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Pseudo Almost Periodic Functions And Their Applications

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PSEUDO ALMOST PERIODIC FUNCTIONS AND THEIR APPLICATIONS

by

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periodic functions. We deal with the pseudo almost periodicity of the primitive of a $\mathcal{PAP}$ function in Section 10. Section 11 is about weak pseudo almost periodicity.

Chapter IV consists of two parts. The first part develops the theory of vector-valued means. A formula is set up between a vector-valued mean and a scalar-valued mean that enables us to translate many important results about scalar-valued means to vector-valued means. We present these results in Sections 12, 13 and 14. Then in Section 15, the second part of the chapter, we return to $\mathcal{WAP}$ functions defined on any semitopological semigroup S. We use the theory established in the first part to investigate the space $\mathcal{WAP}(S, X)$. Also, in Section 15, we answer the question mentioned earlier by using the results in Chapter III.

Notation. The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural numbers, real numbers, and complex numbers respectively. If X and Y are Banach spaces, then $\mathcal{L}(X, Y)$ denotes the vector space of all continuous linear mappings from X into Y. $\mathcal{L}(X, \mathbb{C})$, the dual space of X, is denoted by X^*. $\sigma(X, X^*)$ is called the weak topology on X; it is the weakest topology on X relative to which each member of X^* is continuous. Dually, $\sigma(X^*, X)$ is the weakest topology on X^* relative to which the mapping $x^* \to x^*(x) : X^* \to \mathbb{C}$ is continuous for each $x \in X$; $\sigma(X^*, X)$ is called the w^* topology. We denote the closure of a set $A \subset X$ by $\overline{A}$. spA, coA, and $ccoA$ denote, respectively, the linear span of A, the convex hull of A, and the convex circled hull of A [2, p.291]. If a function $f \in \mathcal{C}(\mathbb{R})$ is uniformly continuous, then the modulus w of uniform continuity of f is defined for $h > 0$ by $w(h) = \inf\{\epsilon \in \mathbb{R} : |f(x') - f(x'')| < \epsilon \text{ for all } x', x'' \in \mathbb{R}, |x' - x''| < h\}$.

vector-valued mean and a scalar-valued mean. As an application of the theory of vector-valued means, a theorem is given to show that the space of vector-valued weakly almost periodic functions is admissible.

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CONTENTS

INTRODUCTION AND NOTATION

The space of almost periodic functions on $\mathbb{R}$ ($\mathcal{AP}(\mathbb{R})$ for short) is the smallest C^*–subalgebra of $C(\mathbb{R})$ containing the continuous periodic functions. To apply the theory of $\mathcal{AP}(\mathbb{R})$ to the solutions of some types of differential equations, one needs to know if the almost periodicity of a function f implies the almost periodicity of its primitive $F(t) = \int_a^t f(x)dx$; a theorem of Bohr shows that $F \in \mathcal{AP}(\mathbb{R})$ if and only if F is bounded. A question arises naturally in $\mathcal{WAP}(\mathbb{R})$, the space of weakly almost periodic functions on $\mathbb{R}$ and one of the generalizations of $\mathcal{AP}(\mathbb{R})$. That is, if $f \in \mathcal{WAP}(\mathbb{R})$, what is a necessary and sufficient condition for F to be in $\mathcal{WAP}(\mathbb{R})$?

Set $\mathcal{PAP}_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \lim_{t\to\infty} 1/2t \int_{-t}^{t} |f(x)|dx = 0\}$. A decomposition theorem for $\mathcal{WAP}(\mathbb{R})$ expresses $f \in \mathcal{WAP}(\mathbb{R})$ as $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\mathbb{R}) \cap \mathcal{WAP}(\mathbb{R})$. The difficult part of answering the question is to handle the function φ. For this purpose, we introduce a new generalization of $\mathcal{AP}(\mathbb{R})$, called pseudo almost periodic functions, $\mathcal{PAP}(\mathbb{R})$ for short. A function $f \in \mathcal{PAP}(\mathbb{R})$ if and only if

$$f = g + \varphi$$

where $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\mathbb{R})$. Obviously, $\mathcal{PAP}(\mathbb{R})$ contains $\mathcal{WAP}(\mathbb{R})$. It is easier to handle a $\mathcal{PAP}$ function than to handle a $\mathcal{WAP}$ function. In fact, after a theorem has been set up which gives a necessary and sufficient condition for the pseudo almost periodicity of the primitive of a $\mathcal{PAP}$ function, the question is answered by an easy corollary.

The question motivated us to propose and investigate $\mathcal{PAP}$ functions. The theory developed for $\mathcal{PAP}$ functions and their applications go far beyond the results and question above. Now we give an outline of the thesis.

The thesis consists of four chapters. In Chapter I, Section 1 is preliminaries. The theory of scalar-valued $\mathcal{PAP}$ functions on $\mathbf{R}$ is developed in Section 2. It is shown that $\mathcal{PAP}(\mathbf{R})$ is a translation invariant C^*-subalgebra of $\mathcal{C}(\mathbf{R})$. The relationship between a $\mathcal{PAP}$ function and its $\mathcal{AP}$ component is discussed. Fourier analysis is carried out on $\mathcal{PAP}(\mathbf{R})$. In Section 3, the theory of $\mathcal{PAP}$ functions depending on parameters is developed; it will be applied in Chapter II to the solutions of some types of differential equations. Finally, in Section 4 we give some examples and make some comments to distinguish between $\mathcal{PAP}(\mathbf{R})$ and $\mathcal{AP}(\mathbf{R})$, as well as between $\mathcal{PAP}(\mathbf{R})$ and $\mathcal{WAP}(\mathbf{R})$.

Chapter II consists of three sections, which present the applications of $\mathcal{PAP}$ functions. In Sections 5, 6 and 7, the solutions of three types of differential equations—ordinary, nonli r parabolic and boundary value problems for harmonic functions—are investigated respectively.

In Chapter III we consider vector-valued $\mathcal{PAP}$ functions $\mathcal{PAP}(\mathbf{J}_a, X)$, where $\mathbf{J}_a = [a, \infty)$ for $a \in \mathbf{R}$ and $\mathbf{J}_a = \mathbf{R}$ when $a = -\infty$.The main result in Section 8 is a decomposition theorem. Section 9 is about the space of vector-valued pseudo almost periodic functions. We deal with the pseudo almost periodicity of the primitive of a $\mathcal{PAP}$ function in Section 10. Section 11 is about weak pseudo almost periodicity.

Chapter IV consists of two parts. The first part develops the theory of vector-valued means. A formula is set up between a vector-valued mean and a scalar-valued mean that enables us to translate many important results about scalar-valued means to vector-valued means. We present these results in Sections 12, 13 and 14. Then in Section 15, the second part of the chapter, we return to $\mathcal{WAP}$ functions defined on any semitopological semigroup S. We use the theory established in the first part to investigate the space $\mathcal{WAP}(S,X)$. Also, in Section 15, we answer the question mentioned earlier by using the results in Chapter III.

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CHAPTER I

NUMERICAL PSEUDO ALMOST PERIODIC FUNCTIONS

In this chapter, we define and investigate a new generalization of the almost periodic functions, which we call the pseudo almost periodic functions. It will be shown that many properties of almost periodic functions can be extended to the pseudo almost periodic functions and Fourier analysis can also be carried out in the space of the pseudo almost periodic functions. As for the space of almost periodic functions and some of its generalizations, the pseudo almost periodic functions have many applications in the theory of differential equations. We will treat the applications in the next chapter.

§1. Preliminaries

Throughout the thesis, $\mathcal{C}(\mathbb{R})$ denotes the C^*–algebra of bounded continuous complex-valued functions on $\mathbb{R}$. For $f \in \mathcal{C}(\mathbb{R})$ and $s \in \mathbb{R}$, the translate of f by s is the function $R_s f(t) = f(t+s)$. A subset $\mathcal{F}$ of $\mathcal{C}(\mathbb{R})$ is said to be translation invariant if $R_s\mathcal{F} \subset \mathcal{F}$ for all $s \in \mathbb{R}$. $\mathcal{F}$ is said to be an algebra if it is a linear space and $fg \in \mathcal{F}$ for all $f, g \in \mathcal{F}$.

A linear functional $\mu : \mathcal{F} \to \mathbb{C}$ is called a mean if $\mu(f) \in \overline{co}f(\mathbb{R})$ for all $f \in \mathcal{F}$. If $\mathcal{F}$ is translation invariant and if a mean μ on $\mathcal{F}$ satisfies

$$\mu(R_s f) = \mu(f) \qquad (s \in \mathbb{R},\ f \in \mathcal{F}),$$

then μ is said to be invariant.

Definition 1.1. *A function $g \in \mathcal{C}(\mathbb{R})$ is called almost periodic if for each $\epsilon > 0$, there exists an $l_\epsilon > 0$ such that every interval of length l_ϵ contains a number τ with the property that*

$$\|R_\tau g - g\| < \epsilon.$$

The number τ is called an ϵ–translation number of g. Denote by $\mathcal{AP}(\mathbb{R})$ the set of all such functions.

$\mathcal{AP}(\mathbb{R})$ is a translation invariant C^*–subalgebra of $\mathcal{C}(\mathbb{R})$ containing the constant functions.

For $f \in \mathcal{C}(\mathbb{R})$, let $\{R_{\mathbb{R}} f\} = \{R_s f : s \in \mathbb{R}\}$. It follows from [6, Theorems 1.9, 1.10, 1.11] that a function $g \in \mathcal{C}(\mathbb{R})$ is in $\mathcal{AP}(\mathbb{R})$ if and only if the set $\{R_{\mathbb{R}} g\}$ is relatively compact in $\mathcal{C}(\mathbb{R})$. By [6, Theorem 1.3], $g \in \mathcal{AP}(\mathbb{R})$ is uniformly continuous.

We call a function $f \in \mathcal{C}(\mathbb{R})$ weakly almost periodic if the set $\{R_{\mathbb{R}} f\}$ is weakly relatively compact in $\mathcal{C}(\mathbb{R})$. Denote by $\mathcal{WAP}(\mathbb{R})$ the set of all weakly almost periodic functions.(The spaces $\mathcal{AP}(\mathbb{R})$ and $\mathcal{WAP}(\mathbb{R})$ can be defined on any semitopological semigroup in terms of these compactness conditions [2]).

It was shown in [6, Theorem 1.12] that, for $g \in \mathcal{AP}(\mathbb{R})$, $\lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} g(x)dx$ exists. Therefore we can define $M : \mathcal{AP}(\mathbb{R}) \to \mathbb{C}$, for each $g \in \mathcal{AP}(\mathbb{R})$, by

$$M(g) = \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} g(x)dx.$$

It is easy to show that M is an invariant mean on $\mathcal{AP}(\mathbb{R})$. For the details on the theory of means, see Chapter IV or [2]. For each $g \in \mathcal{AP}(\mathbb{R})$, the set $\{\lambda :$

$M(ge^{-i\lambda\cdot}) \neq 0\}$ is countable ([6, Theorem 1.15]); denote it by $\{\lambda_k : k \in \mathbb{N}\}$ with $A_k = M(ge^{-i\lambda_k\cdot})$. Then we have the Fourier series of g,

$$g(x) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k x};$$

the numbers $\lambda_k \in \mathbb{R}$ and $A_k \in \mathbb{C}$ are known as the Fourier exponents and coefficients respectively of the function g. It is known [6, Theorem 1.18] that, for $g \in \mathcal{AP}(\mathbb{R})$, Parseval's equality holds:

$$M(|g|^2) = \sum_{k=1}^{\infty} |A_k|^2.$$

Proposition 1.2 [6, Theorem 1.19]. *Distinct almost periodic functions have distinct Fourier series.*

Proposition 1.3 [1, 2.3][6, Theorem 1.24]. *For each $g \in \mathcal{AP}(\mathbb{R})$ with the Fourier series*

$$g(x) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k x},$$

there exists a sequence $\{\sigma_m\}$ of trigonometric polynomials,

$$\sigma_m(x) = \sum_{k=1}^{n(m)} r_{k,m} A_k e^{i\lambda_k x},$$

such that $\|\sigma_m - g\| \to 0$ as $m \to \infty$. The numbers $r_{k,m}$ are rational, depend on λ_k and m, but not on A_k, and satisfy

$$\lim_{m\to\infty} r_{k,m} = 1, \tag{1.1}$$

and

$$0 \le r_{k,m} \le 1. \tag{1.2}$$

§2. Pseudo Almost Periodic Functions on $\mathbb{R}$

In §1, we defined an invariant mean M on $\mathcal{AP}(\mathbb{R})$ by

$$M(g) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} g(x)dx \qquad (g \in \mathcal{AP}(\mathbb{R})).$$

Now, we extend the domain of definition of M.

Set

$$\mathcal{PAP}_0(\mathbb{R}) = \left\{\varphi \in \mathcal{C}(\mathbb{R}) : \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(x)|dx = 0\right\}.$$

$\mathcal{PAP}_0(\mathbb{R})$ is a translation invariant closed ideal of $\mathcal{C}(\mathbb{R})$.

Definition 2.1. *A function $f \in \mathcal{C}(\mathbb{R})$ is called pseudo almost periodic if*

$$f = g + \varphi,$$

where $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\mathbb{R})$. g and φ are called the almost periodic component and the ergodic perturbation respectively of the function f. Denote by $\mathcal{PAP}(\mathbb{R})$ the set of all such functions f.

For the motivation of the terminology for ergodic perturbation, see Remark 4.4 (1).

One can see that $\mathcal{PAP}(\mathbb{R})$ is a translation invariant, conjugate closed, subalgebra of $\mathcal{C}(\mathbb{R})$ containing the constant functions. Let $f \in \mathcal{PAP}(\mathbb{R})$. Since $fe^{-i\lambda\cdot} = ge^{-i\lambda\cdot} + \varphi e^{-i\lambda\cdot}$ and $\lim_{t\to\infty} 1/2t \int_{-t}^{t} f(x)e^{-i\lambda x}dx = M(ge^{-i\lambda\cdot})$, for all $\lambda \in \mathbb{R}$, Proposition 1.2 implies that

$$\mathcal{PAP}(\mathbb{R}) = \mathcal{AP}(\mathbb{R}) \oplus \mathcal{PAP}_0(\mathbb{R}).$$

Thus we can extend M to an invariant mean on $\mathcal{PAP}(\mathbb{R})$ by $M(f) = M(g)$, since each $f \in \mathcal{PAP}(\mathbb{R})$ has an associated Fourier series, namely the Fourier series of its almost periodic component g.

Lemma 2.2. *If $f \in \mathcal{PAP}(\mathbb{R})$ and if g is its almost periodic component, then we have*

$$g(\mathbb{R}) \subset \overline{f(\mathbb{R})}.$$

Therefore $\|f\| \geq \|g\| \geq \inf_{x\in\mathbb{R}} |g(x)| \geq \inf_{x\in\mathbb{R}} |f(x)|$.

Proof. If $g(\mathbb{R}) \not\subset \overline{f(\mathbb{R})}$, then there is an $x_0 \in \mathbb{R}$ such that $\inf_{s\in\mathbb{R}} |g(x_0) - f(s)| > \epsilon$. Since g is continuous at x_0, there is a $\delta > 0$ such that $|x| < \delta$ implies that $\inf_{s\in\mathbb{R}} |R_{x_0}g(x) - R_{x_0}f(s)| > \epsilon$. Since $\mathcal{AP}(\mathbb{R})$ is translation invariant, $R_{x_0}g$ is in $\mathcal{AP}(\mathbb{R})$. By Definition 1.1, for $\epsilon > 0$ there exists $l_{\epsilon/2} > 0$ such that every interval of length $l_{\epsilon/2}$ contains a number τ with the property that $\|R_\tau R_{x_0}g - R_{x_0}g\| \leq \epsilon/2$. If $x \in (-\delta, \delta)$, then $x + \tau \in (\tau - \delta, \tau + \delta)$. Therefore

$$\begin{aligned}
|R_{x_0}\varphi(x+\tau)| &= |R_{x_0}f(x+\tau) - R_{x_0}g(x+\tau)| \\
&\geq |R_{x_0}f(x+\tau) - R_{x_0}g(x)| - |R_{x_0}g(x) - R_{x_0}g(x+\tau)| \\
&\geq \inf_{s\in\mathbb{R}} |R_{x_0}f(s) - R_{x_0}g(x)| - \|R_{x_0}g - R_\tau R_{x_0}g\| \\
&> \frac{\epsilon}{2}.
\end{aligned}$$

This means that $M(|R_{x_0}\varphi|) \geq \delta \cdot \epsilon / l_\epsilon > 0$, a contradiction.

Theorem 2.3. *$\mathcal{PAP}(\mathbb{R})$ is a translation invariant C^*-subalgebra of $\mathcal{C}(\mathbb{R})$ containing the constant functions. Furthermore $\mathcal{PAP}(\mathbb{R})/\mathcal{PAP}_0(\mathbb{R}) \cong \mathcal{AP}(\mathbb{R})$.*

Proof. Noting the paragraph before Lemma 2.2, we see that the only remaining part of the proof is to show that $\mathcal{PAP}(\mathbb{R})$ is a closed subspace of $\mathcal{C}(\mathbb{R})$.

Let a sequence $\{f_n\} \subset \mathcal{PAP}(\mathbb{R})$ be Cauchy. By Lemma 2.2, the sequence $\{g_n\} \subset \mathcal{AP}(\mathbb{R})$ is Cauchy too. So is $\{\varphi_n\} \subset \mathcal{PAP}_0(\mathbb{R})$. Since $\mathcal{AP}(\mathbb{R})$ and $\mathcal{PAP}_0(\mathbb{R})$ are closed in $\mathcal{C}(\mathbb{R})$, there are $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\mathbb{R})$ such th $\|g_n - g\| \to 0$ and $\|\varphi_n - \varphi\| \to 0$, as $n \to \infty$. Set $f = g + \varphi$. $f \in \mathcal{PAP}(\mathbb{R})$ and $\|f_n - f\| \to 0$, as $n \to \infty$.

Let X be the set of all multiplicative means on $\mathcal{PAP}(\mathbb{R})$. With respect to the weak* topology of $\mathcal{PAP}(\mathbb{R})^*$, X is compact ([2, 2.1.8]). Since $\mathcal{PAP}(\mathbb{R})$ is a C^*-algebra, $\mathcal{PAP}(\mathbb{R})$ is isometrically isomorphic to $\mathcal{C}(X)$ ([2, 2.1.9] or [8, 1.4.1]).

Corollary 2.4. *Let Φ be a uniformly continuous function on $\mathcal{U} \subset \mathbf{C}^n$. If $f = (f_1, f_2, \cdots, f_n) \in \mathcal{PAP}(\mathbb{R})^n$ is such that $(f_1(x), f_2(x), \cdots, f_n(x)) \in \mathcal{U}$ for all $x \in \mathbb{R}$, then the function $\Phi \circ f \in \mathcal{PAP}(\mathbb{R})$.*

Proof. Since f is bounded, we can assume without loss of generality that $\mathcal{U}$ is bounded and closed. By the Stone–Weierstrass theorem, for $\epsilon > 0$ there is a polynomial $P_\epsilon \in \mathcal{C}(\mathcal{U})$ such that $\|\Phi - P_\epsilon\| < \epsilon$. So $\|\Phi \circ f - P_\epsilon \circ f\| < \epsilon$. By Theorem 2.3, $P_\epsilon \circ f \in \mathcal{PAP}(\mathbb{R})$. Again by Theorem 2.3, $\Phi \circ f \in \mathcal{PAP}(\mathbb{R})$.

If $f \in \mathcal{PAP}(\mathbb{R})$ has an inverse f^{-1} in $\mathcal{C}(\mathbb{R})$, then there is a number $m > 0$ such that $\inf_{x\in\mathbb{R}} |f(x)| \geq m$. The next corollary shows that the condition is necessary and sufficient for f^{-1} to be in $\mathcal{PAP}(\mathbb{R})$.

Corollary 2.5. *A function $f \in \mathcal{PAP}(\mathbb{R})$ has an inverse f^{-1} in $\mathcal{PAP}(\mathbb{R})$ if and only if there is a number $m > 0$ such that $\inf_{x\in\mathbb{R}} |f(x)| \geq m$. If $f \in \mathcal{PAP}(\mathbb{R})$ has an* *inverse in $C(\mathbb{R})$, so does its almost periodic component g; in this case, $g^{-1} \in \mathcal{AP}(\mathbb{R})$ and the following formula holds*

$$f^{-1} = g^{-1} + \phi,$$

where $\phi = -\varphi g^{-1} f^{-1} \in \mathcal{PAP}_0(\mathbb{R})$ and φ is the ergodic perturbation of f.

Proof. Suppose that $\inf_{x\in\mathbb{R}} |f(x)| \geq m > 0$. Since f is bounded, there is a number $M \in \mathbb{R}$ such that $0 < m \leq |f(x)| \leq M$. The function $\Phi(z) = 1/z$ is uniformly continuous in $m \leq |z| \leq M$, and according to Corollary 2.4, $f^{-1} \in \mathcal{PAP}(\mathbb{R})$. With Lemma 2.2, the second statement can be proved similarly. By checking directly, one shows that the formula holds.

Corollary 2.6. *If $f \in \mathcal{PAP}(\mathbb{R})$ and its derivative f' is uniformly continuous on $\mathbb{R}$, then $f' \in \mathcal{PAP}(\mathbb{R})$.*

Proof. Let $f = h + i\phi$, where h and ϕ are real functions on $\mathbb{R}$, so that $f' = h' + i\phi'$. Consider the functions

$$\phi_n = n(R_{1/n} f - f), \qquad n = 1, 2, \cdots.$$

By Theorem 2.3, $\phi_n \in \mathcal{PAP}(\mathbb{R})$, $n = 1, 2, \cdots$. Now

$$\begin{aligned} \phi_n(x) &= n[h(x + \frac{1}{n}) - h(x)] + in[\phi(x + \frac{1}{n}) - \phi(x)] \\ &= h'(x + \frac{\xi_n}{n}) + i\phi'(x + \frac{\theta_n}{n}), \end{aligned}$$

where $0 < \xi_n, \theta_n < 1$. By the hypothesis of uniform continuity of f', $\|\phi_n - f'\| \to 0$, as $n \to \infty$. Again by Theorem 2.3, $f' \in \mathcal{PAP}(\mathbb{R})$.

Recall the way we defined the Fourier series for functions in $\mathcal{PAP}(\mathbb{R})$. We can similarly define the Fourier series for some other functions in $\mathcal{C}(\mathbb{R})$. Let $f \in \mathcal{C}(\mathbb{R})$. Suppose that for all $\lambda \in \mathbb{R}$ $\lim_{t\to\infty} 1/2t \int_{-t}^{t} f(x)e^{-i\lambda x}dx$ exists and also that the set $\{\lambda : \lim_{t\to\infty} 1/2t \int_{-t}^{t} f(x)e^{-i\lambda x}dx \neq 0\}$ is countable; denote this set by $\{\lambda_k : k \in \mathbb{N}\}$ and put $A_k = \lim_{t\to\infty} 1/2t \int_{-t}^{t} f(x)e^{-i\lambda_k x}dx$. Then we call $\sum_{k=1}^{\infty} A_k e^{i\lambda_k x}$ the Fourier series of f. We will say f satisfies Parseval's equality if $\lim_{t\to\infty} 1/2t \int_{-t}^{t} |f(x)|^2 dx = \sum_{k=1}^{\infty} |A_k|^2$.

The next theorem will give necessary and sufficient conditions for a function $f \in \mathcal{C}(\mathbb{R})$ to be in $\mathcal{PAP}(\mathbb{R})$. First we need a lemma.

Lemma 2.7. *A function $\varphi \in \mathcal{C}(\mathbb{R})$ is in $\mathcal{PAP}_0(\mathbb{R})$ if and only if φ^2 is.*

Proof. The sufficiency follows since

$$\begin{aligned}\frac{1}{2t}\int_{-t}^{t}|\varphi(x)|dx &\leq \frac{1}{2t}\left[\int_{-t}^{t}|\varphi(x)|^2dx\right]^{1/2}\left[\int_{-t}^{t}1dx\right]^{1/2}\\ &= \left[\frac{1}{2t}\int_{-t}^{t}|\varphi(x)|^2dx\right]^{1/2}.\end{aligned}$$

The necessity follows from the fact that $\mathcal{PAP}_0(\mathbb{R})$ is an ideal of $\mathcal{C}(\mathbb{R})$.

Theorem 2.8. *A function $f \in \mathcal{C}(\mathbb{R})$ is in $\mathcal{PAP}(\mathbb{R})$ if and only if there exists a function $g \in \mathcal{AP}(\mathbb{R})$ such that f and g have the same Fourier series and f satisfies Parseval's equality.*

Proof. Necessity.

It follows from Theorem 2.3 that $|f|^2 \in \mathcal{PAP}(\mathbf{R})$. Therefore

$$\begin{aligned} M\left(|f|^2\right) &= M\left((g+\varphi)(\overline{g}+\overline{\varphi})\right) \\ &= M\left(|g|^2\right) + M\left(\varphi\overline{g}\right) + M\left(g\overline{\varphi}\right) + M\left(|\varphi|^2\right) \\ &= M\left(|g|^2\right). \end{aligned}$$

We have stated in the paragraph before Lemma 2.2 that each $f \in \mathcal{PAP}(\mathbf{R})$ has a Fourier series, namely that of its almost periodic component g.

Sufficiency.

To show that the conditions are sufficient, by Lemma 2.7 we show that $(f-g)^2 \in \mathcal{PAP}_0(\mathbf{R})$, i.e., $\lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} |f(x)-g(x)|^2 dx = 0$. By the hypothesis,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} |f(x)-g(x)|^2 dx \\ (2.1) \qquad &= \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} (f(x)-g(x))(\overline{f(x)}-\overline{g(x)})dx \\ &=2M(|g|^2) - \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} f(x)\overline{g(x)}dx - \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} \overline{f(x)}g(x)dx. \end{aligned}$$

Now, we need to discuss two cases.

(I). g has only a finite number of non-zero coefficients A_k, $k = 1, 2, \cdots, n$. In this case, by Proposition 1.2

$$g(x) = \sum_{k=1}^{n} A_k e^{i\lambda_k x}.$$

It is easy to calculate from (2.1) that $\lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} |f(x)-g(x)|^2 dx = 0$ for this case.

(II). g has infinitely many non-zero coefficients A_k. In this case, we can assume that, in Proposition 1.3, $n(m) \to \infty$ when $m \to \infty$.

By Proposition 1.3, $\|\sigma_m - g\| \to 0$ when $m \to \infty$. Therefore

$$\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t} f(x)\overline{g(x)}dx = \lim_{m\to\infty}\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t} f(x)\overline{\sigma_m(x)}dx = \lim_{m\to\infty}\sum_{k=1}^{n(m)} r_{k,m}|A_k|^2.$$

Similarly, we have

$$\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t}\overline{f(x)}g(x)dx = \lim_{m\to\infty}\sum_{k=1}^{n(m)} r_{k,m}|A_k|^2.$$

Noting (1.2) and the fact that $n(m) \to \infty$ as $m \to \infty$, we have

$$\begin{aligned}
0 &\le \sum_{k=1}^{\infty}|A_k|^2 - \sum_{k=1}^{n(m)} r_{k,m}|A_k|^2 \\
&= \sum_{k=1}^{N}(1-r_{k,m})|A_k|^2 + \sum_{k=N+1}^{n(m)}(1-r_{k,m})|A_k|^2 + \sum_{k=n(m)+1}^{\infty}|A_k|^2 \\
&\le \sum_{k=1}^{N}(1-r_{k,m})|A_k|^2 + \sum_{k=N+1}^{\infty}|A_k|^2.
\end{aligned}$$

By (1.1), $\sum_{k=1}^{N}(1-r_{k,m})|A_k|^2 \to 0$ as $m \to \infty$ for each $N \in \mathbf{N}$. Therefore

$$\lim_{m\to\infty}\sum_{k=1}^{n(m)} r_{k,m}|A_k|^2 = \sum_{k=1}^{\infty}|A_k|^2,$$

which implies from (2.1) that $\lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t}|f(x)-g(x)|^2dx = 0$ because, by Parseval's equality,

$$M(|g|^2) = \sum_{k=1}^{\infty}|A_k|^2.$$

The proof is finished.

A function $f \in \mathcal{C}(\mathbb{R})$ is said to have the Fourier-Parseval property (FP, for short) if there exists a function $g \in \mathcal{AP}(\mathbb{R})$ such that f and g have the same Fourier series and f satisfies Parseval's equality.

Theorem 2.8 demonstrates that $\mathcal{PAP}(\mathbf{R})$ is the largest among C^*–subalgebras of $C(\mathbf{R})$ with the FP property. We mention here two other C^*–algebras with the FP property.

In [2], ten classes of functions of almost periodic type on a semitopological semigroup S are investigated; one of them is $\mathcal{WAP}(S)$, the C^*–algebra of weakly almost periodic functions. A decomposition theorem for $\mathcal{WAP}(S)$ is given in [2, 4.3.13]. For the case that $S = \mathbf{R}$, it states that

$$\mathcal{WAP}(\mathbf{R}) = \mathcal{AP}(\mathbf{R}) \oplus \mathcal{F}_0,$$

where $\mathcal{F}_0 = \mathcal{WAP}(\mathbf{R}) \cap \mathcal{PAP}_0(\mathbf{R})$. Therefore

$$\mathcal{WAP}(\mathbf{R}) \subset \mathcal{PAP}(\mathbf{R}),$$

so $\mathcal{WAP}(\mathbf{R})$ has the FP property. It is easier to handle a pseudo almost periodic function than to handle a weakly almost periodic function, which is one of our motivations to introduce the pseudo almost periodic functions.

In Chapter III, we will consider vector–valued pseudo almost periodic functions on $\mathbf{R}$ and on $\mathbf{J}_a = [a, \infty)$, $a \in \mathbf{R}$. $\mathcal{PAP}(\mathbf{J}_a)$ contains many known function spaces, for example, the C^*–algebra of asymptotically almost periodic functions, which therefore has the FP property on $x \geq a$ (see [19, 21] or Chapter III).

§3. Pseudo Almost Periodic Functions Depending on Parameters

Let $\Omega \subset \mathbf{C}^n$ and denote points in Ω by $Z = (z_1, z_2, \ldots, z_n)$.

Definition 3.1. *A function $g \in C(\Omega \times \mathbf{R})$ is called almost periodic in x, uniformly with respect to $Z \in \Omega$, if for any $\epsilon > 0$ corresponds a number $l_\epsilon > 0$ such that any* *interval in $\mathbb{R}$ of length l_ϵ contains at least one number τ for which*

$$|g(Z, x+\tau) - g(Z, x)| < \epsilon \qquad (Z \in \Omega, x \in \mathbf{R}).$$

Denote by $\mathcal{AP}(\Omega \times \mathbb{R})$ the set of all such functions.

Definition 3.2. *Set $\mathcal{PAP}_0(\Omega \times \mathbb{R}) = \{\varphi \in \mathcal{C}(\Omega \times \mathbf{R}) : \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} |\varphi(Z, x)| dx$ $= 0$ uniformly with respect to $Z \in \Omega\}$. Let $\mathcal{PAP}(\Omega \times \mathbf{R})$ denote all the functions f of the form*

$$f = g + \varphi,$$

where $g \in \mathcal{AP}(\Omega \times \mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\Omega \times \mathbb{R})$.

As for $\mathcal{PAP}(\mathbf{R})$, we have $\mathcal{PAP}(\Omega \times \mathbb{R}) = \mathcal{AP}(\Omega \times \mathbf{R}) \oplus \mathcal{PAP}_0(\Omega \times \mathbf{R})$.

Since $\mathcal{AP}(\Omega \times \mathbb{R})$ is a Banach space ([6, Theorem 2.4]), the following theorem can be proved in much the same way as Theorem 2.3.

Theorem 3.3. *$\mathcal{PAP}(\Omega \times \mathbb{R})$ is a C^*- subalgebra of $\mathcal{C}(\Omega \times \mathbf{R})$ that is translation invariant in the $\mathbf{R}$-variable. Furthermore, $\mathcal{PAP}(\Omega \times \mathbf{R})/\mathcal{PAP}_0(\Omega \times \mathbf{R}) \cong \mathcal{AP}(\Omega \times \mathbf{R})$.*

Remark 3.4. *Results analogous to Corollaries 2.4 and 2.5 can be proved for $\mathcal{PAP}(\Omega \times \mathbf{R})$; we don't state them.*

A function $f \in \mathcal{C}(\Omega \times \mathbf{R})$ is called Ω-uniformly continuous on $\mathbf{R}$, if to any $\epsilon > 0$ corresponds a $\delta > 0$ such that

$$|f(Z, x_1) - f(Z, x_2)| < \epsilon \qquad (Z \in \Omega)$$

for all $x_1, x_2 \in \mathbf{R}$ with

$$|x_2 - x_1| < \delta.$$

The proof of the next theorem is similar to the proof of Corollary 2.6, so we omit it.

Theorem 3.5. *If $f \in \mathcal{PAP}(\Omega \times \mathbf{R})$ and $\partial f/\partial x$ is Ω-uniformly continuous, then $\partial f/\partial x \in \mathcal{PAP}(\Omega \times \mathbf{R})$.*

For $H = (h_1, h_2, \cdots, h_n) \in \mathcal{C}(\mathbf{R})^n$, suppose that $H(x) \in \Omega$ for all $x \in \mathbf{R}$. Define $H \times \iota : \mathbf{R} \to \Omega \times \mathbf{R}$ by

$$H \times \iota(x) = (h_1(x), h_2(x), \cdots, h_n(x), x) \qquad (x \in \mathbf{R}).$$

For $F = (f_1, f_2, \cdots, f_n) \in \mathcal{PAP}(\mathbf{R})^n$, let $G = (g_1, g_2, \cdots, g_n)$ and $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$, where g_i and φ_i are the almost periodic component and the ergodic perturbation respectively of f_i, $i = 1, 2, \cdots, n$. The following theorem will be used in the next chapter to investigate the solutions of some differential equations.

Theorem 3.6. *For $i = 1, 2, \cdots, n$, let $\Omega_i \subset \mathbf{C}$ be closed; set $\Omega = \Pi_{i=1}^n \Omega_i \subset \mathbf{C}^n$. Let $f \in \mathcal{PAP}(\Omega \times \mathbf{R})$ satisfy*

$$|f(Z', x) - f(Z'', x)| \le L \sum_{i=1}^{n} |z_i' - z_i''| \qquad (Z', Z'' \in \Omega, \quad x \in \mathbf{R}),$$

where $L > 0$. If $F \in \mathcal{PAP}(\mathbf{R})^n$ and $F(x) \in \Omega$ for all $x \in \mathbf{R}$, then $f \circ (F \times \iota) \in \mathcal{PAP}(\mathbf{R})$.

Proof. Let $f = g + \varphi$ and $F = G + \Phi$ with $G = (g_1, g_2, \cdots, g_n) \in \mathcal{AP}(\mathbf{R})^n$, as above.

Since $F(x) \in \Omega$ when $x \in \mathbf{R}$, it follows from Lemma 2.2 that $g_i(x) \in \Omega_i$, $i = 1, 2, \cdots, n$, i.e., $G(x) \in \Omega$ for $x \in \mathbf{R}$. Now we have

$$f \circ (F \times \iota) = g \circ (G \times \iota) + f \circ (F \times \iota) - g \circ (G \times \iota).$$

By [6, Theorem 2.8], $g\circ(G\times\iota)\in\mathcal{AP}(\mathbb{R})$. To show that $f\circ(F\times\iota)\in\mathcal{PAP}(\mathbb{R})$, we need to show that the function $h=f\circ(F\times\iota)-g\circ(G\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$. First we show that $\varphi\circ(G\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$.

Since $G(\mathbb{R})$ is bounded in $\Omega\subset\mathbb{C}^n$ and $g\in\mathcal{AP}(\Omega\times\mathbb{R})$ is uniformly continuous, for $\epsilon>0$ one can find a finite number, say m, of open balls O_k with center $Z^{(k)}=(z_1^{(k)},z_2^{(k)},\cdots,z_n^{(k)})\in\Omega$, $k=1,2,\cdots,m$, and radius less than $\epsilon/3nL$ such that $G(\mathbb{R})\subset\bigcup_{k=1}^m O_k$ and

$$|g(Z,x)-g(Z^{(k)},x)|<\frac{\epsilon}{3}\qquad(Z\in O_k,\ x\in\mathbb{R}).\tag{3.1}$$

The set

$$B_k=\{x\in\mathbb{R}:\ G(x)\in O_k\}\tag{3.2}$$

is open and $\mathbb{R}=\bigcup_{k=1}^m B_k$. Let $E_k=B_k\setminus\bigcup_{j=1}^{k-1}B_j$, then $E_k\cap E_j=\phi$ when $k\neq j$, $1\leq k,j\leq m$.

Since each $\varphi(Z^{(k)},\cdot)\in\mathcal{PAP}_0(\mathbb{R})$, there is a number $t_0>0$ such that

$$\sum_{k=1}^m\frac{1}{2t}\int_{-t}^t|\varphi(Z^{(k)},x)|dx<\frac{\epsilon}{3}\qquad(t\geq t_0).\tag{3.3}$$

Since f satisfies the Lipschitz condition, and since

$$\varphi\circ(G\times\iota)=f\circ(G\times\iota)-g\circ(G\times\iota)$$

and

$$\varphi(Z^{(k)},\cdot)=f(Z^{(k)},\cdot)-g(Z^{(k)},\cdot),$$

it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned}
&\frac{1}{2t}\int_{-t}^{t}|\varphi(G(x),x)|dx \\
\leq&\frac{1}{2t}\sum_{k=1}^{m}\int_{E_k\cap[-t,t]}|\varphi(G(x),x)-\varphi(Z^{(k)},x)|+|\varphi(Z^{(k)},x)|dx \\
\leq&\frac{1}{2t}\sum_{k=1}^{m}\int_{E_k\cap[-t,t]}|f(G(x),x)-f(Z^{(k)},x)|+|g(G(x),x)-g(Z^{(k)},x)| \\
&\qquad+|\varphi(Z^{(k)},x)|dx \\
\leq&\frac{1}{2t}\sum_{k=1}^{m}\int_{E_k\cap[-t,t]}L\sum_{i=1}^{n}|g_i(x)-z_i^{(k)}|+|g(G(x),x)-g(Z^{(k)},x)|dx \\
&\qquad+\sum_{k=1}^{m}\frac{1}{2t}\int_{-t}^{t}|\varphi(Z^{(k)},x)|dx \\
\leq&\frac{1}{2t}\sum_{k=1}^{m}\int_{E_k\cap[-t,t]}nL|G(x)-Z^{(k)}|+|g(G(x),x)-g(Z^{(k)},x)|dx \\
&\qquad+\sum_{k=1}^{m}\frac{1}{2t}\int_{-t}^{t}|\varphi(Z^{(k)},x)|dx<\epsilon.
\end{aligned}$$

This shows that $\varphi\circ(G\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$. The Lipschitz hypothesis for $f\in\mathcal{PAP}(\Omega\times\mathbb{R})$ tells us that

$$\begin{aligned}
|h(x)|=&|f(F(x),x)-g(G(x),x)| \\
\leq&|f(F(x),x)-f(G(x),x)|+|\varphi(G(x),x)| \\
\leq&L\sum_{i=1}^{n}|\varphi_i(x)|+|\varphi(G(x),x)|.
\end{aligned}$$

Since each $\varphi_i\in\mathcal{PAP}_0(\mathbb{R})$ and $\varphi\circ(G\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$, it follows from the inequality above that $h\in\mathcal{PAP}_0(\mathbb{R})$. The proof is finished.

§4. Examples and Comments

In Section 2, we showed that the pseudo almost periodic functions enjoy many properties of the almost periodic functions and that $\mathcal{PAP}(\mathbb{R})$ is the largest among C^*-subalgebras $\mathcal{F}$ of $C(\mathbb{R})$ with the FP property.

In this section, by giving some examples and making comments, we point out some distinctions between $\mathcal{PAP}(\mathbb{R})$ and $\mathcal{AP}(\mathbb{R})$, and also between $\mathcal{PAP}(\mathbb{R})$ and $\mathcal{WAP}(\mathbb{R})$.

Recall that the evaluation map ϵ is the map from $\mathbb{R}$ to $\mathcal{F}^*$ defined by

$$\epsilon(s)f = f(s) \qquad (s \in \mathbb{R}, \quad f \in \mathcal{F}).$$

$\epsilon(\mathbb{R})$ is relatively w^*-compact since $\epsilon(\mathbb{R}) \subset B_1^*$, the unit ball of $\mathcal{F}^*$. Denote by $M(\mathcal{F})$ the w^* closure of $co\epsilon(\mathbb{R})$ in $\mathcal{F}^*$. So $M(\mathcal{F})$ is w^*-compact.

The first difference is that $\mathcal{PAP}(\mathbb{R})$ is not admissible, while $\mathcal{AP}(\mathbb{R})$ and many of its generalizations are (including $\mathcal{WAP}(\mathbb{R})$, see [2]).

Definition 4.1. *Let $\mathcal{F}$ be a translation invariant linear subspace of $\mathcal{B}(\mathbb{R})$, the space of bounded complex-valued functions on $\mathbb{R}$. For $\mu \in \mathcal{F}^*$, the introversion operator determined by μ is the mapping $T_\mu : \mathcal{F} \to \mathcal{B}(\mathbb{R})$ defined by*

$$(T_\mu f)(s) = \mu(R_s f) \qquad (f \in \mathcal{F}, \; s \in \mathbb{R}).$$

$\mathcal{F}$ is said to be introverted if $T_\mu \mathcal{F} \subset \mathcal{F}$ for all $\mu \in M(\mathcal{F})$.

We will say a subspace of $\mathcal{B}(\mathbb{R})$ is admissible if it is a norm closed, conjugate closed, translation invariant, introverted subspace of $\mathcal{B}(\mathbb{R})$ containing the constant functions.

The next example, which comes from [2, 2.2.6], shows that $\mathcal{PAP}(\mathbb{R})$ is not admissible.

Example 4.2. Let μ be a $\sigma(\mathcal{C}(\mathbb{R})^*, \mathcal{C}(\mathbb{R}))$–limit point of the sequence $\{\epsilon(n) : n \in \mathbb{N}\}$ and f be any member of $\mathcal{C}(\mathbb{R})$ such that $f(t) = f(-t)$, for $t \in (-\infty, 0)$, $f = 0$ on interval $I_n = [n + n^{-1}, n + 1 - (n+1)^{-1}]$ and $f(n) = 1$, $n = 2, 3, \cdots$.

It is easy to check that $f \in \mathcal{PAP}_0(\mathbb{R})$. Now we show that $T_\mu f \notin \mathcal{C}(\mathbb{R})$. For any $s \in (-1, 0) \cup (0, 1)$, $s + n \in I_n$ for all sufficiently large n, hence $\mu(R_s f) = 0$. On the other hand, $\mu(R_0 f) = \mu(f) = 1$. Therefore $T_\mu f$ is not continuous at $s = 0$.

Sometimes, one needs to consider functions that are not almost periodic or weakly almost periodic . Let us give an example. Let a function $f \in \mathcal{C}(\mathbb{R})$ be absolutely integrable on $\mathbb{R}$, i.e., such that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty. \tag{4.1}$$

Then, the Fourier transform of the function f is

$$F(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iux} dx.$$

It is obvious that a function $f \in \mathcal{C}(\mathbb{R})$ satisfying (4.1) is in $\mathcal{PAP}_0(\mathbb{R})$. However, it is easy to give examples of functions $f \in \mathcal{C}(\mathbb{R})$ satisfying (4.1) that are neither almost periodic nor weakly almost periodic .

The next example , which appears in Theorem 3.19 of [5], shows that there are functions f in $\mathcal{PAP}(\mathbb{R})$, which are uniformly continuous, even Lipschitz, but not weakly almost periodic .

Example 4.3. We first construct f on $\mathbb{N}$. Define two sequences thus:

$$n_i = 2^{2i+1}, \quad m_j = 2^{2j}, \quad i, j = 1, 2, \cdots . \tag{4.2}$$

We claim that the representation of an integer as $n_i + m_j$ is unique. Indeed suppose

$$n_i + m_j = n_{i'} + m_{j'},$$

where say, $i > i'$. So $j < j'$. Then

$$2^{2i+1} + 2^{2j} = 2^{2i'+1} + 2^{2j'}$$

$$2^{2i+1} - 2^{2i'+1} = 2^{2j'} - 2^{2j}.$$

Thus

$$2^{2i'+1}(2^{2i-2i'} - 1) = 2^j(2^{2j'-2j} - 1).$$

But now by unique factorization and the fact that $2^{2i-2i'} - 1$ and $2^{2j'-2j} - 1$ are odd integers we must have $2^{2i'+1} = 2^j$ whence $2i' + 1 = 2j$, an obvious contradiction.

With uniqueness in mind we can define f on $\mathbb{N}$

(4.3) $$f(n) = \begin{cases} \frac{1}{i+j} & \text{if } n = n_i + m_j \\ 0 & \text{otherwise.} \end{cases}$$

Let $N \in \mathbb{N}$. If $n_i + m_j \le N$, then $2^{2i+1} + 2^{2j} \le N$ and 2^{2i+1}, $2^{2j} \le N$ and $2i + 1$, $2j \le \log N / \log 2$ and $i, j \le \log N / 2\log 2 = \log N / \log 4$. Therefore if $z(N)$ is the number of integers of the form $n_i + m_j$ less than or equal to N, we shall clearly have $z(N) \le (\log N / \log 4)^2$. Hence since $0 \le f(n) < 1$ we have

(4.4) $$\frac{1}{N}\sum_{n=1}^{N} f(n) < \frac{1}{N} z(N) \le \frac{1}{N}\left(\frac{\log N}{\log 4}\right)^2 = \frac{4}{(\log 4)^2}\left(\frac{\log \sqrt{N}}{\sqrt{N}}\right)^2 \to 0.$$

Extend f to $\mathbb{R}^+$ by linearity between integers, then to $\mathbb{R}$ by $f(-t) = f(t)$. f is uniformly continuous, even Lipschitz, and

$$\int_n^{n+1} |f| \le \max\{f(n), f(n+1)\} \le f(n) + f(n+1)$$

$$\frac{1}{N}\int_0^N |f| \leq \frac{1}{N}\sum_{n=0}^{N-1}[f(n) + f(n+1) \leq \frac{2}{N}\sum_{n=1}^{N} f(n) \to 0,$$

by (4.4). Clearly, $T^{-1}\int_0^T |f| \to 0$. Therefore $f \in \mathcal{PAP}_0(\mathbb{R})$. However

$$\lim_{i\to\infty}\lim_{j\to\infty} f(n_i + m_j) = \lim_{i\to\infty}\lim_{j\to\infty}\frac{i}{i+j} = 0,$$

while

$$\lim_{j\to\infty}\lim_{i\to\infty} f(n_i + m_j) = \lim_{j\to\infty}\lim_{i\to\infty}\frac{i}{i+j} = 1.$$

By [2, 4.2.3], $f \notin \mathcal{WAP}(\mathbb{R})$.

Remarks 4.4.

(1) Following [9, VIII.4], we may regard $t \in \mathbb{R}$ as a time variable and so $\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r} R_t f(x)dt = \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r} R_x f(t)dt$ as a time average when the limit exists. For all $f \in \mathcal{PAP}(\mathbb{R})$, the limit does exist and equals $M(f)$, hence we call the C^*-algebra $\mathcal{PAP}(\mathbb{R})$ ergodic. The subalgebra $\mathcal{AP}(\mathbb{R})$ of $\mathcal{PAP}(\mathbb{R})$ is distinguished by the property that $M(|g|) > 0$ for $g \in \mathcal{AP}(\mathbb{R})$, $g \neq 0$, while $M(|\varphi|) = 0$ for all $\varphi \in \mathcal{PAP}_0(\mathbb{R})$. This is why we call a function $\varphi \in \mathcal{PAP}_0(\mathbb{R})$ an ergodic perturbation.

(2) A function f in $\mathcal{PAP}(\mathbb{R})$ is the sum of a function $g \in \mathcal{AP}(\mathbb{R})$ and a function $\varphi \in \mathcal{PAP}_0(\mathbb{R})$. It is φ that makes the function f not almost periodic (if $\varphi \neq 0$). This is the irregular aspect of f. However, f does show some regular aspects; for example, f has the same time average as its almost periodic component g. Furthermore, we showed in Lemma 2.2 that $\|f\| \geq \|g\|$.

(3) Though the concept of almost periodicity wasn't made precise until this century, the idea for it can be traced back a long time. For instance, as

was pointed out in [11] (for details, see [18],[22]), it was an idea of Ptolemy and Copernicus to show that the motion of a planet (or the moon, sun, etc.) is described by a function we now call almost periodic. The almost periodic functions give a mathematical description of many natural and social phenomena. By introducing the pseudo almost periodic functions, we want to widen the range of the description of such phenomena. We will give an example in the next chapter.

It follows from Lemma 2.2 and comments preceding it that

$$\mu(f) = g(0), \qquad f = g + \varphi \in \mathcal{PAP}(\mathbb{R}),$$

defines a mean $\mu \in \mathcal{PAP}(\mathbb{R})^*$. For the C^*-subalgebra $\mathcal{F} = \mathcal{WAP}(\mathbb{R}) \subset \mathcal{PAP}(\mathbb{R})$ (which was mentioned in Introduction and Notation, Sections 1 and 2), we can identify $\mu|_{\mathcal{F}}$ in a different way. To do this, we note first that $\mathcal{F}$ has the property called m-introversion which $\mathcal{PAP}(\mathbb{R})$ does not have (Example 4.2 can also be used to show this, see [2, 2.2.6]). With this property, the spectrum $\mathbb{R}^{\mathcal{F}}$ of $\mathcal{F}$ can be given a natural semigroup operation derived from addition in $\mathbb{R}$, and becomes a compact abelian semitopological semigroup. As such, it has a unique minimal idempotent e, that is, a member $e \in \mathbb{R}^{\mathcal{F}}$ such that $e^2 = e$ and if $ee_1 = e_1$ for any other idempotent $e_1 \in \mathbb{R}^{\mathcal{F}}$, then $e_1 = e$. Then $\mu|_{\mathcal{F}} = e$. See [2] for more details.

CHAPTER II

THE SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS

In this chapter, we apply the theory established in the previous chapter to investigate the pseudo almost periodic solutions of some differential equations. In Section 5, we consider ordinary differential equations. Section 6 is about certain parabolic equations. Finally, in Section 7 we investigate some kinds of boundary value problems. All the theorems here are generalizations of theorems in [6] about almost periodic functions.

§5. The Solutions of Ordinary Differential Equations

For column vector $G = (g_1, g_2, \cdots, g_n)' \in \mathcal{C}(\mathbf{R})^n$, define $\|G\| = \max\{\|g_i\|, 1 \leq i \leq n\}$.

In this section we will first consider systems of the form

$$\frac{dY}{dx} = AY + F, \tag{5.1}$$

where $A = (a_{ij})$ is a complex $n \times n$ matrix and $F = (f_1, f_2, \cdots, f_n)' \in \mathcal{PAP}(\mathbf{R})^n$.

A solution $Y = (y_1, y_2, \cdots, y_n)'$ of system (5.1) will be called pseudo almost periodic if $Y \in \mathcal{PAP}(\mathbf{R})^n$.

To find the pseudo almost periodic solutions of equations such as (5.1), we must of course look among the bounded solutions.

Theorem 5.1. *Let* $F = (f_1, f_2, \cdots, f_n)' \in \mathcal{PAP}(\mathbb{R})^n$. *If the matrix* $A = (a_{ij})$ *has no eigenvalues with real part zero, then the system (5.1) admits a unique solution* $Y = (y_1, y_2, \cdots, y_n)' \in \mathcal{PAP}(\mathbb{R})^n$. *Moreover*

$$\|Y\| \leq K\|F\|, \tag{5.2}$$

where $K > 0$ *depends only on the matrix* A.

Proof. By the same discussion as in [6, Theorem 4.2], we can assume that the matrix A is triangular, i.e

$$\begin{aligned} \frac{dy_1}{dx} &= \lambda_1 y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1 \\ \frac{dy_2}{dx} &= \lambda_2 y_2 + \cdots + a_{2n}y_n + f_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dx} &= \lambda_n y_n + f_n. \end{aligned} \tag{5.3}$$

Let us first consider the equation

$$\frac{dy}{dx} = \lambda y + f, \tag{5.4}$$

whose general solution is

$$y(x) = e^{\lambda x}[C + \int_0^x e^{-\lambda t} f(t)dt], \tag{5.5}$$

where C is an arbitrary constant. For $\lambda = u + iv \in \mathbf{C}$ with $u \neq 0$ and $f \in \mathcal{PAP}(\mathbb{R})$, let us show that any bounded solution of (5.4) is pseudo almost periodic.

If $u > 0$, $|e^{\lambda x}| = e^{ux} \to \infty$ as $x \to \infty$. For y to be bounded on $\mathbb{R}$, we must have $C + \int_0^x e^{-\lambda t} f(t)dt \to 0$ as $x \to \infty$. This means that we must take in (5.5)

$$C = -\int_0^\infty e^{-\lambda t} f(t)dt. \tag{5.6}$$

We note that the improper integral in (5.6) is convergent, since

$$|e^{-\lambda t} f(t)| \le \|f\| e^{-ut}, \qquad t \ge 0.$$

Thus, the unique bounded solution of equation (5.4) can only be

$$y_0(x) = -\int_x^\infty e^{\lambda(x-t)} f(t)dt, \tag{5.7}$$

and we do have

$$|y_0(x)| \le \|f\| e^{ux} \int_x^\infty e^{-ut} dt = \frac{\|f\|}{u}, \tag{5.8}$$

so that y_0 is bounded.

Now we show that $y_0 \in \mathcal{PAP}(\mathbb{R})$. Since $f \in \mathcal{PAP}(\mathbb{R})$, $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\mathbb{R})$. (5.7) becomes

$$\begin{aligned} y_0(x) = &-\int_x^\infty e^{\lambda(x-t)} g(t)dt \\ &-\int_x^\infty e^{\lambda(x-t)} \varphi(t)dt \\ =& \, I(x) + II(x). \end{aligned} \tag{5.9}$$

It is known that $I \in \mathcal{AP}(\mathbb{R})$ ([6, Theorem 4.2]). To show $y_0 \in \mathcal{PAP}(\mathbb{R})$, we need to show that $II \in \mathcal{PAP}_0(\mathbb{R})$, i.e., we need to show that $\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^r |II(x)|dx = 0$. In fact

$$\begin{aligned} &\frac{1}{2r}\int_{-r}^r |II(x)|dx \\ \le &\frac{1}{2r}\int_{-r}^r dx \int_x^\infty |\varphi(t)| e^{u(x-t)} dt \\ = &\frac{1}{2r}\int_{-r}^r |\varphi(t)|dt \int_{-r}^t e^{u(x-t)}dx + \frac{1}{2r}\int_r^\infty |\varphi(t)|dt \int_{-r}^r e^{u(x-t)}dx \\ = &I_1 + I_2. \end{aligned}$$

Now

$$I_1 = \frac{1}{2r}\int_{-r}^{r} |\varphi(t)|\frac{1}{u}[1 - e^{-u(t+r)}]dt.$$

$[1 - e^{-u(t+r)}]/u$ is bounded because $-r \leq t$ and $u > 0$. Since $\varphi \in \mathcal{PAP}_0(\mathbf{R})$, $I_1 \to 0$ as $r \to \infty$. Also

$$\begin{aligned} I_2 &= \frac{1}{2r}\int_{r}^{\infty} |\varphi(t)|\frac{1}{u}e^{-ut}[e^{ur} - e^{-ur}]dt \\ &\leq \frac{1}{2r}\frac{1}{u}\|\varphi\|[e^{ur} - e^{-ur}]\int_{r}^{\infty} e^{-ut}dt \\ &= \frac{1}{2r}\frac{1}{u^2}\|\varphi\|[1 - e^{-2ur}]. \end{aligned}$$

So $I_2 \to 0$ as $r \to \infty$. This completes the proof in the one–dimensional setting (5.4) when $u = Re\lambda > 0$.

When $u < 0$, we proceed analogously. We have that

$$y_0(x) = \int_{-\infty}^{x} e^{\lambda(x-t)} f(t)dt$$

is the unique bounded solution of (5.4) and $y_0 \in \mathcal{PAP}(\mathbf{R})$. At the same time we have the similar estimate

$$|y_0(x)| \leq \frac{\|f\|}{|u|}. \tag{5.8'}$$

Thus, the existence and uniqueness of the solutions for equations of the form (5.4) have been proved.

Now, let us return to the system (5.3). The last equation of (5.3) has the form of (5.4). Therefore, there exists a unique pseudo almost periodic solution for the last equation, namely

$$y_n(x) = -\int_{x}^{\infty} e^{\lambda_n(x-t)} f_n(t)dt \tag{5.10}$$

if $Re\lambda_n > 0$ and

$$y_n(x) = \int_{-\infty}^{x} e^{\lambda_n(x-t)} f_n(t)dt \tag{5.11}$$

if $Re\lambda_n < 0$. Substituting this y_n in the second last equation, we get for y_{n-1} an equation of the form (5.4). Since $Re\lambda_{n-1} \neq 0$, it follows that we have a unique solution of the second last equation. Applying successively the assertion proved above for (5.4), we get the first assertion of Theorem 5.1.

Let us prove inequality (5.2), which it suffices to do for a system of the form (5.3).

Let $0 < d = \min\{|Re\lambda_i| : 1 \leq i \leq n\}$, and put $K_n = 1/d$. From (5.8) or from (5.8'), it follows that

$$\|y_n\| \leq \frac{\|F\|}{d} = K_n\|F\|.$$

Let us consider the equation which gives us y_{n-1}:

$$\frac{dy_{n-1}}{dx} = \lambda_{n-1}y_{n-1} + a_{n-1,n}y_n + f_{n-1}.$$

Expressing y_{n-1} by a formula of the form (5.10) or (5.11) and estimating the integral again, we set $K_{n-1} = (|a_{n-1,n}|/d + 1)/d$ and obtain

$$\|y_{n-1}\| \leq \frac{1}{d}(|a_{n-1,n}|\frac{\|F\|}{d} + \|F\|) = K_{n-1}\|F\|.$$

Proceeding in the same manner, we get

$$\|y_i\| \leq K_i\|F\| \qquad (1 \leq i \leq n-1).$$

With $K = \max\{K_i : 1 \leq i \leq n\}$ we have

$$\|Y\| \leq K\|F\|.$$

We have finished the proof.

Remark 5.2. The equation $dy/dx = iy + e^{ix}$ has general solution $y(x) = (x + C)e^{ix}$, but no bounded solution. This shows that if the matrix A has an eigenvalue λ_k with $Re\lambda_k = 0$, then the system (5.1) may have no bounded solution.

Let $\Omega \subset \mathbf{C}^n$. Recall that $f \in \mathcal{PAP}(\Omega \times \mathbf{R})$ if $f = g + \varphi$, where $g \in \mathcal{AP}(\Omega \times \mathbf{R})$ and $\varphi \in \mathcal{PAP}_0(\Omega \times \mathbf{R})$ (Definition 3.2). For $H = (h_1, h_2, \cdots, h_n) \in \mathcal{C}(\mathbf{R})^n$ with $H(x) \in \Omega$ for all $x \in \mathbf{R}$, $H \times \iota : \mathbf{R} \to \Omega \times \mathbf{R}$ is defined by

$$H \times \iota(x) = (h_1(x), h_2(x), \cdots, h_n(x), x) \qquad (x \in \mathbf{R}).$$

Now, consider a system of the form

$$\frac{dY}{dx} = AY + F + \mu G \circ (Y \times \iota), \tag{5.12}$$

where $\mu \in \mathbf{C} \setminus \{0\}$, A is a complex $n \times n$ matrix, $F \in \mathcal{PAP}(\mathbf{R})^n$, and $G \in \mathcal{PAP}(\Omega \times \mathbf{R})^n$. Such a system is called quasi-linear [6].

We get the generating system of (5.12) by putting $\mu = 0$. We have the following theorem.

Theorem 5.3. *Let F and A be as in Theorem 5.1. Let $Y^{(0)}$ be the unique solution in $\mathcal{PAP}(\mathbf{R})^n$ of the generating system of (5.12), let $a_i > \|Y^{(0)}\|, i = 1, 2, \cdots, n$, and let $\Omega = \{Z \in \mathbf{C}^n : |z_i| \le a_i\}$. Assume that*

(1) *$G \in \mathcal{PAP}(\Omega \times \mathbf{R})^n$ such that*

$$\|G(Z', \cdot) - G(Z'', \cdot)\| \le L \sum_{i=1}^{n} |z_i' - z_i''|, \qquad (Z', Z'' \in \Omega), \tag{5.13}$$

where $L > 0$;

(2) *$0 < |\mu| \le \min\{1/nKL,\ (a_i - \|Y^{(0)}\|)/K\|G\|, 1 \le i \le n\}$, where $K > 0$ in (5.2) depends only the matrix A.*

Then there exists a unique solution $Y = (y_1, y_2, \cdots, y_n)' \in \mathcal{PAP}(\mathbf{R})^n$ of the system (5.12) such that $Y(x) \in \Omega$ for all $x \in \mathbf{R}$. Furthermore, $\|Y - Y^{(0)}\| \to 0$ as $\mu \to 0$.

Proof. We construct a sequence of approximations by induction, starting with $Y^{(0)}$, and taking as $Y^{(k)}$ the bounded solution of the system

$$\frac{dY^{(k)}}{dx} = AY^{(k)} + F + \mu G \circ (Y^{(k-1)} \times \iota). \tag{5.14}$$

First, we show that $Y^{(k)}$ exists, $Y^{(k)} \in \mathcal{PAP}(\mathbf{R})^n$, and $Y^{(k)}(\mathbf{R}) \subset \Omega$, $k = 0, 1, 2, \cdots$. The conclusion holds for $k = 0$ by the hypothesis. Let us assume that the conclusion holds for $k-1$. Then we show the conclusion for k. Since $G \circ (Y^{(k-1)} \times \iota) \in \mathcal{PAP}(\mathbf{R})^n$ (Theorem 3.6), there is a unique solution $Y^{(k)} \in \mathcal{PAP}(\mathbf{R})^n$ of (5.14) (Theorem 5.1). It follows from (5.1) and (5.14) that

$$\frac{d[Y^{(k)} - Y^{(0)}]}{dx} = A[Y^{(k)} - Y^{(0)}] + \mu G \circ (Y^{(k-1)} \times \iota).$$

According to (5.2), we have

$$\begin{aligned} \|Y^{(k)}\| &\le \|Y^{(k)} - Y^{(0)}\| + \|Y^{(0)}\| \\ &\le |\mu| K \|G\| + \|Y^{(0)}\|. \end{aligned}$$

Therefore, $Y^{(k)}(\mathbf{R}) \subset \Omega$ since $|\mu| \le \min\{1/nKL,\ (a_i - \|Y^{(0)}\|)/K\|G\|, 1 \le i \le n\}$.

Now, with a proof similar to that of [6, Theorem 4.4], it follows that the sequence of approximations is Cauchy and the limit of the sequence is the required solution of (5.12); the limit of the sequence is a o pseudo almost periodic (Theorem 2.3).

§6. The Solutions of Some Partial Differential Equations

In this section, we shall use some knowledge from ordinary differential equations to establish some properties of solutions of the nonlinear parabolic partial differential equation

(6.1) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + f(x,t,u)$$

for

$$(x,t) \in \Delta = \mathbb{R} \times [0,T],$$

where $T > 0$. We assume that the function $f \in C(\Delta_1)$, where

$$\Delta_1 = \Delta \times [-A, A]$$

with $A > 0$, and that f satisfies the Lipschitz condition

(6.2) $$|f(x,t,v') - f(x,t,v'')| \leq L|v' - v''| \qquad ((x,t) \in \Delta,\ v', v'' \in [-A,A]).$$

Fix $t_0 \in [0,T]$ and let $h > 0$. Replacing $\partial u/\partial t$ in (6.1) by a difference quotient, we have

(6.1') $$\frac{\partial^2 u}{\partial x^2} = h^{-1}[u(x,t) - u(x,t_0)] + f(x,t_0,u(x,t_0)) \quad (t \in [0,T],\ |t - t_0| < h).$$

(6.1') is clearly different from (6.1). However, we will show in the next lemma how to use (6.1') to approximate a solution of (6.1). This will enable us to give a pseudo almost periodicity criterion for solutions of (6.1) in Theorem 6.3, the main result of this section. It follows from (6.1') that

$$\frac{\partial^2 u}{\partial x^2} - h^{-1}u = -h^{-1}u(x,t_0) + f(x,t_0,u(x,t_0)).$$

This leads us to consider the following ordinary differential equation

(6.3) $$\frac{d^2 y}{dx^2} - \alpha^2 y = r,$$

where $\alpha > 0$ and $r \in \mathcal{C}(\mathbb{R})$. Note that (6.3) admits a unique bounded solution given by

$$y_0(x) = -\frac{1}{2\alpha}\left\{e^{\alpha x}\int_x^{\infty} e^{-\alpha t}r(t)dt + e^{-\alpha x}\int_{-\infty}^{x} e^{\alpha t}r(t)dt\right\}, \tag{6.4}$$

for which

$$\|y_0\| \leq \frac{1}{\alpha^2}\|r\|. \tag{6.5}$$

In much the same way as for y_0 in (5.7) we can show that y_0 in (6.4) is in $\mathcal{PAP}(\mathbb{R})$ if r is.

Let us now consider the system of ordinary differential equations

$$\frac{d^2u_k}{dx^2} = h^{-1}[u_k - u_{k-1}] + f(x, t_k, u_{k-1}), \quad k = 1, 2, \cdots, n, \tag{6.6}$$

where $h = T/n \in \mathbb{R}$, $t_k = kh$ and $u_0(x) = u(x, 0)$, where u is a solution of (6.1). We need to consider (6.6) for variable n; note that the partition $\{t_k\}_{k=0}^{n}$ of $[0, T]$ depends on n, as do all the functions in a solution $(u_1, u_2, \cdots, u_n)$ of (6.6).

Lemma 6.1. *Let $f \in \mathcal{PAP}(\Delta_1)$ satisfy the Lipschitz condition (6.2). Suppose u is a solution of equation (6.1) such that $\|u\| < A$, u and $\partial u/\partial t$ are uniformly continuous, and $u_0 = u(\cdot, 0) \in \mathcal{PAP}(\mathbb{R})$. Then there exist $n_0 \in \mathbf{N}$ and $M \geq 0$ such that (6.6) has a unique solution $(u_1, u_2, \cdots, u_n) \in \mathcal{PAP}(\mathbb{R})^n$ for $n \geq n_0$; it satisfies*

(1) *$\|u_k\| \leq A$, $1 \leq k \leq n$, and*

(2) *the functions $\varepsilon_0 = 0$, $\varepsilon_k = u(\cdot, t_k) - u_k$ are in $\mathcal{C}(\mathbb{R})$ with*

$$\|\varepsilon_k\| \leq M\omega(h), \tag{6.7}$$

where $h = T/n$, $t_k = kh$, and ω is the modulus of uniform continuity of u and $\partial u/\partial t$.

Proof. Set $M = (1+L)L^{-1}(e^{LT}-1)$ and let $n_0 \in \mathbf{N}$ be such that $M\omega(h) \le A - \|u\|$ when $n \ge n_0$. Fixed $n \ge n_0$, and note that $u_0 \in \mathcal{PAP}(\mathbb{R})$ and $\|u_0\| < A$ by hypothesis. So, suppose we have $u_{k-1} \in \mathcal{PAP}(\mathbb{R})$ with $\|u_{k-1}\| \le A$. Therefore the function: $x \to f(x, t_{k-1}, u_{k-1}(x))$ is in $\mathcal{PAP}(\mathbb{R})$ (Theorem 3.6) and by (6.3) and (6.4) there is a unique solution $u_k \in \mathcal{PAP}(\mathbb{R})$ of the kth equation of (6.6). We show (6.7) for k.

If we set $t = t_k$ in (6.1), and then subtract (6.6), we obtain

$$\frac{d^2\epsilon_k}{dx^2} - h^{-1}\epsilon_k = -h^{-1}\epsilon_{k-1} + r_k,$$

where

$$r_k(x) = \frac{\partial u(x,t_k)}{\partial t} - h^{-1}[u(x,t_k) - u(x,t_{k-1})]$$
$$+ f(x,t_k,u(x,t_k)) - f(x,t_k,u_{k-1}(x)).$$

Thus the equation for ϵ_k has the form of (6.3), so ϵ_k has corresponding form of (6.4) and

$$\|\epsilon_k\| \le \|\epsilon_{k-1}\| + h\|r_k\|$$

by (6.5). Considering the facts that f satisfies the Lipschitz condition and that u and $\partial u/\partial t$ are uniformly continuous, we have

$$\|\epsilon_k\| \le (1+hL)\|\epsilon_{k-1}\| + (1+L)hw(h).$$

Therefore

$$\|\epsilon_k\| \le w(h)(1+L)L^{-1}[(1+hL)^k - 1]$$
$$\le w(h)(1+L)L^{-1}[e^{LT} - 1]$$
$$= Mw(h).$$

So

$$\|u_k\| \leq \|u\| + M\omega(h) \leq A.$$

The proof is finished.

Recalling that for $g \in \mathcal{AP}(\mathbb{R})$ and $\epsilon > 0$, a number $\tau \in \mathbb{R}$ is called an ϵ–translation number of g if $\|R_\tau g - g\| < \epsilon$ (Definition 1.1), we state a lemma we need to prove this section's main theorem, which gives a pseudo almost periodicity criterion for solutions of (6.1).

Lemma 6.2. [6, Proof of Theorem 6.9]. *For every finite set in $\mathcal{AP}(\mathbb{R})$ and $\epsilon > 0$, there exists an $l_\epsilon > 0$ with the property that any interval of length l_ϵ contains a common ϵ-translation number for the finite set.*

Theorem 6.3. *Let f and u satisfy the conditions of Lemma 6.1. Then u is in $\mathcal{PAP}(\Delta)$.*

Proof. Let n_0 be as in Lemma 6.1, and fix $n \geq n_0$ and $t \in [0, T]$. Then there is a $k_0 \leq n$ such that $|t - t_{k_0}| < h$. Recall that $h = T/n$ and $t_k = kn$. If $B_n = \{u_k : k = 1, 2, \cdots, n\} \subset \mathcal{PAP}(\mathbb{R})$ is the solution of (6.6) given by Lemma 6.1, the uniform continuity of u gives

$$\begin{aligned} &|u(x,t) - u_{k_0}(x)| \\ &\leq |u(x,t) - u(x,t_{k_0})| + |u(x,t_{k_0}) - u_{k_0}(x)| \\ &< (1 + M)\omega(h) \qquad (x \in \mathbb{R}). \end{aligned} \tag{6.8}$$

It follows that the function $u(\cdot, t)$ is in the norm closure of $\bigcup_{n=n_0}^{\infty} B_n$; hence $u(\cdot, t) \in \mathcal{PAP}(\mathbb{R})$ (Theorem 2.3).

Let $g(\cdot,t)$ and $\varphi(\cdot,t)$, g_k and φ_k be the almost periodic components and the ergodic perturbations respectively of $u(\cdot,t)$, u_k, $k = 1,2,\cdots,n$.

To show that $u \in \mathcal{PAP}(\triangle)$, we need to show that $g \in \mathcal{AP}(\triangle)$ and $\varphi \in \mathcal{PAP}_0(\triangle)$.

Since $u(\cdot,t'), u(\cdot,t'') \in \mathcal{PAP}(\mathbb{R})$ for any $t', t'' \in [0,T]$, so is $u(\cdot,t') - u(\cdot,t'')$ and also $u(\cdot,t) - u_k$ for $t \in [0,T]$ and $k = 1,2,\cdots,n$. Lemma 2.2 tells us that

$$\|g(\cdot,t') - g(\cdot,t'')\| \le \|u(\cdot,t') - u(\cdot,t'')\| \tag{6.9}$$

and

$$\|g(\cdot,t) - g_k\| \le \|u(\cdot,t) - u_k\| \qquad k = 1,2,\cdots,n. \tag{6.10}$$

Let $\epsilon > 0$. To show that $g \in \mathcal{AP}(\triangle)$, we must show that there exists an $l_\epsilon > 0$ such that any interval on $\mathbb{R}$ of length l_ϵ contains a translation number τ,

$$\|R_\tau g(\cdot,t) - g(\cdot,t)\| < \epsilon \quad (t \in [0,T]). \tag{6.11}$$

Choose $n \ge n_0$ such that $h = T/n$ implies

$$4(M+1)\omega(h) < \epsilon. \tag{6.12}$$

For $t \in [0,T]$ there exists a t_{k_0} such that $|t - t_{k_0}| < h$. If τ is a common $(\epsilon/2)$-translation number of the functions g_k, $k = 1,2,\cdots,n$ (Lemma 6.2), then noting

(6.7), (6.9), (6.10) and (6.12), we have

$$\begin{aligned}
&\|R_\tau g(\cdot,t) - g(\cdot,t)\| \\
\le&\|R_\tau g(\cdot,t) - R_\tau g(\cdot,t_{k_0})\| \\
&\qquad + \|R_\tau g(\cdot,t_{k_0}) - R_\tau g_{k_0}\| + \|R_\tau g_{k_0} - g_{k_0}\| \\
&\qquad + \|g_{k_0} - g(\cdot,t_{k_0})\| + \|g(\cdot,t_{k_0}) - g(\cdot,t)\| \\
\le&\|u(\cdot,t) - u(\cdot,t_{k_0})\| \\
&\qquad + \|u(\cdot,t_{k_0}) - u_{k_0}\| + \|R_\tau g_{k_0} - g_{k_0}\| \\
&\qquad + \|u(\cdot,t_{k_0}) - u_{k_0}\| + \|u(\cdot,t_{k_0}) - u(\cdot,t)\| \\
\le&2(M+1)\omega(h) + \|R_\tau g_{k_0} - g_{k_0}\| < \epsilon,
\end{aligned}$$

which means that (6.11) holds. Therefore $g \in \mathcal{AP}(\Delta)$.

To show that $\varphi \in \mathcal{PAP}_0(\Delta)$, we need to show that for $\epsilon > 0$ there exists a number $r_0 > 0$ such that

$$I = \frac{1}{2r}\int_{-r}^{r} |\varphi(x,t)|dx < \epsilon \quad (t \in [0,T],\ r \ge r_0). \tag{6.13}$$

Since $\varphi_k \in \mathcal{PAP}_0(\mathbb{R})$, $k = 1,2,\cdots,n$, there exists an $r_0 > 0$ such that

$$\frac{1}{2r}\int_{-r}^{r} |\varphi_k(x)|dx < \epsilon/2 \qquad (k = 1,2,\cdots,n,\ r \ge r_0). \tag{6.14}$$

Now, if $|t - t_{k_0}| < h$,

$$\begin{aligned}
I &\le \frac{1}{2r}\int_{-r}^{r} |\varphi(x,t) - \varphi_{k_0}(x)|dx + \frac{1}{2r}\int_{-r}^{r} \varphi_{k_0}(x)|dx \\
&= I_1 + I_2,
\end{aligned}$$

and it follows from (6.14) that $I_2 < \epsilon/2$.

Using (6.8), (6.10) and (6.12), we have

$$\begin{aligned}|\varphi(x,t)-\varphi_{k_0}(x)| &= |(u(x,t)-g(x,t))-(u_{k_0}(x)-g_{k_0}(x))| \\ &\le |u(x,t)-u_{k_0}(x)|+|g(x,t)-g_{k_0}(x)| \\ &\le 2\|u(\cdot,t)-u_{k_0}\| \\ &< 2(M+1)\omega(h) < \epsilon/2.\end{aligned}$$

It follows that $I_1 < \epsilon/2$. The proof is complete.

Remark 6.4. In physics equation (6.1) is called the heat equation. The meaning of the preceding theorem is the following: If the temperature u in an infinite rod at an initial moment $t = 0$ is a pseudo almost periodic function of x and the heat sources inside the rod have pseudo almost periodic distribution f, then the temperature is given by a pseudo almost periodic function of (x, t).

§7. Pseudo Almost Periodic Solutions for Boundary Value Problems

In this section, we show the existence of pseudo almost periodic solutions for the Dirichlet problem in the half-plane $(0,\infty)\times\mathbb{R}$.

Theorem 7.1. *Let $f \in \mathcal{PAP}(\mathbb{R})$ and suppose that the ergodic perturbation φ of f satisfies $1/2t\int_{-t}^{t}|\varphi(s+y)|dy \to 0$ uniformly in $s \in \mathbb{R}$. Then there exists a harmonic function u, bounded and continuous in the half-plane $x > 0$, such that*

$$\lim_{x\to 0^+\ y\to y_0} u(x,y) = f(y_0) \qquad (y_0 \in \mathbb{R}) \tag{7.1}$$

and $u \in \mathcal{PAP}((0,\infty)\times\mathbb{R})$.

Proof. Following [23], we define u by

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\eta)\frac{x\,d\eta}{x^2+(y-\eta)^2} \tag{7.2}$$

for $x > 0$. Let us show that u has the required properties. First observe that the integral converges uniformly for (x, y) in any compact subset of $(0,\infty)\times\mathbf{R}$. This, and also the boundedness of u, follows immediately since f is bounded and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x\,d\eta}{x^2+(y-\eta)^2} = 1 \qquad ((x,y)\in(0,\infty)\times\mathbf{R}). \tag{7.3}$$

Since

$$u_0(x,y) = \frac{x}{x^2+(y-\eta)^2}$$

is harmonic on $(0,\infty)\times\mathbf{R})$, one establishes that u is also harmonic by differentiating under the integral sign.

Let us prove that the boundary condition (7.1) is satisfied. We fix $y_0 \in \mathbf{R}$, and use (7.3) to write

$$u(x,y) - f(y_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} [f(\eta)-f(y_0)]\frac{x\,d\eta}{x^2+(y-\eta)^2}.$$

Substituting $y-\eta = -tx$ we obtain

$$u(x,y) - f(y_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} [f(tx+y)-f(y_0)]\frac{dt}{1+t^2}. \tag{7.4}$$

Let $\epsilon > 0$, let $\eta_1 > 0$ be such that

$$\frac{2\|f\|}{\pi\eta_1} < \frac{\epsilon}{3},$$

and divide up (7.4) as follows:

$$u(x,y) - f(y_0) = \frac{1}{\pi}\left\{\int_{-\infty}^{-\eta_1} + \int_{-\eta_1}^{\eta_1} + \int_{\eta_1}^{\infty}\right\} = I_1 + I_2 + I_3. \tag{7.5}$$

Since f is continuous at y_0, we can find a $\delta > 0$ such that $|f(Y) - f(y_0)| < \epsilon/3$ if $|Y - y_0| < \delta$. Let $y \in (y_0 - \delta/2, y_0 + \delta/2)$ and $x \in (0, \delta/2\eta_1)$. Then, for $t \in [-\eta_1, \eta_1]$, $tx + y \in (y_0 - \delta, y_0 + \delta)$, this means that

$$|I_2| < \frac{\epsilon}{3}. \tag{7.6}$$

Also

$$|I_1| \leq \frac{2\|f\|}{\pi} \int_{-\infty}^{-\eta_1} \frac{dt}{1+t^2} < \frac{\epsilon}{3}, \tag{7.7}$$

and similarly

$$|I_3| < \frac{\epsilon}{3}. \tag{7.8}$$

This shows that

$$\lim_{x \to 0^+ \; y \to y_0} u(x,y) = f(y_0),$$

i.e., (7.1) holds.

Finally, let us show that $u \in \mathcal{PAP}((0,\infty) \times \mathbb{R})$.

Since $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\mathbb{R})$, it follows from (7.2) that

$$\begin{aligned} u(x,y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} g(\eta) \frac{x\,d\eta}{x^2 + (y-\eta)^2} + \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\eta) \frac{x\,d\eta}{x^2 + (y-\eta)^2} \\ &= G(x,y) + \Phi(x,y). \end{aligned}$$

If τ is an ϵ-translation number of the function g, then

$$|G(x, y+\tau) - G(x,y)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} |g(u+\tau) - g(u)| \frac{x\,du}{x^2 + (y-u)^2} < \epsilon,$$

so $G \in \mathcal{AP}((0,\infty) \times \mathbb{R})$.

Now we show that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\Phi(x,y)| dy = 0$$

uniformly with respect to $x > 0$.

Again, substituting $y - \eta = -tx$ in (7.2), we get

$$\Phi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(tx+y) \frac{dt}{1+t^2}.$$

Since φ is bounded,

$$\begin{aligned}
&\frac{1}{2r} \int_{-r}^{r} |\Phi(x,y)| dy \\
&\leq \frac{1}{2r} \int_{-r}^{r} dy \frac{1}{\pi} \int_{-\infty}^{\infty} |\varphi(tx+y)| \frac{dt}{1+t^2} \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} \frac{1}{2r} \int_{-r}^{r} |\varphi(tx+y)| dy,
\end{aligned}$$

and the assumption on φ gives the desired conclusion.

CHAPTER III

VECTOR–VALUED PSEUDO ALMOST PERIODIC FUNCTIONS

In this chapter, we develop the theory of vector–valued pseudo almost periodic functions. The chapter consists of four sections. We will investigate the pseudo almost periodic functions with values in a Banach space in the first three sections. In the last section, we will consider functions with values in a locally convex space. Our definition of a vector–valued pseudo almost periodic function looks quite different from Definition 2.1. The main result in Section 8 is a unique decomposition theorem, from which we will see that the definition here reduces to Definition 2.1 in the scalar–valued case. We investigate the properties of the space of the pseudo almost periodic functions with values in a Banach space in Section 9. Section 10 is about the integration of vector–valued pseudo almost periodic functions. Finally, in Section 11 we treat weak almost periodicity. Some of the results in Section 11 will be used in Section 15 of Chapter IV.

§8. A Unique Decomposition Theorem

Throughout this chapter, X denotes a Banach space and $\mathbf{J}_a$ stands for $[a,\infty)$ when $a \in \mathbb{R}$ and for $\mathbb{R}$ when $a = -\infty$; $C(\mathbf{J}_a, X)$ denotes the space of all bounded continuous functions from $\mathbf{J}_a$ to X. Also, m denotes Lebesgue measure on $\mathbb{R}$.

Definition 8.1. *A subset P of $\mathbf{J}_a$ is said to be relatively dense in $\mathbf{J}_a$ if there exists a number $l > 0$ such that*

$$[t, t+l] \cap P \neq \phi \qquad (t \in \mathbf{J}_a).$$

Definition 8.2. *A closed subset C of $\mathbf{J}_a$ is said to be an ergodic zero set in $\mathbf{J}_a$ if $m(C \cap [a,t])/(t-a) \to 0$ as $t \to \infty$ ($m(C \cap [-t,t])/2t \to 0$ as $t \to \infty$, when $a = -\infty$).*

Since $\lim_{t\to\infty} m(C \cap [a,t])/(t-a) = \lim_{t\to\infty} m(C \cap [a,t])/t$ for $a \in \mathbf{R}$, we will use the latter limit.

Proposition 8.3. *Let C be an ergodic zero set in $\mathbf{J}_a$. Then for any $\delta > 0$ and $L > 0$, there exists an interval $(u,v) \subset \mathbf{J}_a$ with the properties that $v - u > L$ and*

$$m(C \cap (u,v)) < \delta.$$

Proof. If such a (u,v) does not exist, one sees readily that $\liminf_{t\to\infty} m(C \cap [a,t])/t \geq \delta/2L$ ($\liminf_{t\to\infty} m(C \cap [-t,t])/2t \geq \delta/2L$ when $a = -\infty$).

Proposition 8.4. *Let P be relatively dense in $\mathbf{J}_a$ and let C be an ergodic zero set in $\mathbf{J}_a$. Then for any given interval $[c,d] \subset \mathbf{R}$ and $\delta > 0$, there exist $(u,v) \subset \mathbf{J}_a$ and $\tau \in P$ such that*

$$[c,d] + \tau \subset (u,v),$$

and

$$m(C \cap (u,v)) < \delta.$$

Proof. Let $l > 0$ be the number for P as in Definition 8.1 and let $L = l+(d-c)$. By Proposition 8.3, there exists an interval $(u, v) \subset \mathbf{J}_a$ such that $m(C \cap (u, v)) < \delta$ and $L < v-u$. Since we can assume that $u-c \in \mathbf{J}_a$, we can choose $\tau \in [u-c, u-c+l] \cap P$. If $t \in [c, d]$,

$$u < c + \tau \leq t + \tau \leq d + \tau \leq d + u - c + l < v,$$

that is, $[c, d] + \tau \subset (u, v)$.

Definition 8.5. *A function $f \in C(\mathbf{J}_a, X)$ is said to be pseudo almost periodic if for each $\epsilon > 0$, there are a number $\delta > 0$, a relatively dense subset $P(\epsilon)$ of $\mathbf{J}_a$, and an ergodic zero subset C_ϵ of $\mathbf{J}_a$ such that*

$$(8.1) \qquad \|f(t) - f(t + \tau)\| < \epsilon \qquad (\tau \in P(\epsilon),\ t, t + \tau \in \mathbf{J}_a \setminus C_\epsilon),$$

and

$$(8.2) \qquad \|f(t') - f(t'')\| < \epsilon \qquad (t', t'' \in \mathbf{J}_a \setminus C_\epsilon,\ |t' - t''| < \delta).$$

$\mathcal{PAP}(\mathbf{J}_a, X)$ will denote the set of all pseudo almost periodic functions from $\mathbf{J}_a$ to X and $\mathcal{PAP}_0(\mathbf{J}_a, X)$ is defined to be the set of all the functions $f \in C(\mathbf{J}_a, X)$ with the property that $1/t \int_a^t \|f(x)\| dx \to 0$ as $t \to \infty$ $(1/2t \int_{-t}^t \|f(x)\| dx \to 0$ as $t \to \infty$, when $a = -\infty)$.

Remarks 8.6. Under some restrictions on a and C_ϵ in Definition 8.5, the functions defined there reduce to familar ones which have been extensively investigated. For example,

(1) when $a = -\infty$, so $\mathbf{J}_a = \mathbf{R}$, and $C_\epsilon = \phi$, $\mathcal{PAP}(\mathbf{R}, X) = \mathcal{AP}(\mathbf{R}, X)$, the space of almost periodic functions [1, 3, 4, 6, 16].

(2) when $a = 0$ and $C_\epsilon = \phi$, $\mathcal{PAP}(\mathbf{J}_0, \mathbf{C}) = \mathcal{SAP}(\mathbf{J}_0)$, the space of strongly almost periodic functions [2, 7].

(3) when $a = 0$ and C_ϵ is bounded, $\mathcal{PAP}(\mathbf{J}_0, \mathbf{C}) = \mathcal{AP}(\mathbf{J}_0)$, the space of almost periodic functions [2, 7].

(4) when $a \in \mathbf{R}$ and C_ϵ is bounded, $\mathcal{PAP}(\mathbf{J}_a, X) = \mathcal{AAP}(\mathbf{J}_a, X)$, the space of asymptotically almost periodic functions [19, 20, 24].

In all the cases mentioned in Remarks 8.6, (8.2) in Definition 8.5 is a consequence of (8.1). However, we will show in Example 8.14 that (8.2) is independent of (8.1).

The proofs of the following two propositions are straightforward, we omit them.

Proposition 8.7. *A function $\varphi \in \mathcal{C}(\mathbf{J}_a, X)$ is in $\mathcal{PAP}_0(\mathbf{J}_a, X)$ if and only if, for $\epsilon > 0$, the set $C_\epsilon = \{t \in \mathbf{J}_a : \|\varphi(t)\| \geq \epsilon\}$ is an ergodic zero set in $\mathbf{J}_a$.*

Proposition 8.8. *Let C_i, $i = 1, 2, \cdots, n$, be ergodic zero sets. Then $C = \bigcup_{i=1}^n C_i$ is also an ergodic zero set in $\mathbf{J}_a$.*

Let $g \in \mathcal{C}(\mathbf{R}, X)$ and let $\epsilon > 0$. Set

$$P(\epsilon) = \{\tau \in \mathbf{R} : \|g(t) - g(t + \tau)\| < \epsilon \text{ for all } t \in \mathbf{R}\}.$$

Then, from Remark 8.6 (1), $g \in \mathcal{AP}(\mathbf{R}, X)$ if and only if $P(\epsilon)$ is relatively dense in $\mathbf{R}$.

If $g \in \mathcal{AP}(\mathbf{R}, X)$ and $\varphi \in \mathcal{PAP}_0(\mathbf{J}_a, X)$, set $f = g|_{\mathbf{J}_a} + \varphi$. Then $f \in \mathcal{PAP}(\mathbf{J}_a, X)$. For, the almost periodicity of g implies that there is a relatively dense subset $P(\epsilon/3) \subset \mathbf{R}$ such that

$$\|g(t) - g(t + \tau)\| < \frac{\epsilon}{3} \qquad (t \in \mathbf{R},\ \tau \in P(\epsilon/3)).$$

The uniform continuity of g [6, Theorem 6.2] implies that there is a number $\delta > 0$ such that

$$\|g(t') - g(t'')\| < \frac{\epsilon}{3} \qquad (t', t'' \in \mathbb{R},\ |t' - t''| < \delta).$$

Set

$$C_\epsilon = \left\{t \in \mathbb{J}_a : \|\varphi(t)\| \geq \frac{\epsilon}{3}\right\};$$

by Proposition 8.7, C_ϵ is an ergodic zero set of $\mathbf{J}_a$. Now it is easy to show that f satisfies Definition 8.5.

The next theorem shows the converse: any function $f \in \mathcal{PAP}(\mathbf{J}_a, X)$ has a unique decomposition like this. As in Chapter I, we will call g the almost periodic component and φ the ergodic perturbation respectively of f. Before stating the theorem, we need the following lemmas.

Lemma 8.9. *Let P be relatively dense in $\mathbf{J}_a$ and let C be an ergodic zero set in $\mathbf{J}_a$. For each $\tau \in P$, set $B_\tau = \{t \in \mathbb{R} : t + \tau \in C \cup (\mathbb{R} \setminus \mathbb{J}_a)\}$ ($B_\tau = \{t \in \mathbb{R} : t + \tau \in C\}$ when $a = -\infty$) and*

$$B = \bigcap_{\tau \in P} B_\tau. \tag{8.3}$$

Then $m(B) = 0$.

Proof. To show that $m(B) = 0$, it suffices to show that for any interval $[c, d] \subset \mathbb{R}$ and $\delta > 0$, $m([c, d] \cap B) < \delta$. Note that $t \in \mathbb{R} \setminus B$ if and only if there is a $\tau \in P$ such that $t + \tau \in \mathbf{J}_a \setminus C$. By Proposition 8.4, there [illegible] $u, v) \subset \mathbf{J}_a$ and $\tau \in P$ such that

$$[c, d] + \tau \subset (u, v),$$

and

$$m(C \cap (u,v)) < \delta.$$

This means that $m([c,d] \cap B) < \delta$.

Lemma 8.10. *Let P be relatively dense in $\mathbb{J}_a$, let C be an ergodic zero set in $\mathbb{J}_a$, and let $t_i \in \mathbb{R}$, $i = 1,2,\cdots,n$. Then for any $\delta > 0$, there exist a $\tau \in P$ and a $\Delta t \in [0,\delta)$ such that $t_i + \Delta t + \tau \in \mathbb{J}_a \setminus C$, $i = 1,2,\cdots,n$.*

Proof. Suppose that $t_1 \le t_2 \le \cdots \le t_n$. Consider the interval $[t_1, t_n + \delta]$. Proposition 8.4 shows that there exist an interval $(u,v) \subset \mathbb{J}_a$ and a $\tau \in P$ such that $[t_1, t_n + \delta] + \tau \subset (u,v)$ and $m((u,v) \cap C) < \delta/n$. Set

$$F_i = \{0 \le t < \delta : t_i + t + \tau \in C\},$$

and

$$F = \bigcup_{i=1}^{n} F_i.$$

Since $m(F_i) \le m((u,v) \cap C)$, $i = 1,2,\cdots,n$, $m(F) < \delta$. Therefore $[0,\delta) \setminus F \ne \phi$. We can choose $\Delta t \in [0,\delta) \setminus F$ as required.

We are now going to prove the main result of the section. Since the result for $\mathbb{R} = \mathbb{J}_{-\infty}$ is a simple corollary of that for $\mathbb{J}_a$, $a \in \mathbb{R}$ (see Remark 8.12 (3)), we will discuss only the latter.

Theorem 8.11. *A function $f \in C(\mathbb{J}_a, X)$ is pseudo almost periodic if and only if there is a unique function $g \in \mathcal{AP}(\mathbb{R}, X)$ such that $f - g|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$.*

Proof. We only need to show the only if part.

Choose a sequence of positive numbers $\{\epsilon_n\}$ decreasing to zero. Let δ_n, $P(\epsilon_n/7)$ and C_n be for ϵ_n as in Definition 8.5. For $P(\epsilon_n/7)$ and C_n, we have $B_n \subset \mathbb{R}$ from (8.3) of Lemma 8.9 with $m(B_n) = 0$. Without loss of generality, we may assume that $C_n \subset C_{n+1}$ for all $n \in \mathbb{N}$ since we can replace C_{n+1} by $C_n \cup C_{n+1}$, which still satisfies Definitions 8.2 and 8.5. Set $Q(\epsilon_n) = P(\epsilon_n/7) \cup P'(\epsilon_n/7)$, where $P'(\epsilon_n/7) = \{\tau : -\tau \in P(\epsilon_n/7)\}$. $Q(\epsilon_n)$ is relatively dense in $\mathbb{R}$.

In the proof of Lemma 8.9, we pointed out that for each $t \in \mathbb{R} \setminus B_n$, we can choose a $\tau_{n,t} \in P(\epsilon_n/7)$ such that $t + \tau_{n,t} \in \mathbb{J}_a \setminus C_n$. Define a function f_n on $\mathbb{R} \setminus B_n$ by

$$f_n(t) = f(t + \tau_{n,t}). \tag{8.4}$$

f_n is well-defined on $\mathbb{R} \setminus B_n$.

Set

$$B = \bigcup_{n=1}^{\infty} B_n.$$

Since $m(B_n) = 0$, $n = 1, 2, \cdots$, $m(B) = 0$.

We will show that the sequence $\{f_n\}$ converges uniformly to a function $g \in \mathcal{AP}(\mathbb{R}, X)$ on $\mathbb{R} \setminus B$. First of all, we show that each f_n satisfies

$$\|f_n(t) - f_n(t + \tau)\| < \epsilon_n, \qquad (\tau \in Q(\epsilon_n),\ t, t + \tau \in \mathbb{R} \setminus B_n) \tag{8.5}$$

and

$$\|f_n(t') - f_n(t'')\| < \epsilon_n, \qquad (t', t'' \in \mathbb{R} \setminus B_n,\ |t' - t''| < \delta_n). \tag{8.6}$$

We show (8.6) first. According to (8.4),

$$\|f_n(t') - f_n(t'')\| = \|f(t' + \tau_{n,t'}) - f(t'' + \tau_{n,t''})\|, \tag{8.7}$$

where $t' + \tau_{n,t'}$, $t'' + \tau_{n,t''} \in \mathbb{J}_a \setminus C_n$. Lemma 8.10, along with the facts that C_n is closed and f is continuous at $t' + \tau_{n,t'}$, $t'' + \tau_{n,t''} \in \mathbb{J}_a \setminus C_n$, implies that there are a $\tau \in P(\epsilon_n/7)$ and $\Delta t \in [0, \delta_n)$ such that

$$t' + \tau_{n,t'} + \Delta t + \tau,\ t'' + \tau_{n,t'} + \Delta t + \tau,$$
$$t'' + \Delta t + \tau,\ t'' + \tau_{n,t''} + \Delta t + \tau,$$
$$t' + \tau_{n,t'} + \Delta t,\ t'' + \tau_{n,t''} + \Delta t \in \mathbb{J}_a \setminus C_\epsilon$$

and

$$\begin{aligned} &\|f(t' + \tau_{n,t'}) - f(t' + \tau_{n,t'} + \Delta t)\| < \epsilon_n/7, \\ &\|f(t'' + \tau_{n,t''}) - f(t'' + \tau_{n,t''} + \Delta t)\| < \epsilon_n/7. \end{aligned} \tag{8.8}$$

It follows from (8.1), (8.2) and (8.8) that

$$\begin{aligned} &\|f(t' + \tau_{n,t'}) - f(t'' + \tau_{n,t''})\| \\ \leq &\|f(t' + \tau_{n,t'}) - f(t' + \tau_{n,t'} + \Delta t)\| + \\ &\|f(t' + \tau_{n,t'} + \Delta t) - f(t' + \tau_{n,t'} + \Delta t + \tau)\| + \\ &\|f(t' + \tau_{n,t'} + \Delta t + \tau) - f(t'' + \tau_{n,t'} + \Delta t + \tau)\| + \\ &\|f(t'' + \tau_{n,t'} + \Delta t + \tau) - f(t'' + \Delta t + \tau)\| + \\ &\|f(t'' + \Delta t + \tau) - f(t'' + \tau_{n,t''} + \Delta t + \tau)\| + \\ &\|f(t'' + \tau_{n,t''} + \Delta t + \tau) - f(t'' + \tau_{n,t''} + \Delta t)\| + \\ &\|f(t'' + \tau_{n,t''} + \Delta t) - f(t'' + \tau_{n,t''})\| \\ < &\epsilon_n. \end{aligned} \tag{8.9}$$

Similarly, we can show (8.5) in the case that $\tau \in P(\epsilon_n/7)$ and t, $t + \tau \in \mathbb{R} \setminus B_n$.

If $\tau \in P'(\epsilon_n/7)$ and $t, t+\tau \in \mathbb{R} \setminus B_n$, set $T = t+\tau$ and $\tau' = -\tau$. Then $\tau' \in P(\epsilon_n/7)$ and $t = T + \tau'$. Therefore

$$\|f_n(t) - f_n(t+\tau)\| = \|f_n(T) - f_n(T+\tau')\| < \epsilon_n.$$

Now we show that the sequence $\{f_n\}$ converges uniformly on $\mathbb{R} \setminus B$. In fact, for $t \in \mathbb{R}\setminus B$, by (8.4) $f_m(t) = f(t+\tau_{m,t})$ and $f_n(t) = f(t+\tau_{n,t})$, where $t+\tau_{m,t} \in \mathbf{J}_a \setminus C_m$ and $t+\tau_{n,t} \in \mathbf{J}_a \setminus C_n$. Say, $m > n$, so $C_m \supset C_n$ and $\mathbf{J}_a \setminus C_m \subset \mathbf{J}_a \setminus C_n$. Note that $\epsilon_n > \epsilon_m$. In (8.9), replace t', t'' by t, $\tau_{n,t'}$ and $\tau_{n,t''}$ by $\tau_{m,t}$ and $\tau_{n,t}$ respectively, and ϵ_n by ϵ_m, and choose $\tau \in P(\epsilon_m/7)$; we get

$$\begin{aligned} \|f_m(t) - f_n(t)\| &= \|f(t+\tau_{m,t}) - f(t+\tau_{n,t})\| \\ &< \frac{4\epsilon_m}{7} + \frac{3\epsilon_n}{7} < \epsilon_n. \end{aligned} \tag{8.10}$$

Thus there is a function g on $\mathbb{R} \setminus B$ such that $f_n \to g$ uniformly on $\mathbb{R} \setminus B$ as $n \to \infty$. For $\epsilon > 0$, we choose j_0 such that $\epsilon_{j_0} < \epsilon/5$ and

$$\|g(t) - f_{j_0}(t)\| < \frac{\epsilon}{5} \qquad (t \in \mathbb{R} \setminus B). \tag{8.11}$$

Now we show three assertions.

(i) If a sequence $\{t_n\} \subset \mathbb{R} \setminus B$ is Cauchy, so is $\{g(t_n)\}$. For, by (8.6) and (8.11)

$$\begin{aligned} \|g(t_n) - g(t_m)\| \le \|g(t_n) - f_{j_0}(t_n)\| + \|f_{j_0}(t_n) - f_{j_0}(t_m)\| + \\ \|f_{j_0}(t_m) - g(t_m)\| < \epsilon. \end{aligned}$$

This implies that g is continuous on $\mathbb{R} \setminus B$ and extends uniquely to $\mathbb{R}$ by continuity.

(ii) $g \in \mathcal{AP}(\mathbb{R}, X)$. By (8.5) and (8.11) one can similarly show that, for all $t \in \mathbb{R}$

and $\tau \in Q(\epsilon_{j_0})$,

$$
\begin{aligned}
&\|g(t) - g(t+\tau)\| \\
\le&\|g(t) - g(t+\Delta t)\| + \|g(t+\Delta t) - f_{j_0}(t+\Delta t)\| \\
&+ \|f_{j_0}(t+\Delta t) - f_{j_0}(t+\Delta t+\tau)\| + \|f_{j_0}(t+\Delta t+\tau) - g(t+\Delta t+\tau)\| \\
&+ \|g(t+\Delta t+\tau) - g(t+\tau)\| < \epsilon,
\end{aligned}
$$

where, as before, a small number $\Delta t > 0$ is chosen such that $t+\Delta t, t+\Delta t+\tau \in \mathbb{R}\backslash B$, $\|g(t) - g(t+\Delta t)\| < \epsilon/5$, and $\|g(t+\tau) - g(t+\Delta t+\tau)\| < \epsilon/5$. Since $Q(\epsilon_{j_0})$ is relatively dense in $\mathbb{R}$, $g \in \mathcal{AP}(\mathbb{R}, X)$.

(iii) $f - g|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. In fact, if $x \in \mathbb{J}_a \setminus (C_{j_0} \cup B)$, then by (8.1), (8.4) and (8.11)

$$
\begin{aligned}
\|f(x) - g(x)\| &\le \|f(x) - f_{j_0}(x)\| + \|f_{j_0}(x) - g(x)\| \\
&= \|f(x) - f(x+\tau_{j_0,x})\| + \|f_{j_0}(x) - g(x)\| \\
&< \epsilon.
\end{aligned}
$$

Set $M_0 = \sup_{s\in\mathbb{J}_a} \|f(s) - g(s)\|$, it follows from the inequality above that when t is sufficiently large

$$
\begin{aligned}
\frac{1}{t}\int_a^t \|f(x) - g(x)\|dx &\le \frac{1}{t}\Big\{(t-a)\epsilon + \int_{C_{j_0}\cup B\cap[a,t]} \|f(x) - g(x)\|dx\Big\} \\
&\le \frac{1}{t}\Big\{(t-a)\epsilon + M_0 m(C_{j_0} \cup B \cap [a,t])\Big\} < 2\epsilon,
\end{aligned}
$$

because $m(C_{j_0} \cap [a,t])/t \to 0$ as $t \to \infty$ and $m(B) = 0$.

Finally, the decomposition is unique. Note that, for $g \in \mathcal{AP}(\mathbb{R}, X)$, $g|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X) \Leftrightarrow \big\|g|_{\mathbb{J}_a}(\cdot)\big\| \in \mathcal{PAP}_0(\mathbb{J}_a, \mathbb{C}) \Leftrightarrow \|g(\cdot)\| \in \mathcal{PAP}_0(\mathbb{R}) \Leftrightarrow g = 0$, where

$\|g(\cdot)\|$ is the function $t \in \mathbb{R} \to \|g(t)\|$. Therefore if there are two functions g_1, $g_2 \in \mathcal{AP}(\mathbb{R}, X)$ such that $f - g_i|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$, $i = 1, 2$, then $g_1|_{\mathbb{J}_a} - g_2|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. So $g_1 = g_2$.

The proof is complete.

As a consequence of Theorem 8.11, we have

$$\mathcal{PAP}(\mathbb{J}_a, X) = \mathcal{AP}(\mathbb{R}, X) \oplus \mathcal{PAP}_0(\mathbb{J}_a, X).$$

In case $X = \mathbb{C}$, we will omit X from our notation and write, for example, $\mathcal{PAP}(\mathbb{J}_a)$ for $\mathcal{PAP}(\mathbb{J}_a, X)$.

Remarks 8.12. (1) and (2) are known decomposition theorems; we have them as corollaries of Theorem 8.11.

(1) For a function $f \in \mathcal{AP}(\mathbb{J}_0)$ (as in Remark 8.6 (3)), it is known that $f = g|_{\mathbb{J}_0} + \varphi$, where $g \in \mathcal{AP}(\mathbb{R})$ and $\varphi : \mathbb{J}_a \to \mathbb{C}$ is continuous and has limit of zero when $t \to \infty$; see, for example, [2, 4.3.14].

(2) For a function $f \in \mathcal{AAP}(\mathbb{J}_a, X)$ (as in Remark 8.6 (4)), it is shown in [19, Theorem 3.4] and [24] that $f = g|_{\mathbb{J}_a} + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi : \mathbb{J}_a \to X$ is continuous and vanishes at ∞.

(3) When the functions of (1) and (2) in Remarks 8.6 are scalar-valued, there is no essential difference between them because by Theorem 8.11 each function of type (2) has a unique extension a function of type (1).

Remark 8.13. Let $\mathcal{WRC}(\mathbb{J}_0, X)$ be the space of vector-valued weakly almost periodic functions with totally bounded ranges (for the definition of vector-valued weakly almost periodic functions see Section 15). It follows from [12, Theorem 4.17] and [17, Theorem 7] that f is in $\mathcal{WRC}(\mathbb{J}_0, X)$ if and only if $f = g|_{\mathbb{J}_0} + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{WRC}_0(\mathbb{J}_0, X)$, the space of 'flight vectors', those members of $\mathcal{WRC}(\mathbb{J}_0, X)$ that have 0 in the weak closure of the set of translates. With a proof similar to that of Corollary 4.19 in [12], one can show that $\mathcal{WRC}_0(\mathbb{J}_0, X) \subset \mathcal{PAP}_0(\mathbb{J}_0, X)$. Thus, $\mathcal{WRC}(\mathbb{J}_0, X) \subset \mathcal{PAP}(\mathbb{J}_0, X)$.

Now we give an example to show that (8.2) is independent of (8.1) in Definition 8.5.

Example 8.14. For $n \geq 4$, define a function f on $[n, n+1)$ as follows:

$$f(t) = \begin{cases} -\frac{1/2+1/n}{1/n}(t-n)+1 & t \in [n, n+\frac{1}{n}) \\ -(t-n)+1/2 & t \in [n+\frac{1}{n}, n+\frac{1}{2}) \\ 0 & t \in [n+\frac{1}{2}, n+1-\frac{1}{n+1}) \\ (n+1)[t-(n+1-\frac{1}{n+1})] & t \in [n+1-\frac{1}{n+1}, n+1). \end{cases}$$

The graph of the function f in each interval $[n, n+1)$ consists of four segments, and $f : [4, \infty) \to [0, 1]$ is continuous. For each $\epsilon > 0$, set $P(\epsilon) = \{n : n = 4, 5, \ldots, \}$ and $C_\epsilon = [4, 4+1/4] \cup \{\bigcup_{n=5}^{\infty}[n-1/n, n+1/n]\}$; then the function f satisfies all the conditions in Definition 8.5 except (8.2) since

$$(8.12) \qquad \lim_{n\to\infty} f(n+1/n) = \frac{1}{2}, \quad \text{while} \quad f(n-1/n) = 0, \quad n = 5, 6, \cdots.$$

(8.12) also shows that the function f can not have a decomposition as in Theorem 8.11.

§9. The Space of Vector–Valued Pseudo Almost Periodic Functions

For each $f \in \mathcal{PAP}(\mathbb{J}_a, X)$, define $\|f\| = \sup_{t\in\mathbb{J}_a} \|f(t)\|$. $\mathcal{PAP}(\mathbb{J}_a, X)$ is a normed subspace of $\mathcal{C}(\mathbb{J}_a, X)$.

Theorem 9.1. *$\mathcal{PAP}(\mathbf{J}_a, X)$ is a Banach space.*

Proof. We only need to show that $\mathcal{PAP}(\mathbf{J}_a, X)$ is closed in $C(\mathbf{J}_a, X)$. We will use Definition 8.5 to show this. Let f be in the closure of $\mathcal{PAP}(\mathbf{J}_a, X)$.

Let $\epsilon > 0$. There is a function $f_0 \in \mathcal{PAP}(\mathbf{J}_a, X)$ such that

$$\|f_0 - f\| < \epsilon/3. \tag{9.1}$$

Since $f_0 \in \mathcal{PAP}(\mathbf{J}_a, X)$, we can find a number δ, a relatively dense subset $P(\epsilon)$ of $\mathbf{J}_a$ and an ergodic zero subset C_ϵ of $\mathbf{J}_a$ such that

$$\|f_0(t) - f_0(t+\tau)\| < \epsilon/3 \qquad (\tau \in P(\epsilon),\ t, t+\tau \in \mathbf{J}_a \setminus C_\epsilon), \tag{9.2}$$

and

$$\|f_0(t') - f_0(t'')\| < \epsilon/3 \qquad (t', t'' \in \mathbf{J}_a \setminus C_\epsilon,\ |t' - t''| < \delta). \tag{9.3}$$

Now $f \in \mathcal{PAP}(\mathbf{J}_a, X)$ because of (9.1), (9.2), and (9.3). The proof is finished.

As for the numerical case, each $f \in \mathcal{PAP}(\mathbf{J}_a, X)$ has a Fourier series, i.e.

$$f(x) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k x},$$

where $\lambda_k \in \mathbb{R}$, and $A_k = M(fe^{-\lambda_k \cdot}) \in X$, $A_k \neq 0$ for all $k \in \mathbf{N}$, and $M(fe^{-\lambda \cdot}) = 0$ if $\lambda \notin \{\lambda_k : k \in \mathbf{N}\}$; it is the Fourier series of the almost periodic component of f (see [6] for details).

Theorem 9.2. *Let $f_i \in \mathcal{PAP}(\mathbf{J}_a, X)$, $i = 1, 2, \cdots, n$. For each $\epsilon > 0$, there are a $\delta > 0$, a relatively dense subset $P(\epsilon)$ of $\mathbf{J}_a$, and an ergodic zero subset C_ϵ of $\mathbf{J}_a$ such that for $i = 1, 2, \cdots, n$,*

$$\|f_i(t) - f_i(t+\tau)\| < \epsilon \qquad (\tau \in P(\epsilon),\ t, t+\tau \in \mathbf{J}_a \setminus C_\epsilon), \tag{9.4}$$

and

$$\text{(9.5)} \qquad \|f_i(t') - f_i(t'')\| < \epsilon \qquad (t', t'' \in \mathbf{J}_a \setminus C_\epsilon,\ |t' - t''| < \delta).$$

Proof. We know from Theorem 8.11 that $f_i = g_i|_{\mathbf{J}_a} + \varphi_i$, where $g_i \in \mathcal{AP}(\mathbf{R}, X)$ and $\varphi_i \in \mathcal{PAP}_0(\mathbf{J}_a, X)$, $i = 1, 2, \cdots, n$. Therefore there exists a relatively dense subset $P(\epsilon)$ of $\mathbf{J}_a$ from [6, Proof of Theorem 6.9] such that for $i = 1, 2, \cdots, n$

$$\|g_i(t) - g_i(t + \tau)\| < \epsilon/3 \qquad (t \in \mathbf{R},\ \tau \in P(\epsilon)).$$

Since an almost periodic function is uniformly continuous on $\mathbf{R}$ [6, Theorem 6.2], there exists a $\delta > 0$ such that

$$\|g_i(t') - g_i(t'')\| < \epsilon/3 \qquad (i = 1, 2, \cdots, n,\ |t' - t''| < \delta).$$

Set $C_i = \{t \in \mathbf{J}_a : \|\varphi_i(t)\| \geq \epsilon/3\}$, $i = 1, 2, \cdots, n$ and

$$C_\epsilon = \bigcup_{i=1}^{n} C_i.$$

By Propositions 8.7 and 8.8, C_ϵ is an ergodic zero set in $\mathbf{J}_a$. The proof is finished.

Let $f = g|_{\mathbf{J}_a} + \varphi \in \mathcal{PAP}(\mathbf{J}_a, X)$. The function $t \to \|f(t)\|$ is in $\mathcal{PAP}(\mathbf{J}_a)$. For, $\|f(t)\| = \|g(t)\| + \|\|f(t)\| - \|g(t)\|$ for $t \in \mathbf{J}_a$. One can check that the function $t \to \|g(t)\|$ is in $\mathcal{AP}(\mathbf{R})$ and the function $t \to \|\varphi(t)\|$ is in $\mathcal{PAP}_0(\mathbf{J}_a)$. The function $t \to \|f(t)\| - \|g(t)\|$ is in $\mathcal{PAP}_0(\mathbf{J}_a)$ because $\big|\|f(t)\| - \|g(t)\|\big| \leq \|\varphi(t)\|$ for $t \in \mathbf{J}_a$. So, the following theorem is a consequence of Lemma 2.2.

Theorem 9.3. *Let $f \in \mathcal{PAP}(\mathbb{J}_a, X)$ and let g be its almost periodic component. Then $\|f\| \geq \|g\| \geq \inf_{t\in\mathbb{J}_a} \|g(t)\| \geq \inf_{t\in\mathbb{J}_a} \|f(t)\|$.*

Let $\Omega' \subset \mathbb{C}^n$. In Definition 3.1, we defined the space $\mathcal{AP}(\Omega' \times \mathbb{R})$, and in Definition 3.2 we defined the spaces $\mathcal{PAP}(\Omega' \times \mathbb{R})$ and $\mathcal{PAP}_0(\Omega' \times \mathbb{R})$. Replacing $\mathbb{R}$ by $\mathbb{J}_a$ in Definition 3.2, we have the spaces $\mathcal{PAP}(\Omega' \times \mathbb{J}_a)$ and $\mathcal{PAP}_0(\Omega' \times \mathbb{J}_a)$; thus $f \in \mathcal{PAP}(\Omega' \times \mathbb{J}_a)$ if and only if $f = g|_{\mathbb{J}_a} + \varphi$, where $g \in \mathcal{AP}(\Omega' \times \mathbb{R})$ and $\varphi \in \mathcal{PAP}_0(\Omega' \times \mathbb{J}_a)$.

In [6], a definition of functions almost periodic in the mean is given. We now give a definition of functions pseudo almost periodic in p mean; these functions can be considered as pseudo almost periodic functions with values in a Banach space.

Definition 9.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $p \geq 1$. Let $f : \Omega \times \mathbb{J}_a \to \mathbb{C}$ be such that for each $t \in \mathbb{J}_a$, the function $f_t : x \to f(x,t)$ is measurable on Ω. Then f is called pseudo almost periodic in p mean if to any $\epsilon > 0$ corresponds an $h \in \mathcal{PAP}(\Omega \times \mathbb{J}_a)$ such that*

$$\int_\Omega |f(x,t) - h(x,t)|^p dx < \epsilon \qquad (t \in \mathbb{J}_a).$$

If $f : \Omega \times \mathbb{J}_a \to \mathbb{C}$ is pseudo almost periodic in p mean, then $f_t \in L_p(\Omega)$ for all $t \in \mathbb{J}_a$. Consider the map $H : t \to f_t$, $\mathbb{J}_a \to L_p(\Omega)$. One can check that $H(h) \in \mathcal{PAP}(\mathbb{J}_a, L_p(\Omega))$ for all $h \in \mathcal{PAP}(\Omega \times \mathbb{J}_a)$. Thus, $H(f) \in \overline{H(\mathcal{PAP}(\Omega \times \mathbb{J}_a))} \subset \mathcal{PAP}(\mathbb{J}_a, L_p(\Omega))$. So, $H(f) \in \mathcal{PAP}(\mathbb{J}_a, L_p(\Omega))$. Therefore, functions pseudo almost periodic in p mean may be viewed as examples of pseudo almost periodic functions with values in a Banach space.

Summarizing, we have

Theorem 9.5. *The space of pseudo almost periodic functions in p mean on $\Omega \times \mathbf{J}_a$ is a closed subspace of $\mathcal{PAP}(\mathbf{J}_a, L_p(\Omega))$.*

A function f on $\Omega \times \mathbf{J}_a$ is said to be uniformly continuous in p mean if for $\epsilon > 0$, there is $\delta > 0$ such that

$$\int_\Omega |f(x, t + \Delta t) - f(x, t)|^p dx < \epsilon \qquad (t \in \mathbf{J}_a,\ 0 \le \Delta t \le \delta).$$

The proof of the following theorem is similar to the proof of Corollary 2.6; we omit it.

Theorem 9.6. *If $f : \Omega \times \mathbf{J}_a \to \mathbb{C}$ is pseudo almost periodic in p mean and $\partial f / \partial t$ is uniformly continuous in p mean, then $\partial f / \partial t$ is pseudo almost periodic in p mean.*

§10. Integration of Vector–Valued Pseudo Almost Periodic Functions

For an $f \in \mathcal{AP}(\mathbf{R})$, the classical Bohl-Bohr theorem (cf, [6]) asserts that the indefinite integral $F : x \to \int_0^x f(u)du$ is in $\mathcal{AP}(\mathbf{R})$ whenever F is bounded. The conclusion is no longer valid if $f \in \mathcal{AP}(\mathbf{R}, X)$ (for a counterexample, see [6]). In this section, we deal with the indefinite integral of a function $f \in \mathcal{PAP}(\mathbf{J}_a, X)$.

Theorem 10.1. *Let $a \in \mathbf{R}$ and let $f \in \mathcal{PAP}_0(\mathbf{J}_a, X)$. Define $F : \mathbf{J}_a \to X$ by $F(t) = \int_a^t f(u)du$.. Then $F \in \mathcal{PAP}(\mathbf{J}_a, X)$ if and only if there is a vector $A \in X$ such that $F - A \in \mathcal{PAP}_0(\mathbf{J}_a, X)$.*

Proof. The sufficiency follows from the sufficiency of Theorem 8.11. Now we show the necessity.

Since $F \in \mathcal{PAP}(\mathbb{J}_a, X)$, by Theorem 8.11 there are functions $G \in \mathcal{AP}(\mathbb{R}, X)$ and $\Phi \in \mathcal{PAP}_0(\mathbb{J}_a, X)$ such that $F = G|_{\mathbb{J}_a} + \Phi$. To show the necessity, we need to show that G is a constant vector in X.

If it is not, there are $t', t'' \in \mathbb{R}$ with $t' < t''$ such that

$$\|G(t') - G(t'')\| = \epsilon > 0.$$

Since $G \in \mathcal{AP}(\mathbb{R}, X)$, for any $\tau \in P(\epsilon/4)$

$$\|G(t') - G(t' + \tau)\| < \epsilon/4,$$

and

$$\|G(t'') - G(t'' + \tau)\| < \epsilon/4.$$

Combining these three inequalities, we have

(10.1) $$\|G(t' + \tau) - G(t'' + \tau)\| > \epsilon/2 \qquad (\tau \in P(\epsilon/4)).$$

Φ is uniformly continuous on $\mathbb{J}_a$ since F and G are. Let $\delta > 0$ be such that $\|f\|\delta < \epsilon/8$ and

(10.2) $$\|\Phi(t_1) - \Phi(t_2)\| < \epsilon/16 \qquad (t_1, t_2 \in \mathbb{J}_a,\ |t_1 - t_2| < \delta).$$

Set

$$C_1 = \{t \in \mathbb{J}_a : \|\Phi(t)\| \geq min\{\frac{\epsilon}{8(t'' - t')}, \frac{\epsilon}{16}\}\},$$
$$C_2 = \{t \in \mathbb{J}_a : \|f(t)\| \geq min\{\frac{\epsilon}{8(t'' - t')}, \frac{\epsilon}{16}\}\}$$

and

$$C_\epsilon = C_1 \cup C_2.$$

By Propositions 8.7 and 8.8, C_ϵ is an ergodic zero subset in $\mathbf{J}_a$. By Proposition 8.4, there exist a $\tau_0 \in P(\epsilon/4)$ and $(u,v) \subset \mathbf{J}_a$ such that $[t',t''] + \tau_0 \subset (u,v)$ and $m((u,v) \cap C_\epsilon) < \delta$.

We claim that $\|\Phi(t'+\tau_0)\| < \epsilon/8$, $\|\Phi(t''+\tau_0)\| < \epsilon/8$. In fact, if $t'+\tau_0 \in (u,v)\setminus C_\epsilon$, then $\|\Phi(t'+\tau_0)\| < \epsilon/16$; if $t'+\tau_0 \in (u,v) \cap C_\epsilon$, then by (10.2)

$$\|\Phi(t'+\tau_0)\| \le \|\Phi(t'+\tau_0) - \Phi(t)\| + \|\Phi(t)\| < \epsilon/8,$$

where $t \in (u,v) \setminus C_\epsilon$ is such that $|t - (t'+\tau_0)| < \delta$.

Similarly we can show that $\Phi(t''+\tau_0) < \epsilon/8$

Now

$$\begin{aligned}
&\|G(t'+\tau_0) - G(t''+\tau_0)\| \\
\le &\left\| \int_a^{t'+\tau_0} - \int_a^{t''+\tau_0} f(u)du \right\| + \|\Phi(t'+\tau_0)\| + \|\Phi(t''+\tau_0)\| \\
< &\left\| \int_{t'+\tau_0}^{t''+\tau_0} f(u)du \right\| + \frac{\epsilon}{4} \\
= &\int_{[t'+\tau_0, t''+\tau_0]\setminus C_\epsilon} \|f(u)\|du + \int_{[t'+\tau_0, t''+\tau_0]\cap C_\epsilon} \|f(u)\|du + \frac{\epsilon}{4} \\
\le &|t''-t'| \cdot \frac{\epsilon}{8(t''-t')} + \|f\|\delta + \frac{\epsilon}{4} \\
< &\frac{\epsilon}{2},
\end{aligned}$$

which contradicts (10.1).

Recall that $\mathcal{AAP}(\mathbf{J}_a, X)$ stands for the set of asymptotically almost periodic functions (Remark 8.6 (4)) and $\mathcal{C}_0(\mathbf{J}_a, X)$ denotes the set of functions $f \in \mathcal{C}(\mathbf{J}_a, X)$ such that $f(t) \to 0$ as $t \to \infty$, $\mathcal{C}_0(\mathbf{J}_a, X) \subset \mathcal{PAP}_0(\mathbf{J}_a, X)$. We have the following corollary.

Corollary 10.2 [20, 4.2]. *Let $a \in \mathbf{R}$, let $f \in \mathcal{C}_0(\mathbf{J}_a, X)$, and let $F : \mathbf{J}_a \to X$ be defined by $F(t) = \int_a^t f(u)du$. Then $F \in \mathcal{AAP}(\mathbf{J}_a, X)$ if and only if $\lim_{t\to\infty} F(t)$ exists.*

Proof. Necessity.

Since $F \in \mathcal{AAP}(\mathbf{J}_a, X) \subset \mathcal{PAP}(\mathbf{J}_a, X)$ (Remark 8.6 (4)), $F = A + \Phi$, where $A \in X$ and $\Phi \in \mathcal{PAP}_0(\mathbf{J}_a, X)$ (Theorem 10.1). By Remark 8.12 (2), $F = G + \Phi'$, where $G \in \mathcal{AP}(\mathbf{J}_a, X)$ and $\Phi' \in \mathcal{C}_0(\mathbf{J}_a, X)$. The uniqueness of the decomposition implies that $G = A$ and $\Phi \in \mathcal{C}_0(\mathbf{J}_a, X)$. Therefore $F(t) = \int_a^t f(u)du \to A$ as $t \to \infty$.

The sufficiency is easy to prove.

The following example of [20, 4.1] can also be used here to show that the bounded integral of a pseudo almost periodic function may fail to be pseudo almost periodic even in the numerical case.

Example 10.3. Consider the function $f : \mathbf{J}_1 \to \mathbf{R}$ defined by

$$f(t) = (\frac{1}{t})\cos(\log t) \qquad (t \in \mathbf{J}_1).$$

Since $f(t) \to 0$ as $t \to \infty$, $f(t) \in \mathcal{PAP}_0(\mathbf{J}_1)$. The corresponding indefinite integral

$$F(t) = \int_1^t f(u)du = \sin(\log t) \qquad (t \in \mathbf{J}_1)$$

defines a bounded function on $\mathbf{J}_1$. However, $F \notin \mathcal{PAP}(\mathbf{J}_a)$. That is because $\frac{1}{r}\int_1^r |\sin(\log t)|dt \not\to 0$ as $r \to \infty$ and neither does $\frac{1}{r}\int_1^r |\sin(\log t) - A|dt$ for any $A \in \mathbf{C}$. But, if $F \in \mathcal{PAP}(\mathbf{J}_a)$, F differs from some member of $\mathcal{PAP}_0(\mathbf{J}_a)$ by a constant (Theorem 10.1).

Here we have adopted the definition in [17] of a mean on $\mathcal{A}$. [12] gives a definition of a mean in terms of a scalar-valued mean on $\overline{sp}(X^* \circ \mathcal{A}) = \overline{sp}\{x^*\mathcal{A} : x^* \in X^*\}$. In the next lemma, we set up a connection like this, and we will show in Theorem 12.7 that the definitions of a mean in [12] and [17] are equivalent. We will deal with other applications in §15.

Lemma 12.4. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S,X)$. A mapping $\mu : \mathcal{A} \to X$ is in $M(\mathcal{A})$ if and only if, for each $x^* \in X^*$, there is a $\varphi_{\mu,x^*} \in M(x^*\mathcal{A})$ such that*

$$x^*\mu(f) = \varphi_{\mu,x^*}(x^*f) \qquad (f \in \mathcal{A}).$$

If $\mathcal{A}$ is right (left) translation invariant, then μ is right (left) invariant if and only if the φ_{μ,x^*}'s are right (left) invariant. Furthermore, the set $\varphi_\mu = \{\varphi_{\mu,x^*} : x^* \in X^*\}$ is uniquely determined by μ, i.e., $\varphi_{\mu,x^*} = \varphi_{\mu',x^*}$ for all $x^* \in X^*$ if and only if $\mu = \mu'$.

Proof. Sufficiency. First, μ is a linear mapping from $\mathcal{A}$ to X. In fact, for $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} x^*\mu(\alpha f + \beta g) &= \varphi_{\mu,x^*}(x^*(\alpha f + \beta g)) \\ &= \varphi_{\mu,x^*}(x^*(\alpha f)) + \varphi_{\mu,x^*}(x^*(\beta g)) \\ &= \alpha\varphi_{\mu,x^*}(x^*f) + \beta\varphi_{\mu,x^*}(x^*g) \\ &= \alpha x^*\mu(f) + \beta x^*\mu(g) \\ &= x^*(\alpha\mu(f) + \beta\mu(g)). \end{aligned}$$

The equality is true for all $x^* \in X^*$, therefore

$$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g).$$

and $(u,v) \subset \mathbb{J}_a$ such that $[a,t_0] + \tau \subset (u,v)$ and $m((u,v) \cap C) < \delta$. Now,

$$\begin{aligned}
\epsilon &= \min_{y \in 2\overline{cco}(F(\mathbb{J}_a))} |x^*(G_a(t_0) - y)| \\
&\le \left| x^*\{G_a(t_0) - [F(t_0+\tau) - F(a+\tau)]\}\right| \\
&\le \left| x^*\{G_a(t_0) - \int_a^{t_0} g(u+\tau)du\}\right| + \left| x^*\{\int_a^{t_0} g(u+\tau)du - \int_{a+\tau}^{t_0+\tau} f(u)du\}\right| \\
&\le \|x^*\| \int_a^{t_0} \|g(u) - g(u+\tau)\| du + \left| x^*\{\int_a^{t_0} \varphi(u+\tau)du\}\right| \\
&\le \|x^*\|(t_0 - a)\delta + \int_a^{t_0} \left| x^*\{\varphi(u+\tau)\}\right| du \\
&= \|x^*\|(t_0 - a)\delta + \int_{[a+\tau, t_0+\tau]\setminus C} \left| x^*\{\varphi(u)\}\right| du + \int_{[a+\tau, t_0+\tau]\cap C} \left| x^*\{\varphi(u)\}\right| du \\
&< \|x^*\|(t_0 - a)\delta + \delta\|x^*\|(t_0 - a) + \delta\|\varphi\|\|x^*\| \\
&< \epsilon,
\end{aligned}$$

a contradiction.

Lemma 10.5 [20, 4.5]. *Let $a \in \mathbb{R}$ and let $g \in \mathcal{AP}(\mathbb{R}, X)$. Put $G_a(t) = \int_a^t g(u)du$ for $t \in \mathbb{J}_a$, and set*

$$G(t) = \begin{cases} \int_0^t g(u)du, & t \in \mathbb{J}_0 \\ -\int_t^0 g(u)du, & t \in \mathbb{R} \setminus \mathbb{J}_0. \end{cases}$$

Then

$$G(\mathbb{R}) \subset G([0, |a|]) + 2\overline{cco}(G_a(\mathbb{J}_a)).$$

Taken together, Lemmas 10.4 and 10.5 yield the following result.

Lemma 10.6. *Let a, f, g, φ, F and G_a be as in Lemma 10.4. If $F(\mathbb{J}_a)$ is bounded [weakly relatively compact] {relatively compact} in X, then the same is true for $G(\mathbb{R})$.*

Theorem 10.7. *Let a, f, g, φ, and F be as in Lemma 10.4. Suppose either*

(1) $F \in \mathcal{B}(\mathbb{J}_a, X)$ *and X does not contain an isomorphic copy of c_0,*

or

(2) $F(\mathbb{J}_a)$ *is weakly relatively compact in X.*

Then $F \in \mathcal{PAP}(\mathbb{J}_a, X)$ if and only if there is an $A \in X$ such that Φ, defined by

$$\Phi(x) = \int_0^x \varphi(u)du - A$$

is in $\mathcal{PAP}_0(\mathbb{J}_a, X)$.

Proof. Necessity.

We define the indefinite integral $G_a : \mathbb{J}_a \to X$ and $G : \mathbb{R} \to X$ of g as in Lemma 10.5.

Since (1) or (2) is satisfied, Lemma 10.6 shows that at least one of the following holds: $c_0 \not\subset X$, $G(\mathbb{R})$ is bounded, and $G(\mathbb{R})$ is weakly relatively compact in X. By [14, Theorem 1] in case (1), or by [14, Theorem 2] in case (2), G is almost periodic. Therefore, $G_a = G|_{\mathbb{J}_a} - G(a) \in \mathcal{PAP}(\mathbb{J}_a, X)$ and so is ψ, where

$$\psi(t) = \int_a^t \varphi(u)du = F(t) - \int_a^t g(u)du.$$

Now the necessity follows from Theorem 10.1.

The sufficiency is easy to prove; we omit the proof.

§11. Weak Pseudo Almost Periodicity

In this section, we turn our attention to weakly pseudo almost periodic functions.

Let X_{lc} be a Hausdorff locally convex space over $\mathbb{C}$. A function $f \in C(\mathbf{J}_a, X_{lc})$ is called pseudo almost periodic if for any neighborhood V of $0 \in X_{lc}$ there are a number $\delta > 0$, a relatively dense subset $P(V)$ of $\mathbf{J}_a$ and an ergodic zero subset C_V of $\mathbf{J}_a$ such that

$$f(t) - f(t+\tau) \in V \qquad (\tau \in P(V),\ t, t+\tau \in \mathbf{J}_a \setminus C_V),$$

and

$$f(t') - f(t'') \in V \qquad (t', t'' \in \mathbf{J}_a \setminus C_V,\ |t' - t''| < \delta).$$

Let us now consider a Banach space X, and denote by X_W the set X with the weak topology $\sigma(X, X^*)$.

A function $f : \mathbf{J}_a \to X$ is called weakly pseudo almost periodic, if the function $f : \mathbf{J}_a \to X_W$ is pseudo almost periodic.

Noting Theorem 9.2, we can use the proof of [6, 6.17] almost verbatim to prove the following theorem.

Theorem 11.1. *A necessary and sufficient condition for a function $f : \mathbf{J}_a \to X$ to be weakly pseudo almost periodic is that for each $x^* \in X^*$, the composed function $x^* f$ is in $\mathcal{PAP}(\mathbf{J}_a)$.*

Denote by $\mathcal{WPAP}(\mathbf{J}_a, X)$ the set of all functions f such that $x^* f \in \mathcal{PAP}(\mathbf{J}_a)$ for all $x^* \in X^*$. Define analogously $WAP(\mathbf{R}, X)$ using $\mathcal{AP}(\mathbf{R})$ and $\mathcal{WPAP}_0(\mathbf{J}_a, X)$ using $\mathcal{PAP}_0(\mathbf{J}_a)$. By Theorem 11.1, $\mathcal{WPAP}(\mathbf{J}_a, X)$ is the set of all weakly pseudo almost periodic functions.

Remark 11.2. Here we distinguish between $WAP(\mathbf{R}, X)$ and $\mathcal{WAP}(\mathbf{R}, X)$. $WAP(\mathbf{R}, X)$ is the set of functions which are defined as above, and are called weakly almost periodic functions in [1, 6, 16]. $\mathcal{WAP}(\mathbb{R}, X)$ is the set of functions for which $\{R_{\mathbb{R}} f\}$ is weakly relatively compact in $C(\mathbb{R}, X)$ as in the introduction and Sections 1, 2 and 4; we will investigate $\mathcal{WAP}(\mathbb{R}, X)$ further in Chapter IV.

The following theorem is an extension of Lemma 2.2.

Theorem 11.3. *Let $a \in \mathbb{R}$, $g \in \mathcal{WAP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{WPAP}_0(\mathbf{J}_a, X)$, and set $f = g|_{\mathbf{J}_a} + \varphi$. Then*

$$g(\mathbb{R}) \subset \overline{co} f(\mathbf{J}_a).$$

Hence $\inf_{t \in \mathbf{J}_a} \|f(t)\| \leq \inf_{t \in \mathbb{R}} \|g(t)\| \leq \|g\| \leq \|f\|$.

Proof. We use ideas from Lemma 10.4.

Suppose, on the contrary, that there is a $t_0 \in \mathbb{R}$ such that $g(t_0) \notin \overline{co} f(\mathbf{J}_a)$. By the Hahn-Banach theorem, we can find an $x^* \in X^*$ such that

$$\inf_{y \in \overline{co} f(\mathbf{J}_a)} |x^*(g(t_0) - y)| = \epsilon > 0. \tag{11.1}$$

Since $x^* g \in \mathcal{AP}(\mathbb{R})$, there is a relatively dense subset $P(\epsilon/4)$ of $\mathbb{R}$ such that

$$|x^* g(t_0) - x^* g(t_0 + \tau)| < \epsilon/4 \qquad (\tau \in P(\epsilon/4)). \tag{11.2}$$

Let $l > 0$ be the number for $P(\epsilon/4)$ in Definition 8.1. Again by the almost periodicity, $x^* g$ is uniformly continuous on $\mathbb{R}$. We can find a number $0 < \delta < l/2$ such that

$$|x^* g(t_0 + \tau + t) - x^* g(t_0 + \tau)| < \epsilon/4 \qquad (t \in (-\delta, \delta)). \tag{11.3}$$

It follows from (11.1), (11.2) and (11.3) that

$$|x^* f(t_0 + \tau + t) - x^* g(t_0 + \tau + t)| > \epsilon/2,$$

as long as $\tau \in P(\epsilon/4)$, $t \in (-\delta, \delta)$ and $t_0 + \tau + t \in \mathbb{J}_a$.

Set $a_n = 2nl$ and $b_n = a_n + l$, $n = 1, 2, \cdots$, and choose $\tau_n \in [a_n, b_n] \cap P(\epsilon/4)$. There is an integer $n_0 > 0$ such that $t_0 + \tau_n - \delta \in \mathbb{J}_a$, whenever $n > n_0$. Thus, if we put

$$n(t) = \{n \in \mathbb{N} : t_0 + \tau_{n_0} + \delta \leq t_0 + \tau_n - \delta \leq t_0 + \tau_n + \delta \leq t\},$$

then $n(t) \geq \frac{1}{2l}[t - (t_0 + \tau_{n_0} - \delta)]$. Therefore

$$\begin{aligned}
&\frac{1}{t}\int_a^t |x^*\varphi(u)|du \\
=&\frac{1}{t}\int_a^t |x^*f(u) - x^*g(u)|du \\
\geq&\frac{1}{t}\sum_{n(t)}\int_{t_0+\tau_n-\delta}^{t_0+\tau_n+\delta} |x^*f(u) - x^*g(u)|du \\
\geq&\frac{\epsilon}{2}\cdot\frac{1}{t}\cdot\frac{\delta}{2l}[t - (t_0 + \tau_{n_0} - \delta)],
\end{aligned}$$

which contradicts the fact that $x^*\varphi \in \mathcal{PAP}_0(\mathbb{J}_a)$.

If $g \in WAP(\mathbb{R}, X)$ and $\varphi \in \mathcal{WPAP}_0(\mathbb{J}_a, X)$, then $f = g + \varphi \in \mathcal{WPAP}(\mathbb{J}_a, X)$ by Theorem 11.1. We do not know if every weakly pseudo almost periodic function has a decomposition like this. However, if X is separable and reflexive, we do have the decomposition. To show this, we need the definition of a pseudo almost periodic function from $\mathbb{J}_a$ to a metric space and the next theorem.

Let (Y, d) be a metric space and let $\mathcal{C}(\mathbb{J}_a, Y)$ be all of the bounded continuous functions from $\mathbb{J}_a$ to Y. In Definition 8.5, if we replace X and the norm $\|\cdot\|$ by the metric space Y and the distance $d(\cdot,\cdot)$, we have the definition of a pseudo almost periodic function from $\mathbb{J}_a$ to Y. Then Remark 8.6 (1) shows how to define an almost periodic function from $\mathbb{R}$ to Y.

The following theorem is modelled on Theorem 8.11.

Theorem 11.4. *Let (Y, d) be a complete metric space. Then $f \in C(\mathbf{J}_a, Y)$ is pseudo almost periodic if and only if there is a unique $g \in \mathcal{AP}(\mathbf{R}, Y)$ such that the function $t \to d(f(t), g(t))$ is in $\mathcal{PAP}_0(\mathbf{J}_a)$.*

Proof. The proof for the "if" part is similar to its counterpart in Theorem 8.11; we show the "only if" part. The existence of g can be shown similarly as for Theorem 8.11. For the uniqueness, let $g_k \in \mathcal{AP}(\mathbf{R}, Y)$, $k = 1, 2$, be such that the function $t \to d(f(t), g_k(t)) \in \mathcal{PAP}_0(\mathbf{J}_a)$; we show that $g_1 = g_2$. Suppose, on the contrary, that there is $t' \in \mathbf{R}$ such that $g_1(t') \neq g_2(t')$. Let $y = g_1(t')$. For $k = 1, 2$, the function $\varphi_k : t \to d(g_k(t), y)$ is in $\mathcal{AP}(\mathbf{R})$ since

$$
\begin{aligned}
|\varphi_k(t) - \varphi_k(t+\tau)| &= |d(g_k(t), y) - d(g_k(t+\tau), y)| \\
&\leq d(g_k(t), g_k(t+\tau)).
\end{aligned}
$$

So $\varphi = \varphi_1 - \varphi_2 \in \mathcal{AP}(\mathbf{R})$. At the same time, $\varphi|_{\mathbf{J}_a} \in \mathcal{PAP}_0(\mathbf{J}_a)$ because

$$
\begin{aligned}
|\varphi(t)| &= |\varphi_1(t) - \varphi_2(t)| \\
&\leq d(g_1(t), g_2(t)) \\
&\leq d(f(t), g_1(t)) + d(f(t), g_2(t)).
\end{aligned}
$$

It follows from Theorem 8.11 that $\varphi = 0$, i.e., $\varphi_1 = \varphi_2$. This gives

$$
\begin{aligned}
0 &= d(g_1(t'), g_1(t')) = d(g_1(t'), y) = \varphi_1(t') \\
&= \varphi_2(t') = d(g_2(t'), y) = d(g_2(t'), g_1(t')) \neq 0,
\end{aligned}
$$

a contradiction.

Theorem 11.5. *Assume that X is a separable, reflexive Banach space. Then an $f \in \mathcal{B}(\mathbb{J}_a, X)$ is in $\mathcal{WPAP}(\mathbb{J}_a, X)$ if and only if there is a unique $g \in WAP(\mathbf{R}, X)$ such that*

$$f - g|_{\mathbb{J}_a} \in \mathcal{WPAP}_0(\mathbb{J}_a, X);$$

$\mathcal{WPAP}(\mathbb{J}_a, X)$ is a Banach space (under the supremum norm).

Proof. We only need to show that the condition is necessary for the first statement.

By the hypothesis, $\overline{co}f(\mathbb{J}_a)$ is weakly compact. Since X is separable, $\sigma(\overline{co}f(\mathbf{J}_a), X^*)$, the weak topology of X restricted to $\overline{co}f(\mathbf{J}_a)$, is metrizable in the following way [15, Theorem 10.3.2]: if $\{x_n : n \in \mathbf{N}\}$ is a dense subset in X, and for each $n \in \mathbf{N}$, $x_n^* \in X^*$ is such that $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$, then

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n^*(x - y)| \qquad (x, y \in \overline{co}f(\mathbf{J}_a))$$

defines a metric on $\overline{co}f(\mathbf{J}_a)$ whose topology is $\sigma(\overline{co}f(\mathbf{J}_a), X^*)$. Since $(\overline{co}f(\mathbf{J}_a), d)$ is compact, it is complete.

We claim that $f \in \mathcal{PAP}(\mathbf{J}_a, \overline{co}f(\mathbf{J}_a))$. In fact, for a given $\epsilon > 0$, there is an $n_0 > 0$ such that

$$\sum_{n=n_0+1}^{\infty} 2^{-n}|x_n^*(x - y)| < \frac{\epsilon}{2} \qquad (x, y \in \overline{co}f(\mathbf{J}_a)). \tag{11.4}$$

Let δ, $P(\epsilon/2)$ and $C_{\epsilon/2}$ be for ϵ in Theorem 9.2. Then for $n = 1, 2, \cdots, n_0$,

$$|x_n^* f(t) - x_n^* f(t + \tau)| < \frac{\epsilon}{2} \qquad (\tau \in P(\epsilon/2),\ t, t + \tau \in \mathbf{J}_a \setminus C_{\epsilon/2}), \tag{11.5}$$

and

$$(11.6) \qquad |x_n^* f(t') - x_n^* f(t'')| < \frac{\epsilon}{2} \qquad (t', t'' \in \mathbf{J}_a \setminus C_{\epsilon/2},\ |t' - t''| < \delta).$$

It follows from (11.4), (11.5) and (11.6) that $f \in \mathcal{PAP}(\mathbf{J}_a, \overline{co}f(\mathbf{J}_a))$. By Theorem 11.4, there is a unique $g \in \mathcal{AP}(\mathbf{R}, \overline{co}f(\mathbf{J}_a))$ such that the function $t \to d(f(t), g(t))$ is in $\mathcal{PAP}_0(\mathbf{J}_a)$.

We claim that $g \in WAP(\mathbf{R}, X)$. In fact, let $x^* \in X^*$. Since the topology $\sigma(\overline{co}f(\mathbf{J}_a), X^*)$ is given by the metric d above, for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$(11.7) \qquad |x^*(x - y)| < \epsilon \qquad (x, y \in \overline{co}f(\mathbf{J}_a),\ d(x, y) < \delta).$$

Since $g \in \mathcal{AP}(\mathbf{R}, \overline{co}f(\mathbf{J}_a))$, there is a relatively dense subset $P(\delta)$ of $\mathbf{R}$ such that

$$d(g(t), g(t + \tau)) < \delta \qquad (\tau \in P(\delta),\ t \in \mathbf{R}).$$

It follows from (11.7) that

$$|x^* g(t) - x^* g(t + \tau)| < \epsilon \qquad (\tau \in P(\delta),\ t \in \mathbf{R}),$$

i.e., $x^* g \in \mathcal{AP}(\mathbf{R})$.

Similarly we can show that $x^*(f - g|_{\mathbf{J}_a}) \in \mathcal{PAP}_0(\mathbf{J}_a)$. In fact, for $\delta > 0$, set $C_\delta = \{t \in \mathbf{R} : d(f(t), g(t)) \geq \delta\}$. Since the function $t \to d(f(t), g(t))$ is in $\mathcal{PAP}_0(\mathbf{J}_a)$, Proposition 8.7 tells us that $m(C_\delta \cap [a, t])/t \to 0$ as $t \to \infty$. It follows from (11.7) that, for $t \notin C_\delta$,

$$|x^*(f(t) - g(t))| < \epsilon.$$

Now it is routine to show that $x^*(f - g|_{\mathbf{J}_a}) \in \mathcal{PAP}_0(\mathbf{J}_a)$.

The uniqueness is easy to prove.

A modification of the proof of Theorem 2.3 proves the second statement.

We give two more results to end this section. We can now consider the indefinite integral of a function in either $\mathcal{PAP}(\mathbf{J}_a, X)$ or $\mathcal{WPAP}(\mathbf{J}_a, X)$ (whose definition follows Theorem 11.1) and give criteria for the weak pseudo almost periodicity of the indefinite integrals of such functions.

Theorem 11.6. *Let X be a separable, reflexive Banach space and let a function $f \in \mathcal{WPAP}(\mathbf{J}_a, X)$ have weakly relatively compact range $f(\mathbf{J}_a)$. Define $F : \mathbf{J}_a \to X$ by $F(t) = \int_a^t f(u)du$. Then $F \in \mathcal{WPAP}(\mathbf{J}_a, X)$ if and only if $F \in \mathcal{B}(\mathbf{J}_a, X)$ and the ergodic perturbation φ of f satisfies the following condition: for each $x^* \in X^*$, there exist $A_{x^*} \in \mathbb{C}$ and a function $\Phi_{x^*} \in \mathcal{PAP}_0(\mathbf{J}_a)$ such that*

$$\int_a^t x^*\varphi(u)du = A_{x^*} + \Phi_{x^*}(t).$$

Proof. Let $x^* \in X$. Note that $x^*F(t) = \int_a^t x^*f(u)du$ for $t \in \mathbf{J}_a$, that $x^*f \in \mathcal{PAP}(\mathbf{J}_a)$ and that $x^*\varphi$ is the ergodic perturbation of x^*f. The conclusion is now an immediate consequence of Theorem 10.7.

Remark 11.7. In Theorem 11.6, the requirement that X is separable and reflexive is used only to guarantee that f has the decomposition of Theorem 11.5. In case f is known to have the decomposition from another consideration, for example, if $f \in \mathcal{WAP}(\mathbf{J}_0, X)$ (for which the decomposition apears in Section 15), the requirement is not needed. However, the condition (1) or (2) in Theorem 10.7 is needed. We will use this observation to answer a question in Chapter IV.

Theorem 11.8. *Let X be a Banach space and let a, f, g, φ and F be as in Lemma 10.4. Then $F \in \mathcal{WPAP}(\mathbf{J}_a, X)$ if and only if $F \in \mathcal{B}(\mathbf{J}_a, X)$ and for each $x^* \in X$ there are a constant A_{x^*} and a function $\Phi_{x^*} \in \mathcal{PAP}_0(\mathbf{J}_a)$ such that*

$$\int_a^t x^* \varphi(u)du = A_{x^*} + \Phi_{x^*}(t) \qquad (t \in \mathbf{J}_a).$$

Proof. The conclusion follows from Theorems 10.7 and 11.1.

CHAPTER IV

VECTOR–VALUED MEANS AND WEAKLY ALMOST PERIODIC FUNCTIONS

Scalar–valued means have been much studied. However, little has been done on the vector–valued means. In this chapter we develop the theory of vector–valued means.

In Lemma 12.4, we set up a formula between a vector–valued mean and scalar-valued means, by which we will be able to translate many important results about scalar–valued means developed in [2] to vector–valued means. We present these results in Sections 12, 13 and 14. As an application of the theory established in these sections, we investigate vector–valued weakly almost periodic functions in Section 15.

§12. Means on a Linear Subspace of $\mathcal{B}(S, X)$

Throughout this chapter, S denotes a semigroup which need not have an identity, X denotes a Banach space and X^* is the dual space of X. $\mathcal{B}(S, X)$ denotes all of the bounded functions from S to X. When $X = \mathbb{C}$, we simply write $\mathcal{B}(S)$ for $\mathcal{B}(S, X)$. $\mathcal{A}$ denotes a linear subspace of $\mathcal{B}(S, X)$ containing the constant functions. $\mathcal{L}(\mathcal{A}, X)$ denotes all of the bounded linear mappings from $\mathcal{A}$ to X.

Let $f \in \mathcal{B}(S, X)$. Then the right (respectively, left) translate $R_s f$ of f by $s \in S$ is the map $R_s f(t) = f(ts)$ (respectively, $L_s f(t) = f(st)$) for all $t \in S$.

$\mathcal{A}$ is said to be right (respectively, left) translation invariant if $R_S\mathcal{A} = \{R_s f : s \in S,\ f \in \mathcal{A}\} \subset \mathcal{A}$ (respectively, $L_S\mathcal{A} = \{L_s f :\ s \in S,\ f \in \mathcal{A}\} \subset \mathcal{A}$). $\mathcal{A}$ is said to be translation invariant if it is both right and left translation invariant.

Definition 12.1 [17]. *A linear mapping $\mu : \mathcal{A} \to X$ is called a mean on $\mathcal{A}$ provided $\mu(f) \in \overline{co}f(S)$, for all $f \in \mathcal{A}$. Denote by $M(\mathcal{A})$ the set of all means on $\mathcal{A}$.*

If $\mathcal{A}$ is right (respectively, left) translation invariant, μ is said to be right (respectively, left) invariant if $\mu(R_s f) = \mu(f)$ (respectively, $\mu(L_s f) = \mu(f)$) for all $s \in S$ and $f \in \mathcal{A}$.

Remark 12.2. It follows from [2, 2.1.2] that Definition 12.1 will reduce to the definition of a scalar–valued mean when $X = \mathbf{C}$.

Of course, the evaluation mapping $\epsilon : S \to \mathcal{L}(\mathcal{A}, X)$, defined by

$$\epsilon(s)(f) = f(s) \qquad (s \in S,\ f \in \mathcal{A})$$

is in $M(\mathcal{A})$, and if $\mu \in M(\mathcal{A})$ and $f \in \mathcal{A}$ is a constant function, then $\mu(f)$ is the constant.

The following proposition is obvious.

Proposition 12.3. *If $\mathcal{A}$ is a linea· subspace of $\mathcal{B}(S, X)$ containing the constant functions, then each $\mu \in M(\mathcal{A})$ is in $\mathcal{L}(\mathcal{A}, X)$ with $\|\mu\| = 1$.*

For each $x^* \in X^*$,

$$x^*\mathcal{A} = \{x^* f = x^* \circ f :\ f \in \mathcal{A}\}$$

is a linear subspace of $\mathcal{B}(S)$.

Here we have adopted the definition in [17] of a mean on $\mathcal{A}$. [12] gives a definition of a mean in terms of a scalar-valued mean on $\overline{sp}(X^* \circ \mathcal{A}) = \overline{sp}\{x^*\mathcal{A} : x^* \in X^*\}$. In the next lemma, we set up a connection like this, and we will show in Theorem 12.7 that the definitions of a mean in [12] and [17] are equivalent. We will deal with other applications in §15.

Lemma 12.4. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, X)$. A mapping $\mu : \mathcal{A} \to X$ is in $M(\mathcal{A})$ if and only if, for each $x^* \in X^*$, there is a $\varphi_{\mu,x^*} \in M(x^*\mathcal{A})$ such that*

$$x^*\mu(f) = \varphi_{\mu,x^*}(x^*f) \qquad (f \in \mathcal{A}).$$

If $\mathcal{A}$ is right (left) translation invariant, then μ is right (left) invariant if and only if the φ_{μ,x^*}'s are right (left) invariant. Furthermore, the set $\varphi_\mu = \{\varphi_{\mu,x^*} : x^* \in X^*\}$ is uniquely determined by μ, i.e., $\varphi_{\mu,x^*} = \varphi_{\mu',x^*}$ for all $x^* \in X^*$ if and only if $\mu = \mu'$.

Proof. Sufficiency. First, μ is a linear mapping from $\mathcal{A}$ to X. In fact, for $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbf{C}$,

$$\begin{aligned} x^*\mu(\alpha f + \beta g) &= \varphi_{\mu,x^*}(x^*(\alpha f + \beta g)) \\ &= \varphi_{\mu,x^*}(x^*(\alpha f)) + \varphi_{\mu,x^*}(x^*(\beta g)) \\ &= \alpha\varphi_{\mu,x^*}(x^*f) + \beta\varphi_{\mu,x^*}(x^*g) \\ &= \alpha x^*\mu(f) + \beta x^*\mu(g) \\ &= x^*(\alpha\mu(f) + \beta\mu(g)). \end{aligned}$$

The equality is true for all $x^* \in X^*$, therefore

$$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g).$$

We claim that $\mu(f) \in \overline{co}f(S)$, for all $f \in \mathcal{A}$. If it is not true for some $f \in \mathcal{A}$, by the Hahn–Banach theorem there is an $x^* \in X^*$ such that

$$|x^*\mu(f)| > \sup_{s \in S} |x^* f(s)| = \|x^* f\|.$$

It follows from Remark 12.2 and Proposition 12.3 that $\varphi_{\mu,x^*} \in M(x^*\mathcal{A})$ is in $(x^*\mathcal{A})^*$ with $\|\varphi_{\mu,x^*}\| = 1$. So

$$|x^*\mu(f)| = |\varphi_{\mu,x^*}(x^*f)| \leq \|x^*f\|,$$

a contradiction.

Necessity. For each $x^* \in X^*$, define $\varphi_{\mu,x^*} \in (x^*\mathcal{A})^*$ by

$$\varphi_{\mu,x^*}(x^*f) = x^*\mu(f) \qquad (f \in \mathcal{A}).$$

φ_{μ,x^*} is well-defined on $x^*\mathcal{A}$. For, if $x^*f = 0$ for some $f \in \mathcal{A}$, then $f(S) \subset N(x^*)$, the null subspace of x^*, so $\varphi_{\mu,x^*}(x^*f) = x^*\mu(f) = 0$ since $\mu(f) \in \overline{co}f(S)$ (Definition 12.1). Clearly φ_{μ,x^*} is linear on $x^*\mathcal{A}$. Furthermore

$$\varphi_{\mu,x^*}(x^*f) = x^*\mu(f) \in x^*\overline{co}(f)(S) \subset \overline{co}x^*f(S),$$

so φ_{μ,x^*} is in $M(x^*\mathcal{A})$.

The rest of the lemma is clear.

We can furnish $\mathcal{L}(\mathcal{A}, X)$ with two topologies, both of which make $\mathcal{L}(\mathcal{A}, X)$ a locally convex topological space. One is the strong operator topology τ_s, which is the weakest topology of $\mathcal{L}(\mathcal{A}, X)$ relative to which the mapping $U \to Uf$: $\mathcal{L}(\mathcal{A}, X) \to X$ is continuous for each $f \in \mathcal{A}$, and the other is the weak operator topology τ_w, which is the weakest topology of $\mathcal{L}(\mathcal{A}, X)$ relative to which the mapping $U \to x^*Uf : \mathcal{L}(\mathcal{A}, X) \to \mathbb{C}$ is continuous for each $f \in \mathcal{A}$ and $x^* \in X^*$. These topologies can be relativized to $M(\mathcal{A}) \subset \mathcal{L}(\mathcal{A}, X)$.

Proposition 12.5. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, X)$. Then, for τ_s*

(1) *$M(\mathcal{A})$ is convex and closed in $\mathcal{L}(\mathcal{A}, X)$;*

(2) *$co(\epsilon(S))$ is dense in $M(\mathcal{A})$;*

(3) *if S is a topological space and $\mathcal{A} \subset \mathcal{C}(S, X)$, then $\epsilon : S \to M(\mathcal{A})$ is continuous.*

Furthermore, if the range $f(S)$ of f is relatively compact in X for each $f \in \mathcal{A}$, then $M(\mathcal{A})$ is τ_s-compact.

Proof.

(1) The convexity of $M(\mathcal{A})$ follows directly from Definition 12.1. To show that $M(\mathcal{A})$ is closed, let $\{\mu_\alpha\} \subset M(\mathcal{A})$ converge to $\mu \in \mathcal{L}(\mathcal{A}, X)$ for τ_s. Then $\mu_\alpha(f) \to \mu(f)$ for each $f \in \mathcal{A}$, and since $\mu_\alpha(f) \in \overline{co}f(S)$ for all α, $\mu(f) \in \overline{co}f(S)$. Therefore, $\mu \in M(\mathcal{A})$.

(2) Clearly, $co(\epsilon(S)) \subset M(\mathcal{A})$. If there is a $\mu \in M(\mathcal{A})$ such that $\mu \notin \overline{co}(\epsilon(S))$, the closure being taken in τ_s, then there is an $f \in \mathcal{A}$ such that $\mu(f) \notin \overline{co}(\epsilon(S)f) = \overline{co}f(S)$, which contradicts Definition 12.1.

(3) is obvious.

The proof of the compactness of $M(\mathcal{A})$, if $\mathcal{A}$ satisfies the compactness condition, is similar to that of its counterpart in the following proposition, so we omit it.

Proposition 12.6. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, X)$. Then the conclusions (1)–(3) of the previous proposition are true for τ_w. Furthermore, if $\mathcal{A}$ is such that the range $f(S)$ of f is weakly relatively compact in X for each $f \in \mathcal{A}$, then $M(\mathcal{A})$ is τ_w-compact.*

Proof. Using Lemma 12.4. we can prove (1)–(3) in much the same way that (1)–(3) of Proposition 12.5 we proved.

We now show that $M(\mathcal{A})$ is τ_w–compact when $\mathcal{A}$ satifies the weak compactness condition. For each $x^* \in X^*$, $M(x^*\mathcal{A})$ is weak* compact [2, 2.1.8]. Therefore, the product space

$$\prod := \prod\{M(x^*\mathcal{A}) : \ x^* \in X^*\}$$

is compact in the product topology.

By Lemma 12.4, the mapping $\mu \to \varphi_\mu = \{\varphi_{\mu,x^*} : x^* \in X^*\} : M(\mathcal{A}) \to \prod$ is 1-1, and it is homeomorphism when $M(\mathcal{A})$ has the topology τ_w. To show that $M(\mathcal{A})$ is τ_w–compact, it suffices to show that the image of $M(\mathcal{A})$ in $\prod$ is closed.

Let $\varphi = \{\varphi_{x^*} : x^* \in X^*\} \in \prod$ and let the image $\{\varphi_{\mu_\alpha}\}$ of $\{\mu_\alpha\}$ converge to φ in $\prod$. We show that there is a $\mu \in M(\mathcal{A})$ such that φ is the image of μ and $\mu_\alpha \to \mu$ in τ_w.

Since $f(S)$ is weakly relatively compact in X for each $f \in \mathcal{A}$, by the Krein–Smulian theorem [2, A.10] $\overline{co}f(S)$ is weakly compact in X for each $f \in \mathcal{A}$. Since $\mu_\alpha(f) \in \overline{co}f(S)$ for all α and $x^*\mu_\alpha(f) \to \varphi_{x^*}(x^*f)$ for all $x^* \in X^*$, there is a $\mu(f) \in \overline{co}f(S)$ such that $x^*\mu(f) = \varphi_{x^*}(x^*f)$ for all $x^* \in X^*$. The map $f \to \mu(f)$ is clearly linear, so $\mu \in M(\mathcal{A})$. Thus $\mu_\alpha \to \mu$ in τ_w, and the proof is complete.

The following theorem shows that the definition of a mean in [12] is equivalent to that in [17].

Theorem 12.7. *A mapping $\mu : \mathcal{A} \to X$ is in $M(\mathcal{A})$ if and only if there is a unique $\varphi_\mu \in M(\overline{sp}(X^* \circ \mathcal{A}))$ such that*

$$x^*\mu(f) = \varphi_\mu(x^* f) \qquad (x^* \in X^*,\ f \in \mathcal{A}). \tag{12.1}$$

Proof. The sufficiency comes from the sufficiency in the first statement of Lemma 12.4.

Necessity. By Lemma 12.4, if μ is in $M(\mathcal{A})$, then for each $x^* \in X^*$ there is a φ_{μ,x^*} in $M(x^*\mathcal{A})$ such that

$$x^*\mu(f) = \varphi_{\mu,x^*}(x^* f) \qquad (f \in \mathcal{A}).$$

We show first that φ_{μ,x^*} is independent of $x^* \in X^*$, i.e., if x_1^*, $x_2^* \in X^*$ and f_1, $f_2 \in \mathcal{A}$ are such that $x_1^* f_1 = x_2^* f_2$, then $\varphi_{\mu,x_1^*}(x_1^* f_1) = \varphi_{\mu,x_2^*}(x_2^* f_2)$.

Since $\mu \in M(\mathcal{A})$, by Proposition 12.6 (2) there is a net $\{\sum_{s\in S} \lambda_\alpha(s)\epsilon(s)\}$ converging to μ for τ_w; here each $\lambda_\alpha : S \to [0,1]$ has finite support and satisfies $\sum_{s\in S} \lambda_\alpha(s) = 1$. Next, $x_1^*(\sum_{s\in S} \lambda_\alpha(s) f_1(s)) = x_2^*(\sum_{s\in S} \lambda_\alpha(s) f_2(s))$ because $x_1^* f_1 = x_2^* f_2$, so

$$\begin{aligned}\varphi_{\mu,x_1^*}(x_1^* f_1) = x_1^*\mu(f_1) &= \lim_\alpha x_1^* \sum_{s\in S} \lambda_\alpha(s) f_1(s) \\ &= \lim_\alpha x_2^* \sum_{s\in S} \lambda_\alpha(s) f_2(s) = x_2^*\mu(f_2) = \varphi_{\mu,x_2^*}(x_2^* f_2).\end{aligned}$$

Therefore we can define φ_μ for $\sum_{i=1}^m \alpha_i x_i^* f_i \in sp(X^* \circ \mathcal{A})$ by

$$\varphi_\mu\Big(\sum_{i=1}^m \alpha_i x_i^* f_i\Big) = \sum_{i=1}^m \alpha_i \varphi_{\mu,x_i^*}(x_i^* f_i).$$

It is easy to see that φ_μ is in $M(sp(X^* \circ \mathcal{A}))$. Therefore φ_μ has a unique extension to $\overline{sp}(X^* \circ \mathcal{A})$ and satisfies (12.1).

The uniqueness is clear. The proof is finished.

By Theorem 12.7, we can write φ_μ for φ_{μ,x^*} in Lemma 12.4.

§13. Introversion and Semigroups of Vector–Valued Means

Definition 13.1. *Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S,X)$. For a linear map μ from $\mathcal{A}$ to X, define the left introversion operator $T_\mu : \mathcal{A} \to \mathcal{B}(S,X)$ by*

$$T_\mu f(s) = \mu(L_s f) \qquad (f \in \mathcal{A},\ s \in S)$$

and analogously define the right introversion operator $U_\mu : A \to \mathcal{B}(S,X)$ by

$$U_\mu f(s) = \mu(R_s f) \qquad (f \in \mathcal{A},\ s \in S).$$

If $T_\mu \mathcal{A} \subset \mathcal{A}$ for all $\mu \in M(\mathcal{A})$, we will say that $\mathcal{A}$ is left introverted; we will say that $\mathcal{A}$ is right introverted if $U_\mu \mathcal{A} \subset \mathcal{A}$. $\mathcal{A}$ is introverted if it is both left and right introverted.

A semitopological semigroup S is a semigroup and a Hausdorff topological space in such a way that multiplication is separately continuous, i.e., the maps $s \to ts$ and $s \to st$ from S into S are continuous for all $t \in S$. $\mathcal{C}(S,X)$ denotes the Banach space of all continuous members of $\mathcal{B}(S,X)$.

Example 13.2. $\mathcal{C}(S,X)$ is introverted if S is a compact semitopological semigroup.

We note first that the consideration of constant functions shows that $x^*\mathcal{C}(S,X) = \mathcal{C}(S)$ for any $x^* \in X^*$, $x^* \neq 0$.

For $\mu \in M(\mathcal{C}(S,X))$ and $f \in \mathcal{C}(S,X)$, we must show that $T_\mu f$ is continuous. It follows from Theorem 12.7 that we have a $\varphi_\mu \in M(x^*\mathcal{C}(S,X))$ satisfying (12.1). We claim that $T_\mu f$ is weakly continuous. For, suppose that $s_\alpha \to s$ in S and recall that $\mathcal{C}(S)$ is introverted [2, 2.2.5]. Then for any $x^* \in X^*$

$$(T_{\varphi_\mu} x^* f)(s_\alpha) \to (T_{\varphi_\mu} x^* f)(s).$$

Since

$$x^* T_\mu f(s_\alpha) = x^* \mu(L_{s_\alpha} f) = \varphi_\mu(x^* L_{s_\alpha} f) = \varphi_\mu(L_{s_\alpha} x^* f) = (T_{\varphi_\mu} x^* f)(s_\alpha)$$

and

$$x^* T_\mu f(s) = (T_{\varphi_\mu} x^* f)(s),$$

$(T_\mu f)(s_\alpha) \to (T_\mu f)(s)$ weakly.

Since $f(S)$ is compact and $(T_\mu f)(S) = \mu(L_S f) \subset \overline{co} f(S)$, which is also compact (Mazur's theorem [2, A.1]), we can assume without loss of generality that $\{(T_\mu f)(s_\alpha)\}$ converges to an $x \in X$. Therefore $(T_\mu f)(s) = x$. This shows that $T_\mu f \in \mathcal{C}(S,X)$.

Similarly $U_\mu f \in \mathcal{C}(S,X)$. The proof is finished.

Proposition 13.3. *Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S,X)$ containing the constant functions and let $\epsilon : S \to M(\mathcal{A})$ be the evaluation mapping. Then*

(1) *for each $\mu \in M(\mathcal{A})$, $T_\mu : \mathcal{A} \to \mathcal{B}(S,X)$ is a bounded linear transformation with $\|T_\mu\| \leq \|\mu\|$;*

(2) *the mapping* $\mu \to T_\mu : M(\mathcal{A}) \to \mathcal{L}(\mathcal{A}, \mathcal{B}(S,X))$ *is a bounded transformation;*

(3) *if* $\mu \in M(\mathcal{A})$, *then* $T_\mu(x) = x$, $x \in X$;

(4) *for all* $s \in S$ *and* $\mu \in M(\mathcal{A})$

$$T_\mu L_s = L_s T_\mu$$

$$T_\mu R_s = T_{R_s^* \mu}$$

$$T_{\varepsilon(s)} = R_s,$$

where $R_s^* : M(\mathcal{A}) \to M(\mathcal{A})$ *is the adjoint of* R_s;

(5) *if* $f \in \mathcal{A}$, *then* $\{T_\mu f : \mu \in M(\mathcal{A})\}$ *is the closure in* $\mathcal{B}(S,X)$ *of* $co(R_S f)$ *in the topology of pointwise convergence on* S.

The proof of the proposition above is like that for [2, 2.2.3], so we omit it.

Definition 13.4. *Let* $\mathcal{A}$ *be a translation invariant linear subspace of* $\mathcal{B}(S,X)$ *containing the constant functions, and define*

$$Z_T = \{\nu \in \mathcal{L}(\mathcal{A}, X) : \; T_\nu \mathcal{A} \subset \mathcal{A}\}$$

and

$$Z_U = \{\mu \in \mathcal{L}(\mathcal{A}, X) : \; U_\mu \mathcal{A} \subset \mathcal{A}\}.$$

If $\mu \in \mathcal{L}(\mathcal{A}, X)$ *and* $\nu \in Z_T$, *define* $\mu\nu$: $\mathcal{A} \to X$ *by*

$$\mu\nu(f) = \mu(T_\nu f) \qquad (f \in \mathcal{A}).$$

If $\mu \in Z_U$ *and* $\nu \in \mathcal{L}(\mathcal{A}, X)$, *define* $\mu * \nu$: $\mathcal{A} \to X$ *by*

$$\mu * \nu(f) = \nu(U_\mu f) \qquad (f \in \mathcal{A}).$$

Definition 13.5. *An admissible subspace $\mathcal{A}$ of $\mathcal{B}(S,X)$ is a norm closed, translation invariant, left introverted subspace of $\mathcal{B}(S,X)$ containing the constant functions. In the case that $X = \mathbb{C}$, an admissible subspace $\mathcal{A} \subset \mathcal{B}(S)$ is also required to be conjugate closed.*

Let S be a semigroup. Define $\rho_t : S \to S$ and $\lambda_t : S \to S$ by

$$\rho_t = st, \quad \lambda_t = ts \qquad (s \in S).$$

S is called a right topological semigroup if it is a topological space and ρ_t is continuous for all $t \in S$. Set

$$\Lambda(S) = \{s \in S : \lambda_s \text{ is continuous}\}.$$

An affine semigroup S is a semigroup and a convex subset of a vector space in such a way that ρ_t and λ_t are affine mappings for each $t \in S$. The requirement that ρ_t and λ_t be affine means that if r, $s \in S$ and a, $b \in [0,1]$ with $a + b = 1$ then

$$(ar + bs)t = art + bst \text{ and } t(ar + bs) = atr + bts,$$

where $(+)$ denotes vector addition.

The following lemma summarizes the properties of the operation $(\mu,\nu) \to \mu\nu$. The proof is similar to that of [2, 2.2.9]. We omit the statements of the corresponding properties of the operation $(\mu,\nu) \to \mu * \nu$.

Lemma 13.6. *Let $\mathcal{A}$ be as in Definition 13.4 and let $\epsilon : \mathcal{A} \to X$ be the evaluation mapping. Then*

(1) *Z_T is a linear subspace of $\mathcal{L}(\mathcal{A}, X)$ containing $\epsilon(S)$;*

(2) $\mu\nu \in \mathcal{L}(\mathcal{A}, X)$ *for all* $\mu \in \mathcal{L}(\mathcal{A}, X)$ *and* $\nu \in Z_T$;

(3) *if* $\mu \in \mathcal{L}(\mathcal{A}, X)$, $\nu \in Z_T$ *and* $s \in S$, *we have*

$$T_{\mu\nu} = T_\mu \circ T_\nu,$$

$$\varepsilon(s)\nu = L_s^*\nu,$$

$$\mu\varepsilon(s) = R_s^*\mu, \text{ and}$$

$$\|\mu\nu\| \leq \|\mu\|\|\nu\|,$$

where $L_s^* : M(\mathcal{A}) \to M(\mathcal{A})$ *is the adjoint of* L_s;

(4) Z_T *is a right topological semigroup.*

The following result is essentially a consequence of the preceding lemma and Propositions 12.5 and 12.6.

Theorem 13.7.

(1) *If* $\mathcal{A}$ *is an admissible subspace of* $\mathcal{B}(S, X)$, *then for* τ_s *or* τ_w, *and multiplication* $(\mu, \nu) \to \mu\nu$, $M(\mathcal{A})$ *is a right topological affine subsemigroup of* $\mathcal{L}(\mathcal{A}, X)$, $co(\epsilon(S)) \subset \Lambda(M(\mathcal{A}))$ *and* $\epsilon : S \to M(\mathcal{A})$ *is a homomorphism.*

(2) *If we also assume that* $f(S)$ *is (weakly) relatively compact for all* $f \in \mathcal{A}$, *then* $M(\mathcal{A})$ *is also compact for* (τ_w) τ_s.

Let S be a compact semitopological semigroup. By Example 13.2, $\mathcal{C}(S, X)$ is introverted. Hence $\mu\nu$, $\mu * \nu \in M(\mathcal{C}(S, X))$; indeed, they are equal.

Proposition 13.8. *Let* S *be a compact semitopological semigroup and let* $\mathcal{A} = \mathcal{C}(S, X)$. *Then*

(1) $\mu\nu = \mu * \nu$ *for all* $\mu, \nu \in M(\mathcal{A})$;

(2) for τ_s *and multiplication* $(\mu, \nu) \to \mu\nu$, $M(\mathcal{A})$ *is a compact semitopological affine semigroup;*

(3) if S *is also a topological semigroup, so is* $M(\mathcal{A})$ *in* τ_s.

Proof. (1). By Theorem 12.7, the equality of $\mu\nu$ and $\mu * \nu$ is equivalent to the equality of $\varphi_{\mu\nu}$ and $\varphi_{\mu * \nu}$. Since on $\mathcal{C}(S) = M(\overline{sp}(X^* \circ \mathcal{A}))$, $\varphi_\mu \varphi_\nu = \varphi_\mu * \varphi_\nu$ [2, 2.2.12], it suffices to show that

$$\varphi_{\mu\nu} = \varphi_\mu \varphi_\nu.$$

$$\varphi_{\mu * \nu} = \varphi_\mu * \varphi_\nu.$$

In fact, by (12.1)

$$\varphi_{\mu\nu}(x^* f) = x^*(\mu\nu f) = x^* \mu(T_\nu f) = \varphi_\mu(x^*(T_\nu f))$$

and

$$\varphi_\mu \varphi_\nu(x^* f) = \varphi_\mu(T_{\varphi_\nu} x^* f),$$

while

$$(T_{\varphi_\nu} x^* f)(t) = \varphi_\nu(L_t x^* f) = \varphi_\nu(x^* L_t f) = x^* \nu(L_t f) = x^* T_\nu f(t).$$

This shows that

$$x^* T_\nu f = T_{\varphi_\nu} x^* f.$$

So $\varphi_{\mu\nu} = \varphi_\mu \varphi_\nu$. Analogously, we see that $\varphi_{\mu * \nu} = \varphi_\mu * \varphi_\nu$.

(2) is a consequence of Theorem 13.7 (1) and (2).

To verify (3), let $f \in \mathcal{A}$ and define a mapping $V : M(\mathcal{A}) \to \mathcal{A}$ by $V(\mu) = T_\mu f$. Note that for any μ , ν, μ_0 , $\nu_0 \in M(\mathcal{A})$ and $f \in \mathcal{A}$,

$$\|\mu\nu(f) - \mu_0\nu_0(f)\|$$

$$\leq \|\mu\nu(f) - \mu\nu_0(f)\| + \|\mu\nu_0(f) - \mu_0\nu_0(f)\|$$

$$\leq \|T_\nu f - T_{\nu_0} f\| + \|\mu(T_{\nu_0} f) - \mu_0(T_{\nu_0} f)\|.$$

To show that $M(\mathcal{A})$ is a topological semigroup it suffices to show the mapping V is continuous.

Since S is a topological semigroup, the mapping $s \to R_s f : S \to \mathcal{A}$ is continuous, so $R_S f$ is norm compact in $\mathcal{A}$, as is $\overline{co} R_S f$. Since V is continuous if $M(\mathcal{A})$ has τ_s and $\mathcal{A}$ has the topology of pointwise convergence, $V(M(\mathcal{A}))$ is pointwise compact and equals $\overline{co} R_S f$ (Proposition 13.3 (5)), so V is continuous.

§14. Invariant Vector–Valued Means

S denotes a semigroup which need not have an identity and $\mathcal{A}$ denotes a linear subspace of $\mathcal{B}(S, X)$ containing the constant functions. Let $LIM(\mathcal{A})$ ($RIM(\mathcal{A})$) denotes the set of left (right) invariant means on $\mathcal{A}$. $\mathcal{A}$ is said to be left (right) amenable if $LIM(\mathcal{A}) \neq \phi$ ($RIM(\mathcal{A}) \neq \phi$). If $\mathcal{A}$ is translation invariant , we set

$$IM(\mathcal{A}) = LIM(\mathcal{A}) \cap RIM(\mathcal{A})$$

and call members of $IM(\mathcal{A})$ invariant means. $\mathcal{A}$ is said to be amenable if $IM(\mathcal{A}) \neq \phi$.

As in the scalar case, we have the following proposition, whose proof is similar to that of [2, 2.3.5]; so we omit it.

Proposition 14.1. *Let $\mathcal{A}$ be an admissible subspace of $\mathcal{B}(S,X)$ and let $\epsilon : S \to \mathcal{L}(\mathcal{A},X)$ be the evaluation mapping.*

(1) *$LIM(\mathcal{A})$ is the set of right zeros of $M(\mathcal{A})$; hence if $\mathcal{A}$ is left amenable , then $LIM(\mathcal{A})$ is a closed ideal of $M(\mathcal{A})$ contained in every right ideal.*

(2) *If $\mathcal{A}$ is right amenable, then $RIM(\mathcal{A})$ is a closed left ideal of $M(\mathcal{A})$.*

Corollary 14.2. *Let $\mathcal{A}$ be an admissible subspace of $\mathcal{B}(S,X)$. If $\mathcal{A}$ is left and right amenable, then it is amenable.*

Proof. If $\mu \in LIM(\mathcal{A})$ and $\nu \in RIM(\mathcal{A})$, then $\mu\nu \in IM(\mathcal{A})$.

Corollary 14.3. *Let $\mathcal{A}$ be an admissible right introverted subspace of $\mathcal{B}(S,X)$ such that $\mu\nu = \mu * \nu$ for all $\mu, \nu \in M(\mathcal{A})$. Then $\mathcal{A}$ has at most one invariant mean.*

Proof. By the proposition and its right introverted analog, if $\mu, \nu \in IM(\mathcal{A})$, then $\nu = \mu\nu = \mu * \nu = \mu$.

Theorem 14.4. *Let $\mathcal{A}$ be an admissible subspace of $\mathcal{B}(S,X)$ suc' 'at, for each $f \in \mathcal{A}$, the range $f(S)$ of f is relatively weakly compact. Let $K(f)$ denote the closure in $\mathcal{B}(S,X)$ of $co(R_S f)$ for the pointwise topology. The following assertions are equivalent:*

(1) *$\mathcal{A}$ is left amenable;*

(2) *for each $f \in \mathcal{A}$, $K(f)$ contains a constant function;*

(3) *for each $f \in \mathcal{A}$, and $s \in S$, $0 \in K(f - L_s f)$.*

Furthermore, if (1) holds then, for each $f \in \mathcal{A}$, $\{\mu(f) : \mu \in LIM(\mathcal{A})\}$ is the set of constant functions in $K(f)$.

Proof. We omit the proofs that (1)⇒ (2) ⇒ (3) which do not use weak compactenss hypothesis. Here we show that (3) ⇒ (1).

For each $f \in \mathcal{A}$ and $s \in S$, let

$$M(f,s) = \{\mu \in M(A) : T_\mu(f - L_s f) = 0\}.$$

The sets $M(f,s)$ are τ_w-closed, and therefore τ_w-compact. For, let $\{\mu_\alpha\} \subset M(f,s)$ converge to $\mu \in M(\mathcal{A})$. We want to show that $\mu \in M(f,s)$, i.e.,

$$T_\mu(f - L_s f) = 0.$$

Note that

$$T_\mu(f - L_s f)(t) = \mu(L_t f - L_{ts} f) \qquad (t \in S)$$

and $\mu_\alpha(L_t f - L_{ts} f) = T_{\mu_\alpha}(f - L_s f)(t) = 0$ for all α. Since $\mu_\alpha(L_t f - L_{ts} f) \to \mu(L_t f - L_{ts} f)$ weakly, $\mu(L_t f - L_{ts} f) = 0$. That is, $T_\mu(f - L_s f) = 0$.

As in the proof of [2, 2.3.11], we can show that the family $\{M(f,s) : f \in \mathcal{A},\ s \in S\}$ has the finite intersection property. By Proposition 12.6 $M(\mathcal{A})$ is τ_w-compact. So

$$\bigcap\{M(f,s) : f \in \mathcal{A}, s \in S\} \neq \phi.$$

Let μ is any member of this section, then $\mu^2 \in LIM(\mathcal{A})$.

Let S be a group and let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S,X)$. For each $f \in A$ define $\check{f} : S \to X$ by

$$\check{f}(s) = f(s^{-1}) \qquad (s \in S),$$

and set

$$\check{\mathcal{A}} = \{\check{f} : f \in \mathcal{A}\}.$$

If $\mu \in M(\mathcal{A})$, define $\check{\mu} \in M(\check{\mathcal{A}})$ by

$$\check{\mu}(\check{f}) = \mu(f) \qquad (f \in \mathcal{A}).$$

If $\check{\mathcal{A}} = \mathcal{A}$ and $\check{\mu} = \mu$, then μ is said to be inversion invariant.

Theorem 14.5. *Let G be a compact Hausdorff topological group. Then $C(G, X)$ has a unique invariant mean μ. Furthermore μ is inversion invariant.*

Proof. The mean μ can be expressed as

$$\mu(f) = \int_G f d\nu \qquad (f \in C(G, X)),$$

where ν is normalized Haar measure on G; the properties of μ follows from those of ν.

The scalar version of the next theorem is [2, 2.3.14]; a similar result has appeared in [12], but there S is required to have an identity. A small modification of the proof of [2, 2.3.14] yields a proof of the present theorem.

Theorem 14.6. *Let S be a compact Hausdorff semitopological semigroup. Then the following assertions hold:*

(1) *$C(S, X)$ is left (respectively right) amenable if and only if S has a unique minimal right (respectively, left) ideal;*

(2) *$C(S, X)$ is amenable if and only if the minimal ideal of S is a compact topological group.*

§15. Vector–Valued Weakly Almost Periodic Functions

Let S be a semitopological semigroup; we do not assume S has an identity.

Let $\mathcal{WAP}(S,X)$ consist of those members f of $\mathcal{C}(S,X)$ for which the right orbit $R_S f = \{R_s f : s \in S\}$ is weakly relatively compact in $\mathcal{C}(S,X)$.

With a proof similar to that for [2, 4.2.5], one sees that the space $\mathcal{WAP}(S,X)$ is a closed translation invariant subspace of $\mathcal{C}(S,X)$. When $X = \mathbf{C}$, $\mathcal{WAP}(S,X)$ is just $\mathcal{WAP}(S)$, the C^*-algebra of weakly almost periodic functions on S. We note that

$$x^* \circ \mathcal{WAP}(S,X)) = \mathcal{WAP}(S) \qquad (x^* \in X^*,\ x^* \neq 0).$$

Recall that $\epsilon : S \to \mathcal{L}(\mathcal{A}, X)$ is the evaluation mapping $\epsilon(s)f = f(s), f \in \mathcal{WAP}(S,X)$. When $X = \mathbf{C}$ we denote this mapping by ϵ'.

Let $aS^{\mathcal{WAP}}$ denote the w^* closure in $\mathcal{WAP}(S)^*$ of $co\epsilon'(S)$; $aS^{\mathcal{WAP}}$ is a compact affine semitopological semigroup [2, 4.2.11].

Theorem 15.1. *Let S be a semitopological semigroup and let $\mathcal{A} = \mathcal{WAP}(S,X)$. The following assertions hold:*

(1) *$\mathcal{A}$ is an admissible subspace of $\mathcal{B}(S,X)$;*

(2) *for τ_w and multiplication $(\mu,\nu) \to \mu\nu$, $M(\mathcal{A})$ is an affine semitopological semigroup;*

(3) *if $f(S)$ is weakly relatively compact in X for each $f \in \mathcal{A}$, then $M(\mathcal{A})$ is τ_w-compact; in this case $\mathcal{A}$ is left amenable if and only if $\mathcal{WAP}(S)$ is left amenable.*

Proof. (1) Since $\mathcal{A}$ is a closed translation invariant subspace of $\mathcal{C}(S,X)$, to show that $\mathcal{A}$ is admissible we need to show that $\mathcal{A}$ is left introverted, i.e., if $f \in \mathcal{A}$ then $T_\mu f \in \mathcal{A}$ for all $\mu \in M(\mathcal{A})$.

Define $V : M(\mathcal{A}) \to \mathcal{B}(S,X)$ by

$$V(\mu) = T_\mu f \qquad (\mu \in M(\mathcal{A})).$$

By Proposition 13.3 (5)

$$(15.1) \qquad V(M(\mathcal{A})) = \overline{co}(R_S f),$$

the closure being taken in the pointwise topology. Since $f \in \mathcal{A}$, $co(R_S f)$ is weakly relatively compact in $\mathcal{A}$; in view of (15.1) this implies that $V(M(\mathcal{A}))$ is the weak closure in $\mathcal{A}$ of $co(R_S f)$. So $T_\mu f \in \mathcal{A}$ for all $\mu \in M(\mathcal{A})$.

(2) By Theorem 13.7 (1), for τ_w and multiplication $(\mu, \nu) \to \mu\nu$, $M(\mathcal{A})$ is a right topological affine semigroup. It follows from Theorem 12.7 that the mapping $\Pi : \mu \to \varphi_\mu$ is a τ_w–w^* homeomorphism of $M(\mathcal{A})$ into $aS^{\mathcal{WAP}}$. In much the same way as for $\varphi_{\mu\nu}$ in the proof of Theorem 13.8 (1), one can show that $\varphi_{\mu\nu} = \varphi_\mu\varphi_\nu$. Since $\Pi(\mu\nu) = \varphi_{\mu\nu}$, Π is a homomorphism too. So $M(\mathcal{A})$ is an affine semitopological semigroup because $aS^{\mathcal{WAP}}$ is.

(3) When $\mathcal{A}$ satisfies the compactness condition, the τ_w–compactness of $M(\mathcal{A})$ is a consequence of Theorem 13.7 (2). In this case, $M(\mathcal{A}) \cong aS^{\mathcal{WAP}}$. So we get the last statement.

The proof is complete.

Remark 15.2. For $f \in \mathcal{WAP}(S,X)$, in general $f(S) \subset X$ is not weakly relatively compact. However, if S admits an identity, it follows from the double limit property (e.g., [17, Theorem 3]) that $f(S)$ is weakly relatively compact. Of course, if X is reflexive then $f(S)$ is weakly relatively compact.

The following theorem was shown in [20, Theorem 4.11].

Theorem 15.3. *Let $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}, X)$. Define $F : \mathbb{R} \to X$ by $F(t) = \int_0^t f(u)du$. Then F is in $\mathcal{WAP}(\mathbb{R}, X)$ if and only if either*

(1) *$F(\mathbb{R})$ is weakly relatively compact in X or*

(2) *$c_0 \not\subset X$ and $F \in \mathcal{B}(\mathbb{R}, X)$,*

and the following limits exist and satisfy

$$\lim_{t \to \infty} \int_0^t \varphi(u)du = \lim_{t \to -\infty} \int_0^t \varphi(u)du. \tag{15.2}$$

Let $f \in \mathcal{WAP}(\mathbb{R}, X)$. Then $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{WAP}_0(\mathbb{R}, X)$, the subspace of $\mathcal{WAP}(\mathbb{R}, X)$ whose members have the zero function in the weak closure of the set of right translates [7, Theorem 4.11].

Since $\mathcal{C}_0(\mathbb{R}, X) \subsetneq \mathcal{WAP}_0(\mathbb{R}, X)$, Ruess and Summers pointed out in [20, p.33] that Theorem 15.3 does not answer the question: when is the integral of an $f \in \mathcal{WAP}(\mathbb{R}, X)$ again in $\mathcal{WAP}(\mathbb{R}, X)$?

The next theorem answers this question.

Theorem 15.4. *Let $f \in \mathcal{WAP}(\mathbb{R}, X)$. Define $F : \mathbb{R} \to X$ by $F(t) = \int_0^t f(u)du$. Then F is in $\mathcal{WAP}(\mathbb{R}, X)$ if and only if either*

(1) *$F(\mathbb{R})$ is weakly relatively compact in X, or*

(2) *$c_0 \not\subset X$ and $F \in \mathcal{B}(\mathbb{R}, X)$,*

and there is an $A \in X$ such that

$$\psi - A \in \mathcal{WAP}_0(\mathbb{R}, X), \tag{15.3}$$

where $\psi(t) = \int_0^t \varphi(u)du$ for $t \in \mathbb{R}$.

Proof. Set $G(t) = \int_0^t g(u)du$ for $t \in \mathbf{R}$. With a proof similar to that for Lemma 10.6, we can show that if $F(\mathbf{R})$ is bounded (weakly relatively compact) in X, then the same is true for $G(\mathbf{R})$.

Necessity. Since $F \in \mathcal{WAP}(\mathbf{R}, X)$, by Remark 15.2 $F(\mathbf{R})$ is weakly relatively compact, as is $G(\mathbf{R})$. So $G \in \mathcal{AP}(\mathbf{R}, X)$ [14, Theorem 2].

Now we show that (15.3) holds for some $A \in X$. Since $\psi \in \mathcal{WAP}(\mathbf{R}, X)$,

$$\psi = G_1 + \Phi_1, \tag{15.4}$$

where $G_1 \in \mathcal{AP}(\mathbf{R}, X)$ and $\Phi_1 \in \mathcal{WAP}_0(\mathbf{R}, X)$ ([7, Theorem 4.11]). We claim that G_1 is a constant function. For, suppose that there are t_1, $t_2 \in \mathbf{R}$ such that $G_1(t_1) \neq G_1(t_2)$. The almost periodicity of G_1 makes it possible to assume that t_1 and t_2 are in $\mathbf{J}_0$. Then we can find a $x^* \in X^*$ such that $x^*G_1(t_1) \neq x^*G_1(t_2)$. At the same time, since $\psi|_{\mathbf{J}_0} \in \mathcal{WAP}(\mathbf{J}_0, X) \subset \mathcal{WPAP}(\mathbf{J}_0, X)$, by Theorem 11.6 and Remark 11.7 there are an $A_{x^*} \in \mathbf{C}$ and a $\Phi_{x^*} \in \mathcal{PAP}_0(\mathbf{J}_0)$ such that

$$x^*\psi(t) = \int_0^t x^*\varphi(u)du = A_{x^*} + \Phi_{x^*}(t) \qquad (t \in \mathbf{J}_0). \tag{15.5}$$

Comparing (15.5) with (15.4) and using the uniqueness aspect of Theorem 11.6, we conclude that x^*G_1 is a constant function, a contradiction.

Sufficiency. It is obvious that $\psi \in \mathcal{WAP}(\mathbf{R}, X)$ if (15.3) holds for some $A \in X$. By the first paragraph in the proof and [14, Theorems 1, 2], either of (1) and (2) is a sufficient condition for G to be in $\mathcal{AP}(\mathbf{R}, X)$. So $F \in \mathcal{WAP}(\mathbf{R}, X)$.

Remark 15.5. Theorem 15.3 is a corollary of Theorem 15.4. In fact, in Theorem 15.3 $A = \lim_{t\to\infty} \int_0^t \varphi(u)du$.

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