_j w(x) = 0, \tag{2.27}$$

the delta function in the integrand can be written as

$$\prod_t \delta(\dot{x}^i + g^{ij}\partial_j w) = \sum_s \frac{\prod_t \delta(x^i - x^i_s)}{|det(\mathcal{D})|}. \tag{2.28}$$

Here we have used the identity

$$\frac{\delta}{\delta x^j}(\dot{x}^i + g^{ij}\partial_j w)|_{x^i = x^i_s} = \mathcal{D}^i_j|_{x^i = x^i_s}. \tag{2.29}$$

Substituting (2.28) into (2.26), we derive

$$Z = \sum_s \frac{det(\mathcal{D})}{|det(\mathcal{D})|}. \tag{2.30}$$

where $\mathcal{D}$ are defined with background fields $x_s^i(t)$. To proceed further, we have to examine these $x_s^i(t)$. Consider the inequality

$$\oint dt \dot{x}_s^i \dot{x}_s^j g_{ij} \geq 0 \tag{2.31}$$

Since x^i satisfies (2.27), the l.h.s. of (2.31) actually vanishes:

$$\oint dt \dot{x}_s^i \dot{x}_s^j g_{ij} = -\oint dt \dot{x}_s^i \partial_i w(x_s(t)) = 0. \tag{2.32}$$

This means that $x_s(t)$ is a constant configuration, independent of the parameter t. Letting x_s^i be the constants and setting $\dot{x}^i$ to zero in (2.27), we find that (2.27) becomes

$$\partial_i w(x_s) = 0. \tag{2.33}$$

In other words, x_s^i is a critical point of the function w, and different s labels different critical points. The determinant of $\mathcal{D}$ at $x_s^i(t) = x_s^i$ is now reduced to

$$det(\mathcal{D})\mid_s = det(\partial_i \partial_j w)\mid_s . \tag{2.34}$$

For simplicity, we here consider the non-degenerate case, which means that the matrix $\partial_i \partial_j w$ has no zero eigenvalue. This implies that w has only isolated critical points. If we denote λ_s the number of the negative eigenvalues of $\partial_i \partial_j w$ at s, the partition function (2.30) becomes

$$Z = \sum_s (-1)^{\lambda_s}, \tag{2.35}$$

with s running over the whole set of (finite number of) isolated critical points. λ_s so defined is called the index of w at s.

We recall that one of the standard results of the Morse theory is that if w is a function with a finite number of non-degenerate critical points each of which has an index λ_s, the sum $\sum_s(-1)^{\lambda_s}$ is a topological invariant, equal to the Euler characteristic of the manifold, $\chi(M)$. Since this is exactly the r.h.s. of (2.35), we conclude that the partition function Z is equal to $\chi(M)$. As expected, it is indeed a topological invariant.

The partition function Z can also be calculated from (2.18) with $\alpha \neq 0$ and $w = 0$, which corresponds to the Feynman gauge. Consider the change of the variables

$$b_i = \frac{1}{\alpha}\tilde{b}_i - \frac{i}{2\alpha}(\dot{x}^j g_{ij}) \qquad (2.36a)$$

$$\phi^i = \alpha^{-\frac{1}{2}}\tilde{\phi}^i \qquad (2.36b)$$

$$\psi_i = \alpha^{-\frac{1}{2}}\tilde{\psi}_i. \qquad (2.36c)$$

In terms of these new variables the original action (2.18) becomes

$$I = \frac{1}{\alpha}\int dt(b_i b_j g^{ij} + \frac{1}{4}(g_{ij}\dot{x}^i\dot{x}^j) + \psi_i \mathcal{D}^i_j \phi^j + \frac{1}{2}R_{kl}{}^{ij}\phi^k\phi^l\psi_i\psi_j), \qquad (2.37)$$

where we have suppressed the tildes of all variables. Such a change of variables is allowed because the Jacobi of the transformation (2.36) is equal to one. Equation (2.37) suggests that α plays the role of Plank constant $\hbar$. The fact that the semiclassical approximation is exact (see (2.17)) permits us to consider $\alpha \to 0$. In this case, the b_i-integration is gaussian and can be easily performed,

$$\int \frac{[Db]}{(2\pi)^n} \exp -\frac{1}{\alpha} b_i b_j g^{ij} = g^{\frac{1}{2}} \left(\frac{\alpha}{(4\pi)}\right)^{\frac{n}{2}}. \qquad (2.38)$$

To carry out the rest of the functional integrations just notice that in the $\alpha \to 0$ limit there is a contribution from the constant configurations and another from the ratio of the determinants for non-constant modes which is known to be 1 due to supersymmetry [30]. Therefore, using the result (2.38), the partition function Z reduces to the computation of

$$Z = \left(\frac{\alpha}{(4\pi)}\right)^{\frac{n}{2}} \int\limits_M \sqrt{g} d^n x \int d^n \psi d^n \phi exp(-\frac{1}{2\alpha} R^{ij}_{lk} \phi^l \phi^k \psi_i \psi_j), \qquad (2.39)$$

where the only integrations left are the ones for the constant configurations [20]. In the expansion of the exponential in (2.39), we see that only the term in which each ϕ^i and ψ_i appears exactly once contributes. This implies that for compact manifolds of odd dimension $Z = 0$, in agreement with the standard result for the Euler characteristic. If n is even, $n = 2m$, we obtain,

$$Z = \frac{(-1)^m}{(4\pi)^m m! 2^m} \int_M \sqrt{g} d^n x \epsilon^{k_1 l_1 \dots k_m l_m} \epsilon^{i_1 j_1 \dots i_m j_m} R_{k_1 l_1 i_1 j_1} \dots R_{k_m l_m i_m j_m}, \quad (2.40)$$

which, together with the Gauss-Bonnet theorem, gives once again the Euler characteristic of the manifold M. The cancellation of the α dependence in (2.40) is a crucial test for the correctness of our arguments.

Having computed Z, one may naturally want to calculate other topological invariants that are not the partition function. It turns out that for the model in consideration, there are no such invariants; the partition function is the only topological invariant of interest [35]. This is because for the 1-dimensional model, the Atiyah-Singer index of the operator $\mathcal{D}$ is always zero, which means that the expectation value of any BRST-invariant operator with a non-zero ghost number necessarily vanishes. Therefore the Q-cohomology group, which forms the physical subspace, contains only constants and is thus trivial. Only in higher dimensional models can a non-trivial physical Hilbert space emerge.

Chapter 3

The Hamiltonian Approach And The G-Valued Index

The supersymmetric quantum mechanics considered in Chapter 2 is based on the action (2.18). As we have seen, the topological nature of the model allows the action of a topological model to be varied in certain way without changing the physics. In fact, any change that can be made by adding a BRST exact term in the action leads to the same values for the correlation functions for Q-invariant operators. However, the expressions for a correlation function may look very different for different V. The different expressions correspond to different topological indices, and the equality of these indices means that we have an index theorem. However, this index theorem becomes mathematically useful only when the corresponding indices have natural topological interpretations. As an example, we have seen in the Chapter 2 that the partition function Z for two different choices of V, corresponding respectively to $\alpha = 0$ and $w = 0$, has two different expressions. They are the well known algebraic index and the analytic index for the Euler characteristic. The index theorem which guarantees the equality of these two indices is nothing but the Gauss-Bonnet theorem.

The question we ask now is whether a generic V may always result in a natural topological index. The answer to this question is no. As a matter of fact, for all of the known topological theories, although there are an infinite number of gauge fixing functions V available for selection, only very few of them have been seen to have natural topological interpretations. For example, in the topological gauge theory, the desirable gauge choice is to set the self dual part of $F_{\mu\nu}$ to zero, and this leads to the Donaldson's invariants [5]. Though other gauge functionals also seem to be allowed, they have not received serious consideration because we do not know what topological indices are explicitly associated with them. In other words, from the point of view of a topological theory, only those gauges that have explicit topological interpretations leading to mathematically meaningful indices have received attention.

In terms of the Hamiltonian formalism, this statement comes from the requirement that the Hamiltonian H of a topological theory be proportional to $\{Q, Q^\dagger\}$, where $Q^\dagger$ is the adjoint of Q under a properly defined inner-product. Witten argued that this requirement is necessary because it yields an 1-1 correspondence between bosonic and fermionic modes with non-zero energy so that, by Hodge theory, there is a cancellation between them [10,30]. Obviously, not all of gauge fixing functions lead to Hamiltonians of this type. To be specific, upon writing $H = \{Q, U\}$ for some gauge fixing functional V, the above criterion amounts to asking whether one is able to find an inner-product under which $U = Q^\dagger$.

In the case of the 1-dimensional supersymmetric quantum mechanics, we can for instance change V from (2.23) to

$$V = i \int dt \psi_i(\dot{x}^i + f^i(x) + \alpha b_j g^{ij}). \tag{3.1}$$

Notice that we have replaced $g^{ij}\partial_j w$ by a more general tangent field f^i. The Hamiltonian with the new V takes the form $H = \{Q, U\}$ with $Q = -i\psi_i p^i$ and $U = \frac{1}{2}\phi_i(p^i + 2if^i(x))$. It can be readily verified that $U^2 \neq 0$ unless $f_i \equiv g_{ij}f^j$ satisfying $df = 0$. Therefore a generic f^i may not be able to serve as $Q^\dagger$ because $Q^\dagger$ has to be nilpotent. Of course $U^2 = 0$ is only a necessary condition for U to be $Q^\dagger$. It is only when f is an exact form, $f = dw$, that one can globally define an inner-product $< u, v >= \int e^{-w}u^*v$, under which U can be identified as $Q^\dagger$. For a curved background manifold M with nontrivial $H^1(M)$, $df = 0$ does not imply $f = dw$. This is one of the reasons of why so far all considerations of the supersymmetric quantum mechanics in curved background have been restricted to the case leading to the Morse theory, in which f^i is a gradient of some function w [20].

Knowing that U cannot be identified as a $Q^\dagger$ when $df \neq 0$, we have to look for a new topological interpretation. To be more specific, we want to see if there is a mathematically meaningful index that can be associated with the partition function even when df does not vanish. Given the argument presented above, such an index, if exists, would not be exactly of the Witten type.

It turns out that, at least when f^i is a Killing vector of the background manifold M, such an index does exist. It is the character valued Euler character with G generated by f^i. We shall show that with some nontrivial observations this index can actually be derived straightforwardly via the usual canonical quantization. The idea is that we can make a decomposition of the gauge fixing U such that $H = \{Q, Q^\dagger\} + L$, where $Q^\dagger$ is with respect to some well defined inner-product. Obviously, if the gauge choice is "good", the second term L should disappear. When L does not vanish, it is actually possible to reinterpret it as the Lie derivative associated with the Killing vector field generating G. The exponential of L becomes a group element, so that the partition function naturally produces a character valued index.

3.1 1-D Topological Model

We start with the action (2.22) in which V is given by (3.1). (2.22) can be expanded and rewritten in terms of components

$$I = \oint_{s^1} dt \left(\left(ib_i(\dot{x}^i + f^i) + \psi_i \mathcal{D}^i_j(f)\phi^j \right) + \alpha(b_i b_j g^{ij} + \frac{1}{2} R^{ij}{}_{kl}\phi^k \phi^l \psi_i \psi_j) \right), \quad (3.2)$$

$$\mathcal{D}^i_j(f;t) = \delta^i_j(d/dt) + \dot{x}^k \Gamma^i_{jk} + D_j f^i,$$

where f^i is an arbitrary section in T^*M, not necessarily closed. As it stands, the action (3.2) is defined in the euclidean spacetime in the sense that the partition function is given by $\int e^{-I}$ [20]. To obtain a minkowski action I_M, we define $I_M \equiv iI$. The canonical quantization is carried out with respect to I_M. For example, the time evolution operator is e^{iTH}, etc.

We now quantize this model. For the sake of illumination, let us temporarily ignore the terms proportional to α. (These terms will be restored later on.) The action thus is in the Landau gauge. Following the standard rules of canonical quantization, we obtain the canonical commutation relations $[\pi_j, x^i] = -i\delta^i_j$; $\{\phi^i, \psi_j\} = \delta^i_j$, where π_j, the canonical momentum of x^i, is given by $\pi_j = -b_j + i\psi_i\phi^k\Gamma^i_{jk}$. Making the replacement $\pi_j \leftrightarrow -i\frac{\partial}{\partial x^j}$, we observe that

$$b_j = i(\partial_j + \psi_i\phi^k\Gamma^i_{jk}). \tag{3.3}$$

Interpreting ψ and ϕ as (co-)tangent vector basis (see below), b_i behaves like a covariant tensor. This is consistent with the covariance of the classical b^i. The classical Hamiltonian can be conveniently read off from (3.2) (since we have ignored the α dependent terms, we denote the Hamiltonian without α as $H(0)$ and a Hamiltonian with α as $H(\alpha)$)

$$H(0) = b_i f^i - i\psi_i(D_j f^i)\phi^j. \tag{3.4}$$

To quantize it, we substitute (3.3) into (3.4) and observe that the two Riemannian connection terms in (3.4) cancel each other. After the replacement of all the fields by their operators, $H(0)$ becomes

$$H(0) = i(f^j\partial_j + \phi^j(\partial_j f^i)\psi_i). \tag{3.5}$$

We stress that although (3.5) may not look like a covariant expression, it is essentially covariant because (3.5) is derived from (3.4) and the connection terms in (3.4) have been cancelled in (3.5).

In the standard canonical quantization, the order ambiguities of operators need to be treated with care. The operator order seen in the Hamiltonian (3.5) is obtained as follows. When we replace the classical fields in (3.4) by quantum operators, we make the following substitutions: $\pi_i f^i \to (1/2)(\pi_i f^i + f^i \pi_i)$; $\psi_i \phi^j \to (1/2)(\psi_i \phi^j - \phi^j \psi_i)$. Due to the supersymmetry, the zero point energies of bosonic and fermionic parts cancel each other exactly. (3.5) thus follows directly by normal ordering.

According to Witten, ψ_i, ϕ^i can be respectively interpreted as tangent vector basis $-\partial/\partial x^i$ and the exterior derivatives dx^i [10]. The Hilbert space P associated with this Hamiltonian may be recognized as the space $\Omega^*(M)$ of all differential forms defined on the background space M. In other words, associated with a n-form $A = \sum A_{i_1 \ldots i_n}(x) dx^{i_1} \ldots dx^{i_n}$ is a vector $|A\rangle = \sum A_{i_1 \ldots i_n}(x) \phi^{i_1} \ldots \phi^{i_n} |0\rangle$ in P.

We now have the Hamiltonian operate on this state. Before presenting our main observation, let us look at an example. Take an 1-form $A = A_i dx^i$; the associated state is $|A\rangle = A_i \phi^i |0\rangle$. The action of $-iH(0)$ on this state is

$$
\begin{aligned}
|A'\rangle = -iH(0)|A\rangle &= (f^j \partial_j + \phi^j (\partial_j f^i) \psi_i) A_k \phi^k |0\rangle \\
&= (f^j (\partial_j A_i) + (\partial_i f^j) A_j) \phi^i |0\rangle. \qquad (3.6)
\end{aligned}
$$

The state $|A'\rangle$ can of course be expressed by a differential form, which is

$$
A' = -iH(0)A = (f^j (\partial_j A_i) + (\partial_i f^j) A_j) dx^i. \qquad (3.7)
$$

If we now think of f^i as an infinitesimal variation of the coordinates x^i, it is easy to see that the 1-form A' is exactly the variation of the 1-form $A = A_i dx^i$ induced by the diffeomorphism $x^i \to x^i + f^i(x)$. Since such an infinitesimal transformation is generated by a Lie derivative, we thus observe that $-iH(0)$ acts on an 1-form state as a Lie derivative. In general, the following statement is true: the Hamiltonian

$H(0)$ given by (3.5) is identical to, up to a constant, the Lie derivative associated with vector field f^i. (The identification is of course understood with the 1-1 correspondence between states and forms). To see the general case, we compute the action of $-iH(0)$ on a generic $|A\rangle$

$$\begin{aligned} -iH(0)|A\rangle &= \sum (f^j \partial_j + \phi^i(\partial_i f^j)\psi_j) A_{j_1 \dots j_n} \phi^{j_1} \dots \phi^{j_n} |0\rangle \\ &= \sum (f^i \partial_i A_{j_1 \dots j_n} + (\partial_{j_k} f^j) A_{j_1 \dots j \dots j_n}) \phi^{j_1} .. \phi^{j_k} .. \phi^{j_n} |0\rangle \\ &= L_f(A_{j_1 \dots j_n}) \phi^{j_1} \dots \phi^{j_n} |0\rangle, \end{aligned} \quad (3.8)$$

where we have used the fermionic commutation relations, and in the last line we have used the standard definition of L_f [33].

With $H(0)$ being identified as iL_f, it follows that the time evolution operator $e^{-iTH(0)}$ becomes e^{TL_f}, where the time T is the perimeter of the base manifold S^1. e^{TL_f} has an explicit geometric meaning as being the 1-parameter diffeomorphism integrated from f^i. This can be viewed intuitively as follows. We divide the interval T into N pieces each with a small width $t = T/N$. e^{TL_f} thus becomes a multiplication of N identical operators each of which has the form $e^{tL_f} = 1 + tL_f + \mathcal{O}(t^2)$. When acting on a differential form A, $A' = e^{tL_f} A$ is simply the infinitesimal homomorphism induced by $x^{i\prime} = x^i + tf^i(x)$. The entire operator e^{TL_f} can be determined as the consecutive operations of these homomorphisms. Let $X^i(t, x)$ be the solution of the evolution equation

$$\frac{d}{dt} X^i = f^i(X), \quad (3.9)$$

with the initial condition $X^i(t = 0, x) = x^i$. Since this is a first order differential equation, for any x^i given, it has a unique solution at $t = T$, $X^i(t = T, x) \equiv G^i(x)$. This defines an 1-1 map $G : x^i \to G^i(x)$ which maps a point x to $G(x)$. Furthermore, a point x', nearby x^i, is obviously mapped to a point $G(x')$, nearby

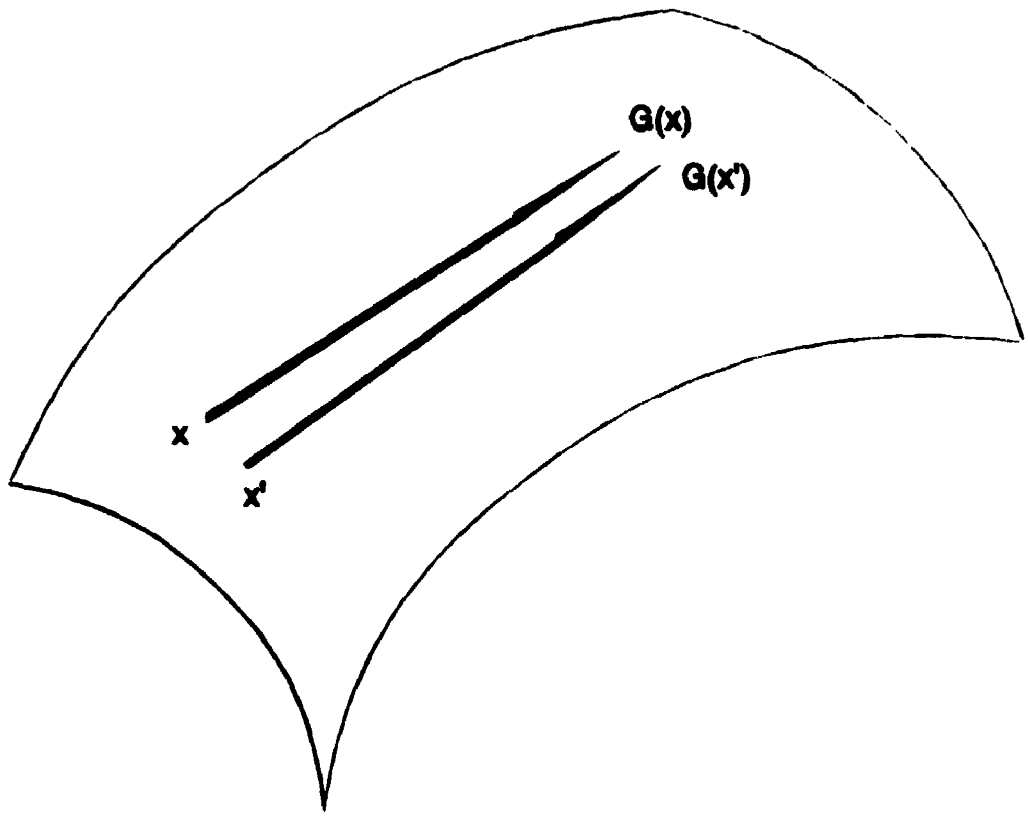
$G^i(x)$; thus the map is actually a homomorphism of M (see fig.(1)). It is not difficult to see that $G^* \equiv e^{TL_f}$ is the homomorphism in $\Omega^*(M)$ induced by G. This homomorphism is called exact homomorphism in the mathematics literature [11,14].

Another way to view this is to consider e^{TL_f} in the path-integral formalism. Let $|x\rangle$ be a "δ-function like" vector in P , whose support is concentrated at x. Then the set of $\{|x,\phi\rangle \equiv \phi^{i_1}...\phi^{i_k}|x\rangle\}$ for all x in M conveniently forms a basis of P. The transfer matrix between $|x_i,\phi\rangle$ and $|x_f,\phi'\rangle$ can be expressed in terms of path-integral as

$$\langle\psi, x_f|e^{TL_f}|x_i,\phi\rangle \sim \int [DxDbD\phi D\psi]e^{-I(\alpha=0)}, \tag{3.10}$$

while holding the initial and final values of x, ϕ fixed. The δ-function which results from the integration over b_i imposes a constraint leaving only the matrix element $\langle x_f,\psi|e^{TL_f}|x_i,\phi\rangle$ for x_i and x_f that satisfy $x_f = G(x_i)$. In other words, the transfer matrix defines a homomorphism that maps an initial x_i to a *unique* final x_f given by (3.9). It is easy to see that this homomorphism is precisely the exact homomorphism G described above.

In terms of G, the partition function Z of the model can be expressed as $Z = Tr((-1)^F e^{TL_f}) = Tr((-1)^F G^*)$, where $(-1)^F$ comes from the periodic boundary condition for fermions. "Tr" is understood with the usual inner product between differential forms. One immediately recognizes that Z is nothing but a character valued index - the usual alternate trace is taken after being twisted by G^*. However, as is well known, the character valued index so given is not well defined unless a regularization suppressing high frequency modes is introduced. Usually, the regularization is done by an insertion of $e^{-s\Delta}$, where Δ is the Laplacian operator on $\Omega^*(M)$, into Z so as to damp the higher modes. It turns out that this insertion can be naturally done as one takes into account the α-dependent terms in (3.2), the terms we have ignored so far.



Fig(1): Eq(3.9) defines an 1-1 map G which maps a point x to G(x). Furthermore, a point x' nearby x is mapped to a point G(x'), nearby G(x); thus the map is actually a homomorphism of M.

Since no time derivatives are involved in these terms, restoring them cannot bring about changes of the canonical commutation relations obtained earlier. The change of the Hamiltonian $\alpha H_1 = H(\alpha) - H(0)$ may be evaluated through replacing the classical fields by the corresponding quantum operators. Noting from (3.3) that b^i actually is the covariant derivative, we derive

$$\alpha H_1 = -i\alpha(b_i b_j g^{ij} + \frac{1}{2} R^{ij}_{kl} \phi^k \phi^l \psi_i \psi_j)$$

$$= i\alpha\{d, d^\dagger\} = -i\alpha\Delta, \quad (3.11)$$

where $d = \phi^i \partial_i$; $d^\dagger = \psi_j g^{ij}(\partial_i + \psi_l \phi^k \Gamma^l_{ik})$ are the exterior derivative in $\Omega^*(M)$ and its adjoint, respectively [31, 32]. With H_1 the partition function Z becomes

$$Z = Tr((-1)^F e^{TL_f - T\alpha\Delta}). \quad (3.12)$$

Note that the regularized character valued index we want is [39]

$$Tr((-1)^F G^* e^{-\beta\Delta}) = Tr((-1)^F e^{TL_f} e^{-\beta\Delta}). \quad (3.13)$$

Compare (3.12) and (3.13), we need $e^{TL_f - T\alpha\Delta} = e^{TL_f} e^{-T\alpha\Delta}$. A sufficient condition for this to happen is that f^i be a Killing field of M, because for a Killing field f, L_f commutes with Δ. Therefore with this condition and $\beta = T\alpha$ the partition function Z becomes (3.13), and Z is indeed the usual regularized character valued index [39].

We have just proved that the partition function of theory (3.2) is a well defined character valued index (at least when f^i is a Killing field). Since the inner-product used in "Tr" is the usual inner-product of exterior forms on M, it is obviously well defined. Though the whole procedure of canonical quantization is carried out within the minkowski framework, the final expression is euclidean. This is because our minkowski action is so defined that $e^{iI_M} = e^{-I}$, therefore the partition function is essentially euclidean. The value of the index may be evaluated by taking advantage of the topological invariance of the model, so that continuously changing the parameters does not affect the final result [20]. As a consistent check, we can let f^i be zero. The calculations in the last chapter shows that the topological index becomes the Euler characteristic $\chi(M)$ which is, as a standard result, equal to the character valued Euler index of M.

We conclude this section by making a brief comparison between our formulation and the one given by Witten in [10]. First of all, We observe that the condition that f^i being a Killing field is exactly the condition Witten had to use to carry out his proof. In some sense, our formulation may be seen as a twisted version of Witten's supersymmetric model. To see this, we compare the Hamiltonians of these two models. Using the notation of exterior derivative, the Hamiltonian $H(\alpha)$ can be written as $H(\alpha) = i\{Q, V\} = i\{d, \alpha d^\dagger + i_f\}$. Witten's Hamiltonian is $H_s = \{d_s, d_s^*\}$ with $d_s = d + s i_f$ and d_s^* its adjoint; $d_s^2 = d_s^{*2} = s i_f$. Therefore d_s in Witten's model is not nilpotent and cannot be viewed as a BRST operator. To obtain a nilpotent operator, we may have to twist d_s by properly adding a term perhaps in a manner similar to Eguchi and Yang [34] and Witten[16], in dealing with other topological models. The resulting nilpotent operator should be d. A price to pay could be that d_s^* also has to be changed to $d^\dagger + 2s i_f$, which is not the adjoint of d. Letting $\alpha = \frac{1}{2}$ and $s = 1$, we see that the Hamiltonian so twisted is precisely $H(\alpha)$. Although we no longer have an hermitean Hamiltonian such as H_s, what we have done in this section has just shown that the mathematical results derived from the two hamiltonians are identical.

3.2 Topological σ Model

The generalization of this idea to the 2-D topological σ model [6] is straightforward. We only sketch the main steps without going into details. As a matter of fact, the whole procedure almost parallels that of 1-dimensional model described in the last section. For simplicity, we only consider the models in a Kahler background manifold M with real dimension $2n$. For the purpose of canonical quantization, we take the world sheet Σ to be a torus. The configuration space at a fixed time is the loopspace $LM = \{x^i(\sigma)|\sigma \in S^1\}$. The (minkowski) action is

$$I_M = \int_\Sigma d^2\sigma(-b_i^\alpha(\partial_\alpha x^i + f_\alpha^i + \epsilon_{\alpha\beta}J_j^i(\partial^\beta x^j + f^{\beta j}) + i\psi_i^\alpha(D_\alpha\phi^i + D_j f_\alpha^i\phi^j$$

$$+\epsilon_{\alpha\beta}J_j^i(D^\beta\phi^j + D_k f^{\beta j}\phi^k)) + \alpha(b_i b_j g^{ij} + \frac{1}{2}R_{ij}^{lk}\phi^i\phi^j\psi_l^\alpha\psi_{\alpha k})), \tag{3.14}$$

where $b^{\alpha i}$, $f^{\alpha i}$ and $\psi^{\alpha i}$ satisfy the self-dual conditions

$$b^{\alpha i} = \epsilon_\beta^\alpha J_j^i b^{\beta j}; \quad \psi^{\alpha i} = \epsilon_\beta^\alpha J_j^i \psi^{\beta j}; \quad f^{\alpha i} = \epsilon_\beta^\alpha J_j^i f^{\beta j}. \tag{3.15}$$

We can choose for M the complex coordinates $x^i, \overline{x}^{\bar{i}}$ with $i; \bar{i} = 1,..,n$. The complex structure J_j^i thus takes the canonical form $J_j^i = i\delta_j^i; J_{\bar{j}}^{\bar{i}} = -i\delta_{\bar{j}}^{\bar{i}}$. The action (3.14) can be obtained by $I = \{Q, V\}$ with the gauge fixing function V being chosen as

$$V = (\partial x^i + f^i) + \overline{\partial}x^{\bar{i}} + \overline{f}^{\bar{i}} \tag{3.16}$$

where f^i and $\overline{f}^{\bar{i}}$ are defined by

$$f^i \equiv f_+^i, \qquad \overline{f}^{\bar{i}} \equiv f_-^{\bar{i}}, \tag{3.17}$$

and, in terms of the 2n real coordinates, Q is given exactly by (2.20). Due to the self dual condition, these are the only non-vanishing components of $f^{\alpha i}$. We sometimes omit anti-holomorphic components. They should be incorporated in the final expressions. The difference to the usual gauge fixing [6] is that the role of the usual Ginzberg-Landau potential w has been replaced by a pair of f^i and $\overline{f}^{\bar{i}}$ through $f^i \to g^{i\bar{j}}\partial_{\bar{j}}w$ and $\overline{f}^{\bar{i}} \to g^{\bar{i}j}\partial_j w$. Of course, an arbitrary f may still result in a non-hermitean Hamiltonian. We can again resort to a reinterpretation identifying the pathological part as an (infinite dimensional) Lie derivative in the loopspace LM.

The canonical variables can also be obtained via the usual canonical quantization. It turns out that all the commutation relations are the same as those in the 1-dimensional model, provided the following replacements are made:

$$
\begin{aligned}
&b_{+i}(\sigma) \rightarrow b_i,\ b_{-\bar{i}}(\sigma) \rightarrow b_{\bar{i}+n} \\
&\psi_{+i}(\sigma) \rightarrow \psi_i,\ \psi_{-\bar{i}}(\sigma) \rightarrow \psi_{\bar{i}+n} \\
&x^i(\sigma) \rightarrow x^i,\ x^{\bar{i}}(\sigma) \rightarrow x^{\bar{i}+n} \\
&\phi^i(\sigma) \rightarrow \phi^i,\ \phi^{\bar{i}}(\sigma) \rightarrow \phi^{\bar{i}+n}. \\
&(i = 1, ..., n;\ \bar{i} = 1, ...n.)
\end{aligned}
\tag{3.18}
$$

This is so because the BRST transformation rules for both theories are same. The Hamiltonian thus has a structure similar to the one in 1-dimensional model. In particular, the α independent terms can be written as

$$
H(0) \sim \oint_{S^1} d\sigma R^i(x(\sigma))\frac{\delta}{\delta x^i(\sigma)} + \oint_{S^1}\oint_{S^1} d\sigma d\sigma' \phi_j(\sigma)\frac{\delta}{\delta x^i(\sigma')} R^j(x(\sigma))\psi^i(\sigma') + h.c.,
\tag{3.19}
$$

where $R^i(x(\sigma)) \equiv \partial_\sigma x^i + f^i(x(\sigma))$, and $h.c.$ stands for the anti-holomorphic part. Obviously, this equation is an analog of (3.5), and R^i is a generalization of f in (3.5). Since all the commutation relations are same, the arguments used to derive (3.8) may still apply. As a result, the Hilbert space can also be identified with $\Omega^*(LM)$, the space of differential forms on LM, and $H(0)$ can be regarded as L_R, the Lie derivative in $\Omega^*(LM)$, associated with R^i via $L_R = di_R + i_R d$, where d and i_R are defined in $\Omega^*(LM)$ as well. This Lie derivative generates 1-parameter diffeomorphism in LM given by $x_T^{i}(\sigma) = G_T^i(x) = X^i(\sigma; T, x)$, where $X^i(\sigma; T, x)$ is the solution of following equation at $t = T$,

$$
\partial_t X^i(\sigma; t, x) = -R^i(X); \quad X^i(\sigma; 0, x) = x^i(\sigma).
\tag{3.20}
$$

Note that in (3.19) f^i has been chosen to satisfy the holomorphic condition $\partial_{\bar{j}} f^i = 0$. This is to make sure that the diffeomorphism $G^i_T(x^i)$ preserves the complex structure J^i_j. The partition function thus becomes $Z = Tr((-1)^{F+\overline{F}} G^* \overline{G}^*)$, an analog of the unregularized character valued index in the 1-dimensional model. Finally, taking into account the α dependent terms, and provided that G is a Killing vector of LM, we obtain the regularized index

$$Ind_G \equiv Z = Tr((-1)^{F+\overline{F}} G^* \overline{G}^* e^{-T\alpha\Delta}), \tag{3.21}$$

where Δ is the Laplacian in loopspace LM. All these results are direct generalizations of the 1-dimensional model. The topological meaning of this index is easily seen as the Euler index of the loopspace LM twisted by the group action G^* given above. Therefore it is a character valued index.

A question we have to address now is whether Ind_G could be trivially zero due to a violation of ghost-number conservation. (An Ind_G that is trivially zero is not welcome, because it will provide us less information about the topology). This may happen when the index of the fermion kinetic operator $ind(\mathcal{D})$ does not vanish. (In 1-dimensional case, because $ind(\mathcal{D})$ is always zero. Ind_G can never be trivially zero [20]). For the case at hand, as far as the partition function is concerned, $ind(\mathcal{D})$ is also zero. To see this, we notice that a partition function is defined on a torus, on which one can always deform an arbitrary metric to the trivial metric $\delta_{\alpha\beta}$ without changing the values of any topologically invariant quantities. With respect to this metric, the zero mode equation for ψ becomes identical to the zero mode equation for ϕ, because each component of ψ behaves on the torus exactly like a world-sheet scalar. Furthermore, due to the self dual condition (3.15), the number of the independent degrees of freedom for ψ is equal to that for ϕ. Thus the number of the zero modes of ψ is same as that of ϕ. This means $ind(\mathcal{D}) = 0$. Therefore, Ind_G cannot be trivially zero.

Witten's topological σ model [6] corresponds to our formulation with $f^{\alpha i} = 0$. We then have $R^i = i\partial_\sigma x^i$; $R^{\bar{i}} = -i\partial_\sigma x^{\bar{i}}$. The diffeomorphism G becomes a constant shift of σ: $\delta\sigma = a$ for holomorphic components and $\delta\sigma = -a$ for antiholomorphic components. This is a simple case of Floer's diffeomorphism [11]. (Floer's diffeomorphism is defined with a "twisted" boundary condition $x^i(\sigma) = h^i(x(\sigma + 2\pi))$. G is obviously a Killing vector because it preserves the metric of LM invariant. Therefore the resulting partition function is a character valued index.

3.3 Topological Gauge Theory

It was first observed by Atiyah that Donaldson's invariants may be related to Floer's group by a generalized Morse theory, whose Morse functional is the Chern-Simons functional [8]. In terms of this Morse functional, the anti-self dual Yang-Mills equation in the $A_0 = 0$ gauge,

$$\partial_t A_i^a = -\epsilon_i^{jk} F_{jk}^a, \tag{3.22}$$

can be interpreted as the instanton equation describing the tunneling effect in the sense of Witten [10]. However, this equation can also be thought of as the equation defining an 1-parameter diffeomorphism in the configuration space. This enables us to identify the topological gauge theory as a character valued index theory in a manner similar to those in the previous sections.

Again, the key point is to manage to write the Hamiltonian as some sort of Lie derivative on a cotangent bundle of the configuration space. However, unlike those models described in the previous sections, the topological gauge theory has redundant bosonic and fermionic gauge symmetries [5]. To fix these symmetries we have to introduce gauge fixing functions. It can be proven that the usual gauge fixing [5, 35] would bring about complicated interactions between gauge ghosts and configuration variables, thus destroying the simple interpretation of the partition function to a character valued index. To avoid this, a judiciously chosen gauge fixing that can result in a decoupling between them is needed. Here we choose the gauge fixing as follows. For the bosonic symmetry, since (3.22) is written in the temporal gauge, we are led to consider the gauge fixing function $\bar{c}A_0$ [40]. For the fermionic gauge symmetry, we choose the gauge $\Psi_0 = 0$, corresponding to the gauge fixing functional $\bar{\phi}\Psi_0$. This can be thought as a fermionic temporal gauge. Now with respect to these gauge choices, the gauge fixed lagrangian can be written as $L = \{Q, V\}$ with

$$V = \chi_+^{\mu\nu} F_{+\mu\nu} + \bar{c}A_0 + \bar{\phi}\psi_0 + \alpha\chi_{+\mu\nu}b_+^{\mu\nu}, \tag{3.23}$$

where $b_{+\mu\nu}$, $\chi_+^{\mu\nu}$ and $F_{+\mu\nu}$ are the self dual parts of $b_{\mu\nu}$, $\chi^{\mu\nu}$ and $F_{\mu\nu}$, which satisfy

$$\begin{aligned} F_{+\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\tau}F_+^{\rho\tau} &= 0 \\ \chi_{+\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\tau}\chi_+^{\rho\tau} &= 0 \\ b_{+\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\tau}b_+^{\rho\tau} &= 0 \end{aligned} \tag{3.24}$$

The operator Q is defined by [32]

$$\begin{aligned} \{Q, A_\mu\} = \psi_\mu + D_\mu c, \quad \{Q, \psi_\mu\} = D_\mu\phi - [\psi_\mu, c], \\ \{Q, \phi\} = [\phi, c], \quad \{Q, c\} = -\phi - \frac{1}{2}\{c, c\}. \\ \{Q, \chi_{+\mu\nu}\} = b_{+\mu\nu} - \{\chi_{+\mu\nu}, c\}, \quad \{Q, b_{+\mu\nu}\} = [b_{+\mu\nu}, c] + [\chi_{+\mu\nu}, \phi], \\ \{Q, \bar{c}\} = b, \quad \{Q, b\} = 0 \\ \{Q, \bar{\phi}\} = \eta + [\bar{\phi}, c], \quad \{Q, \eta\} = [\bar{\phi}, \phi] - \{\eta, c\}, \end{aligned} \tag{3.25}$$

where ψ_μ, $\chi_{+\mu\nu}$, c, $\bar{c}$, η are fermions, and A_μ, $b_{+\mu\nu}$, ϕ, $\bar{\phi}$, b are bosons. The gauge fixed action becomes

$$I = \int d^4x Tr(b_+^{\mu\nu} F_{+\mu\nu} - \chi_+^{\mu\nu} D_\mu \psi_\nu + bA_0 - \bar{c}(\psi_0 + D_0 c) + \eta\psi_0 + \bar{\phi} D_0 \phi$$

$$+\alpha b_{+\mu\nu} b_+^{\mu\nu} + \alpha\phi[\chi_{+\mu\nu}, \chi_+^{\mu\nu}]). \tag{3.26}$$

As before, we set α to zero, and compute the partition function

$$Z = \int [DX] e^{-I}, \tag{3.27}$$

where $[DX]$ stands for the integration of all the fields. We first integrate over b and η. This gives rise to the constraints $A_0 = 0; \psi_0 = 0$. Due to the anti-self dual conditions (3.24), there are only three independent variables for $b_{+\mu\nu}$ and three for $\psi_{+\mu\nu}$. They can be conveniently chosen as $b^i \equiv b^{0i}$ and $\psi^i \equiv \psi^{0i}$ with $i = 1, 2, 3$. The action (3.26) can be reduced to

$$I = \int d^4x (b^i(\partial_t A_i + \epsilon_{ijk} F^{jk}) - \chi^i(\partial_t \psi_i + \epsilon_{ijk} D^j \psi^k) - \bar{c}\partial_t c + \bar{\phi}\partial_t \phi. \tag{3.28}$$

Now the power of the gauge $A_0 = \psi_0 = 0$ becomes very clear: After integrating over the gauge auxiliary fields, the ghost part of the reduced lagrangian is so simple that, not only is it completely decoupled from the configuration variables, but also the Hamiltonian for ghosts is actually zero. In fact, the total Hamiltonian associated with the action (3.28) is

$$H = \int d^3x (b^i \epsilon_{ijk} F^{jk} + \chi^i \epsilon_{ijk} D^j \psi^k), \tag{3.29}$$

along with the canonical commutation relations,

$$[b^i(x'), A_j(x)] = \delta^i_j \delta^3(\vec{x} - \vec{x}'); \quad \{\chi^i(x'), \psi_j(x)\} = \delta^i_j \delta^3(\vec{x} - \vec{x}');$$

$$\{\bar{c}(x'), c(x)\} = \delta^3(\vec{x} - \vec{x}'); \quad [\bar{\phi}(x'), \phi(x)] = \delta^3(\vec{x} - \vec{x}'). \tag{3.30}$$

As usual, χ^i and ψ_i can be viewed respectively as basis for tangent and cotangent bundles of the configuration space $M_A = \{A_i(x)|i = 1,2,3\}$. Repeating the arguments in the derivation of (3.8), one sees that the Hamiltonian given above can be thought as the Lie derivative L on the (infinite dimensional) cotangent bundle $\Omega^*(M_A)$ associated with the vector $-\epsilon_i^{jk}F_{jk}^a$. Therefore the time evolution operator e^{-TH} is the homomorphism on $\Omega^*(M_A)$, induced by the diffeomorphism (3.22).

However, the identification of the trace of this operator with Z is not as obvious as it might first be thought. Since Z is defined in (3.27) by integrating over all fields, its Hilbert space consists of the states generated not only by ψ, but also by the ghost fields c and ϕ. In other words, the partition function Z should be written as

$$Z = \sum_n \langle n|(-1)^{N_\psi + N_c} e^{-TL}|n\rangle, \tag{3.31}$$

where the state $|n\rangle$ has the following form

$$|n\rangle = \sum_{n_1 n_2 n_3 n_4 n_5} F^{(n)}_{n_1 n_2 n_3 n_4 n_5}(A)\psi_1^{n_1}\psi_2^{n_2}\psi_3^{n_3}c^{n_4}\phi^{n_5}|0\rangle. \tag{3.32}$$

The powers of fermionic fields should be understood as the multiplication of these fields either at different positions or with different indices. The vacuum $|0\rangle$ is so chosen that $\psi_i, \bar{c}$ and $\bar{\phi}$ eliminate it separately. In defining $\langle n|$, we have assumed that the conjugates of c and ϕ are $\bar{c}$ and $\bar{\phi}$, respectively. The same condition was also assumed in [6] in a Hamiltonian treatment.

Observe that all the (anti)-ghost fields; $c, \bar{c}, \phi, \bar{\phi}$, commute with the Hamiltonian (3.29). This allows us to factorize the partition function Z as

$$Z = Tr((-1)^{N_\psi} e^{-TL})Z_{ghost}, \tag{3.33}$$

where Z_{ghost} is the partition function for the ghost components. Since the Hamiltonian of the ghost components vanishes, to calculate Z_{ghost} we need a regularization.

By inserting $q^{\bar{\phi}\phi+\bar{c}c} = q^{N_\phi+N_c}$ into (3.31) for every point x' in M the contribution to the regularized Z_{ghost} is $Z_b(x,q)Z_f(x,q)$. Here $Z_b(x,q)$ is the partition function generated by $\bar{\phi}(x)$ and $\phi(x)$:

$$Z_b(x,q) = \sum_{n=0}^{\infty} \langle 0|\bar{\phi}^n(x)q^{N_\phi}\phi^n(x)|0\rangle$$

$$= \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}. \tag{3.34}$$

Likewise, $Z_f(x,q)$ is the fermionic part generated by $\bar{c}(x)$ and $c(x)$:

$$Z_f(x,q) = \langle 0|q^{N_c}|0\rangle - \langle 0|\bar{c}(x)q^{N_c}c(x)|0\rangle = 1 - q. \tag{3.35}$$

We see the ghost partition function at x' is $lim_{q\to 1} Z_b(x,q)Z_f(x,q) = 1$. The total contribution to Z_{ghost} is the multiplication of $Z_b(x)Z_f(x)$ over all x. Therefore $Z_{ghost} = 1$. We conclude that $Z = Tr((-1)^{N_\psi}e^{-TL_R})$ is a character valued index.

Of course, this is an index with no damping term. By turning on α, we wish to recover the damping term, as we did in previous sections. Unfortunately, with the α dependent terms given in (3.26), one can check by direct computation that

$$[H, b^i b_i + \phi\{\psi_i, \psi^i\}] \neq 0. \tag{3.36}$$

where $b_i b^i + \phi\{\psi_i, \psi^i\}$ is the α dependent Hamiltonian. This means that we cannot factor this term in the exponential from the whole Hamiltonian. In other words, the α-dependent terms so given cannot serve as the damping term. To this end, we hope that another choice of α-dependent terms in (3.23) may provide the damping term, but we do not know whether such terms exist.

Finally, since all the ingredients of Z are (space-dependent) gauge covariant, we can trivially divide out this residual gauge degree of freedom, and obtain an index defined in the true configuration space M_A/L_3G.

Chapter 4

Spontaneous Breaking of Topological Symmetry

In this chapter we discuss the issue of spontaneous breaking of topological symmetry. As mentioned in the Introduction, the existence of such a symmetry breaking will serve as an important confirmation of the idea of topological phase transition. However, as was pointed out by Witten, breaking the topological symmetry turns out to be the most difficult part in studies of TQFT. So far, the models that have been investigated on this issue [26] all contain gauge regulating terms. (A gauge regulating term is a gauge fixing term quadratic in anti-ghosts which, in the 1-dimensional supersymmetric quantum mechanics for example, is the α dependent term in (2.18)). Such gauge regulating terms smear the δ-function gauge-fixing, and thus correspond to Feynman gauges. The result of these investigations is somewhat discouraging - no spontaneous breaking of topological symmetries has yet been found [26]. (An exception might be the symmetry breaking attributed to an instanton in a noncompact base manifold [35]. However, we will only consider partition functions defined on a compact manifold in which such instantons do not exist). On the other hand, models without regulating terms (corresponding to the Landau gauge), owing to their simplicity, have been extensively used as substitutes to models with regulating terms to study various properties of the theories. The presumed interchangeability of the two classes of models is based on the argument that they belong to two different gauges of the same model, and the only difference between them - a gauge regulating term which is a BRST exact - does not affect any of the properties of the theories.

However, this assumption is not unconditionally true. In fact, if a model possesses Gribov zero modes [36], the model in the Landau gauge cannot be smoothly deformed via a gauge transformation to the corresponding model in the Feynman gauge [36,37]. This may be understood as follows [37]. Let $V(X)$ be the gauge-fixing functional in the Landau gauge. According to the standard Faddeev-Popov formalism, this picks out a configuration X that is a solution of $V(X) = 0$. Now let us try to transfer to the Feynman gauge. To do this, we replace the $V(X) = 0$ by $V(X) = P$, where P is an arbitrary function of the coordinates of the base space in which X lies. If the path-integration does not depend on P, we may average it with weight $\exp(-\alpha P^2)$ and arrive at the Feynman gauge. However, such P-independence is subject to the condition that, for any infinitesimal variation δP, there exists an associated gauge transformation v that will transform the solution to $V(X) = P$ to the solution to $V(X) = P + \delta P$. This condition can be expressed as

$$T(\delta v) \equiv \int (\frac{\delta V}{\delta X} \cdot \frac{\delta X}{\delta v}) \delta v = \delta P. \tag{4.1}$$

Obviously, if $T(\delta v)$ has zero modes (Gribov zero modes), (4.1) is not always satisfied, which means that we can no longer go from the Landau gauge to the Feynman gauge without changing physics. In other words, in this particular case these two models are in fact not equivalent, and a statement true for one model, such as the absence of spontaneous symmetry breaking, may well be not true for the other. It is therefore reasonable to speculate that, when Gribov zero modes exist, topological symmetries in models in the Landau gauge might be spontaneously broken. It turns out that this is indeed the case, and later we shall show how this happens. Since in this case the two gauges correspond to two different theories, from now on we shall avoid the terminology "gauge" and refer to the two types of models, respectively, as models with and without regulating terms.

Let us start with stating the criterion for the spontaneous breaking of topological symmetry. Since the topological field theory is realized via the BRST procedure, the symmetry generator is the BRST charge Q. A vacuum invariant under a BRST transformation satisfies the following equation

$$Q|0\rangle = 0. \tag{4.2}$$

Note that the results in Chapter 2 are based on this property of the vacuum. On the other hand, a theory with spontaneously broken topological symmetry means that its vacuum cannot be annihilated by Q. Therefore, there should be some well-defined F for which $< Q(F) >$ is not zero, as opposed to (2.1). This serves as our criterion for spontaneous symmetry breaking.

To calculate the expectation values in a theory expected to have spontaneous symmetry breaking, we use a method well known in ferromagnetic theory. We add a symmetry-breaking term ϵI_1 to the topological action and examine the vacuum expectation value $\langle Q(F)\rangle$ of the BRST charge Q acting on a judiciously chosen F in the limit $\epsilon \to 0$. $\langle Q(F)\rangle$ not vanishing in that limit will be taken as evidence for spontaneous symmetry breaking. Since the structure of the vacuum is what we want to determine, the calculations are to be carried out in the path-integral framework. The advantage of the path-integral formalism is that the vacuum is detected, rather than specified as input, as a result of the calculation.

At first sight, one might wonder how spontaneous symmetry breaking could happen this way, since Witten has shown in the path-integral formalism that any BRST-exact term has zero expectation value [5]. We present Witten's argument here. Consider the following path-integral

$$Z_\eta(F) = \int [D\Phi] e^{\eta Q} e^{-I} F. \tag{4.3}$$

Z_η is independent of the infinitesimal parameter η, because the path-integral measure is assumed to be invariant under BRST transformation. Expanding this out, and using the fact that the action is BRST invariant ($\{Q, I\} = 0$), we see that

$$Z_\eta(F) = \int [D\Phi] e^{-I} (F + \eta\{Q, F\}). \tag{4.4}$$

The assertion that $Z_\eta(F)$ is independent of η thus means that

$$0 =< \{Q, F\} >= \int [D\Phi] e^{-I} \{Q, F\}. \tag{4.5}$$

Note that this result is in agreement with its operator counterpart (2.1), in which a BRST invariant vacuum is assumed.

There are reasons believing that the above derivation is valid only in a formal sense. First, it is essentially based on the assumption that the measure is BRST invariant. This however needs to be checked. Secondly, and more importantly, in deriving (4.4) we made an expansion for $e^{\eta Q}$. The validity of such an expansion requires not only η to be small, but also Q to be "not too big" , (this should be understood in the context of Q acting on normalizable states). Generally, the validity of the above two assumptions can only be verified with a direct computation of the correlation functions. It turns out that, for some F, a direct calculation of $\langle Q(F) \rangle$ yields a result that could be multivalued. We shall view the existence of such a multivalued result as an signature of spontaneous symmetry breaking. (The existence of multivaluedness is expected in a theory with spontaneous symmetry breaking, because in such a theory the vacuum is not defined.) One may remove such a multivaluedness by adding to the action a symmetry-breaking term which picks out a specific vacuum. Obviously, Witten's argument still holds where such an multivaluedness does not exist. It will be shown that the condition of a theory being multivalued turns out to be identical to the condition of it having Gribov zero modes. Later we shall briefly discuss the relation of this condition with the existence of reducible configurations.

4.1 The 0-D Topological Model

To understand this idea more concretely, we first consider a 0-dimensional model that will exhibit all the main features of the idea. The construction of such a 0-dimensional model is based on the observation that the (basic) BRST transformation is spacetime independent, therefore suppressing the spacetime coordinates should not change the BRST nature of Q. Let X, B be commuting variables (bosons) and Ψ, Φ be anticommuting variables (fermions). The BRST operator Q is defined by

$$Q(X) = -\Phi, \quad Q(\Phi) = 0, \quad Q(\Psi) = B, \quad Q(B) = 0. \tag{4.6}$$

Q obviously satisfies the nilpotent condition $Q^2 = 0$.

The gauge-fixed action is

$$I = Q(\Psi V(X)) = BV(X) + \Psi V'\Phi, \tag{4.7}$$

where $V(X)$ is the gauge-fixing function and prime denotes derivative with respect to X. Note the absence of the gauge regulating term in the action. Let I_1 be the symmetry breaking term, that is, the term which is not BRST invariant. Let ϵ be a small parameter. The expectation value of $Q(V)$ in the presence of ϵI_1 is

$$\langle Q(F)\rangle_\epsilon = \int dXdBd\Psi d\Phi Q(F)\exp\big(i(I + \epsilon I_1)\big). \tag{4.8}$$

If ϵ is set to zero from the outset, then, by direct expansion, one could formally show that $\langle Q(F)\rangle_0$ would be zero for any F. As we stressed before, this result may not coincide with $\epsilon \to 0$ limit of $\langle Q(F)\rangle_\epsilon$. Indeed, we shall show that under certain conditions $\langle Q(F)\rangle \equiv \lim_{\epsilon\to 0} \langle Q(F)\rangle_\epsilon \neq \langle Q(F)\rangle_0$.

Let

$$I_1 = Bf - \Psi f'\Phi, \tag{4.9}$$

where f is a (bounded) function of X. The opposite sign between the two terms on the right-hand side of (4.9) manifests the BRST non-invariance of I_1. Owing to the conservation of ghost number we only need to consider $F = \Psi H(X)$, where H is an arbitrary (bounded) function. Then

$$\langle Q(\Psi H)\rangle_\epsilon = \int dXdBd\Psi d\Phi (BH + \Psi H'\Phi)\exp\big(iB(V + \epsilon f) + i\Psi(V' - \epsilon f')\Phi\big)$$

$$= \int dX \delta(V + \epsilon f)\left[2\epsilon f' H'/(V' + \epsilon f') + H \frac{d}{dX}(2\epsilon f'/(V' + \epsilon f'))\right]. \qquad (4.10)$$

The δ-function in (4.10) comes from the integration over B, after B in the first term of $Q(\Psi H)$ has been replaced by a derivative with respect to $V + \epsilon f$. Observe that the quantity inside the square bracket will vanish in the limit $\epsilon \to 0$, provided V' does not vanish simultaneously with ϵ. On the other hand, the presence of the aforementioned δ-function dictates that the integrand takes value only at points X that are solutions of $V + \epsilon f = 0$. Thus, so long as V' and V do not vanish simultaneously with ϵ, $\lim_{\epsilon \to 0} \langle Q(\Psi H) \rangle_\epsilon = 0$, so that Witten's conclusion that there is no spontaneous breaking of symmetry stands.

However, if V and V' vanish simultaneously at some point, say, X_c (this is the degenerate case referred to in [20]), then the integrand in (4.10) will in general not vanish in the limit $\epsilon \to 0$, and a nonzero integral may be expected. Notice that in this case one may not directly put V' to zero in the expression $(2\epsilon f')/(V' + \epsilon f')$ in (4.10), because the argument in the δ-function has been shifted by ϵf so that the function needs to be evaluated at the shifted position.

This can be done by expanding all quantities in powers of ϵ about X_c. Suppose the argument of the δ-function vanishes at $X_c + x_c$, then to $\mathcal{O}(\epsilon)$,

$$(1/2)V''(X_c)x_c^2 + \epsilon f(X_c) = 0, \qquad \text{when} \quad f(X_c) \neq 0; \qquad (4.11a)$$

$$(1/2)V''(X_c)x_c + \epsilon f'(X_c) = 0, \qquad \text{when} \quad f(X_c) = 0. \qquad (4.11b)$$

Here one has to deal with these two cases separately because of their different behaviours for small x. One sees that as long as $V''(X_c)$ and $H(X_c)$ do not vanish, the term proportional to H dominates the integrand in (4.10):

$$\delta(V + \epsilon f) H \frac{d}{dx}\left(\frac{2\epsilon f'}{V' + \epsilon f'}\right) = -\sum_c \delta(X - X_c - x_c) \cdot$$

$$\cdot 2\epsilon f'(X_c) V''(X_c) |V''(X_c)x_c + \epsilon f'(X_c)|^{-3} H(X_c) + \text{ less divergent terms.} \qquad (4.12)$$

Upon solving for X in (4.11a) and (4.11b) and substituting them into (4.12), one sees that in both cases $\langle Q(\Psi H)\rangle \sim \epsilon^{-2}$ with a non-vanishing coefficient, so that it approaches infinity, rather than zero, in the limit $\epsilon \to 0$. In other words, when $V(X)$ and $V'(X)$ vanish simultaneously at X_c, it is possible that some $Q(\Psi H)$ have non-vanishing expectation value, thereby meeting the criterion for spontaneous breaking of symmetry. There may be more than one such X_c. Since H is an arbitrary function, its values for different X_c's are not correlated. Therefore one need not worry about possible cancellation among different $\langle Q(\Psi H)\rangle$'s.

The infinity of $\langle Q(\Psi H)\rangle$ suggests the need of a renormalization procedure. An alternative way to obtain a finite value may be formulated as follows. Note that the choice of (4.9) as the symmetry breaking term is not unique. One may, for example, replace the action I_1 in (4.9) by a more general

$$I_1 = Bf_1(X,\epsilon) + \Psi f_2'(X,\epsilon)\Phi, \qquad f_1'(X,\epsilon) \neq f_2'(X,\epsilon), \tag{4.13}$$

where the second condition, holding for sufficiently small ϵ, is necessary for symmetry breaking. It can be shown that if I_1 is of this form, then with the exception of pathological choices of f_1 and f_2, $\langle Q(\Psi H)\rangle$ will be non-vanishing. For our purpose we choose them in such a way that, specifying to (4b), $f_1'(X_c,\epsilon) = f_2'(X_c,\epsilon) + \epsilon f_3(X_c) + O(\epsilon^2)$, and $H(X_c) = 0$, $H'(X_c) \neq 0$ for each zero X_c of V. It follows directly that

$$\langle Q(\Psi H)\rangle = \sum_{X_c} \left(2f_3/(f_1')^2\right) H'|_{X_c}, \tag{4.14}$$

which is finite. In this case, it would be reasonable to assume that there should be a renormalization procedure that will yield the result given by this choice of I_1.

Another reason to choose I_1 as given above is the following. The existence of X_c is exactly the condition for the original partition function $Z_0 = \langle 1\rangle_0$ to be multivalued. The integrand of Z_0 is proportional to $V'/|V'|$ evaluated at the zeroes of V. It becomes multivalued when V' also vanishes at at least one of these zeroes, which renders the vacuum unspecified. The vacuum is specified after an I_1 is chosen. Since the Hilbert spaces for path-connected vacua are isomorphic so that the vacuum energies must be degenerate (otherwise one would be able to specify the vacuum by choosing one having the minimum energy), the partition function Z, unlike other physical observables, has the same value for different vacua. This property imposes a very stringent restriction on the choice of I_1. It requires that, in the notation of (4.13), I_1 must be such that f_1 and f_2 may differ only at the $\mathcal{O}(\epsilon)$ level. This is exactly the I_1 we chose for finiteness in the last paragraph. The coincidence of these two conditions is an important justification of our interpretations described above. Note that with I_1 so chosen, the limit $\epsilon \to 0$ becomes a double scaling limit: $\epsilon \to 0$ and $f_1' \to f_2'$. The significance of this double scaling limit is so far beyond our understanding.

4.2 1-Dimensional Supersymmetric Quantum Mechanics

The model considered in the last section has no dynamics. For non-trivial dynamics one needs to consider models with higher dimensionalities. The next simplest model is the supersymmetric quantum mechanics in one dimension. Since the base manifold of this model is 1-dimensional, it is basically a quantum mechanics, and therefore has no propagating modes. One might still think that this model is too trivial. However, it is sufficiently non-trivial to have interesting features. In particular, it illustrates how the condition for symmetry breaking coincides with the condition for the existence of Gribov modes.

The BRST operator and the action of the model are given by (2.20) and (3.2), respectively. Note that I is what we call to be without regulating term only when $\alpha = 0$, which corresponds to the gauge fixing functional $V^i(t) = \dot{x}^i + f^i$. With this choice of V, the operator T in (4.1) becomes

$$T = \mathcal{D}, \tag{4.15}$$

where $\mathcal{D} \equiv \mathcal{D}^i_j(f,t)$ is given by (3.2). The zero modes of $\mathcal{D}$, if any, are precisely the Gribov zero modes in this model. We want to see under what condition would $\langle Q(F)\rangle$ be non-vanishing for some F.

Since the structure of the action is analogous to the one in the 0-dimensional model so long as we identify $\dot{x}^k + f^k$ here with the $V(X)$ in that model, we expect the expression for $\langle Q(F)\rangle_\epsilon$ to be similar to (4.10). In particular, we expect the quantity V' in (4.10) to be replaced by the derivative of $\dot{x}^k + f^k$, which is exactly $\mathcal{D}$. However, in order to have a non-vanishing $\langle Q(F)\rangle$, it is sufficient that $\mathcal{D}$ have some zero-modes at x_c; it is not necessary for $\mathcal{D}$ to be identically zero at x_c, as V' is. In particular, the presence of higher modes of $\mathcal{D}$ is irrelevant.

As before, we add a symmetry-breaking term (the choice is not unique)

$$I_1 = \oint dt\,(iB_i v^i - \Psi_i (D_j v^i)\Phi^j) \tag{4.16}$$

to the action, where $v^i(x)$ is a section of the bundle $T\mathcal{M}$ and is not necessarily related to the section f^i. Once again, the opposite signs for the two terms in (4.16) guarantee the explicit violation of BRST symmetry. Let $F = \oint dt \Psi_i H^i(x)$. Then

$$\langle Q(\Psi H)\rangle_\epsilon = \int [DX] \oint dt\,(iB_i H^i + \Psi_i (D_j H^i)\Phi^j)\exp(-I - \epsilon I_1), \tag{4.17}$$

where $[DX]$ stands for the measure for all of x^i, B_i, Ψ_i, Φ^i. We are interested in $\langle Q(\Psi H)\rangle \equiv \lim_{\epsilon\to 0}\langle Q(\Psi H)\rangle_\epsilon$. To calculate (4.17), we notice that the first part of the integrand is proportional to B, and can therefore be computed using the identity

$$\int [DB] B_i \exp\int dt iB_i(\dot{x}^i + f^i + \epsilon v^i) = \int dt' \Delta^j_i(t,t',\epsilon)\frac{\delta}{\delta x^j(t')}\delta(\dot{x}^i + f^i + \epsilon v^i) \tag{4.18}$$

where $\Delta^i_j(t,t',\epsilon)$ is the Green's function satisfying

$$\mathcal{D}^j_i(f+\epsilon v;t)\Delta^k_j(t,t';\epsilon)=\delta(t-t')\delta^k_i. \tag{4.19}$$

The second part in (4.17) can be evaluated directly by integrating B_i, yielding a δ function. After integration by parts, and expanding in the Riemann normal coordinates,

$$\langle Q(\Psi H)\rangle_\epsilon=\int[Dx]\prod_k\delta(x^k(\epsilon))\frac{\det\mathcal{D}(-\epsilon)}{|\det\mathcal{D}(\epsilon)|}\left\{\oint dsD_iH^j(x(s))(\Delta^i_j(s,s,\epsilon)\right.$$

$$\left.-\Delta^i_j(s,s,-\epsilon))+\oint ds\oint ds'H^m(x(s))\Delta^l_m(s,s',\epsilon)(\Sigma_l(s',\epsilon)-\Sigma_l(s',-\epsilon))\right\},$$

$$\Sigma_l(s,\epsilon)=\Delta^i_j(s,s,\epsilon)\left(\dot{x}^n R_{in}{}^j{}_l+D_lD_i(f^j+\epsilon v^j)\right)(s). \tag{4.20}$$

The ultraviolet divergence involved in this expression is harmless because it is related with the higher modes of $\mathcal{D}$, and will be suppressed when the $\epsilon\to 0$ limit is taken. Expanding Δ^i_j in powers of ϵ, we see immediately that the right-hand side of (4.20) vanishes when ϵ goes to zero, unless $\Delta^i_j(t,t',\epsilon)$ has poles in ϵ. Let us examine under what condition may poles occur.

First of all, we observe that such poles would not exist if $\mathcal{D}(f)$ has no zero modes. To see this, we let $|m\rangle$ be the m-th eigenvector of the operator $\mathcal{D}^i_j(f+\epsilon v;t)$. In terms of a complete set of $\{|m\rangle\}$, $\Delta(t,t';\epsilon)$ can be expanded as

$$\Delta(t,t';\epsilon)=\sum_n|n\rangle\frac{1}{\lambda_n}\langle n|, \tag{4.21}$$

where λ_n is the eigenvalue associated with $|n\rangle$. In the case of a $\mathcal{D}(f)$ with no zero modes, we have for λ_n the first order approximation: $\lambda_n=\lambda_{n0}+\epsilon\lambda_{n1}$ with $\lambda_{n1}=\langle n|Dv|n\rangle$. It is very clear that, since all λ_{n0} are not zero, the right-hand side of (4.21) is a well defined function of ϵ for sufficiently small ϵ. This means that $\Delta(t,t';\epsilon)$ has no poles in ϵ. However, if $\mathcal{D}(f)$ has zero modes, i.e., if some

λ_{n0} is zero, the corresponding eigenvalues λ_α will behave like $\lambda_\alpha \sim \epsilon$ when $\epsilon \to 0$. $\Delta(t,t';\epsilon)$ diverges as $1/\epsilon$. In fact, we can write Δ for small ϵ as

$$\Delta = \frac{1}{\epsilon}A + B, \tag{4.22}$$

where the operator B is regular in ϵ, and the operator A can in principal be determined by matching (4.21) with (4.22). As a result, A lies entirely in the zero mode subspace and, to the first order of approximation, it can be expanded as

$$A = |\alpha\rangle(P^{-1})_{\alpha\beta}\langle\beta|, \tag{4.23}$$

where $|\alpha\rangle, |\beta\rangle$ are zero modes and P^{-1} is the inverse matrix of P with $P_{\alpha\beta} = \langle\alpha|Dv|\beta\rangle$, which we assume to be non-singular.

To verify that in this case $\langle Q(F)\rangle$ indeed does not vanish, we only need to evaluate the most divergent term. We consider the case of vanishing $v(x_c)$ (but $Dv(x_c) \equiv P \neq 0$, this corresponds to the case (4.11b) in the 0-dimensional model) and assume for simplicity that there is only one zero mode whose wave function is $u^i(t)$ for every x_c satisfying $\dot{x}^i + f^i(x) = 0$. Then the operator A becomes

$$A^j_i(t,t') = -u^j(t)u_i(t')\left(\int dt u^m(\mathcal{D}_m v^n)u_n\right)^{-1}, \tag{4.24}$$

Assuming that the second leading term in the expansion of $\Delta^i_j(t,t',\epsilon)$ to be non-degenerate (this is analogous to assuming $f'' \neq 0$ in the 0-dimensional case), we finally obtain

$$\langle Q(\Psi H)\rangle_\epsilon = \epsilon^{-2}\sum_c 2u^i u^l u_j(\dot{x}^k R_{ki}{}^j{}_l + D_i D_j f^j)\,(u^m u_n(D_m v^n))^{-2}\, u^i(H_i). \tag{4.25}$$

As expected, $\langle Q(\Psi H)\rangle$ is indeed not zero. In general, we conclude that $\langle Q(\Psi H)\rangle$ does not vanish as long as $\mathcal{D}(f)$ has zero modes.

In the 1-dimensional model, the index theorem predicts that there are in general no zero modes [20]. Nevertheless, ***accidental*** zero modes may occur as a result of a judicious choice of the parametric functions F and f, namely

$$\ker(\mathcal{D}(0)|_{x=x_c}) \neq 0. \qquad (4.26)$$

As can be seen, this is precisely what we expect: it is the condition for the existence of Gribov zero modes.

Perhaps the most interesting question now is whether the expectation values of (some of) the physical observables in a topological theory would, as a result of $\langle Q(F)\rangle \neq 0$, become metric-dependent. To answer this question, one could study the expectation value of the energy-momentum tensor $T_{ij} \equiv \partial(I + \epsilon I_1)/\partial g^{ij}$. A straightforward calculation would show that $\langle T_{ij}\rangle$ thus defined vanishes. (Note that in this case T_{ij} is no longer a Q-exact form). This, however, would not be the right way to approach the problem. Since ones goal is to obtain the variation of the vacuum expectation value, the correct approach would be to take the $\epsilon \to 0$ limit *before* varying g^{ij}. In the present model, focusing on the topologically non-trivial partition function, we have, for $I_1 = Bv(x,\epsilon) + \Psi g'(x,\epsilon)\Phi$,

$$\delta Z = \delta(\lim_{\epsilon\to 0} Z_\epsilon) = \delta\left(\lim_{\epsilon\to 0}\sum_c \det(\mathcal{D}(f+\epsilon v))|\det(\mathcal{D}(F+\epsilon g)|^{-1}\right). \qquad (4.27)$$

In the $\epsilon \to 0$ limit, all the higher modes cancel, while the zero modes contribute a ratio of A_f/A_g, where A_f is given by (4.23) and A_g is the same with g replacing f. This result can of course be made to be dependent on g_{ij}.

4.3 Other Models

The generalization to 2-dimensional sigma-models is straightforward. Since in this case the topological index of T is in general not zero, one should consider $\langle Q(F)\rangle$ in which F has the appropriate ghost number that allows it to absorb the topological zero modes. The condition for having non-vanishing $\langle Q(F)\rangle$ becomes that of the existence of accidental zero modes - zero modes which appear in pairs and are therefore not effected by the index theorem. Actually this statement is in general true for any topological model. Thus the problem of whether spontaneous symmetry breaking can happen becomes the problem of finding such accidental zero modes.

It is important to notice that in the topological gauge theory [5] there are zero modes that are associated with reducible connections, namely connections that are invariant under some gauge transformation. Since such degrees of freedom cannot be removed by any well defined gauge-fixing, they give rise to Gribov zero modes [36]. The analysis given above concerning the 0-dimensional and 1-dimensional models suggests that these reducible connections are responsible for topological symmetry breaking, at least in the Landau gauge. The same argument applies to 2-dimensional topological gravity [15], where a reducible configuration is a metric that admits Killing vectors. Since in this model the Landau gauge is a natural gauge, it seems that our approach could be used to study it. In this case, however, a natural gauge-breaking term which may well be non-local in the moduli space needs to be found.

Chapter 5

Conclusion

In this thesis we have studied the properties of topological field theories relating to both mathematics and physics. We demonstrated how the partition functions of cohomology models can be interpreted as character-valued indices. From a mathematics point of view, this establishes an equivalence between the first Donaldson's invariants that are identified with the partition functions of TQFTs and various character-valued indices, thereby enriching our limited understanding of some topological invariants and their inter-relationships. The main problem in this context is that the definition of some indices is not mathematically rigorous. For example, the character-valued index for the topological gauge theory is defined without a regulating term; it is not therefore mathematically well defined. Also, certain mathematical structures of the models, such as that relating to the inner-product of the Hilbert space, need to be better understood.

We investigated the possibility of a topological phase transition by studying in detail a mechanism for the spontaneous breaking of topological symmetry. The latter is achieved by adding a vanishingly small symmetry-breaking term to the action. We showed how the presence of reducible configurations in topological field theories can make the vacuum expectation values of BRST exact terms not vanish, thus spontaneously breaking topological symmetry. In this regard, a number of problems remain to be solved. First, the mechanism for the spontaneous breaking of topological symmetry discussed in Chapter 4 has to be confirmed for models in a space-time dimension higher that 1. The consideration of such models is vital as they contain propagating modes, and a coupling between the propagating modes of gravitation and matter is a crucial signature of a phase with broken topological symmetry. However, in these models, owing to the admixture of topological zero modes and accidental zero modes, an exact evaluation of the correlation functions is so complicated that so far we have been unable to work it out.

An interesting idea related to the topological phase transition is the reconstruction of topological field theory from a conventional field theory. This may be viewed as the reverse process of the standard symmetry breaking. The idea, still at a primitive stage, is based on the observation that the phase with which the topological symmetry is associated usually contains higher energy modes described by fluctuations with short wavelengths, whereas the symmetry broken phase has low energy modes described by fluctuations with long wavelengths. Therefore, it should be possible to formulate a field theory that is topological at small distances and nontopological at large distances. Such a formulation can in principle be carried out with the help of Wilson's spin-block technique. This is conceivable because in this technique fluctuations with small wavelengths are integrated out, resulting in the loss of degrees of freedom that, as mentioned earlier, can be naturally interpreted as topological degrees of freedom. We intend to pursue this line of investigation further, and we hope that it will help us solve the mystery of the topological phase transition.

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