1991

The Lambda-structure Of The Representation Rings Of The Classical Weyl Groups

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THE λ-STRUCTURE OF THE REPRESENTATION RINGS OF THE CLASSICAL WEYL GROUPS

by

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Submitted in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy

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The University of Western Ontario
London, Ontario
March 1991

ISBN 0-315-66324-3

ABSTRACT

First, we introduce a class of operations, called ϕ-operations, on the representation rings of the classical Weyl groups $W(B_k)$ and $W(D_k)$. These operations are shown to generate the exterior power operations in the representation rings $R(W(B_k))$ and $R(W(D_k))$. Given integers l, h satisfying $l + h = k$, let β be a partition of l and α a partition of h. The main theorem shows that induced representations of the form

$$Ind_{W_{\beta,\alpha}}^{W(B_k)}1,$$

where $W_{\beta,\alpha} = \prod W(B_\beta) \times \prod W(A_\alpha)$, can be expressed as an algebraic combination of ϕ-operations acting on the two canonical induced representations

$$X_k = Ind_{W(B_{k-1})\times W(B_1)}^{W(B_k)}1$$
$$Y_k = Ind_{W(B_{k-1})}^{W(B_k)}1.$$

Next, we show that the set

$$\left\{1 \otimes Ind_{W_{\beta,\alpha}}^{W(B_k)}1\right\}$$

is a basis of $\mathbf{Q} \otimes R(W(B_k))$. Since the ϕ-operations generate the λ-operations, one can deduce that $\mathbf{Q} \otimes R(W(B_k))$ is generated as a λ-ring over $\mathbf{Q}$ by the elements $1 \otimes X_k$ and $1 \otimes Y_k$. By applying a result of Lusztig which characterizes the irreducible representations of the Weyl groups $W(B_k)$ and $W(D_k)$ it follows, as a corollary, that $\mathbf{Q} \otimes R(W(D_k))$ is generated by two elements as a λ-ring over $\mathbf{Q}$.

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my gratitude and sincere thanks to my supervisor Dr. Richard M. Kane for all the time he spent and all the effort he made on my behalf.

I would also like to thank Dr. S.A. Rankin for all the hours he spent with me, explaining the operation of the TEX-compiler.

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INTRODUCTION

The representation ring of the symmetric group, $R(S_k)$, can be generated as a λ-ring by the induced representation $X_k = Ind_{S_{k-1}}^{S_k} 1$. This was proved by Boorman in [Bo]. The representation X_k turns out to be the sum of the reflection representation and the trivial representation. Moreover for any Weyl group, if the reflection representation is irreducible then all of its exterior powers are irreducible as well. Lastly, the reflection representation can be realized over **Q** and it has been proved that all representations of $\mathcal{W}(B_k)$ can be realized over **Q**. All of this suggests that the reflection representation may play a preeminent role in the λ-ring theory of the representation rings of the other Weyl groups.

Therefore, a natural question to ask is whether or not the reflection representation can be used to generate the representation rings of the Weyl groups $\mathcal{W}(B_k)$ and $\mathcal{W}(D_k)$ as λ-rings. However, when k is even

$$\mathcal{W}(B_k) \approx \mathcal{W}(D_k) \times \mathbf{Z}/2.$$

and hence

$$R(\mathcal{W}(B_k)) \approx R(\mathcal{W}(D_k)) \odot R(\mathbf{Z}/2).$$

Since $R(\mathcal{W}(D_k))$ must have at least one λ-generator, and $R(\mathbf{Z}/2)$ has one λ-generator, it follows that $R(\mathcal{W}(B_k)$ must have at least two λ-generators when k is even. Hence, the reflection representation cannot generate $R(\mathcal{W}(B_k))$ in general.

So we must look elsewhere for the λ-generators of $R(\mathcal{W}(B_k))$. We do not study the λ-ring structure of $R(\mathcal{W}(B_k))$ dirertly. We define another class of operations which we call ϕ-operations, which generalize Boorman's S-operations. These operations can be shown to generate the λ-operations in $R(\mathcal{W}(B_k))$. Moreover, by applying these operations to two appropriately chosen representations X_k and Y_k we obtain a linearly independent set in $R(\mathcal{W}(B_k))$. We cannot prove that these elements form a **Z**-basis of $R(\mathcal{W}(B_k))$. To overcome this problem, we extend $R(\mathcal{W}(B_k))$ by scalars to

$\mathbf{Q}$. Our results, then, are that the λ-ring $\mathbf{Q} \otimes R(W(B_k))$, as well as $\mathbf{Q} \otimes R(W(D_k))$, is generated by two elements.

The thesis is divided into two parts. Part I is expositional. The expository sections of Part I have two main purposes. Sections I.1 & I.2 are used to show that the symmetric power operations generate the λ-operations and conversely that λ-operations generate the symmetric power operations. Secondly, Sections I.4 & I.5 are used to compute the rank of $R(W(B_k))$. The proof that $\mathbf{Q} \otimes R(W(B_k))$ and $\mathbf{Q} \otimes R(W(D_k))$ are generated by two elements as λ-rings over $\mathbf{Q}$ is the subject of Part II. Sections II.3 - II.6 contains my work which proves these results.

Section I.1 is an adaptation of the material on λ-rings which can be found in Knutson [Kn]. Both the λ-operations and the Adams operations, Ψ^n, are defined and discussed. The main result of this section relates the Adams operations to the λ-operations, as follows:

Let R be ring which contains a subring isomorphic to $\mathbf{Q}$. Suppose that R possesses operations $\Psi^n : R \longrightarrow R$ which satisfy:

$$\Psi^n(1) = 1, \Psi^n(ab) = \Psi^n(a)\Psi^n(b), \text{ and } \Psi^n(\Psi^m(a)) = \Psi^{nm}(a) \ \forall \ a, b \in R.$$

Suppose further that there are operations $\lambda^i : R \longrightarrow R$ satisfying properties

(1) $\lambda^i(1) = 0$ for $i > 1$,

(2) $\lambda^0(x) = 1$, $\lambda^1(x) = x$,

(3) $\lambda^n(x+y) = \sum_i \lambda^i(x)\lambda^{n-i}(y)$.

and such that

$$d/dt(\log \lambda_t(a)) = \sum_n (-1)^n . \Psi^{n+1}(a) t^n$$

for $a \in R$. Then R is a λ-ring.

Section I.2 is concerned with the representation rings of finite groups. The material in this section can be found in Knutson [Kn] and Serre [Se]. First, we use the main result of I.1 to show that the representation ring of any finite group is a λ-ring. Secondly, we show that the symmetric power operations generate the λ-operations and that the λ-operations generate the symmetric power operations.

Section I.3 is a discussion of root systems and their associated reflection groups which follows Humphreys [Hu] and Kane [Ka2].

Let E be a Euclidean space. An invertible linear transformation which fixes some hyperplane pointwise and which sends any vector orthogonal to that hyperplane to its negative is called a reflection. Any non zero vector $\alpha \in E$ determines a reflection σ_α defined by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

A root system (Φ, E) is a finite subset, Φ, of $E\backslash 0$ that spans E as a vector space over $\mathbf{R}$, and that satisfies

(1) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(2) $\sigma_\alpha(\Phi) = \Phi$ for any reflection σ_α, where $\alpha \in \Phi$.

Suppose that Φ is a root system in E. Let $\mathcal{W}(\Phi)$ denote the subgroup of $GL(E)$ which is generated by the reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$. $\mathcal{W}(\Phi)$ will be called the reflection group associated to the root system Φ. We show that for any root system there always exists $\Delta \subset \Phi$, called a base, such that $\mathcal{W}(\Phi)$ is generated by the set $\{\sigma_\alpha \mid \alpha \in \Delta\}$.

Next we consider Coxeter groups. A Coxeter group is any group with a presentation of the form

$$\mathcal{W} = \langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where S is some set and

(1) $m_{ii} = 1$.

(2) $m_{ij} \in \{2, 3, \dots\} \cup \{\infty\}$ if $i \neq j$.

We define the Coxeter graph associated with any such group and indicate how to represent reflection groups by Coxeter graphs.

Section I.4 contains results concerning PSH-algebras which are due to Bernstein and Zelevinski and which have been adapted from [Ze].

Suppose that H is a Hopf algebra which also has the structure of a free $\mathbf{Z}$-module with basis Ω. If the morphisms m, m^*, e, e^* are positive then H is a positive Hopf algebra.

Define an inner product on H so that $\langle \omega, \omega' \rangle = \delta_{\omega,\omega'}$ for ω, $\omega' \in \Omega$. It is symmetric, non-degenerate, and positive definite. Such a Hopf algebra in which m, m^* and e, e^* are adjoints of each other under the inner products on H and $H \otimes H$, is called self-adjoint.

A PSH-algebra is a commutative, connected, positive, self-adjoint Hopf algebra.

The principle result of this section states that any PSH-algebra with unique primitive irreducible element is isomorphic to a polynomial algebra over $\mathbf{Z}$. Furthermore, any two such PSH-algebras, must be isomorphic.

Section I.5 describes the PSH-algebra structure of

$$\bigoplus_{n \geq 0} R(\mathcal{W}(B_k)),$$

given by Zelevinski [Ze]. We are primarily interested in the rank of $R(\mathcal{W}(B_k))$. If $\#(i)$ denotes the number of partitions of the integer i, then

$$Rank(R(\mathcal{W}(B_k)) = \sum_{i+j=k} \#(i).\#(j).$$

Section I.6 describes Boorman's theorem for the representation ring of the symmetric group S_k. This theorem states that $R(S_k)$ is generated by one element as a λ-ring. This description is meant to provide motivation for my work in Part II.

Section II.1 is an exposition of Solomon's Theorem which says that if ε is the alternating representation of $\mathcal{W}(B_k)$ then

$$\varepsilon = \sum (-1)^k Ind_{W_K}^{\mathcal{W}(B_k)} 1,$$

where W_K ranges over all subgroups of $\mathcal{W}(B_k)$ which are generated by subsets of the reflections which generate $\mathcal{W}(B_k)$.

Section II.2 is a description of the j-operation or truncated induction:

THEOREM II.2.1 (MACDONALD). *Let $S_N(V_1)$ be the space of homogeneous polynomial functions on V_1 of degree N. Let U be an absolutely irreducible W_1-submodule of $S_N(V_1)$ which does not occur in $S_i(V_1)$ if $0 \leq i < N$. Regard U as a subspace of $S_N(V)$ and consider the W-submodule $j(U)$ of $S_N(V)$ generated by U. Then:*

(1) *$j(U)$ is an irreducible W-module.*

(2) *$j(U)$ occurs with multiplicity 1 in $S_N(V)$.*

(3) *$j(U)$ does not occur in $S_i(V)$ if $0 \leq i < N$.*

Using this result Macdorald goes on to prove that all the irreducible represen-taitons of the symmetric group can be obtained as j-operations acting on the alter-nating representations of certain subgroups.

Lusztig's Theorem, which we state in this section, extends Macdonald's work to Weyl groups of type B_k, D_k and shows that all irreducible representations of $W(B_k)$, $W(D_k)$ can be obtained via the j-operation.

In Section II.3, we first define the generalized ϕ-operations. Suppose that W is a $W(B_n)$ module, and let T be any $\mathbb{C}$ vector space. Define $\phi_w(T)$ to be the $W(B_n)$ invariants of $W \otimes T^{\otimes n}$ under the $W(B_n)$-action which factors through S_n. Then For any set $S \subset R(W(B_k))$, let

$\langle S\rangle_\phi$:= the subring of $R(W(B_k))$ which is generated by the set obtained by applying all ϕ-operations to the elements of S.

and let

$\langle S\rangle_\lambda$:= the subring of $R(W(B_k))$ generated by the set obtained by applying λ-operations to the elements of S.

We use Macdonald's Theorem to show that

$$\left\{Ind^{W(B_k)}_{\prod W(B_\beta)\times W(D_\alpha)}\varepsilon\right\}$$

is basis for for $R(W(B_k))$. Solomon's Theorem then implies that

$$\left\{Ind^{W(B_k)}_{\prod W(B_\beta)\times W(D_\alpha)\times W(A_\gamma)}1\right\}$$

is a generating set for $R(W(B_k))$. We use this fact to prove that $\langle S\rangle_\phi = \langle S\rangle_\lambda$.

In Section II.4, we discuss the representations X_k, Y_k and present the main the-orem. This theorem shows that any element in S is an element of $\langle X_k, Y_k\rangle_\phi$, where

$$X_k = Ind^{W(B_k)}_{W(B_{k-1})\times W(B_1)}1$$
$$Y_k = Ind^{W(B_k)}_{W(B_{k-1})}1.$$

The proof of this theorem is by induction. Consider the representation

$$Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1$$

where $\mathcal{W}_{\beta,\alpha} = \prod \mathcal{W}(B_\beta) \times \prod \mathcal{W}(A_\alpha)$. We show that

$$Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 = \phi_{M_l}(X_k) \otimes \phi_{M_h}(Y_k) - \sum_\gamma Ind_{H_\gamma}^{\mathcal{W}(B_k)} 1,$$

and that each induced representation $Ind_{H_\gamma}^{\mathcal{W}(B_k)} 1$ satisfies the induction hypothesis. Since $\phi_{M_l}(X_k) \otimes \phi_{M_h}(Y_k) \in \langle X_k, Y_k \rangle_\phi$, $Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \in \langle X_k, Y_k \rangle_\phi$.

In Section II.5 we prove that the conjugacy classes of $\mathcal{W}(B_k)$ can be represented by the "double partitions" of k. This leads us to conclude that the set

$$\mathcal{S} = \left\{ Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \mid \beta \vdash i, \alpha \vdash j, i + j = k \right\}.$$

is a linearly independent subset of $R(\mathcal{W}(B_k))$. It follows that $1 \otimes \mathcal{S}$ is basis of $\mathbf{Q} \otimes R(\mathcal{W}(B_k))$.

Finally, in Section II.6 we show that

$$\mathbf{Q} \otimes R(\mathcal{W}(B_k)) = \langle X_k, Y_k \rangle_\lambda.$$

The results of Section II.3 show that $\langle 1 \otimes X_k, 1 \otimes Y_k \rangle_\phi = \langle 1 \otimes X_k, 1 \otimes Y_k \rangle_\lambda$. Then, by the Main Theorem, $1 \otimes \mathcal{S} \subset \langle 1 \otimes X_k, 1 \otimes Y_k \rangle_\lambda$. Since $1 \otimes \mathcal{S}$ is a $\mathbf{Q}$ basis of $\mathbf{Q} \otimes R(\mathcal{W}(B_k))$ the result follows.

By Lusztig's Theorem (Section II.2) the restriction map from $R(\mathcal{W}(B_k))$ to $R(\mathcal{W}(D_k))$ is onto. It follows immediately that $\mathbf{Q} \otimes R(\mathcal{W}(D_k))$ is generated by two elements as a λ-ring over $\mathbf{Q}$.

PART I

SECTION 1

λ-RINGS

Let R be a commutative ring with identity. Define the polynomials

$$s_i \in R[\xi_1, \xi_2, \dots, \xi_q], \qquad \sigma_j \in R[\eta_1, \eta_2, \dots, \eta_r]$$

by:

$$(1 + s_1 t + s_2 t^2 + \dots) = \prod (1 + \xi_i t)$$
$$(1 + \sigma_1 t + \sigma_2 t^2 + \dots) = \prod (1 + \eta_j t).$$

Then the s_i's and σ_j's are the elementary symmetric functions of the ξ_i's and η_j's respectively.

DEFINITION 1.1. *Define the universal polynomial* $P_n(s_1, s_2, \dots, s_n; \sigma_1, \dots, \sigma_n)$ *to be the coefficient of* t^n *in* $\prod_{i,j}(1 + \xi_i \eta_j t)$. *Similarly, define the universal polynomial* $P_{n,d}(s_1, \dots, s_{nd})$ *to be the coefficient of* t^n *in the product*

$$\prod_{1 \le i_1 < \dots < i_d \le q} (1 + \xi_{i_1} \xi_{i_2} \dots \xi_{i_d} t).$$

By the fundamental theorem of symmetric functions $P_n, P_{n,d}$ are polynomials with integer coefficients and are independent of q and r as long as $q \ge n$, $r \ge n$ in the case of P_n and $q \ge nd$ in the case of $P_{n,d}$.

DEFINITION 1.2. *A* λ*-ring* R *is a commutative ring with identity possessing operations* $\lambda^i : R \longrightarrow R \ \forall i \in \mathbb{Z}^+$ *such that given any* $x, y \in R$,

(1) $\lambda^i(1) = 0$ *for* $i > 1$,
(2) $\lambda^0(x) = 1$, $\lambda^1(x) = x$,
(3) $\lambda^n(x + y) = \sum_i \lambda^i(x) \lambda^{n-i}(y)$,
(4) $\lambda^n(xy) = P_n(\lambda^1(x), \lambda^2(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y))$,
(5) $\lambda^m(\lambda^n(x)) = P_{n,m}(\lambda^1(x), \dots, \lambda^{mn}(x))$.

Thus there is a formal power series

$$\lambda_t(x) = \lambda^0(x) + \lambda^1(x)t + \cdots + \lambda^n(x)t^n + \ldots$$

satisfying $\lambda^0(x) = 1$, $\lambda^1(x) = x$, and $\lambda_t(x + y) = \lambda_t(x)\lambda_t(y)$. Observe that a λ-ring R necessarily has characteristic zero, since for any positive integer $m > 1$, $\lambda_t(m) = \lambda_t(1 + \cdots + 1) = (\lambda_t(1))^m = (1 + t)^m = 1 + mt + \cdots + t^m \neq 1$.

EXAMPLE 1.3.

The simplest example of a λ-ring is the ring of integers where the λ-operation is defined by $\lambda^n(m) = \binom{m}{n}$. Similarly there is a λ-ring structure on $\mathbf{R}$ where the λ-operations are defined to be the coefficients of the binomial series of $(1+t)^r$. That is, $\lambda^n(r) = \frac{r(r-1)\ldots(r-n+1)}{n!}$.

DEFINITION 1.4.

(1) A morphism of λ-rings $f : R_1 \longrightarrow R_2$ is a ring homomorphism such that $\lambda^n(f(r)) = f(\lambda^n(r))$.

(2) An augmented λ-ring is a λ-ring R together with a map of λ-rings $\epsilon : R \longrightarrow \mathbf{Z}$.

(3) Given λ-rings R and S, define the product of R and S to be the direct product $R \times S$ as a ring and with λ-operations defined by $\lambda_t(r, s) = \lambda_t(r, 0).\lambda_t(0, s)$, where $\lambda_t(r, 0) = (1, 1) + \sum(\lambda^n(r), 0)t^n$, $\lambda_t(0, s) = (1, 1) + \sum(0, \lambda^n(s))t^n$. That is $\lambda^n(r, s) = \sum(\lambda^i(r), \lambda^{n-i}(s))$.

(4) The tensor product of two λ-rings is constructed by taking the tensor product $R \otimes S$ of the rings R and S where $\lambda^n(a \otimes 1) = \lambda^n(a) \otimes 1$, $\lambda^m(1 \otimes b) = 1 \otimes \lambda^m(b)$. Then define $\lambda^n(a \otimes b) = \lambda^n(a \otimes 1.1 \otimes b)$ by property (4) of definition 1.2. That is, define

$$\begin{aligned}\lambda^n(a \otimes b) &= \lambda^n(a \otimes 1.1 \otimes b) \\ &= P_n(\lambda^1(a) \otimes 1, \ldots, \lambda^n(a) \otimes 1, 1 \otimes \lambda^1(b), \ldots, 1 \otimes \lambda^n(b))\end{aligned}$$

Now observe that an arbitrary element of $R \otimes S$ has the form $\sum_i a_i \otimes b_i$. Define the λ^n for a general element in $R \otimes S$ by

$$\sum_{m_1 + \cdots + m_t = n} \lambda^{m_1}(a_1 \otimes b_1) \ldots \lambda^{m_t}(a_t \otimes b_t).$$

DEFINITION 1.5. *Let x be an element of a λ-ring R. If $\lambda_t(x)$ is a polynomial of degree n, then x is said to have degree n.*

PROPOSITION 1.6 [KN]. *Let R be a λ-ring and $R[x]$ the ring of polynomials in one variable over R. Then there is a unique λ-structure on $R[x]$ which requires that x have degree 1.*

PROOF: For all $f(x) = \sum a_i x^i$, define

$$\lambda_t(f(x)) = \lambda_t(\sum_i a_i x^i) = \prod_i \lambda_t(a_i x^i) = \prod_i \lambda_{x^i t}(a_i).$$

From this definition it follows that

$$\lambda_t(f(x) + g(x)) = \lambda_t(f(x))\lambda_t(g(x)).$$

It is clear that $\lambda_t(x) = 1 + xt$. To prove property (4) of definition 1.2, it suffices to prove the polynomial identities in the case of products of monomials.

(1) $\lambda_t(rs) = 1 + \sum P_n(\ldots, \lambda^i(r), \ldots, \lambda^i(s), \ldots)t^j, \ r, s \in R.$
(2) $\lambda_t(x^p x^q) = 1 + \sum P_n(\ldots, \lambda^i(x^p), \ldots, \lambda^i(x^q), \ldots)t^j.$
(3) $\lambda_t(rx^p) = 1 + \sum P_n(\ldots, \lambda^i(r), \ldots, \lambda^i(x^p), \ldots)t^j.$

(1) is true by hypothesis, while (2) and (3) follow immediately from the definition of λ_t. Property (5) follows in a similar manner to this. ∎

Observe that if R is augmented then so is $R[x]$. First define a λ-ring morphism $R[x] \longrightarrow R$ by evaluation of the polynomial $f(x)$ at 0 and then compose this with the augmentation $\epsilon : R \longrightarrow \mathbf{Z}$ of R.

Let $\xi_1, \ldots, \xi_n, \ldots$ be independent indeterminates, and set

$$\Omega_0 = \mathbf{Z}, \qquad \Omega_n = \mathbf{Z}[\xi_1, \ldots, \xi_n] \qquad n \geq 1.$$

Then Ω_n is a λ-ring by defining $\lambda_t(\xi_i) = 1 + \xi_i t$. For any $r > n$, define a λ-ring morphism $\phi_{r,n} : \Omega_r \longrightarrow \Omega_n$ by

$$\phi_{r,n}(\xi_i) = \begin{cases} \xi_i & i \leq n \\ 0 & i > n \end{cases}$$

Now, let $\Omega = \varprojlim \Omega_r$ in the category of rings. Recall that this is the set of all sequences $(\ldots, w_n, w_{n-1}, \ldots, w_1, w_0)$ with $w_i \in \Omega_i$ such that $\phi_{n+1,n}(w_{n+1}) = w_n$. Ω has the structure of a λ-ring, with the λ-operation defined by $\lambda_\Omega(\ldots, w_n, \ldots, w_0) = (\ldots, \lambda(w_n), \ldots, \lambda(w_0))$.

For $n \leq r$, let a_n be the nth elementary symmetric function in the independent indeterminates $\xi_1, \ldots, \xi_r$. Also, let a_n denote the associated element in Ω. Then $\lambda^n_\Omega(a_1) = a_n$, for all n, since this is true in each Ω_r. Furthermore, observe that the $a_n \in \Omega$ are algebraically independent. This follows because any polynomial identity involving $a_1, \ldots, a_r$ would force a polynomial identity involving $a_1(\xi_{i_1}, \ldots, \xi_{i_r}), \ldots, a_r(\xi_{i_1}, \ldots, \xi_{i_r})$ in Ω_r.

Let Λ denote the λ-subring of Ω generated by a_1. Then $\Lambda = \mathbf{Z}[a_1, a_2, \ldots]$. Since any λ-subring of Ω containing a_1 must contain $a_n = \lambda^n(a_1)$, it follows that $\Lambda \supset \mathbf{Z}[a_1, a_2, \ldots]$. Now let $f(a_1, a_2, \ldots) \in \mathbf{Z}[a_1, a_2, \ldots]$. By definition, $\lambda^n_\Omega(f(a_1, a_2, \ldots))$ can be expressed in terms of the universal polynomials. Since the universal polynomials can be written as a polynomial in the a_i, it follows that $\Lambda \subset \mathbf{Z}[a_1, a_2, \ldots]$. Thus $\Lambda = \mathbf{Z}[a_1, a_2, \ldots]$. Λ is called the free λ-ring on one generator.

Now let F be the forgetful functor from the category of λ-rings to the category of sets. Then the natural transformations of F are morphisms of sets $a_R : R \longrightarrow R$ such that for any λ-ring morphism $f : R \longrightarrow S$, $fa_R = a_S f$. Addition, and multiplication are defined componentwise:

$$(\alpha_R + \beta_R)(r) = \alpha_R(r) + \beta_R(r)$$

$$\alpha_R \beta_R(r) = \alpha_R(r)\beta_R(r).$$

Proposition 1.7 (Atiyah, Tall). *The set of natural operations T is a λ-ring and is isomorphic to Λ, the free λ-ring on one generator.*

Proof: Define a λ-ring morphism $\Phi : \Lambda \longrightarrow T$ by

$$\Phi(f(\lambda^1, \lambda^2, \ldots))(x) = f(\lambda^1(x), \lambda^2(x), \ldots),$$

for some x in some λ-ring R. In particular x can be chosen to be an element from Λ, and so if $\Phi(f(\lambda^1, \lambda^2, \ldots)) = 0$, then $\Phi(f(\lambda^1, \lambda^2, \ldots))(a_1) = f(\lambda^1(a_1), \lambda^2(a_1), \ldots) =$

$f(a_1, a_2, \dots) = 0$. Since the a_i are algebraically independent elements, it follows that f is the zero polynomial and consequently Φ must be injective.

If $\mu \in \mathcal{T}$, then $\mu(a_1) \in \Lambda$ and $\mu(a_1) = f(a_1, a_2, \dots)$ for some polynomial f. If R is any λ-ring and $x \in R$, there is a λ-ring morphism $u_x : \Lambda \longrightarrow R$ defined by $u_x(a_n) = \lambda^n(x)$. Because μ is natural, it commutes with u_x and hence

$$\begin{aligned}
\mu(x) &= \mu u_x(a_1) \\
&= u_x \mu(a_1) \\
&= u_x f(a_1, a_2, \dots) \\
&= f(\lambda^1(x), \lambda^2(x), \dots) \\
&= \Phi(f(\lambda^1, \lambda^2, \dots))(x).
\end{aligned}$$

Thus $\mu = \Phi(f(\lambda^1, \lambda^2, \dots))$, which implies that Φ is onto.

Define the λ-operations on $\mathcal{T}$ by $\lambda_{\mathcal{T}}(\mu) = \Phi(\lambda_n(f(a_1, a_1, \dots,)))$. It now follows that Φ is a λ-ring isomorphism. ∎

Proposition 1.7 shows that any natural operation $\mu \in \mathcal{T}$, can be expressed as a polynomial in the λ-operations. Given an arbitrary polynomial $f(\lambda^1, \lambda^2, \dots)$ and an operation μ, to verify that $\mu = f(\lambda^1, \lambda^2, \dots)$ it suffices to show $\mu(a_1) = f(a_1, a_2, \dots)$. Since $\Omega \supsetneq \Lambda$, it is only necessary to check this in each Ω_r. That is, it is sufficient to show

$$\mu(\xi_1 + \cdots + \xi_r) = f(a_1(\xi_1, \dots, \xi_r), \dots, a_n(\xi_1, \dots, \xi_r)).$$

This gives rise to the following verification principle.

Verification Principle (Atiyah). *If μ is a λ-ring operation, then μ is uniquely a polynomial in the λ-operations. The operation μ is equal to $f(\lambda^1, \lambda^2, \dots)$, if and only if, the identity holds operating on a sum $\xi_1 + \cdots + \xi_r$ of elements of degree 1, for all $r > 0$.*

The Adams Operations

Let Λ be the free λ-ring on one generator and let $\alpha \in \Lambda$. Then by the proof of Proposition 1.6, given a λ-ring R, α becomes an operation on R. Moreover, for each $r \in R$, $\alpha(r) = f(r, \lambda^2(r), \lambda^3(r), \dots)$. Let $Q_n = \sum \xi_i^n$ denote the symmetric power sums. We can express each Q as a polynomial in the elementary symmetric polynomials $\{a_1, \dots, a_n\}$ of the indeterminates $\xi_1, \xi_2, \dots, \xi_q$. (See [Bou] Appendix to V.2 Proposition 3.) For example we have

$$Q_1 = \xi_1 + \xi_2 + \cdots = a_1$$
$$Q_2 = \xi_1^2 + \xi_2^2 + \cdots = a_1^2 - 2a_2$$
$$Q_3 = \xi_1^3 + \xi_2^3 + \cdots = a_1^3 - 3a_1a_2 + 3a_3.$$

DEFINITION 1.8. *Let Q_n denote the element in $\Lambda = \mathbf{Z}[a_1, a_2, \dots]$ corresponding to the element $Q_n(\xi_1, \dots, \xi_n) \in \Omega_r$. Define a class of operations $\Psi^n : R \longrightarrow R$ on the category of λ-rings by $\Psi^n(a) = Q_n(a, \lambda^2(a), \dots)$. Ψ^n is called the nth Adams operation on R.*

PROPOSITION 1.9 (ADAMS). *Let a, b be elements in a λ-ring R, and n, m integers ≥ 1. Then.*

(1) $\Psi^1(a) = a$
(2) $\Psi^n(1) = 1$
(3) $\Psi^n(a + b) = \Psi^n(a) + \Psi^n(b)$
(4) $\Psi^n(ab) = \Psi^n(a)\Psi^n(b)$
(5) $\Psi^n(\Psi^m(a)) = \Psi^{nm}(a) = \Psi^{mn}(a)$
(6) $\Psi^n(\lambda^m(a)) = \lambda^m(\Psi^n(a))$.

PROOF: By the verification principle. ∎

Thus each Ψ^n is a λ-ring endomorphism of R, and the morphism $\gamma : \mathbf{N} \longrightarrow End(R)$ given by, $\gamma(n) = \Psi^n$, is a morphism of monoids. A ring satisfying properties (1)-(6) of Proposition 1.9 is called a Ψ-ring.

Let R be a ring and let $R[[t]]$ denote the power series ring of R in the indeterminate t. Define $c : R[[t]] \longrightarrow R$ by

$$c\left(\sum_{i=0}^{\infty} r_i t^i\right) = r_0.$$

This is a surjective ring morphism, and induces an epimorphism on the group of units $R[[t]]^*$.

$$c^* : R[[t]]^* \longrightarrow R^*.$$

Observe that

$$\begin{aligned} ker\, c^* &= \{1 + tf \mid f \in R[[t]]\} \\ &= 1 + tR[[t]]. \end{aligned}$$

Define a λ-ring structure on $1 + tR[[t]]$ by:

(1) Addition is defined to be the power series multiplication in $R[[t]]$, and is denoted by $\boxplus$.

(2) Multiplication is given by

$$(1 + \sum a_n t^n) \boxtimes ((1 + \sum b_m t^m) = 1 + \sum P_n(a_1, \ldots, a_n; b_1, \ldots, b_n)t^n$$

(3) The λ-ring structure is given by

$$\Lambda^m(1 + \sum a_n t^n) = 1 + \sum P_{n,m}(a_1, \ldots, a_{mn})t^n.$$

Lemma 1.10 (Witt). $1+tR[[t]]$ *is a ring, called the Universal Ring of Witt Vectors, and satisfies properties (1)-(3) of Definition 1.2*

Proposition 1.11 (Witt). *R is a λ-ring if and only if λ_t is a homomorphism of 'pre'-λ rings (i.e. rings which satisfy properties (1)-(3) of Definition 1.2).*

Proof: This follows immediately from the definition of the operations in $1 + tR[[t]]$. ∎

Theorem 1.12 (Grothendieck). *Let R be a ring which satisfies properties (1)-(3) of Definition 1.2. Furthermore, suppose that R satisfies the following three conditions:*

(1) $\lambda_t(1) = 1 + t$.

(2) $\forall\, r \in R,\ r = \sum \pm a_i$, *where a_i is an element of degree 1,*

(3) *The product of two elements of degree 1 is again an element of degree 1.*

Then R is a λ-ring.

Proof: We prove that λ_t commutes with the λ-operations. λ_t is additive by definition. Let $x, y \in R$. Since x and y can be written as the sum of elements of degree 1, it suffices to prove the theorem when x and y have degree 1. In this case, the product xy has degree 1. Thus

$$\begin{aligned}\lambda_t(xy) &= 1 + xyt \\ &= (1 + xt) \boxtimes (1 + yt) \\ &= \lambda_t(x) \boxtimes \lambda_t(y).\end{aligned}$$

To prove that $\lambda_t(\lambda^n(x)) = \Lambda^n(\lambda_t(x))$, suppose, for the moment, that the identity holds for the elements a, b. Let $\sum\limits_*^*$ denote a summation in $1 + tR[[t]]$. Then

$$\begin{aligned}\lambda_t(\lambda^n(a+b)) &= \lambda_t\left(\sum_*^* \lambda^i(a)\lambda^{n-i}(b)\right) \\ &= \sum_*^* \lambda_t(\lambda^i(a)\lambda^{n-i}(b)) \\ &= \sum_*^* (\lambda_t(\lambda^i(a)) \boxtimes \lambda_t(\lambda^{n-i}(b))) \\ &= \sum_*^* (\Lambda^n(\lambda_t(a)) \boxtimes \Lambda^{n-i}(\lambda_t(b))) \\ &= \Lambda^n(\lambda_t(a) \boxtimes \lambda_t(b)) \\ &= \Lambda^n(\lambda_t(a+b)).\end{aligned}$$

So again it suffices to prove this identity for elements of degree 1. Let $x \in R$ have degree 1. Then

$$\begin{aligned}\lambda_t(\lambda^n(x)) &= \lambda_t(1) \\ &= 1 + t \\ &= \Lambda^n(1 + t) \\ &= \Lambda^n(\lambda_t(x)).\end{aligned}$$

Thus by Proposition 1.11 R is a λ-ring. ∎

THEOREM 1.13 (GROTHENDIECK). *$1+tR[[t]]$ is a λ-ring.*

PROOF: First observe that in $1+tR[[t]]$

$$\lambda_t(1) = 1 + t.$$

Furthermor: in any λ-ring where the product of elements of degree 1 is again an element of degree 1, then if $x = \sum x_i$, $y = \sum y_i$ are sums of elements of degree 1,

$$\lambda_t(xy) = \lambda_t\left(\sum x_i y_j\right) = \prod(1 + x_i y_j t) = \lambda_t(x)\lambda_t(y).$$

Similarly, $\lambda_t(\lambda^n(x)) = \Lambda^n(\lambda_t(x))$.

Let A denote the subring of $1+tR[[t]]$ which is generated by the elements of degree 1. Then in the ring $1+tA[[t]]$ the product of elements of degree 1 is again of degree 1, since $(1+at)(1+bt) = 1+abt$. Let $x = \prod(1+a_i t)$, $y = \prod(1+b_i t)$. Then all such x, y satisfy properties (4) and (5) of definition 1.2 and therefore these properties hold universally on $1+tR[[t]]$. ∎

DEFINITION 1.14. *Let R be a ring, and let $W(R)$ denote the set of countably infinite sequences $[r_1, r_2, \ldots]$, $r_i \in R$. $W(R)$ has the structure of a ring when addition and multiplication are defined componentwise. For each $n \in \mathbf{Z}^+$ define the Adams operations*

$$\Psi^n : W(R) \longrightarrow W(R)$$

by $\Psi^n([r_1, r_2, \ldots]) = [r_n, r_{2n}, \ldots]$.

PROPOSITION 1.15 (WITT). *$W(R)$ is a Ψ-ring. If R is a λ-ring which satisfies properties (1)-(3) of Proposition 1.9 then R is a Ψ-ring if and only if the morphism $\hat{\Psi} : R \longrightarrow W(R)$, defined by $\hat{\Psi}(r) = [\Psi^1(r), \Psi^2(r), \ldots]$ is a morphism which preserves the Ψ-operations.*

DEFINITION 1.16. *A ring R is called torsion free if for any $0 \neq r \in R$ and any integer $n \geq 1$, $n.r = r + \cdots + r \neq 0$.*

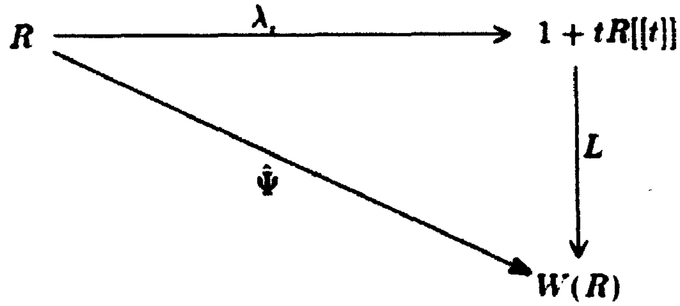
Define a morphism $L : 1+tR[[t]] \longrightarrow W(R)$ by:

$$L(1 + \sum a_i t^i) = (r_1, r_2, \ldots),$$

where the elements r_i are defined by

$$d/dt\left(log(1+\sum a_i t^i)\right) = \sum (-1)^n r_{n+1} t^n.$$

Then L makes the following diagram commute.



PROPOSITION 1.17 (WITT). *If R contains a subring isomorphic to $\mathbf{Q}$, then L is an isomorphism of Ψ-rings.*

PROOF: Let $r, s \in 1 + tR[[t]]$. That $L(r \boxplus s) = L(r) + L(s)$ and $L(1) = 1$, follow immediately from the definitions. To verify that $L(r \boxtimes s) = L(r)L(s)$ and $L(\Psi^n(r)) = \Psi^n L(r)$, it suffices to choose elements r,s which have degree 1. Then since L is additive, the two identities follow directly from the definitions. Finally, define

$$K : W(R) \longrightarrow 1 + tR[[t]]$$

by $K([r_1, r_2, \dots]) = exp(g(t))$, where $g(t) = \sum (-1)^n r_n t^n$. Since R is torsion free and contains $\mathbf{Q}$, K is well defined. It is clear that $K \circ L = 1$ and $L \circ K = 1$. Thus, L is a Ψ-ring isomorphism. ∎

THEOREM 1.18 [KN]. *Let R be a torsion free ring which contains a subring isomorphic to $\mathbf{Q}$. Suppose that R possesses operations $\Psi^n : R \longrightarrow R$ which satisfy:*

$$\Psi^n(1) = 1, \Psi^n(ab) = \Psi^n(a)\Psi^n(b), \text{ and } \Psi^n(\Psi^m(a)) = \Psi^{nm}(a) \ \forall\ a, b \in R.$$

Suppose further that there are operations $\lambda^i : R \longrightarrow R$ satisfying properties (1)-(3) of definition 1.2, and such that

$$(1.19) \qquad d/dt(\log \lambda_t(a)) = \sum_n (-1)^n . \Psi^{n+1}(a) t^n$$

for $a \in R$. Then R is a λ-ring.

PROOF: Since $d/dt(\log \lambda_t(x)) = \sum_n (-1)^n \Psi^{n+1}(x) t^n$, $\lambda_t'/\lambda_t = (-1)^n \sum \Psi^{n+1} t^n$. Since $\lambda_t = \sum_n \lambda^n t^n$ it follows that $\sum_n (n+1)\lambda^{n+1} t^n = \left(\sum_n (-1)^n \Psi^{n+1} t^n\right)\left(\sum_n \lambda^n t^n\right)$.

$$\Rightarrow (-1)^n \Psi^{n+1} + (-1)^{n-1}\Psi^n \lambda^1 + (-1)^{n-2}\Psi^{n-1}\lambda^2 + \cdots + \Psi^1 \lambda^n = (n+1)\lambda^{n+1}.$$

From this recursion relation we obtain the following set of equations;

$$\begin{aligned} \Psi^1 &= \lambda^1 \\ \lambda^1\Psi^1 - \Psi^2 &= 2\lambda^2 \\ \lambda^2\Psi^1 - \lambda^1\Psi^2 + \Psi^3 &= 3\lambda^3 \\ \lambda^3\Psi^1 - \lambda^2\Psi^2 + \lambda^1\Psi^3 - \Psi^4 &= 4\lambda^4 \\ &\vdots \end{aligned}$$

Solving this system for λ^n by Cramer's rule we find that:

$$\lambda^n = \frac{\det \begin{vmatrix} 1 & 0 & 0 & \dots & \Psi^1 \\ \Psi^1 & -1 & 0 & \dots & -\Psi^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \Psi^{n-1} & -\Psi^{n-2} & \Psi^{n-3} & \dots & (-1)^{n+1}\Psi^n \end{vmatrix}}{\det \begin{vmatrix} 1 & 0 & \dots & 0 \\ \Psi^1 & 2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \Psi^{n-1} & -\Psi^{n-2} & \dots & n \end{vmatrix}}$$

and hence,

$$\lambda^n = \frac{1}{n!}\det \begin{vmatrix} 1 & 0 & \dots & \Psi^1 \\ \Psi^1 & -1 & \dots & -\Psi^2 \\ \vdots & \vdots & & \vdots \\ \Psi^{n-1} & -\Psi^{n-2} & \dots & (-1)^{n+1}\Psi^n \end{vmatrix}.$$

Since $L \circ \lambda_t = \hat{\Psi}$ and L is a Ψ-ring isomorphism by Proposition 1.17, λ_t is a Ψ-ring morphism as well. Therefore, because the λ-operations can be expressed as an algebraic combination of the Adams operations and λ_t preserves the Adams operations, λ_t must preserve the λ-operations. Thus, by Proposition 1.11, R is a λ-ring. ∎

SECTION 2

REPRESENTATION RINGS

The material contained in this section can be found in [Ku] and [Se].

Let G be a finite group. Let V_1, V_2 be finite dimensional G-modules over $\mathbf{C}$. Then $V_1 \otimes V_2$ is a G-module under the action $g(v_1 \otimes v_2) = gv_1 \otimes gv_2$. The representation ring of G, $R(G)$, consists of all finite formal sums $\sum_{n_i} n_i[V_i]$, $n_i \in \mathbf{Z}$, of G-modules V_i, modulo the relations

(1) $[V_1] = [V_2]$, if $V_1 \approx V_2$ as G-modules,

(2) $[V_1 \oplus V_2] = [V_1] + [V_2]$.

$R(G)$ is clearly an abelian group. Define a multiplication on $R(G)$ by $[V_1].[V_2] = [V_1 \otimes V_2]$. This gives $R(G)$ the structure of a commutative ring with identity. Let $\mathrm{Irred}(G)$ denote the set of isomorphism classes of irreducible G-modules. Then $R(G)$ is a free $\mathbf{Z}$-module with basis $\{[V] : V \in \mathrm{Irred}(G)\}$. The rank of $R(G)$ is equal to the number of conjugacy classes of G (see p. 21 for the proof of this staement).

Let R be a ring, and let M be a R-module. Let $T^n(M)$ be the n-fold tensor product $M \otimes \cdots \otimes M$, for $n \geq 1$. When $n = 0$, let $T^0(M) = R$. Then

$$T(M) = \bigoplus_{n \geq 0} T^n(M)$$

is a noncommutative R-algebra, called the tensor algebra of M. Define the symmetric algebra

$$S(M) = \bigoplus_{n \geq 0} S^n(M)$$

of M to be the quotient of $T(M)$ by the two sided ideal generated by all expressions of the form $x \otimes y - y \otimes x$, $\forall\, x, y \in M$. Observe that $S(M)$ is a commutative R-algebra. Its component $S^n(M)$ in degree n is called the nth symmetric power of M. If M is a free R-module of rank r, then note that $S(M) \approx R[x_1, \ldots, x_r]$ for any basis $\{x_1, \ldots, x_r\}$ of M.

Define the exterior algebra

$$\Lambda(M) = \bigoplus_{n \geq 0} \Lambda^n(M)$$

of M to be the quotient of $T(M)$ by the two sided ideal $I(M)$ generated by all expressions $x \otimes x$, $\forall\, x \in M$. The component $\wedge^n(M)$ in degree n is called the nth exterior power of M. The equivalence class of $x_1 \otimes \cdots \otimes x_r$ in $\wedge^r(M)$ by $x_1 \wedge \cdots \wedge x_r$. If M is a free R module with basis $\{x_1, \ldots, x_k\}$ then $\{x_{i_1} \wedge \cdots \wedge x_{i_n} \mid i_1 < \cdots < i_n\}$ is a R basis of $\wedge^n(M)$. Observe that $I(M)$ contains all expressions of the form $x \otimes y + y \otimes x$ so that $\wedge(M)$ is a skew commutative graded R-algebra. That is, if $u \in \wedge^r(M)$ and $v \in \wedge^s(M)$, then $u \wedge v = (-1)^{rs} v \wedge u$.

Definition 2.1. *Let V be a G-module. Define a class of operations on $R(G)$ by $\lambda^n([V]) = [\wedge^n V]$. The G-action on $\Lambda^n V$ is given by $g(v_1 \wedge \cdots \wedge v_n) = gv_1 \wedge \cdots \wedge gv_n$.*

Remarks 2.2.

(1) Take $\lambda^0(V)$ to be the trivial one-dimensional representation. Observe that $\lambda^1 V = V$. Furthermore if $n = \dim V$, then $\lambda^n V$ is the one-dimensional representation assigning to each $g \in G$, the determinant of the matrix assigned to g in the representation V.

(2) If $W = U \oplus V$ then $\Lambda^n W = \oplus_i (\Lambda^i U \otimes \Lambda^{n-i} W)$ and so $\lambda^n(x + y) = \sum \lambda^i(x)\lambda^{n-i}(y)$. Thus properties (1)-(3) of definition 1.2 are satisfied.

Let S be a set and let K be a field of characteristic zero. Let $K(S)$ denote the set of maps $f : S \longrightarrow K$. Define addition and multiplication on $K(S)$ component wise. Namely, $(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$. Then $K(S)$ is a commutative ring over K with identity. Suppose that there is a collection of maps $\sigma_n : S \longrightarrow S$ satisfying (1) $\sigma_1 = 1_s$, (2) $\sigma_n \sigma_m = \sigma_{nm}$. Then we can define operations $\Psi^n : K(S) \longrightarrow K(S)$ as follows: given $f : S \longrightarrow K$ and $s \in S$ let

$$\Psi^n(f(s)) = f(\sigma_n(s)).$$

These operations satisfy properties $1-5$ of Proposition 1.9. Next, define operations $\lambda^n : K(S) \longrightarrow K(S)$ by:

$$\lambda^n = \frac{1}{n!} \det \begin{vmatrix} \Psi^1 & 1 & \ldots & 0 \\ \Psi^2 & \Psi^1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \Psi^n & \Psi^{n-1} & \ldots & \Psi^1 \end{vmatrix}.$$

Let $\lambda_t(f) = \sum_i \lambda^i(f)t^i$. The definition of the λ^n produces a system of equations which is equivalent to the system of equations given in Theorem 1.18. This implies that λ_t satisfies:

$$d/dt(\log \lambda_t(f)) = \sum_n (-1)^n \Psi^{n+1}(f)t^n.$$

Therefore

$$\begin{aligned} d/dt(\log \lambda_t(f+g)) &= \sum_n (-1)^n \Psi^{n+1}(f+g)t^n \\ &= \sum_n (-1)^n \Psi^{n+1}(f)t^n + \sum_n (-1)^n \Psi^{n+1}(g)t^n \\ &= d/dt(\log \lambda_t(f)) + d/dt(\log \lambda_t(g)) \\ &= d/dt(\log[\lambda_t(f)\lambda_t(g)]). \end{aligned}$$

and consequently $\lambda_t(f+g) = \lambda_t(f)\lambda_t(g)$. Furthermore since $\Psi^n(1) = 1$.

$$\begin{aligned} d/dt(\log \lambda_t(1)) &= \sum_n (-1)^n \Psi^{n+1}(1)t^n \\ &= \sum_n (-1)^n t^n \\ \Longrightarrow \lambda_t(1) &= 1 + t. \end{aligned}$$

Thus, $\lambda^i(1) = 0$, $\forall\, i > 1$. Finally observe that $\lambda^1(f) = \Psi^1(f) = f$. Thus the λ^i are the λ-operations corresponding to the Adams operations Ψ^i, and hence $K(S)$ is a λ-ring by 1.18.

Given a finite group G, a class function $f : G \longrightarrow \mathbb{C}$ is a morphism of sets such that $f(gg') = f(g'g)$. Let $\mathbb{C}(G)$ denote the set of class functions on G. Let S be the set of conjugacy classes of G, and let $K = \mathbb{C}$. Define a collection of maps $\sigma_n : S \longrightarrow S$ by $\sigma_n(x) = x^n$. Then $\mathbb{C}(G) = K(S)$ is a λ-ring with operations defined by:

(1) $(f_1 + f_2)(a) = f_1(a) + f_2(a)$

(2) $(f_1.f_2)(a) = f_1(a).f_2(a)$

(3) $f^*(a) = (f(a))^*$ - the complex conjugate of $f(a)$

(4) $0(a) = 0, 1(a) = 1$

(5) $(\Psi^n f)(a) = f(a^n)$.

and whose λ-operations are defined via the Adams operations as at the bottom of page 19.

Observe that $\mathbf{C}(G)$ is a $\mathbf{C}$-algebra. We can construct a natural basis for $\mathbf{C}(G)$ as follows: Denote the conjugacy classes of G by $K_1, \ldots, K_n$, and for each conjugacy class K_i, define $\kappa_i : G \longrightarrow \mathbf{C}$ by:

$$\kappa_i(a) = \begin{cases} 1 & a \in K_i \\ 0 & a \notin K_i \end{cases}$$

It is clear that the κ_i are class functions and that given any $f \in \mathbf{C}(G)$, $f = \sum c_i \kappa_i$ for $c_i \in \mathbf{C}$. It follows that $\{\kappa_i \mid i = 1, \ldots, n\}$ is a basis of $\mathbf{C}(G)$. Hence the dimension of $\mathbf{C}(G)$ is the number of conjugacy classes of G.

Define an inner product on $\mathbf{C}(G)$ by

$$(f_1, f_2) = \frac{1}{|G|} \sum_g f_1^*(g) f_2(g).$$

LEMMA 2.3. *The set $\{\kappa_i \mid i = 1, \ldots, n\}$ forms an orthogonal basis with respect to the inner product $(\_,\_)$, defined above.*

PROOF: This is clear from the definition of the inner product, since

$$(\kappa_i, \kappa_j) = \begin{cases} \dfrac{|K_j|}{|G|} & i = j \\ 0 & i \neq j \end{cases} \quad \blacksquare$$

DEFINITION 2.4. *Given representations V_1, V_2 of G let $Hom^G(V_1, V_2)$ denote the set of $C[G]$-homomorphisms from V_1 to V_2, and define*

$$\langle V_1, V_2 \rangle_G = dim\left(Hom^G(V_1, V_2)\right).$$

The map $\langle \_, \_ \rangle_G : R(G) \longrightarrow \mathbf{C}$ of 2.4 defines an inner product on $R(G)$.

The character χ_V of a representation $\rho : G \longrightarrow Aut(V)$ is defined by $\chi_\rho(g) = Tr(\rho(g))$. Since χ depends only on the isomorphism type of V, and since $\chi_{V \oplus W} = \chi_V + \chi_W$, there is a morphism $\Gamma : R(G) \longrightarrow \mathbf{C}(G)$.

THEOREM 2.5. *Both* $\mathbf{C}(G)$ *and* $R(G)$ *are* λ*-rings. Furthermore* $\Gamma : R(G) \longrightarrow \mathbf{C}(G)$ *is a monomorphism of* λ*-rings which preserves the inner product. The irreducible characters of* G *form an orthonormal basis of* $\mathbf{C}(G)$.

PROOF: $\mathbf{C}(G)$ is a λ-ring by Theorem 1.11. Define an operation λ on $R(G)$ via the exterior power operations, that is, $\lambda^i(V) = \Lambda^i(V)$. This operation satisfies properties (1)-(3) of definition 1.2. The morphism $\Gamma : R(G) \longrightarrow \mathbf{C}(G)$ is a ring monomorphism (see p. 21 for the proof of this fact). If Γ commutes with the λ-operations, then because identities (4) and (5) of definition 1.2 hold in $\mathbf{C}(G)$ they must hold in $R(G)$ as well. This would then imply that $R(G)$ is a λ-ring. Let Ψ^n denote the Adams operations corresponding to the exterior powers of $R(G)$. To verify that Γ preserves the λ-structure it suffices to show it preserves the Adams operations. Let V be a G-module, then by definition $\Psi^k(V) = Q_k(\lambda^1 V, \lambda^2 V, \dots, \lambda^k V)$. We must prove that $\Gamma(\Psi^k(V))(g) = \chi_{\Psi^k(V)}(g) = \Psi^k(\chi_V)(g) = \Psi^k(\Gamma(V)(g))$. So given $g \in G$, choose a basis of V so that the corresponding matrix is the diagonal matrix

$$\rho_V(g) = \begin{pmatrix} g_1 & 0 & 0 & \dots & 0 \\ 0 & g_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_n \end{pmatrix}$$

Then

$$\begin{aligned} & 1 + \mathrm{Tr}(g)T + \mathrm{Tr}(\lambda^2(g))T^2 + \dots \\ & = \det(I + gT) \\ & = \det \begin{vmatrix} 1 + g_1 T & 0 & \dots & 0 \\ 0 & 1 + g_2 T & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 + g_n T \end{vmatrix} \\ & = \prod (1 + g_i T) \end{aligned}$$

Thus $\mathrm{Tr}(\lambda^i g)$ is the *ith* symmetric polynomial in the g_i*'s*. It follows that,

$$\begin{aligned} \chi_{\Psi^k(V)}(g) &= Q_k(\mathrm{Tr}\lambda^1 g, \mathrm{Tr}\lambda^2 g, \dots, \mathrm{Tr}\lambda^k g) \\ &= g_1^k + \dots + g_n^k \end{aligned}$$

$$= \mathrm{Tr}(\rho(g)^k)$$

$$= \chi_V(g^k)$$

$$= \Psi^k{}_{\chi_V}(g).$$

Hence $\Gamma : R(G) \longrightarrow \mathbf{C}(G)$ preserves the λ-structure.

Now, let V and W be representations of G. Then

$$(\chi_V, \chi_W) = \frac{1}{|G|}\sum_g \chi_V(g)^* \chi_W(g)$$

$$= \frac{1}{|G|}\sum_g \chi_{V^*}(g)\chi_W(g)$$

$$= \frac{1}{|G|}\sum_g \chi_V(g)\chi_W(g)$$

$$= \frac{1}{|G|}\sum_g \chi_{V\otimes W}(g)$$

$$= \frac{1}{|G|}\sum_g \chi_{Hom(V,W)}(g)$$

$$= \dim_{\mathbb{C}} \mathrm{Hom}(V, W)^G$$

$$= \dim_{\mathbb{C}} \mathrm{Hom}_G(V, W)$$

$$= \langle V, W\rangle_G.$$

Thus Γ preserves the inner product. It follows that Γ is injective and that the set of irreducible characters is orthonormal.

Finally, to show that the set of irreducible characters spans $\mathbf{C}(G)$, let $f \in \mathbf{C}(G)$ and let $\rho : G \longrightarrow Aut(V)$ be an irreducible representation of degree n. Define $\rho_f \in End(V)$ by $\rho_f = \sum_g f(g)\rho(g)$. Then

$$\rho(g)^{-1}\rho_f\rho(g) = \sum_{g'} f(g')\rho(g)^{-1}\rho(g')\rho(g)$$

$$= \sum_{g'} f(g')\rho(g^{-1}g'g)$$

$$= \sum_h f(gh^{-1}g)\rho(h)$$

$$= \sum_h f(h)\rho(h)$$
$$= \rho_f.$$

Hence, by Schur's lemma there is a constant c so that $\rho_f = cI$. $\Longrightarrow \mathrm{Tr}(cI) = nc = \sum_g f(g)\mathrm{Tr}(\rho(g)) = \sum_g f(g)\chi_V(g) = (f^*, \chi_V)$.

Now suppose that f is a class function which is orthogonal to all irreducible characters, that is, $(f, \chi_V) = 0$ for all irreducible G-modules V. If ρ is the representation corresponding to χ_V then $\rho_{f^*} = \sum f(g)^*\chi_V = (f, \chi_V) = 0$. Since each representation is a linear combination of irreducibles, it follows that $\rho_{f^*} = 0$ for any representation ρ. In particular, this is true for the regular representation.

If we let ρ denote the regular representation, then for any basis vector e, $0 = \rho_{f^*}(e) = \sum_g f(g)^*\rho_g(e) = \sum_g f(g)^* g$. Since the set of all elements in G form a basis of the regular representation, it follows that $f(g) = 0$ for all $g \in G$, that is $f = 0$. Thus for any class function f, $f - \sum(f, \chi_V)\chi_V$ is orthogonal to all irreducible characters and so must be zero. Thus $f = \sum(f, \chi_V)\chi_V$. ∎

Definition 2.6. *Let R be a ring, and let $f_1, \ldots, f_r \in R$. Define the Koszul complex $K.(f_1, \ldots, f_r)$ as follows: K_1 is the free R-module of rank r with basis $e_1, \ldots, e_r$. For $p = 0, \ldots, r$, let $K_p = \Lambda^p K_1$. Define the boundary map $d : K_p \longrightarrow K_{p-1}$ by its action on the basis vectors:*

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum (-1)^{j-1} f_{i_j} . e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \ldots e_{i_p}.$$

Since $d \circ d = 0$, $K.(f_1, \ldots, f_r)$ is a homological complex of R-modules. If M is any R-module, then set $K.(f_1, \ldots, f_r; M) = K.(f_1, \ldots, f_r) \otimes_R M$.

Theorem 2.7. *Let E be a G-module. Then in the representation ring $R(G)$, for each integer n there exist integers $m_1, \ldots, m_t$ and a polynomial f_n such that $\lambda^n(E) = f_n(S^{m_1}(E), \ldots, S^{m_t}(E))$. Similarly, given an integer m there exist integers $n_1, \ldots, n_t$ and a polynomial g_m such that $S^m(E) = g_m(\lambda^{n_1}(E), \ldots, \lambda^{n_t}(E))$.*

Proof: Let E be a G-module of dimension r. It suffices to prove that

$$\left(\sum_{n=0}^{\infty}[S^n(E)]t^n\right)\left(\sum_{n=0}^{r+1}(-1)^n[\Lambda^n(E)]t^n\right) = 1.$$

If $e_0, \ldots, e_r$ is a basis of E, then $S(E) = \mathbf{C}[e_0, \ldots, e_r]$. Since $H^p(K.(1,...,1; S(M))) = 0 \ \forall p > 0$ [La XVI 10.4] the Koszul complex $K.(1, \ldots, 1; S(E))$ is exact. Each d_p maps $\wedge^p(E) \otimes S^q(E)$ into $\wedge^{p-1}(E) \otimes S^{q+1}(E)$, and so the graded components of $K.(1, \ldots, 1; S(E))$ decompose into direct sums. The exactness of $K.(1, \ldots, 1; S(E))$ then yields the exact sequences:

$$0 \longrightarrow \wedge^{r+1}(E) \otimes S^{n-r-1}(E) \longrightarrow \cdots \longrightarrow \wedge^1(E) \otimes S^{n-1}(E) \longrightarrow S^n(E) \longrightarrow 0,$$

for all integers $n \geq 1$. Since the exact sequence

$$0 \longrightarrow Im(d_2) \longrightarrow \wedge^1(E) \otimes S^{n-1}(E) \longrightarrow S^n(E) \longrightarrow 0$$

splits,

$$[\wedge^1(E) \otimes S^{n-1}(E)] = [Im(d_2)] + [S^n(E)] \in R(G),$$

by definition. Moreover each short exact sequence

$$0 \longrightarrow Im(d_{k+1}) \longrightarrow \wedge^k(E) \otimes S^{n-k}(E) \longrightarrow Coker(d_{k+1}) \longrightarrow 0,$$

splits, and since

$$Coker(d_{k+1}) = \wedge^k(E) \otimes S^{n-k}(E)/Im(d_{k+1}) \approx \wedge^k(E) \otimes S^{n-k}(E)/ker(d_k) \approx Im(d_k)$$

it follows that

$$0 \longrightarrow Im(d_{k+1}) \longrightarrow \wedge^k(E) \otimes S^{n-k}(E) \longrightarrow Im(d_k) \longrightarrow 0.$$

is split exact. Thus, by induction $[Im(d_2)]$ can be expressed as an alternating sum of products of exterior and symmetric powers. ∎

Suppose that V is a representation of G. Given any subgroup H of G, let $Res^G_H(V)$ denote the restriction of V to H. Observe that the operation of restriction defines a ring homomorphism

$$Res^G_H : R(G) \longrightarrow R(H).$$

Now define an adjoint map

$$Ind^G_H : R(H) \longrightarrow R(G).$$

It is the map defined by $V \longrightarrow \mathbf{C}[G] \otimes_{\mathbf{C}[H]} V$. Res^G_H and Ind^G_H are adoint in the following sense:

THEOREM 2.8 (FROBENIUS RECIPROCITY). *Suppose that W is a H-module and E is a G-module. Then*

$$\langle W, Res_H^G E\rangle_H = \langle Ind_H^G W, E\rangle_G$$

PROOF: It suffices to prove that

$$dim\,(Hom^H\,(W, Res_H^G E)) = dim\,(Hom^H\,(Ind_H^G W, E))\,.$$

This statement follows from the fact that

$$Hom^H\,(W, E) \approx Hom^G\,(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, E)\,. \quad \blacksquare$$

THEOREM 2.9 (FROBENIUS RECIPROCITY 'STRONG FORM'). $Ind_H^G(W) \otimes E \approx Ind_H^G(W \otimes Res_H^G(E))$.

COROLLARY 2.10. *Let $K \subset H \subset G$ be finite groups and let V be a K-module. Then*

$$Ind_H^G\,(Ind_K^H V) \approx Ind_K^G V.$$

PROOF: This result follows from the fact that Ind and Res are adjoint, and because

$$Res_K^H(Res_H^G V) = Res_K^G V. \quad \blacksquare$$

LEMMA 2.11. *Let V be a $\mathbb{C}[G]$-module which is a direct sum $V = \bigoplus_{i\in I} W_i$ of vector subspaces of V permuted transitively by the action of G. Let $i_0 \in I$ and set $W = W_{i_0}$. Let H be the isotropy group of W, that is, the set of elements s in G such that $sW = W$. Then W is stable under the action of the subgroup H and furthermore,*

$$V \approx \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

PROOF: Let $\{w_i\}$ be a basis of W_{i_0}, and let $\{g_j\}$ denote a set of coset representatives of G/H. Then the elements $g_j \otimes w_i$ form a $\mathbb{C}$ basis of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W_{i_0}$. Define a map

$$\imath : \mathbb{C}[G] \times W_{i_0} \longrightarrow V$$

by $\imath(g_j, w_l) = g_j.w_l$. This induces a map

$$\hat{\imath} : \mathbf{C}[G] \otimes_{\mathbf{C}[H]} W_{i_0} \longrightarrow V.$$

Let $\{v_i\}$ be a basis of V and let $v = \sum c_i v_i$ be an arbitrary element of V. Because G acts as a permutation group on the subspaces W_i, there exists $g_j \in G$, $w_l \in W_{i_0}$ such that $g_j.w_l = v_i$. Therefore, $\hat{\imath}(\sum c_i g_j \otimes w_l) = \sum c_i v_i = v$, and $\hat{\imath}$ is an epimorphism. Furthermore since the G-action on V depends only on the coset of H,

$$dim(V) = [G : H].dim(W_{i_0}) = dim(\mathbf{C}[G] \otimes_{\mathbf{C}[H]} W_{i_0}).$$

Thus, $\hat{\imath}$ is an isomorphism which is clearly G-equivariant. ∎

SECTION 3
REFLECTION GROUPS

ROOT SYSTEMS

Let E be a Euclidean space. An invertible linear transformation which fixes some hyperplane pointwise and which sends any vector orthogonal to that hyperplane to its negative is called a reflection. Any non-zero vector $\alpha \in E$ determines a reflection σ_α defined by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

This section studies subgroups of $GL(E)$ which are generated by such reflections. The results contained in this section can be found in [Hu] and [Ka].

DEFINITION 3.1. *A root system (Φ, E) is a finite subset, Φ, of $E\backslash 0$ that spans E as a vector space over $\mathbf{R}$, and that satisfies*

(1) *If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.*

(2) *$\sigma_\alpha(\Phi) = \Phi$ for any reflection σ_α, where $\alpha \in \Phi$.*

If the root system Φ has the additional property that given $\alpha, \beta \in \Phi$

$$\langle \alpha, \beta \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbf{Z},$$

then the root system is called crystallographic.

DEFINITION 3.2. *Suppose that Φ is a root system in E. Let $W(\Phi)$ denote the subgroup of $GL(E)$ which is generated by the reflections σ_α $\forall\, \alpha \in \Phi$. $W(\Phi)$ will be called the reflection group associated to the root system Φ. If in addition the root system Φ is crystallographic then $W = W(\Phi)$ will be called the Weyl group associated to Φ.*

Since different elements of W define different permutations of the set Φ, W may be identified with a subgroup of the symmetric group on Φ. Thus W is a finite group.

Observe that the inner product on E is invariant under W, that is, $(\omega(x), \omega(y)) = (x, y)$, for $\omega \in W$. This follows because for any reflection σ_α, $(\sigma_\alpha(x), \sigma_\alpha(y)) = (x, y)$. The proof of this statement is clear.

LEMMA 3.3. *Let Φ be a crystallographic root system in E with associated reflection group W. If $\sigma \in W$ is such that $\sigma\Phi = \Phi$ then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$.*

PROOF: Let $\alpha, \beta \in \Phi$. Then

$$\sigma\sigma_\alpha\sigma^{-1}(\sigma(\beta)) = \sigma\sigma_\alpha(\beta) \in \Phi,$$

since $\sigma_\alpha(\beta) \in \Phi$. But then

$$\sigma\sigma_\alpha(\beta) = \sigma(\beta - \langle\beta, \alpha\rangle\,\alpha) = \sigma(\beta) - \langle\beta, \alpha\rangle\,\sigma(\alpha).$$

Therefore $\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ invariant while fixing pointwise the hyperplane $\sigma(H_\alpha)$ and sending $\sigma(\alpha)$ to $-\sigma(\alpha)$. Hence $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$. ∎

DEFINITION 3.4. *Let Φ, Φ' be root systems of the Euclidean spaces E, E'. The root systems (Φ, E), and (Φ', E') are said to be isomorphic if there exists a vector space isomorphism*

$$\psi : E \longrightarrow E'$$

such that $\psi : \Phi \to \Phi'$ is a bijection and $\langle\psi(\beta), \psi(\alpha)\rangle = \langle\beta, \alpha\rangle$ for each pair of roots $\beta, \alpha \in \Phi$.

Observe that $\sigma_{\psi(\alpha)}(\psi(\beta)) = \psi(\sigma_\alpha(\beta))$. Thus an isomorphism of root systems induces an isomorphism

$$\sigma \longmapsto \psi \circ \sigma \circ \psi^{-1}$$

of reflection groups.

DEFINITION 3.5. *Let Δ be a subset of Φ which is a basis of E such that $\forall\, \beta \in \Phi$, $\beta = \sum k_\alpha \alpha$, $\alpha \in \Delta$, where the coefficients k_α are either all non-negative or non-positive. The subset Δ is called a base, and the elements of Δ are called simple roots. Define the height of a root with respect to Δ by $h(\beta) = \sum k_\alpha$. When*

$k_\alpha \geq 0$ (respectively $k_\alpha \leq 0$) β is positive (respectively negative) and is written $\beta \succ 0$ (respectively $\beta \prec 0$). The collection of positive roots is denoted Φ^+, and the negative roots by Φ^-.

Definition 3.6. *For all $\gamma \in E$, let*

$$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\},$$

$$\Phi^-(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) < 0\}.$$

The ***reflecting hyperplane*** of the reflection σ_α is $H_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$. Call $\gamma \in E$ ***regular*** if $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ and singular otherwise. When γ is regular $\Phi = \Phi^+(\gamma) \cup \Phi^-(\gamma)$.

Lemma 3.7. *Let γ be regular. If $\Delta \subset \Phi^+(\gamma)$ satisfies,*

(1) *every element in $\Phi^+(\gamma)$ is a linear combination of elements from Δ with non negative coefficients.*

(2) *no subset of Δ satisfies (1).*

then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ in Δ.

Proof: Suppose to the contrary that $(\alpha, \beta) > 0$.

Case(i) $\sigma_\alpha(\beta) \in \Phi^+(\gamma)$. Then

$$\beta - \langle \alpha, \beta \rangle \alpha = \sigma_\alpha(\beta) = \sum d_i \delta_i,$$

where $d_i \geq 0$, $\delta_i \in \Delta$. Write $\beta = \langle \alpha, \beta \rangle \alpha + \sum d_i \delta_i$. Observe that $\beta = \delta_{i_0}$, for some i_0. If the coefficient of δ_{i_0} is less than 1 then β is a positive linear combination of the remaining elements of Δ. This contradicts (2). On the other hand, if the coefficient of δ_{i_0} is greater than or equal to 1, then 0 is a positive linear combination of elements of Δ. Evaluate the inner product of this linear combination and γ. Since $\Delta \subset \Phi^+(\gamma)$ this inner product must be positive. It follows that this positive linear combination cannot be zero. This is a contradiction.

Case(ii) $-\sigma_\alpha(\beta) \in \Phi^+(\gamma)$. Observe that

$$-\beta + \langle \alpha, \beta \rangle \alpha = -\sigma_\alpha = \sum d_i \delta_i.$$

Write $\langle\alpha,\beta\rangle\alpha = \beta + \sum d_i\delta_i$. There is some i_0 such that $\alpha = \delta_{i_0}$. As in Case(i) there are two subcases: when the coefficient of δ_{i_0} is less than $\langle\alpha,\beta\rangle$ and when the coefficient of δ_{i_0} is greater than or equal to $\langle\alpha,\beta\rangle$. The proof is similar to the proof of Case(i). ∎

THEOREM 3.8. *Any root system Φ has a base.*

PROOF: Call $\alpha \in \Phi^+$ *decomposable* if $\alpha = \beta_1 + \beta_2$, $\beta_i \in \Phi^+(\gamma)$, and *indecomposable* otherwise. It suffices to prove that for a regular element $\gamma \in E$, the set $\Delta(\gamma)$, of all indecomposable roots in $\Phi^+(\gamma)$, is a base of Φ, and every base is obtainable in this manner.

Each root in $\Phi^+(\gamma)$ is a non-negative $\mathbb{Z}$-linear combination of elements from $\Delta(\gamma)$. Suppose to the contrary that $\alpha \in \Phi^+(\gamma)$ cannot be expressed as such a linear combination. Choose α such that (γ,α) is minimal. Since $\alpha \notin \Delta(\gamma)$, $\alpha = \beta_1 + \beta_2$, $\beta_1, \beta_2 \in \Phi^+(\gamma)$. Hence

$$(\gamma,\alpha) = (\gamma,\beta_1+\beta_2) = (\gamma,\beta_1) + (\gamma,\beta_2).$$

Since $(\gamma,\beta_1), (\gamma,\beta_2) > 0$, β_1 and β_2 are non-negative $\mathbb{Z}$-linear combinations in $\Delta(\gamma)$, otherwise this contradicts the minimality of (γ,α). But $\alpha = \beta_1 + \beta_2$. So α is also a non-negative $\mathbb{Z}$-linear combination. This is a contradiction. Thus, $\Delta(\gamma)$ satisfies property (1) of Lemma 3.7. That $\Delta(\gamma)$ satisfies property (2) of Lemma 3.7 is clear.

Suppose that $\sum r_\alpha \alpha = 0$, $r_\alpha \in \mathbb{R}$, $\alpha \in \Delta(\gamma)$ and assume there exists α such that $r_\alpha \neq 0$. Separate the indices α for which $r_\alpha > 0$ from those which $r_\alpha < 0$. Denote the positive coefficients by s_α and the negative coefficients by t_β. Then

$$v = \sum_\alpha s_\alpha \alpha = \sum_\beta -t_\beta \beta.$$

Since $(v,v) = \sum s_\alpha(-t_\beta)(\alpha,\beta) \leq 0$, by Lemma 3.7, it follows that $v = 0$. This implies that $(\gamma,v) = \sum s_\alpha(\gamma,\alpha)$ which forces $s_\alpha = 0\ \forall\, \alpha$. Similarly, $t_\beta = 0$. It follows that $\Delta(\gamma)$ is a base of Φ.

Now suppose that Δ is any base of Φ. Choose $\gamma \in E$ such that $(\gamma,\alpha) > 0$ for all $\alpha \in \Delta$. By definition γ is regular and $\Phi^+ \subset \Phi^+(\gamma)$, $\Phi^- \subset -\Phi(\gamma)$. Since

$\Phi^+ = \Phi^+(\gamma)$, Δ consists of indecomposable elements. This implies that $\Delta \subset \Delta(\gamma)$. But $|\Delta| = |\Delta(\gamma)|$, so $\Delta = \Delta(\gamma)$. ∎

DEFINITION 3.9. *The connected components of $E \setminus \bigcup_\alpha H_\alpha$ are called the Weyl chambers of E.*

Each regular element $\gamma \in E$ belongs to precisely one Weyl chamber, which is denoted $\mathcal{C}(\gamma)$ or $\mathcal{C}(\Delta)$. If $\mathcal{C}(\gamma) = \mathcal{C}(\gamma')$, then $\Phi^+(\gamma) = \Phi^+(\gamma')$ and $\Delta(\gamma) = \Delta(\gamma')$. This implies that the Weyl chambers are in one-to-one correspondence with bases. If $\Delta = \Delta(\gamma)$, call $\mathcal{C}(\gamma)$ the fundamental Weyl chamber relative to Δ. It is clear that $\mathcal{C}(\gamma)$ is the open convex set

$$\{\delta \in E \mid (\delta, \alpha) > 0 \ \forall\, \alpha \in \Delta\}.$$

Observe that the action of the reflection group on the Weyl chambers is to permute them, since $\sigma(\mathcal{C}(\gamma)) = \mathcal{C}(\sigma\gamma)$, for $\sigma \in \mathcal{W}$, γ a regular element in E.

LEMMA 3.10. *Let α be a simple root. Then σ_α permutes all positive roots other than α, and sends α to $-\alpha$.*

PROOF: Let $\beta \in \Phi^+ \setminus \{\alpha\}$. Then

$$\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma \qquad k_\gamma \in \mathbf{Z}^+$$

Since β is not proportional to α, $k_\gamma \neq 0$ for some $\gamma \neq \alpha$. This implies that the coefficient of γ in $\sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha$ is k_γ. This means that $\sigma_\alpha(\beta)$ has at least one positive coefficient. This forces it to be positive. Finally observe that $\sigma_\alpha(\beta) \neq \alpha$, since α is the image of $-\alpha$. ∎

COROLLARY 3.11. *Let $\delta = \frac{1}{2} \sum_{\beta \succ 0} \beta$. Then $\sigma_\alpha(\delta) = \delta - \alpha$, for all $\alpha \in \Delta$.*

PROOF: This result follows immediately from Lemma 3.10. ∎

THEOREM 3.12. *Let Δ be a base of Φ. Then*

(1) *The reflection group $\mathcal{W}$ acts transitively on the Weyl chambers.*

(2) *Given any other base Δ' of Φ, $\exists\, \sigma \in \mathcal{W}$ such that $\sigma(\Delta) = \Delta'$.*

(3) *$\mathcal{W}$ is generated by the σ_α such that $\alpha \in \Delta$.*

PROOF: Let W' be the subgroup of W generated by all σ_α such that $\alpha \in \Delta$.

(1) Suppose that $\gamma \in E$ is regular. Let $\delta = \frac{1}{2}\sum_{\beta \succ 0} \beta$, and choose $\sigma \in W'$ such that $(\sigma(\gamma), \delta)$ is maximal. If α is simple then $\sigma_\alpha \sigma \in W'$. Thus the choice of σ implies that

$$(\sigma(\gamma),\delta) \geq (\sigma_\alpha\sigma(\gamma),\delta) = (\sigma(\gamma),\sigma_\alpha(\delta)) = (\sigma(\gamma),\delta-\alpha) = (\sigma(\gamma),\delta) - (\sigma(\gamma),\alpha).$$

where the second last equality follows from Corollary 3.11. This gives

$$(\sigma(\gamma),\alpha) \geq 0 \qquad \forall\, \alpha \in \Delta.$$

Because γ is a regular element, $(\sigma(\gamma),\alpha) \neq 0\ \forall\ \alpha$, otherwise γ would be orthogonal to $\sigma^{-1}\alpha$. This implies that $\sigma(\gamma)$ lies in the fundamental Weyl chamber $C(\Delta)$ and σ sends $C(\gamma)$ to $C(\Delta)$.

(2) Since W' permutes the Weyl chambers transitively it also permutes the bases of Φ transitively.

(3) To prove that $W' = W$, it suffices to prove that $\sigma_\alpha \in W'$, $\forall\ \alpha \in \Phi$. Choose σ such that for any $\beta \in \Delta$ $\sigma(\alpha) = \beta$. Then $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$, so $\sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in W'$. ∎

DEFINITION 3.13. *A root system Φ is called irreducible if it cannot be partitioned into the union of two proper subsets which are mutually orthogonal.*

A base Δ of Φ is irreducible if it satisfies this same condition. Observe that Φ is irreducible if and only if Δ is irreducible.

LEMMA 3.14. *If Φ is irreducible, then W acts irreducibly on E. In particular the W-orbit of a root α spans E.*

PROOF: The span of the W-orbit of any root is a non zero W- invariant subspace of E, so the second statement follows from the first. Let E' be a non-zero subspace of E, invariant under the W. The orthogonal complement E'' of E' is also W-invariant. For $\alpha \in \Phi$, either $\alpha \in E'$ or else $E' \subset H_\alpha$, since $\sigma_\alpha(E') = E'$. Thus if $\alpha \notin E'$ then $\alpha \in E''$. This partitions Φ into two orthogonal subsets. But this forces one of these two subsets to be the empty set. Thus $E' = E$ since Φ spans E. ∎

DEFINITION 3.15. *A Coxeter group is any group which has a presentation of the form*

$$W = \langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where

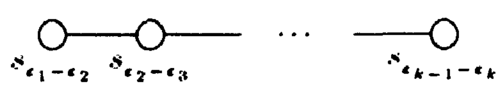
(1) $m_{ii} = 1$.
(2) $m_{ij} \in \{2, 3, \dots\} \cup \{\infty\}$ *if* $i \neq j$.

The presentation (W, S) is referred to as a Coxeter system.

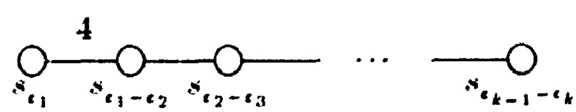
DEFINITION 3.16. *A Coxeter graph is a graph with each edge labelled by an integer* ≥ 3.

To every Coxeter system (W, S), a Coxeter graph can be assigned. Let the vertices be S. There is an edge between s_i and s_j if $m_{ij} \geq 3$. The edge is then labelled by the integer m_{ij}. Such a representation of (W, S) is said to be irreducible as a Coxeter system if and only if the graph is connected. The finite Coxeter groups can be classified by the following Coxeter diagrams. We introduce the following convention: Unlabelled edges will be considered to be labelled by the integer 3.

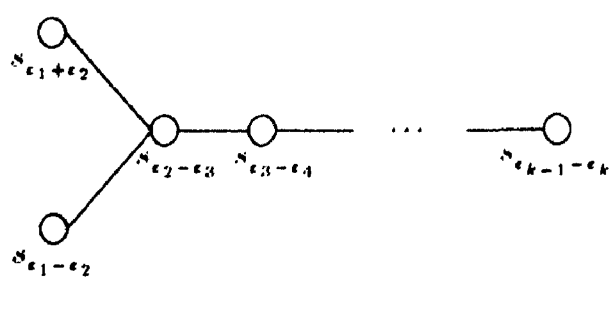
(A_k)



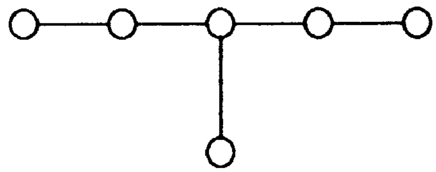
$(C_k = B_k)$



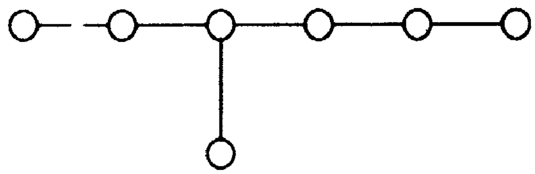
(D_k)



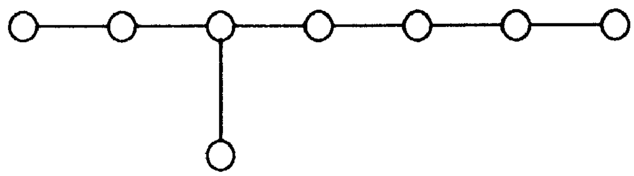
(E_6)



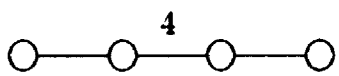
(E_7)



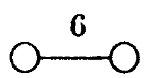
(E_8)



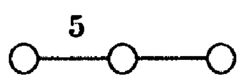
(F_4)



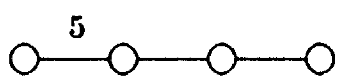
(G_2)



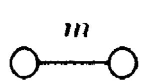
(H_3)



(H_4)



$(I_2(m), m = 5, m \geq 7)$



Each Coxeter graph, with the exception of $I_2(m)$, H_3, H_4 represents a Weyl group. Finally, we describe the root sytems which correspond to the classical Weyl groups.

A_l. Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l+1}\}$ be the standard basis of $\mathbb{R}^{l+1}$. Let E be the following subspace of $\mathbb{R}^{l+1}$

$$E = \left\{ \sum r_i \varepsilon_i \mid \sum r_i = 0 \right\}.$$

Then $\mathbf{R}^{l+1} = E \oplus W$ where W is the orthogonal compliment of E and is generated by $\sum_{i=1}^{l+1} \varepsilon_i$. Let $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, and let $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq l\}$. It is clear that Φ is a root system. The vectors of Δ are linearly independent and since $\varepsilon_i - \varepsilon_j = \sum_{p=i}^{j} \varepsilon_p - \varepsilon_{p+1}$ for $i < j$, Δ is the ***standard base*** of Φ. Observe that the reflection with respect to α_i sends α_i to $-\alpha_i$ and leaves the hyperplane orthogonal to α_i pointwise invariant. Thus there is a bijective correspondence between the reflections $\sigma_{\alpha_i} \in \mathcal{W}(A_l)$ and the transpositions $(i, i+1)$ of the symmetric group S_{l+1}. Therefore there is an isomorphism between $\mathcal{W}(A_l)$ and S_{l+1}.

B_l. Let $E = \mathbf{R}^l$ with standard orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_l\}$. Let $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\} \cup \{\pm\varepsilon_i\}$. It is clear that Φ is a root system. The l vectors $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l$ are linearly independent. Observe that $\varepsilon_i = \sum_{k=i}^{l} (\varepsilon_k - \varepsilon_{k+1}) + \varepsilon_l$ and $\varepsilon_i - \varepsilon_j = \sum_{p=i}^{j-1} \varepsilon_p - \varepsilon_{p+1}$. This collection of vectors form the ***standard base*** of Φ. The Weyl group $\mathcal{W}(B_l)$ acts as the group of all permutations and sign changes of the set $\{\varepsilon_1, \ldots, \varepsilon_l\}$. Thus $\mathcal{W}(B_l) \approxeq S_l \wr \mathbf{Z}/2$.

C_l. C_l may be viewed as the root system dual to B_l. The dual root system of a root system Φ is defined as the set $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi, \check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}\}$. The Weyl group of $\mathcal{W}(\check{\Phi})$ is canonically isomorphic to $\mathcal{W}(\Phi)$. Therefore $\mathcal{W}(C_l)$ is isomorphic to $\mathcal{W}(B_l)$. Alternatively C_l can be described as the root system $\Phi = \{\pm 2\varepsilon_i\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}$ having base $\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_l\}$.

D_l. Let $E = \mathbf{R}^l$, $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}$, $\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_{l-1} + \varepsilon_l\}$. The Weyl group $\mathcal{W}(D_l)$ is the group of permutations and even numbers of sign changes of the set $\{\varepsilon_1, \ldots, \varepsilon_l\}$. Therefore, $\mathcal{W}(D_l)$ is isomorphic to the semi direct product of S_l and $\mathbf{Z}^{l-1}$.

For future reference we make the following definition.

Definition 3.17. *A Coxeter element of a Weyl group is any product of the reflections corresponding to any complete set of simple roots.*

Coxeter has shown that all such elements are conjugate, irrespective of the base chosen or the order in which the reflections occur.

The ***standard Coxeter element*** of the Weyl group of type A_l is the product of the reflections corresponding to the standard base of A_l defined above. That is, the standard Coxeter element of type A_l is defined to be the product $s_{\epsilon_1-\epsilon_2}\cdots s_{\epsilon_{l-1}-\epsilon_l}$. Similarly, the ***standard Coxeter element*** of the Weyl group of type B_l is the product of the reflections corresponding to the standard base of B_l. That is, the standard Coxeter element of type B_l is $s_{\epsilon_1} s_{\epsilon_1-\epsilon_2}\cdots s_{\epsilon_{l-1}-\epsilon_l}$.

Remark 3.18: In what follows, by the standard Coxeter element of a Weyl group with root system

$$\left(\coprod_i B_i\right)\coprod\left(\coprod_j A_j\right),$$

we mean the product of the standard Coxeter elements of the irreducible constituents of the root system.

Section 4

PSH-ALGEBRAS

The Structure of PSH Algebras

Definition 4.1. *Let K be a commutative ring with unit. A Hopf algebra H over K is a graded K-module, together with morphisms of graded K-modules*

$$m : H \otimes H \longrightarrow H \qquad e : K \longrightarrow H$$
$$m^* : H \longrightarrow H \otimes H \qquad e^* : H \longrightarrow K.$$

such that

(1) *H is an algebra under m with unit e.*

(2) *H is a coalgebra under m^* with augmentation (counit) e^*.*

(3) *$m : H \otimes H \longrightarrow H$ is a morphism of coalgebras while $m^* : H \longrightarrow H \otimes H$ is a morphism of algebras.*

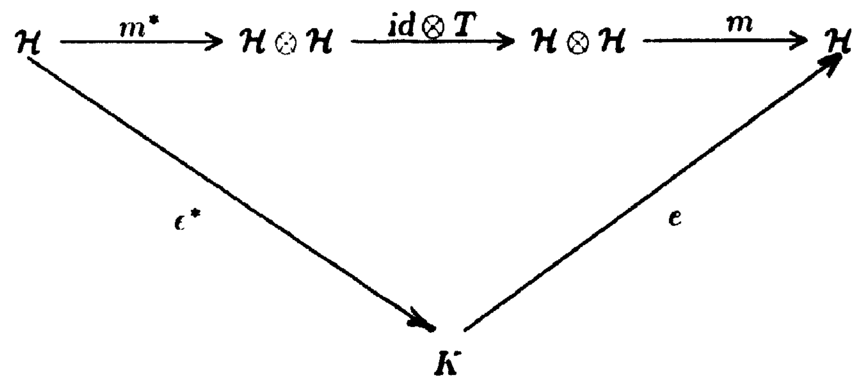
(4) *H is both associative and coassociative.*

Let $\tau : H \otimes H \longrightarrow H \otimes H$ denote the twisting map, defined by $\tau(x \otimes y) = y \otimes x$. H is said to be commutative if $m = m \circ \tau$. Furthermore H is called connected if $H_0 = K$ and $H_\alpha = 0$, for all $\alpha < 0$.

Let J be an algebra ideal of H. If $m^*(J) \subset J \otimes H + H \otimes J$ then J is a Hopf ideal. Consider the Hopf ideal $I = \oplus_{n>0} H_n$. Then for all $x \in I$, $m^*(x) = x \otimes 1 + 1 \otimes x + m_+^*(x)$, where $m_+^*(x) \in I \otimes I$. An element $x \in I$ is called primitive if $m^*(x) = x \otimes 1 + 1 \otimes x$, that is, if $m_+^*(x) = 0$. Let $\mathcal{P}$ denote the subgroup of primitive elements in H and set $I^2 = m(I \otimes I)$.

Definition 4.2. *Let $\mathcal{H}$ be a commutative, connected Hopf algebra over K.*

Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be the morphism of graded K-modules which satisfies the following commutative diagram.



T is an involutive Hopf algebra automorphism called the conjugation of $\mathcal{H}$.

From now on we deal with Hopf algebras over $\mathbf{Z}$.

For any free abelian group F with basis Ω, define a $\mathbf{Z}$-valued bilinear form $\langle -, - \rangle$ on F by

$$\langle \omega, \omega' \rangle = \delta_{\omega,\omega'}, \qquad \forall\, \omega, \omega' \in \Omega,$$

where $\delta_{\omega,\omega'}$ denotes the Kronecker delta. The form $\langle -, - \rangle$ is symmetric, non degenerate and positive. Therefore it is an inner product on F. The elements of Ω will be called the irreducible elements of F. Let

$$F^+ = \left\{ \sum m_\omega \omega \mid m_\omega \geq 0 \right\}.$$

The elements of F^+ are called positive. A morphism between two free abelian groups is called positive if it maps positive elements to positive elements. If $\pi = \sum m_\omega \omega \in F^+$ then the elements $\omega \in \Omega$ such that $m_\omega > 0$ are called the irreducible constituents of π.

DEFINITION 4.3. *Consider the pair (H, Ω) where H is a Hopf algebra over $\mathbf{Z}$, and which is a free $\mathbf{Z}$-module with a distinguished basis Ω. If the morphisms m, m^*, ϵ, ϵ^* are positive, with respect to Ω, then H is a positive Hopf algebra. Such a Hopf algebra in which m, m^* and ϵ, ϵ^* are adjoints of each other under the inner products on H and $H \otimes H$, defined previously, is called self-adjoint. (When there is no danger of confusion about the basis Ω, we simply say that H is a positive, self-adjoint Hopf algebra.)*

DEFINITION 4.4. *A PSH-algebra is a commutative, connected, positive, self-adjoint Hopf algebra.*

The Borel-Hopf Theorem shows that a Hopf algebra over a field of characteristic zero decomposes (as an algebra) into a product $\otimes H_\alpha$ where each Hopf algebra H_α has only one primitive element. In this section we shall describe the analogue of this theorem for PSH-algebras.

Let $\{H_\alpha \mid \alpha \in \Upsilon\}$ be a family of PSH-algebras. Define the tensor product of this family to be the direct limit

$$H = \varinjlim_{S} \bigotimes_{\alpha \in S} H_\alpha$$

where S ranges over all finite subsets of Υ, and when $S \subset S'$ the maps

$$\iota_{S,S'} : \bigotimes_{\alpha \in S} H_\alpha \longrightarrow \bigotimes_{\alpha' \in S'} H_{\alpha'}$$

of the direct system are given by $h_{\alpha_1} \otimes \cdots \otimes h_{\alpha_s} \mapsto h_{\alpha_1} \otimes \cdots \otimes h_{\alpha_s} \otimes 1$. Observe that H is a PSH-algebra whose irreducible elements are

$$\Omega(H) = \coprod_S \left\{ \bigotimes_{\alpha \in S} b_\alpha \mid b_\alpha \in \Omega(H_\alpha) \right\}.$$

Let $\mathcal{C} = \mathcal{P} \cap \Omega$ be the set of irreducible primitive elements in H. Let $S(\mathcal{C}, \mathbf{Z}^+)$ denote the semi-group of functions from $\mathcal{C}$ to $\mathbf{Z}^+$ with finite support. For all $\varphi \in S(\mathcal{C}, \mathbf{Z}^+)$ define

$$\pi_\varphi = \prod_{\rho \in \mathcal{C}} \rho^{\varphi(\rho)}.$$

Since the multiplication map m is positive, $\pi_\varphi \in H^+$. Let

$$\Omega(\varphi) = \{\omega \in \Omega \mid \omega \text{ is an irreducible constituent of } \pi_\varphi\}$$

and let

$$H(\varphi) = \bigoplus_{\omega \in \Omega(\varphi)} \mathbf{Z}\omega.$$

LEMMA 4.5 (BERSTEIN-ZELEVINSKI). *Given $\varphi, \varphi' \in S(\mathcal{C}, \mathbf{Z}^+)$ with disjoint supports, then the multiplication morphism*

$$m : H(\varphi) \otimes H(\varphi') \longrightarrow H(\varphi + \varphi')$$

is an isomorphism of abelian groups.

PROOF: To prove that m is injective it suffices to show that

$$\langle \omega_1\omega_1', \omega_2\omega_2'\rangle = \delta_{\omega_1,\omega_2}\delta_{\omega_1',\omega_2'}.$$

Observe that

$$\begin{aligned}\langle \omega_1\omega_1', \omega_2\omega_2'\rangle &= \langle \omega_1\omega_1', m(\omega_2\otimes\omega_2')\rangle \\ &= \langle m^*(\omega_1\omega_1'), \omega_2\otimes\omega_2'\rangle \\ &= \langle m^*(\omega_1)m^*(\omega_1'), \omega_2\otimes\omega_2'\rangle \\ &= \langle \omega_1\otimes\omega_1', \omega_2\otimes\omega_2'\rangle \\ &= \langle \omega_1, \omega_2\rangle\langle \omega_1', \omega_2'\rangle \\ &= \delta_{\omega_1,\omega_2}\delta_{\omega_1',\omega_2'}.\end{aligned}$$

Any element of $\Omega(\varphi+\varphi')$ must have the form $\omega\omega'$, $\omega\in\Omega(\varphi)$, $\omega'\in\Omega(\varphi')$. For, let

$$\pi_\varphi = \sum \omega_i, \qquad \pi_{\varphi'} = \sum \omega_j'$$

be the decomposition of π_φ and $\pi_{\varphi'}$ into irreducible elements. Then

$$\pi_{\varphi+\varphi'} = \sum \omega_i\omega_j',$$

and the elements $\omega_i\omega_j'$ exhaust all the irreducible constituents of $\pi_{\varphi+\varphi'}$.

Since $|\Omega(\varphi)\otimes\Omega(\varphi')| = |\Omega(\varphi+\varphi')|$, the result follows. ∎

THEOREM 4.6 (BERSTEIN-ZELEVINSKI). *Let H be a PSH-algebra and let C denote its set of irreducible primitive elements. For $\rho\in C$ let*

$$\Omega(\rho) = \{\omega\in\Omega(H) \mid \exists\, n \text{ with } \langle\omega,\rho^n\rangle \neq 0\}$$

$$H(\rho) = \bigoplus_{\omega\in\Omega(\rho)} \mathbf{Z}\omega.$$

Then $H(\rho)$ is a PSH-subalgebra of H whose unique irreducible primitive element is ρ, and whose set of irreducible elements is $\Omega(\rho)$. Furthermore,

$$H \approx \varinjlim_{\mathcal{F}} \bigotimes_{\rho\in\mathcal{F}} H(\rho),$$

where the direct limit is taken over the finite subsets $\mathcal{F}$ of C.

For a proof of this theorem see [Ze] sections 2.2-2.7.

Universal PSH-Algebras With Unique Irreducible Primitive Element

Given any $x \in H$ let x^* denote the adjoint operator to multiplication, that is,

$$\langle x^*(y), z\rangle = \langle y, xz\rangle \qquad \forall\, y, z \in H.$$

Lemma 4.7 (Berstein-Zelevinski). *The operator x^* satisfies the following properties:*

(1) *Let $x \in H_k$. Then $\forall\, n > 0$, $x^*(H_n) \subset H_{n-k}$. In particular, $x^*(H_n) = 0$ for $n < k$.*

(2) *Let $I : H \otimes \mathbf{Z} \longrightarrow \mathbf{Z}$ denote the canonical isomorphism. Then $x^* = I \circ \mathrm{id} \otimes \langle x, \ \rangle \circ m^*$.*

(3) *For any $x, y \in H$, $(xy)^* = y^* \circ x^*$. In particular, since H is commutative all operators having the form x^* commute with each other.*

(4) *If $x \in H^+$ then x^* is positive.*

(5) *If $x, y, z \in H$, and $m^*(x) = \sum a_i \otimes b_j$, then*

$$x^*(yz) = \sum a_i^*(y) b_j^*(z).$$

(6) *If $\rho \in H$ is primitive then, ρ^* is a derivation.*

(7) *If $\rho \in H_n$ is primitive, $0 < k < n$, and $x \in H_k$, then $x^*(\rho) = 0$.*

Proof: (1)-(4) Each of these statements follows immediately from the appropriate definition.

(5) It suffices to show that $\langle x^*(yz), u\rangle = \left\langle \sum_i a^*{}_i(y) b^*{}_i(z), u \right\rangle$.

$$\begin{aligned}\langle x^*(yz), u\rangle &= \langle yz, xu\rangle \\ &= \langle y \otimes z, m^*(xu)\rangle \\ &= \langle y \otimes z, m^*(x) m^*(u)\rangle\end{aligned}$$

$$= \sum_i \langle y \otimes z, (a_i \otimes b_i) m^*(u) \rangle$$

$$= \sum_i \langle a^*{}_i(y) \otimes b^*{}_i(z), m^*(u) \rangle$$

$$= \sum_i \langle m\left[a^*{}_i(y) \otimes b^*{}_i(z)\right], u \rangle$$

$$= \left\langle \sum_i a^*{}_i(y) b^*{}_i(z), u \right\rangle.$$

(6) This statement is a special case of (5).

(7) This follows from the definition of x^*. ∎

LEMMA 4.8. *Let $\epsilon_1, \ldots, \epsilon_p$ be a collection of elements in a PSH-algebra H. Then $\epsilon_1, \ldots, \epsilon_p$ is a basis of a subgroup of H only if*

$$det(\langle \epsilon_i, \epsilon_j \rangle) = 1.$$

PROOF: Suppose that $\sum a_i \epsilon_i = 0$. Then

$$\left\langle \sum a_i \epsilon_i, \epsilon_j \right\rangle = \sum a_i \left\langle \epsilon_i, \epsilon_j \right\rangle = \left\langle 0, \epsilon_j \right\rangle = 0 \qquad \forall\, j = 1, \ldots, p.$$

Because $det(\langle \epsilon_i, \epsilon_j \rangle) = 1$, the matrix $(\langle \epsilon_i, \epsilon_j \rangle)$ is invertible and therefore this system of equations has a unique solution, and by Cramer's rule $a_i = 0\ \forall\, i$. Thus, $\epsilon_1, \ldots, \epsilon_p$ are linearly independent. ∎

THEOREM 4.9 (BERSTEIN-ZELEVINSKI). *Let H be a PSH-algebra with unique irreducible primitive element ρ of degree 1.. Then*

(1) *The element ρ^2 is the sum of two distinct irreducible elements $x_2, y_2 \in H_2$.*

(2) *For any $n \geq 0$ there exist unique irreducible elements x_n, y_n such that*

$$x_2^*(y_n) = 0, \qquad y_2^*(x_n) = 0.$$

(3) *For $0 \leq k \leq n$,*

$$x_k^*(x_n) = x_{n-k}, \qquad y_k^*(y_n) = y_{n-k}$$

(4) *For $n \geq 1$,*

$$m^*(x_n) = \sum_k x_k \otimes x_{n-k}, \qquad m^*(y_n) = \sum_k y_k \otimes y_{n-k}.$$

(5) *The PSH-algebra H has a non-trivial PSH-algebra automorphism, t, which satisfies the property:*

$$t(x_n) = y_n, \qquad t(y_n) = x_n \qquad \forall\, n \geq 1.$$

(6) *The ring H is isomorphic to the polynomial algebra $\mathbf{Z}[x_1, x_2, \ldots] = \mathbf{Z}[y_1, y_2, \ldots]$. Moreover, when $n \geq 1$ $\{x_n\}$, $\{y_n\}$ satisfy the relations:*

$$\sum_{k=0}^{n} (-1)^k x_k y_{n-k} = 0.$$

(7) *Any PSH-algebra, with unique irreducible primitive element of degree 1, is isomorphic to H as a PSH-algebra.*

Note: This theorem immediately generalizes to the case where ρ has degree d simply by dividing all degrees which occur by d.

PROOF: (1) Observe that $\rho^2 \in H^+$. This follows because the multiplication morphism is positive. Thus $\rho^2 = \sum_\omega m_\omega \omega$, and $m_\omega \geq 0 \;\forall\, \omega$. Since

$$\langle \rho^2, \rho^2 \rangle = \langle \rho^*(\rho^2), \rho \rangle = \langle \rho\rho^*(\rho) + \rho^*(\rho)\rho, \rho \rangle = \langle 2\rho, \rho \rangle = 2,$$

it follows that $\sum_\omega [m_\omega^2] =< \rho^2, \rho^2 >= 2$. But this is possible only if exactly two of the m_ω are non-zero and both are equal to 1. Thus ρ^2 must be the sum of two elements, x_2 and y_2.

(2) We prove by induction on k that $\rho^*(x_k) = x_{k-1}$, $\rho^*(y_k) = y_{k-1}$. Assume that x_n, y_n exist, are unique and satisfy the required properties. First, we show that $\rho.x_{k-1}$ is the sum of two distinct irreducible elements. As in (1), it suffices to show that $\langle \rho.x_{k-1}, \rho.x_{k-1} \rangle = 2$. Observe that

$$\begin{aligned} \langle \rho.x_{k-1}, \rho.x_{k-1} \rangle &= \langle \rho^*(\rho.x_{k-1}), x_{k-1} \rangle \\ &= \langle x_{k-1} + \rho.x_{k-2}, x_{k-1} \rangle \\ &= 1 + \langle \rho.x_{k-2}, x_{k-1} \rangle \\ &= 1 + \langle x_{k-2}, \rho^*(x_{k-1}) \rangle = 1 + \langle x_{k-2}, x_{k-2} \rangle = 2. \end{aligned}$$

Recall that $m^*(y_2) = 1\otimes y_2 + y_2\otimes 1 + m^*_+(y_2)$. Since $\left\langle m^*_+(y_2), \rho\otimes\rho \right\rangle = \langle x_2, \rho^2\rangle = 1$ and $\rho\otimes\rho$ is an irreducible element of $H\otimes H$, it follows that $m^*_+(y_2) = \rho\otimes\rho$. Thus, by applying Lemma 4.7 to y_2,

$$\begin{aligned} y_2^*(\rho.x_{k-1}) &= y_2^*(\rho).x_{k-1} + \rho^*(\rho).\rho^*(x_{k-1}) + \rho.y_2^*(x_{k-1}) \\ &= \rho^*(\rho).\rho^*(x_{k-1}). \end{aligned}$$

The first and third terms of this expression are zero by Lemma 4.7. Whence, by the induction hypothesis, $y_2^*(\rho.x_{k-1}) = x_{k-2}$. Now, because $\rho.x_{k-1}$ is the sum of two distinct irreducible elements and since $y_2^*(\rho.x_{k-1})$ is an irreducible element, it follows from the preceeding paragraph that y_2^* takes one irreducible constituent of $\rho.x_{k-1}$ to 0 while it takes the other to x_{k-2}. Let x_k denote the irreducible constituent of $\rho.x_{k-1}$ which is sent to 0 under y_2^* (i.e., $y_2^*(x_k) = 0$). One can define y_k in a similar manner, and show that $x_2^*(y_k) = 0$.

Finally, to prove that $\rho^*(x_k) = x_{k-1}$, observe that since x_k is an irreducible component of $\rho.x_{k-1}$, $\langle \rho^*(x_k), x_{k-1}\rangle = \langle x_k, \rho.x_{k-1}\rangle = 1$. But this implies that $\rho^*(x_k) = x_{k-1}$, by the uniqueness and irreducibility of x_{k-1}. The uniqueness of x_k follows from this last statement. For if u_k is another element which satisfies the condition $y_2^*(u_k) = 0$ then, $\rho^*(u_k) = x_{k-1}$.

$$\begin{aligned} \Longrightarrow\quad \left\langle \rho^*(u_k), x_{k-1}\right\rangle &= \left\langle u_k, \rho.x_{k-1}\right\rangle = 1 \\ &= \left\langle x_k, \rho.x_{k-1}\right\rangle \\ &= \left\langle \rho^*(x_k), x_{k-1}\right\rangle. \\ \Longrightarrow\quad u_k &= x_k. \end{aligned}$$

Thus the induction is complete.

(3) By (2) $(\rho^k)^*(x_n) = (\rho^*)^k(x_n) = x_{n-k}$. Therefore

$$(\rho^{k-n})^*(y^*{}_k(x_n)) = y_k^*((\rho^{k-n})^*)(x_n) = y_k^*(x_k) = 0.$$

By uniqueness of x_k, $x_k^*(x_n) = x_{k-n}$.

(4) Observe that

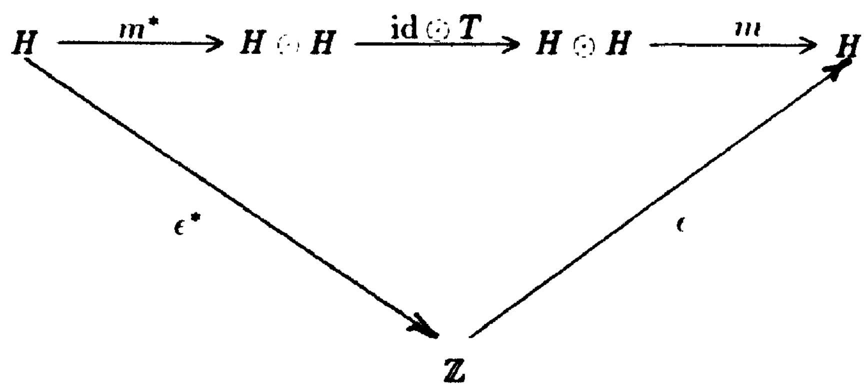
$$\begin{aligned}\langle m^*(x_n), x_k \otimes x_{n-k}\rangle &= \langle x_n, m(x_n \otimes x_{n-k})\rangle \\ &= \langle x_n, x_k . x_{n-k}\rangle \\ &= \langle x_k^*(x_n), x_{n-k}\rangle = 1. \\ \Longrightarrow \left\langle m^*(x_n), \sum x_k \otimes x_{n-k}\right\rangle &= \sum \langle x_k^*(x_n), x_{n-k}\rangle = n.\end{aligned}$$

Since each of the terms $x_k \otimes x_{n-k}$ is irreducible in $H \otimes H$ and since

$$\langle m^*(x_n), x_k \otimes x_{n-k}\rangle = 1 \ \forall\ k$$

, it follows that every elelment of the form $x_k \otimes x_{n-k}$ must be an irreducible constituent of $m^*(x_n)$. Now, because $m^*(x_n)$ has exactly n irreducible components we deduce that $m^*(x_n) = \sum_k x_k \otimes x_{n-k}$.

(5) Let $T : H \longrightarrow H$ be the conjugation automorphism of H, and define $t : H \longrightarrow H$ by $t = (-1)^n T$. Since T is an involutive automorphism of H, it follows that t is as well.It follows from the definition that T is uniquely determined by the following commutative diagram.



The dual diagram is obtained by replacing T by its adjoint T^*. After comparing the above diagram with its dual, it follows by uniqueness, that $T = T^*$. Since T is an involution, it follows that $T = T^* = T^{-1}$. Thus T is an isometry. $\Longrightarrow$ t is an isometry as well. From this fact we may deduce that

$$(4.10) \qquad (t(a))^* = t \circ a^* \circ t^{-1}.$$

By definition of T, $T(\rho) = -\rho$ and hence $t(\rho) = \rho$. $\implies$ $t(\rho^n) = \rho^n$ for $n \geq 0$. Observe that $\omega \in H_n$ is irreducible if and only if $\langle \omega, \omega \rangle = 1$, $\langle \omega, \rho^n \rangle > 0$. Therefore, if $\omega \in H_n$ is irreducible then

$$\langle t(\omega), t(\omega) \rangle = \langle \omega, \omega \rangle = 1,$$

and furthermore

$$\langle t(\omega), \rho^n \rangle = \langle t(\omega), t(\rho^n) \rangle = \langle \omega, \rho^n \rangle > 0.$$

Thus if ω is irreducible, then $t(\omega)$ is irreducible as well.

Now, since $m \circ (id \odot T) \circ m^*(x_2) = 0$, the proof of (4) implies that

$$0 = m \circ (id \odot T)(1 \odot x_2 + \rho \odot \rho + x_2 \odot 1) = T(x_2) - \rho^2 + x_2.$$

Therefore $t(x_2) = T(x_2) = \rho^2 - x_2 = y_2$. $\implies (t(x_2))^*(x_n) = y_2^*(x_n)$. Apply 4.10 to this last expression and deduce that $(x_2^* \circ t)x_n = (t \circ y_2^*)(x_n) = 0$. We have proved that t maps irreducible elements to irreducible elements. Thus, it follows from (2), by uniqueness, that $t(x_n) = y_n$. Similarly, $t(y_n) = x_n$.

(6) Let $\mathcal{P}_n$ denote the set of partitions of n. For all $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_n$ define

$$x_\lambda = x_1 x_2 \ldots, x_r, \qquad y_\lambda = y_1 y_2 \ldots, y_r.$$

To prove that $H = \mathbb{Z}[x_1, x_2, \ldots]$ it suffices to show that the monomials $\{x_\lambda \mid \lambda \in \mathcal{P}_n\}$ form a basis of H_n. Order the monomial elements in H_n in such a way that for $i < j$ $\lambda_i^t \prec \lambda_j^t$, and set $e_i = x_{\lambda_i}$, $e_i' = y_{\lambda_i}^t$. Observe that the matrix $\left(\left\langle e_i, e_j' \right\rangle\right)$ is triangular with 1's along the diagonal, and hence $det\left(\left\langle e_i, e_j' \right\rangle\right) = 1$.

By (4), $m^*(x_n) = \sum_k x_k \odot x_{n-k}$, $\forall\, n \geq 1$. Since $m \circ (id \odot t) \circ m^*(x_n) = 0$, it follows that

$$0 = m \circ (id \odot T) \circ m^*(x_n) = m \circ (id \odot (-1)^n t) \sum_k x_k \odot x_{n-k}$$

Thus,

$$y_n = \sum_k (-1)^{k-1} x_k y_{n-k} \qquad \forall\, n \geq 1.$$

Using an induction argument on n, we can deduce that every element y_n can be expressed as a linear combination of the x_λ with integer coefficients. But this implies that

$$e'_j = \sum_i a_{ij} e_j, \qquad a_{ij} \in \mathbf{Z},$$

and so the matrix $\left(\left\langle e_i, e'_j \right\rangle\right)$ is equal to $(\langle e_i, e_j \rangle)\,(a_{ij})$. Whence

$$1 = det\left(\left\langle e_i, e'_j \right\rangle\right) = det\,(\langle e_i, e_j \rangle)\, det(a_{ij}).$$

Since the factors on the right-hand side must be integers, and since the Gram determinant is always non-negative it follows that $det\,(\langle e_i, e_j \rangle) = 1$. Thus, $\{x_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a $\mathbf{Z}$-basis of a subgroup $H' \subset H_n$. To prove that $H' = H_n$ it suffices to verify that no irreducible element $\omega \in H_n$ is orthogonal to $\{e_i \mid i = 1, \ldots, |\mathcal{P}_n|\}$. But since

$$\langle \omega, e_p \rangle = \left\langle \omega, x^n_{(1)} \right\rangle = 0,$$

this is impossible since, by 4.7, $\left\langle \omega, x^n_{(1)} \right\rangle \neq 0$

(7) Any PSH-algebra H' which has a unique irreducible primitive element is a polynomial algebra $\mathbf{Z}[x'_1, x'_2, \ldots]$ by (6). Define a morphism $\gamma : H \longrightarrow H'$ by $\gamma(x_n) = x'_n$. It suffices to prove that the image under γ of an irreducible in H is irreducible in H'. The proof of this fact is similar to the proof that the morphism t in part (5) sends irreducibles to irreducibles. ∎

SECTION 5

PSH-ALGEBRAS AND REPRESENTATION THEORY

Let G be a finite group, and let $G \wr S_n$ denote the wreath product of G with S_n. For any ordered partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of n set

$$S_\alpha[G] = G \wr S_{\alpha_1} \times G \wr S_{\alpha_2} \times \cdots \times G \wr S_{\alpha_r}.$$

Observe that if $G = 1$ or $\mathbf{Z}/2$ then $G \wr S_n$ is, respectively, the symmetric group or the Weyl group of type B_n.

Consider the graded group

$$R(S[G]) = \bigoplus_{n \geq 0} R(G \wr S_n).$$

$R(S[G])$ is clearly a free abelian group with set of irreducible elements

$$\coprod \Omega(R(G \wr S_n)).$$

THEOREM 5.1 (BERSTEIN-ZELEVINSKI). *$R(S[G])$ is a PSH-algebra and its set of irreducible primitive elements is $\Omega(R(G))$.*

Let $(k,l) \vdash n$ and identify $R(G \wr S_k) \odot R(G \wr S_l)$ with $R(S_{k,l}[G])$. Define the multiplication morphism

$$m : R(G \wr S_k) \odot R(G \wr S_l) \longrightarrow R(G \wr S_{k+l})$$

in $R(S[G])$ by:

$$m = Ind_{S_{k,l}[G]}^{G \wr S_n}.$$

Now define the comultiplication

$$m^* : R(G \wr S_n) \longrightarrow \bigoplus_{k+l=n} R(G \wr S_k) \odot R(G \wr S_l)$$

by

$$m^* = \sum_{k+l=n} Res_{S_{k,l}[G]}^{G \wr S_n}.$$

An irreducible primitive element in $R(S[G])_n$ is an irreducible representation of $G \wr S_n$ whose restriction to any subgroup $G \wr S_k \times G \wr S_l$ such that $k + l = n$, is zero by definition. Therefore all irreducible primitive elements of $R(S[G])$ occur in $R(G \wr S_1) = R(G)$ and so its set of irreducible primitive elements is $\Omega(R(G))$.

When $G = 1$ let $R(S)$ denote the PSH-algebra $R(S[G])$. Theorem 5.1 implies that $R(S)$ has exactly one irreducible primitive element. Thus, it follows from Theorem 4.10(6) that there is a ring isomorphism

$$\Theta : \Lambda = \mathbf{Z}[x_1, x_2, \ldots] \longrightarrow R(S).$$

Recall that Λ is the free λ-ring on one generator. By requiring Θ to be a λ-ring isomorphism there is an induced λ-ring structure on $R(S)$.

When $G = \mathbf{Z}/2$, $R(S[\mathbf{Z}/2])$ as a PSH-algebra decomposes into the tensor product of subalgebras $R(1_{\mathbf{Z}/2}) \otimes R(\varepsilon)$, by Theorem 4.7 where $1_{\mathbf{Z}/2}$, ε denote the trivial and alternating representations of $\mathbf{Z}/2$. Since both $R(1_{\mathbf{Z}/2})$, $R(\varepsilon)$ each have exactly one irreducible primitive element, $\Lambda \approx R(1_{\mathbf{Z}/2}) \approx R(\varepsilon) \approx R(S)$ by Theorem 4.10(7).

REMARK 5.2.

Since $R(S[\mathbf{Z}/2]) \approx R(1_{\mathbf{Z}/2}) \otimes R(\varepsilon) \approx R(S) \otimes R(S)$, it follows immediately that as groups

$$R(\mathcal{W}(B_k)) \approx \bigoplus_{i+j=k} R(S_i) \otimes R(S_j).$$

Therefore, if $\#(i)$ denotes the number of partitions of i, then

$$Rank\,(R(\mathcal{W}(B_k))) = \sum_{i+j=k} \#(i).\#(j).$$

This may also be deduced in a much more elementary way from the conjugacy class count in Lemma 5.3.

Section 6
BOORMAN's THEOREM

Definition. *Let W_k be an S_k-module, and let T be any $\mathbb{C}$-vector space. Define the S-Operation $\phi_{W_k}(T)$ to be the set of invariants of $W_k \otimes T^{\otimes k}$ under the S_k-module action given by*

$$s(w \otimes t_1 \otimes \cdots \otimes t_k) = sw \otimes t_{s^{-1}(1)} \otimes \cdots \otimes t_{s^{-1}(k)})$$

for $s \in S_k$, $w \in W_k$, $t_1, \ldots, t_k \in T$.

Suppose that T is a G-module with G-action $\mu : G \times T \longrightarrow T$. Define

$$\overline{\mu} : G \times (W \otimes T^{\otimes n}) \longrightarrow W \otimes T^{\otimes n}.$$

by

$$\mu(w \otimes t_1 \otimes \cdots \otimes t_n) = (w \otimes \mu(g, t_1) \otimes \cdots \otimes \mu(g, t_n))$$

Now let $\hat{\mu} = \mu|_{\phi_W(T)}$. $\hat{\mu}$ defines a G-action on $\phi_W(T)$.

We introduce the notation $P(S_k)$ to denote the subring of $R(S_k)$ generated by the permutation representations of S_k. For any S_k-module T, let $\langle T \rangle_\phi$ denote the subring of $R(S_k)$ generated by the set $\{\phi(T) \mid \forall$ S-operations $\phi\}$.

Theorem 6.1 (Boorman). *If X_k denotes the canonical S_k-module $Ind_{S_{k-1}}^{S_k} 1$, then $P(S_k)$ is a subring of $\langle X_k \rangle_\phi$.*

Sketch of proof: It is enough to show that if H is a subgroup of S_k which is conjugate to $M \times S_m$ for $M \subset S_{k-m}$, $0 \leq m \leq k$, then $Ind_H^{S_k} 1 \in \langle X_k \rangle_\phi$. The proof is by descending induction on m:

(1) If $m = k$, then $H = S_k$ and $Ind_H^{S_k} 1 = 1 = \phi(X_k) \in \langle X_k \rangle_\phi$.

(2) Let $m < k$ and assume for $m < m' \leq k$ that if H is conjugate to $M' \times S'_m$, then $Ind_H^{S_k} 1 \in \langle X_k \rangle_\phi$. Suppose that H is conjugate to $M \times S_m$ for $M \subset S_{k-m}$. Let $W_{k-m} = Ind_H^{S_{k-m}} 1$. Boorman then proves that $\phi_{W_{k-m}}(X_k)$ is a direct sum of a family of permutation representations $Ind_{H_\alpha}^{S_k} 1$, such that $H_{\alpha_0} = H$, and H_α satisfies

the induction hypothesis for all $\alpha \neq \alpha_0$. That is, she shows that $\phi_{w_{k-m}}(X_k) = \sum_\alpha Ind_{H_\alpha}^{S_k} 1$, and for all $\alpha \neq \alpha_0$, $Ind_{H_\alpha}^{S_k} 1 \in \langle X_k \rangle_\phi$ Thus

$$Ind_H^{S_k} 1 = Ind_{M \times S_m}^{S_k} 1 = \phi_{w_{k-m}}(X_k) - \sum_{\alpha \neq \alpha_0} Ind_{H_\alpha}^{S_k} 1 \in \langle X_k \rangle_\phi .$$

This proves that $P(S_k) \subset \langle X_k \rangle_\phi$. ∎

We have the inclusion $P(S_k) \subset \langle X_k \rangle_\phi \subset R(S_k)$. Since $\{Ind_{\Pi S_\alpha}^{S_k} 1 \mid \alpha \vdash k\}$ is a **Z**-module basis for $R(S_k)$ [At1], it follows that $P(S_k) = R(S_k)$. Thus $R(S_k) = \langle X_k \rangle_\phi$. This proves that $R(S_k)$ is unigenerated as a ring by S-operations. The final step of Boorman's proof shows that the λ-operations generate the S-operations, and consequently $R(S_k)$ is unigenerated as a λ-ring.

The representation X_k is the reflection representation of S_k. Thus it is natural to ask whether or not the same is true of the other Weyl groups. That is, are $R(W(B_k)) = R(W(C_k))$, $R(W(D_k))$ unigenerated as λ-rings? This is untrue in general. For example, two elements are required to generate $R(W(B_3))$ as a λ-ring. Moreover, my original analysis of the PSH-algebra structure of $R(W(B))$ suggests that in general $R(W(B_k))$ can be λ-generated by two elements. At the present time I have been unable to confirm this, and it now seems unlikely. However, I have obtained an intermediate result which shows that $\mathbf{Q} \otimes R(W(B_k))$ is a λ-ring generated by two elements. The demonstration of this result is the subject of Part II.

PART II

SECTION 1

RELATIONS BETWEEN CHARACTERS

In all that follows let $\Phi \subset V$ be a root system and $W \subset GL(V)$ be its associated finite reflection group. Let π be any subset of the base $\Delta = \{s_i\}$ of the root system Φ, and let W_π denote the subgroup of W generated by the reflections corresponding to the elements of the subset π. For each character χ of W let $\hat{\chi}$ denote the associated character $\hat{\chi}(w) = \varepsilon(w)\chi(w)$. Here ε denotes the alternating representation of W which is defined as the homomorphism of W into the group $\{+1, -1\}$ such that $\varepsilon(s_i) = -1$.

THEOREM 1.1 (SOLOMON). *Let $\Phi \subset V$ be a root system with base Δ and let W be its associated finite reflection group. For each $\pi \subset \Delta$, let $\phi_\pi = Ind_{W_\pi}^{W} 1$. Then*

$$\sum (-1)^{|\pi|}\phi_\pi = \varepsilon.$$

PROOF: Let $\{C_i\}$ denote the Weyl chambers of W and let $F_i = C_i \cap S^{n-1}$. Fix $F \in \{F_i\}$. Since W acts simply and transitively on the Weyl chambers, it follows that $\forall\, i$, $F_i = wF$, for a unique $w \in W$. Thus

$$\mathcal{F} = \bigcup_{w \in W} wF$$

is a disjoint union. Furthermore observe that each $F_i = wF$ is a simplex. Therefore the F_i and their faces form a simplicial decomposition of S^{n-1} and the closure of $\mathcal{F}$ is S^{n-1}. Denote this finite simplicial complex by K and let K_p denote its p-skeleton. If $e_1, \ldots, e_n$ are the vertices of F then write $F = (e_1, \ldots, e_n)$. All the simplices of K_{p-1} have the form $(we_{i_1}, \ldots, we_{i_p})$, where $\alpha = \{i_1, \ldots, i_p\}$ is a subset of $\{1, \ldots\ldots, n\}$ and w ranges over W. Let

$$L_\alpha = \bigcup_{w \in W} (we_{i_1}, \ldots, we_{i_p}).$$

It was shown in [**Wi**] that if $we_i = e_j$ then $i = j$, and hence $(we_{i_1}, \ldots, we_{i_p}) = (we_{j_1}, \ldots, we_{j_p})$ forces $e_{i_k} = e_{j_k}$. This fact implies that K_{p-1} is the disjoint union of the L_α, where α is a subset of $\{1, \ldots, n\}$ containing p elements.

Let I_α denote the isotropy group of $(e_{i_1}, \ldots, e_{i_p})$. Since e_i, e_j are equivalent under the action of W if and only if $i = j$, it follows that I_α is the subgroup of W which fixes each of the vertices $e_{i_1}, \ldots, e_{i_p}$. Define

$$\lambda_\alpha = Ind_{I_\alpha}^W 1.$$

Observe that there is a 1:1 correspondence between the vertices e_i of K_{p-1} and the roots r_i of the Weyl chamber. It was shown in [Wi] that if $\alpha = \{i_1, \ldots i_p\}$ and $\pi = \{r_{j_1}, \ldots, r_{j_q}\}$ where α and $\{j_1, \ldots, j_q\}$ are complimentary subsets of $\{1, \ldots, n\}$ then $I_\alpha = W_\pi$. Therefore $\lambda_\alpha = Ind_{I_\alpha}^W 1 = Ind_{W_\pi}^W 1 = \phi_\pi$.

Next, let $I(K_{p-1})$, be the isotropy group of K_{p-1} and define

$$\kappa_{p-1} = Ind_{I(K_{p-1})}^W 1.$$

By definition of induction, κ_{p-1} is a direct sum of W-modules. The reflection group W acts on K_{p-1} by permuting its simplexes. Since the orbits of K_{p-1} under this action are just the L_α, it is clear that the summands of κ_{p-1} are the monomial representations λ_α. Thus

$$\kappa_{p-1} = \sum_\alpha \lambda_\alpha.$$

Given any simplex $\sigma \in K_{p-1}$ there exists a unique subset $\alpha = \{i_1, \ldots, i_p\}$ and some element $w \in W$, such that $\sigma = (we_{i_1}, \ldots, we_{i_p})$. Reindex the e_i if necessary so that the indices i_k are listed in ascending order $i_1, \ldots, i_p$, and define an order on the vertices of σ such that $we_{i_1} \prec \cdots \prec we_{i_p}$. This gives an orientation of K.

Let $C_p(K)$ be the group of p-chains with real coefficients and $H_p(K)$ its p-th homology group with real coefficients. Each element $w \in W$ gives a non-singular linear transformation of the real vector space $C_p(K)$ and this in turn induces a non-singular transformation of $H_p(K)$. Let ψ_p be the character of the representation afforded by the W-action on $C_p(K)$, and similarly let θ_p denote the character of the W-module $H_p(K)$. The Hopf trace formula states that

$$\sum_{p=0}^{k}(-1)^p\psi_p = \sum_{p=0}^{k}(-1)^p\theta_p.$$

where k is the dimension of K.

With the given orientation on K every element $w \in W$ maps positively oriented simplexes to positively oriented simplexes. This implies that $\psi_{p-1} = \kappa_{p-1} = \sum_\alpha \lambda_\alpha$ [Wi]. Applying Hopf's formula we conclude that

$$-\sum_{\alpha \neq \emptyset} (-1)^{|\alpha|} \lambda_\alpha = \sum_{p=0}^{n-1} (-1)^p \theta_p.$$

Since K is a simplicial subdivision of S^{n-1}, $H_i(K) = 0$ for all $i \neq 0, n-1$. This implies that $\theta_1 = \cdots = \theta_{n-2} = 0$. Note that θ_0 is the trivial character by definition. To evaluate θ_{n-1}, let γ be an $n-1$ cycle. Given any simplicial map w, $w\gamma = a\gamma$ where $a \in \mathbf{Z}$ is the degree of w. Since w is a homeomorphism $a = \pm 1$. If w preserves the orientation of S^{n-1} then $w\gamma = \gamma$, otherwise $w\gamma = -\gamma$. It follows that $\theta_{n-1}(w) = det(w) = \varepsilon(w)$. Hence

$$-\sum_{\alpha \neq \emptyset} (-1)^{|\alpha|} \lambda_\alpha = 1 + (-1)^{n-1} \varepsilon. \tag{1.2}$$

This formula in turn becomes

$$-\sum_{\pi \neq \Delta} (-1)^n (-1)^{|\pi|} \phi_\pi = 1 + (-1)^{n-1} \varepsilon$$

and hence

$$\sum_{\pi \neq \Delta} (-1)^{|\pi|} \phi_\pi = (-1)^{n-1} + \varepsilon.$$

Thus

$$\sum_{\pi} (-1)^{|\pi|} \phi_\pi = \varepsilon. \quad \blacksquare \tag{1.3}$$

Corollary 1.4 (Solomon). *Let W be a finite Euclidean reflection group. Let χ be a character of W and let χ_π be the character of W induced by restriction of χ to W_π. Then*

$$\hat{\chi} = \sum_\pi (-1)^{|\pi|} \chi_\pi.$$

PROOF: By the definition of induced character

$$\chi_\pi(w) = \frac{1}{|W_\pi|} \sum_{zwz^{-1} \in W_\pi} \chi(zwz^{-1}) \tag{1.5}$$
$$= \chi(w) \sum_{zwz^{-1} \in W_\pi} 1(zwz^{-1})$$
$$= \chi(w)\phi_\pi(w).$$

Multiply 1.3 by χ. Then it follows at once from 1.5 that

$$\hat{\chi} = \varepsilon\chi = \sum(-1)^{|\pi|}\phi_\pi\chi = \sum(-1)^{|\pi|}\chi_\pi. \quad \blacksquare$$

SECTION 2

REPRESENTATIONS OF WEYL GROUPS OF CLASSICAL TYPE AND THE j-OPERATION

In all that follows, let $\Phi \subset V$ be a root system and $W \subset GL(V)$ its associated finite reflection group. Given any subgroup W_1 of W which is generated by reflections let

$$V_{w_1} = \{v \in V \mid w(v) = v \ \forall w \in W_1\}$$

Let V_1 denote the orthogonal complement of V_{w_1}. Then there is a decomposition $V = V_1 \oplus V_{w_1}$, where V_1 is a W_1-module, which has no non-zero W_1-invariants.

THEOREM 2.1 (MACDONALD). *Let $S_N(V_1)$ be the space of homogeneous polynomial functions on V_1 of degree N. Let U be an absolutely irreducible W_1-submodule of $S_N(V_1)$ which does not occur in $S_i(V_1)$ if $0 \leq i < N$. Regard U as a subspace of $S_N(V)$ and consider the W-submodule $j(U)$ of $S_N(V)$ generated by U. Then:*

(1) *$j(U)$ is an irreducible W-module.*

(2) *$j(U)$ occurs with multiplicity 1 in $S_N(V)$.*

(3) *$j(U)$ does not occur in $S_i(V)$ if $0 \leq i < N$.*

Observe that because all representations are assumed to be complex representations, Schur's Lemma implies that all irreducible representations are absolutely irreducible.

PROOF: Let $\phi : j(U) \longrightarrow S_i(V)$ be a homomorphism of W-modules and let ϕ_1 denote the homomorphism $\phi|_U : U \longrightarrow S_i(V)$ of W_1-modules. Observe that as a W_1-module $S_i(V) \approx \bigoplus_{0 \leq j \leq i} (S_j(V_1) \otimes S_{i-j}(V_{w_1}))$. Suppose that $i < N$. Then U does not occur in $S_j(V_1)$ for $0 \leq j \leq i$. Since W_1 acts trivially on $S_{i-j}(V_{w_1})$, U does not occur in $S_j(V_1) \otimes S_{i-j}(V_{w_1})$ and so $\phi_1 = 0$. But since ϕ is a homomorphism of W-modules and since U generates $j(U)$ as a W-module, $\phi = 0$. This implies that $j(U)$ does not occur in $S_i(V)$ if $i < N$.

Now suppose that $i = N$. Then U does not occur in $S_j(V_1) \otimes S_{i-j}(V_{w_1})$ for $0 \leq j < N$ and so ϕ_1 maps U into $S_N(V_1) \otimes S_0(V_{w_1}) = S_N(V_1)$. Because U occurs with multiplicity one in $S_N(V_1)$ it follows that ϕ_1 must be a scalar multiple of the identity. Since U generates $j(U)$ as a W-module, ϕ must also be a scalar multiple of the identity. Thus $j(U)$ must be irreducible, and occur with multiplicity one in $S_N(V)$. ∎

PROPOSITION 2.2. *Let U be an absolutely irreducible W_1-submodule of $S_N(V_1)$ which occurs with multiplicity one in $S_N(V_1)$ and does not occur in $S_i(V)$ if $i < N$. Then*

(1) *$j(U)$ occurs as a component of the induced module $U \otimes_{CW_1} CW$ with multiplicity one.*

(2) *All irreducible components of the induced module which are not isomorphic to $j(U)$ do not occur in $S_i(V_1)$ if $0 \leq i \leq N$.*

PROOF: Since U is a W_1-submodule of $j(U)$, it follows from the Frobenius reciprocity theorem that $j(U)$ occurs as a component of the induced module. Now let M be an irreducible W-submodule of the induced module $U \otimes_{CW_1} CW$, and suppose M is isomorphic to a submodule of $S_i(V)$. Again, by Frobenius reciprocity, M contains a W_1-submodule isomorphic to U, and because U does not appear in $S_i(V_1)$ for $0 \leq i < N$, it follows that $i \geq N$. Finally if $i = N$, then M is an irreducible W-submodule of $S_N(V)$ which contains a W_1-submodule equal to U and hence M is equal to $j(U)$. Thus if M is not isomorphic to $j(U)$ then M cannot occur in $S_i(V)$ when $0 \leq i \leq N$. ∎

Let Φ be a root system and let $W \subset GL(V)$ be its associated reflection group. Let W_1 be a reflection subgroup of W with root system Φ_1. Let V_1 be W_1-submodule of V such that $V = V_1 \dotplus V^{W_1}$. For each root $\alpha \in \Phi$ let κ_α be the element of the dual space of V_1 given by $\kappa_\alpha(v) = (\alpha, v)$. Let f denote the element of $S_N(V_1)$ given by

$$f = \prod_{\alpha \in \Phi^+} \kappa_\alpha .$$

Observe that $\sigma_\beta(\kappa_\alpha) = \kappa_{\sigma_\beta(\alpha)}$ for all $\beta \in \Phi_1$. This follows because $\sigma_\beta(\kappa_\alpha)(v) = \kappa_\alpha(\sigma_\beta(v)) = (\alpha, \sigma_\beta(v)) = (\sigma_\beta(\alpha), v) = \kappa_{\sigma_\beta(\alpha)}(v)$. Now let β be a simple root of Φ_1.

Then $\sigma_\beta f(v) = \prod \sigma_\beta \kappa_\alpha(v) = \prod \kappa_{\sigma_\beta(\alpha)}(v) = -f(v)$, and so $\sigma_\beta(f) = -f$. For, σ_β transforms β to $-\beta$ and all other positive roots to positive roots, by Lemma I.3.11. Thus for all $w \in W_1$,

$$w(f) = \det w . f$$

Hence $U = \mathbf{C}f$ is a W_1-module which affords the sign representation.

Since any polynomial function p satisfying $wp = \det w . p$ for all $w \in W_1$ must be the product of f with a W_1-invariant polynomial [Ca2 9.4.6], there can be no such polynomial $p \in S_i(V_1)$ for $i < N$, and the only such polynomials $p \in S_N(V_1)$ are those in $\mathbf{C}f$. Hence the sign representation occurs with multiplicity one in $S_N(V_1)$ and does not occur in $S_i(V_1)$ for $i < N$. Thus by Theorem 2.1, the W-submodule $j(U)$ of $S_N(V)$ generated by U is irreducible, occurs with multiplicity one in $S_N(V)$, and does not occur in $S_i(V)$ if $i < N$. This discussion implies:

COROLLARY 2.3 (MACDONALD). *Let Φ be a root system W be its associated reflection group. Let W_1 be a reflection subgroup of W and V_1 a W_1-submodule of V such that $V = V_1 \oplus V_{W_1}$. Let $N = |\Phi^+|$ denote the number of positive roots of W_1, and let U be the alternating representation of W_1. The alternating representation U is a W_1-submodule of $S_N(V_1)$. Moreover, the W-submodule $j(U)$ of $S_N(V)$ generated by U is irreducible, occurs with multiplicity 1 in $S_N(V)$, and does not occur in $S_i(V)$ if $i < N$.*

The irreducible representations obtained in this manner are called the Macdonald representations of W.

PROPOSITION 2.4. *All irreducible representations of $W(A_k)$ may be obtained as Macdonald representations. More precisely, let α be a partion of $k+1$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Let α^* be the dual partition of α. Let Φ_1 be a subroot sustem of $\Phi(A_k)$ of type*

$$A_{\alpha_1^*-1} \cup A_{\alpha_2^*-1} \cup \ldots$$

and denote its Weyl group by W_1. Then each irreducible character of W has the form $j_{W_1}^{W} \varepsilon$.

Proof: It follows from Corollary 2.3 that each representation of the form $j_{W_1}^{W}\varepsilon$ is irreducible. Since each of these occurs in a different grading of the symmetric algebra and since there is a 1:1 correspondence between the number of partitions of k and the number of irreducible representations obtained from the j-operations, each Macdonald representation must be unique. Thus each irreducible representation must be a Macdonald representation. ∎

Lusztig has proved similar results for Weyl groups of type B_k and D_k.

Proposition 2.5 Lusztig. *The Weyl group $W(B_k)$ has order $2^k k!$ and all of its irreducible representations may be obtained as Macdonald representations. There is one irreducible representation of $W(B_k)$ associated to each ordered pair of partitions (α, β) with $|\alpha| + |\beta| = k$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_q)$ with*

$$0 \le \alpha_1 \le \alpha_2 \ldots, \alpha_p, 0 \le \beta_1 \le \beta_2 \ldots, \beta_q.$$

If α^*, β^* are the dual partitions of α, β, then $\Phi(B_k)$ has a subrootsystem Φ_1 of type

$$B_{\alpha_1^*} \cup B_{\alpha_2^*} \cup \cdots \cup B_{\alpha_s} \cup D_{\beta_1^*} \cup D_{\beta_2^*} \cup \cdots \cup D_{\beta_t}.$$

Let W_1 be the Weyl group associated to Φ_1. Then each irreducible representation of $W(B_k)$ is obtained as a Macdonald representation of the form $\psi_{\alpha,\beta} = j_{W_1}^{W}(\varepsilon)$. Furthermore the representation $\psi_{\alpha,\beta}$ remains irreducible on restriction to $W(D_k)$ if $\alpha \neq \beta$. In this case $\psi_{\alpha,\beta}$ and $\psi_{\beta,\alpha}$ coincide on restriction to $W(D_k)$. If $\alpha = \beta$ then $\psi_{\alpha,\alpha}$ decomposes into two irreducible components $\psi_{\alpha,\alpha'}$, $\psi_{\alpha,\alpha''}$ of $W(D_k)$. All irreducible representations of $W(D_k)$ are obtained in this way.

It should be emphasized that the main result of this thesis depends very directly on this proposition, whose proof seems quite difficult. For a proof of this theorem see [Lu].

Lemma 3.6. *Let H be a subgroup of $W(B_l)$ of the form $W(A_{t-1}) \times W(D_n) \times W(B_m)$ where $t + n + m = l$, and let T be a $W(B_k)$-module. If $M_l = Ind_H^{W(B_l)} 1$, then*

$$\phi_{M_l}(T) = \phi_{1_{W(B_t)}}(T) \otimes \phi_{1_{W(B_n)}}(T) \otimes \phi_{1_{W(B_m)}}(T)$$

PROOF:

$$\begin{aligned}
\phi_{M_l}(T) &= (Ind_H^{W(B_l)} 1 \otimes T^{\otimes l})_{W(B_l)} \\
&\approx (1_H \otimes ResT^{\otimes l})_H \qquad \text{(by Lemma 3.6)} \\
&\approx (1_{W(A_{t-1})} \otimes 1_{W(D_n)} \otimes 1_{W(B_m)} \otimes ResT^{\otimes l})_{W(A_{t-1}) \times W(D_n) \times W(B_m)} \\
&\approx (1 \otimes ResT^{\otimes t})_{W(A_{t-1})} \otimes (1 \otimes ResT^{\otimes n})_{W(D_n)} \otimes (1 \otimes ResT^{\otimes m})_{W(B_m)}.
\end{aligned}$$

Since the $W(B_l)$-action on $M_l \otimes T^{\otimes l}$ factors through S_l, it follows that

$$\begin{aligned}
\phi_{1_{W(B_t)}}(T) &= (1 \otimes ResT^{\otimes t})_{W(A_{t-1})} \\
\phi_{1_{W(B_n)}}(T) &= (1 \otimes ResT^{\otimes n})_{W(D_n)}.
\end{aligned}$$

Therefore,

$$\phi_{M_l}(T) \approx \phi_{1_{W(B_t)}}(T) \otimes \phi_{1_{W(B_n)}}(T) \otimes \phi_{1_{W(B_m)}}(T). \quad \blacksquare$$

Theorem 3.7. *Let S be any subset of $R(W(B_k))$ consisting of images of $W(B_k)$-modules under the inclusion ι. Then $\langle S\rangle_{\otimes} = \langle S\rangle_{\lambda}$.*

Proof: Let $T \in S$. Suppose that $[M] \in R(W(B_k))$. Then there are representations M', M'' of $W(B_k)$ such that $[M] = [M'] - [M'']$. By Proposition 3.3

$$\begin{aligned}
M' &= \sum a_{\alpha,\beta,\gamma} Ind_{W_{(\alpha,\beta,\gamma)}}^{W(B_k)} 1, \\
M'' &= \sum b_{\rho,\sigma,\tau} Ind_{W_{(\rho,\sigma,\tau)}}^{W(B_k)} 1,
\end{aligned}$$

where $W_{(\alpha,\beta,\gamma)}$ denotes the Weyl group associated to $(\coprod A_{\alpha-1}) \coprod (\coprod D_\beta) \coprod (\coprod B_\gamma)$. Since $[M'] = [M] + [M''] = [M \oplus M'']$, $M' \approx M \oplus M''$. Therefore, by Lemma 3.4

$$\phi_{M'}(T) = \phi_M(T) \otimes \phi_{M''}(T).$$

$\langle S\rangle_\lambda$:= the subring of $R(\mathcal{W}(B_k))$ generated by all elements of the form $\lambda^n(T)$,

for all $n > 0$, as T ranges over S.

Let ι denote the inclusion map from the semi-group of representations of $\mathcal{W}(B_k)$ into $R(\mathcal{W}(B_k))$. We shall prove that $\langle S\rangle_\bullet = \langle S\rangle_\lambda$ for any set S whose elements are images of modules under ι.

THEOREM 3.2. *The set* $\{Ind^{\mathcal{W}(B_k)}_{\prod \mathcal{W}(B_{\beta'}) \times \prod \mathcal{W}(D_{\delta'})} \varepsilon\}$ *forms a basis of* $R(\mathcal{W}(B_k))$.

PROOF: Let τ denote the number of double partitions of k. By Theorem 2.5 the irreducible representations of $\mathcal{W} = \mathcal{W}(B_k)$ are given by the Macdonald representations $j_i = j^{\mathcal{W}}_{W_i}\varepsilon$ for $1 \le i \le \tau$. Each W_i is the Weyl group associated to a subroot system of B_k of the form $B_{\alpha_1} \cup \cdots \cup B_{\alpha_s} \cup D_{\beta_1} \cup \cdots \cup D_{\beta_t}$. By Theorem II.2.1, j_i occurs with multiplicity one in $S_i(V)$. Furthermore, by Proposition II.2.2 j_i occurs with multiplicity one in $Ind^{\mathcal{W}}_{W_i}\varepsilon$, and all the irreducible components of $Ind^{\mathcal{W}}_{W_i}\varepsilon$ which are not isomorphic to j_i must occur in $S_q(V)$ for $q > i$. This allows us to order the set of Macdonald representations so that each $j_p \in \{j_1 = 1, \ldots, j_\tau = \varepsilon\}$ is not a component of any $Ind^{\mathcal{W}}_{W_i}\varepsilon$ for all $i < p$. We prove by induction on τ that each element of the basis $\{j_1, j_2, \ldots, j_\tau\}$ can be exchanged for an element of the set $\{Ind^{\mathcal{W}(B_k)}_{\prod \mathcal{W}(B_{\beta'}) \times \prod \mathcal{W}(D_{\delta'})}\varepsilon\}$.

(1) Let $W_1 = \{e\}$. Then $\{Ind^{\mathcal{W}(B_k)}_{W_1}\varepsilon, j_2, \ldots, j_\tau\}$ is a basis of $R(\mathcal{W}(B_k))$.

(2) As the induction hypothesis assume that for some p the set

$$\{Ind^{\mathcal{W}(B_k)}_{W_1}\varepsilon, \ldots, Ind^{\mathcal{W}(B_k)}_{W_{p-1}}\varepsilon, j_p, \ldots, j_\tau\}$$

is a basis of $R(\mathcal{W}(B_k))$. As noted in the first paragraph of the proof, Propositon II.2.2 implies that for all $i < p$, j_i does not occur in $Ind^{\mathcal{W}}_{W_p}\varepsilon$. Furthermore j_p has multiplicity one in $Ind^{\mathcal{W}}_{W_p}\varepsilon$ and all of its other components are members of the set $\{j_{p+1}, j_{p+2}, \ldots, j_\tau\}$. Thus it follows that

$$\{Ind^{\mathcal{W}(B_k)}_{W_1}\varepsilon, \ldots, Ind^{\mathcal{W}(B_k)}_{W_p}\varepsilon, j_{p+1}, \ldots, j_\tau\}$$

is a basis of $R(\mathcal{W}(B_k))$. ∎

Proposition 3.3. *Let K range over the subroot systems*

$$(\coprod D_\delta)\coprod(\coprod B_\beta)\coprod(\coprod A_\alpha)$$

of B_k, and let W_K denote the Weyl group associated to K. Then the set $\{Ind_{W_K}^{W}1\}$ generates $R(W(B_k))$ as a group.

PROOF: Let $\Phi_K = Ind_{W_K}^{\prod W(B_{\beta'})\times\prod W(D_{\delta'})}1$. Then by Theorem I.1.1

$$Ind_{\prod W(B_{\beta'})\times\prod W(D_{\delta'})}^{W(B_k)}\varepsilon = Ind_{\prod W(B_{\beta'})\times\prod W(D_{\delta'})}^{W(B_k)}(\sum(-1)^{|K|}\Phi.$$
$$= \sum(-1)^{|K|}Ind_{W_K}^{W(B_k)}1.$$

The last equality follows because induction is transitive. Since the elements

$$Ind_{\prod W(B_{\beta'})\times\prod W(D_{\delta'})}^{W(B_k)}\varepsilon$$

form a basis of $R(W(B_k))$, by Theorem 3.2, the result follows. ∎

Lemma 3.4. *Let T be a $W(B_k)$-module. Then for any representations M, M' of $W(B_n)$.*

$$o_{M+M'}(T) = o_M(T) + o_{M'}(T).$$

PROOF:

$$o_{M+M'}(T) = ((M+M')\otimes T^{\otimes n})_{W(B_n)}$$
$$\approx (M\otimes T^{\otimes n})_{W(B_n)} + (M'\otimes T^{\otimes n})_{W(B_n)}$$
$$= o_M(T) + o_{M'}(T).\ ∎$$

Lemma 3.5. *Let T be a $W(B_k)$-module.. For $n \leq k$ let W be a subgroup of $W(B_n)$. If $M_n = Ind_W^{W(B_n)}1$, then*

$$o_{M_n}(T) = (Res_W^{W(B_n)}T^{\otimes n})_W.$$

PROOF:

$$o_{M_n}(T) = (Ind_W^{W(B_n)}1\otimes T^{\otimes n})_{W(B_n)}$$
$$\approx (Ind_W^{W(B_n)}(1\otimes Res_W^{W(B_n)}T^{\otimes n}))_{W(B_n)} \qquad \text{(Frobenius Reciprocity)}$$
$$\approx (Res_W^{W(B_n)}T^{\otimes n})_W.\ ∎$$

LEMMA 3.6. *Let H be a subgroup of $W(B_l)$ of the form $W(A_{t-1})\times W(D_n)\times W(B_m)$ where $t+n+m=l$, and let T be a $W(B_k)$-module. If $M_l = Ind_H^{W(B_l)}1$, then*

$$\phi_{M_l}(T) = \phi_{1_{W(B_t)}}(T)\odot\phi_{1_{W(B_n)}}(T)\odot\phi_{1_{W(B_m)}}(T)$$

PROOF:

$$\begin{aligned}\phi_{M_l}(T) &= (Ind_H^{W(B_l)}1\odot T^{\otimes l})_{W(B_l)}\\ &\approx (1_H\odot ResT^{\otimes l})_H \qquad \text{(by Lemma 3.6)}\\ &\approx (1_{W(A_{t-1})}\odot 1_{W(D_n)}\odot 1_{W(B_m)}\odot ResT^{\otimes l})_{W(A_{t-1})\times W(D_n)\times W(B_m)}\\ &\approx (1\odot ResT^{\otimes t})_{W(A_{t-1})}\odot(1\odot ResT^{\otimes n})_{W(D_n)}\odot(1\odot ResT^{\otimes m})_{W(B_m)}.\end{aligned}$$

Since the $W(B_l)$-action on $M_l\otimes T^{\otimes l}$ factors through S_l, it follows that

$$\begin{aligned}\phi_{1_{W(B_t)}}(T) &= (1\odot ResT^{\otimes t})_{W(A_{t-1})}\\ \phi_{1_{W(B_n)}}(T) &= (1\odot ResT^{\otimes n})_{W(D_n)}.\end{aligned}$$

Therefore,

$$\phi_{M_l}(T)\approx\phi_{1_{W(B_t)}}(T)\odot\phi_{1_{W(B_n)}}(T)\odot\phi_{1_{W(B_m)}}(T). \quad \blacksquare$$

THEOREM 3.7. *Let S be any subset of $R(W(B_k))$ consisting of images of $W(B_k)$-modules under the inclusion ι. Then $\langle S\rangle_\phi = \langle S\rangle_\lambda$.*

PROOF: Let $T\in S$. Suppose that $[M]\in R(W(B_k))$. Then there are representations M', M'' of $W(B_k)$ such that $[M]=[M']-[M'']$. By Proposition 3.3

$$\begin{aligned}M' &= \sum a_{\alpha,\beta,\gamma}Ind_{W_{(\alpha,\beta,\gamma)}}^{W(B_k)}1,\\ M'' &= \sum b_{\rho,\sigma,\tau}Ind_{W_{(\rho,\sigma,\tau)}}^{W(B_k)}1.\end{aligned}$$

where $W_{(\alpha,\beta,\gamma)}$ denotes the Weyl group associated to $(\coprod A_{\alpha-1})\coprod(\coprod D_\beta)\coprod(\coprod B_\gamma)$. Since $[M']=[M]+[M'']=[M\oplus M'']$, $M'\approx M\oplus M''$. Therefore, by Lemma 3.4

$$\phi_{M'}(T)=\phi_M(T)\oplus\phi_{M''}(T).$$

This implies that

(3.8)
$$[\phi_M(T)] = [\phi_{M'}(T)] - [\phi_{M''}(T)],$$

and hence

$$[\phi_M(T)] = [\phi_{\sum a_{\alpha,\beta,\gamma} Ind_{W(\alpha,\beta,\gamma)}^{W(B_k)} 1}(T)] - [\phi_{\sum b_{\rho,\sigma,\tau} Ind_{W(\rho,\sigma,\tau)}^{W(B_k)} 1}(T)].$$

Now observe that

$$\begin{aligned}\phi_{M'}(T) &= \phi_{\sum a_{\alpha,\beta,\gamma} Ind_{W(\alpha,\beta,\gamma)}^{W(B_k)} 1}(T)\\ &= \sum a_{\alpha,\beta,\gamma}\, \phi_{Ind_{W(\alpha,\beta,\gamma)}^{W(B_k)} 1}(T)\\ &= \sum a_{\alpha,\beta,\gamma}\, \phi_{1_{W(B_{\alpha_1})}}(T) \odot \cdots \odot \phi_{1_{W(B_{\gamma_t})}}(T)\\ &= \sum a_{\alpha,\beta,\gamma}\, S^{\alpha_1}(T) \odot \cdots \odot S^{\gamma_t}(T) \qquad (3.9)\end{aligned}$$

where $S^{\alpha_1}, \ldots, S^{\gamma_t}$ denote the symmetric power operations. Similarly,

(3.10)
$$\phi_{M''}(T) = \sum b_{\rho,\sigma,\tau}\, S^{\rho_1}(T) \odot \cdots \odot S^{\tau_t}(T).$$

It follows from 3.8, 3.9, and 3.10 that $[\phi_M(T)]$ is a polynomial in the symmetric power operations on T. Therefore, by Theorem I.2.10 $[\phi_M(T)]$ is a polynomial in the exterior power operations applied to T. Thus $\langle S\rangle_\phi \subset \langle S\rangle_\lambda$.

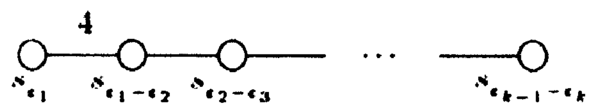
If $T \in \langle S\rangle_\lambda$, then by Theorem I.2.10 T is a polynomial in the symmetric power operations applied to S. Since any symmetric power operation can be expressed as a ϕ-operation with respect to the trivial representation, it follows that $\langle S\rangle_\lambda \subset \langle S\rangle_\phi$. Therefore, $\langle S\rangle_\phi = \langle S\rangle_\lambda$ ∎

SECTION 4

THE CANONICAL REPRESENTATIONS X_k, Y_k OF $W(B_k)$, AND GENERALIZED ϕ-OPERATIONS

The object of this section is to study two $W(B_k)$ modules X_k and Y_k. In particular we will describe the $W(B_k)$-modules obtained by applying certain ϕ-operations to X_k and Y_k.

The Weyl group $W(B_k)$ was described in Part I § 3. It is the group of all permutations and sign changes of the vectors $\{\varepsilon_1, \ldots, \varepsilon_k\}$. It is characterized by the Coxeter graph



The reflections $\{s_{\epsilon_1}, s_{\epsilon_1-\epsilon_2}, \ldots, s_{\epsilon_{k-1}-\epsilon_k}\}$ generate $W(B_k)$. To describe any $W(B_k)$-module M it suffices to describe the action of $\{s_{\epsilon_1}, s_{\epsilon_1-\epsilon_2}, \ldots, s_{\epsilon_{k-1}-\epsilon_k}\}$ on M.

THE REPRESENTATION X_k

Let $X_k = \mathrm{Ind}_{W(B_{k-1})\times W(B_1)}^{W(B_k)} 1$. The cosets of $W(B_k)/W(B_{k-1}) \times W(B_1)$ can be represented by

$$e_1 = [s_{\epsilon_1-\epsilon_k}],$$
$$e_2 = [s_{\epsilon_2-\epsilon_k}],$$
$$\vdots$$
$$e_k = [1]$$

Let $I = \{1, 2, \ldots, k\}$. By Lemma I.2.11

$$X_k = \bigoplus_{i\in I} \mathbf{C}e_i.$$

The $W(B_k)$-module structure of X_k is determined by the following

THEOREM 4.1. *Let $\epsilon_1, \ldots, \epsilon_k$ be the cosets of $\mathcal{W}(B_k)/\mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1)$. Then*

(1) $s_{\epsilon_j} \epsilon_i = \epsilon_i \qquad \forall\, i, j.$

(2) $s_{\epsilon_j \pm \epsilon_{j+1}} \epsilon_i = \epsilon_i \qquad i \neq j, j+1.$

(3) $s_{\epsilon_j - \epsilon_{j+1}}$ *interchanges* ϵ_j *and* ϵ_{j+1}.

PROOF: Since a, b lie in the same coset if and only if $a^{-1}b \in \mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1)$, it follows that

$$\phi\epsilon_i = \epsilon_j \iff s_{\epsilon_j - \epsilon_k}{}^{-1} \phi s_{\epsilon_i - \epsilon_k} \in \mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1).$$

Given a root system Φ in $\mathbf{R}^k$, if $\phi \in Gl(k, \mathbf{R})$ leaves Φ invariant then $\phi\sigma_\alpha\phi^{-1} = \sigma_{\phi(\alpha)}$ for all $\alpha \in \Phi$ by Lemma I.3.2. Therefore

$$\phi^{-1}\sigma_\alpha\phi = \sigma_{\phi^{-1}(\alpha)}.$$

(1) Let $\phi = s_{\epsilon_i - \epsilon_k}$ and $\sigma_\alpha = s_{\epsilon_j}$. Then

$$s_{\epsilon_i - \epsilon_k}(\varepsilon_j) = \begin{cases} \varepsilon_j & j \neq i, k \\ \varepsilon_k & j = i \\ \varepsilon_i & j = k \end{cases}$$

Hence,

$$s_{\epsilon_i - \epsilon_k}{}^{-1} s_{\epsilon_j} s_{\epsilon_i - \epsilon_k} = \begin{cases} s_{\epsilon_j} & j \neq i, k \\ s_{\epsilon_k} & j = i \\ s_{\epsilon_i} & j = k \end{cases}$$

In all cases $s_{\epsilon_i - \epsilon_k}{}^{-1} s_{\epsilon_j} s_{\epsilon_i - \epsilon_k} \subset \mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1)$ and so $s_{\epsilon_j}\epsilon_i = \epsilon_i$ for all j.

(2) By a similar argument,

$$s_{\epsilon_i - \epsilon_k}(\varepsilon_j - \varepsilon_{j+1}) = \begin{cases} \varepsilon_j - \varepsilon_{j+1} & i \neq j, j+1, k \neq j+1 \\ \varepsilon_{k-1} - \varepsilon_i & i \neq j, j+1, k = j+1 \end{cases}.$$

This tells us that $s_{\epsilon_i - \epsilon_k}{}^{-1} s_{\epsilon_j - \epsilon_{j+1}} s_{\epsilon_i - \epsilon_k} \in \mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1)$ when $i \neq j, j+1$. Hence $s_{\epsilon_j - \epsilon_{j+1}}\epsilon_i = \epsilon_i$ if $i \neq j, j+1$.

(3) Since

$$s_{\epsilon_{i+1} - \epsilon_k}{}^{-1} s_{\epsilon_i - \epsilon_{i+1}} s_{\epsilon_i - \epsilon_k} = s_{\epsilon_i - \epsilon_{i+1}} \in \mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1),$$

it follows that $s_{\epsilon_i - \epsilon_{i+1}}\epsilon_i = \epsilon_{i+1}$. ∎

THE REPRESENTATION Y_k

Let $Y_k = \mathrm{Ind}_{W(B_{k-1})}^{W(B_k)} 1$. The cosets of $W(B_k)/W(B_{k-1})$ are

$$\begin{array}{ll} f_1 = [s_{\epsilon_1-\epsilon_k}] & f_{-1} = [s_{\epsilon_1-\epsilon_k} s_{\epsilon_k}] \\ f_2 = [s_{\epsilon_2-\epsilon_k}] & f_{-2} = [s_{\epsilon_2-\epsilon_k} s_{\epsilon_k}] \\ \vdots & \vdots \\ f_k = [1] & f_{-k} = [s_{\epsilon_k}]. \end{array}$$

Let $J = \{-k,\dots,-2,-1\} \coprod \{1,2,\dots,k\}$. By Lemma I.2.11

$$Y_k = \bigoplus_{j \in J} \mathbf{C} f_j .$$

The $W(B_k)$-module structure of Y_k is given by

THEOREM 4.2. *The reflections $s_{\epsilon_j \mp \epsilon_{j+1}}, s_{\epsilon_j}$ associated to the root system B_k fix the cosets f_i, f_{-i} unless j or $j+1 = i$ in which case:*

(1) $s_{\epsilon_i-\epsilon_{i+1}}(f_i) = f_{i+1}$.

(2) $s_{\epsilon_i-\epsilon_{i+1}}(f_{-i}) = f_{-(i+1)}$.

(3) $s_{\epsilon_j}(f_i) = \begin{cases} f_i & \text{if } i \neq j \\ f_{-i} & \text{if } i = j. \end{cases}$

PROOF: Suppose that Φ is a root system in $\mathbf{R}^k$. If $\varphi \in GL(k,\mathbf{R})$ leaves Φ invariant, then $\sigma_{\varphi^{-1}(\alpha)} = \varphi^{-1}\sigma_\alpha\varphi$ for all $\alpha \in \Phi$, by Lemma I 3.2. Let $\varphi = s_{\epsilon_i-\epsilon_k}$, $\sigma_\alpha = s_{\epsilon_j-\epsilon_{j+1}}$. Then

$$\begin{aligned} &(s_{\epsilon_i-\epsilon_k})^{-1} s_{\epsilon_j-\epsilon_{j+1}} (s_{\epsilon_i-\epsilon_k}) \\ &= s_{s_{\epsilon_i-\epsilon_k}(\epsilon_j-\epsilon_{j+1})} \\ &= s_{\epsilon_j-\epsilon_{j+1}} \in W(B_{k-1}) \qquad \forall\, i \neq j, j+1 \neq k. \end{aligned}$$

and hence

$$s_{\epsilon_j-\epsilon_{j+1}} f_i = f_i \qquad \forall\, i \neq j, j+1 \neq k.$$

Thus the generators $s_{\epsilon_j - \epsilon_{j+1}}$ of $W(B_k)$ fix f_i whenever $j, j+1 \neq i$, $j+1 \neq k$. Similarly,

$$(s_{\epsilon_i - \epsilon_k} s_{\epsilon_k})^{-1}(s_{\epsilon_j - \epsilon_{j+1}})(s_{\epsilon_i - \epsilon_k} s_{\epsilon_k}) = s_{\epsilon_j - \epsilon_{j+1}} \in W(B_{k-1}).$$

Hence, $s_{\epsilon_j - \epsilon_{j+1}}$ leave f_{-i} invariant whenever $j, j+1 \neq i$, $j+1 \neq k$.

Furthermore observe that:

(1) $s_{\epsilon_i - \epsilon_{i+1}} f_i = f_{i+1}$, since

$$(s_{\epsilon_{i+1} - \epsilon_k})^{-1}(s_{\epsilon_i - \epsilon_{i+1}})(s_{\epsilon_i - \epsilon_k}) = s_{\epsilon_i - \epsilon_{i+1}} \in W(B_{k-1}).$$

(2) $s_{\epsilon_i - \epsilon_{i+1}} f_{-i} = f_{-(i+1)}$, since

$$(s_{\epsilon_{i+1} - \epsilon_k} s_{\epsilon_k})^{-1}(s_{\epsilon_i - \epsilon_{i+1}})(s_{\epsilon_i - \epsilon_k} s_{\epsilon_k}) = s_{\epsilon_i - \epsilon_{i+1}} \in W(B_{k-1}).$$

(3) $s_{\epsilon_j}(f_i) = \begin{cases} f_i & \text{if } i \neq j \\ f_{-i} & \text{if } i = j \end{cases}$, since

$$(s_{\epsilon_i - \epsilon_k})^{-1}(s_{\epsilon_i})(s_{\epsilon_i - \epsilon_k}) = s_{\epsilon_i - \epsilon_k(\epsilon_i)} = s_{\epsilon_i} \in W(B_{k-1}),$$

$s_{\epsilon_i} f_i = f_{-i}$. Similarly, $s_{\epsilon_j} f_i = f_i$ whenever $i \neq j$. ∎

THE REPRESENTATION $\phi_M(X_k)$

Let $\beta = \{\beta_1 \geq \beta_2 \geq \cdots \geq \beta_p\}$ be an ordered partition of l and let

$$W(B_\beta) = \prod_{\beta_i \in \beta} W(B_{\beta_i})$$

be the subgroup of $W(B_l)$ determined by β. It is the Weyl subgroup of $W(B_l)$ whose root system is the disjoint union $\coprod_{\beta_i \in \beta} B_{\beta_i}$. Define the representation associated to $W(B_\beta)$ to be

$$M = Ind_{W(B_\beta)}^{W(B_l)} 1.$$

By Lemma II.3.5 $\phi_M(X_k) = \left(X_k^{\otimes l}\right)_{W(B_\beta)}$, the $W(B_\beta)$ invariants of $X_k^{\otimes l}$, where $W(B_\beta) \subset W(B_l)$ acts on $X_k^{\otimes l}$ in the fashion described in definition II.3.1. Recall from the description given in section II.3 that $\phi_M(X_k)$ is a $W(B_k)$-module.

We now set out to study this $W(B_k)$-module structure. However, rather than dealing with the $W(B_s)$ invariants of $X_k^{\otimes l}$ we will study $\phi_M(X_k)$ via the $W(B_s)$ orbits of $X_k^{\otimes l}$. This approach will provide a convenient description which will be needed for the proof of the Main Theorem 4.18 of this chapter.

First of all, the $W(B_s)$-orbits can be used to determine a basis of $\phi_M(X_k)$. As before let $I = \{1, \ldots, k\}$. Then

$$\mathcal{E}_l = \left\{ e_{j_1} \otimes \cdots \otimes e_{j_l} \mid (j_1, \ldots, j_l) \in I^l \right\}$$

is a basis of $X_k^{\otimes l}$. Since $W(B_s) \subset W(B_k)$ acts by permuting the components in $X_k^{\otimes l}$ it follows that $\mathcal{E}_l$ is mapped to itself by $W(B_s)$. For each W_s-orbit γ in $\mathcal{E}_l$ the elements $\sum_{x \in \gamma} x$ is a $W(B_s)$ invariant. Moreover, the $W(B_s)$ orbits of $\mathcal{E}_l$ produced in this fashion give a basis of $\phi_M(X_k) = \left(X_k^{\otimes l}\right)_{W(B_s)}$.

Secondly, there is a $W(B_k)$-action on the $W(B_s)$-orbits. For the action of $W(B_k)$ on X_k (described in 4.1) induces an action on $\left(X_k^{\otimes l}\right)$ by the rule

$$g.(e_{j_1} \otimes \cdots \otimes e_{j_l}) = g.e_{j_1} \otimes \cdots \otimes g.e_{j_l}$$

Since this (internal) action commutes with the (external) action of $W(B_s) \subset W(B_k)$ considered above it follows that there is an induced $W(B_k)$-action on the $W(B_s)$-orbits.

Lastly, if we identify the $W(B_s)$-orbits with a basis of $\left(X_k^{\otimes l}\right)_{W(B_s)}$ as above, then the resulting induced action on $\phi_M(X_k) = \left(X_k^{\otimes l}\right)_{W(B_s)}$ agrees with the $W(B_k)$-action given in II.3.

Henceforth, to simplify notation we will identify $\mathcal{E}_l$ with I^l under the correspondence

$$e_{j_1} \otimes \cdots \otimes e_{j_l} \longleftrightarrow (j_1, \ldots, j_l).$$

Then the external $W(B_s)$-action on $X_k^{\otimes l}$ defined in II.3.1 can be described as follows: For $m = st \in W(B_l)$, $s \in S_l$, $t \in (\mathbf{Z}/2)^l$

$$m.(j_1, \ldots, j_l) = (j_{s(1)}, \ldots, j_{s(l)}).$$

Thus $W(B_\beta)$ acts on I^l through the action of $W(B_l)$.

To describe the $W(B_\beta)$ orbits of I^l, write $I^l = I^{\beta_1} \times \cdots \times I^{\beta_p}$. Then for any $j \in I^l$, $j = (j^1, \ldots, j^p)$ where $j^i = (j^i{}_1, \ldots, j^i{}_{\beta_i}) \in I^{\beta_i}$.

LEMMA 4.3. *Let $j, \bar{j} \in I^l$. Then j and $\bar{j}$ are in the same orbit if and only if the components of j^i and $\bar{j}^i$ are permutations of each other, for all $1 \le i \le p$.*

PROOF: This follows because $W(B_\beta)$ permutes the components of each j^i. ∎

So, in each $W(B_\beta)$-orbit the order of the components of each tuple j^r is not important. It suffices to know which elements of I occur as a component of the tuple j^r and the number of times that they do occur (i.e. their multiplicity). So j^r can be described by a multiplicity function

$$m : I \longrightarrow \mathbf{Z}^+.$$

We will adopt the convention of representing such a function by the notation

$$\left\{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\right\},$$

where m_i is the multiplicity of i in j^r, and instead of speaking of functions, we will speak of *sets from I with multiplicities*. The $W(B_\beta)$ orbits of I^l can now be represented by tuples $(V_{\beta_1}, \ldots, V_{\beta_p})$ where for each $1 \le r \le p$,

$$V_{\beta_r} = \left\{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\right\}$$

and $m_1 + m_2 + \cdots + m_k = \beta_r$.

The action of $W(B_k)$ on I induces an action on sets from I with multiplicities in the following manner: For $g \in W(B_k)$,

$$g.\left\{1^{m_1}, \ldots, k^{m_k}\right\} = \left\{(g.1)^{m_1}, \ldots, (g.k)^{m_k}\right\}$$

The action can now be extended to the class of all tuples $\left\{\left(V_{\beta_1}, \ldots, V_{\beta_p}\right)\right\}$ by

$$g.\left(V_{\beta_1}, \ldots, V_{\beta_p}\right) = \left(g.V_{\beta_1}, \ldots, g.V_{\beta_p}\right).$$

Since the tuples $\left\{\left(V_{\beta_1},\ldots,V_{\beta_p}\right)\right\}$ form a basis of $\phi_M(X_k)$ this describes the $\mathcal{W}(B_k)$-action on $\phi_M(X_k)$.

Our object will be to calculate the isotropy group, $\mathcal{I}_\gamma$, of each $\mathcal{W}(\mathcal{B}_\beta)$-orbit $\gamma = \left(V_{\beta_1},\ldots,V_{\beta_p}\right)$. We will show in every case that there is some partition $\delta \vdash l' \leq l$ such that $\mathcal{I}_\gamma = \mathcal{W}_\delta \times \mathcal{W}(B_s)$.

Lemma 4.4. *Let $\gamma = \left(V_{\beta_1},\ldots,V_{\beta_p}\right)$, and let $\mathcal{I}_\gamma$ be the stabilizer of γ. Then there is some ordered partition $\delta \vdash l' \leq l$ such that*

$$\mathcal{I}_\gamma = \mathcal{W}_\delta \times \mathcal{W}(B_s), \qquad s = k - l'.$$

Proof: Let $I(V_{\beta_i})$ be the stabilizer of V_{β_i}. Then

$$\mathcal{I}_\gamma = \bigcap_{\beta_i \in \beta} I(V_{\beta_i}).$$

This is clear, for if $w \in \mathcal{I}_\gamma$ then because w preserves the $\mathcal{W}(\mathcal{B}_\beta)$-orbit γ, $w.V_{\beta_i} = V_{\beta_i}$, for all $\beta_i \in \beta$. Hence $\mathcal{I}_\gamma \subset \bigcap I(V_{\beta_i})$. If $w \in \bigcap I(V_{\beta_i})$, then because $w.V_{\beta_i} = V_{\beta_i}$ for all $\beta_i \in \beta$, w preserves the $\mathcal{W}(\mathcal{B}_\beta)$-orbit γ. Hence $\mathcal{I}_\gamma = \bigcap I(V_{\beta_i})$.

Let $m(j_i)$ be the multiplicity of $j_i \in V_{\beta_p}$ and define

$$V_{\beta_p,k} = \left\{j_i \in V_{\beta_p} \mid m(j_i) = k\right\}.$$

Let $I(V_{\beta_p,k})$ denote the stabilizer of $V_{\beta_p,k}$. Observe that the $V_{\beta_p,k}$ are ordinary subsets of I. We no longer have to keep track of multiplicities. Since $w.V_{\beta_p} = V_{\beta_p}$ if and only if $w.V_{\beta_p,k} = V_{\beta_p,k}$ for all k,

$$I(V_{\beta_p}) = \bigcap_{k=1} I(V_{\beta_p,k}).$$

Since $I_\gamma = \bigcap_p I(V_{\beta_p})$, it follows that

$$I_\gamma = \bigcap_{p,k} I(V_{\beta_p,k}).$$

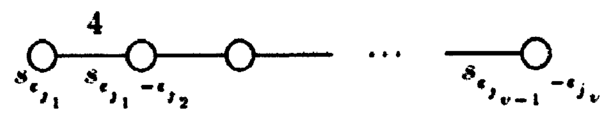
In order to further simplify the calculation of the stabilizer of γ_β, observe that if $V, V' \subset I$ then

$$I(V) \cap I(V') = I(V \setminus V') \cap I(V' \setminus V) \cap I(V \cap V').$$

Consequently, if we begin with the collection of sets $\left\{V_{s_p,k}\right\}$ and then take all set theoretic differences and intersections from this collection, we can find a collection of disjoint sets $\mathcal{U} = \{U\}$ so that

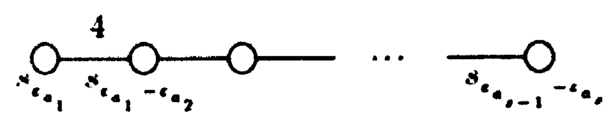
$$(4.5) \qquad I_\gamma = \bigcap_{U \in \mathcal{U}} I(U).$$

For any subset $U = \{j_1, \ldots, j_v\}$ of I let $W(B_U) \subset W(B_k)$ be the group of permutations and sign changes of the vectors $\varepsilon_{j_1}, \ldots, \varepsilon_{j_v}$. That is, $W(B_U)$ is the Weyl group characterized by the Coxeter diagram



Since the generators $s_{\varepsilon_{j_i} - \varepsilon_{j_{i+1}}}$ of $W(B_U)$ simply permute the elements of U and $s_{\varepsilon_{j_1}}$ fixes U, it follows that $W(B_U).U = U$.

Let $S = I \setminus U$. Suppose that $S = \{a_1, \ldots, a_s \mid a_1 < \cdots < a_s\}$. Define $W(B_s)$ to be the group of all permutations and sign changes of the vectors $\varepsilon_{a_1}, \ldots, \varepsilon_{a_s}$, so that $W(B_s)$ is the Weyl group characterized by the Coxeter diagram



Since $S \cap U = \emptyset$, $W(B_s).U = U$. It follows that

$$W(B_U) \times W(B_s) \subset I(U).$$

Suppose that $w \notin W(B_U) \times W(B_s)$. Then since $v + s = k$, w sends some $u \in U$ to S. This implies that $w \notin I(U)$. Thus

$$(4.6) \qquad I(U) = W(B_U) \times W(B_s).$$

Now if $U, U' \in \mathcal{U}$ then because $U \cap U' = \emptyset$,

$$(W(B_U) \times W(B_{I \setminus U})) \bigcap (W(B_{U'}) \times W(B_{I \setminus U'}))$$
$$= W(B_U) \times W(B_{U'}) \times W(B_{I \setminus (U \cup U')}).$$

Thus, if $\hat{S} = I \setminus (\bigcup_{U_i \in \mathcal{U}} U_i)$ then it follows from 4.5 and 4.6 that

$$(4.7) \qquad \mathcal{I}_\gamma = \prod_{U_i \in \mathcal{U}} \mathcal{W}(B_{U_i}) \times \mathcal{W}(B_{\hat{S}}).$$

Now let $s = |\hat{S}|$, and let δ be the partition of $l' = k - s$ such that $\delta_i = |U_i|$. Then

$$\begin{aligned} \mathcal{I}_\gamma &= \prod_{\delta_i \in \delta} \mathcal{W}(B_{\delta_i}) \times \mathcal{W}(B_s) \\ &= \mathcal{W}_\delta \times \mathcal{W}(B_s). \blacksquare \end{aligned}$$

For each γ, every $j = (j_1, \ldots, j_l) \in I^l$ lying in γ has the same number of distinct components. So we can speak of the number of distinct components of γ. Further analysis of the proof of Lemma 4.4 yields:

Lemma 4.8. *The integer l' is equal to the number of distinct components of γ. Moreover, if $l' = l$ then $\delta = \beta$.*

Proof: The proof of Lemma 4.4 shows that there exists a collection of disjoint subsets $\mathcal{U} = \{U_i\}$ of I such that $\mathcal{I}_\gamma = \prod_{U_i \in \mathcal{U}} \mathcal{W}(B_{U_i}) \times \mathcal{W}(B_{\hat{S}})$, (see 4.7). If $j = (j_1, \ldots, j_l) \in \gamma$ then by 4.5, each distinct component of j appears in one and only one of the U_i. Therefore the number of distinct components of γ is equal to $\sum_{U_i \in U} |U_i|$. The partition $\delta \vdash l'$ is defined by $\delta_i = |U_i|$. Therefore $l' = \sum \delta_i = \sum |U_i|$ which is equal to the number of distinct components of γ.

By definition $\gamma = (V_{\beta_1}, \ldots, V_{\beta_p})$, where each V_{β_i} is a set from I with multiplicities. If $l = l'$ then all the components of γ are distinct. Therefore, the elements of V_{β_i} must have multiplicity 1 and the V_{β_i} must be disjoint from each other. But this implies that the collection of sets $\{U_i\}$ must be identical to the collection of sets $\{V_{\beta_i}\}$. Since $|V_{\beta_i}| = \beta_i$ it follows that $\delta = \beta$. $\blacksquare$

The Representation $\phi_{M'}(Y_k)$

Let $\alpha = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_q\}$ be an ordered partition of h and let

$$\mathcal{W}(\mathcal{A}_\alpha) = \prod_{\alpha_i \in \alpha} \mathcal{W}(\mathcal{A}_{\alpha_i})$$

be the subgroup of $W(B_h)$ determined by α. It is the Weyl subgroup of $W(B_h)$ whose root system is the disjoint union $\coprod_{\alpha_i \in \alpha} A_{\alpha_i}$. Let

$$M' = Ind_{W(\mathcal{A}_\alpha)}^{W(B_h)} 1.$$

We now turn to the study of the $W(B_k)$-module structure of $\phi_{M'}(Y_k)$. Our treatment is analogous to that given above for $\phi_M(X_k)$. In particular we will work with the $W(\mathcal{A}_\alpha)$-orbits in $Y_k^{\otimes h}$.

As in the case of $X_k^{\otimes h}$, if $J = \{-k, \ldots, -1\} \coprod \{1, \ldots, k\}$ then

$$\mathcal{F}_h = \left\{ f_{x_1} \otimes \cdots \otimes f_{x_h} \mid (x_1, \ldots, x_h) \in J^h \right\}$$

is a basis of $Y_k^{\otimes h}$. Therefore, under the correspondence

$$f_{x_1} \otimes \cdots \otimes f_{x_h} \longleftrightarrow (x_1, \ldots, x_h),$$

the basis $\mathcal{F}_h$ can be identified with J^h. As before there is a $W(\mathcal{A}_\alpha)$-action on the basis J^h defined as follows: For $m = st \in W(\mathcal{A}_\alpha)$, $s \in S_h$, $t \in (\mathbb{Z}/2)^h$,

$$m.(x_1, \ldots, x_h) = (x_{s(1)}, \ldots, x_{s(h)}).$$

Write $J^h = J^{\alpha_1} \times \cdots \times J^{\alpha_q}$. Given any $x \in J^h$, write $x = (x^1, \ldots, x^q)$, where $x^i = (x^i_1, \ldots, x^i_{\alpha_q}) \in J^{\alpha_i}$.

LEMMA 4.9. *Let $x, \tilde{x} \in J^h$. Then x and $\tilde{x}$ are in the same orbit if and only if the components of x^i and $\tilde{x}^i$ are permutations of each other, for all $1 \leq i \leq q$.*

PROOF: As in Lemma 4.3, this follows because $W(\mathcal{A}_\alpha)$ permutes the components of each x^i ∎

So the effect of passing to $W(\mathcal{A}_\alpha)$-orbits is to ignore the order of the components within each tuple x^r. We need only know which elements of J occur in x^r and the number of times they do occur. Therefore each tuple x^r can be thought of as the set

$$V_{\alpha_r} = \left\{ -k^{m_{(-k)}}, \ldots, -1^{m_{(-1)}}, 1^{m_1}, \ldots, k^{m_k} \right\}.$$

where m_i is the multiplicity of $\pm i$ in x^r. Thus the $\mathcal{W}(\mathcal{A}_a)$ orbits of J^h can be represented by tuples $(V_{a_1}, \ldots, V_{a_q})$ where for each $1 \leq r \leq q$,

$$V_{a_r} = \left\{-k^{m_{(-k)}}, \ldots, -1^{m_{(-1)}}, 1^{m_1}, \ldots, k^{m_k}\right\}$$

and $m_{-k} + \cdots + m_k = a_r$.

The action of $\mathcal{W}(B_k)$ on J described in Theorem 4.2 induces an action on sets from J with multiplicities in the following manner: For $g \in \mathcal{W}(B_k)$,

$$g.\left\{-k^{m_{(-k)}}, \ldots, k^{m_k}\right\} = \left\{(g.-k)^{m_{(-k)}}, \ldots, (g.k)^{m_k}\right\}$$

The action can now be extended to the class of all tuples $\left\{\left(V_{a_1}, \ldots, V_{a_p}\right)\right\}$ by

$$g.\left(V_{a_1}, \ldots, V_{a_p}\right) = \left(g.V_{a_1}, \ldots, g.V_{a_p}\right).$$

Since the tuples $\left\{\left(V_{a_1}, \ldots, V_{a_p}\right)\right\}$ form a basis of $o_{M'}(Y_k)$ this defines the $\mathcal{W}(B_k)$-action on $o_{M'}(Y_k)$.

As in the case of $o_M(X_k)$ we calculate the isotropy groups, $\mathcal{I}_\gamma$, of each $\mathcal{W}(\mathcal{A}_a)$-orbit $\gamma = \left(V_{a_1}, \ldots, V_{a_q}\right)$.

By an argument analogous to that used to prove Lemma 4.4

$$\mathcal{I}_\gamma = \bigcap_{U \in \mathcal{U}} I(U) \tag{4.10}$$

where $\mathcal{U}$ is a collection of disjoint subsets of J. Each set U has a further decomposition. This decomposition of U is obtained by using the concept of parity.

DEFINITION 4.11. *Given a set $U \in \mathcal{U}$ define the parity for each $a \in U$ by*

$$\mathcal{P}(a) = \begin{cases} -1 & \text{if } -a \in U' \text{ for some } U' \in \mathcal{U} \\ 1 & \text{if } -a \notin U' \ \forall\, U' \in \mathcal{U} \end{cases}.$$

Given $U \in \mathcal{U}$ let

$$U^+ = \{a \in U \mid \mathcal{P}(a) = 1\}, \qquad U^- = \{a \in U \mid \mathcal{P}(a) = -1\}.$$

Then $U = U^+ \coprod U^-$ and any $g \in \bigcap_{U \in \mathcal{U}} I(U)$ must preserve this decomposition.

Suppose that $\mathcal{U} = \{U_1, \ldots, U_N\}$. Then for each $1 \leq r, s \leq N$, $r \neq s$, define

$$U^-_{r,s} = \{a \in U^-_r \mid -a \in U^-_s\}.$$

Observe that $U^-_{r,s} = -U^-_{s,r}$. That is, $a \in U^-_{r,s}$ if and only if $-a \in U^-_{s,r}$. Since $U^-_r = \coprod_{1 \leq s \leq N} U^-_{r,s}$ and any $g \in \bigcap_{U \in \mathcal{U}} I(U)$ must preserve this decomposition, it follows that

$$\mathcal{I}_{\mathcal{S}} = \bigcap_{U \in \mathcal{U}} I(U) = \bigcap I(\mathcal{M}), \tag{4.12}$$

where $\mathcal{M} = U^+$ or $\mathcal{M} = U^-_{r,s}$ for $r \leq s$. The restriction $r \leq s$ follows from the fact that the condition $U^-_{r,s} = -U^-_{s,r}$ forces $I(U^-_{r,s}) = I(U^-_{s,r})$.

In the proof of the main theorem it is possible to replace $\mathcal{I}_{\mathcal{S}}$ by $\mathcal{I}_{g,\mathcal{S}} = g\mathcal{I}_{\mathcal{S}}g^{-1}$. This motivates the following :

LEMMA 4.13. *There exists $g \in \mathcal{W}(B_k)$ such that*

$$\mathcal{I}_{g,\mathcal{S}} = \bigcap_{\mathcal{N} \in \mathcal{V}} I(\mathcal{N})$$

where $\mathcal{V} = \{\mathcal{N}\}$ is a collection of disjoint subsets of J which satisfy either

(1) $\mathcal{N} \subset \{1, \ldots, k\}$, *or*

(2) $\mathcal{N} = -\mathcal{N}$, *that is, $a \in \mathcal{N}$ if and only if $-a \in \mathcal{N}$.*

PROOF: Observe that sets of the form $U^-_{r,r}$ already satisfy condition (2). Now consider those $\mathcal{M}$ such that $\mathcal{M} = U^+$ or $\mathcal{M} = U^-_{r,s}$, $r < s$. In either case let $T = \{-a_1, \ldots, -a_t\}$ be the negative elements of $\mathcal{M}$, and let

$$g = s_{e_{a_1}} \cdots s_{e_{a_t}}. \tag{4.14}$$

By Theorem 4.2, $g : \{-a_1, \ldots, -a_t\} \leftrightarrow \{a_1, \ldots, a_t\}$. All other elements of J are left unaltered.

By the choice of $\mathcal{M}$ neither the set $\mathcal{M}$ nor any other $\mathcal{M}'$ can contain any element of $\{a_1, \ldots, a_t\}$. It follows that

$$g.\mathcal{M} = (\mathcal{M} \setminus \{-a_1, \ldots, -a_t\}) \coprod \{a_1, \ldots, a_t\}$$

$$g.\mathcal{M}' = \mathcal{M}', \qquad \mathcal{M}' \neq \mathcal{M}.$$

Now, given the entire collection $\{\mathcal{M}_1, \dots, \mathcal{M}_m\}$, let g_i be the element constructed in 4.14 which corresponds to $\mathcal{M}_i$. For each i, let $\mathcal{N}_i = g_i.\mathcal{M}_i$ and let $\mathcal{V} = \{\mathcal{N}_1, \dots, \mathcal{N}_m\}$. If $g = g_1 \cdots g_m$, then g_g^{-1} is an inner automorphism from $\mathcal{I}_\gamma$ to $\mathcal{I}_{g\gamma}$. The result now follows from 4.12. ∎

Remark 4.15. The effect of the element g determined in Lemma 4.13 is easy to describe. The sets $\mathcal{V} = \{\mathcal{N}\}$ are obtained from the sets $\left\{U^+\right\} \coprod \left\{U^-_{r,s} \mid r \leq s\right\}$ in a very simple fashion. For $\mathcal{M} = U^+$ or $\mathcal{M} = \left\{U^-_{r,s}\right\}$, $r < s$, we replace $\mathcal{M} = \{x_1, \dots, x_N\}$ by $\mathcal{N} = \{|x_1|, \dots, |x_N|\}$. For $\mathcal{M} = U^-_{r,r}$ simply let $\mathcal{N} = \mathcal{M}$.

Lemma 4.16. *Let $\gamma = \left(V_{\alpha_1}, \dots, V_{\alpha_q}\right)$, and let $\mathcal{I}_\gamma$ be the stabilizer of γ. Then there exists $g \in W(B_k)$ plus positive integers a, b, where $a + b = h' \leq h$ and ordered partitions $\rho \vdash a$, $\tau \vdash b$ such that*

$$g\mathcal{I}_\gamma g^{-1} = \mathcal{I}_{g\gamma} = W_\rho \times W_\tau \times W(B_s).$$

where $W_\rho = \prod W(A_{\rho_i})$, $W_\tau = \prod W(B_{\tau_i})$ and $s = k - h'$.

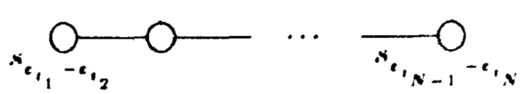
Proof: By 4.13 there exists $g \in W(B_k)$ such that

$$g\mathcal{I}_\gamma g^{-1} = \mathcal{I}_{g\gamma} = \bigcap_{\mathcal{N} \in \mathcal{V}} I(\mathcal{N})$$

where $\mathcal{N}$ satisfies (1) or (2) of 4.13. The first step is to determine each stabilizer $I(\mathcal{N})$.

<u>(I) Determination of $I(\mathcal{N})$, for $\mathcal{N} \subset \{1, \dots, k\}$.</u>

Let $W(A_{\mathcal{N}})$ be the group of permutations of the vectors $\{\varepsilon_i \mid i \in \mathcal{N}\}$. Then $W(A_{\mathcal{N}})$ is the Weyl group which is characterized by the Coxeter diagram

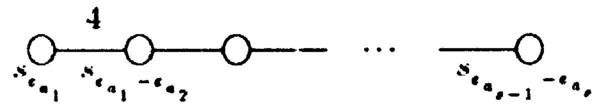


Since the generators of $W(A_{\mathcal{N}})$ permute the elements of $\mathcal{N}$,

$$W(A_{\mathcal{N}}).\mathcal{N} = \mathcal{N}.$$

Furthermore, observe that $s_{\varepsilon_i}.\mathcal{N} \neq \mathcal{N}$, for all $i \in \mathcal{N}$.

Let $S = \{1, \ldots, k\} \setminus \mathcal{N}$, and let $\mathcal{W}(B_s)$ be the group of permutations and sign changes on the vectors $\{\varepsilon_a \mid a \in S\}$. Then $\mathcal{W}(B_s)$ is the Weyl group characterized by the Coxeter diagram



where $s = |S|$. Since $S \cap \mathcal{N} = \emptyset$, it is clear that $\mathcal{W}(B_s).\mathcal{N} = \mathcal{N}$. It follows as in 4.6 that

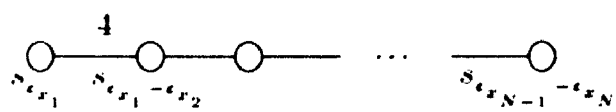
$$I(\mathcal{N}) = \mathcal{W}(A_{\mathcal{N}}) \times \mathcal{W}(B_s).$$

<u>(II) Determination of $I(\mathcal{N})$ for $\mathcal{N} = -\mathcal{N}$.</u>

Let

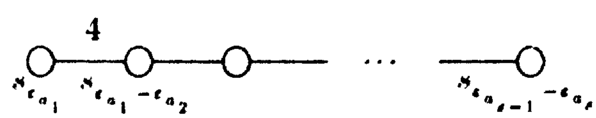
$$\tilde{\mathcal{N}} = \{a \in \mathcal{N} \mid 1 \leq a \leq k\}$$

Suppose that $\tilde{\mathcal{N}} = \{x_1, \ldots, x_N\}$. Let $\mathcal{W}\left(B_{\tilde{\mathcal{N}}}\right)$ be the Weyl group with Coxeter diagram



Then $s_{\varepsilon_{x_{i-1}}-\varepsilon_{x_i}}.\mathcal{N} = \mathcal{N}$, for all $x_i \in \tilde{\mathcal{N}}$. Furthermore since $\mathcal{P}(x_i) = -1$, $s_{\varepsilon_{x_i}}.\mathcal{N} = \mathcal{N}$, for all $x_i \in \tilde{\mathcal{N}}$. It follows that $\mathcal{W}(B_{\tilde{\mathcal{N}}}).\mathcal{N} = \mathcal{N}$.

Let $S = \{1, \ldots, k\} \setminus \tilde{\mathcal{N}}$ and as before let $\mathcal{W}(B_s)$ be the Weyl group characterized by the Coxeter diagram



Since $S \cap \mathcal{N} = \emptyset$, it follows that $\mathcal{W}(B_s).\mathcal{N} = \mathcal{N}$. Therefore, it follows as in 4.6 that

$$I(\mathcal{N}) = \mathcal{W}\left(B_{\tilde{\mathcal{N}}}\right) \times \mathcal{W}(B_s).$$

Apply step (I) to the collection of sets $\mathcal{V}' = \{\mathcal{N}_i \in \mathcal{V} \mid \mathcal{N}_i \neq -\mathcal{N}_i\}$, and apply step (II) to $\mathcal{V}'' = \mathcal{V} \setminus \mathcal{V}'$. Since $\mathcal{V}$ is a disjoint collection of sets, an argument analogous to that employed to derive 4.7 implies that

$$\mathcal{I}_\gamma = \prod_{\mathcal{N}_i \in \mathcal{V}'} \mathcal{W}(A_{\mathcal{N}_i}) \times \prod_{\mathcal{N}_i \in \mathcal{V}''} \mathcal{W}(B_{\tilde{\mathcal{N}}_i}) \times \mathcal{W}(B_{\hat{S}}),$$

where $\hat{S} = I \setminus \left(\coprod_{\mathcal{N}_i \in \mathcal{V}'} \mathcal{N}_i \cup \coprod_{\mathcal{N}_i \in \mathcal{V}''} \tilde{\mathcal{N}}_i \right)$. Let

$$a = \sum_{\mathcal{N}_i \in \mathcal{V}'} |\mathcal{N}_i|$$

and define a partition $\rho \vdash a$ by $\rho_i = |\mathcal{N}_i|$. Similarly, for the integer

$$b = \sum_{\mathcal{N}_i \in \mathcal{V}''} |\tilde{\mathcal{N}}_i|,$$

let $\tau \vdash b$ such that $\tau_i = |\tilde{\mathcal{N}}_i|$. Thus, if $s = |\hat{S}|$ then

$$\mathcal{I}_\gamma = \mathcal{W}_\rho \times \mathcal{W}_\tau \times \mathcal{W}(B_s). \quad \blacksquare$$

LEMMA 4.17. *In Lemma 4.16 the integer h' equals $c + d$ where c is the number of distinct components of γ of parity 1, and $2d$ is the number of distinct components of γ of parity -1. If $h' = h$ then $h' = a = c$ and $\rho = \alpha$.*

PROOF: By the definition of h' in 4.16

$$\begin{aligned} h' &= a + b \\ &= \sum_{\mathcal{N}_i \in \mathcal{V}'} |\mathcal{N}_i| + \sum_{\mathcal{N}_i \in \mathcal{V}''} |\tilde{\mathcal{N}}_i|. \end{aligned}$$

It follows from Remark 4.15 that

$$h' = \sum |U^+| + \sum_{r<s} |U^-_{r,s}| + \frac{1}{2} \sum |U^-_{r,r}|.$$

Moreover by definition of c and d

$$\begin{aligned} c &= \sum |U^+| \\ d &= \sum_{r<s} |U^-_{r,s}| + \frac{1}{2} \sum |U^-_{r,r}|. \end{aligned}$$

Thus $h' = c + d$.

Furthermore, from the definition of c and d, $c + 2d$ is equal to the number of distinct components of γ, and so $c + 2d \leq h$. Therefore if $h = h'$ then

$$c + d = h' = h \geq c + 2d.$$

This forces $d = 0$, and since $b \leq d$, $b = 0$. It follows that $h' = a = c$.

Now consider the sets V_{α_i} from J with multiplicities. If all the components of γ are distinct and have parity 1, then the elements of V_{α_i} must have multiplicity 1 and the V_{α_i} must be disjoint from each other. If $h = h'$ then, by Lemma 4.13, $g.X_i = V_{\alpha_i}$. Since $\alpha_i = |V_{\alpha_i}| = |X_i| = \rho_i$, it follows that $\rho = \alpha$. ∎

Main Theorem

We know turn to the proof of the main result. It concerns the subring $\langle X_k, Y_k \rangle_\phi$ of $\mathcal{R}(W(B_k))$, which is generated by the set obtained by applying λ-operations to the representations X_k and Y_k.

Theorem 4.18. *For integers l, h satisfying $l + h = n \leq k$, let β be a partition of l and α a partition of $h - 1$. Let $\mathcal{W}(\mathcal{B}_\beta)$, $\mathcal{W}(\mathcal{A}_\alpha)$ denote the following subgroups of $\mathcal{W}(B_k)$:*

$$\mathcal{W}(\mathcal{B}_\beta) = \prod \mathcal{W}(B_{\beta_q}) \qquad \beta_q \in \beta \vdash l$$
$$\mathcal{W}(\mathcal{A}_\alpha) = \prod \mathcal{W}(A_{\alpha_p}) \qquad \alpha_p \in \alpha \vdash h - 1.$$

and let $W_{\beta,\alpha} = \mathcal{W}(\mathcal{B}_\beta) \times \mathcal{W}(\mathcal{A}_\alpha)$. If $H = W_{\beta,\alpha} \times \mathcal{W}(B_r)$, for $r = k - n$, then

$$Ind_H^{\mathcal{W}(B_k)} 1 \in \langle X_k, Y_k \rangle_\phi.$$

Observe that when $n = k$, Theorem 4.18 yields:

Corollary 4.19. *Let (β, α) be a double partition of k. Then $Ind_{W_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \in \langle X_k, Y_k \rangle_\phi$.*

The remainder of the chapter will be devoted to the proof of Theorem 4.18. The proof is by induction on n. Alternatively, the proof is by descending induction on r.

(1) If $r = k$ then, $Ind_{W(B_k)}^{W(B_k)}1 = o_0(X_k) \in \langle X_k, Y_k\rangle_\phi$.

(2) If $r < r' < k$, then assume that $Ind_{W' \times W(B_{r'})}^{W(B_k)}1 \in \langle X_k, Y_k\rangle_\phi$, where $W' = W_{\beta'} \times W_{\alpha'}$ is a subgroup of $W(B_l) \times W(A_{h-1})$ and $W_{\beta'} \subset W(B_l)$, $W_{\alpha'} \subset W(A_{k-1})$ for some l, h such that $l + h = n'$, where $n' = k - r'$.

Let r be a fixed integer. Given l and h such that $l + h = n = k - r$, let $\beta \vdash l$ and $\alpha \vdash h - 1$. As before let

$$M = Ind_{W(B_\beta)}^{W(B_l)}1$$
$$M' = Ind_{W(A_\alpha)}^{W(B_k)}1.$$

The proof depends on the analysis of the $W(B_k)$-module $\phi_M(X_k) \bigotimes \phi_{M'}(Y_k)$. The results which were obtained for $\phi_M(X_k)$ and $\phi_{M'}(Y_k)$ will be used in this calculation.

The $W(B_k)$-module $\phi_M(X_k) \bigotimes \phi_{M'}(Y_k)$ has a basis which consists of the $W_{\beta,\alpha}$ orbits on $I^l \times J^h$. The $W_{\beta,\alpha}$-orbits of $I^l \times J^h$ can be described as the set

$$\mathcal{O}_{W_{\beta,\alpha}} = \left\{ \left(V_{\beta_1}, \ldots, V_{\beta_p}, V_{\alpha_1}, \ldots, V_{\alpha_q} \right) \right\},$$

where V_{β_i} and V_{α_i} are sets from I and J, respectively, with multiplicities. Let $\mathcal{J}$ be the set of $W(B_k)$-orbits on $\mathcal{O}_{W_{\beta,\alpha}}$. It follows from Lemma I.2.11 that

$$\phi_M(X_k) \bigotimes \phi_{M'}(Y_k) = \bigoplus_{\gamma \in \mathcal{J}} \mathrm{Ind}_{H_\gamma}^{W(B_k)}1$$

where $H_\gamma \subset W(B_k)$ is the isotropy group of some representative

$$\ell = \left(V_{\beta_1}, \ldots, V_{\beta_p}, V_{\alpha_1}, \ldots, V_{\alpha_q} \right)$$

of the $W(B_k)$-orbit γ. If ℓ and ℓ' are two different representatives of γ, then their isotropy groups are conjugate. Therefore, the $W(B_k)$-module $\mathrm{Ind}_{H_\gamma}^{W(B_k)}1$ is independent of the choice of representative ℓ.

In order to prove the theorem it suffices to show that there exists a unique $\gamma_0 \in \mathcal{J}$ such that

$$H_{\gamma_0} = W_{\beta,\alpha} \times W(B_r) \tag{4.20}$$

while for $\gamma \neq \gamma_0$

$$(4.21) \qquad H_\gamma = W_{\sigma,\rho} \times W(B_{r'})$$

where $r' > r$ and (σ, ρ) is a double partition of $n' = k - r'$. For, then $Ind_{H_\gamma}^{W(B_k)} 1 \in \langle X_k, Y_k \rangle_\phi$ for all $\gamma \neq \gamma_0$ by the induction hypothesis, and therefore

$$Ind_{H_{\gamma_0}}^{W(B_k)} 1 = \phi_M(X_k) \bigotimes \phi_{M'}(Y_k) - \sum_{\gamma \neq \gamma_0} Ind_{H_\gamma}^{W(B_k)} 1 \in \langle X_k, Y_k \rangle_\phi .$$

Let $\gamma \in \mathcal{J}$, and let $\left(V_{\beta_1}, \ldots, V_{\beta_p}, V_{\alpha_1}, \ldots, V_{\alpha_q}\right)$ be a representative of γ. Let $\mathcal{I}_{\gamma_\beta}$ be the isotropy group of $\gamma_\beta = \left(V_{\beta_1}, \ldots, V_{\beta_p}\right)$ and let $\mathcal{I}_{\gamma_\alpha}$ be the isotropy group of $\gamma_\alpha = \left(V_{\alpha_1}, \ldots, V_{\alpha_q}\right)$. If $w \in H_\gamma$, then w preserves γ_β and γ_α simultaneously. Therefore

$$H_\gamma \subset \mathcal{I}_{\gamma_\beta} \cap \mathcal{I}_{\gamma_\alpha} .$$

If $w \in \mathcal{I}_{\gamma_\beta} \cap \mathcal{I}_{\gamma_\alpha}$ then w preserves the $W_{\beta,\alpha}$-orbit γ. This implies that $w \in H_\gamma$. Therefore

$$(4.22) \qquad H_\gamma = \mathcal{I}_{\gamma_\beta} \cap \mathcal{I}_{\gamma_\alpha} .$$

To determine γ_0, let

$$(j, x) = (j_1, \ldots, j_l, x_1, \ldots, x_h) \in I^l \times J^h$$

be an $l + h$-tuple in which all the components are distinct and have parity equal to 1. Any two such tuples lie in the same $W(B_k)$-orbit. Since the components of (j, x) have parity 1, Theorem 4.2 permits (j, x) to be replaced by another tuple in the same $W(B_k)$-orbit which belongs to $I^{l+h} \subset I^l \times J^h$. That is, each component of (j, x) is in I. Since the symmetric group S_k is known to be k-fold transitive, it follows from Theorems 4.1 and 4.2 that the $l + h$-tuple $(1, \ldots, l, l + 1, \ldots, l + h)$ represents the $W(B_k)$-orbit γ_0. Now apply Lemma 4.8 and Lemma 4.17 to 4.22. It follows that

$$\begin{aligned} H_{\gamma_0} &= W(\mathcal{B}_\beta) \times W(\mathcal{A}_\alpha) \times W(B_r) \\ &= W_{\beta,\alpha} \times W(B_r), \end{aligned}$$

where $r = k - (l + h)$.

Define the integers a, b and the partitions $\rho \vdash a$, $\tau \vdash b$ as in Lemma 4.16. Apply Lemma 4.4 and Lemma 4.16 to 4.22. Then, for all $\gamma \neq \gamma_0$,

$$H_\gamma = W_\delta \times W_\rho \times W_\tau \times W(B_{r'}),$$

where $r' = k - (l' + a + b)$. Now let σ be the partition $(\delta, \tau) \vdash l' + a$, and let $W_\sigma = W_\delta \times W_\tau$ Then

$$\begin{aligned} H_\gamma &= W_\sigma \times W_\rho \times W(B_{r'}) \\ &= W_{\sigma,\rho} \times W(B_{r'}). \end{aligned}$$

Moreover, $l' \leq l$, $a+b \leq h$ and equality holds if and only if $\sigma = \beta$, $h = a$, and $\rho = \alpha$. Thus for all $\gamma \neq \gamma_0$, either $l' < l$ or $a + b < h$. Consequently, $r' > k - (l + h) = r$.

Conditions 4.20 and 4.21 have now been verified and therefore the proof is complete.

Section 5

A BASIS FOR $\mathbf{Q} \otimes R(W(B_k))$

Let $W(B_k)$ act on the vector space V as a group of reflections. Recall from section 1.2 that the elements of $W(B_k)$ act on the orthonormal basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ of V by permuting the basis vectors and changing the sign of an arbitrary subset of them. If we ignore the sign changes, each element $w \in W(B_k)$ determines a permutation of $1, 2, \ldots, k$ and this permutation can be expressed as a product of disjoint cycles.

DEFINITION 5.1. *Let $w \in W(B_k)$. As above w determines a permutation of $1, 2 \ldots, k$ and this permutation is the product $s_1 \cdots s_u$ of the disjoint cycles $s_1, \ldots, s_u$. Consider an arbitrary cycle $s_i = (k_1 k_2 \ldots k_{r_i})$. Define the length of the cycle to be the positive integer r_i. The sign of the cycle s_i is positive if $w^{r_i}(\varepsilon_{k_1}) = \varepsilon_{k_1}$ and negative if $w^{r_i}(\varepsilon_{k_1}) = -\varepsilon_{k_1}$. Now, the signed cycle type of w is defined to be the orbit of the u-tuple*

$$(\pm r_1, \ldots, \pm r_u).$$

under the permutation action of S_u.

LEMMA 5.2. *Two elements of $W(B_k)$ are conjugate if and only if they have the same signed cycle type.*

PROOF: Suppose that $w, w' \in W(B_k)$ are conjugate. Then there exists t such that $w' = twt^{-1}$. According to Definition 5.1 w determines a permutation which is the product $s_1 \cdots s_u$, of disjoint cycles $s_1, \ldots, s_u$. Conjugation by t preserves the lengths of the cycles s_i. Moreover because the sign change involutions cancel in pairs, if at all, the sign of each cycle is also preserved.

Conversely suppose that w, w' have the same signed cycle type. Let $s_1 \cdots s_u$ be the permutation associated to w and $s'_1 \cdots s'_u$ the permutation associated to w'.

Define $t' \in S_k$ to be the permutation which maps the entries of the cycles of the top line of the following display to the corresponding entries of the bottom line,

$$s'_1 \cdots s'_u = (l^1_1 \dots l^u_{r_1}) \dots (l^u_1 \dots l^u_{r_u})$$
$$s_1 \cdots s_u = (k^1_1 \dots k^1_{r_1}) \dots (k^u_1 \dots k^u_{r_u}).$$

That is, t' is the permutation:

$$\begin{pmatrix} l^1_1 \dots l^1_{r_1} \dots l^u_1 \dots l^u_{r_u} \\ k^1_1 \dots k^1_{r_1} \dots k^u_1 \dots k^u_{r_u} \end{pmatrix}.$$

Then $t'(s'_1 \cdots s'_u)t'^{-1} = s_1 \cdots s_u$. It is now evident that there exist sign change involutions whose product with t' gives an element $t \in W(B_k)$ such that $w = tw't^{-1}$. ∎

Given non-negative integers i, j such that $i + j = k$, let β be a partition of i and let α be a partition of j. The ordered pairs of partitions (β, α) are called double partitions. Define a total order on the set of all double partitions. First order all partitions of i, j lexicographically so that $(k_1, \dots, k_t) \succ (k'_1, \dots, k'_t)$ if there exists l such that $k_i = k'_i$ for all $i < l$ and $k_l > k'_l$. Then $(\beta', \alpha') \prec (\beta, \alpha)$ if $\beta' \prec \beta$ or if $\beta' = \beta$ and $\alpha' \prec \alpha$.

Let $W_{\beta,\alpha} = \prod W(B_\beta) \times \prod S_\alpha$ and let $\Theta_{\beta,\alpha}$ denote the standard Coxeter element of $W_{\beta,\alpha}$ as defined in Remark 3.18. Observe that the signed cycle type of $\Theta_{\beta,\alpha}$ is $(-\beta, \alpha)$. Hence the conjugacy classes of the Coxeter elements $\Theta_{\beta,\alpha}$ can be represented by the double partitions (β, α).

Lemma 5.3. *The elements $\Theta_{\beta,\alpha}$ defined above represent all conjugacy classes of $W(B_k)$.*

Proof: Since each conjugacy class can be represented by a double partition, as noted in the line above, the number of conjugacy classes is equal to the number of double partitions of k. By Lemma 5.2 each $\Theta_{\beta,\alpha}$ represents a distinct conjugacy class of $W(B_k)$. Since the number of such conjugacy classes is equal to

$$\# \text{ of double partitions of } k,$$

the $\Theta_{\beta,\alpha}$ represent all conjugacy classes of $W(B_k)$. ∎

Proposition 5.4. *Let*

$$\mathcal{S} = \left\{ Ind_{W_{\beta,\alpha}}^{W(B_k)} 1 \mid \beta \vdash i, \alpha \vdash j, i+j=k \right\}.$$

Then $\mathcal{S}$ is a linearly independent subset of $R(W(B_k))$.

Proof: For all (β,α) let $\chi_{\beta,\alpha}$ denote the character of $Ind_{W_{\beta,\alpha}}^{W(B_k)}1$. Suppose that

$$C = \sum_{(\beta,\alpha)\in\mathcal{D}} a^{\beta,\alpha}\chi_{\beta,\alpha} = 0,$$

where $\mathcal{D}$ is a subset of the set of all double partitions of k. Let (ρ,σ) be the maximal element in the total order $\mathcal{D}$.

By Lemma 5.3 each conjugacy class of $W(B_k)$ is represented by one of the elements $\Theta_{\rho,\sigma}$. Represent the conjugacy class of $\Theta_{\rho,\sigma}$ by (ρ,σ). Observe that each conjugacy class of $W(B_k)$ intersecting $W_{\beta,\alpha}$ non-trivially must be represented by a double partition $\preceq(\beta,\alpha)$. Thus, it follows that $\forall\, (\beta,\alpha) \prec (\rho,\sigma)$,

$$t\Theta_{\rho,\sigma}t^{-1} \notin W_{\beta,\alpha} = \prod W(B_\beta) \times \prod S_\alpha \qquad \forall\, t \in W(B_k).$$

Let $w_{\rho,\sigma} = |W_{\rho,\sigma}|$. Then

$$\chi_{\beta,\alpha}(\Theta_{\rho,\sigma}) = \frac{1}{w_{\rho,\sigma}} \sum_{t\Theta t^{-1}\in W_{\rho,\sigma}} 1(t\Theta t^{-1}) = 0 \qquad \forall\, (\beta,\alpha) \prec (\rho,\sigma).$$

It follows that the matrix $(\chi_{\beta,\alpha}(\Theta_{\gamma,\delta}))$ is upper triangular with non zero entries along the diagonal. Consequently, $(\chi_{\beta,\alpha}(\Theta_{\gamma,\delta}))$ is non-singular. Thus $\{\chi_{\beta,\alpha}\}$ is a linearly independent set. ∎

Corollary 5.5. *Let $\langle\mathcal{S}\rangle$ denote the free abelian group generated by $\mathcal{S}$ as defined in 5.5. Then*

$$Rank(\langle\mathcal{S}\rangle) = Rank\,(R(W(B_k))\,.$$

Proof: It follows from the proof of 5.3 that

$$\begin{aligned} Rank\,(R(W(B_k))) &= \sum_{i,j} \#(i).\#(j) \\ &= \#\text{ of double partitions of } k \\ &= |\mathcal{S}|. \end{aligned}$$ ∎

Corollary 5.6. *The set* $\{1 \otimes x \mid x \in \mathcal{S}\}$ *is a basis of* $\mathbf{Q} \bigotimes R(\mathcal{W}(B_k))$.

Proof: It is clear that $\{1 \otimes x \mid x \in \mathcal{S}\}$ is a linearly independent subset of $\mathbf{Q} \bigotimes R(\mathcal{W}(B_k))$. Since

$$dim\left(\mathbf{Q} \bigotimes R(\mathcal{W}(B_k))\right) = Rank\,(R(\mathcal{W}(B_k))) = |\mathcal{S}|,$$

$\{1 \otimes x \mid x \in \mathcal{S}\}$ is a basis. ∎

Section 6

The λ-STRUCTURE of $\mathbf{Q} \otimes R(\mathcal{W}(B_k))$

Proposition 6.1. *Consider the λ-subring, $\langle \mathcal{S} \rangle_\lambda$, of $R(\mathcal{W}(B_k))$ generated by*

$$\mathcal{S} = \left\{ Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \mid \mathcal{W}_{\beta,\alpha} = \prod \mathcal{W}(B_\beta) \times \prod S_\alpha \right\}.$$

Then,

$$\langle \mathcal{S} \rangle_\lambda = \langle \Gamma \rangle_\lambda,$$

where

$$\Gamma = \{ X_k = Ind_{\mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1)}^{\mathcal{W}(B_k)} 1, Y_k = Ind_{\mathcal{W}(B_{k-1})}^{\mathcal{W}(B_k)} 1 \}.$$

PROOF: Since $\Gamma \subset \mathcal{S}$, it is clear that $\langle \Gamma \rangle_\lambda \subset \langle \mathcal{S} \rangle_\lambda$. Corollary 4.19 implies that every element $Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \in \mathcal{S}$ can be written as an algebraic combination of λ-powers on Γ. By Theorem 3.7 $\langle \mathcal{S} \rangle_\phi = \langle \mathcal{S} \rangle_\lambda$. Thus $\langle \mathcal{S} \rangle_\lambda = \langle \mathcal{S} \rangle_\phi \subset \langle \Gamma \rangle_\phi = \langle \Gamma \rangle_\lambda$, and consequently $\langle \mathcal{S} \rangle_\lambda = \langle \Gamma \rangle_\lambda$. ∎

Lemma 6.2. $\mathbf{Q} \otimes R(\mathcal{W}(B_k))$ *is a λ-ring.*

PROOF: This is immediate from Example 1.3 and Definition 1.4 of Part I, because the λ-operations on $\mathbf{R}$ map $\mathbf{Q}$ to $\mathbf{Q}$. ∎

Theorem 6.3. $\mathbf{Q} \otimes R(\mathcal{W}(B_k))$ *is generated by two elements as a λ-ring over* $\mathbf{Q}$. *More precisely,*

$$\mathbf{Q} \otimes R(\mathcal{W}(B_k)) = \langle 1 \otimes X_k, 1 \otimes Y_k \rangle_\lambda.$$

PROOF: By Corollary II.5.6 the set

$$\{ 1 \otimes Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \mid \beta \vdash i, \alpha \vdash j, i + j = k \}$$

is a $\mathbf{Q}$-basis of $\mathbf{Q} \otimes R(\mathcal{W}(B_k))$. Furthermore, it follows from Proposition 6.1, that every element $Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \in \mathcal{S}$ is a polynomial in the λ-operations applied to the elements of

$$\Gamma = \left\{ X_k = Ind_{\mathcal{W}(B_{k-1}) \times \mathcal{W}(B_1)}^{\mathcal{W}(B_k)} 1, Y_k = Ind_{\mathcal{W}(B_{k-1})}^{\mathcal{W}(B_k)} 1 \right\}.$$

That is, there exists a polynomial $f_{\beta,\alpha}$ in countably many variables involving only finitely many of them, such that

$$Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 = f_{\beta,\alpha}\,(\lambda^1(X_k), \lambda^1(Y_k), \lambda^2(X_k), \lambda^2(Y_k), \dots)\,.$$

Thus for all $r \in \mathbf{Q} \otimes R(\mathcal{W}(B_k))$.

$$\begin{aligned} r &= \sum q_{\beta,\alpha} \otimes Ind_{\mathcal{W}_{\beta,\alpha}}^{\mathcal{W}(B_k)} 1 \\ &= \sum q_{\beta,\alpha} \otimes f_{\beta,\alpha}\,(\lambda^1(X_k), \lambda^1(Y_k), \lambda^2(X_k), \lambda^2(Y_k), \dots) \\ &= \sum q_{\beta,\alpha} f_{\beta,\alpha}\,(1 \otimes \lambda^1(X_k), 1 \otimes \lambda^1(Y_k), 1 \otimes \lambda^2(X_k), 1 \otimes \lambda^2(Y_k), \dots) \\ &= \sum q_{\beta,\alpha} f_{\beta,\alpha}\,(\lambda^1(1 \otimes X_k), \lambda^1(1 \otimes Y_k), \lambda^2(1 \otimes X_k), \lambda^2(1 \otimes Y_k), \dots)\,. \end{aligned}$$

This implies that $\mathbf{Q} \otimes R(\mathcal{W}(B_k)) = \langle 1 \otimes X_k, 1 \otimes Y_k \rangle_\lambda$. ∎

COROLLARY 6.4. *$\mathbf{Q} \otimes R(\mathcal{W}(D_k))$ is generated by two elements as a λ-ring over $\mathbf{Q}$.*

PROOF: By Proposition II.2.5 the restriction $Res : R(\mathcal{W}(B_k)) \longrightarrow R(\mathcal{W}(D_k))$ is a λ-ring epimorphism. It follows that $1 \otimes Res : \mathbf{Q} \otimes R(\mathcal{W}(B_k)) \longrightarrow \mathbf{Q} \otimes R(\mathcal{W}(D_k))$ is a λ-ring epimorphism. Thus $\mathbf{Q} \otimes R(\mathcal{W}(D_k))$ has two λ-generators. ∎

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