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Contributions To The Analysis Of Familial Data

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CONTRIBUTIONS TO THE ANALYSIS OF FAMILIAL DATA

by

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ABSTRACT

A unified noniterative approach to point and interval estimation of interclass and intraclass correlations is presented in the context of family studies where there may be more than one individual in each of two classes. The procedure involves a generalization of the Pearson product-moment correlation coefficient, where one permits different weights for the pairs of scores. Unlike the maximum likelihood approach, these estimators are not derived under the assumption of a particular parametric form nor do they require an iterative solution.

The asymptotic distributions of the generalized product-moment estimator and of the maximum likelihood estimator are derived under the assumption of normality. Subsequently, a Monte Carlo study is carried out to examine the asymptotic and small sample properties of these estimators under different weighting schemes. Also, several methods for constructing confidence intervals about the interclass correlation parameter are discussed, and the effectiveness of these methods is evaluated by Monte Carlo simulation.

It is recommended that for family studies, the individual-weighted estimator be used as a point estimator of interclass correlations and the method based upon a

modification of Fisher's Z-transformation be used for interval estimation. In addition, it is recommended that the weighted pairwise estimator using the proposed weighting scheme replace the analysis of variance estimator in the estimation of intraclass correlations.

Although the focus of this dissertation is on the analysis of familial data, the methods discussed are applicable to more general situations, including the assessment of correlations between any two variables where each variable is replicated a different number of times for each sample unit.

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To the loving memory of my mother

"PROGRESS DOES NOT CONSIST IN REPLACING A THEORY THAT IS WRONG WITH ONE THAT IS RIGHT. IT CONSISTS IN REPLACING A THEORY THAT IS WRONG WITH ONE THAT IS MORE SUBTLY WRONG"

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CHAPTER 1 - GENERAL REVIEW

1.1 Introduction to Familial Correlations

A major goal in the analysis of familial data arising from studies of continuous attributes such as blood pressure, cholesterol, and pulmonary function is to estimate the degree of resemblance among family members. Correlation analysis has played an important role in assessing family resemblance, dating back more than one hundred years. Historically, Sir Francis Galton is credited with conceptualizing and computing the first family 'co-relation' In his Presidential Address to the Anthropological Section of the British Association (Galton, 1885) he used a method he called 'reversion' (regression) to measure the genetic relationship between a son's stature and his father's. However, it was Karl Pearson (1895) who derived the mathematical formula for Galton's co-relation measure; and thus, the Pearson product-moment correlation coefficent.

In general, familial correlations may be classified into two broad categories. The first category consists of correlations that measure the degree of resemblance, for a particular trait, between two members from the same class of individuals in a family; we refer to these as *intraclass correlations*. The sib-sib correlation and cousin-cousin correlation are examples. The second category consists of those correlations that measure the degree of resemblance between two members from different classes of individuals in a family. We call such correlations, *interclass correlations;* where father-mother, mother-daughter, sister-brother are examples.

The primary interest in this dissertation focuses on the interclass correlation. Nevertheless, the intraclass correlation is discussed in some detail because of its important role in interclass distribution theory.

1.2 Review of Parent-Offspring Correlations

Research into the estimation and hypothesis testing of interclass correlations has only flourished in the past fifteen years. Even so, most work has focussed on the special case of assessing correlations between a sole member in one class (e.g. mother) and a variable number of individuals in the other (e.g. children). This type of correlation has often been referred to as the *mother-sib*, *parent-child* or *parent-offspring* correlation. In a study consisting only of single-child families, the simple correlation coefficient is clearly the appropriate estimator of the parent-offspring correlation, since each family contributes only one pair of measurements.

However in practice data are collected from families of varying numbers of children, and thus the procedure for estimating a familial correlation becomes unclear. It has been proposed by several authors, including Higgins and Keller (1975), to pair each child's score with the parent's score and compute a product-moment correlation over all such pairs. This method is often referred to as the *pairwise* procedure. Despite its simple approach and intuitive appeal, the accuracy of this correlation estimator has often been questioned. It has been pointed out by Rosner et al. (1977) that the resulting pairs are not statistically independent since the parent's value is repeated for all children and because the siblings in the same family are, in general, correlated. Smith (1980a), in the same spirit, remarks that this procedure tends to disproportionately emphasize large families.

As a consequence, alternative estimators, including the sib-mean, random-sib, ensemble, and maximum likelihood have been suggested. Rosner et al. (1977) used Monte Carlo simulation to compare the relative efficiencies of the first three of these estimators to the pairwise estimator. Their results show that the pairwise and ensemble estimators are more efficient than the sib-mean and random-sib estimators in terms of mean square error. In particular, the pairwise estimator was found to be superior in the case of low sib-sib correlation (ρ_{ss}), whereas the ensemble estimator is superior when ρ_{SS} is high. Rosner (1979), in a further simulation study, showed that the pairwise estimator is roughly equivalent in mean square error to the maximum likelihood estimator for small values of ρ_{SS}, although the former loses efficiency as ρ_{SS} increases. By contrast, the mean square error of the ensemble estimator is approximately equal to that of the maximum likelihood for large values of ρ_{SS}.

For equal number of siblings per family, the pairwise is the maximum likelihood estimator. In the case of a variable number of offspring per family, the maximum likelihood estimator requires an iterative solution. Rosner (1979) proposed an algorithm for finding this estimator which involves maximization of an implicit function of two parameters. Mak and Ng (1981) used a linear model approach to reduce the problem to iterative maximization of a function of one parameter. Srivastava (1984) introduced a transformation which also simplified the function to only one unknown parameter.

The asymptotic properties of the above-mentioned estimators have been derived only recently. Konishi (1982) derived expressions for the asymptotic variances of the pairwise and ensemble estimators, while Srivastava and Katapa (1986) derived the asymptotic variance of the maximum likelihood estimator. Using these results, Konishi (1982) and O'Neill et al. (1987) were able to confirm the results from the earlier simulations of Rosner et al. (1977) and Rosner (1979).

Weighted estimators have been proposed to provide a more unified approach to estimation problems, so that many of the existing estimators would be merely special cases of a generalized theory. Karlin et al. (1981) developed their estimator by assigning weights to paired data points in the pairwise estimator. Defining the weights as the reciprocal of the number of pairs contributed by each family yielded a new estimator, the family-weighted estimator. Srivastava and Keen (1988) derived their estimator utilizing the method of weighted sums of squares originally developed by Smith (1956). A special case of this estimator for the choice of uniform weights is the interclass estimator proposed by Srivastava (1984).

Procedures for testing the statistical significance of a parent-offspring correlation coefficient were discussed by Rosner et al. (1979). They proposed an adjusted pairwise test, in which the effective degrees of freedom is determined as a function of both family size and sib-sib correlation. In a subsequent Monte Carlo simulation, this test was shown to be more powerful than three other procedures that had been previously cited in the literature.

In subsequent research, Donner and Bull (1984) recommended comparing the pairwise estimator to the value of its large sample standard error computed under the null hypothesis (yielding an approximate standard normal deviate) as an appropriate significance testing method. Konishi (1985) provided a generalization of this procedure to test the hypothesis that an interclass correlation is equal to a specified value. He also recommended that hypothesis testing should be based on the pairwise estimator if the sib-sib correlation is small and on the ensemble estimator if the sib-sib correlation is large.

Confidence interval estimation, based upon the asymptotic distributions of the pairwise and ensemble estimators, was first described by Konishi (1982, 1985). However, Donner and Eliasziw (1988) provide a detailed Monte Carlo simulation showing that a modified form of Fisher's Z-transformation as applied to the pairwise estimator produces a confidence interval narrower than ones proposed by Konishi (1982, 1985) or those based upon maximum likelihood procedures.

In summary, the literature tends to recommend the pairwise estimator for both estimation and hypothesis testing. Although the ensemble estimator, and its variants, is preferable to the pairwise method if the sib-sib correlation is at least moderately large, this situation arises only infrequently in human family studies. However, Konishi (1985) pointed out that this may not be case in animal studies where measurements taken on litters have been found to be highly correlated.

1.3 Review of the General Interclass Correlation

Interest in assessing correlations between two classes of individuals was increased with the work of Elston (1975) and Smith (1980a, 1980b). Elston (1975) considered the special case of equal number of individuals in each class, over the entire sample of families, and derived expressions for the relevant maximum likelihood estimators and corresponding asymptotic variances. Dealing with the case of variable class size, Smith (1980a, 1980b) proposed a method for estimation of interclass correlation coefficients using the approach of weighted sums of squares given in Smith (1956), also describing an iterative technique by which one may obtain the estimates. He further showed that under the assumption of normality, his method will yield the maximum likelihood estimators and their variances. However, this procedure offers no substantial reduction in computing intensity over the method of maximum likelihood discussed by Donner and Koval (1981), and Rosner (1982).

An alternative approach to the above iterative methods was suggested by Karlin et al. (1981). Their method involves forming all possible pairs of scores between the classes within a family, assigning some relative weight to those pairs, and then computing a Pearson product-moment correlation over all the weighted pairs. For a certain weighting scheme, this estimator reduces to the pairwise estimator derived by Rosner (1982). In addition, for samples containing equal number of individuals in each class, this estimator is the maximum likelihood estimator. An adjusted pairwise test for the general interclass correlation, extending the one proposed by Rosner et al. (1979), is given by Rosner (1982).

1.4 Overview

The purpose of this thesis is to provide a unified noniterative approach to point and interval estimation of familial correlations. Both the interclass and intraclass correlations are developed as generalizations to the Pearson product-moment correlation. The effectiveness of this approach is assessed in comparison to maximum likelihood procedures.

The general interclass correlation model, along with the underlying assumptions and restrictions, is defined in Chapter 2.

Chapter 3 is devoted to clarifying the relationships among several recent estimators of parent-offspring correlation, specifically, the ensemble, the modified sib-mean (Konishi, 1982), the family weighted, and the estimator proposed by Srivastava (1984).

In Chapter 4, the maximum likelihood estimator and the generalized product-moment estimator are introduced. In addition, their asymptotic distributions are derived under the assumptions of normality.

Chapter 5 deals with the accuracy of these point estimators in a Monte Carlo study. The primary basis for comparison are their small sample and asymptotic relative efficiencies.

A new consistent estimator of intraclass correlation is proposed in Chapter 6 as an extension of the weighted pairwise estimator of Karlin et al. (1981). Moreover, the asymptotic variance for the weighted pairwise estimator is derived. Through a Monte Carlo study, both the small sample and asymptotic properties are compared to those of the maximum likelihood and analysis of variance estimators of intraclass correlation.

Procedures for interval estimation are outlined in Chapter 7. The quality of these methods are evaluated by a subsequent Monte Carlo study.

Chapter 8 provides an analysis of a real data set, summarizes the results of the thesis, and suggests directions for future research.

CHAPTER 2 - THE INTERCLASS CORRELATION MODEL

Suppose a sample of measurements from N families is taken, and that two classes of individuals are being studied with each family having a variable number of individuals in each class. Let

$$\underset{\sim}{z}_i = (x_{i1}, x_{i2}, \ldots, x_{ia_i}, y_{i1}, y_{i2}, \ldots, y_{ib_i})'$$

represent the observations from the i^{th} family, where $x_{i1}, x_{i2}, \ldots, x_{ia_i}$ are the scores on the a_i individuals in class A, and $y_{i1}, y_{i2}, \ldots, y_{ib_i}$ are the scores on the b_i individuals in class B. It is assumed that the $\underset{\sim}{z}_i$ have a (a_i+b_i)-variate normal distribution with mean vector $\underset{\sim}{\mu}_i = (\mu_a, \mu_a, \ldots, \mu_a, \mu_b, \mu_b, \ldots \mu_b)'$ and covariance matrix

$$\underset{\sim}{\Sigma}_i = \begin{bmatrix} \underset{\sim}{\Sigma}_{i11} & \underset{\sim}{\Sigma}_{i12} \\ \underset{\sim}{\Sigma}_{i12}' & \underset{\sim}{\Sigma}_{i22} \end{bmatrix}$$

where $\underset{\sim}{\Sigma}_{i11} = \sigma_a^2\left\{(1-\rho_a)\underset{\sim}{I}_{a_i} + \rho_a \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}'\right\}$;

$$\underset{\sim}{\Sigma}_{i22} = \sigma_b^2\left\{(1-\rho_b)\underset{\sim}{I}_{b_i} + \rho_b \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}'\right\} ;$$

$$\underset{\sim}{\Sigma}_{i12} = \sigma_{ab} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{b_i}' = \rho_{ab}\sigma_a\sigma_b \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{b_i}' ;$$

$\underset{\sim}{\ell}_{a_i}$ = (1,...,1)' is a a_i-vector of ones;

$\underset{\sim}{\ell}_{b_i}$ = (1,...,1)' is a b_i-vector of ones;

and $\underset{\sim}{I}_{a_i}$ and $\underset{\sim}{I}_{b_i}$ are $a_i \times a_i$ and $b_i \times b_i$ identity matrices, respectively.

In this model, it is assumed that individuals in class A have common mean μ_a and variance σ_a^2, while individuals in class B have mean μ_b and variance σ_b^2. Moreover, the interclass correlation denoted by ρ_{ab} and the intraclass correlations ρ_a and ρ_b are assumed to remain constant over all family sizes. The necessary and sufficient conditions for the covariance matrix, $\underset{\sim}{\Sigma}_i$, to be positive definite for all a_i and b_i are

$$\sigma_a^2 > 0 \quad , \quad \sigma_b^2 > 0 \quad ;$$

$$\frac{-1}{a_0-1} \le \rho_a < 1 \quad , \quad \frac{-1}{b_0-1} \le \rho_b < 1 \quad ;$$

and
$$\rho_{ab}^2 < \left[\rho_a + a_0^{-1}(1-\rho_a)\right]\left[\rho_b + b_0^{-1}(1-\rho_b)\right]$$

where $a_0 = \max\{a_i\}$ and $b_0 = \max\{b_i\}$. An approximate form of the latter condition is $\rho_{ab}^2 \le \rho_a\rho_b$.

Without loss of generality, a restriction is also placed on the number of individuals in classes A and B so that $a_i \ge 1$ and $b_i \ge 1$ for all i.

CHAPTER 3 - COMPARISON OF RECENT ESTIMATORS OF PARENT-OFFSPRING CORRELATION

In this chapter, the algebraic and asymptotic properties of the family-weighted estimator and the estimator proposed by Srivastava (1984) are compared to more traditional estimators of parent-offspring correlation, such as the ensemble estimator and the modified sib-mean estimator. The model adopted for these estimators is a special case of the one described in Chapter 2 by letting class A have exactly one individual. In the parent-offspring model, x_{i1} represents the parent's score and $y_{i1}, y_{i2}, \ldots, y_{ib}$ are the scores on their b_i offspring (siblings). The parent-offspring and sib-sib correlations are denoted by ρ_{ab} and ρ_b, respectively.

3.1 Estimators of Parent-offspring Correlation

3.1.1 *Ensemble estimator*

The ensemble estimator was derived by Rosner et al. (1977) by computing the 'expected value' of the random-sib estimator over all possible choices of random sibs from each family, and is expressed as

$$\hat{\rho}_{ab,e} = \frac{\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)(\bar{y}_{is}-\bar{y}_s)}{\left[\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)^2\right]^{\frac{1}{2}}\left[\frac{N-1}{N}\sum_{i=1}^{N}\frac{1}{b_i}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{is})^2+\sum_{i=1}^{N}(\bar{y}_{is}-\bar{y}_s)^2\right]^{\frac{1}{2}}}$$

where $\bar{x}_m = \frac{1}{N}\sum_{i=1}^{N}x_{i1}$, $\bar{y}_{is} = \frac{1}{b_i}\sum_{k=1}^{b_i}y_{ik}$, $\bar{y}_s = \frac{1}{N}\sum_{i=1}^{N}\bar{y}_{is}$.

The asymptotic variance of the ensemble estimator was first derived by Konishi (1982). Using notation given by O'Neill et al. (1987), the asymptotic variance of the ensemble estimator, denoted by $AV(\hat{\rho}_{ab,e})$, is

$$\frac{1}{N}\left[b_h^{-1}(1-\rho_{ab}^2)^2 + (1-b_h^{-1})\left\{(\rho_b-\rho_{ab}^2)(1-\rho_{ab}^2) + \tfrac{1}{2}\rho_{ab}^2(1-\rho_b)^2\right\}\right]$$

where $b_h = \left\{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{b_i}\right\}^{-1}$ is the harmonic mean family size.

3.1.2 *Family-weighted pairwise estimator*

Karlin et al. (1981) suggested weighting the pairs in the pairwise estimator by the inverse of the number of pairs contributed by each family, so as to eliminate the disproportionate effect of large families in the final estimate. Their estimator is of the form

$$\hat{\rho}_{ab,f} = \frac{\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)\frac{1}{b_i}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_s)}{\left[\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{N}\frac{1}{b_i}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_s)^2\right]^{\frac{1}{2}}}$$

3.1.3 *Modified sib-mean estimator*

The sib-mean estimator, obtained by pairing each parent's score with the mean of their b_i sibling's scores and calculating the Pearson product-moment correlation, can be expressed as

$$\hat{\rho}_{ab,m} = \frac{\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)(\bar{y}_{is}-\bar{y}_s)}{\left[\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{N}(\bar{y}_{is}-\bar{y}_s)^2\right]^{\frac{1}{2}}}$$

Lush (1947) and Konishi (1982) have both pointed out that this estimator is not consistent for the parameter ρ_{ab} since, asymptotically,

$$E(\hat{\rho}_{ab,m}) = \left[\bar{b}_h^{-1} + (1-\bar{b}_h^{-1})\rho_b\right]^{-1/2} \rho_{ab} .$$

The modified sib-mean estimator (Konishi, 1982) adjusts $\hat{\rho}_{ab,m}$ in such a way that the new estimator is consistent. It is written as

$$\hat{\rho}_{ab,mm} = \left[\bar{b}_h^{-1} + (1-\bar{b}_h^{-1})\rho_b\right]^{1/2} \hat{\rho}_{ab,m} ,$$

where ρ_b may be estimated by any consistent estimator of intraclass correlation.

3.1.4 *Srivastava's estimator*

Srivastava (1984) proposed an estimator which is given by

$$\hat{\rho}_{ab,s} = \frac{\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)(\bar{y}_{is}-\bar{y}_s)}{\tilde{\sigma}_b\left[N-1\right]^{\frac{1}{2}}\left[\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)^2\right]^{\frac{1}{2}}}$$

where
$$\tilde{\sigma}_b^2 = \frac{\sum_{i=1}^{N}(\bar{y}_{is}-\bar{y}_s)^2}{\left[N-1\right]\left[\bar{b}_h^{-1} + (1-\bar{b}_h^{-1})\tilde{\rho}_b\right]},$$

$$\tilde{\rho}_b = \frac{\tilde{\sigma}_b^2 - \tilde{\gamma}_b^2}{\tilde{\sigma}_b^2},$$

and
$$\tilde{\gamma}_b^2 = \sum_{i=1}^{N}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{is})^2 \Big/ \sum_{i=1}^{N}(b_i-1)$$

Substituting $\tilde{\sigma}_b$ into the above expression for $\hat{\rho}_{ab,s}$ results in the form of the estimator given later by Srivastava and Keen (1988):

$$\hat{\rho}_{ab,s} = \frac{\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)(\bar{y}_{is}-\bar{y}_s)}{\left[\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)^2\right]^{\frac{1}{2}}\left[N^{*}\sum_{i=1}^{N}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{is})^2 + \sum_{i=1}^{N}(\bar{y}_{is}-\bar{y}_s)^2\right]^{\frac{1}{2}}}$$

where $N^{*} = (1-\bar{b}_h^{-1})(N-1) \Big/ \sum_{i=1}^{N}(b_i-1)$

Srivastava & Katapa (1986) showed that the large-sample variance of $\hat{\rho}_{ab,s}$ is

$$AV(\hat{\rho}_{ab,s}) = \frac{1}{N}\left[\rho_{ab}^{4} + \rho_{ab}^{2}\left\{\tfrac{1}{2}c^{2} - 2\lambda - \tfrac{1}{2}\right\} + \lambda\right],$$

where

$$c^{2} = 1 - 2(1-\rho_{b})(1-\bar{b}_{h}^{-1}) + (1-\rho_{b})^{2}\left[\frac{1}{N}\sum_{i=1}^{N}\left\{1-\frac{1}{b_{i}}\right\}^{2} + \frac{(1-\bar{b}_{h}^{-1})^{2}}{(\bar{b}-1)}\right]$$

$\lambda = 1 - (1-\rho_{b})(1-\bar{b}_{h}^{-1})$, and $\bar{b} = \frac{1}{N}\sum_{i=1}^{N} b_{i}$ is the mean sibship size.

It is shown in Appendix 6 that $\tilde{\rho}_{b}$ is just the estimator of intraclass correlation based upon unweighted group means, as first proposed by Smith (1956) and explicitly expressed by Donner & Koval (1983). Thus, $\hat{\rho}_{ab,s}$ is algebraically identical to the modified sib-mean estimator, given by $\hat{\rho}_{ab,mm}$, when ρ_{b} is estimated by $\tilde{\rho}_{b}$.

3.2 <u>Comparison of Estimators</u>

3.2.1 *Equal sibship sizes*

It is easily shown that when the number of siblings per family is the same (i.e. $b_{i} = b$ for $i = 1,\ldots,N$), the estimators $\hat{\rho}_{ab,e}$, $\hat{\rho}_{ab,f}$ and $\hat{\rho}_{ab,s}$ have large-sample mean and variance equivalent to those of the maximum likelihood estimator. Thus if $b_i = b$; $i = 1,\ldots,N$, all the estimators mentioned above are asymptotically efficient. It is also noted here that whereas the ensemble and the family-weighted pairwise both reduce to the Pearson product-moment correlation coefficient in the case of one child per family (i.e. $b_i=1$, $i=1,\ldots,N$), Srivastava's estimator is undefined. This is because $\hat{\rho}_{ab,s}$ is a function of the intraclass correlation based upon unweighted group means, which cannot be computed for this case.

The subsequent sections consider the case of unequal sibship sizes.

3.2.2 *Comparison between* $\hat{\rho}_{ab,f}$ *and* $\hat{\rho}_{ab,e}$

Using the definitions in Section 3.1.1, it is noted that

$$\frac{1}{b_i}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_s) = (\bar{y}_{is}-\bar{y}_s)$$

and

$$\sum_{i=1}^{N}\frac{1}{b_i}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_s)^2 = \sum_{i=1}^{N}\frac{1}{b_i}\left[\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{is})^2 + \sum_{k=1}^{b_i}(\bar{y}_{is}-\bar{y}_s)^2\right]$$

$$= \sum_{i=1}^{N}\frac{1}{b_i}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{is})^2 + \sum_{i=1}^{N}(\bar{y}_{is}-\bar{y}_s)^2$$

Using these results, the family-weighted pairwise estimator can be written as

$$\frac{\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)(\bar{y}_{is}-\bar{y}_s)}{\left[\sum_{i=1}^{N}(x_{i1}-\bar{x}_m)^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{N}\frac{1}{\bar{b}_i}\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{is})^2+\sum_{i=1}^{N}(\bar{y}_{is}-\bar{y}_s)^2\right]^{\frac{1}{2}}}$$

Presented in this form, it is observed that the family-weighted pairwise estimator is essentially the same as the ensemble estimator, since $\frac{N-1}{N}$ in $\hat{\rho}_{ab,e}$ tends to unity as N increases.

Thus, the asymptotic variance for $\hat{\rho}_{ab,f}$, under the assumption of normality, can be taken to be the same as for the ensemble estimator. That is

$$AV(\hat{\rho}_{ab,f}) = \frac{1}{N}\left[\bar{b}_h^{-1}(1-\rho_{ab}^2)^2 + (1-\bar{b}_h^{-1})\left\{(\rho_b-\rho_{ab}^2)(1-\rho_{ab}^2) + \tfrac{1}{2}\rho_{ab}^2(1-\rho_b)^2\right\}\right]$$

3.2.3 *Comparison between* $\hat{\rho}_{ab,s}$ *and* $\hat{\rho}_{ab,e}$

The asymptotic variance of $\hat{\rho}_{ab,s}$, given in Section 3.1.4 can be equivalently written as

$$\frac{1}{N}\left[\bar{b}_h^{-1}(1-\rho_{ab}^2)^2 + (1-\bar{b}_h^{-1})(\rho_b-\rho_{ab}^2)(1-\rho_{ab}^2) + \tfrac{1}{2}\rho_{ab}^2(1-\rho_b)^2T\right]$$

where $T = s_{-1}^2 + \left[\frac{\bar{b}_h-1}{\bar{b}_h}\right]^2 \times \frac{\bar{b}}{\bar{b}-1}$ and

$s_{-1}^2 = \frac{1}{N}\sum_{i=1}^{N}(b_i^{-1} - \bar{b}_h^{-1})^2$ is the variance of family size reciprocals.

The difference between the asymptotic variances of Srivastava's estimator and the ensemble estimator, given a particular probability distribution for b_i and values for ρ_{ab}, ρ_b, can be expressed as:

$$\Delta(b_i, \rho_{ab}, \rho_b) = [AV(\hat{\rho}_{ab,s}) - AV(\hat{\rho}_{ab,e}) \mid b_i, \rho_{ab}, \rho_b]$$

$$= \frac{1}{N}\left[\tfrac{1}{2}\rho^2_{ab}(1-\rho_b)^2\left\{T - (1-\bar{b}_h^{-1})\right\}\right]$$

$$= \frac{1}{N}\left[\tfrac{1}{2}\rho^2_{ab}(1-\rho_b)^2\left\{s^2_{-1} - \frac{\bar{b}_h-1}{\bar{b}-1}\times\frac{\bar{b}-\bar{b}_h}{\bar{b}_h^2}\right\}\right]$$

Srivastava and Keen (1988) have shown that this difference is always negative without, however, evaluating its magnitude. In the section below, the asymptotic variances of the two estimators are compared.

3.2.4 *Comparison of asymptotic variances*

A Monte Carlo simulation was undertaken to compare $AV(\hat{\rho}_{ab,s})$ to $AV(\hat{\rho}_{ab,e})$. In designing the simulation study, the truncated negative binomial distribution was used to generate the b_i given by

$$Pr(b_i=r) = \frac{(m + r - 1)!\; Q^{-m}\; (P/Q)^r}{(m - 1)!\; r!\; (1 - Q^{-m})}, \quad Q = 1+P, \quad r = 1,2,...$$

Brass (1958) has shown that the above distribution fits the observed distribution of sibship sizes very well in a variety of human populations for appropriate choices of parameters m and P.

Estimates of m and P were derived by Brass (1958) for 29 different countries, based on data available in the United Nations Demographic Yearbook. These estimates correspond to values of mean family size ranging from 2.55 in England & Wales to 7.27 in Brazil and Malta. For each of the 29 countries, 500 sibship sizes were generated on each of 1015 independent runs. On each of these runs, the percentage reduction in variance associated with Srivastava's estimator was calculated, and is given by

$$\Lambda(b_i,\rho_{ab},\rho_b) = \frac{\Delta(b_i,\rho_{ab},\rho_b)}{AV(\hat{\rho}_{ab,e})} \times 100$$

The median of these values over the 1015 runs was selected as an appropriate estimator of the percentage reduction in variance for a particular sibship size distribution and parameters ρ_{ab}, ρ_b. The results are shown in Table 1.1.

The U.S. sibship size distribution (m=2.84, P=.93) is considered first since it is the one most often cited in the literature. It was found that the largest reduction in variance over all parameter combinations of ρ_{ab} and ρ_b was 0.9%. This occured for $\rho_{ab} = 0.6$ and $\rho_b = 0.4$. For the same family correlation parameters, and the remainder of countries listed in Brass (1958) Table 1.1 shows that the reduction in variance ranges from 0.6% (England & Wales) to 1.7% (Brazil and Malta).

3.3 Summary

In summary, it has been shown that the family-weighted pairwise estimator is, aside from a factor of (N-1)/N, identical to the ensemble estimator. Thus, Karlin et al.'s (1981) criticism of the ensemble estimator not being a true correlation, in contrast to the family-weighted pairwise estimator, is not merited. Further, it was demonstrated that the estimator proposed by Srivastava (1984) is:
a) identical to the modified sib-mean estimator when the sib-sib correlation is estimated by the method of unweighted group means, b) is only slightly more efficient than the ensemble estimator, and c) is undefined when the data consist of one sibling per family. Thus, the results indicate that the asymptotic properties of the above-mentioned estimators are essentially indistinguishable.

TABLE 1.1

Mean sibship size and percentage reduction in variance for the 29 countries reported by Brass (1958)

Country	Mean Sibship Size	% Reduction in Variance
Brazil	7.27	1.74
Malta	7.27	1.74
Puerto Rico	6.69	1.73
Venezuela	6.26	1.72
Jamaica	5.73	1.68
Windward Islands	5.72	1.66
Ceylon	5.62	1.49
Italy	5.44	1.48
North Borneo	5.43	1.54
Mauritius	5.32	1.53
Japan	5.29	1.43
Hawaii	5.26	1.54
Trinidad & Tobago	5.21	1.58
British Guiana	5.18	1.55
Sarawak	5.04	1.52
Leeward Islands	4.93	1.51
Canada	4.77	1.45
Barbados	4.65	1.46
Czechoslovakia	4.21	1.24
Mozambique	4.21	1.21
Bermuda	3.95	1.23
Portuguese Guinea	3.70	1.02
Singapore	3.51	1.09
Switzerland	3.29	0.94
Australia	3.19	0.87
U.S.A.	3.13	0.90
Scotland	2.94	0.78
Norway	2.68	0.64
England & Wales	2.55	0.63

CHAPTER 4 - ESTIMATORS OF INTERCLASS CORRELATION AND THEIR ASYMPTOTIC DISTRIBUTION

The remainder of the thesis deals with the general interclass correlation model introduced in Chapter 2, as characterized by a variable number of individuals in each class. Two methods for estimating the model's interclass correlation parameter are discussed first.

4.1 The Maximum Likelihood Estimator

A well-established approach to estimating the interclass correlation is the method of maximum likelihood (ML). If $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_N$ denotes a sample from the model, the conditional likelihood (L) of this sample is given by

$L(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i | \underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_N, a_i, b_i)$

$$= (2\pi)^{-\sum^{N}(a_i+b_i)/2} \prod_{i=1}^{N} |\underset{\sim}{\Sigma}_i|^{-1/2} \exp\left\{-\frac{1}{2} \sum_{i=1}^{N} (\underset{\sim}{z}_i - \underset{\sim}{\mu}_i)' \underset{\sim}{\Sigma}_i^{-1} (\underset{\sim}{z}_i - \underset{\sim}{\mu}_i)\right\}$$

The estimates of μ_a, μ_b, σ_a^2, σ_b^2, ρ_a, ρ_b and ρ_{ab} are then obtained by maximizing L, or equivalently minimizing -2logL.

In the case of varying a_i and b_i, the maximum likelihood estimates cannot be expressed in closed form, but must be obtained through Newton-Raphson iterative methods. The ZXMIN subroutine from the International Mathematical and

Statistical Libraries (1977) and the method of Atwood and Foster (1973), to handle the complication that five of the seven parameters are bounded, is a standard procedure for function minimization. It is noted that the ZXMIN subroutine is based upon the quasi-Newton method of Fletcher (1972), whereby the first and second derivatives of the likelihood function are not supplied but are approximated at each iteration. The maximum likelihood estimates of the correlation parameters will be denoted as $\hat{\rho}_{a,ML}$, $\hat{\rho}_{b,ML}$ and $\hat{\rho}_{ab,ML}$.

It follows directly from the properties of maximum likelihood estimation (Cox and Hinkley, 1974, pgs. 294-295; Mood et al., 1974, pg. 359) that $(\hat{\rho}_{ab,ML} - \rho_{ab})$ is asymptotically normal with mean zero and variance $AV(\hat{\rho}_{ab,ML})$. Furthermore,

$$\frac{\hat{\rho}_{ab,ML} - \rho_{ab}}{[AV(\hat{\rho}_{ab,ML})]^{1/2}}$$

is an approximate pivotal quantity (Mood et al., 1974, pgs. 393-394). Inversion of this quantity leads directly to confidence interval construction about the parameter ρ_{ab}.

The asymptotic variance is derived by the usual method of inverting the information matrix for the parameters μ_a, μ_b, σ_a^2, σ_b^2, ρ_a, ρ_b and ρ_{ab} . The computations required for obtaining the elements of this matrix are shown in Appendix 1.

If we let

$$V_i = 1 + (a_i - 1)\rho_a$$

$$W_i = 1 + (b_i - 1)\rho_b$$

$$T_i = V_i W_i - \rho_{ab}^2 a_i b_i$$

$$\Psi_1 = \sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i + \frac{\rho_{ab}^2}{2} \sum_{i=1}^{N} a_i \sum_{i=1}^{N} \frac{a_i b_i}{T_i} + \frac{\rho_{ab}^2}{2} \sum_{i=1}^{N} b_i \sum_{i=1}^{N} \frac{a_i b_i}{T_i}$$

$$\Psi_2 = \frac{1}{2^{1/2}(1-\rho_a)} \sum_{i=1}^{N} \frac{a_i(a_i-1)(W_i\rho_a - \rho_{ab}^2 b_i)}{T_i}$$

$$\Psi_3 = \frac{1}{2^{1/2}(1-\rho_b)} \sum_{i=1}^{N} \frac{b_i(b_i-1)(V_i\rho_b - \rho_{ab}^2 a_i)}{T_i}$$

$$u_1 = \begin{cases} 1 \text{ if } a_i = 1 \text{ for all } i = 1,2,\ldots,N \\ \displaystyle\sum_{i=1}^{N} \frac{(a_i-1)\left\{T_i^2 + W_i^2(a_i-1)(1-\rho_a)^2\right\}}{2(1-\rho_a)^2 T_i^2} - \frac{\Psi_2^2}{\Psi_1}\left\{\sum_{i=1}^{N} b_i + \frac{\rho_{ab}^2}{2} \sum_{i=1}^{N} \frac{a_i b_i}{T_i}\right\} \end{cases}$$

$$u_2 = \frac{\rho_{ab}^2}{2}\left\{\sum_{i=1}^{N} \frac{(a_i-1)(b_i-1)a_i b_i}{T_i^2} - \frac{\Psi_2\Psi_3}{\Psi_1} \sum_{i=1}^{N} \frac{a_i b_i}{T_i}\right\}$$

$$u_3 = -\rho_{ab}\left[\!\left[\sum_{i=1}^{N} \frac{W_i a_i b_i (a_i-1)}{T_i^2} + \frac{\Psi_2}{2^{1/2}\Psi_1}\left[\sum_{i=1}^{N} b_i \sum_{i=1}^{N} \frac{a_i b_i}{T_i} + \left\{\rho_{ab} \sum_{i=1}^{N} \frac{a_i b_i}{T_i}\right\}^2\right]\right]\!\right]$$

$$u_4 = \begin{cases} 1 \text{ if } b_i = 1 \text{ for all } i = 1,2,\ldots,N \\ \displaystyle\sum_{i=1}^{N} \frac{(b_i-1)\left\{T_i^2 + V_i^2(b_i-1)(1-\rho_b)^2\right\}}{2(1-\rho_b)^2 T_i^2} - \frac{\Psi_3^2}{\Psi_1}\left\{\sum_{i=1}^{N} a_i + \frac{\rho_{ab}^2}{2} \sum_{i=1}^{N} \frac{a_i b_i}{T_i}\right\} \end{cases}$$

$$u_5 = -\rho_{ab}\left\|\sum_{i=1}^{N}\frac{V_i a_i b_i(b_i-1)}{T_i^2} + \frac{\Psi_3}{2^{1/2}\Psi_1}\left[\sum_{i=1}^{N} a_i \sum_{i=1}^{N}\frac{a_i b_i}{T_i} + \left\{\rho_{ab}\sum_{i=1}^{N}\frac{a_i b_i}{T_i}\right\}^2\right]\right\|$$

$$u_6 = \sum_{i=1}^{N}\frac{a_i b_i(V_i W_i + \rho_{ab}^2 a_i b_i)}{T_i^2} - \left\|\frac{1}{\Psi_1}\left[\sum_{i=1}^{N}\frac{a_i b_i}{T_i}\right]\right.$$

$$\left.\times\left[\Psi_1 - \sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i + \left\{\rho_{ab}^2\sum_{i=1}^{N}\frac{a_i b_i}{T_i}\right\}^2\right]\right\|$$

then the asymptotic variance of $\hat{\rho}_{ab,ML}$ is

$$AV(\hat{\rho}_{ab,ML}) = \frac{u_1 u_4 - u_2^2}{u_6(u_1 u_4 - u_2^2) + 2u_2 u_3 u_5 - u_3^2 u_4 - u_1 u_5^2}$$

4.2 <u>Generalized Product-moment Estimator</u>

A reasonable noniterative alternative to the maximum likelihood estimator is given by a generalized form of the Pearson product-moment correlation coefficient, as proposed by Karlin et al. (1981). This estimator is defined analogously to the simple correlation coefficient, as

$$\hat{\rho}_{ab,GP} = \frac{\hat{\sigma}_{ab}}{\hat{\sigma}_a \hat{\sigma}_b}$$

where $\hat{\sigma}_{ab} = \sum_{i=1}^{N} w_i \sum_{j=1}^{a_i}(x_{ij}-\bar{x}_{..}) \sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{..})$

$$\hat{\sigma}_a^2 = \sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i}(x_{ij}-\bar{x}_{..})^2$$

$$\hat{\sigma}_b^2 = \sum_{i=1}^{N} w_i a_i \sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{..})^2$$

Here $\bar{x}_{..} = \sum_{i=1}^{N} w_i a_i b_i \bar{x}_{i.}$; $\bar{x}_{i.} = \frac{1}{a_i} \sum_{j=1}^{a_i} x_{ij}$;

$\bar{y}_{..} = \sum_{i=1}^{N} w_i a_i b_i \bar{y}_{i.}$; $\bar{y}_{i.} = \frac{1}{b_i} \sum_{k=1}^{b_i} y_{ik}$;

and the weights, w_i, are chosen so that the constraint,

$\sum_{i=1}^{N} w_i a_i b_i = 1$, is satisfied.

For example, if we let $w_i = 1 / \sum_{i=1}^{N} a_i b_i$ then $\hat{\rho}_{ab,GP}$ takes on the form of the pairwise estimator proposed by Rosner (1982). If $a_i = 1$ for all $i = 1,2,\ldots,N$ then $\hat{\rho}_{ab,GP}$ is the usual pairwise estimator discussed in Chapter 1.2. In the case where $a_i = b_i = 1$, $\hat{\rho}_{ab,GP}$ reduces to the simple Pearson product-moment correlation.

Another choice of weights that satisfies the above constraint is $w_i = 1 / N a_i b_i$. When these weights are substituted into $\hat{\rho}_{ab,GP}$, one obtains an estimator which is analogous to the family-weighted pairwise in the parent-offspring case.

Two other weights, not considered by Karlin et al. (1981), that are plausible and also satisfy the above constraint are:

$$w_i = 1 / a_i \sum_{i=1}^{N} b_i \quad \text{and} \quad w_i = 1 / b_i \sum_{i=1}^{N} a_i \; .$$

The first of these de-emphasizes contributions from families with large class A sizes, whereas the latter de-emphasizes those with large class B sizes.

In the subsequent chapters, the following weighting schemes will be considered:

Weighting Scheme	Value of w_i
Individual (IND)	$1 / \sum_{i=1}^{N} a_i b_i$
Family (FAM)	$1 / N a_i b_i$
Class-A (CL-A)	$1 / a_i \sum_{i=1}^{N} b_i$
Class-B (CL-B)	$1 / b_i \sum_{i=1}^{N} a_i$

From the work of Krewski and Rao (1981), it can be established that $(\hat{\rho}_{ab,GP} - \rho_{ab})$ is asymptotically normal with mean zero and variance $AV(\hat{\rho}_{ab,GP})$. Similarly,

$$\frac{\hat{\rho}_{ab,GP} - \rho_{ab}}{[AV(\hat{\rho}_{ab,GP})]^{1/2}}$$

is also an approximate pivotal quantity.

Here, $AV(\hat{\rho}_{ab,GP})$ is the first-order approximation to the asymptotic variance of $\hat{\rho}_{ab,GP}$, derived through a linear Taylor series expansion. This procedure is discussed in detail by Wolter (1985), pgs. 222-227. The δ (delta) Method

(Rao, 1973 pg. 388) that is often mentioned in the literature is practically equivalent to that of Krewski and Rao (1981).

From the computations in Appendix 2A, the asymptotic variance of $\hat{\rho}_{ab,GP}$ is given by

$$
\begin{aligned}
AV(\hat{\rho}_{ab,GP}) = \; & (1-\rho_{ab}^2)^2 \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 \\
& + (1-\rho_a)(1-\rho_b) \sum_{i=1}^{N} w_i^2 a_i (a_i-1) b_i (b_i-1) \\
& + (1-\rho_a)\left\{\tfrac{3}{2}\rho_{ab}^2 - \tfrac{1}{2}\rho_{ab}^2\rho_a - 1\right\} \sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i-1) \\
& + (1-\rho_b)\left\{\tfrac{3}{2}\rho_{ab}^2 - \tfrac{1}{2}\rho_{ab}^2\rho_b - 1\right\} \sum_{i=1}^{N} w_i^2 a_i^2 b_i (b_i-1)
\end{aligned}
$$

For completeness, a first-order approximation to the bias of $\hat{\rho}_{ab,GP}$ is derived in Appendix 3 as

$$
\begin{aligned}
Bias(\hat{\rho}_{ab,GP}) = \; & -\tfrac{1}{2}\rho_{ab}(1-\rho_{ab}^2) \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 \\
& - \tfrac{1}{4}\rho_{ab}(1-\rho_a)(3\rho_a-1) \sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i-1) \\
& - \tfrac{1}{4}\rho_{ab}(1-\rho_b)(3\rho_b-1) \sum_{i=1}^{N} w_i^2 a_i^2 b_i (b_i-1)
\end{aligned}
$$

It is clearly seen that $Bias(\hat{\rho}_{ab,GP})$ is a negative value. Thus on average, $\hat{\rho}_{ab,GP}$ will underestimate ρ_{ab}.

CHAPTER 5 - ACCURACY OF THE ESTIMATORS OF INTERCLASS CORRELATION

5.1 Asymptotic Relative Efficiency of the Product-moment Estimator

The accuracy of a point estimator may be assessed using a number of different criteria. One important criterion is the asymptotic variance of the estimator, under which the estimator with the smallest asymptotic variance is preferred. In practice, the asymptotic relative efficiency (in percent), defined in our case as,

$$ARE(\hat{\rho}_{ab,GP}, \hat{\rho}_{ab,ML}) = \lim_{N\to\infty}\left\{\frac{AV(\hat{\rho}_{ab,ML})}{AV(\hat{\rho}_{ab,GP})}\right\} \times 100$$

is a ratio measure frequently adopted because of its boundedness property. In particular, since it is implied by optimality properties of maximum likelihood theory that $\hat{\rho}_{ab,ML}$ has the smallest asymptotic variance of all possible estimators that can be constructed, it is clear that the ARE is a number between zero and one hundred.

Although explicit expressions for the asymptotic variances have been derived in Chapter 4, the mathematics for computing this limiting ratio analytically are for the most part intractable. Thus the only recourse is to perform a Monte Carlo simulation.

In designing the simulation study, one immediate difficulty is the selection of aggregate class sizes $\{a_i+b_i\}$; in any given investigation, there is an infinite number of combinations that could be considered. This difficulty is overcome by generating the $\{a_i+b_i\}$, at random, from a distribution of aggregate class sizes which are reflective of those found in human populations. The negative binomial distribution truncated below 1, given by

$$Pr(a_i+b_i=R) = \frac{(m + R - 1)!\ Q^{-m}\ (P/Q)^R}{(m - 1)!\ R!\ (1 - Q^{-m})}, \quad Q = 1+P, \quad R = 1,2,\ldots$$

has been shown by Brass (1958) to fit the observed distribution of sibship sizes very well in a variety of human populations for appropriate choices of parameters m and P.

This simulation study was performed with parameter values m=2.84 and P=.93, that correspond to the U.S. sibship size distributions in 1950 reported by Brass (1958). It is noted that only families with aggregate class sizes greater than or equal to two were included in this study, since it has been assumed previously that $a_i \geq 1$ and $b_i \geq 1$ for all i. Under these conditions, the mean class size is 3.83.

Given an aggregate class size, $R=a_i+b_i$, the a_i was then generated from a binomial distribution with probability one-half. This implies that the occurrence of a male or female offspring within each family is of equal probability.

Uniform pseudo-random deviates were generated using the RANDOM algorithm written by Wichmann and Hill (1982).

On each of t=1015 independent runs at N=300 and for selected values of ρ_a, ρ_b, ρ_{ab}, the ARE was computed. The median of these values over the 1015 runs was chosen as the estimate of relative effectiveness of the non-iterative estimator $\hat{\rho}_{ab,GP}$ to the maximum likelihood estimator in large samples. The values for ρ_a, ρ_b and ρ_{ab} were selected to satisfy the model constraints. Further, the sample size of 300 families was chosen on the basis that it was well within the range of large sample theory (Donner and Koval, 1983).

All computer programs were written in FORTRAN and executed on three SUN 3/50 Workstations under a UNIX operating system at the University of Western Ontario.

The results are shown in Table 5.1 for the case when the two intraclass correlations are assumed to be equal. The individual-weighted (IND) estimator performs very well at $\rho_a = \rho_b = 0.0$ but declines rapidly in relative effectiveness as ρ_a and ρ_b increase. In contrast, the family-weighted (FAM) estimator's relative effectiveness increases with increasing intraclass correlation. At $\rho_a = \rho_b = 0.3$, the AREs are similar. It is also noted that for values of intraclass correlation greater than 0.5, the family-weighted estimator has the highest relative efficiency.

The class-weighted (CL-A, CL-B) estimators are, in general, less variable in terms of relative effectiveness. Except in the case of independent observations, the ARE for both class-weighted estimators lie in the range of 80-90%, reaching a maximum when the intraclass correlation is around 0.3. In addition, their relative efficiencies are higher than those for the individual-weighted estimator at all intraclass correlation values greater than 0.1

In Table 5.2, the intraclass correlations are allowed to be different in each class. It is evident from the results that the relative effectiveness of both the individual-weighted and family-weighted estimators are dependent upon the average value of the intraclass correlation between the two classes. For example, the ARE at ρ_{ab}=0.1, ρ_a=0.3, ρ_b=0.5 is similar to the ARE at ρ_{ab}=0.1, ρ_a=0.4, ρ_b=0.4 in Table 5.1; also in the case for ρ_{ab}=0.5, ρ_a=0.5, ρ_b=0.7 and ρ_{ab}=0.5, ρ_a=0.6, ρ_b=0.6.

It is also observed that the relative effectiveness of the class-weighted estimators are influenced by the size of the intraclass correlations. In particular, the class-A weighted estimator performs better when ρ_a is large, whereas the class-B weighted estimator is more efficient when ρ_b is large. However, for the parameter combinations shown in Table 5.2, the class-weighted estimators do just as well or better than the individual-weighted estimator.

5.2 Empirical Relative Efficiency and Bias of the Product-moment Estimator

The variances presented in the previous section are all first-order approximations to the true mean square error. In the case of sufficiently large sample sizes, these expressions provide reliable results. This does not, however, necessarily mean that these results hold in small samples. The Monte Carlo study was therefore extended to calculate the small sample relative efficiencies and biases for the estimators.

Adding onto the Monte Carlo study in Chapter 5.1, observations x_{ij} and y_{ik} for each family were generated from a multivariate normal distribution with covariance structure as defined in Chapter 2, using the GGNRM subroutine from the International Mathematical and Statistical Libraries (1977).

On each of t=1015 independent runs at N = 25, 50, 100 and for selected values of ρ_a, ρ_b, ρ_{ab}, the values of each of the estimators $\hat{\rho}_{ab,GP}$ and $\hat{\rho}_{ab,ML}$ were computed. The number of families, N, was chosen to reflect a range of sample sizes commonly encountered in practice. If for any reason the maximum likelihood procedure failed to converge in a particular 'run', then that 'run' was completely replaced by another.

The estimated small sample relative efficiency (in percent) of the generalized product-moment estimator to the maximum likelihood estimator will be defined as the ratio of mean square errors. That is,

$$SRE(\hat{\rho}_{ab,GP}, \hat{\rho}_{ab,ML}) = \frac{MSE(\hat{\rho}_{ab,ML})}{MSE(\hat{\rho}_{ab,GP})} \times 100$$

where $$MSE(\hat{\rho}_{ab,ML}) = \frac{1}{1015} \sum_{t=1}^{1015} (\hat{\rho}_{ab,MLt} - \rho_{ab})^2$$

and $$MSE(\hat{\rho}_{ab,GP}) = \frac{1}{1015} \sum_{t=1}^{1015} (\hat{\rho}_{ab,GPt} - \rho_{ab})^2$$

Here. $\hat{\rho}_{ab,MLt}$, $\hat{\rho}_{ab,GPt}$ are the values of $\hat{\rho}_{ab,ML}$, $\hat{\rho}_{ab,GP}$ on the t^{th} iteration.

The bias of $\hat{\rho}_{ab,ML}$ and $\hat{\rho}_{ab,GP}$ are defined respectively as, $\left\{\frac{1}{1015} \sum_{t=1}^{1015} \hat{\rho}_{ab,MLt} - \rho_{ab}\right\}$ and $\left\{\frac{1}{1015} \sum_{t=1}^{1015} \hat{\rho}_{ab,GPt} - \rho_{ab}\right\}$.

Preference is given to the estimator with largest SRE and smallest bias.

The patterns of the relative efficiencies in Tables 5.3-5.6 are not unlike of those for large samples. One observes however, that in smaller samples the generalized product-moment estimator gains efficiency over the maximum likelihood estimator, and at times exceeding 100%. This is particularly true for the individual-weighted estimator. As expected, this gain in efficiency is dampened with increasing sample size.

The small sample biases are presented in Tables 5.7-5.10. In general, the biases are small and negative, with the estimators becoming less biased as the sample size increases. There is no clear pattern for the biases, although the estimators tend to become more biased with increasing intraclass correlations. Overall, these results affirm that estimator bias encountered in practice is negligible.

TABLE 5.1

Asymptotic relative efficiencies for estimators of interclass correlation when $\rho_a = \rho_b$

ρ_{ab}	ρ_a	ρ_b	IND	FAM	CL-A	CL-B
0.0	0.0	0.0	100.0	47.1	70.3	75.1
0.0	0.1	0.1	94.7	61.0	82.7	85.3
0.0	0.2	0.2	85.2	71.6	88.9	89.6
0.0	0.3	0.3	75.7	79.9	90.9	90.3
0.0	0.4	0.4	67.0	86.3	90.4	88.9
0.0	0.5	0.5	59.5	91.2	88.2	86.3
0.0	0.6	0.6	53.0	94.8	85.2	83.2
0.0	0.7	0.7	47.5	97.3	81.5	79.6
0.1	0.1	0.1	91.9	59.7	80.9	83.7
0.1	0.2	0.2	84.3	70.8	87.9	88.7
0.1	0.3	0.3	75.4	79.3	90.4	89.9
0.1	0.4	0.4	67.0	85.9	90.[illegible]	88.6
0.1	0.5	0.5	59.6	90.9	88.2	86.3
0.1	0.6	0.6	53.1	94.6	85.2	83.2
0.1	0.7	0.7	47.6	91.2	81.6	79.6
0.3	0.3	0.3	71.4	73.9	85.2	85.5
0.3	0.4	0.4	66.4	82.2	87.7	86.8
0.3	0.5	0.5	60.2	88.4	87.4	85.6
0.3	0.6	0.6	54.1	93.1	85.3	83.4
0.3	0.7	0.7	48.5	96.4	82.1	80.1
0.5	0.5	0.5	58.6	80.9	82.7	82.1
0.5	0.6	0.6	55.3	88.3	84.1	82.5
0.5	0.7	0.7	50.4	93.8	82.7	80.8

TABLE 5.2

Asymptotic relative efficiencies for estimators of interclass correlation when $\rho_a \neq \rho_b$

ρ_{ab}	ρ_a	ρ_b	IND	FAM	CL-A	CL-B
0.0	0.1	0.3	85.9	69.2	83.7	93.2
0.0	0.3	0.1	85.6	71.4	92.4	84.4
0.0	0.3	0.5	67.2	85.1	87.0	92.3
0.0	0.5	0.3	67.6	86.3	93.4	85.4
0.0	0.5	0.7	53.0	94.1	82.9	85.8
0.0	0.7	0.5	53.6	94.8	87.6	80.6
0.1	0.1	0.3	84.7	68.3	82.6	92.3
0.1	0.3	0.1	84.6	70.6	91.3	83.5
0.1	0.3	0.5	67.2	84.7	86.7	92.1
0.1	0.5	0.3	67.6	85.9	93.2	85.2
0.1	0.5	0.7	53.1	93.9	82.9	85.8
0.1	0.7	0.5	53.7	94.6	87.6	80.7
0.3	0.3	0.5	66.4	80.6	83.9	90.2
0.3	0.5	0.3	66.9	82.1	90.9	83.0
0.3	0.5	0.7	54.1	92.3	82.7	86.2
0.3	0.7	0.5	54.7	93.1	87.9	80.6
0.5	0.5	0.7	55.2	87.2	80.9	85.6
0.5	0.7	0.5	56.0	88.3	87.1	79.3
0.7	0.7	0.9	47.6	92.0	76.8	81.9
0.7	0.9	0.7	48.5	93.2	83.7	75.0

TABLE 5.3

Small sample (N=25) relative efficiencies for estimators of interclass correlation when $\rho_a = \rho_b$

ρ_{ab}	ρ_a	ρ_b	IND	FAM	CL-A	CL-B
0.0	0.0	0.0	118.4	58.2	85.5	85.7
0.0	0.1	0.1	110.5	62.4	86.3	87.1
0.0	0.3	0.3	106.8	75.3	95.2	91.5
0.0	0.5	0.5	93.9	87.3	97.5	94.4
0.0	0.7	0.7	77.7	95.4	93.4	90.4
0.1	0.1	0.1	110.0	64.5	88.7	87.9
0.1	0.3	0.3	105.1	77.2	95.8	92.9
0.1	0.5	0.5	94.4	88.8	98.0	95.7
0.1	0.7	0.7	78.9	94.9	93.7	91.5
0.3	0.3	0.3	94.1	82.1	94.1	95.4
0.3	0.5	0.5	88.4	89.8	96.4	95.9
0.3	0.7	0.7	78.3	95.8	93.3	92.9
0.5	0.5	0.5	74.9	94.3	90.6	94.6
0.5	0.7	0.7	71.9	96.6	89.8	91.5
0.7	0.7	0.7	56.6	100.4	82.6	85.5
0.7	0.9	0.9	55.9	98.8	77.5	82.9

TABLE 5.4

Small sample (N=50) relative efficiencies for estimators of interclass correlation when $\rho_a = \rho_b$

ρ_{ab}	ρ_a	ρ_b	IND	FAM	CL-A	CL-B
0.0	0.0	0.0	115.9	45.3	70.8	74.0
0.0	0.1	0.1	118.3	57.3	82.6	85.1
0.0	0.3	0.3	110.4	76.2	96.4	95.7
0.0	0.5	0.5	93.2	88.5	97.6	97.5
0.0	0.7	0.7	76.0	96.1	93.3	93.1
0.1	0.1	0.1	114.6	61.2	87.9	89.3
0.1	0.3	0.3	105.2	77.1	94.4	97.4
0.1	0.5	0.5	91.4	88.6	96.3	98.3
0.1	0.7	0.7	76.6	96.0	92.6	94.5
0.3	0.3	0.3	89.3	81.0	94.3	96.7
0.3	0.5	0.5	78.2	91.1	91.8	93.9
0.3	0.7	0.7	69.4	96.9	88.7	89.8
0.5	0.5	0.5	67.1	88.6	86.0	89.7
0.5	0.7	0.7	63.4	96.1	84.5	88.9
0.7	0.7	0.7	49.4	93.2	77.2	80.6
0.7	0.9	0.9	49.1	99.1	73.9	78.1

TABLE 5.5

Small sample (N=100) relative efficiencies for estimators of interclass correlation when $\rho_a = \rho_b$

ρ_{ab}	ρ_a	ρ_b	IND	FAM	CL-A	CL-B
0.0	0.0	0.0	111.0	42.9	70.3	71.9
0.0	0.1	0.1	111.4	53.8	80.2	80.1
0.0	0.3	0.3	102.5	76.1	90.5	96.9
0.0	0.5	0.5	82.9	89.0	91.6	96.3
0.0	0.7	0.7	66.1	97.3	86.6	89.5
0.1	0.1	0.1	108.6	57.3	84.1	84.4
0.1	0.3	0.3	96.4	76.1	92.2	93.6
0.1	0.5	0.5	82.6	88.7	90.7	96.2
0.1	0.7	0.7	68.5	96.8	86.8	90.5
0.3	0.3	0.3	77.9	79.1	90.4	91.3
0.3	0.5	0.5	69.7	88.1	86.5	92.7
0.3	0.7	0.7	63.6	95.4	84.6	88.7
0.5	0.5	0.5	58.7	86.7	81.4	87.3
0.5	0.7	0.7	55.1	94.8	79.0	86.3
0.7	0.7	0.7	55.1	94.8	79.0	86.3
0.7	0.9	0.9	43.8	89.9	72.9	77.5

TABLE 5.6

Small sample (N=50) relative efficiencies for estimators of interclass correlation when $\rho_a \neq \rho_b$

ρ_{ab}	ρ_a	ρ_b	IND	FAM	CL-A	CL-B
0.0	0.1	0.3	114.3	66.3	86.0	94.9
0.0	0.3	0.1	117.2	65.7	94.0	87.3
0.0	0.3	0.5	102.9	82.0	94.4	100.4
0.0	0.5	0.3	101.9	82.1	101.0	93.3
0.0	0.5	0.7	84.8	92.4	93.1	98.4
0.0	0.7	0.5	85.4	91.6	98.9	93.1
0.1	0.1	0.3	109.4	67.2	86.1	96.7
0.1	0.3	0.1	109.9	71.1	97.0	91.1
0.1	0.3	0.5	98.7	82.5	92.3	101.2
0.1	0.5	0.3	99.4	82.9	99.2	95.7
0.1	0.5	0.7	84.8	92.1	91.7	99.4
0.1	0.7	0.5	85.3	92.4	98.3	94.1
0.3	0.3	0.5	83.5	87.9	89.1	100.0
0.3	0.5	0.3	85.0	87.2	97.1	94.6
0.3	0.5	0.7	73.8	93.5	88.7	94.2
0.3	0.7	0.5	74.2	94.4	93.3	90.5
0.5	0.5	0.7	64.5	92.6	88.2	91.8
0.5	0.7	0.5	66.6	93.2	88.3	88.1
0.7	0.7	0.9	50.3	96.7	74.6	81.6
0.7	0.9	0.7	53.0	98.0	78.7	81.9

TABLE 5.7

Small sample (N=25) biases for estimators of interclass correlation when $\rho_a = \rho_b$ ($\times 10^4$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B
0.0	0.0	0.0	- 41.4	- 31.7	- 11.4	- 56.7	- 63.9
0.0	0.1	0.1	2.1	2.3	27.9	15.3	5.2
0,0	0.3	0.3	- 55.7	- 37.1	- 40.9	- 51.4	- 42.2
0.0	0.5	0.5	- 82.9	- 79.0	- 64.7	- 78.3	- 80.5
0.0	0.7	0.7	- 63.8	- 57.8	- 58.4	- 74.4	- 56.6
0.1	0.1	0.1	- 47.8	- 97.4	47.5	- 19.3	- 15.2
0.1	0.3	0.3	-151.1	-216.5	- 67.5	-153.9	-130.4
0.1	0.5	0.5	-180.4	-265.4	-131.4	-207.7	-190.0
0.1	0.7	0.7	-136.9	-229.9	-117.5	-184.0	-169.7
0.3	0.3	0.3	-248.5	-457.2	- 83.3	-253.4	-252.4
0.3	0.5	0.5	-265.3	-500.8	-161.9	-325.2	-308.0
0.3	0.7	0.7	-216.9	-462.7	-171.9	-318.6	-297.8
0.5	0.5	0.5	-282.2	-630.1	-106.1	-354.8	-327.3
0.5	0.7	0.7	-274.5	-618.1	-204.0	-394.3	-381.7
0.7	0.7	0.7	-285.4	-676.2	-177.3	-396.3	-380.0
0.7	0.9	0.9	-221.9	-550.2	-197.4	-369.4	-339.8

TABLE 5.8

Small sample (N=50) biases for estimators of interclass correlation when $\rho_a = \rho_b$ ($\times 10^4$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B
0.0	0.0	0.0	- 12.7	- 17.7	15.6	- 5.7	1.8
0.0	0.1	0.1	8.5	0.7	27.6	15.2	5.2
0.0	0.3	0.3	- 29.3	- 12.3	- 38.3	- 27.7	- 44.6
0.0	0.5	0.5	- 37.8	- 26.6	- 31.0	- 32.5	- 45.8
0.0	0.7	0.7	- 53.3	- 68.5	- 45.7	- 66.9	- 59.7
0.1	0.1	0.1	- 58.1	-130.6	76.6	- 24.8	- 13.8
0.1	0.3	0.3	- 67.5	-165.2	12.8	- 65.9	- 86.4
0.1	0.5	0.5	- 69.9	-169.4	- 29.8	-100.7	-103.9
0.1	0.7	0.7	- 81.0	-207.9	- 60.2	-134.4	-133.9
0.3	0.3	0.3	-134.4	-376.2	47.2	-148.0	-137.5
0.3	0.5	0.5	-101.9	-386.3	1.8	-175.0	-179.6
0.3	0.7	0.7	- 98.4	-387.2	- 55.2	-212.4	-209.8
0.5	0.5	0.5	-130.2	-488.6	- 1.2	-215.6	-205.3
0.5	0.7	0.7	-102.3	-472.1	- 44.8	-242.1	-227.7
0.7	0.7	0.7	-140.4	-524.6	- 63.4	-257.0	-243.6
0.7	0.9	0.9	- 96.5	-438.6	- 75.9	-240.0	-229.9

TABLE 5.9

Small sample (N=100) biases for estimators of interclass correlation when $\rho_a = \rho_b$ ($\times 10^4$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B
0.0	0.0	0.0	- 5.4	- 8.2	- 11.6	- 3.6	- 17.3
0.0	0.1	0.1	- 24.7	- 26.1	- 18.6	- 24.8	- 26.3
0.0	0.3	0.3	- 34.0	- 29.0	- 29.6	- 35.8	- 34.4
0.0	0.5	0.5	- 32.4	- 37.5	- 19.4	- 35.3	- 37.5
0.0	0.7	0.7	- 35.0	- 38.8	- 31.9	- 42.3	- 44.1
0.1	0.1	0.1	- 7.7	- 78.6	159.1	50.2	40.4
0.1	0.3	0.3	- 37.5	-151.5	68.5	- 41.4	- 40.9
0.1	0.5	0.5	- 45.4	-154.6	- 8.6	- 88.1	- 80.0
0.1	0.7	0.7	- 37.8	-152.7	- 23.9	- 92.4	- 86.4
0.3	0.3	0.3	- 50.2	-287.6	128.5	- 66.3	- 61.4
0.3	0.5	0.5	- 42.8	-328.6	49.6	-129.4	-120.6
0.3	0.7	0.7	- 47.1	-315.2	- 15.0	-164.7	-149.4
0.5	0.5	0.5	- 50.6	-370.2	60.3	-133.6	-125.2
0.5	0.7	0.7	- 42.8	-395.5	12.2	-176.1	-165.8
0.7	0.7	0.7	- 61.7	-370.5	- 1.0	-160.1	-147.8
0.7	0.9	0.9	- 30.0	-373.6	- 8.4	-174.9	-158.2

TABLE 5.10

Small sample (N=50) biases for estimators of interclass correlation when $\rho_a \neq \rho_b$ ($\times 10^4$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B
0.0	0.1	0.3	- 20.6	- 21.6	- 13.4	- 15.1	- 26.5
0.0	0.3	0.1	- 23.9	- 22.5	- 41.0	- 23.3	- 50.3
0.0	0.3	0.5	- 38.4	- 18.9	- 40.5	- 37.9	- 42.8
0.0	0.5	0.3	- 15.6	0.3	- 16.6	- 7.6	- 29.6
0.0	0.5	0.7	- 64.6	- 45.4	- 60.9	- 64.6	- 60.3
0.0	0.7	0.5	- 41.1	- 47.1	- 31.8	- 49.4	- 44.2
0.1	0.1	0.3	- 43.2	-136.3	- 51.2	- 30.5	- 35.7
0.1	0.3	0.1	- 68.0	-161.9	- 47.0	- 42.7	- 65.3
0.1	0.3	0.5	- 81.7	-185.9	- 16.9	- 89.5	-110.1
0.0	0.5	0.3	- 52.8	-162.3	9.0	- 69.1	- 84.5
0.1	0.5	0.7	- 64.8	-175.0	- 31.7	- 97.4	-110.3
0.1	0.7	0.5	- 73.8	-195.9	- 41.0	-119.4	-119.1
0.3	0.3	0.5	-124.9	-400.9	- 24.8	-167.9	-172.2
0.3	0.5	0.3	-121.5	-391.6	- 30.3	-165.0	-157.1
0.3	0.5	0.7	-104.5	-398.2	- 31.2	-192.2	-206.8
0.3	0.7	0.5	- 92.3	-382.3	- 22.5	-196.1	-182.1
0.5	0.5	0.7	-116.4	-491.7	- 20.2	-230.9	-221.2
0.5	0.7	0.5	-130.4	-502.8	- 30.6	-250.2	-225.7
0.7	0.7	0.9	-107.1	-473.5	- 53.1	-232.3	-229.4
0.7	0.9	0.7	-106.6	-462.1	- 56.3	-239.2	-217.3

CHAPTER 6 - CONSISTENT ESTIMATORS OF INTRACLASS CORRELATION

Use of the asymptotic variances of Chapter 4 for hypothesis testing and confidence interval construction requires that the value for the intraclass correlations, ρ_a and ρ_b, be known. This will generally not be the case and thus these parameters will usually be estimated empirically from the sample data. This chapter discusses two such estimation procedures.

6.1 Analysis of Variance Estimator

The analysis of variance (ANOVA) estimator of intraclass correlation has been most frequently recommended and adopted in the literature to estimate ρ_a and ρ_b (e.g. Rosner, 1982; Konishi, 1985; Donner, 1986; Donner and Eliasziw, 1988). In a Monte Carlo simulation study, Donner and Koval (1980a) demonstrated that the ANOVA estimator tends to be as effective as the maximum likelihood estimator ($\hat{\rho}_{a,ML}$) for values of intraclass correlation that arise naturally in family studies. In addition, the ANOVA estimator is computationally simpler than the iterative maximum likelihood procedure.

For class A, the ANOVA estimator will be denoted by

$\hat{\rho}_{a,AN}$ and for class B, by $\hat{\rho}_{b,AN}$. They are defined and described fully in Appendix 4.

6.2 Weighted Pairwise Estimator

An alternative to the ANOVA estimator has been proposed by Karlin et al. (1981), referred to as the weighted pairwise estimator. It is written in its most general form as (for class A),

$$\hat{\rho}_{a,W} = \frac{\sum_{i=1}^{N} W_i \sum_{\substack{j=1 \\ j \neq \ell}}^{a_i} \sum_{\ell=1}^{a_i} (x_{ij} - \tilde{\bar{x}}_{..})(x_{i\ell} - \tilde{\bar{x}}_{..})}{\sum_{i=1}^{N} W_i (a_i - 1) \sum_{j=1}^{a_i} (x_{ij} - \tilde{\bar{x}}_{..})^2}$$

where $\tilde{\bar{x}}_{..} = \sum_{i=1}^{N} W_i (a_i - 1) \sum_{j=1}^{a_i} x_{ij}$, and the weights W_i are chosen so that the constraint, $\sum_{i=1}^{N} W_i a_i (a_i - 1) = 1$, is satisfied.

This estimator has several advantages over the ANOVA estimator. First, $\hat{\rho}_{a,W}$ is presented directly as a form of Pearson's product-moment correlation coefficient. This is in contrast to the ANOVA estimator, derived through equating moments from a components of variance model (Searle, 1971, pgs. 376-472). Secondly, in the case of constant class sizes, the weighted pairwise estimator yields the maximum likelihood estimator of intraclass correlation; a property not inherent in the ANOVA estimator. Therefore, the weighted pairwise estimator of intraclass correlation suggests a more unified approach to the analysis of familial data, in that the procedures and properties closely resemble those for the generalized product-moment estimator of interclass correlation.

For the weights W_i, Karlin et al. (1981) suggested three possible choices. If the weights are taken to be $1 / \sum_{i=1}^{N} a_i(a_i-1)$, then $\hat{\rho}_{a,W}$ is just the simple pairwise estimator of intraclass correlation (Donner and Koval, 1980a). This is also referred to as the sib-pair method. In the family method, each pair of observations is assigned equal weight, but independent of the number of observations. This is done by setting $W_i = 1 / Na_i(a_i-1)$. An intermediate method, known as the individual method, weights each pair according to the number of pairs that an individual appears. In this case, $W_i = 1 / (a_i-1)\sum_{i=1}^{N} a_i$.

In examining each of the above weighting schemes, several points arise. As described by Fieller and Smith (1951), the sib-pair method tends to give too much weight to large-sized classes. Futhermore, Donner and Koval (1980a) have shown that the relative efficiency is poor for values of intraclass correlations that are likely to arise in family studies. For these reasons, the sib-pair estimator is rarely used in practice. It is also observed that both the family-weighted and individual-weighted methods are limited to family data with class sizes of at least two. This is because whenever a family has only one individual in either class (i.e. $a_i=1$ or $b_i=1$), W_i is undefined.

The following weighting scheme is proposed to alleviate the aforementioned problems. The weights are defined as : $W_i = 1 / a_i \sum_{i=1}^{N} (a_i-1)$. This choice of weights de-emphasizes contributions from larger classes and yet are valid when some of the class sizes are equal to one.

An estimator, $\hat{\rho}_{b,W}$ is obtained in an analogous way by replacing j with k, a_i with b_i, x_{ij} with y_{ik}, ρ_a with ρ_b in the expressions of Chapter 6.2.

6.3 Accuracy of the Weighted Pairwise Estimator

A Monte Carlo study was conducted to determine whether the weighted pairwise estimator with the proposed weights is comparable to the ANOVA estimator of intraclass correlation. For completeness, a further comparison was made between the ANOVA estimator and the other weighting schemes.

The random samples were generated following the algorithm in Donner and Koval (1980a). In particular, the observations were generated from a multivariate normal distribution. Class sizes were generated from a negative binomial distribution (m=2.84, P=.93) truncated below at 2 to assure that each of the weighting schemes were defined. Truncating below at 2 instead of below at 1, as in Donner and Koval (1980a), leads to no loss of generality.

For each of t=1015 independent runs at N=50, the values of each estimator $\hat{\rho}_{a,W}$, $\hat{\rho}_{a,AN}$ and $\hat{\rho}_{a,ML}$ were computed at ρ equal to 0.0 thru 0.9 . It is noted that negative values of intraclass correlations are usually not considered plausible in human studies and therefore the estimates were truncated (set to zero) if their computed values resulted in a negative quantity. Truncated estimators will be denoted as $\hat{\rho}^{\mapsto}_{a,W}$, $\hat{\rho}^{\mapsto}_{a,AN}$ and $\hat{\rho}^{\mapsto}_{a,ML}$, respectively. As a measure of the effectiveness, small sample relative efficiences and small sample biases were computed.

In addition, the large sample (N=300) properties of these estimators were examined by computing the asymptotic relative efficiencies. The asymptotic variances of $\hat{\rho}_{a,AN}$ and $\hat{\rho}_{a,ML}$ were derived by Smith (1956), and Donner and Koval (1980b), respectively. The large sample variance of $\hat{\rho}_{a,W}$ is derived in Appendix 5 and is given by

$$AV(\hat{\rho}_{a,W}) = 2(1-\rho_a)^2 \sum_{i=1}^{N} W_i^2 a_i (a_i-1)\left\{1 + (a_i-1)\rho_a\right\}^2$$

A closed form expression for the asymptotic variance of $\hat{\rho}_{a,W}$ also lends itself naturally to hypothesis testing and confidence interval construction about the intraclass correlation parameter.

The small sample and asymptotic relative efficiencies are given in Tables 6.1 and 6.2, respectively. As expected, the sib-pair (simple pairwise) estimator performs well at $\rho_a = 0.0$ but declines rapidly in relative efficiencies as ρ_a increases. Donner and Koval (1980a) found an identical result. The family-weighted estimator, on the other hand, does poorly at $\rho_a = 0.0$ and increases in relative efficiency as ρ_a increases. The relative efficiencies of the individual-weighted estimator mimic the pattern of the ANOVA estimator, but tend to be lower than those of the latter.

Provided that $\rho_a \le 0.2$, the estimator based upon the proposed weighting has relative efficiencies greater than those of the ANOVA estimator. As ρ_a increases beyond 0.2, the difference between the relative efficiencies of the proposed and ANOVA estimator slowly increases, but at no point is this difference greater than 10%, provided $\rho_a < 0.7$.

The biases of the five estimators are shown in Table 6.3. It is observed that the ANOVA estimator is the least biased provided $\rho_a > 0.2$, whereas the simple pairwise estimator is least biased when $\rho_a \leq 0.2$. The other three methods of weighting yield estimators with similar biases. Overall, the size of the biases are negligible in practice.

The preceeding results indicate that the estimator based upon the proposed method of weighting is a reasonable alternative to the ANOVA estimator of familial intraclass correlations. Although the estimator using the individual method of weighting is also quite satisfactory, its use is limited to samples with class sizes of at least two individuals each. Relative efficiencies were also computed when the negative binomial distribution was truncated below at one. The results shown in Table 6.4 remain supportive of the conclusions above.

TABLE 5.1

Small sample (N=50) relative efficiencies for estimators of intraclass correlation with negative binomial distribution truncated below at 2

ρ	Sib-pair	Individual	Family	Proposed	ANOVA
0.0	111.5	84.6	62.3	97.3	80.3
0.1	99.9	94.5	74.4	102.3	92.9
0.2	91.3	96.8	79.6	101.4	99.4
0.3	84.0	101.0	87.3	100.9	102.8
0.4	75.8	99.8	89.2	97.1	102.6
0.5	70.6	98.7	92.2	93.8	101.8
0.6	64.6	97.7	94.1	90.7	101.4
0.7	59.2	95.1	94.3	86.5	99.6
0.8	54.6	93.6	94.9	83.8	98.8
0.9	50.6	92.4	95.8	81.7	98.5

TABLE 6.2

Asymptotic relative efficiencies for estimators of intraclass correlation with negative binomial distribution truncated below at 2

ρ	Sib-pair	Individual	Family	Proposed	ANOVA
0.0	100.0	59.2	29.8	76.6	68.6
0.1	86.9	83.4	48.8	95.7	91.9
0.2	72.5	92.1	61.0	97.1	98.5
0.3	62.2	94.3	69.2	93.7	98.9
0.4	54.8	93.6	74.8	89.2	97.0
0.5	49.4	91.7	78.6	84.8	94.3
0.6	45.2	89.5	81.2	80.8	91.4
0.7	42.0	87.1	83.0	77.2	88.7
0.8	39.3	84.8	84.3	74.1	86.1
0.9	37.2	82.7	85.2	71.4	83.7

TABLE 6.3

Small sample (N=50) biases for estimators of intraclass correlation with negative binomial distribution truncated below at 2 ($\times 10^4$)

ρ	Sib-pair	Individual	Family	Proposed	ANOVA
0.0	362.8	436.2	508.5	403.1	458.5
0.1	47.4	88.2	102.4	81.1	154.2
0.2	-105.3	- 53.9	- 62.9	- 57.1	23.1
0.3	-172.1	-102.3	-115.3	-109.4	- 22.2
0.4	-204.7	-115.3	-123.6	-126.8	- 36.7
0.5	-203.9	-106.3	-112.5	-119.7	- 33.9
0.6	-234.5	-127.2	-127.9	-143.2	- 61.5
0.7	-292.5	-103.1	-100.9	-119.2	- 48.0
0.8	-171.9	- 86.6	- 81.2	-101.2	- 46.8
0.9	-106.7	- 52.5	- 47.2	- 62.2	- 31.1

TABLE 6.4

Small sample (N=50) and asymptotic relative efficiences for estimators of intraclass correlation with negative binomial distribution truncated below at 1

	Small sample		Asymptotic	
ρ	Proposed	ANOVA	Proposed	ANOVA
0.0	113.8	64.6	79.9	59.2
0.1	112.0	83.3	96.5	85.2
0.2	105.5	92.9	96.4	95.2
0.3	98.9	101.9	92.1	97.9
0.4	92.1	108.8	86.8	97.2
0.5	83.2	111.6	81.5	94.7
0.6	72.4	108.3	76.4	91.3
0.7	62.2	103.2	71.7	87.6
0.8	53.7	97.9	67.4	83.7
0.9	46.1	93.1	63.3	79.9

CHAPTER 7 - ACCURACY OF INTERVAL ESTIMATION

The pivotal quantities defined in Chapter 4 are inverted to construct approximate 100(1-α)% two-sided confidence intervals for ρ_{ab}. Prior to inversion, the variances $AV(\hat{\rho}_{ab,ML})$ and $AV(\hat{\rho}_{ab,GP})$ are replaced by their consistent estimators $\hat{A}\tilde{V}(\hat{\rho}_{ab,ML})$ and $\hat{A}\tilde{V}(\hat{\rho}_{ab,GP})$, where the unknown parameters are replaced by appropriate estimators. It is known from normal theory that the accuracy of the asymptotic distribution of pivotal quantities is not appreciably affected by substituting consistent estimators.

Three general methods for obtaining two-sided confidence intervals for ρ_{ab} are described, followed by an assessment of their quality. It is noted that quality is measured in terms of coverage probabilities and interval widths, where the method yielding the narrowest interval with an acceptable coverage probability is preferred.

7.1 A Method Based on Maximum Likelihood Theory (Method ML)

A consistent estimator $\hat{A}\tilde{V}(\hat{\rho}_{ab,ML})$ of $AV(\hat{\rho}_{ab,ML})$ is obtained by replacing ρ_{ab}, ρ_a and ρ_b by the maximum likelihood estimators $\hat{\rho}_{ab,ML}$, $\hat{\rho}^{\mapsto}_{a,ML}$ and $\hat{\rho}^{\mapsto}_{b,ML}$, respectively.

Again, $\hat{\rho}^{\mapsto}_{a,ML}$ and $\hat{\rho}^{\mapsto}_{b,ML}$ denote truncated (non-negative) estimators.

Thus, from the pivotal quantity method of constructing confidence intervals, an approximate 100(1-α)% two-sided confidence interval for ρ_{ab} is given by

$$\left[\hat{\rho}_{ab,ML} \quad \pm \quad z_{1-\alpha/2}\left\{\widehat{AV}(\hat{\rho}_{ab,ML})\right\}^{1/2}\right]$$

where $z_{1-\alpha/2}$ is the 100(1-α/2) percentile point of the standard normal distribution.

7.2 A Method Based on the Standard Error of the Generalized Product-moment Estimator (Methods IND, FAM, CL-A, CL-B)

Similar reasoning to above leads to confidence intervals based on the estimator $\hat{\rho}_{ab,GP}$, given by

$$\left[\hat{\rho}_{ab,GP} \quad \pm \quad z_{1-\alpha/2}\left\{\widehat{AV}(\hat{\rho}_{ab,GP})\right\}^{1/2}\right]$$

A consistent estimator of $\widehat{AV}(\hat{\rho}_{ab,GP})$ of $AV(\hat{\rho}_{ab,GP})$ is obtained by substituting $\hat{\rho}_{ab,GP}$ for ρ_{ab}, and $\hat{\rho}^{\mapsto}_{a,AN}$ and $\hat{\rho}^{\mapsto}_{b,AN}$ or $\hat{\rho}^{\mapsto}_{a,W}$ and $\hat{\rho}^{\mapsto}_{b,W}$ for ρ_a and ρ_b, respectively.

7.3 A Method Based on a Modified Fisher's Z-Transformation (Methods ZIND, ZFAM, ZCL-A, ZCL-B

Following the methodology of Donner and Eliasziw (1988), an alternative confidence interval can be constructed by considering a generalization of the well-known variance stabilizing transformation for a simple correlation coefficient proposed by Fisher (1921). That is,

$$Z(\hat{\rho}_{ab,GP}) = \tfrac{1}{2}\log \frac{1 + \hat{\rho}_{ab,GP}}{1 - \hat{\rho}_{ab,GP}}$$

is approximately normally distributed with mean $Z(\rho_{ab})$ and variance $1/(D-3)$; where D is defined as the aggregate degrees of freedom over N families (Rosner, 1982), given by

$$D = \sum_{i=1}^{N} \frac{a_i b_i}{\left\{1 + (a_i-1)\rho_a\right\}\left\{1 + (b_i-1)\rho_b\right\}} ,$$

where ρ_a and ρ_b are replaced by consistent truncated estimators. Moreover, the quantity

$$\left\{Z(\hat{\rho}_{ab,GP}) - Z(\rho_{ab})\right\}\left\{\frac{1}{D-3}\right\}^{-1/2}$$

is pivotal and has a standard normal distribution.

Therefore, an approximate $100(1-\alpha)\%$ two-sided confidence interval for ρ_{ab} is given by

$$I\left[Z(\hat{\rho}_{ab,GP}) \pm z_{1-\alpha/2}\left\{\frac{1}{D-3}\right\}^{1/2}\right]$$

where I denotes the inverse transformation

$$I(\theta) = \frac{e^{2\theta} - 1}{e^{2\theta} + 1}$$

7.4 Evaluation of the Methods

In order to assess the quality of the intervals constructed by the various methods, a Monte Carlo simulation was conducted since it was not possible to obtain exact coverage probabilities using analytic techniques. The simulation study was conducted as part of the algorithm for investigating the accuracy of the point estimators in Chapter 5.

For various combinations of the parameter values (ρ_{ab}, ρ_a, ρ_b), 95% two-sided confidence intervals were calculated for each of the three methods and four interclass weighting schemes. The number of runs, t=1015, was chosen so that a reduction to 0.925 of the true coverage probability for a nominal 95% interval could be detected with 90% power (two-tailed).

As summary statistics over the 1015 independent runs, for each parameter combination, the empirical coverage probabitility (converted to percentages) and mean interval width were computed.

Estimated coverage probabilities for N = 25, 50, 100 are given in Tables 7.1-7.3, respectively. Corresponding mean interval widths are shown in Tables 7.4-7.6.

As would be expected, the coverage levels for all methods are as close or closer to nominal with increasing values of N. In larger samples (N=100), all methods except

ZIND and ZFAM provide coverage levels close to nominal for all parameter combinations. Method ZIND provides acceptable coverage levels at values of $\rho_a, \rho_b \le 0.3$, whereas ZFAM yields acceptable coverage at values of $\rho_a, \rho_b > 0.3$.

At N=50, method ZFAM no longer gives overall acceptable coverage probabilities, in particular at values of $\rho_{ab} = 0.1$, $\rho_a, \rho_b > 0.1$.

In smaller samples (N=25) there is a general tendency for all the methods to underestimate the nominal coverage probability. Moreover, none of the methods give adequate coverage at $\rho_{ab} = 0.3$, $\rho_a, \rho_b \le 0.5$.

In terms of precision, methods based upon the modified Fisher's Z-transformation have mean interval widths as small or smaller then their corresponding methods based upon the standard errors. In addition, the widths are very similar to method ML. Method IND yields mean widths very similar to method ML for parameter combinations $\rho_a, \rho_b \le 0.3$. Methods FAM and ML provide similar mean widths at values of $\rho_a, \rho_b > 0.3$. On the other hand, methods CL-A and CL-B give interval widths that are consistently wider than those of method ML.

Turning now to the case of unequal intraclass correlations, Tables 7.7 and 7.8 show the estimated coverage probabilities and mean interval widths at N=50. These results are found to be similar to those in Tables 7.2 and 7.5, in the sense that the respective coverages and widths are fairly robust to departures from intraclass correlation equality.

It is noted that the results for the above simulation study were reported solely for the case where the weighted pairwise estimator with weights $w_i = 1 / a_i \sum_{i=1}^{N} (a_i - 1)$ was used to estimate the intraclass correlations. The reason for this is that confidence intervals employing the ANOVA estimator as an estimate of ρ_a and ρ_b yield, on one hand, very similar coverage probabilities but on the other hand consistently wider intervals. This observation provides yet a third advantage in favour of the weighted pairwise estimator, in addition to those mentioned in Chapter 6.2. For illustrative purposes, Tables 7.9 and 7.10 have been enclosed and may be compared to Tables 7.2 and 7.5.

TABLE 7.1

95% two-sided confidence intervals for the interclass correlation
Coverage percentages at N=25 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: ($\rho_a = \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	94.4	96.7	93.1	96.1	95.5	96.8	87.1	94.5	94.5
0.0	0.1	0.1	94.0	96.5	92.6	95.0	94.4	96.8	87.8	94.6	93.8
0.0	0.3	0.3	91.8	95.1	92.4	93.8	92.2	95.0	90.0	94.3	92.9
0.0	0.5	0.5	91.3	94.4	91.7	92.9	91.5	93.2	91.7	94.4	93.3
0.0	0.7	0.7	91.5	92.3	91.7	91.7	91.0	90.2	93.7	93.5	92.9
0.1	0.1	0.1	94.0	95.3	93.3	94.4	93.6	95.9	88.7	94.4	93.7
0.1	0.3	0.3	92.1	94.9	92.2	93.5	92.3	94.4	90.2	94.1	92.8
0.1	0.5	0.5	91.1	94.2	91.5	93.3	91.5	93.5	92.3	94.4	92.7
0.1	0.7	0.7	91.8	93.2	91.8	92.2	91.3	90.6	93.4	94.4	93.1
0.3	0.3	0.3	90.8	91.9	91.6	91.7	92.5	91.6	90.3	91.6	91.6
0.3	0.5	0.5	90.9	92.6	91.0	92.3	91.9	91.6	91.3	92.2	91.6
0.3	0.7	0.7	91.9	93.0	90.4	92.2	92.2	90.5	93.8	94.1	92.8
0.5	0.5	0.5	92.1	93.2	93.0	92.9	93.7	90.3	93.8	93.2	93.6
0.5	0.7	0.7	91.4	94.9	91.1	93.0	93.2	91.5	93.6	94.5	93.8
0.7	0.7	0.7	93.9	95.0	93.6	94.5	95.4	89.7	95.4	94.6	94.1
0.7	0.9	0.9	93.5	97.2	93.1	94.3	95.4	92.5	95.8	94.4	95.3

TABLE 7.2

95% two-sided confidence intervals for the interclass correlation
Coverage percentages at N=50 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: ($\rho_a = \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	95.2	97.4	93.2	94.7	96.2	97.4	82.5	93.1	93.9
0.0	0.1	0.1	93.5	97.2	93.0	94.0	94.7	97.0	84.7	92.5	93.2
0.0	0.3	0.3	92.3	96.3	92.7	93.4	93.2	95.2	89.6	93.1	93.1
0.0	0.5	0.5	92.6	96.2	92.3	93.6	92.8	93.6	92.4	93.2	92.8
0.0	0.7	0.7	93.2	95.1	92.7	94.1	93.9	89.5	94.0	93.8	93.5
0.1	0.1	0.1	93.9	96.7	92.8	94.2	95.3	96.5	86.3	92.1	93.8
0.1	0.3	0.3	92.5	95.8	92.2	93.5	94.1	94.9	89.7	92.7	93.3
0.1	0.5	0.5	93.0	96.2	91.7	93.1	94.0	93.3	91.6	93.3	94.1
0.1	0.7	0.7	92.7	95.0	91.7	93.9	94.0	90.0	93.6	93.4	93.7
0.3	0.3	0.3	92.6	94.3	94.0	94.3	94.5	92.4	91.2	93.5	92.6
0.3	0.5	0.5	92.7	94.9	92.9	93.2	94.7	91.5	93.8	93.9	94.1
0.3	0.7	0.7	92.6	95.3	92.4	93.5	94.6	90.1	93.9	94.4	93.2
0.5	0.5	0.5	94.5	95.4	94.5	94.6	95.6	91.5	95.1	93.4	95.3
0.5	0.7	0.7	94.1	96.6	93.0	94.0	95.1	90.9	95.2	94.3	94.5
0.7	0.7	0.7	95.0	97.1	94.4	95.5	95.7	89.3	95.7	94.4	95.3
0.7	0.9	0.9	93.8	97.5	93.1	94.8	96.1	88.7	95.6	93.7	94.2

TABLE 7.3

95% two-sided confidence intervals for the interclass correlation
Coverage percentages at N=100 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: $(\rho_a = \rho_b)$

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	95.9	97.4	92.5	94.3	94.7	97.4	80.7	90.4	91.3
0.0	0.1	0.1	95.0	97.0	92.8	94.5	94.1	96.7	83.7	92.3	92.3
0.0	0.3	0.3	94.4	96.4	93.4	94.5	94.8	94.7	89.3	93.5	94.3
0.0	0.5	0.5	94.3	96.3	93.6	95.2	95.8	93.3	93.0	94.6	95.0
0.0	0.7	0.7	94.3	97.0	94.0	95.5	95.5	88.6	94.4	94.4	94.2
0.1	0.1	0.1	93.6	96.1	93.6	95.0	94.1	95.7	85.2	92.5	91.5
0.1	0.3	0.3	94.8	97.4	93.6	94.5	95.0	95.6	90.6	94.1	94.1
0.1	0.5	0.5	94.7	96.7	93.7	95.2	96.0	94.2	93.1	94.4	95.0
0.1	0.7	0.7	94.4	96.7	93.8	95.4	96.2	90.2	94.0	94.0	94.6
0.3	0.3	0.3	95.2	95.4	94.8	95.2	96.1	93.1	93.1	94.5	94.9
0.3	0.5	0.5	95.5	97.0	94.6	95.2	96.5	92.3	94.3	93.7	95.4
0.3	0.7	0.7	95.5	97.1	94.6	95.6	96.4	90.2	95.2	94.7	94.9
0.5	0.5	0.5	95.3	95.3	94.7	95.3	96.1	90.9	95.2	94.4	95.8
0.5	0.7	0.7	95.4	97.4	94.2	95.5	96.8	90.6	95.5	94.5	95.6
0.7	0.7	0.7	96.3	96.7	94.5	96.1	97.0	89.7	96.2	95.3	95.6
0.7	0.9	0.9	95.1	98.7	94.3	96.8	97.3	88.0	96.3	93.4	94.1

TABLE 7.4

95% two-sided confidence intervals for the interclass correlation
Mean interval widths at N=25 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: ($\rho_a = \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	.45	.45	.57	.49	.49	.44	.44	.44	.44
0.0	0.1	0.1	.47	.48	.57	.50	.50	.47	.46	.47	.47
0.0	0.3	0.3	.53	.57	.60	.55	.55	.53	.52	.52	.52
0.0	0.5	0.5	.60	.68	.63	.62	.62	.59	.59	.59	.59
0.0	0.7	0.7	.66	.82	.67	.70	.70	.66	.66	.66	.66
0.1	0.1	0.1	.46	.48	.57	.50	.50	.46	.46	.46	.46
0.1	0.3	0.3	.52	.57	.59	.55	.55	.52	.52	.52	.52
0.1	0.5	0.5	.59	.68	.62	.62	.62	.59	.59	.59	.59
0.1	0.7	0.7	.66	.81	.66	.70	.70	.65	.65	.66	.66
0.3	0.3	0.3	.47	.52	.54	.50	.50	.49	.48	.48	.48
0.3	0.5	0.5	.54	.63	.57	.57	.57	.55	.55	.55	.55
0.3	0.7	0.7	.61	.76	.61	.65	.64	.62	.61	.62	.62
0.5	0.5	0.5	.46	.52	.47	.47	.47	.48	.46	.47	.47
0.5	0.7	0.7	.50	.64	.51	.54	.54	.54	.52	.53	.53
0.7	0.7	0.7	.33	.45	.35	.37	.37	.41	.37	.39	.39
0.7	0.9	0.9	.39	.56	.38	.44	.44	.45	.42	.44	.44

TABLE 7.5

95% two-sided confidence intervals for the interclass correlation
Mean interval widths at N=50 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: $(\rho_a = \rho_b)$

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	.30	.30	.40	.34	.34	.30	.30	.30	.30
0.0	0.1	0.1	.33	.34	.41	.35	.35	.32	.32	.32	.32
0.0	0.3	0.3	.38	.42	.43	.40	.40	.38	.38	.38	.38
0.0	0.5	0.5	.44	.52	.46	.46	.46	.43	.43	.43	.43
0.0	0.7	0.7	.48	.63	.49	.53	.53	.48	.48	.48	.48
0.1	0.1	0.1	.31	.33	.40	.35	.35	.32	.32	.32	.32
0.1	0.3	0.3	.38	.41	.43	.40	.40	.38	.38	.38	.38
0.1	0.5	0.5	.43	.51	.45	.45	.45	.43	.43	.43	.43
0.1	0.7	0.7	.48	.63	.49	.52	.52	.48	.48	.48	.48
0.3	0.3	0.3	.34	.38	.39	.36	.36	.35	.34	.35	.35
0.3	0.5	0.5	.39	.47	.41	.41	.41	.40	.39	.40	.40
0.3	0.7	0.7	.44	.58	.44	.48	.48	.45	.44	.45	.45
0.5	0.5	0.5	.31	.39	.33	.34	.34	.35	.33	.34	.34
0.5	0.7	0.7	.36	.48	.36	.39	.39	.39	.37	.38	.38
0.7	0.7	0.7	.23	.33	.24	.26	.26	.29	.26	.27	.27
0.7	0.9	0.9	.27	.41	.27	.31	.31	.31	.29	.30	.30

TABLE 7.6

95% two-sided confidence intervals for the interclass correlation
Mean interval widths at N=100 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: ($\rho_a = \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	.21	.21	.28	.23	.23	.20	.20	.20	.20
0.0	0.1	0.1	.23	.23	.29	.25	.25	.23	.23	.23	.23
0.0	0.3	0.3	.27	.30	.31	.29	.29	.27	.27	.27	.27
0.0	0.5	0.5	.31	.38	.33	.33	.33	.31	.31	.31	.31
0.0	0.7	0.7	.35	.47	.35	.38	.38	.35	.35	.35	.35
0.1	0.1	0.1	.22	.23	.29	.24	.24	.22	.22	.22	.22
0.1	0.3	0.3	.27	.30	.30	.28	.25	.27	.27	.27	.27
0.1	0.5	0.5	.31	.38	.32	.33	.33	.31	.31	.31	.31
0.1	0.7	0.7	.34	.46	.35	.37	.37	.34	.34	.34	.34
0.3	0.3	0.3	.24	.28	.27	.26	.26	.25	.24	.25	.25
0.3	0.5	0.5	.28	.35	.29	.30	.30	.29	.28	.29	.29
0.3	0.7	0.7	.31	.42	.32	.34	34	.32	.32	.32	.32
0.5	0.5	0.5	.22	.28	.24	.24	.24	.24	.23	.24	.24
0.5	0.7	0.7	.25	.34	.26	.28	.28	.27	.26	.27	.27
0.7	0.7	0.7	.16	.23	.17	.18	.18	.20	.18	.19	.19
0.7	0.9	0.9	.19	.30	.19	.22	.22	.21	.20	.20	.20

TABLE 7.7

95% two-sided confidence intervals for the interclass correlation
Coverage percentages at N=50 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: ($\rho_a \neq \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.1	0.3	92.7	97.2	92.6	94.0	94.6	96.7	86.2	92.5	94.1
0.0	0.3	0.1	92.8	96.5	92.2	93.3	93.4	95.8	86.4	93.0	92.5
0.0	0.3	0.5	93.2	96.4	92.5	93.8	93.3	94.8	91.5	93.6	93.7
0.0	0.5	0.3	91.4	96.4	92.2	92.8	92.9	95.1	90.5	93.2	92.4
0.0	0.5	0.7	92.6	95.5	92.6	93.7	93.3	91.6	93.0	93.2	93.7
0.0	0.7	0.5	93.1	96.1	92.4	94.5	93.9	91.5	93.4	94.8	93.0
0.1	0.1	0.3	92.7	96.7	92.7	93.2	95.3	96.0	87.4	92.8	94.1
0.1	0.3	0.1	93.0	96.2	92.7	93.8	94.2	95.6	88.0	93.2	92.8
0.1	0.3	0.5	92.7	96.2	92.6	93.5	93.8	93.9	91.5	93.0	94.2
0.1	0.5	0.3	92.2	95.6	92.6	92.8	93.8	93.7	91.0	93.3	93.0
0.1	0.5	0.7	92.8	95.8	91.8	93.5	93.8	91.6	92.5	93.1	94.3
0.1	0.7	0.5	92.7	96.3	91.7	93.3	93.7	90.6	92.4	93.7	93.3
0.3	0.3	0.5	93.5	94.3	93.8	92.9	95.0	92.0	92.9	93.1	94.4
0.3	0.5	0.3	92.6	95.3	93.9	93.8	95.6	92.0	92.9	94.1	94.0
0.3	0.5	0.7	93.3	95.5	92.4	94.0	94.3	91.5	94.0	93.9	94.3
0.3	0.7	0.5	92.3	94.6	92.1	93.0	93.8	90.6	93.1	94.1	93.1
0.5	0.5	0.7	94.6	96.3	94.3	94.7	95.6	92.2	95.7	94.6	95.2
0.5	0.7	0.5	94.3	95.8	93.7	96.0	93.7	90.5	94.9	94.3	95.1
0.7	0.7	0.9	93.8	97.8	93.2	95.4	95.6	90.3	95.9	94.2	95.6
0.7	0.9	0.7	94.5	98.2	93.9	95.3	96.0	91.1	95.9	94.5	95.0

TABLE 7.8

95% two-sided confidence intervals for the interclass correlation
Mean interval widths at N=50 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: ($\rho_a \neq \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.1	0.3	.35	.37	.42	.38	.37	.35	.35	.35	.35
0.0	0.3	0.1	.35	.37	.42	.37	.38	.35	.35	.35	.35
0.0	0.3	0.5	.41	.46	.44	.43	.42	.40	.40	.40	.40
0.0	0.5	0.3	.41	.47	.44	.42	.43	.40	.40	.40	.40
0.0	0.5	0.7	.46	.57	.47	.50	.48	.46	.46	.46	.46
0.0	0.7	0.5	.46	.57	.47	.48	.50	.46	.46	.46	.46
0.1	0.1	0.3	.35	.37	.41	.38	.36	.35	.34	.35	.35
0.1	0.3	0.1	.35	.37	.42	.36	.38	.35	.35	.35	.35
0.1	0.3	0.5	.40	.46	.44	.43	.41	.40	.40	.40	.40
0.1	0.5	0.3	.40	.46	.44	.41	.43	.40	.40	.40	.40
0.1	0.5	0.7	.46	.57	.47	.49	.48	.45	.45	.45	.45
0.1	0 7	0.5	.46	.57	.47	.48	.49	.45	.45	.45	.45
0.3	0.3	0.5	.36	.42	.40	.39	.38	.38	.37	.37	.37
0.3	0.5	0.3	.36	.42	.40	.38	.39	.38	.37	.37	.37
0.3	0.5	0.7	.41	.52	.43	.45	.43	.43	.42	.42	.42
0.3	0.7	0.5	.41	.52	.43	.43	.45	.42	.42	.42	.42
0.5	0.5	0.7	.33	.43	.35	.37	.35	.37	.35	.36	.36
0.5	0.7	0.5	.33	.43	.35	.35	.36	.37	.35	.36	.36
0.7	0.7	0.9	.25	.36	.25	.29	.27	.30	.27	.28	.28
0.7	0.9	0.7	.25	.36	.25	.27	.28	.30	.27	.28	.28

TABLE 7.9

95% two-sided confidence intervals for the interclass correlation
Coverage percentages at N=50 with $\hat{\rho}_{a,AN}$ and $\hat{\rho}_{b,AN}$: ($\rho_a = \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	95.2	97.9	93.7	95.3	96.2	97.8	84.4	93.3	94.8
0.0	0.1	0.1	93.5	97.5	93.2	94.9	94.9	97.4	86.5	93.5	94.1
0.0	0.3	0.3	92.3	97.1	93.1	94.5	94.3	96.0	91.0	94.0	93.7
0.0	0.5	0.5	92.6	96.8	92.9	94.2	93.4	94.1	93.0	94.2	93.7
0.0	0.7	0.7	93.2	95.5	93.0	94.5	94.3	89.9	94.5	94.0	93.8
0.1	0.1	0.1	93.9	97.2	93.1	94.8	95.4	97.1	87.2	93.5	94.2
0.1	0.3	0.3	92.5	96.7	92.8	94.1	94.8	95.5	91.2	93.7	94.8
0.1	0.5	0.5	93.0	96.8	92.1	94.2	94.2	93.9	92.4	94.0	94.3
0.1	0.7	0.7	92.7	95.7	91.9	94.2	94.5	90.2	94.3	94.1	94.1
0.3	0.3	0.3	92.6	94.8	94.5	95.3	95.1	93.2	92.2	94.7	94.3
0.3	0.5	0.5	92.7	96.2	93.7	93.5	95.7	93.0	94.1	94.1	94.7
0.3	0.7	0.7	92.6	95.6	92.7	93.9	95.0	90.5	94.2	94.7	93.6
0.5	0.5	0.5	94.5	96.6	94.8	95.6	96.0	92.4	95.6	94.8	95.8
0.5	0.7	0.7	94.1	97.0	93.2	94.8	96.0	91.8	95.4	94.8	95.1
0.7	0.7	0.7	95.0	97.7	94.4	96.3	96.0	90.3	96.1	94.9	95.5
0.7	0.9	0.9	93.8	97.8	93.5	95.2	96.3	89.0	95.6	93.8	94.5

TABLE 7.10

95% two-sided confidence intervals for the interclass correlation
Mean interval widths at N=50 with $\hat{\rho}_{a,AN}$ and $\hat{\rho}_{b,AN}$: ($\rho_a = \rho_b$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.0	0.0	.30	.32	.40	.34	.34	.31	.31	.31	.31
0.0	0.1	0.1	.33	.35	.41	.36	.36	.33	.33	.33	.33
0.0	0.3	0.3	.38	.44	.44	.41	.41	.39	.39	.39	.39
0.0	0.5	0.5	.44	.54	.47	.47	.47	.44	.44	.44	.44
0.0	0.7	0.7	.48	.65	.49	.53	.53	.49	.49	.49	.49
0.1	0.1	0.1	.31	.35	.41	.36	.36	.33	.33	.33	.33
0.1	0.3	0.3	.38	.43	.43	.41	.41	.39	.39	.39	.39
0.1	0.5	0.5	.43	.53	.46	.46	.46	.44	.44	.44	.44
0.1	0.7	0.7	.48	.64	.49	.53	.53	.48	.48	.48	.48
0.3	0.3	0.3	.34	.40	.39	.37	.37	.36	.36	.36	.36
0.3	0.5	0.5	.39	.49	.42	.42	.42	.41	.41	.41	.41
0.3	0.7	0.7	.44	.60	.45	.49	.49	.45	.45	.45	.45
0.5	0.5	0.5	.31	.41	.34	.34	.34	.36	.34	.34	.34
0.5	0.7	0.7	.36	.50	.37	.40	.40	.39	.37	.38	.38
0.7	0.7	0.7	.23	.34	.24	.27	.26	.29	.26	.27	.27
0.7	0.9	0.9	.27	.42	.27	.31	.31	.31	.29	.30	.30

CHAPTER 8 - DISCUSSION

8.1 An Example

An application of some of the methods described in the thesis is obtained by analyzing data taken from a study of familial aggregation of blood pressure reported by Tishler et al. (1977), as previously analyzed by Donner and Koval (1980). One relationship of interest in this study is that between the daughter's diastolic blood pressure and her brother's. The blood pressure for a given individual is standardized for age by computing a z-score; This quantity is defined as the difference between the individual's blood pressure score and the mean score in his/her 10-year age group, divided by the value of the corresponding standard deviation. There are 88 families with at least one daughter and at least one son. The distribution of daughters and sons is as follows:

Number of daughters	Number of sons	Number of families	Aggregate degrees of freedom*
1	1	48	48.00
1	2	11	22.00
1	3	2	6.00
2	1	15	22.90
2	2	4	12.21
2	3	4	18.32
3	1	2	3.70
3	2	2	7.41
		N = 88	D = 140.54

* see Chapter 7.3 for definition

The estimated correlations and corresponding variances are given as follows:

Type of correlation	Correlation value	Asymptotic variance
sister-sister, $\hat{\rho}^{**}_{a,W}$	0.3101	0.0265
sister-sister, $\hat{\rho}_{a,AN}$	0.3521	0.0256
sister-sister, $\hat{\rho}_{a,ML}$	0.3545	0.0231
brother-brother, $\hat{\rho}^{**}_{b,W}$	0.0000 (-0.0275)	0.0297
brother-brother, $\hat{\rho}_{b,AN}$	0.0000 (-0.0433)	0.0523
brother-brother, $\hat{\rho}_{b,ML}$	0.0651	0.0299
sister-brother, IND	0.1659	0.0068
sister-brother, FAM	0.2619	0.0075
sister-brother, CL-A	0.1979	0.0068
sister-brother, CL-B	0.2141	0.0074
sister-brother, ML	0.1796	0.0069

** using proposed weighting scheme

It is observed that the individual-weighted (IND) and the maximum likelihood (ML) estimators of the sister-brother correlation yield similar estimates and asymptotic variances.

Approximate 95% confidence intervals about the true sister-brother correlation, ρ_{ab}, for each of the three methods discussed in Chapter 7 are given in the following chart. The weighted pairwise estimator using the proposed weighting scheme is used to compute the intraclass correlations required in the interval estimates.

Method	Lower Limit	Upper Limit	Width
IND	0.0036	0.3283	0.3247
FAM	0.0918	0.4320	0.3402
CL-A	0.0361	0.3598	0.3237
CL-B	0.0452	0.3830	0.3378
ZIND	0.0003	0.3226	0.3223
ZFAM	0.1007	0.4097	0.3090
ZCL-A	0.0334	0.3520	0.3185
ZCL-B	0.0503	0.3667	0.3164
ML	0.0169	0.3424	0.3255

These results show that methods ZIND, ZCL-A, ZCL-B and ML provide interval estimates for the sister-brother correlation that are very similar and slightly shorter than those provided by the other methods.

8.2 Summary and Recommendations

The main objective of this dissertation has been to present a simple noniterative procedure for estimating interclass correlations in the context of family data, where there may be more than one individual in each class.

The procedure involves a generalization of the Pearson product-moment correlation coefficient, where one permits different weights for the pairs of scores. Unlike the maximum likelihood approach, this estimator is not derived under the assumption of a particular parametric form nor does it require an iterative solution. However, for the purpose of this thesis, it was assumed that the attributes of interest were normally distributed. On the basis of this assumption, the asymptotic distribution of the generalized product-moment estimator and of the maximum likelihood estimator were derived. Subsequently, a Monte Carlo study was carried out to examine the properties of these estimators.

For sibship size distributions typical of North American populations, the simulation results indicate that the individual-weighted (IND) estimator is preferred for the estimation of interclass correlations when the average value of the intraclass correlation between the two classes is small to moderate (≤ 0.3). It is further noted for this case that the relative efficiency of this estimator exceeds 100%

in small samples. In terms of interval estimation, the method based upon a modification of Fisher's Z-transformation (ZIND) is recommended since it yields interval widths very similar to those for the method of maximum likelihood, while being far simpler to compute. However, neither method is valid for sample sizes smaller than 50 families because of poor coverage levels.

If the value of the average intraclass correlation is known from previous studies to be fairly large (>0.5), it is recommended that the family-weighted (FAM) estimator be used for point estimation and the corresponding modified Fisher's Z-transformation (ZFAM) for confidence interval construction. Nevertheless, the family-weighted estimator will have fairly limited utility in family studies, where large values of intraclass correlation are uncommon. For values between 0.3 and 0.5, the IND and FAM estimators appear equally effective.

The class-weighted estimators (CL-A, CL-B) were found to have fairly high relative efficiences and to produce confidence intervals comparable in quality to those yielded by the other two estimators. However, the small sample properties are slightly inferior to those of the individual-weighted estimator, and it is on this basis that the latter is recommended for family studies.

Although interval estimation is often an important aspect in the analysis of family data, significance testing is also of interest. Tests of hypotheses of the form H_0: $\rho_{ab} = \rho_0$ where $\rho_0 \geq 0$ may be easily derived from the approximate pivotal quantities, since they follow a standard normal distribution. Because the testing of hypotheses is closely related to the problem of interval estimation, it is felt that whatever recommendations are made pertaining to the methods of confidence interval construction would also be appropriate for the corresponding testing procedures.

It may be argued that the results for the Monte Carlo study may be influenced by the magnitude of the mean aggregate class size. To investigate this, the simulation study was repeated with parameter values m=2.82 and P=2.13 for the truncated negative binomial distribution. These values correspond to the distribution of sibship sizes found in Venezuela, as quoted by Brass (1958). This alternative distribution was chosen because the mean sibship size (5.69) is larger than for the U.S. (3.87). Again, aggregate class sizes equal to one were excluded in the study. Selected small sample (N=50) results are shown in Tables 8.1-8.4. All the findings reported for the U.S distribution are confirmed in the samples from the Venezuela distribution.

Alternative estimators of interclass correlation could also be derived as generalizations of the ensemble estimator and weighted sums of squares estimator of a parent-

2 OF/DE 2

1.0 1.1 1.25 1.4 1.6 1.8 2.0 2.2 2.5 2.8 3.2 3.6 4.0 4.5 5.0 5.6

offspring correlation. However, it would be expected that these generalizations would have properties very similar to the family-weighted estimator (FAM).

Among the consistent estimators of intraclass correlation, the weighted pairwise estimator using the proposed weighting scheme is recommended as a replacement for the analysis of variance estimator in family studies. As a generalization to the Pearson product-moment correlation, the weighted pairwise estimator together with the individual-weighted estimator of interclass correlation yield a unified noniterative approach to point and interval estimation.

Although this dissertation is directed towards the analysis of familial data, the methods discussed are applicable to more general situations, including the assessment of correlations between any two variables where each variable is replicated a different number of times for each sample unit. For example, these variables could be replicate cholesterol and blood pressure readings ascertained on the same individuals at one point in time.

The properties of the estimators were evaluated under the assumption that the attributes measured are normally distributed. It would be useful to investigate the appropriateness of the various methods in the non-normal case, more specifically when the attributes have a discrete distribution; for example, a 5-point ordinal scale. Likewise, it may be of interest to consider distribution-free measures of interclass correlation. One such estimator was proposed by Shirahata (1982) for the parent-offspring case, as an extension to Kendall's measure of dependence. Other estimators could also be developed, specifically if one regards a Spearman Rank correlation coefficient to be a nonparametric analogue of the generalized product-moment estimator discussed here.

As a final summary, the algebraic expressions for the noniterative estimators of interclass and intraclass correlation are once again presented, along with their asymptotic variances and recommended weighting scheme for family studies on the subsequent pages.

Generalized Product-moment Estimator of Interclass Correlation and its Asymptotic Variance

$$\hat{\rho}_{ab,GP} = \frac{\sum_{i=1}^{N} w_i \sum_{j=1}^{a_i}(x_{ij}-\bar{x}_{..}) \sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{..})}{\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i}(x_{ij}-\bar{x}_{..})^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{N} w_i a_i \sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{..})^2\right]^{\frac{1}{2}}}$$

$$\begin{aligned}
AV(\hat{\rho}_{ab,GP}) = \; & (1-\rho_{ab}^2)^2 \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 \\
& + (1-\rho_a)(1-\rho_b) \sum_{i=1}^{N} w_i^2 a_i(a_i-1) b_i(b_i-1) \\
& + (1-\rho_a)\left\{\tfrac{3}{2}\rho_{ab}^2 - \tfrac{1}{2}\rho_{ab}^2\rho_a - 1\right\} \sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i-1) \\
& + (1-\rho_b)\left\{\tfrac{3}{2}\rho_{ab}^2 - \tfrac{1}{2}\rho_{ab}^2\rho_b - 1\right\} \sum_{i=1}^{N} w_i^2 a_i^2 b_i (b_i-1)
\end{aligned}$$

Recommended weights for family studies : $w_i = 1 / \sum_{i=1}^{N} a_i b_i$

Weighted Pairwise Estimator of Intraclass Correlation and its Asymptotic Variance (Class A)

$$\hat{\rho}_{a,W} = \frac{\sum_{i=1}^{N} W_i \sum_{\substack{j=1 \\ j\neq\ell}}^{a_i} \sum_{\ell=1}^{a_i} (x_{ij}-\tilde{\bar{x}}_{..})(x_{i\ell}-\tilde{\bar{x}}_{..})}{\sum_{i=1}^{N} W_i (a_i-1) \sum_{j=1}^{a_i} (x_{ij}-\tilde{\bar{x}}_{..})^2}$$

$$AV(\hat{\rho}_{a,W}) = 2(1-\rho_a)^2 \sum_{i=1}^{N} W_i^2 a_i (a_i-1)\left\{1 + (a_i-1)\rho_a\right\}^2$$

Recommended weights for family studies : $W_i = 1 \,/\, a_i \sum_{i=1}^{N} (a_i-1)$

TABLE 8.1

Small sample (N=50) relative efficiencies for estimators of interclass correlation when $\rho_a = \rho_b$ using the Venezuela sibship size distribution

ρ_{ab}	ρ_a	ρ_b	IND	FAM	CL-A	CL-B
0.0	0.1	0.1	124.6	42.5	78.2	83.7
0.0	0.3	0.3	111.0	65.3	99.4	98.9
0.0	0.5	0.5	85.1	83.4	100.1	98.2
0.0	0.7	0.7	68.1	93.7	93.5	91.7
0.1	0.1	0.1	116.2	44.2	87.2	86.7
0.1	0.3	0.3	98.9	65.3	98.5	96.8
0.1	0.5	0.5	78.3	84.7	98.7	95.8
0.1	0.7	0.7	66.8	94.3	92.4	90.7
0.3	0.3	0.3	68.1	76.1	93.7	95.8
0.3	0.5	0.5	63.2	87.7	90.7	89.7
0.3	0.7	0.7	55.1	95.1	84.4	84.3
0.5	0.5	0.5	48.1	85.1	79.9	81.4
0.5	0.7	0.7	46.9	95.6	78.5	78.2
0.7	0.7	0.7	33.3	91.2	66.7	68.7
0.7	0.9	0.9	38.3	99.8	68.8	69.4

TABLE 8.2

Small sample (N=50) biases for estimators of interclass correlation when $\rho_a = \rho_b$ using the Venezuela sibship size distribution ($\times 10^4$)

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B
0.0	0.1	0.1	- 3.9	- 12.3	7.0	1.6	- 2.5
0.0	0.3	0.3	- 3.8	8.6	- 3.4	2.0	- 4.8
0.0	0.5	0.5	5.8	5.1	14.9	5.8	6.6
0.0	0.7	0.7	50.1	48.9	50 6	46.1	51.7
0.1	0.1	0.1	- 12.6	-138.2	208.5	43.2	27.6
0.1	0.3	0.3	- 15.7	-196.5	95.7	- 45.2	- 64.8
0.1	0.5	0.5	- 28.2	-214.9	23.3	- 88.4	-108.0
0.1	0.7	0.7	27.2	-142.4	42.4	- 45.0	- 45.0
0.3	0.3	0.3	- 90.9	-517.2	144.2	-175.7	-177.9
0.3	0.5	0.5	- 73.7	-523.4	43.2	-225.9	-239.6
0.3	0.7	0.7	- 44.9	-500.2	- 0.2	-235.0	-236.2
0.5	0.5	0.5	- 93.9	-658.2	48.2	-267.7	-271.5
0.5	0.7	0.7	- 89.3	-659.9	- 23.3	-315.4	-319.3
0.7	0.7	0.7	-130.5	-676.9	- 53.4	-315.9	-311.3
0.7	0.9	0.9	-104.7	-622.7	- 78.8	-325.6	-326.9

TABLE 8.3

95% two-sided confidence intervals for the interclass correlation using the Venezuela sibship size distribution
Coverage percentages at N=50 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: $(\rho_a = \rho_b)$

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.1	0.1	93.9	97.2	91.8	91.4	93.3	95.9	73.9	88.3	89.7
0.0	0.3	0.3	94.3	97.9	92.4	95.2	94.7	95.0	86.1	94.4	94.2
0.0	0.5	0.5	93.3	97.5	92.5	94.6	95.1	93.0	92.0	94.1	94.9
0.0	0.7	0.7	94.0	97.0	92.9	95.7	95.0	88.8	93.9	94.5	94.2
0.1	0.1	0.1	94.4	95.6	94.2	94.2	95.0	94.8	76.6	91.4	92.5
0.1	0.3	0.3	94.2	97.5	92.2	95.1	95.1	94.7	86.9	94.8	93.8
0.1	0.5	0.5	93.8	96.9	92.7	95.0	94.2	91.6	91.9	94.8	93.2
0.1	0.7	0.7	93.7	97.0	93.1	95.4	94.9	89.6	94.0	94.0	93.9
0.3	0.3	0.3	92.8	91.2	95.0	92.9	94.6	86.0	90.5	91.7	92.7
0.3	0.5	0.5	94.0	94.9	93.8	94.2	94.6	88.0	93.4	92.9	93.4
0.3	0.7	0.7	94.1	96.6	93.4	95.5	95.0	87.1	95.0	93.1	94.0
0.5	0.5	0.5	92.2	92.5	93.0	91.8	93.4	83.3	92.8	90.4	92.1
0.5	0.7	0.7	94.0	96.6	93.5	94.5	95.0	85.7	94.4	92.8	92.5
0.7	0.7	0.7	93.9	95.6	94.2	94.3	94.9	83.0	96.2	92.1	92.2
0.7	0.9	0.9	93.6	98.7	92.8	96.0	95.7	84.0	95.1	92.1	91.9

TABLE 8.4

95% two-sided confidence intervals for the interclass correlation using the Venezuela sibship size distribution
Mean interval widths at N=50 with $\hat{\rho}_{a,W}$ and $\hat{\rho}_{b,W}$: $(\rho_a = \rho_b)$

ρ_{ab}	ρ_a	ρ_b	ML	IND	FAM	CL-A	CL-B	ZIND	ZFAM	ZCL-A	ZCL-B
0.0	0.1	0.1	.21	.22	.31	.23	.23	.20	.20	.20	.20
0.0	0.3	0.3	.30	.35	.35	.31	.31	.29	.29	.29	.29
0.0	0.5	0.5	.38	.49	.40	.40	.40	.37	.37	.37	.37
0.0	0.7	0.7	.45	.63	.45	.49	.49	.44	.44	.44	.44
0.1	0.1	0.1	.21	.22	.30	.23	.23	.20	.20	.20	.20
0.1	0.3	0.3	.29	.35	.34	.31	.31	.29	.29	.29	.29
0.1	0.5	0.5	.37	.49	.39	.39	.39	.37	.37	.37	.37
0.1	0.7	0.7	.44	.63	.45	.49	.49	.44	.44	.44	.44
0.3	0.3	0.3	.27	.32	.31	.28	.28	.27	.27	.27	.27
0.3	0.5	0.5	.33	.45	.35	.36	.36	.34	.34	.34	.34
0.3	0.7	0.7	.40	.58	.40	.45	.45	.41	.40	.41	.41
0.5	0.5	0.5	.27	.37	.28	.29	.29	.30	.28	.29	.29
0.5	0.7	0.7	.32	.49	.32	.36	.36	.36	.34	.35	.35
0.7	0.7	0.7	.24	.33	.21	.24	.24	.27	.23	.25	.25
0.7	0.9	0.9	.26	.45	.25	.31	.31	.31	.28	.29	.29

APPENDIX 1 - COMPUTATION OF THE ASYMPTOTIC VARIANCE FOR THE MAXIMUM LIKELIHOOD ESTIMATOR

In order to simplify the notation in the following appendix, we partition the vector $\underset{\sim}{z}_i$ such that

$\underset{\sim}{z}_i = (\underset{\sim}{x}_i | \underset{\sim}{y}_i)'$; where $\underset{\sim}{x}_i = (x_{i1}, x_{12}, \ldots, x_{ia_i})$

and $\underset{\sim}{y}_i = (y_{i1}, y_{i2}, \ldots, y_{ib_i})$. In a similar way, $\underset{\sim}{\mu}_i$ is

partitioned as $\underset{\sim}{\mu}_i = (\underset{\sim}{\mu}_i^x | \underset{\sim}{\mu}_i^y)' = (\mu_a, \mu_a, \ldots, \mu_a | \mu_b, \mu_b, \ldots, \mu_b)'$.

We define a vector $\underset{\sim}{\mathfrak{z}}_i$ as the difference between $\underset{\sim}{z}_i$ and $\underset{\sim}{\mu}_i$;

i.e $\underset{\sim}{\mathfrak{z}}_i = (\underset{\sim}{z}_{i1} | \underset{\sim}{z}_{i2})' = (\underset{\sim}{x}_i - \underset{\sim}{\mu}_i^x | \underset{\sim}{y}_i - \underset{\sim}{\mu}_i^y)' = (\underset{\sim}{x}_i - \mu_a \underset{\sim}{\ell}_{a_i} | \underset{\sim}{y}_i - \mu_b \underset{\sim}{\ell}_{b_i})'$.

For the model of Chapter 2, the inverse of the covariance matrix, $\underset{\sim}{\Sigma}_i$, may be written (Graybill, 1983, pgs. 184,190) as

$$\underset{\sim}{\Sigma}_i^{-1} = \begin{bmatrix} \underset{\sim}{\Sigma}_{i11}^* & \underset{\sim}{\Sigma}_{i12}^* \\ \underset{\sim}{\Sigma}_{i12}^{*\prime} & \underset{\sim}{\Sigma}_{i22}^* \end{bmatrix}$$

where
$$\underset{\sim}{\Sigma}_{i11}^* = \frac{1}{\sigma_a^2(1-\rho_a)} \left[\underset{\sim}{I}_{a_i} - \frac{W_i \rho_a - \rho_{ab}^2 b_i}{T_i} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right]$$

$$\underset{\sim}{\Sigma}_{i22}^* = \frac{1}{\sigma_b^2(1-\rho_b)} \left[\underset{\sim}{I}_{b_i} - \frac{V_i \rho_b - \rho_{ab}^2 a_i}{T_i} \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}' \right]$$

$$\underset{\sim}{\Sigma}_{i12}^* = - \frac{\rho_{ab}}{\sigma_a \sigma_b T_i} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{b_i}'$$

and $V_i = \left[1 + (a_i - 1)\rho_a\right]$

$W_i = \left[1 + (b_i - 1)\rho_b\right]$

$T_i = \left[V_i W_i - \rho_{ab}^2 a_i b_i\right]$

The determinant of $\underset{\sim}{\Sigma}_i$ can be expressed (Graybill, 1983, pg. 184) as $|\underset{\sim}{\Sigma}_i| = \sigma_a^{2a_i} \sigma_b^{2b_i} (1-\rho_a)^{a_i-1} (1-\rho_b)^{b_i-1} T_i$.

Using the above results, the log likelihood function from Chapter 4.1 can be written as

$$\log L = -\tfrac{1}{2}\log(2\pi)\sum_{i=1}^{N}(a_i+b_i) - \tfrac{1}{2}\log\sigma_a^2\sum_{i=1}^{N}a_i - \tfrac{1}{2}\log\sigma_b^2\sum_{i=1}^{N}b_i$$

$$-\tfrac{1}{2}\log(1-\rho_a)\sum_{i=1}^{N}(a_i-1) - \tfrac{1}{2}\log(1-\rho_b)\sum_{i=1}^{N}(b_i-1)$$

$$-\frac{1}{2\sigma_a^2(1-\rho_a)}\sum_{i=1}^{N}\underset{\sim}{z}_{i1}'\left[\underset{\sim}{I}_{a_i} - \frac{W_i\rho_a - \rho_{ab}^2 b_i}{T_i}\underset{\sim}{\ell}_{a_i}\underset{\sim}{\ell}_{a_i}'\right]\underset{\sim}{z}_{i1}$$

$$-\frac{1}{2\sigma_b^2(1-\rho_b)}\sum_{i=1}^{N}\underset{\sim}{z}_{i2}'\left[\underset{\sim}{I}_{b_i} - \frac{V_i\rho_b - \rho_{ab}^2 a_i}{T_i}\underset{\sim}{\ell}_{b_i}\underset{\sim}{\ell}_{b_i}'\right]\underset{\sim}{z}_{i2}$$

$$+\frac{\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\frac{\underset{\sim}{z}_{i1}'\underset{\sim}{\ell}_{a_i}\underset{\sim}{z}_{i2}'\underset{\sim}{\ell}_{b_i}}{T_i} - \tfrac{1}{2}\sum_{i=1}^{N}\log T_i$$

The partial derivatives of logL are computed as follows:

$$\frac{\partial \log L}{\partial \mu_a} = \frac{1}{\sigma_a^2}\sum_{i=1}^{N}\underset{\sim}{z}_{i1}'\underset{\sim}{\ell}_{a_i}\frac{W_i}{T_i} - \frac{\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\frac{a_i\underset{\sim}{z}_{i2}'\underset{\sim}{\ell}_{b_i}}{T_i}$$

$$\frac{\partial \log L}{\partial \mu_b} = \frac{1}{\sigma_b^2}\sum_{i=1}^{N}\underset{\sim}{z}_{i2}'\underset{\sim}{\ell}_{b_i}\frac{V_i}{T_i} - \frac{\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\frac{b_i\underset{\sim}{z}_{i1}'\underset{\sim}{\ell}_{a_i}}{T_i}$$

$$\frac{\partial \log L}{\partial \sigma_a^2} = -\frac{1}{2\sigma_a^2}\sum_{i=1}^{N} a_i - \frac{\rho_{ab}}{2\sigma_a^3\sigma_b}\sum_{i=1}^{N}\frac{\mathbf{z}_{i1}'\boldsymbol{\ell}_{a_i}\mathbf{z}_{i2}'\boldsymbol{\ell}_{b_i}}{T_i}$$

$$+ \frac{1}{2\sigma_a^4(1-\rho_a)}\sum_{i=1}^{N}\mathbf{z}_{i1}'\left[\mathbf{I}_{a_i} - \frac{W_i\rho_a - \rho_{ab}^2 b_i}{T_i}\boldsymbol{\ell}_{a_i}\boldsymbol{\ell}_{a_i}'\right]\mathbf{z}_{i1}$$

$$\frac{\partial \log L}{\partial \sigma_b^2} = -\frac{1}{2\sigma_b^2}\sum_{i=1}^{N} b_i - \frac{\rho_{ab}}{2\sigma_a\sigma_b^3}\sum_{i=1}^{N}\frac{\mathbf{z}_{i1}'\boldsymbol{\ell}_{a_i}\mathbf{z}_{i2}'\boldsymbol{\ell}_{b_i}}{T_i}$$

$$+ \frac{1}{2\sigma_b^4(1-\rho_b)}\sum_{i=1}^{N}\mathbf{z}_{i2}'\left[\mathbf{I}_{b_i} - \frac{V_i\rho_a - \rho_{ab}^2 a_i}{T_i}\boldsymbol{\ell}_{b_i}\boldsymbol{\ell}_{b_i}'\right]\mathbf{z}_{i2}$$

$$\frac{\partial \log L}{\partial \rho_a} = \frac{1}{2(1-\rho_a)}\sum_{i=1}^{N}(a_i-1) - \frac{1}{2}\sum_{i=1}^{N}\frac{W_i}{T_i}(a_i-1)$$

$$- \frac{1}{2\sigma_a^2(1-\rho_a)^2}\sum_{i=1}^{N}\mathbf{z}_{i1}'\left[\mathbf{I}_{a_i} - \left\{\frac{W_i\rho_a - \rho_{ab}^2 b_i}{T_i}\right.\right.$$

$$\left.\left. + \frac{W_i(W_i - \rho_{ab}^2 b_i)(1-\rho_a)}{T_i^2}\right\}\boldsymbol{\ell}_{a_i}\boldsymbol{\ell}_{a_i}'\right]\mathbf{z}_{i1}$$

$$- \frac{1}{2\sigma_b^2(1-\rho_b)}\sum_{i=1}^{N}\mathbf{z}_{i2}'\left[\mathbf{0} - \left\{\frac{\rho_b(a_i-1)}{T_i}\right.\right.$$

$$\left.\left. - \frac{W_i(a_i-1)(V_i\rho_b - \rho_{ab}^2 a_i)}{T_i^2}\right\}\boldsymbol{\ell}_{b_i}\boldsymbol{\ell}_{b_i}'\right]\mathbf{z}_{i2}$$

$$- \frac{\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\frac{W_i(a_i-1)}{T_i^2}\,\mathbf{z}_{i1}'\boldsymbol{\ell}_{a_i}\mathbf{z}_{i2}'\boldsymbol{\ell}_{b_i}$$

$$\frac{\partial \log L}{\partial \rho_b} = \frac{1}{2(1-\rho_b)} \sum_{i=1}^{N} (b_i - 1) - \frac{1}{2} \sum_{i=1}^{N} \frac{V_i}{T_i}(b_i - 1)$$

$$- \frac{1}{2\sigma_a^2(1-\rho_a)} \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \left[\underset{\sim}{\emptyset} - \left\{ \frac{\rho_a(b_i-1)}{T_i} \right.\right.$$

$$\left.\left. - \frac{V_i(b_i-1)(W_i\rho_a - \rho_{ab}^2 b_i)}{T_i^2} \right\} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right] \underset{\sim}{z}_{i1}$$

$$- \frac{1}{2\sigma_b^2(1-\rho_b)^2} \sum_{i=1}^{N} \underset{\sim}{z}_{i2}' \left[\underset{\sim}{I}_{b_i} - \left\{ \frac{V_i\rho_b - \rho_{ab}^2 a_i}{T_i} \right.\right.$$

$$\left.\left. + \frac{V_i(V_i - \rho_{ab}^2 a_i)(1-\rho_b)}{T_i^2} \right\} \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}' \right] \underset{\sim}{z}_{i2}$$

$$- \frac{\rho_{ab}}{\sigma_a\sigma_b} \sum_{i=1}^{N} \frac{V_i(b_i-1)}{T_i^2} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial \log L}{\partial \rho_{ab}} = \rho_{ab} \sum_{i=1}^{N} \frac{a_i b_i}{T_i} + \frac{\rho_{ab}}{\sigma_a^2} \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \left[\underset{\sim}{\emptyset} - \frac{b_i W_i}{T_i^2} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right] \underset{\sim}{z}_{i1}$$

$$+ \frac{\rho_{ab}}{\sigma_b^2} \sum_{i=1}^{N} \underset{\sim}{z}_{i2}' \left[\underset{\sim}{\emptyset} - \frac{a_i V_i}{T_i^2} \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}' \right] \underset{\sim}{z}_{i2}$$

$$+ \frac{1}{\sigma_a\sigma_b} \sum_{i=1}^{N} \left[\frac{V_i W_i + \rho_{ab}^2 a_i b_i}{T_i^2} \right] \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}$$

The second partial derivatives of logL are computed as

$$\frac{\partial^2 \log L}{\partial \mu_a^2} = -\frac{1}{\sigma_a^2} \sum_{i=1}^{N} \frac{a_i W_i}{T_i}$$

$$\frac{\partial^2 \log L}{\partial \mu_b \partial \mu_a} = \frac{\rho_{ab}}{\sigma_a\sigma_b} \sum_{i=1}^{N} \frac{a_i b_i}{T_i}$$

$$\frac{\partial^2 \log L}{\partial \sigma_a^2 \partial \mu_a} = \frac{\rho_{ab}}{2\sigma_a^3\sigma_b} \sum_{i=1}^{N} \frac{a_i}{T_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i} - \frac{1}{\sigma_a^4} \sum_{i=1}^{N} \frac{W_i}{T_i} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i}$$

$$\frac{\partial^2 \log L}{\partial\sigma_b^2\partial\mu_a} = \frac{\rho_{ab}}{2\sigma_a\sigma_b^3}\sum_{i=1}^{N}\frac{a_i}{T_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}b_i$$

$$\frac{\partial^2 \log L}{\partial\rho_a\partial\mu_a} = \frac{1}{\sigma_a^2(1-\rho_a)^2}\sum_{i=1}^{N}\left[1 - a_i\left\{\frac{W_i\rho_a-\rho_{ab}^2 b_i}{T_i}\right.\right.$$

$$\left.\left.+\frac{W_i(W_i-\rho_{ab}^2 b_i)(1-\rho_a)}{T_i^2}\right\}\right]\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}a_i$$

$$+\frac{\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\frac{W_i a_i(a_i-1)}{T_i^2}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}b_i$$

$$\frac{\partial^2 \log L}{\partial\rho_b\partial\mu_a} = \frac{1}{\sigma_a^2(1-\rho_a)}\sum_{i=1}^{N}\left[\varnothing - a_i\left\{\frac{\rho_a(b_i-1)}{T_i}\right.\right.$$

$$\left.\left.-\frac{V_i(b_i-1)(W_i\rho_a-\rho_{ab}^2 b_i)}{T_i^2}\right\}\right]\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}a_i$$

$$+\frac{\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\frac{V_i a_i(b_i-1)}{T_i^2}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}b_i$$

$$\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\mu_a} = -\frac{2\rho_{ab}}{\sigma_a^2}\sum_{i=1}^{N}\left\{\varnothing - \frac{a_i b_i W_i}{T_i^2}\right\}\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}a_i$$

$$-\frac{1}{\sigma_a\sigma_b}\sum_{i=1}^{N}a_i\left\{\frac{V_i W_i+\rho_{ab}^2 a_i b_i}{T_i^2}\right\}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}b_i$$

$$\frac{\partial^2 \log L}{\partial\mu_b^2} = -\frac{1}{\sigma_b^2}\sum_{i=1}^{N}\frac{b_i V_i}{T_i}$$

$$\frac{\partial^2 \log L}{\partial\sigma_a^2\partial\mu_b} = \frac{\rho_{ab}}{2\sigma_a^3\sigma_b}\sum_{i=1}^{N}\frac{b_i}{T_i}\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}a_i$$

$$\frac{\partial^2 \log L}{\partial\sigma_b^2\partial\mu_b} = \frac{\rho_{ab}}{2\sigma_a\sigma_b^3}\sum_{i=1}^{N}\frac{b_i}{T_i}\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}a_i - \frac{1}{\sigma_b^4}\sum_{i=1}^{N}\frac{V_i}{T_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}b_i$$

$$\frac{\partial^2 \log L}{\partial \rho_a \partial \mu_b} = \frac{1}{\sigma_b^2(1-\rho_b)} \sum_{i=1}^{N} \left[\emptyset - b_i \left\{ \frac{\rho_b(a_i-1)}{T_i} - \frac{W_i(a_i-1)(V_i\rho_b - \rho_{ab}^2 a_i)}{T_i^2} \right\} \right] \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i} + \frac{\rho_{ab}}{\sigma_a \sigma_b} \sum_{i=1}^{N} \frac{W_i b_i (a_i-1)}{T_i^2} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i}$$

$$\frac{\partial^2 \log L}{\partial \rho_b \partial \mu_b} = \frac{1}{\sigma_b^2(1-\rho_b)^2} \sum_{i=1}^{N} \left[1 - b_i \left\{ \frac{V_i\rho_b - \rho_{ab}^2 a_i}{T_i} + \frac{V_i(V_i - \rho_{ab}^2 a_i)(1-\rho_b)}{T_i^2} \right\} \right] \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i} + \frac{\rho_{ab}}{\sigma_a \sigma_b} \sum_{i=1}^{N} \frac{V_i b_i (b_i-1)}{T_i^2} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i}$$

$$\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \mu_b} = - \frac{2\rho_{ab}}{\sigma_b^2} \sum_{i=1}^{N} \left\{ \emptyset - \frac{a_i b_i V_i}{T_i^2} \right\} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i} - \frac{1}{\sigma_a \sigma_b} \sum_{i=1}^{N} b_i \left\{ \frac{V_i W_i + \rho_{ab}^2 a_i b_i}{T_i^2} \right\} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i}$$

$$\frac{\partial^2 \log L}{\partial \sigma_a^2 \partial \sigma_a^2} = \frac{1}{2\sigma_a^4} \sum_{i=1}^{N} a_i + \frac{3\rho_{ab}}{4\sigma_a^5 \sigma_b} \sum_{i=1}^{N} \frac{1}{T_i} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i} - \frac{1}{\sigma_a^6(1-\rho_a)} \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \left\{ \underset{\sim}{I}_{a_i} - \frac{W_i\rho_a - \rho_{ab}^2 b_i}{T_i} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right\} \underset{\sim}{z}_{i1}$$

$$\frac{\partial^2 \log L}{\partial \sigma_b^2 \partial \sigma_a^2} = \frac{1}{4\sigma_a^3 \sigma_b^3} \sum_{i=1}^{N} \frac{1}{T_i} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial \rho_a \partial \sigma_a^2} = \frac{1}{2\sigma_a^4(1-\rho_a)^2} \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \left[\underset{\sim}{I}_{a_i} - \left\{ \frac{W_i\rho_a - \rho_{ab}^2 b_i}{T_i} \right.\right.$$

$$\left.\left. + \frac{W_i(W_i - \rho_{ab}^2 b_i)(1-\rho_a)}{T_i^2} \right\} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right] \underset{\sim}{z}_{i1}$$

$$+ \frac{\rho_{ab}}{2\sigma_a^3 \sigma_b} \sum_{i=1}^{N} \frac{W_i(a_i - 1)}{T_i^2} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial \rho_b \partial \sigma_a^2} = \frac{1}{2\sigma_a^4(1-\rho_a)} \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \left[\underset{\sim}{\emptyset} - \left\{ \frac{\rho_a(b_i - 1)}{T_i} \right.\right.$$

$$\left.\left. - \frac{V_i(b_i - 1)(W_i\rho_a - \rho_{ab}^2 b_i)}{T_i^2} \right\} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right] \underset{\sim}{z}_{i1}$$

$$+ \frac{\rho_{ab}}{2\sigma_a^3 \sigma_b} \sum_{i=1}^{N} \frac{V_i(b_i - 1)}{T_i^2} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b}$$

$$\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \sigma_a^2} = - \frac{\rho_{ab}}{\sigma_a^4} \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \left\{ \underset{\sim}{\emptyset} - \frac{b_i W_i}{T_i^2} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right\} \underset{\sim}{z}_{i1}$$

$$- \frac{1}{2\sigma_a^3 \sigma_b} \sum_{i=1}^{N} \left\{ \frac{V_i W_i + \rho_{ab}^2 a_i b_i}{T_i^2} \right\} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial \sigma_b^2 \partial \sigma_b^2} = \frac{1}{2\sigma_b^4} \sum_{i=1}^{N} b_i + \frac{3\rho_{ab}}{4\sigma_a \sigma_b^5} \sum_{i=1}^{N} \frac{1}{T_i} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}$$

$$- \frac{1}{\sigma_b^6(1-\rho_b)} \sum_{i=1}^{N} \underset{\sim}{z}_{i2}' \left\{ \underset{\sim}{I}_{b_i} - \frac{V_i\rho_b - \rho_{ab}^2 a_i}{T_i} \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}' \right\} \underset{\sim}{z}_{i2}$$

$$\frac{\partial^2 \log L}{\partial \rho_a \partial \sigma_b^2} = \frac{1}{2\sigma_b^4(1-\rho_b)} \sum_{i=1}^{N} \underset{\sim}{z}_{i2}' \left[\underset{\sim}{\emptyset} - \left\{ \frac{\rho_b(a_i - 1)}{T_i} \right.\right.$$

$$\left.\left. - \frac{W_i(a_i - 1)(V_i\rho_b - \rho_{ab}^2 a_i)}{T_i^2} \right\} \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}' \right] \underset{\sim}{z}_{i2}$$

$$+ \frac{\rho_{ab}}{2\sigma_a\sigma_b^3} \sum_{i=1}^{N} \frac{W_i(a_i-1)}{T_i^2} \underset{\sim}{z}'_{i1}\underset{\sim}{\ell}_{a_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial\rho_b\partial\sigma_b^2} = \frac{1}{2\sigma_b^4(1-\rho_b)^2} \sum_{i=1}^{N} \underset{\sim}{z}'_{i2}\Big[\underset{\sim}{I}_{b_i} - \Big\{\frac{V_i\rho_b-\rho_{ab}^2 a_i}{T_i}$$

$$+ \frac{V_i(V_i-\rho_{ab}^2 a_i)(1-\rho_b)}{T_i^2}\Big\}\underset{\sim}{\ell}_{b_i}\underset{\sim}{\ell}'_{b_i}\Big]\underset{\sim}{z}_{i2}$$

$$+ \frac{\rho_{ab}}{2\sigma_a\sigma_b^3} \sum_{i=1}^{N} \frac{V_i(b_i-1)}{T_i^2} \underset{\sim}{z}'_{i1}\underset{\sim}{\ell}_{a_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\sigma_b^2} = -\frac{\rho_{ab}}{\sigma_b^4} \sum_{i=1}^{N} \underset{\sim}{z}'_{i2}\Big\{\varnothing - \frac{a_i V_i}{T_i^2}\underset{\sim}{\ell}_{b_i}\underset{\sim}{\ell}'_{b_i}\Big\}\underset{\sim}{z}_{i2}$$

$$- \frac{1}{2\sigma_a\sigma_b^3} \sum_{i=1}^{N} \Big\{\frac{V_i W_i+\rho_{ab}^2 a_i b_i}{T_i^2}\Big\}\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}_{a_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial\rho_a\partial\rho_a} = \frac{1}{2(1-\rho_a)^2} \sum_{i=1}^{N} (a_i-1) + \frac{1}{2} \sum_{i=1}^{N} \frac{W_i^2(a_i-1)^2}{T_i^2}$$

$$- \frac{1}{\sigma_a^2(1-\rho_a)^3} \sum_{i=1}^{N} \underset{\sim}{z}'_{i1}\Big[\underset{\sim}{I}_{a_i} - \Big\{\frac{W_i\rho_a-\rho_{ab}^2 b_i}{T_i} + \frac{W_i(W_i-\rho_{ab}^2 b_i)(1-\rho_a)}{T_i^2}$$

$$- \frac{W_i^2(a_i-1)(W_i-\rho_{ab}^2 b_i)(1-\rho_a)^2}{T_i^3}\Big\}\underset{\sim}{\ell}_{a_i}\underset{\sim}{\ell}'_{a_i}\Big]\underset{\sim}{z}_{i1}$$

$$- \frac{1}{\sigma_b^2(1-\rho_b)} \sum_{i=1}^{N} \underset{\sim}{z}'_{i2}\Big[\varnothing - \Big\{- \frac{\rho_b W_i(a_i-1)^2}{T_i^2}$$

$$+ \frac{W_i^2(a_i-1)^2(V_i\rho_b-\rho_{ab}^2 a_i)}{T_i^3}\Big\}\underset{\sim}{\ell}_{b_i}\underset{\sim}{\ell}'_{b_i}\Big]\underset{\sim}{z}_{i2}$$

$$+ \frac{2\rho_{ab}}{\sigma_a\sigma_b} \sum_{i=1}^{N} \frac{W_i^2(a_i-1)^2}{T_i^3} \underset{\sim}{z}'_{i1}\underset{\sim}{\ell}_{a_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial\rho_b\partial\rho_a} = \frac{\rho_{ab}^2}{2}\sum_{i=1}^{N}\frac{a_i b_i(a_i-1)(b_i-1)}{T_i^2}$$

$$-\frac{1}{\sigma_a^2(1-\rho_a)}\sum_{i=1}^{N}\underset{\sim}{z}'_{i1}\left[\underset{\sim}{\emptyset} - \left\{\frac{W_i(b_i-1)}{T_i^2}\right.\right.$$

$$\left.\left. - \frac{V_i W_i(b_i-1)(W_i-\rho_{ab}^2 b_i)}{T_i^3}\right\}\underset{\sim}{\ell}_{a_i}\underset{\sim}{\ell}'_{a_i}\right]\underset{\sim}{z}_{i1}$$

$$-\frac{1}{\sigma_b^2(1-\rho_b)}\sum_{i=1}^{N}\underset{\sim}{z}'_{i2}\left[\underset{\sim}{\emptyset} - \left\{\frac{V_i(a_i-1)}{T_i^2}\right.\right.$$

$$\left.\left. - \frac{V_i W_i(a_i-1)(V_i-\rho_{ab}^2 a_i)}{T_i^3}\right\}\underset{\sim}{\ell}_{b_i}\underset{\sim}{\ell}'_{b_i}\right]\underset{\sim}{z}_{i2}$$

$$-\frac{\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\left\{\frac{(a_i-1)(b_i-1)}{T_i^2} - \frac{2V_i W_i(a_i-1)(b_i-1)}{T_i^3}\right\}\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}_{a_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\rho_a} = -\rho_{ab}\sum_{i=1}^{N}\frac{a_i b_i W_i(a_i-1)}{T_i^2}$$

$$+\frac{\rho_{ab}}{\sigma_a^2}\sum_{i=1}^{N}\underset{\sim}{z}'_{i1}\left[\underset{\sim}{\emptyset} - \left\{-\frac{2b_i(a_i-1)W_i^2}{T_i^3}\right\}\underset{\sim}{\ell}_{a_i}\underset{\sim}{\ell}'_{a_i}\right]\underset{\sim}{z}_{i1}$$

$$+\frac{\rho_{ab}}{\sigma_b^2}\sum_{i=1}^{N}\underset{\sim}{z}'_{i2}\left[\underset{\sim}{\emptyset} - \left\{\frac{a_i(a_i-1)}{T_i^2} - \frac{2a_i(a_i-1)V_i W_i}{T_i^3}\right\}\underset{\sim}{\ell}_{b_i}\underset{\sim}{\ell}'_{b_i}\right]\underset{\sim}{z}_{i2}$$

$$+\frac{1}{\sigma_a\sigma_b}\sum_{i=1}^{N}\left\{\frac{W_i(a_i-1)}{T_i^2} - \frac{2W_i(a_i-1)(V_i W_i+\rho_{ab}^2 a_i b_i)}{T_i^3}\right\}\underset{\sim}{z}'_{i1}\underset{\sim}{\ell}_{a_i}\underset{\sim}{z}'_{i2}\underset{\sim}{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial\rho_b\partial\rho_b} = \frac{1}{2(1-\rho_b)^2}\sum_{i=1}^{N}(b_i-1) + \frac{1}{2}\sum_{i=1}^{N}\frac{V_i^2(b_i-1)^2}{T_i^2}$$

$$- \frac{1}{\sigma_a^2(1-\rho_a)}\sum_{i=1}^{N} \tilde{z}_{i1}'\left[\tilde{\emptyset} - \left\{- \frac{\rho_a V_i(b_i-1)^2}{T_i^2}\right.\right.$$

$$\left.\left. + \frac{V_i^2(b_i-1)^2(W_i\rho_a-\rho_{ab}^2 b_i)}{T_i^3}\right\}\tilde{\ell}_{a_i}\tilde{\ell}_{a_i}'\right]\tilde{z}_{i1}$$

$$- \frac{1}{\sigma_b^2(1-\rho_b)^3}\sum_{i=1}^{N} \tilde{z}_{i2}'\left[\tilde{I}_{b_i} - \left\{\frac{V_i\rho_b-\rho_{ab}^2 a_i}{T_i} + \frac{V_i(V_i-\rho_{ab}^2 a_i)(1-\rho_b)}{T_i^2}\right.\right.$$

$$\left.\left. - \frac{V_i^2(b_i-1)(V_i-\rho_{ab}^2 a_i)(1-\rho_b)^2}{T_i^3}\right\}\tilde{\ell}_{b_i}\tilde{\ell}_{b_i}'\right]\tilde{z}_{i2}$$

$$+ \frac{2\rho_{ab}}{\sigma_a\sigma_b}\sum_{i=1}^{N}\frac{V_i^2(b_i-1)^2}{T_i^3}\tilde{z}_{i1}'\tilde{\ell}_{a_i}\tilde{z}_{i2}'\tilde{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\rho_b} = -\rho_{ab}\sum_{i=1}^{N}\frac{a_i b_i V_i(b_i-1)}{T_i^2}$$

$$+ \frac{\rho_{ab}}{\sigma_a^2}\sum_{i=1}^{N}\tilde{z}_{i1}'\left[\tilde{\emptyset} - \left\{\frac{b_i(b_i-1)}{T_i^2} - \frac{2b_i(b_i-1)V_iW_i}{T_i^3}\right\}\tilde{\ell}_{a_i}\tilde{\ell}_{a_i}'\right]\tilde{z}_{i1}$$

$$+ \frac{\rho_{ab}}{\sigma_b^2}\sum_{i=1}^{N}\tilde{z}_{i2}'\left[\tilde{\emptyset} - \left\{- \frac{2a_i(b_i-1)V_i^2}{T_i^3}\right\}\tilde{\ell}_{b_i}\tilde{\ell}_{b_i}'\right]\tilde{z}_{i2}$$

$$+ \frac{1}{\sigma_a\sigma_b}\sum_{i=1}^{N}\left\{\frac{V_i(b_i-1)}{T_i^2} - \frac{2V_i(b_i-1)(V_iW_i+\rho_{ab}^2 a_i b_i)}{T_i^3}\right\}\tilde{z}_{i1}'\tilde{\ell}_{a_i}\tilde{z}_{i2}'\tilde{\ell}_{b_i}$$

$$\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \rho_{ab}} = \sum_{i=1}^{N} \frac{a_i b_i (V_i W_i + \rho_{ab}^2 a_i b_i)}{T_i^2}$$

$$+ \frac{1}{\sigma_a^2} \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \left[\underset{\sim}{\emptyset} - \left\{ \frac{b_i W_i (V_i W_i + 3\rho_{ab}^2 a_i b_i)}{T_i^3} \right\} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right] \underset{\sim}{z}_{i1}$$

$$+ \frac{1}{\sigma_b^2} \sum_{i=1}^{N} \underset{\sim}{z}_{i2}' \left[\underset{\sim}{\emptyset} - \left\{ \frac{a_i V_i (V_i W_i + 3\rho_{ab}^2 a_i b_i)}{T_i^3} \right\} \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}' \right] \underset{\sim}{z}_{i2}$$

$$+ \frac{1}{\sigma_a \sigma_b} \sum_{i=1}^{N} \left\{ \frac{2\rho_{ab} a_i b_i (3V_i W_i + \rho_{ab}^2 a_i b_i)}{T_i^3} \right\} \underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}$$

The following equalities are useful in simplifing the calculation of the information matrix:

$$E(\underset{\sim}{z}_{i1}) = \underset{\sim}{\emptyset} \; ; \quad E(\underset{\sim}{z}_{i2}) = \underset{\sim}{\emptyset}$$

where $\underset{\sim}{\emptyset}$ is the zero vector.

Next (Graybill, 1983, pg. 303),

$$E\left\{ \sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \underset{\sim}{G}_i \underset{\sim}{z}_{i1} \right\} = \sum_{i=1}^{N} \operatorname{tr}\left\{ \underset{\sim}{G}_i E(\underset{\sim}{z}_{i1} \underset{\sim}{z}_{i1}') \right\}$$

$$E\left\{ \sum_{i=1}^{N} \underset{\sim}{z}_{i2}' \underset{\sim}{H}_i \underset{\sim}{z}_{i2} \right\} = \sum_{i=1}^{N} \operatorname{tr}\left\{ \underset{\sim}{H}_i E(\underset{\sim}{z}_{i2} \underset{\sim}{z}_{i2}') \right\}$$

where tr is the trace of a matrix, and $\underset{\sim}{G}_i$ and $\underset{\sim}{H}_i$ are matrices of the form $\underset{\sim}{G}_i = g_{1i} \underset{\sim}{I}_{a_i} + g_{2i} \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}'$ and $\underset{\sim}{H}_i = h_{1i} \underset{\sim}{I}_{b_i} + h_{2i} \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}'$, respectively. From the model,

$$E(\underset{\sim}{z}_{i1} \underset{\sim}{z}_{i1}') = \underset{\sim}{\Sigma}_{i11} = \sigma_a^2 \left\{ (1-\rho_a) \underset{\sim}{I}_{a_i} + \rho_a \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{a_i}' \right\} \; ;$$

$$E(\underset{\sim}{z}_{i2} \underset{\sim}{z}_{i2}') = \underset{\sim}{\Sigma}_{i22} = \sigma_b^2 \left\{ (1-\rho_b) \underset{\sim}{I}_{b_i} + \rho_b \underset{\sim}{\ell}_{b_i} \underset{\sim}{\ell}_{b_i}' \right\}$$

$$\text{Thus} \quad E\left\{\sum_{i=1}^{N} \underset{\sim}{z}_{i1}' \underset{\sim}{G}_i \underset{\sim}{z}_{i1}\right\} = \sigma_a^2 \sum_{i=1}^{N} a_i(g_{1i} + g_{2i}V_i)$$

$$E\left\{\sum_{i=1}^{N} \underset{\sim}{z}_{i2}' \underset{\sim}{H}_i \underset{\sim}{z}_{i2}\right\} = \sigma_b^2 \sum_{i=1}^{N} b_i(h_{1i} + h_{2i}W_i)$$

$$\text{Also,} \quad E\left\{\underset{\sim}{z}_{i1}' \underset{\sim}{\ell}_{a_i} \underset{\sim}{z}_{i2}' \underset{\sim}{\ell}_{b_i}\right\} = \underset{\sim}{\ell}_{a_i}' E(\underset{\sim}{z}_{i1} \underset{\sim}{z}_{i2}') \underset{\sim}{\ell}_{b_i}$$

$$= \underset{\sim}{\ell}_{a_i}' \rho_{ab}\sigma_a\sigma_b \underset{\sim}{\ell}_{a_i} \underset{\sim}{\ell}_{b_i}' \underset{\sim}{\ell}_{b_i}$$

$$= \rho_{ab}\sigma_a\sigma_b a_i b_i$$

The information matrix for the maximum likelihood estimators, $\hat{\mu}_a$, $\hat{\mu}_b$, $\hat{\sigma}_a^2$, $\hat{\sigma}_b^2$, $\hat{\rho}_a$, $\hat{\rho}_b$ and $\hat{\rho}_{ab}$ is defined by the following elements:

$$E\left\{-\frac{\partial^2 \log L}{\partial\sigma_a^2 \partial\mu_a}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\sigma_b^2 \partial\mu_a}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\rho_a \partial\mu_a}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\mu_b \partial\mu_a}\right\}$$

$$= E\left\{-\frac{\partial^2 \log L}{\partial\rho_{ab} \partial\mu_a}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\sigma_a^2 \partial\mu_b}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\sigma_b^2 \partial\mu_b}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\rho_a \partial\mu_b}\right\}$$

$$= E\left\{-\frac{\partial^2 \log L}{\partial\rho_b \partial\mu_b}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\rho_{ab} \partial\mu_b}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\rho_b \partial\sigma_a^2}\right\} = E\left\{-\frac{\partial^2 \log L}{\partial\rho_a \partial\sigma_b^2}\right\} = \emptyset$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\mu_a \partial\mu_a}\right\} = \frac{1}{\sigma_a^2} \sum_{i=1}^{N} \frac{a_i W_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\mu_b \partial\mu_b}\right\} = \frac{1}{\sigma_b^2} \sum_{i=1}^{N} \frac{b_i V_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\mu_a \partial\mu_b}\right\} = -\frac{\rho_{ab}}{\sigma_a\sigma_b} \sum_{i=1}^{N} \frac{a_i b_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\sigma_a^2 \partial\sigma_a^2}\right\} = \frac{1}{2\sigma_a^4} \sum_{i=1}^{N} a_i + \frac{\rho_{ab}^2}{4\sigma_a^4} \sum_{i=1}^{N} \frac{a_i b_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\sigma_b^2\partial\sigma_b^2}\right\} = \frac{1}{2\sigma_b^4}\sum_{i=1}^{N} b_i + \frac{\rho_{ab}^2}{4\sigma_b^4}\sum_{i=1}^{N}\frac{a_i b_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\sigma_a^2\partial\sigma_b^2}\right\} = -\frac{\rho_{ab}^2}{4\sigma_a^2\sigma_b^2}\sum_{i=1}^{N}\frac{a_i b_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_a\partial\sigma_a^2}\right\} = -\frac{1}{2\sigma_a^2(1-\rho_a)}\sum_{i=1}^{N}\frac{a_i(a_i-1)(W_i\rho_a-\rho_{ab}^2 b_i)}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_b\partial\sigma_b^2}\right\} = -\frac{1}{2\sigma_b^2(1-\rho_b)}\sum_{i=1}^{N}\frac{b_i(b_i-1)(V_i\rho_b-\rho_{ab}^2 a_i)}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\sigma_a^2}\right\} = -\frac{\rho_{ab}}{2\sigma_a^2}\sum_{i=1}^{N}\frac{a_i b_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\sigma_b^2}\right\} = -\frac{\rho_{ab}}{2\sigma_b^2}\sum_{i=1}^{N}\frac{a_i b_i}{T_i}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_a\partial\rho_a}\right\} = \frac{1}{2(1-\rho_a)^2}\sum_{i=1}^{N}\frac{(a_i-1)\left\{T_i^2 + W_i^2(a_i-1)(1-\rho_a)^2\right\}}{T_i^2}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_b\partial\rho_b}\right\} = \frac{1}{2(1-\rho_b)^2}\sum_{i=1}^{N}\frac{(b_i-1)\left\{T_i^2 + V_i^2(b_i-1)(1-\rho_b)^2\right\}}{T_i^2}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_a\partial\rho_b}\right\} = \frac{\rho_{ab}^2}{2}\sum_{i=1}^{N}\frac{a_i b_i(a_i-1)(b_i-1)}{T_i^2}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\rho_a}\right\} = -\rho_{ab}\sum_{i=1}^{N}\frac{a_i b_i(a_i-1)W_i}{T_i^2}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\rho_b}\right\} = -\rho_{ab}\sum_{i=1}^{N}\frac{a_i b_i(b_i-1)V_i}{T_i^2}$$

$$E\left\{-\frac{\partial^2 \log L}{\partial\rho_{ab}\partial\rho_{ab}}\right\} = \sum_{i=1}^{N}\frac{a_i b_i(V_i W_i+\rho_{ab}^2 a_i b_i)}{T_i^2}$$

Inverting the information matrix, one obtains the asymptotic variance-covariance maxtrix for the maximum likelihood estimators. In particular, an explicit form of the asymptotic variance for the estimator $\hat{\rho}_{ab,ML}$ can be expressed as:

$$AV(\hat{\rho}_{ab,ML}) = \frac{u_1 u_4 - u_2^2}{u_6(u_1 u_4 - u_2^2) + 2u_2 u_3 u_5 - u_3^2 u_4 - u_1 u_5^2}$$

where u_1 thru u_6 are defined as

$$u_1 = \begin{cases} 1 \text{ if } a_i = 1 \text{ for all } = 1,2,\ldots,N \\ E\left\{-\dfrac{\partial^2 \log L}{\partial \rho_a \partial \rho_a}\right\} + \dfrac{4\sigma_a^4 \sigma_b^4}{C} E\left\{-\dfrac{\partial^2 \log L}{\partial \rho_a \partial \sigma_a^2}\right\}^2 E\left\{-\dfrac{\partial^2 \log L}{\partial \sigma_b^2 \partial \sigma_b^2}\right\} \end{cases}$$

$$u_2 = E\left\{-\frac{\partial^2 \log L}{\partial \rho_a \partial \rho_b}\right\} + \left[\frac{\rho_{ab}\sigma_a^3 \sigma_b^3}{C} E\left\{-\frac{\partial^2 \log L}{\partial \rho_a \partial \sigma_a^2}\right\} E\left\{-\frac{\partial^2 \log L}{\partial \rho_b \partial \sigma_b^2}\right\} \times E\left\{-\frac{\partial^2 \log L}{\partial \mu_a \partial \mu_b}\right\}\right]$$

$$u_3 = E\left\{-\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \rho_a}\right\} - \left[\!\left[\frac{2\sigma_a^2 \sigma_b^2}{C}\left[E\left\{-\frac{\partial \log L}{\partial \rho_{ab} \partial \sigma_b^2}\right\} \sum_{i=1}^{N} b_i - 2\rho_{ab}\sigma_b^2 \, E\left\{-\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \sigma_b^2}\right\}^2\right] E\left\{-\frac{\partial^2 \log L}{\partial \rho_a \partial \sigma_a^2}\right\}\right]\!\right]$$

$$u_4 = \begin{cases} 1 \text{ if } b_i = 1 \text{ for all } = 1,2,\ldots,N \\ E\left\{-\dfrac{\partial^2 \log L}{\partial \rho_b \partial \rho_b}\right\} + \dfrac{4\sigma_a^4 \sigma_b^4}{C} E\left\{-\dfrac{\partial^2 \log L}{\partial \rho_b \partial \sigma_b^2}\right\}^2 E\left\{-\dfrac{\partial^2 \log L}{\partial \sigma_a^2 \partial \sigma_a^2}\right\} \end{cases}$$

$$u_5 = E\left\{-\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \rho_b}\right\} - \left[\!\left[\frac{2\sigma_a^2 \sigma_b^2}{C} \left[E\left\{-\frac{\partial \log L}{\partial \rho_{ab} \partial \sigma_a^2}\right\} \sum_{i=1}^{N} a_i \right.\right.\right.$$

$$\left.\left.\left. - 2\rho_{ab}\sigma_a^2 \, E\left\{-\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \sigma_a^2}\right\}^2 \right] E\left\{-\frac{\partial^2 \log L}{\partial \rho_b \partial \sigma_b^2}\right\} \right]\!\right]$$

$$u_6 = E\left\{-\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \rho_{ab}}\right\} + \frac{\sigma_a \sigma_b}{C \cdot \rho_{ab}} E\left\{-\frac{\partial^2 \log L}{\partial \mu_a \partial \mu_b}\right\} \left[C - \sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i \right.$$

$$\left. + \rho_{ab}^2 \sigma_a^2 \sigma_b^2 \, E\left\{-\frac{\partial^2 \log L}{\partial \mu_a \partial \mu_b}\right\}^2 \right]$$

and $$C = \sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i - \sigma_a^2 \, \rho_{ab} \, E\left\{-\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \sigma_a^2}\right\} \sum_{i=1}^{N} a_i$$

$$- \sigma_b^2 \, \rho_{ab} \, E\left\{-\frac{\partial^2 \log L}{\partial \rho_{ab} \partial \sigma_b^2}\right\} \sum_{i=1}^{N} b_i$$

In the above expressions, $E\left\{-\frac{\partial^2 \log L}{\partial \cdot\cdot \partial \cdot\cdot}\right\}^2$ means $\left[E\left\{-\frac{\partial^2 \log L}{\partial \cdot\cdot \partial \cdot\cdot}\right\}\right]^2$.

APPENDIX 2A - COMPUTATION OF THE ASYMPTOTIC VARIANCE FOR THE GENERALIZED PRODUCT-MOMENT ESTIMATOR

For computational considerations, we express the components of $\hat{\rho}_{ab,GP}$ as follows:

$$\hat{\sigma}_{ab} = \left\{\sum_{i=1}^{N} w_i a_i b_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..})\right\} / \sum_{i=1}^{N} C_i$$

$$\hat{\sigma}_a^2 = \left\{\sum_{i=1}^{N} w_i a_i b_i (\bar{x}_{i.} - \bar{x}_{..})^2 + \sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i} (x_{ij} - \bar{x}_{i.})^2\right\} / \sum_{i=1}^{N} C_i$$

$$\hat{\sigma}_b^2 = \left\{\sum_{i=1}^{N} w_i a_i b_i (\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^{N} w_i a_i \sum_{k=1}^{b_i} (y_{ik} - \bar{y}_{i.})^2\right\} / \sum_{i=1}^{N} C_i$$

where $C_i = w_i a_i b_i$ and $\sum_{i=1}^{N} C_i = 1$

One component of the Taylor series method is computing the covariance matrix of the parameters $\hat{\sigma}_{ab}$, $\hat{\sigma}_a^2$ and $\hat{\sigma}_b^2$. For example, the variance of $\hat{\sigma}_{ab}$ can be computed by the standard method of 'taking expectations':

$$\mathrm{var}(\hat{\sigma}_{ab}) = E\left[\left[\left\{\sum_{i=1}^{N} w_i a_i b_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..})\right\} / \sum_{i=1}^{N} C_i\right] - \sigma_{ab}\right]^2$$

$$= E\left[\left[\sum_{i=1}^{N}\left\{w_i a_i b_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..}) - C_i \sigma_{ab}\right\}\right] / \sum_{i=1}^{N} C_i\right]^2$$

$$= \sum_{i=1}^{N} E\left\{w_i a_i b_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..}) - C_i \sigma_{ab}\right\}^2 / \left\{\sum_{i=1}^{N} C_i\right\}^2$$

where $E[\cdot]^2$ means $\{E[\cdot]\}^2$.

Similarly,

$$\mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_a^2) = \sum_{i=1}^{N} E\left[\left\{w_i a_i b_i(\bar{x}_{i.}-\bar{x}_{..})(\bar{y}_{i.}-\bar{y}_{..}) - C_i\sigma_{ab}\right\}\right.$$

$$\left.\times \left\{w_i a_i b_i(\bar{x}_{i.}-\bar{x}_{..})^2+w_i b_i\sum_{j=1}^{a_i}(x_{ij}-\bar{x}_{i.})^2 - C_i\sigma_a^2\right\}\right] \Big/ \left\{\sum_{i=1}^{N} C_i\right\}^2$$

In order to simplify the notation in Appendix 2A, the following expressions are defined to replace the terms in the above variance (,covariance) formulas.

Let $t_i = w_i a_i b_i(\bar{x}_{i.}-\bar{x}_{..})(\bar{y}_{i.}-\bar{y}_{..}) - C_i\sigma_{ab}$

$= z_{i1}z_{i2} - C_i\sigma_{ab}$

$u_i = w_i a_i b_i(\bar{x}_{i.}-\bar{x}_{..})^2 + w_i b_i\sum_{j=1}^{a_i}(x_{ij}-\bar{x}_{i.})^2 - C_i\sigma_a^2$

$= z_{i1}^2 + z_{i3} - C_i\sigma_a^2$

$v_i = w_i a_i b_i(\bar{y}_{i.}-\bar{y}_{..})^2 + w_i a_i\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{i.})^2 - C_i\sigma_b^2$

$= z_{i2}^2 + z_{i4} - C_i\sigma_b^2$

where

$z_{i1} = (w_i a_i b_i)^{1/2}(\bar{x}_{i.}-\bar{x}_{..})$

$z_{i2} = (w_i a_i b_i)^{1/2}(\bar{y}_{i.}-\bar{y}_{..})$

$z_{i3} = w_i b_i\sum_{j=1}^{a_i}(x_{ij}-\bar{x}_{i.})^2$

$z_{i4} = w_i a_i\sum_{k=1}^{b_i}(y_{ik}-\bar{y}_{i.})^2$

Under the assumption of normality, z_{i1} and z_{i2} are distributed as bivariate normal variables, independently of z_{i3} and z_{i4} with mean vector $(\emptyset\ \emptyset)'$ and covariance matrix

$$\mathrm{cov}(z_{i1},z_{i2}) = C_i \begin{bmatrix} \overset{*}{\sigma}{}_a^2 & \sigma_{ab} \\ \sigma_{ab} & \overset{*}{\sigma}{}_b^2 \end{bmatrix}$$

where $\overset{*}{\sigma}{}_a^2 = w_i b_i \left\{a_i - (a_i-1)(1-\rho_a)\right\}\sigma_a^2 \ / \ C_i$

and $\overset{*}{\sigma}{}_b^2 = w_i a_i \left\{b_i - (b_i-1)(1-\rho_b)\right\}\sigma_b^2 \ / \ C_i$

In addition, the moment generating function of z_{i1} and z_{i2} is : $m(s_1,s_2) = \exp\left[C_i^2\left\{\tfrac{1}{2}s_1^2\overset{*}{\sigma}{}_a^2 + s_1 s_2 \sigma_{ab} + \tfrac{1}{2}s_2^2\overset{*}{\sigma}{}_b^2\right\}\right]$

The variables z_{i3} and z_{i4} have a bivariate chi-square distribution, independent of z_{i1} and z_{i2}, with mean vector $\left\{w_i b_i (a_i-1)(1-\rho_a)\sigma_a^2 \ \vdots\ w_i a_i (b_i-1)(1-\rho_b)\sigma_b^2\right\}'$ and covariance matrix

$$\mathrm{cov}(z_{i3},z_{i4}) = \begin{bmatrix} 2(a_i-1)w_i^2 b_i^2 (1-\rho_a)^2\sigma_a^4 & \emptyset \\ \emptyset & 2(b_i-1)w_i^2 a_i^2 (1-\rho_b)^2 \sigma_b^4 \end{bmatrix}$$

For large N, it can be shown that $E(t_i) = E(u_i) = E(v_i) = \emptyset$ (see Appendix 2B), and

$$E(t_i^2) = E(z_{i1}^2 z_{i2}^2) - C_i^2\sigma_{ab}^2$$

$$E(u_i^2) = \mathrm{var}(u_i) = \mathrm{var}(z_{i1}^2) + \mathrm{var}(z_{i3})$$

$$E(v_i^2) = \mathrm{var}(v_i) = \mathrm{var}(z_{i2}^2) + \mathrm{var}(z_{i4})$$

$$E(t_i u_i) = E(z_{i1}^3 z_{i2}) + E(z_{i1} z_{i2} z_{i3}) - C_i w_i a_i b_i \sigma_{ab} \sigma_a^2$$

$$E(t_i \;\; {}_i) = E(z_{i1} z_{i2}^3) + E(z_{i1} z_{i2} z_{i4}) - C_i w_i a_i b_i \sigma_{ab} \sigma_b^2$$

$$E(u_i v_i) = E(z_{i1}^2 z_{i2}^2) + E(z_{i1}^2 z_{i4}) + E(z_{i2}^2 z_{i3})$$

$$+ E(z_{i3} z_{i4}) - C_i w_i a_i b_i \sigma_a^2 \sigma_b^2$$

Using the moment generating function of z_{i1}, z_{i2}, the joint moments, $E\{z_{i1}^p z_{i2}^q\}$, are obtained by differentiating $m(s_1,s_2)$ p times with respect s_1 and q times with respect to s_2 and then setting s_1 and s_2 equal to zero in the resulting expression (Mood et al., 1974, pgs. 159-160).

Now,

$$E(z_{i1}^2 z_{i2}^2) = \{\overset{*}{\sigma}{}_a^2 \overset{*}{\sigma}{}_b^2 + 2\sigma_{ab}^2\} C_i^2$$

$$\mathrm{Var}(z_{i1}^2) = 2w_i^2 b_i^2 \{a_i - (a_i-1)(1-\rho_a)\}^2 \sigma_a^4$$

$$\mathrm{Var}(z_{i3}) = 2(a_i-1) w_i^2 b_i^2 (1-\rho_a)^2 \sigma_a^4$$

$$\mathrm{Var}(z_{i2}^2) = 2w_i^2 a_i^2 \{b_i - (b_i-1)(1-\rho_b)\}^2 \sigma_b^4$$

$$\mathrm{Var}(z_{i4}) = 2(b_i-1) w_i^2 a_i^2 (1-\rho_b)^2 \sigma_b^4$$

$$E(z_{i1}^3 z_{i2}) = 3C_i^2 \sigma_{ab} \overset{*}{\sigma}{}_a^2$$

$$E(z_{i1} z_{i2} z_{i3}) = C_i w_i b_i (a_i-1)(1-\rho_a) \sigma_{ab} \sigma_a^2$$

$$E(z_{i1} z_{i2}^3) = 3C_i^2 \sigma_{ab} \overset{*}{\sigma}{}_b^2$$

$$E(z_{i1} z_{i2} z_{i4}) = C_i w_i a_i (b_i-1)(1-\rho_\nu) \sigma_{ab} \sigma_b^2$$

$$E(z_{i1}^2 z_{i4}) = w_i^2 a_i b_i \left\{a_i - (a_i-1)(1-\rho_a)\right\}(b_i-1)(1-\rho_b)\sigma_a^2\sigma_b^2$$

$$E(z_{i2}^2 z_{i3}) = w_i^2 a_i b_i \left\{b_i - (b_i-1)(1-\rho_b)\right\}(a_i-1)(1-\rho_a)\sigma_a^2\sigma_b^2$$

$$E(z_{i3} z_{i4}) = w_i^2 a_i b_i (a_i-1)(b_i-1)(1-\rho_a)(1-\rho_b)\sigma_a^2\sigma_b^2$$

Hence,

$$E(t_i^2) = C_i^2 \sigma_a^{*2} \sigma_b^{*2} + C_i^2 \sigma_{ab}^2$$

$$E(u_i^2) = 2w_i^2 b_i^2 \sigma_a^4 \left\{a_i^2 - 2a_i(a_i-1)(1-\rho_a) + a_i(a_i-1)(1-\rho_a)^2\right\}$$

$$E(v_i^2) = 2w_i^2 a_i^2 \sigma_b^4 \left\{b_i^2 - 2b_i(b_i-1)(1-\rho_b) + b_i(b_i-1)(1-\rho_b)^2\right\}$$

$$E(t_i u_i) = 2\left\{C_i w_i a_i b_i - C_i w_i b_i (a_i-1)(1-\rho_a)\right\}\sigma_{ab}\sigma_a^2$$

$$E(t_i v_i) = 2\left\{C_i w_i a_i b_i - C_i w_i a_i (b_i-1)(1-\rho_b)\right\}\sigma_{ab}\sigma_b^2$$

$$E(u_i v_i) = 2C_i^2 \sigma_{ab}^2$$

Thus,

$$\text{var}(\hat{\sigma}_{ab}) = \sum_{i=1}^{N} E(t_i^2) \Big/ \left\{\sum_{i=1}^{N} C_i\right\}^2$$

$$= \sum_{i=1}^{N}\left[w_i^2 a_i b_i \left\{a_i-(a_i-1)(1-\rho_a)\right\}\left\{b_j-(b_i-1)(1-\rho_b)\right\}\right]\sigma_a^2\sigma_b^2$$

$$+ \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 \sigma_{ab}^2$$

$$\text{var}(\hat{\sigma}_a^2) = \sum_{i=1}^{N} E(u_i^2) \Big/ \left\{\sum_{i=1}^{N} C_i\right\}^2$$

$$= 2\sigma_a^4 \left\{\sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 - 2\sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i-1)(1-\rho_a)\right.$$

$$\left. + (1-\rho_a)^2 \sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i-1)\right\}$$

$$\mathrm{var}(\hat{\sigma}_b^2) = \sum_{i=1}^{N} E(v_i^2) \Big/ \left\{\sum_{i=1}^{N} c_i\right\}^2$$

$$= 2\sigma_b^4\left\{\sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 - 2\sum_{i=1}^{N} w_i^2 a_i^2 b_i(b_i-1)(1-\rho_b) + (1-\rho_b)^2 \sum_{i=1}^{N} w_i^2 a_i^2 b_i(b_i-1)\right\}$$

$$\mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_a^2) = \sum_{i=1}^{N} E(t_i u_i) \Big/ \left\{\sum_{i=1}^{N} c_i\right\}^2$$

$$= 2\left\{\sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 - \sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i-1)(1-\rho_a)\right\}\sigma_{ab}\sigma_a^2$$

$$\mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_b^2) = \sum_{i=1}^{N} E(t_i v_i) \Big/ \left\{\sum_{i=1}^{N} c_i\right\}^2$$

$$= 2\left\{\sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 - \sum_{i=1}^{N} w_i^2 a_i^2 b_i (b_i-1)(1-\rho_b)\right\}\sigma_{ab}\sigma_b^2$$

$$\mathrm{cov}(\hat{\sigma}_a^2,\hat{\sigma}_b^2) = \sum_{i=1}^{N} E(u_i v_i) \Big/ \left\{\sum_{i=1}^{N} c_i\right\}^2$$

$$= 2\sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 \sigma_{ab}^2$$

Letting $\underset{\sim}{d} = \left[\dfrac{\partial\rho_{ab}}{\partial\sigma_{ab}} \;\Big|\; \dfrac{\partial\rho_{ab}}{\partial\sigma_a^2} \;\Big|\; \dfrac{\partial\rho_{ab}}{\partial\sigma_b^2}\right]' = \left[\dfrac{\rho_{ab}}{\sigma_{ab}} \;\Big|\; -\dfrac{\rho_{ab}}{2\sigma_a^2} \;\Big|\; -\dfrac{\rho_{ab}}{2\sigma_b^2}\right]'$

and $\mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_a^2,\hat{\sigma}_b^2) = \begin{bmatrix} \mathrm{var}(\hat{\sigma}_{ab}) & \mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_a^2) & \mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_b^2) \\ \mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_a^2) & \mathrm{var}(\hat{\sigma}_a^2) & \mathrm{cov}(\hat{\sigma}_a^2,\hat{\sigma}_b^2) \\ \mathrm{cov}(\hat{\sigma}_{ab},\hat{\sigma}_b^2) & \mathrm{cov}(\hat{\sigma}_a^2,\hat{\sigma}_b^2) & \mathrm{var}(\hat{\sigma}_b^2) \end{bmatrix}$

then the first-order approximation of the asymptotic variance of $\hat{\rho}_{ab,GP}$ is given by

$$AV(\hat{\rho}_{ab,GP}) = \underset{\sim}{d}' \operatorname{cov}(\hat{\sigma}_{ab}, \hat{\sigma}_a^2, \hat{\sigma}_b^2)\, \underset{\sim}{d}$$

$$= (1-\rho_{ab}^2)^2 \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2$$

$$+ (1-\rho_a)(1-\rho_b) \sum_{i=1}^{N} w_i^2 a_i (a_i - 1) b_i (b_i - 1)$$

$$+ (1-\rho_a)\left\{\tfrac{3}{2}\rho_{ab}^2 - \tfrac{1}{2}\rho_{ab}^2\rho_a - 1\right\} \sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i - 1)$$

$$+ (1-\rho_b)\left\{\tfrac{3}{2}\rho_{ab}^2 - \tfrac{1}{2}\rho_{ab}^2\rho_b - 1\right\} \sum_{i=1}^{N} w_i^2 a_i^2 b_i (b_i - 1)$$

APPENDIX 2B - FIRST MOMENTS OF THE COVARIANCE AND VARIANCE COMPONENTS FOR THE GENERALIZED PRODUCT-MOMENT ESTIMATOR

From Chapter 4.2,

$$\hat{\sigma}_{ab} = \sum_{i=1}^{N} w_i \sum_{j=1}^{a_i} (x_{ij} - \bar{x}_{..}) \sum_{j=1}^{b_i} (y_{ik} - \bar{y}_{..})$$

$$\hat{\sigma}_a^2 = \sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i} (x_{ij} - \bar{x}_{..})^2$$

$$\hat{\sigma}_b^2 = \sum_{i=1}^{N} w_i a_i \sum_{k=1}^{b_i} (y_{ik} - \bar{y}_{..})^2$$

where $\bar{x}_{..} = \sum_{i=1}^{N} w_i a_i b_i \bar{x}_{i.}$; $\bar{x}_{i.} = \frac{1}{a_i} \sum_{j=1}^{a_i} x_{ij}$;

$\bar{y}_{..} = \sum_{i=1}^{N} w_i a_i b_i \bar{y}_{i.}$; $\bar{y}_{i.} = \frac{1}{b_i} \sum_{k=1}^{b_i} y_{ik}$;

and the weights are chosen so that $\sum_{i=1}^{N} w_i a_i b_i = 1$.

For the variables x_{ij} and y_{ik}, we adopt the one-way random effects model (Snedecor and Cochran, 1980, pgs. 238-248) given by

$$x_{ij} = \mu_a + A_i + e_{ij}$$

$$y_{ik} = \mu_b + B_i + \varepsilon_{ik}$$

respectively. It is assumed that the family effects $\{A_i\}$, $\{B_i\}$ are normally and identically distributed with means zero and variances σ_A^2 and σ_B^2, respectively. Further, the residual errors $\{e_{ij}\}$, $\{\varepsilon_{ik}\}$ are normally and identically distributed with means zero and variances σ_e^2 and σ_ε^2, respectively. Also, the $\{A_i\}$, $\{e_{ij}\}$ are completely independent; along with the $\{B_i\}$, $\{\varepsilon_{ik}\}$.

Thus the variance of x_{ij} and y_{ik} are respectively, $\sigma_a^2 = \sigma_A^2 + \sigma_e^2$ and $\sigma_b^2 = \sigma_B^2 + \sigma_\varepsilon^2$. The covariance between x_{ij} and y_{ik} is denoted by σ_{ab}.

We define the means of the model as follows:

$$\bar{x}_{i.} = \mu_a + A_i + \bar{e}_{i.} \quad ; \quad \bar{x}_{..} = \sum_{i=1}^{N} w_i a_i b_i (\mu_a + A_i + \bar{e}_{i.})$$

$$\bar{y}_{i.} = \mu_b + B_i + \bar{\varepsilon}_{i.} \quad ; \quad \bar{y}_{..} = \sum_{i=1}^{N} w_i a_i b_i (\mu_b + B_i + \bar{\varepsilon}_{i.})$$

Consider the first moment of $\hat{\sigma}_a^2$. That is,

$$E\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i} (x_{ij} - \bar{x}_{..})^2\right]$$

$$= E\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i} (\bar{x}_{i.} - \bar{x}_{..})^2 + \sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i} (x_{ij} - \bar{x}_{i.})^2\right]$$

$$= E\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i} \left\{\mu_a + A_i + \bar{e}_{i.} - \sum_{i=1}^{N} w_i a_i b_i (\mu_a + A_i + \bar{e}_{i.})\right\}^2\right]$$

$$+ E\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i} \left\{\mu_a + A_i + e_{ij} - (\mu_a + A_i + \bar{e}_{i.})\right\}^2\right]$$

$$= E\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i}\left\{A_i - \sum_{i=1}^{N} w_i a_i b_i A_i\right\}^2\right]$$

$$+ E\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i}\left\{\bar{e}_{i.} - \sum_{i=1}^{N} w_i a_i b_i \bar{e}_{i.}\right\}^2\right]$$

$$+ E\left[\sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i}\left\{e_{ij} - \bar{e}_{i.}\right\}^2\right]$$

$$= \sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i}\left\{\sigma_A^2 - 2w_i a_i b_i \sigma_A^2 + \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 \sigma_A^2\right\}$$

$$+ \sum_{i=1}^{N} w_i b_i \sum_{j=1}^{a_i}\left\{\frac{\sigma_e^2}{a_i} - \frac{2w_i a_i b_i \sigma_e^2}{a_i} + \sum_{i=1}^{N} w_i^2 a_i b_i^2 \sigma_e^2\right\}$$

$$+ \sum_{i=1}^{N} w_i b_i (a_i - 1)\sigma_e^2$$

$$= \sum_{i=1}^{N}\left[\left\{w_i a_i b_i - 2w_i^2 a_i^2 b_i^2 + w_i a_i b_i \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2\right\}\sigma_A^2\right.$$

$$\left. + \left\{- 2w_i^2 a_i b_i^2 + w_i a_i b_i \sum_{i=1}^{N} w_i^2 a_i b_i^2 + w_i a_i b_i\right\}\sigma_e^2\right]$$

$$= \left\{1 - \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2\right\}\sigma_A^2 + \left\{1 - \sum_{i=1}^{N} w_i^2 a_i b_i^2\right\}\sigma_e^2$$

Thus,

$$E(\hat{\sigma}_a^2) = \left\{1 - \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2\right\}\sigma_a^2 + \left\{\sum_{i=1}^{N} w_i^2 a_i^2 b_i^2 - \sum_{i=1}^{N} w_i^2 a_i b_i^2\right\}\sigma_e^2$$

Noting that the w_i^2's are of order N^{-2} implies that $\lim_{N\to\infty} E(\hat{\sigma}_a^2) = \sigma_a^2$. An identical procedure can be adopted for the variable y_{ik} to show that $\lim_{N\to\infty} E(\hat{\sigma}_b^2) = \sigma_b^2$.

In determining $E(\hat{\sigma}_{ab})$, we define $E(x_{ij}y_{ik}) = \sigma_{ab}$

and proceed with the computations, similar to the ones for $E(\hat{\sigma}_a^2)$. In the end,

$$E(\hat{\sigma}_{ab}) = \left\{1 - \sum_{i=1}^{N} w_i^2 a_i^2 b_i^2\right\}\sigma_{ab}$$

and thus $\lim_{N\to\infty} E(\hat{\sigma}_{ab}) = \sigma_{ab}$.

Defining as in Appendix 2A,

$$t_i = w_i a_i b_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..}) - C_i \sigma_{ab}$$

$$u_i = w_i a_i b_i (\bar{x}_{i.} - \bar{x}_{..})^2 + w_i b_i \sum_{j=1}^{a_i} (x_{ij} - \bar{x}_{i.})^2 - C_i \sigma_a^2$$

$$v_i = w_i a_i b_i (\bar{y}_{i.} - \bar{y}_{..})^2 + w_i a_i \sum_{k=1}^{b_i} (y_{ik} - \bar{y}_{i.})^2 - C_i \sigma_b^2$$

where $C_i = w_i a_i b_i$, it is straightforward to show, by summing both sides over N and using the above results, that:

$$\lim_{N\to\infty} E(t_i) = 0$$

$$\lim_{N\to\infty} E(u_i) = 0$$

$$\lim_{N\to\infty} E(v_i) = 0$$

APPENDIX 3 - COMPUTATION OF THE ASYMPTOTIC BIAS FOR THE GENERALIZED PRODUCT-MOMENT CORRELATION

The bias of an estimator is also found by a simple linear Taylor series expansion of the estimator. In particular,

$$\text{Bias}(\hat{\rho}_{ab,GP}) = \tfrac{1}{2}\underset{\sim}{C}'\underset{\sim}{D}\cdot\text{cov}(\hat{\sigma}_{ab},\hat{\sigma}^2_a,\hat{\sigma}^2_b)\underset{\sim}{C}$$

where $\underset{\sim}{C} = [\ 1\ 1\ 1\ ]'$,

$\underset{\sim}{D}$ is a matrix of partial second derivatives given by

$$\underset{\sim}{D} = \begin{bmatrix} \dfrac{\partial^2\rho_{ab}}{\partial^2\sigma^2_{ab}} & \dfrac{\partial^2\rho_{ab}}{\partial\sigma_{ab}\partial\sigma^2_a} & \dfrac{\partial^2\rho_{ab}}{\partial\sigma_{ab}\partial\sigma^2_b} \\ \dfrac{\partial^2\rho_{ab}}{\partial\sigma_{ab}\partial\sigma^2_a} & \dfrac{\partial^2\rho_{ab}}{\partial^2\sigma^4_a} & \dfrac{\partial^2\rho_{ab}}{\partial\sigma^2_a\partial\sigma^2_b} \\ \dfrac{\partial^2\rho_{ab}}{\partial\sigma_{ab}\partial\sigma^2_b} & \dfrac{\partial^2\rho_{ab}}{\partial\sigma^2_a\partial\sigma^2_b} & \dfrac{\partial^2\rho_{ab}}{\partial^2\sigma^4_b} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dfrac{\rho_{ab}}{2\sigma_{ab}\sigma^2_a} & -\dfrac{\rho_{ab}}{2\sigma_{ab}\sigma^2_b} \\ -\dfrac{\rho_{ab}}{2\sigma_{ab}\sigma^2_a} & \dfrac{3\rho_{ab}}{4\sigma^2_a} & \dfrac{\rho_{ab}}{4\sigma^2_a\sigma^2_b} \\ -\dfrac{\rho_{ab}}{2\sigma_{ab}\sigma^2_b} & \dfrac{\rho_{ab}}{4\sigma^2_a\sigma^2_b} & \dfrac{3\rho_{ab}}{4\sigma^4_b} \end{bmatrix}$$

and $\text{cov}(\hat{\sigma}_{ab},\hat{\sigma}_a,\hat{\sigma}_b)$ is the covariance matrix given in

Appendix 2A. Here, * signifies a Hadamard product of two matrices (Rao, 1973, pg. 30).

After some matrix manipulations, we get

$$\text{Bias}(\hat{\rho}_{ab},\text{GP}) = -\tfrac{1}{2}\rho_{ab}(1-\rho_{ab}^2)\sum_{i=1}^{N} w_i^2 a_i^2 b_i^2$$

$$-\tfrac{1}{4}\rho_{ab}(1-\rho_a)(3\rho_a-1)\sum_{i=1}^{N} w_i^2 a_i b_i^2 (a_i-1)$$

$$-\tfrac{1}{4}\rho_{ab}(1-\rho_b)(3\rho_b-1)\sum_{i=1}^{N} w_i^2 a_i^2 b_i (b_i-1)$$

APPENDIX 4 - THE ANALYSIS OF VARIANCE ESTIMATOR OF INTRACLASS CORRELATION

The analysis of variance (ANOVA) estimator of intraclass correlation is based upon the one-way random effects model given by

$$x_{ij} = \mu_a + A_i + e_{ij}$$

(discussed in Appendix 2B), and defined as

$$\rho_{a,AN} = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_e^2}$$

Noting that $\sigma_A^2 \geq 0$ and $\sigma_e^2 > 0$ implies that $0 \leq \rho_{a,AN} < 1$.

Since this model focuses on the estimation of components σ_A^2 and σ_e^2, it is also referred to as the components of variance model.

Unbiased estimates of σ_A^2 and σ_e^2 are computed from the following ANOVA table

Source	d.f.	Sum of squares	Mean square	Expected mean square
Among subjects	$N-1$	SSA	$MSA = \frac{SSA}{N-1}$	$\sigma_e^2 + \left\{\frac{\Sigma a_i - \Sigma a_i^2/\Sigma a_i}{N-1}\right\}\sigma_A^2$
Within subjects	$\Sigma a_i - N$	SSW	$MSW = \frac{SSW}{\Sigma a_i - N}$	σ_e^2

where $SSA = \sum_{i=1}^{N} a_i(\tilde{x}_{i.}-\tilde{x}_{..})^2$ and $SSW = \sum_{i=1}^{N}\sum_{j=1}^{a_i}(x_{ij}-\tilde{x}_{i.})^2$

with $\tilde{x}_{i.} = \frac{1}{a_i}\sum_{j=1}^{a_i} x_{ij}$ and $\tilde{x}_{..} = \sum_{i=1}^{N}\sum_{j=1}^{a_i} x_{ij} \Big/ \sum_{i=1}^{N} a_i$

They are given by $\hat{\sigma}_A^2 = \frac{MSA-MSW}{a_0}$ and $\hat{\sigma}_e^2 = MSW$, respectively.

Therefore, the ANOVA estimate of ρ_a is

$$\hat{\rho}_{a,AN} = \frac{MSA - MSW}{MSA + (a_0-1)MSW}$$

where $a_0 = \frac{1}{N-1}\left[\sum_{i=1}^{N} a_i - \left\{\sum_{i=1}^{N} a_i^2 \Big/ \sum_{i=1}^{N} a_i\right\}\right]$

Donner and Koval (1980a) provide additional discussion of this estimator, which is consistent for ρ_a and asymptotically normally distributed. In practice, $\hat{\rho}_{a,AN}$ is set to zero if MSW > MSA, since negative intraclass correlations are inadmissable in the context of a random effects model.

An estimator, $\hat{\rho}_{b,AN}$ is obtained in an analogous way, by replacing j with k, a_i with b_i, x_{ij} with y_{ik}, μ_a with μ_b, A_i with B_i, e_{ij} with ε_{ik}, σ_A^2 with σ_B^2, σ_a^2 with σ_b^2, ρ_a with ρ_b, and a_0 with b_0 in Appendix 4.

APPENDIX 5 - COMPUTATION OF THE ASYMPTOTIC VARIANCE FOR THE WEIGHTED PAIRWISE ESTIMATOR OF INTRACLASS CORRELATION

From Chapter 6, the weighted pairwise estimator of intraclass correlation is given by

$$\hat{\rho}_{a,W} = \hat{\sigma}_C \;/\; \hat{\sigma}_V^2$$

where $\hat{\sigma}_C = \sum_{i=1}^{N} W_i \sum_{\substack{j=1 \\ j\neq\ell}}^{a_i} \sum_{\ell=1}^{a_i} (x_{ij}-\tilde{\bar{x}}_{..})(x_{i\ell}-\tilde{\bar{x}}_{..})$

$$\hat{\sigma}_V^2 = \sum_{i=1}^{N} W_i(a_i-1) \sum_{j=1}^{a_i} (x_{ij}-\tilde{\bar{x}}_{..})^2$$

and $\tilde{\bar{x}}_{..} = \sum_{i=1}^{N} W_i a_i (a_i-1)\tilde{\bar{x}}_{i.}$; $\tilde{\bar{x}}_{i.} = \frac{1}{a_i} \sum_{j=1}^{a_i} x_{ij}$

Following the methods for linear Taylor series expansion, (similar to those in Appendix 2A) we first re-express $\hat{\sigma}_C$ and $\hat{\sigma}_V^2$ as:

$$\hat{\sigma}_C = \left\{ \sum_{i=1}^{N} W_i a_i (a_i-1)(\tilde{\bar{x}}_{i.}-\tilde{\bar{x}}_{..})^2 - \sum_{i=1}^{N} W_i \sum_{j=1}^{a_i} (x_{ij}-\tilde{\bar{x}}_{i.})^2 \right\} \Big/ \sum_{i=1}^{N} D_i$$

$$\hat{\sigma}_V^2 = \left\{ \sum_{i=1}^{N} W_i a_i (a_i-1)(\tilde{\bar{x}}_{i.}-\tilde{\bar{x}}_{..})^2 + \sum_{i=1}^{N} W_i (a_i-1) \sum_{j=1}^{a_i} (x_{ij}-\tilde{\bar{x}}_{i.})^2 \right\} \Big/ \sum_{i=1}^{N} D_i$$

where $D_i = W_i a_i (a_i-1)$ and $\sum_{i=1}^{N} D_i = 1$

We now let,

$$r_i = W_i a_i(a_i-1)(\tilde{\bar{x}}_{i.}-\tilde{\bar{x}}_{..})^2 - W_i \sum_{j=1}^{a_i}(x_{ij}-\tilde{\bar{x}}_{i.})^2 - D_i \rho_a \sigma_a^2$$

$$= z_{i5}^2 - \frac{z_{i6}}{a_i-1} - D_i \rho_a \sigma_a^2$$

$$s_i = W_i a_i(a_i-1)(\tilde{\bar{x}}_{i.}-\tilde{\bar{x}}_{..})^2 + W_i(a_i-1)\sum_{j=1}^{a_i}(x_{ij}-\tilde{\bar{x}}_{i.})^2 - D_i \sigma_a^2$$

$$= z_{i5}^2 + z_{i6} - D_i \sigma_a^2$$

where

$$z_{i5} = [W_i a_i(a_i-1)]^{1/2}(\tilde{\bar{x}}_{i.}-\tilde{\bar{x}}_{..})$$

$$z_{i6} = W_i(a_i-1)\sum_{j=1}^{a_i}(x_{ij}-\tilde{\bar{x}}_{i.})^2$$

Under the assumption of normality, z_{i5} is distributed as a normal variable, independently of z_{i6}, with mean zero and variance $W_i(a_i-1)\{a_i - (a_i-1)(1-\rho_a)\}\sigma_a^2$. The variable z_{i6} has a chi-square distribution with mean $W_i(a_i-1)^2(1-\rho_a)\sigma_a^2$ and variance $2(a_i-1)^3 W_i^2(1-\rho_a)^2\sigma_a^4$.

Noting that $E(r_i) = E(s_i) = 0$ for large N,

$$E(r_i^2) = E(z_{i5}^4) + \frac{E(z_{i6}^2)}{(a_i-1)^2} + D_i^2\rho_a^2\sigma_a^4 - \frac{2}{a_i-1}E(z_{i5}^2 z_{i6})$$

$$- 2D_i\rho_a\sigma_v^2 E(z_{i5}^2) + \frac{2}{a_i-1}D_i\rho_a\sigma_a^2 E(z_{i6})$$

$$E(s_i^2) = \text{var}(s_i) = \text{var}(z_{i5}^2) + \text{var}(z_{i6})$$

$$E(r_i s_i) = E(z_{i5}^4) + E(z_{i5}^2 z_{i6}) - \frac{1}{a_i-1}E(z_{i5}^2 z_{i6}) - \frac{1}{a_i-1}E(z_{i6}^2)$$

$$- \rho_a W_i a_i (a_i-1) D_i \sigma_a^4$$

Similar to the methods of Appendix 2A, we obtain

$$\mathrm{var}(z_{i5}^2) = 2W_i^2(a_i-1)^2\left\{a_i - (a_i-1)(1-\rho_a)\right\}^2 \sigma_a^4$$

$$\mathrm{var}(z_{i6}) = 2(a_i-1)^3 W_i^2 (1-\rho_a)^2 \sigma_a^4$$

$$E(z_{i5}^4) = 3W_i^2(a_i-1)^2\left\{a_i - (a_i-1)(1-\rho_a)\right\}^2 \sigma_a^4$$

$$E(z_{i5}^2 z_{i6}) = W_i^2(a_i-1)^3(1-\rho)\left\{a_i - (a_i-1)(1-\rho_a)\right\}\sigma_a^4$$

$$E(z_{i6}^2) = W_i^2(1-\rho_a)^2(a_i-1)^3(a_i+1)\sigma_a^4$$

Hence,

$$E(r_i^2) = 2\sigma_a^4\left[W_i^2(a_i-1)^2\left\{1 + (a_i-1)\rho_a\right\}^2 + W_i^2(a_i-1)(1-\rho_a)^2\right]$$

$$E(s_i^2) = 2\sigma_a^4\left[W_i^2(a_i-1)^2\left\{1 + (a_i-1)\rho_a\right\}^2 + W_i^2(a_i-1)^3(1-\rho_a)^2\right]$$

$$E(r_i s_i) = \sigma_a^4\Big[3W_i^2(a_i-1)^2\left\{1 + (a_i-1)\rho_a\right\}^2$$

$$+ (a_i-2)W_i^2(a_i-1)^2(1-\rho_a)\left\{1 + (a_i-1)\rho_a\right\}$$

$$- W_i^2(a_i-1)^2(1-\rho_a)^2(a_i+1) - \rho_a W_i^2 a_i^2 (a_i-1)^2\Big]$$

Thus,

$$\mathrm{var}(\hat{\sigma}_c) = \sum_{i=1}^{N} E(r_i^2) \Big/ \left\{\sum_{i=1}^{N} D_i\right\}^2$$

$$= 2\sigma_a^4\left\{\sum_{i=1}^{N} W_i^2(a_i-1)^2a_i^2 - 2\sum_{i=1}^{N} W_i^2a_i(a_i-1)^3(1-\rho_a)\right.$$

$$\left. + (1-\rho_a)^2\sum_{i=1}^{N} W_i^2a_i(a_i-1)(a_i^2 - 3a_i + 3)\right\}$$

$$\mathrm{var}(\hat{\sigma}_v^2) = \sum_{i=1}^{N} E(s_i^2) \Big/ \left\{\sum_{i=1}^{N} D_i\right\}^2$$

$$= 2\sigma_a^4\left\{\sum_{i=1}^{N} W_i^2(a_i-1)^2a_i^2 - 2\sum_{i=1}^{N} W_i^2a_i(a_i-1)^3(1-\rho_a)\right.$$

$$\left. + (1-\rho_a)^2\sum_{i=1}^{N} W_i^2a_i(a_i-1)^3\right\}$$

$$\mathrm{cov}(\hat{\sigma}_c,\hat{\sigma}_v^2) = \sum_{i=1}^{N} E(r_is_i) \Big/ \left\{\sum_{i=1}^{N} D_i\right\}^2$$

$$= \sigma_a^4\left[3\sum_{i=1}^{N} W_i^2(a_i-1)^2\left\{1 + (a_i-1)\rho_a\right\}^2\right.$$

$$+ \sum_{i=1}^{N}(a_i-2)W_i^2(a_i-1)^2(1-\rho_a)\left\{1 + (a_i-1)\rho_a\right\}$$

$$- \sum_{i=1}^{N} W_i^2(a_i-1)^2(1-\rho_a)^2(a_i+1)$$

$$\left. - \sum_{i=1}^{N} \rho_a W_i^2a_i^2(a_i-1)^2\right]$$

We note here that $E(\hat{\sigma}_c) = \rho_a\sigma_a^2$ and $E(\hat{\sigma}_v^2) = \sigma_a^2$, and

therefore, $\underset{\sim}{d} = \left[\dfrac{\partial\rho_a}{\partial\sigma_c} \;\middle|\; \dfrac{\partial\rho_a}{\partial\sigma_v^2}\right]' = \left[\dfrac{1}{\sigma_a^2} \;\middle|\; -\dfrac{\rho_a}{\sigma_a^2}\right]'$

Letting $\text{cov}(\hat{\sigma}_C, \hat{\sigma}_V^2) = \begin{bmatrix} \text{var}(\hat{\sigma}_C^2) & \text{cov}(\hat{\sigma}_C, \hat{\sigma}_V^2) \\ \text{cov}(\hat{\sigma}_C, \hat{\sigma}_V^2) & \text{var}(\hat{\sigma}_V^2) \end{bmatrix}$

then the first-order approximation of the asymptotic variance of $\hat{\rho}_{a,W}$ is given by

$$\begin{aligned} AV(\hat{\rho}_{a,W}) &= \underset{\sim}{d}' \,\text{cov}(\hat{\sigma}_C, \hat{\sigma}_V^2)\, \underset{\sim}{d} \\ &= 2(1-\rho_a)^2 \sum_{i=1}^{N} W_i^2 a_i(a_i-1)\left\{1 + (a_i-1)\rho_a\right\}^2 \end{aligned}$$

The asymptotic variance of $\hat{\rho}_{b,W}$, $AV(\hat{\rho}_{b,W})$, is obtained in an analogous way by replacing j with k, a_i with b_i, x_{ij} with y_{ik}, ρ_a with ρ_b in Appendix 5.

APPENDIX 6 - THE UNWEIGHTED GROUP MEANS ESTIMATOR OF INTRACLASS CORRELATION

From Chapter 3.1, an estimator of intraclass correlation derived by Srivastava (1984) is given by

$$\tilde{\rho}_b = \frac{\tilde{\sigma}_b^2 - \tilde{\gamma}_b^2}{\tilde{\sigma}_b^2}$$

where $$\tilde{\sigma}_b^2 = \frac{\sum_{i=1}^{N} (\bar{y}_{is} - \bar{y}_s)^2}{\left[N-1\right]\left[\bar{b}_h^{-1} + (1-\bar{b}_h^{-1})\tilde{\rho}_b\right]} ,$$

$$\tilde{\gamma}_b^2 = \sum_{i=1}^{N} \sum_{k=1}^{b_i} (y_{ik} - \bar{y}_{is})^2 \Big/ \sum_{i=1}^{N} (b_i - 1)$$

and $$\bar{b}_h = \left\{\frac{1}{N} \sum_{i=1}^{N} \frac{1}{\bar{b}_i}\right\}^{-1}, \quad \bar{y}_{is} = \frac{1}{\bar{b}_i} \sum_{k=1}^{b_i} y_{ik}, \quad \bar{y}_s = \frac{1}{N} \sum_{i=1}^{N} \bar{y}_{is}$$

Upon substituting $\tilde{\rho}_b$ into $\tilde{\sigma}_b^2$, one gets:

$$\tilde{\sigma}_b^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{y}_{is} - \bar{y}_s)^2 + (1-\bar{b}_h^{-1})\gamma_b^2$$

A subsequent substitution of $\tilde{\sigma}_b^2$ and $\tilde{\gamma}_b^2$ into $\tilde{\rho}_b$ results in:

$$\tilde{\rho}_b = \frac{\frac{1}{N-1} \sum_{i=1}^{N} (\bar{y}_{is} - \bar{y}_s)^2 + (1-\bar{b}_h^{-1})\gamma_b^2 - \gamma_b^2}{\frac{1}{N-1} \sum_{i=1}^{N} (\bar{y}_{is} - \bar{y}_s)^2 + (1-\bar{b}_h^{-1})\gamma_b^2}$$

Defining,

$$V_B' = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{y}_{is} - \bar{y}_s)^2 - \bar{b}_h^{-1} \sum_{i=1}^{N} \sum_{k=1}^{b_i} (y_{ik} - \bar{y}_{is})^2 \Big/ \sum_{i=1}^{N} (b_i - 1)$$

and $$V_W = \sum_{i=1}^{N} \sum_{k=1}^{b_i} (y_{ik} - \bar{y}_{is})^2 \Big/ \sum_{i=1}^{N} (b_i - 1)$$

one obtains the expression of $\tilde{\rho}_b$ that appears in Donner and Koval (1983), namely,

$$\tilde{\rho}_b = \frac{V_B'}{V_B' + V_W}$$

Smith (1956) was first to derive this estimator using a weighted sums of squares approach. It was later referred to as the estimator of intraclass correlation based upon the unweighted group means by Donner and Koval (1983).

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