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Essays On Negotiation And Renegotiation

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by

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Abstract

This dissertation consists of two essays related to negotiation and renegotiation in game theory. They investigate the renegotiation-proof equilibria in finitely repeated games and subgame perfect equilibria in negotiation games, respectively.

The renegotiation-proof equilibria in finitely repeated games with many players are studied in the first essay. Renegotiation-proof equilibrium requires not only subgame perfectness but also subgame efficiency. The main result of Benoît and Krishna (1988) who studied the renegotiation-proof equilibria in two-player finitely repeated games does not apply to the games with more than two players. One sufficient condition for renegotiation-proof equilibria to be Pareto optimal in finitely repeated games with a sufficient long horizon is provided. An example shows that this sufficient condition cannot be weakened. The set of payoffs which can be approximated by renegotiation-proof equilibria in repeated games with a sufficiently long horizon is characterized such that it must be either Pareto optimal or dimensionally "small". We also show by way of an example that renegotiation-proof equilibria may lead to very different outcomes even in the games whose stage games have identical sets of feasible and individually rational payoffs as well as identical Nash equilibria.

In the standard bargaining game of Rubinstein (1982), the disagreement payoff is independent of players' past strategies. The model of negotiation proposed in the second essay merges ideas from bargaining theory and the repeated games literature. If no agreement has been reached in any period, players must play a stage game in normal form to determine that period payoffs. This model allows us to analyse the importance of strategic behaviour during periods without an agreement in the negotiations. The set of all perfect equilibria in the negotiation model is characterized. Quite generally, many feasible outcomes of the negotiation games can be sustained as subgame perfect equilibria. Particularly, many Pareto inefficient outcomes are sustainable even under the presence of perfect information and full rationality.

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Contents

Chapter 1

Renegotiation-Proof equilibria in Finitely Repeated Games

1.1 Introduction

The results of the folk theorem for repeated games have long been recognized. A repeated game consists of a sequence of identical stage games such that the stage game will be played countably many times by the same players. The folk theorem states that almost all feasible and individually rational outcomes of a stage game may arise as equilibrium outcomes in the corresponding repeated game. The reason seems obvious; since the players anticipate playing the stage game in the future, every deviation by a single player from a proposed equilibrium path will be followed by the worst possible equilibrium for the deviator for the rest of the game. Each of the other players is willing to enforce such a punishment equilibrium, since otherwise he will be punished. Thus no player in the game has any incentives to deviate individually. It is also possible to construct an equilibrium such that it induces an equilibrium as well in every subgame. Therefore, the proposed equilibrium satisfies the subgame perfection criterion given by Selten (1975). In infinitely repeated games, a number of folk theorems have been developed under different models, such as those in Friedman (1971), Rubinstein (1979), and Fudenberg and Maskin (1986).

These arguments are valid not only for infinitely repeated games but also for finitely repeated games in which the stage game will be played many but only a finite number of times. Since the publications of Friedman (1985) and Benoît and Krishna (1985), it has been recognized that subgame perfect equilibria in finitely repeated games need not be formed only by the repetitions of one-shot Nash equilibria. In fact, the concept of subgame perfect equilibrium does not refine that of Nash equilibrium for a wide class of finitely repeated games. In finitely repeated games, the perfect folk theorem given by Benoît and Krishna (1985) states that if the set of feasible and individually rational payoffs of the stage game is full dimensional and each player has multiple equilibrium payoffs in the stage game, then every feasible and individually rational payoff of the stage game can be approximated by a subgame perfect equilibrium in the finitely repeated game with a sufficiently long horizon.

The positive aspect of the perfect folk theorem is that many Pareto optimal outcomes may arise as subgame perfect equilibria in a repeated game, although they may not be the equilibria in the stage game. However, the perfect folk theorem is also criticized for several reasons. First, it demonstrates a lack of predictability of subgame perfect equilibria in repeated game models. Second, because the players can cooperate without any binding contracts in a repeated game, we would expect that the equilibrium outcomes should be more efficient than some of the equilibrium outcomes. If we assume that players can communicate, then no Pareto dominated perfect equilibria should be played. Also, Farrell (1983) argued that if there were two equilibria under a given theory, neither of them should be Pareto dominated by the other. Lastly, a subgame perfect equilibrium is vulnerable to the possibility of renegotiation, if it is supported by Pareto inefficient punishment equilibria[1]. Recently, Osborne (1990) applied a different approach to deal with a similar issue. He argue that many perfect equilibria in a finitely repeated game are not stable in the sense that they could be upset by convincing deviations. Therefore, many subgame perfect

[1]See the example in Benoît and Krishna (1988).

equilibria in finitely repeated games could be upset by either convincing deviations or the possibility of renegotiation.

Among these criticisms, the most serious one would seem to be that a Pareto dominated outcome can arise as the result of players following perfect equilibrium strategies. Recently, it has been suggested that if players could communicate in a game, such a Pareto inefficient outcome would not arise as an equilibrium outcome. The idea is that, the players called upon to play strategies leading to a Pareto dominated outcome would communicate their common interest in renegotiation to achieve a set of strategies that made all players better off. An equilibrium that is immune to such suggested revisions is called *renegotiation-proof equilibrium*. Such equilibria have been studied in infinitely repeated games by a number of people, including Farrell and Maskin (1987), Bernheim and Ray (1987), and van Damme (1989). In their papers, a perfect equilibrium is said to be renegotiation-proof if every induced perfect equilibrium in every subgame is not strictly Pareto dominated by any other induced perfect equilibrium. Certain difficulties arise in formulating the idea of renegotiation-proof equilibrium in infinitely repeated games. As Benoît and Krishna (1988) pointed out, "an acceptable notion of renegotiation-proof equilibria remains elusive" for infinitely repeated games. Recently, a different approach has been taken by Pearce (1987) and Abreu, Pearce and Stacchetti (1989). They define a perfect equilibrium to be renegotiation-proof if the lowest continuation payoff from the perfect equilibrium is not strictly less than that from any other perfect equilibrium. Bergin and Macleod (1987) studied renegotiation-proof equilibria in continuous time games.

However, a relatively straightforward notion of renegotiation-proof equilibria in finitely repeated games is available. As we argued, if the players can communicate but cannot sign binding contracts, then the equilibria in such a game model must be both subgame perfect and subgame Pareto efficient. In other words, the equilibrium should induce a Pareto efficient perfect equilibrium in every subgame. Here, we should clarify the notions of Pareto efficient and Pareto optimal outcomes. A Pareto efficient [optimal] outcome is an equilibrium [feasible] outcome that is not Pareto dominated by any other equilibrium [feasible] outcome. A Pareto efficient outcome is an equilibrium outcome, but a Pareto optimal outcome may not be. A set of outcomes is said to be Pareto optimal if every outcome in the set is Pareto optimal. In a stage game, renegotiation-proof equilibria are simply all Pareto efficient Nash equilibria. A formal definition of renegotiation-proof equilibrium was given by Benoît and Krishna (1988) for finitely repeated games with two players. They also proved that the set of payoffs which can be approximated by renegotiation-proof equilibria in a two-player repeated game is either singleton or weakly Pareto optimal. Thus, the concept of renegotiation-proof equilibrium leads to a sharp refinement of perfect equilibrium in two-player finitely repeated games.

This essay studies renegotiation-proof equilibria in finitely repeated games with many players. Section 1.2 sets up the repeated game model and extends the definition of renegotiation-proof equilibrium to games with more than two players. Section 1.3 investigates renegotiation-proof equilibria in repeated games. An example is provided to show that the main result in Benoît and Krishna (1988) does not hold when there are more than two players in the game. A sufficient condition for the renegotiation-proof equilibria to be Pareto optimal in finitely repeated games with a sufficiently long horizon is provided. The example also demonstrates that the sufficient condition cannot be weakened. Renegotiation-proof equilibrium outcomes are characterized as follows: if there exists an equilibrium outcome which is not Pareto optimal, then the set of payoffs which can be approximated by renegotiation-proof equilibria must be dimensionally "small". We also demonstrate by way of an example that renegotiation-proof equilibria may lead to different outcomes even in games whose stage games have identical feasible and individually rational outcomes as well as identical Nash equilibria. Conclusion is given in Section 1.4.

1.2 The Model and Definitions

An n-player one-shot (stage) game in normal form consists of a set of n players, their strategy sets and payoff functions,

$$G = \{(A_i, u_i(\cdot))_{i\in N}\}$$

where $N = \{1, 2, \ldots, n\}$ is the set of players, A_i is player i's *strategy* (*action*) *set*[2], and $u_i(\cdot) : A = \times_{j=1}^{n} A_j \rightarrow \mathbf{R}$ is his *payoff function* for $i \in N$. $\forall\, i \in N$, the strategy set A_i is assumed to be compact. The set A, which is compact by our assumptions on A_i for $i \in N$, can be interpreted as the set of outcomes of G. The payoff function is assumed to be continuous on A. Let $u(\cdot) : A \rightarrow \mathbf{R}^n$ be the function whose i-th component is $u_i(\cdot)$. The *feasible set* of G is the convex hull of $u(A)$, i.e. $Co[u(A)]$ which is both compact and convex in $\mathbf{R}^n$. These assumptions can be replaced by compactness of the set $u(A)$ without affecting the results in this essay. $\forall\, i \in N$, we decompose every generic element of A, $a \in A$, as $a = (a_i, a_{-i})$, where $a_i \in A_i$ and $a_{-i} \in \times_{j\neq i} A_j$. Let m_i be player i's *minimax payoff*,

$$m_i = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i})$$

and $m = (m_1, \ldots, m_n)$ be the *minimax vector of the game G*. m may not be feasible. $\forall\, x \in \mathbf{R}^n$, $SD(x)$ and $D(x)$ are the sets whose elements strictly and weakly dominate[3] x, respectively.

$$SD(x) = \{\, y \in \mathbf{R}^n \mid y \gg x \,\} \tag{1.1}$$

$$D(x) = \{\, y \in \mathbf{R}^n \mid y \geq x \,\} \tag{1.2}$$

[2]In this essay, we study pure strategy equilibria, It is difficult to judge deviations from a mixed strategy unless the players can observe not only the outcome of the game but also the randomization of the mixed strategy itself.

[3]$\forall\, x, y \in \mathbf{R}^n$, $y \gg x$ means that $y_i > x_i$ and $y \geq x$ means that $y_i \geq x_i$, for $i = 1, \ldots, n$.

Every vector in the set $D(m)$ is *individually rational* in the game G. The set of both feasible and individually rational payoffs is, therefore, the intersection of $Co[u(A)]$ and $D(m)$, $F = Co[u(A)] \cap D(m)$ which is both compact and convex in $\mathbf{R}^n$. Finally, it is assumed that G has at least one Nash equilibrium in pure strategies.

Let $G(T)$ be the *finitely repeated game* in which G is played successively T times, where T is a positive integer. An *outcome path* of $G(T)$ is defined as $\pi(T) = (a^1, \ldots, a^T) \in A^T$. The *total payoff* of the players from an outcome path is the sum of the payoffs from all T periods. It can be written as $U(\cdot) : A^T \rightarrow \mathbf{R}^n$, where

$$U(\pi(T)) = \sum_{t=1}^{T} u(a^t). \tag{1.3}$$

$\frac{1}{T}U(\pi(T))$ is players' average payoffs from the outcome path $\pi(T)$.

Player i's *strategy* in game $G(T)$ is a functions, f_i, which map from the set of all possible histories into the set of all possible actions,

$$f_i(\cdot) : \bigcup_{t=1}^{T} A^{t-1} \rightarrow A_i.$$

A^{t-1} may be referred to as the set of $(t-1)$-period histories in period t when G has been played $(t-1)$ times. $A^0 = \emptyset$ denotes the null history and $f_i(\emptyset) \in A_i$. Every *strategy combination* $f = (f_1, \ldots, f_n)$ induces a unique outcome path $\pi(f) = (a^1(f), \ldots, a^T(f)) \in A^T$; that is

$$\begin{aligned} a^1(f) &= (\ f_1(\emptyset), \ldots, f_n(\emptyset)\ ) \quad \text{and} \quad \forall\ 1 < t \leq T \\ a^t(f) &= (\ f_1(a^1(f), \ldots, a^{t-1}(f)), \ldots, f_n(a^1(f), \ldots, a^{t-1}(f))\ ) \end{aligned}$$

The payoff from a strategy combination f is determined by the outcome path induced by f through (1.3).

For a *t-period history* $h(t) = (a^1, \ldots, a^t) \in A^t$, $f|_{h(t)}$ is *the strategy combination induced by* f on the subgame $G(T-t)$ following history $h(t)$. $f_i|_{h(t)}(\cdot) = f_i(h(t), \cdot)$ for $t < T$ and $i \in N$. $U(\pi(f|_{h(i)}))$ is the *continuation payoff* prescribed by the strategy combination f on the subgame $G(T-t)$ following the history $h(t)$. A strategy combination f is a ***subgame perfect equilibrium*** if $f|_{h(t)}$ is a Nash equilibrium in the subgame $G(T-t)$ following the history $h(t)$ for $0 \leq t < T$ and for every possible t-period history $h(t)$. The perfect folk theorem in finitely repeated games given by Benoît and Krishna (1985) states that every payoff vector in the set F can be approximated by the average payoff vector from a subgame perfect equilibrium of the game $G(T)$ with a sufficiently large T, if F is full dimensional and each player has different equilibrium payoffs in the game G. In other words, if $P(T)$ is the set of total payoffs from subgame perfect equilibria of $G(T)$, then $P(T)/T$ converge to F in Hausdorff metric[4] as T goes to infinity.

In the repeated game $G(T)$, we assume that the players can communicate but cannot sign binding contracts in every period before they choose their actions for that period. As we argued, a renegotiation-proof equilibrium is a subgame perfect equilibrium which induces Pareto efficient equilibria in all subgames. Before giving the formal definition of renegotiation-proof equilibrium, we introduce the strongly and weakly Pareto efficient frontiers of a set in $\mathbf{R}^n$. For a set $S \subseteq \mathbf{R}^n$, the sets of *strongly and weakly Pareto efficient points* of the set S, $Eff(S)$ and $WEff(S)$, are defined as

$$\begin{aligned} Eff(S) &= \{ \, x \in S | \; \nexists \, y \in S \; s.t. \; y \geq x \text{ and } y \neq x \, \} \\ &= \{ \, x \in S | \; D(x) \cap S = \{x\} \, \} \\ WEff(S) &= \{ \, x \in S | \; \nexists \, y \in S \; s.t. \; y \gg x \, \} \\ &= \{ \, x \in S | \; SD(x) \cap S = \emptyset \, \}. \end{aligned}$$

It is obvious that these two operators retain some properties of set S, including closedness and boundedness. In other words, if S is compact, then both $Eff(S)$ and $WEff(S)$ are compact. But, if S is convex, then both $Eff(S)$ and $WEff(S)$ are connected. The definition of renegotiation-proof equilibrium is given as the following,

[4]See Hildenbrand (1974) for details.

Definition 1.1 *A perfect equilibrium f of $G(T)$ is said to be renegotiation-proof if $U(\pi(f)) \in R(T)$, where $R(1)$ is the set of all Pareto efficient one-shot Nash equilibrium payoffs, and for $T > 1$*

$$Q(T) = \{U(\pi(f)) \in P(T) \mid \forall\, h(1) \in A,\ U(\pi(f|_{h(1)})) \in R(T-1)\ \}$$
$$R(T) = Eff(Q(T)).$$

In the definition, the only additional requirement to that of perfect equilibrium is that all continuation equilibria are Pareto efficient. In fact, the list $(R(1), \ldots, R(T))$ is weakly renegotiation-proof under the definition of Farrell and Maskin (1987). No other weakly renegotiation-proof list in finitely repeated games is less controversial than $(R(1), \ldots, R(T))$. Definition 1.1 is the direct generalization of that given by Benoît and Krishna (1988). Renegotiation-proof equilibria are coalition-proof equilibria by Bernheim, Peleg and Whinston (1987) in the repeated games with only two but not more than two players. In a renegotiation-proof equilibrium in a game with more than two players, a sub-coalition of the players may improve their payoffs by changing their own strategies. However, there is no guarantee that coalition-proof equilibrium exists in games with more than two players. Certainly, the equilibrium concept given above is weaker than that of coalition-proof equilibrium in finitely repeated games with more than two players. Due to these considerations, we also assume that every player has veto power for any changes on future equilibrium strategy profile. This assumption implies that no proper sub-coalition of players can change the equilibrium outcome by changing their own strategies.

In this essay, we investigate the equilibria by studying the payoffs rather than the strategies. Since the equilibrium is defined in terms of payoffs, and the equilibrium strategies can be easily recovered from the equilibrium payoffs. Because $R(T)/T$ is a compact subset of F for every finite T, for each player $i \in N$, there is at least one renegotiation-proof equilibrium in $G(T)$ in which player i's payoff is less than or equal to that from any other renegotiation-proof equilibrium in the game $G(T)$. Such an equilibrium is called ***an optimal punishment equilibrium*** for player i in the game $G(T)$. Let $w_i(T)$ be player i's average payoff from his optimal punishment equilibria in game $G(T)$; $\forall\, i \in N$ and T,

$$w_i(T) = \min_{x \in R(T)/T} x_i \tag{1.4}$$

To conclude this section, we give the following proposition which can serve as an equivalent definition of renegotiation-proof equilibrium. In fact, we will use the following Proposition 1.1 more frequently than Definition 1.1 itself.

Proposition 1.1 *R(1)=Eff(P(1)) and for* $T > 1$*;*

$$\begin{aligned} Q(T) &= \{u(a) + x \mid x \in R(T-1), \ \textit{and} \ \ \forall\, i \in \ N, \\ &\qquad \max_{a_i' \in A_i} u_i(a_i', a_{-i}) + (T-1)\cdot w_i(T-1) \leq u_i(a) + x_i \ \} \\ R(T) &= Eff(Q(T)) \end{aligned}$$

Proof: From Definition 1.1, the subgame perfectness and efficiency. **Q.E.D.**

1.3 Renegotiation-Proof Equilibria

Some fundamental properties of renegotiation-proof equilibria are investigated in section 1.3.1. In section 1.3.2, an example is provided to show that Theorem 1 in Benoît and Krishna (1988) cannot be applied directly in the finitely repeated games with more than two players. Renegotiation-proof equilibrium outcomes are characterized in the finitely repeated games with many players in section 1.3.3. Section 1.3.4 demonstrates by an example that renegotiation-proof equilibria may lead to different outcomes even in games whose stage games have identical sets of feasible and individually rational outcomes as well as identical Nash equilibria. Most of the results are proved by using backward induction.

1.3.1 Properties

In the definition of renegotiation-proof equilibrium, we eliminate all Pareto inefficient perfect equilibria. Therefore, not all Pareto optimal outcomes can be approximated by renegotiation-proof equilibria, because Pareto inefficient punishments are no longer valid. The next proposition shows that renegotiation-proof equilibria have the following periodic property. Furthermore, if there is only one payoff vector can be supported as renegotiation-proof equilibrium in a repeated game with a fixed horizon, then it is the only payoff which can be approximated as renegotiation-proof equilibria.

Proposition 1.2 *If there exists a finite T_0 such that $R(T_0)/T_0$ is singleton, then $\forall\, t \geq 0$, $R(T_0 + t) = R(T_0) + R(t)$, and $R(T)/T$ converges to $R(T_0)/T_0$ in Hausdorff metric as T goes to infinity.*

Proof: We prove the first part by induction. Since $R(T_0)/T_0$ is a singleton set, we must have

$$R(T_0)/T_0 = \{w(T_0)\} = \{\ (w_1(T_0), w_2(T_0), \ldots, w_n(T_0))\ \}$$

From Proposition 1.1, $\forall\, i \in N$ and $T_0 \cdot w(T_0) + u(a) \in Q(T_0 + 1)$,

$$\begin{aligned} \max_{a_i' \in A_i} u_i(a_i', a_{-i}) + T_0 \cdot w_i(T_0) &\leq u_i(a) + T_0 \cdot w_i(T_0) \\ \Rightarrow \quad \max_{a_i' \in A_i} u_i(a_i', a_{-i}) &\leq u_i(a) \end{aligned}$$

$a \in A$ must be a Nash equilibrium. Therefore, $Q(T_0 + 1) \subseteq R(T_0) + P(1)$. Also from Proposition 1.1, $R(T_0) + P(1) \subseteq Q(T_0 + 1)$. Hence, $Q(T_0 + 1) = R(T_0) + P(1)$. Under $R(T_0)$ is singleton,

$$R(T_0 + 1) = Eff(Q(T_0 + 1)) = Eff(R(T_0) + P(1)) = R(T_0) + R(1)$$

Suppose we have, for $t \geq 0$,

$$R(T_0 + t) = R(T_0) + R(t) \tag{1.5}$$

Since $R(T_0)$ is singleton and (1.5), we have $(T_0+t)\cdot w_i(T_0+t) = T_0\cdot w_i(T_0) + t\cdot w_i(t)$. From Proposition 1.1, $u(a) + x + T_0 \cdot w(T_0) \in Q(T_0 + t + 1)$ if and only if

$$\begin{aligned} \max_{a_i' \in A_i} u_i(a_i', a_{-i}) + (T_0 + t)\cdot w_i(T_0 + t) &\leq u_i(a) + T_0 \cdot w_i(T_0) + x_i \\ \text{iff} \quad \max_{a_i' \in A_i} u_i(a_i', a_{-i}) + t \cdot w_i(t) &\leq u_i(a) + x_i \\ \text{iff} \quad u(a) + x &\in Q(t+1). \end{aligned}$$

$Q(T_0 + t + 1) = R(T_0) + Q(t+1)$. Therefore, (1.5) holds for $(t+1)$ since $R(T_0)$ is singleton. By induction, (1.5) holds for every positive t.

Every positive integer T can be decomposed as $T = K \cdot T_0 + t$, where $1 \leq t < T_0$. Applying equation (1.5) repeatedly K times, we have

$$\begin{aligned} R(T) &= K \cdot R(T_0) + R(t) \\ \frac{R(T)}{T} &= \frac{K\cdot R(T_0)}{K\cdot T_0 + t} + \frac{R(t)}{T} = \frac{R(T_0)}{T_0 + t/K} + \frac{R(t)}{T} \end{aligned} \tag{1.6}$$

Let $d(\cdot,\cdot) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ be the metric defined on $\mathbf{R}^n$ as; $\forall\, x,\ y \in \mathbf{R}^n$

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Let $\mathcal{C}$ be the space of all the closed subset of $\mathbf{R}^n$. $\forall\, S \subseteq \mathcal{C}$ and $c \in \mathbf{R}_+$, define

$$S + c = \bigcup_{x\in S} \{y \in \mathbf{R}^n \mid d(x,y) \leq c\} = \{y \in \mathbf{R}^n \mid \exists\, x \in S \text{ s.t. } d(x,y) \leq c\}$$

The Hausdorff metric $\delta(\cdot,\cdot)$ on $\mathcal{C}$ is defined as follows; $\forall\, S_1$ and $S_2 \in \mathcal{C}$,

$$\delta(S_1, S_2) = \inf\{\ c \mid S_1 \subseteq S_2 + c \ \text{ and } \ S_2 \subseteq S_1 + c\ \}$$

Since the set $\bigcup_{s=1}^{T_0} R(s)$ is bounded, $\exists\, c > 0$, such that $\forall\, t < T_0$

$$\begin{aligned} R(t) &\subseteq \bigcup_{s=1}^{T_0} R(s) \subseteq \{0\} + c \ \text{ and } \ \{0\} \subseteq R(t) + c \\ \frac{R(t)}{T} &\subseteq \{0\} + \frac{c}{T} \ \text{ and } \ \{0\} \subseteq \frac{R(t)}{T} + \frac{c}{T} \end{aligned}$$

$\delta(\frac{R(t)}{T}, \{0\}) \leq \frac{c}{T}$. Therefore,

$$\lim_{T\to\infty} \delta(\frac{R(t)}{T}, \{0\}) = 0, \quad \text{i.e. } \lim_{T\to\infty} \frac{R(t)}{T} = \{0\} \tag{1.7}$$

$$\text{also} \quad \lim_{T\to\infty} \frac{K \cdot R(T_0)}{T} = R(T_0) \lim_{K\to\infty} \frac{1}{T_0 + t/K} = \frac{R(T_0)}{T_0}. \tag{1.8}$$

From (1.6), (1.7) and (1.8),

$$\lim_{T\to\infty} \frac{R(T)}{T} = \lim_{T\to\infty} \frac{R(t)}{T} + \lim_{K\to\infty} \frac{R(T_0)}{T_0 + t/K} = \{0\} + \frac{R(T_0)}{T_0} = \frac{R(T_0)}{T_0}.$$

Q.E.D.

Note that, although $R(T_0)/T_0$ is singleton, the game $G(T_0)$ may still have many renegotiation-proof equilibrium strategies that result in the same payoffs. Under the condition, if G is going to be played for more than T_0 periods, then no player can be punished in the last T_0 periods in the sense that all the equilibria have identical payoffs. Players should treat the (T_0+1)-th last period in the same way as they treat the last period and follow the rules to determine the strategies in the earlier periods. The first part of the proposition follows. Furthermore, since all renegotiation-proof equilibria result in the same payoff for every T_0 periods, the only payoff which can be approximated by renegotiation-proof equilibria must be $R(T_0)/T_0$.

Proposition 1.2 implies that renegotiation-proof equilibria do not always result in Pareto optimal payoffs. For instance, $R(T_0)/T_0$ is singleton but not Pareto optimal. One special case is obtained if G has only one Pareto dominant equilibrium, for example in the repeated prisoners' dilemma game, $R(T)$ is a singleton set for every T. In fact, the only renegotiation-proof equilibrium in every finitely repeated game is the repetition of the Pareto dominant Nash equilibrium. Therefore, we may only be interested in the repeated games whose stage games have more than one Pareto dominant equilibria. A repeated game whose stage game has multiple Pareto dominant Nash equilibria may still have only one equilibrium payoff[5].

[5] Benoit and Krishna (1988) provided Example 2 in the paper such that $R(2)$ is singleton.

Suppose the sequence of sets $\{R(T)/T\}_{T=1}^{\infty}$ converges to a closed set R in Hausdorff metric as T goes to infinity. Every payoff vector in R can be approximated by an equilibrium in the repeated game with a sufficiently long horizon. We restate Theorem 1 of Benoît and Krishna (1988) as the following theorem.

Theorem 1.1 *For a two-player finitely repeated game, R is either a singleton set or a subset of WEff(F)*

One goal of this essay is to extend this result to games with more than two players. We first present some fundamental properties of sets $R(T)/T$ and R.

Proposition 1.3 *If both x and $y \in R(T)/T$, then neither $x \geq y$ nor $y \geq x$. If both x and $y \in R$, then neither $x \gg y$ nor $y \gg x$.*

Proof: The first part is from Definition 1.1. Now we prove the second part by contradiction. Suppose $\exists\ x, y \in R \subseteq \mathbf{R}^n$, such that $x \gg y$, then

$$\epsilon_0 = \min_{i \in N}\{(x_i - y_i)/3\} > 0.$$

By the definition of R, we have two sequences $\{x(T)\}_{T=1}^{\infty}$ and $\{y(T)\}_{T=1}^{\infty}$ such that $x(T),\ y(T) \in R(T)/T$ for every T, and

$$\lim_{T\to\infty} x(T) = x \quad \text{and} \quad \lim_{T\to\infty} y(T) = y$$

For such a $\epsilon_0 > 0$, $\exists\ T_0 > 0$ such that for $T \geq T_0$ and $i \in N$

$$d(x(T), x) < \epsilon_0 \quad \Rightarrow \quad |x_i(T) - x_i| < \epsilon_0$$

$$d(y(T), y) < \epsilon_0 \quad \Rightarrow \quad |y_i(T) - y_i| < \epsilon_0$$

It follows that $\forall\ i \in N$,

$$x_i(T) > x_i - \epsilon_0 = y_i + 3\epsilon_0 - 2\epsilon_0 > y_i(T)$$

Therefore, $\forall\ T \geq T_0$, we have $x(T) \gg y(T)$ which contradicts the first part of the proposition. **Q.E.D.**

Both $R(T)$ and R are, at most, $(m-1)$ dimensional if F is m dimensional ($m \leq n$). The sufficient condition for the existence of renegotiation-proof equilibrium in a finitely repeated game is that the stage game has pure strategy Nash equilibria. From the definition of renegotiation-proof equilibrium, $R(1) + R(T-1) \subseteq Q(T) \subseteq R(T-1) + u(A)$. Hence, every repetition of pure strategy Nash equilibria of the stage game is weakly Pareto dominated by some renegotiation-proof equilibria in the repeated game with a sufficiently long horizon.

Proposition 1.4 $\forall\, x \in Co[R(1)]$, $D(x) \cap R \neq \emptyset$.

Proof: $\forall\, x \in Co[R(1)]$, we have a sequence $\{x^t\}_{t=1}^{\infty} \subseteq R(1)$ such that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} x^t = x.$$

From Proposition 1.1, $\forall\, T$, $R(1) + R(T-1) \subseteq Q(T)$. Therefore, there exists a sequence $\{y^T\}_{T=1}^{\infty}$ such that $y^T \in R(T)/T$ and

$$y^T \geq \frac{1}{T} \sum_{t=1}^{T} x^t. \tag{1.9}$$

Since F is compact, the sequence $\{y^T\}_{T=1}^{\infty} \subseteq F$ has a convergent subsequence. Without loss of generality, we assume that $\lim_{T\to\infty} y^T = y \in R$. Also $y \in D(x)$, since $y \geq x$ from (1.9). Therefore, $D(x) \cap R$ is not empty. **Q.E.D.**

Proposition 1.4 is important under the absence of a complete theory on the renegotiation-proof equilibrium in finitely repeated games. By using Proposition 1.4, we can study the set R based on the stage game Proposition 1.4 also implies Theorem 2 of Benoît and Krishna (1988). Under the conditions of their Theorem 2, there are $x' \neq x''$, and both x' *and* $x'' \in R(1)$, such that $D(x') \cap D(x'') \cap F = \emptyset$. So R must not be singleton by Proposition 1.4. Therefore, $R \subseteq WEff(F)$ can be concluded from Theorem 1.1. Once we have Proposition 1.4, the next two corollaries follow immediately. Corollary 1.1 states that optimal repetitions of one-shot Nash equilibria are renegotiation-proof equilibria. The repetition of a Pareto optimal one-shot Nash equilibrium is the only possible stationary equilibrium in a finitely repeated game.

Corollary 1.1 *If $B \subseteq R(1)$ and $Co[B] \subseteq Eff(F)$, then $Co[B] \subseteq R$.*

Proof: $\forall\, x \in Co[B] \subseteq Co[R(1)]$, $D(x) \cap R \neq \emptyset$ by Proposition 1.4. $Co[B] \subseteq Eff(F)$ implies that $\emptyset \neq D(x) \cap R \subseteq D(x) \cap F = \{x\}$. Hence $x \in R$. **Q.E.D.**

Using Proposition 1.4, we can determine some payoffs which can be approximated by renegotiation-proof equilibria. Certainly, we can also determine the payoffs which cannot be the equilibrium payoffs. Propositions 1.3 and 1.4 imply that an outcome which is Pareto dominated by a one-shot Nash equilibrium cannot be approximated by a renegotiation-proof equilibrium. Hence not all one-shot Nash equilibrium outcomes are in the set R.

Corollary 1.2 *For a two-player game, if $R(1)$ is neither singleton nor Pareto optimal, then $Co[R(1)] \cap R = \emptyset$.*

Proof: Under $n = 2$, Theorem 1.1 states that R is either singleton or Pareto optimal. In the later case, the result follows since $R(1)$ is not Pareto optimal. If R is singleton, say $R = \{x\}$, x has to dominate every one-shot Nash equilibrium outcome by Proposition 1.4. Hence, $x \notin R(1)$, because $R(1)$ is not a singleton set. **Q.E.D.**

These propositions and corollaries give some predictions for renegotiation-proof outcomes. Theorem 1.1 captured a key characteristics of renegotiation-proof equilibria in two-player finitely repeated games. Unfortunately, renegotiation-proof equilibria in games with more than two players do not have such a sharp characteristics.

1.3.2 An Example with Three players

Example 1.1: Consider a three-player game in normal form. Player 1 has three strategies, player 2 has four and player 3 has two. The payoffs are given by the following two matrices where player 1 chooses the rows, player 2 the columns and player 3 the matrices.

$(1,1,0^*)$	(0,0,0)	(0,0,0)	(0,0,0)
(0,0,0)	(3,0,0)	(0,3,0)	(3,0,0)
(0,0,0)	(0,3,0)	(3,0,0)	(0,0,3)

$(0,0,0^*)$	(0,0,0)	$(0,0,0^*)$	(0,0,0)
(0,0,0)	$(\frac{1}{2},\frac{1}{2},1^*)$	(0,0,0)	(0,0,0)
(0,0,0)	(0,0,0)	(0,0,0)	(1,1,1)

$F = \{x \in \mathbf{R}^3_+ | \ x_1 + x_2 + x_3 \leq 3\}$ is the set of feasible and individually rational outcomes of this game. There are two Pareto dominant Nash equilibria in the game, $(1,1,0)$ and $(\frac{1}{2},\frac{1}{2},1)$. The set of F and Nash equilibria are illustrated in Figure 1.1. Because of the perfect folk theorem, every payoff vector in F can be approximated by the average payoff from a perfect equilibrium in the repeated game with a sufficiently long horizon. However, not all feasible and individually rational outcomes can be approximated by renegotiation-proof equilibria. Following the definition, we can prove that

$$
\begin{aligned}
R(1) &= \{\ (1,1,0),(\frac{1}{2},\frac{1}{2},1)\ \} \\
\frac{R(2)}{2} &= \{\ (1,1,0),(\frac{3}{4},\frac{3}{4},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},1)\ \} \\
\forall T &\geq 3 \\
\frac{R(T)}{T} &= \{\ (1,1,0),(1-\frac{1}{2T},1-\frac{1}{2T},\frac{1}{T}),(1-\frac{1}{T},1-\frac{1}{T},1)\ \}
\end{aligned}
$$

It follows that $R = \{(1,1,0),(1,1,1)\}$. Theorem 1.1 fails in this example, since the set R is neither singleton nor Pareto optimal. In the example, players 1 and 2 share a same optimal punishment equilibrium in every finitely repeated game. However, no Pareto optimal outcomes of the stage game can satisfy both players 1 and 2 at the same time except $(1,1,1)$. Therefore, renegotiation-proof equilibrium outcomes must consist of outcomes either $(1,1,1)$ or $(1,1,0)$ in all but a fixed number of periods. In other words, every payoff other than $(1,1,1)$ and $(1,1,0)$ may appear for only a fixed number of periods in a renegotiation-proof equilibrium. Hence, the payoff which can be approximated by a renegotiation-proof equilibrium must be either $(1,1,1)$ or

(1, 1, 0). If we replace 3 by 4 in the example, then no renegotiation-proof equilibria are Pareto optimal.

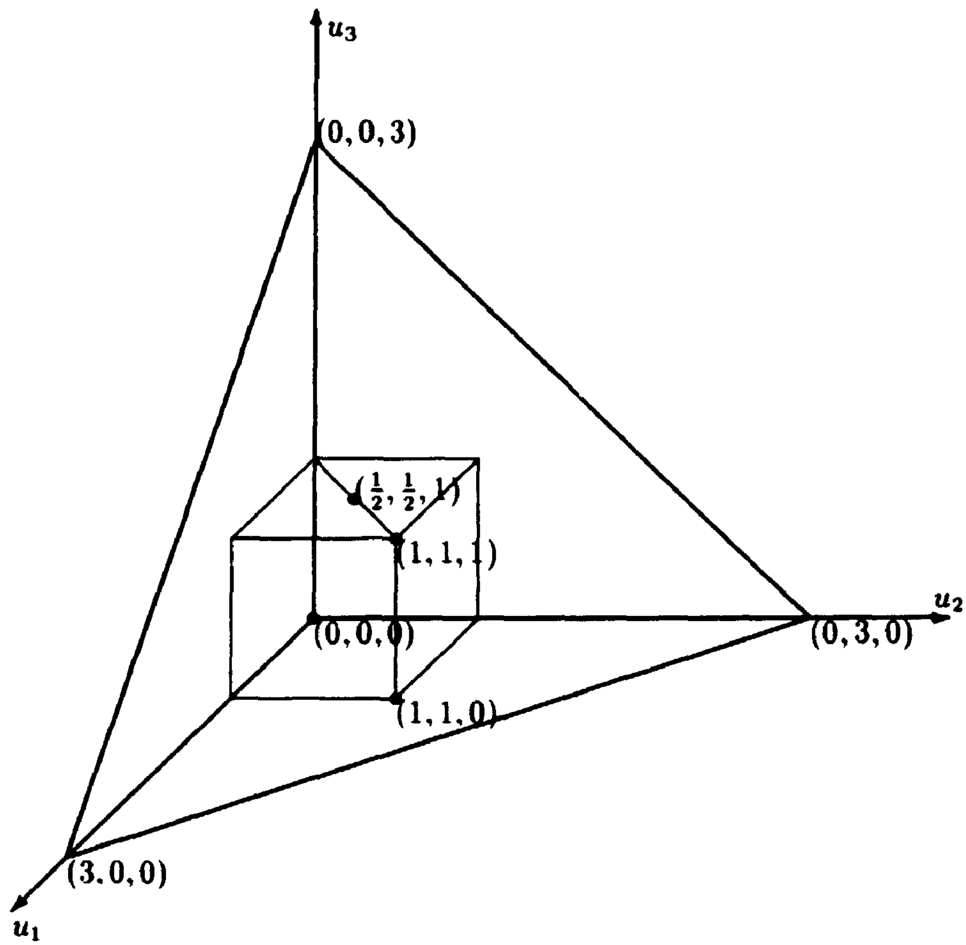

Figure 1.1: The Set of F and Nash Equilibria in Example 1.1

1.3.3 Optimality

Example 1.1 demonstrates that, without modifications, Theorem 1.1 cannot be generalized to repeated games with more than two players . To ensure the optimality of renegotiation-proof equilibria, we should not have the problem as we had in the example. Therefore, it should be required that each player can be punished individually.

Since R is a closed subset of F which is compact, R is compact. Therefore, $\forall\ i \in N$, $w_i = \min_{x \in R} x_i$ is well defined. In fact, w_i is the limit of $w_i(T)$ as T goes to infinity for $i \in N$. w denotes the point in $\mathbf{R}^n$ whose i-th coordinate is w_i. A sufficient condition for the set R to be Pareto optimal is $SD(w) \cap R \neq \emptyset$. Under this condition, each player can be punished based on the equilibrium whose payoff is in the set $SD(w) \cap R$.

Theorem 1.2 *If $R \cap SD(w) \neq \emptyset$, then $Eff(F) \cap SD(w) \subseteq R$.*

Proof: $\forall\, y \in Eff(F) \cap SD(w)$, we need to show that $y \in R$. Under the condition of $R \cap SD(w) \neq \emptyset$, there exists a $x \in R \cap SD(w)$. Because both x and y are in set $SD(w)$,

$$\epsilon_0 = \min_{i \in N}\{(x_i - w_i)/3, (y_i - w_i)/3\} > 0 \tag{1.10}$$

Since $y \in F$, there exists a sequence $\{a^t\}_{t=1}^{\infty} \subseteq A$ such that

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} u(a^t) = y$$

For such a $\epsilon_0 > 0$, $\exists\, T_1$ such that for $T \geq T_1$,

$$d(\frac{1}{T} \sum_{t=1}^{T} u(a^t), y) < \epsilon_0 \tag{1.11}$$

Since $x \in R$, there exists a sequence $\{x(T)\}_{T=1}^{\infty}$ such that $x(T) \in R(T)/T$ and $\lim_{T \to \infty} x(T) = x$. Also $\lim_{T \to \infty} w_i(T) = w_i$ for $i \in N$. Let

$$\bar{d} = \max_{i \in N}[\max_{a \in A} u_i(a_i', a_{-i}) - u_i(a)],$$

$\bar{d}$ is finite since A is compact and $u(\cdot)$ is continuous. For $\epsilon_0 > 0$, $\exists\, T_2 \geq \frac{T_1 \cdot \bar{d}}{\epsilon_0}$ such that for $T \geq T_2$

$$d(x(T), x) < \epsilon_0 \quad \text{and} \quad |w_i(T) - w_i| < \epsilon_0 \tag{1.12}$$

We are going to prove by induction that, $\forall\, T$, $\exists\, z(T_2 + T) \in R(T_2 + T)/(T_2 + T)$ such that

$$T_2 \cdot x(T_2) + \sum_{t=1}^{T} u(a^t) \leq (T_2 + T) \cdot z(T_2 + T) \tag{1.13}$$

Consider $T_2 \cdot x(T_2) + u(a^1)$. Since $T_2 \cdot x(T_2) \in R(T_2)$, and $\forall\, i \in N$, due to (1.12),

$$\begin{aligned} T_2 \cdot x_i(T_2) - T_2 \cdot w_i(T_2) &= T_2 \cdot (x_i(T_2) - w_i(T_2)) \\ &\geq T_2 \cdot \epsilon_0 \geq T_1 \cdot \bar{d} \geq \bar{d} \geq \max_{a_i' \in A_i} u_i(a_i', a_{-i}^1) - u_i(a^1) \end{aligned}$$

$T_2 \cdot x(T_2) + u(a^1) \in Q(T_2 + 1)$. Therefore, $\exists\, z(T_2 + 1) \in R(T_2 + 1)/(T_2 + 1)$ such that (1.13) holds for $T = 1$.

Now suppose that $\exists\, z(T_2 + T) \in R(T_2 + T)/(T_2 + T)$ such that (1.13) holds for T. Consider $(T_2 + T) \cdot z(T_2 + T) + u(a^{T+1})$. Since $z(T_2 + T) \in R(T_2 + T)/(T_2 + T)$, and $\forall\, i \in N$, if $T \leq T_1 - 1$,

$$\begin{aligned} (T_2 + T) \cdot z_i(T_2 + T) &- (T_2 + T) \cdot w_i(T_2 + T) \\ \text{by (1.13)} \quad &\geq T_2 \cdot x_i(T_2) + \sum_{t=1}^{T} u_i(a^t) - (T_2 + T) \cdot w_i(T_2 + T) \\ \text{by (1.12)} \quad &\geq T_2 \cdot (x_i - \epsilon_0) + \sum_{t=1}^{T} u_i(a^t) - (T_2 + T) \cdot (w_i + \epsilon_0) \\ &\geq T_2 \cdot (x_i - w_i - 2\epsilon_0) - T \cdot \bar{d} \geq T_2 \cdot \epsilon_0 - T \cdot \bar{d} \geq \bar{d} \\ &\geq \max_{a_i' \in A_i} u_i(a_i', a_{-i}^{T+1}) - u_i(a^{T+1}), \end{aligned}$$

and if $T \geq T_1$,

$$\begin{aligned} (T_2 + T) \cdot z_i(T_2 + T) &- (T_2 + T) \cdot w_i(T_2 + T) \\ \text{by (1.13)} \quad &\geq T_2 \cdot x_i(T_2) + T \cdot \Big(\frac{1}{T} \sum_{t=1}^{T} u_i(a^t)\Big) - (T_2 + T) \cdot w_i(T_2 + T) \\ \text{by (1.11) and (1.12)} \quad &\geq T_2 \cdot (x_i - \epsilon_0) + T \cdot (y_i - \epsilon_0) - (T_2 + T) \cdot (w_i + \epsilon_0) \\ &\geq T_2 \cdot (x_i - w_i - 2 \cdot \epsilon_0) + T \cdot (y_i - w_i - 2 \cdot \epsilon_0) \\ &\geq (T_2 + T) \cdot \epsilon_0 \geq \bar{d} \geq \max_{a_i' \in A_i} u_i(a_i', a_{-i}^{T+1}) - u_i(a^{T+1}) \end{aligned}$$

$(T_2 + T) \cdot z(T_2 + T) + u(a^{T+1}) \in Q(T_2 + T + 1)$. Therefore, (1.13) holds for $(T + 1)$. By induction. there is a sequence $\{z(T_2 + T)\}_{T=1}^{\infty}$ in which $z(T_2 + T) \in R(T_2 + T)/(T_2 + T)$ satisfies (1.13) for all $T \geq 0$.

Since F is compact in $\mathbf{R}^n$, $\{z(T_2 + T)\}_{T=1}^{\infty}$ has a convergent subsequence. Without loss of generality, we assume that $\lim_{T\to\infty} z(T_2 + T) = z$. Therefore, $z \in R \subseteq F$. From (1.13), $z \geq y$. But $y \in Eff(F)$, we must have $y = z \in R$. **Q.E.D.**

Under $R \cap SD(w) \neq \emptyset$, every Pareto optimal outcome which strictly dominates the vector w can be approximated by a renegotiation-proof equilibrium in finitely repeated game with a sufficiently long horizon. Proposition 1 in Benoît and Krishna (1988) holds under $R \cap SD(w) \neq \emptyset$, that is

Corollary 1.3 *If $WEff(F) = Eff(F)$ and $R \cap SD(w) \neq \emptyset$, then R is connected.*

Proof: In fact, under these two conditions, $R = Eff(F) \cap D(w) = Eff(F \cap D(w))$ by Theorem 1.2. Hence, R is connected, since $F \cap D(w)$ is convex. **Q.E.D.**

Theorem 1.3 *If $R \cap SD(w) \neq \emptyset$, $R \subseteq WEff(F)$*

Proof: $\forall\, x \in R$, we prove $x \in WEff(F)$ by contradiction. Suppose that $x \notin WEff(F)$, then $SD(x) \cap Eff(F) \neq \emptyset$. Therefore, $\exists\, y \in Eff(F)$ such that $y \gg x \geq w$, i.e. $y \in SD(w)$. Theorem 1.2 implies that $y \in R$. Hence, both x and $y \in R$. However, $y \gg x$ contradicts Proposition 1.3. **Q.E.D.**

Example 1.1 also implies that the condition $R \cap SD(w) \neq \emptyset$ cannot be relaxed in Theorems 1.2, 1.3 and Corollary 1.3. The difficulty with the results is that the condition of $R \cap SD(w) \neq \emptyset$ cannot be verified. Applying Propositions 1.3 and 1.4, however, we are able to develop a condition which can be easily applied.

Renegotiation-proof equilibria of a stage game are all Pareto efficient Nash equilibria. Given game G, $R(1) = Eff(P(1))$ can be easily found. We define $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n) \in \mathbf{R}^n$ in which

$$\bar{w}_i = \min_{x \in Co[R(1)]} \max_{y \in F \cap D(x)} y_i \qquad \text{for } i \in N \tag{1.14}$$

Proposition 1.5 *$\bar{w}$ is well defined, and $w \leq \bar{w}$.*

Proof: $\forall\, i \in N$, $\bar{w}_i$ is well defined, since $F \cap D(x)$ a compact-valued continuous correspondence, and the set $Co[R(1)]$ is compact. $\forall\, i \in N$, there exists a $x \in Co[R(1)]$ such that

$$\bar{w}_i = \max_{y \in F \cap D(x)} y_i \tag{1.15}$$

From Proposition 1.4, $R \cap D(x) \neq \emptyset$. Suppose that $z \in R \cap D(x)$. Therefore, $\forall\, i \in N$, $w_i \leq z_i \leq \bar{w}_i$. **Q.E.D.**

Proposition 1.5 proves that w is bounded above by $\bar{w}$. Furthermore, since every renegotiation-proof equilibrium is subgame perfect, w is also bounded below by the minimax vector of the stage game. Therefore, $m \leq w \leq \bar{w}$.

Theorem 1.4 *If $Co[R(1)] \cap SD(\bar{w}) \neq \emptyset$, then $R \cap SD(\bar{w}) \neq \emptyset$.*

Proof: Since $Co[R(1)] \cap SD(\bar{w}) \neq \emptyset$, there is a $x \in Co[R(1)]$ such that $x \gg \bar{w}$. With Proposition 1.5, $x \gg w$. Proposition 1.4 states that $D(x) \cap R \neq \emptyset$. Therefore, $R \cap SD(w) \neq \emptyset$, since $D(x) \cap R \subseteq SD(w) \cap R$. **Q.E.D.**

Corollary 1.4 *If $Co[R(1)] \cap SD(\bar{w}) \neq \emptyset$, then $Eff(F) \cap SD(\bar{w}) \subseteq R$ and $R \subseteq WEff(F)$.*

Proof: From Theorems 1.2, 1.3 and 1.4. **Q.E.D.**

In finitely repeated games with only two players, if R is not singleton, then at least one player who has different equilibrium payoffs can be punished. For instance, player 1 can be punished based on an equilibrium. One can construct a renegotiation-proof equilibrium which consists two phases. The second phase is the equilibrium such that player 1 can be punished. The first phase consists of a sequence of outcomes in which player 2 gets the highest payoff in every period. Neither players will deviate in the second phase, since it is a renegotiation-proof equilibrium. Player 2 has no incentive to deviate in the first phase. If the second phase is long enough, player 1 will not deviate either. These arguments support that these two phases form a renegotiation-proof equilibrium. When we select such an equilibrium carefully, the condition of Theorem 1.3 can be satisfied if R is not singleton. But in games with more than two players, a so-called optimal punishment may punish another player at the same time. Hence the same conclusion cannot follow. By following similar ideas, however, we find that if there exists an equilibrium in which at least $(n-1)$ players could be punished, then the result of Theorem 1.3 emerges.

Lemma 1.1 *Let F be a convex and compact subset of $\mathbf{R}^n$. $\forall\ v \in F$, define*

$$G(v) = \{\ x \in D(v) \cap F \mid \nexists y \in D(v) \cap F, y \gg x\}$$

If $v \notin WEff(F)$, then $\exists\ x^* \in G(v)$, $y \in F$, $\epsilon_0 > 0$ and an open interval $O \subseteq [0,1]$ such that $\forall\ x \in B_{\epsilon_0}(x^*)$ and $\alpha \in O$, $\alpha \cdot x_j + (1-\alpha) \cdot y_j > v_j + \epsilon_0$ for $j = 1, \ldots, n$.

Proof: Under the conditions, we can select that $x^* \in G(v)$, in which $x_i^* = v_i$ and $x_{-i}^* \gg v_{-i}$ for some $i \in N$, and $y \in \arg\max_{z \in F} z_i$, in which $y_i > v_i$. Let $\epsilon_1 = \min_{j \neq i}\{(x_j - v_j)/3\} > 0$. It follows that, $\forall\ x \in B_{\epsilon_1}(x^*)$, $x_j \geq v_j + 2\epsilon_1$ for $j \neq i$. Since

$$\lim_{\alpha \to 1}[\alpha(w_j + 2\epsilon_1) + (1-\alpha)y_j] = v_j + 2\epsilon_1$$

For such a $\epsilon_1 > 0$, $\exists\ \alpha_*$ such that for $\alpha \in (\alpha_*, 1]$, $\alpha(v_j + 2\epsilon) + (1-\alpha)y_j \geq v_j + \epsilon_1$. It turns out that, $\forall\ x \in B_\epsilon(x^*)$ in which $\epsilon < \epsilon_1$, for all $\alpha \in (\alpha_*, 1]$, we have

$$\alpha x_j + (1-\alpha)y_j \ \ \geq \ \ v_j + \epsilon$$

Let $\alpha^* = (1+\alpha_*)/2$, $\epsilon_0 = \min\{\epsilon_1, \frac{1-\alpha^*}{1+\alpha^*}(y_i - v_i)\}$ and $O = (\alpha_*, \alpha^*)$. One may verify that the lemma holds for this $\epsilon_0 > 0$ and the open set O. **Q.E.D.**

Theorem 1.5 *If there exists a $x^* \in R$ such that $x_{-i}^* \gg w_{-i}$ for some $i \in N$, then $R \subseteq WEff(F)$.*

Proof: We prove the theorem by contradiction. Suppose that the result does not hold. First, $w \notin WEff(F)$. For i, $\exists\ a \in A$ such that

$$u_i(a) = \max_{x \in F} x_i \quad \text{and} \quad u_i(a) > w_i$$

From Lemma 1.1, $\exists\ \epsilon_0 > 0$ and an open set O such that for $x \in B_{\epsilon_0}(x^*)$ and $\alpha \in O$

$$\alpha \cdot x_j + (1 - \alpha) \cdot u_j(a) > w_j + \epsilon_0, \quad \text{for } j = 1, ..., n \tag{1.16}$$

Second, $R \cap SD(w) = \emptyset$. It implies that $R \subseteq G(w)$. For such a ϵ_0, $\exists\ T_0$ such that for $T \geq T_0$,

$$\frac{R(T)}{T} \subseteq G(w) + \epsilon_0 \tag{1.17}$$

It is possible to select T_0 such that $T_0 \cdot \epsilon_0 \geq \bar{d}$. Since the open set O contains a fraction, say $\frac{T_1}{T_1+T_2} \in O$ where $T_1 \geq T_0$. Conditions imply that $\exists\ x(T_1) \in B_{\epsilon_0}(x^*) \cap R(T_1)/T_1$.

We are going to show, by induction for $t \leq T_2$, $\exists\ z(T_1 + t) \in R(T_1 + t)/(T_1 + t)$ such that

$$(T_1 + t) \cdot z(T_1 + t) \geq T_1 \cdot x(T_1) + t \cdot u(a) \tag{1.18}$$

For player $j \neq i$,

$$T_1 \cdot x_j(T_1) - T_1 \cdot w_j(T_1) \geq T_1 \cdot \epsilon_0 \geq \bar{d} \geq \max_{a'_j \in A_j} u_j(a'_j, a_{-j}) - u_j(a)$$

For player i,

$$T_1 \cdot x_i(T_1) - T_1 \cdot w_i(T_1) \geq 0 = \max_{a'_i \in A_i} u(a'_i, a_{-i}) - u_i(a).$$

Hence (1.18) holds for $t = 1$.

Now suppose that we have (1.18) for $t \leq T_2 - 1$. $\forall\ j \neq i$, if $x_j(T_1) \geq u_j(a)$, then $z_j(T_1 + t) \geq \frac{T_1}{T_1+T_2} x_j(T_1) + \frac{T_2}{T_1+T_2} u_j(a) \geq w_j + 2\epsilon_0$ and, if $x_j(T_1) < u_j(a)$, then $z_j(T_1 + t) > x_j(T_1) \geq w_j + 2\epsilon_0$. In either cases, $z_j(T_1 + t) - w_j(T_1 + t) \geq \epsilon^*$. It follows that, for player $j \neq i$

$$(T_1 + t) \cdot [z_j(T_1 + t) - w_j(T_1 + t)] \geq (T_1 + t) \cdot \epsilon_0 \geq \bar{d} \geq \max_{a'_j \in A_j} u_j(a'_j, a_{-j}) - u_j(a)$$

for player i,

$$(T_1 + t) \cdot [z_i(T_1 + t) - w_i(T_1 + t)] \geq 0 = \max_{a_i' \in A_i} u_i(a_i', a_{-i}) - u_i(a)$$

At the end of induction, together with (1.16), we get, $\forall\, i \in N$

$$z_i(T_1 + T_2) \geq \frac{T_1}{T_1 + T_2} \cdot x_i(T_1) + \frac{T_2}{T_1 + T_2} u_i(a) > w_i + \epsilon_0$$

which contradicts (1.17). **Q.E.D.**

Theorem 1.5 and $n = 2$ imply Theorem 1 in Benoît and Krishna (1988). Renegotiation-proof equilibrium outcomes are characterized by the following theorem. Theorem 1.6 describes that the outcomes which can be approximated by renegotiation-proof equilibria either are Pareto optimal or form an object whose dimension is less than $(n - 1)$. In other words, if there exists an equilibrium which is not Pareto optimal, then the set of payoffs which can be so approximated must be relatively small.

Theorem 1.6 *For n-player repeated games, either* $R \subseteq WEff(F)$ *or* $dim R < (n-1)$.

Proof: If R is $(n - 1)$ dimensional, then either Theorem 1.3 or 1.5 holds. Therefore, $R \subseteq WEff(F)$. **Q.E.D.**

1.3.4 Determination

Subgame perfect equilibria lead to the same outcomes in the repeated games whose stage games have identical sets of feasible and individually rational outcomes, as long as every player has multiple equilibrium payoffs in the stage games. We do not have the same conclusion for renegotiation-proof equilibria. Furthermore, even if the stage games have identical Nash equilibria, the payoffs which can be approximated by renegotiation-proof equilibria could be quite different in the repeated games as time horizon tends to infinity. Example 1.2 demonstrates such a characteristics for renegotiation-proof equilibrium.

Example 1.2: Consider a two-player symmetric game G_ϵ with parameter $\epsilon \in [0,6]$. Every player in the game G_ϵ has six pure strategies, $A_1 = \{\alpha_1, \ldots, \alpha_6\}$ and $A_2 = \{\beta_1, \ldots, \beta_6\}$. The payoff functions are given by the following payoff matrix.

1\2	β_1	β_2	β_3	β_4	β_5	β_6
α_1	(0,0)	(0,0)	$(4,2^*)$	(0,0)	(0,0)	(0,0)
α_2	(0,0)	$(3,3^*)$	(0,0)	(0,0)	$(\epsilon,0)$	(0,0)
α_3	$(2,4^*)$	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
α_4	(0,0)	(0,0)	(0,0)	(0,0)	(0,6)	(0,0)
α_5	(0,0)	$(0,\epsilon)$	(0,0)	(6,0)	(0,0)	(0,0)
α_6	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(0,0^*)$

Let the sets with subscript ϵ be the sets corresponding to the game G_ϵ. For $\epsilon \in [1,6]$, G_ϵ has identical set of feasible and individually rational outcomes, $F = \{(x_1, x_2) \in \mathbf{R}^2_+ | x_1 + x_2 \leq 6\}$. $\forall\, \epsilon \in [0,6]$, G_ϵ has a Pareto dominated Nash equilibrium (α_6, β_6) which leads to the minimax vector $m = (0,0)$. R_ϵ is Pareto optimal and connected by Corollaries 1.3 and 1.4 for $\epsilon \in [0,6]$. Also, $\bar{w}_\epsilon \leq (2,2)$ for game G_ϵ. Therefore, $Co[\{(4,2),(2,4)\}] \subseteq R_\epsilon$ for all $\epsilon \in [0,6]$. Here, we are going to consider three cases, $\epsilon = 0, 1$ and $\epsilon \in [2,6]$.

First, for $\epsilon = 0$, game G_0 has six Nash equilibria. We have that $\bar{w} = m = (0,0)$. Therefore, $w = (0,0)$ due to $m \leq w \leq \bar{w}$.

Proposition 1.6 *For G_0, $w = (0,0)$ and $R_0 = Co[\{(6,0),(0,6)\}]$.*

Proof: From $w = (0,0)$ and Theorem 1.2. **Q.E.D.**

For $\epsilon \in (0,6]$, game G_ϵ has four Nash equilibria, and $\bar{w} = (2,2)$. Therefore, we have to find the optimal punishments in order to determine the set R_ϵ for $\epsilon \in (0,6]$. Second, for G_1, we have

Proposition 1.7 *For G_1, $w = (1,1)$ and $R_1 = Co[\{(5,1),(1,5)\}]$.*

Proof: Since the game is symmetric, we only consider for player 1. We are going to show that total payoff from the optimal punishment for player 1 in game $G_1(T)$

is $(T+1)$. For G_1, $R_1(1) = \{(2,4),(3,3),(4,2)\}$. $(2,4)$ is the optimal punishment equilibrium outcome for player 1. We assume that $\{(T,5T-6),(T+1,5T-7)\} \subseteq R_1(T-1)$. $(T,5T-6)$ is the optimal punishment outcome for player 1 in $G_1(T-1)$. Following the definition, paths $(0,6)+(T+1,5T-7)$ and $(2,4)+(T,5T-6)$ form two renegotiation-proof equilibria in game $G_1(T)$. Also, path $(0,6)+(T,5T-6)$ cannot be a renegotiation-proof equilibrium in game $G_1(T)$. Hence, the optimal punishment for player 1 leads to payoff $(T+1)$ for player 1 in game $G_1(T)$. By induction, $w_1(T) = \frac{T+1}{T}$. Therefore, $w = (1,1)$. Theorem 1.3 implies that $R_1 = Co[\{(5,1),(1,5)\}]$. **Q.E.D.**

Last, for G_ϵ with $\epsilon \in [2,6]$, player 1 can gain $\epsilon \geq 2$ by deviating from $(0,6)$. Therefore, player 1 can guarantee himself 2 as average payoff in every renegotiation-proof equilibrium in repeated game. As we argue, player 1's payoff from the optimal punishment is less than 2. Therefore, the optimal punishment for player 1 must lead to him average payoff 2.

Proposition 1.8 *For G_ϵ with $\epsilon \in [2,6]$, $w = (2,2)$ and $R_\epsilon = Co[\{(4,2),(2,4)\}]$.*

Proof: Again, we consider the optimal punishment for player 1 in the repeated game. In the stage game G_ϵ with $\epsilon \in [2,6]$, $w_1(1) = 2$. Now suppose that $(T-1)(2,4)$ is the optimal punishment outcome for player 1 in game $G_\epsilon(T-1)$. Since an outcome in $R(T)$ has the form $x + u(a)$ where $x \in R(T-1)$. The only possible outcome which may lead to player 1's payoff less than 2 is (α_4, β_5) with payoff (0,6). For $x + u(a) \in R(T+1)$, the following inequality must be satisfied.

$$\begin{aligned}
\max_{\alpha \in A_1} u_1(\alpha, \beta_5) &+ (T-1) \cdot w_1(T-1) \leq u_1(\alpha_4, \beta_5) + x_1 \\
\Rightarrow x_1 &\geq \max_{\alpha \in A_1} u_1(\alpha, \beta_5) - u_1(a) + (T-1) \cdot w_1(T-1) \\
&= \epsilon + 2 \cdot (T-1) \geq 2 \cdot T
\end{aligned}$$

By induction, $w_1(T) = 2$ for $T \geq 1$. Therefore, the results follow. **Q.E.D.**

We have seen that $R_2 \subset R_1 \subset R_0$. Although stage games G_1 and G_2 have identical sets of feasible and individually rational payoffs as well as identical sets of Nash equilibria, the outcomes which may approximated by renegotiation-proof equilibria in the repeated games are not the same. Thus, whether an outcome can be approximated by a renegotiation-proof equilibrium depends on something more than its feasibility and rationality. One interesting feature of Example 1.2 is that if we eliminate the dominated strategy for every player in the game, α_6 and β_6, then $R_\epsilon = Eff(F_\epsilon)$ for $\epsilon = 0, 1$ and $\epsilon \in [2, 6]$.

For perfect equilibria, subadditivity of the optimal punishments ensures the convergence of the punishment payoffs. However, the optimal punishments in renegotiation-proof equilibria do not have such a nice property. The sum of two subgame perfect equilibria forms a new subgame perfect equilibrium in the repeated game with the sum of the periods. However, the sum of two renegotiation-proof equilibria may not be renegotiation-proof.

Since we are always looking for Pareto efficient payoffs, one might expect that renegotiation-proof equilibria have a weak monotonicity, i.e. if $\exists\ x \in R(T)/T$ and $y \in R(t)/t$, then there is $z \in R(T+t)/(T+t)$ such that

$$(T+t) \cdot z \geq T \cdot x + t \cdot y \tag{1.19}$$

Suppose we have two renegotiation-proof equilibria with average payoffs x and y, for T and t periods repeated game respectively. When the game G is going to be played $(T+t)$ times, x could be the average payoff in the last T periods, since $T \cdot x$ is an equilibrium outcome in the game $G(T)$. y can be the average payoff when the game G is played t times without a punishment in the last period. With a punishment threat in t, the equilibrium with average payoff y can be played in the first t periods, if it is still Pareto efficient. This weak monotonicity seems to be true without any additional restriction. It certainly holds when either T or t is equal to 1. However, this weak monotonicity of renegotiation-proof equilibria holds only under (1.20).

Proposition 1.9 *Suppose $x \in R(T)/T$ for some fixed T. If x satisfies*

$$(T + t) \cdot w_i(T + t) \leq T \cdot x_i + t \cdot w_i(t) \;\; \forall\, i \in N \;\; \text{and } t > 0. \tag{1.20}$$

and $\{a^t\}_{t=1}^{\infty}$ is a renegotiation-proof equilibrium path, i.e. $\forall\, t$

$$\sum_{s=1}^{t} u(a^s) \in R(t),$$

then $\forall\, t$, $\exists\, z(T+t) \in R(T+t)/(T+t)$ such that

$$(T + t) \cdot z(T + t) \geq T \cdot x + \sum_{s=1}^{t} u(a^s) \tag{1.21}$$

Proof: It is easy to see that such a $z(T + 1)$ exists. Now suppose that (1.21) holds up to t. $\forall\, i \in N$,

$$\begin{aligned}
\max_{a_i' \in A_i} u_i(a_i', a_{-i}^{t+1}) - u_i(a^{t+1}) &\leq \sum_{s=1}^{t} u_i(a^s) - t \cdot w_i(t) \\
\text{by (1.20)} \qquad &\leq \sum_{s=1}^{t} u_i(a^s) + T \cdot x_i - (T + t) \cdot w_i(T + t) \\
\text{by (1.21)} \qquad &\leq (T + t) \cdot z_i(T + t) - (T + t) \cdot w_i(T + t),
\end{aligned}$$

It implies that there is a $(T + t) \cdot z(T + t) + u(a^{t+1}) \in Q(T + t + 1)$ such that (1.21) holds for $(t + 1)$. By induction, (1.21) holds for every positive t. **Q.E.D.**

1.4 Conclusion

This essay studies renegotiation-proof equilibria in finitely repeated games. An equilibrium in finitely repeated games in which the players can communicate but cannot sign any binding contracts should be both subgame perfect and subgame efficient. The concept of renegotiation-proof equilibrium provides a sharp refinement for that of perfect equilibrium in finitely repeated games. Indeed, in generalizing Benoît and Krishna (1988), we found that either all renegotiation-proof equilibria are Pareto optimal, or a good deal of predictability is available, namely, the set of average equilibrium payoffs is of lower dimension than the Pareto optimal frontier. Optimal punishment equilibria are the key to understand renegotiation-proof equilibria. Some intuitively plausible conjectures fail to be correct, and this leads to stand in the way of smoother progress. Nevertheless, we have proved as well a number of sufficient conditions for renegotiation-proof equilibria to be Pareto optimal. Unlike subgame perfect equilibria, an example demonstrates that renegotiation-proof equilibria may lead to different outcomes even in games whose stage games have identical sets of feasible and individually rational outcomes as well as Nash equilibria.

Chapter 2

Perfect Equilibria in Negotiation Games

2.1 Introduction

What features characterize negotiations? Focusing on their outcome, one answer is that they take time and may never succeed. Short reflection on the experience with economic negotiation problems, such as takeover or contract negotiations, confirms this point. Concentrating on the situation faced by the negotiators, features that are directly related to the question of how to model negotiations come to mind. Typically, two parties who interact strategically are trying to agree to terms governing their future behaviour. This involves a change in their relationship, in that any such agreement must either cover a new relationship or must modify the current relationship by at least restricting strategic behaviour — otherwise there would not be an incentive to engage negotiations. Finally, the parties are expected to continue their original relationship until an agreement is reached (note that this covers the case where one of the parties may refuse to participate in any activities until agreement is obtained).

A simple way to capture these realities of negotiations is via a model that has two parties trying to split a surplus (the returns from agreement) while playing a repeated game (the current relationship.) Such a model will be presented in this essay. First, however, consider why *both* these elements have to be modeled explicitly.

The issue of 'surplus splitting' in a non-cooperative setting has been addressed very successfully by the bargaining model of Rubinstein (1982). Since a large part of the interest in negotiations stems from the question of how the available surplus will be allocated, we will first investigate why the bargaining model may fail to address negotiations.

The bargaining model makes one key assumption: The two parties involved in bargaining have no strategic actions other than offers, counter-offers, and rejection. This implies that there are no strategic payoffs to the parties from their relationship other than a share of the surplus. Thus, the *status quo* point in bargaining is stationary and given exogenously, and only the (additional) payoff from agreement matters to the parties. While this is a reasonable approximation for many situations, it contradicts a characteristic feature of negotiations: In negotiations the parties are already in a strategic relationship. They have actions available which will affect the payoffs they receive concurrently with their efforts to reach an agreement. The *status quo* point (and the cost of delay) is therefore a function of the actions taken during the negotiations and is thus endogenous. This would not be a problem if the solution of the bargaining problem did not depend on the *status quo* point and on the fact that it is exogenous and stationary. Since it does, the standard bargaining model cannot be used to analyse negotiations, where we observe an endogenous and non-stationary *status quo* point.

A model that has proven to be successful in analysing repeated strategic interaction between parties is the repeated game model. We will briefly investigate why this framework alone is not sufficient to analyse negotiations either.

The two-player repeated game model arrives at its results by assuming a fixed time horizon during which a given relationship exists. This time frame is exogenously given as either finite or as infinite horizon. Even if one were to ignore the bargaining aspect of the negotiation problem, this assumption contradicts one of the main features of negotiations, namely that the time horizon is inherently endogenous. As a matter of fact, the time taken to reach an agreement, that is to the end of the current relationship, is one of the predictions sought from a model. If one were, in order to circumvent this restriction, to impose different time frames on the model and solve for each one in turn, one is left without any guidance as to which outcome would be chosen by rational players in an equilibrium. This leads to the conclusion that two-player repeated games cannot be used to model the negotiation problems, since they cannot accommodate the endogenous time horizon natural to the latter.

The discussion sofar points to the difference in the assumptions underlying bargaining models and repeated game models on the one hand, and what one thinks of the features of negotiations on the other. Define a *Negotiation Problem* as a situation where two rational parties are involved in a repeated strategic relationship which yields periodic payoffs, and where a surplus is available to the parties if they can agree on how to share it. Agreement ends the repeated relationship. The question asked about a negotiation problem is what the equilibrium outcomes will be.

In this essay, we propose a model of the negotiation problem. The model merges ideas from bargaining theory and repeated games, accounting for the strategic behaviour and payoffs received during the negotiations, the importance attached to the agreement sought, and the endogeneity of the decision to abandon the current relationship in favour of the new agreement. For each player, we will find his worst (or optimal punishment) equilibrium payoff, and show that this worst payoff can be characterized very simply in terms of the stage game which is the strategic relationship between the players. Finally, we will characterize all subgame perfect equilibrium payoffs of the negotiation problem by showing that every feasible payoff of the negotiation game which strictly dominates the punishment payoffs can be supported as the average payoff from a subgame perfect equilibrium for a large enough discount factor.

This chapter is structured as follows: Section 2.2 describes the model and defines all necessary concepts. Section 2.3 presents two examples and derives the main results of the essay. In section 2.4, we provide a discussion of the model's contribution to issues raised in the literature, in particular the question of delay in bargaining and the question of the robustness of folk theorems. Section 2.5 offers some concluding remarks, focusing on areas for further development.

2.2 The Model and Definitions

Consider the following example situation for negotiations: There are two Cournot duopolists in a market. Both face a relatively large fixed cost. If no side-payments are possible but the formation of one monopoly is allowed, they could gain by forming a monopoly with only one plant. Negotiations are held over the allocation of the resulting surplus, but the two still produce and serve the market while these negotiations go on (behaving as Cournot quantity setters). We are interested in the possible allocations of the surplus and the strategies followed during the negotiations.

We model this situation as follows: There are two players, indexed by $i = 1, 2$. Time is discrete and indexed by $t = 1, 2, 3, \ldots$ The time horizon is infinite. Both players discount the future, with their (common) discount factor being denoted by $\delta \in (0, 1)$. In every period t in which no agreement has yet been reached, the players play the following *constituent game*: At the beginning of each period, one player makes a offer to the other player. The offer is in terms of a share of the surplus resulting from agreement. The other player can either agree or disagree with this offer. Should he agree, both players receive their agreed upon shares of the surplus from this period onward and their prior strategic relationship (and the game) is over. Should he not agree, both players play a simultaneous move game in normal form, denoted by G, and receive a payoff from it. Time then advances, and the constituent game is repeated. A schematic of the game is given in the following Figure 2.1.

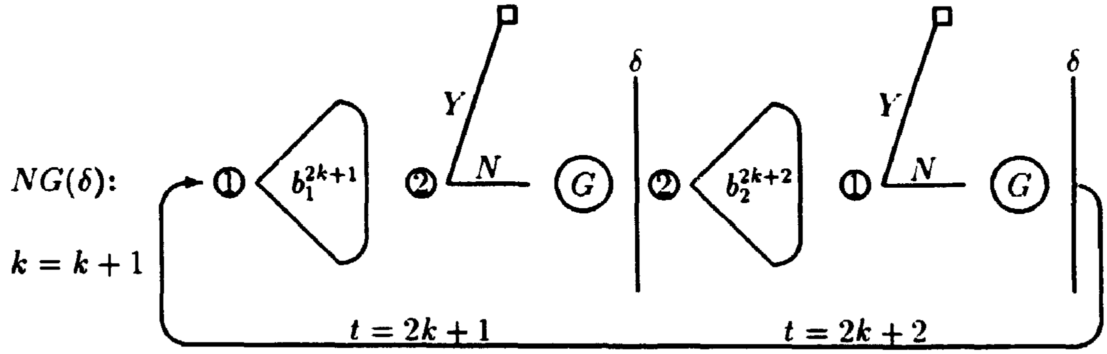

Figure 2.1: Diagram of the Negotiation Game

We will now formally define all concepts and variables. First, consider the exit offer game within each period. We employ Rubinstein's (1982) method to formulate the offer and agreement. A proposal by a player is a vector in the unit simplex of $\mathbf{R}^2$, say $(b, 1-b)$, where b is player 1's share and $(1-b)$ is player 2's share of the surplus. We will denote a proposal just by its first coordinate, b. A player's response to a proposal is either rejection or acceptance, indicated by N and Y, respectively. Players reach an agreement if one player accepts the other one's proposal. The negotiation game ends when an agreement is reached. The players then obtain the same proportion of the surplus in each of the subsequent periods. In this essay we will only consider the case in which the two players make proposals alternately, with player 1 proposing in odd periods and 2 proposing in even periods.

Next, recall some definitions and notation for two-player infinitely repeated games with discounting.

A two-player one-shot (stage) game in normal form consists of a set of two players, their strategy (action) sets, and their payoff functions. The *stage game* is denoted by $G = \{A_1, A_2, u_1(\cdot), u_2(\cdot)\}$. Here, A_i is player i's *strategy (action) set* which is assumed to be compact and $u_i(\cdot) : A \rightarrow \mathbf{R}$ is his *payoff function* which is assumed to be continuous, where $A = A_1 \times A_2$ for $i = 1, 2$.

The set A can also be interpreted as the set of outcomes of the stage game G. A generic element of the set A is denoted $a = (a_1, a_2)$. Let $u(\cdot) = (u_1(\cdot), u_2(\cdot))$. The set of *feasible payoffs* of the stage game G is given by the convex hull of $u(A)$, $Co[u(A)]$, which is compact by the assumptions. Let mx^i, $i = 1,2$, denote the strategy pair leading to player i's minimax payoff. F, the set of *feasible and individually rational payoffs*, is the intersection of $Co[u(A)]$ and $\{v \in \mathbf{R}^2 | v_1 \geq u_1(mx^1),\ v_2 \geq u_2(mx^2)\}$.

In this essay, we will make the following assumption: The players' strategies in G are mixed and correlated strategies, and deviations from a mixed or a correlated strategy by either player are publicly observable. Therefore, the set A_i is convex for $i = 1,2$, and for a feasible payoff vector v, $\exists\ a \in A$ such that $v = u(a)$, and the stage game G has at least one Nash equilibrium.

Let $G^\infty(\delta)$ denote the *infinitely repeated game* in which the stage game G is played by the same players for infinitely many periods and the players discount future payoff by a factor $\delta \in (0,1)$.

An *outcome path* of $G^\infty(\delta)$ is defined as $\pi = (a^1, \ldots, a^t, \ldots) \in A^\infty$. The *average payoff function* of the players, $U_i(\cdot) : A^\infty \rightarrow \mathbf{R}$, is defined over the set of all outcome paths and is given by

$$U_i(\pi) \ = \ (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t)\,, \qquad \text{for } i = 1,2 \tag{2.1}$$

Players' payoffs in the first period are not discounted. Note that the players' average payoffs in $G^\infty(\delta)$ are of the same scale as those in the first period.

A *strategy* for player i in $G^\infty(\delta)$ is a function $f_i(\cdot)$ which maps from the set of all possible histories into the set of all possible actions, i.e.

$$f_i(\cdot) \ : \ H = \bigcup_{t=0}^{\infty} A^t \ \rightarrow A_i$$

where A^t may be referred to as the set of all *t-period histories* in period $(t+1)$ when G has been played t times. $A^0 = \emptyset$ denotes the null history, and $f_i(\emptyset) \in A_i$.

Given a ***strategy combination*** $f = (f_1, f_2)$, $\pi(f) \in A^\infty$ is the unique ***outcome path*** induced by f, where $\pi(f) = (a^1(f), \ldots, a^t(f), \ldots)$. The payoffs from a strategy combination f are calculated from the outcome path induced by f and equation (2.1), and are given by $U(\pi(f)) = (\, U_1(\pi(f)), U_2(\pi(f))\,)$.

In order to simplify the analysis, we normalize the surplus from agreement to be 1 and the stage game G such that $u_i(mx^i) = 0$ for $i = 1, 2$. We also assume that $\forall\, a \in A$, $u_1(a) + u_2(a) \leq 1$ which does entail a loss of generality and will be discussed further in the last section.

We now define the ***negotiation game*** $NG(\delta)$ to be the game where two players with discount factor δ play a sequence of constituent games until agreement, where a constituent game is an offer game followed by the stage game G after rejection. Let $NG_i(\delta)$ be the game in which player i makes the proposal in the first period (note that the first period in $NG_2(\delta)$ is an even period). In this essay, we will present the results explicitly for only $NG_1(\delta)$. It is not difficult to state the similar results for $NG_2(\delta)$ by using analogous arguments.

A *type 1 t-period history* in the game $NG_1(\delta)$ is a finite sequence denoted by $h_1(t) = (b^1, a^1, \ldots, b^t, a^t)$, in which b^s is the proposal made in period s and $a^s \in A$ is the outcome of G in period s after the proposal b^s has been rejected, for $s = 1, \ldots, t$. Let $h_1(0) = \emptyset$. A type 1 t-period history can be decomposed as $h_1(t) = b(t) \oplus a(t)$ where

$$b(t) = (b^1, \ldots, b^t) \in [0,1]^t; \quad a(t) = (a^1, \ldots, a^t) \in A^t$$

A *type 2 t-period history* is $h_2(t) = h_1(t) \oplus b^{t+1}$, indicating that following the type 1 t-period history $h_1(t)$, b^{t+1} has been proposed in period $(t+1)$.

A *type 3 t-period history* is denoted by $h_3(t) = h_2(t) \oplus \{N\}$ indicating that the proposal b^{t+1} has been rejected in period $(t+1)$. We do not define a path with an acceptance as a history, since the game ends with acceptance.

The sets of all possible histories of three types, H_1, H_2 and H_3, can be written as

$$\begin{aligned} H_1 &= \bigcup_{t=0}^{\infty} H_1(t) = \bigcup_{t=0}^{\infty} ([0,1]^t \times A^t) \\ H_2 &= \bigcup_{t=0}^{\infty} H_2(t) = \bigcup_{t=0}^{\infty} ([0,1]^{t+1} \times A^t) \\ H_3 &= \bigcup_{t=0}^{\infty} H_3(t) = \bigcup_{t=0}^{\infty} ([0,1]^{t+1} \times A^t \times \{N\}) \end{aligned}$$

A *strategy combination* $f = (f_1, f_2)$ for the game $NG_1(\delta)$ consists of two functions which map from the sets of all appropriate histories into the sets of all appropriate actions, i.e.

$$f_i : H_1 \bigcup H_2 \bigcup H_3 \rightarrow [0,1] \bigcup \{N, Y\} \bigcup A_i \quad \text{for } i = 1,2$$

such that

$$\begin{aligned} (f_1, f_2) &: H_1(t) \times H_2(t) \rightarrow [0,1] \times \{N, Y\} & \text{if } t \text{ is even} \\ (f_1, f_2) &: H_2(t) \times H_1(t) \rightarrow \{N, Y\} \times [0,1] & \text{if } t \text{ is odd} \\ (f_1, f_2) &: H_3 \rightarrow A \end{aligned}$$

The strategy combination f gives players' instructions how to play the game in every period, conditional on a history. For example, in an odd period $(t+1)$ after a type 1 t-period history $h_1(t)$, when player 1 will make the proposal , $f_1(h_1(t))$ is player 1's proposal, $f_2(h_2(t))$ is player 2's response to player 1's proposal b^{t+1} and ($f_1(h_3(t)), f_2(h_3(t))$) is the one-shot play of the stage game G in period $(t+1)$ after the proposal b^{t+1} has been rejected. The players' actions in an even period are specified also by f analogously.

An *outcome path* of $NG_1(\delta)$, $\pi(T) = (b^1, a^1, b^2, a^2, \ldots, b^T, \{Y\})$ can be interpreted to indicate that the proposal b^t has been rejected and the stage game outcome a^t has been played in period t for $1 \leq t < T$, and the proposal b^T has been accepted in period T. By convention, T is set to be infinity in an outcome path in which the two players never reach an agreement. An outcome path of $NG_1(\delta)$ can be decomposed as $\pi(T) = b(T) \oplus a(T-1) \oplus \{Y\}$.

The *payoff* to the players from an outcome path is determined by the payoff from the stage game in all the periods before the agreement is reached and the agreement itself. The average payoffs to the players from the outcome path $\pi(T)$ are given by[1]

$$U_1(\pi(T)) = (1-\delta)\sum_{t=1}^{T-1}\delta^{t-1}u_1(a^t) + \delta^{T-1}b^T \quad (2.2)$$

$$U_2(\pi(T)) = (1-\delta)\sum_{t=1}^{T-1}\delta^{t-1}u_2(a^t) + \delta^{T-1}(1-b^T) \quad (2.3)$$

The payoffs from a strategy combination f can be calculated directly from (2.2) and (2.3) and the unique outcome path induced by f.

2.3 The Subgame Perfect Equilibria

In this section, we will examine the subgame perfect equilibria (SPE) of the negotiation game. The section has four subsections which deal with the three issues at hand: First we calculate the perfect proposals for every player when the continuation disagreement payoff path is fixed. A stationary SPE of the negotiation game can be constructed by using the perfect proposals and a Nash equilibrium of the stage game. Second, we provide two examples which show two extreme cases respectively; every feasible and individually rational payoff and only Nash equilibrium payoff of the stage game can serve as the (average) disagreement payoff of a SPE in negotiation games, respectively. An optimal punishment equilibrium for each of the players is constructed in the third section by employing ideas gained from the examples. Lastly, we characterize all SPEs in the negotiation game using the optimal punishment equilibrium payoffs. Most of the proofs are constructive.

[1]Note that in general superscript indicates the period to which a variable belongs and not a power. The exceptions are the discount factor δ and strategy sets. The distinction should always be clear from the context.

2.3.1 Perfect Proposals

In the negotiation game, the proposals which are made in every period certainly depend upon players' payoffs from all the periods thereafter. In the bargaining games, if the payoffs to the players in the periods without any agreements are fixed but may not be the same for all the periods, then the bargaining game has a SPE. In the SPE, players' proposals are certainly not stationary, and they are the best actions of the players given the continuation disagreement payoffs. We shall call the proposals in the SPE of the bargaining game with fixed continuation disagreement payoffs path as *the perfect proposals.*

Following Shaked and Sutton (1984), we are able to calculate the perfect proposals given any continuation disagreement payoff path. Later, we will use these perfect proposals to construct some SPE in the negotiation game.

Theorem 2.1 *In $NG_1(\delta)$, if players' payoffs are fixed to be $u(a^t)$ in period t whenever there is no agreement, then b^t is the unique perfect proposal of the player who makes the proposal and accepting a proposal which is preferred to the perfect proposal by the player who makes the response is his best response in period t , where*

$$b^t = \frac{1}{1+\delta} + (1-\delta)\sum_{s=0}^{\infty} \delta^{2s}[\delta u_1(a^{t+2s+1}) - u_2(a^{t+2s})] \quad \textit{for odd } t$$

$$b^t = \frac{\delta}{1+\delta} + (1-\delta)\sum_{s=0}^{\infty} \delta^{2s}[u_1(a^{t+2s}) - \delta u_2(a^{t+2s+1})] \quad \textit{for even } t$$

Proof: In the negotiation game $NG_1(\delta)$ with the fixed disagreement payoff path $\{u(a^t)\}_{t=1}^{\infty}$, we first suppose that the set of the average payoffs from perfect equilibria is not empty, and then show that the proposals we find will be the perfect proposals in the game with fixed continuation disagreement payoff path $\{u(a^t)\}_{t=1}^{\infty}$. Let M_i^t and m_i^t be the supremum and infimum of player i's average equilibrium payoffs in the subgame that starts from period t in $NG_1(\delta)$, for $i = 1, 2$.

First, consider players' strategies in an odd period t in which player 1 makes the proposal and player 2 makes the response. If player 2 rejects the proposal, his payoff will be $u_2(a^t)$ in period t and a SPE payoff from period $(t+1)$ on which is bounded between by m_2^{t+1} and M_2^{t+1}. Therefore, player 2 will always reject a proposal if his average payoff from the proposal is less than $(1-\delta)u_2(a^t)+\delta m_2^{t+1}$, and always accept a proposal if his average payoff is more than $(1-\delta)u_2(a^t)+\delta M_2^{t+1}$. Subgame perfectness requires that any proposal made by player 1, b^t, should satisfy

$$(1-\delta)u_2(a^t)+\delta m_2^{t+1} \leq 1-b^t \leq (1-\delta)u_2(a^t)+\delta M_2^{t+1}$$

which implies that m_1^t and M_1^t satisfy the following inequalities,

$$1-(1-\delta)u_2(a^t)-\delta M_2^{t+1} \leq m_1^t \leq M_1^t \leq 1-(1-\delta)u_2(a^t)-\delta m_2^{t+1} \tag{2.4}$$

Considering players' strategies in the following even period $(t+1)$, we obtain by similar arguments that

$$1-(1-\delta)u_1(a^{t+1})-\delta M_1^{t+2} \leq m_2^{t+1} \leq M_2^{t+1} \leq 1-(1-\delta)u_1(a^{t+1})-\delta m_1^{t+2} \tag{2.5}$$

(2.4) and (2.5) can be rewritten as

$$\begin{aligned}
M_1^t &\leq 1-(1-\delta)u_2(a^t)-\delta[1-(1-\delta)u_1(a^{t+1})-\delta M_1^{t+2}] \\
m_1^t &\geq 1-(1-\delta)u_2(a^t)-\delta[1-(1-\delta)u_1(a^{t+1})-\delta m_1^{t+2}] \\
M_2^{t+1} &\leq 1-(1-\delta)u_1(a^{t+1})-\delta[1-(1-\delta)u_2(a^{t+2})-\delta M_2^{t+3}] \\
m_2^{t+1} &\geq 1-(1-\delta)u_1(a^{t+1})-\delta[1-(1-\delta)u_2(a^{t+2})-\delta m_2^{t+3}]
\end{aligned}$$

Substituting iteratively then yields

$$\begin{aligned}
b^t \leq\ & m_1^t \leq M_1^t \leq b^t \quad \text{for odd } t \\
1-b^t \leq\ & m_2^t \leq M_2^t \leq 1-b^t \quad \text{for even } t
\end{aligned}$$

where b^t is as given in the theorem. Therefore, $M_i^t = m_i^t$ for $i = 1,2$. Hence, if the perfect equilibrium exists in $NG_1(\delta)$ with the fixed disagreement payoff path $\{u(a^t)\}_{t=1}^{\infty}$, it must be unique in terms of payoff.

Consider the following strategies in the game $NG_1(\delta)$ with the fixed disagreement payoff path $\{u(a^t)\}_{t=1}^{\infty}$: in period t, the player who makes the proposal will propose b^t, and the player who makes the response will accept all proposals that he weakly prefers to b^t and reject all others. After a rejection (which will not occur in the equilibrium), players' payoffs are determined by $u(a^t)$.

These strategies are subgame perfect as the following considerations show. By the assumption $u(a^t)$ is fixed for period t, the player who makes the response in period t is just indifferent between accepting the proposal b^t and waiting to propose b^{t+1} in the next period, collecting $u_i(a^t)$ in the meantime. Therefore, rejecting the proposals which are not preferred to b^t and accepting those which are is his best response. This implies that any proposal which is preferred to b^t by the player who makes the proposal will be rejected. Since

$$\begin{aligned} b^t &\geq (1-\delta)u_1(a^t) + \delta b^{t+1} && \text{for odd } t \\ (1-b^t) &\geq (1-\delta)u_2(a^t) + \delta(1-b^{t+1}) && \text{for even } t \end{aligned}$$

the proposing player prefers the proposal to deviating and waiting for one period. We conclude that the strategy profile we offered is a subgame perfect equilibrium in $NG_1(\delta)$ with the fixed disagreement payoff path $\{u(a^t)\}_{t=1}^{\infty}$. **Q.E.D.**

Theorem 2.1 gives the proposals of the players in the unique SPE if the continuation disagreement payoff path is fixed. It also shows that the proposal in a SPE is uniquely determined by the continuation disagreement payoffs. Let $P_i(\cdot)$ denote player i's perfect proposal function which maps from the set of all continuation disagreement payoff paths into unit interval $[0,1]$.

Note that this theorem implies that, with a common discount factor, the non-stationarity of the *status quo* point does not affect Rubinstein's uniqueness result. As with a stationary *status quo* point, there exists a unique perfect proposal in every period which will be accepted. Note also that if $a^t = a \in A$ for $t \geq 1$, it follows from

Theorem 2.1 that player i will propose b_i in the SPE, where

$$b_1 = \frac{1 + \delta u_1(a) - u_2(a)}{1+\delta} \quad \text{and} \quad b_2 = \frac{\delta + u_1(a) - \delta u_2(a)}{1+\delta}$$

In particular, if $u_1(a) = u_2(a) = 0$, then $\{b_1, b_2\}$ will be just the Rubinstein solution in the strategic bargaining game.

The next theorem proves the existence of a subgame perfect equilibrium in the negotiation game. The equilibrium in the theorem is stationary in the sense that the disagreement payoffs and proposals made by players are independent of histories. In the equilibrium, players always play a Nash equilibrium in the stage game whenever there is no agreement, and the player who makes the proposal will always make his perfect proposal.

Theorem 2.2 *Suppose that $a^* \in A$ is a Nash equilibrium in the stage game G. $\forall\, \delta \in (0,1)$, $NG_i(\delta)$ has a perfect equilibrium whose outcome is that player i's perfect proposal $P_i(\pi^*)$ is accepted by player $j \neq i$ in the first period of the game. Here $\pi^* = (a^*, \ldots, a^*, \ldots)$ and*

$$\begin{aligned} P_1(\pi^*) &= \frac{1 + \delta u_1(a^*) - u_2(a^*)}{1+\delta} \\ P_2(\pi^*) &= \frac{\delta + u_1(a^*) - \delta u_2(a^*)}{1+\delta} \end{aligned}$$

Proof: Since the disagreement outcome in every period is a Nash equilibrium of the stage game and all the continuation payoffs are independent of histories, neither player will deviate from a^* individually in the stage game in every period when no agreement has been reached. Therefore, playing a^* are indeed the best strategies of the players in every period.

The equilibrium strategies in $NG_1(\delta)$ are given as follows. $\forall\, h_1(t) \in H_1$, $h_2(t) = h_1(t) \oplus b^{t+1} \in H_2$ and $h_3(t) \in H_3$; for an odd period $(t+1)$,

$$f_1(h_1(t)) = \frac{1 + \delta u_1(a^*) - u_2(a^*)}{1+\delta}$$

$$
\begin{aligned}
f_2(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \le f_1(h_1(t)) \\ N & \text{otherwise} \end{cases} \\
f(h_3(t)) &= a^* \in A
\end{aligned}
$$

and for an even period $(t+1)$,

$$
\begin{aligned}
f_2(h_1(t)) &= \frac{\delta + u_1(a^*) - \delta u_2(a^*)}{1+\delta} \\
f_1(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \ge f_2(h_1(t)) \\ N & \text{otherwise} \end{cases} \\
f(h_3(t)) &= a^* \in A
\end{aligned}
$$

It is not difficult to see that these strategies are indeed a subgame perfect equilibrium of the negotiation game $NG_1(\delta)$. **Q.E.D.**

2.3.2 Two Examples

As we discussed in the last section, the proposals in a SPE are determined by the future disagreement payoff path. If some payoffs of the stage game can serve as the disagreement payoffs, then it is not difficult to find the SPE which is supported by these disagreement payoffs. This section considers two extreme examples, in which every feasible and individually rational payoff of the stage game can be the disagreement payoffs in a SPE of the negotiation game in the first example, but only the Nash equilibrium payoff in the second example. In both examples, the stage games are prisoners' dilemma games which are different in payoffs. Prisoners' dilemma games have a unique Nash equilibrium which is also the minimax strategy combination. For expositional convenience, we will not normalize the negotiation games.

Example 2.1: Consider the negotiation game which consists of a negotiation surplus $(10 + 5\epsilon)$ with $\epsilon \ge 0$, and the stage game G with the following payoff matrix.

1\2	C	D
C	$(5,5)$	$(-4,14)$
D	$(14,-4)$	$(0,0)^*$

Both players have a common discount factor $\delta = 2/3$.

The payoff vector (0,0) is both the minimax and the Nash equilibrium payoff. According to Theorem 2.2, $(6+3\epsilon, 4+2\epsilon)$ and $(4+2\epsilon, 6+3\epsilon)$ can be supported as SPEs in the negotiation games when the Nash equilibrium payoff (0,0) serves as the disagreement payoff, note that $6+3\epsilon = \frac{1}{1+\delta}(10+5\epsilon)$ and $4+2\epsilon = \frac{\delta}{1+\delta}(10+5\epsilon)$. However, every feasible and individually rational payoff of the stage game in Example 2.1 can serve as the disagreement payoff in a SPE of the negotiation games. The following proposition shows that the negotiation game has two SPEs in which two very extraordinary payoffs of the stage game, (0,10) and (10,0), serve as the disagreement payoffs respectively. These SPEs are illustrated in the following Figure 2.2.

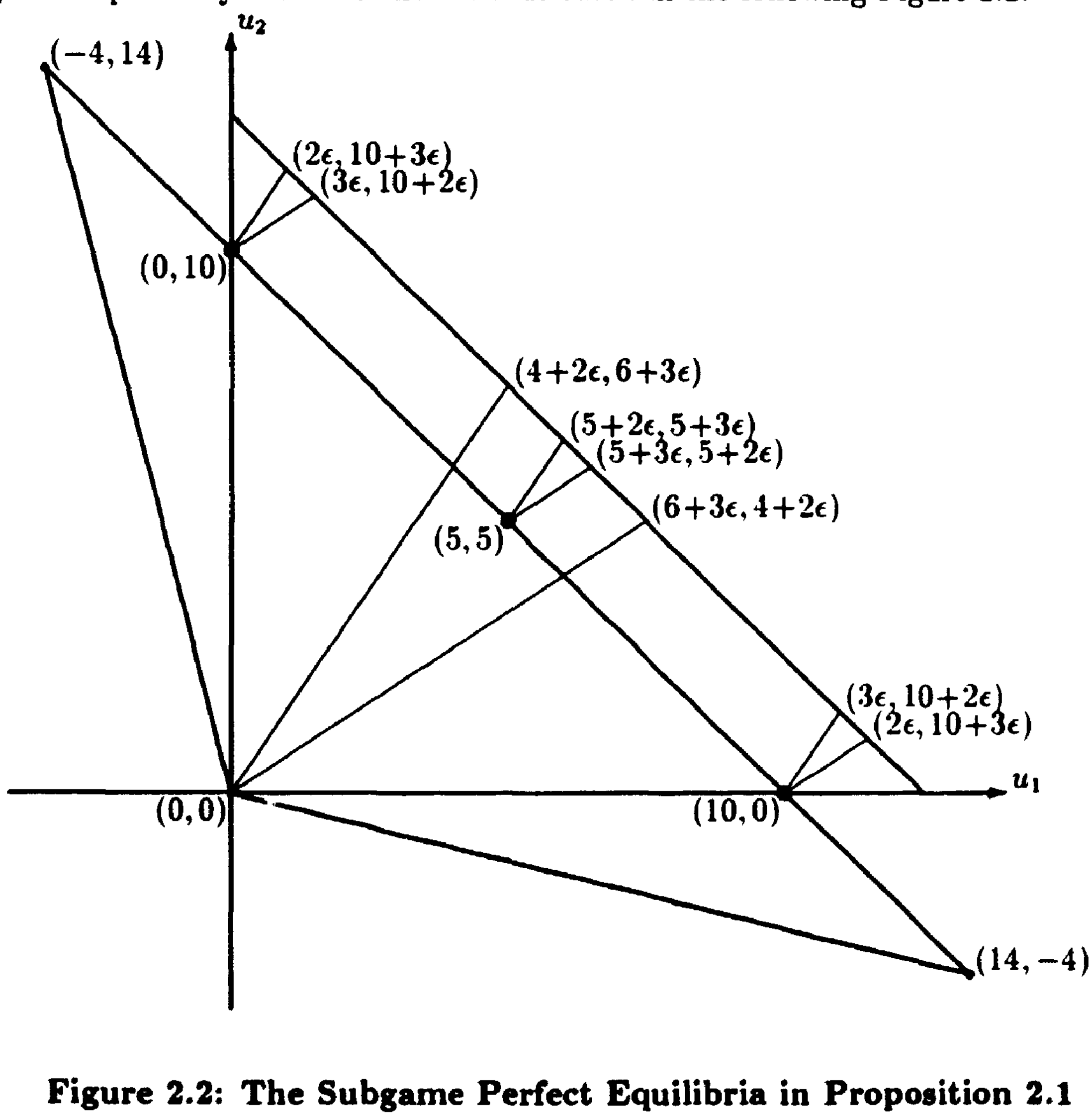

Figure 2.2: The Subgame Perfect Equilibria in Proposition 2.1

Proposition 2.1 $(3\epsilon, 10+2\epsilon)$ *and* $(10+3\epsilon, 2\epsilon)$ [$(2\epsilon, 10+3\epsilon)$ *and* $(10+2\epsilon, 3\epsilon)$] *can be supported as the average payoffs from subgame perfect equilibria in* $NG_1(\frac{2}{3})$ [$NG_2(\frac{2}{3})$].

Proof: Consider the infinitely repeated game with the stage game G and common discount factor $\delta = 2/3$, $G^\infty(\frac{2}{3})$. The simple strategy profile[2] $\sigma(\pi_0, \pi_1, \pi_2)$ forms a (renegotiation-proof, see van Damme (1989)) subgame perfect equilibrium in $G^\infty(\frac{2}{3})$, where

$$\pi_0 = (C, C)^\infty \ ; \ \pi_1 = (C, D)^2 \oplus (C, C)^\infty \ ; \ \pi_2 = (D, C)^2 \oplus (C, C)^\infty$$

Let $g = (g_1, g_2)$ denote the equilibrium strategy functions which are induced by $\sigma(\pi_0, \pi_1, \pi_2)$. $\forall\ a(t) \in A^t$, let $\pi(g|_{a(t)})$ be the continuation path induced by strategy g after a t-period history $a(t)$ in $G^\infty(\delta)$.

Consider the following strategies in $NG_1(\frac{2}{3})$. $\forall\ h_1(t) = a(t) \oplus b(t) \in H_1$, $h_2(t) = h_1(t) \oplus b^{t+1}$ and $h_3(t) = h_2(t) \oplus \{N\} \in H_3$; for an odd period $(t+1)$,

$$\begin{aligned} f_1(h_1(t)) &= P_1(\pi(g|_{a(t)})) \\ f_2(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \le f_1(h_1(t)) \\ N & \text{otherwise} \end{cases} \\ f(h_3(t)) &= g(a(t)) \in A \end{aligned}$$

and for an even period $(t+1)$,

$$\begin{aligned} f_2(h_1(t)) &= P_2(\pi(g|_{a(t)})) \\ f_1(h_2(t)) &= \begin{cases} Y & \text{if } b^{t+1} \ge f_2(h_1(t)) \\ N & \text{otherwise} \end{cases} \\ f(h_3(t)) &= g(a(t)) \in A \end{aligned}$$

Since the proposals are the perfect proposals, if neither player deviates from (f_1, f_2) individually after every type 3 history in the stage game G, then neither will deviate from $f = (f_1, f_2)$ individually. Therefore, the given strategy combination f forms a

[2] According to Abreu (1988), a simple strategy profile $\sigma(\pi_0, \pi_1, \pi_2)$ indicates that path π_0 will be played in G^∞. A deviation by player i from the on going path will be followed by restaring path π_i.

subgame perfect equilibrium in $NG_1(2/3)$. The rest of this proof verifies that neither player will deviate from (f_1, f_2) individually after every type 3 history.

Let $(t+1)$ be an odd period. $\forall\, h_3(t) = a(t) \oplus b(t+1) \oplus \{N\}$ and even t, there are three cases that have to be verified:

Case 1: $(f_1, f_2)(h_3(t)) = (C, C)$.

If player 1 deviated from (C, C), player 2 would propose 2ϵ in period $(t+2)$, the perfect proposal associated with the continuation disagreement payoff path π_1, instead of $5 + 2\epsilon$, the perfect proposal from the path π_0, because π_1 would be the continuation payoff path associated with this deviation. Since

$$(1-\delta)u_1(D,C) + 2\delta\epsilon \;=\; \frac{14+4\epsilon}{3} \;<\; \frac{15+4\epsilon}{3} \;=\; (1-\delta)u_1(C,C) + \delta(5+2\epsilon),$$

player 1 will not deviate from (C, C).

If player 2 deviated from (C, C), player 2 would have to propose $10 + 2\epsilon$, the perfect proposal for the path π_2, instead of $5 + 2\epsilon$ in period $(t+2)$, because π_2 would be the continuation disagreement payoff path. Since

$$(1-\delta)u_2(C,D) + 3\delta\epsilon \;=\; \frac{14+6\epsilon}{3} \;<\; \frac{15+6\epsilon}{3} \;=\; (1-\delta)u_2(C,C) + \delta(5+3\epsilon),$$

player 2 will not deviate from (C, C) either.

Case 2: $(f_1, f_2)(h_3(t)) = (C, D)$.

Player 2 will not deviate from (C, D) individually, since he has the highest payoff from (C, D) among all the outcomes of the stage game. If player 1 deviated from (C, D), player 2 would propose 2ϵ in period $(t+2)$, which is the perfect proposal for path π_1. But, if player 1 follows the equilibrium strategies, player 2 will at least propose $2 + 2\epsilon$, since the worst possible continuation disagreement payoff path to player 1 is $(C, D) \oplus (C, C)^\infty$. Since

$$(1-\delta)u_1(D,D) + 2\delta\epsilon \;=\; \frac{4\epsilon}{3} \;=\; (1-\delta)u_1(C,D) + \delta(2+2\epsilon).$$

player 1 will not deviate from (C, D).

Case 3: $(f_1, f_2)(h_3(t)) = (D, C)$.

Player 1 will not deviate from (D, C) individually, since he has the highest payoff from (D, C) among all the outcomes of the stage game. If player 2 deviated from (D, C), player 2 would have to propose $10+2\epsilon$. But, if player 2 follows the equilibrium strategies, player 2 will, at most, propose $8+2\epsilon$, since the worst possible continuation disagreement payoff path for player 2 is $(D, C) \oplus (C, C)^{\infty}$. Since

$$(1-\delta)u_2(D, D) + 3\delta\epsilon \;=\; \frac{6\epsilon}{3} \;=\; (1-\delta)u_2(C, D) + \delta(2+3\epsilon),$$

player 2 will not deviate from (D, C).

Similar arguments show that neither player will deviate from $(f_1, f_2)(h_3(t))$ in every even period. Therefore, the strategy combination $f = (f_1, f_2)$ is a subgame perfect equilibrium in $NG_1(\frac{2}{3})$, and the strategy profile which is induced by f in every subgame is also the subgame perfect equilibrium in that subgame.

Consider the following histories given by

$$h_1^1(1) = 1 \oplus (D, C) \;\; ; \;\; h_1^1(2) = 1 \oplus (C, C) \oplus 0 \oplus (D, C)$$
$$h_1^2(1) = 1 \oplus (C, D) \;\; ; \;\; h_1^2(2) = 1 \oplus (C, C) \oplus 0 \oplus (C, D)$$

In $h_1^i(j)$, player i is the last deviator in the stage game in period j, player i will be punished from period $(j+1)$ on for $i, j = 1, 2$. Since

$$\begin{aligned} NG_1(\frac{2}{3}) &= NG_1(\frac{2}{3})|_{h_1^1(2)} = NG_1(\frac{2}{3})|_{h_1^2(2)} \\ NG_2(\frac{2}{3}) &= NG_1(\frac{2}{3})|_{h_1^1(1)} = NG_1(\frac{2}{3})|_{h_1^2(1)} \end{aligned}$$

the induced strategies $f|_{h_1^1(2)}$ and $f|_{h_1^2(2)}$ [$f|_{h_1^1(1)}$ and $f|_{h_1^2(1)}$] are two subgame perfect equilibria of $NG_1(\frac{2}{3})$ [$NG_1(\frac{2}{3})$] whose outcomes are; player 1's [2's] perfect proposals 3ϵ and $10+3\epsilon$ [2ϵ and $10+2\epsilon$] are accepted by player 2 [1] in the first period of the games. **Q.E.D.**

Proposition 2.1 also holds for $\delta > \frac{2}{3}$. The result of Proposition 2.1 implies that many feasible and individually rational payoffs of the stage game can serve as disagreement payoffs in subgame perfect equilibria of the negotiation games. Unlike infinitely repeated games, there may not be the opportunity to punish a player for infinitely many periods in the negotiation games. However, a player can still be punished by changing the continuation disagreement payoff path. In the SPEs, making a non-perfect proposal does not have to be considered as a deviation. The proof of Proposition 2.1 demonstrates that the perfect proposals are the same as those with the average payoff of the continuation disagreement payoff path as the disagreement payoff for every period, when the continuation disagreement payoff path consists of playing one outcome for a certain number of periods and playing another outcome thereafter.

However, the negotiation game in the next example has a unique SPE which is given in Theorem 2.2, even the stage game is also a prisoners' dilemma game.

Example 2.2: Consider the negotiation game which consists of the surplus $(10+\epsilon)$ with $\epsilon \geq 0$ and the following stage game:

$1\backslash 2$	C	D
C	$(5,5)$	$(-10,10)$
D	$(10,-10)$	$(0,0)^*$

Both players have a common discount factor $\delta \in (0,1)$.

Like Example 2.1, (0,0) is both the minimax and the unique Nash equilibrium payoff in the stage game. The negotiation game $NG_1(\delta)$ $[NG_2(\delta)]$ has a SPE in which the Nash equilibrium payoff (0,0) serves as the disagreement payoff. Unlike Example 2.1, Proposition 2.2 shows that the negotiation game has a unique SPE given by Theorem 2.2 for any discount factor. This SPE is illustrated in Figure 2.3.

Proposition 2.2 $\forall\ \delta \in (0,1)$ *and* $\epsilon \geq 0$, *the negotiation game* $NG_1(\delta)$ [$NG_2(\delta)$] *has a unique subgame perfect equilibrium with average payoffs* $(\frac{1}{1+\delta}(10+\epsilon), \frac{\delta}{1+\delta}(10+\epsilon))$ [$(\frac{\delta}{1+\delta}(1+\epsilon), \frac{1}{1+\delta}(1+\epsilon))$].

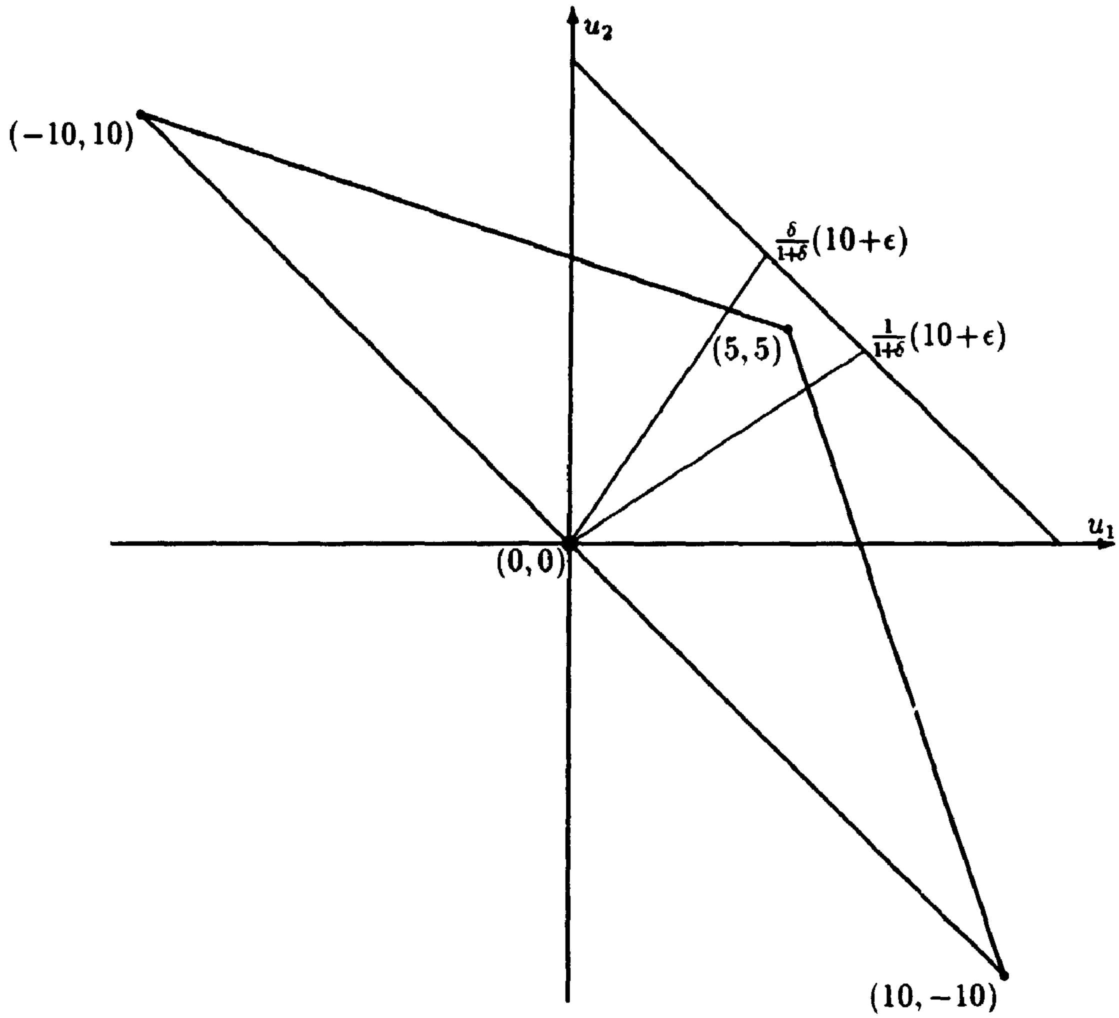

Figure 2.3: The Subgame Perfect Equilibrium in Proposition 2.2

Proof: Consider mixed strategies of the stage game, let α and β be the probabilities of players 1 and 2 playing pure strategy C respectively. A mixed strategy combination is denoted by $(\alpha, \beta) \in [0,1]^2$. The payoff functions, therefore, are; for $\alpha, \beta \in [0,1]$,

$$u_1(\alpha,\beta) = 5\alpha\beta - 10\alpha + 10\beta$$

$$u_2(\alpha,\beta) = 5\alpha\beta + 10\alpha - 10\beta$$

From Theorem 2.2, the sets of SPE payoffs in $NG_1(\delta)$ and $NG_2(\delta)$ are not empty. Let $M_i(\delta)$ and $m_i(\delta)$ be the supremum and the infimum of player i's average SPE payoffs in $NG_i(\delta)$ for a given $\delta \in (0,1)$ and $i = 1,2$.

In the first period of $NG_1(\delta)$, suppose that if player 2 rejects player 1's proposal, they will play (α, β) in the stage game in this period and player 2 will propose $b(\alpha, \beta)$ in the next period. However, player 1 should not deviate from (α, β) if player 2 does reject is proposal. Subgame perfectness requires that

$$\begin{aligned}(1-\delta)\max_{\alpha,\beta} u_1(\alpha,\beta) + \delta^2 m_1(\delta) &= (1-\delta)10\beta + \delta^2 m_1(\delta) \\ &\leq (1-\delta)(5\alpha\beta - 10\alpha + 10\beta) + \delta b(\alpha,\beta) \\ \text{i.e.} \qquad \delta b(\alpha,\beta) &\geq \delta^2 m_1(\delta) - (1-\delta)(5\beta - 10)\alpha\end{aligned}$$

Also, player 1 will not make a proposal in which player 2's payoff is more than the maximum of player 2's continuation payoffs after player 2 rejects player 1's proposal. Therefore,

$$\begin{aligned}10+\epsilon-m_1(\delta) &\leq \max_{\alpha,\beta}[(1-\delta)u_2(\alpha,\beta) + \delta(10+\epsilon-b(\alpha,\beta))] \\ &\leq (10+\epsilon)\delta - \delta^2 m_1(\delta) + (1-\delta)\max_{\alpha,\beta}[u_2(\alpha,\beta) + (5\beta-10)\alpha] \\ &= (10+\epsilon)\delta - \delta^2 m_1(\delta) + (1-\delta)\max_{\alpha,\beta}[5\alpha\beta + 10\alpha - 10\beta + 5\alpha\beta - 10\alpha] \\ &= (10+\epsilon)\delta - \delta^2 m_1(\delta) + (1-\delta)\max_{\alpha,\beta}[10\alpha\beta - 10\beta] \\ &= (10+\epsilon)\delta - \delta^2 m_1(\delta)\end{aligned}$$

Hence, $m_1(\delta) \geq \frac{1}{1+\delta}(10+\epsilon)$, together with Theorem 2.2, $m_1(\delta) = \frac{1}{1+\delta}(10+\epsilon)$. A similar argument proves that $m_2(\delta) = \frac{1}{1+\delta}(10+\epsilon)$. Therefore, the negotiation game has a unique SPE as given in Theorem 2.2. **Q.E.D.**

Although both the stage games in Example 2.1 and 2.2 are prisoners' dilemma games which are different only in payoffs, the corresponding negotiation games have very different equilibrium outcomes. They demonstrate that the equilibrium outcomes of a negotiation game depend on the payoff structure of the stage game. These two examples are very helpful in finding the optimal punishment equilibrium which will be done in the next section.

2.3.3 Optimal Punishments

In this section, we investigate the optimal punishment equilibria for the players. ***An optimal punishment equilibrium*** for player i is a SPE in which player i's equilibrium payoff is less than or equal to all SPE payoffs of the game. We will find an optimal punishment equilibrium for player 1. Similarly, we can then find an optimal punishment equilibrium for player 2. Therefore, we are able to characterize all SPE's of the negotiation game in the next section.

From Theorem 2.1, the perfect proposals in SPE of the negotiation games are determined by the continuation disagreement path. A perfect proposal made by a player depends only on the disagreement payoffs to the other player in the periods when he makes the proposals and his disagreement payoff in the periods when the other player makes the proposal. In an optimal punishment equilibrium of player 1, players should play minimax strategy against player 1 in every even period, and play an outcome of the stage in every odd period in which the outcome gives player 2 the highest possible payoff while playing this outcome is still the best for player 1. It turns out that the highest possible (average) payoff to player 2 in every odd period is

$$y_1^* = \max_{a \in A} \{ \ u_1(a) + u_2(a) - \max_{a_1' \in A_1} u_1(a_1', a_2) \ \} \tag{2.6}$$

y_1^* has the following interesting interpretation; in every period without an agreement, players may play any outcome in the stage game, say $a \in A$, although each player has incentive to deviate from this outcome $a \in A$. In SPEs, at least, player 1 should not deviate from a even player 1 can gain

$$\max_{a_1' \in A_1} u_1(a_1', a_2) - u_1(a)$$

Player 2 may change the proposal in the next period such that, if player 1 does not deviate from a in this period, the proposal in the next period will pay player 1 more which is equal to what player 1 can gain by deviating from a. Equivalently, player

2's payoff becomes

$$u_2(a) - [\max_{a_1' \in A_1} u_1(a_1', a_2) - u_1(a)] = u_1(a) + u_2(a) - \max_{a_1' \in A_1} u_1(a_1', a_2)$$

Then, the maximum payoff to player 2 when player 1 is fully compensated in the next proposal is, therefore, equal to y_1^*. In the next two theorems, an optimal punishment equilibrium for player 1 is constructed. Following the similar idea as from Example 2.2, we find a lower bound of player 1's equilibrium payoffs in the negotiation game for a given discount factor. As in Example 2.1, a SPE in which player 1's payoff reaches this lower bound is constructed in Theorem 2.4. Therefore, the SPE in Theorem 2.4 is indeed an optimal punishment equilibrium for player 1 in the negotiation game.

Theorem 2.3 *In the game $NG_1(\delta)$ $[NG_2(\delta)]$, player 1's average payoff in any subgame perfect equilibrium is bounded below by $\frac{1}{1+\delta}(1 - y_1^*)$ $[\frac{\delta}{1+\delta}(1 - y_1^*)]$.*

Proof: From Theorem 2.2, $NG_i(\delta)$ has, at least, one subgame perfect equilibrium for $\delta \in (0,1)$. Since F, the set of feasible and individually rational payoffs of the stage game G, is compact, the set of average payoffs of the SPEs in the negotiation game $NG_i(\delta)$ is not empty and bounded, $\forall \delta \in (0,1)$ and $i = 1,2$.

Given $\delta \in (0,1)$, let $m_1(\delta)$ be the infimum of player 1's average equilibrium payoffs in $NG_1(\delta)$. In $NG_2(\delta)$, since player 1 can guarantee himself a payoff of 0 in the current (even) period and his average payoff from the next period on cannot be less than $m_1(\delta)$, player 1's average equilibrium payoffs in the game $NG_2(\delta)$ are bounded below by $\delta m_1(\delta)$.

By the definition of the infimum, $\forall \epsilon > 0$, the game $NG_1(\delta)$ has a subgame perfect equilibrium with average payoff (x_1, y_1) such that

$$m_1(\delta) \leq x_1 \leq m_1(\delta) + \epsilon \tag{2.7}$$

If this SPE is inefficient, i.e. $x_1 + y_1 < 1$, then it must be the case that player 1's proposal is rejected in the first period of $NG_1(\delta)$. Consider a new strategy combination which is the same as the equilibrium strategy combination which leads to average payoff (x_1, y_1) in every subgame except when player 2 rejects player 1's proposal x_1 in the first period. The strategies in that subgame are the same as those in the subgame where player 2 follows his old strategies to reject player 1's proposal given by the old strategies. Under these new strategies, when player 1 proposes x_1, if player 2 rejects this proposal, his average payoff will be y_1 which is less than $1 - x_1$. Hence, player 2 will accepts player 1's proposal x_1. The new strategy combination is, therefore, a SPE of the negotiation game with average payoff $(x_1, 1 - x_1)$. Hence, we can select an efficient SPE that satisfies (2.7) and

$$x_1 + y_1 = 1 \tag{2.8}$$

In a SPE of $NG_1(\delta)$, if player 2 rejects player 1's proposal in the first period, players must play one stage game outcome, say $a \in A$, and a SPE's in $NG_2(\delta)$, with average payoff $(x_2(a), y_2(a))$, where $x_2(a) + y_2(a) \leq 1$. Therefore, if player 2 rejects player 1's proposal in the first period of $NG_1(\delta)$, player 2's average payoffs are bounded above by the maximum of all possible continuation payoffs. In any SPE of $NG_1(\delta)$, player 1 will not propose such that player 2's payoff is more than his maximum continuation payoffs. Therefore, the equilibrium payoff (x_1, y_1) should satisfies

$$y_1 \leq \sup_{a \in A}\{(1-\delta)u_2(a) + \delta y_2(a)\} \tag{2.9}$$

On the other hand, if player 2 does reject player 1's proposal, player 1 should not deviate from $a \in A$ in the stage game. Subgame perfectness then requires that

$$\begin{aligned} (1-\delta)\max_{a_1' \in A_1} u_1(a_1', a_2) + \delta^2 m_1(\delta) &\leq (1-\delta)u_1(a) + \delta x_2(a) \\ &\leq (1-\delta)u_1(a) + \delta(1-y_2(a)) \end{aligned}$$

which implies that

$$\delta y_2(a) \leq \delta(1-\delta m_1(\delta)) - (1-\delta)(\max_{a_1' \in A_1} u_1(a_1', a_2) - u_1(a)) \tag{2.10}$$

Substituting (2.10) into (2.9), with (2.6), we have

$$\begin{aligned} y_1 &\le (1-\delta)\sup_{a\in A}\{u_1(a)+u_2(a)-\max_{a_1'\in A_1} u_1(a_1',a_2)\}+\delta(1-\delta m_1(\delta)) \\ &= (1-\delta)\max_{a\in A}\{u_1(a)+u_2(a)-\max_{a_1'\in A_1} u_1(a_1',a_2)\}+\delta(1-\delta m_1(\delta)) \\ &= (1-\delta)y_1^* + \delta(1-\delta m_1(\delta)) \end{aligned}$$

with (2.7) and (2.8),

$$\begin{aligned} 1-m_1(\delta)-\epsilon &\le 1-x_1 = y_1 \le (1-\delta)y_1^* + \delta(1-\delta m_1(\delta)) \\ \Rightarrow \quad m_1(\delta) &\ge \frac{1-y_1^*}{1+\delta} - \frac{\epsilon}{1-\delta^2} \end{aligned}$$

Since ϵ can be chosen arbitrarily small, the last inequality implies that $m_1(\delta)$ is bounded below by $\frac{1}{1+\delta}(1-y_1^*)$. **Q.E.D.**

In the following theorem, we construct a subgame perfect equilibrium of $NG_1(\delta)$ [$NG_2(\delta)$] in which player 1's average equilibrium payoff is $\frac{1}{1+\delta}(1-y_1^*)$ [$\frac{\delta}{1+\delta}(1-y_1^*)$] if the discount factor δ is large enough. From Theorem 2.3 then, this SPE is an optimal punishment equilibrium for player 1 in $NG_1(\delta)$ [$NG_2(\delta)$]. In the optimal punishment equilibrium, the outcome is the same as the SPE when $(0, y_1^*)$ is fixed to be the disagreement payoffs to the players in every period Refer to Figure 2.4 for diagrammatic representation.

Theorem 2.4 *There exists a* $\underline{\delta} \in (0,1)$ *such that, for* $\delta \in (\underline{\delta}, 1)$, *the average payoff vector*

$$\left(\frac{1-y_1^*}{1+\delta}, \frac{\delta+y_1^*}{1+\delta}\right) \qquad \left[\left(\frac{\delta(1-y_1^*)}{1+\delta}, \frac{1+\delta y_1^*}{1+\delta}\right)\right]$$

can be supported by a subgame perfect equilibrium in the game $NG_1(\delta)$ [$NG_2(\delta)$].

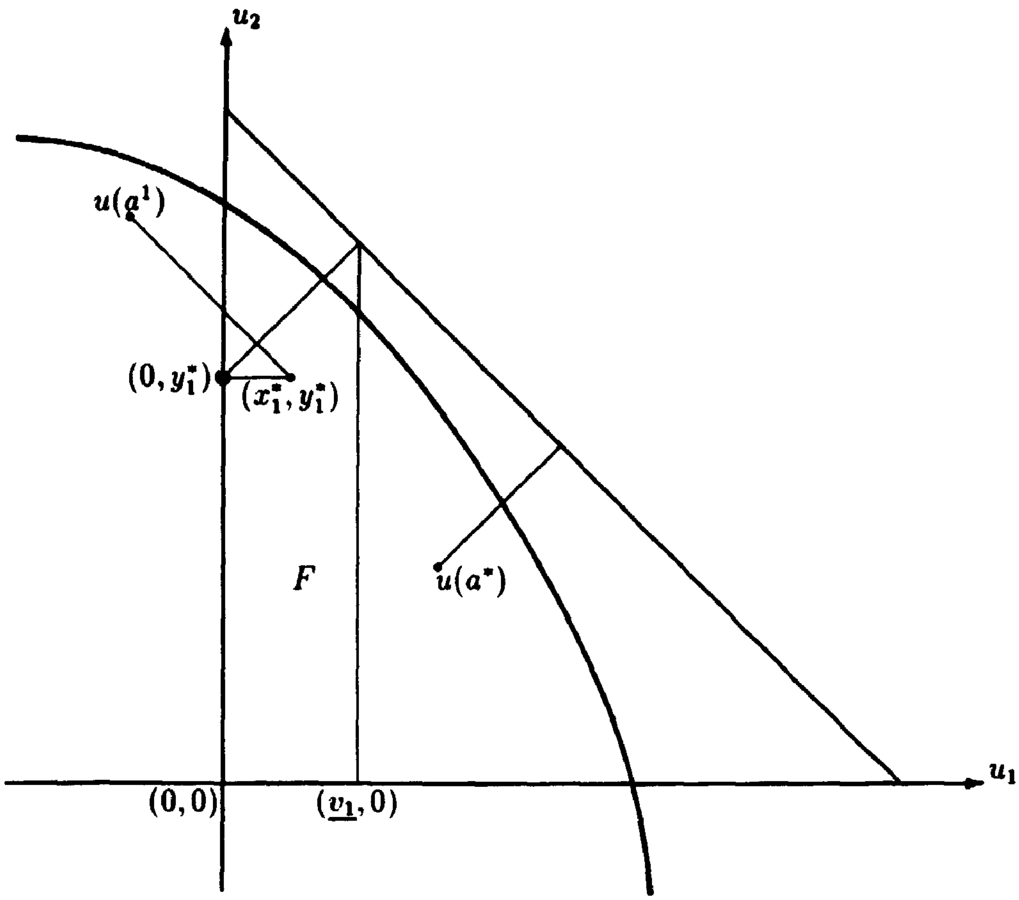

Figure 2.4: An Optimal Punishment Equilibrium For Player 1

Proof: Note that the payoffs correspond to the perfect offers for a disagreement path with payoff $(0, y_1^*)$ in every period. If $(0, y_1^*)$ is a Nash equilibrium of the stage game G, then the result follows from Theorem 2.2 and no further proof is necessary. Therefore, suppose that $(0, y_1^*)$ is not a Nash equilibrium of G. Under this assumption the proof is lengthy. We will proceed in three steps. First we construct $\underline{\delta}$. Then, imposing a regularity condition on G which guarantees the existence of the required disagreement path, we give equilibrium strategies attaining the punishment payoffs. Finally, we modify the strategies to hold when the regularity condition is relaxed. Strictly speaking, only the first and third step are required for a complete proof, and the reader may skip the second step. However, additional insight is generated by the arguments in the second step, facilitating understanding of the third.

Step 1: Suppose a^* is a Nash equilibrium in the stage game G. From (2.6), we have $y_1^* \geq u_2(a^*)$. Since $u_1(a^*) \geq 0$ and $(0, y_1^*) \neq u(a^*)$, we must have $y_1^* > u_2(a^*) - u_1(a^*)$. Therefore, $\epsilon_0 = [y_1^* - u_2(a^*) + u_1(a^*)]/3$ is strictly positive. From the definition of y_1^*, $\exists\, a^1 \in A$ such that

$$x_1^* + y_1^* = u_1(a^1) + u_2(a^1) \quad \text{and} \quad x_1^* = \max_{a_1' \in A_1} u_1(a_1', a_2^1) \tag{2.11}$$

Let $d = \max[u_i(a') - u_i(a'')]$, $\forall a', a'' \in A$ and $i = 1, 2$. Since the set $u(A)$ is compact, d must be finite. $\forall a = (a_1, a_2) \in A$ and $i = 1, 2$, we have

$$\max_{a_i' \in A_i} u_i(a_i', a_{-i}) - u_i(a) \leq d$$

Consider the following four functions of $\delta \in (0, 1]$,

$$\begin{aligned}
c_1(\delta) &= \epsilon_0 \frac{\delta^2}{1+\delta} - (1-\delta)d \\
c_2(\delta) &= \frac{\delta}{1+\delta}[\delta(y_1^* - 2\epsilon_0 - u_2(a^*)) + u_1(a^*)] - (1-\delta)d \\
c_3(\delta) &= \frac{\delta}{1+\delta}[y_1^* - 2\epsilon_0 - u_2(a^*) + \delta u_1(a^*)] - (1-\delta)d \\
c_4(\delta) &= 2\epsilon_0\delta^2 - (1-\delta^2)(u_2(a^1) - y_1^*)
\end{aligned}$$

Since these four functions are positive and continuous at $\delta = 1$, there must exist $\underline{\delta} \in (0, 1)$ such that, $\forall \delta \in (\underline{\delta}, 1)$, the functions $c_1(\delta)$, $c_2(\delta)$, $c_3(\delta)$ and $c_4(\delta)$ are all positive. Equivalently, $\forall \delta \in (\underline{\delta}, 1)$, the following four inequalities hold

$$(1-\delta)d \leq \delta\frac{\delta}{1+\delta}[1 - y_1^* + \epsilon_0] - \delta\frac{\delta}{1+\delta}[1 - y_1^*] \tag{2.12}$$

$$(1-\delta)d \leq \delta\frac{1+\delta(y_1^* - 2\epsilon_0)}{1+\delta} - \delta\frac{1 - u_1(a^*) + \delta u_2(a^*)}{1+\delta} \tag{2.13}$$

$$(1-\delta)d \leq \delta\frac{\delta + y_1^* - 2\epsilon_0}{1+\delta} - \delta\frac{\delta - \delta u_1(a^*) + u_2(a^*)}{1+\delta} \tag{2.14}$$

$$\delta^2(y_1^* - 2\epsilon_0) \leq y_1^* - (1-\delta^2)u_2(a^1) \tag{2.15}$$

This concludes the derivation of $\underline{\delta}$.

Step 2: For a given $\delta \in (\underline{\delta}, 1)$, we now construct a subgame perfect equilibrium under the following regularity condition.

Regularity condition: *There exists an outcome $a \in A$ such that (x_1^*, y_1^*) is a strictly convex combination of $u(a^1)$ and $u(a)$.*

Under this regularity condition, $\exists\, T < \infty$ and $a^2 \in A$ such that

$$y_1^* - 2\epsilon_0 \leq u_2(a^2) \leq y_1^* - \epsilon_0 \text{ and } (x_1^*, y_1^*) = (1-\delta^T)u(a^1) + \delta^T u(a^2) \qquad (2.16)$$

Consider the *modified simple strategy profile* $\sigma(\pi_0, \pi_1^1, \pi_1^2, \pi_2^1, \pi_2^2)$ in the infinitely repeated game $G^\infty(\delta)$, where $\pi_0 = \pi_1^1 = \{\hat{a}^t\}_{t=1}^\infty$ with

$$\hat{a}^t = \begin{cases} mx^1 & \text{if t is even} \\ a^1 & \text{if t is odd and } t \leq T \\ a^2 & \text{if t is odd and } t > T \end{cases}$$

and $\pi_1^2 = \{\bar{a}^t\}_{t=1}^\infty$ with

$$\bar{a}^t = \begin{cases} mx^1 & \text{if t is odd} \\ a^1 & \text{if t is even and } t \leq T \\ a^2 & \text{if t is even and } t > T \end{cases}$$

and $\pi_2^1 = \pi_2^2 = (a^*)^\infty$.

A modified simple strategy profile is defined as follows: At the beginning of the game, players play the stage game G according to the initial path π_0. If player i deviates from the current path in an even (odd) period, then the players will start to play the stage game according to the path π_i^1 (π_i^2) from the next period on, for $i = 1, 2$.

Let $g = (g_1, g_2)$ be the strategy functions induced by $\sigma(\pi_0, \pi_1^1, \pi_1^2, \pi_2^1, \pi_2^2)$. Consider the following strategies in th. negotiation game $NG_1(\delta)$. $\forall\, h_1(t) = a(t) \oplus b(t) \in H_1$, $h_2(t) = h_1(t) \oplus b^{t+1} \in H_2$ and $h_3(t) = h_2(t) \oplus \{N\} \in H_3$; for an odd period $(t+1)$,

$$\begin{aligned} f_1(h_1(t)) &= P_1(\pi(g|_{a(t)})) \\ f_2(h_2(t)) &= \begin{cases} Y & \text{if } b^t \leq f_1(h_1(t)) \\ N & \text{otherwise} \end{cases} \\ f(h_3(t)) &= g(a(t)) \end{aligned}$$

and for an ever period $(t+1)$,

$$f_2(h_1(t)) = P_2(\pi(g|_{a(t)}))$$

$$
\begin{aligned}
f_1(h_2(t)) &= \begin{cases} Y & \text{if } b^t \geq f_2(h_1(t)) \\ N & \text{otherwise} \end{cases} \\
f(h_3(t)) &= g(a(t))
\end{aligned}
$$

We will now verify subgame perfectness of the strategy profile f. Since all proposals are perfect by construction, f is a perfect equilibrium in $NG_1(\delta)$ if players do not deviate after every type 3 history. Given $h_3(t) = a(t) \oplus b(t+1) \oplus \{N\} \in H_3$, there are four cases which have to be considered.

Case 1: $(t+1)$ *is odd and* $f(h_3(t)) = a^1$

If player 1 deviated from a^1, according to the strategies above, player 2 would propose

$$\frac{\delta}{1+\delta}(1 - y_1^*)$$

in period $(t + 2)$, since the continuation disagreement payoff path is π_1^2. However, if player 1 follows his strategies, player 2 will propose at least

$$\frac{1-\delta}{\delta}u_2(a^1) + \frac{\delta^2 - y_1^*}{\delta(1+\delta)}$$

in period $(t+2)$, the perfect proposal for path $\{a^1\}^{T-1} \oplus \{a^2\}^\infty$. Since

$$
\begin{aligned}
(1-\delta)u_1(a) + \delta P_2(\pi(g|_{a(t)\oplus a^1})) &= (1-\delta)u_1(a) + (1-\delta)u_2(a^1) + \frac{\delta^2 - y_1^*}{1+\delta} \\
= (1-\delta)(x^* + y_1^*) + \frac{\delta^2 - y_1^*}{1+\delta} &= (1-\delta)x^* + \frac{\delta^2}{1+\delta}(1 - y_1^*) \\
&= (1-\delta)\max_{a_1' \in A_1} u_1(a_1', a_2^1) + \delta\frac{\delta}{1+\delta}(1 - y_1^*)
\end{aligned}
$$

player 1 will not deviate from a^1.

Consider player 2 next. If player 2 deviated from a^1, according to the strategies above he would have to propose $\frac{1}{1+\delta}(\delta + u_1(a^*) - \delta u_2(a^*))$, because the continuation disagreement payoff path would be π_1^2. Player 2's payoff from such a proposal would be

$$\frac{1 - u_1(a^*) + \delta u_2(a^*)}{1+\delta}$$

However, if player 2 follows his strategies, he will at most propose $\frac{\delta}{1+\delta}(1-u_2(a^2))$ in the next period. Then player 2's payoff will be at least

$$\frac{1}{1+\delta}(1+\delta u_2(a^2))$$

Inequality (2.13) and $u_2(a^2) \geq y_1^* - 2\epsilon_0$ imply that

$$(1-\delta)[\max_{a_2' \in A_2} u_2(a_1^1, a_2') - u_2(a^1)] \leq (1-\delta)d \leq \delta\frac{1+\delta u_2(a^2)}{1+\delta} - \delta\frac{1-u_1(a^*)+\delta u_2(a^*)}{1+\delta}$$

Therefore, player 2 will not deviate from a^1 either.

Case 2: $(t+1)$ *is odd and* $f(h_3(t)) = a^2$

If player 1 deviated from a^2, player 2 would, according to the strategies, propose $\frac{\delta}{1+\delta}(1-y_1^*)$ in period $(t+2)$. If player 1 follows the strategies, however, player 2 will propose

$$\frac{\delta}{1+\delta}(1-u_2(a^2))$$

in period $(t+2)$. Inequality (2.12) and $u_2(a^2) \leq y_1^* - \epsilon_0$ imply that

$$(1-\delta)[\max_{a'_1 \in A_1} u_1(a'_1, a_2^2) - u_1(a^2)] \leq (1-\delta)d \leq \frac{\delta^2}{1+\delta}(1-u_2(a^2)) - \frac{\delta^2}{1+\delta}(1-y_1^*)$$

Therefore, player 1 will not deviate from a^2.

If player 2 deviated from a^2, as in Case 1, his average payoff from the next period on would be $\frac{1}{1+\delta}(1-u_1(a^*)+\delta u_2(a^*))$. If player 2 follows the strategies, his payoff will be $\frac{1}{1+\delta}(1+\delta u_2(a^2))$. Inequality (2.13) and $u_2(a^2) \geq y_1^* - 2\epsilon_0$ imply that

$$(1-\delta)[\max_{a_2' \in A_2} u_2(a_1^2, a_2') - u_2(a^2)] \leq (1-\delta)d \leq \delta\frac{1+\delta u_2(a^2)}{1+\delta} - \delta\frac{1-u_1(a^*)+\delta u_2(a^*)}{1+\delta}$$

Therefore, player 2 will not deviate from a^2 either.

Case 3: $(t+1)$ *is even and* $f(h_3(t)) = mx^1$

Player 1 will not deviate, because mx^1 is the minimax strategy against player 1. If player 2 deviated from mx^1, according to the strategies, player 1 would make a proposal yielding

$$\frac{\delta - \delta u_1(a^*) + u_2(a^*)}{1+\delta}$$

to player 2 in period $(t+2)$. However, if player 2 follows the strategies, player 1 will make a proposal yielding, at least,

$$\frac{\delta + u_2(a^2)}{1+\delta}$$

to player 2. Inequality (2.14) and $u_2(a^2) \geq y_1^* - 2\epsilon_0$ imply that

$$(1-\delta)[\max_{a_2' \in A_2} u_2(mx_1^1, a_2') - u_2(mx^1)] \leq (1-\delta)d \leq \delta\frac{\delta + u_2(a^2)}{1+\delta} - \delta\frac{\delta - \delta u_1(a^*) + u_2(a^*)}{1+\delta}$$

Therefore, player 2 will not deviate from mx^1.

Case 4: $f(h_3(t)) = a^*$

Since a^* is a Nash equilibrium in the stage game, $\forall i = 1,2$, if player i deviates from a^*, his payoff cannot be increased in the current period and his payoff thereafter may only be decreased. Therefore, player i should not deviate from a^* for $i = 1,2$.

This concludes the proot that (under the regularity condition) f is a subgame perfect equilibrium strategy profile for the game $NG_1(\delta)$, and that the outcome of this equilibrium is that player 1's perfect proposal is accepted by player 2 in the first period, yielding a payoff vector $(\frac{1}{1+\delta}(1-y_1^*), \frac{1}{1+\delta}(\delta + y_1^*))$.

Step 3: In the equilibrium under the regularity condition, however, we never observe the stage game outcome a^2. Therefore, we may construct a payoff equivalent subgame perfect equilibrium without the regularity condition.

Consider the following strategies for $NG_1(\delta)$. They are defined recursively. In the first period, players' strategies are

$$\begin{aligned} f_1(\emptyset) &= \frac{1-y_1^*}{1+\delta} \\ f_2(b^1) &= \begin{cases} Y & \text{if } b^1 \leq \frac{1}{1+\delta}(1-y_1^*) \\ N & \text{otherwise} \end{cases} \\ f(b^1 \oplus \{N\}) &= \begin{cases} a^* & \text{if } b^1 \leq \frac{1}{1+\delta}(1-y_1^*) \\ a^1 & \text{otherwise} \end{cases} \end{aligned}$$

$\forall h_1(t) = a(t) \oplus b(t) \in H_1$, $h_2(t) = a(t) \oplus b(t+1) \in H_2$ and $h_3(t) = a(t) \oplus b(t+1) \oplus \{N\} \in$

H_3; for an odd period $(t+1)$,

$$f_1(h_1(t)) = \begin{cases} \frac{1}{1+\delta}(1+\delta u_1(a^*)-u_2(a^*)) & \text{if either } f(h_3(t-1)) = a^* \\ & \text{or } a_1^t = f_1(h_3(t-1)), a_2^t \neq f_2(h_3(t-1)) \\ \frac{1}{1+\delta}(1-y_1^*) & \text{otherwise} \end{cases}$$

$$f_2(h_2(t)) = \begin{cases} Y & \text{if } b^{t+1} \leq f_1(h_1(t)) \\ N & \text{otherwise} \end{cases}$$

$$f(h_3(t)) = \begin{cases} a^* & \text{if either } f(h_3(t-1)) = a^* \text{ or } b^{t+1} \leq f_1(h_1(t)) \\ & \text{or } a_1^t = f_1(h_3(t-1)),\ a_2^t \neq f_2(h_3(t-1)) \\ a^1 & \text{otherwise} \end{cases}$$

and for an even period $(t+1)$,

$$f_2(h_1(t)) = \begin{cases} \frac{1}{1+\delta}(\delta+u_1(a^*)-\delta u_2(a^*)) & \text{if either } f(h_3(t-1)) = a^* \\ & \text{or } a_1^t = f_1(h_3(t-1)), a_2^t \neq f_2(h_3(t-1)) \\ \frac{\delta}{1+\delta}(1-y_1^*) & \text{if } a_1^t \neq f_1(h_3(t-1)), a_2^t = f_2(h_3(t-1)) \\ \frac{1-\delta}{\delta}u_2(a^1) + \frac{\delta^2-y_1^*}{\delta(1+\delta)} & \text{otherwise} \end{cases}$$

$$f_1(h_2(t)) = \begin{cases} Y & \text{if } b^{t+1} \geq f_2(h_1(t)) \\ N & \text{otherwise} \end{cases}$$

$$f(h_3(t)) = \begin{cases} a^* & \text{if either } a_1^t = f_1(h_3(t-1));\ a_2^t \neq f_2(h_3(t-1)) \\ & \text{or } b^{t+1} < f_2(h_1(t)) \text{ or } f(h_3(t-1)) = a^* \\ mx^1 & \text{otherwise} \end{cases}$$

We will now prove the subgame perfectness of the strategy combination f. Since the proposals are no longer perfect, we have to consider every part of the strategies for every player in every period.

In an odd period $(t+1)$, there are two cases to be considered.

Case 1: *either* $a_1^t = f_1(h_3(t-1)), a_2^t \neq f_2(h_3(t-1))$; *or* $f(h_3(t-1)) = a^*$; *or* $b^{t+1} < f_1(h_1(t))$

Player 2 is the last deviator, i.e. player 2 either deviated in the stage game or rejected a proposal which should have been accepted. The disagreement payoff will be $u(a^*)$ for ever after. Since a^* is a Nash equilibrium in the stage game, Theorem 2.2 implies that players should not deviate from f.

Case 2: *otherwise, i.e. not Case 1*

$f(h_3(t)) = a^1$. If player 1 deviated from a^1, according to the strategies, player 2 would then propose $\frac{\delta}{1+\delta}(1-y_1^*)$ instead of $\frac{1-\delta}{\delta}u_2(a^1)+\frac{1}{\delta(1+\delta)}(\delta^2-y_1^*)$. Since

$$\begin{aligned}(1-\delta)u_1(a^1)+(1-\delta)u_2(a^1)+\frac{\delta^2-y_1^*}{1+\delta} &= (1-\delta)[u_1(a^1)+u_2(a^1)]+\frac{\delta^2-y_1^*}{1+\delta}\\ =(1-\delta)(x^*+y_1^*)+\frac{\delta^2-y_1^*}{1+\delta} &= (1-\delta)x^*+\frac{\delta^2}{1+\delta}(1-y_1^*)\\ &= (1-\delta)\max_{a_1'\in A_1}u_1(a_1',a_2^1)+\delta\frac{\delta}{1+\delta}(1-y_1^*),\end{aligned}$$

player 1 will not deviate from a^1.

If player 2 deviated from a^1, player 2 would have to propose $\frac{1}{1+\delta}(1-u_1(a^*)+\delta u_2(a^*))$ instead of $\frac{\delta+y_1^*}{\delta(1+\delta)}-\frac{1-\delta}{\delta}u_2(a^1)$. Inequalities (2.13) and (2.15) imply that

$$\begin{aligned}(1-\delta)[\max_{a_2'\in A_2}u_2(a_1^1,a_2') &- u_2(a^1)] \le (1-\delta)d\\ &\le \delta\frac{1+\delta(y_1^*-2\epsilon_0)}{1+\delta}-\delta\frac{1-u_1(a^*)+\delta u_2(a^*)}{1+\delta}\\ &= \frac{\delta}{1+\delta}+\frac{\delta^2}{1+\delta}(y_1^*-2\epsilon_0)-\frac{\delta}{1+\delta}(1-u_1(a^*)+\delta u_2(a^*))\\ &\le \frac{\delta+y_1^*}{1+\delta}-(1-\delta)u_2(a^1)-\frac{\delta}{1+\delta}(1-u_1(a^*)+\delta u_2(a^*))\end{aligned}$$

Therefore, player 2 will not deviate from a^1.

Player 2 will also not deviate from f_2 in responding player 1's proposal, because his payoffs from rejecting the wrong and right proposal are

$$\frac{\delta+y_1^*}{1+\delta}\quad\text{and}\quad(1-\delta)u_2(a^1)+\frac{\delta}{1+\delta}(1-u_1(a^*)+\delta u_2(a^*))<\frac{\delta+y_1^*}{1+\delta}$$

respectively.

Finally, player 1 should not deviate from f_1 in making the proposal, because wrong proposals will be rejected and player 2 will propose $\frac{1-\delta}{\delta}u_2(a^1)+\frac{1}{\delta(1+\delta)}(\delta^2-y_1^*)$ in the next period. Since $x_1^*+y_1^*\le 1$,

$$\begin{aligned}(1-\delta)u_1(a^1)+(1-\delta)u_2(a^1)+\frac{\delta^2-y_1^*}{1+\delta} &= (1-\delta)(x_1^*+y_1^*)+\frac{\delta^2-y_1^*}{1+\delta}\\ \le(1-\delta)+\frac{\delta^2-y_1^*}{1+\delta} &= \frac{1-y_1^*}{1+\delta}\end{aligned}$$

Therefore, player 1 will not deviate. This concludes the checks for an odd period.

In an even period $(t+1)$, when either $f(h_3(t-1)) = a^*$ or $\delta^{t+1} < f_2(h_1(t))$ or $a_1^t = f_1(h_3(t-1)), a_2^t \neq f_2(h_3(t-1))$, Theorem 2.2 implies that players should not deviate, because the disagreement payoff in every period is the Nash equilibrium payoff $u(a^*)$. Otherwise, that is if 2 has not deviated last, there are two cases that have to be considered.

Case 1: $a_1^t \neq a_1^1$ *and* $a_2^t = a_2^1$

Player 1 will not deviate from mx^1, because mx^1 is his minimax strategy. If player 2 deviated from mx^1, the disagreement payoff would be $u(a^*)$ in every period thereafter, and player 1 will propose $\frac{1}{1+\delta}(\delta - \delta u_1(a^*) + u_2(a*))$ to player 2 instea[illegible] $\frac{\delta+y_1^*}{1+\delta}$. Inequality (2.14) implies that

$$(1-\delta)[\max_{a_2'\in A_2} u_2(mx_1^1, a_2') - u_2(mx^1)] \leq (1-\delta)d \leq \delta\frac{\delta+y_1^*}{1+\delta} - \delta\frac{\delta-\delta u_1(a^*)+u_2(a^*)}{1+\delta}$$

Therefore, player 2 will not deviate from mx^1.

Player 1 should not deviate from $f_1(h_2(t))$, because player 1's payoff is $\frac{\delta}{1+\delta}(1-y_1^*)$ from rejecting player 2's proposal. Player 2 should not deviate from $f_2(h_1(t))$, because wrong proposals will be rejected and player 1 will offer $\frac{1}{1+\delta}(\delta - \delta u_1(a^*) + u_2(a^*))$ to player 2 in the next period. Since $\delta u_2(a^*) - u_1(a^*) < u_2(a^*) - u_1(a^*) < y_1^*$,

$$(1-\delta)u_2(a^*) + \delta\frac{\delta-\delta u_1(a^*)+u_2(a^*)}{1+\delta} < \frac{1-u_1(a^*)+\delta u_2(a^*)}{1+\delta} < \frac{1+y_1^*}{1+\delta}$$

Therefore, player 2 will not deviate from $f_2(h_1(t))$.

Case 2: $a^t = a^1$, *i.e. not Case 1*

For the same reasons as in Case 1, players should not deviate from $f(h_3(t))$. Player 1 should only accept an equilibrium proposal, since his payoff from rejecting such a proposal is equal to $\frac{\delta}{1+\delta}(1-y_1^*)$ which is less than $\frac{1-\delta}{\delta}u_2(a^1) + \frac{1}{\delta(1+\delta)}(\delta^2 - y_1^*)$ due to

$$\begin{aligned} y_1^* \leq u_2(a^1) &\Rightarrow (1-\delta^2)y_1^* \leq (1-\delta^2)u_2(a^1) \\ &\Rightarrow \delta^2(1-y_1^*) \leq (1-\delta^2)u_2(a^1) + \delta^2 + y_1^* \\ &\Rightarrow \frac{\delta}{1+\delta}(1-y_1^*) \leq \frac{1-\delta}{\delta}u_2(a^1) + \frac{1}{\delta(1+\delta)}(\delta^2 - y_1^*) \end{aligned}$$

Also, since $u_2(a^*) - \frac{1}{\delta}u_1(a^*) < u_2(a^*) - u_1(a^*) < y_1^*$, player 1 should reject a wrong proposal, because his payoff from rejecting a wrong proposal is equal to

$$
\begin{aligned}
(1-\delta)u_1(a^*) + \frac{\delta}{1+\delta}(1+\delta u_1(a^*)-u_2(a^*)) &= \frac{\delta}{1+\delta}(1-u_2(a^*) + \frac{1}{\delta}u_1(a^*)) \\
&> \frac{\delta}{1+\delta}(1-y_1^*)
\end{aligned}
$$

Therefore, player 1 will not deviate from $f_1(h_2(t))$.

Player 2 should make the equilibrium proposal, because a wrong proposal will be rejected, and player 2's payoff will be

$$
\begin{aligned}
(1-\delta)u_2(a^*) \;+\; \frac{\delta}{1+\delta}(\delta-\delta u_1(a^*)+u_2(a^*)) &< \frac{1}{1+\delta}(1-u_1(a^*)+\delta u_1(a^*)) \\
&< \frac{\delta+y_1^*}{\delta(1+\delta)} - \frac{1-\delta}{\delta}u_2(a^1) \\
&= 1 - \frac{1-\delta}{\delta}u_2(a^1) + \frac{\delta^2-y_1^*}{\delta(1+\delta)}
\end{aligned}
$$

Therefore, player 2 will not deviate from $f_2(h_1(t))$.

We conclude that the strategy profile f constitutes a subgame perfect equilibrium of the negotiation game $NG_1(\delta)$. The equilibrium outcome is that player 1's proposal is accepted by player 2 in the first period, yielding average payoffs of $(\frac{1}{1+\delta}(1-y_1^*), \frac{1}{1+\delta}(\delta+y_1^*))$. Finally, consider the one period history $h_1(1) = 1 \div (a_1', a_2^1)$ where $a_1' \neq a_1^*$. $f|_{h_1(1)}$ is a subgame perfect equilibrium of $NG_1(\delta)|_{h_1(1)}$ which is $NG_2(\delta)$, and the equilibrium outcome is that player 2's proposal is accepted by player 1 in the first period, yielding average payoffs of $(\frac{\delta}{1+\delta}(1-y_1^*), \frac{1}{1+\delta}(1+\delta y_1^*))$. **Q.E.D.**

Combining the results of Theorem 2.3 and Theorem 2.4, we may conclude that equilibria in Theorem 2.4 are the optimal punishment equilibria for player 1 in $NG_1(\delta)$ and $NG_2(\delta)$. Player 1's worst punishment payoffs in $NG_1(\delta)$ and $NG_2(\delta)$ are thus $\frac{1}{1+\delta}(1-y_1^*)$ and $\frac{\delta}{1+\delta}(1-y_1^*)$ respectively for δ large enough.

By analogous arguments, we can obtain that optimal punishment equilibria for player 2 exist in $NG_1(\delta)$ and $NG_2(\delta)$ and his punishment equilibrium payoffs are

$\frac{\delta}{1+\delta}(1-x_2^*)$ and $\frac{1}{1+\delta}(1-x_2^*)$ respectively. Here x_2^* is given by

$$x_2^* = \max_{a\in A}\{u_1(a) + u_2(a) - \max_{a_2'\in A_2} u_2(a_1, a_2')\} \tag{2.17}$$

Finally, define

$$\underline{v_1} = \frac{1}{2}(1-y_1^*) \quad \text{and} \quad \underline{v_2} = \frac{1}{2}(1-x_2^*). \tag{2.18}$$

From Theorem 2.3, $\forall\ \delta \in (0,1)$, player i's average payoffs in all subgame perfect equilibrium of $NG_i(\delta)$ are bounded below by $\underline{v_i}$.

2.3.4 Equilibrium Outcomes

In this section, we characterize the set of subgame perfect equilibrium of the negotiation game as the discount factor gets large. The theorem proves that any feasible payoff which dominates $(\underline{v_1}, \underline{v_2})$ can be supported as a SPE outcome in the negotiation game when the discount factor is large enough.

Theorem 2.5 *For a given feasible payoff vector* (v_1, v_2) *of the negotiation game* $NG_1(\delta)$ $[NG_2(\delta)]$ *such that* $(v_1, v_2) > (\underline{v_1}, \underline{v_2})$, *there is* $\underline{\delta} \in (0,1)$ *such that* $\forall\ \delta \in (\underline{\delta}, 1)$, $NG_1(\delta)$ $[NG_2(\delta)]$ *has a subgame perfect equilibrium with average payoff* (v_1, v_2).

Before we formally prove the theorem, we first briefly discuss the equilibrium strategy profile. First, we find an outcome path which leads to the average payoff (v_1, v_2). It consists of the agreement players reach in some period T, and the outcomes of the stage game G they play in every period before the agreement is reached. The outcome path also has the feature that in every period both players have continuation average payoffs above their minimum equilibrium payoffs. We then implement the outcome path by the following strategies. In every period before the last, the player who makes the proposal demands the whole surplus for himself. Any other offer will be considered as a deviation by him and he will be punished by his punishment equilibrium starting from the next period. The other player accepts a proposal made before the last period only if the proposal pays him more than he could obtain if the other player is punished. In the stage game, players play strategies leading to the appropriate outcome for the period (as specified in the outcome path). If a player deviates from his strategy in the stage game, he is again punished from the next period on. As is common, simultaneous deviations by the players will be ignored.

With the above strategies, after a player has deviated from the proposed equilibrium path, the strategies are subgame perfect, since optimal punishment equilibria are SPEs. Therefore, we do not need to verify the perfectness of the strategies out off the equilibrium path. Thus, if neither player can benefit by deviating from the proposed path, the overall strategies are subgame perfect.

For an outcome path of the negotiation game $\hat{\pi}(T) = \hat{b}(T) \dotplus \hat{a}(T-1) \dotplus \{Y\}$, where $\hat{b}(T) = \{\hat{b}^t\}_{t=1}^{T}$ and $\hat{a}(T-1) = \{\hat{a}^t\}_{t=1}^{T-1}$, the indicator function

$$ID(\cdot) : H_1 \cup H_2 \cup H_3 \rightarrow \{0\} \cup \{(i,t) | i = 1,2;\ 1 \leq t \leq T\}$$

is defined recursively as follows: at the beginning of period 1, the history is the empty set and the indicator function takes the value 0, i.e. $ID(\emptyset) = 0$. Thereafter, $\forall\, h_1(t) = h_3(t-1) \dotplus (a_1^t, a_2^t) \in H_1$, $h_2(t) = h_1(t) \dotplus b^{t+1} \in H_2$, and $h_3(t) = h_2(t) \dotplus \{N\} \in H_3$,

$$ID(h_1(t)) = \begin{cases} (1,t) & \text{if } a_1^t \neq \hat{a}_1^t;\ a_2^t = \hat{a}_2^t \text{ and } ID(h_3(t-1)) = 0 \\ (2,t) & \text{if } a_1^t = \hat{a}_1^t;\ a_2^t \neq \hat{a}_2^t \text{ and } ID(h_3(t-1)) = 0 \\ ID(h_3(t-1)) & \text{otherwise} \end{cases}$$

$$ID(h_2(t)) = \begin{cases} (1,t+1) & \text{if } b^{t+1} \neq \hat{b}^{t+1} \text{ and } (t+1) \text{ is odd and } ID(h_1(t)) = 0 \\ (2,t+1) & \text{if } b^{t+1} \neq \hat{b}^{t+1} \text{ and } (t+1) \text{ is even and } ID(h_1(t)) = 0 \\ ID(h_1(t)) & \text{otherwise} \end{cases}$$

$$ID(h_3(t)) = \begin{cases} (1,t+1) & \text{if } t+1 = T \text{ and } T \text{ is odd and } ID(h_2(t)) = 0 \\ (2,t+1) & \text{if } t+1 = T \text{ and } T \text{ is even and } ID(h_2(t)) = 0 \\ ID(h_2(t)) & \text{otherwise} \end{cases}$$

The indicator function takes two types of possible values, 0 and (i,t). The value 0 means that no player has deviated from the path $\hat{\pi}(T)$, and (i,t) means that player i first deviated from the path in period t, where $1 \leq t \leq T$. We now prove Theorem 2.5.

Proof: We prove the theorem for $NG_1(\delta)$. Let a^* be a Nash equilibrium of the stage game G. Since $(v_1, v_2) > (\underline{v_1}, \underline{v_2})$, $\exists\ \epsilon_0 = \min\{v_1 - \underline{v_1}, v_2 - \underline{v_2}\}/2 > 0$. According to Theorem 2.4, $\exists\ \underline{\delta}$ such that, $\forall\ \delta \in (\underline{\delta}, 1)$, the game $NG_j(\delta)$ has an optimal punishment equilibrium for player i with strategy f^{ij}, and

$$\underline{v_1}+\epsilon_0 \ \geq\ \max\left\{\frac{1-y_1^*}{1+\delta}; (1-\delta)u_1(a^*)+\delta\frac{1-y_1^*}{1+\delta}; 1-(1-\delta)u_2(a^*)-\delta\frac{\delta+y_1^*}{1+\delta}\right\} \quad (2.19)$$

$$\underline{v_2}+\epsilon_0 \ \geq\ \max\left\{\frac{1-x_2^*}{1+\delta}; (1-\delta)u_2(a^*)+\delta\frac{1-x_2^*}{1+\delta}; 1-(1-\delta)u_1(a^*)-\delta\frac{\delta+x_2^*}{1+\delta}\right\} \quad (2.20)$$

$$\frac{1-\delta}{\delta}d \ \leq\ \epsilon_0 = v_i - (\underline{v_i} + \epsilon_0) \qquad \text{for } i = 1, 2 \quad (2.21)$$

$\forall\ \delta \in (\underline{\delta}, 1)$, $\exists\ \hat{a} \in A$, $\hat{b} \in [0, 1]$ and a positive T (which may or may not be finite), such that

$$(v_1, v_2) = (1 - \delta^T)u(\hat{a}) + \delta^T(\hat{b}, 1 - \hat{b}) \quad \text{and} \quad (\hat{b}, 1 - \hat{b}) > v > u(\hat{a}) \quad (2.22)$$

Consider the outcome path $\hat{\pi}(T) = \hat{b}(T) \oplus \hat{a}(T-1) \oplus \{Y\}$ of $NG_1(\delta)$, where

$$\hat{a}(T-1) = \{\hat{a}\}_{t=1}^{T-1} \in A^{T-1} \quad \text{and} \quad \hat{b}(T) = (1, 0, 1, 0, \ldots, \hat{b}).$$

Inequality (2.22) implies that players' average payoffs from the outcome path $\hat{\pi}(T)$ are (v_1, v_2), i.e. $U(\pi(T)) = (v_1, v_2)$. Let $ID(\cdot)$ be the indicator function for the outcome path $\hat{\pi}(T)$ as defined before. We decompose a type k t-period history $h_k(t) \in H_k$ as $h_k(t) = h_1(s) \oplus h_k(t-s)$, for $k = 1, 2, 3$ and $s \leq t$.

Consider the following strategies in the game $NG_1(\delta)$. $\forall\ h_1(t) \in H_1$, $h_2(t) = h_1(t) \oplus b^{t+1} \in H_2$, and $h_3(t) = h_2(t) \oplus \{N\} \in H_3$: for an odd period $(t+1)$

$$f_1(h_1(t)) \ = \ \begin{cases} \hat{b} & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 = T^* \\ 1 & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 < T^* \\ f_1^{11}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1, s) \text{ for even } s \\ f_1^{12}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1, s) \text{ for odd } s \\ f_1^{21}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2, s) \text{ for even } s \\ f_1^{22}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2, s) \text{ for odd } s \end{cases}$$

$$f_2(h_2(t)) = \begin{cases} Y & \text{if } t+1 = T \text{ and } ID(h_2(t)) = 0 \text{ or if } ID(h_2(t)) = \\ & (1, t+1) \text{ and } 1 - b^{t+1} \geq (1-\delta)u_2(a^*) + \frac{\delta}{1+\delta}(\delta + y_1^*) \\ f_2^{11}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1, s) \text{ for even } s \leq t \\ f_2^{12}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1, s) \text{ for odd } s \leq t \\ f_2^{21}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2, s) \text{ for even } s \leq t \\ f_2^{22}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2, s) \text{ for odd } s \leq t \\ N & \text{otherwise} \end{cases}$$

and for an even period $(t+1)$,

$$f_1(h_2(t)) = \begin{cases} Y & \text{if } t+1 = T \text{ and } ID(h_2(t)) = 0 \text{ or if } ID(h_2(t)) = \\ & (2, t+1) \text{ and } b^{t+1} \geq (1-\delta)u_1(a^*) + \frac{\delta}{1+\delta}(\delta + x_2^*) \\ f_1^{11}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1, s) \text{ for even } s \leq t \\ f_1^{12}(h_2(t-s)) & \text{if } ID(h_2(t)) = (1, s) \text{ for odd } s \leq t \\ f_1^{21}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2, s) \text{ for even } s \leq t \\ f_1^{22}(h_2(t-s)) & \text{if } ID(h_2(t)) = (2, s) \text{ for odd } s \leq t \\ N & \text{otherwise} \end{cases}$$

$$f_2(h_1(t)) = \begin{cases} \hat{b} & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 = T^* \\ 0 & \text{if } ID(h_1(t)) = 0 \text{ and } t+1 < T^* \\ f_2^{11}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1, s) \text{ for even } s \\ f_2^{12}(h_1(t-s)) & \text{if } ID(h_1(t)) = (1, s) \text{ for odd } s \\ f_2^{21}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2, s) \text{ for even } s \\ f_2^{22}(h_1(t-s)) & \text{if } ID(h_1(t)) = (2, s) \text{ for odd } s \end{cases}$$

for both odd and even periods $(t+1)$.

$$f(h_3(t)) = \begin{cases} \hat{a} & \text{if } ID(h_3(t)) = 0 \\ a^* & \text{if } ID(h_3(t)) = (i, t+1) \text{ for } i = 1, 2. \\ f^{11}(h_3(t-s)) & \text{if } ID(h_3(t)) = (1, s) \text{ for even } s \leq t \\ f^{12}(h_3(t-s)) & \text{if } ID(h_3(t)) = (1, s) \text{ for odd } s \leq t \\ f^{21}(h_3(t-s)) & \text{if } ID(h_3(t)) = (2, s) \text{ for even } s \leq t \\ f^{22}(h_3(t-s)) & \text{if } ID(h_3(t)) = (2, s) \text{ for odd } s \leq t \end{cases}$$

It remains to verify that the strategy profile $f = (f_1, f_2)$ constitutes a SPE of $NG_1(\delta)$.

$\forall\, h_1(t) \in H_1$, if $ID(h_1(t)) \neq 0$, the induced strategy profile $f|_{h_1(t)}$ is one of the four subgame perfect equilibrium strategy profiles f^{11}, f^{12}, f^{21} or f^{22}. Therefore, the strategy profiles given above are subgame perfect if $ID(h_1(t)|\pi(T)) \neq 0$. We now verify the strategy profiles along the proposed path $\hat{\pi}(T)$, i.e. for $ID(h_1(t)) = 0$ and $(t+1) \leq T$. Due to the symmetry in profiles, we only consider an odd period $(t+1)$ before period $(T+1)$.

$\forall\, h_1(t) \in H_1$ such that $ID(h_1(t)) = 0$, player 1 should follow his strategy $f_1(h_1(t))$ to propose 1 in period $(t+1) < T$ and $\hat{b}$ in period T. If player 1 follows the strategies, by (2.22) above, his average payoff will be

$$(1-\delta^{T-t})u(\hat{a}) + \delta^{T-t}\hat{b} \;\geq\; v_1.$$

However, if player 1 deviates from $f_1(h_1(t))$, due to (2.19) his average payoff is either

$$(1-\delta)u_1(a^*) + \delta\frac{\delta(1-y_1^*)}{1+\delta} \;<\; (1-\delta)u_1(a^*) + \delta\frac{1-y_1^*}{1+\delta} \;\leq\; v_1$$

or

$$1-(1-\delta)u_2(a^*) - \frac{\delta}{1+\delta}(\delta + y_1^*) \;\leq\; \underline{v_1} + \epsilon_0 \;<\; v_1$$

Therefore, player 1 will not deviate from $f_1(h_1(t))$.

$\forall\, h_2(t) \in H_2$. If $ID(h_2(t)) = (1, t+1)$, i.e. player 1 has deviated in period $(t+1)$, player 2's payoff from rejecting player 1's deviating proposal is

$$(1-\delta)u_2(a^*) + \delta\frac{\delta + y_1^*}{1+\delta} \tag{2.23}$$

Therefore, player 2 should accept the proposal only if his share is not less than (2.23). If $ID(h_2(t)) = 0$, player 2 should reject the proposal in period $(t+1) < T$, since his payoff from accepting the proposal is 0 which is certainly less than that from rejecting, and he should accept the proposal in period T, since his payoff from accepting the proposal is more than that from rejecting the proposal. (2.20) and (2.22) imply that

$$1-\hat{b} > v_2 > (1-\delta)u_2(a^*) + \delta\frac{1-x_2^*}{1+\delta}.$$

Therefore, player 2 will not deviate from $f_2(h_2(t))$.

$\forall\, h_3(t) \in H_3$. If $ID(h_3(t)) = 0$, $f(h_3(t)) = \hat{a}$, and neither players should deviate from $\hat{a}$, since

$$\begin{aligned}\frac{1-\delta}{\delta}\left[\max_{a_1' \in A_1} u_1(a_1', \hat{a}_2) - u_1(\hat{a})\right] \;&\leq\; \frac{1-\delta}{\delta}d \;\leq\; \epsilon_0 \;=\; v_1 - (\underline{v_1} + \epsilon_0) \\ &\leq\; (1-\delta^{T-t-1})u_1(\hat{a}) + \delta^{T-t-1}\hat{b} - \frac{1-y_1^*}{1+\delta}\end{aligned}$$

and

$$\frac{1-\delta}{\delta}\left[\max_{a_2'\in A_2} u_2(\hat{a}_1,a_2') - u_2(\hat{a})\right] \leq \frac{1-\delta}{\delta}d \leq \epsilon_0 = v_2 - (\underline{v_2}+\epsilon_0)$$
$$\leq (1-\delta^{T-t-1})u_2(\hat{a}) + \delta^{T-t-1}\hat{b} - \frac{1-x_2^*}{1+\delta}$$

due to (2.19), (2.21) and (2.22). If $ID(h_3(t)) \neq 0$, $f(h_3(t)) = a^*$, and since a^* is a Nash equilibrium in the stage game, a player cannot increase his payoff in period $(t+1)$ and thereafter by deviating from a^*. Therefore, players will not deviate from f individually.

(v_1, v_2) is, therefore, supported by the strategy profile f as a subgame perfect equilibrium payoff from the outcome path $\hat{\pi}(T)$ in the negotiation game $NG_1(\delta)$.

Q.E.D.

2.4 Discussion

We have presented a model of negotiations and derived its equilibrium set. The equilibrium outcomes can be characterized simply in terms of players' punishment payoffs — which depend on the payoff structure of the stage game played in periods when no agreement has been reached. We utilize the value for normal form games which gives a measure of the maximal payoff of one player net of compensation to the other for not deviating from a strategy which is not best response.

The negotiation model has been introduced as an alternative to the bargaining model. The difference between the two is to be found in the players' ability to affect periodic payoffs during disagreement in the negotiation model. In the introduction, we have argued that the potential for strategic actions in disagreement periods, which is an inherent part of 'real' negotiations, makes the *status quo* point endogenous, and that the bargaining model cannot accommodate such an endogenous *status quo* point.

As the results and examples show, our assessment was generally correct. Although Rubinstein is easily adapted to deal with an exogenous non-stationary *status quo* point (as shown in Theorem 2.1), Example 2.1 and Theorems 2.4 and 2.5 show that endogeneity can affect the equilibrium set significantly. Example 2.2, however, points to the fact that the mere existence of strategic options during disagreement is not sufficient to destroy Rubinstein's results. That is, immediate agreement on Rubinstein shares is the unique equilibrium outcome.

The existence of strategic opportunities therefore may or may not matter to the equilibrium set. The essay provides a simple test if it does: If both y_1^* and x_2^* are 0, the bargaining game is sufficient to analyse the situation. The negotiation model is thus a superset of the bargaining model: If the stage game is trivial (i.e. no payoffs or no actions) or if it is 'inconsequential', the model can be simplified to Rubinstein's without affecting the equilibrium set.

The negotiation model also has implications for the analysis of delay in negotiations. It is well known that the bargaining model cannot generate delay. Several extensions have been proposed in the past in order to deal with this 'failure' (actually, Rubinstein was too successful by having a unique equilibrium, a fact that surprised even himself (p.99).) One approach to explain delay has been to relax the assumption of perfect and complete information. Models with both one- and two-sided incomplete information have been brought forward and seem to feature delay. The general argument of these types of model is that delay (often called strike) is necessary to separate types, that is, the willingness of a party to incur the cost of delay serves as a signal of its type, when type is private information.

Recently, a different approach has been forwarded by Haller and Holden (1990), and by Fernandez and Glazer (1991). Their models derive from the realization that delay does not imply strike — contrary to popular usage of the terms in the literature. They model the strike decision explicitly by allowing the union to decide after an offer has been rejected whether to keep working (and to earn income) or whether to strike (and to get nothing) in that period. Both sets of authors show that multiple equilibria exist in such a framework and that real delay can be supported.

The negotiation model provides an even stronger result, in a sense. In both of the above mentioned models, the payoff point when there is no strike is on the bargaining frontier. Thus, there is no surplus to be gained from agreement. The negotiation model, in contrast, supports real delay even though agreement gives access to a surplus, and allows for infinite delay — no agreement — to be an equilibrium. Applied to wage negotiations, the negotiation model is also much more general than either Haller and Holden (1990) or Fernandez and Glazer (1991). In particular, it is easy to incorporate such labour action as work to rule or the employment of strike breakers.

The negotiation model is not limited to these types of application, however. To show this point, we will now consider the links of this model to the repeated games literature. To a large part, the current investigation derived from an attempt to endogenize the time horizon of a relationship modeled by a repeated game. The current model can also be interpreted as a *Repeated Game with Exit.*[3] One implication of the model in this context is that a relationship may be inefficient in equilibrium. In other words, the model explains how parties may continue within a relationship even if they both could gain by agreeing to end it.

Another implication is for the equilibrium set of repeated games. One interpretation of the repeated game framework is that of implicit contracts, that is, the question which allocations can be supported in a non-cooperative framework. The standard example is the Prisoners' Dilemma. In a one shot game the unique Nash equilibrium is Pareto dominated. This can be interpreted as implying that cooperation in such situations is 'not possible' — with implications for the threat of collusion in Cournot Duopolies, for example. Since collusion is observed, it has been realized that

[3]We have a specification of the model in this context in which in each period the stage game is played first. Then an exit offer is made which, if accepted, starts to determine the payoffs from the next period onwards. The stream of payments thus stays aligned. The model has precisely the same features as the one presented, and, but for minor and obvious changes, the equilibrium strategies and payoffs are the same. The robustness of our results to this change in game form increases our confidence in the model as a vehicle to modelling dynamic allocation problems of either sort.

this simplified approach is not sufficient as a model of non-cooperative cooperation. This realization has spawned the folk theorem literature, and it is now known that in infinitely repeated games every feasible and individually rational payoff can be supported. This result also holds for finitely repeated games of sufficient length, if a dimensionality condition is satisfied.

If one interprets the negotiation model as a model of a repeated game with exit, our result indicate that the folk theorems collapse if exit is possible. Although for certain games infinite cooperation can still be supported, the equilibrium set in the negotiation model is significantly smaller than that of the corresponding repeated game. In particular, the model implies that a high degree of cooperation may have to be present for the relationship not to have been abandoned. In other words, if a duopoly exists even though there are no impediments to its dissolution, it is likely that the duopolists collude. The results also limit the degree of asymmetry in payoffs which may be supported in equilibria, ruling out equilibria where one player gets close to his minimax while the other player has a payoff close to the payoff frontier.

On the other hand, the model reduces the importance of Nash equilibria in the stage game. In particular, a relationship of finite length may be observed to feature cooperation even if there are no Nash equilibria in the stage game — an impossible outcome in finitely repeated game models.

In summary, the negotiation model unifies the bargaining and the repeated game approaches to allocation in dynamic settings, and both of the latter are special cases of the former. The negotiation model provides new results in both areas and allows for the influence of repeated payoffs versus repeated offers to be explored.

2.5 Conclusion

In the presentation of the model we have made some assumptions, for analytical convenience, which entail a loss of generality. While some of these are relatively minor, some others may be the subject of future work in this area.

The least significant assumption which affects some of our results is that of a common discount factor. In particular, Theorem 2.1 will not go through as it stands (the reason is that players may evaluate the future very differently and that a perfect offer may not be made under such circumstances). The main results in Theorems 2.4 and 2.5, however, have us choose a sufficiently high discount rate beforehand and will not be affected, although the statement of the offers will become more cumbersome. Overall, the effect of a common discount factor on the results is minimal.

As indicated in the model section, we made two assumptions on the payoff space which entail a loss of generality. One is the restriction that every payoff in G is weakly dominated by the total surplus. This assumption can be defended by realizing that the result that the parties may remain in the relationship forever is more surprising under it. There is an obvious extension, however: The exit payoff could intersect the stage game payoffs. This will require much further work. In particular, we conjecture that it will matter greatly if the payoffs to players from punishing a player are still outside the stage game payoffs or not.

The other assumption on the payoff space is that bargaining is over the unit simplex. It will be interesting to allow the exit frontier to have an arbitrary shape, and in particular to have it coincide with the boundary of the stage game payoffs. This latter assumption is reminiscent of the work by Okada (1986), although he used a very different framework. Such an extension would require a different approach in proving the results. In particular, we would not be able to construct a path of stage game payoffs parallel to the exit border (as we do now). The main idea, however, is that a one period non-best-reply strategy by the punished player in the stage game can be supported by compensating him sufficiently in the following exit offer, and that this may yield a 'surplus' to the punisher. This type of argument should clearly also hold for a concave exit frontier. Further work is needed, however, to confirm this conjecture.

Mimicking the development of the bargaining model, other avenues for future research are to investigate continuous time versions of the negotiation model and to introduce incomplete information. Finally, there is the question of non-static payoffs. It is conceivable that players' actions during the negotiations not only affect current payoffs but also the structure of future payoffs (for example the size of the surplus.) A model incorporating such dynamic payoffs will be another step closer to being a complete model of dynamic allocation.

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