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Transient Heat Conduction Due To A Circular Pipe Buried In A Half-space

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by

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ABSTRACT

This thesis considers the two-dimensional transient heat conduction problems due to a circular cylinder held at one constant temperature and immersed in a conducting medium with its axis parallel to a plane surface held at another constant temperature. In the conducting medium each phase is assumed to have constant properties.

Approximate perturbation solutions are obtained for: (i) the "simple" problem, in which the boundary temperatures are such that no change-of-phase takes place in the medium and (ii) the non-linear "two-phase" problem where change-of-phase occurs at a temperature between the boundary temperatures.

These analyses are mathematical idealisations of the physical problem of, say, a pipe containing a "warm" liquid flowing through a "cold" environment (or its' converse) and, in case (ii), melting the frozen material near the pipe.

Although numerical methods of solution for both of these cases are available, the convenience and usefulness of simple analytical approximations is obvious. In the case of the non-linear problem, the accuracy of the methods employed is questionable.

Acknowledgements

I would like to express my thanks to Prof. J. H. Blackwell for suggesting the topic of this thesis, and for his continued interest, encouragement and guidance throughout the course of this work. I would also like to thank Mrs. Blackwell for putting up with my frequent, work-related invasions of their home. Finally, I am indebted to my parents for affording me the opportunity they never had, to obtain a university education.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

T ; temperature

T^S, T^L ; solid and liquid temperatures

T_i ; initial temperature

T_f ; fusion temperature

T_a ; pipe temperature

R_0 ; pipe radius

ρ ; density

c ; specific heat

K ; thermal conductivity

$\kappa = \mathrm{K}/\rho c$; thermal diffusivity

$\bar{\kappa} = \kappa_L/\kappa_S$; ratio of thermal diffusivities of liquid and solid

$\bar{\mathrm{K}} = \mathrm{K}_L/\mathrm{K}_S$; ratio of thermal conductivities of liquid and solid

$\psi_a = \bar{\mathrm{K}}(T_a - T_f)/(T_i - T_f)$

L ; latent heat

$\varepsilon = c_L(T_i - T_f)/L$; Stefan Number

x', y' ; rectangular coordinates (dimensional)

x_i' ; interface location in rectangular coordinates (dimensional)

t ; time

x, y ; dimensionless rectangular coordinates

x_i ; interface location in rectangular coordinates

τ, $\bar{\tau}$, γ ; dimensionless time

α, β ; bipolar coordinates (dimensionless)

α_i ;interface location in bipolar coordinates

z ; bilinear coordinate

z_i ;interface location in bilinear coordinates

ξ, η ; dimensionless distance coordinates

ψ, $\bar{\psi}$, ϕ, $\bar{\phi}$; dimensionless temperatures of the solid and liquid

σ, $\bar{\sigma}$; dimensionless interface position

δ ; pipe immersion depth (dimensionless)

s ; Laplace Transform variable

INTRODUCTION

An exact solution to the steady-state, limiting case of the problems just described is available, but an exact analytic treatment of the transient heat conduction problem has not yet been derived.

W.W. Martin and S.S. Sadhal [1] place bounds on the solution to the problem in case (i) of the abstract. We obtain a perturbation series approximation to the solution of this problem, with the depth of the buried cylinder as a perturbation parameter. An approximation to the solution of the linear, transient heat conduction problem is obtained, for deep and shallow cylinder depths.

The ecology of the Arctic is dependent upon the integrity of the permafrost underlying most of that region. The fast pace of research and development of our Northern resources is disturbing the fragile equilibrium of this environment (Mackay [2]). Due to the harsh nature of the Northern environment these resources are transported mainly via pipelines. For example consider a cold gas pipeline. The gas is liquefied by refrigeration for transport and is maintained at about $-100°C$ by several pumping stations along a route. In the far North this will not pose a problem but as the pipe moves further to the south it encounters so-called "discontinuous" permafrost which is unfrozen for periods of time. A cold gas pipeline passing through a discontinuous area would immediately freeze the surrounding medium. The frozen zone will act as a barrier to water and thus may have a serious effect on the drainage of the local area. Additional problems are also encountered with oil pipelines. For example an enormous discovery of petroleum was made in the early 1970's near Prudhoe Bay on the North Arctic slope of Alaska. The most practical ice-free port, was 800 miles away, on the southern shore of Alaska at a port called Valdez. The Trans Alaska Pipeline System (TAPS) found that a pipeline, 4 feet in diameter with a throughput of about 2 million barrels/day (as proposed by the petroleum industry) would result in maintaining a high temperature of the oil from the Northern point of entry to the south-coast port of Valdez. The estimates of the oil temperature ranged from about $65°C$ to about $80°C$. The rapid thawing that would result was cause for grave concern since it would disrupt the local environment and possibly threaten the security of the pipeline itself with devastating consequences.

In view of the possibility for environmental damage that could result it is important to first of all develop a rational engineering "design" to model the predominant features of this complex problem. Based upon experiment and field research, engineers have used the following assumptions in modelling change of state occuring in permafrost: (i) Heat conduction is the dominant mode of heat transfer. Heat conduction is the most inefficient method of heat transfer mechanisms. Although this is virtually the only mechanism possible in frozen materials, heat can be transferred much more efficiently in unfrozen material if it is moving(or if the interstitial water is). In the initial stages of say, a change of phase involving melting, convection can be neglected. Later in the melting process convection may be neglected for interfacial velocities of the order of a few meters/year. If the interfacial velocity increases to a few tenths of a metre/day thawing could be increased by an order of magnitude and for a few metres/day by a thousand fold. The controlling mechanism for the interfacial velocity is the moisture content of the soil by volume which is about 30% for permafrost. A high moisture content results in a large latent heat of fusion (treated as a source of heat in the energy balance across the interface location) which greatly lowers the frontal speed. A low frontal velocity then reduces the importance of the effects of convection relative to conduction (see survey by Lachenbruch [3]). (ii) Each phase is assumed to have constant properties. If assumption (i) is correct, then a simple heat conduction model is used and the thermal conductivity must account for heat flow, in each phase, from mechanisms such as: conduction, convection in the pore air and/or pore water, radiation through the voids, evaporation and condensation, and moisture-vapour migration. Field measurements have been carried out to measure the effective values of the coefficients of conductivity: (i) by using thermal probes (Lachenbruch [4], Makowski and Machinski [5], Van Herzen and Maxwell [6]) and (ii) by temperature and heat flow measurements (Lettau [7], Pearce and Gold[8]). It has been found by Penner [9], that the horizontal conductivity is greater than the vertical conductivity but the ratio is usually less than two. The effective values of thermal conductivity coefficients may also change significantly with only small variations in the four phase nature of frozen soil (solids, water, ice, and gas). To obtain a reasonable estimate of the specific heat one must have a knowledge of dry unit weight, moisture content and (unfrozen) water content. Since not all the moisture in a soil changes phase at one temperature (Linell and Kaplar [10], Lovell [11]) the primary source of difficulty is to determine the effective (unfrozen) water content, particularly at temperatures just below freezing (Williams [12], Penner [9], Yong [13]) (see survey by C T. Hwang, D. W.

Murray and E. W. Brooker [14]). (iii) The medium changes phase at one temperature. Although a mushy region does exist between the two phases, engineers have chosen to use a single design temperature which is representative of the temperatures over which the change of phase takes place.

In view of assumptions (i), (ii) and (iii) it is reasonable then to refer to the frozen material as the "solid" and the unfrozen material as the "liquid".

A large number of important problems, in addition to the problem of interest here, involve solution of the equations describing diffusion of heat, mass or some other quantity subject to moving boundaries that are unknown a priori. Therefore much work has been done and has been cited in literature surveys by, Boley [15], Bankoff[16], Muehlbauer and Sunderland [17], Rubinstein [18] and Fox[19], among others. The main difficulty in the analysis of such problems is the non-linear nature of the energy balance across the interface location which forces the discovery of special solution techniques. One of the few exact solutions available was developed in the 1860's by Franz Neumann [20] and involves freezing/melting in a semi-infinite region. Problems involving finite domains, (as in this thesis) are more difficult to treat than problems involving semi-infinite domains, since they preclude the use of self-similar solutions. To date most multiphase free boundary problems have been highly simplified to lend themselves to analytical treatment. Two of the main simplifications involve: (i) the assumption of steady temperature profiles which is usually referred to as the quasi-steady assumption and (ii) the restriction of the initial temperature to be equal to the fusion temperature of the medium (equivalent to restricting the initial phase to be at the surface concentration in diffusion controlled moving boundary problems) which results in a so-called "one-phase" problem. Some examples of research that employ either of these two assumptions are: Riley, Smith and Poots [21], Tao [22], and Pedroso and Domoto [23] who investigated the inward solidification of cylinders and spheres and (ii) Jiji [24] and Shih and Tsay [25] the outward growth and decay of a solid phase on a cylindrical surface, Duda and Vrentas [26] the growth and dissolution of a spherical bubble and Cho and Sunderland [27], Shih and Cho [28], and Theofanos and Lim [29] the solidification of a saturated liquid outside a sphere. Weinbaum and Jiji [30] and Jiji and Weinbaum, [31] using the method of matched asymptotic expansions, have developed inner and outer expansions, for small Stefan Numbers (see pg. vii) for finite one-dimensional, and axisymmetric regions involving more than one phase.

Melting or freezing around pipes buried in permafrost has been the subject of numerous studies devoted to developing either, simplified, quasi-steady approximate analytical methods, Porkahayev [32], Thornton [33], Hwang, Seshadri and Krishnayya [34], Seshadri and Krishnayya [35], Lunardini [36], Zhang, Weinbaum and Jiji [37] or two-dimensional, finite difference or finite element numerical solutions, Wheeler [38], Gold, Johnston, Slusarchuk and Goodrich [39], Lachenbruch [31] Hwang, Murray and Brooker [40]. In this study we use the approach of Jiji and Weinbaum, in attacking the problem of freezing (or melting) due to a pipe buried in the half-space beneath a surface maintained at a constant temperature. A one-term approximation is found to the interface location for finite pipe depths, while higher approximations are obtained, for deep and shallow pipe depths, by using a two-parameter perturbation series, where the second parameter is the depth of the pipe.

G.S.H. Lock [41] demonstrates clearly the physical significance of a small Stefan Number. Since the assumption of a small Stefan Number is appropriate for most materials and situations of interest, the case of large Stefan Numbers (which has the linear problem as a limiting case) has not been considered.

Since the formulation of case (ii) of the abstract is symmetric with respect to melting or freezing, we will assume, for the sake of argument, that the change of phase involves freezing of the liquid medium surrounding the pipe.

Chapter 1

1.1 STATEMENT OF THE LINEAR PROBLEM.

The temperature distribution satisfies the Fourier heat conduction equation:

$$\frac{\partial^2 T}{\partial x'^2} + \frac{\partial^2 T}{\partial y'^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1.1.1}$$

$$(x'-d')^2+y'^2>R_0^2 : x'>0 : t>0$$

$$(x'-d')^2+y'^2=R_0^2 \quad T=T_a \quad t>0$$

$$x'=0 \quad T=T_i \quad t\geq 0$$

$$t=0 \quad T=T_i \quad (x'-d')^2+y'^2>R_0^2 : x'>0.$$

The quantities that appear in (1.1.1), are made non-dimensional by putting,

$$x=\frac{x'}{R_0},\ y=\frac{y'}{R_0},\ d=\frac{d'}{R_0},\ \phi=\frac{T-T_i}{T_a-T_i},\ \tau=\frac{\kappa t}{R_0^2}:$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial \tau} \tag{1.1.2}$$

$$(x-d)^2+y^2>1 : x>0 : t>0$$

$$(x-d)^2+y^2=1 \quad \phi=1 \quad t>0$$

$$x=0 \quad \phi=0 \quad t\geq 0$$

$$\tau=0 \quad \phi=0 \quad (x-d)^2+y^2>1 : x>0.$$

A transformation is made to bipolar coordinates (see Moon and Spencer [42] and Figure 1.1.1):

$$x=\frac{\sinh\alpha_0\ \sinh\alpha}{\cosh\alpha-\cos\beta}\ ,\quad y=\frac{\sinh\alpha_0\ \sin\beta}{\cosh\alpha-\cos\beta}.$$

The non-dimensional, linear problem in bipolar coordinates is then:

$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\beta^2} = \frac{\sinh^2\alpha_0}{(\cosh\alpha - \cos\beta)^2}\frac{\partial\phi}{\partial\tau} \tag{1.1.3}$$

$$0<\alpha<\alpha_0 : |\beta| \le \pi : \tau > 0$$

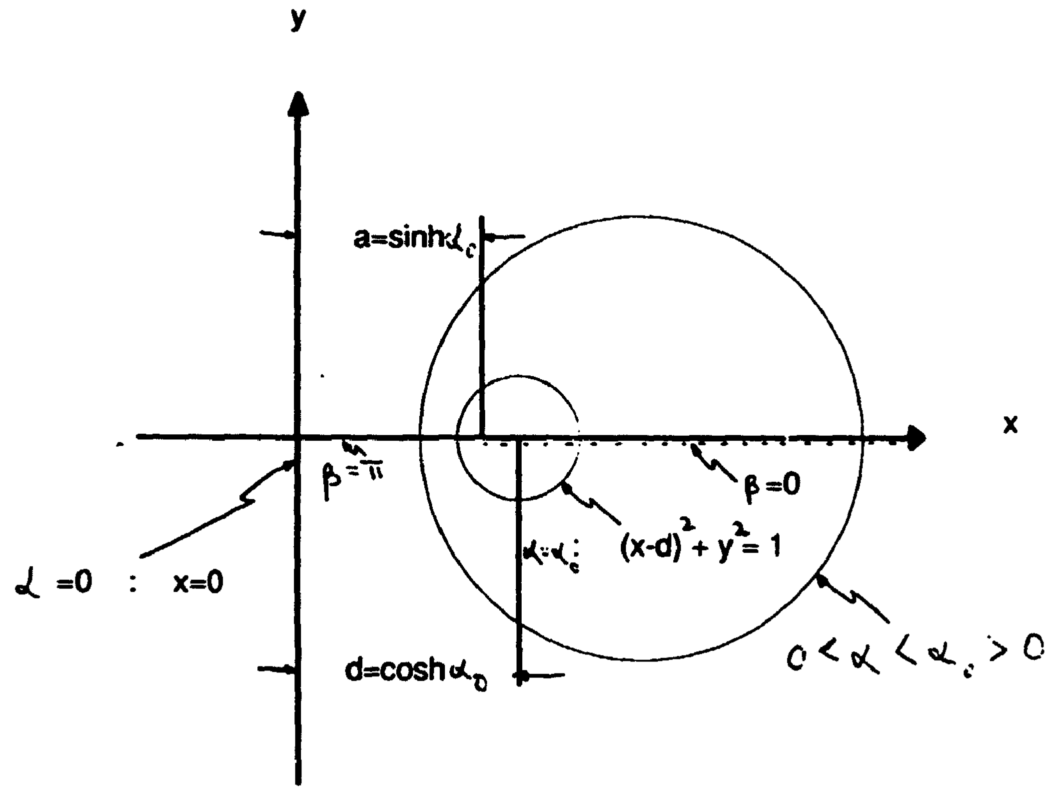
$$\alpha = \alpha_0 \quad \phi = 1 \quad \tau > 0$$

$$\alpha = 0 \quad \phi = 0 \quad \tau \ge 0$$

$$\tau = 0 \quad \phi = 0 \quad 0<\alpha<\alpha_0 : |\beta| \le \pi.$$

The parameter α_0 represents the proximity of the pipe and the plane surface. An approximation to the solution of (1.1.3) is considered for: (i) $\alpha_0 \gg 1$ and (ii) $0 < \alpha_0 \ll 1$.

Figure 1.1.1 Bipolar Coordinates



1.2 CASE I: $\alpha_0 \gg 1$

Put:

$$z = \frac{e^{-\alpha}}{\delta}, \quad \delta = \exp(-\alpha_0), \quad 0 < \delta \ll 1, \tag{1.2.1}$$

in equation (1.1.3):

$$\frac{\partial^2\phi}{\partial z^2} + \frac{1}{z}\frac{\partial\phi}{\partial z} + \frac{1}{z^2}\frac{\partial^2\phi}{\partial\beta^2} = \frac{(1-\delta^2)^2}{(1-2\delta z\cos\beta+\delta^2 z^2)^2}\frac{\partial\phi}{\partial\tau} \tag{1.2.2}$$

$$1 < z < \frac{1}{\delta} \;:\; |\beta| \le \pi \;:\; \tau > 0$$

$$z = 1 \quad \phi = 1 \quad \tau > 0$$

$$z = \frac{1}{\delta} \quad \phi = 0 \quad \tau \ge 0$$

$$\tau = 0 \quad \phi = 0 \quad 1 < z < \frac{1}{\delta} \;:\; |\beta| \le \pi.$$

A one-term approximation to the solution of (1.2.2) is found using matched asymptotic expansions.

OUTER SOLUTION

Put:

$$\phi^o = \phi_0 + M_0, \tag{1.2.3}$$

where M_0 is an asymptotic correction. If: (i) we substitute (1.2.3) into (1.2.2), (ii) expand for $\delta \ll 1$, (iii) drop the boundary condition at $z = 1/\delta$ and (iv) equate coefficients of δ^0 we have

$$\frac{\partial^2\phi_0}{\partial z^2} + \frac{1}{z}\frac{\partial\phi_0}{\partial z} + \frac{1}{z^2}\frac{\partial^2\phi_0}{\partial\beta^2} = \frac{\partial\phi_0}{\partial\tau}$$

$$z = 1 \quad \phi_0 = 1$$

$$\tau = 0 \quad \phi_0 = 0. \tag{1.2.4}$$

If we put

$$\bar{\phi}_0(z,\beta,s) = L_\tau \phi_0(z,\beta,\tau)$$

where L_τ denotes the Laplace Transform of ϕ_0, then a one-term approximation to ϕ^o is:

$$\phi^o = L^{-1}\left[\frac{K_0(\sqrt{s}\, z)}{s K_0(\sqrt{s})}\right] \qquad (1.2.5)$$

where L^{-1} denotes the inverse Laplace Transform of $\bar{\phi}_0$.

INNER SOLUTION

Put: (i) $w = \ln(\delta z)/g(\delta)$ (i.e. stretch near $z = 1/\delta$) and (ii) expand for $\delta \ll 1$ in (1.2.2):

$$\frac{\partial^2 \phi^i}{\partial w^2} + O(g^2) = \frac{g^2}{4\delta^2(1-\cos\beta)^2}\frac{\partial \phi^i}{\partial \tau}(1+\ldots) \qquad (1.2.6)$$

$$w = 0 \quad \phi^i = 0$$

where the boundary condition at $z = 1$ has been dropped. If: (i) we assume that $g = o(\delta)$ (i.e. the first order problem is steady) and (ii) as $\delta \to 0$:

$$\frac{g^2}{\delta^2}\frac{1}{A}\frac{dA}{d\tau} \to 0,$$

then we may write;

$$\phi^i = A(\delta,\tau)\,\phi_0 + M_1, \qquad (1.2.7)$$

where M_1 is an asymptotic correction.

If: (i) we substitute (1.2.7) into (1.2.6) and (ii) equate coefficients of δ^0 we have:

$$\frac{\partial^2 \phi_0}{\partial w^2} = 0 \qquad (1.2.8)$$

$$w = 0 \quad \phi_0 = 0.$$

Thus a one-term approximation to ϕ^i is:

$$\phi^i \approx A(\delta,\tau)\, w \qquad (1.2.9)$$

where $A(\delta,\tau)$ is found by matching ϕ^i with ϕ^o. Since we assumed that the temperature distribution was steady the inner solution is not valid in the neighbourhood of: (i) $\beta = 0$ and (ii) $\tau = 0$.

MATCH

Putting: (i) ϕ^i in terms of outer coordinates, (ii) expanding for $\delta \ll 1$ and (iii) keeping the lead term, we have

$$(\phi^i)^o \approx \frac{A \ \ln\delta}{g}. \tag{1.2.10}$$

Putting: (i) ϕ^o in terms of inner coordinates, (ii) expanding for $\delta \ll 1$ and (iii) keeping the lead term, we have

$$(\phi^o)^i \approx L^{-1}\left[\frac{K_0(\sqrt{s}/\delta)}{sK_0(\sqrt{s})}\right] \tag{1.2.11}$$

Solving for $A(\delta,\tau)$ yields a one-term approximation to ϕ^i as:

$$\phi^i \approx \frac{g(\delta)w}{\ln\delta} L^{-1}\left[\frac{K_0(\sqrt{s}/\delta)}{sK_0(\sqrt{s})}\right], \tag{1.2.12}$$

where $g = o(\delta)$ is sufficient for the one-term match considered here. A one-term, uniformly valid composite solution is then:

$$\phi \approx L^{-1}\left[\frac{K_0(\sqrt{s}\,z)}{sK_0(\sqrt{s})} - \frac{\ln z}{|\ln\delta|}\frac{K_0(\sqrt{s}/\delta)}{sK_0(\sqrt{s})}\right] \tag{1.2.13}$$

RESULTS

Figs. (1.2.1)-(1.2.3) show plots of the temperature distribution of (1.2.13), at $\tau = 10,1,0.1$ respectively, where $\delta = \exp(-\alpha_0)$, $\alpha_0 = 2$. Each figure shows the temperature distribution of (1.2.13) compared to the numerical solution of (1.1.3) (see Appendix) for different values of β. Figs. (1.2.1)-(1.2.3) show that (1.2.13) is closest to the numerical solution when $\beta = \pi/2$ and when $\tau \ll 1$ or $\tau \gg 1$.

As $\delta \to 0$ in (1.2.13), ϕ of (1.2.13) reduces exactly, to the case of a pipe in an infinite medium

Fig. 1.2.1 Temperature field for deep pipe depths at $\tau = 10$

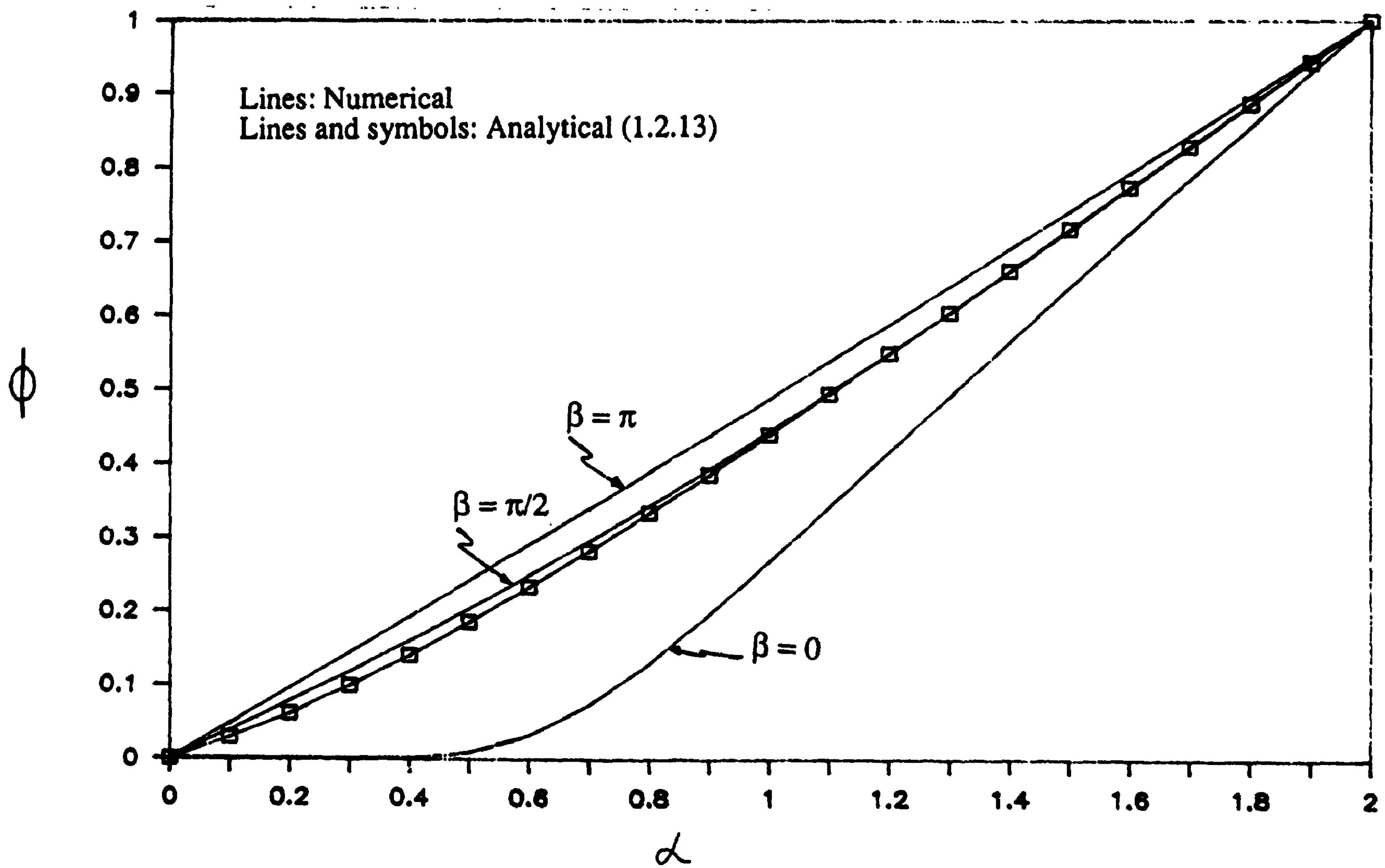


Fig. 1.2.2 Temperature field deep pipe depths at $\tau = 1$

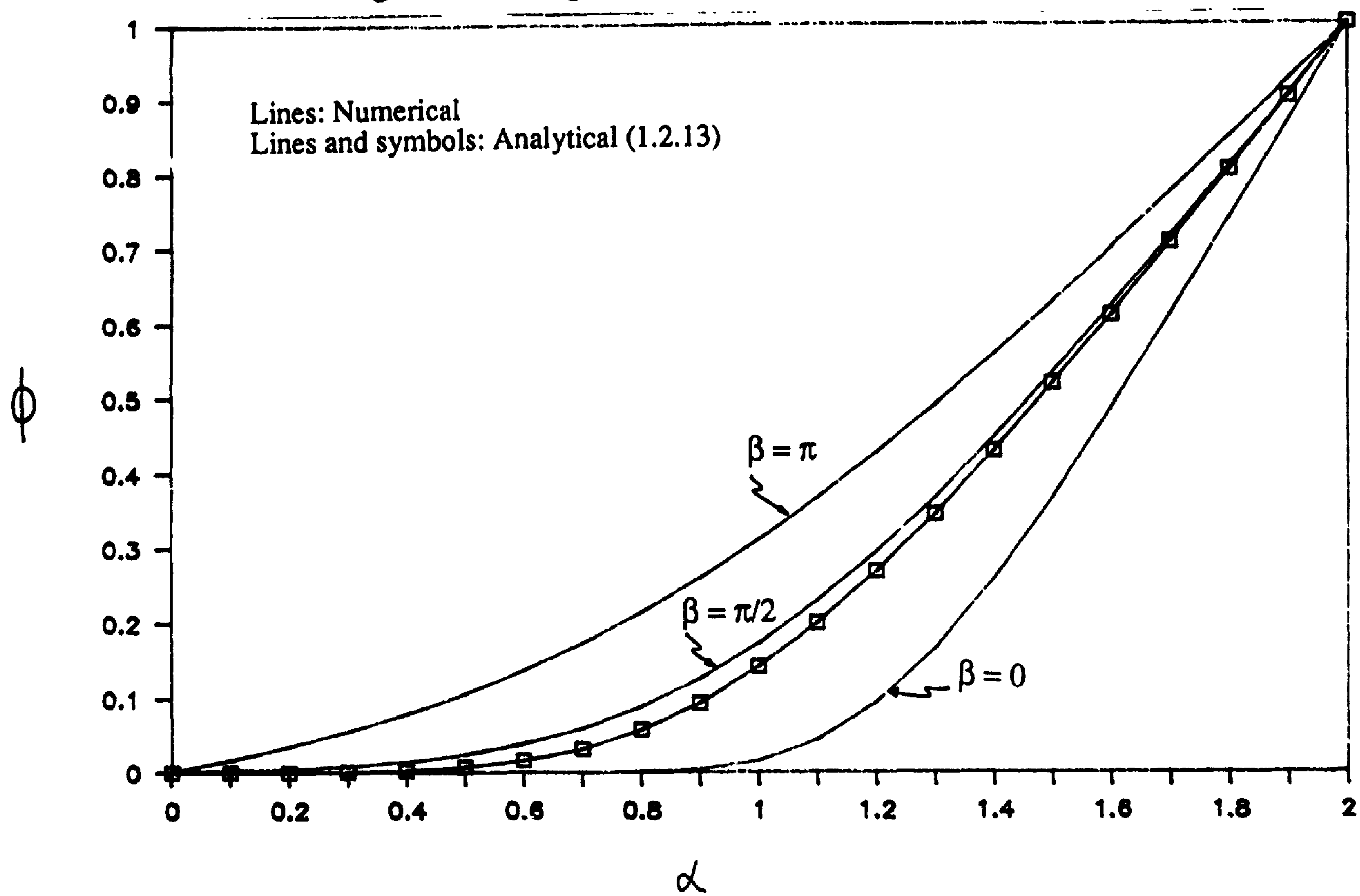
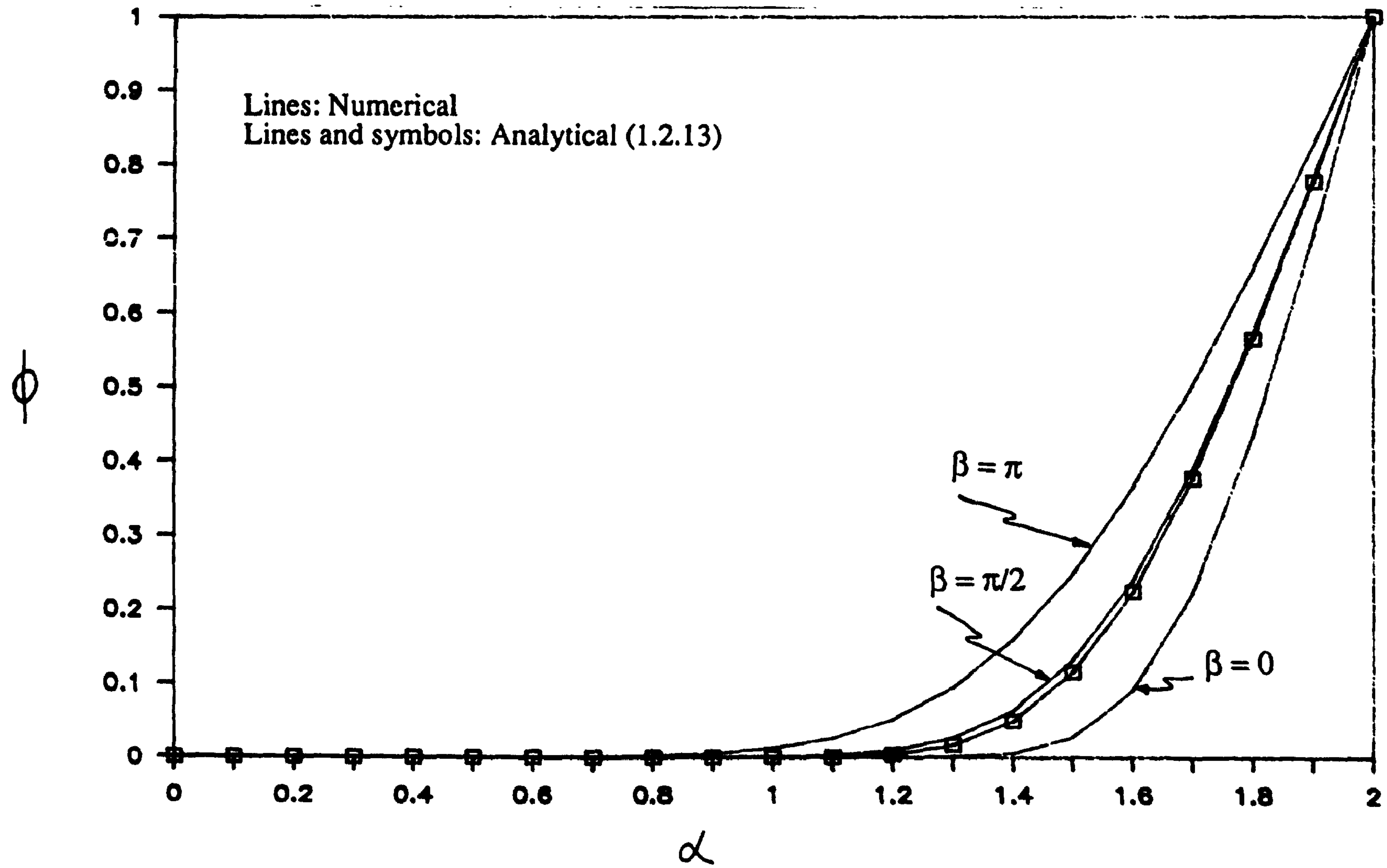


Fig. 1.2.3 Temperature field for deep pipe depths at $\tau = 0.1$



1.3 CASE II: $0 < \alpha_0 \ll 1$

An approximation to the solution of (1.1.3) is considered for: (I) $\beta \neq 0$ and (II) $\beta = O(\alpha_0)$.

(I) : $\beta \neq 0$

Put: (i) $\nu = \alpha/\delta$, $\delta = \alpha_0$, $0 < \delta \ll 1$, (ii) $\bar{\tau} = \tau/\delta^4$ and (iii) expand for $\delta \ll 1$, in (1.1.3):

$$\frac{\partial^2\phi}{\partial\nu^2} + \delta^2 \frac{\partial^2\phi}{\partial\beta^2} = \frac{1}{(1-\cos\beta)^2} \frac{\partial\phi}{\partial\bar{\tau}} (1 + O(\delta^2)) \tag{1.3.1}$$

$$0 < \nu < 1 \ : \ |\beta| \leq \pi,\ \beta \neq 0 : \bar{\tau} > 0$$

$$\nu = 0 \quad \phi = 0 \quad \bar{\tau} > 0$$

$$\nu = 1 \quad \phi = 1 \quad \bar{\tau} > 0$$

$$\bar{\tau} = 0 \quad \phi = 0 \quad 0 < \nu < 1 \ : \ |\beta| \leq \pi,\ \beta \neq 0.$$

For ϕ we assume:

$$\phi = \phi_0 + \delta^2\phi_1 + \dots . \tag{1.3.2}$$

Substitute (1.3.2) into (1.3.1) and equate coefficients of δ^0 to find:

$$\frac{\partial^2\phi_0}{\partial\nu^2} = \frac{1}{(1-\cos\beta)^2} \frac{\partial\phi_0}{\partial\bar{\tau}} \tag{1.3.3}$$

$$\nu = 0 \quad \phi_0 = 0$$

$$\nu = 1 \quad \phi_0 = 1$$

$$\bar{\tau} = 0 \quad \phi_0 = 0.$$

A one-term approximation to ϕ is then, either of:

$$\phi \approx \nu + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2\pi^2(1-\cos\beta)^2\bar{\tau}} \sin(n\pi\nu) \tag{1.3.4}$$

$$\phi \approx \sum_{n=0}^{\infty}\left\{\operatorname{erfc}\frac{2n+1-\nu}{2(1-\cos\beta)\sqrt{\bar{\tau}}} - \operatorname{erfc}\frac{2n+1+\nu}{2(1-\cos\beta)\sqrt{\bar{\tau}}}\right\}. \tag{1.3.5}$$

(II): $\beta = O(\alpha_0)$

Put $\nu = \alpha/\delta$, $\theta = \beta/\delta$ where $\delta = \alpha_0 \ll 1$, in (1.1.3):

$$\frac{\partial^2\phi}{\partial\nu^2} + \frac{\partial^2\phi}{\partial\theta^2} = \frac{\delta^2 \sinh^2\delta}{[\cosh\{\delta(1-\nu)\} - \cos(\delta\theta)]^2}\frac{\partial\phi}{\partial\tau} \quad (1.3.6)$$

$$\nu = 0 \quad \phi = 1$$

$$\nu = 1 \quad \phi = 0$$

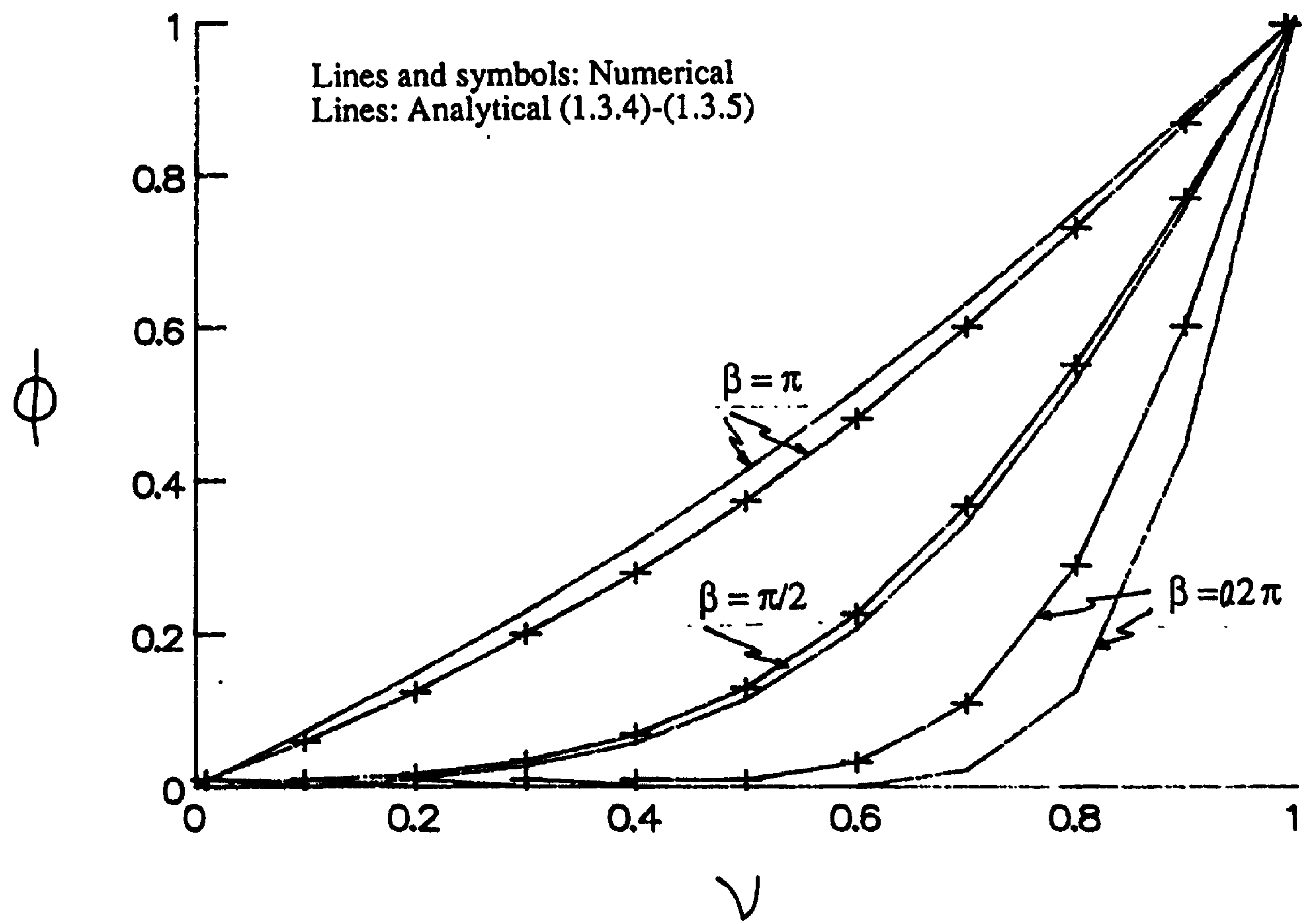
$$\tau = 0 \quad \phi = 0.$$

Assuming a straightforward perturbation expansion ϕ based on $\delta \ll 1$, results in a zeroth approximation that is unable to satisfy both boundary conditions. Therefore an attempt was made to develop a uniformly valid approximation to ϕ using matched asymptotic expansions. Unfortunately we were unable to obtain the stretching that would allow a match between an appropriate inner and outer solution.

RESULTS

Fig. (1.3.1) shows the comparison of equations (1.3.4) and (1.3.5) to the numerical solution of (1.3.1)(see Appendix), where $\bar{\delta} = 0.85$, $|\psi_a| = 1$ and $\bar{\tau} = 0.05$. Fig. (1.3.1) shows the expected decrease in the accuracy of (1.3.4) and (1.3.5) as $\beta \to 0$. The high accuracy of (1.3.4) and (1.3.5) away from $\beta = 0$, is partially due to the correction to (1.3.4) and (1.3.5) being $O(\delta^2)$.

Fig. 1.3.1 Temperature field for shallow pipe depths at $\bar{\tau}=0.05$



Chapter 2

2.1 STATEMENT OF THE NONLINEAR PROBLEM.

The temperature distributions of the solid and of the liquid, both satisfy the Fourier heat conduction equation:

$$\frac{\partial^2 T^S}{\partial x'^2} + \frac{\partial^2 T^S}{\partial y'^2} = \frac{1}{\kappa_S}\frac{\partial T^S}{\partial t} \tag{2.1.1a}$$

$$(x'-d')^2 + y'^2 > R_0^2 : x' > 0 : t > 0$$

$$\frac{\partial^2 T^L}{\partial x'^2} + \frac{\partial^2 T^L}{\partial y'^2} = \frac{1}{\kappa_L}\frac{\partial T^L}{\partial t} \tag{2.1.1b}$$

$$(x'-d')^2 + y'^2 > R_0^2 : x' > 0 : t > 0$$

$$(x'-d')^2 + y'^2 = R_0^2 \qquad T^S = T_a < T_f \qquad t > 0 \tag{2.1.1c}$$

$$t = 0 \qquad T^L = T_i > T_f \qquad (x'-d')^2 + y'^2 > R_0^2 : x' > 0 \tag{2.1.1d}$$

$$x' = 0 \qquad T^L = T_i > T_f \qquad t \geq 0 \tag{2.1.1e}$$

$$x' = x_i'(y', t) \qquad T^L = T^S = T_f \tag{2.1.1f}$$

$$\mathrm{K}_S \frac{\partial T^S}{\partial x'} - \mathrm{K}_L \frac{\partial T^L}{\partial x'} = \rho L \frac{\partial x_i'/\partial t}{1 + (\partial x_i'/\partial y')^2} \tag{2.1.1g}$$

The independent variables x', y', t' that appear in (2.1.1) are made non-dimensional by putting:

$$x = \frac{x'}{R_0},\ y = \frac{y'}{R_0},\ d = \frac{d'}{R_0},\ \tau = \frac{\kappa_L\, t}{R_0^2},\ x_i = \frac{x_i'}{R_0}.$$

The dependent variables are made non-dimensional by putting:

$$\bar{\psi} = \bar{K}\frac{T^S - T_f}{T_i - T_f},\ \bar{\phi} = \frac{T^L - T_f}{T_i - T_f}.$$

Non-dimensionalizing (2.1.1) we have:

$$\frac{\partial^2\bar{\psi}}{\partial x^2} + \frac{\partial^2\bar{\psi}}{\partial y^2} = \bar{\kappa}\frac{\partial\bar{\psi}}{\partial\tau} \tag{2.1.2a}$$

$$(x-d)^2 + y^2 > 1 : x > 0 : \tau > 0$$

$$\frac{\partial^2\bar{\phi}}{\partial x^2} + \frac{\partial^2\bar{\phi}}{\partial y^2} = \frac{\partial\bar{\phi}}{\partial\tau} \tag{2.1.2b}$$

$$(x-d)^2 + y^2 > 1 : x > 0 : \tau > 0$$

$$(x-d)^2 + y^2 = 1 \qquad \bar{\psi} = \psi_a \qquad \tau > 0 \tag{2.1.2c}$$

$$x = 0 \qquad \bar{\phi} = 1 \qquad \tau \geq 0 \tag{2.1.2d}$$

$$t = 0 \qquad \bar{\phi} = 1 \qquad (x-d)^2 + y^2 > 1 : x > 0 \tag{2.1.2e}$$

$$x = x_i(y,\tau) \qquad \bar{\phi} = \bar{\psi} = 0 \tag{2.1.2f}$$

$$\varepsilon\left(\frac{\partial\bar{\psi}}{\partial x} - \frac{\partial\bar{\phi}}{\partial x}\right) = \frac{\partial x_i/\partial\tau}{1 + (\partial x_i/\partial y)^2} \tag{2.1.2g}$$

where: $\bar{\kappa} = \kappa_L/\kappa_S$, $\bar{K} = K_S/K_L$ and we have defined

$$\psi_a = \bar{K}\frac{T_a - T_f}{T_i - T_f}\ ,\quad \varepsilon = \frac{c_L}{L}(T_i - T_f) \tag{2.1.3}$$

where $|\psi_a| \geq O(1)$, $\psi_a < 0$, $T_i \geq T_f$. The present analysis: (i) does not treat $|\psi_a| \ll 1$ since this problem is not as physically interesting and (ii) assumes that $c_L(T_a - T_f)/L < 1$. Finally a transformation is made to bipolar coordinates (see Section 1.1).

Transforming (2.1.2a), (2.1.2c) and (2.1.2f) to bipolar coordinates, we have:

$$\frac{\partial^2\bar{\psi}}{\partial\alpha^2} + \frac{\partial^2\bar{\psi}}{\partial\beta^2} = \frac{\bar{\kappa}\sinh^2\alpha_0}{(\cosh\alpha - \cos\beta)^2}\frac{\partial\bar{\psi}}{\partial\tau} \tag{2.1.4}$$

$$\alpha_i < \alpha < \alpha_0 : |\beta| \leq \pi : \tau > 0$$

$$\alpha = \alpha_0 \quad \bar{\psi} = \psi_* \quad \tau > 0$$

$$\alpha = \alpha_i(\beta,\tau) \quad \bar{\psi} = 0.$$

Transforming (2.1.2b), (2.1.2d), (2.1.2e), and (2.1.2f) we have:

$$\frac{\partial\,\bar{\phi}}{\partial\alpha^2} + \frac{\partial^2\bar{\phi}}{\partial\beta^2} = \frac{\sinh^2\alpha_0}{(\cosh\alpha - \cos\beta)^2}\frac{\partial\bar{\phi}}{\partial\tau} \tag{2.1.5}$$

$$0 < \alpha < \alpha_i : |\beta| \leq \pi : \tau > 0$$

$$\alpha = 0 \quad \bar{\phi} = 1 \quad \tau \geq 0$$

$$\alpha = \alpha_i(\beta,\tau) \quad \bar{\phi} = 0$$

$$\tau = 0 \quad \bar{\phi} = 1 \quad 0 < \alpha < \alpha_0 : |\beta| \leq \pi.$$

Transforming the energy balance across the surface of separation between the solid and liquid phases, equation (2.1.2g), gives:

$$\varepsilon\left(\frac{\partial\bar{\psi}}{\partial\alpha} - \frac{\partial\bar{\phi}}{\partial\alpha}\right) = \frac{\sinh^2\alpha_0}{(\cosh\alpha_i - \cos\beta)^2}\,\frac{\partial\alpha_i/\partial\tau}{1 + (\partial\alpha_i/\partial\beta)^2} \tag{2.1.6}$$

$$\tau = 0 \quad \alpha_i = \alpha_0.$$

To transform (2.1.2g) to bipolar coordinates consult Moon and Spencer [43]. Although the non-linear problem is normally not stated as we have in (2.1.4)-(2.1.6), this is done here since it facilitates the understanding and application of the method that is used to approach the problem.

Existing approximate analytical methods, based on *ad hoc* assumptions in which certain terms are eliminated from the governing differential equations and boundary conditions, fall into two categories: (i) the quasi-stationary approximation and (ii) the quasi-steady approximation. In the latter, the unsteady term is omitted in the differential equation and the gradient at the phase boundary determined by solving the steady-state diffusion equation (quasi-steady-state solutions in general are not capable of satisfying initial conditions). In the quasi-stationary approximation the unsteady term is retained in the diffusion equation and the latter solved assuming that the interface is stationary. The quasi-stationary approximation in retaining the unsteady term in the diffusion equation is capable of satisfying an initial temperature profile (however, the solution will not in general be valid for all time).

In the present analysis the Stefan Number, "ε" is assumed to be small, i.e. $0 < \varepsilon \ll 1$. A composite series approximation to the interface location, based on the Stefan Number and uniformly valid in time, is found by using the method of matched asymptotic expansions in which the quasi-steady and quasi-stationary approximations serve as lowest order generating functions in a scheme of successive asymptotic approximations. The inner solution is valid for small-time (i.e. $\tau = O(1)$) and the outer for large-time (i.e. $\tau \gg 1$). On the short-time scale one is able to simplify the interface condition and satisfy the initial conditions whereas on the long-time scale one is able to neglect the initial conditions and simplify the governing differential equations but satisfy the non-linear interface condition. These two separate solutions are matched term by term in a region of overlapping validity and when taken together, provide a composite solution which is uniformly valid for all time (Weinbaum and Jiji [31]).

Inner solutions require different stretchings for the ranges: (i) $\alpha_0 \geq O(1)$ (Section 2.2) and (ii) $0 < \alpha_0 \ll 1$, (Section 2.3 - this case appears to be less important experimentally). Although the outer solution of Section 2.2 is valid for the full range of α_0 it is simpler algebraically to construct a separate outer solution in Section 2.3.

2.2 CASE I: $\alpha_0 \geq O(1)$

If we put

$$z = \frac{e^{-\alpha}}{\delta}, \ \delta = \exp(-\alpha_0), \ \bar{\sigma} = \frac{\exp(-\alpha_i)}{\delta} - 1, \ 0 \leq \delta < 1, \tag{2.2.1}$$

in (2.1.4), (2.1.5) and (2.1.6) respectively we have: (i) for the temperature in the solid

$$\frac{\partial^2 \bar{\psi}}{\partial z^2} + \frac{1}{z}\frac{\partial \bar{\psi}}{\partial z} + \frac{1}{z^2}\frac{\partial^2 \bar{\psi}}{\partial \beta^2} = \frac{\bar{\kappa}(1-\delta^2)^2}{(1-\delta 2z\cos\beta + \delta^2 z^2)^2}\frac{\partial \bar{\psi}}{\partial \tau} \tag{2.2.2}$$

$$1 < z < 1 + \bar{\sigma} : |\beta| \leq \pi : \tau > 0$$

$$z = 1 \quad \bar{\psi} = \psi_e \quad \tau > 0$$

$$z = 1 + \bar{\sigma} \quad \bar{\psi} = 0,$$

(ii) for the temperature in the liquid

$$\frac{\partial^2 \bar{\phi}}{\partial z^2} + \frac{1}{z}\frac{\partial \bar{\phi}}{\partial z} + \frac{1}{z^2}\frac{\partial^2 \bar{\phi}}{\partial \beta^2} = \frac{(1-\delta^2)^2}{(1-\delta 2z\cos\beta + \delta^2 z^2)^2}\frac{\partial \bar{\phi}}{\partial \tau} \tag{2.2.3}$$

$$1 + \bar{\sigma} < z < \frac{1}{\delta} : |\beta| \leq \pi : \tau > 0$$

$$z = \frac{1}{\delta} \quad \bar{\phi} = 1 \quad \tau \geq 0$$

$$z = 1 + \bar{\sigma} \quad \bar{\phi} = 0$$

$$\tau = 0 \quad \bar{\phi} = 1 \quad 1 < z < \frac{1}{\delta} : |\beta| \leq \pi,$$

and (iii) the energy balance across the surface of separation requires

$$\varepsilon\left(\frac{\partial \bar{\psi}}{\partial z} - \frac{\partial \bar{\phi}}{\partial z}\right) = \frac{(1-\delta^2)^2}{(1-2\delta(1+\bar{\sigma})\cos\beta + \delta^2(1+\bar{\sigma})^2)^2}\,\frac{\partial \bar{\sigma}/\partial \tau}{1 + (\partial \bar{\sigma}/\partial \beta)^2/(1+\bar{\sigma})^2}$$

$$\tau = 0 \quad \bar{\sigma} = 0. \tag{2.2.4}$$

We remove $\bar{\sigma}$ from the boundary conditions: (i) on $\bar{\psi}$ by putting

$$\xi = \frac{z-1}{\bar{\sigma}} \tag{2.2.5}$$

in (2.2.2) and (ii) on $\bar{\phi}$ by putting

$$\eta = \frac{(1-\delta)z - \bar{\sigma}}{p}, \; p = 1 - \delta(1+\bar{\sigma}) \tag{2.2.6}$$

in (2.2.3).

If we substitute (2.2.5) and (2.2.6) into (2.2.2), (2.2.3) and (2.2.4) we have: (i) for the temperature in the solid

$$\frac{\partial^2\bar{\psi}}{\partial\xi^2} + \frac{\bar{\sigma}}{1+\bar{\sigma}\xi}\frac{\partial\bar{\psi}}{\partial\xi} + \frac{1}{(1+\bar{\sigma}\xi)^2}\{\bar{\sigma}^2\frac{\partial^2\bar{\psi}}{\partial\beta^2} - 2\bar{\sigma}\xi\frac{\partial\bar{\sigma}}{\partial\beta}\frac{\partial^2\bar{\psi}}{\partial\xi\partial\beta} - \bar{\sigma}\xi\frac{\partial^2\bar{\sigma}}{\partial\beta^2}\frac{\partial\bar{\psi}}{\partial\xi} +$$

$$2\xi\left(\frac{\partial\bar{\sigma}}{\partial\xi}\right)^2\frac{\partial\bar{\psi}}{\partial\xi} + \xi^2\left(\frac{\partial\bar{\sigma}}{\partial\beta}\right)^2\frac{\partial^2\bar{\psi}}{\partial\xi^2}\} = F_0(\xi,\beta,\tau)\left[\bar{\sigma}^2\frac{\partial\bar{\psi}}{\partial\tau} - \bar{\sigma}\xi\frac{\partial\bar{\sigma}}{\partial\tau}\frac{\partial\bar{\psi}}{\partial\xi}\right] \tag{2.2.7}$$

$$0<\xi<1 \; : \; |\beta|\leq\pi \; : \; \tau>0$$

$$\xi=0 \quad \bar{\psi}=\psi_a \quad \tau>0$$

$$\xi=1 \quad \bar{\psi}=0$$

where $F_0(\xi,\beta,\tau)$ is given by,

$$F_0(\xi,\beta,\tau) = \frac{(1-\delta^2)^2}{[1-2\delta(1+\bar{\sigma}\xi)\cos\beta+\delta^2(1+\bar{\sigma}\xi)^2]^2},$$

(ii) for the temperature in the liquid

$$\frac{\partial^2\bar{\phi}}{\partial\eta^2} + \frac{p}{p\eta+\bar{\sigma}}\frac{\partial\bar{\phi}}{\partial\eta} + \frac{1}{(p\eta+\bar{\sigma})^2}\{p^2\frac{\partial^2\bar{\phi}}{\partial\beta^2} + 2p(\delta\eta-1)\frac{\partial\bar{\sigma}}{\partial\beta}\frac{\partial^2\bar{\phi}}{\partial\eta\partial\beta}$$

$$+ 2\delta(\delta\eta-1)\left(\frac{\partial\bar{\sigma}}{\partial\beta}\right)^2\frac{\partial\bar{\phi}}{\partial\eta} + p(\delta\eta-1)\frac{\partial^2\bar{\sigma}}{\partial\beta^2}\frac{\partial\bar{\phi}}{\partial\eta}$$

$$+ p^2(\delta\eta-1)^2\left(\frac{\partial\bar{\sigma}}{\partial\beta}\right)^2\frac{\partial^2\bar{\phi}}{\partial\eta^2}\} = F_1(\eta,\beta,\tau)\left[\frac{\partial\bar{\phi}}{\partial\tau} + p(\delta\eta-1)\frac{\partial\bar{\sigma}}{\partial\tau}\frac{\partial\bar{\phi}}{\partial\eta}\right] \tag{2.2.8}$$

$$1<\eta<\frac{1}{\delta} \; : \; |\beta|\leq\pi \; : \; \tau>0$$

$$\eta = 1 \quad \bar{\phi} = 0$$

$$\eta = \frac{1}{\delta} \quad \bar{\phi} = 1 \quad \tau \geq 0$$

$$\tau = 0 \quad \bar{\phi} = 1 \quad 1 < \eta < \frac{1}{\delta} \ : \ |\beta| \leq \pi$$

where $F_1(\eta,\beta,\tau)$ is given by,

$$F_1(\eta,\beta,\tau) = (1+\delta)^2 \left\{ 1 - \frac{2\delta}{1-\delta}(p\eta + \bar{\sigma})\cos\beta + \frac{\delta^2}{(1-\delta)^2}(p\eta + \bar{\sigma})^2 \right\}^{-2},$$

and (iii) across the surface of separation between the two phases we have,

$$\varepsilon\left(\frac{\partial\bar{\psi}}{\partial\xi} - \frac{(1-\delta)}{1-\delta(1+\bar{\sigma})}\bar{\sigma}\frac{\partial\bar{\phi}}{\partial\eta}\right) = \frac{(1-\delta^2)^2}{[1-\delta 2(1+\bar{\sigma})\cos\beta + \delta^2(1+\bar{\sigma})^2]^2}\ \frac{\bar{\sigma}\ \partial\bar{\sigma}/\partial\tau}{1+(\partial\bar{\sigma}/\partial\beta)^2/(1+\bar{\sigma})^2} \tag{2.2.9}$$

$$\tau = 0 \quad \bar{\sigma} = 0.$$

A perturbation approximation to the problem will now be found. The Stefan Number, "ε" is assumed to be small, i.e. $0 < \varepsilon \ll 1$. An approximation to the surface of separation between the two phases is found using the method of matched asymptotic expansions. The inner solution is valid for small-time (i.e. $\tau = O(1)$) and the outer for large-time (i.e. $\tau \gg 1$).

INNER SOLUTION

For the inner solution, we treat $\delta \neq O(1)$ and $\delta \ll 1$ separately, since we can calculate higher order terms when $\delta \ll 1$.

(I): $\delta \neq O(1)$

For $\bar{\psi}$ and $\bar{\sigma}$ we assume:

$$\bar{\psi} = \bar{\psi}_0 + M_0, \tag{2.2.10}$$

$$\bar{\sigma} = \sqrt{\varepsilon}\ \bar{\sigma}_0 + M_1 \tag{2.2.11}$$

where M_0, M_1 are asymptotic corrections (note that the first order problem is independent of $\bar{\phi}$). Putting the above into equation (2.2.6), equating coefficients of ε^0 and solving the resulting differential equation gives

$$\bar{\psi} \sim \psi_a(1-\xi). \tag{2.2.12}$$

If: (i) we substitute $\bar{\psi}$ and $\bar{\sigma}$ of the above into (2.2.9), (ii) equate coefficients of ε^0 and solve the resulting differential equation, we have:

$$\bar{\sigma} \sim \frac{1-\delta 2\cos\beta+\delta^2}{1-\delta^2}\sqrt{2\,|\,\psi_a\,|\,\varepsilon\tau}. \tag{2.2.13}$$

(II) : $\delta \ll 1$

For $\bar{\psi}$ and $\bar{\sigma}$ we assume:

$$\bar{\psi}(\xi,\beta,\varepsilon,\delta) = \bar{\psi}_0(\xi,\beta,\tau) + \sqrt{\varepsilon}\,\bar{\psi}_1(\xi,\beta,\tau) + \delta\,\bar{\psi}_2(\xi,\beta,\tau) + \ldots + \text{EST}, \tag{2.2.14}$$

$$\bar{\sigma}(\beta,\tau,\varepsilon,\delta) = \sqrt{\varepsilon}\,[\bar{\sigma}_1(\beta,\tau) + \sqrt{\varepsilon}\,\bar{\sigma}_2(\beta,\tau) + \delta\,\bar{\sigma}_3(\beta,\tau) + \ldots + \text{EST}]. \tag{2.2.15}$$

To find $\bar{\psi}$: (i) put (2.2.14) and (2.2.15) into (2.2.7) and (ii) equate coefficients of like powers of ε and δ. Solving the differential equations that result gives:

ε^0: $\quad \bar{\psi}_0 = \psi_a(1-\xi)$

$\sqrt{\varepsilon}$: $\quad \bar{\psi}_1 = \frac{\psi_a}{2}\bar{\sigma}_1\xi(\xi-1)$

δ: $\quad \bar{\psi}_2 = 0$

combining terms,

$$\bar{\psi} \sim \psi_a(1-\xi) + \sqrt{\varepsilon}\,\frac{\psi_a}{2}\bar{\sigma}_1\xi(\xi-1). \tag{2.2.16}$$

A one-term approximation is found for $\bar{\phi}$ using matched asymptotic expansions. The outer solution is found by putting: (i) $u = (\eta-1)/g(\varepsilon,\delta)$, $q = \tau/g^2(\varepsilon,\delta)$, (recall $\bar{\sigma}$ is valid for $\tau \ll 1$) (ii)

$$\bar{\phi}^o = \phi_0 + M_0$$

where M_0 is an asymptotic correction and (iii) (2.2.15), in (2.2.8). If we then: (i) expand for $(\varepsilon,\delta) \ll 1$ and (ii) equate coefficients of (ε^0,δ^0) we find:

$$\frac{\partial^2\phi_0}{\partial u^2} = \frac{\partial\phi_0}{\partial q}$$

$$u = 0 \qquad \phi_0 = 0$$

$$q = 0 \qquad \phi_0 = 1,$$

A one-term approximation to $\bar{\phi}^o$ is then:

$$\bar{\phi}^o \approx \operatorname{erf}\frac{u}{2\sqrt{q}}$$

The inner solution is found by putting: (i) $w = \ln(\delta\eta)/h(\varepsilon,\delta)$ (i.e. stretch near $\eta = 1/\delta$) in (2.2.8). If we then: (i) assume that $h = o(\delta)$ (i.e. the first order problem is steady) and (ii) that as $(\varepsilon,\delta) \to 0$:

$$\frac{h^2}{\delta^2}\frac{1}{A}\frac{dA}{d\tau} \to 0 \qquad (2.2.17)$$

then we may write:

$$\bar{\phi}^i = 1 + A(\varepsilon,\delta,\tau)\,\phi_0 + M_1 \qquad (2.2.18)$$

where M_1 is an asymptotic correction. If we then: (i) substitute (2.2.15) and (2.2.18) into (2.2.8), (ii) expand for $(\varepsilon,\delta) \ll 1$, and (iii) equate coefficients of (ε^0,δ^0) we find:

$$\frac{\partial^2\phi_0}{\partial w^2} = 0$$

$$w = 0 \quad \phi_0 = 0$$

A one-term approximation to $\bar{\phi}^i$ is then:

$$\bar{\phi}^i \approx 1 + A(\varepsilon,\delta,\tau)\,w$$

Since ϕ_0 of $\bar{\phi}^i$ is steady, $\bar{\phi}^i$ is not valid in the neighbourhood of: (i) $\beta = 0$ and (ii) $\tau = 0$.

After matching we find a one-term, uniformly valid approximation to be:

$$\bar{\phi} \approx \operatorname{erf}\frac{\eta-1}{2\sqrt{\tau}} + \frac{\ln\eta}{|\ln\delta|}\operatorname{erfc}\frac{1-\delta}{2\delta\sqrt{\tau}}.$$

For convenience write:

$$\bar{\phi}_{00}: \qquad \bar{\phi}_{00} = \operatorname{erf}\frac{\eta-1}{2\sqrt{\tau}}$$

$$\bar{\phi}_{01}: \qquad \bar{\phi}_{01} = \frac{\ln\eta}{|\ln\delta|}\operatorname{erfc}\frac{1-\delta}{2\delta\sqrt{\tau}}. \tag{2.2.19}$$

where: (i) we have returned to (η,τ) and (ii) equation (2.2.17) requires $h = o(\delta^2)$. To find $\bar{\sigma}$ put (2.2.16), (2.2.19) and (2.2.15) into (2.2.9). Simplifying,

$$\begin{aligned}\frac{\partial\bar{\psi}_0}{\partial\xi} &+ \sqrt{\varepsilon}\left(\frac{\partial\bar{\psi}_1}{\partial\xi} - \bar{\sigma}_1\frac{\partial\bar{\phi}_{00}}{\partial\eta}\right) + \delta\left(\frac{\partial\bar{\psi}_2}{\partial\xi} - \frac{\partial\bar{\psi}_0}{\partial\xi} - 4\cos\beta\frac{\partial\bar{\psi}_0}{\partial\xi}\right)\\ &- \sqrt{\varepsilon}\,\bar{\sigma}_1\frac{\partial\bar{\phi}_{01}}{\partial\eta} + \dots = \bar{\sigma}_1\frac{\partial\bar{\sigma}_1}{\partial\tau} + \sqrt{\varepsilon}\,\frac{\partial(\bar{\sigma}_1\bar{\sigma}_2)}{\partial\tau}\\ &+ \delta\left[\frac{\partial(\bar{\sigma}_1\bar{\sigma}_3)}{\partial\tau} - \bar{\sigma}_1\frac{\partial\bar{\sigma}_1}{\partial\tau}\right] + \dots\end{aligned} \tag{2.2.20}$$

where the derivatives of $\bar{\psi}$ and $\bar{\phi}$ are evaluated at $\xi = \eta = 1$. Putting (2.2.14) and (2.2.19) into (2.2.20) we have:

$$\begin{aligned}-\psi_a &+ \sqrt{\varepsilon}\left(\frac{\psi_a\bar{\sigma}_1}{2} - \frac{\bar{\sigma}_1}{\sqrt{\pi\tau}}\right) + \delta\,\psi_a(1+4\cos\beta)\\ &- \frac{\sqrt{\varepsilon}}{|\ln\delta|}\bar{\sigma}_1\operatorname{erfc}\frac{1-\delta}{2\delta\sqrt{\tau}} + \dots = \bar{\sigma}_1\frac{\partial\bar{\sigma}_1}{\partial\tau} + \sqrt{\varepsilon}\,\frac{\partial(\bar{\sigma}_1\bar{\sigma}_2)}{\partial\tau}\\ &+ \delta\left[\frac{\partial(\bar{\sigma}_1\bar{\sigma}_3)}{\partial\tau} - \bar{\sigma}_1\frac{\partial\bar{\sigma}_1}{\partial\tau}\right] + \dots\end{aligned} \tag{2.2.21}$$

To find $\bar{\sigma}_n$ equate coefficients quantities of equal order in ε and δ, and solve the resulting differential equations for $\bar{\sigma}_n$:

$$\varepsilon^0: \qquad \bar{\sigma}_1 = \sqrt{2|\psi_a|\tau}$$

$$\sqrt{\varepsilon}: \qquad \bar{\sigma}_2 = \frac{\psi_a\tau}{3} - \sqrt{\frac{\tau}{\pi}} \tag{2.2.22}$$

$$\delta: \qquad \bar{\sigma}_3 = -2\sqrt{2|\psi_a|\tau}\cos\beta.$$

Combining terms:

$$\bar{\sigma} \approx \sqrt{\varepsilon}\left\{\sqrt{2|\psi_a|\tau} + \sqrt{\varepsilon}\left[\frac{\psi_a\tau}{3} - \sqrt{\frac{\tau}{\pi}}\right] - 2\delta\sqrt{2|\psi_a|\tau}\cos\beta\right\} \tag{2.2.23}$$

OUTER SOLUTION

To find the outer solution put: $\gamma = \varepsilon\tau$ in equations (2.2.2), (2.2.3) and (2.2.4). For σ we assume:

$$\sigma = \sigma_0 + M, \tag{2.2.24}$$

where M is an asymptotic correction. To obtain a one-term approximation for ψ and ϕ change independent variables in equations (2.2.2) and (2.2.3) by putting: (i) $u = lnz/g(\varepsilon)$, $g = o(1)$ in (2.2.2) and (ii) $u = \ln(\delta z)/h(\varepsilon)$, $h = o(1)$ in (2.2.3). Matched asymptotic expansions are then used to create a one-term approximation to ψ and ϕ, where the outer variable is u. One-term, constant temperature approximations, not valid in the neighbourhood of the interface boundary, are found to ψ^{o} and ϕ^{o}. Linear stretches are made, near the interface boundary, and one-term approximations are found to ψ^{i} and ϕ^{i}. Matching and combining these to form a one-term composite solution for ψ and ϕ respectively gives:

$$\psi \approx \psi_a\left[1 - \frac{\ln z}{\ln(1+\sigma_0)}\right], \tag{2.2.25}$$

$$\phi \approx \frac{\ln[(1+\sigma_0)/z]}{\ln\delta(1+\sigma_0)}, \tag{2.2.26}$$

where we have returned to z. If: (i) we substitute (2.2.24), (2.2.25) and (2.2.26) into (2.2.4) and (ii) equate coefficients of ε^0 we find:

$$\frac{-\psi_a}{\ln(1+\sigma_0)} + \frac{1}{\ln\delta(1+\sigma_0)} = \frac{(1-\delta^2)^2}{[1-\delta 2(1+\sigma_0)\cos\beta+\delta^2(1+\sigma_0)^2]^2}\,\frac{(1+\sigma_0)\,\partial\sigma_0/\partial\gamma}{1+(\partial\sigma_0/\partial\beta)^2/(1+\sigma_0)^2}\,. \tag{2.2.27}$$

It is easier to perform the match of the inner solution to the outer solution via (2.2.27). In addition, the construction of the solution to equation (2.2.27) is best left until after the match since the match yields information that is useful in setting up σ_0.

MATCH

The outer expansion of $\bar{\sigma}$ is found by putting $\tau=\gamma/\varepsilon$ in equation (2.2.13) expanding for $\varepsilon \ll 1$:

$$(\bar{\sigma})^o \approx \frac{1-\delta 2\cos\beta+\delta^2}{1-\delta^2}\sqrt{2\,|\,\psi_a\,|\,\gamma}. \tag{2.2.28}$$

Now we find the outer expansion of the inner solution valid for $\delta \ll 1$. Put $\gamma=\varepsilon\tau$ in equation (2.2.23) and expand for $(\varepsilon,\delta) \ll 1$:

$$(\bar{\sigma})^o \approx \sqrt{2\,|\,\psi_a\,|\,\gamma} + \frac{\psi_a\gamma}{3} - \sqrt{\varepsilon\gamma/\pi} - 2\delta\sqrt{2\,|\,\psi_a\,|\,\gamma}\,\cos\beta. \tag{2.2.29}$$

Keeping lead terms in (2.2.29) we find:

$$(\bar{\sigma})^o \approx \sqrt{2\,|\,\psi_a\,|\,\gamma} + \frac{\psi_a\gamma}{3}. \tag{2.2.30}$$

To find $(\sigma)^i$ we return to equation (2.2.27). Since (2.2.27) is not suitable for matching we assume that

$$\sigma_0 = a_0(\beta) + a_1(\beta)\sqrt{\gamma} + a_2(\beta)\gamma + \dots\,. \tag{2.2.31}$$

Since $(\bar{\sigma})^o$ vanishes as $\gamma \to 0$ in each of equations (2.2.28), (2.2.30) and (2.2.31), matching with $(\sigma)^i$ requires that

$$a_0(\beta) = 0. \tag{2.2.32}$$

With result (2.2.33), we substitute equation (2.2.32) into (2.2.27) and equate coefficients of $\gamma^{n/2}$. Equating coefficients of $\sqrt{\gamma}$ and γ we have:

$$a_1(\beta) = \frac{1-\delta 2\cos\beta+\delta^2}{1-\delta^2}\sqrt{2\,|\,\psi_e\,|}, \tag{2.2.33}$$

$$a_2(\beta) = \frac{\psi_e}{3}\frac{(1-\delta 2\cos\beta+\delta^2)^2}{(1-\delta^2)^2}\left[1 - \frac{2}{\psi_e\,|\ln\delta\,|} + \frac{8\delta(\cos\beta-\delta)}{1-2\delta\cos\beta+\delta^2}\right] \tag{2.2.34}$$

where we have assumed that

$$\frac{\sigma_0}{|\,\psi_e \ln\delta\,|} < 1. \tag{2.2.35}$$

Note that equation (2.2.35) is the source of the restriction $\delta \neq O(1)$. Thus for $\delta \neq O(1)$, we substitute $a_1(\beta)$ of (2.2.33) into equation (2.2.31) to find

$$(\sigma)^i \approx \frac{1-\delta 2\cos\beta+\delta^2}{1-\delta^2}\sqrt{2\,|\,\psi_e\,|\,\gamma}. \tag{2.2.36}$$

If we compare $(\sigma)^i$ equation (2.2.36) with $(\tilde{\sigma})^o$ of equation (2.2.28) we see that the one-term outer solution matches the one-term inner solution.

The inner solution of equation (2.2.23) is based on $\delta \ll 1$ so we expand equations (2.2.33) and (2.2.34) for $\delta \ll 1$ and substitute the results into equation (2.2.32) to obtain:

$$(\sigma)^i \approx \sqrt{2\,|\,\psi_e\,|\,\gamma} + \frac{\psi_e\gamma}{3}. \tag{2.2.37}$$

If we compare $(\sigma)^i$ of equation (2.2.37) with $(\tilde{\sigma})^o$ of equation (2.2.30) we find that the two-term inner solution matches the one-term outer solution.

SOLUTION TO EQUATION (2.2.27)

Rewrite equation (2.2.27) so that $\gamma = \gamma(\sigma_0,\beta)$:

$$f(\sigma_0)[1+\delta^2(1+\sigma_0)^2 - 2\delta(1+\sigma_0)\cos\beta]^2\left[(1+\sigma_0)^2\left(\frac{\partial\gamma}{\partial\sigma_0}\right)^2 + \left(\frac{\partial\gamma}{\partial\beta}\right)^2\right]$$
$$= (1-\delta^2)^2(1+\sigma_0)^3\,\frac{\partial\gamma}{\partial\sigma_0} \qquad (2.2.38)$$

where

$$f(\sigma_0) = \frac{-\psi_a}{\ln(1+\sigma_0)} + \frac{1}{\ln\delta(1+\sigma_0)}.$$

To find $\gamma(\sigma_0,\beta)$ we subtract off a particular solution $\bar{\gamma}$:

$$\gamma(\sigma_0,\beta) = \bar{\gamma}(\sigma_0,\beta) + G(\sigma_0,\beta). \qquad (2.2.39)$$

Since: (i) (2.2.27) has a constant steady state and (ii) we have result (2.2.31) and (2.2.32),

$$\partial\sigma_0/\partial\beta \rightarrow 0 \text{ as } \gamma \rightarrow (0,\infty).$$

Therefore $\bar{\gamma}$ is fou..d by dropping $\partial\sigma_0/\partial\beta$ in equation (2.2.27) or, $(\partial\gamma/\partial\beta)/(\partial\gamma/\partial\sigma_0)$ in equation (2.2.38):

$$\bar{\gamma} = (1-\delta^2)^2\int_0^{\sigma_0}\frac{(1+u)du}{f(u)[1-2\delta(1+u)\cos\beta+\delta^2(1+u)^2]^2} \qquad (2.2.40)$$

where: (i) $f(u)$ is defined in equation (2.2.38) and (ii) we choose $\bar{\gamma}(0,\beta)=0$ since we have (2.2.31) and (2.2.32). Since $\partial\sigma_0/\partial\beta$ of (2.2.27) does not contribute to $(\sigma)^i$ of (2.2.36) and (2.2.37), equation (2.2.40) contains all of the necessary information for the match, to the order considered here. To find $G(\sigma_0,\beta)$ we put (2.2.39), (along with (2.2.40)) into equation (2.2.38) and solve the resulting problem. It is possible to construct a convergent series solution for $G(\sigma_0,\beta)$, asymptotic in δ, as $\delta \rightarrow 0$, for which we can derive the following properties by considering the first term of G:

(i) $\sigma_0 \rightarrow 0,\ \gamma \rightarrow \bar{\gamma}$

(ii) $\sigma_0 \rightarrow$ steady state, $\gamma \rightarrow \bar{\gamma}$

(iii) $\beta \rightarrow (0,\pi),\ \gamma \rightarrow \bar{\gamma}$

(iv) $\delta \rightarrow 0,\ \gamma \rightarrow \bar{\gamma}$.

Numerical results (see "Results" section) show that $G(\sigma_0,\beta)$ can be neglected.

RESULTS

The outer expansion of the inner solution ignores the second phase as a first approximation to the solution in the overlap region, and thus for the inner expansion of the outer solution to do the same in the overlap region, we must apply the condition: $\sigma_0 / |\psi_a \ln\delta| < 1$. Results presented here, are for $|\psi_a| = 1, \varepsilon = 0.2$.

Equation (2.2.39) represents a one-term composite approximation to the interface location, and numerical calculations show that $G(\sigma_0,\beta)$ contributes only a small percentage to the interface location. For example, at $\delta = 0.75$, $G(\sigma_0,\beta)$ contributes less than 2% to the interface location. Thus, since equation (2.2.40) also contains all of the necessary information for the match to the order considered here, (see "Solution to equation (2.2.27)") it will be used to represent σ_0.

Fig. 2.2.1 shows the overlap region between the outer and inner solutions of equations (2.2.40) (see above paragraph) and (2.2.23) for deep pipe depths, where $\delta = 0.135$ ($\beta = \pi/2$ in (2.2.40)). At finite pipe depths, for example $\delta = 0.5$, Fig. 2.2.2 shows the one-term composite approximation to the interface location, i.e. (2.2.40) (see above paragraph), for different values of the angle-like variable, β.

To consider $|\psi_a| \gg 1$ equations (2.1.4) and (2.1.6) indicate that we must have $|\varepsilon\psi_a| = c_L |T_a - T_f| / L < 1$ in order for a quasi-stationary or quasi-steady assumption to be valid. However, C.T. Hwang [43] published numerical results where $c_L |T_a - T_f| / L > 1$, which he then compared to quasi-static approximations (private communication indicates C. T. Hwang believes this comparison is a valid one). For the sake of comparison, Fig. 2.2.3 (from C. T. Hwang [43]) shows that (2.2.40) performs much better (under dubious conditions) than do the analytical results that C. T. Hwang has shown. The information defining the problem considered in Fig. 2.2.3, is as follows:

Pipe temperature: $T_a = 65.5°\,C (150°\,F)$

Pipe radius: $R_0 = 0.61$ m (2 ft)

Burial depth: $d' = 0.91$ m (3 ft)

Initial ground temperature: $T_i = -1°C(30.2°F)$

Fusion temperature: $T_f = 0°C(32°F)$

Soil type: silty clay

Total water content: W=25%

Unfrozen water content: $W_u = 5\%$

Dry density: $\rho_{dry} = 1.6\,g/cm^3(100\,lb/ft^3)$

Thermal conductivity: $K_L = 3.7 \times 10^{-3}\,cal^1 s^{-1} cm^{-1}\,°C^{-1}$

Thermal conductivity: $K_S = 4.8 \times 10^{-3}\,cal^1 s^{-1} cm^{-1}\,°C^{-1}$

Volumetric specific heat: $C_L = 0.68\,cal\,cm^{-3}\,°C^{-1}$

Volumetric specific heat: $C_S = 0.50\,cal\,cm^{-3}\,°C^{-1}$

Volumetric latent heat: $L = 25.6\,cal/cm^3$

Fig. 2.2.1 Interface location for deep pipe depths: $|\psi_a| = 1$, $\varepsilon = 0.2$, $\delta = 0.135$

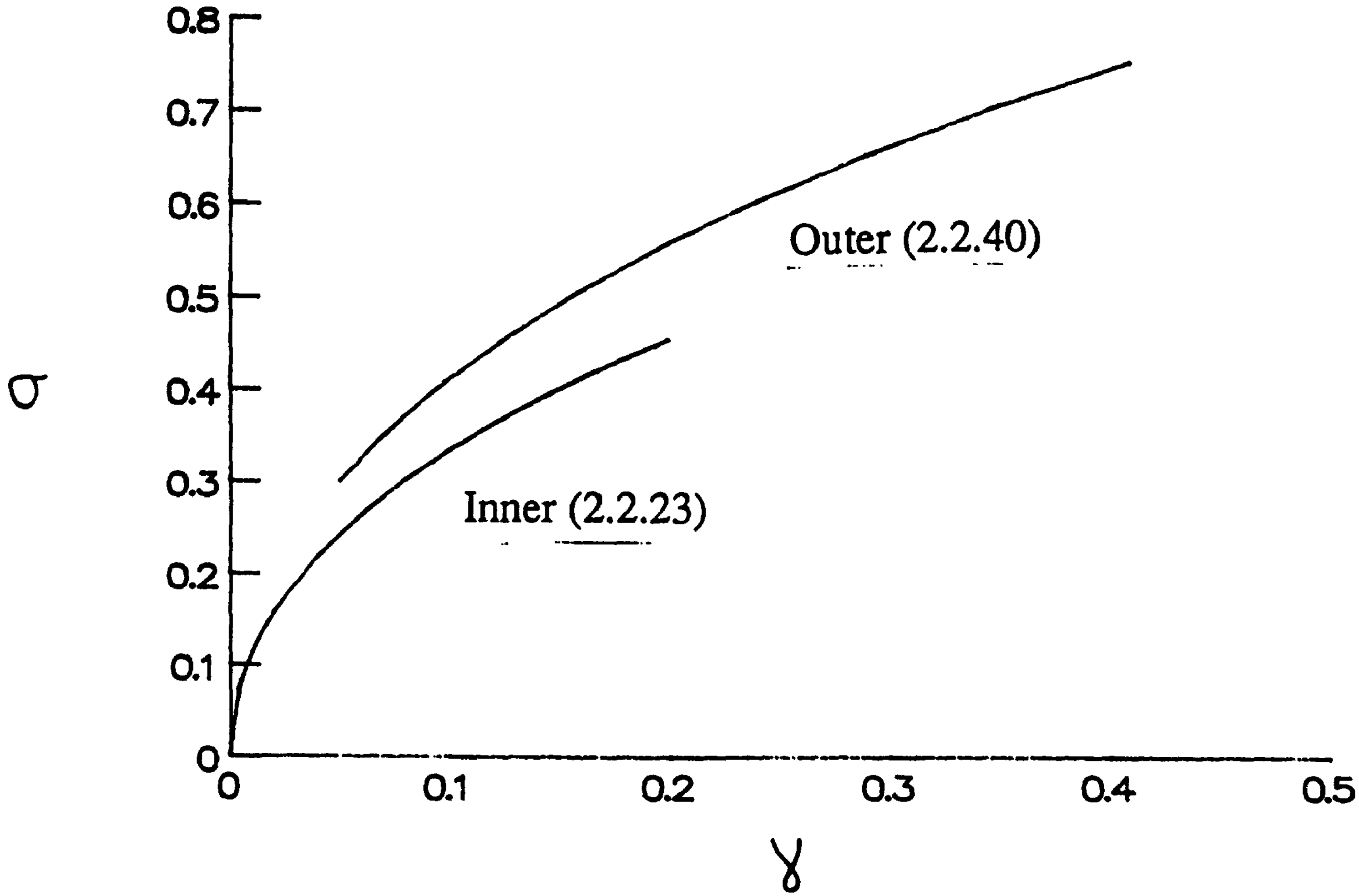


Fig. 2.2.2 One-term composite approximation to interface location: $|\gamma_k| = 1$, $\varepsilon = 0.2$, $\delta = 0.5$

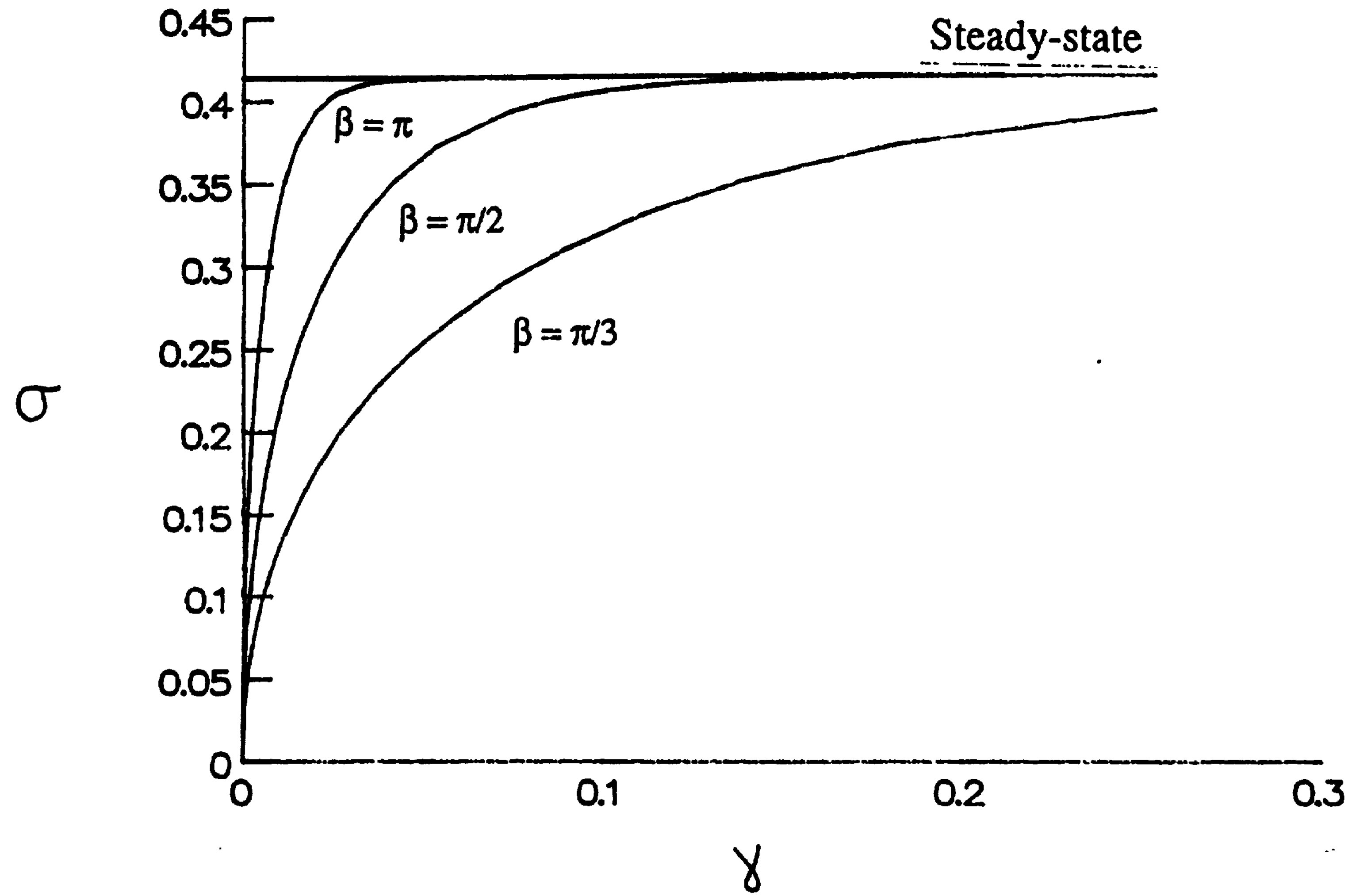
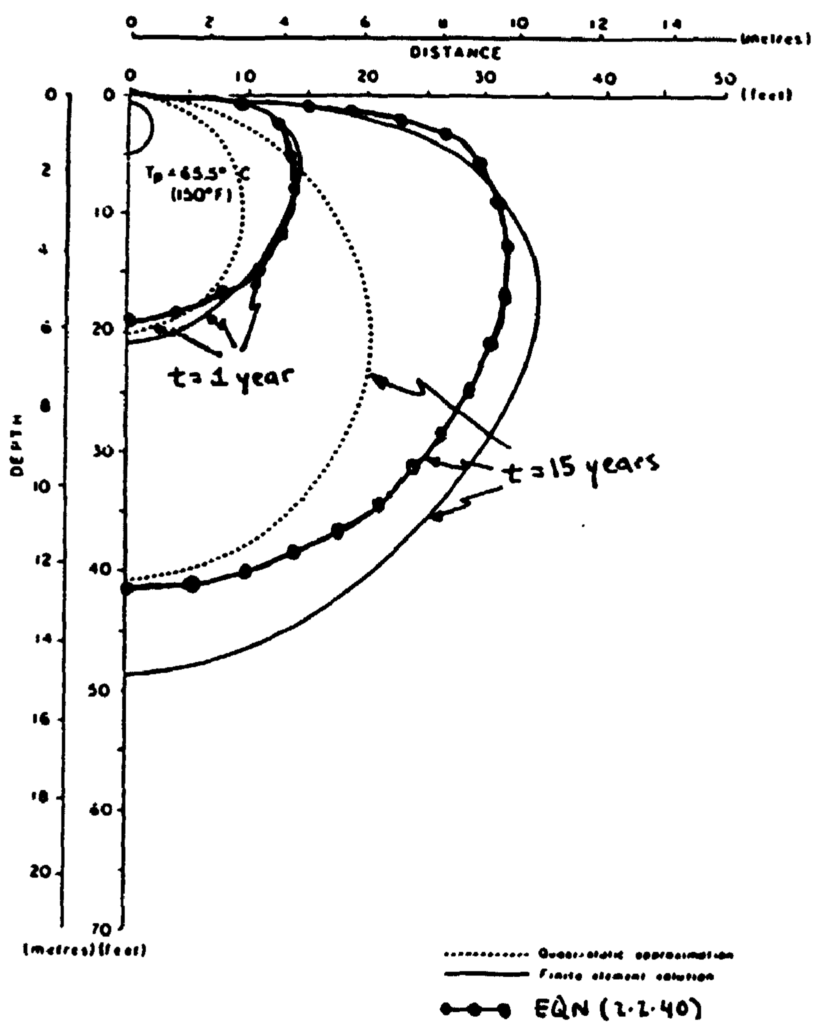


Fig. 2.2.3 Comparison of one-term composite approximation to interface location to finite element numerical solution



2.3 CASE II: $0 < \alpha_0 \ll 1$

Consider (2.1.4), (2.1.5) and (2.1.6). To remove $\bar{\sigma}$ from the boundary conditions: (i) on $\bar{\psi}$ we put

$$\xi = \frac{1 - \alpha/\delta}{\bar{\sigma}} \tag{2.3.1}$$

in (2.1.4) and (ii) on $\bar{\phi}$ we put

$$\eta = \frac{\delta(1-\bar{\sigma}) - \alpha}{\delta(1-\bar{\sigma})} \tag{2.3.2}$$

in (2.1.5) where: (i) $\delta = \alpha_0$, $0 < \delta \ll 1$, and (ii) $\bar{\sigma} = 1 - \alpha_i/\delta$. Substituting (2.3.1) and (2.3.2) into (2.1.4), (2.1.5) and (2.1.6) respectively, we have: (i) for the temperature in the solid,

$$\frac{\partial^2\bar{\psi}}{\partial\xi^2} + \delta^2\left[\bar{\sigma}^2\frac{\partial^2\bar{\psi}}{\partial\beta^2} - 2\bar{\sigma}\xi\frac{\partial\bar{\sigma}}{\partial\beta}\frac{\partial^2\bar{\psi}}{\partial\xi\partial\beta} + 2\xi\left(\frac{\partial\bar{\sigma}}{\partial\beta}\right)^2\frac{\partial\bar{\psi}}{\partial\xi} - \bar{\sigma}\xi\frac{\partial^2\bar{\sigma}}{\partial\beta^2}\frac{\partial\bar{\psi}}{\partial\xi} + \xi^2\left(\frac{\partial\bar{\sigma}}{\partial\beta}\right)^2\frac{\partial^2\bar{\psi}}{\partial\xi^2}\right]$$

$$= \frac{\bar{\kappa}\delta^2\sinh^2\delta}{[\cosh\{\delta(1-\bar{\sigma}\xi)\} - \cos\beta]^2}\left[\bar{\sigma}^2\frac{\partial\bar{\psi}}{\partial\tau} - \bar{\sigma}\xi\frac{\partial\bar{\sigma}}{\partial\tau}\frac{\partial\bar{\psi}}{\partial\xi}\right] \tag{2.3.3}$$

$$0 < \xi < 1 : |\beta| \le \pi : \tau > 0$$

$$\xi = 0 \quad \bar{\psi} = \psi_* \quad \tau > 0$$

$$\xi = 1 \quad \bar{\psi} = 0$$

(ii) for the temperature in the liquid

$$\frac{\partial^2\bar{\phi}}{\partial\eta^2} + \delta^2\,[(1-\bar{\sigma})^2\frac{\partial^2\bar{\phi}}{\partial\beta^2} + 2(\eta-1)(1-\bar{\sigma})\frac{\partial\bar{\sigma}}{\partial\beta}\frac{\partial^2\bar{\phi}}{\partial\eta\partial\beta} + 2(\eta-1)\left(\frac{\partial\bar{\sigma}}{\partial\beta}\right)^2\frac{\partial\bar{\phi}}{\partial\eta}$$

$$+ (\eta-1)(1-\bar{\sigma})\frac{\partial^2\bar{\sigma}}{\partial\beta^2}\frac{\partial\bar{\phi}}{\partial\eta} + (\eta-1)^2\left(\frac{\partial\bar{\sigma}}{\partial\beta}\right)^2\frac{\partial^2\bar{\phi}}{\partial\eta^2}]$$

$$= \frac{\delta^2\sinh^2\delta}{[\cosh\{\delta(1-\eta)(1-\bar{\sigma})\} - \cos\beta]^2}\left[\frac{\partial\bar{\phi}}{\partial\tau} + (\eta-1)(1-\bar{\sigma})\frac{\partial\bar{\phi}}{\partial\eta}\frac{\partial\bar{\sigma}}{\partial\tau}\right] \tag{2.3.4}$$

$$0 < \eta < 1 : |\beta| \le \pi : \tau > 0$$

$$\eta = 0 \quad \bar{\phi} = 0$$

$$\eta = 1 \quad \bar{\phi} = 1 \quad \tau \geq 0$$

$$\tau = 0 \quad \bar{\phi} = 1 \quad 0 < \eta < 1 : |\beta \leq \pi$$

and (iii)

$$\varepsilon\left((1-\bar{\sigma})\frac{\partial\bar{\psi}}{\partial\xi} - \bar{\sigma}\frac{\partial\bar{\phi}}{\partial\eta}\right) = \frac{\delta^2 \sinh^2\delta}{[\cosh\{\delta(1-\bar{\sigma})\} - \cos\beta]^2} \frac{\bar{\sigma}\,(1-\bar{\sigma})\,\partial\bar{\sigma}/\partial\tau}{1+\delta^2(\partial\bar{\sigma}/\partial\beta)^2} \tag{2.3.5}$$

$$\tau = 0 \quad \bar{\sigma} = 0.$$

A perturbation approximation to this problem will now be found. The Stefan Number, "ε" is assumed to be small, i.e. $0 < \varepsilon \ll 1$. An approximation to the surface of separation between the two phases is found using the method of matched asymptotic expansions. The inner solution is valid for small-time (i.e. $\tau = O(1)$) and the outer solution for large-time (i.e. $\tau \gg 1$). Equations (2.3.3), (2.3.4) and (2.3.5) show that we must consider: (I) $\beta \neq 0$ and (II) $\beta = O(\delta)$.

(I) : $\beta \neq 0$

Put:

$$\bar{\tau} = \frac{\tau}{\delta^2 \sinh^2\delta}$$

in (2.3.3), (2.3.4) and (2.3.5). To reduce the effort in finding $\bar{\sigma}$, "$\sinh^2\delta$" instead of "δ^2" is used in the definition of $\bar{\tau}$.

INNER SOLUTION

A two parameter expansion is adopted for each of $\bar{\psi}$, $\bar{\phi}$ and $\bar{\sigma}$:

$$\bar{\psi} = \bar{\psi}_0 + \sqrt{\varepsilon}\,\bar{\psi}_1 + \delta^2\,\bar{\psi}_2 + \ldots, \tag{2.3.6}$$

$$\bar{\phi} = \bar{\phi}_0 + M_0, \tag{2.3.7}$$

$$\bar{\sigma} = \sqrt{\varepsilon}\,(\bar{\sigma}_1 + \sqrt{\varepsilon}\,\bar{\sigma}_2 + \delta^2\,\bar{\sigma}_3 + \ldots) \tag{2.3.8}$$

where M_0 is an asymptotic correction. If: (i) we put (2.3.6) and (2.3.8) into (2.3.3), (ii) equate coefficients of like powers of ε and δ and (iii) solve the resulting differential equations, we have for $\bar{\psi}$:

$$\bar{\psi} \approx \psi_a(1-\xi) . \tag{2.3.9}$$

To find $\bar{\phi}$ the same procedure as above is followed with (2.3.4), (2.3.7) and (2.3.8) to obtain a one-term approximation to $\bar{\phi}$ as:

$$\bar{\phi} \approx \bar{\phi}_0 = 1 - L^{-1}\left\{\frac{\sinh[\sqrt{s}(1-\eta)/(1-\cos\beta)]}{s\sinh[\sqrt{s}/(1-\cos\beta)]}\right\} \tag{2.3.10}$$

where $\bar{\phi}_0$ can be written in one of the following two forms:

$$\bar{\phi}_0 = \eta + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2(1-\cos\beta)^2\bar{\tau}}\sin(n\pi(1-\eta))$$

$$\bar{\phi}_0 = 1 - \sum_{n=0}^{\infty}\left\{\operatorname{erfc}\frac{2n+\eta}{2(1-\cos\beta)\sqrt{\bar{\tau}}} - \operatorname{erfc}\frac{2(n+1)-\eta}{2(1-\cos\beta)\sqrt{\bar{\tau}}}\right\}.$$

Put (2.3.9) and (2.3.10) in (2.3.8):

$$\frac{\partial\bar{\psi}_0}{\partial\xi}(1-\cos\beta)^2 - \sqrt{\varepsilon}\,\bar{\sigma}_1(1-\cos\beta)^2\left(\frac{\partial\bar{\psi}_0}{\partial\xi} + \frac{\partial\bar{\phi}_0}{\partial\eta}\right) + \delta^2(1-\cos\beta)\frac{\partial\bar{\psi}_0}{\partial\xi}$$

$$+ \dots = \bar{\sigma}_1\frac{\partial\bar{\sigma}_1}{\partial\bar{\tau}} + \sqrt{\varepsilon}\left[\frac{\partial(\bar{\sigma}_1\bar{\sigma}_2)}{\partial\bar{\tau}} - \bar{\sigma}_1^2\frac{\partial\bar{\sigma}_1}{\partial\bar{\tau}}\right] + \delta^2\frac{\partial(\bar{\sigma}_1\bar{\sigma}_3)}{\partial\bar{\tau}} + \dots$$

where the derivatives of $\bar{\psi}_0$ and $\bar{\phi}_0$ of (2.3.14) and (2.3.15) are evaluated at $\xi = 1$ and $\eta = 0$ respectively. If: (i) equate coefficients of like powers of ε and δ and (ii) solve the resulting differential equations, we find

ε^0: $$\bar{\sigma}_1 = \sqrt{2|\psi_a|\bar{\tau}}\,(1-\cos\beta)$$

$\sqrt{\varepsilon}$: $$\bar{\sigma}_2 = \frac{-(1-\cos\beta)}{\sqrt{\bar{\tau}}}\int_0^{\bar{\tau}}\sqrt{u}\,du\,L^{-1}\left\{(1-\cos\beta)\frac{\partial\bar{\phi}_0}{\partial\eta}\Big|_{\eta=0}\right\} \tag{2.3.11}$$

δ^2: $$\bar{\sigma}_3 = \delta^2\sqrt{|\psi_a|\bar{\tau}/2}.$$

Combining terms:

$$\bar{\sigma} \approx \sqrt{\varepsilon}\ \{\sqrt{2\,|\,\psi_a\,|\,\bar{\tau}}\ (1-\cos\beta) - \frac{\sqrt{\varepsilon}\ (1-\cos\beta)}{\sqrt{\bar{\tau}}} \tag{2.3.12}$$

$$\int_0^{\bar{\tau}} \sqrt{u}\, du L^{-1}\left[\frac{\cosh(\sqrt{s}/(1-\cos\beta))}{\sqrt{s}\sinh[\sqrt{s}/(1-\cos\beta)]}\right] + \delta^2\sqrt{|\,\psi_a\,|\,\bar{\tau}/2}\}$$

where the inverse transform of the function found in equation (2.3.12) can be written in one of the following two forms:

$$(i)\ (1-\cos\beta)\left[1+2\sum_{n=1}^{\infty} e^{-n^2\pi^2(1-\cos\beta)^2 u}\right]$$

$$(ii)\ \frac{1}{\sqrt{\pi u}}\left[1+2\sum_{n=1}^{\infty} e^{-n^2/(1-\cos\beta)^2/u}\right].$$

OUTER SOLUTION

To find the outer solution put $\gamma=\varepsilon\bar{\tau}$ in equations (2.3.3), (2.3.4) and (2.3.5) respectively to find: (i) for the temperature in the solid,

$$\frac{\partial^2\psi}{\partial\xi^2} + \delta^2[\sigma^2\frac{\partial^2\psi}{\partial\beta^2} - 2\sigma\xi\frac{\partial\sigma}{\partial\beta}\frac{\partial^2\psi}{\partial\xi\partial\beta} + 2\xi\left(\frac{\partial\sigma}{\partial\beta}\right)^2\frac{\partial\psi}{\partial\xi} - \sigma\xi\frac{\partial^2\sigma}{\partial\beta^2}\frac{\partial\psi}{\partial\xi}$$

$$+ \xi^2\left(\frac{\partial\sigma}{\partial\beta}\right)^2\frac{\partial^2\psi}{\partial\xi^2}] = \frac{\varepsilon\bar{\kappa}[\ \sigma^2\,\partial\psi/\partial\gamma - \sigma\xi\ (\partial\sigma/\partial\gamma)(\partial\psi/\partial\xi)]}{[\cosh\{\delta(1-\sigma\xi)\}-\cos\beta]^2} \tag{2.3.13}$$

$$\xi=0 \qquad \psi=\psi_a$$

$$\xi=1 \qquad \psi=0$$

(ii) for the temperature in the liquid

$$\frac{\partial^2\phi}{\partial\eta^2} + \delta^2[(1-\sigma)^2\frac{\partial^2\phi}{\partial\beta^2} + 2(\eta-1)(1-\sigma)\frac{\partial\sigma}{\partial\beta}\frac{\partial^2\phi}{\partial\eta\partial\beta} + 2(\eta-1)\left(\frac{\partial\sigma}{\partial\beta}\right)^2\frac{\partial\phi}{\partial\eta} + (\eta-1)(1-\sigma)\frac{\partial^2\sigma}{\partial\beta^2}\frac{\partial\phi}{\partial\eta}$$

$$+ (\eta-1)^2\left(\frac{\partial\sigma}{\partial\beta}\right)^2\frac{\partial^2\phi}{\partial\eta^2}] = \frac{\varepsilon\,[\partial\phi/\partial\gamma + (\eta-1)(1-\sigma)(\partial\phi/\partial\eta)(\partial\sigma/\partial\gamma)]}{[\cosh\{\delta(1-\eta)(1-\sigma)\}-\cos\beta]^2}$$

(2.3.14)

$$\eta=0 \quad \phi=0$$

$$\eta=1 \quad \phi=1$$

and (iii),

$$(1-\sigma)\frac{\partial\psi}{\partial\xi} - \sigma\frac{\partial\phi}{\partial\eta} = \frac{1}{[\cosh\{\delta(1-\sigma)\}-\cos\beta]^2}\,\frac{\sigma\,(1-\sigma)\,\partial\sigma/\partial\gamma}{1+\delta^2(\partial\sigma/\partial\beta)^2}.$$

(2.3.15)

Put:

$$\psi = \psi_0 + M_0, \qquad (2.3.16)$$

$$\phi = \phi_0 + M_1, \qquad (2.3.17)$$

$$\sigma = \sigma_0 + M_2, \qquad (2.3.18)$$

where M_0, M_1 and M_2 are asymptotic corrections. If: (i) put (2.3.16)-(2.3.18) into (2.3.13)-(2.3.14) and (ii) equate coefficients of (ε^0,δ^0) we get:

$$\psi \approx \psi_s(1-\xi). \qquad (2.3.19)$$

$$\phi \approx \eta. \qquad (2.3.20)$$

To get a one-term approximation to σ : (i) put (2.3.19) and (2.3.20) into (2.3.15) and (ii) equate coefficients of (ε^0,δ^0)

$$\psi_s(1-\sigma_0) + \sigma_0 = \frac{-\sigma_0(1-\sigma_0)}{(1-\cos\beta)^2}\,\frac{\partial\sigma_0}{\partial\gamma}. \qquad (2.3.21)$$

Integrating (2.3.21) yields

$$\frac{\sigma_0^2}{2(1-\psi_*)} - \frac{\sigma_0}{(1-\psi_*)^2} + \frac{\psi_*}{(1-\psi_*)^3}\ln\left[1 + \frac{\sigma_0(1-\psi_*)}{\psi_*}\right]$$
$$= \gamma(1-\cos\beta)^2 + C(\beta) \qquad (2.3.22)$$

where $C(\beta)$ is unknown and is found by matching to the inner solution.

MATCH

To find $(\bar{\sigma})^o$ put $\gamma = \varepsilon\bar{\tau}$ in (2.3.12):

$$(\bar{\sigma})^o \approx \sqrt{2|\psi_*|\gamma}\,(1-\cos\beta) - \frac{(1-\cos\beta)^2}{\sqrt{\gamma}}\int_0^\gamma \sqrt{u}\,du$$
$$\left[1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2(1-\cos\beta)^2u/\varepsilon}\right] + \delta^2\sqrt{|\psi_*|\gamma/2}\,. \qquad (2.3.23)$$

where ε has been removed from the integration limit. Expanding for $\varepsilon \ll 1$ in equation (2.3.28) and keeping lead terms gives

$$(\bar{\sigma})^o \approx \sqrt{2|\psi_*|\gamma}\,(1-\cos\beta) - \frac{(1-\cos\beta)^2}{\sqrt{\gamma}}\int_0^\gamma \sqrt{u}\,du[1+\ldots]\,. \qquad (2.3.24)$$

Integrating,

$$(\bar{\sigma})^o \approx \sqrt{2|\psi_*|\gamma}\,(1-\cos\beta) - \frac{2\gamma}{3}(1-\cos\beta)^2. \qquad (2.3.25)$$

To find $(\sigma)^i$ we return to (2.3.22). Since (2.3.22) is unsuitable for matching we write

$$\sigma_0 = a_0(\beta) + a_1(\beta)\sqrt{\gamma} + a_2(\beta)\gamma + \ldots\,. \qquad (2.3.26)$$

Since $(\bar{\sigma})^o$ of equation (2.3.25) vanishes as $\gamma \rightarrow 0$ we are required to put

$$a_0(\beta) = 0 \qquad (2.3.27)$$

which means that

$$C(\beta) = 0. \qquad (2.3.28)$$

With result (2.3.27) we substitute equation (2.3.26) into equation (2.3.22) and equate coefficients of $\sqrt{\gamma}$ and γ to find:

$$(\sigma)^i = \sqrt{2|\psi_*|\gamma}\,(1-\cos\beta) - \frac{2\gamma}{3}(1-\cos\beta)^2. \qquad (2.3.29)$$

Comparing $(\bar{\sigma})^o$ of equation (2.3.25) to $(\sigma)^i$ of equation (2.3.29) we see that the one-term outer solution of equation (2.3.27) matches the two-term inner solution of equation (2.3.12).

(II) : $\beta = O(\delta)$

To get an approximation to the solution near $\beta = 0$: (i) we put

$$\theta = \frac{\beta}{\delta} \qquad (2.3.30)$$

and (ii) we reduce the work involved in finding $\bar{\sigma}$, by putting

$$\bar{\tau} = \frac{\delta^2 \tau}{4\sinh^2\delta} \qquad (2.3.31)$$

in (2.3.3), (2.3.4) and (2.3.5) respectively to get: (i) for the temperature in the solid,

$$\frac{\partial^2\bar{\psi}}{\partial\xi^2} + \bar{\sigma}^2\frac{\partial^2\bar{\psi}}{\partial\theta^2} - 2\bar{\sigma}\xi\frac{\partial\bar{\sigma}}{\partial\theta}\frac{\partial^2\bar{\psi}}{\partial\xi\partial\theta} + 2\xi\left(\frac{\partial\bar{\sigma}}{\partial\theta}\right)^2\frac{\partial\bar{\psi}}{\partial\xi} - \bar{\sigma}\xi\frac{\partial^2\bar{\sigma}}{\partial\theta^2}\frac{\partial\bar{\psi}}{\partial\xi}$$

$$+ \xi^2\left(\frac{\partial\bar{\sigma}}{\partial\theta}\right)^2\frac{\partial^2\bar{\psi}}{\partial\xi^2} = \frac{\bar{\kappa}\delta^4[\bar{\sigma}^2\,\partial\bar{\psi}/\partial\bar{\tau} - \bar{\sigma}\xi\,(\partial\bar{\sigma}/\partial\bar{\tau})(\partial\bar{\psi}/\partial\xi)]}{4[\cosh\{\delta(1-\bar{\sigma}\xi)\} - \cos(\delta\theta)]^2}$$

(2.3.32)

$$\xi = 0 \qquad \bar{\psi} = \psi_*$$

$$\xi = 1 \qquad \bar{\psi} = 0$$

and (ii) for the temperature in the liquid,

$$\frac{\partial^2\bar{\phi}}{\partial\eta^2} + (1-\bar{\sigma})^2\frac{\partial^2\bar{\phi}}{\partial\theta^2} + 2(\eta-1)(1-\bar{\sigma})\frac{\partial\bar{\sigma}}{\partial\theta}\frac{\partial^2\bar{\phi}}{\partial\eta\partial\theta} + 2(\eta-1)\left(\frac{\partial\bar{\sigma}}{\partial\theta}\right)^2\frac{\partial\bar{\phi}}{\partial\eta} + (\eta-1)(1-\bar{\sigma})\frac{\partial^2\bar{\sigma}}{\partial\theta^2}\frac{\partial\bar{\phi}}{\partial\eta}$$

$$+ (\eta - 1)^2 \left(\frac{\partial \bar{\sigma}}{\partial \theta}\right)^2 \frac{\partial^2 \bar{\phi}}{\partial \eta^2} = \frac{\delta^4 [\partial \bar{\phi}/\partial \bar{\tau} + (\eta - 1)(1 - \bar{\sigma})(\partial \bar{\phi}/\partial \eta)(\partial \bar{\sigma}/\partial \bar{\tau})]}{4[\cosh\{\delta(1-\eta)(1-\bar{\sigma})\} - \cos(\delta\theta)]^2}$$

(2.3.33)

$$\eta = 0 \quad \bar{\phi} = 0$$

$$\eta = 1 \quad \bar{\phi} = 1$$

$$\bar{\tau} = 0 \quad \bar{\phi} = 1$$

and (iii) across the surface of separation between the two phases,

$$\varepsilon \left((1 - \bar{\sigma}) \frac{\partial \bar{\psi}}{\partial \xi} - \bar{\sigma} \frac{\partial \bar{\phi}}{\partial \eta} \right) = \frac{\delta^4}{4[\cosh\{\delta(1 - \bar{\sigma})\} - \cos(\delta\theta)]^2} \frac{\bar{\sigma}(1 - \bar{\sigma})\, \partial \bar{\sigma}/\partial \bar{\tau}}{1 + (\partial \bar{\sigma}/\partial \theta)^2}$$

(2.3.34)

$$\bar{\tau} = 0 \quad \bar{\sigma} = 0.$$

INNER SOLUTION

Recall in Section 1.3 we were unable to develop a one-term uniformly valid approximation to that part of the linear problem. Since the zeroth approximation to the liquid temperature distribution,(as in Section 2.3 (I) and Section 2.2) will be the linear problem we will be unable to include a higher order correction to the interface location from the liquid temperature gradient. However, we will be able to develop a one-term approximation to the non-linear problem since the one-term approximation to the interface location is independent to first order of the liquid temperature gradient.

For $\bar{\psi}$ and $\bar{\sigma}$ we assume:

$$\bar{\psi} = \bar{\psi}_0 + M_0 \qquad (2.3.35)$$

$$\bar{\sigma} = \sqrt{\varepsilon}\, \bar{\sigma}_1 + M_1. \qquad (2.3.36)$$

To find a one-term approximation to $\bar{\psi}$: (i) put (2.3.35) and (2.3.36) into (2.3.32), (ii) equate coefficients of ε^0 and δ^0 and (iii) solve the resulting differential equation to get:

$$\bar{\psi} \approx \psi_a (1 - \xi). \qquad (2.3.37)$$

If we: (i) put (2.3.35), (2.3.36) and (2.3.37) into (2.3.34), (ii) equate coefficients of ε^0 and δ^0 and (iii) solve the resulting differential equations, we find:

$$\tilde{\sigma} \approx \sqrt{\varepsilon}\,\sqrt{2\,|\,\psi_*\,|\,\bar{\tau}}\;\;(1+\theta^2) \tag{2.3.38}$$

OUTER SOLUTION

Putting $\gamma=\varepsilon\bar{\tau}$ in (2.3.32), (2.3.33) and (2.3.34) respectively we have: (i) for the temperature in the solid,

$$\frac{\partial^2\psi}{\partial\xi^2} + \sigma^2\frac{\partial^2\psi}{\partial\theta^2} - 2\sigma\xi\frac{\partial\sigma}{\partial\theta}\frac{\partial^2\psi}{\partial\xi\partial\theta} + 2\xi\left(\frac{\partial\sigma}{\partial\theta}\right)^2\frac{\partial\psi}{\partial\xi} - \sigma\xi\frac{\partial^2\sigma}{\partial\theta^2}\frac{\partial\psi}{\partial\xi}$$

$$+\,\xi^2\left(\frac{\partial\sigma}{\partial\theta}\right)^2\frac{\partial^2\psi}{\partial\xi^2} = \frac{\varepsilon\bar{\kappa}\delta^4[\sigma^2\,\partial\psi/\partial\gamma - \sigma\xi\,(\partial\sigma/\partial\gamma)(\partial\psi/\partial\xi)]}{4[\cosh\{\delta(1-\sigma\xi)\}-\cos(\delta\theta)]^2}$$

(2.3.39)

$$\xi=0 \qquad \psi=\psi_*$$

$$\xi=1 \qquad \psi=0$$

(ii) for the temperature in the liquid,

$$\frac{\partial^2\phi}{\partial\eta^2} + (1-\sigma)^2\frac{\partial^2\phi}{\partial\theta^2} + 2(\eta-1)(1-\sigma)\frac{\partial\sigma}{\partial\theta}\frac{\partial^2\phi}{\partial\eta\partial\theta} + 2(\eta-1)\left(\frac{\partial\sigma}{\partial\theta}\right)^2\frac{\partial\phi}{\partial\eta} + (\eta-1)(1-\sigma)\frac{\partial^2\sigma}{\partial\theta^2}\frac{\partial\phi}{\partial\eta}$$

$$+\,(\eta-1)^2\left(\frac{\partial\sigma}{\partial\theta}\right)^2\frac{\partial^2\phi}{\partial\eta^2} = \frac{\varepsilon\delta^4[\partial\phi/\partial\gamma + (\eta-1)(1-\sigma)(\partial\phi/\partial\eta)(\partial\sigma/\partial\gamma)]}{4[\cosh\{\delta(1-\eta)(1-\sigma)\}-\cos(\delta\theta)]^2}$$

(2.3.40)

$$\eta=0 \qquad \phi=0$$

$$\eta=1 \qquad \phi=1$$

and (iii),

$$(1-\sigma)\frac{\partial\psi}{\partial\xi} - \sigma\frac{\partial\phi}{\partial\eta} = \frac{\delta^4}{4[\cosh\{\delta(1-\sigma)\}-\cos(\delta\theta)]^2}\,\frac{\sigma\,(1-\sigma)\,\partial\sigma/\partial\gamma}{1+(\partial\sigma/\partial\theta)^2}.$$

(2.3.41)

For σ we assume:

$$\sigma = \sigma_0 + M , \tag{2.3.42}$$

where M is an asymptotic correction.

To obtain a one-term approximation for ψ and ϕ we change independent variables in both, (2.3.39) and (2.3.40), by putting: (i) $u = \xi/g(\varepsilon,\delta)$, $g = o(1)$ in (2.3.39) and (ii) $u = (1-\eta)/h(\varepsilon,\delta)$, $h = o(1)$ in (2.3.40). A one-term approximation to $\tilde{\phi}$ is found using matched asymptotic expansions, where the outer variable is u. One-term, constant temperature approximations, not valid in the neighbourhood of the interface boundary, are found for ψ^o and ϕ^o. Linear stretches are made, near the interface boundary, and one-term approximations are found to ψ^i and ϕ^i. Matching and combining these to form a one-term composite solution for ψ and ϕ respectively gives:

$$\psi \approx \psi_a(1-\xi), \tag{2.3.43}$$

$$\phi \approx \eta, \tag{2.3.44}$$

where we have returned ψ and ϕ to dependence on, ξ and η. If: (i) put (2.3.42), (2.3.43), and (2.3.44) into (2.3.41) and (ii) equate coefficients of (ε^0,δ^0) we have:

$$\psi_a(1-\sigma_0) + \sigma_0 = \frac{-\sigma_0(1-\sigma_0)}{[(1-\sigma_0)^2+\theta^2]^2}\frac{\partial\sigma_0/\partial\gamma}{1+(\partial\sigma_0/\partial\theta)^2}. \tag{2.3.45}$$

As before it is easier to perform the match of the inner solution to the outer solution via equation (2.3.45). In addition the construction of the solution to equation (2.3.45) is best left until after the match since the match yields information that is useful in setting up σ_0.

MATCH

The outer expansion of $\tilde{\sigma}$ is found by putting $\gamma = \varepsilon\bar{\tau}$ in equation (2.3.38):

$$(\tilde{\sigma})^o \approx \sqrt{2\,|\,\psi_a\,|\,\gamma}\;(1+\theta^2). \tag{2.3.46}$$

The inner expansion of the outer solution found by putting:

$$\sigma_0 = a_0(\theta) + a_1(\theta)\sqrt{\gamma} + a_2(\theta)\gamma + \ldots \tag{2.3.47}$$

in (2.3.45).

Since $(\bar{\sigma})^{*} \to 0$ as $\gamma \to 0$ in (2.3.46) we must put

$$a_0(\theta) = 0. \tag{2.3.48}$$

With (2.3.48) we substitute (2.3.47) into (2.3.45) and equate coefficients of $\gamma^{n/2}$ to get:

$$(\sigma)^i = \sqrt{2\,|\,\psi_a\,|\,\gamma}\,(1+\theta^2) - \frac{2\gamma}{3}\left[1 - \frac{4\psi_a}{1+\theta^2}\right](1+\theta^2)^2. \tag{2.3.49}$$

Comparing $(\bar{\sigma})^{*}$ of equation (2.3.46) with $(\sigma)^i$ of equation (2.3.49) shows that the one-term inner solution matches the one-term outer solution.

A SOLUTION TO EQUATION (2.3.45)

Put $\gamma = f(\sigma_0, \beta)$ in (2.3.45):

$$[(1-\sigma_0)^2 + \theta^2]^2\left[\left(\frac{\partial\gamma}{\partial\sigma_0}\right)^2 + \left(\frac{\partial\gamma}{\partial\theta}\right)^2\right] = f(\sigma_0)\,\frac{\partial\gamma}{\partial\sigma_0} \tag{2.3.50}$$

where

$$f(\sigma_0) = \frac{-\sigma_0(1-\sigma_0)}{\psi_a(1-\sigma_0)+\sigma_0}.$$

To find $\gamma(\sigma_0,\theta)$ we subtract off a particular solution $\bar{\gamma}$:

$$\gamma(\sigma_0,\theta) = \bar{\gamma}(\sigma_0,\theta) + G(\sigma_0,\theta). \tag{2.3.51}$$

Since: (i) (2.3.45) has a constant steady state and (ii) we have result (2.3.47) and (2.3.48),

$$\partial\sigma_0/\partial\theta \to 0 \text{ as } \gamma \to (0,\infty).$$

Therefore $\bar{\gamma}$ is found by dropping $\partial\sigma_0/\partial\theta$ in equation (2.3.45) or, $(\partial\gamma/\partial\theta)/(\partial\gamma/\partial\sigma_0)$ in equation (2.3.50).

$$\bar{\gamma} = \int_0^{\sigma_0} \frac{f(u)\,du}{[(1-u)^2+\theta^2]^2} \tag{2.3.52}$$

where: (i) $f(u)$ is defined in equation (2.3.52) and (ii) $\bar{\gamma}(0,\theta)=0$ from (? 3.47) and (2.3.48). Since $\partial\sigma_0/\partial\theta$ of (2.3.45) does not contribute to $(\sigma)^i$ of (2.3.47) and (2.3.48), equation (2.3.52) contains all of the necessary information for the match, to the order considered here.

If we substitute (2.3.51) into (2.3.50) a new problem for G is found, which when solved numerically, is shown (see "Results" section) to be a negligible correction to the interface location. By assuming a convergent series expansion for $G(\sigma_0,\theta)$, we can deduce the following properties:

(i) $\sigma_0 \rightarrow 0,\ \gamma \rightarrow \bar{\gamma}$

(ii) $\sigma_0 \rightarrow$ *steady state*, $\gamma \rightarrow \bar{\gamma}$

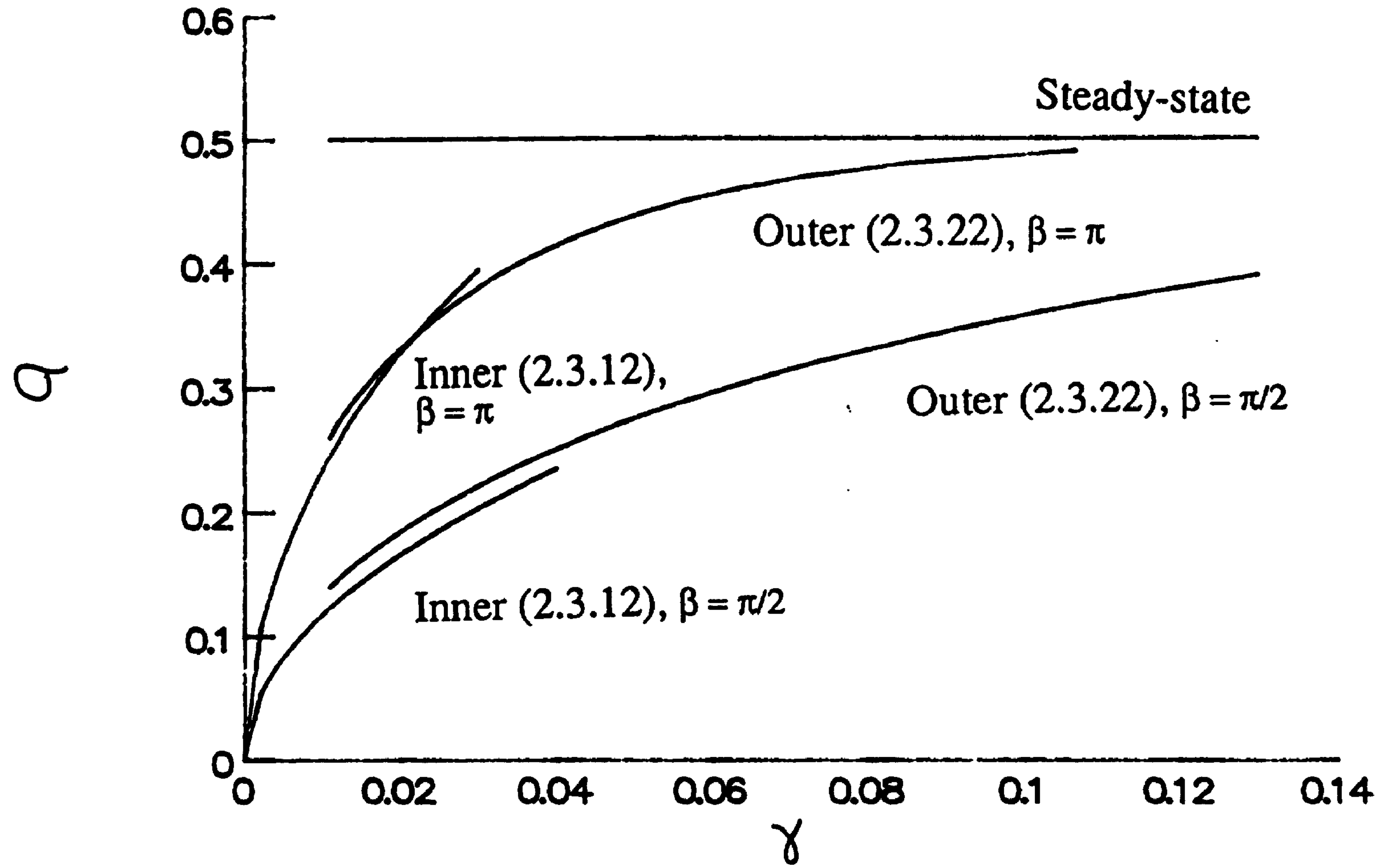
(iii) $\theta \rightarrow 0,\ \gamma \rightarrow \bar{\gamma}$.

RESULTS

The one-term approximation to the inner solution found in Section 2.3 (I) contributes no new information to the one-term approximation to the outer solution. Results shown here assume that $|\psi_a|=1$, $\delta=0.135$ and $\varepsilon=0.2$. Figure 2.3.1 shows the overlap region between the inner and outer solutions of equations (2.3.12) and (2.3.45).

Numerical calculations show that (2.3.51) is well represented by (2.3.52), since $G(\sigma_0,\beta)$ represents less than a 2% correction to the one-term, large-time interface location. Thus, since equation (2.3.52) also contains all of the necessary information for the match to the order considered here, (see "A solution to equation (2.3.45)") it will be used to represent σ_0. Since the one-term approximations to the inner solutions found in Sections 2.3 (I) and (II) contribute no new information to a one-term approximation to the outer solution, the one-term results of Section 2.2 are uniformly valid for all depths.

Fig. 2.3.1 Interface location for shallow pipe depths:
$|\psi_a| = 1,\ \varepsilon = 0.2,\ \delta = 0.135$



CONCLUSIONS

The analytical results derived here, for both the linear and the non-linear problem, (i.e., case (i) and (ii) of the abstract) are perturbation approximations valid near t=0 and which also approach the exact steady solution as $t \to \infty$, in addition they: (i) provide insight into the interaction between various parameters and (ii) are easy to use.

In chapter 1 it was possible to derive a one-term perturbation approximation to the solution of the linear problem (i.e., case (i) of the abstract) for deep and shallow, pipe depths. The one-term analytical approximations found for the temperature field have shown appropriate agreement with numerical results for representative values of the parameters involved. Martin and Sadhal [1] have considered the same problem for finite depths by solving a simplified version of the problem. This method tends to conceal the interaction between various parameters of the problem, unlike the perturbation method.

In chapter 2 case (ii) of the abstract is considered, where the medium is initially liquid and the pipe surface temperature is such that it freezes the surrounding liquid. Since the formulation is symmetric with respect to freezing or melting, the analytical results found here can be used for the melting case.

Unlike all of the previous analytical approaches to this problem, which are based on *ad hoc* assumptions, the results presented here, have been constructed using a systematic mathematical method: the method of matched asymptotic expansions. For example: (i) the "patching" [42] of solutions has been avoided and (ii) it has not been assumed that the interface location is described by the isotherms of the steady-state solution i.e. independent of the angle-like variable β (a questionable assumption).

For finite pipe depths equation (2.2.39) is a one-term composite approximation to the interface location, and since (2.2.40) (which contains all of the necessary information for the match), was shown to be a good approximation to (2.2.39), we have a compact solution which: (i) shows a continuous transition from a "small-time" interface location, concentric with the pipe to a circular but non-concentric "steady" interface location and (ii) approaches the quasi-steady approximation by Carslaw and Jaeger [20], to the problem of infinite pipe depth.

For deep and shallow pipe immersion depths, we were able to obtain higher order approximations to the interface location, which introduce a correction to account for the effect of the liquid thermal gradient on the interface location at small times.

Furthermore, for all cases: (i) the temperature distributions used to construct the inner and outer approximations to the interface location, also match; which is consistent and (ii) the limiting case of a liquid at the fusion temperature is described.

Although numerical approaches may be used, (Wheeler [38], Gold, Johnston, Slusarchuk and Goodrich [39], Lachenbruch [3]), which give a good approximation to the solution of this problem, these methods are complicated, cumbersome to use.

In constructing an approximation to a physical problem there are two sources of error: (i) the assumptions that one must make about the physics of the problem itself in constructing a mathematical model for the problem and (ii) the mathematical approximations that one makes, barring an exact solution, in deriving an analytical or numerical approximation to the model. In the case of permafrost, assessing the errors made in (i) of the above is extraordinarily difficult due to, for example, the harsh climate involved. However, it is possible to quantify the errors in (ii) by comparing analytical results to available numerical results. Figure 2.2.3 shows a comparison of numerical results and previous analytical approximations (to the problem considered here) with analytical results derived here for the same problem. Although the numerical results appear to have been found for a particular problem out of the range of validity of any analytical approximation, a comparison is still of interest. It is quite clear in Figure 2.2.3 that the analytical results found here, show a substantial gain in accuracy over previous analytical results. It should be remembered also that the numerical results in Figure 2.2.3, derived by C. T. Hwang [43], are in error by 5-10%.

It is intended in future work to use a numerical solution of this problem to estimate: (i) the errors in the approximations to the interface location and (ii) the range of values of the defining parameters over which these approximations may be used. Numerical solutions presented in the literature either: (i) do not treat the full problem since they first make the quasi-static assumption before beginning the numerical analysis, Zhang, Weinbaum and Jiji [37], or (ii) present results for which an analytical approximation has not been found; in fact, C.T. Hwang [43] used a general finite-element method developed for this model which he compared with the quasi-static approximation, (unfortunately in cases for which that approximation appears not to be valid).

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Appendix

An iterative numerical approximation was generated to equation (1.1.3). For convenience, the independent variable α, in equation (1.1.3) was changed from α to $\nu = \alpha/\alpha_0$. A second-order, fully implicit, finite difference scheme was then constructed. The implicit approximation was iterated to convergence, for each time level, using a relative error criterion for convergence on a particular time level. Apparent convergence in both space and time was used to assure the convergence of the results to within the thickness of a line(code available on request).