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Three Essays On The Acquisition Of Information

Jacques Robert

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THREE ESSAYS ON THE ACQUISITION OF INFORMATION

by

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of the requirements for the degree of
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ABSTRACT

This thesis consists of three essays related to the problem of acquisition of information by economic agents.

The first essay, entitled "Bundled Insurance", considers the problem of screening good- and bad-risk individuals in an insurance market. It argues that bundled insurance or private compulsory insurance programs are natural outcomes when insurance markets suffer from adverse selection. Insurance markets under adverse selection are known to generate incomplete risk sharing. This essay shows that it is optimal, in the context of reactive equilibria, for a monopolist in some unrelated market to bundle its product with a compulsory insurance policy. This essay may help us to understand why employers often provide compulsory group insurance.

The purpose of the second essay, entitled "Search and Price Advertising", is to study the role and implications of price advertising when the acquisition of price information is costly to consumers. Advertising is introduced into a random sequential search model. Under the assumptions of the model, we are able to construct a unique equilibrium exhibiting price dispersion. The model generates interesting predictions about the shape of the price distribution on the advertising behavior of the firms and the interactions between informative advertising and competition.

In auctions, the seller's problem derives from the fact that he has imperfect information about the buyer's willingness to pay for the object on sale. However, when the bidders' private valuations are not statistically independent, the auctioneer would be able to extract all the surplus (Cremer and McLean (1988)). The intent of the third essay, entitled "Continuity in Auction Design", is to see how limited liability and/or risk aversion affect the above result. Invoking the Maximum Theorem (Berge, 1963), we show that the optimal expected gain attainable by the auctioneer becomes continuous and the set of optimal auctions is upper hemi-continuous in the set of possible auctions.

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This dissertation is dedicated to my parents, Ginette and Pierre Robert, from whom I have learned the value of effort, education and love.

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CHAPTER A:
BUNDLED INSURANCE

A.1. INTRODUCTION

Upon employment, every faculty member at the University of Western Ontario is either required to buy or provided with some dental, medical and disability insurance plans. This is not an isolated practice. According to the American Council of Life Insurance, "group protection amounted to 46.2% of life insurance in force in the U.S. at the end of '82. Group life insurance is a near-universal employee benefit in the United States."[1] Similarly, "about 80% of private health insurance (in the U.S.) is purchased through the workplace, and in the great majority of groups there is no choice by the individual employee."[2] A recent Canadian survey[3] found that 97.7 % of employees had access to a group life insurance plan; participation was voluntary in only 6% of these cases. Supplementary Health Care coverage was compulsory or heavily subsidized (by 50% or more) for approximately 83.3. % of the employees. In 1986, group protection amounted to 53.8 % of the life insurance in force in Canada[4].

Tax incentives are often proposed as the main rationale for group insurance. The taxation approach alone cannot, however, account for some basic stylized facts concerning group insurance. The following questions remain unanswered: Why would employers and employees choose insurance plans instead of some other forms of group benefits? Why would group insurance be made compulsory? Why would legislators allow such tax evasion in the first place? The intent of this essay is to explore the non-tax rationale for private compulsory insurance (or insurance bundled with employment).

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1 p.27, "1983-Life Insurance Fact Book", American Council of Life Insurance.

2 Pauly (1986), p.647.

3 Pay Research Bureau, "Benefits and Working Conditions - 1986", Canada.

4 Author's calculation using data from the "1987 Annual Review of Statistics, Canadian Insurance, Canadian Insurance Agent & Broker Statistics".

This essay considers an insurance market suffering from adverse selection. It examines whether some monopolist (monopsonist) selling (buying) a product, say good *G*, will benefit from integrating the insurance market and bundling its product *G* with some insurance contract.

The strategy of product-bundling by a multiproduct monopolist has been argued in previous papers (Adams and Yellin (1976), Schmalensee (1984), McAfee, McMillan and Whinston (1989)). These papers indicate that bundling can serve as a useful price discrimination device when the consumers' valuations for the products are correlated or when the seller has market power in the sale of both bundled goods.

The idea of bundling a good with an insurance policy has also been analyzed before. Warranties, i.e. insurance against product failure, are often bundled with the related product. Braverman, Guash and Salop (1983) show that a monopolist can use optional warranties as a mean to discriminate between heavy users (high-risk) and light users (low-risk). Note that a customer would value a warranty only if he buys the related good. Schlensiger and Venezian(1986) also examine the case where insurance protection and loss-prevention services are bundled. Here, the consumer's valuation for the service bundled with insurance closely depends on how he values the risk and the insurance contract.

In this essay, we propose to consider the case where the customer's valuations for product *G* and insurance are totally unrelated and where the firm selling bundled insurance has no market power in the insurance market. In this context, the rationale for insurance-product bundling is different from the previous models.

In an economic environment with adverse-selection à la Rothschild and Stiglitz (1976), the competitive insurance market fails to achieve first-best risk-sharing because of informational asymmetries[5]. Riley (1979) shows that a reactive equilibrium for the

[5] Many equilibrium concepts have been suggested to examine the market equilibrium in a competitive insurance market. Rothschild and Stiglitz (1976) demonstrated that a pure strategy Nash equilibrium may fail to exist. If, however, it does exist, it is fully separating; i.e. low-risk individuals will choose to separate themselves from high-risk types by accepting lower coverage. Wilson (1977) proposed an Anticipatory equilibrium concept where firms are deterred from introducing a new offer (deviation) by the possibility that existing offers may be withdrawn from the market. The Wilson Anticipatory Equilibrium may allow at equilibrium some pooling

competitive insurance market will always exist and involve full separation of risk types and incomplete risk-sharing.

A monopolist facing a downward sloping demand for its product cannot extract all the consumers' surplus, i.e. some consumers may strictly benefit from buying from the monopolist. The firm can use this to partially resolve the insurance adverse-selection problem. In the context of *reactive equilibria*, this essay shows that it is optimal for the monopolist to offer to its clients a compulsory insurance coverage; by doing so, it exploits the inefficiencies of the insurance market in spite of free-entry into the market.

In *section A.2* of this essay, we examine a competitive insurance market with asymmetric information à la Rothschild and Stiglitz (1976). We establish, following Riley (1979), that there exists a unique Reactive Insurance Equilibrium for this economy. In *Section A.3*, we show that a monopolist facing a continuous downward-sloping demand (with strictly decreasing marginal revenue and constant marginal cost) can always increase its profits by offering a product-insurance bundle. *Section A.4* confirms that the monopolist's involvement in the insurance market is resistant against admissible reaction from other insurers (in the sense specified by the reactive equilibrium concept). *Section A.5* discusses the welfare impact of bundled insurance. We argue, in *section A.6*, that firms in an oligopolistic market can also exploit the inefficiencies of the insurance market as long as their products are not valued identically by consumers.

A.2. INSURANCE MARKET EQUILIBRIUM UNDER ADVERSE SELECTION

The intent of this section is to define a simple competitive market for insurance and characterize its equilibrium. The economic environment examined here is similar to the one described by Rothschild and Stiglitz (1976). There exists asymmetric information; i.e. each consumer's probability of accident is private knowledge. In order to resolve the well-known problem of existence of (pure strategy) Nash equilibrium in the presence of asymmetric information, we propose to use the reactive equilibrium concept proposed by

contracts. The Reactive Equilibrium concept has been proposed by Riley(1979). We shall define and use this concept in the course this paper.

Riley (1979). Let us, in this section, restate the main characteristics of the reactive equilibrium for the competitive insurance market.

A.2.1. *Description of the economy*

We shall consider an economy where risk-averse individuals (or consumers) can buy insurance from risk-neutral insurers (insurance companies). The market for insurance is competitive in that there is free entry. Consumers possess the same state-independent, increasing, strictly concave (VonNeumann-Morgenstern) utility function. Every consumer faces the possibility of suffering a loss L normalized to 1. Individuals differ only in their probability of suffering such a loss. There are two types of individuals: those at high-risk whose probability of accident is defined by Θ_H and those low-risk with an accident probability $\Theta_L<\Theta_H$. Each individual's specific risk type is private knowledge. I assume that the proportion of low-risk individuals in the population is given by δ. Parameters Θ_L, Θ_H and δ are assumed to be common knowledge.

Let us, for the sake of simplicity, assume that consumers have a constant absolute risk aversion. This assumption is fairly restrictive as it implies that all consumers have exponential wealth-dependent utility function; $U(w)= 1-e^{-kw}$. The main ideas of this essay do not depend on this assumption. Constant absolute risk aversion implies that one's preference over various insurance contracts is independent of wealth. More specifically, it is independent both of the price and the valuation for product *G* which will be introduced in the following sections of this essay. Furthermore, under constant risk aversion, we need not assume (as most previous papers did) that consumers are endowed with identical wealth.

An insurance contract is defined as a pair (p,w). By selecting a pair (p,w), a consumer agrees to pay the insurance company a premium p in exchange for compensation w received should an accident occur. The insurance companies which tender the contract (p,w) earn expected profits $[p - \Theta w]$ on each contract sold to type-Θ individuals. Similarly, a type-Θ consumer who enters into an insurance contract (p,w) has expected utility $\Theta(w-1-p) + (1-\Theta)(-p) - B(w,\Theta) = [-\Theta(1-w) -p]- B(w,\Theta)$, where $[-\Theta(1-w) -p]$ is the consumer's expected wealth under such contract and $\beta(w,\Theta)$[6] denotes the risk premium associated with a consumer losing $(1-w)$ with probability Θ. $\beta(w,\Theta)$ can be interpreted as the monetary cost imposed on a type-Θ consumer by an incomplete risk coverage w. Given the opportunity to buy fair (zero-profit) insurance, risk-averse consumers will wish to be fully insured. The risk premium $\beta(w,\Theta)$ attains its minimum at $w=1$, when $\beta(1,\Theta)=0$.

Although insurers are uninformed about the type of a given consumer, they are able to assess every individual's riskiness from the kind of insurance contract purchased. Indeed, the opportunity cost of decreasing the level of coverage increases with Θ, so that $[\Theta_H - \beta_W(w,\Theta_H)] > [\Theta_L - \beta_W(w,\Theta_L)]$ for $\forall\ \Theta_H > \Theta_L$. For some $0 \leq w_1 < w_2 \leq 1$ and some $p_1 < p_2$ if

$$[-\Theta_H(1-w_1) - p_1] - \beta(w_1,\Theta_H) = [-\Theta_H(1-w_2) - p_2] - \beta(w_2,\Theta_H)$$

$$[-\Theta_L(1-w_1) - p_1] - \beta(w_1,\Theta_L) > [-\Theta_L(1-w_2) - p_2] - \beta(w_2,\Theta_L)$$

High-risk individuals will accept to pay higher premiums than low-risk individuals to obtain a better coverage. Insurers can consequently offer a menu of insurance policies which will separate low-risk from high-risk individuals[7].

A.2.2. *Market equilibrium*

An insurance company's strategy consists in a set of contracts S_i of the form $\{(p_i,w_i); i\in[L,H]\}$. A consumers will select among the available contracts the one that maximizes his expected utility.

We use the reactive equilibrium concept proposed by Riley (1979) in answer to the well-known problem of existence of pure strategy Nash equilibrium in screening games. The essence of the reactive equilibrium notion is that some firms are deterred from introducing certain offers (deviations) in their belief that others will introduce additional offers (or reactions) which will render the initial deviation unprofitable. The reactive

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6 Using Pratt's (1964) definition of risk premium, we can define β so that $U(W+\Theta(w-1)-p-\beta) = \Theta\, U(W+w-1-p) + (1-\Theta)\, U(W-p)$.

Under constant risk aversion, β has the interesting property of depending only on w, the level of coverage provided in the contract, and Θ, the probability that the bad state occurs. For $U(W)= 1 - e^{-kW}$, we have $\beta(w,\Theta)= (1/k) \ln[\Theta e^{k(1-w)}+(1-\Theta)] + \Theta(1-w)$.

7 In signalling and screening games, the above condition is often referred to as the single-crossing condition.

equilibrium concept postulates a less myopic behavior on the part of the various agents in the economy.

DEFINITION: (Riley,1979) *A set of offers is a* reactive equilibrium *if, for any additional offer which generates an expected gain to the agent making the offer, there is another which yields a gain to a second agent and losses to the first. Moreover, no further addition to the set of offers generates losses to the second agent*[8].

In a free-entry and free-exit economy, insurance companies in equilibrium will make zero profits overall. Firms must make in particular zero profit on every contract sold. Should an insurance company expect losses overall, it will be easy for it to withdraw from the market. Now, suppose that a company i makes positive profits from some customers. Since firms can separate low-risk and high-risk individuals by offering a menu of insurance coverage a competitor j will be able to make a more attractive offer to lure away the potentially profitable consumers. As shown by Riley(1979), firm j can always design its offer to earn profits from all its clients. No counter-reaction can therefore induce firm j to incur losses. Hence, at equilibrium, no firm can make either losses or strict positive profits on any contract it sells. Further, competition guarantees that no equilibrium market offering will be Pareto-dominated by a separating, zero-profit set of contracts.

THEOREM A.1: (Riley,1979)[9]*There exists a unique Reactive Insurance*

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8 Let S_j be a feasible strategy for agent j. Also let $O_j(S_1,S_2,...,S_n)$ be the outcome for agent j when the strategies adopted by agents 1, ...,n, are $S_1,S_2,...,S_n$. Then $S^*=(S^*_1,S^*_2,...,S^*_n)$ is a reactive equilibrium if for all j, and all S_j, such that

$$O_j(S^*_1,...,S^*_{j-1},S_j,S^*_{j+1},...,S^*_n) >_j O_j(S^*)$$

there exists an agent k and feasible strategy S_k, such that

(i) $O_k(S^*_1,S^*_2...,S_j,...,S^*_k,...,S^*_n) \leq_k O_k(S^*)$

(ii) $O_k(S^*_1,S^*_2...,S_j,...,S_k,...,S^*_n) >_k O_k(S^*_1,S^*_2...,S_j,...,S^*_k,...,S^*_n)$

(iii) $O_j(S^*_1,S^*_2...,S_j,...,S_k,...,S^*_n) <_j O_j(S^*)$

(iv) $\forall$ $l\neq j,k$ and all feasible S_l, $O_k(S^*_1,S^*_2...,S_j,...,S_k,...,S_l,...,S^*_n) \geq_k O_k(S^*_1,S^*_2...,S_j,...,S^*_k,...,S^*_l,...,S^*_n)$

9 Riley (1979) proves theorem 1 for an economy with a continuum of types, its proof can be readily adapted to a two-type case.

Equilibrium which is the (unique) Pareto-dominating member of the family of the separating, zero-profit market offering.

The reactive equilibrium has interesting properties: it is unique, fully separating and insurance companies will always earn zero profit on every contract sold. Furthermore, following the Revelation Principle[10], we may assume, without loss of generality, that the contracts offered will be such that consumers will report their true type. We can imagine insurance companies asking the consumers to report directly their respective type, say Θ_i, before awarding them a contract (p_i,w_i) accordingly. These properties can be used to characterize the equilibrium; leading us to consider the (unique) insurance market reactive equilibrium as the Pareto-dominating contract offering $[(p_L,w_L);(p_H,w_H)]$ subject to the zero-profit, individual rationality and self-selection constraints. Thus, we obtain:

LEMMA A.1: The contract offering $[(p_L,w_L);(p_H,w_H)]$ is the unique reactive equilibrium for the insurance market if it satisfies the following conditions:

(A.1) $\Theta_H w_H = p_H$; (A.2) $\Theta_L w_L = p_L$; (A.3) $w_H=1$;

(A.4) $[\Theta_H U(w_L-1-p_L)+(1-\Theta_H)U(-p_L)] = [\Theta_H U(w_H-1-p_H)+(1-\Theta_H)U(-p_H)] = U(-\Theta_H)$

or $\{-\Theta_H + [\Theta_H-\Theta_L]w_L\} - ß(w_L;\Theta_H) = -\Theta_H$

A proof is contained in the section A.8.

Lemma A.1 fully characterizes the reactive equilibrium insurance contract offering in the competitive insurance market[11]. (A.1) and (A.2) ensure that all insurance companies will make zero profit on all contracts sold. Conditions (A.3) and (A.4) are straightforward results for self-selection games. (A.3) states that high-risk individuals are fully insured (optimal risk-sharing); (A.4) implies that insurance contracts are such that a high-risk

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10 The Revelation Principle (for useful expositions of the Revelation Principle, see Harris & Townsed 1985 and Myerson 1985) implies that for any arbitrary market offering there exists an alternative market offering so that it is in every consumer's best interest to report honestly his type.

11 Lemma A.1 also characterizes the unique pure strategy Nash equilibrium when such equilibria exist. Indeed, every pure strategy Nash equilibrium is a reactive equilibrium; thus if a Nash equilibrium exists it must correspond to the reactive equilibrium described in lemma A.1.

consumer is indifferent to telling the truth or claiming to be a low-risk individual.

It is important to note that the equilibrium market offering is independent of the proportion of risk types among the population. So long as the support of risk types remains the same, the equilibrium outcome will be unchanged. This property follows from the fact that the reactive equilibrium is always fully separating; this premise will be exploited to validate the major conclusion of this essay.

For future reference, let the parameter w_L denote the equilibrium level of coverage tendered to all low-risk individuals in the competitive insurance market. From lemma A.1, w_L is such that $[\Theta_H - \Theta_L]w_L - \beta(w_L;\Theta_H) = 0$.

Hence, the pair of contracts $\{(\Theta_L w_L, w_L);(\Theta_H, 1)\}$ denotes the equilibrium market contracts tendered in the competitive insurance market. Note that $\beta(w_L,\Theta_L)$ can be seen as a measure of the inefficiencies of the competitive insurance market, estimating the monetary cost imposed on low-risk individuals by the informational asymmetry.

The main assumptions and results of this section can be summarized as follows:

i) $\partial\beta(y,\Theta)/\partial y \equiv \beta_y(y,\Theta) < 0$ if and only if $y<1$.

ii) $\beta(y,\Theta) > 0$ if and only if $y \neq 0$, $\forall\ \Theta \in (0,1)$.

iii) $[(\Theta_H - \Theta_L) - \beta_y(y,\Theta_H) + \beta_y(y,\Theta_L)] > 0$ for $\forall$ y and $\forall\ \Theta_H > \Theta_L$

iv) $[\Theta_H - \Theta_L]w_L - \beta(w_L;\Theta_H) = 0$.

v) $[(\Theta_H - \Theta_L)y - \beta(y,\Theta_H) + \beta(y,\Theta_L) - \beta(w_L,\Theta_L)] > 0$ if and only if $w_L < y$.

i) and ii) state that the risk burden imposed on an individual decreases as the level of coverage goes to 1 with a unique minimum at $\beta(1,\Theta) = 0$. iii) corresponds to the single-crossing condition, it states that the cost, in monetary terms, of decreasing the level of coverage is greater for higher-risk individuals. iv) follows from condition (4). Finally, $[(\Theta_H - \Theta_L)y - \beta(y,\Theta_H) + \beta(y,\Theta_L) - \beta(w_L,\Theta_L)] = 0$ when $y = w_L$, which following iii) brings us to v). These results will be useful in the remaining sections of this essay.

A.3. BUNDLED INSURANCE PROVIDED BY A OUTSIDE MONOPOLIST

The next two sections of this essay examine how the insurance market's reactive equilibrium is altered when we allow a firm with some market power in selling a product

G (unrelated to insurance) to bundle its product together with an insurance policy. For the remainder of the essay, the firm selling product *G* will be referred to as the monopolist and firms selling only insurance will be called insurance companies.

We shall argue that any reactive equilibria for the insurance market with a monopolist from some unrelated market involves a product-insurance bundling by the monopolist. It will be useful to consider first what a monopolist would do when taking the actions of insurance companies as given and characterized by the solution attained in lemma A.1. Possible reactions to the monopolist's action will be analyzed in the next section.

Let us consider a monopolist selling a product, namely good *G*, which is produced at a constant marginal cost, c, and assume that the monopolist faces a continuum of consumers with different valuations for good *G*. Each consumer desires at most one unit of product *G* and is endowed with some valuation v for *G*. The specific reservation of each individual is private knowledge. The reservation value v and the risk of accident are taken to be independently distributed in the population[12]. F(v) denotes the cross-sectional distribution function over v with density f(v) defined over the support $[v_L, v_H]$. We further suppose that the function $J(v)= v - \{[1-F(v)] / f(v)\}$ is strictly increasing in v. Under the above assumption, marginal revenue increases strictly in price while the profit function has a unique maximum[13].

If the monopolist restricts itself to sell only product *G*, its strategy will be to set its price to maximize its expected profit given by [P-c] [1-F(P)]. At the optimum, we shall have $(P^*-c) f(P^*) = [1-F(P^*)]$ or $c = P^* - \{[1-F(P^*)] / f(P^*)\}$. There will exist a unique optimal price as $P - \{[1-F(P)] / f(P)\}$ is strictly decreasing in P.

Now, let us allow the monopolist to offer some bundled insurance. We assume that the monopolist can require its consumers to buy, with product *G*, one of its insurance policies and that resale of product *G* is impossible. When bundled insurance is feasible, the monopolist can, in principle, offer a menu of insurance contracts to its potential consumers. Subject to the self-selection constraints, the monopolist will be able to offer a

12 Constant absolute risk aversion also guarantees that the marginal value of insurance coverage is independent of v. Allowing for the contrary will unnecessarily complicate the analysis.

13 For further discussion, see Maskin & Riley (1984)

set of insurance contracts of the form (t_R, y_R), where t_R represents the price for which a consumer reporting his type as Θ_R receives a unit of product *G* as well as a guarantee of compensation y_R in case of an accident.

The set of possible strategies facing the monopolist may be quite large. In particular, the monopolist can always, under such a scheme, attain the same level of profit achieved under no bundling (no insurance provision). Indeed, let the monopolist sell its product, *G*, bundled with a choice of insurance contracts identical to the one offered in the competitive market; that is $\{(t_L,y_L);(t_H,y_H)\} = \{(\Theta_L w_L+P^*, w_L); (\Theta_H+P^*,1)\}$. This is the equivalent, both for the monopolist and the consumers, to the case where the monopolist offers product *G* without insurance at price P* and consumers buy their insurance directly from the competitive insurance market. Alternatively, let the monopolist offer a uniform insurance package such that $\{(t_L,y_L);(t_H,y_H)\} = \{(P',1); (P',1))\}$. Here, the price of *G* includes de facto the purchase of a full coverage insurance policy; that is, all individuals buying *G* must also pay for a full coverage insurance.

A consumer will purchase a product-insurance package from the monopolist if, by doing so, he attains greater utility than otherwise by declining to purchase product *G* and buying insurance policy directly from the competitive insurance companies. Thus, a consumer of type u_i and valuation v will purchase from the monopolist if the latter offers a product-insurance package (t_R,y_R) so that

(A.5) $v-t_R-\Theta_i(1-y_R)-\beta(y_R,\Theta_i) \geq -\Theta_i-b(w_i,\Theta_i)$ or $v \geq t_R-\Theta_i y_R + \beta(y_R,\Theta_i)-\beta(w_i,\Theta_i)$

Again, $\beta(y_R,\Theta_i)$ denotes the monetary cost of the risk imposed by an incomplete coverage y_R on a type Θ_i-consumer.

Now, if we let (t_R,y_R) be the contract purchased by a type Θ_i-consumer reporting type Θ_R, the value $Z(\Theta_R|\Theta_i) \equiv t_R-\Theta_i y_R + \beta(y_R,\Theta_i)-\beta(w_i,\Theta_i)$ may then be interpreted as the effective price paid by the consumer to purchase a unit of product *G*. From the Revelation Principle, we may assume, without loss of generality, that consumers will reveal their true type; that is, the set of contracts offered by the monopolist satisfies the self-selection constraints. Let (t_i,y_i) denote the contract selected by a consumer of type Θ_i, we must have $Z(\Theta_i|\Theta_i) \geq Z(\Theta_R|\Theta_i)$. When there is truthful reporting, we set $Z(\Theta_i|\Theta_i) \equiv Z_i$. So that at equilibrium:

(A.6) $Z_L \equiv t_L - \Theta_L y_L + \beta(y_L,\Theta_L) - \beta(w_L,\Theta_L)$

(A.7) $Z_H \equiv t_H - \Theta_H y_H + \beta(y_H,\Theta_H)$

Where Z_i represents the effective price a type Θ_i-consumer must pay at equilibrium to purchase a unit of product *G*.

Taking the actions of other insurance companies as given, the monopolist's problem consist in selecting a set of contracts that will maximize profits subject to the self-selection constraints, i.e. select a pair of contracts $\{(t_L,y_L);(t_H,y_H)\}$ in order to solve the following:

(A.8) Max $\delta\{(t_L - y_L\Theta_L - c)[1-F(t_L - y_L\Theta_L - \beta(y_L,\Theta_L)+\beta(w_L,\Theta_L))]\}$ +

$(1-\delta)\{(t_H - y_H\Theta_H - c)[1-F(t_H - y_H\Theta_H - \beta(y_H,\Theta_H))]\}$

subject to

(A.9) $t_L - t_H + \Theta_H(y_H - y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H) \geq 0$

(A.10) $t_H - t_L + \Theta_L(y_L - y_H) - \beta(y_L,\Theta_L) + \beta(y_H,\Theta_L) \geq 0$

$(t_i - y_i\Theta_i - c)$ corresponds to the net expected profit earned by the monopolist on every good-insurance package sold to a type Θ_i-consumer. $[1-F(Z_i)] = [1-F(t_i - y_i\Theta_i - \beta(y_i,\Theta_i)+\beta(w_i,\Theta_i))]$ corresponds to the probability that an individual of type Θ_i accepts the product-insurance package (t_i,y_i), i.e. that $v \geq Z_i$. Thus for some market offering $\{(t_L,y_L);(t_H,y_H)\}$, the expected profit function for the monopolist is given by (A.8). (A.9) and (A.10) are the self-selection constraints.

Using Z_L and Z_H, we can rewrite the monopolist's problem as

(A.8') Max $\Pi = \{\delta\,(Z_L - c)[1-F(Z_L)] + (1-\delta)\,(Z_H - c)[1-F(Z_H)]\}$

$+ \{\delta\,[\beta(w_L,\Theta_L) - \beta(y_L,\Theta_L)][1-F(Z_L)] + (1-\delta)\,[-\beta(y_H,\Theta_H)][1-F(Z_H)]\}$

(A.9') s.t. $Z_L - Z_H \geq [(\Theta_H - \Theta_L)y_L - \beta(y_L,\Theta_H) - \beta(w_L,\Theta_L) + \beta(y_L,\Theta_L)]$.

(A.10') $Z_L - Z_H \leq [(\Theta_H - \Theta_L)y_H\ \beta(y_H,\Theta_H) + \beta(y_H,\Theta_L)]$.

The first term of the profit function is maximized when $Z_L = Z_H = P^*$, where P^* is assumed to be the unique maximum of the profit function. The first term in (A.8') can therefore attain its maximum only if $y_L = z_L$, that is, when $[(\Theta_H - \Theta_L)y_L - \beta(y_L,\Theta_H) - \beta(w_L,\Theta_L) + \beta(y_L,\Theta_L)] = 0$ or $Z_L = Z_H$. If the monopolist increases y_L above z_L, we then have $Z_L - Z_H \geq [(\Theta_H - \Theta_L)y_L - \beta(y_L,\Theta_H) - \beta(w_L,\Theta_L) + \beta(y_L,\Theta_L)] > 0$, the effective price charged by the monopolist will vary according to risk-types. This will in turn force the monopolist to follow a suboptimal pricing strategy. On the other hand, the second term of the profit function is maximized when y_L is equal to $1>w_L$, i.e. when individuals are fully protected against the risk of accident; and as such allowing the monopolist to extract full rent from risk-sharing. Consequently, the monopolist's problem is to set y_L optimally according to two conflicting objectives: optimal risk-sharing and optimal pricing.

Solving for $\{(t_L,y_L);(t_H,y_H)\}$, we obtain the following conditions (see section A.8 for a derivation):

(A.11) $y_H=1$

(A.12) $t_L=t_H-\Theta_H(1-y_L)-\beta(y_L,\Theta_H)$

where t_H and $y_L \leq y_H=1$ solve

(A.13) $\delta[1-F(t_L-\Theta_L-\beta(y_L,\Theta_L)+\beta(w_L,\Theta_L))]\,\beta_y(y_L,\Theta_L)= \lambda\,[(\Theta_H-\Theta_L)-\beta_y(y_L,\Theta_H)+\beta_y(y_L,\Theta_L)]$

(A.14) $\{[1-F(t_L-\Theta_L-\beta(y_L,\Theta_L)+\beta(w_L,\Theta_L)) - (t_L-y_L\Theta_L-c)\, f(t_L-\Theta_L-\beta(y_L,\Theta_L)+\beta(w_L,\Theta_L))\} = -\lambda$

(A.15) $(1-\delta)\{\, [1-F(t_H-\Theta_H)]-(t_H-\Theta_H-c)\, f(t_H-\Theta_H)\, \} = \lambda$

Conditions (A.11) and (A.12) are familiar results in a self-selection game. (A.11) states that it will always be optimal to provide full coverage to high-risk individuals. Condition (A.12) states that high-risk types will be indifferent between either contracts, so that at equilibrium, constraint (A.9) is always binding. The value l in (A.14) to (A.15) reflects the marginal cost of increasing the effective price differential between low- and high-risk individuals, Z_L-Z_H. At equilibrium, we have $\lambda>0$. Condition (A.13) states that, at the optimum, the marginal cost of increasing Z_L-Z_H further above its optimal value must correspond to the marginal gain of increasing the coverage to low-risk individuals.

Using conditions (A.11) to (A.15), we arrive at

LEMMA A.2: For the problem in (A.8) to(A.10), we have at the optimum $w_L<y_L<y_H=1$. (see proof in section A.8).

Lemma A.2 states that the optimal level of coverage provided to the low-risk individuals will remain below that of high-risk individuals but (strictly) higher than that provided in the competitive insurance market. Lemma A.2 constitutes an important result. It shows that at $w_L = y_L$, it will be advantageous for the monopolist to increase locally the coverage tendered to low-risk individuals.

This result is easy to understand: while at $y_L=w_L$, the monopolist achieves optimal pricing, so that varying y_L has no effect marginally on the first term of equation (A.8'). On the contrary, increasing y_L will improve risk-sharing and increase the second term of (A.8'). The monopolist is able to use profitably the tradeoff that exists between optimal monopolist rent seeking and optimal risk-sharing[14]. Thus, taking the actions of other insurance companies as given, it will always be in the monopolist's best interest to provide some bundled insurance with "above-market" coverage for low-risk individuals.

A.4. REACTIONS AGAINST BUNDLED INSURANCE

Under the reactive-equilibrium paradigm, we require that each firm, before making any new offers, consider the possible reactions of other firms in the market. To remain consistent with the reactive-equilibrium paradigm, we must question whether any involvement by the monopolist in the insurance market can be discouraged by some (admissible) reactions.

As argued in section A.2, no ordinary insurance company can exploit the inefficiencies of the insurance market. Any new pooling offer will be rendered unprofitable by some profitable reactions that attracts away (cream-skimming) all the consumers from whom the pooling contract earns positive profits. We shall demonstrate that the monopolist, unlike ordinary insurance companies, can exploit the insurance market inefficiencies in spite of potential reactions from other firms in the market.

Suppose that the monopolist offers an insurance-good package where low-risk individuals subsidize high-risk customers. A reaction aimed at attracting low-risk clients will face two main difficulties. First, some (if not most) low-risk clients will, in their desire to continue buying product *G*, resist changing their insurance policy. Secondly, any attempt to lure the monopolist's low-risk clients may also attract the high-risk non-

14 The above is essentially a marginal argument. If the demand for good G were discontinuous around P*, i.e.the optimal price without bundling, lemma 2 would not be guaranteed to hold. In the extreme case where all consumers have the same valuation for good G, the monopolist would behave like any other insurance company in the insurance market.

client population.

Under a broad condition, insurance companies will not be able to react to the monopolist's actions. Suppose that $v_L < t_H - \Theta_H$. A strictly positive proportion of high-risk types ($F(t_H-\Theta_H)>0$) and of low-risk types ($F(t_L-\Theta_L y_L + \beta(y_L,\Theta_L)-\beta(w_L,\Theta_L)) > F(t_H-\Theta_H) >0$) will still prefer to purchase their insurance policy directly in the competitive market. In this case, the reactive equilibrium is unaffected by the monopolist's action since the support of types of individuals purchasing insurance in the competitive market remains unchanged. That is, taking the monopolist's strategy as given, there will exist, for any profit-increasing reaction by an insurer, k, a counter-reaction by an other insurer, l, which would render the initial reaction unprofitable.

Theorem A.2 extends this intuitive result proving that even if an admissible reaction exists, it will not be sufficient to deter the monopolist from offering bundled insurance.

THEOREM A.2: *A reactive equilibrium for the insurance market with an outside monopolist always involves the monopolist offering bundled insurance. That is, even considering the potential reactions from other insurance companies, it will always be advantageous for the monopolist to offer some product-insurance package to its customers.*

PROOF OF THEOREM A.2: Lemma A.2 shows that the monopolist can always increase profits by offering bundled insurance, such that $y_L>z_L$. We need to show that no admissible reaction (as defined by the reactive equilibrium concept) can deter the monopolist from offering such product-insurance package.

A valid reaction S_k to some deviation by the monopolist (firm j: S_j) must be such that no further addition to the set of offers can generate losses to firm k. In particular, firm k must always make non-negative profits on every contract sold in S_k; if not, there will exist some additional set of offers (S_l) that will lure away profitable clients and leave k with a deficit. This condition clearly limits the scope of the possible reactions against the monopolist.

An insurer k will be able to make a better offer to low-risk individuals, in answer to the monopolist's offer, only if no high-risk individuals will accept this new offer. A new offer to low-risk individuals, (p_L,x_L), constitutes an admissible reaction only if it generates non-negative profits and if:

(A.16) $\max(v_L - t_H; -\Theta_H) \geq [-p_L + \Theta_H x_L - \Theta_H - b(x_L, \Theta_H)]$.

Hence, for any contract offering $\{(t_L,y_L);(t_H,y_H)\}$ by the monopolist, the most attractive contract (p_L,x_L) satisfying both condition (A.A.16) and the non-negative profit condition is such that $p_L = x_L\Theta_L$ and $\max(v_L - t_H + \Theta_H; 0) = [-p_L + \Theta_H x_L - ß(x_L, \Theta_H)]$. This contract constitutes a *bound*[15] on the worst possible reaction that the monopolist can expect from other insurance companies.

Now, suppose that the monopolist expects such an (extreme) reaction and behaves accordingly as a Stackelberg leader. The monopolist's problem then becomes the selection a set of contracts $\{(t_L,y_L);(t_H,y_H)\}$ and some induced x_L so as to maximize its expected profits

(A.17) $$\text{Max } \delta\{(t_L - y_L\Theta_L - c)[1 - F(t_L - y_L\Theta_L - ß(y_L,\Theta_L) + ß(x_L,\Theta_L))]\}$$
$$+ (1-\delta)\{(t_H - y_H\Theta_H - c)[1 - F(t_H - y_H\Theta_H - ß(y_H,\Theta_H))]\}$$

subject to (A.9), (A.10) and

(A.18) $\max(0; v_L - t_H + \Theta_H) = [(\Theta_H - \Theta_L)x_L - ß(x_L, \Theta_H)]$

Given such a conjecture, it will still be advantageous for the monopolist to offer some bundled insurance so that $y_L < w_L$. This problem is not much different from the problem (A.8) to (A.10). Evaluated at $y_L = w_L$, constraint (A.18) is either not binding or at worst just binding; so that at the margin there is still no cost of increasing y_L above w_L, while the marginal gain from improved risk-sharing remains strictly positive.

Specifically, if at equilibrium, we have $v_L < t_H - \Theta_H$, then by (A.18) and (A.4) we have $x_L = w_L$. That is, if at equilibrium some high-risk individuals do not buy from the monopolist, the latter can take the actions of insurance companies as given. This can occur whenever v_L is sufficiently small[16]. Thus, even when the monopolist conjectures the

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15 A reaction S_k must yield some additional profits to firm k, so that all admissible reactions will be less attractive to low-risk individuals than the above contract, and would constitute a lesser threat to the monopolist.

16 In particular, $v_L < t_H - \Theta_H$ will always hold at equilibrium if $v_L < c - \delta(\Theta_H - \Theta_L)$. We have $Z_H > P' - \Theta_H$, where P' denotes the price charged by the monopolist for the insurance-good package if it were to offer a (full coverage) pooling contract. The cost of such a package to the monopolist is given by $[c + \delta u_L + (1-\delta)\Theta_H]$, so that $P' \geq [c + \delta u_L + (1-\delta)\Theta_H]$. It follows that if $v_L \leq [c + \delta u_L + (1-\delta)\Theta_H]$ we must have $v_L \leq [c + \delta u_L + (1-\delta)\Theta_H] \leq P' - \Theta_H < Z_H \equiv t_H - \Theta_H$.

worst possible reaction from other insurance companies, such a threat will not be enough to deter it from introducing bundled insurance. Q.E.D.

A.5. ON WELFARE

We shall discuss in this section the welfare effects of bundled insurance, state general theorems and present an example[17].

Bundled insurance may affect welfare in two ways. First, in the competitive insurance market, the gains from risk-sharing are not all exhausted because of adverse selection; bundled insurance allows the monopolist to exploit these inefficiencies and provide "above-market" coverage to low-risk individuals. Secondly, the market for product *G* is inefficient since it is served by a monopolist. Bundled insurance affects the effective price paid by each consumer for a unit of product *G*; thus, good *G* may become "more affordable" to consumers, thereby increasing consumers' surplus.

THEOREM A.3: *Bundled insurance reduces the overall inefficiencies of the insurance market.*

Theorem A.3 follows directly from the results arrived at in sections A.3 and A.4 in which it was shown that $y_L > w_L$ and $y_H = w_H = 1$. The monopolist will provide to all its low-risk customers a level of coverage superior to that offered by the insurance companies, while providing the same coverage to high-risk customers. Overall, the gains from risk-sharing increase with bundled insurance.

With bundled insurance, low-risk individuals are offered a superior insurance coverage than the one provided by other insurance companies. To prevent high-risk individuals from claiming to be of low-risk type, the monopolist must ask low-risk individuals to pay a higher fee for their insurance-product package and to subsidize high-

17 To keep the analysis simple, I assume here that v_L is sufficiently small so that some high-risk individuals will always prefer to buy their insurance policy directly in the competitive insurance market. We can assume, in this case, that the monopolist takes the actions of other firms in the insurance market as given. The optimal behavior for the monopolist is described in section A.3.

risk customers. At equilibrium, high-risk customers will find product *G* more affordable, i.e. $Z_H<P^*$. Hence, although high-risk individuals do not gain from increased coverage, they will always benefit from bundled insurance.

THEOREM A.4: *High-risk individuals will always benefit from bundled insurance. That is, the effective cost to high-risk individuals of purchasing a unit of product G decreases as the monopolist tenders bundled insurance,* $Z_H<P^*$.

PROOF OF THEOREM A.4: Let us suppose on the contrary that $Z_H \geq P^*$, where $[1-F(P^*)]-(P^*-c)f(P^*)=0$. We have $Z_H - [1-F(Z_H)]/f(Z_H) \geq c$ or $[1-F(Z_H)]-(Z_H-c)f(Z_H) \leq 0$. Since $y_L>w_L$, we have $Z_L>Z_H$. This proposition implies that $Z_L - [1-F(Z_L)]/f(Z_L) > c > c-[\beta(w_L,\Theta_L)-\beta(y_L,\Theta_L)]$ or $[1-F(Z_L)]-(Z_L-c+[\beta(w_L,\Theta_L)-\beta(y_L,\Theta_L)])f(Z_L)<0$. But this contradicts the condition that $\delta\{[1-F(Z_L)]-(Z_L-c+[\beta(w_L,\Theta_L)-\beta(y_L,\Theta_L)])f(Z_L)\} + (1-\delta)\{[1-F(Z_H)]-(Z_H-c)f(Z_H)\} =0$. Q.E.D.

Theorem A.4 establishes clearly the welfare implications of bundled insurance for high-risk individuals. There exists, however, no equivalent theorem for low-risk individuals. The effects of bundled insurance on low-risk individuals remain ambiguous: improved coverage is compensated by higher fees. It is possible, as illustrated in the following example, that low-risk individuals may also benefit from the introduction of bundled insurance[18]; this, however, it is not guaranteed.

Consider the case where the reservation value for product *G* is uniformly distributed between $[v_L,v_H]$ (linear demand). One can check that the optimal price for the monopolist without bundled insurance is given by

(A.19) $P^*=(v_H+c)/2$

We can determine the effective price charged to individuals of types Θ_L and Θ_H. We have

(A.20) $Z_H = P^* - \delta/2\,[\beta(w_L,\Theta_L)-\beta(y_L,\Theta_L)] - \delta[(\Theta_H-\Theta_L)y_L-\beta(y_L,\Theta_H)-\beta(w_L,\Theta_L)+$

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18 A Pareto improvement should not be too surprising. If the proportion of high risk individuals is sufficiently small or if the separating cost is prohibitive, the pooling contract often Pareto-dominates the separating contract. The competitive insurance market cannot, however, provide the pooling contract because of cream-skimming.

$\beta(y_L,\theta_L)]$

(A.21) $Z_L = P^* - \delta/2\,[\beta(w_L,\theta_L)-\beta(y_L,\theta_L)] + (1-\delta)\,[(\theta_H-\theta_L)y_L-\beta(y_L,\theta_H)-\beta(w_L,\theta_L)+\beta(y_L,\theta_L)]$

Recall that both $[(\theta_H-\theta_L)y_L-\beta(y_L,\theta_H)-\beta(w_L,\theta_L)+\beta(y_L,\theta_L)]$ and $[\beta(w_L,\theta_L)-\beta(y_L,\theta_L)]$ are strictly greater than zero if and only if $y_L>w_L$.

This example illustrates clearly the impact of bundled insurance on the effective price paid by high-risk individuals for a unit of product *G*. Z_H is always smaller than P^*. On the other hand, the effect of bundled insurance on the effective price charged to low-risk individuals remains ambiguous. It is possible (not guaranteed) that at the optimum $Z_L<P^*$. This can occur when d is sufficiently close to 1. In this case, both high- and low-risk individuals can benefit from bundled insurance.

Furthermore, one can check that the total production of product *G* increases with y_L; that is

$$\{\delta\,[1-F(Z_L)] + (1-\delta)\,[1-F(Z_H)]\} = [1-F(P^*)] + \delta[\beta(w_L,\theta_L)-\beta(y_L,\theta_L)]/[2(v_H-v_L)].$$

Similarly, the average consumer surplus, evaluated in monetary terms, also increases with y_L.

In the above example, we see that bundled insurance increases the general efficiency of the market for product *G*. The monopolist fails to take into account these welfare effects so that the equilibrium level of coverage y_L will be suboptimal from the point of view of welfare. Thus, legislators can improve welfare by setting tax incentives to induce greater y_L. Further, a scheme encouraging compulsory private insurance could be more desirable than an universal public insurance program. A public program offer, in general, superior risk protection to all individuals; on the other hand, private bundled insurance may induce superior performance in the market for the product bundled with insurance.

A.6. THE ROLE OF IMPERFECT COMPETITION

We assumed, so far, that the firm selling bundled insurance is a monopolist in the market for product *G*. This is a quite strong assumption, since pure monopolies are rare in the "real" world economy. The above assumption can be relaxed without changing the main conclusion of this essay. Indeed, we may show that oligopolists will also offer bundled insurance at equilibrium whenever they sell differentiated products.

Consider a market served by an oligopoly (a set of n firms), where each firm produces a differentiated version of product *G* and where each consumer can buy at most one unit of the product from one of the firms. The valuation for a consumer for the good produced by firm j is denoted by v^j.

If every firm, $k \neq j$, offers its product separate from insurance at some price P^k, we can define each consumer's "pseudo-reservation value" for firm j's product as

$$v^{*j} = v^j - \max\{0, v^1-P^1, \ldots, v^{j-1}-P^{j-1}, v^{j+1}-P^{j+1}, \ldots, v^n-P^n\}$$

Accordingly, firm j may act as a monopolist relative to the demand induced by valuations v^{*j}. The arguments of the previous sections will apply, so that it will be optimal for firm j to offer some form of bundled insurance. Furthermore, if other firms also offer bundled insurance, we can calculate the "pseudo-reservation values" using Z_L^k and Z_H^k, the respective prices charged to low- and high-risk consumers by every firm k. Since $Z_L^k > Z_H^k$, low-risk individuals will, on average, be willing to pay more for the product sold by firm j. In this case, offering bundled insurance is even more advantageous since it allows firm j both to increase the price charged to low-risk individuals relative to the price charged to those of higher-risk and to increase rents extracted from risk-sharing.

Competition would matter only in the extreme case where firms produce perfectly identical products (from the consumers' point of view). Consider two firms, A and B, producing an identical product and behaving according to the reactive equilibrium conjecture. Neither firm A nor firm B can use bundled insurance to offer increased coverage to their low-risk customers for the same reasons that ordinary insurance company cannot increase the coverage tendered theirs. Suppose that firm A offers a product-insurance package where low-risk individuals subsidize high-risk individuals; firm B can then offer the same product with a different insurance package in order to attract all low-risk customers, leaving firm A with a deficit.

Bundled insurance requires at least some market power, i.e. barriers to entry in the market for the product sold with the insurance.

A.7. CONCLUSION

The problem of adverse selection in an insurance market is a well-known problem in informational economics. It has been shown that asymmetric information may lead to inefficiencies in a competitive insurance market.

To reduce the competitive market inefficiencies, the government can argue for a public insurance program. A social insurance program would generally involve subsidies from the low- to the high-risk groups. To prevent cream-skimming, a social insurance program must be made compulsory; that is, low-risk individuals cannot opt out from the program to purchase private insurance. The government can also regulate the indus' y by restricting either entry into the industry or the kind of insurance contracts available on the market.

There exists however a less interventionist means to mitigate the inefficiencies of the insurance market. It can be referred to as private compulsory insurance or bundled insurance. The idea is essentially that of a monopolist (monopsonist) selling (buys) a product, namely good *G*. Good *G* may or may not be related to insurance. The monopolist may require that each of its customers also purchase an insurance policy (or be requested to choose from a menu of insurance policies).

The clearest example of compulsory private insurance provision is through employment benefits[19]. In general, employers offer well-differentiated job opportunities, which cannot be resold. Although the exchange of labor for wages is in general unrelated with the insurance market, labor contracts often include provision for health, dental or life insurance schemes. In practice, a large proportion of insurance is provided through employment benefits.

This essay shows that it is optimal for a monopolist in some unrelated market to integrate the insurance market and require its clients to purchase an insurance policy. By bundling its product with an insurance policy, the monopolist can, de facto, differentiate its

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19 There exist some other less convincing examples of bundled insurance. Some credit card companies offer their members travel related insurance such as flight insurance, cash rebate and medical-assistance services. Québec's credit unions offer a discount life insurance to their members proportional to the amount of savings in their accounts.

insurance services from those supplied elsewhere. It can use its market power to partially solve the adverse-selection problem, thereby making it possible for the monopolist to exploit the inefficiencies of the insurance market even when other insurance companies cannot do so. This may help us to understand why bundled insurance is a relatively important phenomenon.

Bundled insurance has two types of welfare implications. First, it increases the level of coverage provided to low-risk individuals. Second, it affects the performance of the (monopolistic) market for product G. Indeed, bundled insurance may increase the consumers' surplus in the market for product G. Since the monopolist will not take into account these effects when offering its product-insurance package, it may be advantageous for the legislators to offer tax incentives to promote product-insurance bundling.

This essay does not tell the entire story concerning group insurance: tax incentives, union preferences, the degree of monopolistic competition and types of insurance packages offered elsewhere will affect both the likelihood and form of group insurance. Nevertheless, this essay suggests a possible avenue for investigating the role and importance of compulsory insurance while presenting a new perspective for understanding how private insurance is provided in a market where asymmetric information exists. We demonstrate that compulsory private insurance programs are a natural outcome when insurance markets suffer from adverse selection. Bundled insurance may be, indeed, the most efficient way in which the private sector can supply insurance under adverse-selection conditions. For these reasons, bundled insurance schemes should not be neglected by the insurance market theory.

A.8. PROOFS

PROOF OF LEMMA A.1: We need to show that the contract offering described in lemma A.1 is the unique Pareto-dominating contract offering $[(p_L,w_L);(p_H,w_H)]$ subject to the zero-profit, individual rationality and self-selection constraints.

First we show that the contract $\{(p_L,w_L);(p_H,w_H)\}$ described in lemma A.1 satisfies the zero-profit , individual rationality and self-selection constraints. The zero-profit constraints are directly satisfied by conditions (A.1) and (A.2). The individual rationality constraint are trivially satisfied since $w_L>0$ and $w_H>0$. (A.4) states that the insurance coverage is such that high-risk consumers are indifferent between telling the truth and claiming to be low-risk individuals. Using fact (iii) and (A.4) we have $\{-\Theta_L - ß(w_L;\Theta_L)\} > -\Theta_H$; i.e. low-risk individuals will prefer to reveal their true type. The self-selection constraints also hold.

Secondly, we need to show that any alternative contract set $\{(p'_L,w'_L);(p'_H,w'_H)\}$ satisfying constraint zero-profit and self-selection constraints is Pareto dominated by the contract set solving (A.1) to (A.4). Assume that $w'_H \neq w_H=1$. Since consumers prefer complete coverage given actuarial odds, high-risk individuals will be worse off. Thus, to guarantee that they will not lie about their type, we must make the contract offered to low-risk individuals less attractive than before. Such a scheme implies that low-risk individuals would also be worse off.

Now assume that $w'_H=w_H=1$ but $w'_L \neq w_L$. We must have $w'_L<w_L$. If not , that is if $w'_L>w_L$, high-risk individuals will prefer contract $(\Theta_L w'_L,w'_L)$. This violates the incentive-compatibility constraint. But if $w'_L<w_L$ then low-risk individuals are worse off under $\{(p'_L,w'_L);(p'_H,w'_H)\}$.

In either case, the initial set of contracts described in lemma A.1 Pareto-dominates all possible alternative sets of contracts. Q.E.D.

PROOF OF EQUATIONS (A.11) TO (A.15): Using the Langragian method to solve the problem stated by (A.8) to (A.10) with multiplier l and π for constraints (A.9) and (A.10) respectively, we obtain the following first order conditions:

(A.22) t_L: $\delta\{[1-F(Z_L)] - (t_L-\Theta_L y_L-c)f(Z_L)\} = (\pi-\lambda)$

(A.23) y_L: $-\Theta_L\delta[1-F(Z_L)] + \delta[\Theta_L-ß_y(y_L,\Theta_L)]\,(t_L-\Theta_L y_L-c)f(Z_L)$

$$= \lambda[\Theta_H-ß_y(y_L,\Theta_H)] - \pi[\Theta_L-ß_y(y_L,\Theta_L)]$$

(A.24) t_H: $(1-\delta)\,\{[1-F(Z_H)] - (t_H-\Theta_H y_H-c)f(Z_H)\} = (\lambda-\pi)$

(A.25) y_H: $-\Theta_H(1-\delta)[1-F(Z_H)] + (1-\delta)[\Theta_H-ß_y(y_H,\Theta_H)]\,(t_H-\Theta_H y_H-c)f(Z_H)$

$$= \pi[\Theta_L-ß_y(y_H,\Theta_L)] - \lambda[\Theta_H-ß_y(y_H,\Theta_H)]$$

(A.26) $\lambda\quad[t_L-t_H+\Theta_H(y_H-y_L)-ß(y_H,\Theta_H)+ß(y_L,\Theta_H)] = 0$

(A.27) $\pi \quad [t_H - t_L + \Theta_L(y_L - y_H) - \beta(y_L,\Theta_L) + \beta(y_H,\Theta_L)] = 0$

(A.28) (A.29) $\lambda \geq 0$, $\pi \geq 0$ and constraint (A.10) and (A.11)

Substituting (A.22) into (A.23) and (A.24) into (A.25) we obtain

(A.30) $-\beta_y(y_L,\Theta_L)\delta[1-F(Z_L)] = \lambda\, [\Theta_H - \Theta_L - \beta_y(y_L,\Theta_H) + \beta_y(y_L,\Theta_L)]$

(A.31) $-\beta_y(y_H,\Theta_H)(1-\delta)[1-F(Z_H)] = -\pi\, [\Theta_H - \Theta_L - \beta_y(y_H,\Theta_H) + \beta_y(y_H,\Theta_L)]$

Let us recall first that $[\Theta_H - \Theta_L - \beta_y(y_H,\Theta_H) + \beta_y(y_H,\Theta_L)] > 0$ and that $[\Theta_H(y_H - y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H)] > [\Theta_L(y_H - y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H))]$ if and only if $y_H > y_L$.

We now need to show that $\pi=0$. By way of contradiction, suppose that $\pi>0$ and $\lambda>0$. Conditions (A.26) and (A.27) would imply that $[\Theta_H(y_H-y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H)] = [\Theta_L(y_H-y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H)]$ or $y_H=y_L$. But then condition (A.30) implies that $-\beta_y(y_L,\Theta_L)>0$ or $y_L=y_H<1$ and condition (A.31) implies that $-\beta_y(y_H,\Theta_H)<0$ or $y_H=y_L>1$: contradiction. Now, suppose that $\pi>0$ and $\lambda=0$. From conditions (A.30) and (A.31), we have $-\beta_y(y_L,\Theta_L)=0$ and $-\beta_y(y_H,\Theta_H)>0$ or $y_L=1>y_H$. But, this violates constraint (A.9).

Given that $\pi=0$, we have $-\beta_y(y_H,\Theta_H)=0$ or $y_H=1$. Furthermore, we can check that constraint (A.9) must always be binding. If $\lambda>0$, then, by condition (A.26) we have $[t_L - t_H + \Theta_H(y_H-y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H)] = 0$. Now let $\lambda=0$, by condition (A.30) we have $-\beta_y(y_L,\Theta_L)=0$ or $y_L=1=y_H$; then, by constraints (A.9) and (A.10) we have $t_L \geq t_H$ and $t_L \leq t_H$, so that $t_L - t_H = [t_L - t_H + \Theta_H(y_H-y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H)] = 0$. In either case, constraint (A.9) must hold with equality.

Finally, we must show that $y_L \leq 1 = y_H$. Constraints (A.9) and (A.10) imply that $[\Theta_H(y_H-y_L) - \beta(y_H,\Theta_H) + \beta(y_L,\Theta_H)] \geq [\Theta_L(y_H-y_L) - \beta(y_H,\Theta_L) + \beta(y_L,\Theta_L)]$. Whenever $y_L>1=y_H$, the above condition is violated since $[(\Theta_H-\Theta_L)(1-y_L) - \beta(y_L,\Theta_H) + \beta(y_L,\Theta_L)]<0$ as $y_L>1$. Q.E.D.

PROOF OF LEMMA A.2: i) $y_L < y_H = 1$: We have already shown that $y_L \leq 1$. We need to show that $y_L \neq 1$. By way of contradiction suppose that $y_L=1$. Then from (A.13), we must have $\lambda=0$. But also, as $y_L=1$ we have $t_L=t_H$. We get $Z_L - Z_H = (\Theta_H - \Theta_L) - b(w_L,\Theta_L) > 0$. Hence, if $\{ [1-F(Z_H)] - (Z_H - c)f(Z_H) \} = \lambda = 0$, as the marginal revenu is strictly decreasing we obtain $Z_L - [1-F(Z_L)]/f(Z_L) > c > c - b(w_L,\Theta_L)$ or $\delta \{ [1-F(Z_L)] - (Z_L -$

$c+\beta(w_L,\Theta_L)-\beta(y_L,\Theta_L))f(Z_L)\} \neq 0=\lambda$. This contradicts condition (A.15).

ii) $y_L \geq w_L$: By way of contradiction suppose that $y_L < w_L$. It is easy to show that it can not be an optimal solution for the monopolist. In particular, we can show that the monopolist can earn strictly greater profits as $y_L = w_L$. Consider the profit function

$$\text{(A.8')} \qquad \Pi = \{\delta\,(Z_L-c)[1-F(Z_L)] + (1-\delta)\,(Z_H-c)[1-F(Z_H)]\} + \delta\,[\beta(w_L,\Theta_L)-b(y_L,\Theta_L)][1-F(Z_L)]$$

The first term is maximized as $Z_L=Z_H=P^*$; that is when $y_L=w_L$. Furthermore - $b(y_L,\Theta_L)$ is increasing with y_L so that $\delta\,[\beta(w_L,\Theta_L)-\beta(y_L,\Theta_L)][1-F(Z_L)] < \delta\,[\beta(w_L,\Theta_L)-\beta(w_L,\Theta_L)][1-F(Z_L)]=0$ if $y_L<w_L$. It follows that having $y_L<w_L$ can not be an optimal strategy for the monopolist.

iii) Finally we need to show that $y_L \neq w_L$. By way of contradiction suppose that $y_L=w_L$. We have $Z_L=Z_H$ and and $Z_H=t_L-\Theta_L y_L$; so that $\{\,[1-F(Z_L)]-(t_L-\Theta_L y_L-c)f(Z_L)\,\} =\{\,[1-F(Z_H)]-(Z_H-c)f(Z_H)\,\}$. Then by condition (A.14) and (A.15) we must have $\lambda=0$. But it violates condition (A.13), since we have $\delta[1-F(Z_L)]\,\beta_y(w_L,\Theta_L) > 0 = \lambda\,[(\Theta_H-\Theta_L)-\beta_y(w_L,\Theta_H)+\beta_y(w_L,\Theta_L)]$. This latter result shows that at $y_L=w_L$, the monopolist can improve its profits by increasing locally y_L above w_L.

CHAPTER B:
SEARCH AND PRICE ADVERTISING

B.1. INTRODUCTION

When it is costly for consumers to gather information about prices, stores offering relatively lower prices would gain by informing consumers; thereby attracting a number of previously uniformed consumers. A casual glance at a daily newspaper or at the pile of advertising junk mail, indicates that stores often advertise sales. The purpose of this essay is to study the role and implications of price advertising when the acquisition of price information is costly to consumers. To do so, we introduce advertising into a random sequential search model[20]. The interest of this essay is twofold.

First, this essay provides an interesting contribution to the literature of search and dispersed price equilibria. Consumers' information on price is crucial in determining the nature of market equilibria when the search is costly ; price advertising has therefore important strategic implications on the market equilibria when price information is imperfect. In an environment where firms have identical costs and where consumers must bear the same sequential search cost, a price dispersion will arise in equilibrium only if there exists some heterogeneity in consumer information. If all consumers were to observe only one price, the unique equilibrium would be the monopoly price equilibrium (see Diamond (1971)). If all consumers were to observe at least two prices, the unique

[20] To our knowledge, the only attempt to incorporate search and price advertising is in Butters(1977) section 3. Butters, however, did not assume that uninformed buyers search randomly among sellers, an assumption that would "lead to certain unpalatable complications". Rather, it is assumed that "the probability that a search yields an offer from any particular seller is equal to the(seller's) share in total sales". Under, the above assumption, an informed consumer (one who observed at least one ad) is neither more nor less likely to purchase the good from a low-priced store than a uninformed consumer. In his model, uninformed consumers do not suffer from their lack of information, nor do informed consumers benefit from their superior information.

equilibrium would be the competitive price equilibrium (see Burdett and Judd's lemma 2 (1983)). Heterogeneity in consumer information has driven many of the previous price-dispersion models. Varian (1981) assumed that an exogenous proportion of the population is informed about all the prices in the market. Salop and Stiglitz (1976) and Wilde and Schwartz (1977) rely on different search cost or propensities to search among consumers to explain why some consumers will be ex-post more informed than others. The stochastic nonsequential search in Wilde (1977) and the sequential noisy search in Burdett and Judd (1983) explain why chances are that some agents learn about two or more prices.

By introducing advertising into a search model, we can generate endogenously the heterogeneity in consumer information necessary to induce price dispersion. Stores may have the incentive to diffuse information, thus providing consumers with free price information. We believe it is a realistic way to approach price-dispersion models.

Second, the essay offers a benchmark model to study the relationship between price advertising and competition. The relation between advertising(in its various forms) and competition has been the subject of a number of papers (see Comanor and Wilson (1979) for a survey of the subject). These papers provide empirical findings and intuitive arguments focusing on the effect of advertising on profitability and the impact of profitability on advertising expenditures. The view taken by this essay is that both profitability and advertising are endogenous outcomes[21] of a complex market equilibrium where consumers and firms interact strategically according to the industry's intrinsic technological structure. When the information on prices i imperfect, there is little or no incentive for firms to cut prices and engage in Bertrand-like competition; but, when price advertising is available to firms, some stores may wish to advertise price cuts in order to attract more consumers. Price advertising affects non-trivially the dynamics of competition between firms. Conversely, when there is a large number of firms in the market, ads will have little impact on each firm's individual demand; and competition will therefore reduce the incentive for advertising. This essay offers a game theoretical model to study the interactions between informative *price* advertising and profitability and competitiveness[22].

21 This view is not unique to this paper. Schmalensee (1972), among others, stresses that profitability and advertising are simultaneously determined.

We state in section B.2, the basic assumptions of our model. Consumers search sequentially and it costs them some fixed c to visit a store, either to get some new price information and/or to purchase the good. Stores advertise (with endogenous intensity) when it is in their interest to do so. Under these assumptions we are able to construct, in section B.3, a unique equilibrium exhibiting price dispersion. The model generates interesting predictions about the shape of the price distribution and on the advertising behavior of the firms. In section B.4, we analyze the interactions between informative advertising and competition. We show that total advertising expenditures decrease as the number of firms increases. At the limit, when the entry cost becomes small the total advertising expenditure will converge to zero. Further, advertising costs (and the search cost) will affect the degree of competition between firms. At the limit, when advertising cost (and search cost) become small, the equilibrium converges to Bertrand competition.

B.2. THE MODEL

We assume that there exists an infinite supply of potential firms (retailers, sellers or stores); and that each firm sells the same homogeneous good. Before selling any good, a shop must bear some fixed sunk cost k. The existence of k conveys the idea that average cost decreases with the amount of good sold: k acts as a barrier to entry and will help determine the number of firms that will actually enter the market.

The cost of producing each unit of good is constant and the same for all stores. Retailers face no capacity constraints. Without loss of generality, we may assume that the marginal cost is zero. This simply means that in the following analysis all prices have to be interpreted as deviations from marginal cost or as price mark ups. If unit cost were not zero, then price p would have to be replaced by price minus cost in the subsequent analysis.

22 Grossman and Shapiro (1984) introduce informative advertising into a model of product differentiation. Advertising conveys full and accurate information about the characteristics of products and improves the match between products and the heterogeneous consumers. The paper considers, in particular, the effect of changes in the advertising cost on the market equilibrium. Our model is different in that we consider instead price advertising by firms selling homogeneous products.

Finally, stores can inform consumers about the price they intend to charge. We assume that ads never lie; i.e. that stores are legally bound to offer the price quoted in the ad. Advertising is assumed to be costly. The advertising process is not explicitly modeled in this essay. Stores can use many different ways to advertise their prices: mouth-to-ear advertising, posting a sign on the shop-window, mailing, newspaper- and TV-advertising, etc. They will, presumably, select the most efficient advertising medium combination in order to reach the desired proportion of the population. Let A(b) denote the advertising cost of informing a proportion b of consumers. We say that a store selects an advertising intensity b when it chooses to spend A(b) to reach a proportion b of the consumer population. A(.) is assumed to be twice continuously differentiable, strictly convex and increasing. Convexity of A(.) reflects the presumption that it is increasingly costly to reach an extra proportion of the population. It is also assumed that not advertising is costless, $A(0)=0$; that the per-capita cost to reach a small fraction of the population is arbitrarily small, $A'(0)=0$; and that $A'(1)$ is arbitrarily high so that it is excessively costly to reach the entire population.

On the other side of the market there is a large number of households (consumers or buyers). We assumed that the set of all households is an atomless space with measure normalized to 1. Customers will buy at most one unit of good and have an identical valuation, $v>0$, for the good, v being the maximum price a consumer will ever be ready to pay for the good.

It is common knowledge that consumers know the structure of the market, i.e. the number of stores, their respective location, the production and advertisement costs; and are highly rational. However, households are a priori uninformed about the prices charged by each respective store but may learn about the prices in the economy either through advertising or through search.

It is assumed that consumers must bear a cost $c>0$ for every store visited either to search for a better price and/or to buy a good. If a customer is dissatisfied with the prices offered in the observed ads or in the previously visited stores, she[23] can visited a new shop at cost c. Note that she must bear cost c, even if she chooses to purchase an

23 The feminine gender will be used whenever we refer to a consumer or buyer.

advertised good. It is further assume that consumers can always freely recall an offer from some previously visited store[24].

Finally, consumers cannot engage in information acquisition activities other than search (i.e. visiting stores); they cannot exchange ad information nor influence the probability of seeing some ads. It is assumed that all consumers have the same probability of reading a given ad. Furthermore, the probability that a given consumer receives a message from one firm is independent of the probability of receiving a message from any other firm. Thus, if a firm tries to reach a proportion b of the population through advertising, every consumer has an independent probability b of seeing the ad.

The game proceeds as follows. Each firm first decides whether it will enter the retailing market. Given n firms entered in the market, shops choose simultaneously their pricing and advertising strategies .

The (mixed) price strategy for some firm j is a price distribution of the form $F_j(.)$, where $F_j(p)$ denotes the probability that it offers a price smaller than or equal to p. This representation is sufficiently general to allow for mass points in the price distribution function. Stores will never charge above the consumer' valuation v, nor will they charge below zero, the marginal cost. An advertising strategy for some firm j is a function $b_j(.)$ [25] defined over the support of $F_j(.)$ such that $0 \leq b_j(p) \leq 1$ and where $b_j(p)$ denotes the proportion of consumers reached by j's ad, given that j charges price p. Given $F_j(.)$ and $b_j(.)$, we let $T_j(p) \equiv \int_0^p b_j(x) dF_j(x)$. $T_j(p)$ is the probability that a consumer observes an ad offering a price at p or below from store j. Similarly, we let $G_j(p) \equiv [\int_0^p (1-b_j(x))\, dF_j(x)\,] / [1-\int_0^v b_j(x)\, dF_j(x)\,] \equiv [F_j(p)-T_j(p)]/[1-T_j(v)]$. $G_j(p)$ represents the probability the customer places on the event that j charges price p or below conditional on not observing an

24 Free recall is not a necessary assumption. If we assume rather that a consumer can recall an offer only if she goes back to that store at cost c>0, the market equilibrium will remain unchanged. Such generalized result requires however a much longer proof.

25 A firm advertising strategy may be seen as a conditional distribution H(.|.), where H(b|p) denotes the probability of advertising with an intensity smaller than or equal to b given price p is charged. This will allow for a mixed advertising strategy at every price. Such a notation is not used, since there is a unique optimal advertising intensity, b(p), at every price p.

ad from store j.

Finally, upon observing various ads, a customer must decide whether to go to the shop offering the lowest advertised price, or pick at random a non-advertising store and begin a search process. The consumers' search strategy must be optimal, given their (correct) beliefs concerning firms' price and advertising strategies.

B.3. THE EQUILIBRIUM

In order to construct the equilibrium, we will first restrict our attention to symmetric equilibria (equilibria where firms play the same strategy) and to cases where consumers' valuation is high enough not to be binding. These assumptions are later relaxed to show the existence and uniqueness of the equilibrium for this particular game.

In sections B.3.1 to B.3.3, we will show, for a given number n of stores, the existence of a unique symmetric market equilibrium. In section B.3.1, we compute the optimal search strategy for each consumer given some a priori belief about the symmetric strategy of firms. In section B.3.2, we construct the n-firm symmetric equilibrium pricing and advertising strategy for all n stores, given some consumers' effective reservation price. A n-firm symmetric equilibrium will specify the price dispersion and advertising intensity at each possible price and will determine the level of profits attained by all n firms at equilibrium. In section B.3.3, we show the existence of a unique n-firm symmetric market equilibrium, where the firms' behavior is optimal given their (correct) belief about the strategy of consumers and consumers' behavior is optimal given their (correct) belief about the actions of firms. Section B.3.4 extends this result to show that all market equilibria must be symmetric; while section B.3.5 considers more specifically the case in which v is charged with positive probability.

Finally, given the post-entry profits generated at equilibrium for each given number of stores, we can determine the actual number of stores that will accept to enter into the retail market given the fixed entry cost k. Section B.3.6 confirms the existence of a unique market equilibrium with entry.

B.3.1 *Consumer's Strategy*

We consider here the optimal search strategy for buyers, given that all firms follow the same strategy. The optimal strategy for a consumer in our model is a straightforward generalization of sequential search with a finite number of stores and free recall[26]. Before choosing the store, a consumer may observe some price ads. Let p_j be the price advertised by store j. The consumer can purchase j's good at price p_j providing she pays c to go to store j. Throughout her search, she can use the option of buying j's good at cost p_j+c. Let r be the maximum price a consumer will accept to pay *once in a store*, she will then accept to buy j's good if $p_j+c \leq r$. Alternatively, if $p_j+c>r$, she will prefer to continue prospecting the market.

The maximum price r a consumer will accept to pay can be obtained using the sequential search theory. Let z be the best offer received so far by some consumer. z can be interpreted as the lowest advertised price plus search cost, c, or the lowest price from all previously visited store. Assuming that there is one unprospected store,i.e. an unvisited store from which no ad has been observed. The consumer can accept price z or visit this extra store. If she samples the extra offer she will bear some search cost c but may gain by obtaining a better price. If she receives a worse offer, she will pay z for the good; otherwise, she will pay whatever price is offered. Let G(.) denote the conditional price distribution of a store from which no ad has been observed. Symmetry guarantees that G(x) is the same for all these stores. The consumer will choose to visit the unprospected store if and only if

(B.1) $$z > z \int_z^v dG(x) + \int_0^z x\, dG(x) + c$$

If v is such that $v < \int_0^v x\, dG(x) + c$; consumers will accept any price lower or equal to v. If not, note that $z - z \int_z^v dG(x) + \int_0^z x\, dG(x) + c$ is strictly increasing in z. There would exist a unique $r<v$ such that:

(B.2) $$r = r \int_r^v dG(x) + \int_0^r x\, dG(x) + c\,.$$ The consumer will continue to search as long as its best offer is above r, since $z > z \int_z^v dG(x) + \int_0^z x\, dG(x) + c$ if and only if $z>r$.

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26 See Lippman and McCall (1976) for a complete exposition.

The above argument will also hold when there exist many unprospected stores, so that consumers will continue to search for better prices if and only if the previous offers exceed r (or v if $v < \int_0^v x\, dG(x) + c$) . Similarly, a consumer will accept to pay c to go to some store which advertises, only if it offers a price $p \le r - c$.

We can now state the following theorem:

THEOREM B.1: *Consumer search strategy may be summarized by a scalar $r \le v$, where r is the highest price a consumer will accept to pay for the good once in a store. Moreover, a consumer will go to the store advertising the lowest price only if its price does not exceed (r-c). If r<v, then*

$$\text{(B.2)} \qquad r = r\int_r^v dG(x) + \int_0^r x\, dG(x) + c$$

We will assume through most of this essay that the consumer's valuation v is high enough, so that the effective reservation r is strictly lower than v and given by eq.(B.2). In section B.3.5, we argue that such a high v exists; the case where r=v at equilibrium is also considered in section B.3.5.

B.3.2 *The n-firm symmetric equilibrium*

A store must, in order to compute its optimal strategy, consider the price distribution and advertising strategy selected by other stores and the effective reservation price used by consumers. A firm's expected profit will depend on i) its price p and advertising intensity b, ii) the number n of firms in the industry, iii) the strategies played by all other n-1 firms and iv) the effective reservation price used by consumers. Let $\Pi(p,b)$ denote a firm's expected profit when it charges p, advertises with intensity b when ii), iii) and iv) are well-defined. An n-firm symmetric equilibrium is such that all firms play identical strategies and that each firm maximizes its expected profits given the pricing and advertising strategies of the other n-1 firms and the reservation price r used by consumers.

DEFINITION: Given the effective reservation price r and some number n of firm in the market, an *n-firm symmetric equilibrium* is a triple $(F(.), b(.), \pi)$, where F(.) is a distribution function, b(.) is a function mapping price into the unit interval and π is a scalar, such that:

a) $b(p) \in \text{Argmax}\, \Pi(p,b)$

b) $\pi = \Pi(p,b(p))$ for all p in the support of F(.)

c) $\pi \geq \Pi(p,b(p))$ for all p.

Condition a) guarantees that the level of advertising intensity chosen by the firm at each price is optimal. Condition b) implies that all firms earn the same expected profits at an n-firm symmetric equilibrium, where c) implies there is no incentive for any firm to change its price.

LEMMA B.1: *If* (F(.),b(.),π) *is a n-firm symmetric equilibrium associated with the reservation price* r < v, *then*

i) F(r)=1

ii) $$r = \int_0^r x\, dG(x) + c = \frac{\left[\int_0^r x\,(1-b(x))\, dF(x) \right]}{\left[1-\int_0^r b(x)\, dF(x) \right]} + c$$

PROOF: Condition i) states that no store will charge above r. By Theorem B.1, consumers will go from store to store until they find a price offer at or below r. Hence, a firm charging above r will sell its product only if every stores happen to charge strictly above r. In this case, consumers will have visited every stores and will be fully informed about all prices in the market. Using their free recall ability, consumers will be able to purchase their good from the lowest-priced store.

Suppose that F(r)<1 and consider the highest price, p'>r, ever charged in this proposed equilibrium. If stores charge p' with discrete probability, there will be a discrete probability of a tie at p'. Thus firms will have an incentive to charge ε below p' to break the tie and attract more consumers. However if the price distribution is continuous at p', stores pricing p' have a zero probability of selling their product and will make zero-expected profits. Since by assumption F(r)<1, the profits earned by a firm charging r is at least $r[1-F(r)]^{n-1}/n > 0$. Stores will make strictly more profits charging r rather than p'. This contradicts the assumption that p'>r is in the support of F(.).

From lemma B.1, r is such that $r = r\int_r^{\infty} dG(x) + \int_0^r x\, dG(x) + c$. But since firms never charge above r, we have $r = \int_0^r x\, dG(x) + c$, proving condition ii). Q.E.D.

By lemma B.1, each consumer, at any equilibrium, will visit only one store since all observed prices are no greater than r.

LEMMA B.2: *If* (F(.),b(.),π) *is a n-firm symmetric equilibrium associated with the reservation price r, then:*

i) b(p)=0 *for all* p>r-c

ii) F(p)=F(r-c) *for all* r-c ≤p <r

iii) b(p)>0 *for all* p≤r-c

iv) *The support of* F(.) *below* r-c *is connected with* r-c *being in the support.*

v) F(.) *is continuous when* p≤r-c.

PROOF: i): From Theorem B.1, a consumer not observing an ad from store j believes that j offers a price equal on average to r-c. Thus, store j advertising some p_j>r-c, informs consumers that its price is higher than what they would expect on average. When p_j>r-c, advertising is counterproductive as it pushes away consumers rather than attracting them. A consumer will buy from a store advertising some price p_j above r-c, only if she observes ads from all other stores and they all offer prices above p_j. But even in this case, the consumer will buy from store j if j does not advertise. Consequently, at equilibrium, no store will advertise a price above r-c.

The term [1-T(r-c)] denotes the probability that a consumer observes an ad from a given store. ii) states that no store will charge a price between r-c and r. If a store does not advertise, its probability of attracting a given consumer is independent of its price. Since they do not advertise, stores charging above r-c will be better off charging r, the highest acceptable price, rather than some lower price.

iii) A consumer will accept an advertising offer p≤r-c only if she observes no cheaper offer. This means that for a store offering price p≤r-c and advertising with intensity b, a proportion b of the population will purchase its good with probability $[1-T(p)]^{n-1}$. A consumer observing no ad will select randomly among the n stores. Thus, for stores advertising with intensity b, a proportion (1-b) of consumers will purchase their good with probability $[1-T(r-c)]^{n-1}/n$. We have:

$$\text{(B.3)}\quad \Pi(p,b) = p\left\{ b[1-T(p)]^{n-1} + (1-b)\frac{[1-T(r-c)]^{n-1}}{n}\right\} - A(b) \quad \text{for every } p \leq r-c$$

Thus, for some given $p \leq r-c$, the optimal level of advertising intensity satisfies

$$\text{(B.4)}\qquad A'(b(p)) = p\left\{[1-T(p)]^{n-1} - \frac{[1-T(r-c)]^{n-1}}{n}\right\} > 0$$

The convexity of A(.) implies that there is a unique solution to the above problem. Furthermore, since $A'(b(p)) > 0$, we must have $b(p)>0$ (recall that $A'(0)=0$ and $A''>0$).

iv) Since $b(p)>0$ for $p \leq r-c$, we have $T(p_1)=T(p_2)$ if and only if $F(p_1)=F(p_2)$. To prove iv), we must show that for any distinct p_1 and p_2 in the support of F(.), so that $p_1 < p_2 \leq r-c$, we must have $T(p_1) < T(p_2)$. Suppose in the contrary that $T(p_1) < T(p_2)$ for some $p_1 < p_2$. Consider a price p', so that $p_1 < p' < p_2$; p' would succeed in being the lowest price with the same probability as p_1. A deviant store offering p' with positive probability will have the same expected demand as would a firm charging p_1 but at a higher price, thus earning a greater profit. This contradicts the assumption that p_1 is in the support of F(.). Using the same argument for $p_2=r-c$, we can show that r-c must be in the support of F(.).

v) Finally, we must show that there exists no mass point in the pricing strategy at each price $p < r-c$. Suppose that some price p were advertised with positive probability, there would then be a positive probability of a tie. If a deviant store charged a slightly lower price, $p-\varepsilon$, with the same probability with which other stores charged p. It would be able, for some arbitrarily small loss of the magnitude of ε, to break the tie and make some discrete gain through increased demand. Thus for a small ε, the deviant store will increase its profits, contradicting the assumption of equilibrium. Q.E.D.

The basic structure of the mixed pricing strategy can be summarized as follows; for some effective reservation price r, stores will never charge above r. Furthermore, r can be a mass point in the equilibrium price distribution; this result is possible because it does not pay to advertise a price marginally lower than r. Since consumers will remain unaware of any price reduction in the neighborhood of r, stores will never charge a price strictly between r and r-c. [1-F(r-c)] will therefore denote the probability of a store ever charging price r and [1-T(r-c)] can be interpreted as the probability of a consumer observing no ad from a given store.

Stores will begin to advertise whenever they decide to charge below r-c. Because some consumers at least will be informed about the store's price, any price reduction will affect demand. Since it will always pay to decrease marginally the advertised price to break a potential tie, there will be no mass in the price distribution below r-c. Furthermore, everywhere below r-c, the price-distribution function will be a strictly increasing continuous function.

Following the above discussion, the profit functions are given by

$$\text{(B.5)}\qquad \Pi(p,b) = r\left[\frac{[1-T(r-c)]^{n-1}}{n}\right] \qquad \text{if } p=r$$

$$= p\left[b\,[1-T(r-c)]^{n-1} + (1-b)\,\frac{[1-T(r-c)]^{n-1}}{n}\right] - A(b) \qquad \text{if } p\leq r-c$$

We use this to further characterize the n-firm symmetric equilibria.

LEMMA B.3: *If* (F(.),b(.),π) *is a n-firm symmetric equilibrium given some r and such that* $0<F(r-c)<1$, *then it is the unique triple* (F(.),b(.),π) *which satisfies the following conditions*:

(B.6) F(.) *has support defined by* $S \equiv \{p \mid p=r \text{ or } p \in [p^*, r-c]\}$

(B.7) $b(p)=0$ *for all* $p > r-c$

(B.8) $A'(b(p)) = p\left\{[1-T(p)]^{n-1} - \frac{[1-T(r-c)]^{n-1}}{n}\right\}$ *for all* $p \in [p^*, r-c]$

(B.9) $r\,\frac{[1-T(r-c)]^{n-1}}{n} = \pi$; *with* (B.10) $[1-T(r-c)]^{n-1} < 1$ *or* $r > n\pi$.

(B.11) $p\,b(p)\left\{[1-T(p)]^{n-1} + (1-b(p))\,\frac{[1-T(r-c)]^{n-1}}{n}\right\} - A(b(p)) = \pi$ *for all* $p \in [p^*, r-c]$

The lowest price in the support S, denoted by price p, is the lowest price which satisfies eq.*(B.11) *and the highest p so that T(p)=0. We have:*

(B.12) $p^*\left\{b(p^*) + (1-b(p^*))\,\frac{[1-T(r-c)]^{n-1}}{n}\right\} - A(b(p^*)) = \pi$ for all $p \in [p^*, r-c]$

where by definition, $T(p) = \int_0^p b(x)\,dF(x)$

PROOF OF LEMMA B.3: We first show that if $(F(.),b(.),\pi)$ is a n-firm symmetric equilibrium then it must necessarily satisfy conditions (B.6) to (B.12). If $0<F(r-c)<1$, then stores must charge r with discrete probability as well as some prices below r-c; furthermore, by lemma B.2, the support of F(.) below r-c is connected with r-c being in the support. S must necessarily define the support of F(.), where (B.12) defines the minimum price p^* that would ever be charged where $[1-T(p^*)]=1$. Conditions (B.7) and (B.8) define the optimal level of advertising at each price, as required by definition 1. Conditions (B.9) and (B.11) are required by definition, i.e. firms attain the same level of profit when charging any price in the support of F(.). Finally, $T(r-c)=\int_0^{r-c} b(x)\, dF(x)>0$, since $b(p)>0$ for every $p\leq r-c$ and by assumption $F(r-c)>0$. Thus condition (B.10) follows, i.e. $[1-T(r-c)]^{n-1} \leq [1-T(r-c)] <1$.

We show now, by construction, that for every r there exists at most one triple $(F^*(.),b^*(.),\pi^*)$ which satisfies (B.6) to (B.12).

We first show that there exists a unique function $b^{o}(.|r,\pi)$ and a unique function $F^{o}(.|r,\pi)$ for any given pair r and π. By (B.9) we have $[1-T(r-c)]^{n-1}/n = \pi/r$. Substituting the above into (B.8) we obtain

$$\text{(B.13)} \qquad A'(b(p)) = p\ \{[1-T(p)]^{n-1} - \pi/r\},$$

Substituting (B.13) into (B.11) we have $b(p)A'(b(p)) + p\,\pi/r - A(p) = \pi$ or

$$\text{(B.14)} \qquad b(p)A'(b(p)) - A(p) = \pi\,(r-p)/r$$

Equation (B.14) fully characterizes b(p) in terms of A(.), r ,π and p. We let $b^{o}(.|r,\pi)$ denote the function b(.) solving equation (B.14), at every p given some r and π.

$b^{o}(p|r,\pi)$ and $A'(b^{o}(p|r,\pi))$ are continuously differentiable in p, π and r. As shown in section B.6, the optimal level of advertising decreases in price (B.33 in section B.6), and increases with π (B.34 in section B.6) and r (B.32 in section B.6).

Transforming (B.13), we obtain

$$\text{(B.15)} \qquad [1-T(p)]^{n-1} = \left[\frac{A'(b^{o}(p|r,\pi))}{p} + \frac{\pi}{r}\right] \qquad \text{and}$$

$$\text{(B.16)} \qquad [1-T(p)] = \left[\frac{A'(b^{o}(p|r,\pi))}{p} + \frac{\pi}{r}\right]^{1/n-1}$$

Given a pair r and π, there exists, by (B.14), a unique $b^{o}(p|r,\pi)$ for every p. Given r, π and $b^{o}(p|r,\pi)$, there is, from (B.16) a unique T(p) for every p. We let $T^{o}(.|r,\pi)$ denote the function T(.) solving equation (B.16), at every p given some r and π.

From $T^\circ(.|r,\pi))$, we may obtain the corresponding distribution of F(.), by differentiating equation (B.16) on both sides; we obtain for every $p \in [p^*,r\text{-}c]$:

(B.17) $$b^\circ(p|r,\pi)\, f(p) = \frac{1}{n-1}\left[\frac{A'(b^\circ(p|r,\pi))}{p} + \frac{\pi}{r}\right]^{(2-n)/n-1}\left[\frac{A'(b^\circ(p|r,\pi))}{p^2} + \frac{\pi}{pb^\circ(p|r,\pi)r}\right]$$

Note that f(p) is a well-defined, strictly positive and continuously differentiable function. Given every pair r and π, we can construct a unique price distribution F(.). Let $F^\circ(.|r,\pi)$ denote the unique price distribution constructed, given some r and π. Note from (B.17) that $F^\circ(p|r,\pi))$ is continuously differentiable in r and π. Since by assumption $F(r\text{-}c)>0$, we must have $[1-T^\circ(r\text{-}c|r,\pi)]<1$, also note that $A'(b^\circ(p|r,\pi))/p$ goes to infinity as p goes to zero. Thus, since $[1-T^\circ(p|r,\pi)]$ is strictly decreasing in p, there is a unique $p^* \in (0,r\text{-}c)$ such that

(B.18) $$[1-T^\circ(p^*|r,\pi)] = \left[\frac{A'(b^\circ(p^*|r,\pi))}{p^*} + \frac{\pi}{r}\right] = 1.$$

Next, we show that given some r, there is at most one π^* which guarantees that firms are indifferent between charging r and advertising r-c. Let π denote the profit attained by firms charging price r. We have $r\,[1-T(r\text{-}c)]^{n-1}/n = \pi$. Now let us define I suo that

(B.19) $$I = (r-c)\ (n-1)\,\pi/r - A'(b^\circ(r-c|r,\pi)) .$$

Where by (B.14), $b^\circ(r\text{-}c|r,\pi)$ is such that

(B.20) $$b^\circ(r-c|r,\pi)A'(b^\circ(r-c|r,\pi)) - A(b^\circ(r-c|r,\pi)) = \pi c/r$$

Let π^{r-c} denote the profits earned by a firm charging r-c. We have $\pi^{r-c} = (r-c)\,\{\, b^\circ(r-c|r,\pi)\,[1-T(r-c)]^{n-1} + (1-b^\circ(r-c|r,\pi))\,[1-T(r-c)]^{n-1}/n\} - A(b^\circ(r-c|r,\pi))$ where $b^\circ(r\text{-}c|r,\pi)$ is the optimal level of advertising intensity given r and π. Using $n\pi/r = [1-T(r-c)]^{n-1}$, we have

(B.21) $$\pi^{r-c} = (r-c)\, b^\circ(r-c|r,\pi)\,(n-1)\pi/r + (r-c)\,\pi/r - A(b^\circ(r-c|r,\pi))$$

Rewriting (20) we obtain

(B.22) $$b^\circ(r-c|r,\pi)\, A'(b^\circ(r-c|r,\pi)) + (r-c)\pi/r - A(b^\circ(r-c|r,\pi)) = \pi$$

Firms will strictly prefer to charge r-c than r, i.e. $\pi^{r-c}>\pi$, if and only if $I= (r-c)\ (n-1)\,\pi/r - A'(b^\circ(r-c|r,\pi)) >0$. Firms will be indifferent to charging r and/or advertising r-c if and only if $I(r,\pi)=0$; this condition must hold in equilibrium whenever $0<F(r-c)<1$. In lemma B.7 (see section B.6), we show that if $I(r,\pi)=0$ then $\partial I/\partial\pi>0$. Thus for any given r, there is at most one π^* which satisfies $I(r,\pi^*)=0$.

This ends the proof of lemma B.3. For every r, there is at most one π^* so that firms are indifferent to charging r and/or advertising r-c, i.e. $I(r,\pi^*)=0$. Given the pair r and π^*, we construct according to (B.14) and (B.17) a unique pair of function $b^*(p) \equiv b^o(p|r,\pi^*)$ and $F^*(p) \equiv F^o(p|r,\pi^*)$. The triple $(F^*(.),b^*(.), \pi^*)$, if it exists, is the unique triple satisfying the necessary conditions (B.6) to (B.12). Q.E.D.

We are now ready to state the following theorem.

THEOREM B.2: *There is some* $\underline{r} > c$ *and some* $\overline{r} > \underline{r}$ *such that :*

i) For every $r \leq \underline{r}$ *the n-firm symmetric equilibrium is such that all firms charge r with probability one; i.e.* $F(r-c)=0$.

ii) For every $r \geq \overline{r}$, *every n-firm symmetric equilibrium is such that firms always charge below r ; i.e.* $F(r-c)=1$.

iii) For every r, $\underline{r} < r < \overline{r}$, *there exists a unique n-firm symmetric equilibrium* $(F(.),b(.),\pi)$ *associated with the reservation price r and it is such that* $0<F(r-c)<1$.

The intuitive argument of Theorem B.2 is illustrated in figure 1. First, define π' as the profit a store would earn if it charges price r, $\pi' \equiv r\ ([1-T(r-c)]^{n-1}/n)$. If r is charged with positive probability, we have $\pi'=\pi$. In figure 1, we consider various combination of π' and r.

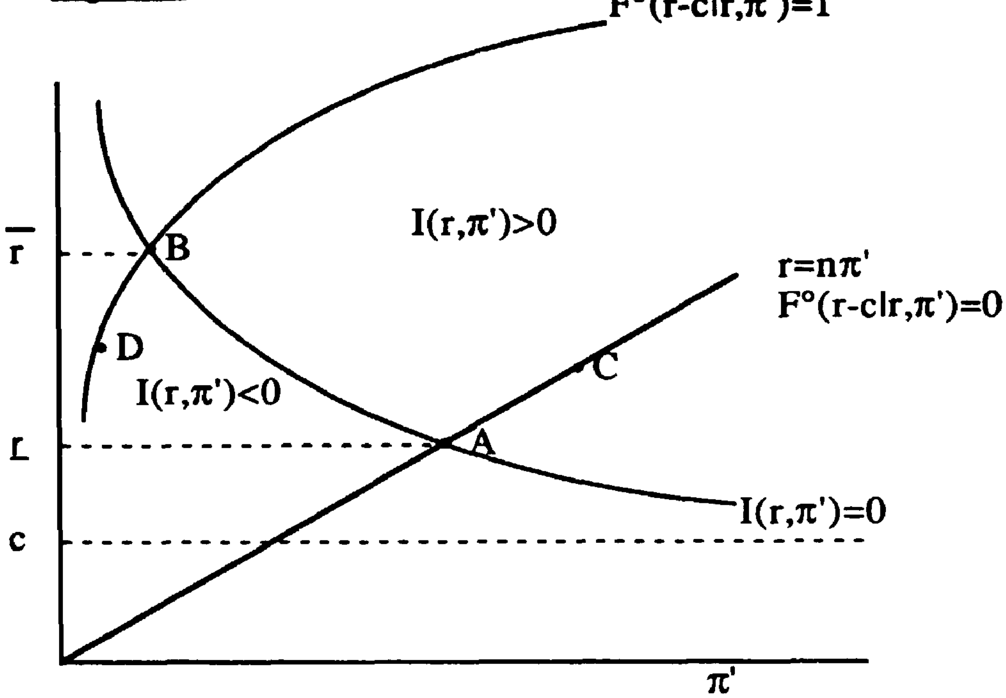
By definition, $[1-T(r-c)]^{n-1}=n\pi'/r$. Thus, the line $r=n\pi'$ represents the relation between r and π' so that consumers will never observe an ad and all firms will charge r with probability 1, i.e $F(r-c)=0$.

The relation $I(r,\pi')=0$ in figure 1, establishes the relation between r and π' so that firms are indifferent to charging r and/or advertising prices lower or equal to r-c. The above equality must hold if at equilibrium $0<F(r-c)<1$. If $F(r-c)=0$, firms charge r with probability one, anf must then prefer to charge r rather than advertising r-c. It follows that if $F(r-c)=0$ then $I\leq 0$. If $F(r-c)=1$, firms always advertise prices lower or equal to r-c, they must prefer to advertise r-c then charging r. It follows that if $F(r-c)=1$ then $I\geq 0$.

Finally, given some pair (r, π') we can construct from (B.17) some price distribution $F^o(.|r,\pi')$. Accordingly, we can construct a relation between r and π' such that

$F^o(r\text{-}c|r,\pi')=1$. We can use the above relationship to prove Theorem B.2. We argue that the unique equilibrium π' given any r is that corresponding to the bold line in figure 1.

Figure 1



CLAIM B.1: There exists a unique pair $(\underline{r}, \underline{\pi})$ (point A in figure 1) such that $\underline{r} = n\,\underline{\pi}$ and $I(\underline{r}, \underline{\pi})=0$. Further, $I(r,\pi)\geq 0$ and $r < \underline{r}$ imply $r<n\pi$; also, $r = n\pi$ and $r > \underline{r}$ imply $I(r,\pi)>0$.

PROOF OF CLAIM B.1: Consider any pair (r,π) such that $r=n\pi$. $b^o(r\text{-}c|r,\pi)$ is such that $b^o(r\text{-}c|r,\pi)\, A'(b^o(r\text{-}c|r,\pi)) - A(b^o(r\text{-}c|r,\pi)) = \pi\, c/r = c/n$. $b^o(r\text{-}c|r,\pi)$ is unique, it depends only on c and n and is independent of r and π. Let β denote the above advertising intensity, so that $\beta A'(\beta)-A(\beta)=c/n$. If $I(\underline{r}, \underline{\pi})=0$ and $\underline{r}=n\underline{\pi}$, we must have $(\underline{r}\text{-}c)\,(n\text{-}1)(\underline{\pi}/\underline{r})= (\underline{r} - c)\,(n\text{-}1)/n = A'(\beta)$. There is a unique $\underline{r}$ which satisfies the above condition. Note that $\underline{r} > c$. The latter part of the proof follows immediately from the facts that $I_r>0$ (FACT 6 in section B.6.1) and $I_\pi>0$ whenever $I(r,\pi)=0$ (FACT 5 in section B.6.1). If $I(r,\pi)\geq 0$ and $r < \underline{r}$ then $\pi > \underline{\pi}$, it follows that $r<n\pi$. Similarly, if $r=n\pi$ and $r > \underline{r}$ then $\pi > \underline{\pi}$, it follows that $I(r,\pi) >0$.

We can use claim B.1 to show that if $(F(.),b(.),\pi)$ is a n-firm symmetric equilibrium, given the effective reservation price r, then $F(r-c)>0$ if and only if $r>\underline{r}$. Suppose that $r \leq \underline{r}$ and that $F(r-c)>0$. It follows that firms must weakly prefer to charge r-c rather than charging r, we must have therefore $I(r,\pi)\geq 0$. But, by claim B.1, this implies that $r<n\pi$, contradicting condition (11). Now, suppose that $r > \underline{r}$ and that $F(r-c)=0$; i.e. firms never charge below r and $r=n\pi$. Then by claim 1 we must have $I(r,\pi)>0$ (point C in figure 1). This implies that firms will strictly prefer to charge r-c rather than charging r, contradicting the assumption that $F(r-c)=0$. This establishes part i) of Theorem B.2.

CLAIM B.2: For every $r>\underline{r}$ there always exists a unique π' solving $I(r,\pi')=0$. Also, let $F^{\circ}(.|r,\pi')$ denote the cumulative price distribution function constructed from (B.17) given some pair r and π'. Then there exists a unique pair $(\bar{r}, \bar{\pi})$ (point B in figure 1) such that $F^{\circ}(\bar{r}-c \mid \bar{r}, \bar{\pi})=1$ and $I(\bar{r}, \bar{\pi})=0$. Further, $I(r,\pi)\leq 0$ and $r \geq \bar{r}$ imply that $F^{\circ}(r-c|r,\pi)\geq 1$; and, $F^{\circ}(r-c|r,\pi)=1$ and $r < \bar{r}$ imply that $I(r,\pi)<0$. (See proof in section B.6).

We can use claim B.2 to show that if $(F(.),b(.),\pi)$ is a n-firm equilibrium given the effective reservation price r, $F(r-c)<1$ if and only if $r<\bar{r}$. Suppose that $r \geq \bar{r}$ and that $F(r-c)<1$. Let π' denote the profit earned by firms charging r consistent with $F(r-c)=F^{\circ}(r-c|r,\pi')$. Since $F(r-c)<1$, firms must either prefer to charge r at worst be indifferent to charging r and/or r-c, we must have $I(r,\pi')\leq 0$. But, by claim B.2, $I(r,\pi')\leq 0$ and $r < \bar{r}$ implies $F^{\circ}(r-c|r,\pi')=F(r-c)\geq 1$, contradicting the initial assumption that $F(r-c)<1$. Now, suppose that $r<\bar{r}$ and that $F^{\circ}(r-c|r,\pi')=F(r-c)=1$; i.e. firms never charge r. By claim B.2, we must have $I(r,\pi')<0$ (point D in figure 1). This implies that firms will strictly prefer to charge r rather than charging r-c contradicting the assumption that $F(r-c)=1$. This establishes part ii) of Theorem B.2.

It follows from the above that if $(F(.),b(.),\pi))$ is a n-firm equilibrium given some $\underline{r} < r < \bar{r}$ then we must have $0<F(r-c)<1$. By lemma B.3 , there is at most one such equilibrium. To prove part iii) of Theorem B.2, it remains to show that a triple $(F^*(.),b^*(.),\pi^*)$ satisfying equations (B.6) to (B.12) always exists and is a n-firm equilibrium for the effective reservation price r. Existence of the triple $(F^*(.),b^*(.),\pi^*)$ can be proven by construction. Claim B.2 establishes that for every $r > \underline{r}$ there always exists some $\pi^*(r)$ solving $I(r,\pi^*(r))=0$. Using π^*, we let $F^*(.|r)=F^o(.|r,\pi^*(r))$ and $b^*(.|r)=b^o(.|r,\pi^*(r))$. As argued in lemma B.3 the function $F^*(.|r)$ and $b^*(|r)$ so constructed satisfy equations (B.6) to (B.12).

Now suppose that firms play the mixed strategy $(F^*(.|r); b^*(.|r);\pi^*(r))$ defined by (B.6) to (B.12). No store will wish to deviate and charge some price p, $r-c<p<r$, since it will make less profit than charging r. No store will charge $p < p^*$; since p will succeed in being the lowest price with the same probability as p^*, it follows therefore that the store will earn less than by charging p^*. (B.7) and (B.8) imply that b(p) is the optimal advertising intensity at price p. By conditions (B.9) and (B.11), stores will be indifferent to charging all prices in the support $S \equiv \{[p^*,r-c]; r\}$. Firms have no incentive to deviate from the proposed strategy. Finally, following claim 2 for every r smaller than $\bar{r}$, the constructed price distribution is, as assumed, such that $F^*(r-c|r)<1$. This ends the proof of part iii) and Theorem B.2. Q.E.D.

According to Theorem B.2, if r is low enough, smaller than some $\underline{r}$, no store has the incentive to offer and advertise r-c even when all other stores charge r with probability 1. If r is high enough, higher than some $\bar{r}$, stores will always want to advertise and offer the price r-c in order to increase their clientele even when all other stores advertise with probability one. For every $r \in (\underline{r}, \bar{r})$, the n-firm symmetric equilibrium involves price dispersion and discrete probability of charging price r. The n-firm symmetric equilibrium in this case is characterized by conditions (B.6) to (B.12).

B.3.3 *Market equilibrium*

We have shown that for every consumers' effective reservation price, r, $\underline{r} < r < \bar{r}$, there exists a unique symmetric n-firm equilibrium strategy such that $0<F(r-c)<1$. Furthermore, for a given price distribution and firm advertising strategy and a search cost $c>0$, there exists a unique effective reservation r which summarizes consumers' behavior. We must now show that there exists some mutually consistent strategies for both firms and consumers. We are looking for a market equilibrium where firms maximize their expected profits given their (correct) beliefs about consumer's behavior, and consumers minimize their expected cost of purchasing one unit of good given their (correct) beliefs about the distribution of prices and advertising in the market.

DEFINITION : For any given number n of firms, any convex advertising cost and any cost of search $c>0$, a *n-firm symmetric market equilibrium* is a quadruple $<(F(.),b(.),\pi); r>$ where (a) $(F(.),b(.),\pi)$ is a n-firm symmetric equilibrium given the effective reservation price r and (b) r is the effective reservation price given F(.) and b(.).

Theorem B.2 established that there is a unique n-firm symmetric equilibrium with a dispersed price equilibrium for every effective reservation price r, $\underline{r} < r < \overline{r}$. Let $(F^*(.|r); b^*(.|r); \pi^*(r))$ denote the unique firm equilibrium for any $\underline{r} < r < \overline{r}$. If a consumer is faced with the distribution function $F^*(.|r) \equiv F^{\circ}(x|r,\pi^*(r))$ and the advertising intensity $b^*(.|r) \equiv b^{\circ}(x|r,\pi^*(r))$, let $R^*(r)$ indicate the reservation price used by the consumer for a fixed $c>0$. Using lemma B.1(ii), we have:

$$(B.23) \quad R^*(r) \equiv R^{\circ}(r,\pi^*(r)) \equiv \left[\frac{\int_0^r x(1-b^{\circ}(x|r,\pi^*(r)))\,dF^{\circ}(x|r,\pi^*(r))}{1-\int_0^r b^{\circ}(x|r,\pi^*(r))\,dF^{\circ}(x|r,\pi^*(r))}\right] + c$$

$$(B.24) \quad R^*(r) = r + \left[\frac{\int_0^{r-c} x(1-b^{\circ}(x|r,\pi^*(r)))\,dF^{\circ}(x|r,\pi^*(r))}{1-T^{\circ}(r-c|r,\pi^*(r))}\right] - r\,\frac{[F^{\circ}(r-c|r,\pi^*(r))-T^{\circ}(r-c|r,\pi^*(r))]}{[1-T^{\circ}(r-c|r,\pi^*(r))]} + c$$

At equilibrium, we must have $R^*(r)=r$.

To verify the existence of an equilibrium, two extreme positions are considered. Suppose that $r \leq \underline{r}$, where $F^*(r-c|r)=0$. Then all firms will charge price r with probability one and the consumer's effective reservation price $R^*(r) = r+c > r$. Next consider the case where $r \geq \overline{r}$, we have $F(r-c)=1$. From (B.24) we can show that :

$$(B.25) \quad R^*(r) < r + (r-c)\frac{[F(r-c)-T(r-c)]}{[1-T(r-c)]} - r\frac{[F(r-c)-T(r-c)]}{[1-T(r-c)]} + c = r .$$

Finally, note that $f^*(.|r)$, $b^*(.|r)$ are continuously differentiable in r, so that $R^*(r)$ is clearly continuous. It follows that there exists at least one market equilibrium for some r, $\underline{r} < r < \overline{r}$.

The following claim establishes the uniqueness of the n-firm symmetric market equilibrium.

CLAIM B.3: If $R^*(r)=r$ for some $r \in (\underline{r}, \bar{r})$, then $R^{*\prime}(r)<1$.

See proof in section B.6.2. We may now state the following theorem.

THEOREM B.3: *For some fixed number of firms, n, convex advertising cost, A(.), and search cost, c>0; there exists a unique n-firm market equilibrium, where $0<F(r-c)<1$. The market equilibrium involves price dispersion with discrete mass point at $p=r$.*

PROOF: Existence follows immediately from the previous discussion and uniqueness follows from Claim B.3. Further, we can show that $0<F(r-c|r)<1$, by appealing to the fact that $R(r)=r+c > r$ when $F(r-c|r)=0$; and $R(r) < r$, when $F(r-c|r)=1$. Recall that $[1-F(r-c)]$ denotes the probability that firms charge r; so that r must be charged with strict positive probability.

B.3.4 *On the existence of asymmetric equilibria.*

So far, we have only considered symmetric equilibria where all firms choose the same pricing and advertising strategies. We show, here, that the symmetry assumption is imposed without loss of generality, i.e. that all equilibria must be symmetric.

LEMMA B.4: *Consumers observing no ads must be indifferent, at equilibrium, between going to any of the n stores.*

Let $r_i=\int_0^v p\,(1-b_i(p))\,dF_i(p)/[1-T_i(v)]$, *be the expected price charged by store i conditional on not observing an ad from i. Then* $r_i=r_j$, *for every* $i\neq j \in [1,...,n]$.

PROOF: By contradiction. Let $r_i \equiv \min(r_1,...,r_n) < r_j \equiv \max(r_1,...,r_n)$. All consumers will prefer to go to store i (or some other store k such that $r_k=r_i$) unless she observes an ad offering a price lower or equal to r_i. Thus store j, in order to sell its good, must always advertise and charge a price at least equal to r_i. But this contradicts the assumption that $r_j \equiv \int_0^v p\,(1-b_j(p))\,dF_j(p)/[1-T_j(v)] > r_i$. Q.E.D.

LEMMA B.5: At equilibrium, all firms will earn the same post-entry profits.

PROOF: According to lemma B.4, consumers will apply the same effective reservation price rule to every firm. Following lemma B.1, no store will charge above this common effective reservation price r, nor will it advertise any price above r-c.

Let $\prod_{i\neq j}[1-T_i(r-c)]$ be the probability that a consumer observes no ad from any store $i\neq j$. If store j charges r, its profits are given by :

$\pi_j \equiv r\{\prod_{i\neq j}[1-T_i(r-c)]\}/n$. A basic equilibrium condition requires that firms be indifferent between charging r and advertising r-c. That is:

$r[\prod_{i\neq j}[(1-T_i(r-c)]/n)] = \max^b\{(v-c)[b\prod_{i\neq j}[1-T_i(r-c)]+(1-b)(\prod_{i\neq j}[1-T_i(r-c)]/n)]-A(b)\} = \pi_j$ Substituting $\prod_{i\neq j}[1-T_i(r-c)]$ for $n\pi_j/r$, and using the first order condition on the optimal advertising intensity, we obtain, much like for eq.(B.19) and (B.20), the following equilibrium condition

(B.26) $\quad I = (r-c)\ (n-1)\ \pi_j/r - A'(b_j(r-c)) = 0$

Where by (16), $b_j(r-c)$ is such that:

(B.27) $\quad b_j(r-c)A'(b_j(r-c)) - A(b_j(r-c)) = \pi_j c/r$

Following the proof of lemma B.3, for any given r and function A(.), there exists a unique π_j, which solves equations (B.26) and (B.27). Thus for every firm j and i, we must have $\pi_j=\pi_i$ Q.E.D.

Having shown lemma B.4 and B.5, it is now easy to show that firms must use the same advertising and pricing strategies. We have, for every j:

$\{\ p[b_j(p)\prod_{i\neq j}[1-T_i(p)] + (1-b_j(p))\ (\pi/r)] - A(b_j(p))\ \} = \pi$

where b is such that $p[\prod_{i\neq j}[1-T_i(p)] - (\pi/r)] = A'(b_j(p))$.

By transformation, we obtain, for every j and k:

(B.28) $\quad b_j(p)A'(b_j(p)) - A(b_j(p)) = \pi(r-p)/r$

(B.29) $\quad \prod_{i\neq j}[1-T_i(p)] = [(A'(b_j(p))/p) + (\pi/r)]$

It follows that $b_i(p)=b_j(p)$ and $[1-T_i(p)]=[1-T_j(p)]$ for every p and every $i\neq j$ $\epsilon[1,...,n]$. This establishes the following theorem:

THEOREM B.4: Every n-firm market equilibrium must be symmetric. Let

$<(F_i(.),b_i(.),\pi_i);\ i\in[1,...,n]>$ be some n-firm equilibrium consistent with some consumer's optimal search strategy then for every $i\neq j\in[1,...,n]$, we must have $F_i(.)=F_j(.)$; $b_i(.)=b_j(.)$ and $\pi_i=\pi_j$.

B.3.5 *Cases where r=v.*

In section B.3.3, we have shown the existence of a finite endogenous reservation price r. Note that the reservation price r, such that $R^*(r)=r$, is independent of v so long as v>r. We have restricted our attention so far to cases where the endogenous effective reservation price r was below the consumer's valuation for the good. This need not, in general, be the case. We consider here more specifically the case where v is binding, i.e. $R^*(r)=r$ only if r>v and $R^*(v)>v$.

Since no consumer will pay more than v, no store will offer a price above v. Besides, since a consumer must pay c in order to go to a store, no consumer will go to a store which charges surely above v-c. It follows that no store will advertise a price above v-c.

At equilibrium, consumers observing no ads must be indifferent to entering and/or not entering the market; and they will mix between the two alternatives. Suppose that consumers observing no ads never enter the market; then only stores advertising a price below v-c will make any profits. The stores will therefore never charge above v-c. But this will induce consumers to enter the market. Suppose that consumers observing no ads enter the market with probability one; the optimal firm strategy will then be given as in section B.3.2; but since by assumption $R^*(v)$: consumers observing no ads will not wish to enter the market due to the expected cost for a good exceeding v.

Let α be the probability that a consumer observing no ads enters the market. We have:

(B.30) $$v\left[\frac{\alpha[1-T(v-c)]^{n-1}}{n}\right]=\pi \qquad \text{or} \qquad \frac{\pi}{v}=\left[\frac{\alpha[1-T(v-c)]^{n-1}}{n}\right]$$

Firms will be indifferent between charging v and advertising v-c if and only if:

$$v\left[\alpha\frac{[1-T(r-c)]^{n-1}}{n}\right]=\max{}^{b}\,(v-c)\left[b[1-T(p)]^{n-1}+(1-b)\,\alpha\frac{[1-T(v-c)]^{n-1}}{n}\right]-A(b)$$

or (B.31) $$I(v,\pi,\alpha)=(v-c)\,((n/\alpha)-1)\,\pi/r-A'(b(v-c))=0.$$

When v is binding, we can construct the equilibrium in the following way. First, find $F^{\circ}(.|v,\pi)$ and $b^{\circ}(.|v,\pi)$, namely the price distribution and advertising strategy construct from v and π. Then, find π such that

$$R^{\circ}(v,\pi)=\frac{\left[\int_0^r x\,(1-b^{\circ}(x|v,\pi))\,dF^{\circ}(x|v,\pi)\right]}{\left[1-\int_0^r b^{\circ}(x|v,\pi)\,dF^{\circ}(x|v,\pi)\right]}+c=v.$$

Finally, find α in order to satisfy $I(v,\pi,\alpha)=0$. (see figure 2)

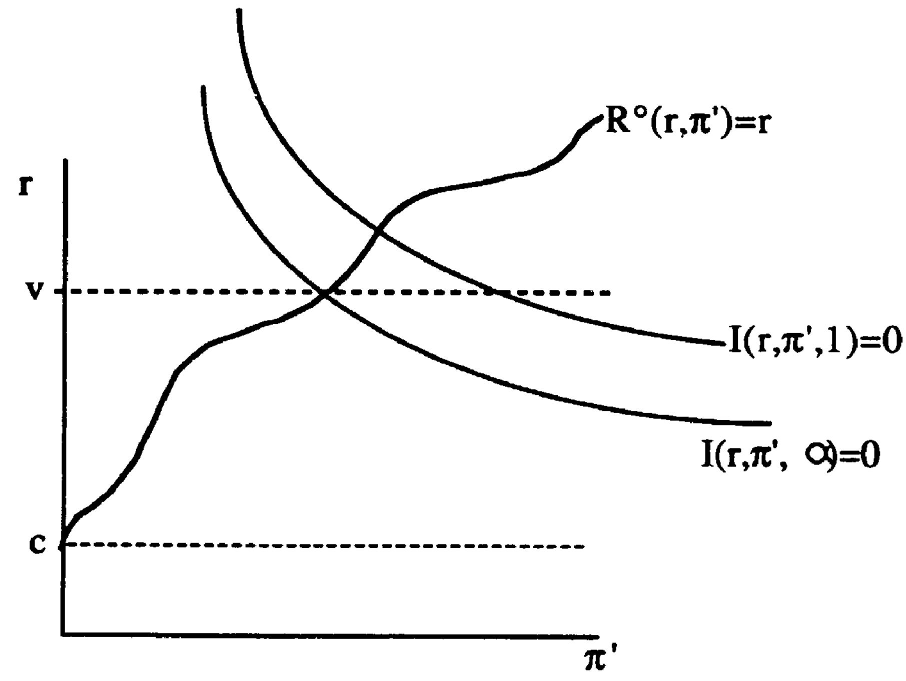


FIGURE 2

B.3.6 *Entry decision and monopolistic competition:*

We consider now the entry decision of firms into the retail industry. Firms will enter the market until the expected profits accruing to a firm exceed the fixed entry cost k. Thus we can endogeneize the number of firms into the industry.

LEMMA B.6: *Let* $\langle F(.)^n, b(.)^n, \pi^n \rangle$ *and* $\langle F(.)^{n+1}, b(.)^{n+1}, \pi^{n+1} \rangle$ *denote respectively the n- and n+1-firm market equilibrium; then for some fixed entry cost k, the integer n is* an equilibrium number of firm *in the retailing industry if and only if* $\pi^n \geq k > \pi^{n+1}$.

PROOF: Necessity. Suppose that $k \leq \pi^{n+1}$, it will be in the interest of a $n+1^{th}$ to enter in the retailing industry as its post-entry expected profit will be sufficient to compensate for the entry cost k. Now suppose that $k < \pi^n$, the post-entry profit will not compensate for the entry cost and at least one firm will prefer not to enter rather than lose money.

Sufficiency. Suppose that $\pi^n \geq k > \pi^{n+1}$. n can be supported as an equilibrium number of firm. Indeed, no firm will wish to withdraw from the market since $\pi^n \geq k$, and no firm will wish to enter in the industry since $k > \pi^{n+1}$. Q.E.D.

THEOREM B.5: *There exists a unique equilibrium number of firms.*

Two claims are stated to prove Theorem B.5.

CLAIM B.4: Let $<F(.)^n, b(.)^n, \pi^n>$ and $<F(.)^{n+1}, b(.)^{n+1}, \pi^{n+1}>$ denote respectively the n- and n+1-firm market equilibrium; then $\pi^n > \pi^{n+1}$. (See the full proof in section B.6.2). Claim B.4 establishes uniqueness. Suppose that n and m>n were both an equilibrium number of firms, then by lemma B.4: $\pi^m \geq k > \pi^{n+1}$, which contradicts claim B.4.

CLAIM B.5: *As n increases, π^n converges to zero.* (See proof in section B.6.2).

Claim B.5 establishes existence. It states that for every k>0, there exists at least some n such that $k > \pi^n$. Q.E.D.

B.4. ADVERTISING AND COMPETITION.

Our model offers the opportunity to study the interaction between informative advertising and competition. Advertising expenditure and the number of firms in the industr· are both determined endogenously in the model. We consider here how exogenous changes in the entry cost and advertising technology affect both the firms' and the consumers' behaviors.

In section B.4.1, we consider the effect of greater competition, i.e. lower sunk cost of entry k, on the retail market outcome. Will a greater number of stores induce more advertising or less? What will be the effect of competition on the consumer's welfare? on prices? on the information available to consumers? In section B.4.2, we consider the the way in which the costs of advertising and search will affect the degree of competition among firms in the industry.

B.4.1 *The effects of greater competition:*

We consider here how the retailing market is affected by stronger competition among stores; i.e. when k decreases or when n increases. To sign these effects, we estimate what happens as the number of stores, n, becomes arbitrarily large. Using a limit argument, we are able to establish, for some large enough n, the effect of greater competition (larger n or smaller k) on most endogenous variables of the model.

THEOREM B.6: *As the number of stores becomes arbritarily large (or as fixed sunk cost goes to zero), we have:*

i) $n\pi$, *the total net profits, become arbitrarily small.*

ii) $b(r-c)$ *and* $b(p^*)$, *the advertising intensity, converges to* 0.

ii) p^* *converges to* 0.

iii) $n\int_{p^*}^{r-c} A(p)\, dF(p)$, *the total advertising expenditures, converge to* 0

iv) $E(p_{min})$, *the expected price paid by consumers, converges to* 0

vi) $T(r-c)$, *the probability that a consumer observes an ad from a given store, converges to* 0

vii) $[1-T(r-c)]^n$,*the probability that a consumer observes no ad, converges to* 0.

viii) *Finally, the expected number of ads observed by each consumer becomes arbitrarily large.* (see proof in section B.6.2)

Theorem B.6 lists some of the effects of competition (lower entry cost or larger number of stores) on the consumer's welfare and on the endogenous consumer's information about prices. The best way to understand the effect of k on the retail market in our model is by analogy to models with differentiated products. When consumers are differentiated according to taste and when firms are allowed to choose between different products or locations, firms may act as small monopolies against consumers in their respective neighborhoods. Firms will be able to generate enough post-entry profits to compensate for the sunk entry cost. In our model, consumers are ex-ante indifferent as to the various stores. They may, however, have different preferences after observing the ads. One consumer may prefer to go to store j, while another may prefer to go to store i. This situation can arise when both consumers have different information about the prices charged in every store. We may say that stores are *informationally differentiated.*

In a differentiated location model, as more firms enter the market, the proportion of consumers served by each firm decreases as do the travelling expenditures and as the price paid by consumers converges to marginal cost. The result is quite similar in our model. The total advertising expenditures decrease while each store advertises less intensively. But as the number of stores increases, the probability that a consumer observes at least one ad (or any finite number of ads) increases[27]. Stiffer competition increases both the welfare and information of consumers. As k goes to zero, the equilibrium converges to the full information perfect competition, where each consumer buys her good at marginal cost.

B.4.2. *The role of information costs:*

In a simple location model, the geographic differentiation becomes meaningless if consumers can travel cost-free to buy from the lowest priced store, or if stores can deliver their good at no cost. In our model, the effects of information costs are similarly important. We consider here how the equilibrium is affected as the advertising costs and the search cost c become small.

To do so we introduce a shift parameter, "a", into the advertising cost function such that aA(b) is the cost of reaching proportion b of the population. As "a" decreases, the properties of the advertising cost function remain unchanged but it becomes globally

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27 In models where the proportion of informed customers is set exogenously, the equilibrium number of firms decreases as the proportion of informed individuals increases . More informed individuals increase the incentive to reduce prices, which leads to stronger competition between firms and lower expost profits. Our model shows that the number of firms need not vary negatively with the proportion of informed consumers. The endogenous proportion of informed consumers can increase with the number of firms.

cheaper to advertise.

THEOREM B.7: *For a given number of stores, the equilibrium level of profit* π *goes to zero if*

i) *the search cost, c, goes to zero.*

ii) *the advertising shift parameter,* "a", *goes to zero.* (See proof in section B.6.2)

In our model, if consumers can search for the lowest price at no cost, or if firms can advertise without charge their price, informational differentiation will disappear. For any number of stores, the outcome will converge to perfect Bertrand competition. Search and advertising costs have the equivalent implication travel costs do in a location model, preventing full price competition and allowing firms to extract post-entry rents.

B.5. CONCLUSION

This essay attempts to introduce advertising into a random sequential search model. Information about prices is both gathered by consumers and disseminated by producers. In simple search models, firms offering low prices have the incentive to inform customers about their price, it is therefore natural to presume that they will attempt to advertise prices. Advertising produces endogenously the ex-post information heterogeneity necessary to generate "nondegenerated" price dispersion. We have been able to show that the unique equilibrium exhibits price dispersion. Moreover, the model generates predictions which seem consistent with basic casual observations of the retail market.

At equilibrium, a firm's pricing and advertising strategy can be summarized as follows. The model allows for price dispersion as firms will randomize between charging a high unadvertised price and a low advertised one. Stores charge an endogenous maximum price r with strict positive probability; where r can be interpreted as the list price. Often, stores will offer and advertise sales. The price reduction must be significant, i.e. the rebate must be greater or equal to c. Furthermore, the greater is the price reduction, the greater the advertising intensity and the proportion of the population informed of the sale[28]. In markets with relatively high entry cost and a small number of stores, the total advertising expenditures will be relatively greater than in markets with lower entry cost and a greater number of stores; as well the expected price paid by consumers will be lower in a more competitive market. We believe the model is relevant to understand the retail market.

This essay offers a broader contribution to the field of information economics. Ever since, Stigler(1961)'s seminal paper on "The Economics of Information", economists have been interested in the process by which consumers acquire information on prices. Search models have been developed to help understand how consumers gather their price information. The main conclusion in following this path of research is that the ability to search is not, in general, sufficient to generate price dispersion nor actual searching behavior[29]. The market outcome is much different when we allow the producers (or the owners of the information) to diffuse the information on prices. Our model illustrates that the diffusion of information by the producers, i.e. price advertising, can be as much important (if not more important) than search to understand how consumers acquire market information .

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28 Butters (1977) shows that firms will advertise more when they charge higher prices. This result is contradicted by our model. The difference between his result and ours is that consumers in Butters' model go only to stores from which they observe an ad. High pricing stores spend a lot on advertising in order to attract consumers who observe no other ads.

29 Heterogeneity among producers or consumers is often necessary to induce price dispersion in search models. One exemption is in Burdett and Judd(1983); who obtain price dispersion under certain conditions with homogenous consumers and producers in a non-sequential search framework.

B.6 APPENDICES

B.6.1. *Facts*

Before proving the remaining claims and theorem, let us begin by stating some basic facts. First consider the equation $b^\circ(.|r,\pi)$, where $b^\circ(p|r,\pi)$ solves

(B.14) $\quad b^\circ(p|r,\pi)A'(b^\circ(p|r,\pi)) - A(b^\circ(p|r,\pi)) = \pi(r-p)/r$

The above equation fully characterizes $b^\circ(p|r,\pi)$ in terms of r, π and p for some A(.). We have

(B.32-34) $\dfrac{\partial b^\circ(p|r,\pi)}{\partial r} = \dfrac{\pi p}{r^2 bA''(b)} > 0$; $\dfrac{\partial b^\circ(p|r,\pi)}{\partial p} = \dfrac{-\pi}{r\, bA''(b)} < 0$; $\dfrac{\partial b^\circ(p|r,\pi))}{\partial \pi} = \dfrac{(r-p)}{r\, bA''(b)} > 0$

(B.35-37) $\dfrac{\partial A'(b^\circ(p|r,\pi))}{\partial r} = \dfrac{\pi p}{r^2 b} > 0$; $\dfrac{\partial A'(b^\circ(p|r,\pi))}{\partial p} = \dfrac{-\pi}{r\, b} < 0$; $\dfrac{\partial A'(b^\circ(p|r,\pi))}{\partial \pi} = \dfrac{(r-p)}{r\, b} > 0$

Note, in particular, that:

(B.38) $\quad \pi\dfrac{\partial b^\circ(p|r,\pi)}{\partial \pi} + (r-p)\dfrac{\partial b^\circ(p|r,\pi)}{\partial p} = 0$ and (B.39) $\dfrac{\partial b^\circ(p|r,\pi)}{\partial r} + \dfrac{p}{r}\dfrac{\partial b^\circ(p|r,\pi)}{\partial p} = 0$

LEMMA B.7: Let define $E^\circ(r,\pi) = c\,[1-F^\circ(r-c|r,\pi)] - \int_p^{r-c}[F^\circ(p|r,\pi)-T^\circ(p|r,\pi)]\,dp$. And let $I(r,\pi) = (r-c)(n-1)\,\pi/r - A'(b^\circ(r-c|r,\pi))$, where $b^\circ(r-c|r,\pi)$ is defined by equation (B.14). Then the following facts are true:

(FACT 1) $\partial F^\circ(r-c|r,\pi)/\partial\pi < 0$.

(FACT 2) $\partial F^\circ(r-c|r,\pi)/\partial r > 0$.

(FACT 3) $\partial E^\circ(r,\pi)/\partial\pi < 0$.

(FACT 4) If $E^\circ(r,\pi)=0$, then $\partial E^\circ(r,\pi)/\partial r > 0$.

(FACT 5) If $I(r,\pi)=0$, then $\partial I(r,\pi)/\partial\pi > 0$.

(FACT 6) $\partial I(r,\pi)/\partial r > 0$.

PROOF OF LEMMA B.7: For notational purpose, we let $b(p)=b^\circ(p|r,\pi)$, $T(p)=T^\circ(p|r,\pi)$ and $f(p)=f^\circ(p|r,\pi)$. The functions b(.), f(.) and T(.) are those defined respectively by equations (B.14), (B.16) and (B.17) given some well-defined pair r and π.

From (B.16) we have $[1-T(p)]^{n-1} = [A'(b(p))/p + \pi/r]$. We have therefore:

(B.40) $$\frac{\partial T(p)}{\partial \pi} = -\frac{[1-T(p)]^{2-n}}{n-1}\left[\frac{(r-p)}{rbp}+\frac{1}{r}\right] = \int_{p^*}^{p}\frac{\partial b(x)f(x)}{\partial \pi}dx - b(p^*)f(p^*)\frac{dp^*}{d\pi} < 0$$

Using integration by parts, we can show that:

(B.41) $$-\frac{(r-p)}{\pi}b(p)f(p) + \int_{p^*}^{p}\frac{(r-x)}{\pi}\frac{\partial b(x))f(x)}{\partial x} - \frac{b(x))f(x)}{\pi}\,dx + \frac{(r-p^*)}{\pi}b(p^*)f(p^*) = 0$$

Substituting the above into (B.40), we obtain: (B.42) $\partial T(P)/\partial \pi$ =

$$-\frac{(r-p)}{\pi}b(p)f(p)+\int_{p^*}^{p}\frac{\partial b(x))f(x)}{\partial \pi}+\frac{(r-x)}{\pi}\frac{\partial b(x))f(x)}{\partial x}-\frac{b(x))f(x)}{\pi}\,dx - b(p^*)f(p^*)\left[\frac{dp^*}{d\pi}-\frac{(r-p^*)}{\pi}\right]<0$$

Similarly, we have:

(B.43) $$\left[\frac{\partial F(p)}{\partial \pi}\right] = -\frac{(r-p)}{\pi}f(p) + \int_{p^*}^{p}\left[\frac{\partial f(x)}{\partial \pi}+\frac{(r-x)}{\pi}\frac{\partial f(x)}{\partial x}-\frac{f(x)}{\pi}\right]dx - f(p^*)\left[\frac{dp^*}{d\pi}-\frac{(r-p^*)}{\pi}\right]$$

Now recall from (B.36) that $\partial b(x)/\partial \pi + (r-x/\pi)\,\partial b(x)/\partial x = 0$, we have:

$$\left[\frac{\partial b(x)f(x)}{\partial \pi}+\frac{(r-x)}{\pi}\frac{\partial b(x)f(x)}{\partial x}-\frac{b(x)f(x)}{\pi}\right] = b(x)\left[\frac{\partial f(x)}{\partial \pi}+\frac{(r-x)}{\pi}\frac{\partial f(x)}{\partial x}-\frac{f(x)}{\pi}\right].$$

(B.44) Thus $b(p^*)\left[\frac{\partial F(p)}{\partial \pi}\right] - \left[\frac{\partial T(p)}{\partial \pi}\right]$ =

$$-\frac{(r-p)}{\pi}f(p)\,[b(p^*)-b(p)] + \left[\int_{p^*}^{p}[b(p^*)-b(p)]\left[\frac{\partial f(x)}{\partial \pi}+\frac{(r-x)}{\pi}\frac{\partial f(x)}{\partial x}-\frac{f(x)}{\pi}\right]dx\right. .$$

To show that $b^*(p)\,\partial F(p)/\partial \pi < \partial T(p)/\partial \pi < 0$ for every p. It is sufficient to show that for all p: $\left[\frac{\partial b(p)f(p)}{\partial \pi}+\frac{(r-p)}{\pi}\frac{\partial b(p)f(p)}{\partial p}-\frac{b(p)f(p)}{\pi}\right] < 0$.

(B.45) $$(r-p)\frac{\partial T(p)}{\partial p}+\pi\frac{\partial T(p)}{\partial \pi} = \frac{[1-T(p)]^{2-n}}{n-1}\left\{\left[\frac{A'}{p^2}+\frac{\pi}{rpb}\right](r-p) - \left[\frac{\pi(r-p)}{rpb}+\frac{\pi}{r}\right]\right\}$$

$$= \frac{[1-T(p)]^{2-n}}{n-1}\left\{\left[\frac{A'(r-p)}{p^2}-\frac{\pi}{r}\right]\right\} = \frac{[1-T(p)]^{2-n}}{n-1}\left\{\left[\frac{(r-p)[1-T(p)]^{n-1}}{p}-\frac{\pi}{p}\right]\right\}$$

(B.46) $$\frac{\partial}{\partial p}\left\{(r-p)\frac{\partial T(p)}{\partial p}+\pi\frac{\partial T(p)}{\partial \pi}\right\} = \left[\pi\frac{\partial b(p)f(p)}{\partial \pi}+(r-p)\frac{\partial b(p)f(p)}{\partial p}-b(p)f(p)\right]$$

$$= -\frac{r\,[1-T(p)]}{p^2\,(n-1)} + \frac{\pi\,[1-T(p)]^{2-n}}{p^2\,(n-1)} - \frac{(r-p)\,b(p)f(p)}{p\,(n-1)} - \frac{(n-2)\pi[1-T(p)]^{n-1}b(p)f(p)}{(n-1)p}$$

$$< -\frac{r\,[1-T(p)]}{p^2\,(n-1)} + \frac{\pi\,[1-T(p)]^{2-n}}{p^2\,(n-1)} < -\frac{r\,[1-T(p)]^{2-n}}{p^2\,(n-1)}\left\{[1-T(p)]^{n-1}-\frac{\pi}{r}\right\} < 0$$

Thus, from (B.44), we have shown that $\partial F(p)/\partial \pi = \partial F^{o}(r\text{-}c|r,\pi)/\partial \pi < \partial T(p)/\partial \pi = \partial T^{o}(r\text{-}c|r,\pi)/\partial \pi < 0$. In particular, we have $\partial F(r\text{-}c)/\partial \pi = \partial F^{o}(r\text{-}c|r,\pi)/\partial \pi < 0$. This establishes FACT 1. The above result can also be used to prove FACT 3, i.e that

$\partial E^\circ/\partial\pi<0$. We have:

(B.47) $$\frac{\partial E^\circ}{\partial\pi} = -c\,\frac{\partial F^\circ(r-c|r,\pi)}{\partial\pi} - \int_{p^*}^{r-c}\left[\frac{\partial F^\circ(p|r,\pi)}{\partial\pi} - \frac{\partial T^\circ(p|r,\pi)}{\partial\pi}\right]dp \;>0$$

since we have: $\frac{\partial F^\circ(r-c|r,\pi)}{\partial\pi} < 0$ and $\left[\frac{\partial F^\circ(p|r,\pi)}{\partial\pi} - \frac{\partial T^\circ(p|r,\pi)}{\partial\pi}\right] < 0$.

From (16) we have $[1-T(p)]^{n-1}= [A'(b(p))/p + \pi/r]$. So we have:

(B.48) $$\frac{p}{r}\frac{\partial T(p)}{\partial p} + \frac{\partial T(p)}{\partial r} = \frac{[1-T(p)]^{2-n}}{n-1}\left\{\left[\frac{A'}{p^2} + \frac{\pi}{rpb}\right]\frac{p}{r} - \left[\frac{\pi}{br^2} - \frac{\pi}{r^2}\right]\right\} = \frac{[1-T(p)]}{r(n-1)} > 0.$$

Also,

(B.49) $$\frac{\partial}{\partial p}\left[\frac{p}{r}\frac{\partial T(p)}{\partial p} + \frac{\partial T(p)}{\partial r}\right] = \left[\frac{\partial b(x)f(x)}{\partial r} + \frac{x}{r}\frac{\partial b(x)f(x)}{\partial x} + \frac{b(x)f(x)}{r}\right] = -\frac{b(p)f(p)}{r(n-1)}$$

Using integration by parts, we can show that:

(B.50) $$\frac{-p}{r}b(p)f(p) + \left[\int_{p^*}^{r-c}\left[\frac{x}{r}\frac{\partial b(x)f(x)}{\partial x} + \frac{b(x)f(x)}{r}\right]dx + \frac{p^*}{r}\,b(p^*)f(p^*) = 0$$

Substituting (B.49) and (B.50) into (B.48), we obtain (B.51)

$$\left[\frac{p}{r}\frac{dT(p)}{dp}+\frac{dT(p)}{dr}\right] = \left[\int_{p^*}^{p}\left[\frac{\partial b(x)f(x)}{\partial r} + \frac{x}{r}\frac{\partial b(x)f(x)}{\partial x} + \frac{b(x)f(x)}{r}\right]dx - b(p^*)f(p^*)\left[\frac{dp^*}{dr} - \frac{p^*}{r}\right]\right.$$

$$= \int_{p^*}^{p}\left[-\frac{b(x)f(x)}{(n-1)r}\right]dx - b(p^*)f(p^*)\left[\frac{dp^*}{dr} - \frac{p}{r}\right] = \frac{[1-T(p)]}{r(n-1)}$$

Thus, $\{-b(p^*)f(p^*)\,[dp^*/dr - p/r]\}=1/r(n-1)$. Further recall that $\partial b(x)/\partial r + (x/r)\partial b(x)/\partial x = 0$. So we have:

(B.52) $$\frac{1}{b(x)}\left[\frac{\partial b(x)f(x)}{\partial r} + \frac{x}{r}\frac{\partial b(x)f(x)}{\partial x} + \frac{b(x)f(x)}{r}\right] = \left[\frac{\partial f(x)}{\partial r} + \frac{x}{r}\frac{\partial f(x)}{\partial x} + \frac{f(x)}{r}\right] = -\frac{f(x)}{r(n-1)}\,.$$

It follows that

(B.53) $\left[\frac{p}{r}\frac{\partial F(p)}{\partial p}+\frac{\partial F(p)}{\partial r}\right] = \frac{[1/b(p^*)-F(p)]}{r(n-1)} > 0$. Thus

(B.54) $$\frac{\partial F(p)}{\partial p}+\frac{\partial F(p)}{\partial r} = \frac{(r-p)}{r}f(p) + \left[\frac{p}{r}\frac{\partial F(p)}{\partial p}+\frac{\partial F(p)}{\partial r}\right] = \frac{(r-p)}{r}f(p) + \frac{[1/b(p^*)-F(p)]}{r(n-1)} > 0$$

(B.55) $$\frac{\partial T(p)}{\partial p}+\frac{\partial T(p)}{\partial r} = \frac{(r-p)}{r}\,b(p)f(p) + \frac{[1-T(p)]}{r(n-1)} > 0$$

This establishes, the (FACT 2) i.e.

(B.56) $$\frac{\partial F^\circ(r-c|r,\pi)}{\partial r} = f(r-c)+\int_{p^*}^{r-c}\left[\frac{\partial f(x)}{\partial r}\right]dx - f(p^*)\left[\frac{dp^*}{dr}\right] = \frac{c}{r}f(r-c) + \frac{[1/b(p^*)-F(r-c)]}{r(n-1)} > 0$$

Now, we have:

(B.57) $$\frac{\partial E^o}{\partial r} = -c\,\frac{\partial F(r-c)}{\partial r} - \int_{p^*}^{r-c} \frac{\partial[F(p)-T(p)]}{\partial r} + \frac{\partial[F(p)-T(p)]}{\partial p}\,dp$$

Substituting equations (B.54) to (B.56) into (B.57), we obtain (B.58):

$$\frac{\partial E^o}{\partial r} = -\frac{c}{r}f(r-c) - c\,\frac{[1/b(p^*)-1]}{r(n-1)} - c\,\frac{[1-F(r-c)]}{r(n-1)} - \int_{p^*}^{r-c} \frac{(r-p)}{r}(1-b(p))f(p) + \frac{[1/b(p^*)-1]}{r(n-1)} - \frac{[F(p)-T(p)]}{r(n-1)}\,dp$$

If $E(r,\pi)=0$, we have $c\,[1-F(r-c)] - \int_{p^*}^{r-c}[F(p)-T(p)]\,dp = 0$. Thus, we can establish (FACT 4):

(B.59) $$\frac{\partial E^o}{\partial r} = -\frac{c}{r}f(r-c) - c\,\frac{[1/b(p^*)-1]}{r(n-1)} - \int_{p^*}^{r-c} \frac{(r-p)}{r}(1-b(p))f(p) + \frac{[1/b(p^*)-1]}{r\,(n-1)}\,dp < 0$$

Finally, we can show that if $I(r,\pi)=0=\pi(n-1)(r-c)/r - A'(b(r-c))$, where $b(r-c)A'(r-c) - A(r-c) = \pi c/r$, then for every $\pi>0$, we have:

(B.60) $$\frac{\partial I}{\partial \pi} = \frac{(r-c)n-1}{r} - \frac{c}{rb(r-c)} = \frac{A'(b(r-c))}{\pi} - \frac{A'(b(r-c))}{\pi} + \frac{A(b(r-c))}{\pi\, b(r-c)} > 0$$

Also note that:

(B.61) $$\frac{\partial I}{\partial r} = \frac{(n-1)\pi c}{r^2} + \frac{\pi c}{r^2 b} > 0$$

This establishes FACT 5 and FACT 6 respectively. Q.E.D.

B.6.2. *Proofs*

PROOF OF CLAIM B.2: To prove claim B.2, we first show that for every $r>\underline{r}$ there always exists some π^* solving $I(\cdot,\pi^*)=0$. Consider two extreme positions. Let $\pi=r/n$, we have $(r-c)\ (n-1)/n > A'(b^o(r-c|r,\pi))$ since by assumption $r>\underline{r}$. Now, consider the $\lim_{\pi\to 0} (A'(b^o(r-c|r,\pi))/\pi)$, where $b^o(r-c|r,\pi)$ is such that $b^o(r-c|r,\pi)A'(b^o(r-c|r,\pi)) - A(b^o(r-c|r,\pi)) = \pi c/r$. We have

$$\frac{A'(b^o(r-c|r,\pi))}{\pi} = \frac{c}{rb^o(r-c|r,\pi)} + \frac{A(b^o(r-c|r,\pi))}{\pi b^o(r-c|r,\pi))}$$

Properties of function $A(.)$ imply that $b^o(r-c|r,\pi)$ goes to zero as π goes to zero. Thus we have $\lim_{\pi\to 0} (A'(b^o(r-c|r,\pi))/\pi) = \lim_{\pi\to 0} c/rb^o(r-c|r,\pi)) = \infty$. Thus for every r there always exists some π sufficiently small such that $(r-c)\ (n-1)/r < A'(b^o(r-c|r,\pi))/\pi$. Continuity of I with respect to π guarantees the existence of some π^* so that $I(r,\pi^*)=0$. Uniqueness is guaranteed since $\partial I(r,\pi)/\partial\pi>0$ whenever $I(r,\pi)=0$ (FACT 5). Let $\pi^*(r)$ denote the level of profits so that $I(r,\pi^*(r))=0$. Since $\partial I(r,\pi)/\partial r>0$, $\pi^*(r)$ is decreasing in r.

We must show now that there exists some r sufficiently high that $F^{o}(r-c|r,\pi^*(r))>1$, where $\pi^*(r)$ solves $I(r,\pi^*(r))=0$. Let ß be given by $\beta A'(\beta)-A(\beta)=\pi^*(r) > \pi^*(r)\,(r-p)/r$; we have $\beta>b^{o}(p|r,\pi)$, for every p. Thus $F^{o}(r-c|r,\pi^*(r))> T^{o}(r-c|r,\pi^*(r))/\beta$. Since $\pi^*(r)$ is decreasing with r so that $\lim_{r\to\infty}\pi^*(r)=0$, we have $\lim_{r\to\infty}T^{o}(r-c|r,\pi^*(r)) = \lim_{r\to\infty}[1-(n\pi^*(r)/r)^{1/n-1}]= 1$. Also, we have $\lim_{r\to\infty}[\beta A'(\beta)-A(\beta)]=\lim_{r\to\infty}\pi^*(r)$ or $\lim_{r\to\infty}\beta=0$. Thus there exists an r such that $T^{o}(r-c|r,\pi^*(r))>\beta$, which would imply that $F^{o}(r-c|r,\pi^*(r))>1$.

By lemma B.7, we have $\partial F^{o}(r-c|r,\pi)/\partial\pi<0$ (FACT 1) and $\partial F^{o}(r-c|r,\pi)/\partial r>0$ (FACT 2); also if $I(r,\pi)=0$, we have $\partial I(r,\pi)/\partial\pi>0$ (FACT 5) and $\partial I(r,\pi)/\partial r>0$ (FACT 6). The latter part of claim 2, follows immediately from the above inequalities. Let $(\bar{r},\bar{\pi})$ be a pair such that $F^{o}(\bar{r}-c\,|\,\bar{r},\bar{\pi})=1$ and $I(\bar{r},\bar{\pi})=0$. If $I(r,\pi)\le 0$ and $r\ge\bar{r}$ then $\pi\le\bar{\pi}$, it follows that $F^{o}(r-c|r,\pi)\ge 1$. Similarly, if $F^{o}(r-c|r,\pi)=1$ and $r<\bar{r}$ then $\pi<\bar{\pi}$, it follows that $I(r,\pi)<0$. Q.E.D.

PROOF OF CLAIM B.3: We must show that, if $R^*(r)=r$ for some $r\in(\underline{r}, r^*)$, then $R^{*\prime}(r)<1$. We have defined

(B.62) $$E^{o}(r,\pi) = c\,[1-F^{o}(r-c|r,\pi)] - \int_{p^*}^{r-c}[F^{o}(p|r,\pi)-T^{o}(p|r,\pi)]\,dp$$

Using the results in lemma B.7, if $E(r,\pi)=0$ we have:

(B.63) $$\frac{dE^{o}(r,\pi^*(r))}{dr} = \frac{\partial E^{o}(r,\pi^*(r))}{\partial r} + \frac{d\pi^*(r)}{dr}\bigg|_{I=0}\frac{\partial E^{o}(r,\pi^*(r))}{\partial\pi} < 0$$

Where $\pi^*(r)$ solves $I(r,\pi^*(r))=0$. Now, note that:

(B.64) $$R^{o}(r,\pi^*(r)) = r + \frac{E^{o}(r,\pi^*(r))}{[1-T^{o}(r-c|r,\pi^*(r))]}$$. Therefore:

(B.65) $$\frac{dR^{o}(r,\pi^*(r))}{dr} = 1 + \frac{dE^{o}(r,\pi^*(r))/dr}{[1-T^{o}(r-c|r,\pi^*(r))]} + \frac{E^{o}(r,\pi^*(r))\,[dT^{o}(r-c|r,\pi^*(r))/dr]}{[1-T^{o}(r-c|r,\pi^*(r))]^2}$$

From (B.64), we have that $R^{o}(r,\pi^*(r))=r$ if and only if $E^{o}(r,\pi)=0$. Thus, if $R^{o}(r,\pi^*(r))=r$ and $E^{o}(r,\pi^*(r))=0$, then, from (B.63) and (B.65), $R^{*\prime}(r)<1$. Q.E.D.

PROOF OF CLAIM B.4: We must show that π is decreasing with n. At equilibrium we must have $E^{o}=I=0$. Thus:

(B.66) $\frac{d\pi}{dn} = \frac{(\partial I/\partial n)(\partial E^\circ/\partial r) - (\partial E^\circ/\partial n)(\partial I/\partial r)}{(\partial I/\partial r)(\partial E^\circ/\partial \pi) - (\partial E^\circ/\partial r)(\partial I/\partial \pi)}$

We have already shown that $(\partial E^\circ/\partial r)<0$, $(\partial E^\circ/\partial \pi)>0$, $(\partial I/\partial r)>0$, $(\partial I/\partial \pi)>0$. To show that $d\pi/dn<0$, it is sufficient to that $(\partial E^\circ/\partial n)>0$ and $(\partial I/\partial n)>0$. Recall that:

$$I = (r-c)\ (n-1)\ \pi/r - A'(b(r-c)) = 0$$

further note that $\partial b(p)/\partial n = 0$ for every p. So we have $(\partial I/\partial n) = \pi(r-c)/r > 0$.

To show $(\partial E^\circ/\partial n)>0$, we must first show that $\partial F^\circ(p|r,\pi)/\partial n < \partial T^\circ(p|r,\pi)/\partial n < 0$ for every p. For notational purpose, we let $b(p)=b^\circ(r-c|r,\pi)$, $T(p)=T^\circ(r-c|r,\pi)$ and $f(p)=f^\circ(r-c|r,\pi)$. We have:

(B.67) $\frac{\partial [1-T(p)]^{n-1}}{\partial n} = \ln[1-T(p)][1-T(p)]^{n-1} - (n-1)\,[1-T(p)]^{n-2}\frac{\partial T(p)}{\partial n} = \frac{\partial}{\partial n}\left[\frac{A'}{p} + \frac{\pi}{r}\right] = 0$

(B.68) $\partial[T^\circ(p|r,\pi)]/\partial n = \ln[1-T(p)]\,[1-T(p)]/(n-1) < 0$. Further we have,

(B.69) $\frac{\partial^2[T(p)]}{\partial p\partial n} = \frac{\partial b(p)f(p)}{\partial n} = -\left\{\frac{1+\ln[1-T(p)]}{(n-1)}\right\} b(p)f(p)$

(B.70) $\frac{\partial T(p)}{\partial n} = = \left[\int_{p^*}^{p} -\left\{\frac{1+\ln[1-T(x)]}{(n-1)}\right\} b(x)f(x)\,dx\right] < 0$

Since, $\partial b(x)/\partial n = 0$, we also have

(B.71) $\frac{\partial F(p)}{\partial n} = \left[\int_{p^*}^{p} -\left\{\frac{1+\ln[1-T(x)]}{(n-1)}\right\} f(x)\,dx\right].$

We define p" so that $\{1+\ln[1-T(p")]\}=0$; thus $\{1+\ln[1-T(x)]\}<0$ if and only if $p">x$ or $(b(p")-b(x))<0$. Thus

(B.72) $b(p")\left[\frac{\partial F(p)}{\partial n}\right] - \left[\frac{\partial T(p)}{\partial n}\right] = \left[\int_{p^*}^{p}[b(p")-b(x)]\left[-\left\{\frac{1+\ln[1-T(x)]}{(n-1)}\right\} f(x)\right]dx\right] < 0.$

It follows that $\frac{\partial F^\circ(p|r,\pi) - dT^\circ(p|r,\pi)}{\partial n} < 0$. In particular, $\frac{\partial F^\circ(r-c|r,\pi)}{\partial n} < \frac{\partial T^\circ(r-c|r,\pi)}{\partial n} < 0$.

Therefore

(B.73) $\frac{\partial E^\circ}{\partial n} = -c\,\frac{\partial F^\circ(r-c|r,\pi)}{\partial n} - \int_{p^*}^{p}\left[\frac{\partial F^\circ(p|r,\pi) - \partial T^\circ(p|r,\pi)}{\partial n}\right]dp > 0.$

Having show that $(\partial E^\circ/\partial n)>0$ and $(\partial I/\partial n)>0$, we obtain by (B.66) $d\pi/dn < 0$. Q.E.D.

PROOF OF CLAIM B.5: Note that $r-c \geq E(p_{min})$ where $E(p_{min})$ is the minimum expected price paid by a consumer. We have (B.74)

$$E(p_{min}) = \int_{p^*}^{r-c} pn[1-T(p)]^{n-1}\, b(p)\, f(p)\, dp + r-c\ [1-T(r-c)]^n$$

Using (B.13) and (B.25) we can show that $E(p_{min}) = n\int_{p^*}^{r-c} A(b(p))\, dF(p)$ + $n\pi$. This result should not be surprising as the firm's total gross revenue should equal the consumers' total spending.

It follows that $n\pi \leq r-c$ and (B.75) $n\pi/(n\pi+c) < (r-c)/r$.

Now, let define ß as the advertising intensity such that

(B.76) $\beta A'(\beta) - A(\beta) = c/n > \pi c/r = b^{\circ}(r\text{-}c|r,\pi)(r-c)\, A'(b^{\circ}(r\text{-}c|r,\pi)) - A(b^{\circ}(r\text{-}c|r,\pi))$

Properties of the advertising cost function are such that ß goes to 0 as c/n goes to zero. Furthermore, since $\beta > b^{\circ}(r\text{-}c|r,\pi)$, we have $\lim_{n\to\infty} b^{\circ}(r\text{-}c|r,\pi)=0$ and $\lim_{n\to\infty} A'(b^{\circ}(r\text{-}c|r,\pi)) = A'(0)=0$.

Using equation (I) and the inequality (B.75), we have:

(B.77) $\frac{(n-1)}{n}\frac{[n\pi]^2}{n\pi+c} < \frac{(r-c)}{r}(n-1)\pi = A'(b^{\circ}(r\text{-}c|r,\pi))$

(B.78) $n\pi < [\, A'(b^{\circ}(r\text{-}c|r,\pi)) + [A'(b^{\circ}(r\text{-}c|r,\pi))^2 + 4\, cA'(b^{\circ}(r\text{-}c|r,\pi))]^{1/2}\,] / 2$

(B.79) $n\pi < [\, A'(b^{\circ}(r\text{-}c|r,\pi)) + [cA'(b^{\circ}(r\text{-}c|r,\pi))]^{1/2}\,]$

Thus, $\lim_{n\to\infty} n\pi = \lim_{n\to\infty} A'(b^{\circ}(r\text{-}c|r,\pi)) = A'(0)=0$. This establishes that $\lim_{n\to\infty} \pi=0$. Q.E.D.

PROOF OF THEOREM B.6: We have already established (i), i.e. $\lim_{n\to\infty} n\pi = 0$.

Now, let define b* as the advertising intensity such that

(B.80) $b^* A'(b^*) - A(b^*) \equiv \pi \geq \pi(r-p)/r = b^{\circ}(p|r,\pi)\, A'(b^{\circ}(p|r,\pi)) - A(b^{\circ}(p|r,\pi))$

Properties of the advertising cost function are such that b* goes to 0 as π goes to 0 and that $b^* \geq b^{\circ}(p|r,\pi)$, for every $p \leq r-c$. This implies ii).

T(r-c), the probability that a consumer observe an ad from a given store is defined by

(B.81) $T^{\circ}(r\text{-}c|r,\pi) = \int_{p^*}^{r-c} b^{\circ}(p|r,\pi)\, dF^{\circ}(p|r,\pi) \leq b^* F(r-c) \leq b^*$. Establishing iii).

We can show iv) using equation (B.18). We have $p^* [\, 1-\pi/r\,] = A'(b^{\circ}(p^*|r,\pi)) < A'(b^*)$. Thus, the minimum price p* converges to 0 as A'(b*) goes to 0.

The minimum expected price paid by consumers is given by

(B.82) $E(p_{min}) = \int_{p^*}^{r-c} p\, n\, [1-T^{\circ}(p|r,\pi)]^{n-1}\, b^{\circ}(p|r,\pi)\, dF^{\circ}(p|r,\pi) + r-c\, [1-T^{\circ}(r\text{-}c|r,\pi)]^{n}$

(B.83) $= n\int_{p^*}^{r-c} A(b^{\circ}(p|r,\pi))\, dF^{\circ}(p|r,\pi) + n\pi \leq n\, [\beta^* A'(\beta^*)]$

Where (B.84) $n\pi \equiv n\, [\beta^* A'(\beta^*) - A(\beta^*)] = n\, [\int_0^{\beta^*} x\, A''(x)\, dx\,]$

$$= n [\beta^* A'(\beta^*) - \int_0^{\beta^*} x^2/2 \, (A'(x)-A'(0)) \, dx] \geq n [\beta^* A'(\beta^*)] \; [1-\beta^{*2}/2]$$

So, we have

(B.85) $E(p_{min}) = n [\int_{p^*}^{r-c} A(b^\circ(p|r,\pi)) \, dF^\circ(p|r,\pi)] + n\pi \leq n \; [\beta^* A'(\beta^*)] \leq n\pi / [1-\beta^{*2}/2]$

Also, (B.86) $n [\int_{p^*}^{r-c} A(b^\circ(p|r,\pi)) \, dF^\circ(p|r,\pi)] \leq n\pi [\beta^{*2} / (2-\beta^{*2})]$. Establishing v) and vi).

Finally, we have $[1-T(p)]^n \leq [1-T(p)]^{n-1} = n\pi/r$. Thus as $n\pi$ converges to 0, $[1-T(p)]^n$ also goes to 0. This can be used to show that the expected number of ads observed increases as $n\pi$ goes to zero. $[1-T(p)]^n$ is the probability that a consumer does not observe a single ad among the n stores. Now, consider a finite number of stores m<n. Divide all n stores in m equal groups. The probability that a store observes no ad from each individual group is given by $[1-T(p)]^{n/m} = [n\pi/r]^{n/m(n-1)}$. This goes to zero with $n\pi$. As the probability that the consumer observes at least one ad from each m groups goes to one, so is the probability of observing at least m ads overall. Q.E.D.

PROOF OF THEOREM B.7: The proof of Theorem B.7 is obtained using an argument similar to the one used in the proof of claim B.5. We have

(B.87) $$\frac{(n-1)}{n} \frac{[n\pi]^2}{n\pi+c} < (r-c) \, (n-1)\frac{\pi}{r} = aA'(b^\circ(r-c|r,\pi)) .$$

As a result. $n\pi$ will converges to zero whenever $aA'(b^\circ(r-c|r,\pi))$ goes to zero.

i) Now, let us define β as the advertising intensity such that $\beta A'(\beta) - A(\beta) = c/n$. Properties of the advertising cost function are such that β goes to 0 as c/n goes to zero. Since $b^\circ(r-c|r,\pi) < \beta$, we have $\lim_{c\to 0} b^\circ(r-c|r,\pi)=0$ and $\lim_{c\to 0} A'(b^\circ(r-c|r,\pi)) = A'(0)=0$, implying that $n\pi$ converges to zero as c becomes smaller.

ii) Now, let $\beta(a)$ be such that $\beta a A'(\beta) - aA(\beta) = c/n$. We have

(B.88) $$\frac{d \, aA'(\beta(a))}{d \, a} = A'(\beta(a)) - \frac{c}{na\beta(a)} = \frac{A(\beta(a))}{\beta(a)} > 0$$

Moreover, note that $\beta(a)$ increases as a goes to 0. So $\lim_{a\to 0} aA'(\beta(a)) = 0$. Since $\beta(a) > b^\circ(r-c|r,\pi)$, we also have $\lim_{a\to 0} aA'(b(r-c)) = 0$. Thus $\lim_{a\to 0} n\pi=0$. Q.E.D.

CHAPTER C:
CONTINUITY IN AUCTION DESIGN

C.1. INTRODUCTION

Under some basic assumptions (namely, (A1) risk-neutral bidders; (A2) symmetric bidders and (A3) independent-private values), the English, Dutch, first-price sealed-bid and second-price sealed-bid auctions are optimal selling mechanisms provided they are supplemented with an adequate reservation price (see Myerson, 1981). These simple auctions are not necessarily equivalent nor optimal when we alter the above assumptions. Economic theorists have wondered how alterrations to assumptions (A1) to (A3) affect the auction design problem (see McAfee and McMillan (1987) for a survey on auctions). In that respect, the work by Crémer and McLean (1988) leads to a devastating claim: if the private valuations of bidders are not statistically independent, the auctioneer can extract all the surplus.

To understand Crémer's and McLean's result, consider an auction where bidders can have a finite number of valuations for the good on sale. Let $\pi(v_{-i}|v_i)$ represent the probability that other bidders' have valuations v_{-i} given bidder i has valuation v_i, and let $U(v_i)$ be the expected profits earned by a bidder of valuation v_i in a Vickrey auction (second-price auction). The auctioneer can extract all the economic rents if there exists a vector of participation fee, $f_i(v_{-i})$ conditional on v_{-i}, so that $U(v_i)= \sum_{v_{-i}} \pi(v_{-i}|v_i) f_i(v_{-i})$ for all i and v_i. Such vector f will exist, if the matrix $\pi(v_{-i}|v_i)$ is of full rank and invertible. Since the matrix $\pi(v_{-i}|v_i)$ is invertible for almost every distribution of values (in Lesbegue measure), it will "usually" be possible to extract all the rents. The main result has been extend to auctions with a continuum of valuations (McAfee and Reny, 1988).

This result is troublesome. Conventional wisdom and economic intuition suggest that the possession of private information by bidders prevents the auctioneer from extracting all the rents. In order to generate such a prediction, current auction theory must rely, according to Crémer and McLean, on the (unrealistic) presumption that information is exactly independently distributed. Crémer and McLean (1988) propose some paths for research in order to generate "no full extraction of the surplus" from more "realistic" assumptions. One difficulty in their paper is linked to the fact that the entry fees may have to be quite large. They aknowledge that their result depends on the absence of risk aversion and/or limited liability. This provided the motivation for the present study. This essay intends to show how limited liability and/or risk aversion affect Crémer's and McLean's (1988) result.

We are able to show, invoking the Maximum Theorem (Berge, 1963), that the optimal expected gain attainable by the auctioneer is continuous in the set of possible valuation distributions when we introduce either risk aversion or limited liability. Under risk aversion or limited liability, the asymmetry of information will be economically relevant for some positive measure (in the sense of Lebesgue) of auction environments. Furthermore, the English or the first-price sealed-bid auctions are approximately optimal when the auction environment is approximately close to those satisfying A1 to A3. It is also shown that the set of optimal auctions is upper hemi-continuous in the set of possible valuation distributions.

C.2. AUCTION WITH LIMITED LIABILITY

C.2.1 *Auction Environments*

We describe here a general class of auction-design problems where agents are assumed to have limited liability or a finite budget.

One seller (or auction designer) sells a unique indivisible good. He faces n bidders, or potential buyers, who are indexed by the set N={1, 2, ..., n}. The seller's problem derives from the fact that he has imperfect information about the buyers' willingness to pay for the object. All bidders have some private information about the nature of the auction which affects their willingness to pay for the good. The 'type' of bidder i takes values in a finite set $T_i=[1,...,m_i]$. We will call T the set of all possible bidders' types, i.e. $T=T_1\times...\times T_n$. We also let $T_{-i}=T_1\times...\times T_{i-1}\times T_{i+1}\times...\times T_n$. We shall assume that the seller's uncertainty about the 'type' or 'characteristic' of all bidders can be described by a probability distribution over T. Let $\pi: T\rightarrow \mathbb{R}_+$ be the probability distribution for the vector of characteristics $t=(t_1,...,t_n)\in T$ and assume that $\pi(t)>0$ for all t in the finite set T.

Bidder i's subjective probability distribution over T_{-i} conditional on having type t_i; is assumed consistent with the distribution π: it is given by $\pi(t_{-i}|t_i)=\pi(t_{-i},t_i)/\pi_i(t_i)$ where $\pi_i(t_i)=\sum_{t_{-i}\in T_{-i}} \pi(t_{-i},t_i)$.

The function $w_i: T\rightarrow \mathbb{R}_+$ will be called an individual valuation function for agent i, $w_i(t)$ denotes the good's monetary value for agent i when the state of nature is t. Similarly, we let $w_0(t)$ be the seller's valuation when t occurs. We let $w: T\rightarrow \mathbb{R}_+^{n+1}$ represent the vector of individual valuations, $w(t)=(w_0(t), w_1(t),...,w_n(t))$. w will be called the valuation function.

Limited liability is introduced by assuming that bidders have a limited budget, i.e. there exists a limited amount of money which can be paid by each bidder to the auctioneer. We let B_i denote the maximum amount of money bidder i can offer to the seller. Similarly, we denote by B_0 the maximum payment the auctioneer can offer the bidders. We call B the vector $B=\{B_0,B_1,...,B_n\}$.

Provided that the vector B and the finite set T are well-defined, an auction problem or auction environment is characterized by a pair $[\pi,w]$, where π is the probability distribution function and w is the vector of (n+1) valuation functions, each defined over the set T. For future reference, we will denote a typical auction environment or pair $[\pi,w]$ by the letter z. The set of possible auction environments, z, is denoted by Z. Z is a subset of $\mathbb{R}_+^{|T|}\times \mathbb{R}^{|T|\times n+1}$, where $|T|$ is the cardinality of the set T; i.e. $|T|=\prod_{i\in N} m_i$. Our intent is to examine how alterations in the auction environment change the seller's expected payoffs and the set of optimal auctions.

C.2.2 *Auctions: Definition*

Invoking the Relevation principle, we limit ourselves to direct auctions, i.e. auctions that induce the bidders to truthfully reveal their type or private information. Thus an auction is conducted in the following way: the auctioneer asks each bidder i to submit an element t_i of T_i. If $t=(t_1,...,t_n)\in T$ is the vector of announced types, each bidder i is asked to pay an amount $x_i(t)$ to participate in a lottery in which he wins the object with probability $p_i(t)$. Since each bidder i has a limited budget, the payment $x_i(t)$ must not exceed B_i for all $t\in T$. Similarly, the seller's limited budget limits the payment he can make to all bidders for any $t \in T$, i.e. $\sum_{i\in N} -x_i(t) \geq B_0$.

DEFINITION: An auction is a collection $\langle \{p_i(t), x_i(t)\}_{i\in N, t\in T}\rangle$ where $x_i(t)\leq B_i$ and $p_i(t) \geq 0$ for all i and t and $\sum_{i\in N} -x_i(t) \geq B_0$ and $\sum_{i\in N} p_i(t) \leq 1$ for all t.

We will use throughout the essay the letters y to designate a typical auction and $\overline{Y}$ the set of possible auctions.

FACT 1: The set $\overline{Y}$ of possible auctions is a compact subset fo $\mathbb{R}^{|T|xn} \times \mathbb{R}^{|T|xn}$.

Indeed, we have $0\leq p_i(t)\leq 1$ and $-B_0-\sum_{j\neq i} B_j \leq x_i(t) \leq B_i$. Closure of $\overline{Y}$ follows clearly from the weak inequalities in the definition of auctions. Compactness of a subset of $\mathbb{R}^n$, for any finite n, is implied by closure and boundness. Compactness is key in showing the continuity results.

C.2.3 *Continuity*

Given any auction environment, the seller's objective is to maximize his expected revenue subject to the individual rationality constraints and the Bayesian incentive compatibility constraints. Let $\Pi(y,z)$ represent the expected revenue for the auctioneer given some auction environment z and auction y; and let $\phi(z)$ denote the set of all feasible auctions in $\overline{Y}$ which satisfy the Bayesian incentive compatible and the individual rationality constraints. The auctioneer's problem is to select some $y\in\phi(z)\subset \overline{Y}$ in order to maximize $\Pi(y,z)$ given some $z\in Z$. Let V be the value of this optimal auction, i.e. $V(z) \equiv \sup\{\Pi(y,z): y\in\phi(z)\}$. V(z) denotes the maximum expected revenu the auctioneer can attained given the environment z. Let M be the set of "optimal" auction: $M(z) = \{y: y\in\phi(z),$

$\Pi(y,z)=V(z)\}$. We want to show that V(.) and M(.) are respectively continuous and upper hemi-continuous in Z.

$\Pi(y,z)$, the expected revenue for the auctioneer given some auction environment z and auction y, is given by

(C.1) $\Pi(y,z)=\Pi(\langle\{p_i(t), x_i(t)\};i\in N,t\in T\rangle,[\pi,w]) =\Sigma_{t\in T}\Sigma_{i\in N} [x_i(t)+(1-p_i(t))w_0(t)]\pi(t)$

Given some well-defined auction environment z and some auction y, let $E_i(t_i,s_i|y,z)/\pi_i(t_i)$ denote the expected utility of bidder i when he has type t_i and reports type s_i. We have:

(C.2) $E_i(t_i,s_i|y,z) = \Sigma_{t_{-i}\in T_{-i}} [p_i(t_{-i}, s_i) w_i(t_{-i}, t_i) - x_i(t_{-i}, s_i)] \pi(t_{-i}, t_i)$.

The set of all feasible auctions in $\overline{Y}$ which satisfy the Bayesian incentive compatible and the individual rationality constraints, $\phi(z)$, is formally defined as the set of auctions $y=\langle\{p_i(t), x_i(t)\}i\in N,t\in T\rangle \in \overline{Y}$ such that

(C.3) $E_i(t_i,t_i|y,z) \geq 0$; for each $i \in N$ and each $t_i \in T_i$.

(C.4) $E_i(t_i,t_i|y,z) - E_i(t_i,s_i|y,z) \geq 0$, for each $i \in N$ and each s_i and $t_i \in T_i$.

The inequalities in (C.3) specify the individual rationality constraints. For some i and t_i, the left-hand side of the inequality is the expected payoff to agent i for participating in the auction when i has characteristic t_i and all other bidders' are expected to reveal truthfully their type. (C.3) represents a system of $\Sigma_{i\in N} m_i$ constraints. The inequalities in (C.4) correspond to the Bayesian incentive compatibility constraints. Given player i has characteristic t_i, his expected payoff when he announces type t_i is at least as great as his expected payoff when he announces any other type $s_i \neq t_i$. (C.4) represents a system of $\Sigma_{i\in N} m_i (m_i-1)$ constraints.

We can write $\phi(z) \equiv \{y\in\overline{Y}| E_i(t_i,t_i|y,z) \geq 0$; for each $i \in N$ and each $t_i \in T_i$ and $E_i(t_i,t_i|y,z) - E_i(t_i,s_i|y,z) \geq 0$, for each $i \in N$ and each s_i and $t_i \in T_i\}$ where $\overline{Y}$ is a compact set.

LEMMA C.1: *The following holds*

i) *the auctioneer's expected revenue function* Π: $\overline{Y}\times Z \to \mathbb{R}$, *is a continuous function in* $\overline{Y}\times Z$.

The set-valued function $\phi(z)$ *from* Z *into* $\overline{Y}$ *defined by* (C.3) *and* (C.4) *is*

ii) ***non-empty for all*** $z \in Z$,

iii) ***upper hemi-continuous in*** Z,

iv) ***lower hemi-continuous in*** Z. (see proof in section C.5.2)

THEOREM C.1: ***Let*** Z ***be the space of all possible auction environments, let*** V(z) ***be the maximum expected payoff attainable by the auctioneer under auction environment*** z, ***and let*** M(z) ***be the set of all optimal auctions given*** z. ***Then*** V(z) ***is continuous and*** M(z) ***is upper hemi-continuous in*** Z.

Theorem C.1 follows immediately from lemma C.1 and the Maximum Theorem (Berge, 1963). By lemma C.1, the real-valued function $\Pi(y,z)$ is continuous in $\overline{Y}xZ$, the set-valued function $\phi(z)$ is non-empty for all $z \in Z$ and the mapping ϕ from Z into $\overline{Y}$ is continuous in Z. This implies, by the Maximum Theorem, the results arrived at in Theorem C.1.

C.3. AUCTIONS WITH RISK-AVERSE BIDDERS

C.3.1 *Auction Environments*

The results of section C.2 can be attained if, rather than assuming a limited budget, we assume that agents (the auctioneer and bidders) are risk-averse.

As in section C.2, we let N be the set of bidders and the finite set T be the set of all possible bidders' types. Agents face no liability constraints but are assumed risk-averse. Let $u_i(-x,t)$ be the utility of buyer i winning the good and paying amount x when the state of nature is t. Let $v_i(-x)$ be the utility of buyer i when he looses the good and pays amount x. Similarly, let $u_0(x,t)$ denote the utility of the seller when he keeps the good for himself at state t and receive payment x; and let $v_0(x)$ be the utility of the seller when sells the good and receive payment x.

We assume that for all i, $u_i(.,.)$ and $v_i(.)$ satisfy the following restrictions:

(H1) $u_i(-x, t)$ and $v_i(-x)$ are twice continuously differentiable

(H2) $u_i' > 0$, $v_i' > 0$.

(H3) $v_i(0)=0$.

(H4) $v_i''\leq 0$, and there exists some finite $x\geq 0$ and some $\varepsilon>0$ such that $[v_i''(-y)]<-\varepsilon$ for all $y\geq x$.

(H5) There exists some finite $\overline{w}$ and B such that $[u_i(-x,t)-v_i(-x)]\leq v_i(\overline{w}) < B$ for all i, x and t.

The prime denote the partial derivative with respect to wealth. It is natural to assume that utility is increasing in income; hence H2. Assumption H3 is simply a convenient normalization of preferences. Since we are interested in risk averse bidders, concavity of the function v_i is assumed. Note that in order to prove our result, we need to assume strict concavity (v_i'' is uniformily bounded away from zero) when wealth goes to $-\infty$. (H4) implies that $\lim_{x\to\infty} v(-x)/x = -\infty$ [30]. Assumption H5 implies that there exists a maximum amount of money $\overline{w}$ one is willing to pay in order to get the good. Note that the concavity of each u_i is not required, although by H5, $u_i(-x,t)$ remains within some distance of $v_i(-x)$ which is itself globally concave.

Provided that the set T and the functions u_i and v_i are well-defined, an auction environment is a distribution of types, π. As in section C.2, we let $z=\langle\pi(t),t\in T\rangle$ denote a typical auction environment. The space of auction environment, denoted by Z, is a closed subset of $\mathbb{R}^{|T|}$ such that $\pi(t)\geq\xi>0$ for all $t\in T$ and $\sum_{t\in T}\pi(t)=1$. Note that ξ can be any arbitrarily small positive number[31].

Let us, as before, restrict our attention to direct auction mechanisms, where bidders have the incentive to truthfully reveal their private information. Each bidder i submits an element t_i of T_i. If $t=(t_1,\dots,t_n)\in T$ is the vector of announced types, then the auctioneer assigns buyer i a probability $p_i(t)$ of winning the good and requires him to make payment $\beta_i(t_1,\dots,t_n)$ if he wins and payment $\alpha_i(t_1,\dots,t_n)$ if he loses.

30 Let x be some finite number such that $[v''(-y)]<-\varepsilon<0$ for all $y\geq x$. Then for all $y>x$, we have $v(-y)\leq -\varepsilon y^2+v'(x)(y-x)+v(x)$. Thus, $\lim_{y\to\infty} v(-y)/y \leq \lim_{y\to\infty} -\varepsilon y = -\infty$.

31 The bound ξ on each $\pi(t)$ should not limit the power of our result. Suppose that we wish to evaluate continuity in the neighborhood of some distribution $\pi(t)$, such that $\pi(t)>0$ for every t. We find $\delta=\inf_{t\in T}(\pi(t))$, and let $\xi=\delta/2$, since $\delta>0$ we have $\xi>0$.

If $t\in T$ is the set of announced types and $s\in T$ be the true types, then bidder i has the following expected utility: $[p_i(t)\ u_i(-\beta_i(t); s) + (1-p_i(t))\ v_i(-\alpha_i(t))]$. Similarly, the seller's utility is given by: $[\Sigma_{i\in N}\ p_i(t)\ v_0(\beta_i(t)+\Sigma_{j\neq i}\ \alpha_j(t)) + \Sigma_{i\in N}\ (1-p_i(t))\ u_0(\Sigma_{j\in N}\alpha_j(t); s)]$.

DEFINITION: An auction is a collection $<\{p_i(t), \beta_i(t), \alpha_i(t)\}; i\in N, t\in T>$ where $\beta_i(t)$ and $\alpha_i(t)\in\mathbb{R}$ and $p_i(t)\geq 0$ for all i and t, and $\Sigma_{i\in N}\ p_i(t)\leq 1$ for all t.

Our intent is to examine how alterations in the auction environment $z=<\pi(t),t\in T>$ change the seller's expected payoffs and the set of optimal auctions.

C.3.2 *Compactness*

In order to apply the Maximum Theorem, we need to show that we can restrict our attention to a compact space of auctions. Here, contrary to section C.2, the set of possible auctions is not trivially bounded in the usual topology. We must redefine the space of possible auctions using a new metric such that there will exists a compact set of auctions in the new topology which contains all feasible auctions. Assumptions (H1) to (H5) will allow us to do so without loss of generality.

Before stating the lemmata of this section, we introduce some notation. Let $F_i(s,t\,|y)$ be the expected utility of agent i, when s is the vector of true types and t is the vector of announced types given auction $y=<(p_i(t),\beta_i(t),\alpha_i(t); i\in N, t\in T>$. If $t\in T$ is the vector of announced types and $s\in T$ is the vector of true types, then bidder i has the following expected utility: $F_i(s,t\,|y) = [p_i(t)\ u_i(-\beta_i(t); s) + (1-p_i(t))\ v_i(-\alpha_i(t))]$. Similarly, the seller's utility is given by: $F_0(s,t\,|y) = [\Sigma_{i\in N}\ p_i(t)\ v_0(\beta_i(t)+\Sigma_{j\neq i}\ \alpha_j(t)) + \Sigma_{i\in N}\ (1-p_i(t))\ u_0(\Sigma_{j\in N}\alpha_j(t); s)]$.

We define the following metric over the set of auctions:

$$\text{(C.5)}\quad d_Y(y^\circ,y)=\{\sum_{t\in N}\ \sum_{s\in N}(F_0(s,t\,|y^\circ)-F_0(s,t\,|y))^2+\sum_{i\in N}\ \sum_{t\in T}\ \sum_{s\in T}\ (F_i(s,t\,|y^\circ)-F_i(s,t\,|y))^2\}^{1/2}$$

Note that two auctions, y° and y are equivalent if $d_Y(y^\circ,y)=0$. Indeed, if $d_Y(y^\circ,y)=0$, then every player i will receive exactly the same payoff under auctions y° and y, at every state of the nature s and for every vector of reported types t. For <Y, d_Y> to be a metric space, we must think of <Y, d_Y> as a space of classes of equivalent auctions. It shall lead to no confusion if we simply refer to Y as the space of auctions.

LEMMA C.2: *For some arbitrarily large* M, *define* Y(M) *as the space of auctions* y *such that* $|F_i(s,t|y)| \leq M$; *for all* $i \in \{0,1,...,n\}$, s *and* $t \in T$. *Then the completion of the metric space* <Y(M),d_Y>, *denoted by* <$\overline{Y}$(M),$d_{\overline{Y}}$> *is a compact metric space.*

PROOF: For any element y of Y(M), there exists a unique finite collection of numbers < $(F_i(s,t|y))$, for all $i \in \{0,1,...,n\}$, s and $t \in T$>. Such collection can be represented as an element of $\mathbb{R}^m$, where $m=(n+1)|T||T|$. Let $\sigma(.)$ be the function mapping from Y(M) into $\mathbb{R}^m$ such that $\sigma(y)=$< $(F_i(s,t|y))$, for all $i \in \{0,1,...,n\}$, s and $t \in T$>. Note that by the definition of the metric (C.5) on Y, the distance between two auctions y^o and y is given by the distance between $\sigma(y^o)$ and $\sigma(y)$ using usual euclidean metric on $\mathbb{R}^m$. Thus, $\sigma(.)$ is an isometric isomorphism. Hence, both $\sigma(.)$ and its inverse $\sigma^{-1}(.)$ are continuous. Let S $\equiv$ $\sigma(Y(M))$ be the image of Y(M) in $\mathbb{R}^m$. The set S is a bounded subset of $\mathbb{R}^m$; since by the definition of S and Y(M), if $x=(x_1,....,x_m) \in S$ then $|x_i| < M$ for all $i \in \{1,...,m\}$.

Now, let $\overline{\sigma}$ denote the natural extension of σ to $\overline{Y}$(M). That is, for $y \in \overline{Y}(M)$, $\overline{\sigma}(y) = \lim_{n \to \infty} \sigma(y_n)$ for some $\{y_n\}_{n=1}^{\infty}$ in Y(M) converging to y with respect to $d_{\overline{Y}}$. It is straightforward to show that $\overline{\sigma}$ is an isometric isomorphism from $\overline{Y}$(M) onto $\overline{S}$, the closure of S. Hence $\overline{\sigma}^{-1}(\overline{S}) = \overline{Y}(M)$. That is, $\overline{Y}$(M) is the continuous image of the compact set $\overline{S}$ and is therefore itself compact.. Q.E.D.

Although the space of auctions defined by Y(M) is not itself a compact space, its completion is. It is therefore handy to use $\overline{Y}$(M) instead of Y(M). For the purpose of this essay, we can do so without loss of generality. Consider all the points in the completion of Y(M) but not in Y(M) itself. These points do not correspond to implementable auctions, however, for any the point $y \in [\overline{Y}(M)-Y(M)]$, there exists a sequence of auctions in Y(M) which yields payoffs arbitrarily close to y. Thus, the maximum expected payoff we can acheive over the compact set $\overline{Y}$(M) is equal to the supremum acheived over the set Y(M). This will be used later.

Lemma C.3: *Let $\phi(z)$ denote the set of feasible auctions given auction environment z, then there exists a finite M such that* $\phi(z) \subset Y(M)$, *for all* $z \in Z$, *such that* $\pi(t) > \xi$ *for* $\forall t \in T$.

PROOF: We want to show that there exists some M such that if y is a feasible auction given an auction environment $z \in Z$, where $\pi(t) > \xi$ for $\forall t \in T$, then $|F_i(s,t|y)| \leq M$; for all $i \in \{0,1,...,n\}$, s and $t \in T$. That is, there exists a bound on the expected payoff a agent can earns at every state of the nature, and given any vector of reported types. This follows from risk aversion. If, following some event, an agent were required to pay some very large amount of money, the cost associated with such risk may be large enough to offset any possible gain from participating in the auction.

In order for the seller and all bidders to be willing to participate in the auction, we must have

(C.6) $\sum_{t \in T}\{[\sum_{i \in N} p_i(t)\, v_0(\beta_i(t)+\sum_{j \neq i}\alpha_j(t))]+[\sum_{i \in N}(1-p_i(t))]\, u_0(\sum_{j \in N}\alpha_j(t);\, t)\}\, \pi(t)$

$$\geq \sum_{t \in T} u_0(0;\, t)\pi(t) \geq 0.$$

and (C.7) $\sum_{t \in T} [p_i(t)\, u_i(-\beta_i(t);t) + (1-p_i(t))\, v_i(-\alpha_i(t))]\, \pi(t) \geq v_i(0)=0$; for all $i \in N$.

Using (H5), the following inequalities must hold

(C.8) $\sum_{t \in T}[p_i(t)\, v_i(-\beta_i(t))+(1-p_i(t))\, v_i(-\alpha_i(t))]\, \pi(t) \geq$

$$-\sum_{t \in T} p_i(t)[u_i(-\beta_i(t);t)-v_i(-\beta_i(t))]\pi(t) \geq -\sum_{t \in T} p_i(t)\, v_i(\overline{w})\, \pi(t) \geq -v_i(\overline{w}).$$

Concavity of $v_i(.)$ implies that

(C.9) $\sum_{t \in T} [p_i(t)\, \beta_i(t) + (1-p_i(t))\, \alpha_i(t)])\, \pi(t)\,) \leq \sum_{t \in T} p_i(t)\, \overline{w}\, \pi(t)$, for all $i \in N$, and

(C.10) $\sum_{i \in S}\sum_{t \in T} [p_i(t)\, \beta_i(t) + (1-p_i(t))\, \alpha_i(t)]\, \pi(t) \leq \overline{w}$, for all $S \subset N$.

The value $\overline{w}$ denote the maximum economic surplus which can be obtained from the auction, thus the average profit earned by the auctionner can not exceed $\overline{w}$.

If we apply the same transformations on the seller's expected utility, we have:

(C.11) $\sum_{t \in T}\{[\sum_{i \in N} p_i(t)\, v_0(\beta_i(t)+\sum_{j \neq i}\alpha_j(t))]+[\sum_{i \in N}(1-p_i(t))]\, v_0(\sum_{j \in N}\alpha_j(t))\}\pi(t)$

$$\geq -\sum_{t \in T}[\sum_{i \in N}(1-p_i(t))]\, v_0(\overline{w})\, \pi(t) \quad \text{and}$$

(C.12) $\sum_{t \in T}\{[\sum_{i \in N} p_i(t)\, (\beta_i(t)+\sum_{j \neq i}\alpha_j(t))]+[\sum_{i \in N}(1-p_i(t))]\, \sum_{j \in N}\alpha_j(t)\}\, \pi(t) =$

$$\sum_{t\in T}\sum_{i\in N} [p_i(t)\, \beta_i(t) + (1-p_i(t))\, \alpha_i(t)]\, \pi(t) \geq -\sum_{t\in T}[\sum_{i\in N}(1-p_i(t))]\, \bar{w}\, \pi(t).$$

Thus we must have for all $i\in N$

(C.13) $\sum_{t\in T} [p_i(t)\, \beta_i(t)+(1-p_i(t))\, \alpha_i(t)])\, \pi(t) \geq$

$$- \sum_{t\in T}\sum_{j\neq i} [p_j(t)\, \beta_j(t) + (1-p_j(t))\alpha_j(t)]\, \pi(t) + -\sum_{t\in T}[\sum_{i\in N}(1-p_i(t))]\, \bar{w}\, \pi(t) \geq -\bar{w}.$$

Now, we want to show now that the expression $[p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))]$ is bounded for all $i\in N$ and all $s\in T$. Let define k^* such that $v_i(-k^*)\equiv [p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))]$ for any arbitrary $p_i(s)$, $\beta_i(s)$ and $\alpha_i(s)$. We show that k^* and $v_i(-k^*)$ must be bounded.

Consider the collection $\langle\{p_i(t),\beta_i(t),\alpha_i(t)\}i\in N,t\in T\rangle$ which solves the following optimization problem:

(C.14) maximize $\sum_{t\in T} [p_i(t)\, v_i(-\beta_i(t)) + (1-p_i(t))\, v_i(-\alpha_i(t))]\, \pi(t)$ subject to

(C.15) $\sum_{t\in T} [p_i(t)\beta_i(t) + (1-p_i(t))\alpha_i(t)]\, \pi(t) \geq - \bar{w}$ and

(C.16) $[p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))] = v_i(-k^*)$.

Since $v_i(.)$ is increasing and concave in wealth, the optimal solution for the above problem is such that for $\forall$ $r\neq s$ in T, $\alpha_i(r)=\beta_i(r)= [\bar{w}+ k^*\pi(s)] / [(1-\pi(s))]$ and $\alpha_i(s)=\beta_i(s)=k^*$.

Thus, if $[p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))] = v_i(-k^*)$, then the inequalities (C.8) and (C.13) hold simultaneously only if the following is true:

(C.17) $\pi(s)\, v_i(-k^*) + [1-\pi(s)]\; v_i\left(\frac{[\bar{w}+ k^*\pi(s)]}{[1-\pi(s)]}\right) \geq -v_i(\bar{w})$

If $k^*>0$, then we may rewrite (C.17) as follows:

(C.18) $\frac{[v_i(-k^*)]}{k^*} \geq \frac{1}{\pi(s)k^*}\; [-v_i(\bar{w}) - [1-\pi(s)]\; v_i\left(\frac{[\bar{w}+k^*\pi(s)]}{[1-\pi(s)]}\right)]$

Since by assumption, $1-\xi> \pi(s)>\xi$, we have for all $\pi(s)$:

(C.19) $\frac{[v_i(-k^*)]}{k^*} \geq \frac{1}{\xi k^*}\; [-v_i(\bar{w}) - \xi\; v_i\left(\frac{[\bar{w}+k^*\xi]}{[1-\xi]}\right)]$

Now, suppose that k^* converges to $+\infty$. By (H4), the left-hand of inequality (C.19) will converge to $-\infty$ and we can show that the right-hand side will converge to some finite negative number no smaller than $-(\xi v'(0)/1-\xi)$[32]. This would violate (C.19): k^* must

[32] Consider the limit of the right-hand side of the inequality (C.11) when k^* converges to $+\infty$:

therefore be bounded from above. Now, suppose that $v_i(-k^*)$ converges to $-\infty$, since k^* is finite and the rigth-hand side of (C.19) remains finite, (C.19) would be violated: $v_i(-k^*) \equiv [p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))]$ must therefore be bounded from below. Now let define $h^* = -[\overline{w}+k^*\pi(s)]/[1-\pi(s)]$ and $k^* = -[(1-\pi(s))h^*+\overline{w}]/\pi(s)$. Using a similar argument we can show that h^* cannot converge to $+\infty$, so that k^* can not converge to $-\infty$. k^* must be bounded from below. Now, since $v_i(-k^*) \leq v_i'(0)\,(-k^*)$ for every negative k^*, $v_i(-k^*) \equiv [p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))]$ is bounded from above.

Thus, there exists some M' large enough such that if for some y we have $|\, p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))\,| > M'$ for some i and t, then y cannot be a feasible auction given some z, such that $\pi(t)<\xi$ for $\forall\ t\in T$. If $y\in\phi(z)$ then $|\, p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))\,| \leq M'$ for all i and t. By assumption H5, $|u_i(-x,t)-v_i(-x)| < B$ for every i and t, we have $|\, p_i(s)\, u_i(-\beta_i(s);t) + (1-p_i(s))\, v_i(-\alpha_i(s))\,| \leq |\, p_i(s)\, v_i(-\beta_i(s)) + (1-p_i(s))\, v_i(-\alpha_i(s))\,| + |\, p_i(s)\,(u_i(-\beta_i(s);t) - v_i(-\beta_i(s)))| < M'+B$. Thus, there exists a finite number $M=M'+B$, such that $|\, F_i(t,s|y)\,| < M$ for every bidder i, t and s in T and feasible auction y.

One can apply the same argument using the seller's expected utility. If we fix the following: $\{[\sum_{i\in N} p_i(s)\, v_0(\beta_i(s)+\sum_{j\neq i}\alpha_j(s))]+[\sum_{i\in N}(1-p_i(s))]\, v_0(\sum_{j\in N}\alpha_j(s))\} = v_0(k^*)$ for some $s\in T$, and if we maximize the left-hand side of inequality (C.11) subject to condition (C.10) where $J=N$, we show that the inequalities (C.10) and (C.11) hold simultaneously only if the following is true:

$$\text{(C.20)} \qquad \pi(s)\, v_0(k^*) + [1-\pi(s)]\ v_0\!\left(\frac{[\overline{w}-k^*\pi(s)]}{[1-\pi(s)]}\right) \geq -v_0(\overline{w})$$

We can show that (C.20) implies boundness of k^* and $v_0(k^*)$. Thus, there exists a finite number M, such that $|\, F_0(t,s|y)\,| < M$ for every t and s in T and feasible auction y.

Q.E.D.

We have shown the existence of some set Y(M) such that if y is a feasible auction given some z, then $y\in Y(M)$. Further, since every payoff attained by some auction in the compact space $\overline{Y}(M)$ can be approximated arbitrarily close by some sequence of auctions in

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$$\lim_{k^*\to\infty} \frac{1}{\xi k^*}\left[-v_i(\overline{w}) - \xi\, v_i\!\left(\frac{[\overline{w}+k^*\xi]}{[1-\xi]}\right)\right] = \lim_{k^*\to\infty} \frac{1}{\xi}\left[\frac{-\xi^2}{[1-\xi]}\ v_i'\!\left(\frac{[\overline{w}+k^*\xi]}{[1-\xi]}\right)\right]$$
$$= -v_i'(\infty)\xi/1-\xi \ \geq -v_i'(0)\xi/1-\xi.$$

Y(M), we can define, without loss of generality, the $\overline{Y}(M)$ as the space of auctions used by the auctionner. For future reference will call $\overline{Y}$ the above compact space.

C.3.3 *Continuity*

We consider here the constrained maximization problem where the auctionner takes $\overline{Y}$ as the space of possible auctions. Given any auction environment, the seller's objective is to maximize his expected utility subject to the individual rationality constraints and the incentive compatibility constraints. The auctioneer problem is to select a $y \in \phi(z) \subset \overline{Y}$ in order to maximize $\Pi(y,z)$ given $z \in Z$. We want to show that V(.), the maximum expected payoff the auctioneer can attain given the type distribution and the seller's valuation, and M(.), the set of "optimal" auction, are respectively continuous and upper hemi-continuous in Z.

$\Pi(y,z)$, the expected utility for the auctioneer given some auction environment z and auction y, is given by (C.21)

$$\Pi(y,z)=\Sigma_{t\in T}\{[\Sigma_{i\in N}\, p_i(t)\, v_0(\beta_i(t)+\Sigma_{j\neq i}\alpha_j(t))]+[\Sigma_{i\in N}(1-p_i(t))]\, u_0(\Sigma_{j\in N}\alpha_j(t);\, t)\}\pi(t)$$

Let $E_i(t_i,s_i)/\pi_i(t_i)$ be the expected utility of bidder i when he has type t_i and reports type s_i. We have (C.22): $E_i(t_i,s_i|y,z)$

$$=\Sigma_{t_{-i}\in T_{-i}}[p_i(t_{-i},s_i)\, u_i(-\beta_i(t_{-i},s_i);(t_{-i},t_i))+(1-p_i(t_{-i},s_i))\, v_i(-\alpha_i(t_{-i},s_i))]\, \pi(t_{-i},t_i)$$

The set of all "feasible auctions" in $\overline{Y}$ which satisfy the Bayesian incentive compatible and the individual rationality constraints, $\phi(z)$, is formally defined as follows: $y \in \phi(z)$ if and only if there exists a sequence $\{y^n\}$ in $Y \subset \overline{Y}$ converging to y (with respect to $d_{\overline{Y}}$) such that for every n, $y^n=\langle\{p_i^n(t), \beta_i^n(t), \alpha_i^n(t)\};\, i\in N, t\in T\rangle$ and

(C.23) $E_i(t_i,t_i|y^n,z) \geq 0$; for each $i \in N$ and each $t_i \in T_i$.

(C.24) $E_i(t_i,t_i|y^n,z) - E_i(t_i,s_i|y^n,z) \geq 0$, for each $i \in N$ and each s_i and $t_i \in T_i$.

The inequalities in (C.23) specify the individual rationality constraints. The inequalities in (C.24) correspond to the incentive compatibility constraints.

Let denote the extension of Π to $\overline{Y}xZ$. That is, for $(y,z) \in \overline{Y}xZ$, $\Pi(y,z)= \lim_{n\to\infty} \Pi(y_n,z)$ for some sequence (y_n) in Y converging to y with respect to $d_{\overline{Y}}$. Similarly, for every $i \in N$ and $s,t \in T$, let $\overline{E}_i(s_i,t_i|y,z)$ denote the extension of $E_i(s_i,t_i|y,z)$ to $\overline{Y} \times Z$.

LEMMA C.4: *The following holds*

i) *the auctioneer's expected payoff function* Π: $\overline{Y} \times Z \rightarrow \mathbb{R}$, *is a continuous function in* $\overline{Y} \times Z$.

The set-valued function $\phi(z)$ *from* Z *into* $\overline{Y}$ *defined by* (C.23) *and* (C.24) *is*

ii) *non-empty for all* $z \in Z$,

iii) *upper hemi-continuous in* Z,

iv) *lower hemi-continuous in* Z. (see proof in section C.5.2)

THEOREM C.2: *Let* Z *be the space of all possible auction environments, let* V(z) *be the maximum expected payoff attainable by the auctioneer given the auction environment* z, *and let* M(z) *be the set of all optimal auctions given* z. *Then* V(z) *is continuous and* M(z) *is upper hemi-continuous in* Z.

Theorem C.2, like Theorem C.1, follows from the Maximum Theorem.

COROLLARY: *Define* $\Omega(z) \equiv \sup\{\Pi(y,z)| \ y \in \phi(z) \cap Y\}$, *as the supremum expected utility of the auctionner over the set of auctions satisfying the individual rationality constraints and Bayesian incentive compatibility constraints. Then* $\Omega(z)$ *is continuous in* z.

PROOF: From lemma C.3, there exists some large M such that $\phi(z) \subseteq Y(M)$. Thus $\Omega(z) \equiv \sup\{\Pi(y,z)| \ y \in \phi(z) \cap Y\} = \sup\{\Pi(y,z)| \ y \in Y(M), y \in \phi(z)\}$. Now let $\overline{Y}$ be the completion of Y(M), where is a compact space. Since for every payoffs attainable by some $y \in \overline{Y}$, there exists a sequence in Y(M) which yields payoffs arbitrarily close to y, we have $\sup\{\Pi(y,z)| \ y \in Y(M), y \in \phi(z)\} = \max\{\Pi(y,z)| \ y \in \overline{Y}, y \in \phi(z)\} \equiv V(z)$. It follows that $\Omega(z) = V(z)$, for every z. Since, by Theorem C.2, V(z) is continuous in Z, then $\Omega(z)$ must also be continuous.

C.4 CONCLUSION

According to Crémer and McLean (1988), in "nearly all" auctions, the seller will be able to extract all tne surplus and the asymmetry of information between the seller and the buyer will have no economic relevance. This follows from the discontinuity in the auctioneer optimal payoff function when the valuations of bidders are no longer statistically independent.

The auctioneer is able to extract all the rents, in Crémer's and McLean's framework, in part because he can request and offer arbitrarily large payments. When we bound the set of feasible auctions, the seller's optimal expected payoff becomes continuous in the set of possible auction environment. We show in this essay, that limited liability and/or risk averse bound the set of feasible auctions and guarantee continuity in the auction design problem.

Given limited liability and/or risk averse, if we wish to insure that bidders are able to exploit their private information, we need not presume that information is exactly independently distributed. So long as the distribution of private valuations is close enough to the independence case, the seller's expected payoff will be close to what he can get under independent values. For all pratical purposes, it may still be legitimate to assume independence of private valuations.

C.5. APPENDICES

C.5.1 *Mathematical appendix*

DEFINITIONS: Let ϕ be a mapping from a topological space Z into a topological space Y. We say that ϕ is **lower hemi-continuous at** z_0 (l.h.c.at z_0)if for each open set B meeting $\phi(z_0)$ there is a neighbourhood $N(z_0)$ such that $z \in N(z_0) \rightarrow \phi(x) \cap B \neq \emptyset$. We say that ϕ is **upper hemi-continuous at** z_0 (u.h.c.at z_0)if for each open set B meeting $\phi(z_0)$ there is a neighbourhood $N(z_0)$ such that $z \in N(z_0) \rightarrow \phi(z) \subset B$. We say that ϕ is **lower hemi-continuous** in Z (l.h.c.in Z) if it is lower hemi-continuous at each point of Z. We say that ϕ is **upper hemi-continuous** in Z (u.h.c.in Z) if it is upper hemi-continuous at each point of Z and if, also, $\phi(z)$ is a compact set for each z. If ϕ is both l.h.c. in Z and u.h.c. in Z, then it is called **continuous** in Z.

Define ϕ^- by $\phi^-(B) \equiv \{z \in Z \mid \phi(z) \cap B \neq \emptyset\}$ and define ϕ^+ by $\phi^+(B) \equiv \{z \in Z \mid \phi(z) \subset B\}$

THEOREM A: (Berge 1963, p.109) A necessary and sufficient condition for ϕ to be lower hemi-continuous (l.h.c.) is that for each open set B in Y, the set $\phi^-(B)$ is open.

THEOREM B: (Berge 1963, p.110) A necessary and sufficient condition for ϕ to be upper hemi-continuous (u.h.c.) is that the set $\phi(z)$ is compact for each z and that for each open set B in Y, the set $\phi^+(B)$ is open.

DEFINITION: We say that ϕ is a *closed-mapping* of Z into Y if whenever $z^o \in Z$ and $y^o \in Y$, $y^o \notin \phi(z)$ there exist neighbourhoods $N_Y(y^o)$ and $N_Z(z^o)$ such that $z \in N_Z(z^o)$ imply that $\phi(z) \cap N_Y(y^o) = \emptyset$.

FACT A: (Berge 1963, p.111) If g is a continuous real-valued function in Z×Y, the mapping defined by $\phi(z)=\{y \mid y \in Y, g(z,y) \leq 0\}$ is a closed mapping of Z into Y.

THEOREM C: (Berge 1963, p.111) If $(\phi_i / i \in I)$ is a family of closed mappings of Z into Y, then $\phi = \cap_{i \in I} \phi_i$ is also a closed mapping.

THEOREM D: (Berge 1963, p.112) If ϕ_1 is a closed-mapping of Z into Y and $\overline{Y}$ is a compact subset of Y, the mapping $\phi = \phi_1 \cap \overline{Y}$ is u.h.c.

THEOREM E: If $\langle g_i \mid i \in I \rangle$ is a family of continuous real-valued functions in ZxY, the mapping defined by $\phi(z)=\{ y \mid y \in \overline{Y};\ g_i(y,z) \geq 0 \text{ for } \forall i \in I\}$ is u.h.c. if $\overline{Y}$ is compact.

PROOF: Let us define $\phi_i(z)=\{ y \mid g_i(y,z) \geq 0 \}$ for each $i \in I$ and define $\phi_1(z) = \cap_{i \in I} \phi_i(z)$. Since each g_i is continuous, each mapping ϕ_i is a closed mappings (fact A). By Theorem C, ϕ_1 must also be a closed mapping. Note that $\phi = \phi_1 \cap \overline{Y}$, where $\overline{Y}$ is, by assumption, a compact set. By Theorem D, ϕ is u.h.c.

THEOREM F: Let g be a continuous real-valued function in YxZ, such that $\forall\ y \in Y$, $z \in Z$ and $\forall\ \delta > 0$, if $g(y,z)=0$, there always exists some y^o such that $d_Y(y,y^o) < \delta$ and such that $g(y^o,z)>0$. Then the mapping defined by $\phi(z)=\{ y \mid y\in Y;\ g(y,z)\geq 0\}$ is l.h.c.

PROOF: Let define $\phi(z)=\{ y \mid y\in Y;\ g(y,z)\geq 0 \}$ and $\phi^-(B)=\{z \mid \exists\, y\in B:\ g(y,z)\geq 0 \}$. We want to show that for any open set B in Y if $z\in\phi^-(B)$, then there is some ε such that if $d_Z(z,z^o)<\varepsilon$ then $z^o\in\phi^-(B)$. Suppose that $g(y,z)>0$ for some $y\in B$, then by continuity of $g(y,z)$, there exists some $\varepsilon>0$ such $d_Z(z,z^o)<\varepsilon$ imply that $d(g(y,z),g(y,z^o)) < g(y,z)$ or $g(y,z^o)>0$. In this case, $d_Z(z,z^o)<\varepsilon$ imply that $z^o\in\phi^-(y)\subset \phi^-(B)$. Now, suppose that $g(y,z)=0$ for some $y\in B$. Since B is an open set, there must exist some $\delta>0$ such that $d_Y(y,y^o)<\delta$ implies $y^o\in B$. By assumption, for every $\delta>0$, there exists some y^o such that $d_Y(y,y^o)<\delta$ and $g(y^o,z)>0$. Using an argument as above, we can show that there exists some $\varepsilon>0$ such $d_Z(z,z^o)<\varepsilon$ implies that $g(y^o,z^o)>0$, i.e. $z^o\in\phi^-(y^o)\subset\phi^-(B)$. Thus the set $\phi^-(B)$ is open for all open set B.

MAXIMUM THEOREM: (Berge, 1963) If Π is a continuous real-valued function in $Y\times Z$ and ϕ is a continuous mapping of Z into Y such that, for each z, $\phi(z)\neq\emptyset$, then the real-valued function $V(z)= \max\{ \Pi(y,z) \mid y\in\phi(z)\}$ is continuous in X and the mapping M defined by $M(z)=\{y \mid y\in\phi(z), \Pi(y,x)=V(x)\}$ is a u.h.c. mapping of Z into Y.

DEFINITION: (Friedman 1970, p.92) A subset $\overline{Y}$ of a metric space Y is called **sequentially compact** if every sequence $\{y^n\}$ in $\overline{Y}$ has a subsequence $\{y^{nk}\}$ that converges to a point in $\overline{Y}$; that is, $d_Y(y^{nk},y)\to 0$ as $n_k\to\infty$, for some $y\in\overline{Y}$.

THEOREM G: (Friedman 1970, p.108) A subset of a metric space is compact if and only if it is sequentially compact.

C.5.2 *Proofs*.

PROOF OF LEMMA C.1: PART i): Continuity of $\Pi(y,z)$ in $\overline{Y}\times Z$ should be obvious to the reader. In equation (C.1), $\Pi(.,.)$ is linear in each $p_i(t)$, $x_i(t)$, $\pi(t)$ and $w_0(t)$.

PART ii): We need to show that for every $z=[\pi,w]$ there exists some auction y that satisfies all the feasibility constraints. In particular, consider $y=\langle\{p_i(t), x_i(t)\}; i\in N, t\in T\rangle$ where $p_i(.)=0$ and $x_i(.)=0$ for all i and t. The above auction satisfies trivially the individual ratioanlity constraints, the incentive compatibility constraints and the limited budget constraints for every $B_i \geq 0$.

PART iii): The set of feasible auctions is defined as the set of auctions such that $E_i(t_i,t_i) \geq 0$; for each $i \in N$ and each $t_i \in T_i$ and $E_i(t_i,t_i) - E_i(t_i,s_i) \geq 0$, for each $i \in N$ and each s_i and $t_i \in T_i$.} In lemma C.1, each functions $E_i(t_i,s_i)$ is given by

(C.2) $E_i(t_i,s_i) = \sum_{t\text{-}i \in T\text{-}i} [p_i(t\text{-}i, s_i)\, w_i(t\text{-}i, t_i) - x_i(t\text{-}i, s_i)]\, \pi(t\text{-}i, t_i)/\pi_i(t_i)$

Each $E_i(t_i,s_i)$ is linear in each $p_i(t)$, $x_i(t)$, $\pi(t)$ and $w_i(t)$ and is therefore continuous in $Z \times Y$. This implies that the mapping $\phi(z)$ is of the form $\phi(z)=\{ y \mid y \in \overline{Y};\ g_i(y,z) \geq 0$ for $\forall i \in I\}$ where $<g_i \mid i \in I>$ is a family of continuous real-valued function in $Z \times Y$ and where $\overline{Y}$ is a compact set. By Theorem E, ϕ must be u.h.c.

PART iv): The mapping $\phi(z)$ defining the set of feasible auction given z is of the form $\phi(z)=\{y \mid y \in \overline{Y},\ \inf_{i \in I} g_i(y,z) \geq 0\}$, where $<g_i(.,.);\quad I>$ denotes a family of continuous functions. Since every function g_i is continuous in $\overline{Y} \times Z$, the function $g(y,z) \equiv \inf_{i \in I} g_i(y,z)$ is also continuous in $\overline{Y} \times Z$.

In order to apply Theorem F, we must show that $\forall\ \delta > 0$, if $g(y,z)=0$ then there always exists some y^o such that $d(y,y^o) < \delta$ such that $g(y^o,z) > 0$. That is, there exists an auction y^o arbitrarily close to y such that all constraints are not binding.

One can check that the above condition hold in the context of the auction problem in section C.2. If some individual rationality constraints binding, a decrease in each $x_i(t)$ by some arbitrarily small amount will guarantee that $E_i(t_i,t_i|y,z) > 0$, for each $i \in N$ and each $t_i \in T_i$. If we have $E_i(t_i,t_i|y,z) - E_i(t_i,s_i|y,z) = 0$ and $E_i(s_i,s_i|y,z) - E_i(s_i,t_i|y,z) > 0$, for some $i \in N$ and some s_i and $t_i \in T_i$, a decrease in each $x_i(t_i,t\text{-}_i)$ by some arbitrarily small amount will guarantee that $E_i(t_i,t_i|y^o,z) - E_i(t_i,s_i|y^o,z) > 0$ and $E_i(s_i,s_i|y^o,z) - E_i(s_i,t_i|y^o,z) > 0$. Now suppose that $E_i(t_i,t_i|y,z) - E_i(t_i,s_i|y,z) = 0$ and $E_i(s_i,s_i|y,z) - E_i(s_i,t_i|y,z) = 0$, for some $i \in N$ and some s_i and $t_i \in T_i$. We have,

$$\sum_{t\text{-}i \in T\text{-}i} [(p_i(t\text{-}i, t_i)-p_i(t\text{-}i, s_i))\, w_i(t\text{-}i, t_i) - (x_i(t\text{-}i, t_i)-x_i(t\text{-}i, s_i))]\, \pi(t\text{-}i, t_i)= 0 =$$

$$\sum_{t\text{-}i \in T\text{-}i} [(p_i(t\text{-}i, t_i)-p_i(t\text{-}i, s_i))\, w_i(t\text{-}i, s_i) - (x_i(t\text{-}i, t_i)-x_i(t\text{-}i, s_i))]\, \pi(t\text{-}i, s_i)$$

If $w_i(t_{\text{-}i}, t_i) > w_i(t_{\text{-}i}, s_i)$ for some $t_{\text{-}i}$, we can increase $(p_i(t_{\text{-}i}, t_i)-p_i(t_{\text{-}i}, s_i))$ by some small amount so that $E_i(t_i,t_i|y^o,z) - E_i(t_i,s_i|y^o,z) > E_i(s_i,t_i|y^o,z) - E_i(s_i,s_i|y^o,z)$. Thus there will exist some p_i and x_i such that $E_i(t_i,t_i|y^o,z) - E_i(t_i,s_i|y^o,z) > 0 > E_i(s_i,t_i|y^o\ \ ) - E_i(s_i,s_i|y^o,z)$.

If $\pi(t_{-i}, t_i) > \pi(t_{-i}, s_i)$ for some t_{-i}, we can increase $(x_i(t_{-i}, t_i) - x_i(t_{-i}, s_i))$ by some small amount so that $E_i(t_i,t_i|y^o,z) - E_i(t_i,s_i|y^o,z) > E_i(s_i,t_i|y^o,z) - E_i(s_i,s_i|y^o,z)$. Thus there will exist a x_i such that $E_i(t_i,t_i|y^o,z) - E_i(t_i,s_i|y^o,z) > 0 > E_i(s_i,t_i|y^o,z) - E_i(s_i,s_i|y^o,z)$.

Thus $\forall\ \delta>0$, if $\inf_{i\in I} g_i(y,z)=0$, there always exists some y^o such that $d(y,y^o)<\delta$ such that $\inf_{i\in I} g_i(y^o,z)>0$. By Theorem F, ϕ is l.h.c.

PROOF OF LEMMA C.4: The arguments to prove lemma C.4 are much similar than those presented above. However, we must first show the continuity of $\Pi(y,z)$ in $\overline{Y}\times Z$ and of each function $E_i(t_i,s_i|(y^o,z^o))$ with respect to the metric defined by (C.5) on $\overline{Y}$ and the Euclidean metric defined over Z.

The distance between any two auctions y^o and y in Y is given by:

$$\text{(C.5)}\quad d_Y(y^o,y)= \{\sum_{t\in N}\sum_{s\in N} (F_0(s,t\,|y^o)-F_0(s,t\,|y))^2 + \sum_{i\in N}\sum_{t\in T}\sum_{s\in T} (F_i(s,t\,|y^o)-F_i(s,t\,|y))^2\}^{1/2}$$

Where $F_i(s,t\,|y) = [p_i(t)\ u_i(-\beta_i(t);\, s) + (1-p_i(t))\ v_i(-\alpha_i(t))]$ is the expected utility of bidder i in auction y when $t\in T$ is the vector of announced types and $s\in T$ be the vector of true types. Similarly, $F_0(s,t\,|y) = [\sum_{i\in N} p_i(t)\ v_0(\beta_i(t)+\sum_{j\neq i} \alpha_j(t)) + \sum_{i\in N} (1-p_i(t))\ u_0(\sum_{j\in N}\alpha_j(t);\, s)]$ denote the seller's utility given y, t and s.

Thus $\Pi(y^o,z^o) = \sum_{t\in T} (F_0(t,t\,|y^o)\ \pi^o(t)$ and $E_i(t_i,s_i|(y^o,z^o))=\sum_{t_{-i}\in T_{-i}} (F_i((t_{-i},t_i),(t_{-i},s_i)\,|y^o)\ \pi^o((t_{-i},t_i))$.

We now show that if $d_Y(y^o,y)<\varepsilon$ then $d(\Pi(y^o,z^o),\Pi(y,z^o))$[33] $<\varepsilon$ for every $\varepsilon>0$ and $z^o\in Z$. We have $d(\Pi(y^o,z^o),\Pi(y,z^o)) = |\sum_{t\in T} (F_0(t,t\,|y^o)-(F_0(t,t\,|y)\ \pi^o(t)\,|$

$\leq \{\sum_{t\in T} [(F_0(t,t\,|y^o)-(F_0(t,t\,|y)]^2\}^{1/2} \leq d_Y(y^o,y)$.

Also, if $d(z^o,z) <\varepsilon/M$ then $d(\Pi(y,z^o),\Pi(y,z)) < \varepsilon$ for every $\varepsilon>0$ and $y\in Y$. Indeed,

$d(\Pi(y,z^o),\Pi(y,z)) = |\sum_{t\in T} (F_0(t,t\,|y^o)\ (\pi^o(t)-\pi(t))\,| \leq \max_{t\in T} (F_0(t,t\,|y^o)\ \{\sum_{t\in T} [\pi^o(t)-\pi(t)]^2\}^{1/2} \leq M\, d(z^o,z)$. Thus, if $d((y^o,z^o),(y,z))= \{[d_Y(y^o,y)]^2+[d(z^o,z)]^2\}^{1/2} <\varepsilon/(M+1)$ then $d_Y(y^o,y)<\varepsilon/(M+1)$ and $d(z^o,z)<\varepsilon/(M+1)$. It follows that $\varepsilon > d(\Pi(y^o,z^o),\Pi(y,z^o))+d(\Pi(y,z^o),\Pi(y,z)) \geq d(\Pi(y^o,z^o),\Pi(y,z))$. $\Pi(y,z)$ is continuous in $Y\times Z$. We have established part i) of lemma C.4.

Using a similar argument, we can also shown that each function $E_i(t_i,s_i|(y^o,z^o))$ is continuous in $Y\times Z$. Continuity of the extensions and $\overline{E}_i$ and Π on $\overline{Y}\times Z$ follows form the

[33] The distance d(.,.) denotes the usual Euclidean distance.

definition of $d_{\overline{Y}}$.

PART ii): Consider the auction $y=\langle\{p_i(t), \beta_i(t), \alpha_i(t)\}; i\in N, t\in T\rangle$ where $p_i(t)=0$, $\beta_i(t)=0$ and $\alpha_i(t)=0$ for all i and t. The above auction satisfies trivially the individual rationality constraints and the incentive compatibility constraints. The set of feasible auctions is always non-empty.

PART iii): As argued above, each $\overline{E}_i(t_i,s_i)$ is continuous in $\overline{Y}\times Z$. The mapping $\phi(z)$ is therefore of the form $\phi(z)=\{\ y \mid y\in\overline{Y},\ g_i(y,z)\geq 0 \text{ for } \forall i\in I\}$ where $\langle g_i \mid i\in I\rangle$ is a family of continuous real-valued function in $\overline{Y}\times Z$ and where $\overline{Y}$ is a compact space. By Theorem E, ϕ is u.h.c.

PART iv): Much like in the proof of lemma C.1, we can consider changes in each p_i and simultaneous changes in β_i and α_i which guarantee that for every feasible auction $y\in\overline{Y}$, there will always exist a feasible auction $y^{\circ}\in Y$, arbitrarily close to y, such that all the constraints are non-binding. This is particularly true since the distance between two auctions $y^{\circ}$and y depends of their effect of the payoffs , so long as constraints are non-binding by some very small ϵ, then the distance between $y^{\circ}$and y can be arbitrarily small. This allows us to apply Theorem F: ϕ is l.h.c. Q.E.D.

REFERENCES

Adams, W.J. and J.L. Yellen (1976), "Commodity Bundling and the Burden of Monopoly," *Quarterly Journal of Economics*, XC (1976), 475-98.

American Council of Life Insurance; *"1983-Life Insurance Fact Book"*.

Berge, C.: *"Topological Spaces,"* translated by E.M.Patterson, MacMillan Company, N.Y., 1963.

Braverman A., J.L. Guasch, S. Salop ; "Defects in Disneyland: Quality Control as a Two-Part Tariff", *Review of Economic Studies*, L (1983), 121-31.

Burdett, K. and K.L. Judd: "Equilibrium Price Dispersion," *Econometrica*, 51(1983), 955-969.

Butters, G.R.: "Equilibrium Distributions of Sales and Advertising Prices." *Review of Economics Studies*, 44(1977), 465-491.

Canadian Insurance Agent & Broker Statistics, "1987 Annual Review of Statistics, *Canadian Insurance*, (1987).

Comanor,W.S. and T.A. Wilson: "Advertising, Market Structure and Performance," *Journal of Economic Literature*, Vol. XVII (June 1979), 453-476.

Crémer J. and R.P.McLean: "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56(Nov.88) 1247-1257.

Diamond, P.A.: "A Model of Price Adjustment," *Journal of Economic Theory*, 3(1971), 156-168.

Friedman, A. *"Foundations of Modern Analysis,"* Dover Publications Inc., N.Y., 1970.

Grossman G.M. and C.Shapiro: "Informative Advertising with Differentiated Products," *Review of Economic Studies*, LI (1984), 63-81.

Harris, M. and R.M. Townsed, "Allocation Mechanisms, Asymmetric Information, and the 'Revelation Principle'," in *Issues in contemporary microeconomics and welfare*. Ed. George Fiewel. (Albany: SUNY Press, 1985), 379-94.

Lippman, S. and J.J. McCall: "The Economics of Job Search: A Survey," *Economic Inquiry*, 14(1976), 155-189.

Maskin, E. and J. Riley, "Monopoly under Incomplete Information," *Rand Journal of Economics*, XV (Summer 1984), 171-96.

McAfee, R.P. and J.McMillan: "Auctions and Bidding," *Journal of Economic Literature*, XXV(June 1987), 699-738.

McAfee, R.P., J. McMillan and M.D. Whinston (1989), "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values," *Quarterly Journal of Economics*, forthcoming.

McAfee, R.P. and P.J.Reny: "Correlated Information and Mechanism Design," mimeo University of Western Ontario, July 1988.

Myerson, R.B.: "Optimal Auction Design," *Mathematical of Operations Research*, 6(1981), 58-73.

Myerson, R.B., "Bayesian Equilibrium and Incentive Compatibility: An Introduction," in *Social Goals and Social Organization*. Eds.:Leonid Hurwicz, David Schmeidler, and Hugo Sonnenshein. (Cambridge U. Press, 1985), pp.229-59.

Pauly, M.V., "Taxation, Health Insurance and Market Failure in the Medical Economy," *Journal of Economic Literature*, XXIV (June 1986).

Pay Research Bureau, *"Benefits and Working Conditions - 1986"*, Canada.

Riley, J.G., "Competitive Signalling," *Journal of Economic Theory*, X (April 1975), 174-86.

Riley, J.G., "Informational Equilibrium," *Econometrica*, XLVII (March 1979), 331-59.

Rothschild, M. and J. Stiglitz, "Equilibrium in Insurance Markets: An Essay on the Economics of Imperfect Information," *Quarterly Journal of Economics*, XC (Nov.1976), 629-49.

Salop, S. and J. Stiglitz: "Bargains and Ripoffs: A Model of Monopolistically Competitive Prices," *Review of Economic Studies*, 44(1976), 493-510.

Schlensiger, H. and E. Venezian, "Insurance Market with Loss-Prevention Activity: Profits, Market Structure, and Consumer Welfare," *Rand Journal of Economics*, XVII (Summer 1986).

Schmalensee, R.: *"The Economics of Advertising"*. Amsterdam: North-Holland, 1972.

Schmalensee, R., "Gaussian Demand and Commodity Bundling," *Journal of Business*, LVII (1984), S211-S230.

Spence, M., "Product Differentiation and Performance in Insurance Markets," *Journal of Public Economics*, X (1978), 427-447.

Stigler, G.J: "The Economics of Information." *Journal of Political Economy*, 69 (June 1961), 213-25.

Varian, H.R.: "A Model of Sales," *American Economic Review*, 70(1980), 651-658.

Wilde, L.L.: "Labor Market Equilibrium Under Nonsequential Search," *Journal of Economic Theory*, 16(1977), 373-393.

Wilde, L.L. and A. Schwartz: "Equilibrium Comparison Shopping," *Review of Economic Studies*, 46(1979), 543-554.

Wilson, C., "A Model of Insurance Markets with Incomplete Information," *Journal of Economic Theory*, XII (December 1977), 167-207.