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Sonata In Contracting

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ABSTRACT

This thesis comprises two essays linked by their focus on problems in contracting and by their usage of game theory as the vehicle of analysis.

The first essay addresses the issue of how and why incomplete contracts might arise endogenously. It provides a model of contract formation that focuses on the differential bargaining power that is bestowed upon agents by the procedures implied by different contract settings. The model employs a multi-issue bargaining approach, and distinguishes between issue-by-issue bargaining, where issues are dealt with separately, and single-issue bargaining, where they are combined. Agents are free to bargain over the form of the equilibrium process. It is shown that this structure allows for incomplete contracts, in the form of issue-by-issue, or short-term agreements, to be derived as an equilibrium outcome for some environments. This is in contrast to much of the literature on incomplete contracts, which relies on the roles of unobservability or transaction costs in order to justify the imposition of incomplete contracts as equilibrium contract form by the modeler.

The second essay analyses an alternating offers bargaining game in which the payoff in every period in which no agreement has been reached is the outcome of a normal form stage game. Two insights are gained from this model: *i)* only disagreement period opportunities available to a player when he makes an accept/reject decision can increase his game payoffs, and *ii)* in general such negotiation games have many equilibria which are Pareto inefficient, even though the game is one of complete information and full rationality. There exist, however, stage games which lead to a unique efficient outcome if exit weakly dominates repeated play. An alternative interpretation of this model — relevant to implicit

contracts — is as a repeated game with endogenous exit. In this context the model points to the restrictions imposed on equilibrium payoffs in potentially infinitely repeated games by the existence of the possibility of binding exit agreements. The set of supportable allocations, however, is generally smaller than the Folk Theorem literature would suggest.

In memoriam Gert v. Transehe-Roseneck

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OVERVIEW

In what follows, the two essays in this thesis will be discussed separately and disjointly. While the two essays have no direct topical relation to each other, there do exist unifying aspects between them. One is the use of non-cooperative game theory as the chosen mode of analysis. Both models employ bargaining theory, and thus bear intellectual debt to the work of Rubinstein, whose 1982 paper is referred to extensively and provides the foundation for both models.

The second unifying aspect of the papers is to which problems this technology is applied. Both papers share a common concern about contracting issues. In the first paper, entitled *Endogenous Incomplete Contracts*, this takes the form of an explicit analysis of complete versus incomplete contracting procedures. As will become apparent later, the essay shows that while the intuition that frictions are necessary for incomplete contracts to arise in equilibrium is correct, the existence of differential costs in favour of incomplete contracts is not necessary to cause that outcome. This contradicts apparently widely held beliefs, and serves as a warning to those modelers who would assume a particular kind of incomplete contract after vague reference to "costs". The model allows agents to chose endogenously between two methods of allocating a sequence of two surpluses. One method is to treat both issues as one large allocation problem, and agree at once on an allocation for both the current and the future surplus. The alternative method is a piecemeal allocation rule, whereby first only the first surplus is allocated, and then, once the second surplus arrives, it in turn is allocated. The model shows that in some environments one agent may benefit if the issues are separated, and that, although in general such separate allocations are inefficient, he may probabilistically attain his favoured outcome in equilibrium.

The second paper, *Perfect Equilibria in a Negotiation Model*, is not overtly targeted to the discussion of contracts. It does, however, have implications for implicit contracts. The model is presented as an alternative to standard bargaining models, with the innovative part being that the disagreement payoffs of players are determined endogenously, in contrast to the usual assumption that they are exogenous and fixed. This is formally modelled by having players play a stage game each time they have not reached an agreement in an alternating offers bargaining game. The set of subgame perfect equilibria is characterized, and it is shown that multiple equilibria can exist, confirming the findings of Fernandez and Glazer (1991) and Haller and Holden (1990), who have a similar model. In contrast to their results, sufficient conditions are provided which guarantee a unique equilibrium.

While presented as a negotiation model, this game can be reinterpreted as a model of implicit contracts when binding agreements also exist. In this interpretation the model becomes one of repeated games with endogenous exit, where exit is valuable, but must be mutually agreed. It differs from the model of Okada (1991) by having separate frontiers for the implicit contracts and the explicit contracts. In any case, the model implies that the Folk Theorem results may not hold if the time horizon is endogenous and a valuable outside option exists. Having said this, this aspect of the model will not be stressed in what follows.

Chapter 1

Endogenous Incomplete Contracts

> When there are two objects to negotiate, the decision to negotiate them *simultaneously* or in separate forums *at separate times* is by no means neutral to the outcome, particularly if there is a latent extortionate threat that can be exploited. ... The protection against extortion depends on refusal, unavailability, or inability to negotiate. (Thomas Schelling, "An Essay on Bargaining", 1956. Italics added)

1.1 Introduction

Traditionally, economists have assumed that transactions among individuals take place within the context of a complete set of markets. When idiosyncratic exchange between individuals is at issue, the assumption has been that the exchange is governed by a complete contract. Recently, however, attention has turned to situations in which exchange takes place with less than complete markets and contracts. In the area of contracts, the focus has been on three issues: the impact that different forms of incompleteness have on the allocation of goods (see, for instance, Grossman and Hart (1986), Hart and Moore (1990)), the role that contract renegotiation plays in situations where contracts are incomplete (see Huberman and Kahn (1988), Hart and Moore (1988), Ma (1991)), and the impact on the allocation of goods when agents are unable to commit to long-term contracts (see Crawford (1988), Fudenberg, Holmstrom and Milgrom (1990), Laffont and Tirole (1988)). While dealing with many diverse aspects of exchange when contracts are incomplete, one feature that all of this research has in common is the fact that the nature of the contractual incompleteness is given exogenously. While reference is usually made to various costs, either of third-party verification or of describing complex states of the world, which make complete contracts infeasible, no attempt is made to either model these costs explicitly or to determine their impact on the form of the contract.

Although few would deny the importance of these lines of research, the failure of these models to account adequately for the existence of incomplete contracts does pose problems. If, for instance, the cost of writing state contingent contracts is positive but not "prohibitive", should one expect to see the sorts of incomplete contracts posited by these models? If so, in what environments should one expect them to arise; if not, what form should the contractual incompleteness take (if contracts are indeed incomplete)? Equally important is the question of how the form of the equilibrium contract might be expected to vary as the economic environment varies. It, too, is left unanswered. There is, therefore, no way to test whether these models are useful descriptions of economic reality.

The purpose of this paper is to provide a model of the transactions process between individuals in which the structure of the contract which governs transactions is determined endogenously. Without denying the important role played by the direct costs of writing "complex" contracts, such costs are not the focus of the model. Rather, the model focuses on the impact that the selective inclusion of certain items into contracts has on the ability of an agent to bargain successfully (i.e. to obtain an agreement favourable to the agent). As will be shown, these indirect cost can vary across contract forms since different contract structures imply different costs of "holding out" for a favourable deal. Depending on the characteristics of the agents, certain types of contracts may in this sense be more costly for one agent than for another. An agent will seek to implement a contract that is more favourable (less costly) to that agent, and the interplay between agents seeking the most favourable deal determines the equilibrium contract structure.

In order to highlight the role played played by these differential bargaining costs, the model is purposely simple. The model considers a pure exchange economy in which two agents must decide how to allocate one unit of each of two goods, the endowment of each good occurring at different points in time. The transactions technology by which all decisions are made is an offer-counter-offer bargaining process in the style of Rubinstein (1982). The agents can either adopt a process whereby the allocation of both goods is determined simultaneously (a process observationally equivalent to a complete contract) or employ one in which the allocation of the good arriving first is determined in a bargaining round separate and sequentially prior to that determining the allocation of the good arriving later. In this latter procedure, the success or failure of the later bargain does not affect the implementation of the earlier agreement. This latter procedure is the observational equivalent of an incomplete contract. The form that the contract takes (i.e. the bargaining process adopted) is also determined by an offer-counter-offer bargaining process. Agents are assumed to know the sizes of the endowments, their arrival dates and each other's preferences, and there are no costs to any of the contracting processes other than costs of delay. In its most simple form, the model assumes identical costs of delay for all processes and endowments that occur with certainty.

The model points to several features which are key to obtaining the incomplete contract outcome. First, it is necessary that there be some friction in the transactions process in the sense that delay costs must be positive. If bargaining is frictionless, then the complete contract is always implemented in equilibrium. Second, it is not necessarily the case that the complete contract having larger delay costs (being more costly to implement) than the incomplete contract makes the incomplete contract a more likely outcome. Finally, it is, on the other hand, not necessary that the complete contract procedure involve greater frictions (larger delay costs) than the incomplete contract procedure in order for incomplete contracts to be observed. Even if each is equally costly, differences in the agents' preferences regarding the two goods can produce incomplete contracts. If, for instance, one agent prefers the first good relative to the second (and vice versa for the other agent), the agent preferring the first good will seek an incomplete contract because this procedure allows that agent to extract more of the second good than is possible under a complete contract procedure. The reason that the agent can obtain more of the second good is as follows. Under a complete contract efficiency requires relatively more of an agent's preferred good to be allocated to him. The agent who prefers the first good relatively more can obtain more of the second only if he can delay and hold out for it — but this delays consumption of the first good. This delay is relatively more costly to him than the other agent since he is allocated a larger share. With the incomplete contract, the agent's consumption of the first good is already determined, and independent of an agreement on the second good. Therefore, delaying agreement on the allocation of the second good is less costly to him and he can obtain a larger share.

Several other authors have also modelled the process by which the structure of contracts is determined as an equilibrium phenomenon. Dye (1985) considers a simple cost model in which additional clauses in a contract result in additional costs to the contracting parties. He shows that incomplete contracts arise in this environment. Lipman (1991) has a cost of contracting model in which states can only be determined at some cost, so that a contract specifying an allocation at some state is costly to write because the state must first be determined before the contract can be specified. Lipman shows that, even as these costs of observing states become arbitrarily small, incomplete contracts still arise in equilibrium. Finally, Allen and Gale (1990) show that, if agents cannot write contracts contingent on states of nature (presumably because this is too costly) but only on noisy signals of these states, then it is possible that the agents will choose noncontingent contracts in equilibrium. This outcome arises because of both the ability of one of the agents to manipulate the noisy signal and the existence of incomplete information about this agent's type.

Rather than viewing our model as an alternative to these cost models, we see it as a complementary analysis that supplements our understanding of the role of contracting costs in producing incomplete contracts. That is, not only are the direct costs of writing contracts important but also important are the indirect costs that various types of contracts impose on agents via their effect on the agents' ability to obtain a favourable deal within that framework.

The structure of the paper is as follows. The next section sets out the basic model and solves for the equilibrium contract in the case in which agents' preferences are linear. Section 3 extends the model to more general preferences as well as to situations in which there are different (exogenous) delay costs for the different contracts and uncertainty in the endowment process of the second good. Section 4 contains a discussion and concluding remarks, while an Appendix contains proofs for a number of the results in the text.

1.2 The Contracting Problem: An Example[1]

1.2.1 The Model

Consider a situation in which there are two agents, 1 and 2, endowed jointly with a single unit of each of two distinct goods, X and Y. Further, assume that the endowment process is such that the agents obtain X sequentially prior in time to Y, with the dates of the endowments' arrival known with certainty by both agents and given by t_X and t_Y respectively ($t_X \geq 1$). Agents determine (by a process to be specified below) an allocation of X and Y between them, with agent 1's share of X given by x and his share of Y given by y (agent 2's shares being $(1-x)$ and $(1-y)$ respectively). The agents' preferences over an allocation (x, y) with x consumed at date $t \geq t_X$ and y consumed at date $\tau \geq t_Y$ are given by the utility functions

$$U_1(x, t, y, \tau) = \delta^{(t-1)}ax + \delta^{(\tau-1)}y \tag{1}$$

$$U_2(x, t, y, \tau) = \delta^{(t-1)}(1-x) + \delta^{(\tau-1)}b(1-y) \tag{2}$$

Here a and b are constants assumed to satisfy the conditions $a \geq 1$ and $b \geq 1 + 1/a$, while $\delta \in (0,1)$ is the agents' common discount factor. These functions are standard time separable utility functions with the restriction to $a, b \geq 1$ implying that, were the agents to consume X and Y at the same time, agent 1's marginal utility from X would be larger than from Y (i.e. agent 1 prefers X to Y) while the opposite would be true for agent 2 (i.e. agent 2 prefers Y to X). The implications of the stronger restriction that $b \geq 1 + 1/a$ will become apparent shortly.

All decisions on allocations in this world are assumed to be determined by offer-counter-offer bargaining processes in the style of Rubinstein (1982). Given that

[1]This example draws on work by Herrero (1989)

allocations must be determined for an entire endowment stream, there are a number of possible offer-counter-offer procedures that the players could adopt. From the structure of the endowment stream two natural procedures arise, however, and attention is restricted to these two for the purposes of this example. The first procedure involves bargaining over the allocation of the entire endowment stream at once. More precisely, an offer under this procedure is a pair (x, y) specifying a division of both goods. The two agents make offers and counter-offers of (x, y) until an agreement is reached, with agent 1 making the initial offer if the period t, at which bargaining begins is odd and agent 2 making the initial offer if t is even. Once agreement is reached, the agreed upon allocations of goods X and Y are implemented. Should agreement be reached before X or Y arrive, the agreed upon allocation is implemented once each endowment arrives. No allocations are made until agreement is reached on the division of both goods. This procedure is labelled the complete contract (CC) procedure, although, given that there is no uncertainty in this example, it is perhaps more aptly referred to as the long-term contract procedure. The procedure is described by Figure 1.

The second procedure involves a sequential determination of allocations, with the allocation of good X determined in a separate procedure from that determining the allocation of good Y. Given the structure of the endowment stream, the natural timing involves the two agents bargaining over the allocation of X first and over the allocation of Y subsequently. Agent 1 is assigned odd numbered periods to make his offers, while agent 2 is assigned even numbered periods. Thus, in the second procedure, if the period in which bargaining begins is odd, then it is assumed that agent 1 initially makes an offer, x, which agent 2 can either accept or reject. Should agent 2 reject the offer, then he makes a counter-offer, x', that agent 1 can either accept or reject, and so on (The procedure of offer and counter offer is reversed if the initial period is even.). Once the agents reach an agreement on an allocation of X bargaining ends, and the allocation is implemented. Bargaining between the agents resumes at time t_Y to determine an allocation of Y (Should agreement on X be reached at some time $t \geq t_Y$, then bargaining on Y begins immediately.). Again, agent 1 makes his offers in odd numbered periods while agent 2 makes his offers in even numbered periods. Bargaining on an allocation of Y proceeds in a fashion analogous to that for X (with an offer being a value, $y \in [0,1]$) and the allocation of Y is implemented once an agreement is reached.[2] This procedure is labelled the incomplete contract (IC) procedure, although again, given the lack of uncertainty in this example, it is perhaps more aptly referred to as the short-term-contract procedure. This procedure is illustrated in Figure 2.

Although there are other bargaining procedures one might imagine, CC and IC capture the essential differences between complete and incomplete contracts and are sufficient to illustrate the key issues. The first procedure corresponds to a complete contract in the sense that it requires the agents to commit, within a single contracting process, to allocations for all possible states of the world (In this example states of the world are just different times at which allocations arrive with certainty.). By determining all allocations simultaneously, agents are assured that the allocations will be efficient (i.e. on the Pareto frontier). However, this simultaneous allocation procedure also means that agents are not able to separate their attitude regarding the allocation of any one good from the allocation of the other good. The second procedure corresponds to an incomplete contract in that the allocations of X and

[2]Note that the restriction that bargaining on Y begins at t_Y is innocuous, since there is no cost to delay before t_Y.

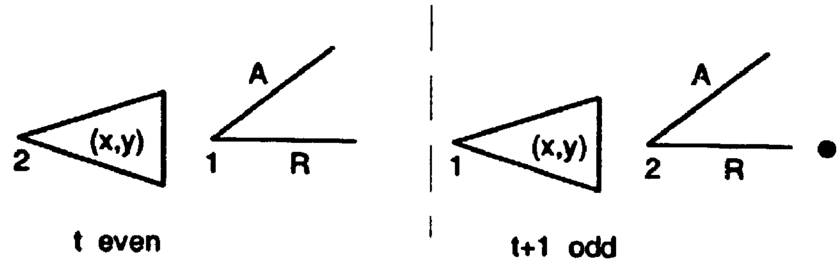

Figure 1: the CC procedure

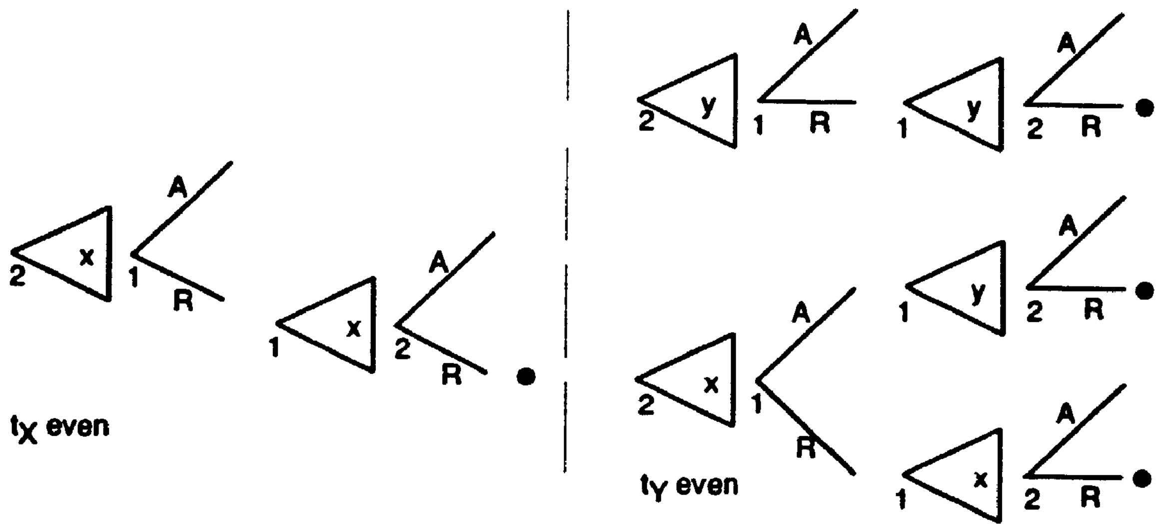

Figure 2: the IC procedure

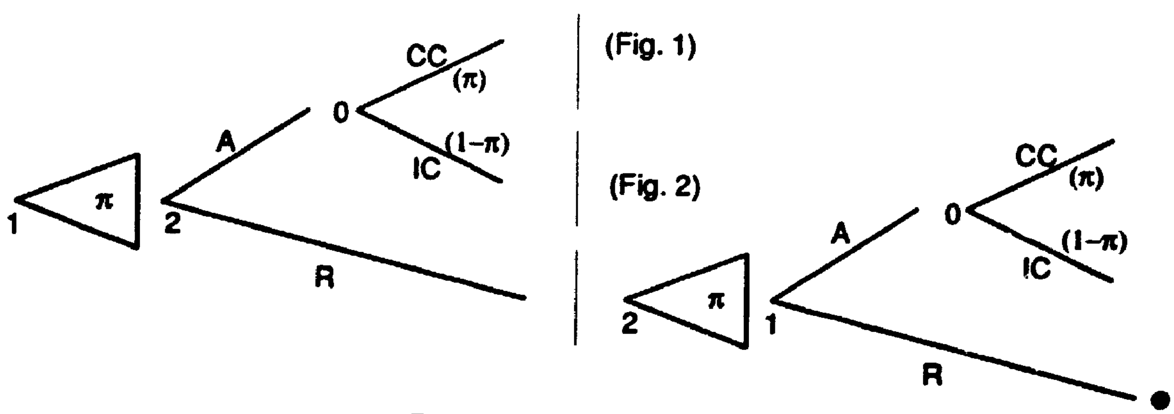

Figure 3: the contract bargain

Y are made piecemeal, with the allocation of Y determined only after the state of nature (in this case t_Y) is determined. Because of this piecemeal determination of allocations, the incomplete contract procedure will generally be inefficient (i.e. interior to the Pareto frontier, although not necessarily in this example). However, this procedure does allow the agents' attitudes toward any proposed allocation of Y to be separated from the allocation of X and so potentially alters (relative to the complete contract) the set of proposals, y, that are acceptable to each player.

Still undetermined, of course, is the means by which one of these two bargaining processes is adopted by the agents as the process by which allocations of X and Y are to be decided. Rather than one of the two being imposed exogenously, it is assumed that which contracting procedure is employed is also something about which the agents bargain. This bargaining takes place sequentially prior to any bargaining over allocations of X and Y. An offer in this bargaining round is a number $\pi \in [0, 1]$, where π represents the probability that the CC procedure is employed. The randomization scheme π is assumed contractable, and its outcome costlessly enforceable. In addition, its outcome is assumed to be known to the agents prior to entering into bargaining over X and Y.[3] As with the other bargaining rounds, it is assumed that the agent making the initial offer is determined by whether the initial period is odd (1 moves first) or even (2 moves first).

For the purposes of this example, it is assumed that the initial bargaining process begins at date $t = 1$ (the first time period) and that $t_X = 2$. Bargaining on allocations of X and Y begin the period after the bargaining over π is completed. Figure 3 provides a diagram that summarizes the complete bargaining process. The

[3] What is envisioned here is a procedure whereby the agents first decide what the subject of a particular bargaining round will be (i.e. what variables will be bargained over), and then decide the allocations for the variables in question.

predicted outcome of this bargaining process is given by the set of subgame perfect Nash equilibrium strategies for the game described by Figure 3. The outcomes associated with these strategies are described below.

1.2.2 Equilibrium Outcomes

Consider first the two possible bargaining procedures that determine allocations of X and Y (the set of potential subgames generated should agreement be reached on a value of π). Suppose that the realization of the randomization mechanism is such that the CC procedure is to be followed and that the current subgame has $t = t_Y$. Then, should agreement be reached in the current period or any subsequent period, the allocation of both X and Y will be made immediately, and the two agents' utilities at the time of agreement will be given by

$$U_1 = ax + y \tag{3}$$

$$U_2 = (1 - x) + b(1 - y) \tag{4}$$

The set of instantaneous utilities achievable from allocations (x, y) is simply given by all (U_1, U_2) consistent with (3) and (4) and is depicted in Figure 4 below. Because of the linearity of the utility functions, the utility frontier has the property that $y = 0$ for $U_1 \leq a$, while, $x = 1$ for $U_1 \geq a$.

Given that all offers (x, y) have corresponding utility offers (U_1, U_2) given by (3) and (4), bargaining in these subgames can be analyzed in terms of (U_1, U_2) offers drawn from the set depicted in Figure 4. More specifically, bargaining must take place over (U_1, U_2) given by the utility frontier $U(CC)$. Letting (U_1^i, U_2^i) be an offer from $U(CC)$ by agent i, an equilibrium offer must satisfy the conditions

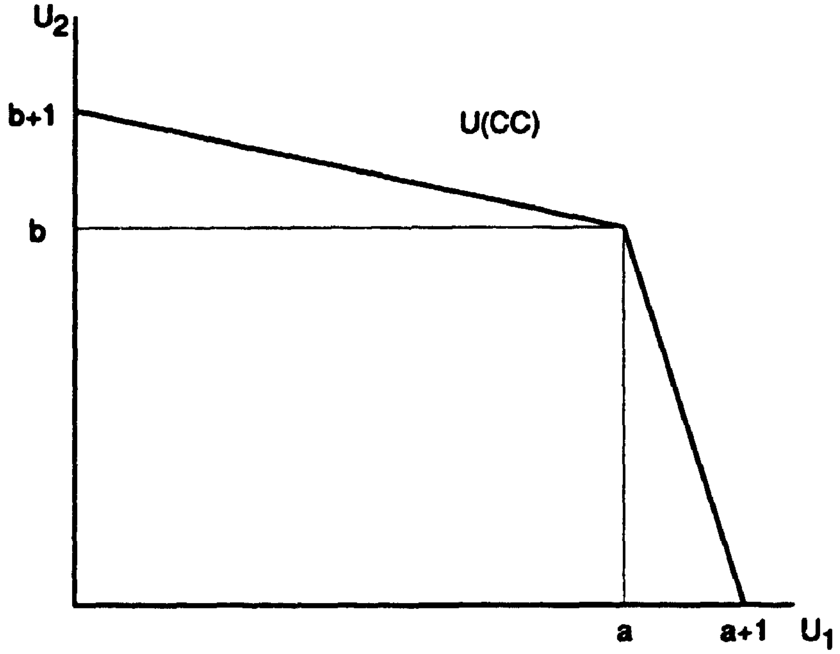

Figure 4: CC utility frontier

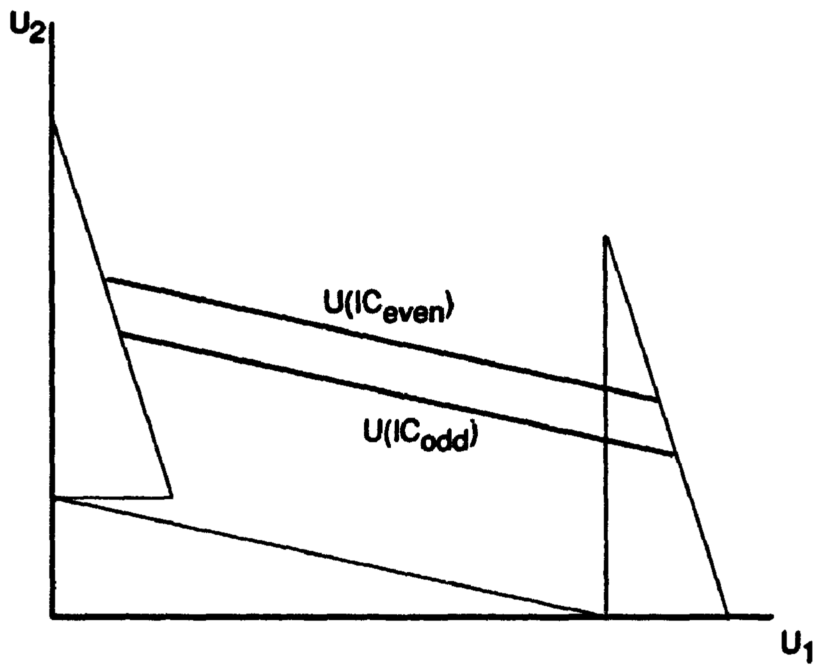

Figure 5: IC utility frontiers

$$U_1^2 = \delta U_1^1 \quad (5)$$

$$U_2^1 = \delta U_2^2 \quad (6)$$

Under the assumption that $a, b > 1$, it can easily be shown that an equilibrium offer by agent 1 involves $x = 1, \quad y > 0$, while an equilibrium offer by 2 involves $y = 0, \quad x < 1$. That is, 1's offer lies on $U(CC)$ to the right of the kink while 2's offer lies on $U(CC)$ to the left of the kink. Employing these facts in (5) and (6) yields the equilibrium offers

$$x^* = 1, \qquad y^* = \frac{a(1-\delta)(b-\delta)}{ab-\delta^2} \quad (7)$$

$$x^{**} = \frac{\delta(b(a+1)-\delta(b+1))}{ab-\delta^2}, \qquad y^{**} = 0 \quad (8)$$

should agent 1 or agent 2 make the offer, respectively.

The situation is somewhat different in subgames for which $t < t_Y$. In these subgames, delay in reaching an agreement is only costly in terms of foregone consumption of X. Delay does not result in foregone consumption of Y because the endowment of Y is not available until t_Y. As a consequence, the analysis of equilibrium offers in these cases must be slightly modified from the above to account for the different cost of delay. The modification simply involves a backward induction process from t_Y employing conditions analogous to (5) and (6) to determine the sequence of equilibrium offers and counter offers. This process yields an initial offer by agent 2 at time $t = t_X$ given by

$$\hat{x} = \delta \sum_{i=0}^{(t_Y-t_X-1)} (-\delta)^i + \delta^{t_Y-t_X} x^{**}, \qquad \hat{y} = 0 \quad (9)$$

$$\hat{x} = \delta \sum_{i=0}^{(t_Y-t_X-2)} (-\delta)^i + \delta^{t_Y-t_X-1} x^{**}, \qquad \hat{y} = 0 \quad (10)$$

if t_Y is even or odd, respectively. (This construction is much the same as that in Shaked and Sutton (1985).) The pair $(\hat{x},\ \hat{y})$ represents the equilibrium allocation of X and Y under the CC procedure.

The analysis of the equilibrium allocations for the IC procedure is similar. Consider first the subgames in which an allocation of X has been determined and $t \geq t_Y$. This case is a simple Rubinstein bargaining problem with the equilibrium allocation, y_I^* given by the usual Rubinstein solution (i.e. $1/(1+\delta)$ if 1 makes the first offer and $\delta/(1+\delta)$ if 2 makes the first offer). Next, consider those subgames for which no agreement on an allocation of X has been reached. As with the CC procedure, this problem can be broken down into two parts: those cases for which $t \geq t_Y$ and those for which $t < t_Y$. In the former case, the utility for the two agents should an agreement be reached on an allocation of X is given by

$$U_1 = ax + \delta y_I^* \tag{11}$$

$$U_2 = (1-x) + \delta b(1-y_I^*) \tag{12}$$

In Figure 5 the set of attainable utilities from bargaining over X are depicted. As in the case of the CC procedure, delay in reaching agreement on an allocation of X imposes costs both in terms of foregone consumption of X and of Y when $t \geq t_Y$. Therefore, equilibrium offers are ones satisfying conditions (5) and (6) above with the set of possible utility offers drawn from the IC frontier in figure 5. These offers result in equilibrium allocations of X given by $x_I^* = (1+b\delta-\delta^2/a)/(1+\delta)$ if t is odd and $b < 1+\delta/a$, and $x_I^* = 1$ if $b > 1+\delta/a$. If t is even the respective allocations are $x_I^{**} = \delta(1+\delta b-1/a)/(1+\delta)$ if $b < 1+\delta/a$ and $x_I^{**} = \delta(a-1+\delta)/a$ if $b > 1+\delta/a$. The initial restriction that $b > 1+1/a$ implies that the equilibrium allocations when

t is either odd or even is given by the latter values, i.e.

$$x_I^* = 1 \qquad \text{and} \qquad x_I^{**} = \frac{\delta(a-1+\delta)}{a}$$

For those cases in which $t < t_Y$, the same modification used in the CC procedure applies. Again, delay in these subgames is only costly in terms of foregone X consumption and not foregone consumption of Y. Therefore, the allocation of X at t_X is determined by the same backward induction process as used in the CC procedure. This process yields an allocation of X at time t_X given by $\hat{x}_I = \delta \sum_{i=0}^{t_Y-t_X-2}(-\delta)^i$ if t_Y is even and by $\hat{x}_I = \delta \sum_{i=0}^{t_Y-t_X-1}(-\delta)^i - \delta^{t_Y-t_X}\frac{(1-\delta)}{a}$ if t_Y is odd. The complete allocation under the IC procedure is then given by

$$\hat{x}_I = \delta \sum_{i=0}^{t_Y-t_X-2}(-\delta)^i, \qquad y_I^* = \delta/(1+\delta) \tag{13}$$

$$\hat{x}_I = \delta \sum_{i=0}^{t_Y-t_X-1}(-\delta)^i - \delta^{t_Y-t_X}\frac{(1-\delta)}{a}, \qquad y_I^* = 1/(1+\delta) \tag{14}$$

if t_Y is even or odd, respectively.

At this point, a comparison of the allocations under the two bargaining procedures may prove informative. Those for the CC procedure are given by one of (9) or (10) above while for the IC procedure they are given by one of (13) or (14). Clearly, agent 1 gains under the IC procedure as regards the allocation of Y. This gain results from the fact that agent 1 has been able to split-off bargaining over X from bargaining over Y, and do so in a way that implies that delay in reaching an agreement on an allocation of Y is not costly to 1 in terms of consumption of X (the good that 1 prefers). As a consequence, it pays agent 1 to hold out for a positive share of Y in the IC procedure whereas such behavior is too costly in terms of foregone consumption of X under the CC procedure.

As regards the allocation of X, the appropriate comparisons are (9) versus (13) and (10) versus (14). A comparison of (10) and (14) shows that the value of $\hat{x}$

(the allocation under the CC procedure) is larger than the value of $\hat{x}_I$ (the allocation under the IC procedure) if t_Y is odd. The reason for this outcome is simple; specifically, the fact that agent 1 receives more of good Y under the IC procedure means that it is less costly for agent 2 to delay agreement on X by holding out for a larger share of X (and correspondingly more costly for agent 1 to delay). As a consequence, agent 2's share of good X increases. A similar force is at work if t_Y is even, tending to make (9) larger than (13). However, countervailing this force is the fact that, should agent 2 reject 1's offer at $t_Y - 1$ and wait until t_Y to make a counter offer, then 2 gives agent 1 the first offer in the good Y bargaining round, whereas else 2 himself would have the first offer. As Y is the good preferred by agent 2 such delay is quite costly to agent 2.[4] Indeed, it is sufficiently costly for agent 2 that agent 1 actually obtains more of good X (as well as more of Y) than under the CC procedure.

Of course, the important comparison is in regard to the utilities that the two agents obtain under each procedure. Clearly, if t_Y is even agent 1's utility is higher under the IC procedure while agent 2's utility is higher under the CC procedure. If t_Y is odd, the agents' preferences are less immediately obvious. Their utilities may be determined by evaluating equations (1) and (2) at the allocations implied by the CC and IC procedures. Such an evaluation reveals that, for sufficiently large values of δ, agent 1's utility is higher under the IC procedure while agent 2's utility is higher under the CC procedure.[5] The implication of this result is that, for large enough δ,

[4]When t_Y is odd, then it is costly for 1 to reject 2's offer in $t_Y - 1$, making it even more costly for 1 to delay, thereby giving 2 additional bargaining power over X.

[5]The exact condition on δ guaranteeing that agent 1 prefers the IC procedure to the CC procedure is that $\delta^2/(1+\delta) \geq (1-\delta)((ba-\delta a)/(ba-\delta^2))$. For agent 2, the condition guaranteeing that CC is preferred to IC is $b/(1+\delta) \geq (1-\delta)/a + (1-\delta)((b-\delta)/(ab-\delta^2))$.

agent 1 always prefers an incomplete contract as the means of allocating X and Y while agent 2 always prefers a complete contract. This conflict in preferences over contracts means that the bargain over the contract form will be a nontrivial process.

Turning to this bargaining phase and letting U_i^I and U_i^C represent agent i's equilibrium utility levels under the IC and CC procedures respectively, an agent's expected utility for any offer π is given by

$$\mathbf{E}U_i = \pi U_i^C + (1 - \pi)U_i^I \tag{15}$$

An equilibrium offer is a pair (π^*, π^{**}), representing an offer and counter offer by agents 1 and 2 respectively, such that conditions analogous to (5) and (6) hold either as equalities if $0 < \pi^*$, $\pi^{**} < 1$ or as a strict inequality (>) for at least one of the two conditions if either of π^* or π^{**} equals 0 or 1.

A moment's reflection on the bargaining process makes it clear that, in fact, π^{**} equals 1. The reason for this outcome is simple. Both the points U_i^C and U_i^I are feasible offers in the CC bargaining procedure, as are all linear combinations of these points (by the convexity of the utility space). The offer U_1^C would be accepted by agent 1 in the CC bargaining procedure, being at least as good as the best counter offer 1 could make in the next period. As the set of possible counter offers in the contract bargaining round is a subset of the set of counter offers in the CC procedure, 1's best counter offer here can be no better than that in the CC procedure. Therefore, 1 will accept an offer of $\pi^{**} = 1$.

Given that $\pi^{**} = 1$, it is easy to calculate agent 1's equilibrium offer, π^*. This offer will be such that agent 2 is just indifferent between accepting it and waiting one period and offering $\pi^{**} = 1$ (as long as $\pi^* > 0$). This value of π is given by the expression

$$\pi^* = \frac{\delta U_2^C(3) - U_2^I(2)}{U_2^C(2) - U_2^I(2)} \tag{16}$$

where the numbers inside the parentheses indicate the period in which the initial offer is made in the subsequent bargaining procedure. This expression is strictly less than one for all $\delta < 1$. Clearly, for small enough values of δ this expression becomes negative, implying that $\pi^* = 0$.[6] In either case, the implication of (16) is that the IC procedure will be employed with positive probability in equilibrium.

To motivate some intuition about these results and which features of the model are driving them, it is helpful to consider the model first in the limit as δ approaches 1 and then when δ is constrained below 1. As δ approaches 1, the equilibrium allocation under the CC procedure approaches the point $x = 1$, $y = 0$; under the IC procedure, the allocation approaches the point $x = 1$, $y = 1/2$. These points are depicted in Figure 6, where the CC and IC frontiers are the ones given in Figures 4 and 5 above when $\delta = 1$. The assumption that $b > 1 + 1/a$ guarantees that $x = 1$ under the IC procedure, and thus the allocation under this procedure lies on the CC frontier (i.e. the IC allocation is efficient). As a result, the CC and IC allocations are Pareto non-comparable. It should also be noted that the CC allocation coincides with the Nash bargaining solution for the case in which the set of feasible utilities is given by the set of points beneath the CC frontier.[7] Finally, from (16), as δ

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[6]Substitution of the equilibrium values for x and y into agent 2's utility function results in a value of π^* of

$$1 - \frac{1}{\delta^{(t_Y-2)}}\left(\frac{\delta b(ab-1)}{(ab-\delta^2)(1-\delta^2)} + \frac{ab}{(ab-\delta^2)}\right) \quad \left[1 - \frac{1}{(\delta^{(t_Y-1)}}\left(\frac{\delta(a+\delta)}{a(ab-\delta^2)} + \frac{b(ab+\delta^2-2)}{(ab-\delta^2)(1-\delta^2))}\right)\right]$$

if t_Y is odd [even]. Given the restrictions $a, b > 1$, $b > 1 + 1/a$, $\pi^* < 1$ for all $\delta < 1$. Further, while it may not be readily apparent, it is possible to construct examples in which δ is both large enough so that agent 1 prefers the IC procedure while agent 2 prefers the CC procedure yet small enough that $\pi^* = 0$.

[7]The IC allocation will coincide with the Nash bargaining solution given the set of feasible

approaches 1, π^* also approaches 1 (recall that $\pi^{**} = 1$) so that the CC allocation $x = 1,\ y = 0$ is implemented with probability 1.

Several points are noteworthy regarding these results. First, in spite of the disagreement between the agents over the preferred bargaining procedure, the CC procedure is always implemented in the limit. That is to say, in a world of no frictions, the alternating offers procedure always produces the complete contract. This result is intuitively appealing and also serves to confirm that the bargaining procedure, by itself, is not the source of contract incompleteness. One explanation for this outcome can be found in the fact that the equilibrium allocation under an alternating offers procedure converges to the Nash bargaining solution as δ approaches 1. Referring to Figure 6, it is clear that the bargain over π converges in the limit to a bargain over a subset of the CC frontier that includes the Nash bargaining solution for the frontier. Since the equilibrium outcome of the bargain on π must converge to the Nash bargaining solution for this subset of the frontier, the only possible outcome is the point $x = 1,\ y = 0$, the Nash bargaining solution for the CC frontier.

Of course, with δ bounded strictly away from 1, the picture becomes different. The fact that $b > 1 + 1/a$ continues to guarantee, at least for some range of δ, that the allocations under the two procedures are Pareto non-comparable. However, the allocation under the IC procedure may no longer be efficient in the sense that it may lie in the interior of the space of utilities achievable under the CC procedure. Further, because bargaining is no longer costless, the IC procedure is implemented with positive probability. That is, because it is costly for agent 2 to delay implementation of the contract, 2 sacrifices some utility to 1 by accepting (with positive probability)

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utilities is defined by the set of points beneath the IC frontier if $b > 2+1/a$. For $b \in [1+1/a, 2+1/a)$, the Nash bargaining solution understates agent 1's utility and overstates agent 2's utility. See Section 3 and the Appendix for a more detailed discussion of this point.

the incomplete contract outcome. The conflict in preferences over contract form between the two agents, which proved irrelevant when bargaining was costless, now leads to the possibility of incomplete contracts being implemented. Moreover, a brief inspection of the expression for π^* (see fn. 6) reveals that, as bargaining becomes more costly, incomplete contracts become more prevalent (likely). On the other hand, the expression for π^* also reveals that the incomplete contract becomes more likely as t_Y increases. This result also accords with intuition: as the second issue (Y) becomes less important, the allocations of X under the two contracting regimes converge. While the agents still have opposing preferences over contracts, the value to 2 of holding out for a complete contract declines relatively faster as t_Y increases than the value to 1 from getting the incomplete contract.

The above example points to two issues that are key to producing incomplete contracts as equilibrium outcomes. One is the requirement that there is disagreement between the agents over which contract form is preferred. In the above example, this was guaranteed by the condition that $a > 1$ and $b > 1 + 1/a$. This restriction made sure that agent 1 sacrificed sufficiently little of good X (the good 1 preferred) in the IC procedure relative to the CC procedure that the gain in good Y under the IC procedure left agent 1 better-off overall. Conversely, agent 2 was left worse off by the IC procedure. The other key feature is the need for a friction in bargaining. This friction was created by the existence of a simple delay cost, δ. As delay costs vanished, the CC procedure was implemented with probability one, even though the agents still disagreed on the preferred procedure. Thus, opposing preferences over contract type alone are not sufficient to produce incomplete contracts. In what remains of this paper, we will explore each of these issues in more detail, seeking to obtain a better understanding of what structure produces these preference differences across contract form, and how various frictions in the transactions process affect the equilibrium contract outcome. As well, we will explore simple ways of introducing uncertainty (and thereby contingent contract structures) into this framework.

1.3 A Generalized Contracting Model

As before, it is assumed that the two agents face an endowment stream of one unit of good X and one unit of good Y with the endowment of X occurring at time t_X and the endowment of Y occurring at time $t_Y > t_X$. However, unlike previously, it is now assumed that the endowment of good Y occurs only probabilistically, with the probability of Y arriving at t_Y given by $s \in (0,1]$. The agents' preferences are again assumed to be representable by time separable utility functions; however, these functions are now assumed to take the general form

$$\mathbf{E}U_1 = \delta^{t-1}V_1(x) + \delta^{\tau-1}sW_1(y) \tag{17}$$

$$\mathbf{E}U_2 = \delta^{t-1}V_2(1-x) + \delta^{\tau-1}sW_2(1-y) \tag{18}$$

where $\mathbf{E}U_i$ is the expected utility of agent i, and, as before, x and y represent agent 1's allocation of goods X and Y respectively, while the times t and τ represent the dates at which each of X and Y are consumed ($t \geq t_X, \tau \geq t_Y$). The functions $V_i(\cdot)$ and $W_i(\cdot)$ are assumed to be increasing, twice continuously differentiable and concave, and such that $V_i(0) = W_i(0) = 0$ while $V_i'(0) = W_i'(0) = \infty$.

Allocations, as in the previous section, are determined through offer-counter-offer bargaining procedures. In addition to the two procedures available previously, it is assumed that a third procedure is available involving, first, a bargain over the allocation of good Y and, once agreement has been reached on this allocation, a

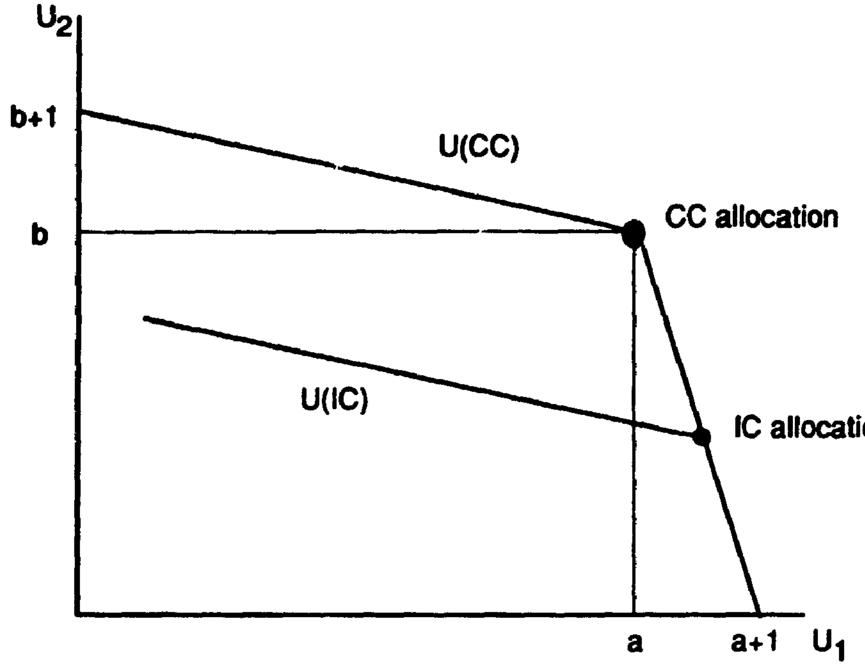

Figure 6: allocations in the limit

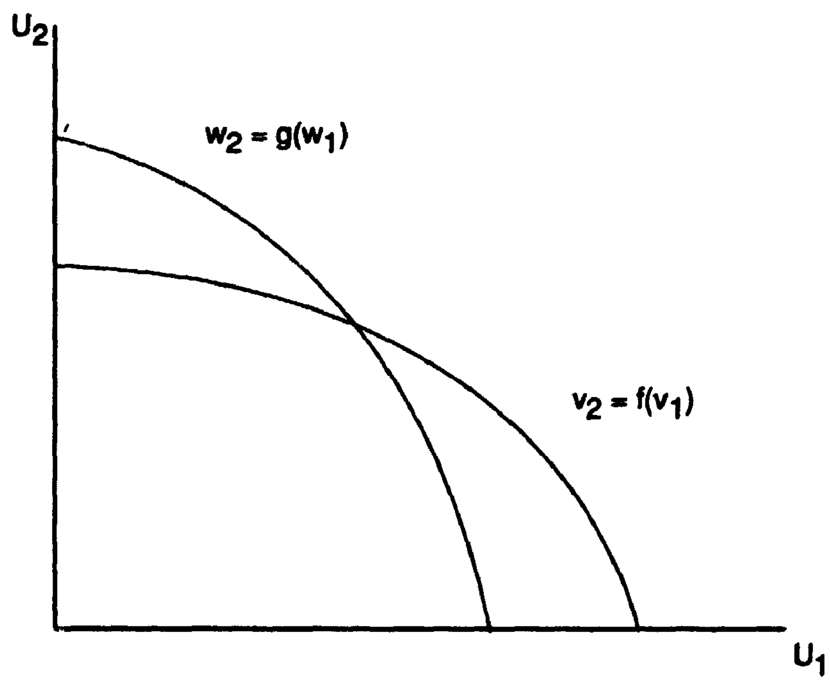

Figure 7: the frontiers for the general case

bargain over the allocation of X (i.e. an offer consists of a value, y, in the first stage bargaining and a value, x, in the second stage bargaining). This procedure will be referred to as IC2 while the sequential offer procedure of the previous section will be referred to as IC1. The simultaneous offer procedure will continue to be referred to as the CC procedure. Which of these bargaining procedures is adopted is also determined by an alternating offers bargaining process, as before, with an offer being a pair $\pi = (\pi_C, \pi_1)$, giving the probability that the CC procedure or the IC1 procedure, respectively, are adopted.

In this section, it is assumed that the length of time between an offer and a counter offer may vary across bargaining procedures, with this length given by Δ_C for the simultaneous offers (CC) procedure and Δ_I for both of the sequential offer (IC) procedures.[8] The length of time between offers in the contract bargaining round is given by Δ_π. This structure is adopted to allow for the possibility that different bargaining procedures may involve different costs, thereby incorporating, in a simple way, the notion that the costs of contracting may vary across contract forms.

Finally, it is assumed, without loss of generality, that the contract bargaining round (i.e. the bargaining over π) begins at time $t_X - \Delta_\pi$, defined to be the date t=1. Also, it is assumed that player 1 makes the initial offer at t=1 and all subsequent offers alternate between players, as in the previous section. Equilibrium allocations are given by the set of subgame perfect equilibrium strategies for the two agents. In what follows, we will examine in turn *i)* how the structure of preferences affects the agents' relative bargaining powers and thereby the allocations that result under

[8]To maintain consistency with the previous section, it is assumed that all Δ are drawn from the set $\{\Delta_n = (2n+1)^{-1},\ n = 0, 1, 2, ...\}$. Thus 2 continues to make his offers in all even (integer) periods, and 1 in all odd (integer) periods.

different bargaining procedures; *ii)* how different bargaining costs associated with different contract forms affect the likelihood of any particular contract being observed; and *iii)* how the analysis of the previous section may be extended to the contingent contract setting, and what role uncertainty plays in determining contract form.

1.3.1 Preferences and Relative Bargaining Power

In order to isolate the role that the differential bargaining power implied by different bargaining procedures plays in the determination of contract structure, it is assumed in this section that $\Delta_C = \Delta_I = 1$ and $s = 1$. Thus, this section generalizes the analysis of Section 2 to the extent that it allows for more general utility functions (and the IC2 bargaining procedure).

As a means of organizing our discussion of the way that the structure of both agent preferences and the bargaining procedure affects the allocation of goods (and the agents' rankings of these allocations), the following definitions will be useful. Agent 1 will be said to *weakly prefer* X to Y if, for all allocations $(x,\ y)$ with $0 < x,\ y < 1$ and such that $x = y$, $V_1'(x) \geq W_1'(y)$, while agent 1 *weakly prefers* Y to X if $V_1'(x) \leq W_1'(y)$. Agent 1 will be said to *strictly prefer* X to Y if, for all allocations $(x,\ y)$ with $0 < x,\ y < 1$ and such that $x = y$, $V_1'(x) > W_1'(y)$, while agent 1 *strictly prefers* Y to X if $V_1'(x) < W_1'(y)$. Analogous definitions apply for agent 2 with x and y replaced by $1 - x$ and $1 - y$. In what follows, the following assumption will be maintained:

> **Assumption 1** — Agent 1 weakly prefers X to Y while agent 2 weakly prefers Y to X.

In the example of the previous section, the stronger assumption of strict preference was maintained through the restriction that $a, b > 1$. The effect of the weaker restriction to $a, b \geq 1$ will become apparent shortly.

Next define the sets of feasible utility pairs associated with allocations of x and allocations of y. These sets are defined by the equations

$$v_2 = V_2[1 - V_1^{-1}(v_1)] \quad (19)$$

$$w_2 = W_2[1 - W_1^{-1}(w_1)] \quad (20)$$

respectively, where $v_1 = V_1(x)$ and $w_1 = W_1(y)$. All offers by the two agents will be chosen from subsets of these two sets.

Finally, define three particular subsets of the set of all feasible utility pairs $(v_1 + w_1,\ v_2 + w_2)$. The first is the set given by the utility pairs on the utility possibility frontier. This set is defined as all $(v_1 + w_1,\ v_2 + w_2)$ satisfying (19) and (20) and the condition $V_2'/V_1' = W_2'/W_1'$. This set will be labelled CC as it corresponds to the set of utility pairs from which the CC offers are drawn. The other two sets are the corresponding sets for the IC1 and IC2 procedures. The first (IC1) is given by all $(v_1 + w_1,\ v_2 + w_2)$ such that v_1 and v_2 are determined by (19) while w_1 and w_2 are determined such that $W_2(w_1) - w_1 W_2'/W_1' = 0$ (the Nash bargaining solution for the set given by (20)). The second (IC2) is given by all $(v_1 + w_1,\ v_2 + w_2)$ such that w_1 and w_2 are determined by (20) while v_1 and v_2 are determined such that $V_2(v_1) - v_1 V_2'/V_1' = 0$ (the Nash bargaining solution for the set given by (19)).

The fact that this model differs from that in Section 2 only in the specification of the utility set means that the analysis of the preceding section continues to apply. Thus, under the CC procedure, equilibrium allocations of X and Y can be thought of as being determined through a sequence of utility offers that satisfy the analogues of (5) and (6), and are drawn from the utility possibility frontier (the CC set). Further, these equilibrium utility offers must converge to the Nash bargaining solution for the utility set defined by this frontier as Δ approaches zero.

The IC1 and IC2 procedures can be analyzed in a sequential fashion analogous to that employed in the analysis of the IC procedure above. One can show that, as Δ approaches 0, the Nash bargaining solution for the utility set defined by the IC1 (IC2) frontier serves as a lower bound on the utility that agent 1 (2) obtains under the IC1 (IC2) procedure and upper bound on the utility that agent 2(1) obtains.[9] This feature of the IC procedures results from the fact that, in the bargaining over an allocation of x, delay on either of the agents' parts is costly not just because of the time cost (the cost captured by the Nash bargaining solution) but also because the agent that delays loses the opportunity to make the first offer in the subsequent bargaining round. Because agent 2 is assumed to (at least weakly) prefer Y and agent 1 X, this latter delay cost is more severe to agent 2 (1) under the IC1 (IC2) procedure than to agent 1(2). As a consequence, agent 1 is able to get at least as much of X, and so at least as much utility, under the IC1 procedure as would be predicted by the Nash bargaining solution for the IC1 frontier. Similarly 2 is able to get at least as much of Y under the IC2 procedure as predicted by the Nash bargaining solution for IC2.

These relationships between the equilibrium allocations under the respective bargaining procedures and the Nash bargaining solutions for the corresponding utility frontiers proves useful in analyzing the agents' rankings of the various allocation procedures. Specifically, should the two agents rank the various Nash bargaining solution outcomes differently, then, as long as δ is close enough to 1, the agents will also rank the equilibrium allocations from the various bargaining procedures

[9]A proof of this fact is contained in the Appendix.

differently. Thus, at least for large δ, an analysis of the conditions under which the two agents disagree on the preferred contracting form can proceed by an analysis of the conditions under which the agents disagree on the ranking of these three Nash bargaining solutions. It is to this analysis that we now turn.

To begin, consider a situation in which each agent views X and Y as identical in the sense that, for all $x = y$, $V_i'(.) = W_i'(.)$, $i = 1,2$ (note that V_1' need **not** equal V_2'). In this case, the frontier defined by (19) coincides with that defined by (20), while the CC frontier is simply a proportional (radial) blow-up of either of these frontiers. It is not difficult to see that, in this case, the Nash bargaining solutions for the CC, IC1, and IC2 sets all coincide. Further, because each agent views X and Y as identical, there is no differential delay cost due to losing the first move in the bargaining over the next good under IC1 or IC2. Thus, the Nash bargaining solutions for the IC1 and IC2 frontiers provide exact limits for each agent's utility under the IC1 and IC2 procedures, respectively. One can conclude, therefore, that, in the limit, both agents are indifferent across all contract structures. That is to say, when goods are identical, agents have, in the limit, no strictly preferred contract structure.

The situation outside of the limit (i.e. when δ is strictly less than 1) can easily be analyzed by noting that this case is essentially the case of the example in Section 2 with $a = b = 1$. A comparison of the utilities of each agent under the CC versus IC1 procedures in this example reveals that agent 1 prefers IC1 to CC when t_Y is even (implying agent 2 prefers CC to IC1) and that 1 prefers CC to IC1 when t_Y is odd (so that 2 prefers IC1 to CC). That the agents' rankings of the two procedures depend on t_Y results from the already noted fact that, in the IC1 procedure, t_Y being even implies a cost of delay to agent 2 at $t_Y - 1$ resulting from agent 2 losing the ability to make the first offer in the bargain over Y. This additional cost places agent 1 at an advantage in the bargain over X and so means that 1 prefers IC1 to CC. When t_Y is odd, agent 1 bears this delay cost and agent 2 is advantaged so that 1 now prefers CC to IC1. In the limit as δ approaches 1, this delay cost vanishes together with the vanishing first mover advantage in the bargain over Y, and so the agents view the two procedures as equivalent.

Next, suppose that agent 1 strictly prefers X to Y while agent 2 strictly prefers Y to X. In this case, the frontiers defined by (19) and (20) appear as in Figure 7. These frontiers have the property that, along any ray from the origin, $V_2'/V_1' < W_2'/W_1'$ (except, of course, along the rays with zero and infinite slope where $V_2'/V_1' = W_2'/W_1'$). Also, let v_i^* and w_i^* be the utilities for agent i under the allocation implied by the Nash bargaining solution for the CC frontier and $\hat{v}_i$, $\hat{w}_i$ be the utilities for agent i under the allocation implied by the Nash bargaining solution for the IC1 frontier. Further, assume that the functions $V_i(\cdot)$ and $W_i(\cdot)$ are scaled such that v_1^*, $w_1^* \geq 1$. Then, the following can be shown to hold.

> **Proposition 1**— Suppose that agent 1 strictly prefers X to Y while agent 2 strictly prefers Y to X. Suppose also that v_1^*, $w_1^* \geq 1$ and that $V_2''/V_2'V_1' + V_1''/V_1' \leq -1$. Then, $\hat{v}_1 + \hat{w}_1 > v_1^* + w_1^*$ while $v_2^* + w_2^* > \hat{v}_2 + \hat{w}_2$.

In words, as long as the curvature condition of the Proposition is satisfied, agent 1 will obtain higher utility at the Nash bargaining solution for the IC1 frontier while agent 2 will obtain higher utility at the Nash bargaining solution for the CC frontier.

How should this result be interpreted? First, from our preceding analysis it can be concluded that, as δ approaches 1, the allocations of X and Y under the CC procedure must converge to those implied by the Nash bargaining solution for the CC frontier. For the IC1 procedure, the allocation of Y converges to the Nash bargaining solution for the utility frontier defined by (20) while the allocation of X converges to a point giving agent 1 at least as much as under the Nash bargaining solution for the IC1 frontier. Further, the utilities of each agent at t_X must converge to $v_i^* + w_i^*$ for the CC procedure and have $\hat{v}_1 + \delta\hat{w}_1$ as a lower bound for agent 1's utility and $\hat{v}_2 + \delta\hat{w}_2$ as an upper bound for agent 2's utility under the IC1 procedure. As $v_1^* > \hat{v}_1$ while $w_1^* < \hat{w}_1$ it can be guaranteed, therefore, that agent 1 prefers the IC1 procedure to the CC procedure as long as δ is close to 1. Agent 2 can be guaranteed to prefer CC to IC1 under similar circumstances.

As to the derivative condition in the proposition, it has an easy interpretation. First, one should note that the allocation of Y to agent 1 is larger under the IC1 procedure than the CC procedure ($w_1^* < \hat{w}_1$). That is, because agent 1 is able to separate bargaining over X (the good 1 prefers) from bargaining over Y (the good 2 prefers) in the IC1 procedure, 1 is able to reduce the cost to himself from a delay in reaching an agreement on Y (1 has already reached agreement on X and guaranteed consumption of x). Therefore, 1's share of Y increases relative to the CC procedure. In short, the IC1 procedure gives agent 1 a superior bargaining position relative to the CC procedure in the bargain over Y.

This increased share of Y that 1 obtains under IC1 reduces the cost to agent 2 of delaying agreement on X under IC1 (1's cost is larger) relative to CC. It is this lower cost for 2 that results in 2's share of X being larger under the Nash bargaining solution for IC1 than under that for CC ($v_1^* > \hat{v}_1$). In short, 1's bargaining position is deteriorated relative to CC in the bargain over X. Whether, on net, agent 1 is better-off under IC1 than CC depends on how badly 1's bargaining position is deteriorated in the X bargain. The derivative condition in the proposition determines the extent of the deterioration. Put simply, it places a restriction on the percentage rate of change of the slope of the IC1 frontier.[10] If, as is assumed, this number is large (in absolute value terms), then increases in 2's utility from increased shares of X (relative to the share under the CC procedure) become increasingly costly, in terms of utility loss, to 1. As a consequence, the extra cost of delay to 1 due to a larger share of Y looms small relative to the gains 1 can obtain from delay in terms of increased utility. Or, looking at the problem from the other side, if the derivative condition is met, then the costs to 2 of extracting some extra utility from 1 become sufficiently large as to quickly offset the reduced delay costs from a smaller share of Y. Therefore 1's bargaining position in the bargaining over X is deteriorated only slightly under the IC1 procedure. As a result 1 is better-off overall. Agent 2, of course, having lost share of Y and only gained slightly in terms of X is worse-off.

From the above discussion one might suspect that if the percentage rate of change of the slope of the IC1 frontier is sufficiently small, then agent 1 is worse-off under the Nash bargaining solution for the IC1 frontier than that for the CC frontier. This conjecture is, in fact, correct as the next proposition shows.

> **Proposition 2**— Assume that agent 1 strictly prefers X to Y while agent 2 strictly prefers Y to X. Let $\tilde{v}_1$ be defined such that $v_1^* + w_1^* = \tilde{v}_1 + \hat{w}_1$ and $\tilde{v}_2 = V_2(\tilde{v}_1)$. If $V_2'(v_1^*)/V_1'(v_1^*) - V_2'(\tilde{v}_1)/V_1'(\tilde{v}_1) < (v_2^* + w_2^*)/(v_1^* + w_1^*) - (\tilde{v}_2 + \hat{w}_2)/(\tilde{v}_1 + \hat{w}_1)$, then $v_i^* + w_i^* > \hat{v}_i + \hat{w}_i$, $i = 1, 2$.

Obviously, the condition in the above proposition will be satisfied if, for all $v_1 \in [\tilde{v}_1, v_1^*]$, $V_2''/V_1'V_2' + V_1''/V_1'$ is sufficiently small (in absolute terms). In this case, the cost to agent 1 of delay due to the larger share of Y that 1 obtains under IC1

[10]Note here, that the slope of the IC1 frontier is only determined by the x share in question. That is the reason no W terms appear in the condition.

looms large relative to any increased gains from delay due to the shifting allocation of X. As a result, 1's bargaining position is deteriorated sufficiently that 1 is worse-off under IC1 relative to CC. Interestingly, agent 2 is worse-off as well.

Because the utilities determined by the Nash bargaining solution for the IC1 frontier represent a lower bound on agent 1's utility under the IC1 procedure and an upper bound for agent 2's utility, it is not possible to conclude from the above that both agents are worse-off under the IC1 procedure relative to the CC procedure. The lower delay cost that agent 1 faces (relative to agent 2) arising from the loss of the first move in the bargaining over Y may still give 1 sufficient bargaining power over X to result in 1 being better-off overall under IC1. Nonetheless, the above proposition is useful in terms of clarifying what may go wrong under the IC1 procedure and so result in its not being adopted. In essence, what happens in this case is that the altered bargaining positions under the IC1 procedure are sufficiently extreme as to allow agent 1 to gain enough of the good that agent 2 prefers (Y) to make 2 worse-off relative to the CC procedure and to allow 2 to gain enough of the good that 1 prefers (X) to make 1 worse-off relative to the CC procedure. The IC1 procedure generally results in an inefficient allocation of the goods if frontiers are strictly concave. In the case of the second proposition, it misallocates X and Y sufficiently badly as to be potentially Pareto dominated by the CC procedure (at least for δ close to 1).

Not surprisingly, analogous results can be obtained for a comparison of the CC and IC2 procedures. In this case, the slope restrictions would be on the IC2 frontier (i.e. on the derivatives of the W_1 and W_2 functions). Conditions on these functions analogous to those in Proposition 1 being satisfied would mean that agent 2 prefers the Nash bargaining solution associated with IC2 to that associated with CC while agent 1 prefers the Nash bargaining solution associated with CC to that associated with IC2. Similarly, conditions on W_1 and W_2 analogous to those in Proposition 2 would imply that both agents prefer the Nash bargaining solution associated with CC to that associated with IC2.

Finally, there is the issue of how these results relate to the results obtained in Section 2. While the situations are not exactly comparable (due to the linearity of the sub-utility frontiers in the example) these results do shed light on the example. Specifically, because a and b are assumed larger than 1, it is relatively more costly for agent 1 to give up X to agent 2 than Y, and relatively more costly for 2 to give up Y to agent 1 than X. As a consequence, the complete contract allocates all of Y to agent 2 and most (in the limit all) of X to agent 1. The incomplete contract allows agent 1 to transfer some of Y to himself, making 1 better-off and 2 worse-off. The fact that $a > 1$ and $b > 1 + 1/a$ means both that agent 2 still receives "enough" utility under the incomplete contract and that it is sufficiently costly for agent 1 to transfer X to agent 2 that the extra Y agent 1 receives does not deteriorate his bargaining position sufficiently to make him worse-off (much as in Proposition 1).

1.3.2 Differential Contracting Costs

One of the common explanations for the observation of incomplete contracts is the existence of cost differentials in constructing complete versus incomplete contracts. In particular, it is argued that the complexity of a complete contract makes it sufficiently costly to write as to make it efficient for the contracting parties to write a simpler, incomplete contract (this would be the explanation offered by Dye (1985)). This sort of exogenous cost differential between complete and incomplete contracts is easily incorporated within the current framework. It is done by allowing the bargaining costs in the CC procedure, Δ_C, to differ from those in the IC1 and IC2 procedures, Δ_I. To capture the notion that complete contracts, being more complex, are more costly to construct than incomplete contracts, one need only assume that $\Delta_C > \Delta_I$. In essence, the time required for the agents to make offers and counter offers when bargaining over a complete contract is longer than under an incomplete contract setting. This is assumed to be due to the complexity of the complete offers. The question, then, is how such a cost differential interacts with the bargaining cost differentials implied by the different contracting structures, and whether it has the anticipated effect of producing incomplete contracts with greater frequency (probability).

To gain some insight into the impact of these exogenous contracting costs on the likelihood of incomplete contracts, we first consider the equilibrium when preferences are as specified in Section 2. Further, we assume contracting costs are such that $\Delta_\pi = 1$ while $\Delta_I < \Delta_C < 1$; that is, bargaining over the structure of the contract is the most costly, bargaining over the parameters of the complete contract is next most costly and bargaining over the parameters of the incomplete contract is least costly. Finally, we only consider the case in which t_Y is even. Under these circumstances, equation (16) exactly specifies the equilibrium value of π with the denominator and numerator of that expression given by

$$U_2^C - U_2^I = \frac{\delta^{\Delta_I} + \delta^{t_Y - 2}}{(1 + \delta^{\Delta_I})} - \frac{\delta^{\Delta_C} - \delta^{\Delta_C}\delta^{t_Y-2}}{(1+\delta^{\Delta_C})} + \delta^{t_Y-2}\left(\frac{b\delta^{\Delta_I}}{1+\delta^{\Delta_I}} - x^{**}\right) \qquad (21)$$

$$\delta U_2^C - U_2^I = \frac{\delta^{\Delta_I} + \delta^{t_Y - 2}}{(1 + \delta^{\Delta_I})} - \frac{\delta - \delta^{\Delta_C}\delta^{t_Y-2}}{(1+\delta^{\Delta_C})} + \delta^{t_Y-2}\left(\frac{b\delta^{\Delta_I}}{1+\delta^{\Delta_I}} - x^{**}\right) - 1 + \delta \qquad (22)$$

It is easy to check from equation (21) that as Δ_C increases the utility differential between the complete and incomplete contract procedures also increases. The situation is less clear for equation (22). It is straightforward to show that the change in the utility differential in (22) is smaller than in (21) and that, indeed, the utility differential may decrease as Δ_C increases. Should (22) decrease as Δ_C increases, then π must decrease; that is, as the complete contract becomes more costly, it is less likely to be utilized. However, it is also possible that (22) increases (by a smaller amount than (21)). In this case, the impact on π of an increase in Δ_C is less clear in the sense that, depending on parameter values, π may either decrease or increase. In this latter case, even though the complete contract is more costly, it is actually more likely to be employed.

To understand what drives the above results, it is helpful to consider how the bargaining equilibrium for the CC procedure is determined in general. For those subgames for which $t \geq t_Y$, the equilibrium offers are determined by the standard set of conditions, which for this problem are given by

$$V_1(x^2) + W_1(y^2) = \delta^{\Delta_C}[V_1(x^1) + W_1(y^1)] \tag{23}$$

$$V_2(1 - x^1) + W_2(1 - y^1) = \delta^{\Delta_C}[V_2(1 - x^2) + W_2(1 - y^2)] \tag{24}$$

As is standard in these problems, the utility that agent 2 obtains in equilibrium when it is his turn to offer (given by the bracketed expression on the right-hand-side of (24)) increases as Δ_C increases. Similarly, the utility that 2 obtains in equilibrium when it is 1's turn to offer (given by the left-hand-side of (24)) decreases as Δ_C increases. Quite simply, because delay is more costly as Δ_C increases, agent 2 is able to extract more x and y from agent 1 when it is 2's turn to offer and conversely 1 is able to extract more when it is 1's turn to offer. An immediate consequence of these facts is that, were bargaining on the value of π to begin at t_X and Δ_π to be such that $t_Y - t_X < \Delta_\pi$, then the equilibrium value of π would always be decreasing in Δ_C.

The possibility that π may increase as Δ_C increases arises when t_Y is outside of the period of delay, Δ_π. This possibility arises because of the fact that, over the period $[t_X, t_Y)$, delay costs are different than after t_Y. Over this period, delay is costly in terms of foregone consumption of X but there is no cost in terms of Y as Y does not become available until t_Y. As a consequence, the conditions (23) and (24) no longer define the equilibrium utilities for the two agents. Rather, equilibrium is defined recursively such that, if t_Y is even, the system takes the form

$$\begin{aligned} V_2(1-x^1) + \delta^{\Delta_C} W_2(1-y^1) &= \delta^{\Delta_C} V_2(1-x^*) + \delta^{\Delta_C} W_2(1-y^*) \\ V_1(x^2) + \delta^{2\Delta_C} W_1(y^2) &= \delta^{\Delta_C} V_1(x^1) + \delta^{2\Delta_C} W_1(y^1) \\ &\vdots \\ V_1(x^2) + \delta^{t_Y - t_X} W_1(y^2) &= \delta^{\Delta_C} V_1(x^1) + \delta^{t_Y - t_X} W_1(y^1) \end{aligned} \tag{25}$$

where x_*, y_* is the offer made by agent 2 at t_Y and is defined by the system (23), (24).

An increase in Δ_C has two effects on the system (25). As previously, an increase in Δ_C makes delay more costly thereby benefiting the agent whose turn it is to offer. In the above case, at t_X, agent 2 makes the offer and so agent 2's utility under the complete contract increases at t_X due to this effect. Similarly, should agent 2 delay one period to make a counter offer in the bargaining over π, then the offer in the complete contract bargaining is turned over to agent 1 and the increase in Δ_C decreases agent 2's utility.

A second effect of an increase in Δ_C is that it decreases the values x_* and y_*, thereby leading to an increase in the initial utility level for agent 2 that defines the recursive system. The effect of this increase is to reduce the utility that agent 1 can get in any period in which 1 makes the offer and to increase the utility 2 can get in any period in which 2 makes the offer. On net then, agent 2's utility at t_X under the complete contract increases both because of the improved initial condition and the increased cost to 1 of delay (Equation (21) is increasing in Δ_C). The change in the utility that 2 receives under the complete contract should 1 offer is ambiguous, with the increased delay cost tending to reduce 2's utility but the improved initial conditions tending to increase it. Should the former effect dominate the latter, then an increase in Δ_C unambiguously decreases π. Should the latter effect dominate then the effect of increases in Δ_C on π is ambiguous.

Obviously, a similar analysis could be performed for the case of t_Y odd. In this case, the affect of an increase in Δ_C on the initial conditions of the recursive system is to increase agent 1's utility, thereby leading to an unambiguous reduction in 2's utility should 1 make the offer. The impact on 2's utility should 2 make the offer is now ambiguous with the increased delay costs increasing it and the worsened initial conditions decreasing it. As long as, on net, 2's utility when 2 makes the offer either rises or falls by less than when 1 makes the offer, then π decreases as Δ_C increases.[11]

To sum up, if bargaining were to begin at t_X (rather than $t_X - \Delta_\pi$) and t_Y were sufficiently close to t_X in the sense of being within the the period of delay Δ_π, then an increase in Δ_C unambiguously reduces the likelihood of a complete contract being employed. When t_Y is farther away from t_X, while increases in Δ_C may still reduce the likelihood of a complete contract being employed, this result is not uniformly true. There are cases in which the complete contract being more costly actually results in an increased likelihood of it being employed. This outcome

[11]The reader should note that this analysis assumes that both x and y are strictly between 0 and 1 (as must be the case given the utility function restrictions). The example in Section 2 fails to satisfy this requirement. As a consequence, the equilibrium when t_Y is odd does not conform exactly to the analysis here. In particular, the analysis of the complete contract when t_Y is odd is the same as that for the t_Y even case except discounted one less period. This outcome is a consequence of the fact that $y = 0$ is part of the equilibrium for this problem.

occurs because the increased delay costs in bargaining over the parameters of the complete contract may mitigate the delay costs in bargaining over the contract form sufficiently as to make the use of a complete contract more likely in equilibrium.

1.3.3 Contingent Contracts

So far, agents have had no uncertainty about the endowment process so that contract choices have been only between complete (long-term) and temporally incomplete (short-term) contracts. However, as suggested at the beginning of this section, the model can be extended to situations in which the agents face an uncertain endowment process and so may wish to utilize complete, contingent contracts. In such settings, complete contracts provide potential efficiency gains relative to incomplete contracts by allowing efficient sharing of risk (as well as efficient intertemporal allocation of consumption.) On the other hand, bargaining considerations of the sort that arise in the certainty case provide incentives for the agents to prefer various incomplete contracts. As before, which contract is utilized depends on the relative magnitudes of these efficiency and bargaining effects. In what follows, the focus of attention will be the bargaining aspects of contract determination when endowments are uncertain, and how the nature of the uncertainty alters the bargaining effects. Risk-sharing considerations will be neutralized by the assumption that agents are risk neutral. Therefore, any efficiency considerations will enter, as before, only through the intertemporal allocation of consumption.

The model employed to analyze these issues is that of Section 2 with the modification that good Y arrives at t_Y with probability $s \in (0,1)$ and fails to arrive with probability $(1-s)$. Both agents are assumed to know the value of s.[12] It is also

[12]This endowment process is just a special case of a more general process in which the amount of Y that arrives at t_Y is uncertain.

assumed that if Y fails to arrive at t_Y it will not arrive at any $t > t_Y$ (and this fact is known by the agents.)

The structure of contracts (bargaining procedures) is modified to allow for contingent contracting. The incomplete contract procedure, IC, consists, as before, of a bargain over X followed by a bargain over Y only after agreement on x has been reached. The share x is non-contingent. The complete contract procedure, CC, on the other hand, is a concurrent bargain over the allocation of both goods in both states of nature. Under this procedure an offer is a triple $(\hat{x}, (x, y))$, giving an allocation of X of $(\hat{x}, (1 - \hat{x}))$ in state 0 when the size of Y is 0, and an allocation of X of $(x, (1 - x))$ and an allocation of Y of $(y, (1 - y))$ in state 1 when the size of Y is 1.[13] The bargain over contract form continues to be a bargain over $\pi \in [0, 1]$, the probability that the CC procedure will be used in determining the allocations.

Equilibrium Outcomes

Consider first the bargaining procedures which determine the allocations of X and Y. Suppose that the agents have adopted the CC procedure and $t \geq t_Y$. Then the state is known and agents either bargain over X alone or over X and Y together. The situation in state 1 is unchanged from that in the first section, so that the state 1 allocation is given by

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[13]These procedures are the extreme cases of possible contracts. In particular, there are 2 other kinds of incomplete contracts. One is incomplete only in the temporal dimension (state contingent x offers, followed by a bargain over Y, once, and if, available), while the other is incomplete only in the contingency dimension (a joint bargain over one x and y, independent of state). As it turns out, the temporally complete but non-contingent contract yields the same utilities as the CC procedure used here. This is due to the risk-neutrality of the agents, making insurance pointless. The contingent but temporally incomplete contract, on the other hand, is worse for 1 than the totally incomplete contract IC since the backward induction starts off a lower expected X share.

$$x_1^* = 1, \qquad y_1^* = \frac{a(1-\delta)(b-\delta)}{ab-\delta^2} \tag{26}$$

$$x_1^{**} = \frac{\delta(b(a+1)-\delta(b+1))}{ab-\delta^2}, \qquad y_1^{**} = 0 \tag{27}$$

depending on whether agent 1 or agent 2 make the offer, respectively. State 0 is a standard Rubinstein game over x only, so that the state 0 allocation is

$$x_0^* = \frac{1}{1+\delta} \qquad x_0^{**} = \frac{\delta}{1+\delta}$$

depending on whether agent 1 or 2 make the offer, respectively.

The situation changes for $t < t_Y$. In these periods the state is not known when shares are offered or the decision to accept or reject is made. Assume for the moment, however, that the correct allocation can be implemented as soon as agreement is reached. Then, as previously, delay is costly only in terms of foregone X consumption and not foregone Y consumption, and offers are derived by a backward induction argument. The backward induction process yields an allocation of Y of $y_C = 0$ for both even and odd t_Y, and an allocation of X of

$$sx_C + (1-s)\hat{x}_C = \delta \sum_{i=0}^{t_Y-t_X-1} (-\delta)^i + \delta^{t_Y-t_X}(sx_1^{**} + (1-s)x_0^{**}) \tag{28}$$

$$sx_C + (1-s)\hat{x}_C = \delta \sum_{i=0}^{t_Y-t_X-2} (-\delta)^i + \delta^{t_Y-t_X}(sx_1^{*} + (1-s)x_0^{*}) + s\delta^{t_Y-t_X}\frac{y_1^*}{a} \tag{29}$$

if t_Y is even or odd, respectively.[14]

The analysis for the IC procedure is similar. For $t \geq t_Y$ the state is again known

[14]Risk neutrality on the part of both agents implies that, while expected utility levels are determined, x and $\hat{x}$ are only jointly determined. Consider, for example, the first induction step for t_Y even. Agent 1 wishes to maximize, by choice of $(\hat{x},(x,y))$, his expected utility: $s(ax+\delta y)+(1-s)a\hat{x}$. This is subject to 2 accepting such an offer, that is: $s(\,(1-x)+\delta b(1-y)\,)+(1-s)(1-\hat{x}) = \delta\,[\,s(\,(1-x_1^{**})+b\,)+(1-s)(1-x_0^{**})]$. The solution to this problem is $y=0$ and $sx+(1-s)\hat{x} = 1-\delta+\delta(sx_1^{**}+(1-s)x_0^{**})$, leaving x and $\hat{x}$ undetermined.

and so the agreement for state 1 is

$$x_1^{\ddagger} = \frac{\delta(a-1+\delta)}{a} \qquad y_1^{\dagger} = \frac{1}{1+\delta} \tag{30}$$

$$x_1^{\dagger} = 1 \qquad y_1^{\ddagger} = \frac{\delta}{1+\delta} \tag{31}$$

if t is even or odd, respectively. For state 0 the agreement is

$$x_0^{\ddagger} = \frac{\delta}{1+\delta} \qquad x_0^{\dagger} = \frac{1}{1+\delta} \tag{32}$$

for t even or odd respectively.

For $t < t_Y$ the state is unknown, and so equilibrium offers for x are determined by a backward induction on expected utilities. This process yields allocations

$$\hat{x}_I = \delta \sum_{i=0}^{t_Y - t_X - 2} (-\delta)^i \tag{33}$$

if t_Y is even, and

$$\hat{x}_I = \delta \sum_{i=0}^{t_Y - t_X - 2} (-\delta)^i + \delta^{t_Y - t_X}(s x_1^{\dagger} + (1-s)x_0^{\dagger}) - \delta^{t_Y - t_X}\frac{(1-\delta)s}{a} \tag{34}$$

if t_Y is odd, with $\hat{y}_I$ given by $y_I^{\ddagger}$ and $y_I^{\dagger}$, respectively.

It is useful to stop at this point and compare the allocations here with those in the certainty case (equations (9) and (10) for the CC procedure, and (13) and (14) for the IC procedure.) For both contract procedures the allocations are of the same form as before, except that the allocations of X at t_Y used to determine allocations at t_X are expected allocations. The logic behind these results is also quite similar. By splitting off the allocation of Y from that of X, agent 1 reduces his cost of holding out for larger Y shares, thereby increasing the amount of Y he can expect to receive. If t_Y is even, under the IC procedure, agent 2 once again faces the additional cost of handing the first mover advantage over Y to agent 1 should

2 reject 1's offer at $t_Y - 1$. This cost is severe enough to 2 that the amount of X 2 obtains actually declines in the incomplete contract. If t_Y is odd then, as before, 1's share of Y increases but only at the expense of a lower share of X.

For the determination of the equilibrium procedure, of course, the utility comparison between procedures is what matters. Clearly, agent 1 gets higher expected utility from the IC procedure if t_Y is even, while agent 2 looses from the IC procedure. However, agent 1 also gains overall from the IC procedure if t_Y is odd and δ is large enough.[15] Thus, the bargain over which procedure is to be implemented is again non-trivial.

Turning to the bargain over contract form, it once again is the case that both the IC and the CC utility allocations are feasible in the CC procedure, as are all linear combinations. The expected utility bargain thus must feature an offer by 2 of $\pi^{**} = 1$, since in the CC procedure this point is an equilibrium offer for 2, and thus 1 can not have a better counter offer in the restricted set under consideration now. Given that $\pi^{**} = 1$, 1's offer is easy to calculate for δ satisfying the restrictions outlined before. It is that π^* which leaves 2 just indifferent between accepting the offered "gamble" and rejecting it in favour of getting the CC procedure implemented for certain — next period. Notice again that not only does the discount factor affect this decision, but also the cost of "surrendering" the first move in whichever procedure is to be implemented, since a first mover advantage exists for discounts strictly less than 1. The equilibrium value of π^* is given by

$$\pi^* = \frac{\delta \mathbf{E}U_2^C(3) - \mathbf{E}U_2^I(2)}{\mathbf{E}U_2^C(2) - \mathbf{E}U_2^I(2)} \tag{35}$$

[15]The exact condition on δ guaranteeing that agent 1 prefers the IC procedure over the CC procedure is precisely the same as before, namely $\delta^2/(1+\delta) \geq (1-\delta)(ab - a\delta)/(ab - \delta^2)$. It is $b/(1+\delta) \geq (1-\delta)/a + (1-\delta)(b-\delta)/(ab-\delta^2)$ for agent 2 preferring the CC procedure.

where numbers in parenthesis indicate the period in which the initial offer is made in the procedure indicated by the superscript. Once again, this expression is strictly less than 1 for all δ and becomes negative for sufficiently small δ, implying $\pi^* = 0$.

Of interest, of course, is the impact on π^* of changes in s. It is easy to show that

$$\frac{\partial \pi^*}{\partial s} \geq 0 \tag{36}$$

that is, the IC procedure becomes more likely the less likely the second surplus becomes.[16] The reasoning is as follows: A decrease in s causes the second good to be less important in the overall allocation for both agents, deteriorating agent 2's overall bargaining power and bringing the X allocations in the two contracting procedures closer and closer (note that $\partial x^*/\partial s > \partial x_1^*/\partial s$). Therefore 2 has to rely increasingly on X to provide his utility, and it becomes less worthwhile for him to hold out for the CC procedure. Agent 1, of course, still benefits from the chance at some additional utility from Y. Balancing the various effects, agent 1's preferred outcome (IC) occurs more often. Thus, in terms of bargaining effects, increasing the probability of Y failing to arrive makes incomplete contracts a more viable option. Notice that this effect is similar to that obtained by increasing t_Y in Section 2. Both have the effect of "shrinking" Y.

Throughout this section it was assumed that the correct allocation of X could be implemented under the CC procedure as soon as agreement was reached (even if $t < t_Y$). As a final note to this analysis, the issue as to how such an implementation might take place is addressed. One possible mechanism would be for the agents to have available an investment technology whereby agents could invest X at a rate of $(1-\delta)/\delta$ until the state of nature is revealed at t_Y. The investment would take place

[16]See appendix

only after agreement on an allocation of X is determined. This sort of technology would precisely offset agents' discounting of consumption, while leaving the cost of delay in agreement unchanged. An alternative mechanism would be one that allowed consumption of an agent's minimum allocation across states, with the remainder of X held until the state is revealed. However, without investment opportunities, the effect of such a scheme is to shrink the effective CC utility frontier (allocations are inefficient, since time passes before consumption can be finalized), thereby further increasing the likelihood of the IC procedure.

1.4 Discussion

In the preceding pages we have provided a simple model of contract formation which derived incomplete contracts as equilibrium outcome. Its focus was on the differential bargaining power bestowed upon agents in the various contracting procedures implied by the contract form. The model has shown that if allocations are arrived at by offer-counter-offer bargaining under the existence of bargaining frictions (a cost of delay), then incomplete contracts can arise endogenously if agents' preferences are sufficiently "skewed", in the sense that the agents have different preferences among issues.

While the main thrust of the paper was towards temporally incomplete contracts (short-term contracts), it was shown that the framework also allows for an analysis of contingent contracts.

It is clear, as in any model employing bargaining theory, that the results depend on the assumptions placed on the bargaining processes. While the assumption on which agent starts the alternating moves process is innocuous[17] the alternating

[17]In the sense that the problem is symmetric, and parametrizations can be found to yield the

moves structure is important.

Another issue that arises is related to the procedure used in choosing contract form. There are various procedures one can imagine for this problem. One possibility is that players never formally bargain over the procedure, but instead make offers or counter-offers consistent with a particular contract form. In particular, agent 2 could always make complete offers, while agent 1 could simply make incomplete offers any time it is his turn. Such a scheme would abolish the driving force behind the results. The reason is the following: Incomplete contracts are, in general, inefficient. As the results above have shown, this does not imply the existence of an equilibrium complete contract which Pareto dominates an equilibrium incomplete contract. However, the equilibrium incomplete contract is chosen from a restricted set of possible offers. In the paper, agent 2 faced a binding constraint in the best possible counter-offer he could make under the restriction that his offer constitute an incomplete contract. This restriction was crucial to the results. Unrestricted bargaining is equivalent to complete contract bargaining in this framework, and the offers by 1 and 2 are known. In particular, although agent 1 was of course free to make an offer which is equivalent to an incomplete contract, he never chose to do so. In short, agent 1 can only gain by exploiting the weakening of agent 2's bargaining power which is brought about by the restriction on 2's offers due to the agreed procedure.

Note in this context that law books imply that the agenda is set by the initial offer and counter-offer.[18] If this "rule" were to be amended to count the initial

same results with changed labelling.

[18]See, for example, Edwards, H, and J. White, *The Lawyer as Negotiator: Problems, Readings and Materials*, WestPublishing Co., 1977, 48-58.

offer only, our results would go through. The law literature also implies that while collective bargaining does not seem to involve agenda bargaining, agent bargaining does (the "real" issues bargaining being referred to as "substantive bargaining".) The assumption that agenda bargaining precedes the allocation bargain is therefore not totally implausible.

While our model has attempted to show how incomplete contracts can arise endogenously without assuming them, we have in effect assumed that agents can not affect side-payments. As will be recalled, the results depended on the asymmetry of utility transferability between goods. Consider then, what would happen if both agents had some initial wealth and could contract on side payments in the bargain over contract type only. Wealth can, by assumption, be transferred one to one. The probability π, on the other hand transfers utility, in general, in some other ratio. It is thus to be expected that if the slope of the line joining the CC and IC equilibrium allocations is less than -1, both agents would only offer complete contracts, $\pi = 1$, but agent 1 would demand a monetary transfer from 2 and would be able to obtain it. For slopes flatter than -1, on the other hand, it is to be expected that while 2 continues to offer the complete contract, 1 would ask for an incomplete contract for certain, but offer monetary compensation to 2. Again it can be seen that the parameterization of the model will play a crucial role.

Overall, then, our analysis points to the following issues: 1) The pure existence of frictions or costs may not be sufficient to cause incomplete contracts — and certainly leaves the form of the incompleteness open to investigation if incompleteness does occur. 2) Incomplete contracts can arise in environments were complete contracts do not have a cost disadvantage. 3) The fact that different procedures (contract types) imply different restrictions on feasible offers and utility transferability can lead to different equilibrium allocations under those procedures. If equilibrium allocations under procedures differ, on the other hand, they may be Pareto non-comparable, and the agent who prefers a certain procedure may be able to have it implemented — even if it leads to an inefficient allocation.

Chapter 2

Perfect Equilibria in a Negotiation Model

2.1 Introduction

The bargaining models of Ståhl (1972) and Rubinstein (1982) were the first models, after Nash's initial analysis of the bargaining problem (1950, 1953), which took the dynamic nature of bargaining and the typical structure of proposals and counterproposals one observes into account. While the model of Rubinstein was considerably more general, the version of it which is most familiar today is the alternating offers, infinite horizon with discounting formulation used extensively in the analysis of the allocation of gains from trade. One of the hallmarks of this particular formulation, and the reason for its popularity, is the fact that it has a unique subgame perfect equilibrium which is efficient and features immediate agreement. This feature has, on the other hand, also been recognized as a limitation of the model, in view of the fact that many circumstances which are thought to feature bargaining also feature observable delay before agreement is reached (if it is reached at all.)

Although many writers have analyzed the contribution of incomplete information to this result,[1] few have studied the influence of the particular assumptions made

[1] A very partial list of papers employing incomplete information to generate delay includes

with respect to the payoff spaces.[2] While Binmore (1987) relaxes Rubinstein's original assumption of a linear bargaining frontier, recent work by Haller and Holden (1990) and Fernandez and Glazer (1991) relaxes the assumption of an exogenous fixed periodic payoff to players if they do not agree. In their model of wage bargaining delay and strike are separated, and the decision to strike in a period of delay is modelled explicitly. Both papers come to the conclusion that multiple equilibria can be supported in such a framework.

Bargaining, however, is not the only dynamic model of allocation. There is a large body of literature concerned with repeated games.[3] Repeated games feature the repeated play of some stage game for some (fixed) period of time. In contrast to bargaining, where players receive their payoffs at the end of the game as a function of their strategies during the game, in repeated games the payoffs accrue during the game, with no strategy dependent payoffs at the end of the game (if the game is finite.) The other difference between the two models is the fixed time of interaction in repeated games versus the endogenous time horizon in bargaining. It is a well known feature of repeated games that for large enough discount factors any individually rational and feasible payoffs can be supported as subgame perfect equilibria, if the game is infinitely repeated or satisfies a dimensionality condition and is "long enough", a fact known as the "Folk Theorem".

The polar predictions of these two models with regard to the set of equilibrium outcomes pose the question which of the features of the models cause their respec-

Admati and Perry (1987), Gul and Sonnenschein (1988), Chaterjee and Samuelson (1987), Ausubel and Deneckere (1989). A survey of related literature can be found in Wilson (1987).

[2]It is known, of course, that different formulations of bargaining costs affect the outcome — see, for example, Rubinstein's original paper (1982)

[3]See, for example, the surveys by Aumann (1986, 1989) and Mertens (1987).

tive results. The example of Haller and Holden (1990) and Fernandez and Glazer (1991) suggests that the fact that bargaining does not allow for strategic payoffs during disagreement might be crucial to the uniqueness result of Rubinstein. They could, on the other hand, not support all payoffs which are feasible and individually rational in the stage game, suggesting that the Folk Theorems do not apply if mutually agreed exit is possible. On the other hand, Okada (1991) does not find a restriction on the Folk Theorems in a repeated game where long run binding agreements can be written, a situation very similar to that addressed here. This paper provides a general model of bargaining under the presence of strategic disagreement period payoffs, or, equivalently, a model of repeated games with endogenous exit. The model sheds further light on the driving forces behind the results of standard bargaining and repeated game models. In particular, the paper will determine the extent to which the Folk Theorem results carry over in this general framework, and to what extent the ability to end the game with mutual consent constrains the Folk Theorem results. The paper will also investigate the circumstances in which uniqueness will occur.

The model employed is a general version of that used by Haller and Holden (1990) and Fernandez and Glazer (1991): Two agents bargain over a surplus of fixed size via a (possibly infinite) sequence of offers and counter offers. In contrast to standard Rubinstein bargaining, the players play a stage game in every period in which no agreement has been reached. The outcome of this stage game is the disagreement period payoff for that period. As in Rubinstein (1982) and some of the infinitely repeated game literature (for example Abreu (1988)), players discount the future.

Although phrased in a bargaining context, the paper utilizes many tools and concepts from the repeated game literature. Indeed, one can think of this model as a repeated game with endogenous exit. In this latter interpretation one can think of a repeated implicit contract situation which may be abandoned only with a mutually acceptable final settlement, which is binding. All results presented in the sequel apply equally to either interpretation.

The outline of the paper is as follows. Section 2 presents the model and defines all necessary concepts. In Section 3 we consider the model under the assumption that bargaining agreement yields a surplus over play of the disagreement stage game. In order to generate some insight into the effects of non-stationary disagreement payoffs, we first investigate whether an exogenously given sequence of disagreement payoffs can upset Rubinstein's uniqueness result. The answer is negative: an arbitrary sequence of disagreement payoffs will result in immediate agreement as long as agreement dominates continued play. The reason is essentially the same as in the original model by Rubinstein: At each point in time the player making the accept/reject decision has some maximum future payoff, consisting of what he can get from his offer next period plus what he can get in the current period as disagreement payoff. He will accept any offer which gives him at least as much as this. Since agents discount and, by assumption, exit offers a surplus over repeated play, the player who makes the offer can make an acceptable offer and collect all of the surplus exit offers over delaying one period. This is true in all periods, and in particular the first.

Following this, the players' optimal punishment payoffs for non-trivial stage games are derived and implemented as subgame perfect equilibria (SPE). Doing so generates two main insights. For one, the presence of a stage game which endogenously generates disagreement payoffs does in general lead to multiple equilibria, but does **not** guarantee multiplicity (and thus delay). Conditions on the stage game which guarantee uniqueness can be derived. This analysis also clarifies why the stage game considered by Haller and Holden (1990) and Fernandez and Glazer (1991) does generate multiplicity.

The second insight is into what "goes wrong" if one tries to support all feasible and individually rational payoffs. Recall the structure of repeated game strategies (e.g. Abreu (1988)). Such strategies consist of an equilibrium path and punishment paths for each player. A one-shot non-optimal strategy in a particular repetition of the stage game is made subgame perfect within the repeated game by punishing a player for a deviation. This is done by starting his punishment path, lowering his payoff in the future. Repeated games have the feature that there always exists a future of a known length. In this model the game ends endogenously, however, and there may not be any future! In particular a player can never be punished for accepting the "wrong" exit offer, since the game is over after he has done so. But the player who makes the offer knows that and may make just such an "irresistible" offer in order to break out of some punishment he is currently facing, thereby limiting the severity of any punishment he can be made to bear. This is particularly true if exit offers a surplus over repeated play.

Finally, section 4 changes the assumption on the payoff spaces. We reconsider the model under the assumption that there do exist stage game payoffs which Pareto dominate some exit payoffs. In this context we show that all stage game payoffs which lie outside the bargaining frontier can always be supported. Multiplicity of equilibria is therefore guaranteed under these circumstances, stressing the importance of the economic rationale for exit, in the form of an available surplus, in the previous model, and in the standard bargaining models.

2.2 The Model

Define a *Negotiation Problem* as a situation where two rational parties who are involved in an ongoing repeated relationship have a surplus available to them if and only if they end their present relationship and can agree on an allocation of the surplus. The questions asked about the negotiation problem are *i*) what agreements are possible in equilibrium, *ii*) how long it will take to reach agreement, and *iii*) what the equilibrium value of such a relationship will be to the parties, taking into account that not only the agreement itself but also the path of play by which it is achieved yields payoffs.

Consider, for example, the following situation: There are two of possibly many Cournot firms in a market. Both face a relatively large fixed cost. Assuming that no side-payments are possible but that a merger into a single firm is allowed, they could both gain if they were to merge and operate only one plant. Negotiations are held over the allocation of the resulting surplus. While these negotiations are under way, however, the two continue to produce and serve the market as separate firms.

This situation is modelled as follows[4]: There are two players, indexed by $i = 1, 2$. Time is discrete and indexed by $t = 1, 2, 3, \ldots$. The time horizon is infinite. Both players discount the future, with their (common) discount factor being denoted by $\delta \in (0, 1)$. In every period t in which no agreement has yet been reached, the players play the following *constituent game*: At the beginning of the period, one player makes an offer to the other player. The offer is in terms of the players' shares of the surplus resulting from agreement. The other player can then either agree or disagree with this offer. Should he agree, both players receive their share of the

[4]For a model of similar flavour see Haller and Holden (1990) or Fernandez and Glazer (1991).

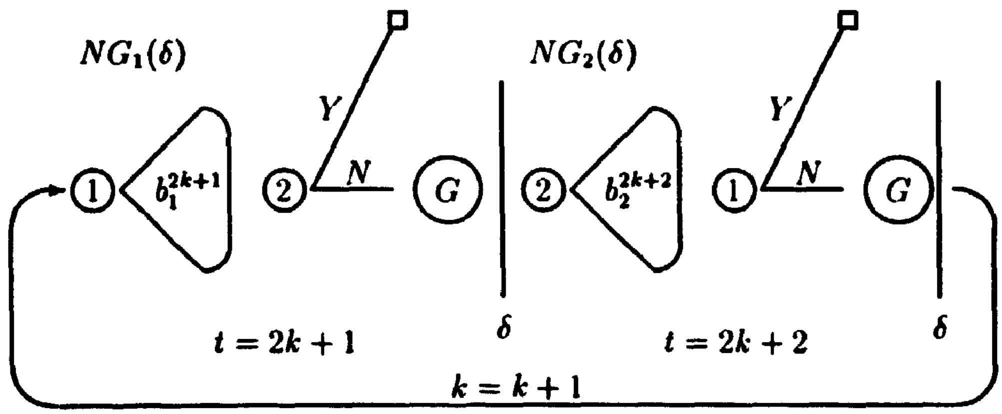

Figure 2.1: Schematic of a Negotiation Game

surplus from this period onward forever, and their prior strategic relationship, and the game, ends. Should he not agree, both players play a simultaneous move game in normal form, called the *stage game* and denoted G, the outcome of which determines the players' payoffs for this period. Time then advances and the constituent game is repeated. The paper only analyses the case where the two players make alternating proposals, with player 1 proposing in odd periods and player 2 proposing in even periods. A schematic of the game is given in Figure 1. We will now define the necessary notation.

First, consider the exit share bargain within each period. The offer and agreement are formulated as in Rubinstein (1982). A proposal by a player is a vector in the unit simplex of $\mathbf{R}^2$, say $(b, 1-b)$, where b is player 1's share of the surplus and $(1-b)$ is player 2's share. A proposal is denoted just by its first coordinate, $b \in [0,1]$. A player's response to a proposal is either rejection or acceptance, indicated by N and Y, respectively. The players are said to *reach agreement* if one player accepts the other one's proposal. The negotiation game ends when players reach agreement, and the players obtain the same proportion of the surplus, which is given by the proposal which was accepted, in each of the subsequent periods.

Next, consider the stage game which is played after a proposal b has been rejected. The stage game is modelled as a two-player one-shot game in normal form. It consists of a set of two players, their strategy (action) sets, and their payoff functions, and is given by $G = \{A_1, A_2, u_1(\cdot), u_2(\cdot)\}$. Here, A_i is player i's ***strategy (action) set***, assumed compact, and $u_i(\cdot) : A \rightarrow \mathbf{R}$ is his ***payoff function***, assumed continuous, where $A = A_1 \times A_2$.

The set A can also be interpreted as the set of outcomes of the stage game G. A generic element of the set A is denoted $a = (a_1, a_2)$. Let $u(\cdot) = (u_1(\cdot), u_2(\cdot)) : A \rightarrow \mathbf{R}^2$. The set of *feasible payoffs* of the stage game G is given by the convex hull of $u(A)$, $Co[u(A)]$. Let m^i, $i = 1, 2$, denote the strategy pair leading to player i's minimax payoff. The set of *feasible and individually rational payoffs* is the intersection of $Co[u(A)]$ and $\{v \in \mathbf{R}^2 | v_1 \geq u_1(m^1),\ v_2 \geq u_2(m^2)\}$. It is denoted by F.

To simplify the analysis, the following assumptions are made:

> **A1:** *The players' strategies in G are correlated mixed strategies, and deviations by either player are publicly observable.*
>
> **A2:** *The surplus from agreement is* 1 *and the stage game* G *is normalized such that* $u_i(m^i) = 0$ *for* $i = 1, 2$.

A1 implies that the set A_i is convex for $i = 1, 2$; that for any feasible payoff vector v, $\exists\ a \in A$ such that $v = u(a)$; and that the stage game G has at least one Nash equilibrium.[5] **A2** is assumed for convenience only.

One further assumption on the payoff spaces is made. It is that the surplus from agreement dominates the payoffs from G, giving rise to gains from trade.

[5]We have chosen not to introduce more notation for the correlated strategies. Formally, we assume that there exists a publicly observable randomization device which players can condition on. Since the device is public, deviations can be observed.

Diagrammatically, this implies that the stage game payoffs are everywhere below the exit payoff line (see Figure 2). This condition is formally stated as:

A3: $\forall\, a \in A,\;\; u_1(a) + u_2(a) \leq 1.$

Define the *negotiation game* $NG(\delta)$ to be the game in which 2 players with discount factor δ play a sequence of constituent games until agreement, where a constituent game is an offer game followed, after rejection, by the stage game G, and agreement is the acceptance of a proposal. Let $NG_i(\delta)$ be the game in which player i makes the proposal in the first period (note that by convention the first period in $NG_2(\delta)$ is an even period). For the sake of brevity, all results will be proven only for $NG_1(\delta)$, the proof for $NG_2(\delta)$ following in an analogous manner. Figure 2 gives a diagrammatical representation of the payoff frontiers in a typical constituent game.

Define a *type 1 t-period history* in the game $NG_i(\delta)$ as a finite sequence denoted by $h_1(t) = (b^1, a^1, \ldots, b^t, a^t)$, in which b^s is the proposal made in period s and $a^s \in A$ is the outcome of G in period s after proposal b^s had been rejected, for $s = 1, \ldots, t$. Let $h_1(0) = \emptyset$. A type 1 t-period history can be decomposed as $h_1(t) = b(t) \oplus a(t)$ where

$$b(t) = (b^1, \ldots, b^t) \in [0,1]^t; \;\; a(t) = (a^1, \ldots, a^t) \in A^t$$

A *type 2 t-period history* in the game $NG_i(\delta)$ is denoted by $h_2(t) = h_1(t) \oplus b^{t+1}$, indicating that following the type 1 t-period history $h_1(t)$, b^{t+1} has been proposed in period $(t+1)$.

A *type 3 t-period history* in the game $NG_i(\delta)$ is denoted by $h_3(t) = h_2(t) \oplus \{N\}$, indicating that the proposal b^{t+1} has been rejected in period $(t+1)$.

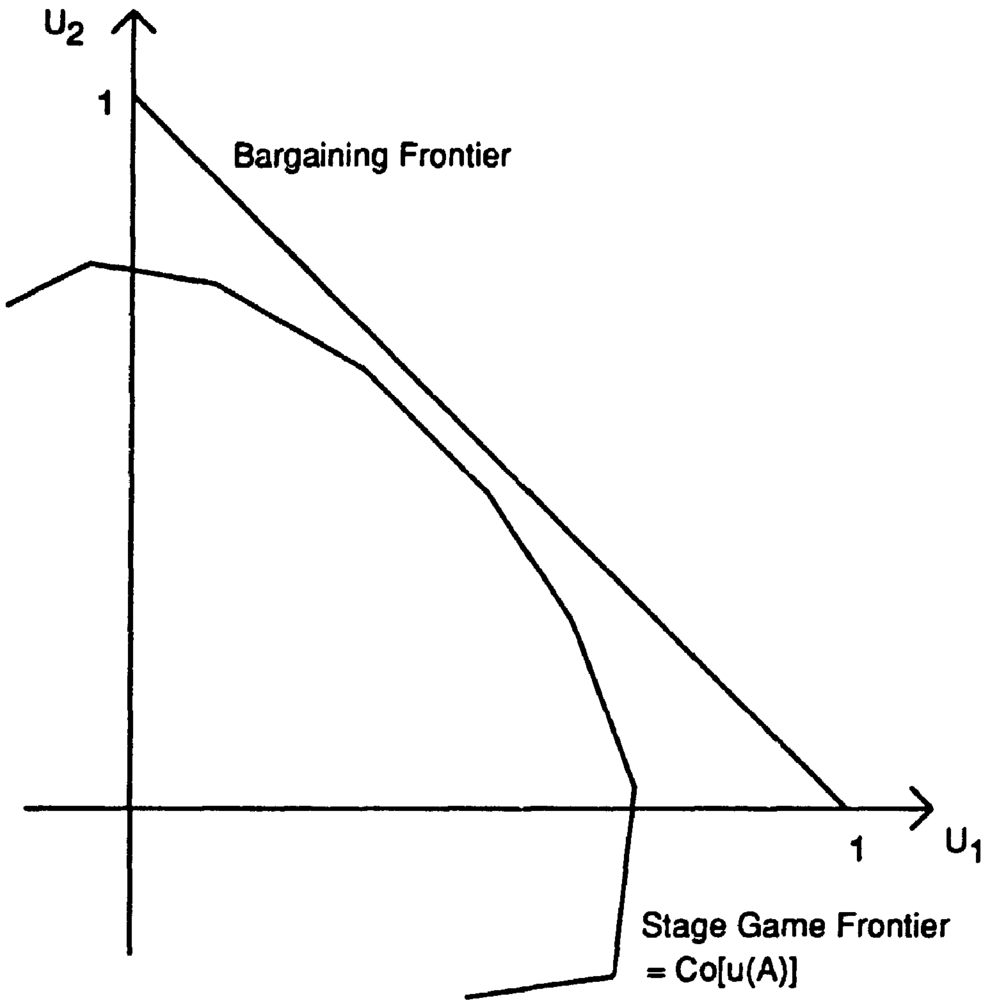

Figure 2.2: Constituent Game Payoff Frontiers

The sets of all possible histories of all three types, H_1, H_2 and H_3, can be written in the usual way by taking the appropriate countably infinite unions over time of the sets of all possible t-period histories. For example for H_1: $H_1 = \cup_{t=0}^{\infty} H_1(t) = \cup_{t=0}^{\infty}([0,1]^t \times A^t))$.

A *strategy combination* $f = (f_1, f_2)$ for the game $NG_1(\delta)$ consists of two functions which map from the sets of all appropriate histories to the sets of all appropriate actions, such that

$$(f_1, f_2) \; : \; H_1(t) \times H_2(t) \rightarrow [0,1] \times \{Y, N\} \quad \text{if } t \text{ is even}$$

$$(f_1, f_2) \; : \; H_2(t) \times H_1(t) \rightarrow \{Y, N\} \times [0,1] \quad \text{if } t \text{ is odd}$$

$$(f_1, f_2) \; : \; H_3 \rightarrow A$$

The strategy combination f gives the players' instructions on how to play the game in every period, conditional on history. For example, in the odd period $(t+1)$ after the type 1 t-period history $h_1(t)$, $f_1(h_1(t))$ gives player 1's proposal b^{t+1}, $f_2(h_2(t))$ gives player 2's response to player 1's proposal, and $(f_1(h_3(t)), f_2(h_3(t)))$ is the one-shot play of the stage game G in period $(t+1)$ after 1's proposal has been rejected by 2.

An *outcome path* of $NG_1(\delta)$, $\pi(T) = (b^1, a^1, b^2, a^2, \ldots, b^T, \{Y\})$, can be interpreted to indicate that the proposal b^t has been rejected and the stage outcome a^t has been played in period t for $1 \leq t < T$, and that the proposal b^T has been accepted in period T. By convention, T is set to infinity in any outcome path in which the two players never reach agreement. An outcome path of $NG_1(\delta)$ can be decomposed as $\pi(T) = b(T) \oplus a(T-1) \oplus \{Y\}$.

The *payoff* to the players from outcome path $\pi(T)$ is determined by the stage game outcomes in all periods before agreement is reached and by the agreement itself. The average payoffs the players receive from the outcome path $\pi(T)$ are given by

$$U_1(\pi(T)) = (1-\delta)\sum_{t=1}^{T-1} \delta^{t-1} u_1(a^t) + \delta^{T-1} b^T \tag{1}$$

$$U_2(\pi(T)) = (1-\delta)\sum_{t=1}^{T-1} \delta^{t-1} u_2(a^t) + \delta^{T-1}(1-b^T) \tag{2}$$

Since a strategy combination f induces a unique outcome path in the game $NG_1(\delta)$, the average payoffs from f can be calculated directly from equations (1) and (2) and the outcome path induced by f.

2.3 Subgame Perfect Equilibria

In what follows the subgame perfect equilibria (SPE) of the negotiation game $NG_1(\delta)$ are characterized. In order to generate some intuition on how the game behaves, two examples are analyzed first.

2.3.1 Existence and Two Examples

Example 1: Consider the negotiation game which consists of a surplus of size 1 and a stage game G with the following payoff matrix:

$1\backslash 2$	C	D
C	$(.4, .4)$	$(-.2, .6)$
D	$(.6, -.2)$	$(0, 0)^*$

Note that the payoff vector $(0, 0)$ is both the minimax and one-shot Nash equilibrium outcome in the stage game G.[6] This payoff is also the same as the status quo payoff in the standard Rubinstein game. The following claim should therefore come as no surprise.

Claim 1: *The negotiation game of Example 1 has a Subgame Perfect Equilibrium in which player 1's proposal of* $1/(1+\delta)$ *is accepted by player 2 in the first period.*

The offer-accept/reject part of the strategies implementing this equilibrium are identical to those implementing the equilibrium in the Rubinstein game. Player 1 proposes $1/(1+\delta)$ irrespective of the history of the game and rejects any proposal less than $\delta/(1+\delta)$. Player 2 rejects any proposal larger than $1/(1+\delta)$ and proposes $\delta/(1+\delta)$ irrespective of the history of the game. The stage game part of the strategies is as follows: Should the stage game G be reached in any period, both players play their Nash equilibrium strategies D, irrespective of history.

[6]To keep the examples simple, we consider only pure strategies. Since the examples have only two pure strategies, one of which is strictly dominated, the results will go through for correlated strategies.

While we will not formally prove the subgame perfection of these strategies, the following arguments should be convincing. Consider the stage game. In every period in which no agreement has been reached the Nash equilibrium of the stage game is played, irrespective of history. The SPE of the negotiation game is constructed using the one-shot Nash payoffs in G as the status quo point in every period. Since the strategies in G are Nash and the proposal/reject strategies are history independent, both players have no incentive to deviate in G. But then the game effectively reduces (for those strategies) to a bargaining game with a fixed status quo point, and subgame perfection of the offers and accept/reject decisions follows. This argument is made formally in the proof to the next theorem, which asserts existence of SPE in negotiation games.

Theorem 1 *Suppose that $a^* \in A$ is a Nash equilibrium in the stage game G. $\forall\ \delta \in (0,1)$, $NG_1(\delta)$ has a subgame perfect equilibrium in which player i's proposal b_i^* is accepted by player $j \neq i$, where*

$$b_1^* = \frac{1 + \delta u_1(a^*) - u_2(a^*)}{1+\delta} \qquad \textit{and} \qquad b_2^* = \frac{\delta + u_1(a^*) - \delta u_2(a^*)}{1+\delta}$$

It should be clear that, should the stage game have more than one Nash equilibrium outcome, any sequence of one shot Nash equilibrium outcomes can be used in the stage game part of the strategies. This observation guarantees a multiplicity of equilibria in negotiation games if the stage game has multiple Nash equilibrium payoffs. In particular, with just two different Nash equilibrium payoffs in the stage game, infinitely many SPE result for the associated negotiation game. The argument is similar to the one before: Every time the stage game is reached, the proposed strategy is to play one of the Nash equilibria. By definition, a player can not gain from a one-shot deviation in the stage game, since the stage game payoff is already Nash, and future strategies are history independent. Deviations in offers are likewise not possible, if the offers are computed according to the rule that the other player is just indifferent between accepting and rejecting. But since there are infinitely many different sequences of Nash equilibrium payoffs possible, there are infinitely many different discounted future payoffs which can be supported. The result follows.

The equilibrium of Claim 1 is not the only one in the negotiation game of Example 1, however. The next Claim demonstrates that multiplicity of equilibria can occur even if G has a unique Nash equilibrium.

Claim 2: *In the negotiation game $NG_1(\delta)$ of Example 1 the average payoffs $((1-.4)/(1+\delta),\ (\delta+.4)/(1+\delta))$ can be supported as a subgame perfect equilibrium.*

This claim states that player 1 can do considerably worse than the "Rubinstein payoffs" of Claim 1. In fact, as will be formally shown later, the equilibrium proposed here yields the worst possible equilibrium payoff for player 1 in the example game. The strategies that support this equilibrium are as follows:

In an odd period player 1 proposes $.6/(1+\delta)$, and player 2 accepts this proposal, unless player 2 has deviated from the equilibrium strategies at any stage. Any higher proposal is rejected by 2 and the players play the strategy pair (C, D), yielding a one period payoff of $(-.2, .6)$. In the following even period, if player 1 has not deviated from C in the last period, player 2 proposes $(.2+.4\delta^2)/\delta(1+\delta)$ and 1 accepts any proposal at least that big. However, if 1 deviated from C, player 2 proposes $.6\delta/(1+\delta)$ instead, and 1 accepts any proposal at least that big. Should 1 not accept these offers by player 2, they play the strategy pair (D, D) in G, yielding a one period payoff of $(0, 0)$. The strategies then repeat. Finally, should 2 not make the required offers, or should 2 not accept 1's equilibrium offer, the players will follow the equilibrium strategies of Claim 1 from the next subgame on. Deviations by 2 in G are ignored.

Interestingly, the average payoffs in Claim 2 are the same as those in a Rubinstein bargaining game with a status quo point of (0, .4). Also note that, considering only the stage game sequence of payoffs, the disagreement period payoff alternates between $(-.2, .6)$ in odd periods and $(0, 0)$ in even periods. Furthermore, player 2 compensates player 1 precisely for not deviating from the not one-shot Nash strategies (C, D), thereby making them subgame perfect. After this compensation 2 is left with a net payoff of .4. Note that 2 will pay the compensation since if he fails to do so play reverts to the strategies of Claim 1, which yield a lower payoff to him than following the equilibrium strategies proposed here. Thus the $(-.2, .6)$ stage game payoff is effectively worth $(0, .4)$ to him, since 1 gets compensated for not deviating to 0, leaving 2 with $.6 - .2$. The significance of these observations will be explained shortly.

Lest it appear that all negotiation games have multiple equilibria, we now present a courter example. Consider the following negotiation game.

Example 2: The surplus is of size 1, and the stage game G has the following payoff matrix:

1\2	C	D
C	$(.4, .4)$	$(-.8, .8)$
D	$(.8, -.8)$	$(0, 0)^*$

Claim 3: *The negotiation game of Example 2 has a unique subgame perfect equilibrium. It is given by history independent offers of* $1/(1+\delta)$ *by player 1 and* $\delta/(1+\delta)$ *by player 2, and the play of* (D, D) *in* G *in all periods in which an offer has been rejected.*

In this example strategies analogous to those used in Claim 2 do not lower 1's payoff. Notice in particular that the payoffs in G do not allow 2 to receive a higher one period disagreement payoff than from simply playing the Nash equilibrium, since any compensation would wipe out his payoffs completely. As indicated before, in a SPE 2 has to effectively compensate 1 for not playing a one shot best response. He can not do so in this game while getting a higher payoff since the deviation gain by 1 for every strategy is at least as much as 2's payoff. This causes his future payoffs (the ones from rejecting a deviating exit proposal) to be 0 in every stage game, and thus 1 can exit at Rubinstein shares, since 2 will accept any such offer. Notice here that we can not get 2 not to accept such offers, since we can never punish him for doing so.

The stage game in this second example has the same structure as that in the first example, both games being variants of the Prisoners' Dilemma game. In particular, both have a unique Nash equilibrium which is also the mutual minimax, and players have only 2 pure strategies, of which one is strictly dominated in both games. Nevertheless, the equilibrium sets of the associated negotiation games apparently differ dramatically. In the remainder of the paper we will characterize the equilibrium set of negotiation games, which will allow us to state conditions which guarantee uniqueness.

2.3.2 Optimal Punishments

The *optimal punishment* for player i in $NG_1(\delta)$ is defined as that SPE in which player i's equilibrium payoff is less than or equal to all SPE payoffs of the game. Since the arguments are analogous for both players, only player 1's optimal punishment will be derived explicitly.

The derivation of the optimal punishment will proceed in two steps. First, the lower bound of player 1's equilibrium payoffs is computed for the game $NG_1(\delta)$. Then, a SPE is constructed such that player 1's average equilibrium payoff achieves this lower bound. It then follows that the equilibrium which has been constructed is in fact the optimal punishment for player 1.

Consider for a moment the type of strategies used as punishments in repeated games, in particular the simple penal codes of Abreu (1988). One feature of these punishments is that the punishment of player 1 is enforced by restarting the same punishment should 1 deviate from it, and by starting a punishment for player 2 should he fail to punish 1. Deviations by more than one player in the same subgame are ignored. This structure makes the strategies particularly simple. We employ strategies of a similar nature here, only complicated by the fact that the game is not symmetric between even and odd periods, and that it has 3 subgames per period. Formally, this implies multiple punishment paths for each player, one for every subgame in odd and even periods. As will be seen later, however, it is sufficient to specify a path starting in the player's offer period. This "generic" punishment is then simply "picked up" in the appropriate subgame, instead of at its literal beginning. As mentioned in the introduction, there is an additional difference to simple penal codes, however. While in repeated games there exists a punishment after every history, in negotiation games player 2 can not be punished for wrongly accepting a deviating exit offer. Contrary to repeated games, in which there always exists a future,[7] a deviating accept decision in negotiations ends the game, precluding punishment of the deviation. Combined with the fact that exit offers a surplus

[7]Notice how in finitely repeated games a long enough time span is required, and how the punishments collapse near the time horizon.

over continued play, this limits the severity of the punishment which a player can be made to suffer in negotiation games. This intuition is borne out in the results which follow. Player 2, for example, will accept any offer which yields at least as much as punishing 1 in the future, and 1 will therefore want to make such an offer, since it leaves him better off than delaying by virtue of exit offering a surplus over continuation (**A3**). It follows that, while one is only concerned with minimizing the punished player's payoffs, subject to the punisher receiving at least as much as in his own punishment, in repeated games, in negotiation games one has to minimize one player's payoff while maximizing the other's at the same time in order for that player to reject the highest possible exit offers.

Before we derive the worst punishment payoffs, we present the following Lemma. It is useful in clarifying the role that disagreement period payoffs play in players' offer strategies. The Lemma concerns Rubinstein type bargaining games with exogenous but time variant status quo points. It upholds Rubinstein's uniqueness result for this class of bargaining games, under the condition of common discount factors.

Lemma 1: *In a Rubinstein type bargaining game with alternating offers over the split of a surplus of size 1 in which players have a common discount factor δ and receive the (exogenous) payoff $(u_1(t), u_2(t))$ in period t if no agreement has yet been reached, and where $u_1(t) + u_2(t) \leq 1$, the unique subgame perfect equilibrium is that the proposal b^t is accepted in period t, where b^t is given as follows:*

If t is odd

$$b^t = \frac{1}{1+\delta} + (1-\delta)\sum_{k=0}^{\infty} \delta^{2k}[\delta u_1(t+2k+1) - u_2(t+2k)]$$

and if t is even

$$b^t = \frac{\delta}{1+\delta} + (1-\delta)\sum_{k=0}^{\infty} \delta^{2k}[u_1(t+2k) - \delta u_2(t+2k+1)]$$

Consider player 1's payoff in an odd period. The Lemma shows that 1's payoff depends positively on 1's disagreement payoffs in even periods, and negatively on 2's disagreement payoffs in odd periods. The interesting point the Lemma makes about equilibrium offers is the fact that only the disagreement payoffs received in periods in which the player does *not* make an offer enter into the computation of the equilibrium offers. A player's disagreement payoffs in periods in which he himself makes an offer are totally irrelevant to his equilibrium payoffs. Thus, only payer 1's even period payoffs and player 2's odd period payoffs affect the equilibrium offers. Therefore, if the choice of the sequence of disagreement payoffs were up to the modeller, then the Lemma shows that it is necessary to minimize 1's payoffs in his accept/reject periods and to maximize 2's payoffs in 1's offer periods in order to minimize 1's game payoff. The question we have to address is therefore how one can support a sequence of stage game strategies which achieve this as part of a subgame perfect equilibrium.

In light of this, consider again the strategies outlined for implementing the equilibrium in Claim 2. The unusual feature of these strategies is that they call for 1 to play C in G in odd periods, a strategy which is not 1-shot optimal in G. In order to make this strategy subgame perfect, player 1 is compensated in the following exit proposal, and his payoff from playing along with the strategy and being compensated is indeed the same as his payoff from deviating to the 1-shot optimal strategy D and not being compensated. In Example 1, this strategy choice leads to a net gain for player 2 in odd periods, since he obtains 0.6 and compensates 1 by 0.2. Player 2's 'effective disagreement payoff' in an odd period is thus 0.4. In even periods player 1 is minimaxed and receives zero. periods, when he is minimaxed.

Using these values, $u_1(even) = 0$, $u_2(odd) = .4$, in Lemma 1, we obtain the offer claimed before.

This provides the motivation for the candidate strategies in 1's punishments. As Lemma 1 has shown, a higher disagreement payoff for 2 in odd periods will decrease the exit offer made by 1, while a lower disagreement payoff to 1 in even periods will also decrease the exit offer. We know that minimaxing 1 in even periods will yield the worst possible payoff to him in even periods. The remaining question is what the highest possible 'effective disagreement payoff' in odd periods is for player 2, given any game G. We will now turn our attention to that issue.

As the examples have indicated, a strategy which is not one-shot optimal in G can nevertheless be supported as part of a SPE strategy profile if the player is compensated for the foregone deviation gain in the future. Notice here that while we are unable to force 2 to reject certain offers, we can force him to make an offer higher than 1's expected future payoffs, as is necessary if he is to compensate 1, due to the fact that 1 can reject and punish 2, should he fail to do so.[8] Let y_1^* denote player 2's highest effective disagreement payoff. It is defined as

$$y_1^* = \max_{a \in A}\{u_2(a) - (\max_{a_1' \in A_1} u_1(a_1', a_2) - u_1(a))\}. \tag{3}$$

For future reference in strategy profiles, also define the strategy combination in G which achieves y_1^* as[9]

$$a^1 = \operatorname{Arg}\max_{a \in A}\{u_2(a) - (\max_{a_1' \in A_1} u_1(a_1', a_2) - u_1(a))\}. \tag{4}$$

---

[8]Notice that the bargaining frontier slope is -1, implying that any equilibrium worse for 1 than the always existing "play Nash and exit" equilibrium of Theorem 1 must simultaneously be better for 2 if it involves exit. Thus, such a punishment for 2 is guaranteed to exist.

[9]This may, of course, not be unique. Since player 1's payoff does not affect the exit offer and exit will occur immediately in equilibrium, any such strategy may be chosen.

The value y_1^* gives the maximum difference between player 2's payoff and player 1's best deviation gain, over all strategies in G. This is, in fact, the maximum payoff 2 can effectively obtain from disagreement in odd periods under the candidate punishment strategies for 1. The value of y_1^* depends only on the structure of G, and **A2** and **A3** imply $0 \leq y_1^* \leq 1$. It remains to be shown that a^1 can be supported in equilibrium and leads to the worst payoff for 1. This is done in two steps. The following theorem derives the lower bounds of player 1's payoffs in negotiation games.

Theorem 2 *Player 1's average equilibrium payoffs in the game $NG_1(\delta)$ $[NG_2(\delta)]$ are bounded below by $\frac{1}{1+\delta}(1 - y_1^*)$ $\left[\frac{\delta}{1+\delta}(1 - y_1^*)\right]$.*

The proof derives the infimum of player 1's equilibrium payoffs, which is a meaningful concept due to Theorem 1. In doing so, the restrictions imposed by subgame perfection and the fact that $u_1(a) + u_2(a) \leq 1$ are exploited.

The next theorem completes the proof of the assertion that a^1 can be supported by demonstrating that the lower bound of theorem 2 can be achieved by a subgame perfect equilibrium of $NG_1(\delta)$ $[NG_2(\delta)]$. Equilibrium strategies are given in which player 1's average equilibrium payoff is $\frac{1}{1+\delta}(1 - y_1^*)$ $\left[\frac{\delta}{1+\delta}(1 - y_1^*)\right]$ for large enough δ. Theorems 2 and 3 together directly imply that this constructed SPE is the optimal punishment equilibrium for player 1 in $NG_1(\delta)$ $[NG_2(\delta)]$.

Theorem 3 *There exists a $\underline{\delta} \in (0,1)$ such that, $\forall\, \delta \in (\underline{\delta}, 1)$, the average payoff vector*

$$\left(\frac{1 - y_1^*}{1+\delta}, \frac{\delta + y_1^*}{1+\delta}\right) \qquad \left[\left(\frac{\delta(1 - y_1^*)}{1+\delta}, \frac{1 + \delta y_1^*}{1+\delta}\right)\right]$$

can be supported by a subgame perfect equilibrium in the game $NG_1(\delta)$ $[NG_2(\delta)]$.

The strategies implementing these payoffs are defined recursively and given below for $NG_1(\delta)$. Strategies for $NG_2(\delta)$ are analogous. In the strategies, a^* refers to a Nash equilibrium strategy in G, and a^1 and m^1 are as defined previously.[10]

In the first period, players' strategies are

$$
\begin{aligned}
f_1(h_1(0)) &= \frac{1-y_1^*}{1+\delta} \\
f_2(b^1) &= \begin{cases} Y & \text{if } b^1 \le \frac{1}{1+\delta}(1-y_1^*) \\ N & \text{otherwise} \end{cases} \\
(f_1, f_2)(b^1 \oplus \{N\}) &= \begin{cases} a^* & \text{if } b^1 \le \frac{1}{1+\delta}(1-y_1^*) \\ a^1 & \text{otherwise} \end{cases}
\end{aligned}
$$

Thereafter, $\forall\, h_1(t) = a(t) \oplus b(t) \in H_1$, $h_2(t) = a(t) \oplus b(t+1) \in H_2$ and $h_3(t) = a(t) \oplus b(t+1) \oplus \{N\} \in H_3$:

For an odd period $(t+1)$

$$
\begin{aligned}
f_1(h_1(t)) &= \begin{cases} \frac{1}{1+\delta}(1+\delta u_1(a^*) - u_2(a^*)) & \text{if either } (f_1, f_2)(h_3(t-1)) = a^* \\ & \text{or } a_1^t = f_1(h_3(t-1)) \text{ and} \\ & \quad a_2^t \ne f_2(h_3(t-1)) \\ \frac{1}{1+\delta}(1-y_1^*) & \text{otherwise} \end{cases} \\
f_2(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \le f_{11}(h_1(t)) \\ N & \text{otherwise} \end{cases} \\
(f_1, f_2)(h_3(t)) &= \begin{cases} a^* & \text{if either } (f_1, f_2)(h_3(t-1)) = a^* \text{ or } b^{t+1} \le f_1(h_1(t)) \\ & \text{or } a_1^t = f_1(h_3(t-1)) \text{ and } a_2^t \ne f_2(h_3(t-1)) \\ a^1 & \text{otherwise} \end{cases}
\end{aligned}
$$

and for an even period $(t+1)$

$$
\begin{aligned}
f_2(h_1(t)) &= \begin{cases} \frac{1}{1+\delta}(\delta + u_1(a^*) - \delta u_2(a^*)) & \text{if either } (f_1, f_2)(h_3(t-1)) = a^* \\ & \text{or } a_1^t = f_1(h_3(t-1)) \text{ and} \\ & \quad a_2^t \ne f_2(h_3(t-1)) \\ \frac{\delta}{1+\delta}(1-y_1^*) & \text{if } a_1^t \ne f_1(h_3(t-1)) \text{ and} \\ & \quad a_2^t = f_2(h_3(t-1)) \\ \frac{1-\delta}{\delta}u_2(a^1) + \frac{\delta^2 - y_1^*}{\delta(1+\delta)} & \text{otherwise} \end{cases} \\
f_1(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \ge f_2(h_1(t)) \\ N & \text{otherwise} \end{cases} \\
(f_1, f_2)(h_3(t)) &= \begin{cases} a^* & \text{if either } a_1^t = f_1(h_3(t-1)) \text{ and } a_2^t \ne f_2(h_3(t-1)) \\ & \text{or } b^{t+1} < f_2(h_1(t)), \text{ or } (f_1, f_2)(h_3(t-1)) = a^* \\ m^1 & \text{otherwise} \end{cases}
\end{aligned}
$$

---

[10] Note that the choice of a^* affects the value of $\underline{\delta}$ needed.

Theorems 2 and 3 together have confirmed the intuition derived from Lemma 1: If y_1^* is positive, then there exist strategies in G which can be supported as part of a SPE that give player 2 an "effective" payoff of y_1^* in odd period stage games. In even periods, player 1 is minimaxed and his payoff is 0. The Lemma suggested that only the payoffs available in periods when a player is called upon to accept or reject an offer matter to the exit offers consistent with equilibrium for exogenously given payoffs. What we have shown is that with the strategies proposed, it is as *if* player 1 were to receive 0 as disagreement value, while player 2 receives y_1^*. The equilibrium exit offers are thus the same as in a Rubinstein game with the disagreement payoff (0, y_1^*). Notice, however, that (0, y_1^*) must not lie in the feasible set of the stage game payoffs, since it is never really played at all.

While complex to write formally, the equilibrium strategies are simple to describe: Unless player 2 has deviated, player 1 always offers the equilibrium offer. Should he fail to do so, the players play the stage game and follow the strategies leading to 2's highest effective disagreement payoff. Player 2 then offers exit at the value corresponding to the value to 1 of being minimaxed once and having to restart the punishment, plus an amount just equal to that foregone by 1 in the stage game from not deviating but following the equilibrium. Should 1 deviate in the stage game this "compensation" is not paid. In any case, should 1 not accept an equilibrium offer by 2, he is minimaxed once, and the punishment is restarted. Any deviation by 2 is punished by reverting to the play of some Nash equilibrium in all future stage games, which implies some exit offers consistent with those strategies.

By arguments analogous to those above, and without proof, the optimal punishment for player 2 can be found. Let

$$x_2^* = \max_{a \in A}\{u_1(a) - (\max_{a_2' \in A_2} u_2(a_1, a_2') - u_2(a))\} \tag{5}$$

and

$$a^2 = \text{Arg}\max_{a \in A}\{u_1(a) - (\max_{a_2' \in A_2} u_2(a_1, a_2') - u_2(a))\} \tag{6}$$

be player 1's highest effective disagreement payoff in even periods in player 2's punishment, and the strategy combination in G implementing it, respectively. As before, $0 \leq x_2^* \leq 1$. The following theorem gives the optimal punishment payoffs for player 2 as a function of x_2^*.

Theorem 4 *There exists a $\underline{\delta} \in (0,1)$ such that, $\forall \delta \in (\underline{\delta}, 1)$, the average payoff vector*

$$\left(\frac{1 + \delta x_2^*}{1 + \delta}, \frac{\delta(1 - x_2^*)}{1 + \delta}\right) \qquad \left[\left(\frac{\delta + x_2^*}{1 + \delta}, \frac{1 - x_2^*}{1 + \delta}\right)\right]$$

can be supported by a subgame perfect equilibrium in the game $NG_1(\delta)$ $[NG_2(\delta)]$.

2.4 Perfect Equilibria in Negotiation Games

In the spirit of the Folk Theorem literature, the characterization of the supportable equilibrium payoffs is for "large enough" discount factors. Define the following limiting values are as δ tends to 1:

$$\underline{v_1} = \frac{1}{2}(1 - y_1^*) \quad \text{and} \quad \underline{v_2} = \frac{1}{2}(1 - x_2^*). \tag{7}$$

The results so far indicate that player i's equilibrium payoffs in the negotiation game $NG_i(\delta)$ are bounded below by $\underline{v_i}$. The outstanding question at this point is if indeed all feasible payoffs above the lower bound $\underline{v_i}$ can be supported as SPE for "large enough" discount factors. This question is answered in the affirmative by the following theorem.

Theorem 5 *For a given feasible payoff vector* (v_1, v_2) *in the negotiation game* $NG_1(\delta)$ $[NG_2(\delta)]$ *such that* $(v_1, v_2) > (\underline{v_1}, \underline{v_2})$*, there exists* $\underline{\delta} \in (0,1)$ *such that* $\forall\ \delta \in (\underline{\delta}, 1)$*,* $NG_1(\delta)$ $[NG_2(\delta)]$ *has a subgame perfect equilibrium with average payoff* (v_1, v_2)*.*

The equilibrium strategy profiles implementing any such equilibrium are straight forward. They have a structure similar to the simple penal codes developed by Abreu (1988). First, an outcome path which leads to the average payoff (v_1, v_2) is found. It consists of the agreement players reach in some period T, and the outcomes of the stage game G in every period before agreement is reached. The outcome path is, in general, not unique. In order for it to be permissible, both players must have a future average payoff above their respective minimum payoffs at every point in the path. The structure of the game guarantees that there exists at least one outcome path with that feature for any average payoff above the lower bound. The following strategies then implement this outcome path.

In every period before the last, the player who makes the proposal demands the whole value of the surplus for himself. Any other offer will be considered a deviation on the part of that player, and he will be punished by implementation of his punishment equilibrium, subject to the fact that the other player accepts a proposal made before the last period if the proposal pays him more than he could obtain if the other player were punished. In the stage game, players play strategies leading to the appropriate outcome for the period as specified in the equilibrium outcome path. If a player deviates from his strategy in the stage game, he is punished by implementation of his punishment equilibrium. Simultaneous deviations by the players are ignored. The strategies are formally given in the appendix.

Theorem 5 characterizes all subgame perfect equilibria of negotiation games. It shows that all payoffs above players' optimal punishment payoffs can be supported in such games. This result is of precisely the same flavour as the Folk Theorems: Any outcome above some lower bound is supportable. The lower bound is determined by what a player can be held to under individual rationality. In negotiation games this is determined by a split of the surplus consistent with minimaxing a player while (in alternating periods) maximizing his punisher's payoff in the stage game.

The results show that negotiation games can have a multiplicity of equilibria. While there will be a range of efficient equilibria along the bargaining frontier, this multiplicity implies, as in Folk Theorems, that Pareto inefficient outcomes can also be an equilibrium. Delay, where the parties do not agree to a split of the surplus right away, can therefore generally be supported in negotiation games. Indeed, infinite delay can be supported for some games. This is true even though there is a surplus over continuation available via exit, and even though players have complete information. The former point contrasts the results in this model from those of Haller and Holden (1990) and Fernandez and Glazer (1991), where exit did not necessarily yield a surplus over continuation (since the Nash equilibrium of their stage game lies on the exit frontier), while the latter contrasts them from the incomplete information bargaining literature, where delay acts as a signal in a world of incomplete information.

The model also shows explicitly how the structure of the stage game and the size of the surplus which agreement yields over continued play of the stage game, the size of the gains from trade, affect the equilibrium set. In particular, Theorem 5 shows that the pure existence of strategic payoffs in disagreement periods is not sufficient to guarantee multiplicity of equilibria. It is necessary to compute x_2^* and y_1^* to determine if multiplicity — and delay — can occur. Only if both are equal to zero will there be a unique equilibrium and no delay. While not sufficient, this implies that any Nash equilibria must be mutual minimax. Furthermore, the game must have the feature that for all strategies a player's payoff is less or equal to the other player's gain from deviation.

2.5 SPE if A3 is violated

In order to stress the role of the availability of a surplus from exit on the results, we will now relax assumption 3. Consider, then, the same model when assumption 3 is violated. What this implies, is that while some exit payoffs may dominate some repeated game payoffs, there are repeated game payoffs which dominate exit. Diagrammatically, this case is represented in Figure 3.

We will first establish that the strategies presented so far can still be supported. The only change is that the set of strategies over which the parameters y_1^* and x_2^* are determined must be explicitly restricted to those which satisfy assumption 3. Otherwise, for stage game payoffs outside the bargaining frontier, we can not guarantee that an exit offer will be made, since continuation yields more than exit. This is borne out by the restrictions assumed in Lemma 1. Lemma 1 does not hold if payoffs outside the bargaining set can be achieved. The reason is the same: While one can still compute the offer which would be accepted by the other player, the offering player will be better off to make an unacceptable offer, and continue with the game, rather than to make that acceptable offer.

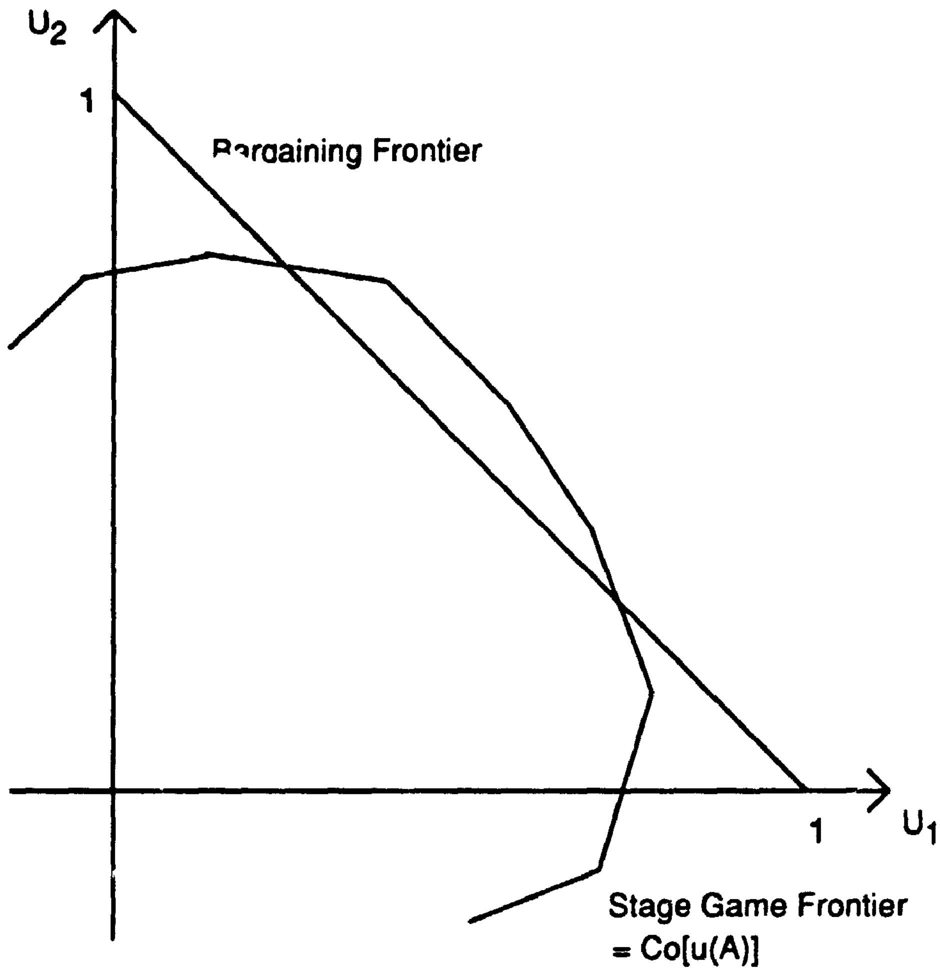

Figure 2.3: Constituent Game Payoff Frontiers, unrestricted

Lemma 2: *For a given feasible payoff vector (v_1, v_2) in the negotiation game $NG_1(\delta)$ $[NG_2(\delta)]$ such that $(v_1, v_2) > (\underline{v_1}, \underline{v_2})$, there exists $\underline{\delta} \in (0,1)$ such that $\forall\ \delta \in (\underline{\delta}, 1)$, $NG_1(\delta)$ $[NG_2(\delta)]$ has a subgame perfect equilibrium with average payoff (v_1, v_2).*

Here (v_1, v_2) is as defined before, with the proviso that y_1^* and x_2^* are computed only over strategies which satisfy $u_1(a) + u_2(a) \leq 1$.

Next, it is established that all repeated game payoffs which strictly dominate exit can be supported. This result is intuitive. Since exit in this case is dominated by continued play, all we need to do is to verify that there exist repeated game punishments for any given payoff which always yield a payoff vector above exit.

While we can no longer use simple penal codes, the strategies are still fairly simple and familiar: A player is minimaxed "sufficiently long" to wipe out any deviation gain, followed by a reward period in which the original payoffs are played again (c.f. Fudenberg and Maskin (1986)).

Lemma 3: *For any* $v \in \{u|u_1(a) + u_2(a) > 1\}$, *there exists a discount factor* $\underline{\delta} < 1$ *such that for all* $\delta \in (\underline{\delta}, 1)$, *there exists a subgame perfect equilibrium of* $NG_1(\delta)$ $[NG_2(\delta)]$ *with average payoffs* (v_1, v_2).

One implication of this last lemma is that negotiation games which violate assumption 3 will always have multiple equilibria. Not all individually rational and feasible payoffs can be supported, however, even then. This is shown in the next Theorem.

Theorem 6: *In the negotiation game* $NG_1(\delta)$ *which violates* **A3**, *all average equilibrium payoffs for player 1 which give him at least* $min[z_1, \frac{(1-y_1^*)}{(1+\delta)}]$ *can be supported as a SPE. Here* $z_1 := \inf\{x_1|(x_1, x_2) \in F, x_1 + x_2 > 1\}$,

2.6 Discussion

The previous sections have presented a model of negotiations and characterized its equilibria. The model built upon two extant models of dynamic allocation, the bargaining model of Rubinstein, and the repeated game model. The set of subgame perfect equilibrium payoffs of the negotiation model was characterized as any payoffs above the optimal punishment equilibrium payoffs for the players, which in turn were shown to depend in a simple way on the payoff structure of the stage game played in disagreement periods. The relevant magnitude is the highest effective disagreement payoff the punisher can obtain in perio' in which it is his turn to accept or reject offers. This was shown to be easily computed as the maximal difference between the punisher's payoff and the punishee's best deviation gain in the stage game.

In the presentation of the model certain assumptions were made for analytical convenience. While some are minor, the relaxation of others could be the subject of future work in this area. The assumption of a common discount factor is clearly minor, since all results require a sufficiently high discount factor beforehand.[11] Most of the cause of bargaining power in this model is furthermore to be suspected in the stage game, not different time preferences, thus diminishing the case for separate discount factors.

There were two assumptions made on the payoff spaces. One is the restriction that every payoff in G is dominated by some exit payoff. This assumption is economically motivated — there are gains from trade in negotiations, not just a redistribution. The result of delay in equilibrium is also stronger under this assumption, due to the implied inefficiency. Relaxation of this assumption has been shown to guarantee multiplicity of equilibria.

The second assumption is on the shape of the bargaining frontier, which is here assumed to be a straight line. This assumption could be relaxed to allow for an arbitrary bargaining frontier — and in particular for a frontier that coincides with that of the stage game. This latter possibility is reminiscent of the work by Okada (1991), although he uses a very different focus and completely different framework of analysis. It may, however, provide an alternative approach to the question of long-term contracting which he addresses.

---

[11]It should be noted, however, that Lemma 1 will not hold for sufficiently disparate discount factors. This is due to the fact that agents may evaluate future payoff paths too differently, and thus an acceptable offer may not be made.

Appendix I

In order to economize on notation, let the utility frontier defined by equation (19) in the text be represented by the relationship $v_2 = f(v_1)$ and that defined by (20) be represented by $w_2 = g(w_1)$. Then, the Nash bargaining solutions for the utility sets defined by (19) and (20) are given by the conditions

$$f(v_1)/v_1 = -f'(v_1) \tag{1}$$

$$g(w_1)/w_1 = -g'(w_1) \tag{2}$$

respectively. The Nash bargaining solutions for the utility sets defined by the CC and IC1 frontiers are given, respectively, by the conditions

$$\begin{aligned} f(v_1) + g(w_1) + (v_1 + w_1)f_1' &= 0 \\ f(v_1) + g(w_1) + (v_1 + w_1)g_1' &= 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} f(v_1) + g(w_1) + (v_1 + w_1)f_1' &= 0 \\ g(w_1) + w_1 g_1' &= 0 \end{aligned} \tag{4}$$

Finally, as in the text, the (v_1, w_1) pair solving conditions (3) is denoted by (v_1^*, w_1^*), while the (v_1, w_1) that solve (1) and (2) are given by $\hat{v}_1$ and $\hat{w}_1$ respectively. The v_1 that solves the conditions (4) is given by $\bar{v}_1$.

Two results regarding the relationships among these various Nash bargaining solutions will prove useful subsequently. These results are given by Lemmas 1 and 2 below.

Lemma 1: If agent 1 strictly prefers X to Y while agent 2 strictly prefers Y to X, then $v_1^* > \hat{v}_1$ and $w_1^* < \hat{w}_1$

Proof: The conditions (3) imply that $f'(v_1) = g'(w_1)$. If $v_1^* = \hat{v}_1$ and $w_1^* = \hat{w}_1$, then the conditions (3), (1) and (2) jointly imply that $f(\hat{v}_1)/\hat{v}_1 = g(\hat{w}_1)/\hat{w}_1 = -f'(\hat{v}_1) = -g'(\hat{w}_1)$, a contradiction given the assumption on preferences. Therefore, it must be either that $f(v_1^*) + v_1^* f'(v_1^*) < 0$ (implying that $g(w_1^*) + w_1^* g'(w_1^*) > 0$) or $f(v_1^*) + v_1^* f'(v_1^*) > 0$ (implying that $g(w_1^*) + w_1^* g'(w_1^*) < 0$). Note, however, that $-g'(\hat{w}_1) > -f'(\hat{v}_1)$, so that if the latter set of inequalities hold then, by the concavity of $f(\cdot)$ and $g(\cdot)$, $-g'(w_1^*) > -f'(v_1^*)$ (the latter set of inequalities imply that $v_1^* < \hat{v}_1, w_1^* > \hat{w}_1$), contradicting the requirement that $f'(v_1^*) = g'(w_1^*)$. Thus, it must be the former set of inequalities that hold, proving the Lemma. ||

Lemma 2: If agent 1 prefers X to Y and agent 2 prefers Y to X then $\hat{v}_1 < \bar{v}_1 < v_1^*$.

Proof: To prove the first part of the inequality, note that $-f'(\bar{v}_1) < -g'(\hat{w}_1)$. To see this, suppose to the contrary that $-f'(\bar{v}_1) > -g'(\hat{w}_1)$. Then, $f(v_1)+v_1 f'(v_1) \geq 0$ from the first equation in condition (4); that is, $\bar{v}_1 \leq \hat{v}_1$. However, as $-f'(\hat{v}_1) < -g'(\hat{w}_1)$, concavity of $f(\cdot)$ implies that $-f'(\bar{v}_1) < -g'(\hat{w}_1)$, a contradiction. Thus, $-f'(\bar{v}_1) < -g'(\hat{w}_1)$, implying that $\bar{v}_1 > \hat{v}_1$.

To prove the second part of the inequality, suppose to the contrary that $\bar{v}_1 = v_1^*$. Then, as $\hat{w}_1 > w_1^*$ (from Lemma 1), the first equation in condition (4) cannot be satisfied. Indeed, the LHS of this condition must be strictly less than zero, implying that it cannot be satisfied for any $v_1 > v_1^*$. Thus, $\bar{v}_1 < v_1^*$. ||

In considering the relationship between the various Nash bargaining solutions and the limiting equilibria of the different bargaining procedures described in the text, it proves helpful to rewrite some of the equilibrium conditions in terms of the frontiers $f(\cdot)$ and $g(\cdot)$. Consider the equilibrium conditions for the IC1 procedure for subgames in which $t \geq t_Y - 1$. These conditions become

$$\begin{aligned} f(v_1^1) + \delta g(w_1^1) &= \delta\left[f(v_1^2) + \delta g(w_1^2)\right] \\ v_1^2 + \delta w_1^2 &= \delta\left[v_1^1 + \delta w_1^2\right] \end{aligned} \tag{5}$$

$$\begin{aligned} g(w_1^1) &= \delta g(w_1^2) \\ w_1^2 &= \delta w_1^1 \end{aligned} \tag{6}$$

where the superscript denotes the agent making the offer. By substituting for v_1^2 in the (5) conditions, one obtains a single condition for v_1^1 given by

$$f(v_1^1) + \delta g(w_1^2)(1 - \delta^2) = \delta f(\delta v_1^1 - w_1^2(1 - \delta^2)) \tag{7}$$

Next, consider a modified version of the IC1 procedure, where the modification involves agent 2 always making the first offer in the bargaining over Y. Then, for this procedure and for all subgames in which $t \geq t_Y - 1$, the equilibrium conditions are (6) above and

$$\begin{aligned} f(v_1^1) + \delta g(w_1^2) &= \delta\left[f(v_1^2) + \delta g(w_1^2)\right] \\ v_1^2 + \delta w_1^2 &= \delta\left[v_1^1 + \delta w_1^2\right] \end{aligned} \tag{8}$$

Again, by substituting for v_1^2 in the (8) conditions, one obtains a condition for v_1^1 given by

$$f(v_1^1) + \delta g(w_1^2)(1 - \delta) = \delta f(\delta v_1^1 - w_1^2(1 - \delta)) \tag{9}$$

By arguments analogous to those in Herrero (1989), one can show that the equilibrium offers defined by (8) and (6) approach the Nash bargaining solution values defined by (4) above as δ approaches 1. Thus, in particular, the value of v_1^1 given by (9) above must converge to $\bar{v}_1$ as δ approaches 1.

As to the relationship between the value of v_1^1 given by condition (7) and $\bar{v}_1$, the convergence result above implies that this relationship can be determined by a comparison of the conditions (7) and (9) in a neighbourhood of $\delta = 1$. Because (6) defines the value of w_1^2 in both equations, the conditions differ only in the weight applied to this offer (and the corresponding utility for agent 2, $g(w_1^2)$) being $(1-\delta^2)$ in the (7) condition and $(1-\delta)$ in the (9) condition. Thus, for any $\delta \in (0,1)$, the v_1^1 offer implied by (7) will be the same offer implied by (9), except with the weight on the utility resulting from Y being larger. Given this fact, one can show the following relationship exists between the limiting value of each agent's utility under the IC1 procedure and the values given by the Nash bargaining solution.

> **Lemma 3:** If agent 1 strictly prefers X to Y and agent 2 strictly prefers Y to X, then $\bar{v}_1$ servers as a lower bound on the value of v_1^1 defined by condition (7) as δ approaches 1.

Proof: Consider the condition (9) in a neighbourhood of $\delta = 1$. As δ approaches 1, w_1^2 approaches $\hat{w}_1$ from below. Further, v_1^1 approaches $\bar{v}_1$. Consider next the impact on this condition of an increase in the weight applied to w_1^2 and $g(w_1^2)$ (i.e. the move to the (7) condition). This increase increases the LHS at a rate $\delta g(w_1^2)$ and the RHS at a rate $-\delta f' w_1^2$. Because w_1^2 approaches $\hat{w}_1$ from below (implying that $g(w_1^2)/w_1^2 > -g'(w_1^2)$) and, from Lemma 2, $-g'(\hat{w}_1) > -f'(\bar{v}_1)$, $g(w_1^2)$ must be greater than $-f' w_1^2$ for δ in a neighbourhood of 1. Therefore, increasing the weight on w_1^2 and $g(w_1^2)$ in a neighbourhood of $\delta = 1$ increases the LHS by more than the the RHS. By the concavity of $f(\cdot)$, the value of v_1^1 that restores equality must be a value larger than that implied by (9). Thus, for δ close to 1, the solution to (7) must be larger than the solution to (9). AS the solution to (9) converges to $\bar{v}_1$, $\bar{v}_1$ must define a lower bound on the solution to (7). ||

There are two immediate corollaries to this Lemma. The first is:

> **Corollary 1:** When each agent views X and Y as identical (as defined in the text), then the solution to (7) converges to $\bar{v}_1$ as δ approaches 1.

The reason for this result is that, with these sorts of preferences, $g'(\hat{w}_1) = f'(\bar{v}_1)$, so that, as δ approaches 1, an increase in the weight on w_1^2 and $g(w_1^2)$ results in equal increases in the right- and left-hand sides of condition (9). Therefore, if $\bar{v}_1$ solves (9) in the limit, it must also solve (7).

> **Corollary 2:** As δ approaches 1, agent 1's utility under the IC1 procedure is bounded below by the Nash bargaining solution for the IC1 frontier, while agent 2's utility is bounded above by the Nash bargaining solution.

The difference between this result and Lemma 3 is that Lemma 3 only applies to subgames with $t > t_Y - 1$. However, as δ approaches 1, any offer at t_X that fails to converge to those defined by (5) and (6) will surely be rejected as both agents could delay until $t_Y - 1$ at almost no cost and be guaranteed these utility levels. Therefore, as δ approaches 1, Lemma 3 implies that agent 1's utility must be at least $\bar{v}_1 + \hat{w}_1$. As gains in agent 1's utility from X must come at the expense of agent 2's utility, agent 2's utility can be no greater than $f(\bar{v}_1) + g(\hat{w}_1)$.

Turning finally to the proofs of propositions 1 and 2, note that the equation $f(v_1) + g(w_1) + (v_1 + w_1)f_1' = 0$ from the (4) condition defines a locus of $(v_1,\ w_1)$ pairs that includes $(v_1^*,\ w_1^*)$ and $(\bar{v}_1,\ \hat{w}_1)$. Note also that the slope of this locus, given by

$$\frac{dv_1}{dw_1} = \frac{-(g'(w_1) + f'(v_1))}{2f'(v_1) + v_1 f''(v_1)} \tag{10}$$

is negative for all w_1 and strictly grater than -1 at (v_1^*, w_1^*). This latter fact implies that an increase in w_1 in a neighbourhood of w_1^* increases agent 1's total utility $v_1 + w_1$. Finally Lemma 1 tells us that w_1^* (the value of w_1 implied by the conditions (3)) is less than $\hat{w}_1$ (the value of w_1 implied by conditions (4)). Therefore, the issue is whether or not the increase in agent 1's utility that results from an increase in w_1 in a neighbourhood of w_1^* persists as w_1 is increased to $\hat{w}_1$. The following Lemma provides a condition guaranteeing that equation (10) is strictly greater than -1 for all $w_1 \in [w_1^*, \hat{w}_1]$, thereby guaranteeing that agent 1's utility increases.

Lemma 4: Suppose that, for all $v_1 \in [\hat{v}_1, v_1^*]$, $f''(v_1)w_1^*/f'(v_1) \geq 1$. Then, $\bar{v}_1 + \hat{w}_1 > v_1^* + w_1^*$.

Proof: The condition that (10) be greater or equal to 1 can be written as the condition that $f'(v_1) - g'(w_1) + (v_1 + w_1)f''(v_1) \leq 0$. Substituting for w_1 from the (4) condition reduces this condition to $f'(v_1) - f''(v_1)f(v_1)/f'(v_1) \leq g'(w_1) + f''(v_1)g(w_1)/f'(v_1)$. Next, if $w_1 \in [w_1^*, \hat{w}_1]$, then $g(w_1) > -g'(w_1)w_1$. Thus, over the interval $[w_1^*, \hat{w}_1]$, (10) will be greater or equal to -1 if $f'(v_1) - f''(v_1)f(v_1)/f'(v_1) \leq g'(w_1)(1 - f''(v_1)w_1/f'(v_1))$ (substituting $-g'(w_1)w_1$ for $g(w_1)$). The LHS of this expression is negative while the RHS will be positive if $f''(v_1)w_1/f'(v_1) > 1$. Because $w_1 \geq w_1^*$ and v_1 must be an element of the interval $[\bar{v}_1, v_1^*]$ as w_1 ranges over the interval $[w_1^*, \hat{w}_1]$, this inequality holds by the assumptions of the Lemma. Therefore, (10) is greater or equal to -1 for all $w_1 \in [w_1^*, \hat{w}_1]$, and the Lemma is proved. ||

Proof of Proposition 1: Prop$^{\text{n}}$ 1 follows as an immediate corollary of Lemma 4.

As for proposition 2 the proof is almost immediate.

Proof of Proposition 2: From (3), $f(v_1^*)+g(w_1^*)+(v_1^*+w_1^*)f'(v_1^*)=0$. Therefore, the condition in the proposition reduces to the condition that at $\bar{v}_1$, $f(\bar{v}_1)+g(\hat{w}_1)+(\bar{v}_1+\hat{w}_1)f'(\bar{v}_1)<0$; or that the solution to (4) results in a value of $v_1<\bar{v}_1$. Therefore, agent 1 must be worse off under the Nash bargaining solution for the IC1 frontier than under the Nash bargaining solution for the CC frontier. That agent 2 is worse off follows from the fact that, for all $w\in[w_1^*,\ \hat{w}_1]$, and all v_1 that solve the condition (4) when w_1 is in this range, $-g'(w_1)>-f'(v_1)$ (see Lemma 1). ||

Proof of $\partial\pi^*/\partial s>0$: Applying the quotient rule and simplifying equation 36:

$$\frac{\partial}{\partial s}\left[\delta \mathbf{E}U_2^C(3)-\mathbf{E}U_2^I(2)\right]>\pi^*\frac{\partial}{\partial s}\left[\mathbf{E}U_2^C(2)-\mathbf{E}U_2^I(2)\right]$$

Since $sx_C+(1-s)\hat{x}_c=(sx'_C+(1-s)\hat{x}'_c)/\delta$, where the prime indicates offers by 1 in t_X+1, $\frac{\partial}{\partial s}\delta\mathbf{E}U_2^C(3)=\frac{\partial}{\partial s}\mathbf{E}U_2^C(2)$, and thus the above simplifies to

$$\frac{\partial}{\partial s}\mathbf{E}U_2^C(2)>\frac{\partial}{\partial s}\mathbf{E}U_2^I(2)$$

For t_Y even, this leads to the following sequence of implications:

$$\begin{aligned}
\delta^{t_Y-t_X}\left[b+\frac{\delta}{1+\delta}-\frac{\delta(b(a+1)-\delta(b+1))}{ab-\delta^2}\right] &> \delta^{t_Y-t_X}\frac{b}{1+\delta}\\
\frac{\delta}{1+\delta}\left[\frac{(1+\delta)(b(a+1)-\delta(b+1))}{ab-\delta^2}-1\right] &< \frac{\delta}{1+\delta}b\\
(1+\delta)(b(a+1)-\delta(b+1))-ab+\delta^2 &< (ab-\delta^2)b\\
(b-\delta) &< ab(b-\delta)
\end{aligned}$$

Since $1<ab$ the assertion is shown. For t_Y odd the sequence of implications is similar, but does not simplify as nicely. In the end one obtains the condition that

$$ab(ab\delta^2-1)>(1-\delta^2)(ab(a\delta-1)-\delta(a-\delta))$$

which is true for δ large enough. Note that this condition is related to that for 2 to prefer the CC contract (see footnote 15), in particular we require

$$\frac{b}{(1+\delta)} \geq \frac{(1-\delta)}{a} + \frac{(1-\delta)(b-\delta)(ab+1)}{(ab-\delta^2)}$$

which is satisfied for δ lower than those required for 2 to prefer CC. Thus the assertion is shown. ||

Appendix II

Proof to Lemma, Section 3.2

Following Shaked and Sutton (1984), the equilibrium proposals in every period are derived for the bargaining game with a fixed sequence of disagreement payoffs $\{u(t)\}_{t=1}^{\infty}$.

Assume that the set of equilibrium payoffs in such a game is not empty. Let M_i^t and m_i^t be the supremum and infimum of player i's average equilibrium payoffs in the subgame that starts in period t, for $i = 1, 2$ and $t \geq 1$.

First, consider players' strategies in an odd period t in which player 1 makes the proposal and player 2 makes the response. Player 2's payoff from rejecting is $u_2(t)$ in period t and a SPE payoff from period $(t+1)$ on which is bounded between m_2^{t+1} and M_2^{t+1}. Therefore, player 2 will always reject if his payoff in the proposal is less than $(1-\delta)u_2(t) + \delta m_2^{t+1}$, and always accept if his payoff in the proposal is more than $(1-\delta)u_2(t) + \delta M_2^{t+1}$. Subgame perfection requires that player 1's proposal, b^t, should satisfy

$$(1-\delta)u_2(t) + \delta m_2^{t+1} \leq 1 - b^t \leq (1-\delta)u_2(t) + \delta M_2^{t+1}$$

which implies that m_1^t and M_1^t satisfy the following inequalities

$$m_1^t \geq 1 - (1-\delta)u_2(t) - \delta M_2^{t+1} \quad (1)$$

$$M_1^t \leq 1 - (1-\delta)u_2(t) - \delta m_2^{t+1} \quad (2)$$

Considering players' strategies in the following even period $(t+1)$, we have

$$m_2^{t+1} \geq 1 - (1-\delta)u_1(t+1) - \delta M_1^{t+2} \quad (3)$$

$$M_2^{t+1} \leq 1 - (1-\delta)u_1(t+1) - \delta m_1^{t+2} \quad (4)$$

Substituting (4) into (1), (3) into (2), (2) into (3), and (1) into (4), with appropriate updating, yields

$$m_1^t \geq 1-(1-\delta)u_2(t)-\delta[1-(1-\delta)u_1(t+1)-\delta m_1^{t+2}] \quad (5)$$

$$M_1^t \leq 1-(1-\delta)u_2(t)-\delta[1-(1-\delta)u_1(t+1)-\delta M_1^{t+2}] \quad (6)$$

$$m_2^{t+1} \geq 1-(1-\delta)u_1(t+1)-\delta[1-(1-\delta)u_2(t+2)-\delta m_2^{t+3}] \quad (7)$$

$$M_2^{t+1} \leq 1-(1-\delta)u_1(t+1)-\delta[1-(1-\delta)u_2(t+2)-\delta M_2^{t+3}] \quad (8)$$

Recursive substitution on equations (5), (6), (7) and (8) yields, for odd t,

$$m_1^t \geq \sum_{k=0}^{\infty}[1-(1-\delta)u_2(t+2k)-\delta[1-(1-\delta)u_1(t+2k+1)]] \quad (9)$$

$$M_1^t \leq \sum_{k=0}^{\infty}[1-(1-\delta)u_2(t+2k)-\delta[1-(1-\delta)u_1(t+2k+1)]] \quad (10)$$

$$m_2^{t+1} \geq \sum_{k=0}^{\infty}[1-(1-\delta)u_1(t+2k+1)-\delta[1-(1-\delta)u_2(t+2k+2)]] \quad (11)$$

$$M_2^{t+1} \leq \sum_{k=0}^{\infty}[1-(1-\delta)u_1(t+2k+1)-\delta[1-(1-\delta)u_2(t+2k+2)]] \quad (12)$$

Further simplification then yields that

$$b^t \leq m_1^t \leq M_1^t \leq b^t \quad \text{if } t \text{ is odd} \quad (13)$$

$$1-b^t \leq m_2^t \leq M_2^t \leq 1-b^t \text{ if } t \text{ is even} \quad (14)$$

where b^t is as given in the Lemma. (13) and (14) imply that the infima and suprema coincide, and thus if an equilibrium exists in the game, it must be unique in terms of payoffs.

It remains to be shown that an equilibrium exists for the game. Consider the following strategies: in period t, the player who makes the proposal will propose b^t, and the player who makes the response will accept all proposals that he weakly prefers to b^t and reject all others.

By construction of b^t, the player i who makes the response in period t is just indifferent between accepting b^t and waiting to propose b^{t+1} in the next period, collecting $u_i(t)$ in the meantime. Therefore, rejecting proposals which are not preferred to b^t and accepting those which are is player i's best strategy. This implies that any proposal which is preferred to b^t by the proposing player will be rejected. It then is easy to show that the assumption that $u_1(t) + u_2(t) \leq 1$ implies that

$$\begin{array}{rcll} b^t & \geq & (1-\delta)u_1(t) + \delta b^{t+1} & \text{if } t \text{ is odd} \\ (1-b^t) & \geq & (1-\delta)u_2(t) + \delta(1-b^{t+1}) & \text{if } t \text{ is even} \end{array}$$

and thus the proposing player prefers the proposal he is to make according to his equilibrium strategy over deviating and waiting for one period. The strategy profile is, therefore, a subgame perfect equilibrium of the bargaining game for the given sequence of disagreement payoffs.

Q.E.D.

Proof to Theorem 1, Section 3.1

The proof will show that the equilibrium claimed in the Theorem is a special case of the Lemma of Section 3.2. First, note that the disagreement outcome in every period is a Nash equilibrium of the stage game and that all proposals are history independent. Therefore, neither player will deviate in the stage game from a^* individually, since he cannot increase his payoff in the current period or thereafter by doing so. Thus, a fixed disagreement payoff is given by $u(a^*)$ for every period without agreement. The Lemma gives the equilibrium proposals in the unique SPE of a bargaining game with a fixed sequence of disagreement payoffs. These proposals are uniquely determined by the disagreement payoffs. Here, $u_i(t) = u_i(a^*)\ \forall\ t \geq 1$.

The equilibrium outcome is that player 1's [2's] proposal

$$b_1^* = \frac{1+\delta u_1(a^*) - u_2(a^*)}{1+\delta} \quad \left[b_2^* = \frac{\delta + u_1(a^*) - \delta u_2(a^*)}{1+\delta}\right]$$

is accepted in the first period of the game $NG_1(\delta)$ [$NG_2(\delta)$].

In equilibrium, player i will always propose b_i^* and only reject proposals which are not preferred to b_j^* for $j \neq i$. After any rejection, players will play the Nash equilibrium a^* in the stage game. The equilibrium strategies for $NG_1(\delta)$ are given as follows. $\forall\, h_1(t) \in H_1$, $h_2(t) = h_1(t) \oplus b^{t+1} \in H_2$ and $h_3(t) \in H_3$:

for an odd period $(t+1)$,

$$\begin{aligned} f_1(h_1(t)) &= \frac{1}{1+\delta}(1+\delta u_1(a^*) - u_2(a^*)) \\ f_2(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \leq f_1(h_1(t)) \\ N & \text{otherwise} \end{cases} \\ f(h_3(t)) &= a^* \in A \end{aligned}$$

for an even period $(t+1)$,

$$\begin{aligned} f_2(h_1(t)) &= \frac{1}{1+\delta}(\delta + u_1(a^*) - \delta u_2(a^*)) \\ f_1(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \geq f_2(h_1(t)) \\ N & \text{otherwise} \end{cases} \\ f(h_3(t)) &= a^* \in A \end{aligned}$$

Q.E.D.

Proof to Theorem 2, Section 3.2

The proof proceeds by deriving the infima of the set of average subgame perfect equilibrium payoffs, taking assumption A2 and subgame perfection into account. Theorem 1 states that $NG_i(\delta)$ has, at least, one subgame perfect equilibrium $\forall\, \delta \in (0,1)$. Therefore, the set of average payoffs of the SPEs in the negotiation game $NG_i(\delta)$ is not empty, $\forall\, \delta \in (0,1)$ and $i = 1,2$. Given $\delta \in (0,1)$, let $m_1(\delta)$ be the infimum of player 1's average equilibrium payoffs in $NG_1(\delta)$. In the game $NG_2(\delta)$, since player 1 can guarantee himself a payoff of 0 in the current period, and his average payoff from the next period on cannot be less than $m_1(\delta)$, player 1's average equilibrium payoffs are bounded below by $\delta m_1(\delta)$.

By the definition of the infimum, $\forall\, \epsilon > 0$, $NG_1(\delta)$ has a SPE with average payoff (x_1, y_1) such that

$$m_1(\delta) \leq x_1 \leq m_1(\delta) + \epsilon \tag{15}$$

If $x_1 + y_1 < 1$, it must be the case that player 1's proposal is rejected in the first period of $NG_1(\delta)$. Construct a new SPE whose strategies are the same as those in the equilibrium with the payoff (x_1, y_1), but in which player 2 only accepts the proposal x_1 in the first period. If player 2 rejects x_1, the strategy is the same as when player 2 rejects player 1's equilibrium proposal in the equilibrium with payoff (x_1, y_1). This new SPE is efficient and the average payoff vector is $(x_1, 1 - x_1)$. Therefore assume without loss of generality that

$$x_1 + y_1 = 1. \tag{16}$$

In a SPE of $NG_1(\delta)$, if player 2 rejects player 1's proposal in the first period, players must play one stage game outcome, say $a \in A$, and one of the SPEs in $NG_2(\delta)$, the payoff of which is, say, $(x_2(a), y_2(a))$, where $x_2(a) + y_2(a) \leq 1$. Therefore, if player 2 rejects player 1's proposal in the first period of $NG_1(\delta)$, player 2's average payoff is bounded above by the maximum of all possible continuation payoffs. Subgame perfection implies that 2 will certainly accept a proposal if his payoff is more than the maximum of his continuation payoffs, and that player 1 will propose x_1 in the first period of $NG_1(\delta)$ only if

$$y_1 \leq \max_{a \in A}\{(1 - \delta)u_2(a) + \delta y_2(a)\} \tag{17}$$

However, if player 2 does reject player 1's proposal, player 1 should not deviate from $a \in A$ in the stage game. Subgame perfection, then, requires that

$$\begin{aligned}(1-\delta)\max_{a_1' \in A_1} u_1(a_1', a_2) + \delta^2 m_1(\delta) &\leq (1-\delta)u_1(a) + \delta x_2(a) \\ &\leq (1-\delta)u_1(a) + \delta(1 - y_2(a)),\end{aligned}$$

which implies that

$$\delta y_2(a) \leq \delta(1 - \delta m_1(\delta)) - (1-\delta)(\max_{a_1' \in A_1} u_1(a_1', a_2) - u_1(a)) \qquad (18)$$

Substituting (18) into (17), and using the definition of y_1^* from the text, one obtains

$$\begin{aligned} y_1 &\leq (1-\delta)\max_{a \in A}\{u_1(a) + u_2(a) - \max_{a_1' \in A_1} u_1(a_1', a_2)\} + \delta(1 - \delta m_1(\delta)) \\ &= (1-\delta)y_1^* + \delta(1 - \delta m_1(\delta))\end{aligned}$$

Together with (15) and (16), this implies

$$\begin{aligned} 1 - m_1(\delta) - \epsilon \leq 1 - x_1 = y_1 &\leq (1-\delta)y_1^* + \delta(1 - \delta m_1(\delta)) \\ \Rightarrow m_1(\delta) &\geq \frac{1 - y_1^*}{1+\delta} - \frac{\epsilon}{1-\delta^2}\end{aligned}$$

Since ϵ can be chosen arbitrarily small, the last inequality implies that $m_1(\delta)$ is greater than or equal to $\frac{1}{1+\delta}(1 - y_1^*)$. Moreover, player 1's average equilibrium payoffs in $NG_2(\delta)$ are bounded below by $\delta m_1(\delta)$, which is greater than or equal to $\frac{\delta}{1+\delta}(1 - y_1^*)$.

Q.E.D.

Proof to Theorem 3, Section 3.2

Note that the payoffs correspond to the perfect equilibrium for a bargaining game with the disagreement payoff $(0, y_1^*)$ in every period. If $(0, y_1^*)$ is a Nash equilibrium outcome of the stage game G, then the result follows from Theorem 1. If $(0, y_1^*)$ is not a Nash equilibrium of G, the proof is lengthy. The necessary $\underline{\delta}$ will be derived first. Then subgame perfection of the given strategy for $\delta \geq \underline{\delta}$ will be verified.

Suppose a^* is a Nash equilibrium in the stage game G. The definition of y_1^* implies that $y_1^* \geq u_2(a^*)$. Since $u_1(a^*) \geq 0$ and $(0, y_1^*) \neq u(a^*)$, it must be that $y_1^* + u_1(a^*) - u_2(a^*) > 0$. Let $a^1 \in A$ such that

$$x_1^* + y_1^* = u_1(a^1) + u_2(a^1) \quad \text{and} \quad x_1^* = \max_{a_1' \in A_1} u_1(a_1', a_2^1) \tag{19}$$

Let $d = \max[u_i(a') - u_i(a'')]$, $\forall\, a', a'' \in A$ and $i = 1, 2$. Since the set $u(A)$ is compact, d must be finite. Consider the following three functions of $\delta \in (0, 1]$,

$$\begin{aligned}
c_1(\delta) &= \frac{\delta}{1+\delta}[y_1^* + \delta u_1(a^*) - \delta^2 u_2(a^*)] - (1-\delta)[d + u_2(a^1)] \\
c_2(\delta) &= \frac{\delta}{1+\delta}[y_1^* + \delta u_1(a^*) - u_2(a^*)] - (1-\delta)d \\
c_3(\delta) &= y_1^* - (1-\delta^2)u_2(a^1) - \delta^2 u_2(a^*) + \delta u_1(a^*)
\end{aligned}$$

Since these three functions are positive and continuous at $\delta = 1$, there must exist $\underline{\delta} \in (0, 1)$ such that, $\forall\, \delta \in (\underline{\delta}, 1)$, the functions $c_1(\delta)$, $c_2(\delta)$ and $c_3(\delta)$ are positive. Equivalently, $\forall\, \delta \in (\underline{\delta}, 1)$, the following three inequalities hold

$$(1-\delta)d \leq \frac{\delta + y_1^*}{1+\delta} - (1-\delta)u_2(a^1) - \frac{\delta}{1+\delta}[1 - u_1(a^*) + \delta u_2(a^*)] \tag{20}$$

$$(1-\delta)d \leq \frac{\delta}{1+\delta}[\delta + y_1^*] - \frac{\delta}{1+\delta}[\delta - \delta u_1(a^*) + u_2(a^*)] \tag{21}$$

$$\frac{1-\delta}{\delta}u_2(a^1) + \frac{\delta^2 - y_1^*}{\delta(1+\delta)} \leq \frac{1}{1+\delta}(\delta + u_1(a^*) - \delta u_2(a^*)) \tag{22}$$

This concludes the derivation of $\underline{\delta}$.

Consider the strategy for $NG_1(\delta)$ which was given in the text. The subgame perfection of the strategy will be proven by exhaustive consideration of all subgames.

In an odd period $(t+1)$, there are two cases to be considered.

Case 1: *either* $f(h_3(t-1)) = a^*$; *or* $a_1^t = f_1(h_3(t-1))$, $a_2^t \neq f_2(h_3(t-1))$; *or* $b^{t+1} \leq f_1(h_1(t))$.

Player 2 is the last deviator, i.e. he either deviated in the negotiation game before period $(t-1)$, or in the stage game in period t, or rejected a proposal which should have been accepted in period $(t+1)$. The disagreement payoff will be $u(a^*)$ for every period thereafter. Since a^* is a Nash equilibrium in the stage game, Theorem 1 implies that the strategy f induces a perfect equilibrium in such a subgame.

Case 2: *otherwise*

$f(h_3(t)) = a^1$. If player 1 were to deviate from a^1, according to the strategy, player 2 will propose $\frac{\delta}{1+\delta}(1-y_1^*)$ instead of $\frac{1-\delta}{\delta}u_2(a^1) + \frac{1}{\delta(1+\delta)}(\delta^2 - y_1^*)$ in period $(t+2)$. Comparing player 1's payoffs, one obtains

$$\begin{aligned}
(1-\delta)u_1(a^1) + (1-\delta)u_2(a^1) + \frac{\delta^2 - y_1^*}{1+\delta} &= (1-\delta)[u_1(a^1) + u_2(a^1)] + \frac{\delta^2 - y_1^*}{1+\delta} \\
= (1-\delta)(x_1^* + y_1^*) + \frac{\delta^2 - y_1^*}{1+\delta} &= (1-\delta)x_1^* + \frac{\delta^2}{1+\delta}(1-y_1^*) \\
&= (1-\delta)\max_{a_1' \in A_1} u_1(a_1', a_2^1) + \frac{\delta^2}{1+\delta}(1-y_1^*)
\end{aligned}$$

Therefore, player 1 will not deviate from a^1.

If player 2 were to deviate from a^1, player 2 will demand $\frac{1}{1+\delta}(1-u_1(a^*)+\delta u_2(a^*))$ instead of $\frac{\delta+y_1^*}{\delta(1+\delta)} - \frac{1-\delta}{\delta}u_2(a^1)$ in period $(t+2)$. Inequality (20) implies that

$$\begin{aligned}
(1-\delta)[\max_{a_2' \in A_2} u_2(a_1^1, a_2') - u_2(a^1)] &\leq (1-\delta)d \\
\leq \frac{\delta + y_1^*}{1+\delta} - (1-\delta)u_2(a^1) &- \frac{\delta}{1+\delta}(1 - u_1(a^*) + \delta u_2(a^*))
\end{aligned}$$

Therefore, player 2 will not deviate from a^1.

Player 2 will also not deviate from $f_2(h_2(t))$: If player 1 were to deviate from $f_1(h_1(t))$, player 2's payoff from rejecting will be $\frac{\delta+y_1^*}{1+\delta}$. Therefore, player 2 will accept the proposal only if player 1 proposes less than $\frac{1-y_1^*}{1+\delta}$. On the other hand, if player

1 follows $f_1(h_1(t))$, player 2's payoff from rejecting will be

$$(1-\delta)u_2(a^*) + \frac{\delta}{1+\delta}(1 - u_1(a^*) + \delta u_2(a^*)) = \frac{1}{1+\delta}[\delta - \delta u_1(a^*) + u_2(a^*)$$

which is less than $\frac{\delta+y_1^*}{1+\delta}$. Therefore, player 2 will not deviate from $f_2(h_2(t))$.

Finally, player 1 will not deviate from $f_1(h_1(t))$: If player 1 were to make a higher proposal player 2 will reject and propose $\frac{1-\delta}{\delta}u_2(a^1) + \frac{1}{\delta(1+\delta)}(\delta^2 - y_1^*)$ in period $(t+2)$. Since $x_1^* + y_1^* \leq 1$ by A2,

$$\begin{aligned} (1-\delta)u_1(a^1) + (1-\delta)u_2(a^1) + \frac{\delta^2 - y_1^*}{1+\delta} &= (1-\delta)(x_1^* + y_1^*) + \frac{\delta^2 - y_1^*}{1+\delta} \\ \leq (1-\delta) + \frac{\delta^2 - y_1^*}{1+\delta} &= \frac{1 - y_1^*}{1+\delta} \end{aligned}$$

Therefore, player 1 will not deviate from $f_1(h_1(t))$. This concludes the checks for an odd period.

In an even period $(t+1)$, when either $f(h_3(t-1)) = a^*$ or $a_1^t = f_1(h_3(t-1)), a_2^t \neq f_2(h_3(t-1))$ or $b^{t+1} < f_2(h_1(t))$, Theorem 1 implies that the induced strategy forms a perfect equilibrium in such a subgame, because the disagreement payoff in every period thereafter is the Nash equilibrium payoff $u(a^*)$. Otherwise, if player 2 has not deviated last, there are two cases that have to be considered.

Case 1: $a_1^t \neq a_1^1$ *and* $a_2^t = a_2^1$

Player 1 deviated in the stage game in period t. Player 1 will not deviate from m^1, because m^1 is his minimax strategy. If player 2 deviates from m^1, the disagreement payoff will be $u(a^*)$ in every period thereafter, and player 1 will propose $\frac{1}{1+\delta}(\delta - \delta u_1(a^*) + u_2(a^*))$ to player 2 instead of $\frac{\delta+y_1^*}{1+\delta}$. Inequality (21) and the definition of d imply that

$$(1-\delta)\max_{a_2' \in A_2} u_2(m_1^1, a_2') + \delta\frac{\delta - \delta u_1(a^*) + u_2(a^*)}{1+\delta} \leq (1-\delta)u_2(m^1) + \delta\frac{\delta + y_1^*}{1+\delta}$$

Therefore, player 2 will not deviate from m^1.

Player 1 will not deviate from $f_1(h_2(t))$: If player 2 follows his strategy, player 1's payoff from rejecting will be $\frac{\delta}{1+\delta}(1-y_1^*)$. So, player 1 will accept. If player 2 deviates, player 1's payoff from rejection will be $\frac{1}{(1+\delta)}[\delta + u_1(a^*) - \delta u_2(a^*)]$, which is more than $\frac{\delta}{1+\delta}(1-y_1^*)$. Player 1 will therefore not deviate from $f_1(h_2(t))$.

Player 2 will not deviate from $f_2(h_1(t))$ either: If player 2 were to demand more, player 1 will reject and propose $\frac{1}{1+\delta}(\delta - \delta u_1(a^*) + u_2(a^*))$ to player 2 in period $(t+2)$. Since $\delta u_2(a^*) - u_1(a^*) < u_2(a^*) - u_1(a^*) < y_1^*$,

$$(1-\delta)u_2(a^*) + \delta\frac{\delta - \delta u_1(a^*) + u_2(a^*)}{1+\delta} < \frac{1 - u_1(a^*) + \delta u_2(a^*)}{1+\delta} < \frac{1+y_1^*}{1+\delta}$$

Therefore, player 2 will not deviate from $f_2(h_1(t))$.

Case 2: $a^t = a^1$

For the same reasons as in Case 1, players will not deviate from $f(h_3(t)) = m^1$.

If player 2 follows his strategy, player 1 will accept, since his payoff from rejecting is equal to $\frac{\delta}{1+\delta}(1-y_1^*)$ which is less than $\frac{1-\delta}{\delta}u_2(a^1) + \frac{1}{\delta(1+\delta)}(\delta^2 - y_1^*)$ due to

$$\begin{aligned} y_1^* \le u_2(a^1) &\Rightarrow (1-\delta^2)y_1^* \le (1-\delta^2)u_2(a^1) \\ &\Rightarrow \delta^2(1-y_1^*) \le (1-\delta^2)u_2(a^1) + \delta^2 - y_1^* \\ &\Rightarrow \frac{\delta}{1+\delta}(1-y_1^*) \le \frac{1-\delta}{\delta}u_2(a^1) + \frac{1}{\delta(1+\delta)}(\delta^2 - y_1^*) \end{aligned}$$

On the other hand, if player 2 were to demand more, player 1's payoff from rejecting is equal to

$$(1-\delta)u_1(a^*) + \frac{\delta}{1+\delta}(1 + \delta u_1(a^*) - u_2(a^*)) = \frac{1}{1+\delta}(\delta + u_1(a^*) - \delta u_2(a^*))$$

which is greater than or equal to $\frac{1-\delta}{\delta}u_2(a^1) + \frac{\delta^2 - y_1^*}{\delta(1+\delta)}$ due to (22). Therefore, player 1 will not deviate from $f_1(h_2(t))$.

Player 2 will follow $f_2(h_1(t))$: If player 2 were to demand more, player 1 will reject and propose $\frac{\delta}{1+\delta}(\delta - \delta u_1(a^*) + u_2(a^*))$ to player 2 in period $(t+2)$. Player 2's payoff then would be

$$\begin{aligned}(1-\delta)u_2(a^*) + \frac{\delta}{1+\delta}(\delta - \delta u_1(a^*) + u_2(a^*)) &< \frac{1}{1+\delta}(1 - u_1(a^*) + \delta u_2(a^*)) \\ < \frac{\delta + y_1^*}{\delta(1+\delta)} - \frac{1-\delta}{\delta}u_2(a^1) &= 1 - \frac{1-\delta}{\delta}u_2(a^1) - \frac{\delta^2 - y_1^*}{\delta(1+\delta)}\end{aligned}$$

Therefore, player 2 will not deviate from $f_2(h_1(t))$.

It has been shown that the strategy profile f constitutes a subgame perfect equilibrium for the negotiation game $NG_1(\delta)$. The equilibrium outcome is that player 1's proposal is accepted by player 2 in the first period, yielding average payoffs of $(\frac{1}{1+\delta}(1 - y_1^*), \frac{1}{1+\delta}(\delta + y_1^*))$.

Finally, consider the one period history $h_1(1) = 1 \oplus (a_1', a_2^1)$ where $a_1' \neq a_1^1$. $f|_{h_1(1)}$ is a perfect equilibrium of $NG_1(\delta)|_{h_1(1)}$ which is $NG_2(\delta)$, and the equilibrium outcome is that player 2's proposal is accepted by player 1 in the first period, yielding average payoffs of $(\frac{\delta}{1+\delta}(1 - y_1^*), \frac{1}{1+\delta}(1 + \delta y_1^*))$. This proves the theorem for the game $NG_2(\delta)$.

Q.E.D.

Proof to Theorem 5, Section 3.3

The theorem is proven for $NG_1(\delta)$ only, but the arguments can easily be adapted to prove the theorem for $NG_2(\delta)$. Let a^* be a Nash equilibrium of G. Since $(v_1, v_2) > (\underline{v_1}, \underline{v_2})$, $\exists\ \epsilon_0 = \min\{v_1 - \underline{v_1}, v_2 - \underline{v_2}\}/2 > 0$. According to the results in section 3.2, $\exists\ \underline{\delta}$ such that, $\forall\ \delta \in (\underline{\delta}, 1)$, the game $NG_j(\delta)$ has an optimal punishment equilibrium for player i with strategy f^{ij}, and

$$\underline{v_1} + \epsilon_0 \geq$$
$$\max\left\{\frac{1-y_1^*}{1+\delta};\ (1-\delta)u_1(a^*) + \delta\frac{1-y_1^*}{1+\delta};\ 1-(1-\delta)u_2(a^*) - \delta\frac{\delta+y_1^*}{1+\delta}\right\} \quad (23)$$
$$\underline{v_2} + \epsilon_0 \geq$$
$$\max\left\{\frac{1-x_2^*}{1+\delta};\ (1-\delta)u_2(a^*) + \delta\frac{1-x_2^*}{1+\delta};\ 1-(1-\delta)u_1(a^*) - \delta\frac{\delta+x_2^*}{1+\delta}\right\} \quad (24)$$
$$\frac{1-\delta}{\delta}d \leq \epsilon_0 \leq v_i - (\underline{v_i} + \epsilon_0) \quad \text{for } i = 1,2 \quad (25)$$

$\forall\, \delta \in (\underline{\delta}, 1)$, $\exists\, \hat{a} \in A$, $\hat{b} \in [0,1]$ and a positive integer T (which may or may not be finite), such that

$$(v_1, v_2) = (1-\delta^T)u(\hat{a}) + \delta^T(\hat{b}, 1-\hat{b}) \ \text{ and } \ (\hat{b}, 1-\hat{b}) > v > u(\hat{a}) \quad (26)$$

Consider the outcome path $\pi(T) = \hat{b}(T) \oplus \hat{a}(T-1) \oplus \{Y\}$ of $NG_1(\delta)$, where

$$\hat{a}(T-1) = \{\hat{a}^t\}_{t=1}^{T-1} \in A^{T-1} \ \text{ and } \ \hat{b}(T) = (1,0,1,0,\ldots,\hat{b}).$$

Inequality (26) implies that players' average payoffs from the outcome path $\pi(T)$ are (v_1, v_2). Let $ID(\cdot)$ be the indicator function for the outcome path $\pi(T)$ as defined. Decompose the type k t-period history $h_k(t) \in H_k$ as $h_k(t) = h_1(s) \oplus h_k(t-s)$, for $k = 1,2,3$ and $s \leq t$.

Consider the given strategy profile $f = (f_1, f_2)$. It remains to verify that f constitutes a SPE for $NG_1(\delta)$.

$\forall\ h_1(t) \in H_1$, if $ID(h_1(t)) \neq 0$, $f|_{h_1(t)}$ is one of the four strategy profiles f^{11}, f^{12}, f^{21} or f^{22}, which are subgame perfect due to Theorems 3 and 4. Therefore, the strategy profiles under consideration are subgame perfect if $ID(h_1(t)) \neq 0$. It remains to verify the strategy profile along the proposed path $\pi(T)$, i.e. for $ID(h_1(t)) = 0$ and $(t+1) \leq T$. Due to symmetry, only an odd period $(t+1)$ before period T needs to be considered.

$\forall\, h_1(t) \in H_1$ such that $ID(h_1(t)) = 0$, player 1 will follow the strategy to propose 1 in period $(t+1) < T$ and $\hat{b}$ in period T. If player 1 follows this strategy his average payoff will be, by (26) above,

$$(1-\delta^{T-t})u(\hat{a}) + \delta^{T-t}\hat{b} \;\geq\; v_1.$$

However, if player 1 deviates, according to the strategy and (23), his average payoff will be either

$$\begin{array}{rlll} & (1-\delta)u_1(a^*) + \delta\dfrac{\delta(1-y_1^*)}{1+\delta} \;<\; (1-\delta)u_1(a^*) + \delta\dfrac{1-y_1^*}{1+\delta} & < & v_1 \\ \text{or} & 1-(1-\delta)u_2(a^*) - \dfrac{\delta}{1+\delta}(\delta+y_1^*) \;\leq\; \underline{v_1}+\epsilon_0 & < & v_1 \end{array}$$

Therefore, player 1 will not deviate from $f_1(h_1(t))$.

$\forall\, h_2(t) \in H_2$. If $ID(h_2(t)) = (1, t+1)$, player 1 has deviated from $f_1(h_1(t))$ in period $(t+1)$. Player 2's payoff from rejecting is

$$(1-\delta)u_2(a^*) + \delta\frac{\delta+y_1^*}{1+\delta} \tag{27}$$

Therefore, player 2 will accept a proposal only if his share is not less than (27) before period T. In period T, (23) and (26) imply that $1-\hat{b}$ is less than (27), so player 2 will reject if player 1 demands more than $\hat{b}$ in period T. If $ID(h_2(t)) = 0$, player 2 will reject in period $(t+1) < T$, since his payoff from accepting is 0 which is certainly less than that from rejecting. In period T, player 2 will accept the proposal if $ID(h_2(T-1)) = 0$. Due to (24) and (26), his payoff from rejecting satisfies

$$(1-\delta)u_2(a^*) + \delta\frac{1-x_2^*}{1+\delta} < v_2$$

which is less than $1-\hat{b}$. Therefore, player 2 will not deviate from $f_2(h_2(t))$.

$\forall\, h_3(t) \in H_3$. If $ID(h_3(t)) = 0$, $f(h_3(t)) = \hat{a}$. Neither players will deviate from $\hat{a}$, since

$$\frac{1-\delta}{\delta}\left[\max_{a_1' \in A_1} u_1(a_1', \hat{a}_2) - u_1(\hat{a})\right] \;\leq\; \frac{1-\delta}{\delta}d \;\leq\; \epsilon_0 \;\leq\; v_1 - (\underline{v_1}+\epsilon_0)$$

$$\leq \quad (1-\delta^{T-t-1})u_1(\hat{a}) + \delta^{T-t-1}\hat{b} - \frac{1-y_1^*}{1+\delta}$$

and

$$\frac{1-\delta}{\delta}\left[\max_{a_2' \in A_2} u_2(\hat{a}_1, a_2') - u_2(\hat{a})\right] \quad \leq \quad \frac{1-\delta}{\delta}d \leq \epsilon_0 \leq v_2 - (\underline{v}_2 + \epsilon_0)$$

$$\leq \quad (1-\delta^{T-t-1})u_2(\hat{a}) + \delta^{T-t-1}\hat{b} - \frac{1-x_2^*}{1+\delta}$$

due to (23), (25) and (26). If $ID(h_3(t)) = (i, t+1)$, $i = 1, 2$, $f(h_3(t)) = a^*$. Since a^* is a Nash equilibrium in the stage game and the continuation payoff is history independent, no player can increase his payoffs in period $(t+1)$ or thereafter by deviating from a^* individually. Therefore, players will not deviate from $f(h_3(t))$.

(v_1, v_2) is, therefore, supported by the strategy profile f as a subgame perfect equilibrium payoff from the outcome path $\pi(T)$ in the negotiation game $NG_1(\delta)$.

Q.E.D.

Lemma 2 is an extension of the proof to Theorem 3. The only difference to the situation in Theorem 3 is that a deviation may leave players outside the bargaining frontier. This difference is not of any importance, however, since only 1's gain is used, and 2's payoff after a deviation by 1 does not enter the equations.

Lemma 3 can be shown by a straight-forward modification of the proof in Fudenberg and Maskin (1986) and is not demonstrated here.

Theorem 6: The proof is immediate. We know that $min[z_1, \frac{(1-y_1^*)}{(1+\delta)}]$, where $z_1 := \inf\{x_1 | (x_1, x_2) \in F, x_1 + x_2 > 1\}$, can be supported as a SPE. We now employ the strategies leading to the minimum as punishment path for player 1, and play equilibrium strategies as in Theorem 5. A path leading to the appropriate payoff vector is played, and any deviation by a player is punished by reversion to that player's punishment path. Simultaneous deviations are ignored.

Appendix III

This appendix provides the strategies supporting Theorem 5.

Let $\pi(T) = \hat{b}(T) \oplus \hat{a}(T-1) \oplus \{Y\}$ be the outcome path of the negotiation game, where $\hat{b}(T) = \{\hat{b}^t\}_{t=1}^{T}$ and $\hat{a}(T-1) = \{\hat{a}^t\}_{t=1}^{T-1}$. Define the indicator function

$$ID(\cdot) : H_1 \cup H_2 \cup H_3 \rightarrow \{0\} \cup \{(i,t) | i = 1,2;\ 1 \leq t \leq T\}$$

recursively as follows: at the beginning of period 1, the history is the empty set and the indicator function takes the value 0, i.e. $ID(\emptyset) = 0$. Thereafter, $\forall\, h_1(t) = h_3(t-1) \oplus (a_1^t, a_2^t) \in H_1$, $h_2(t) = h_1(t) \oplus b^{t+1} \in H_2$, and $h_3(t) = h_2(t) \oplus \{N\} \in H_3$,

$$ID(h_1(t)) = \begin{cases} (1,t) & \text{if } a_1^t \neq \hat{a}_1^t;\ a_2^t = \hat{a}_2^t \text{ and } ID(h_3(t-1)) = 0 \\ (2,t) & \text{if } a_1^t = \hat{a}_1^t;\ a_2^t \neq \hat{a}_2^t \text{ and } ID(h_3(t-1)) = 0 \\ ID(h_3(t-1)) & \text{otherwise} \end{cases}$$

$$ID(h_2(t)) = \begin{cases} (1,t+1) & \text{if } b^{t+1} \neq \hat{b}^{t+1} \text{ and } (t+1) \text{ is odd and } ID(h_1(t)) = 0 \\ (2,t+1) & \text{if } b^{t+1} \neq \hat{b}^{t+1} \text{ and } (t+1) \text{ is even and } ID(h_1(t)) = 0 \\ ID(h_1(t)) & \text{otherwise} \end{cases}$$

$$ID(h_3(t)) = \begin{cases} (1,t+1) & \text{if } t+1 = T \text{ and } T \text{ is odd and } ID(h_2(t)) = 0 \\ (2,t+1) & \text{if } t+1 = T \text{ and } T \text{ is even and } ID(h_2(t)) = 0 \\ ID(h_2(t)) & \text{otherwise} \end{cases}$$

The indicator function takes two types of possible values, 0 and (i,t). The value 0 implies that no player has deviated from the proposed path $\pi(T)$. The value (i,t) implies that player i first deviated from the proposed path in period t, where $1 \leq t \leq T$.

Let f^{ij} denote the strategy combination in the optimal punishment equilibrium for player i in the game $NG_j(\delta)$. The strategies implementing the outcome path $\pi(T) = \hat{b}(T) \oplus \hat{o}(T-1) \oplus \{Y\}$, where

$$\hat{a}(T-1) = \{\hat{a}^t\}_{t=1}^{T-1} \in A^{T-1} \quad \text{and} \quad \hat{b}(T) = (1,0,1,0,\ldots,\hat{b}),$$

in $NG_1(\delta)$ for large enough δ then are:

$\forall\, h_1(t) \in H_1$, $h_2(t) = h_1(t) \oplus b^{t+1} \in H_2$, and $h_3(t) = h_2(t) \oplus \{N\} \in H_3$, for an odd period $(t+1)$

$$f_1(h_1(t)) = \begin{cases} \hat{b} & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 = T^* \\ 1 & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 < T^* \\ f_1^{11}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1,s) \text{ for even } s \\ f_1^{12}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1,s) \text{ for odd } s \\ f_1^{21}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2,s) \text{ for even } s \\ f_1^{22}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2,s) \text{ for odd } s \end{cases}$$

$$f_2(h_2(t)) = \begin{cases} Y & \text{if } t+1 = T \text{ and } ID(h_2(t)) = 0 \text{ or if } ID(h_2(t)) = \\ & (1,t+1) \text{ and } 1-b^{t+1} \geq (1-\delta)u_2(a^*)+\frac{\delta}{1+\delta}(\delta+y_1^*) \\ N & \text{otherwise} \\ f_2^{11}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1,s) \text{ for even } s \leq t \\ f_2^{12}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1,s) \text{ for odd } s \leq t \\ f_2^{21}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2,s) \text{ for even } s \leq t \\ f_2^{22}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2,s) \text{ for odd } s \leq t \end{cases}$$

for an even period $(t+1)$,

$$f_1(h_2(t)) = \begin{cases} Y & \text{if } t+1 = T \text{ and } ID(h_2(t)) = 0 \text{ or if } ID(h_2(t)) = \\ & (2,t+1) \text{ and } b^{t+1} \geq (1-\delta)u_1(a^*)+\frac{\delta}{1+\delta}(\delta+x_2^*) \\ N & \text{otherwise} \\ f_1^{11}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1,s) \text{ for even } s \leq t \\ f_1^{12}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1,s) \text{ for odd } s \leq t \\ f_1^{21}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2,s) \text{ for even } s \leq t \\ f_1^{22}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2,s) \text{ for odd } s \leq t \end{cases}$$

$$f_2(h_1(t)) = \begin{cases} \hat{b} & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 = T^* \\ 0 & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 < T^* \\ f_2^{11}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1,s) \text{ for even } s \\ f_2^{12}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1,s) \text{ for odd } s \\ f_2^{21}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2,s) \text{ for even } s \\ f_2^{22}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2,s) \text{ for odd } s \end{cases}$$

for both odd and even periods $(t+1)$,

$$f(h_3(t)) = \begin{cases} \hat{a} & \text{if } ID(h_3(t)) = 0 \\ a^* & \text{if } ID(h_3(t)) = (*,t+1) \\ f^{11}(h_3(t-s)) & \text{if } ID(h_3(t)) = (1,s) \text{ for even } s \leq t \\ f^{12}(h_3(t-s)) & \text{if } ID(h_3(t)) = (1,s) \text{ for odd } s \leq t \\ f^{21}(h_3(t-s)) & \text{if } ID(h_3(t)) = (2,s) \text{ for even } s \leq t \\ f^{22}(h_3(t-s)) & \text{if } ID(h_3(t)) = (2,s) \text{ for odd } s \leq t \end{cases}$$

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