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Qcd Condensates, The Operator-product Expansion And The Dynamical Quark Mass

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QCD CONDENSATES, THE OPERATOR-PRODUCT EXPANSION AND THE DYNAMICAL QUARK MASS

by

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ABSTRACT

The operator-product expansion will be applied to the two-point Green functions of QCD in order to study the dynamical generation of quark masses. The lowest order quark and gluon condensate contributions to the quark self-energy will be calculated to all orders in the expansions of the quark-quark and gluon-gluon vacuum expectation values. The resulting on-shell gauge independence of the quark self-energy leads self-consistently to a non-perturbative dynamical quark mass in agreement with the phenomenology of up and down constituent mass.

The order g^3 chiral-violating, mixed condensate component of the quark self-energy will be calculated to order m^3 in the expansions of the quark-quark and quark-gluon-quark vacuum expectation values, and then extended to all orders. The resulting self-energy is gauge independent for arbitrary values of the current mass, and the pole position of the quark propagator is unshifted from the value obtained from the quark and gluon condensates.

The lowest order quark condensate projection of the gluon propagator will be calculated to all orders in the quark-quark vacuum expectation value. The self-energy is found to be transverse, satisfying the Slavnov-Taylor identity. Implications for mechanisms which lead to an effective gluon mass will be considered. A brief application of this result to the QCD sum rule for the ρ meson will also be presented.

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CHAPTER ONE

INTRODUCTION

1-1 Quantum Chromodynamics: An Overview

Quantum Chromodynamics (QCD) is the gauge theory generally accepted as describing the interactions of the fundamental fermions known as quarks. The original motivation for the formulation of quark theories arose from high-energy electron-proton scattering experiments which indicated that protons have substructure.[1] Evidence for a kinematic behaviour inconsistent with that of point-like particles was found by measurements of the two structure functions which parametrize deep inelastic scattering.

Experimental results then demonstrated the dependence of these functions upon only one of the two kinematic variables involved, a phenomenon known as Bjorken scaling.[2] Subsequent work led to the Callan-Gross relation[3] which showed that only one of the structure functions was independent. An early explanation of these surprising experimental features was provided by the parton model,[4] which hypothesized that protons were composed of structureless, spin ½ particles whose self-interactions were weak in comparison with the electromagnetic interaction.

An extension of the parton model hypothesis was the flavour quark model which denoted the spin ½ partons as "quarks". This theory was inspired by the Gell-Mann-Nishijima relation[5] which identified the electric charge of a particle as a combination of its hypercharge (baryon number and strangeness) and isospin. The desire to view the Gell-Mann-Nishijima relation as a consequence of a higher symmetry was realized by introducing quarks in three flavours (up, down and strange), which were assumed to transform as the fundamental representation of SU(3). The flavour quark model thus catalogued hadrons as various combinations of up, down and strange (u,d,s) quarks, in a way consistent with SU(3). For example, the higher dimension representation $3^{*}\otimes 3 = 8\oplus 1$ represents the binding of a quark and anti-quark to compose the 0^{-} meson octet along with the η^{o}.[6]

Some insight into the quark mass spectrum can be obtained by noticing that the SU(3) flavour symmetry is not exact, since hadronic multiplets are only approximately degenerate, while the isospin multiplets show a higher degree of degeneracy. Since particles in the same isospin multiplet have the same number of strange quarks but vary the combinations of up and down ones, the u and d must be nearly degenerate in mass while the s is comparatively heavy. This apparent degeneracy may actually be a reflection of the u and d being light (<10MeV) compared to the strong interaction scales, allowing the ratio m_d/m_u to substantially differ from one.

The SU(3) model succeeded in imposing some structure upon the growing particle zoo, but it contained some fundamental difficulties which were eventually overcome by the ideas of QCD. Apart from the failure to observe either free quarks or $3\otimes 3$ hadronic states, the problem of a fundamental contradiction of the Pauli exclusion principle existed for the $3/2^{+}$ baryons. A spin 3/2 object composed of three u quarks cannot exist as a ground state wavefunction without violating the Pauli exclusion principle. To resolve this fundamental issue the concept of colour symmetry was introduced, giving quarks a colour quantum number in addition to flavour (u,d,s).[10] The total number of colours, predicted to be three to resolve the statistics problem mentioned above, can also be determined empirically by fitting the experimental ratio $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \text{muons})$ to the theoretical predictions with N colours. It was found that N=3 provides an excellent agreement with R both above and below the charm threshold.[7] A choice of N=3 is also motivated by electroweak perturbative field theory, since electroweak interactions are anomaly-free for each generation of quarks provided that the total number of colours is three.[8]

The lack of 3 and $3\otimes 3$ hadronic states can now be understood if it is hypothesized that physics is invariant under rotations of the three quark colours. This invariance is realized by a rigid or global $SU_c(3)$ symmetry* of the quark Lagrangian, suggesting that colour singlets are the ground states of nature. Since the representations 3 and $3\otimes 3 = 3^* \oplus 6$ do not contain colour singlets, then if such hadronic states exist, they must necessarily lie at an energy beyond existing experiments.

The interactions between quarks may now be formulated by gauging the $SU_c(3)$ symmetry, replacing global invariance with a local gauge symmetry.[11] This procedure necessarily introduces one gauge boson for each symmetry generator, implying the existence of the eight gluons which are the gauge bosons of $SU_c(3)$. The $SU_c(3)$ gauge theory of quarks whose interactions are mediated by gluons is the non-abelian gauge theory known as Quantum Chromodynamics (QCD).

QCD cannot be considered a viable description of the interactions between quarks unless it correctly models both the weak binding of

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*The "c" represents colour to distinguish between the SU(3) flavour and SU(3) colour symmetries.

quarks observed at large momentum transfer, and the strong binding of quarks into hadrons at low momentum transfer. For example, the proton appears as a point-like particle until a sufficiently high energy is reached and parton-like structure becomes evident. Renormalization of a general gauge theory demands that the effective coupling constant be a function of momentum. For a non-abelian gauge theory the coupling constant decreases with increasing momentum, a property known as asymptotic freedom.[9,12] Thus, the behaviour of the QCD coupling constant is consistent with the appearance of proton substructure at large momentum transfer, with QCD resulting in a complicated and largely unknown low-energy limit (nuclear physics, hadronic mass spectrum) which cannot be studied perturbatively.

This thesis will utilize the technique of operator-product expansions (OPE) which extend the range of QCD into a small part of the non-perturbative region. The problem of generating a gauge independent, dynamical quark mass will be studied by using OPE methods to calculate non-perturbative contributions to the quark self-energy.

1-2 Quark Mass Parameters: Current and Constituent Masses

One of the major complications in QCD is the issue of quark confinement. Experimental evidence supports the hypothesis that quarks may only appear as colour-singlet bound states; a result intuitively consistent with the long distance behaviour of both the QCD coupling constant and the quark/anti-quark potential. Thus if quarks are actually confined, then it is clearly not possible to define a quark mass analagous to the mass of a free particle such as the electron. Two concepts of quark mass exist despite the complication of quark confinement.

The first which is known as the current or Langrangian mass, arises because the electroweak Lagrangian must still contain quark mass parameters. In electroweak theory this mass appears as a result of the Yukawa coupling of quarks to the Higgs field. This interaction becomes a mass parameter when the Higgs acquires a dimension-one vacuum expectation value (vev) $\langle\phi\rangle$ characterizing the spontaneous symmetry breaking of the $SU(2)\otimes U(1)$ electroweak symmetry.[13] It is possible to estimate the current mass through the application of current algebra[14] and PCAC[15] (partial conservation of axial current) techniques, resulting in typical current mass values of $m_u \approx m_d \approx 10$ MeV.[16,17] The current mass for u and d quarks is actually quite small within the strong interaction scale, but large compared with the electron which lies within the same family of fermions. This indicates that the electron to current-quark mass ratio is sensitive to the quark's role in QCD and hence might be a relic of a higher symmetry describing a grand unified theory.[18-20]

The second quark mass parameter, known as the constituent mass, is at least an order of magnitude removed from the current mass for u and d quarks. Constituent masses successfully model the magnetic moments of nucleons (composed entirely of u and d quarks) by assuming that the nucleons are three-quark quantum states with $m_u = m_d = m_{nucleon}/3 \approx 300$ MeV.[21] Another application of these masses is in hadron spectroscopy where excited hadronic states can be well described using parameters consistent with constituent masses.[22]

A large constituent mass cannot be understood by application of perturbation theory, since the renormalization group will only logarithmically correct the current mass,[18,20] implying that non-perturbative QCD interactions are responsible for a large "dynamical" component of the constituent mass. Thus the discrepancy between current and constituent quark masses is a fundamental problem of QCD which may only be resolved using non-perturbative methods.

The mass scales present in QCD are large compared to the 10 MeV up and down current masses, so chiral symmetry (m=0) must be an approximate symmetry of the classical QCD Lagrangian. However, since constituent masses exist in the presence of such small current masses, chiral symmetry must be broken by non-perturbative effects. There is evidence in the near masslessness of the pions to suggest that the mechanism for chiral symmetry breaking lies within the QCD vacuum. If the chiral flavour symmetry ($SU_L(2) \otimes SU_R(2)$) is broken to an SU(2) flavour symmetry by the failure of the vacuum to be annihilated by the axial charges, then the Goldstone mechanism requires a massless boson for each of the three broken generators.[23] The small mass of the three pions is thus explained by identifying them as the Goldstone bosons of chiral symmetry-breaking. The broken $U_A(1)$ generator should also lead to a massless η' particle, but such is not the case ($m_{\eta} \approx 4m_{\pi}$), a result which can be partially understood as a consequence of the anomaly in the axial current.[24]

One of the implications of a QCD vacuum which violates chiral symmetry is the possible existence of non-zero vacuum expectation values for chiral non-invariant operators such as $\langle\bar{\psi}\psi\rangle$. This is indeed the case, since PCAC arguments imply[11] $(m_u+m_d)\langle\bar{u}u+\bar{d}d\rangle = -f_\pi^2 m_\pi^2$ and hence, that the quark condensate is non-zero. Independent confirmation of the non-trivial nature of the quark condensate can be found in QCD sum rules, which presume the existence of $\langle\bar{\psi}\psi\rangle\neq 0$ then self-consistently determine its value from experimental data.[25,26]

The QCD vacuum is parametrized by the gauge invariant vev's known as condensates, two examples being $\langle\bar{\psi}\psi\rangle$ and $\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle$. Condensates are categorized according to their mass dimension and generally decrease in importance with increasing dimension. The two lowest-dimensional condensates which exist in QCD are the dimension-three quark condensate $\langle\bar{\psi}\psi\rangle$ and the dimension-four gluon condensate $\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle$. Values for the quark and gluon condensate are compatible with an intuitive understanding of the scales present in QCD. For example, the magnitude of the quark condensate should be approximated by a quark number density, typically of the order $(\frac{4}{3}\pi r_c^3)^{-1}$, where r_c is a confinement radius of about 0.5 F.[27] This naive estimate of the quark condensate yields $|\langle\bar{\psi}\psi\rangle| \simeq (250\ \text{MeV})^3$ in agreement with evaluations using PCAC and QCD sum rules[25,28-30] techniques.

In perturbation theory, non-trivial vev's for normal-ordered products of fields cannot exist for a Poincare-invariant vacuum. This implies that non-perturbative or long distance effects must alter the structure of the QCD vacuum, producing non-zero values for the QCD condensates. Existence of a non-zero value for $\langle\bar{\psi}\psi\rangle$ implies that chiral symmetry is not respected by the QCD vacuum, which suggests a mechanism for dynamically generating constituent quark masses in the presence of explicit Lagrangian chiral symmetry.

Based on dimensional considerations, the effect of the quark condensate on quark self-energies would correspond to an effective quark mass of the form $M_{eff}(p^2)\approx\langle\bar{\psi}\psi\rangle/p^2$. An effective mass of this form will be negligible in the large momentum regime where current masses are applicable, but significant in the low momentum regime where the constituent mass is dominant. Determining the dynamical mass generated by the dimension-three quark condensate requires a self-consistent application of quantum field theory using operator-product expansion techniques, as contrasted to the Higgs mechanism which generates current masses statically from the dimension-one Higgs vev $\langle\phi\rangle$.

In a gauge theory such as QCD, physical quantitites such as the position of a propagator pole must be gauge independent in order to be physically meaningful. The first calculations of $M_{eff}(p^2)$ revealed a constituent mass scale of approximately 300 Mev for up and down quarks,[31] but the gauge dependence of the result was questioned because the calculation was performed in Landau gauge. A subsequent calculation in a general covariant gauge indicated that the quark condensate component of the quark self-energy was gauge dependent in the $\not{D}\psi=0$ limit,[32] casting further doubt upon the field-theoretical validity of the mass scale generated by the quark condensate.

The issue of gauge independence was then examined by Elias, Scadron and Tarrach[33] who supplemented the field equation by admitting the possibility of a mass scale defined by the $\not{D}\psi=-im\psi$ equation of motion.[34] This complicated the evaluation of the quark self-energy since the quark condensate component of the quark self-energy now appeared as a power series in the parameter $m/\not{p}$. The result of the calculation was a quark self-energy which was gauge invariant at the $\not{p}=m$ mass shell to second order in $m/\not{p}$.[33]

Chapter Two will begin with an introduction to operator-product expansion methods and a brief review of the Elias-Scadron-Tarrach result. The lowest order quark condensate component of the quark self-energy will then be calculated to all orders of the $m/\not{p}$ expansion[35,36] to examine whether gauge independence is still maintained. To further examine gauge independence, the self-energy will also be calculated in the light-cone gauge, and a comparison between the results of covariant and non-covariant gauges will be made by examining the on-shell value of the quark self-energy.

Condensates of dimension larger than three may also contribute to the quark self-energy.[37] The dimension-five mixed condensate $\langle\bar{\psi}\sigma G\psi\rangle$ is particularly important since it is the second lowest dimensional chiral-violating condensate. In Chapter Three, the lowest order mixed condensate contribution to the quark self-energy will be explicitly calculated to order m^3 and then extended to all orders.[36,38] The field-theoretical question of gauge independence will be addressed, and the physical issue of mixed condensate effects on the constituent mass scale will be considered. An interesting feature of the mixed condensate component of the quark self-energy is a dependence on both the Lagrangian (current) and dynamical (constituent) mass scales. Restrictions placed upon the Lagrangian mass by the requirement of gauge independence will be investigated.

The dimension-four gluon condensate can also contribute to the order g^2 corrections to the quark self-energy. Chapter Three will begin with the calculation of this component of the quark self-energy, and will also consider the physical effects of the gluon condensate upon dynamical mass generation.

Higher order corrections to the quark self-energy can arise from the lowest order quark condensate component of the gluon self-energy. Gauge independence of these corrections is guaranteed if the gluon self-energy is transverse, satisfying the Slavnov-Taylor identity. In Chapter Four, the lowest order quark condensate component of the gluon self-energy will be calculated to all orders in the dynamical mass scale m.[39] This result will be applied to the QCD sum rule for the ρ meson, extending the quark condensate portion of the sum rule to all orders in the mass scale m. Physical effects of the gluon self-energy upon the corrected gluon propagator will be investigated in the context of gluon mass generation.

CHAPTER TWO

NON-PERTURBATIVE CONTRIBUTIONS TO THE QUARK SELF-ENERGY

2-1 Operator-Product Expansions

In this chapter the effect of non-trivial QCD condensates on the quark propagator will be calculated. The effect of the chiral symmetry-violating quark condensate is particularly important since it is capable of generating a dynamical component of the quark mass.

To incorporate the existence of condensates into quantum field theory, the operator-product expansion must be employed. The OPE states that for any local operators A, B the time-ordered product may be written as a sum over operators θ [40]

$$T(A(x)B(0)) = \sum_{\theta} C_{\theta}(x)\ \theta \tag{2.1}$$

where $C_{\theta}(x)$ are c-numbers. Only operators which satisfy the symmetries of the underlying quantum field theory can appear in the OPE.

One of the original applications of the OPE was determining the short-distance singularity properties of composite operators. For example, in scalar field theory the OPE for the composite operator $\phi^2(x)\phi^2(0)$ is[41]

$$\langle 0|T(\phi(x/2)\phi(-x/2)\phi^2(0))|0\rangle \underset{x\to 0}{\sim} \left\{\ln(\tfrac{1}{2}m|x|) + O(m^2x^2)\right\} \langle\phi^4\rangle \tag{2.2}$$

Since the OPE studies short distance properties, the coefficients $C_{\theta}(x)$ can be determined perturbatively, independent of the process involved.[42]

The OPE can now be applied to a general Green function by taking a vacuum expectation value of (2.1), leading to the result

$$\langle 0|T(A(x)B(0))|0\rangle = \sum_{\theta} C_{\theta}(x) \ \langle\theta\rangle \qquad (2.3)$$

The operators θ in (2.3) are usually ordered according to increasing mass dimension, generally decreasing in importance as the dimension increases. This behaviour is understood by Fourier-transforming (2.3), resulting in the following expression.

$$\int e^{ip\cdot x} \langle 0|T(A(x)B(0))|0\rangle \, d^4x = \sum_{\theta} C_{\theta}(p) \ \langle\theta\rangle \qquad (2.4)$$

The mass dimension of (2.4) is fixed, so that as the dimension of $\langle\theta\rangle$ increases the dimension of $C_{\theta}(p)$ must decrease. This will contribute power-law corrections to the Green function, with increasing inverse powers of momentum. Thus for a reasonably large momentum, the importance of the coefficients will diminish with increasing operator dimension.

The OPE can only be useful if its coefficients may be calculated in a reasonably simple fashion. For purely perturbative field theories such as $\lambda\phi^4$ or quantum electrodynamics, the existence of the OPE can be proven rigorously and the coefficients may be calculated by taking a convenient matrix element of (2.1).[28,43] Thus for perturbative field theories the OPE factors short and long distance effects into $C_{\theta}(p)$ and $\langle\theta\rangle$, respectively.

The operator-product expansion (2.3) must respect the symmetries of the quantum field theory being considered, placing restrictions upon the operators appearing in the OPE. For an unbroken gauge theory such as QCD, intuition would suggest that only gauge invariant operators could appear. This result has been proven through a number of different techniques,[42,44-46] with different physical interpretations. The proof which has the most physical appeal recognizes that confined quarks must propagate in a background gluon field, leading to an OPE which contains only gauge invariant vevs.

In QCD, the objects $\langle\theta\rangle$ (condensates) still represent long distance effects, but the degree to which $C_\theta(p)$ is determined by perturbative processes is not completely understood. This consideration is known as factorization, and is a fundamental problem in QCD. Despite this difficulty, it is possible to estimate the validity of determining the coefficients perturbatively. The result of this analysis is that for condensates of mass dimension less than ten and for sufficiently large momentum, the coefficients $C_\theta(p)$ are dominated by short distance effects.[25,28] Additional evidence from the successful extraction of hadronic masses from QCD sum rules,[25,28] seems to confirm that the OPE can be applied meaningfully to complicated processes. Hence within certain limits, the coefficients and the condensates themselves factor the short and long distance effects of QCD, enabling the OPE to provide a useful extension of QCD into the non-perturbative regime.

2-2 QCD Condensate Contributions to the Quark Self-Energy: Formulation

The operator-product expansion for the quark propagator is

$$iS_F(p) = \int e^{ip\cdot x} \langle 0|T(\psi(x)\bar{\psi}(0))|0\rangle \, d^4x$$

$$= C_I(p) + C_{\bar{\psi}\psi}(p) \langle\bar{\psi}\psi\rangle + C_{FF}(p) \langle F^a_{\mu\nu} F^a_{\mu\nu}\rangle$$

$$+ \text{ higher dimension condensates} \qquad (2.5)$$

where $C_I(p)$ represents the purely perturbative QCD contribution. The next two terms contain the quark condensate $\langle\bar{\psi}\psi\rangle$ and the gluon condensate $\langle F^a_{\mu\nu} F^a_{\mu\nu}\rangle$, representing the lowest dimensional condensates existing in QCD.

The coefficient $C_{\bar{\psi}\psi}(p)$ in (2.5) is particularly important since it is the coefficient of the lowest-dimension, chiral symmetry-violating condensate $\langle\bar{\psi}\psi\rangle$. As outlined in the introduction, a non-zero value of this condensate makes possible the dynamical generation of a quark mass, even when the QCD Lagrangian is chiral invariant.

The OPE is an operator identity and the coefficient $C_{\bar{\psi}\psi}(p)$ can be determined perturbatively, so any method which obtains this coefficient using perturbative methods is acceptable. The method which will be used to calculate this coefficient allows condensates to directly enter the Feynman-Dyson perturbation series through normal-ordered products occurring in the Wick expansion.[41] This technique has been successfully applied to QCD sum rule calculations[44,46,47] and was the method used by Elias, Scadron and Tarrach for their OPE calculations.[33,37]

Consider the Feynman-Dyson series for the two-point quark amplitude in terms of the free fields.

$$iS_F(p) = \int d^4x\, e^{ip\cdot x} \langle 0|T(\psi(x)\bar{\psi}(0)\, \exp\left[i\int L_{int}(z)\, d^4z\right])|0\rangle \qquad (2.6)$$

The interaction Lagrangian $L_{int}(z)$ represents the deviation from free fields, and is given by

$$L_{int}(z) = \frac{g}{2}\, \bar{\psi}(z)\gamma_\mu \lambda^a A^a_\mu(z)\psi(z) \qquad (2.7)$$

where g is the QCD coupling constant. The gluon-gluon interaction has been ignored in (2.7) since it only contributes to higher order terms in g. The non-trivial contribution of (2.6) which is lowest order in g is given by the second order term of the exponential.

$$iS_F(p) = -\tfrac{1}{2} \int d^4x\, e^{ip\cdot x}\, d^4z d^4y \langle 0|T\left[\psi(x)\bar{\psi}(0)\, \frac{g^2}{4}\, \bar{\psi}(z)\gamma_\mu \lambda^a A^a_\mu(z)\psi(z)\right.$$

$$\left. \times\, \bar{\psi}(y)\gamma_\nu \lambda^b A^b_\nu(y)\psi(y)\right]|0\rangle \qquad (2.8)$$

This expression is easily reduced to products of free field propagators through use of the Wick expansion.[41] In conventional condensate-free perturbation theory, normal-ordered terms arising from the Wick expansion annihilate the vacuum, but QCD generates non-trivial vev's for these objects, thus producing the operator-product expansion.

Wick expanding (2.8) and ignoring the usual perturbative corrections leads to the following second order ($\mathcal{O}(g^2)$) correction to the quark propagator.[36,48]

$$i\Delta S(p) = \frac{g^2}{4} \int d^4x d^4y d^4z \; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma_\mu \lambda^a \langle :\bar{\psi}(z)\psi(y):\rangle$$

$$\times \gamma_\nu \lambda^b \langle T(\psi(z)\bar{\psi}(0))\rangle \langle T(A^a_\mu(y) A^b_\nu(z))\rangle$$

$$- \frac{g^2}{4} \int d^4x d^4y d^4z \; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma_\mu \lambda^a \langle :A^a_\mu(y) A^b_\nu(z):\rangle$$

$$\times \langle T(\psi(y)\bar{\psi}(z))\rangle \gamma_\nu \lambda^b \langle T(\psi(z)\bar{\psi}(0))\rangle \qquad (2.9)$$

The non-perturbative content of (2.9) resides in the non-local vev's $\langle :\bar{\psi}(z)\psi(y):\rangle$ and $\langle :A^a_\mu(y) A^b_\nu(z):\rangle$, both of which are related to the QCD condensates. It is useful to split (2.9) into two separate pieces with the following corrections to the quark propagator.[36,48]

$$i\Delta S_1(p) = \frac{g^2}{4} \int d^4x d^4y d^4z \; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma_\mu \lambda^a \langle :\bar{\psi}(z)\psi(y):\rangle$$

$$\times \gamma_\nu \lambda^b \langle T(\psi(z)\bar{\psi}(0))\rangle \langle T(A^a_\mu(y) A^b_\nu(z))\rangle \qquad (2.10)$$

$$i\Delta S_2(p) = -\frac{g^2}{4} \int d^4x d^4y d^4z \; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma_\mu \lambda^a \langle :A^a_\mu(y) A^b_\nu(z):\rangle$$

$$\times \langle T(\psi(y)\bar{\psi}(z))\rangle \gamma_\nu \lambda^b \langle T(\psi(z)\bar{\psi}(0))\rangle \qquad (2.11)$$

The amplitudes in (2.10) and (2.11) illustrate a feature common to all OPE calculations. Since the vev's do not carry any momentum, the order g^2 amplitudes in (2.10) and (2.11) represent tree-order processes, as opposed to one-loop processes in perturbation theory. This demonstrates that the lowest order OPE corrections to the quark propagator are analagous to the tree-order insertions of the Higgs vev in spontaneous symmetry breaking.

The second order contributions to the quark propagator in (2.10) and (2.11) are represented diagrammatically in Figures One and Two. The "NP" box in the figures represents the non-perturbative vev corresponding to the fields entering the box, and the quark and gluon lines represent the perturbative free-field propagators given by

$$\langle T(\psi(x)\bar{\psi}(y))\rangle \equiv i\int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, S_o(k,m_L)$$

$$= i\int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\,\frac{\not{k}+m_L}{k^2-m_L^2+i\varepsilon} \qquad (2.12)$$

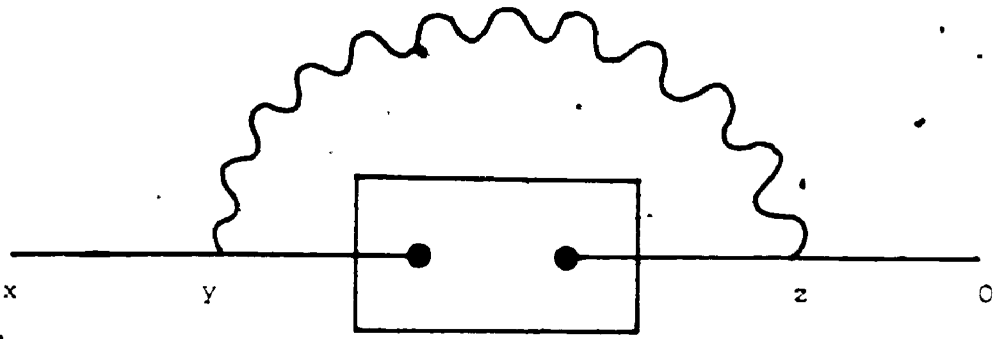
and

$$\langle T(A_\mu^a(x)A_\nu^b(y))\rangle \equiv i\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, D_{\mu\nu}(k) \qquad (2.13)$$

$$\equiv i\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}\left[-g^{\mu\nu}+(1-a)\frac{k^\mu k^\nu}{k^2+i\varepsilon}\right].$$

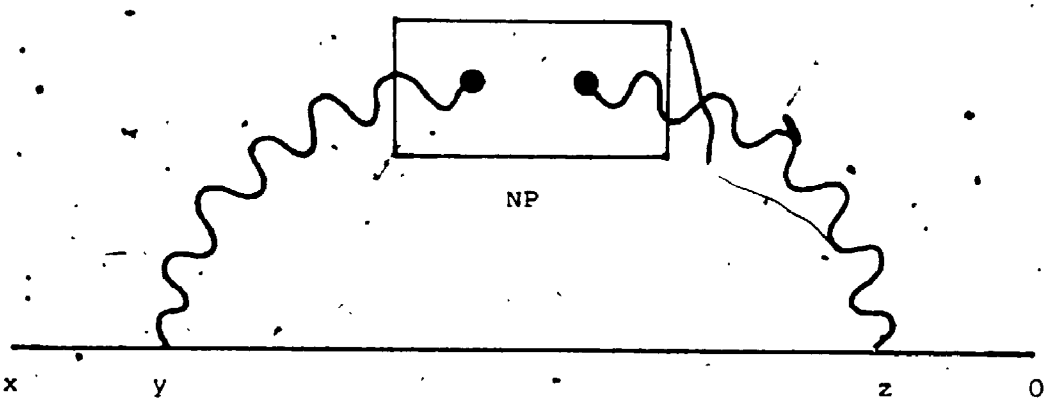
The quark propagator in (2.12) represents a quark with Lagrangian (current) mass m_L, while the gauge parameter "a" in (2.13) corresponds to the gauge degree of freedom in a covariant gauge. Physical results must be independent of the gauge parameter if meaningful physics is to be described, an issue which will be closely examined in subsequent sections.

FIGURE ONE



Order g^2 contribution to the quark self-energy from the quark-quark vev.

FIGURE TWO



Order g^2 contribution to the quark self-energy from the gluon-gluon vev.

2-3 Fixed-Point Gauge Methods

Calculation of the order g^2 $C_{\bar{\psi}\psi}(p)$ coefficient in (2.5) requires an evaluation of the two-point amplitudes in (2.10) and (2.11) in terms of the QCD condensates, which then identifies $C_{\bar{\psi}\psi}(p)$ as the coefficient of the quark condensate. Since the only unknowns in (2.10) and (2.11) are the non-perturbative vev's $<:\bar{\psi}(z)\psi(y):>$ and $<:A^a_\mu(y)A^b_\nu(z):>$, the condensate projections of these objects must be obtained. Only the quark-quark vev will contain components proportional to the quark condensate, a point which will be considered later.

The examination of the condensate content of $<:\bar{\psi}(z)\psi(y):>$ begins by expanding the vev in a Taylor series about y=z=0.

$$
\begin{aligned}
<:\bar{\psi}(z)\psi(y):> &= <:\bar{\psi}(0)\psi(0):> + y^\alpha<\bar{\psi}(0)\partial_\alpha\psi(0)> + z^\alpha<\bar{\psi}(0)\overleftarrow{\partial}_\alpha\psi(0)> \\
&+ \tfrac{1}{2} y^\alpha y^\beta<\bar{\psi}(0)\partial_\alpha\partial_\beta\psi(0)> + \tfrac{1}{2} z^\alpha z^\beta<\bar{\psi}(0)\overleftarrow{\partial}_\alpha\overleftarrow{\partial}_\beta\psi(0)> \\
&+ z^\alpha y^\beta<\bar{\psi}(0)\overleftarrow{\partial}_\alpha\partial_\beta\psi(0)> + \text{higher order terms} \qquad (2.14)
\end{aligned}
$$

The first term in (2.14) is simply the quark condensate with a factor of twelve to account for the implicit colour and Dirac algebra, as will be outlined in the next section. When (2.14) is substituted into (2.10), it will lead to a result for $\Delta S(p)$ which is precisely the form of the OPE. However, a technical problem is posed by the gauge dependence of objects such as $<\bar{\psi}(0)\partial_\alpha\partial_\beta\psi(0)>$ which play the role of condensates. Since only gauge invariant condensates are allowed to enter the OPE, the gauge independent components of these objects must be identified.

The fixed-point gauge allows the gauge invariant contributions of the terms in (2.14) to be extracted by replacing ordinary derivatives with covariant ones, rendering (2.14) gauge invariant.[44,47,48] The fixed-point gauge condition is [49]

$$x \cdot A^a(x) = 0 \quad , \tag{2.15}$$

which upon repeated differentiation implies

$$A^a_\mu(0) = 0 \quad ; \quad \partial_{(\alpha} \ldots \partial_\nu A^a_{\omega)}(0) = 0 \quad . \tag{2.16}$$

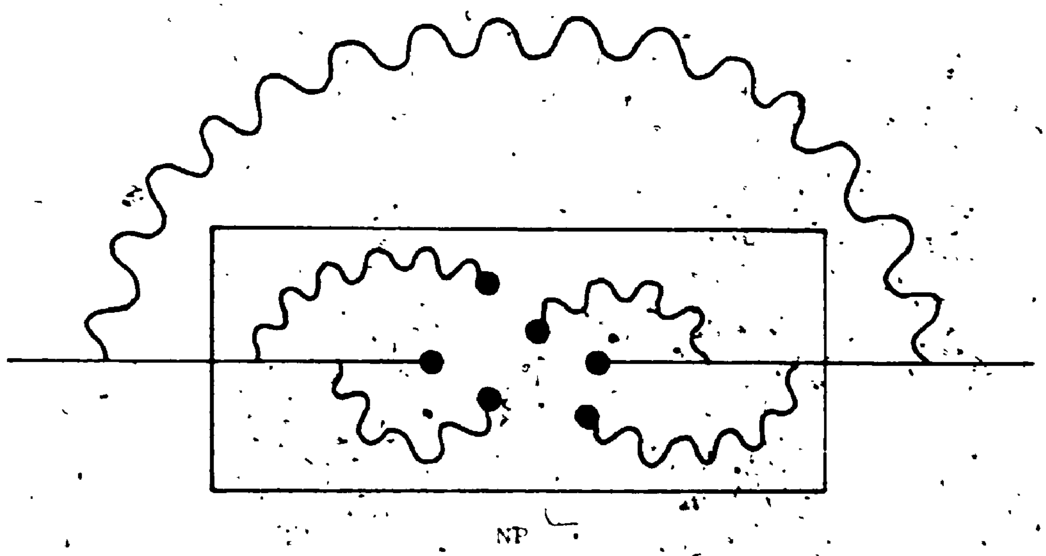
It is easy to see that a series of ordinary derivatives can now be replaced by symmetrized, covariant derivatives in the fixed-point gauge evaluated at x=0, since the difference between the two is composed of the trivial objects in (2.16). Therefore the operators in (2.14) can be replaced as follows.[44,47,48]

$$\partial_\alpha \ldots \partial_\omega \psi(0) = D_{(\alpha} \ldots D_{\omega)} \psi(0) \tag{2.17}$$

The notation in (2.16) for $\psi(0)$ is a reminder that all covariant derivatives are evaluated at x=0 in a consistent fashion. The effect of the fixed-point gauge can be intuitively understood as the introduction of multiple gluon insertions into the NP box of Figure One, a procedure diagrammatically illustrated in Figure Three.

The Taylor expansion of a gluon field can also be written in a gauge invariant manner through use of the fixed-point gauge.[44] The derivation of this result[47] begins with the identity

FIGURE THREE



Multiple gluon insertions into the non-perturbative vev, as generated by the fixed-point gauge.

$$A^a_\mu(x) = \partial_\mu[A^a_\nu(x)x^\nu] - x^\nu\partial_\mu A^a_\nu \quad . \tag{2.18}$$

The field strength $F^a_{\mu\nu}(x)$ is given by (see appendix)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu \quad . \tag{2.19}$$

The first term in (2.18) is zero by the gauge condition (2.14), and the second term can be simplified using (2.15) and (2.19).

$$A^a_\mu(x) + x^\nu\partial_\nu A^a_\mu(x) = x^\nu F^a_{\nu\mu}(x) \tag{2.20}$$

Scaling x to tx turns the L.H.S. of (2.20) into a total derivative of $tA^a_\mu(tx)$ with respect to t, so that $A^a_\mu(x)$ can be expressed in a gauge invariant fashion.[47]

$$A^a_\mu(x) = \int_0^1 dt\; tx^\lambda F^a_{\lambda\mu}(tx) \tag{2.21}$$

The expression for $A^a_\mu(x)$ in (2.21) clearly satisfies the condition $x\cdot A=0$ but it is not obvious that substitution of (2.21) into (2.19) will consistently lead to $F^a_{\mu\nu}(x)$. The reason for this uncertainty lies in the R.H.S. of (2.20) which is completely insensitive to the non-abelian nature of $F^a_{\mu\nu}(x)$. Since the fixed-point gauge is an important tool for the analysis of non-perturbative vev's, the self-consistency of the gauge is extremely important, a point which has been neglected in the literature.

Consider the substitution of (2.21) into (2.19). For (2.19) to be consistent, the following expression must be satisifed:

$$F^a_{\mu\nu}(x) = \int_0^1 dt\ 2t\, F^a_{\mu\nu}(tx) + \int_0^1 dt\ t^2 x^\lambda F^a_{\lambda\nu,\mu}(tx)$$

$$- \int_0^1 dt\ t^2 x^\lambda F^a_{\lambda\mu,\nu}(tx) + gf_{abc} A^b_\mu(x) A^c_\nu(x) \quad . \tag{2.22}$$

The Jacobi identity for $F^a_{\mu\nu}$ is (see appendix)

$$[D_\mu, G_{\lambda\nu}] - [D_\nu, G_{\lambda\mu}] = [D_\lambda, G_{\mu\nu}] \tag{2.23a}$$

$$F^a_{\lambda\nu,\mu} + gf_{abc} A^b_\mu F^c_{\lambda\nu} - F^a_{\lambda\mu,\nu} - gf_{abc} A^b_\nu F^c_{\lambda\mu}$$

$$= F^a_{\mu\nu,\lambda} + gf_{abc} A^b_\lambda F^c_{\mu\nu} \quad . \tag{2.23b}$$

Substituting (2.23b) into (2.22) gives the expression

$$F^a_{\mu\nu}(x) = \int_0^1 dt\ 2t\, F^a_{\mu\nu}(tx) + \int_0^1 dt\ t^2 x^\lambda F^a_{\mu\nu,\lambda}(tx) + gf_{abc} A^b_\mu(x) A^c_\nu(x)$$

$$+ gf_{abc} \int_0^1 dt\ t^2 x^\lambda \Big[A^b_\lambda(tx)\, F^c_{\mu\nu}(tx) - A^b_\mu(tx) F^c_{\lambda\nu}(tx)$$

$$+ A^b_\nu(tx)\, F^c_{\lambda\mu}(tx) \Big] \quad . \tag{2.24}$$

Equation (2.24) may be simplified by recalling that $x \cdot A^a = 0$ and by writing the first two terms as a total derivative.

$$F^a_{\mu\nu}(x) = \int_0^1 \frac{d}{dt} \Big[t^2 F^a_{\mu\nu}(tx) \Big]\, dt + gf_{abc} A^b_\mu(x) A^c_\nu(x)$$

$$- gf_{abc} \int_0^1 dt\ t^2 x^\lambda \Big[A^b_\mu(tx)\, F^c_{\lambda\nu}(tx) - A^b_\nu(tx)\, F^c_{\lambda\mu}(tx) \Big] \tag{2.25}$$

Upon integration of the first term in (2.25) and use of the identity

$$x^{\lambda} F^{a}_{\lambda\nu} = x^{\lambda} A^{a}_{\nu,\lambda} - x^{\lambda} A^{a}_{\lambda,\nu} \tag{2.26}$$

the condition which remains to be satisfied is

$$f_{abc} A^{b}_{\mu}(x) A^{c}_{\nu}(x) = f_{abc} \int_0^1 dt\ t^2 x^{\lambda} \Big[A^{b}_{\mu}(tx) \left(A^{c}_{\nu,\lambda}(tx) - A^{c}_{\lambda,\nu}(tx) \right) - A^{b}_{\nu}(tx) \left(A^{c}_{\mu,\lambda}(tx) - A^{c}_{\lambda,\mu}(tx) \right) \Big] . \tag{2.27}$$

Differentiation of the gauge condition $x \cdot A = 0$ gives the identity

$$A_{\mu}(x) + x^{\lambda} A_{\lambda,\mu}(x) = 0 \ , \tag{2.28}$$

which simplifies (2.27).

$$\begin{aligned} f_{abc} A^{b}_{\mu}(x) A^{c}_{\nu}(x) &= f_{abc} \int_0^1 dt\ t^2 x^{\lambda} \left[A^{b}_{\mu}(tx) A^{c}_{\nu,\lambda}(tx) + A^{c}_{\nu}(tx) A^{b}_{\mu,\lambda}(tx) \right] \\ &\quad + f_{abc} \int_0^1 dt\ 2t\, A^{b}_{\mu}(tx)\, A^{c}_{\nu}(tx) \\ &= f_{abc} \int_0^1 \frac{d}{dt} \left(t^2 A^{b}_{\mu}(tx)\, A^{c}_{\nu}(tx) \right) dt \end{aligned} \tag{2.29}$$

Equation (2.29) is an identity, showing that the fixed-point gauge is self-consistent.

In summary, the fixed-point gauge allows the non-perturbative vev $<:\bar{\psi}(z)\psi(y):>$ to be written as a covariantized Taylor series.

$$<:\bar{\psi}(z)\psi(y):> = <\bar{\psi}(0)\psi(0)> + y^\alpha<\bar{\psi}(0)D_\alpha\psi(0)> + z^\alpha<\bar{\psi}(0)\overleftarrow{D}_\alpha\psi(0)>$$

$$+ \tfrac{1}{2}z^\alpha z^\beta<\bar{\psi}(0)\overleftarrow{D}_\alpha\overleftarrow{D}_\beta\psi(0)> + \tfrac{1}{2}y^\alpha y^\beta<\bar{\psi}(0)D_\alpha D_\beta\psi(0)>$$

$$+ z^\alpha y^\beta<\bar{\psi}(0)\overleftarrow{D}_\alpha D_\beta\psi(0)> + \text{higher order terms} \qquad (2.30)$$

The notation in (2.30) has been condensed, so it should be remembered that all covariant derivatives are evaluated at x=0 and that an implicit normal-ordering occurs. In the next section, the quark condensate component of (2.30) will be calculated to all orders in the covariantized Taylor series.

2-4 Quark Condensate Projection of the Quark-Quark Vacuum Expectation Value

Evaluation of the quark-quark vev $\langle\bar{\psi}(z)\psi(y)\rangle$ requires the calculation of the coefficients in the covariantized Taylor series of (2.30). A typical term which must be calculated is

$$\langle\bar{\psi}(0)\overleftarrow{D}_{(\alpha}\cdots\overleftarrow{D}_{\mu)}D_{(\nu}\cdots D_{\omega)}\psi(0)\rangle \tag{2.31}$$

A gauge covariant vev must be Poincare invariant, implying the identity

$$\partial_\mu\langle\bar{\psi}(x)\overleftarrow{D}_\alpha\cdots\overleftarrow{D}_\lambda D_\nu\cdots D_\omega\psi(x)\rangle$$

$$= \langle\bar{\psi}\overleftarrow{D}_\alpha\cdots\overleftarrow{D}_\lambda\overleftarrow{\partial}_\mu D_\nu\cdots D_\omega\psi\rangle + \langle\bar{\psi}\overleftarrow{D}_\alpha\cdots\overleftarrow{D}_\lambda\partial_\mu D_\nu\cdots D_\omega\psi\rangle \tag{2.32}$$

If the fixed-point gauge is imposed in (2.32) and the limit as x approaches 0 is taken, then an expression for the integration by parts of covariant derivatives is obtained.

$$\langle\bar{\psi}(0)\overleftarrow{D}_\alpha\cdots\overleftarrow{D}_\lambda\overleftarrow{D}_\mu D_\nu\cdots D_\omega\psi(0)\rangle$$

$$= -\langle\bar{\psi}(0)\overleftarrow{D}_\alpha\cdots\overleftarrow{D}_\lambda D_\mu D_\nu\cdots D_\omega\psi(0)\rangle \tag{2.33}$$

Equation (2.33) is easily generalized to the form appearing in (2.31) with the result

$$\langle\bar{\psi}(0)\overleftarrow{D}_{(\alpha_1}\cdots\overleftarrow{D}_{\alpha_n)}D_{(\mu}\cdots D_{\omega)}\psi(0)\rangle$$

$$= (-1)^n\langle\bar{\psi}(0)D_{(\alpha_1}\cdots D_{\alpha_n)}D_{(\mu}\cdots D_{\omega)}\psi(0)\rangle \qquad (2.34)$$

Use of the expression (2.34) simplifies (2.30) to the form

$$\langle\bar{\psi}(z)\psi(y)\rangle = \langle\bar{\psi}(0)\psi(0)\rangle + (y-z)^{\alpha}\langle\bar{\psi}D_{\alpha}\psi\rangle + \tfrac{1}{2}(y^{\alpha}y^{\beta}+z^{\alpha}z^{\beta})\langle\bar{\psi}D_{(\alpha}D_{\beta)}\psi\rangle$$

$$- z^{\alpha}y^{\beta}\langle\bar{\psi}D_{\alpha}D_{\beta}\psi\rangle + \text{higher order terms} \qquad (2.35)$$

where all covariant derivatives are evaluated at x=0.

Consider the zero order term in (2.35), explicitly including the colour algebra indices ε,η and the Dirac indices i,j. Lorentz invariance of a gauge covariant vev implies that $\langle\bar{\psi}_i^{\varepsilon}(0)\psi_j^{\eta}(0)\rangle$ must be written as

$$\langle\bar{\psi}_i^{\varepsilon}(0)\psi_j^{\eta}(0)\rangle = A\delta^{\varepsilon\eta}\delta_{ij} \qquad (2.36)$$

The unknown coefficient A in (2.36) is evaluated in terms of the quark condensate by contracting both sides of (2.35) with $\delta^{\varepsilon\eta}\delta^{ij}$.

$$\langle\bar{\psi}^{\varepsilon}\psi_i^{\varepsilon}\rangle \equiv \langle\bar{\psi}\psi\rangle = 12A \qquad (2.37a)$$

$$A = \frac{1}{12}\langle\bar{\psi}\psi\rangle \qquad (2.37b)$$

Calculation of the higher order terms of (2.35) in the context of the quark self-energy was neglected until the work of Elias, Scadron and Tarrach,[33] whose methods for the first few terms in (2.35) will be reviewed. Consider the first order term in (2.35) which can be written as a Lorentz covariant object since it is a gauge covariant vev.

$$\langle \bar{\psi}^{\varepsilon}_{i} D^{\eta\tau}_{\mu} \psi^{\tau}_{j} \rangle = B\gamma^{\mu}_{ji}\delta^{\varepsilon\eta} \tag{2.38}$$

The unknown coefficient B is determined by contracting (2.38) with $\gamma^{\mu}_{ij}\delta^{\varepsilon\eta}$. The equation of motion $\not{D}\psi=-im\psi$ is then used to self-consistently introduce the possibility of a mass scale m generated by the quark condensate.

$$\langle \bar{\psi}\not{D}\psi \rangle = -im\langle \bar{\psi}\psi \rangle = B\cdot 3\cdot \mathrm{Tr}(\gamma^{\mu}\gamma_{\mu}) = 48B \tag{2.39a}$$

$$B = -\frac{im}{48}\langle \bar{\psi}\psi \rangle \tag{2.39b}$$

The colour averaging factor of three will now be implicitly included in subsequent calculations involving the quark-quark vev.

Now consider the second order terms in (2.35) involving the coefficients $\langle \bar{\psi}D_{(\alpha}D_{\beta)}\psi \rangle$ and $\langle \bar{\psi}D_{\alpha}D_{\beta}\psi \rangle$, the latter of which can be written as the following linear combinations of Lorentz invariants.

$$\langle \bar{\psi}_{i} D_{\alpha} D_{\beta} \psi_{j} \rangle = C\delta_{ij} g_{\alpha\beta} + D\sigma^{\alpha\beta}_{ji} \tag{2.40}$$

Contracting (2.40) in turn with $g^{\alpha\beta}\delta_{ij}$ and $\sigma^{\alpha\beta}_{ij}$ leads to the following two equations.

$$\langle \bar{\psi}D^{2}\psi \rangle = 16C \tag{2.41a}$$

$$\langle\bar{\psi}\sigma^{\alpha\beta}D_{\alpha}D_{\beta}\psi\rangle = D\cdot Tr(\sigma^{\alpha\beta}\sigma_{\alpha\beta}) = 48D \tag{2.41b}$$

Equation (2.41a) can be simplified by using the Dirac matrix identity

$$g_{\alpha\beta} = \gamma_{\alpha}\gamma_{\beta} + i\sigma_{\alpha\beta} \tag{2.42a}$$

$$D^2 = \not{D}\not{D} + \frac{i}{2}\sigma^{\alpha\beta}[D_{\alpha},D_{\beta}] = \not{D}\not{D} - \frac{i}{2}\sigma^{\alpha\beta}G_{\alpha\beta} \tag{2.42b}$$

to write (2.41a) as

$$16C = \langle\bar{\psi}(\not{D}\not{D} - \frac{i}{2}\sigma^{\alpha\beta}G_{\alpha\beta})\psi\rangle = -m^2\langle\bar{\psi}\psi\rangle - \frac{i}{2}\langle\bar{\psi}\sigma^{\alpha\beta}G_{\alpha\beta}\rangle \tag{2.43}$$

The equation of motion $\not{D}\psi=-im\psi$ and the definition of the field strength $G_{\mu\nu}=[D_{\nu},D_{\mu}]$ were used to eliminate the covariant derivatives in (2.43). Equation (2.41b) may also be simplified through the definition of the field strength, leading to the result

$$48D = \frac{1}{2}\langle\bar{\psi}\sigma^{\alpha\beta}[D_{\alpha},D_{\beta}]\psi\rangle = -\frac{1}{2}\langle\bar{\psi}\sigma^{\alpha\beta}G_{\alpha\beta}\psi\rangle \equiv -\frac{1}{2}\langle\bar{\psi}\sigma G\psi\rangle \tag{2.44}$$

The dimension-five mixed condensate $\langle\bar{\psi}\sigma G\psi\rangle$ appearing in (2.43) and (2.44) will be considered in greater detail in Chapter Three.

Collecting the results of (2.43), (2.44), (2.39b), and (2.37b) gives the expressions for the zeroth, first and second order coefficients in (2.40), (2.38), and (2.36).

$$\langle\bar{\psi}_i\psi_j\rangle = \frac{1}{12}\delta_{ij}\langle\bar{\psi}\psi\rangle \tag{2.45}$$

$$\langle\bar{\psi}_i D_\mu \psi_j\rangle = -\frac{im}{48}\langle\bar{\psi}\psi\rangle\,\gamma^\mu_{ji} \tag{2.46}$$

$$\langle\bar{\psi}_i D_\alpha D_\beta \psi_j\rangle = -\frac{m^2}{48}\langle\bar{\psi}\psi\rangle g_{\alpha\beta}\delta_{ij} + \frac{\langle\bar{\psi}\sigma G\psi\rangle}{288}\left(-3i g_{\alpha\beta}\delta_{ij} - \sigma^{\alpha\beta}_{ji}\right) \tag{2.47}$$

Substitution of the quark condensate components of (2.45) through (2.47) into (2.35) gives the quark condensate projection of $\langle\bar{\Psi}(z)\Psi(y)\rangle$ to second order.[33]

$$\langle\bar{\psi}(z)\psi(y)\rangle = \frac{1}{12}\langle\bar{\psi}\psi\rangle - \frac{im}{48}\langle\bar{\psi}\psi\rangle\gamma\cdot(y-z) - \frac{m^2}{96}\langle\bar{\psi}\psi\rangle(y-z)^2 \pm \ldots \tag{2.48}$$

The general form of the quark condensate component of the $\langle\bar{\Psi}(z)\Psi(y)\rangle$ vev can be determined by examination of the QCD equations of motion.

$$\not{D}\psi = -im\psi \tag{2.49}$$

$$[D_\nu, D_\mu] \equiv G_{\mu\nu} = igT^a F^a_{\mu\nu} \tag{2.50}$$

$$[D_\alpha, G_{\alpha\beta}] = -ig^2 T^a \sum_{\text{flavour}} \bar{\psi}\lambda^a\gamma_\beta\psi \equiv j_\beta \tag{2.51}$$

$$[D_\beta, j_\beta] = 0 \text{ (current conservation)} \tag{2.52}$$

$$[D_\mu, G_{\lambda\nu}] + [D_\lambda, G_{\nu\mu}] + [D_\nu, G_{\mu\lambda}] = 0 \quad \text{(Jacobi Identity)} \tag{2.53}$$

The important feature of the QCD equations of motion is the impossibility of non-trivially eliminating a field strength $G_{\mu\nu}$, without increasing the number of fermion fields by the introduction of a current j_β.

A general term in the expansion (2.35) has a Taylor coefficient of the form $\langle\bar{\psi}D_{(\alpha}\ldots D_{\mu)}D_{(\nu}\ldots D_{\omega)}\psi\rangle$. If only the quark condensate component of this object is to be extracted, then covariant derivatives may be freely commuted, since any field strengths generated through commutators cannot be eliminated without leading to higher dimension condensates. Thus if only the completely symmetric part of (2.35) is kept, then the following translation invariant is obtained.[35,36,39]

$$\langle\bar{\psi}(z)\psi(y)\rangle = \sum_{n=0}^{\infty}\frac{(y-z)^{\alpha_1}\ldots(y-z)^{\alpha_{2n}}}{(2n)!}\langle\bar{\psi}D_{(\alpha_1}\ldots D_{\alpha_{2n})}\psi\rangle$$

$$+\sum_{n=0}^{\infty}\frac{(y-z)^{\alpha_1}\ldots(y-z)^{\alpha_{2n+1}}}{(2n+1)!}\langle\bar{\psi}D_{(\alpha}\ldots D_{\alpha_{2n+1})}\psi\rangle \qquad (2.54)$$

Consider a general even term in (2.54), which on grounds of Lorentz invariance of gauge covariant vev's, can be written as

$$\langle\bar{\psi}D_{(\alpha_1}\ldots D_{\alpha_{2n})}\psi\rangle = A\,S_{\alpha_1\ldots\alpha_{2n}} \qquad (2.55)$$

where S is a Lorentz invariant, completely symmetric tensor. Contracting both sides of (2.55) with $g^{\alpha_1\alpha_2}g^{\alpha_3\alpha_4}\ldots g^{\alpha_{2n-1}\alpha_{2n}}$, commuting derivatives freely, using the identity (2.42b), and ignoring terms which do not contribute to the quark condensate, leads to the expression

$$\langle\bar{\psi}\not{D}\not{D}\ldots(2n\text{ times})\ldots\not{D}\not{D}\psi\rangle = A\cdot\mathrm{Tr}\left(S^{\alpha_1}{}_{\alpha_1}\ldots{}^{\alpha_n}{}_{\alpha_n}\right) \qquad (2.56)$$

Using the $\not{D}\psi = -im\psi$ equation of motion, colour averaging, and denoting $S^{\alpha_1}{}_{\alpha_1}\cdots{}^{\alpha_n}{}_{\alpha_n}$ as S_{2n}, determines A in terms of $\langle\bar{\psi}\psi\rangle$.

$$A = \frac{(-im)^{2n}\langle\bar{\psi}\psi\rangle}{3\cdot Tr(S_{2n})} \quad ; \quad S_{2n} \equiv S^{\alpha_1}{}_{\alpha_1}\cdots{}^{\alpha_n}{}_{\alpha_n} \tag{2.57}$$

For an odd term in (2.54) a similar expression is obtained.

$$\langle\bar{\psi}D_{(\alpha_1}\cdots D_{\alpha_{2n+1})}\psi\rangle = B\cdot S_{\alpha_1\cdots\alpha_{2n+1}} \tag{2.58}$$

$$B = \frac{(-im)^{2n+1}\langle\bar{\psi}\psi\rangle}{3\cdot Tr(S_{2n+1})} \tag{2.59}$$

The object S_{2n+1} in (2.59) is defined as the contraction

$$S_{2n+1} \equiv \gamma^{\alpha_1}g^{\alpha_2\alpha_3}\cdots g^{\alpha_{2n}\alpha_{2n+1}}S_{\alpha_1\cdots\alpha_{2n+1}} \tag{2.60}$$

where $S_{\alpha_1\cdots\alpha_{2n+1}}$ is a completely symmetric, Lorentz invariant tensor.

Completely symmetric Lorentz invariant tensors will have the constructions

$$S_{\alpha_1\cdots\alpha_{2n}} = g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_4}\cdots g_{\alpha_{2n-1}\alpha_{2n}} + \text{permutations} \tag{2.61a}$$

$$S_{\alpha_1\cdots\alpha_{2n+1}} = \gamma_{\alpha_1}g_{\alpha_2\alpha_3}\cdots g_{\alpha_{2n}\alpha_{2n+1}} + \text{permutations} \tag{2.61b}$$

respectively containing N_{2n} and N_{2n+1} total permutations.* Substituting the above equations along with (2.59) and (2.57) into (2.54) leads to a

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*For example $N_4 = 3$ since $S_{\mu\nu\lambda\rho} = g_{\mu\nu}g_{\lambda\rho} + g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda}$.

translation invariant quark condensate component of $<\bar{\Psi}(z)\Psi(y)>$.

$$<\bar{\psi}(z)\psi(y)> = \sum_{n=0}^{\infty} \frac{(y-z)^{2n}(-im)^{2n}<\bar{\psi}\psi>}{(2n)!\ 3\cdot Tr(S_{2n})} N_{2n}$$

$$+ \sum_{n=0}^{\infty} \frac{\gamma\cdot(y-z)(y-z)^{2n}(-im)^{2n+1}<\bar{\psi}\psi>}{(2n+1)!\ 3\cdot Tr(S_{2n+1})} N_{2n+1} \qquad (2.62)$$

The coefficients N_n, $Tr(S_n)$ can be evaluated recursively. The tensors $S_{\alpha_1\ldots\alpha_{2n+1}}$ and $S_{\alpha_1\ldots\alpha_{2n+2}}$ can be written as

$$S_{\alpha_1\ldots\alpha_{2n+1}} = \gamma_{\alpha_1} S_{\alpha_2\ldots\alpha_{2n+1}} + \gamma_{\alpha_2} S_{\alpha_1\alpha_3\ldots\alpha_{2n+1}}$$

$$+ \ldots + \gamma_{\alpha_{2n+1}} S_{\alpha_1\ldots\alpha_{2n}} \qquad (2.63)$$

and

$$S_{\alpha_1\ldots\alpha_{2n+2}} = g_{\alpha_{2n+1}\alpha_{2n+2}} S_{\alpha_1\ldots\alpha_{2n}} + g_{\alpha_{2n+1}\alpha_1} S_{\alpha_{2n+2}\alpha_2\ldots\alpha_{2n}}$$

$$+ \ldots + g_{\alpha_{2n+1}\alpha_{2n}} S_{\alpha_1\ldots\alpha_{2n-1}\alpha_{2n+2}} \qquad (2.64)$$

implying the recursive relations

$$Tr(S_{2n+1}) = 2(n+2)\ Tr(S_{2n}) \qquad (2.65a)$$

$$N_{2n+1} = (2n+1)\ N_{2n} \qquad (2.65b)$$

$$Tr(S_{2n+2}) = 2(n+2)\ Tr(S_{2n}) \qquad (2.66a)$$

$$N_{2n+2} = (2n+1)N_{2n} \tag{2.66b}$$

Values of $N_0=1$, $N_2=1$, $Tr(S_0)=4$, $Tr(S_2)=16$ allow the difference equations of (2.66) to be solved, with the results

$$N_{2n} \begin{cases} (2n-1)!! = \dfrac{(2n-1)!}{(n-1)!\, 2^{n-1}} & ; n\neq 0 \\ 1 & ; n=0 \end{cases} \tag{2.67}$$

$$Tr(S_{2n}) = 4(n+1)!2^n \tag{2.68}$$

Finally, substituting the results of (2.67), (2.68) and (2.65) into (2.62) gives the quark condensate component of $\langle\bar{\Psi}(z)\Psi(y)\rangle$ to all orders in its Taylor expansion.[36,39]

$$\langle\bar{\Psi}(z)\Psi(y)\rangle = \frac{1}{3}\langle\bar{\psi}\psi\rangle \sum_{n=0}^{\infty} \frac{(-im)^{2n}(y-z)^{2n}}{n!(n+1)!\; 4^{n+1}}$$

$$+ \frac{1}{3}\langle\bar{\psi}\psi\rangle \sum_{n=0}^{\infty} \frac{(-im)^{2n+1}\gamma\cdot(y-z)(y-z)^{2n}}{2\,(n+2)!n!\; 4^{n+1}}$$

$$+ \text{ higher dimensional condensates} \tag{2.69}$$

The infinite sum in (2.69) has excellent convergence properties, converging uniformly to the function

$$\langle\bar{\Psi}(z)\Psi(y)\rangle = \frac{\langle\bar{\psi}\psi\rangle}{6m}\left\{(x^2)^{-\frac{1}{2}}J_1\left(m(x^2)^{\frac{1}{2}}\right) - i\,\frac{\gamma\cdot x}{x^2}J_2\left(m(x^2)^{\frac{1}{2}}\right)\right\}$$

$$+ \text{ higher dimensional condensates} \tag{2.70}$$

where $x=y-z$, $x^2>0$ and J_1, J_2 are Bessel functions.

The fundamental result of this section has been the calculation of the quark condensate component of $\langle\bar{\psi}(z)\psi(y)\rangle$ to all orders of the covariantized Taylor series. This expression will be used in the next section to calculate the quark condensate component of the quark self-energy.

2-5 Gauge Independence of the Quark Condensate Component of the Quark Self-Energy and the Dynamical Quark Mass

The basic formulation for calculating the quark condensate component of the quark propagator was introduced in Section 2-2, leading to an expression in (2.10) containing the unknown object $\langle\bar{\psi}(z)\psi(y)\rangle$.

Fundamental considerations of the OPE in Section 2-1 revealed that only gauge invariant condensates would contribute to the OPE. This property was exploited in Section 2-3 by introducing the fixed point gauge which covariantizes the Taylor expansion for $\langle\bar{\psi}(z)\psi(y)\rangle$. In Section 2-4 the quark condensate contribution to the resulting Taylor coefficients was then evaluated to all orders, introducing the mass scale m through the QCD equations of motion.

All the information required to evaluate the two-point amplitude in (2.10) has now been collected, and thus the quark condensate component of the quark self-energy can now be calculated. To begin, consider equation (2.10) with the propagators as defined in (2.12) and (2.13).

$$i\Delta S_1(p) = \frac{g^2}{4}\int d^4x\,d^4y\,d^4z\; e^{ip\cdot x}\; i\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{\not{k}+m_L}{k^2-m_L^2+i\varepsilon}$$
$$\times\,\gamma^\mu\lambda^a\,\langle\bar{\psi}(z)\psi(y)\rangle\,\gamma^\nu\lambda^b\; i\int\frac{d^4T}{(2\pi)^4}\,e^{-iT\cdot z}\,\frac{\not{T}+m_L}{T^2-m_L^2+i\varepsilon}$$
$$\times\, i\delta^{ab}\int\frac{d^4q}{(2\pi)^4}\,\frac{e^{-iq\cdot(y-z)}}{q^2+i\varepsilon}\left[-g^{\mu\nu}+(1-a)\frac{q^\mu q^\nu}{q^2+i\varepsilon}\right] \qquad (2.71)$$

The variables of integration are now changed from x,y,z to (x-y),(y-z),z (the Jacobian of this transformation is trivial). Since the quark condensate component of $\langle\bar{\psi}(z)\psi(y)\rangle$ is translation invariant, (2.71) can be factored into three separate integrals.

$$i\Delta S_1(p) = -\frac{ig^2}{4}\lambda^a\lambda^a\int d^4(x-y)\, e^{ip\cdot(x-y)} \int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{\not k + m_L}{k^2-m_L^2+i\varepsilon}$$

$$\times\int d^4(y-z)\, e^{ip\cdot(y-z)}\, \gamma^\mu \langle\bar\psi(z)\psi(y)\rangle\, \gamma^\nu$$

$$\times\int\frac{d^4q}{(2\pi)^4}\, \frac{e^{-iq\cdot(y-z)}}{q^2+i\varepsilon}\left[-g^{\mu\nu} + (1-a)\frac{q^\mu q^\nu}{q^2+i\varepsilon}\right]$$

$$\times\int d^4z\, e^{ip\cdot z}\int\frac{d^4T}{(2\pi)^4}\, e^{-iT\cdot z}\, \frac{\not T + m_L}{T^2-m_L^2+i\varepsilon} \tag{2.72}$$

The (x-y) and z integrals in (2.72) result in delta functions $(2\pi)^4\delta^4(p-k)$ and $(2\pi)^4\delta^4(p-T)$ which upon integration over k and T simplifies (2.72) to the expression

$$i\Delta S_1(p) = -\frac{4}{3}ig^2\, \frac{\not p+m_L}{p^2-m_L^2}\int d^4(y-z)\, e^{ip\cdot(y-z)}\, \gamma^\mu\langle\bar\psi(z)\psi(y)\rangle\gamma^\nu$$

$$\times\int\frac{d^4k}{(2\pi)^4}\, \frac{e^{-ik\cdot(y-z)}}{k^2+i\varepsilon}\left[-g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2+i\varepsilon}\right]\frac{\not p+m_L}{p^2-m_L^2} \tag{2.73}$$

where the $\lambda^a\lambda^a = 16/3$ colour algebra result has been used.

The quark self-energy is defined by the relation

$$iS_F(p) = \frac{i}{\not p-m_L-\Sigma(p)} = \frac{i}{\not p-m_L} + \frac{i}{\not p-m_L}\left(-i\Sigma(p)\right)\frac{i}{\not p-m_L} + \ldots$$

$$= \frac{i}{\not p-m_L} + i\Delta S_1(p) + \ldots \tag{2.74}$$

which identifies the quark self-energy in (2.73)

$$\Sigma(p) = -\frac{4}{3}g^2\int d^4(y-z)\; e^{ip\cdot(y-z)}\gamma^\mu \langle\bar\psi(z)\psi(y)\rangle\gamma^\nu$$

$$\times\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot(y-z)}}{k^2+i\varepsilon}\left[-g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2+i\varepsilon}\right] \tag{2.75}$$

The series expansion for $\langle\bar\psi(z)\psi(y)\rangle$ given in (2.69) is now substituted into (2.75), which permits a term by term integration using the identity

$$\int d^4x\; x^\mu e^{ip\cdot x}\int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} f(k) = -i\frac{\partial}{\partial p_\mu} f(p) \tag{2.76}$$

If only the zero and first order terms in the expansion of the quark-quark vev are considered, then the following expression for $\Sigma(p)$ is obtained.[33]

$$\Sigma(p) = -\frac{4}{3}g^2\left[\frac{1}{12}\langle\bar\psi\psi\rangle\gamma^\mu\gamma^\nu\left[-g^{\mu\nu} + (1-a)\frac{p^\mu p^\nu}{p^2+i\varepsilon}\right]\frac{1}{p^2+i\varepsilon}\right.$$

$$\left. - \frac{im}{48}\langle\bar\psi\psi\rangle\gamma^\mu\gamma^\lambda\gamma^\nu(-i)\frac{\partial}{\partial p_\lambda}\left[\frac{1}{p^2+i\varepsilon}\left[-g^{\mu\nu} + (1-a)\frac{p^\mu p^\nu}{p^2+i\varepsilon}\right]\right]\right]$$

$$+ \text{ higher order terms} \tag{2.77}$$

Evaluating the derivatives in (2.77) leads to the expression

$$\Sigma(p) = -\frac{1}{9}g^2\langle\bar\psi\psi\rangle\left[\gamma^\mu\gamma^\nu\left[-g^{\mu\nu} + (1-a)\frac{p^\mu p^\nu}{p^2+i\varepsilon}\right]\frac{1}{p^2+i\varepsilon}\right.$$

$$- \frac{m}{4}\gamma^\mu\gamma^\lambda\gamma^\nu\left[\frac{2g^{\mu\nu}p^\lambda}{(p^2+i\varepsilon)^2} + (1-a)\frac{p^\nu g^{\mu\lambda} + p^\mu g^{\nu\lambda}}{(p^2+i\varepsilon)^2}\right.$$

$$\left.\left. - 4(1-a)\frac{p^\mu p^\nu p^\lambda}{(p^2+i\varepsilon)^3}\right]\right] \tag{2.78}$$

After simplification of (2.78), the result of Elias, Scadron and Tarrach for the order g^2 quark condensate component of the quark self-energy is obtained.[33]

$$\Sigma(p) = g^2 \frac{\langle\bar{\psi}\psi\rangle}{9p^2}\left[(3+a) - a\frac{m\not{p}}{p^2} + O(\varepsilon) + \text{higher order terms}\right]$$

$$\Sigma(p) = g^2 \frac{\langle\bar{\psi}\psi\rangle}{9p^2}\left[(3+a) - a\frac{m\not{p}}{p^2} + \text{higher order terms}\right] \quad ; \ p^2 \neq 0 \qquad (2.79)$$

The zero order term in the $\langle\bar{\psi}(z)\psi(y)\rangle$ expansion leads to the gauge dependent (3+a) portion of (2.79).[32] This gauge dependence is cancelled at the $\not{p}=m$ mass shell by the $am\not{p}/p^2$ contribution from the linear term of the quark-quark vev.[33] The higher orders in the $\langle\bar{\psi}(z)\psi(y)\rangle$ expansion are now important since they will determine whether the on-shell gauge independence of (2.79) will be maintained to all orders. A related issue is the convergence of the series (2.79) since the expansion parameter $m\not{p}/p^2$ will equal one on the $\not{p}=m$ mass shell, possibly leading to a divergent series.

The contribution to $\Sigma(p)$ from a general even term in expansion (2.69), is proportional to

$$\gamma^\mu\gamma^\nu\left(\frac{\partial^2}{\partial p^2}\right)^n\left[-\frac{g^{\mu\nu}}{p^2+i\varepsilon} + (1-a)\frac{p^\mu p^\nu}{(p^2+i\varepsilon)^2}\right] \qquad (2.80)$$

Evaluating (2.80) for $n\geq 1$ gives the expression

$$\gamma^\mu\gamma^\nu\left(\frac{\partial^2}{\partial p^2}\right)^{n-1}\left[2(1-a)\left(\frac{g^{\mu\nu}}{(p^2+i\varepsilon)^2} - \frac{4p^\mu p^\nu}{(p^2+i\varepsilon)^3}\right) + O(\varepsilon)\right]$$

$$= \left(\frac{\partial^2}{\partial p^2}\right)^{n-1} 8(1-a)\left[(p^2+i\varepsilon)^{-2} - p^2(p^2+i\varepsilon)^{-3} + O(\varepsilon)\right] \qquad (2.81)$$

which is equal to zero for $p^2 \neq 0$, $n \geq 1$. The $i\varepsilon$ prescription for dealing with the pole of the gluon propagator has been employed in order to demonstrate that the potentially troublesome delta functions of momentum ($\delta(p^2)$) do not contribute to the quark self-energy.

A similar result occurs for the odd terms in the expansion of $\langle\bar{\psi}(z)\psi(y)\rangle$. A general odd term in (2.69) will contribute to $\Sigma(p)$ a term proportional to

$$\gamma^\mu\gamma^\lambda\gamma^\nu \left(\frac{\partial^2}{\partial p^2}\right)^n \frac{\partial}{\partial p_\lambda}\left(-\frac{g^{\mu\nu}}{p^2+i\varepsilon} + (1-a)\frac{p^\mu p^\nu}{(p^2+i\varepsilon)^2}\right) \qquad (2.82)$$

Evaluating (2.82) for $n \geq 1$ with the assistance of (2.81) leads to the expression

$$\gamma^\mu\gamma^\lambda\gamma^\nu \left(\frac{\partial^2}{\partial p^2}\right)^{n-1} \frac{\partial}{\partial p_\lambda} 2(1-a)\left(\frac{g^{\mu\nu}}{(p^2+i\varepsilon)^2} - \frac{4p^\mu p^\nu}{(p^2+i\varepsilon)^3} + O(\varepsilon)\right) \qquad (2.83a)$$

$$= 2(1-a)\gamma^\mu\gamma^\lambda\gamma^\nu \left(\frac{\partial^2}{\partial p^2}\right)^{n-1}\left(\frac{-4\left(g^{\mu\nu}p^\lambda + g^{\mu\lambda}p^\nu + g^{\nu\lambda}p^\mu\right)}{(p^2+i\varepsilon)^3} + \frac{24p^\mu p^\nu p^\lambda}{(p^2+i\varepsilon)^4} + O(\varepsilon)\right)$$

$$= 2(1-a)\left(\frac{\partial^2}{\partial p^2}\right)^{n-1}\left(-\frac{24\not{p}}{(p^2+i\varepsilon)^3} + \frac{24\not{p}p^2}{(p^2+i\varepsilon)^4} + O(\varepsilon)\right) \qquad (2.83b)$$

which is equal to zero for $p^2 \neq 0$, $n \geq 1$.

The results of (2.83b) and (2.81) show that all second and higher order terms in the expansion of the quark-quark vev do not contribute to the quark self-energy, truncating the series in (2.79).[35] Thus the order g^2 quark condensate component of the quark self-energy, valid to all orders in the expansion of $\langle\bar{\psi}(z)\psi(y)\rangle$, is given by

$$\Sigma(p) = \frac{g^2}{9p^2}\langle\bar{\psi}\psi\rangle\left((3+a) - a\frac{m\not{p}}{p^2}\right) \qquad (2.84)$$

This result has been independently verified through use of plane-wave techniques, supporting the fixed-point gauge methods which have been employed.[50]

The self-energy in (2.84) is gauge independent at the $\not{p}=m$ mass shell, identifying m as a mass scale of physical importance. An estimate of m can be obtained by considering the effective mass $M(p^2)$ which is defined by the self-energy in (2.74). The inverse propagator is

$$S_F^{-1}(p) = \not{p} - m_L - \Sigma(p) \tag{2.85a}$$

$$S_F^{-1}(p) = \not{p}\left(1 + g^2\langle\bar{\psi}\psi\rangle\frac{am}{9p^4}\right) - m_L - g^2(3+a)\frac{\langle\bar{\psi}\psi\rangle}{9p^2}.$$

$$S_F^{-1}(p) = \left(1 + g^2\langle\bar{\psi}\psi\rangle\frac{am}{9p^4}\right)\left[\not{p} - \left(\frac{m_L + g^2(3+a)\frac{\langle\bar{\psi}\psi\rangle}{9p^2}}{1 + g^2\langle\bar{\psi}\psi\rangle\frac{am}{9p^4}}\right)\right] \tag{2.85b}$$

The pole position of the corrected quark propagator identifies the effective mass $M(p^2)$.

$$M(p^2) = \frac{m_L + g^2(3+a)\frac{\langle\bar{\psi}\psi\rangle}{9p^2}}{1 + g^2\langle\bar{\psi}\psi\rangle\frac{am}{9p^4}} \tag{2.86}$$

In the large p^2 limit where QCD turns off, the Lagrangian (current) mass still defines the pole of the quark propagator, since

$$\lim_{p^2\to\infty} M(p^2) = m_L . \tag{2.87}$$

This result is consistent with an intuitive understanding of current masses. Since the effective mass is gauge independent at the $\not{p}=m$ mass shell, $M(p^2)$ must self-consistently generate the mass scale m, implying that

$$M(m^2) = m \quad ; \quad m = \frac{m_L + g^2|\langle\bar{\psi}\psi\rangle|\frac{(3+a)}{9m^2}}{1 + |\langle\bar{\psi}\psi\rangle|\frac{ag^2}{9m^3}} \tag{2.88}$$

The absolute value of the quark condensate has been taken in (2.88) to ensure the positivity of $M(p^2)$ upon reflection from Euclidean (space-like) to time-like momentum.[33,35] The gauge dependent portions of (2.88) cancel each other, leaving a gauge independent expression to be solved for m.[33,35]

$$m = m_L + g^2|\langle\bar{\psi}\psi\rangle|\frac{1}{3m^2} \tag{2.89}$$

In the limit of unbroken Lagrangian chiral symmetry ($m_L=0$), the solution for m in (2.89) is known as the dynamical mass m_{dyn}. Note that the chiral limit is only relevant to m_{dyn} for up and down quarks.

$$m_{dyn} \equiv \lim_{m_L\to 0} m = g^2|\langle\bar{\psi}\psi\rangle|\frac{1}{3m_{dyn}^2} \tag{2.90a}$$

$$m_{dyn} = \left(\frac{1}{3}g^2|\langle\bar{\psi}\psi\rangle|\right)^{1/3} \tag{2.90b}$$

The expression for m_{dyn} in (2.90b) is not explicitly renormalization group invariant, so a momentum scale at which to evaluate m_{dyn} must be chosen. Since α_s and $\langle\bar{\psi}\psi\rangle$ are both known at $Q^2 = -1$ GeV^2, this will be the momentum scale used to evaluate m_{dyn}.

QCD sum rules estimate that $\langle\bar{\psi}\psi\rangle$ has a value of about $(-250\ \text{MeV})^3$ at the $Q^2=-(1\text{GeV})^2$ energy scale.[25,28,30] Substituting the value[34] $g^2(-(1\text{GeV})^2)\simeq 2\pi$ and the estimate of $\langle\bar{\psi}\psi\rangle$ into (2.90b) gives $m_{dyn}\simeq 320\text{MeV}\simeq m_{nucleon}/3$, in agreement with the constituent mass scale.

To conclude this section, an alternative approach to deriving (2.88) without explicitly constraining m to be the pole position of (2.86), will be examined.[36] Let the pole of (2.86) be at the value μ such that

$$\mu = M(\mu^2) = \frac{m_L + g^2(3+a)\frac{|\langle\psi\psi\rangle|}{9\mu^2}}{1 + g^2|\langle\bar{\psi}\psi\rangle|\frac{am}{9\mu^4}} \tag{2.91}$$

Differentiating both sides of (2.91) with respect to the gauge parameter "a" and demanding that $\frac{\partial\mu}{\partial a} = 0$ leads to the constraint

$$g^2|\langle\bar{\psi}\psi\rangle|\frac{m}{9\mu^3} = g^2|\langle\bar{\psi}\psi\rangle|\frac{1}{9\mu^2} \tag{2.92}$$

which only has the solution μ=m. Substituting μ=m into the constraint (2.91) leads to the expression in (2.88) originally obtained by restricting the pole position to be at μ=m.

In retrospect, the gauge independent pole of the quark condensate component of the quark self-energy has been analyzed through use of the Nielsen identities of QCD.[51] This analysis shows that at the pole position of the corrected propagator the self-energy must be gauge independent,[52] supporting the intuitive expectation of a gauge independent self-energy.

In summary, the order g^2 quark condensate component of the quark self-energy has been calculated to all orders of the $\langle\bar{\psi}(z)\psi(y)\rangle$ expansion. The zeroth and first order terms of the quark-quark vev led to a gauge independent self-energy at the $\not{p}=m$ mass shell, a result preserved to all orders, since the infinite series for $\Sigma(p)$ truncates at second order. An effective mass was defined self-consistently by demanding that the pole position of the corrected propagator is gauge parameter independent, defining a constituent mass scale of approximately 300 MeV. Thus, the concept of a running dynamical mass leading to the constituent mass scale has been given a field-theoretic foundation.

2-6 The Light-Cone Gauge and the Quark Condensate Component of the Quark Self-Energy

In the previous section, the order g^2 quark condensate component of the quark self-energy was found to be gauge independent on the $\not{p}=m$ mass shell. The gauge parameter "a" appearing in this calculation represented the gauge degree of freedom in a covariant gauge. The gauge independence of the quark self-energy will now be investigated in a larger context by evaluating the quark condensate contribution to the quark self-energy in a non-covariant gauge.

The light-cone gauge[53] involves a gauge-fixing condition of the form

$$n \cdot A^a(x) = 0 \quad ; \quad n^2 = 0 \tag{2.93}$$

which leads to the following gluon propagator.

$$\langle T\left(A^a_\mu(x) A^b_\nu(0)\right)\rangle \equiv i\delta^{ab} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} D_{\mu\nu}(k) \tag{2.94a}$$

$$D_{\mu\nu}(p) = -\frac{g^{\mu\nu}}{p^2} + \frac{n^\mu p^\nu + p^\mu n^\nu}{(n\cdot p)p^2} \tag{2.94b}$$

If the gluon propagator of (2.94b) replaces the covariant-gauge propagator in (2.75), then the lowest-order quark condensate correction to the quark self-energy is

$$\Sigma(p) = -\frac{4}{3} g^2 \int d^4(y-z)\, e^{ip\cdot(y-z)}\, \gamma^\mu \langle\bar{\psi}(z)\psi(y)\rangle \gamma^\nu \times \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(y-z)} \left(-\frac{g^{\mu\nu}}{k^2} + \frac{n^\mu k^\nu + k^\mu n^\nu}{(n\cdot k)k^2}\right) \tag{2.95}$$

Substituting the series expansion of (2.69) for the quark-quark vev, and performing integrals via (2.76) leads to an expression for $\Sigma(p)$ in terms of derivatives with respect to momentum.

$$\Sigma(p) = -\frac{4}{9} g^2 \langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{m^{2N}}{N!(N+1)!4^{N+1}} \left(\frac{\partial^2}{\partial p^2}\right)^N \gamma^\mu \gamma^\nu \left(-\frac{g^{\mu\nu}}{p^2} + \frac{n^\mu p^\nu + n^\nu p^\mu}{(n\cdot p)p^2} \right)$$

$$+ \frac{2}{9} g^2 \langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{m^{2N+1}}{N!(N+2)!4^{N+1}} \left(\frac{\partial^2}{\partial p^2}\right)^N \frac{\partial}{\partial p_\lambda} \gamma^\mu \gamma^\lambda \gamma^\nu \left(-\frac{g^{\mu\nu}}{p^2} + \frac{n^\mu p^\nu + n^\nu p^\mu}{(n\cdot p)p^2} \right) \tag{2.96}$$

Equation (2.96) is simplified by performing the Dirac algebra and by differentiating with respect to p^λ in the second term.

$$\Sigma(p) = -\frac{4}{9} g^2 \langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{m^{2N}}{N!(N+1)!4^{N+1}} \left(\frac{\partial^2}{\partial p^2}\right)^N \left(-\frac{2}{p^2} \right)$$

$$+ \frac{2}{9} g^2 \langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{m^{2N+1}}{N!(N+2)!4^{N+1}} \left(\frac{\partial^2}{\partial p^2}\right)^N \left(-\frac{4\not{p}}{p^4} + \frac{4\not{n}}{p^2(n\cdot p)} \right) \tag{2.97}$$

The n-independent terms in (2.97) were evaluated in Section 2-5, where it was found that only the N=0 terms were non-zero. Using this result gives an expression for $\Sigma(p)$.

$$\Sigma(p) = \frac{2}{9} g^2 \langle\bar{\psi}\psi\rangle \left(\frac{1}{p^2} - \frac{m\not{p}}{2p^4} \right)$$

$$+ \frac{8}{9} g^2 \langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{\not{n}\, m^{2N+1}}{N!(N+2)!4^{N+1}} \left(\frac{\partial^2}{\partial p^2}\right)^N \frac{1}{p^2(n\cdot p)} \tag{2.98}$$

To evaluate the derivatives appearing in (2.98), consider the following object required for the N=1 term.

$$\frac{\partial^2}{\partial p^2}\left[\frac{1}{p^2(n\cdot p)}\right] = \frac{\partial}{\partial p_\lambda}\left[-\frac{2p^\lambda}{p^4(n\cdot p)} - \frac{n^\lambda}{p^2(n\cdot p)^2}\right]$$

$$= \frac{4}{p^4(n\cdot p)} + \frac{2n^2}{p^2(n\cdot p)^2} = \frac{4}{p^4(n\cdot p)} \quad ; n^2 = 0 \tag{2.99}$$

This result indicates that since n is a null vector, the inverse power of $n\cdot p$ cannot be increased through repeated differentiation, implying that

$$\left(\frac{\partial^2}{\partial p^2}\right)^N \frac{1}{p^2(n\cdot p)} \equiv \frac{A_N}{p^{2N+2}(n\cdot p)} \tag{2.100}$$

Applying the operator $\frac{\partial^2}{\partial p^2}$ to both sides of (2.100) allows a difference equation for A_N to be obtained.

$$\left(\frac{\partial^2}{\partial p^2}\right)\frac{A_N}{p^{2N+2}(n\cdot p)} \equiv \frac{A_{N+1}}{p^{2N+4}(n\cdot p)} = \frac{4(N+1)^2 A_N}{p^{2N+4}(n\cdot p)} \tag{2.101a}$$

$$A_{N+1} = 4(N+1)^2 A_N \tag{2.101b}$$

Solving the difference equation of (2.101b) with the conditions that $A_1=4$, $A_0=1$ determines A_N in (2.100).

$$\left(\frac{\partial^2}{\partial p^2}\right)^N \frac{1}{p^2(n\cdot p)} = \frac{4^N N!N!}{p^{2N+2}(n\cdot p)} \tag{2.102}$$

Substituting (2.102) into (2.98) determines $\Sigma(p)$ as a power-series in m^2/p^2.

$$\Sigma(p) = \frac{2}{9} g^2 \langle\bar{\psi}\psi\rangle \left(\frac{1}{p^2} - \frac{m\not{p}}{2p^4} + \frac{\not{n}}{m(n\cdot p)} \sum_{N=0}^{\infty} \frac{1}{(N+1)(N+2)} \left(\frac{m^2}{p^2}\right)^{N+1} \right) \qquad (2.103)$$

The series in (2.103) converges for $|p^2| \geq m^2$, and is given by

$$\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)} = 1 + \frac{1}{x}(1-x)\ln(1-x) \quad ; \quad |x| \leq 1 \quad . \qquad (2.104)$$

With the result of (2.104), the quark condensate component of the quark self-energy becomes

$$\Sigma(p) = \frac{2}{9} g^2 \langle\bar{\psi}\psi\rangle \left(\frac{1}{p^2} - \frac{m\not{p}}{2p^4} + \frac{\not{n}}{m(n\cdot p)} \left[1 + \left(\frac{p^2}{m^2} - 1 \right) \ln\left(1 - \frac{m^2}{p^2} \right) \right] \right)$$

$$; \quad |p^2| \geq m^2 \quad . \qquad (2.105)$$

The expression for $\Sigma(p)$ in (2.105) clearly differs from the result obtained in a covariant gauge (see 2.84). However, the results should only agree on-shell,* since this is the physically meaningful value of the quark self-energy. To determine the on-shell behaviour of (2.105), an expectation value of $\Sigma(p)$ must be taken between external quark wave functions denoted by q. The only difficulty in this procedure involves the expectation value of $\not{n}$, which can be determined as follows.

$$\bar{q}\not{n}q = n^\mu \bar{q}\gamma^\mu q = n^\mu \bar{q} \frac{(\not{p}\gamma^\mu + \gamma^\mu \not{p})}{2m} q = \frac{n\cdot p}{m} \quad ; \quad p^2 = m^2 \qquad (2.106)$$

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*It is also anticipated that the large-momentum behaviour is similar, a property obviously satisfied by (2.105) since $\Sigma(p)$ approaches zero for large p^2.

Thus the on-shell value of $\Sigma(p)$ is given by

$$\Sigma(m) = \frac{2}{9} g^2 \langle\bar{\psi}\psi\rangle \left(\frac{1}{m^2} - \frac{1}{2m^2} + \frac{1}{m^2} \right) = g^2 \langle\bar{\psi}\psi\rangle \frac{1}{3m^2} \quad , \qquad (2.107)$$

a value in agreement with the covariant-gauge result of (2.84).

In summary, the lowest-order, quark condensate component of the quark self-energy was evaluated in the light-cone gauge to all orders in the mass parameter m. The on-shell value of the resulting self-energy is in agreement with the covariant-gauge result of Section 2-5, providing further support for the gauge independence of the running dynamical mass.

CHAPTER THREE

HIGHER DIMENSION CONDENSATES AND THE QUARK SELF-ENERGY

3-1 Total Order g^2 OPE Corrections to the Quark Propagator

Condensates of mass dimension greater than three can contribute to the quark self-energy, possibly modifying the quark condensate effects of Chapter Two. The dimension-four gluon condensate is of particular interest since it sets the energy scale of the string tension, identifying the gluon condensate with confinement. Chiral symmetry is respected by the gluon condensate, suggesting that in the chiral limit $\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle$ will not contribute to the effective quark mass, and thus will decouple the order parameter of confinement from that of dynamical mass generation.

One of the basic properties of the OPE in (2.5) is the ability to calculate the coefficients $C_\theta(p)$ perturbatively. The only order g^2 amplitudes are shown in Figures One and Two, corresponding to processes involving the non-perturbative vev's $\langle\bar{\psi}(z)\psi(y)\rangle$ and $\langle A^a_\mu(y)A^b_\nu(z)\rangle$. Additional powers of the coupling constant can be introduced either explicitly by increasing the number of vertices in the amplitudes, or implicitly through the commutator of two covariant derivatives leading to the field strength $[D_\nu,D_\mu]=G_{\mu\nu} = (ig/2)\lambda^a F^a_{\mu\nu}$.[36] Thus the entire set of $\mathcal{O}(g^2)$ contributions to the OPE is encompassed by the condensates generated from (2.10) and (2.11), without introducing additional coupling constants through QCD equations of motion. With this criterion, the quark condensate component of $\Sigma(p)$ evaluated in Chapter Two entirely determines the order g^2 contribution through the $\langle\bar{\psi}(z)\psi(y)\rangle$ vev in (2.10). This leaves only the $\langle A^a_\mu(y)A^b_\nu(z)\rangle$ vev in (2.11) as a source of further order g^2 corrections to $\Sigma(p)$.

Consider the $\langle A_\mu^a(y) A_\nu^b(z)\rangle$ vev. Using the fixed-point gauge result of (2.21), $A_\mu^a(x)$ can be written in terms of the field strength as

$$A_\mu^a(x) = \int_0^1 dt\; t x^\nu F_{\nu\mu}^a(tx) \tag{3.1}$$

Expanding $F_{\mu\nu}^a(tx)$ in a Taylor series about tx=0, leads to an expansion of $A_\mu(x) = (ig/2)\lambda^a A_\mu^a(x)$

$$A_\mu(x) = \frac{1}{2} x^\rho G_{\rho\mu}(0) + \frac{1}{3} x^\rho x^\alpha \partial_\alpha G_{\rho\mu}(0) + \ldots$$

$$+ \frac{1}{n!(n+2)} x^\rho x^{\alpha_1} \ldots x^{\alpha_n} \partial_{\alpha_1} \ldots \partial_{\alpha_n} G_{\rho\mu}(0) + \ldots \tag{3.2}$$

where $G_{\mu\nu} = (ig/2)\lambda^a F_{\mu\nu}^a$. In the fixed point gauge the ordinary derivatives in (3.2) may be written as covariant ones (see 2.16), so that (3.2) becomes the following expression.

$$A_\mu(x) = \frac{1}{2} x^\rho G_{\rho\mu}(0) + \frac{1}{3} x^\rho x^\alpha [D_{(\alpha}, G_{\rho)\mu}(0)] + \ldots$$

$$+ \frac{1}{n!(n+2)} x^\rho x^{\alpha_1} \ldots x^{\alpha_n} [D_{(\alpha_1} \ldots [D_{\alpha_n}, G_{\rho)\mu}(0)] \ldots] + \ldots \; . \tag{3.3}$$

The two-point amplitude of (2.11) may now be written in a different form, through use of the relation $A_\mu = (ig/2)\lambda^a A_\mu^a$.

$$i\Delta S_2(p) = \int d^4x\, d^4y\, d^4z\; e^{ip\cdot x} \langle T(\psi(x)\bar\psi(y))\rangle\, \gamma^\mu \langle :A_\mu(y) A_\nu(z):\rangle$$

$$\times \langle T(\psi(y)\bar\psi(z))\rangle\, \gamma^\nu \langle T(\psi(z)\bar\psi(0))\rangle \tag{3.4}$$

The non-perturbative vev $\langle A_\mu(y)A_\nu(z)\rangle$ can be written as a covariantized Taylor series using (3.3).

$$\langle A_\mu(y)A_\nu(z)\rangle = \frac{1}{4}y^\rho z^\tau \langle G_{\rho\mu}(0)G_{\tau\nu}(0)\rangle + \frac{1}{6}y^\rho z^\alpha z^\tau \langle G_{\rho\mu}[D_{(\alpha},G_{\tau)\nu}]\rangle$$

$$+ \frac{1}{6}y^\rho y^\alpha z^\tau \langle [D_{(\alpha},G_{\rho)\mu}]G_{\tau\nu}\rangle + \text{higher order terms} \tag{3.5}$$

First consider the lowest order term in (3.5).

$$\langle G_{\rho\nu}(0)G_{\tau\nu}(0)\rangle = -\frac{g^2}{4}\lambda^a\lambda^b\langle F^a_{\rho\mu}F^b_{\tau\nu}\rangle \tag{3.6}$$

On grounds of Lorentz invariance, $\langle F^a_{\rho\mu}F^b_{\tau\nu}\rangle$ can be written as

$$\langle F^a_{\rho\mu}F^b_{\tau\nu}\rangle = A\,\delta^{ab}(g_{\rho\tau}g_{\mu\nu} - g_{\rho\nu}g_{\mu\tau}) \tag{3.7}$$

where the symmetries of the vev are respected by the tensor on the R.H.S. of (3.7). Contracting both sides of (3.7) with $g_{\rho\tau}g_{\mu\nu}\delta^{ab}$ gives an equation for the unknown constant A.

$$\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle = A(8)(16-4) = 96A \tag{3.8}$$

Substituting (3.8) into both (3.7) and (3.5) gives the lowest order term in the expansion of $\langle A_\mu(y)A_\nu(z)\rangle$.[46-48]

$$\langle A_\mu(y)A_\nu(z)\rangle = -\frac{g^2}{4}\lambda^a\lambda^a\frac{y^\rho z^\tau}{384}(g_{\rho\tau}g_{\mu\nu} - g_{\rho\nu}g_{\mu\tau})\langle F^c_{\lambda\sigma}F^c_{\lambda\sigma}\rangle$$

$$+ \text{higher order terms} \tag{3.9}$$

All higher order terms in (3.5) will only contribute to condensates of dimension greater than four and will be accompanied by additional powers of the coupling constant.[36] To demonstrate this point, consider a general coefficient of the Taylor series in (3.5).

$$<[D_\alpha \ldots [D_\mu, G_{\tau\lambda}] \ldots][D_\beta \ldots [D_\nu, G_{\eta\omega}] \ldots> \tag{3.10}$$

Covariant derivatives in (3.10) can be integrated by parts as in Section 2-4. This is achieved by recognizing that gauge invariant vev's are Poincare invariant, implying the relation

$$0 = \lim_{x \to 0} \partial_\tau <[D_{\alpha_1} \ldots [D_{\alpha_n}, G_{\mu\lambda}] \ldots][D_{\beta_1} \ldots [D_{\beta_m}, G_{\nu\rho}] \ldots]> \tag{3.11a}$$

which becomes

$$<[D_\tau, [D_{\alpha_1} \ldots [D_{\alpha_n}, G_{\mu\lambda}] \ldots][D_{\beta_1} \ldots [D_{\beta_m}, G_{\nu\rho}] \ldots]>$$

$$= - <[D_{\alpha_1} \ldots [D_{\alpha_n}, G_{\mu\lambda}] \ldots][D_\tau, [D_{\beta_1} \ldots [D_{\beta_m}, G_{\nu\rho}] \ldots]> \tag{3.11b}$$

in the fixed-point gauge. Equation (3.11b) shows that only objects of the form

$$<G_{\mu\nu}[D_{\alpha_1} \ldots [D_{\alpha_n}, G_{\lambda\sigma}] \ldots]> \tag{3.12}$$

need to be considered. Since (3.12) is a Dirac singlet, it must be written as a combination of metrics satisfying the appropriate symmetry requirements. Consequently only an even number of covariant derivatives in (3.12) will lead to a non-trivial condensate, because a Dirac singlet tensor with an odd number of indices cannot be constructed.

The order of the covariant derivatives in (3.12) can be interchanged at the expense of generating a field strength. Using the Jacobi Identity, the covariant derivatives can be interchanged in the following expression.

$$[D_\alpha,[D_\beta,[D_{\lambda_1}\ldots[D_{\lambda_n},G_{\mu\nu}]\ldots] \equiv [D_\alpha,[D_\beta,A]$$

$$= [D_\beta,[D_\alpha,A]] - [G_{\alpha\beta},A] \quad ; \quad A \equiv [D_{\lambda_1}\ldots[D_{\lambda_n},G_{\mu\nu}]\ldots] \qquad (3.13)$$

Through repeated use of (3.13), covariant derivatives in (3.12) can be written in any order desired. After contracting (3.12) with a linear combination of metrics, covariant derivatives are eliminated either through the $[D_\alpha,G_{\alpha\beta}] = j_\beta$ equation of motion or through generating extra factors of field strengths. Thus the higher order terms in (3.9) can only contribute to condensates of mass dimension larger than four, leaving only the lowest order term in (3.5) to determine the gluon condensate component of $\langle A_\mu(y)A_\nu(z)\rangle$.

3-2 Gluon Condensate Component of the Quark Self-Energy and the Dynamical Quark Mass

The order g^2 gluon condensate contribution to the quark self-energy can now be calculated by substituting (3.9) into the amplitude of (3.4). After changing the variables of integration to (x-y),(y-z),z from x,y,z, equation (3.4) becomes

$$i\Delta S_2(p) = \frac{4}{3} i g^2 \frac{\langle FF\rangle}{384} \int d^4(x-y)\, e^{ip\cdot(x-y)} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{\not k+m_L}{k^2-m_L^2}$$
$$\times (g_{\rho\tau}g_{\mu\nu}-g_{\rho\nu}g_{\mu\tau}) \Bigg\{ \int d^4(y-z)\, e^{ip\cdot(y-z)} (y-z)^\rho \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(y-z)}$$
$$\times \gamma^\mu \frac{\not q+m_L}{q^2-m_L^2}\gamma^\nu \int d^4z\, e^{ip\cdot z} z^\tau \int \frac{d^4T}{(2\pi)^4} e^{-iT\cdot z} \frac{\not T+m_L}{T^2-m_L^2} \tag{3.14}$$
$$+ \int d^4(y-z)\, e^{ip\cdot(y-z)} \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(y-z)} \gamma^\mu \frac{\not q+m_L}{q^2-m_L^2}\gamma^\nu$$
$$\times \int d^4z\, e^{ip\cdot z} z^\rho z^\tau \int \frac{d^4T}{(2\pi)^4} e^{-iT\cdot z} \frac{\not T+m_L}{T^2-m_L^2} \Bigg\} \quad ; \quad \langle FF\rangle \equiv \langle F^c_{\lambda\sigma}F^c_{\lambda\sigma}\rangle .$$

Using the techniques of Section 2-5 the integrals in (3.14) can be evaluated.

$$i\Delta S_2(p) = -ig^2 \frac{\langle FF\rangle}{288} \frac{\not p+m_L}{p^2-m_L^2} (g_{\rho\tau}g_{\mu\nu} - g_{\rho\nu}g_{\mu\tau})$$
$$\times \frac{\partial}{\partial p_\rho} \left(\gamma^\mu \frac{\not p+m_L}{p^2-m_L^2} \gamma^\nu \frac{\partial}{\partial p_\tau} \frac{\not p+m_L}{p^2-m_L^2} \right) \tag{3.15}$$

Simplifying (3.15) by evaluating the derivatives and performing the Dirac algebra gives the following expression;[33,36,48]

$$i\Delta S_2(p) = i\,\frac{\not p+m_L}{p^2-m_L^2}\left(\frac{g^2\langle FF\rangle\, m_L \not p(\not p-m_L)}{12(p^2-m_L^2)^3}\right)\frac{\not p+m_L}{p^2-m_L^2} \tag{3.16}$$

where $\langle FF\rangle \equiv \langle F^a_{\lambda\sigma}F^a_{\lambda\sigma}\rangle$. Using equation (2.74) to identify the self-energy in (3.16) leads to the result for the $\mathcal{O}(g^2)$ gluon condensate component of the quark self-energy.[36,50]

$$\Sigma(p) = \frac{g^2\langle FF\rangle\, m_L \not p(\not p-m_L)}{12(p^2-m_L^2)^3} \tag{3.17}$$

As expected from the chiral invariance of the gluon condensate, the self-energy in (3.17) is zero in the $m_L=0$ chiral limit, decoupling the gluon condensate from the dynamical mass.

It was argued at the beginning of the section that the entire order g^2 correction to the OPE in (2.5) is composed of the quark and gluon condensate contributions. Thus by combining the results of (2.84) and (3.17), the total order g^2 correction to the non-perturbative quark self-energy is obtained.

$$\Sigma(p) = g^2\,\frac{|\langle\bar\psi\psi\rangle|}{9p^2}\left((3+a) - a\,\frac{m\not p}{p^2}\right) + \frac{g^2\langle FF\rangle\, m_L(p^2-m_L\not p)}{12(p^2-m_L^2)^3} \tag{3.18}$$

As in Section 2-5 the total coefficient of $\not p$ is removed from the inverse propagator and is absorbed into a wave function renormalization.[54] This procedure extracts the effective quark mass $M_{eff}(p^2)$ from (3.18).[36]

$$S_F^{-1}(p) = \not p - m_L - \Sigma(p)$$

$$= \left[1 + g^2\left(am\,\frac{|\langle\bar\psi\psi\rangle|}{9p^4} + \frac{m_L^2\langle FF\rangle}{12(p^2-m_L^2)^3}\right)\right]\left(\not p - M_{eff}(p^2)\right). \tag{3.19a}$$

$$M_{eff}(p^2) = \frac{36p^4(p^2-m_L^2)^3 m_L + 4g^2|\langle\bar{\psi}\psi\rangle| p^2(p^2-m_L^2)^3(3+a) + 3g^2\langle FF\rangle m_L^2 p^6}{36p^4(p^2-m_L^2)^3 + 4g^2|\langle\bar{\psi}\psi\rangle|(p^2-m_L^2)^3 am + 3g^2\langle FF\rangle m_L^2 p^4} \qquad (3.19b)$$

Following the procedures in Section 2-5, the pole position of (3.19a) is determined.

$$\mu = \lim_{p^2 \to \mu^2} M_{eff}(p^2) \qquad (3.20)$$

This leads to the constraint

$$0 = 36\mu^4(\mu^2-m_L^2)^3(m_L-\mu) + 12g^2|\langle\bar{\psi}\psi\rangle|(\mu^2-m_L^2)^3\mu^2$$

$$+ 3g^2\langle FF\rangle m_L\mu^5(\mu-m_L) + 4g^2 a|\langle\bar{\psi}\psi\rangle|\mu(\mu-m)(\mu^2-m_L^2)^3 \qquad (3.21)$$

Differentiating (3.21) with respect to the gauge parameter a, and demanding that $\frac{\partial\mu}{\partial a} = 0$ implies that μ equals zero, m, or m_L. The $\mu=0$ constraint represents a chiral-restoring solution, and $\mu=m_L$ corresponds to the current mass obtainable from $M_{eff}(p^2)$ in the large p^2 limit. Both these solutions are completely insensitive to the condensates.

Substituting the non-trivial $\mu=m$ solution into (3.21) leads to the following order g^2 expression for m.

$$m \doteq m_L\left[1 + \frac{g^2\langle FF\rangle}{12(m^2-m_L^2)^2}\left(\frac{m}{m+m_L}\right)\right] + g^2\frac{|\langle\bar{\psi}\psi\rangle|}{3m^2} \qquad (3.22)$$

The gluon condensate contribution to m in (3.22) is small for light quarks. The size of this effect can be estimated by setting $m = m_{dyn} + \Delta$ in (3.22), and assuming that Δ is small in comparison with m_{dyn}.

$$m_{dyn} + \Delta = m_L\left(1 + \frac{g^2\langle FF\rangle}{12m_{dyn}^4}\right) + m_{dyn} = 2\Delta$$

$$\Delta = \frac{1}{3}m_L\left(1 + \frac{g^2\langle FF\rangle}{12m_{dyn}^4}\right) \qquad (3.23)$$

The gluon condensate $\alpha_s\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle$ has been determined by several techniques, with estimates ranging between $(425\ \mathrm{MeV})^4$ [25,30] and $\geq$ $(5[illegible]\ \mathrm{MeV})^4$.[55] Thus the bracketed expression in (3.23) is slightly larger than three, so that Δ is approximately m_L, leaving the $m = m_{dyn} \simeq 320$ MeV result of Section 2-5 essentially unaltered.

The difference between up and down quark masses is determined by Δ since the dynamical mass is flavour independent. This leads to an expression for the ratio of current to constituent mass differences for u and d quarks.

$$m_u - m_d = \Delta_u - \Delta_d = \frac{1}{3}(m_L^u - m_L^d)\left(1 + \frac{g^2\langle FF\rangle}{12m_{dyn}^4}\right)$$

$$\frac{m_L^u - m_L^d}{m_u - m_d} = 3\left(1 + \frac{g^2\langle FF\rangle}{12m_{dyn}^4}\right)^{-1} \qquad (3.24)$$

In the absence of the gluon condensate, (3.24) indicates an enhancement by a factor of three in the ratio of u and d current mass difference to constituent mass difference.[36] Some enhancement of this ratio is expected phenomenologically,[56] but with gluon condensate effects included the ratio will be approximately equal to one.

The gluon condensate provides an interesting mass scale in the heavy quark limit. Consider the case of $m = m_L + \delta$; δ/m_L small. In this limit (3.22) becomes

$$m_L + \delta = m_L\left(1 + \frac{g^2\langle FF\rangle (m_L+\delta)}{12\delta^2(2m_L+\delta)^3}\right) + g^2|\langle\bar\psi\psi\rangle|\frac{1}{3(m_L+\delta)^2}$$

$$\delta = g^2 m_L \frac{\langle FF\rangle (m_L+\delta)}{12\delta^2(2m_L+\delta)^3} + g^2|\langle\bar\psi\psi\rangle|\frac{1}{3(m_L+\delta)^2} \qquad (3.25)$$

For heavy quarks the quark and gluon condensates are related by the expression[25,44,57]

$$\frac{g^2\langle FF\rangle}{48\pi^2} = m_L|\langle\bar\psi\psi\rangle| \qquad (3.26)$$

which simplifies (3.25).

$$\delta \simeq \frac{\pi^2}{2\delta^2}|\langle\bar\psi\psi\rangle| + \frac{g^2}{3m_L^2}|\langle\bar\psi\psi\rangle| \simeq \frac{\pi^2}{2\delta^2}|\langle\bar\psi\psi\rangle| \qquad (3.27a)$$

$$\delta^3 \simeq \frac{1}{2}\pi^2|\langle\bar\psi\psi\rangle| \simeq \frac{g^2\langle FF\rangle}{96 m_L} \qquad (3.27b)$$

The expression in (3.27b) is valid for charm and heavier quarks since in this region $\langle FF\rangle$ is much smaller than m_L^4, implying that δ/m_L is small. At the charm mass scale of about one GeV, (3.27b) leads to $\delta \simeq 25$ MeV, demonstrating that the dynamical mass effects are negligible for heavy quarks.

Strange quarks lie in the intermediate zone between heavy and light quarks. This turns out to be the interesting range of (3.22) which is extremely sensitive to values of the condensates. Constituent and current mass scales for the s are about 130 MeV and 500 MeV respectively,[11,28] and $\langle\bar{s}s\rangle$ is approximately $(0.7)\langle\bar{u}u\rangle$.[28] Equation (3.22) fits these parameters, provided that the rather large value of $\langle FF\rangle = (700\ \text{MeV})^4$ is used.

In summary, the quark and gluon condensate provide the entire order g^4 corrections to the quark self-energy. Inclusion of the gluon condensate does not significantly alter the values of m for light quarks, shifting m away from the chiral-limiting value m_{dyn} only by an amount on the order of m_L. The gluon condensate also provides a value for the small shift δ of m away from the Lagrangian mass m_L, illustrating that dynamical mass generation is insignificant for heavy quarks. Finally, by using a rather large value for the gluon condensate, the current and constituent masses of the strange quark can be modelled by the quark and gluon condensates.

3-3 Mixed Condensate Component of the Quark Self-Energy: Formulation

The dimension-five mixed condensate $\langle\bar{\psi}\sigma G\psi\rangle$ is the second-lowest dimensional condensate which violates chiral symmetry. It is thus of interest to examine the lowest order mixed condensate component of the quark self-energy to determine the effect upon dynamical mass generation. The gauge independence of the quark self-energy and the stability of the dynamical mass m_{dyn}, after the inclusion of mixed condensate effects, will be studied in the next few sections.

Mixed condensate contributions to the quark self-energy enter through the explicitly $O(g^2)$ amplitude of Figure One (equation 2.10), and through the order g^3 amplitudes, represented in Figures Four to Six. The mixed condensate can only appear in the $\langle\bar{\psi}(z)\psi(y)\rangle$ vev of (2.10) after covariant derivatives are commuted, generating the field strength $F^a_{\mu\nu}$. However, the $\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ amplitude immediately generates the field strength $F^a_{\mu\nu}$ through use of (3.2). Thus the explicit order g^2 amplitude in (2.10) leads to a mixed condensate correction of the same order as the three-vertex graphs of Figures Four to Six.

One of the fundamental properties of the OPE is the ability to perturbatively calculate the coefficients of the condensates. Since the OPE has this property, it is reasonable to expect a hierarchy of condensates to develop based on the order of perturbation theory at which a condensate may first enter the quark propagator. If this hierarchy of condensates actually exists, then clearly the mixed condensate should be less important than the quark and gluon condensates.

The calculation of the $\langle\bar{\psi}\sigma G\psi\rangle$ component of the quark propagator begins with the expressions for the amplitudes of Figures Four to

FIGURE FOUR

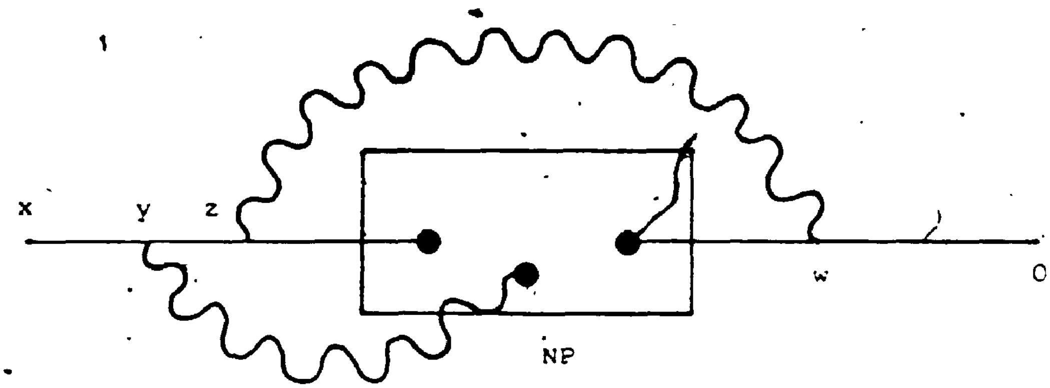


FIGURE FIVE

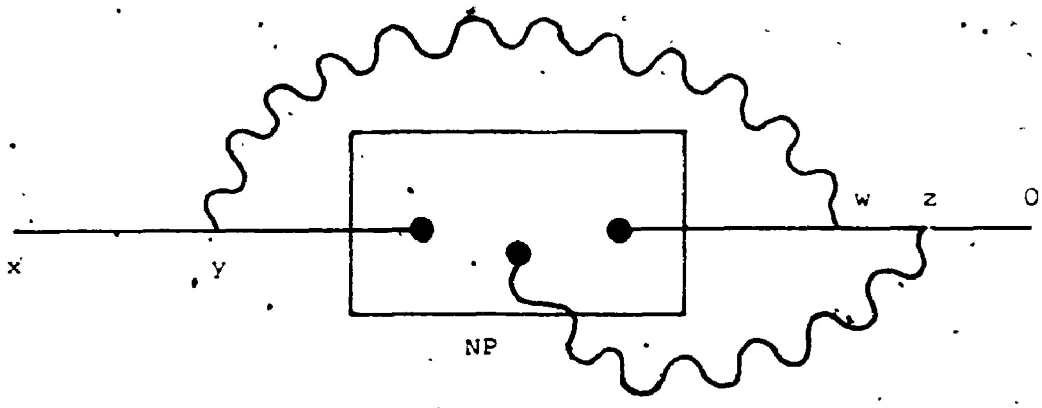
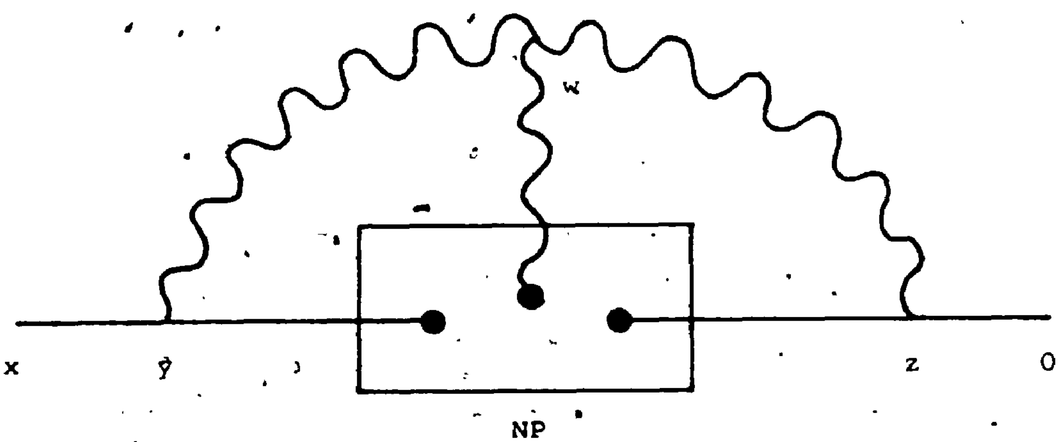


FIGURE SIX



Order g^3 corrections to the quark self-energy from the quark-gluon-quark vev.

Six.[36] These expressions are obtained by Wick-expanding the appropriate third order expansions in (2.6).

$$i\Delta S_4(p) = \frac{g^2}{4}\lambda^a\lambda^b \int d^4x\, d^4y\, d^4z\, d^4w\; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle$$

$$\times\, \gamma^\mu \langle T(\psi(y)\bar{\psi}(z))\rangle\, \gamma^\nu\; i\frac{g}{2}\langle\bar{\psi}(w)A^a_\mu(y)\psi(z)\rangle$$

$$\times\, \lambda^c\gamma^\rho\, \langle T(\psi(w)\bar{\psi}(0))\rangle\, \langle T(A^b_\nu(z)A^c_\rho(w))\rangle \qquad (3.28)$$

$$i\Delta S_5(p) = \frac{g^2}{4}\lambda^a \int d^4x\, d^4y\, d^4z\, d^4w\; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle$$

$$\times\, \gamma^\mu\; i\frac{g}{2}\langle\bar{\psi}(z)A^c_\rho(w)\psi(y)\rangle\, \lambda^b\lambda^c\gamma^\nu \langle T(\psi(z)\bar{\psi}(w))\rangle$$

$$\times\, \gamma^\rho \langle T(\psi(w)\bar{\psi}(0))\rangle\, \langle T(A^a_\mu(y)A^b_\nu(z))\rangle \qquad (3.29)$$

$$i\Delta S_6(p) = -\frac{g^2}{2}\lambda^a f_{ehn} \int d^4x\, d^4y\, d^4z\, d^4w\; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle$$

$$\times\, \gamma^\mu \langle T(A^a_\mu(y)A^h_\rho(w))\rangle \Big\{ \langle T(A^n_\nu(w)A^b_\lambda(z))\rangle$$

$$\times\, i\frac{g}{2}\Big[\frac{\partial}{\partial w_\rho}\langle\bar{\psi}(z)A^e_\lambda(w)\psi(y)\rangle - \frac{\partial}{\partial w_\lambda}\langle\bar{\psi}(z)A^e_\rho(w)\psi(y)\rangle\Big]$$

$$+\, ig\langle\bar{\psi}(z)A^n_\lambda(w)\psi(y)\rangle \Big\{\frac{\partial}{\partial w_\rho}\langle T(A^e_\lambda(w)A^b_\nu(z))\rangle$$

$$-\frac{\partial}{\partial w_\lambda}\langle T(A^e_\rho(w)A^g_\nu(z))\rangle\Big\}\Big\}\, \lambda^b\gamma^\nu \langle T(\psi(z)\bar{\psi}(0))\rangle \qquad (3.30)$$

Equations (3.28) and (3.29) represent the amplitudes in Figures Four and Five which do not contain any non-abelian interaction vertices, while (3.30) represents the Figure Six amplitude containing the non-abelian triple-gluon interaction. To evaluate the order g^3 amplitudes in equations (3.28) to (3.30) the mixed condensate projection of the $\langle\bar{\psi}(z)A_\mu(w)\psi(y)\rangle$ vev is required. The only other graph that contributes to the mixed condensate component of the quark self-energy is given in (2.10) and Figure One, and requires knowledge of the $\langle\bar{\psi}\sigma G\psi\rangle$ projection of the quark-quark vev.

3-4 Mixed Condensate Projection of the Quark-Quark Vacuum Expectation Value

The mixed condensate projection of the non-perturbative vev $\langle\bar{\psi}(z)\psi(y)\rangle$ will be calculated in this section. Consider the covariantized Taylor series for the quark-quark vev in (2.35).

$$\langle\bar{\psi}(z)\psi(y)\rangle = \langle\bar{\psi}(0)\psi(0)\rangle + (y-z)^{\alpha}\langle\bar{\psi}D_{\alpha}\psi\rangle + \frac{1}{2}(y^{\alpha}y^{\beta}+z^{\alpha}z^{\beta})\langle\bar{\psi}D_{(\alpha}D_{\beta)}\psi\rangle$$

$$- z^{\alpha}y^{\beta}\langle\bar{\psi}D_{\alpha}D_{\beta}\psi\rangle + \text{higher order terms} \quad . \qquad (3.31)$$

Since the mixed condensate $\langle\bar{\psi}\sigma^{\mu\nu}G_{\mu\nu}\psi\rangle$ is of dimension five, the first possibility for its occurrence is at second order in (3.31). In Section 2-4 the mixed condensate contribution to the second-order term in (3.31) was calculated (see 2.47).

$$\langle\bar{\psi}_i D_{\mu} D_{\nu}\psi_j\rangle = \frac{1}{288}\langle\bar{\psi}\sigma G\psi\rangle\left(-3ig_{\mu\nu}\delta_{ij} - \sigma^{\mu\nu}_{ji}\right) \qquad (3.32)$$

Substituting (3.32) into (3.31) leads to the following second order contribution to the mixed condensate,[37]

$$\langle\bar{\psi}_i(z)\psi_j(y)\rangle_{II} = \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\left(-\frac{i}{64}(y-z)^2\delta_{ij} - \frac{1}{96}y^{\mu}z^{\nu}\sigma^{\mu\nu}_{ji}\right) \qquad (3.33)$$

where $\langle\bar{\psi}(z)\psi(y)\rangle_{II}$ denotes that only second order terms have been considered.

Explicit calculations for the $\langle\bar{\psi}\sigma G\psi\rangle$ projection of $\langle\bar{\psi}(z)\psi(y)\rangle$ will be presented up to fifth order in the covariantized Taylor expansion. An algorithm for extending the calculation to all orders will then be given, illustrating the general form of the series.

Consider the third order terms in the covariantized expansion of $<\bar{\Psi}(z)\psi(y)>$.

$$<\bar{\psi}(z)\psi(y)>_{III} = -\frac{1}{6}\left(z^{\mu}z^{\nu}z^{\omega} - y^{\mu}y^{\nu}y^{\omega}\right)<\bar{\psi}_i D_{(\mu}D_{\nu}D_{\omega)}\psi_j>$$

$$+ \frac{1}{2}z^{\mu}z^{\nu}y^{\omega}<\bar{\psi}_i D_{(\mu}D_{\nu)}D_{\omega}\psi_j> - \frac{1}{2}z^{\omega}y^{\mu}y^{\nu}<\bar{\psi}_i D_{\omega}D_{(\mu}D_{\nu)}\psi_j> \qquad (3.34)$$

The object $<\bar{\psi}_i D_{(\mu}D_{\nu}D_{\omega)}\psi_j>$ will now be calculated. Using the covariance arguments introduced in Section 2-4, this vev can be written in terms of the Lorentz invariant Dirac matrices.

$$<\bar{\psi}_i D_{(\mu}D_{\nu}D_{\omega)}\psi_j> = A\left(\gamma^{\mu}_{ji}g_{\nu\omega} + \gamma^{\nu}_{ji}g_{\mu\omega} + \gamma^{\omega}_{ji}g_{\mu\nu}\right) \qquad (3.35)$$

Contracting both sides of (3.35) with $\gamma^{\mu}_{ij}g^{\nu\omega}$ gives an expression for A.

$$\frac{1}{3}<\bar{\psi}\left(\not{D}D^2 + D^2\not{D} + D_{\mu}\not{D}D_{\mu}\right)\psi> = 96\,A \qquad (3.36)$$

To simplify (3.36) an identity is required. Consider the equation of motion in (2.51).

$$[D_{\mu}, G_{\mu\nu}] = j_{\nu} = [D_{\mu}, D_{\nu}D_{\mu} - D_{\mu}D_{\nu}] \qquad (3.37a)$$

$$2D_{\mu}D_{\nu}D_{\mu} - D_{\nu}D^2 - D^2D_{\nu} = j_{\nu} \qquad (3.37b)$$

Solving (3.37b) for $D_{\mu}D_{\nu}D_{\mu}$ yields the identity

$$D_\mu D_\nu D_\mu = \frac{1}{2}\left(D_\nu D^2 + D^2 D_\nu\right) + J_\nu \qquad (3.37c)$$

Using (3.37) along with the identity of (2.42b), and ignoring the terms which do not lead to the mixed condensate, simplifies (3.36) to

$$96\, A = \frac{1}{2}\langle\bar{\psi}\left(\not{D}D^2 + D^2\not{D}\right)\psi\rangle = -\frac{m}{2}\langle\bar{\psi}\sigma G\psi\rangle \qquad (3.38a)$$

$$A = -\frac{m}{192}\langle\bar{\psi}\sigma G\psi\rangle \qquad (3.38b)$$

where the $\not{D}\psi = -im\psi$ equation of motion has been used. The above equation for A is now substituted into the expression (3.35) and a colour averaging is performed, leading to the following result.

$$\langle\bar{\psi}_i D_{(\mu}D_\nu D_{\omega)}\psi_j\rangle = -\frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m}{192}\left\{\gamma^\mu_{ji}g_{\nu\omega} + \gamma^\nu_{ji}g_{\mu\omega} + \gamma^\omega_{ji}g_{\mu\nu}\right\}. \qquad (3.39)$$

Now consider the object $\langle\bar{\psi}_i D_{(\mu}D_{\nu)}D_\omega\psi_j\rangle$ in (3.34). Following the now familiar techniques, this is written as a linear combination of Dirac matrices satisfying the symmetry requirements.

$$\langle\bar{\psi}_i D_{(\mu}D_{\nu)}D_\omega\psi_j\rangle = B\, g_{\mu\nu}\gamma^\omega_{ji} + C\left\{\gamma^\mu_{ji}g_{\nu\omega} + \gamma^\nu_{ji}g_{\mu\omega}\right\} \qquad (3.40)$$

Contracting (3.40) with the linearly independent objects $g^{\mu\nu}\gamma^\omega_{ij}$ and $g^{\nu\omega}\gamma^\mu_{ij}$ leads to the following two equations for B and C.

$$\langle\bar{\psi}D^2\not{D}\psi\rangle = 64B + 32C \qquad (3.41a)$$

$$\frac{1}{2}\langle\bar{\psi}(\not{D}D^2 + D_\mu \not{D} D_\mu)\psi\rangle = 16B + 80C \tag{3.41b}$$

The identities (2.42b) and (3.37) are now used along with the $\not{D}\psi=-im\psi$ equation of motion to simplify (3.41a) and (3.41b).

$$4B + 2C = -\frac{m}{32}\langle\bar{\psi}\sigma G\psi\rangle \tag{3.42a}$$

$$B + 5C = \frac{1}{64}\langle\bar{\psi}(3\not{D}D^2 + D^2\not{D})\psi\rangle = -\frac{m}{32}\langle\bar{\psi}\sigma G\psi\rangle \tag{3.42b}$$

The values of B and C are now obtained by solving the equations in (3.42).

$$C = B = -\frac{m}{192}\langle\bar{\psi}\sigma G\psi\rangle \tag{3.43}$$

Substituting the results of (3.43) into (3.40) and including the colour-averaging factor of three, leads to the following completely symmetric tensor.

$$\langle\bar{\psi}_i D_{(\mu}D_{\nu)}D_\omega\psi_j\rangle = -\frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m}{192}\left(\gamma^\mu_{ji}g_{\nu\omega} + \gamma^\nu_{ji}g_{\mu\omega} + \gamma^\omega_{ji}g_{\mu\nu}\right) \tag{3.44}$$

The remaining object which determines the third order term of the mixed condensate projection of $\langle\bar{\psi}(z)\psi(y)\rangle$ is $\langle\bar{\psi}_i D_\omega D_{(\mu}D_{\nu)}\psi_j\rangle$. Writing this as a linear combination of Dirac matrices leads to the expression

$$\langle\bar{\psi}_i D_\omega D_{(\mu}D_{\nu)}\psi_j\rangle = D\, g_{\mu\nu}\gamma^\omega_{ji} + E\left(\gamma^\mu_{ji}g_{\nu\omega} + \gamma^\nu_{ji}g_{\mu\omega}\right) \tag{3.45}$$

Contracting both sides of (3.45) with $g_{\mu\nu}\gamma^{\omega}_{ij}$ and $g_{\nu\omega}\gamma^{\mu}_{ij}$ respectively yields the following two equations.

$$\langle\bar{\psi}\not{D}D^2\psi\rangle = 64D + 32E \tag{3.46a}$$

$$\tfrac{1}{2}\langle\bar{\psi}(D^2\not{D}+D_\mu\not{D}D_\mu)\psi\rangle = 16D + 80E \tag{3.46b}$$

After using the identities (2.42b) and (3.37) the above equations become identical to those of (3.42), indicating that

$$\langle\bar{\psi}_i D_\omega D_{(\mu}D_{\nu)}\psi_j\rangle = -\frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m}{192}\left(\gamma^{\mu}_{ji}g_{\nu\omega} + \gamma^{\nu}_{ji}g_{\mu\omega} + \gamma^{\omega}_{ji}g_{\mu\nu}\right) . \tag{3.47}$$

Collecting the results of (3.47), (3.44) and (3.39) it is observed that the mixed condensate projection of all the third order coefficients in (3.34) are equal to the following completely symmetric object.

$$\langle\bar{\psi}_i D_{(\mu}D_\nu D_{\omega)}\psi_j\rangle = \langle\bar{\psi}_i D_{(\mu}D_{\nu)}D_\omega\psi_j\rangle = \langle\bar{\psi}_i D_\omega D_{(\mu}D_{\nu)}\psi_j\rangle$$

$$= -\frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m}{192}\left(\gamma^{\mu}_{ji}g_{\nu\omega} + \gamma^{\nu}_{ji}g_{\mu\omega} + \gamma^{\omega}_{ji}g_{\mu\nu}\right) \tag{3.48}$$

Substituting the above expression into (3.34) leads to the final result for the third order mixed condensate projection of $\langle\bar{\psi}(z)\psi(y)\rangle$.[36,38]

$$\langle\bar{\psi}_i(z)\psi_j(y)\rangle_{III} = -\frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m}{384}\gamma_{ji}\cdot(y-z)(y-z)^2 \tag{3.49}$$

To evaluate $\langle\bar{\psi}(z)\psi(y)\rangle$ at higher orders an algorithm for generating the appropriate linear combinations of Dirac matrices is helpful. The basic problem is to evaluate objects such as $\langle\bar{\psi}D_{(\alpha}\ldots D_{\mu)}D_{(\nu}\ldots D_{\omega)}\psi\rangle$ which take the form

$$\langle\bar{\psi}D_{(\alpha}\ldots D_{\mu)}D_{(\nu}\ldots D_{\omega)}\psi\rangle = T_{(\alpha\ldots\mu)(\nu\ldots\omega)} \tag{3.50}$$

where T is a Lorentz invariant object with the indicated symmetries. It is possible to generate T iteratively, eliminating one of the complications of higher order terms.

$$\begin{aligned}
T_{(\alpha_1\ldots\alpha_{n+1})(\beta_1\ldots\beta_m)} = {} & A\Big(\gamma_{\alpha_1} T_{(\alpha_2\ldots\alpha_{n+1})(\beta_1\ldots\beta_m)} \\
& \qquad + \ldots + \gamma_{\alpha_{n+1}} T_{(\alpha_1\ldots\alpha_n)(\beta_1\ldots\beta_m)}\Big) \\
& + \text{independent } \gamma \text{ insertions into } A \\
& + B\Big[\gamma_{\beta_1}\Big(T_{(\alpha_1\ldots\alpha_n)(\alpha_{n+1}\beta_2\ldots\beta_m)} \\
& \qquad + T_{(\alpha_2\ldots\alpha_{n+1})(\alpha_1\beta_2\ldots\beta_m)} \\
& \qquad + \ldots + T_{(\alpha_1\ldots\alpha_{n-1}\alpha_{n+1})(\alpha_n\beta_2\ldots\beta_m)}\Big) \\
& \quad + \ldots + \gamma_{\beta_m}\Big(T_{(\alpha_1\ldots\alpha_n)(\beta_1\ldots\beta_{m-1}\alpha_{n+1})} \\
& \qquad + \ldots + T_{(\alpha_1\ldots\alpha_{n-1}\alpha_{n+1})(\beta_1\ldots\beta_{m-1}\alpha_n)}\Big)\Big] \\
& + \text{independent } \gamma \text{ insertions into } B
\end{aligned} \tag{3.51}$$

The notation of independent γ insertions implies that the γ matrix explicitly multiplying the tensor T from the left in (3.51), must also appear in every possible combination with the γ matrices composing T. The tensor T resulting from the above procedure is then reduced to linearly independent combinations of Dirac matrices.

The fourth order term in the covariantized Taylor expansion of $\langle\bar{\psi}(z)\psi(y)\rangle$ will now be evaluated.

$$\langle\bar{\psi}_i(z)\psi_j(y)\rangle_{IV} = \frac{1}{24}\left(y^\alpha y^\beta y^\mu y^\nu + z^\alpha z^\beta z^\mu z^\nu\right)\langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_{\nu)}\psi_j\rangle$$
$$-\frac{1}{6}y^\alpha z^\mu z^\nu z^\lambda\langle\bar{\psi}_i D_{(\mu}D_\nu D_{\lambda)}D_\alpha\psi_j\rangle - \frac{1}{6}z^\alpha y^\mu y^\nu y^\lambda\langle\bar{\psi}_i D_\alpha D_{(\mu}D_\nu D_{\lambda)}\psi_j\rangle$$
$$+\frac{1}{4}y^\mu y^\nu z^\alpha z^\beta\langle\bar{\psi}_i D_{(\alpha}D_{\beta)}D_{(\mu}D_{\nu)}\psi_j\rangle \qquad (3.52)$$

To begin, consider the term $\langle\bar{\psi}D_{(\alpha}D_{\beta)}D_{(\mu}D_{\nu)}\psi\rangle$. The tensor $T_{(\alpha\beta)(\mu\nu)}$ is determined by using the algorithm of (3.51) and equation (3.45).

$$T_{(\alpha\beta)(\mu\nu)} = A\, g_{\mu\nu}\left(\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha\right) + B\left(\gamma_\mu\gamma_\beta g_{\nu\alpha} + \gamma_\mu\gamma_\alpha g_{\nu\beta} + \gamma_\nu\gamma_\alpha g_{\mu\beta}\right.$$
$$\left. + \gamma_\nu\gamma_\beta g_{\mu\alpha}\right) + C\left(\gamma_\beta\gamma_\mu g_{\nu\alpha} + \gamma_\alpha\gamma_\mu g_{\nu\beta} + \gamma_\beta\gamma_\nu g_{\mu\alpha} + \gamma_\alpha\gamma_\nu g_{\mu\beta}\right)$$
$$+ D\left[\gamma_\alpha\left(\gamma_\mu g_{\nu\beta} + \gamma_\nu g_{\mu\beta}\right) + \gamma_\beta\left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha}\right)\right]$$
$$+ E\left[\left(\gamma_\mu g_{\nu\beta} + \gamma_\nu g_{\mu\beta}\right)\gamma_\alpha + \left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha}\right)\gamma_\beta\right]$$
$$+ F\left[\gamma_\mu\left(\gamma_\alpha g_{\nu\beta} + \gamma_\nu g_{\alpha\beta} + \gamma_\beta g_{\nu\alpha}\right)\right.$$

$$+ \gamma_\nu \left(\gamma_\alpha g_{\mu\beta} + 2\gamma_\mu g_{\alpha\beta} + \gamma_\beta g_{\mu\alpha} \right) \Big]$$

$$+ G \Big[\left(\gamma_\alpha g_{\nu\beta} + 2\gamma_\nu g_{\alpha\beta} + \gamma_\beta g_{\nu\alpha} \right) \gamma_\mu$$

$$+ \left(\gamma_\alpha g_{\mu\beta} + 2\gamma_\mu g_{\alpha\beta} + \gamma_\beta g_{\mu\alpha} \right) \gamma_\nu \Big] \tag{3.53}$$

Equation (3.53) is now written as a linearly independent combination of Dirac matrices through use of the identity (2.42a).

$$T_{(\alpha\beta)(\mu\nu)} = A\, g_{\mu\nu} g_{\alpha\beta} + B \left(g_{\nu\beta} g_{\mu\alpha} + g_{\mu\beta} g_{\nu\alpha} \right)$$

$$+ C \left(g_{\nu\beta} \sigma_{\mu\alpha} + g_{\mu\beta} \sigma_{\nu\alpha} + g_{\nu\alpha} \sigma_{\mu\beta} + g_{\mu\alpha} \sigma_{\nu\beta} \right) \tag{3.54}$$

Thus the object $\langle \bar{\Psi}_i D_{(\alpha} D_{\beta)} D_{(\mu} D_{\nu)} \psi_j \rangle$ appearing in (3.52) is written as the following combination of Dirac matrices.

$$\langle \bar{\psi}_i D_{(\alpha} D_{\beta)} D_{(\mu} D_{\nu)} \psi_j \rangle = A\, g_{\mu\nu} g_{\alpha\beta} \delta_{ij} + B \left(g_{\nu\beta} g_{\mu\alpha} + g_{\mu\beta} g_{\nu\alpha} \right) \delta_{ij}$$

$$- C \left[g_{\nu\beta} \sigma^{\mu\alpha}_{ji} + g_{\mu\beta} \sigma^{\nu\alpha}_{ji} + g_{\nu\alpha} \sigma^{\mu\beta}_{ji} + g_{\mu\alpha} \sigma^{\nu\beta}_{ji} \right] \tag{3.55}$$

Contracting (3.55) with $g_{\alpha\beta} g_{\mu\nu} \delta_{ij}$, $g_{\alpha\mu} g_{\nu\beta} \delta_{ij}$ and $g_{\mu\alpha}\, \sigma^{\beta\nu}_{ij}$ in turn, leads to the following three equations.

$$64A + 32B = \langle \bar{\psi} D^2 D^2 \psi \rangle \tag{3.56a}$$

$$16A + 80B = \tfrac{1}{2} \langle \bar{\psi} (D^\alpha D^\mu D_\alpha D_\mu + D^\alpha D^2 D_\alpha) \psi \rangle \tag{3.56b}$$

$$6(48)C = \frac{1}{4} \langle \bar{\psi} \Big(\sigma^{\beta\nu} D^{\alpha} D_{\beta} D_{\alpha} D_{\nu} + D^{\alpha} \sigma^{\beta\nu} D_{\beta} D_{\nu} D_{\alpha}$$

$$+ \sigma^{\beta\nu} D_{\beta} D^2 D_{\nu} + \sigma^{\beta\nu} D_{\beta} D_{\alpha} D_{\nu} D^{\alpha} \Big) \psi \rangle \qquad (3.56c)$$

To simplify the equations in (3.56) a number of identities are required. First consider the combination

$$D^{\alpha} D^{\mu} D_{\alpha} D_{\mu} + D^{\alpha} D^{\mu} D_{\mu} D^{\alpha} = D^{\alpha} D^{\mu} D_{\alpha} D_{\mu} + D^{\alpha} D^{\mu} \left(D_{\alpha} D_{\mu} + G_{\alpha\mu} \right)$$

$$= 2D^{\alpha} D^{\mu} D_{\alpha} D_{\mu} + O(G^2)$$

$$= 2D^2 D^2 + 2D^{\alpha} G_{\alpha\mu} D^{\mu} + O(G^2) \qquad (3.57a)$$

which leads to the identity

$$D^{\alpha} D^{\mu} D_{\alpha} D_{\mu} + D^{\alpha} D^{\mu} D_{\mu} D_{\alpha} = 2D^2 D^2 + O(G^2) + O(Dj) \qquad (3.57b)$$

Now consider the object

$$\sigma^{\beta\nu} D_{\beta} D^2 D_{\nu} = \left(i\gamma^{\beta}\gamma^{\nu} - ig^{\beta\nu} \right) D_{\beta} D^2 D_{\nu} = i\not{D} D^2 \not{D} - iD^{\beta} D^2 D_{\beta} \qquad (3.58a)$$

Using the identity of (3.57) simplifies (3.58a).

$$\sigma^{\beta\nu} D_{\beta} D^2 D_{\nu} = i\not{D} D^2 \not{D} - iD^2 D^2 + O(G^2) + O(Dj) \qquad (3.58b)$$

The next identity required is derived with the assistance of (3.37), and can be further simplified through (3.58).

$$\sigma^{\beta\nu} D_\alpha D_\beta D^\alpha D_\nu \text{[illegible]} \left[D_\beta D^2 + D^2 D_\beta \right] D_\nu + \mathcal{O}(D/J)$$

$$\sigma^{\beta\nu} D_\alpha D_\beta D^\alpha D_\nu = \frac{1}{2}\sigma^{\beta\nu} D_\beta D^2 D_\nu - \frac{1}{4} D^2 \sigma G + \mathcal{O}(D/J) \qquad (3.59)$$

An expression similar to (3.59) is also required.

$$\sigma^{\beta\nu} D_\beta D_\alpha D_\nu D^\alpha = \frac{1}{2}\sigma^{\beta\nu} D_\beta D^2 D_\nu - \frac{1}{4}\sigma G D^2 + \mathcal{O}(D/J) \qquad (3.60)$$

The final identity is

$$D^\alpha \sigma^{\beta\nu} D_\beta D_\nu D_\alpha = -\frac{1}{2} D^\alpha \sigma G D_\alpha$$

$$= -\frac{1}{4} D^\alpha \left[D_\alpha \sigma G + [\sigma G, D_\alpha] \right] - \frac{1}{4} \left[\sigma G D_\alpha + [D_\alpha, \sigma G] \right] D_\alpha \, ,$$

$$D^\alpha \sigma^{\beta\nu} D_\beta D_\nu D_\alpha = -\frac{1}{4}\sigma G D^2 - \frac{1}{4} D^2 \sigma G + \frac{1}{4}[D_\alpha, [D^\alpha, \sigma G]] \qquad (3.61)$$

which after use of the Jacobi Identity on the last term becomes

$$D^\alpha \sigma^{\beta\nu} D_\beta D_\nu D_\alpha = -\frac{1}{4}\sigma G D^2 - \frac{1}{4} D^2 \sigma G + \mathcal{O}(G^2) + \mathcal{O}(D/J) \qquad (3.62)$$

Simplification of the set of equations in (3.56) can now be achieved by using (3.57) to (3.60), (3.62), (2.42b), and the $\not{D}\psi = -im\psi$ equation of motion, leading to

$$64A + 32B = im^2 \langle \bar{\psi} \sigma G \psi \rangle \qquad (3.63a)$$

$$16A + 80B = im^2 \langle \bar{\psi} \sigma G \psi \rangle \qquad (3.63b)$$

$$6(48)C = \frac{1}{2} m^2 \langle\bar{\psi}\sigma G\psi\rangle \qquad (3.63c)$$

where all condensates except for $\langle\bar{\psi}\sigma G\psi\rangle$ have been ignored. The three equations in (3.63) are solved and the values of A,B and C are substituted into (3.55), leading to the following result.

$$\langle\bar{\psi}_i D_{(\alpha}D_{\beta)}D_{(\mu}D_{\nu)}\psi_j\rangle = \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle \frac{im^2}{96}\delta_{ij}\left(g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\mu}g_{\beta\nu} + g_{\beta\mu}g_{\alpha\nu}\right)$$

$$+ \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle \frac{m^2}{576}\left(g_{\beta\mu}\sigma^{\alpha\nu}_{ji} + g_{\alpha\mu}\sigma^{\beta\nu}_{ji} + g_{\beta\nu}\sigma^{\alpha\mu}_{ji} + g_{\alpha\nu}\sigma^{\beta\mu}_{ji}\right) \qquad (3.64)$$

The coefficient $\langle\bar{\psi}D_{(\alpha}D_{\beta}D_{\mu}D_{\nu)}\psi\rangle$ appearing in the fourth order expansion of (3.52) will now be calculated. The algorithm of (3.51) is applied in order to write this object in terms of Lorentz invariants, with the result

$$\langle\bar{\psi}_i D_{(\alpha}D_{\beta}D_{\mu}D_{\nu)}\psi_j\rangle = A\,\delta_{ij}\left(g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu}\right) \qquad (3.65)$$

Contracting both sides of (3.65) with $g^{\alpha\beta}g^{\mu\nu}\delta_{ij}$ gives

$$\frac{1}{3}\langle\bar{\psi}\left(D^2D^2 + D^{\alpha}D_{\mu}D_{\alpha}D^{\mu} + D^{\alpha}D^2D_{\alpha}\right)\psi\rangle = 96A \qquad (3.66)$$

Using (3.57), (2.42b), the $\not{D}\psi = -im\psi$ equation of motion, and ignoring all condensates other than $\langle\bar{\psi}\sigma G\psi\rangle$, simplifies (3.66).

$$im^3\langle\bar{\psi}\sigma G\psi\rangle = 96A \qquad (3.67)$$

Substituting (3.67) into (3.65) and including the colour averaging factor leads to the following result.

$$\langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_{\nu)}\psi_j\rangle = \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle \frac{im^2}{96}\,\delta_{ij}\left\{ g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\mu\beta}\right\} \tag{3.68}$$

In order to evaluate the remaining two fourth-order coefficients in (3.52), the algorithm of (3.51) will now be explicitly applied to obtain $T_{\alpha(\mu\nu\lambda)}$.

$$\begin{aligned}
T_{\alpha(\mu\nu\lambda)} = {} & A\,\gamma_\alpha\left\{\gamma_\mu g_{\nu\lambda} + \gamma_\nu g_{\mu\lambda} + \gamma_\lambda g_{\mu\nu}\right\} \\
& + B\left\{\gamma_\mu g_{\nu\lambda} + \gamma_\nu g_{\mu\lambda} + \gamma_\lambda g_{\mu\nu}\right\}\gamma_\alpha \\
& + C\Big\{\gamma_\mu\left\{\gamma_\alpha g_{\nu\lambda} + \gamma_\nu g_{\lambda\alpha} + \gamma_\lambda g_{\beta\nu}\right\} \\
& \quad + \gamma_\nu\left\{\gamma_\alpha g_{\mu\lambda} + \gamma_\mu g_{\alpha\lambda} + \gamma_\lambda g_{\alpha\mu}\right\} \\
& \quad + \gamma_\lambda\left\{\gamma_\alpha g_{\mu\nu} + \gamma_\nu g_{\mu\alpha} + \gamma_\mu g_{\alpha\nu}\right\}\Big\} \\
& + D\Big\{\left\{\gamma_\alpha g_{\nu\lambda} + \gamma_\nu g_{\alpha\lambda} + \gamma_\lambda g_{\alpha\nu}\right\}\gamma_\mu \\
& \quad + \left\{\gamma_\alpha g_{\mu\lambda} + \gamma_\mu g_{\alpha\lambda} + \gamma_\lambda g_{\alpha\mu}\right\}\gamma_\nu \\
& \quad + \left\{\gamma_\alpha g_{\mu\nu} + \gamma_\nu g_{\alpha\mu} + \gamma_\mu g_{\alpha\nu}\right\}\gamma_\lambda\Big\}
\end{aligned} \tag{3.69}$$

Equation (3.69) can be written as a linearly independent combination of Dirac matrices by using (2.42a).

$$T_{\alpha(\mu\nu\lambda)} = A\left[g_{\alpha\mu}g_{\nu\lambda} + g_{\alpha\nu}g_{\mu\lambda} + g_{\alpha\lambda}g_{\mu\nu} \right]$$

$$+ B\left[\sigma_{\alpha\mu}g_{\nu\lambda} + \sigma_{\alpha\nu}g_{\mu\lambda} + \sigma_{\alpha\lambda}g_{\mu\nu} \right] \qquad (3.70)$$

Using the above result it is now possible to write the following fourth-order coefficient in terms of Lorentz invariant quantities.

$$\langle\bar{\psi}_i D_\alpha D_{(\mu}D_\nu D_{\lambda)}\psi_j\rangle = A\delta_{ij}\left[g_{\alpha\mu}g_{\nu\lambda} + g_{\alpha\nu}g_{\mu\lambda} + g_{\alpha\lambda}g_{\mu\nu} \right]$$

$$+ B\left[\sigma^{\alpha\mu}_{ji}g_{\nu\lambda} + \sigma^{\alpha\nu}_{ji}g_{\mu\lambda} + \sigma^{\alpha\lambda}_{ji}g_{\mu\nu} \right] \qquad (3.71)$$

Contracting (3.71) with $g^{\alpha\mu}g^{\nu\lambda}\delta^{ij}$ and $\sigma^{\alpha\mu}_{ij}g^{\nu\lambda}$ leads respectively to the following two equations.

$$\frac{1}{3}\langle\bar{\psi}\left(D^2D^2 + D_\alpha D^2D^\alpha + D_\alpha D^\mu D^\alpha D_\mu \right)\psi\rangle = 96A \qquad (3.72a)$$

$$\frac{1}{3}\langle\bar{\psi}\left(\sigma^{\alpha\mu}D_\alpha D_\mu D^2 + \sigma^{\alpha\mu}D_\alpha D^2 D_\mu + \sigma^{\alpha\mu}D_\alpha D_\lambda D_\mu D_\lambda \right)\psi\rangle = 6(48)B \qquad (3.72b)$$

The above equations are simplified through use of the identities in (3.57), (3.58), (3.60) and (2.42b), leading to the expressions

$$96A = im^2\langle\bar{\psi}\sigma G\psi\rangle \qquad (3.73a)$$

$$12(48)B = m^2\langle\bar{\psi}\sigma G\psi\rangle \; ; \qquad (3.73b)$$

where the $\not{D}\psi = -im\psi$ equation of motion has been used, and condensates other than $\langle\bar{\psi}\sigma G\psi\rangle$ have been ignored. Substituting the results of (3.73) into (3.71) and including the colour averaging factor, gives the final result for this fourth-order coefficient.

$$<\bar{\psi}_i D_\alpha D_{(\mu} D_\nu D_{\lambda)} \psi_j> = \frac{1}{3} <\bar{\psi}\sigma G\psi> \frac{im^2}{96} \delta_{ij} \left(g_{\alpha\mu} g_{\nu\lambda} + g_{\alpha\nu} g_{\mu\lambda} + g_{\alpha\lambda} g_{\mu\nu} \right)$$

$$+ \frac{1}{3} <\bar{\psi}\sigma G\psi> \frac{m^2}{576} \left(g_{\nu\lambda} \sigma^{\alpha\mu}_{ji} + g_{\mu\nu} \sigma^{\alpha\lambda}_{ji} + g_{\mu\lambda} \sigma^{\alpha\nu}_{ji} \right) \tag{3.74}$$

The only term in (3.52) which remains to be calculated is $<\bar{\psi}_i D_{(\mu} D_\nu D_{\lambda)} D_\alpha \psi_j>$, which can be written as a Lorentz invariant combination identical to (3.71). However after contraction with $g^{\alpha\mu} g^{\nu\lambda} \delta_{ij}$ and $\sigma^{\alpha\mu}_{ij} g^{\nu\lambda}$ the following equations are obtained.

$$\frac{1}{3} <\bar{\psi}(D^2 D^2 + D_\alpha D^2 D_\alpha + D_\alpha D_\mu D^\alpha D^\mu)\psi> = 96A \tag{3.75a}$$

$$-\frac{1}{3} <\bar{\psi}(\sigma^{\alpha\mu} D_\alpha D^2 D_\mu + D^2 D_\alpha D_\mu \sigma^{\alpha\mu} + \sigma^{\alpha\mu} D^\lambda D_\alpha D_\lambda D_\mu)\psi> = 6(48)B \tag{3.75b}$$

The above expressions lead to a mixed condensate component identical to that of (3.73) except for an overall negative sign appearing with the constant B, implying that

$$<\bar{\psi}_i D_{(\mu} D_\nu D_{\lambda)} D_\alpha \psi_j> = \frac{1}{3} <\bar{\psi}\sigma G\psi> \frac{im^2}{96} \delta_{ij} \left(g_{\alpha\mu} g_{\nu\lambda} + g_{\alpha\nu} g_{\mu\lambda} + g_{\alpha\lambda} g_{\mu\nu} \right)$$

$$- \frac{1}{3} <\bar{\psi}\sigma G\psi> \frac{m^2}{576} \left(g_{\nu\lambda} \sigma^{\alpha\mu}_{ji} + g_{\mu\nu} \sigma^{\alpha\lambda}_{ji} + g_{\mu\lambda} \sigma^{\alpha\nu}_{ji} \right) \tag{3.76}$$

Finally, by substituting the expressions for the coefficients of (3.76), (3.74), (3.68), and (3.64) into (3.52), the fourth order mixed condensate projection of $<\bar{\psi}(x)\psi(y)>$ is obtained.[36,38]

$$\langle\bar{\psi}_i(z)\psi_j(y)\rangle_{IV} = \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\left(\frac{im^2}{768}(y-z)^4\delta_{ij} + \frac{m^2}{1152}y^\mu z^\nu \sigma^{\mu\nu}_{ji}(y-z)^2\right) \quad (3.77)$$

Consider the fifth order terms in the covariantized Taylor expansion of $\langle\bar{\psi}(z)\psi(y)\rangle$.

$$\langle\bar{\psi}_i(z)\psi_j(y)\rangle_V = \frac{1}{120}\left(y^\alpha y^\beta y^\mu y^\nu y^\lambda - z^\alpha z^\beta z^\mu z^\nu z^\lambda\right)\langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_\nu D_{\lambda)}\psi_j\rangle$$

$$+ \frac{1}{24}y^\alpha z^\beta z^\mu z^\nu z^\lambda \langle\bar{\psi}_i D_{(\beta}D_\mu D_\nu D_{\lambda)}D_\alpha\psi_j\rangle$$

$$- \frac{1}{24}z^\alpha y^\beta y^\mu y^\nu y^\lambda \langle\bar{\psi}_i D_\alpha D_{(\beta}D_\mu D_\nu D_{\lambda)}\psi_j\rangle$$

$$- \frac{1}{12}y^\alpha y^\beta z^\mu z^\nu z^\lambda \langle\bar{\psi}_i D_{(\mu}D_\nu D_{\lambda)}D_{(\alpha}D_{\beta)}\psi_j\rangle$$

$$+ \frac{1}{12}z^\alpha z^\beta y^\mu y^\nu y^\lambda \langle\bar{\psi}_i D_{(\alpha}D_{\beta)}D_{(\mu}D_\nu D_{\lambda)}\psi_j\rangle \quad (3.78)$$

The expressions for $T_{(\alpha\beta\mu\nu\lambda)}$ and $T_{\alpha(\beta\mu\nu\lambda)}$ in terms of Lorentz invariant quantities are easily obtained.

$$\langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_\nu D_{\lambda)}\psi_j\rangle = A\left[\gamma^\alpha_{ji}\left(g_{\beta\mu}g_{\nu\lambda} + g_{\beta\nu}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\nu}\right)\right.$$

$$+ \gamma^\beta_{ij}\left(g_{\alpha\mu}g_{\nu\lambda} + g_{\alpha\nu}g_{\mu\lambda} + g_{\alpha\lambda}g_{\mu\nu}\right)$$

$$+ \gamma^\mu_{ij}\left(g_{\alpha\beta}g_{\nu\lambda} + g_{\beta\nu}g_{\alpha\lambda} + g_{\beta\lambda}g_{\alpha\nu}\right)$$

$$+ \gamma^\nu_{ij}\left(g_{\beta\mu}g_{\alpha\lambda} + g_{\beta\alpha}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\alpha}\right)$$

$$\left.+ \gamma^\nu_{ij}\left(g_{\beta\mu}g_{\alpha\lambda} + g_{\beta\alpha}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\alpha}\right)\right]. \quad (3.79)$$

$$\langle\bar{\psi}_i D_\alpha D_{(\beta} D_\mu D_\nu D_{\lambda)} \psi_j\rangle = B\,\gamma^\alpha_{ij}\left(g_{\beta\mu}g_{\nu\lambda} + g_{\beta\nu}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\nu}\right)$$

$$+ C\left\{\gamma^\beta_{ji}\left(g_{\alpha\mu}g_{\nu\lambda} + g_{\alpha\nu}g_{\mu\lambda} + g_{\alpha\lambda}g_{\mu\nu}\right)\right.$$

$$+ \gamma^\mu_{ji}\left(g_{\beta\alpha}g_{\nu\lambda} + g_{\beta\nu}g_{\alpha\lambda} + g_{\beta\lambda}g_{\alpha\nu}\right)$$

$$+ \gamma^\nu_{ji}\left(g_{\beta\mu}g_{\alpha\lambda} + g_{\beta\alpha}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\alpha}\right)$$

$$\left. + \gamma^\lambda_{ji}\left(g_{\beta\mu}g_{\nu\alpha} + g_{\beta\nu}g_{\mu\alpha} + g_{\beta\alpha}g_{\mu\nu}\right)\right\} \qquad (3.80)$$

Obtaining the expression for $T_{(\alpha\beta)(\mu\nu\lambda)}$ requires the use of the algorithm (3.51) and equation (3.55).

$$T_{(\alpha\beta)(\mu\nu\lambda)} = A\,g_{\alpha\beta}\left(\gamma_\lambda g_{\mu\nu} + \gamma_\mu g_{\nu\lambda} + \gamma_\nu g_{\mu\lambda}\right)$$

$$+ B\left\{\gamma_\lambda\left(g_{\alpha\mu}g_{\beta\nu} + g_{\beta\mu}g_{\alpha\nu}\right) + \gamma_\mu\left(g_{\alpha\lambda}g_{\beta\nu} + g_{\beta\lambda}g_{\alpha\nu}\right)\right.$$

$$\left. + \gamma_\nu\left(g_{\alpha\mu}g_{\beta\lambda} + g_{\beta\mu}g_{\alpha\lambda}\right)\right\}$$

$$+ C\left\{\gamma_\lambda\left(g_{\alpha\mu}\sigma_{\beta\nu} + g_{\alpha\nu}\sigma_{\beta\mu} + g_{\beta\mu}\sigma_{\alpha\nu} + g_{\beta\nu}\sigma_{\alpha\mu}\right)\right.$$

$$+ \gamma_\mu\left(g_{\alpha\lambda}\sigma_{\beta\nu} + g_{\alpha\nu}\sigma_{\beta\lambda} + g_{\beta\lambda}\sigma_{\alpha\nu} + g_{\beta\nu}\sigma_{\alpha\lambda}\right)$$

$$\left. + \gamma_\nu\left(g_{\alpha\lambda}\sigma_{\beta\mu} + g_{\alpha\mu}\sigma_{\beta\lambda} + g_{\beta\lambda}\sigma_{\alpha\mu} + g_{\beta\mu}\sigma_{\alpha\lambda}\right)\right\}$$

$$+ D\left\{\gamma\text{'s on other side in } C\right\}$$

$$+ E\left[\gamma_\alpha\left(g_{\lambda\beta}g_{\mu\nu} + g_{\mu\beta}g_{\lambda\nu} + g_{\nu\beta}g_{\mu\lambda}\right)\right.$$

$$\left. + \gamma_\beta\left(g_{\lambda\alpha}g_{\mu\nu} + g_{\mu\alpha}g_{\lambda\nu} + g_{\nu\alpha}g_{\mu\lambda}\right)\right]$$

$$+ F\left[\gamma_\alpha\left(g_{\lambda\mu}g_{\beta\nu} + g_{\beta\mu}g_{\lambda\nu} + g_{\beta\lambda}g_{\mu\nu}\right)\right.$$

$$\left. + \gamma_\beta\left(g_{\lambda\mu}g_{\alpha\nu} + g_{\alpha\mu}g_{\lambda\nu} + g_{\alpha\lambda}g_{\mu\nu}\right)\right]$$

$$+ G\left[\gamma_\alpha\left(g_{\lambda\mu}\sigma_{\beta\nu} + g_{\lambda\nu}\sigma_{\beta\mu} + g_{\mu\nu}\sigma_{\beta\lambda}\right)\right.$$

$$\left. + \gamma_\beta\left(g_{\lambda\mu}\sigma_{\alpha\nu} + g_{\lambda\nu}\sigma_{\alpha\mu} + g_{\mu\nu}\sigma_{\alpha\lambda}\right)\right]$$

$$+ H\left(\gamma\text{'s on other side in } G\right) \tag{3.81}$$

The following Dirac matrix identity allows (3.81) to be written as a linearly independent combination of Dirac matrices.

$$\gamma_\alpha\sigma_{\mu\beta} + \gamma_\beta\sigma_{\mu\alpha} = -2ig_{\alpha\beta}\gamma_\mu + i\left(\gamma_\alpha g_{\beta\mu} + \gamma_\beta g_{\alpha\mu}\right) \tag{3.82}$$

$$\langle\bar{\psi}_i D_{(\alpha}D_{\beta)}D_{(\mu}D_\nu D_{\lambda)}\psi_j\rangle = D\, g_{\alpha\beta}\left(\gamma^\lambda_{ji}g_{\mu\nu} + \gamma^\mu_{ji}g_{\nu\lambda} + \gamma^\nu_{ji}g_{\mu\lambda}\right)$$

$$+ E\left[\gamma^\lambda_{ji}\left(g_{\alpha\mu}g_{\beta\nu} + g_{\beta\mu}g_{\alpha\nu}\right) + \gamma^\mu_{ji}\left(g_{\alpha\lambda}g_{\beta\nu} + g_{\beta\lambda}g_{\alpha\nu}\right)\right.$$

$$\left. + \gamma^\nu_{ji}\left(g_{\alpha\mu}g_{\beta\lambda} + g_{\mu\beta}g_{\alpha\lambda}\right)\right]$$

$$+ F\left[\gamma^\alpha_{ji}\left(g_{\lambda\beta}g_{\mu\nu} + g_{\mu\beta}g_{\lambda\nu} + g_{\nu\beta}g_{\mu\lambda}\right)\right.$$

$$+ \gamma^{\beta}_{ji} \left(g_{\lambda\alpha} g_{\mu\nu} + g_{\mu\alpha} g_{\lambda\nu} + g_{\nu\alpha} g_{\mu\lambda} \right) \Bigg] \tag{3.83}$$

Equation (3.79) is now contracted with $\gamma^{\alpha}_{ij} g^{\beta\mu} g^{\nu\lambda}$ to determine A.

$$16(48)A = \frac{1}{15} \langle \bar{\psi} \Big[\not{D} D^2 D^2 + \not{D} D_\mu D^2 D_\mu + \not{D} D_\mu D_\nu D_\mu D_\nu + D_\mu \not{D} D_\mu D^2$$

$$+ D_\mu \not{D} D_\nu D_\mu D_\nu + D_\mu \not{D} D^2 D_\mu + D_\mu D^2 \not{D} D_\mu + D^2 D_\mu \not{D} D_\mu$$

$$+ D_\nu D_\mu D_\nu \not{D} D_\mu + D^2 \not{D} D^2 + D_\nu D_\mu \not{D} D_\mu D_\nu + D_\mu D_\nu \not{D} D_\mu D_\nu$$

$$+ D^2 D^2 \not{D} + D_\mu D^2 D_\mu \not{D} + D_\mu D_\nu D_\mu D_\nu \not{D} \Big] \psi \rangle \tag{3.84}$$

The identities of (2.42b), (3.37) and (3.57) allow the mixed condensate component to be extracted from most of the terms appearing in (3.84). The only exceptions are the following quantitites.

$$\langle \bar{\psi} D_\mu \not{D} D^2 D_\mu \psi \rangle = \langle \bar{\psi} \left(\not{D} D_\mu + \gamma^\lambda G_{\lambda\mu} \right) D^2 D_\mu \psi \rangle$$

$$= \langle \bar{\psi} \not{D} D_\mu D^2 D_\mu \psi \rangle - \langle \bar{\psi} \frac{1}{2} \left(\not{D} D^2 - D^2 \not{D} \right) D^2 \psi \rangle$$

$$= m^3 \langle \bar{\psi} \sigma G \psi \rangle + \text{other condensates} \tag{3.85a}$$

$$\langle \bar{\psi} D_\mu D^2 \not{D} D_\mu \psi \rangle = m^3 \langle \bar{\psi} \sigma G \psi \rangle + \text{other condensates} \tag{3.85b}$$

$$\langle \bar{\psi} D_\mu D_\nu \not{D} D_\mu D_\nu \psi \rangle = \langle \bar{\psi} D_\mu D_\nu \not{D} D_\nu D_\mu \psi \rangle + \langle \bar{\psi} D_\mu D_\nu \not{D} G_{\nu\mu} \psi \rangle$$

$$= \langle \bar{\psi} \frac{1}{2} D_\mu \left(\not{D} D^2 + D^2 \not{D} \right) D_\mu \psi \rangle + \frac{1}{2} \langle \bar{\psi} G_{\nu\mu} \not{D} G_{\nu\mu} \psi \rangle$$

$$\langle\bar{\psi}D_\mu D_\nu \not{D} D_\mu D_\nu \psi\rangle = m^3\langle\bar{\psi}\sigma G\rangle \quad + \text{ other condensates} \tag{3.86}$$

Simplification of (3.84) by using (3.85), (3.86), (2.42b), (3.37), (3.57), the $\not{D}\psi=-im\psi$ equation of motion, and ignoring all condensates except for $\langle\bar{\psi}\sigma G\psi\rangle$, leads to the following expression.

$$m^3\langle\bar{\psi}\sigma G\psi\rangle = A(96)(8) \tag{3.87}$$

Substituting (3.87) into (3.79) and including the colour averaging factor gives the result

$$\begin{aligned}\langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_\nu D_{\lambda)}\psi_j\rangle = \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m^3}{768}\Big[& \gamma^\alpha_{ji}\left(g_{\beta\mu}g_{\nu\lambda} + g_{\beta\nu}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\nu}\right) \\ & + \gamma^\beta_{ji}\left(g_{\alpha\mu}g_{\nu\lambda} + g_{\alpha\nu}g_{\mu\lambda} + g_{\alpha\lambda}g_{\mu\nu}\right) \\ & + \gamma^\mu_{ji}\left(g_{\beta\alpha}g_{\nu\lambda} + g_{\beta\nu}g_{\lambda\alpha} + g_{\beta\lambda}g_{\alpha\nu}\right) \\ & + \gamma^\nu_{ji}\left(g_{\beta\mu}g_{\alpha\lambda} + g_{\beta\alpha}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\alpha}\right) \\ & + \gamma^\lambda_{ji}\left(g_{\beta\mu}g_{\nu\alpha} + g_{\beta\nu}g_{\mu\alpha} + g_{\beta\alpha}g_{\mu\nu}\right)\Big] .\end{aligned} \tag{3.88}$$

Contracting (3.80) with the objects $\gamma^\alpha_{ij}g^{\beta\mu}g^{\nu\lambda}$ and $\gamma^\beta_{ij}g^{\alpha\mu}g^{\nu\lambda}$ leads to the following two equations for B and C.

$$\frac{1}{3}\langle\bar{\psi}\not{D}\left(D_\mu D^2 D_\mu + D^2D^2 + D_\mu D_\nu D_\mu D_\nu\right)\psi\rangle = 4(96)B + 4(96)C \tag{3.89a}$$

$$\frac{1}{12}\langle\bar{\psi}D_\mu\Big(\not{D}D_\mu D^2 + \not{D}D_\nu D_\mu D_\nu + \not{D}D^2 D_\mu + D_\mu\not{D}D^2 + D_\nu\not{D}D_\mu D_\nu$$

$$+ D_\nu\not{D}D_\nu D_\mu + D_\mu D^2\not{D} + D^2 D_\mu\not{D} + D_\nu D_\mu D_\nu\not{D} + D_\mu D_\nu\not{D}D_\nu$$

$$+ D_\nu D_\mu\not{D}D_\nu + D^2\not{D}D_\mu\Big)\psi\rangle = 96B + 7(96)C \qquad (3.89b)$$

The identities required to extract the mixed condensate component of (3.89) are given in (3.85), (3.86), (2.42b), (3.37) and (3.87); leading to the following values for B and C.

$$B = C = \frac{m^3}{768}\langle\bar{\psi}\sigma G\psi\rangle \qquad (3.90)$$

Substituting the values of B and C from (3.90) into (3.80) gives a completely symmetric expression identical to (3.88), implying that

$$\langle\bar{\psi}_i D_\alpha D_{(\beta}D_\mu D_\nu D_{\lambda)}\psi_j\rangle = \langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_\nu D_{\lambda)}\psi_j\rangle \qquad (3.91)$$

where only the mixed condensate projection is considered. A similar analysis for $\langle\bar{\psi}D_{(\beta}D_\mu D_\nu D_{\lambda)}D_\alpha\psi\rangle$ leads to an identical result for the mixed condensate projection.

$$\langle\bar{\psi}_i D_{(\beta}D_\mu D_\nu D_{\lambda)}D_\alpha\psi_j\rangle = \langle\bar{\psi}_i D_\alpha D_{(\beta}D_\mu D_\nu D_{\lambda)}\psi_j\rangle = \langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_\nu D_{\lambda)}\psi_j\rangle \; . \qquad (3.92)$$

Contracting (3.83) with $g^{\alpha\beta}\gamma^\lambda_{ij}g^{\mu\nu}$, $\gamma^\lambda_{ij}g^{\alpha\mu}g^{\beta\nu}$ and $\gamma^\alpha_{ij}g^{\lambda\beta}g^{\mu\nu}$ leads in turn to the following three equations for D, E and F.

$$\frac{1}{3}\langle\bar{\psi}D^2\left[\not{D}D^2 + D^2\not{D} + D_\nu\not{D}D_\nu\right]\psi\rangle = 4(96)D + 2(96)E + 2(96)F \qquad (3.93a)$$

$$\frac{i}{6}\langle\bar{\psi}\left[D_\mu D_\nu D_\mu D_\nu\not{D} + D_\mu D^2 D_\mu\not{D} + D_\mu D_\nu\not{D}D_\mu D_\nu + D_\mu D_\nu\not{D}D_\nu D_\mu\right.$$

$$\left. + D_\mu D_\nu D_\mu\not{D}D_\nu + D_\mu D^2\not{D}D_\mu\right]\psi\rangle = 96D + 5(96)E + 2(96)F \qquad (3.93b)$$

$$\frac{1}{3}\langle\bar{\psi}\not{D}\left(D_\mu D_\nu D_\mu D_\nu + D_\mu D^2 D_\mu + D^2D^2\right)\psi\rangle + \frac{1}{3}\langle\bar{\psi}D^\lambda\not{D}\left(D^2D_\lambda + D_\lambda D^2 + D_\mu D_\lambda D_\mu\right)\psi\rangle$$
$$= 96D + 2(96)E + 5(96)F \qquad (3.93c)$$

The mixed condensate projection of equations (3.93) is extracted through use of the $\not{D}\psi=-im\psi$ equation of motion and identities (2.42b), (3.85), (3.37) and (3.57).

$$\frac{m^3}{96}\langle\bar{\psi}\sigma G\psi\rangle = 4D+2E + 2F = D + 5E + 2F = D + 2E + 5F \qquad (3.94)$$

Solving (3.94) for D, E and F and substituting into (3.83) leads to a completely symmetric expression identical to (3.88), implying that

$$\langle\bar{\psi}_i D_{(\alpha}D_{\beta)}D_{(\mu}D_\nu D_{\lambda)}\psi_j\rangle = \langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_\nu D_{\lambda)}\psi_j\rangle \qquad (3.95)$$

where only the mixed condensate projection has been considered. A result identical to (3.95) is obtained for $\langle\bar{\psi}_i D_{(\mu}D_\nu D_{\lambda)}D_{(\alpha}D_{\beta)}\psi_j\rangle$.

Collecting the results of (3.95), (3.92) and (3.88), leads to the following result for the mixed condensate projection of the fifth order coefficients of (3.78).

$$\langle\bar{\psi}_i D_{(\alpha}D_\beta D_\mu D_\nu D_{\lambda)}\psi_j\rangle = \langle\bar{\psi}_i D_\alpha D_{(\beta}D_\mu D_\nu D_{\lambda)}\psi_j\rangle$$

$$= \langle\bar{\psi}_i D_{(\beta}D_\mu D_\nu D_{\lambda)}D_\alpha\psi_j\rangle = \langle\bar{\psi}_i D_{(\alpha}D_{\beta)}D_{(\mu}D_\nu D_{\lambda)}\psi_j\rangle$$

$$= \langle\bar{\psi}_i D_{(\mu}D_\nu D_{\lambda)}D_{(\alpha}D_{\beta)}\psi_j\rangle = \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m^3}{768}\Big(\gamma^\alpha_{ji}\left[g_{\beta\mu}g_{\nu\lambda} + g_{\beta\nu}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\nu}\right]$$

$$+ \gamma^\beta_{ji}\left[g_{\alpha\mu}g_{\nu\lambda} + g_{\alpha\nu}g_{\mu\lambda} + g_{\alpha\lambda}g_{\mu\nu}\right]$$

$$+ \gamma^\mu_{ji}\left[g_{\beta\alpha}g_{\nu\lambda} + g_{\beta\nu}g_{\lambda\alpha} + g_{\beta\lambda}g_{\alpha\nu}\right]$$

$$+ \gamma^\nu_{ji}\left[g_{\beta\mu}g_{\alpha\lambda} + g_{\beta\alpha}g_{\mu\lambda} + g_{\beta\lambda}g_{\mu\alpha}\right]$$

$$+ \gamma^\lambda_{ji}\left[g_{\beta\mu}g_{\alpha\nu} + g_{\beta\nu}g_{\mu\alpha} + g_{\beta\alpha}g_{\mu\nu}\right]\Big) \qquad (3.96)$$

Finally, substitution of (3.96) into (3.78) leads to the mixed condensate projection of the fifth order expansion of $\langle\bar{\psi}(z)\psi(y)\rangle$.[36,38]

$$\langle\bar{\psi}_i(z)\psi_j(y)\rangle_V = \frac{1}{3}\langle\bar{\psi}\sigma G\psi\rangle\frac{m^3}{6144}\gamma_{ji}\cdot(y-z)(y-z)^4 \qquad (3.97)$$

Collecting the results of (3.97), (3.77), (3.49) and (3.33), gives the mixed condensate projection of $\langle\bar{\psi}(z)\psi(y)\rangle$ valid to fifth order in its covariantized Taylor expansion.[36,38]

$$\langle\bar{\psi}_i(z)\psi_j(y)\rangle = \frac{1}{288}\langle\bar{\psi}\sigma G\psi\rangle\Big(\Big[-\frac{3}{2}i(y-z)^2\delta_{ij} - z^\nu y^\mu\sigma^{\mu\nu}_{ji}\Big]$$

$$-\frac{m}{4}\gamma_{ji}\cdot(y-z)(y-z)^2$$

$$+ m^2 \left[\frac{i}{8} (y-z)^4 \delta_{ij} + \frac{1}{12} (y-z)^2 y^\mu z^\nu \sigma^{\mu\nu}_{ji} \right]$$

$$+ m^3 \left[\frac{1}{64} \gamma_{ji} \cdot (y-z)(y-z)^4 \right] + \text{higher order terms} \Bigg] \quad (3.98)$$

Equation (3.98) suggests that the general form for the $\langle\bar{\psi}\sigma G\psi\rangle$ projection of the quark-quark vev is

$$\langle\bar{\psi}(z)\psi(y)\rangle = \frac{1}{288} \langle\bar{\psi}\sigma G\psi\rangle \left\{ \sum_{n=2}^{\infty} m^{n-2} a_n \left(\gamma \cdot (y-z)\right)^n + \sum_{n=0}^{\infty} m^{2n} b_{2n} y^\mu z^\nu \sigma_{\mu\nu} (y-z)^{2n} \right] \quad (3.99)$$

This expression may be justified by observing that a general tensor $T_{(\alpha\ldots\mu)(\nu\ldots\omega)}$ may be written as a completely symmetric part, plus pieces with anti-symmetries between two or more indices. An anti-symmetry between more than two indices necessarily generates more than one field strength, leading to condensates of dimension larger than five. By construction, the portions of T that are anti-symmetric in one pair of indices are contained in the σ matrices, generating the field strength required for the mixed condensate.

2 of/de 2

3-5 Mixed Condensate Projection of the Quark-Quark-Gluon Vacuum Expectation Value

In this section the mixed condensate projection of the non-perturbative vev $(ig/2)\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ will be calculated to third order in its covariantized Taylor series. This object appears in the amplitudes of (3.28) to (3.30), which are represented in Figures 4 to 6.

Consider the mixed condensate component of the vev $(ig/2)\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$. The gluon field in this quantity can be expanded in a covariantized Taylor series about w=0 (see 3.3),

$$A^a_\mu(w) = \frac{1}{2}w^\lambda F^a_{\lambda\mu}(0) + \frac{1}{3}w^\lambda w^\sigma [D_{(\sigma},F^a_{\lambda)\mu}] + \ldots \qquad (3.100)$$

where all field strengths and covariant derivatives are evaluated at w=0. Only the lead term in this expression can contribute to the mixed condensate, since equations of motion will reduce the commutators to combinations of field strengths and currents contributing to higher-dimensional condensates. This point is actually non-trivial, since in the covariantized Taylor series of the quark-gluon-quark vev, contractions such as $\langle\bar{\psi}D\ldots[D_\alpha,\sigma G]D_\alpha D\ldots\rangle$ would seem to lead to a mixed condensate projection. However, use of the identity (3.61) reduces this quantity to

$$\langle\bar{\psi}D\ldots[D_\alpha,\sigma G]D_\alpha D\ldots\psi\rangle = \frac{1}{2}\langle\bar{\psi}D\ldots\left(D^2\sigma G - \sigma G D^2\right)D\ldots\psi\rangle \qquad (3.101)$$

which fails to yield a mixed condensate projection, a result which has been verified by explicit calculation for several low-order terms.

It is convenient to write the vev $(ig/2)\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ in a slightly different form. The behaviour of this vev under a global SU(3) transformation implies that

$$\frac{ig}{2}\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle = \frac{1}{16}\lambda^a\langle\bar{\psi}(z)A_\mu(w)\psi(y)\rangle \qquad (3.102)$$

This expression is easily verified by contracting both sides with λ^a and using the definitions $Tr(\lambda^a\lambda^a) = 16$ and $A_\mu = (ig/2)\lambda^a A^a_\mu$.

Expanding $(ig/2)\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ in a series about $y=z=w=0$, including only the lowest-order term from (3.100), and taking (3.102) into account, leads to a covariantized Taylor series.

$$\frac{ig}{2}\langle\bar{\psi}_i(z)A^a_\mu(w)\psi_j(y)\rangle = \frac{1}{16}\lambda^a\Big(\frac{1}{2}w^\lambda\langle\bar{\psi}_i G_{\lambda\mu}\psi_j\rangle + \frac{1}{2}w^\lambda z^\alpha\langle\bar{\psi}_i\overleftarrow{D}_\alpha G_{\lambda\mu}\psi_j\rangle$$
$$+ \frac{1}{2}w^\lambda y^\alpha\langle\bar{\psi}_i G_{\lambda\mu}D_\alpha\psi_j\rangle + \frac{1}{4}z^\alpha z^\beta w^\lambda\langle\bar{\psi}_i\overleftarrow{D}_{(\alpha}\overleftarrow{D}_{\beta)}G_{\lambda\mu}\psi_j\rangle$$
$$+ \frac{1}{4}y^\alpha y^\beta w^\lambda\langle\bar{\psi}_i G_{\lambda\mu}D_{(\alpha}D_{\beta)}\psi_j\rangle + \frac{1}{2}z^\alpha w^\lambda y^\beta\langle\bar{\psi}_i\overleftarrow{D}_\alpha G_{\lambda\mu}D_\beta\psi_j\rangle + \ldots\Big)$$

+ contributions leading to higher dimension condensates. (3.103)

A general covariant derivative in (3.103) can then be integrated by parts.

$$\langle\bar{\psi}\overleftarrow{D}\overleftarrow{D}\ldots\overleftarrow{D}_\alpha G_{\lambda\mu}D\ldots\psi\rangle = -\langle\bar{\psi}\overleftarrow{D}\overleftarrow{D}\ldots\overleftarrow{D}[D_\alpha, G_{\lambda\mu}]D\ldots\psi\rangle - \langle\bar{\psi}\overleftarrow{D}\ldots\overleftarrow{D}G_{\lambda\mu}D_\alpha D\ldots\psi\rangle \qquad (3.104)$$

Equation (3.104) is based upon the Poincare invariance of a gauge covariant quantity written in the fixed-point gauge (see Section 2-3).

Using (3.104), and ignoring the commutators which act upon field strengths leading to higher dimensional condensates, simplifies (3.103).

$$\frac{ig}{2}\langle\bar{\psi}_i(z)A^a_\mu(w)\psi_j(y)\rangle = \frac{1}{16}\lambda^a\Bigg[\frac{1}{2}w^\lambda\langle\bar{\psi}_i G_{\lambda\mu}\psi_j\rangle + \frac{1}{2}w^\lambda(y-z)^\alpha\langle\bar{\psi}_i G_{\lambda\mu}D_\alpha\psi_j\rangle$$

$$+\frac{1}{4}w^\lambda\left[z^\alpha z^\beta + y^\alpha y^\beta\right]\langle\bar{\psi}_i G_{\lambda\mu}D_{(\alpha}D_{\beta)}\psi_j\rangle$$

$$-\frac{1}{2}w^\lambda z^\alpha y^\beta\langle\bar{\psi}_i D_\alpha G_{\lambda\mu}D_\beta\psi_j\rangle + \ldots\Bigg]$$

$$+ \text{contributions leading to higher dimension condensates} \qquad (3.105)$$

The evaluation of the mixed condensate projection of the quark-gluon-quark vev begins with a consideration of the lowest order term in (3.105). The coefficient $\langle\bar{\psi}G_{\lambda\mu}\psi\rangle$ is proportional to the following Lorentz invariant quantity.

$$\langle\bar{\psi}_i G_{\lambda\mu}\psi_j\rangle = A\,\sigma^{\lambda\mu}_{ji} \qquad (3.106)$$

Contracting both sides of (3.106) with $\sigma^{\lambda\mu}_{ij}$ allows the constant A to be determined.

$$\langle\bar{\psi}\sigma G\psi\rangle = A(48) \qquad (3.107)$$

Thus the lowest-order mixed condensate projection of the quark-gluon-quark vev is given by[37,48]

$$\frac{ig}{2}\langle\bar{\psi}_i(z)A^a_\mu(w)\psi_j(y)\rangle_{LO} = \frac{1}{1536}\lambda^a w^\lambda\sigma^{\lambda\mu}_{ji}\langle\bar{\psi}\sigma G\psi\rangle \qquad (3.108)$$

where the notation LO designates lowest order, and I,II etc. will denote first, second and higher order terms.

Now consider the quantity $\langle\bar{\psi}_i G_{\lambda\mu} D_\alpha \psi_j\rangle$ which is the coefficient of the linear term in (3.105). As usual the vev is Poincare invariant and is therefore written as a combination of Lorentz invariant quantities satisfying the appropriate symmetry requirements.

$$\langle\bar{\psi}_i G_{\lambda\mu} D_\alpha \psi_j\rangle = A\left(\sigma^{\lambda\mu}\gamma^{\alpha}\right)_{ji} + B\left\{\gamma^{\lambda}_{ji} g_{\mu\alpha} - \gamma^{\mu}_{ji} g_{\alpha\lambda}\right\} \tag{3.109}$$

Contracting (3.109) with $(\sigma^{\lambda\mu}\gamma^{\alpha})_{ij}$ and $\gamma^{\lambda}_{ij} g^{\mu\alpha}$ respectively leads to the two equations for the constants A, B.

$$-im\langle\bar{\psi}\sigma G\psi\rangle = +2i(48)B \tag{3.110a}$$

$$\langle\bar{\psi}(D_\mu \not{D} D_\mu - \not{D}D^2)\psi\rangle = (48)iA + (48)B \tag{3.110b}$$

The identities (3.37) and (2.42b), along with $\not{D}\psi = -im\psi$ are used to extract the mixed condensate component of (3.110b), leading to the expression

$$0 = iA + B \tag{3.111}$$

Solving (3.110a) and (3.111) for A, B and substituting into (3.109) gives an expression for the $\langle\bar{\psi}\sigma G\psi\rangle$ component of $\langle\bar{\psi}_i G_{\lambda\mu} D_\alpha \psi_j\rangle$.

$$\langle\bar{\psi}_i G_{\lambda\mu} D_\alpha \psi_j\rangle = -\frac{1}{96} m\langle\bar{\psi}\sigma G\psi\rangle \left[i\sigma^{\lambda\mu}_{jk}\gamma^{\alpha}_{ki} + \left(\gamma^{\lambda}_{ji} g^{\mu\alpha} - \gamma^{\mu}_{ji} g^{\alpha\lambda}\right)\right] \tag{3.112}$$

Thus the first order mixed-condensate projection of $\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ is obtained by substituting (3.112) into (3.105).[37,48]

$$\frac{ig}{2}\langle\bar{\psi}_i(z)A^a_\mu(w)\psi_j(y)\rangle_I = \frac{1}{3072}m\langle\bar{\psi}\sigma G\psi\rangle w^\lambda \lambda^a$$

$$\times\left\{ i\sigma^{\mu\lambda}_{jk}\gamma_{ki}\cdot(y-z) + (y-z)^\alpha\left(\gamma^\mu_{ji}g^{\alpha\lambda} - \gamma^\lambda_{ji}g^{\alpha\mu}\right)\right\} . \qquad (3.113)$$

Now consider the object $\langle\bar{\psi}_i G_{\lambda\mu}D_\alpha D_\beta\psi_j\rangle$ which contributes to the second order terms in (3.105). In accordance with the usual procedures, the vev is written as a linear combination of Lorentz invariant quantitites.

$$\langle\bar{\psi}_i D_\alpha G_{\lambda\mu} D_\beta \psi_j\rangle = A\, g_{\alpha\beta}\sigma^{\lambda\mu}_{ji} + B\,\sigma^{\lambda\mu}_{jk}\sigma^{\alpha\beta}_{ki} - C\left(g_{\beta\lambda}g_{\alpha\mu} - g_{\mu\beta}g_{\alpha\lambda}\right)\delta_{ij}$$

$$+ D\left(\sigma^{\lambda\beta}_{ji}g^{\alpha\mu} - \sigma^{\mu\beta}_{ji}g^{\alpha\lambda}\right) + E\left(\sigma^{\alpha\lambda}_{ji}g^{\beta\mu} - \sigma^{\alpha\mu}_{ji}g^{\beta\lambda}\right) \qquad (3.114)$$

Contracting (3.114) with $g^{\alpha\beta}\sigma^{\lambda\mu}_{ij}$, $(\gamma^\alpha\sigma^{\lambda\mu}\gamma^\beta)_{ij}$, $g^{\beta\lambda}g^{\alpha\mu}\delta_{ij}$, $\sigma^{\lambda\beta}_{ij}g_{\alpha\mu}$, $\sigma^{\alpha\lambda}_{ij}g^{\beta\mu}$ in turn leads to five linear equations for the unknown constants.

$$\langle\bar{\psi}D_\alpha\sigma G D_\alpha\psi\rangle = 48(4A + 2D - 2E) \qquad (3.115a)$$

$$\langle\bar{\psi}\not{D}\sigma G\not{D}\psi\rangle = 48(2iC - 6D + 6E) \qquad (3.115b)$$

$$\langle\bar{\psi}D_\mu G_{\beta\mu}D_\beta\psi\rangle = 48(-B - C) \qquad (3.115c)$$

$$\langle\bar{\psi}D_\mu\sigma^{\lambda\beta}G_{\lambda\mu}D_\beta\psi\rangle = 48(A + 2iB + 3D - E) \qquad (3.115d)$$

$$\langle\bar{\psi}\sigma^{\alpha\lambda}D_{\alpha}G_{\lambda\mu}D_{\mu}\psi\rangle = 48(-A + 2iB - D + 3E) \qquad (3.115e)$$

In obtaining (3.115) the value of $Tr(\sigma^{\lambda\beta}\sigma_{\lambda\mu}\sigma^{\mu\beta}) = 2i(48)$ is required. The mixed condensate projections of (3.115) are obtained by writing $G_{\mu\nu} = [D_{\nu}, D_{\mu}]$, and then employing identities (2.42b), (3.59) and (3.37) along with $\not{D}\psi = -im\psi$, leading to a set of linear equations.

$$-\frac{m^2}{48}\langle\bar{\psi}\sigma G\psi\rangle = 4A + 2D - 2E \qquad (3.116a)$$

$$-\frac{m^2}{48}\langle\bar{\psi}\sigma G\psi\rangle = 2iC - 6D + 6E \qquad (3.116b)$$

$$0 = B + C \qquad (3.116c)$$

$$0 = A + 2iB + 3D - E, \qquad (3.116d)$$

$$0 = -A + 2iB - D + 3E \qquad (3.116e)$$

The values for A to E are found by solving (3.116) and are then substituted into (3.114), with the result

$$\langle\bar{\psi}_i D_{\alpha}G_{\lambda\mu}D_{\beta}\psi_j\rangle = -\frac{m^2}{96}\langle\bar{\psi}\sigma G\psi\rangle\left[\frac{2}{3}g_{\alpha\beta}\sigma^{\lambda\mu}_{ji} + \frac{1}{6}\left(-\sigma^{\lambda\beta}_{ji}g^{\alpha\mu} + \sigma^{\mu\beta}_{ji}g^{\alpha\lambda} + \sigma^{\alpha\lambda}_{ji}g^{\beta\mu} - \sigma^{\alpha\mu}_{ji}g^{\beta\lambda}\right)\right]. \qquad (3.117)$$

Now consider the object $\langle\bar{\psi}_i G_{\lambda\mu}D_{(\alpha}D_{\beta)}\psi_j\rangle$ which is a coefficient of a second-order term in (3.105). This object is now written as a linear combination of Lorentz invariant quantities.

$$\langle\bar{\psi}_i G_{\lambda\mu} D_{(\alpha} D_{\beta)} \psi_j\rangle = A\, \sigma^{\lambda\mu}_{ji} g_{\alpha\beta}$$

$$+ B \left[\sigma^{\lambda\alpha}_{ji} g_{\mu\beta} + \sigma^{\lambda\beta}_{ji} g_{\mu\alpha} + \sigma^{\alpha\mu}_{ji} g_{\lambda\beta} + \sigma^{\beta\mu}_{ji} g_{\lambda\alpha} \right] \quad (3.118)$$

Contracting (3.118) with $\sigma^{\lambda\mu}_{ij} g^{\alpha\beta}$ and $\sigma^{\beta\mu}_{ij} g^{\alpha\lambda}$ leads respectively to the following two equations.

$$\langle\bar{\psi}\sigma G D^2 \psi\rangle = 4(48)(A + B) \quad (3.119a)$$

$$\tfrac{1}{2}\langle\bar{\psi}\sigma^{\beta\mu} G_{\lambda\mu} (D_\lambda D_\beta + D_\beta D_\lambda)\psi\rangle = 48(A + 4B) \quad (3.119b)$$

The mixed condensate projections of (3.119b) are obtained by writing $G_{\lambda\mu} = D_\mu D_\lambda - D_\lambda D_\mu$ and then using the identities (3.58), (3.59), (3.60) and (3.61). The identity (2.42b) and the $\not{D}\psi = -im\psi$ equation of motion then simplify (3.119).

$$-\frac{m^2}{192}\langle\bar{\psi}\sigma G\psi\rangle = A + B \quad (3.120a)$$

$$0 = A + 4B \quad (3.120b)$$

Solving the two equations in (3.120) and substituting into (3.118) gives an expression for the $\langle\bar{\psi}\sigma G\psi\rangle$ component of $\langle\bar{\psi}G_{\lambda\mu} D_{(\alpha} D_{\beta)}\psi\rangle$.

$$\langle\bar{\psi}_i G_{\lambda\mu} D_{(\alpha} D_{\beta)} \psi_j\rangle = -\frac{m^2}{48}\langle\bar{\psi}\sigma G\psi\rangle \left[\frac{1}{3}\sigma^{\lambda\mu}_{ji} g_{\alpha\beta} \right.$$

$$\left. - \frac{1}{12}\left(\sigma^{\alpha\mu}_{ji} g_{\lambda\beta} + \sigma^{\beta\mu}_{ji} g_{\lambda\alpha} + \sigma^{\lambda\alpha}_{ji} g_{\mu\beta} + \sigma^{\lambda\beta}_{ji} g_{\mu\alpha} \right) \right] \quad (3.121)$$

Substituting the results of (3.121) and (3.117) into (3.105) leads to the expression for the second order mixed condensate projection of the $\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ vev.[36,38]

$$\frac{ig}{2}\langle\bar{\psi}_i(z)A^a_\mu(w)\psi_j(y)\rangle_{II} = -\frac{m^2}{(12)1536}\langle\bar{\psi}\sigma G\psi\rangle\lambda^a\Big[2w^\lambda\sigma^{\lambda\mu}_{ji}(y-z)^2 - (y-z)^\lambda\sigma^{\lambda\mu}_{ji}w\cdot(y-z) - w^\lambda\sigma^{\lambda\varepsilon}_{ji}(y-z)_\varepsilon(y-z)_\mu\Big] \quad (3.122)$$

Now consider the third order terms in the covariantized Taylor expansion of $\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ which is given by the expression

$$\frac{ig}{2}\langle\bar{\psi}_i(z)A^a_\mu(w)\psi_j(y)\rangle_{III} = \frac{1}{16}\lambda^a\Big[\frac{1}{12}w^\lambda\left(y^\alpha y^\beta y^\nu - z^\alpha z^\beta z^\nu\right) \times\langle\bar{\psi}_i G_{\lambda\mu}D_{(\alpha}D_\beta D_{\nu)}\psi_j\rangle + \frac{1}{4}w^\lambda\left(z^\alpha z^\beta y^\nu - y^\alpha y^\beta z^\nu\right)\langle\bar{\psi}_i D_\nu G_{\lambda\mu}D_{(\alpha}D_{\beta)}\psi_j\rangle\Big]$$
$$+ \text{ terms leading to higher dimension condensates.} \quad (3.123)$$

To evaluate the coefficients of third and higher order terms in (3.105), it is useful to have an algorithm similar to that of Section 3-4 for generating the Lorentz structure of the vev's. The problem is to determine the Lorentz invariant tensor T such that

$$\langle\bar{\psi}D_{(\alpha_1}\cdots D_{\alpha_n)}G_{\lambda\mu}D_{(\beta_1}\cdots D_{\beta_m)}\psi\rangle = T_{(\alpha_1\ldots\alpha_n)[\lambda\mu](\beta_1\ldots\beta_m)} \quad (3.124)$$

It is possible to generate T recursively through use of the following algorithm.

$$T_{(\alpha_1\ldots\alpha_{n+1})[\lambda\mu](\beta_1\ldots\beta_m)} = A\Big[\gamma_{\alpha_1} T_{(\alpha_2\ldots\alpha_{n+1})[\lambda\mu](\beta_1\ldots\beta_m)} + \ldots + \gamma_{\alpha_{n+1}} T_{(\alpha_2\ldots\alpha_n)[\lambda\mu](\beta_1\ldots\beta_m)}\Big]$$

$$+ \text{ independent } \gamma \text{ insertions into } A$$

$$+ B\Big[\gamma_{\beta_1}\Big\{T_{(\alpha_1\ldots\alpha_n)[\lambda\mu](\alpha_{n+1}\ldots\beta_m)} + \ldots + T_{(\alpha_2\ldots\alpha_{n+1})[\lambda\mu](\alpha_1\ldots\beta_m)}\Big\} + \ldots + \gamma_{\beta_m}\Big\{T_{(\alpha_1\ldots\alpha_n)[\lambda\mu](\beta_1\ldots\alpha_{n+1})} + \ldots + T_{(\alpha_2\ldots\alpha_{n+1})[\lambda\mu](\beta_1\ldots\alpha_1)}\Big\}\Big]$$

$$+ \text{ independent } \gamma \text{ insertions into } B$$

$$+ C\Big[\gamma_\lambda\Big\{T_{(\alpha_1\ldots\alpha_n)[\alpha_{n+1}\mu](\beta_1\ldots\beta_m)} + \ldots + T_{(\alpha_2\ldots\alpha_{n+1})[\alpha_1\mu](\beta_1\ldots\beta_m)}\Big\} - \gamma_\mu\Big\{T_{(\alpha_1\ldots\alpha_n)[\alpha_{n+1}\lambda](\beta_1\ldots\beta_m)} + \ldots + T_{(\alpha_2\ldots\alpha_{n+1})[\alpha_1\lambda](\beta_1\ldots\beta_m)}\Big\}\Big]$$

$$+ \text{ independent } \gamma \text{ insertions into } C \tag{3.125}$$

The Lorentz structure of $\langle\bar{\psi}G_{\lambda\mu}D_{(\alpha}D_{\beta}D_{\nu)}\psi\rangle$ can now be generated through use of (3.125) and (3.118).

$$\langle\bar{\psi}G_{\lambda\mu}D_{(\alpha}D_{\beta}D_{\nu)}\psi\rangle = A\,\sigma_{\lambda\mu}\left(\gamma_\nu g_{\alpha\beta} + \gamma_\alpha g_{\nu\beta} + \gamma_\beta g_{\nu\alpha}\right)$$

$$+ B\left(\gamma\text{'s on other side in } A\right)$$

$$+ C\left[\gamma_\nu\left(\sigma_{\alpha\mu}g_{\lambda\beta} + \sigma_{\beta\mu}g_{\lambda\alpha} + \sigma_{\lambda\alpha}g_{\mu\beta} + \sigma_{\lambda\beta}g_{\mu\alpha}\right)\right.$$

$$+ \gamma_\beta\left(\sigma_{\alpha\mu}g_{\lambda\nu} + \sigma_{\nu\mu}g_{\lambda\alpha} + \sigma_{\lambda\alpha}g_{\mu\nu} + \sigma_{\lambda\nu}g_{\mu\alpha}\right)$$

$$\left. + \gamma_\alpha\left(\sigma_{\nu\mu}g_{\lambda\beta} + \sigma_{\beta\mu}g_{\lambda\nu} + \sigma_{\lambda\nu}g_{\mu\beta} + \sigma_{\lambda\beta}g_{\mu\nu}\right)\right]$$

$$+ D\left(\gamma\text{'s on other side in } C\right)$$

$$+ E\left[\gamma_\lambda\left(\sigma_{\nu\mu}g_{\alpha\beta} + \sigma_{\alpha\mu}g_{\beta\nu} + \sigma_{\beta\mu}g_{\alpha\nu}\right)\right.$$

$$\left. - \gamma_\mu\left(\sigma_{\nu\lambda}g_{\alpha\beta} + \sigma_{\alpha\lambda}g_{\beta\nu} + \sigma_{\beta\lambda}g_{\alpha\nu}\right)\right]$$

$$+ F\left(\gamma\text{'s on other side in } E\right)$$

$$+ G\left[\gamma_\lambda\left(\sigma_{\alpha\mu}g_{\nu\beta} + \sigma_{\beta\mu}g_{\nu\alpha} + \sigma_{\nu\mu}g_{\alpha\beta}\right)\right.$$

$$\left. - \gamma_\mu\left(\sigma_{\alpha\lambda}g_{\nu\beta} + \sigma_{\beta\lambda}g_{\nu\alpha} + \sigma_{\nu\lambda}g_{\alpha\beta}\right)\right]$$

$$+ H\left(\gamma\text{'s on other side in } G\right) \tag{3.126}$$

Using the Dirac matrix identity (3.82) and the identities

$$\gamma_\alpha \sigma_{\mu\nu} = \sigma_{\mu\nu}\gamma_\alpha + 2i\left(\gamma_\nu g_{\mu\alpha} - \gamma_\mu g_{\nu\alpha}\right) \qquad (3.127a)$$

$$\gamma_\lambda \sigma_{\nu\mu} - \gamma_\mu \sigma_{\nu\lambda} = 2\sigma_{\lambda\mu}\gamma_\nu + i\left(\gamma_\lambda g_{\mu\nu} - \gamma_\mu g_{\nu\lambda}\right) \qquad (3.127b)$$

equation (3.126) can be written as a linearly independent combination of Dirac matrices.

$$\begin{aligned}\langle\bar{\psi}_i G_{\lambda\mu} D_{(\alpha}D_\beta D_{\nu)}\psi_j\rangle &= A\left\{\gamma^\alpha_{jk} g_{\beta\nu} + \gamma^\beta_{jk} g_{\alpha\nu} + \gamma^\nu_{jk} g_{\alpha\beta}\right\}\sigma^{\lambda\mu}_{ki} \\ &+ B\left\{\gamma^\lambda_{ji}\left(g_{\alpha\nu}g_{\beta\mu} + g_{\nu\beta}g_{\alpha\mu} + g_{\alpha\beta}g_{\mu\nu}\right)\right. \\ &\left. - \gamma^\mu_{ji}\left(g_{\alpha\nu}g_{\beta\lambda} + g_{\nu\beta}g_{\alpha\lambda} + g_{\alpha\beta}g_{\lambda\nu}\right)\right\} \qquad (3.128)\end{aligned}$$

Contracting (3.128) with $(\sigma^{\lambda\mu}\gamma^\alpha)_{ij}g^{\beta\nu}$ and $\gamma^\lambda_{ij}g^{\alpha\nu}g^{\beta\mu}$ respectively yields the following pair of linear equations.

$$\frac{1}{3}\langle\bar{\psi}\sigma G(\not{D}D^2 + D^2\not{D} + D^\alpha\not{D}D_\alpha)\psi\rangle = 24(48)A + 12(48)iB \qquad (3.129a)$$

$$\frac{1}{3}\langle\bar{\psi}\gamma^\mu G_{\mu\beta}(D_\alpha D_\beta D_\alpha + D^2 D_\beta + D_\beta D^2)\psi\rangle = -6(48)iA + 6(48)B \qquad (3.129b)$$

Equation (3.129a) is easily simplified using (3.85), (2.42b), (3.57) and $\not{D}\psi = -im\psi$, while (3.129b) requires that $G_{\mu\nu}$ be written as $[D_\nu, D_\mu]$ before application of the same identities. The result of the calculation is a pair of equations for the constants A, B.

$$2A + iB = \frac{1}{576} m^3 \langle\bar{\psi}\sigma G\psi\rangle \qquad (3.130a)$$

$$-iA + B = 0 \qquad (3.130b)$$

Solving (3.130) and substituting the values for A, B into (3.128) leads to the result for the $\langle\bar{\psi}\sigma G\psi\rangle$ projection of the $\langle\bar{\psi}G_{\lambda\mu}D_{(\alpha}D_{\beta}D_{\nu)}\psi\rangle$ vev.

$$\langle\bar{\psi}_i G_{\lambda\mu} D_{(\alpha}D_{\beta}D_{\nu)}\psi_j\rangle = \frac{1}{576} m^3 \langle\bar{\psi}\sigma G\psi\rangle \times \Big[i\left(\gamma^{\alpha}_{jk} g_{\beta\nu} + \gamma^{\beta}_{jk} g_{\alpha\nu} + \gamma^{\nu}_{jk} g_{\alpha\beta}\right)\sigma^{\lambda\mu}_{ki} - \left(\gamma^{\lambda}_{ji}\left(g_{\alpha\nu}g_{\beta\mu} + g_{\beta\nu}g_{\alpha\mu} + g_{\alpha\beta}g_{\mu\nu}\right) - \mu\leftrightarrow\lambda\right)\Big] \qquad (3.131)$$

The coefficient $\langle\bar{\psi}D_{\nu}G_{\lambda\mu}D_{(\alpha}D_{\beta)}\psi\rangle$ in the third order expansion (3.123) can be written as a linear combination of Dirac matrices through use of (3.118) and the algorithm of (3.125).

$$\langle\bar{\psi}D_{\nu}G_{\lambda\mu}D_{(\alpha}D_{\beta)}\psi\rangle = A\,\gamma_{\nu}\sigma_{\lambda\mu}g_{\alpha\beta} + B\,\sigma_{\lambda\mu}\gamma_{\nu}g_{\alpha\beta}$$
$$+ C\left[\gamma_{\alpha}\sigma_{\lambda\mu}g_{\nu\beta} + \gamma_{\beta}\sigma_{\lambda\mu}g_{\nu\alpha}\right] + D\left[\gamma\text{'s on other side in } C\right]$$
$$+ E\left[\gamma_{\lambda}\sigma_{\nu\mu}g_{\alpha\beta} - \gamma_{\mu}\sigma_{\nu\lambda}g_{\alpha\beta}\right] + F\left[\gamma\text{'s on other side in } E\right]$$
$$+ G\,\gamma_{\nu}\left[\sigma_{\alpha\mu}g_{\lambda\beta} + \sigma_{\beta\mu}g_{\lambda\alpha} + \sigma_{\lambda\alpha}g_{\mu\beta} + \sigma_{\lambda\beta}g_{\mu\alpha}\right]$$
$$+ H\left[\gamma\text{'s on other side in } G\right]$$

$$+ I\left[\gamma_\alpha\left(\sigma_{\nu\mu}g_{\lambda\beta}+\sigma_{\beta\mu}g_{\lambda\nu}+\sigma_{\lambda\nu}g_{\mu\beta}+\sigma_{\lambda\beta}g_{\mu\nu}\right)\right.$$

$$\left.+\gamma_\beta\left(\sigma_{\nu\mu}g_{\lambda\alpha}+\sigma_{\alpha\mu}g_{\lambda\nu}+\sigma_{\lambda\nu}g_{\mu\alpha}+\sigma_{\lambda\alpha}g_{\mu\nu}\right)\right]$$

$$+ J\left[\gamma\text{'s on other side in } I\right]$$

$$+ K\left[\gamma_\lambda\left(\sigma_{\alpha\mu}g_{\nu\beta}+\sigma_{\beta\mu}g_{\nu\alpha}+\sigma_{\nu\alpha}g_{\mu\beta}+\sigma_{\nu\beta}g_{\mu\alpha}\right)\right.$$

$$\left.-\gamma_\mu\left(\sigma_{\alpha\lambda}g_{\nu\beta}+\sigma_{\beta\lambda}g_{\nu\alpha}+\sigma_{\nu\alpha}g_{\lambda\beta}+\sigma_{\nu\beta}g_{\lambda\alpha}\right)\right]$$

$$+ L\left[\gamma\text{'s on other side in } K\right] \qquad (3.132)$$

The identities (3.127) and (3.82) allow (3.132) to be written as a linearly independent combination of Dirac matrices.

$$\langle\bar{\psi}_i D_\nu G_{\lambda\mu} D_{(\alpha} D_{\beta)}\psi_j\rangle = A\, g_{\alpha\beta}\sigma^{\lambda\mu}_{jk}\gamma^\nu_{ki} + B\sigma^{\lambda\mu}_{jk}\left(\gamma^\alpha_{ki}g_{\nu\beta}+\gamma^\beta_{ki}g_{\nu\alpha}\right)$$

$$+ C\left[g_{\nu\beta}\left(g_{\lambda\alpha}\gamma^\mu_{ji}-g_{\mu\alpha}\gamma^\lambda_{ji}\right)+g_{\nu\alpha}\left(g_{\lambda\beta}\gamma^\mu_{ji}-g_{\mu\beta}\gamma^\lambda_{ji}\right)\right]$$

$$+ D\left(g_{\alpha\mu}\sigma^{\lambda\beta}_{jk}+g_{\beta\mu}\sigma^{\lambda\alpha}_{jk}-g_{\alpha\lambda}\sigma^{\mu\beta}_{jk}-g_{\beta\lambda}\sigma^{\mu\alpha}_{jk}\right)\gamma^\nu_{ki}$$

$$+ E\, g_{\alpha\beta}\left(\gamma^\lambda_{ji}g_{\nu\mu}-\gamma^\mu_{ji}g_{\nu\lambda}\right) \qquad (3.133)$$

$$+ F\left[g_{\mu\nu}\left(\gamma^\alpha_{ji}g_{\lambda\beta}+\gamma^\beta_{ji}g_{\lambda\alpha}\right)-g_{\nu\lambda}\left(\gamma^\alpha_{ji}g_{\mu\beta}+\gamma^\beta_{ji}g_{\mu\alpha}\right)\right]$$

Contracting (3.133) with $g^{\alpha\beta}(\gamma^\nu\sigma^{\lambda\mu})_{ij}$, $g^{\nu\beta}(\sigma^{\lambda\mu}\gamma^\alpha)_{ij}$, $g^{\nu\beta}g^{\lambda\alpha}\gamma^\mu_{ij}$, $g^{\alpha\mu}(\gamma^\nu\sigma^{\lambda\beta})_{ij}$, $g^{\alpha\beta}\gamma^\lambda_{ij}g^{\nu\mu}$ and $g^{\mu\nu}g^{\lambda\beta}\gamma^\alpha_{ij}$ respectively leads to the next set of expressions.

$$\langle\bar{\psi}\not{D}\sigma G D^2\psi\rangle = 192\left[4A + 2B + iC + 4D - 2iE - iF\right] \qquad (3.134a)$$

$$\frac{1}{2}\langle\bar{\psi}D_\beta\sigma G(\not{D}D_\beta + D_\beta\not{D})\psi\rangle = 96\left[-5iC - 6D + iE - iF\right] \qquad (3.134b)$$

$$\frac{1}{2}\langle\bar{\psi}D_\beta G_{\alpha\nu}\gamma^\nu(D_\alpha D_\beta + D_\beta D_\alpha)\psi\rangle = -48i\left[A + 5B + 5iC + 4D - iE + iF\right] \qquad (3.134c)$$

$$\frac{1}{2}\langle\bar{\psi}\not{D}\sigma^{\mu\beta}G_{\mu\alpha}(D_\alpha D_\beta + D_\beta D_\alpha)\psi\rangle = 96\left[2A + B + 2iC + 8D - iE - 2iF\right] \qquad (3.134d)$$

$$\langle\bar{\psi}D_\nu G_{\mu\nu}\gamma^\mu D^2\psi\rangle = -96i\left[-2A - B - iC - 2D + 2iE + iF\right] \qquad (3.134e)$$

$$\frac{1}{2}\langle\bar{\psi}D_\nu G_{\beta\nu}(D_\beta\not{D} + \not{D}D_\beta)\psi\rangle = -48i\left[-A + B + iC - 4D + iE + 5iF\right] \qquad (3.134f)$$

The mixed condensate component of the equations in (3.134) can be extracted through use of $G_{\mu\nu} = [D_\nu, D_\mu]$, $\not{D}\psi = -im\psi$, (3.61), (3.59) and (3.58), yielding the following set of equations.

$$\frac{1}{192}\, i m^3\langle\bar{\psi}\sigma G\psi\rangle = 4A + 2B + iC + 4D - 2iE - iF \qquad (3.135a)$$

$$-\frac{1}{96}\, m^3\langle\bar{\psi}\sigma G\psi\rangle = -5iC - 6D + iE - iF \qquad (3.135b)$$

$$0 = A + 5B + 5iC + 4D - iE + iF \qquad (3.135c)$$

$$0 = 2A + B + 2iC + 8D - iE - 2iF \qquad (3.135d)$$

$$0 = -2A - B - iC - 2D + 2iE + iF \qquad (3.135e)$$

$$0 = -A + B + iC - 4D \pm iE + 5iF \tag{3.135f}$$

The solution to (3.135) is

$$A = B = -iC = iE = \frac{1}{576} i m^3 \langle\bar{\psi}\sigma G\psi\rangle \tag{3.136a}$$

$$F = D = 0 \tag{3.136b}$$

Substituting the results of (3.136) into (3.133) leads to the expression for the $\langle\bar{\psi}\sigma G\psi\rangle$ component of the $\langle\bar{\psi}D_\nu G_{\lambda\mu}D_{(\alpha}D_{\beta)}\psi\rangle$ vev.

$$\langle\bar{\psi}_i D_\nu G_{\lambda\mu} D_{(\alpha}D_{\beta)}\psi_j\rangle = \frac{1}{576} m^3 \langle\bar{\psi}\sigma G\psi\rangle \Big[i g_{\alpha\beta}\sigma^{\lambda\mu}_{jk}\gamma^\nu_{ki} + i\sigma^{\lambda\mu}_{jk}\left(\gamma^\alpha_{ki} g_{\nu\beta} + \gamma^\beta_{ki} g_{\nu\alpha}\right)$$
$$- \left(g_{\nu\beta}\left(g_{\nu\alpha}\gamma^\mu_{ji} - g_{\mu\alpha}\gamma^\lambda_{ji}\right) + g_{\nu\alpha}\left(g_{\beta\lambda}\gamma^\mu_{ji} - g_{\beta\mu}\gamma^\lambda_{ji}\right)\right)$$
$$+ g_{\alpha\beta}\left(\gamma^\lambda_{ji} g_{\nu\mu} - \gamma^\mu_{ji} g_{\nu\lambda}\right)\Big] \tag{3.137}$$

Substituting the results of (3.131) and (3.137) into (3.123) leads to the third order mixed condensate projection of the quark-gluon-quark vev,[36,38]

$$\frac{ig}{2}\langle\bar{\psi}_i(z)A^a_\mu(w)\psi_j(y)\rangle_{III} = \frac{1}{1536}\lambda^a\langle\bar{\psi}\sigma G\psi\rangle\frac{m^3}{24}\Big[i w^\lambda \sigma^{\lambda\mu}_{jk}(y-z)^2\gamma_{ki}\cdot(y-z)$$
$$+ \gamma_{ji}\cdot w(y-z)^2(y-z)_\mu - \gamma^\mu_{ji}(y-z)^2 w\cdot(y-z)\Big] \tag{3.138}$$

Collecting the results of (3.138), (3.122), (3.113) and (3.108) gives the final expression for the mixed condensate projection of the

$<\bar{\psi}(z)A_{\mu}^{a}(w)\psi(y)>$ vev, valid to third order in the covariantized Taylor expansion.[36,38]

$$\frac{ig}{2}<\bar{\psi}_i(z)A_{\mu}^{a}(w)\psi_j(y)> = \frac{i}{1536}\lambda^a<\bar{\psi}\sigma G\psi>\Bigg[w^{\lambda}\sigma_{ji}^{\lambda\mu}$$

$$+ \frac{m}{2}\left[-iw^{\lambda}\sigma_{jk}^{\lambda\mu}\gamma_{ki}\cdot(y-z) + w^{\lambda}(y-z)^{\beta}\left(\gamma_{ji}^{\mu}g_{\beta\lambda} - \gamma_{ji}^{\lambda}g_{\mu\beta}\right)\right]$$

$$- \frac{m^2}{12}\Big(2w^{\lambda}\sigma_{ji}^{\lambda\mu}(y-z)^2 - (y-z)^{\lambda}\sigma_{ji}^{\lambda\mu}w\cdot(y-z)$$

$$- w^{\lambda}\sigma_{ji}^{\lambda\varepsilon}(y-z)^{\varepsilon}(y-z)_{\mu}\Big)$$

$$+ \frac{m^3}{24}\Big[iw^{\lambda}\sigma_{jk}^{\lambda\mu}(y-z)^2\gamma_{ki}\cdot(y-z) + \gamma_{ji}\cdot w(y-z)^2(y-z)_{\mu}$$

$$- \gamma_{ji}^{\mu}(y-z)^2 w\cdot(y-z)\Big] + O(m^4)\Bigg] \qquad (3.139)$$

3-6 Gauge Independence of the Mixed Condensate Component of the Quark Self-Energy and the Dynamical Quark Mass

The lowest order at which the mixed condensate can contribute to the quark self-energy is at order g^3. As outlined in Section 3-3, the amplitudes which contribute to the mixed condensate at this order are given by (3.28), (3.29), (3.30) and (2.10), corresponding to the diagrams in Figures One, Four, Five and Six. To evaluate the mixed condensate component of these amplitudes the $\langle\bar{\psi}\sigma G\psi\rangle$ projection of the non-perturbative vev's $\langle\bar{\psi}(z)\psi(y)\rangle$ and $\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ was required. In Sections 3-4 and 3-5 the mixed condensate projection of these vev's was explicitly calculated to order m^3 in the dynamical mass parameter, respectively corresponding to the fifth and third order expansions in the covariantized Taylor series.

The mixed condensate component of the quark self-energy can now be calculated by substituting the results of Sections 3-4 and 3-5 into the amplitudes of (3.28) to (3.30) and (2.10). First consider the amplitude of (2.10), corresponding to Figure One. Substituting the expansions (3.98) and (3.99) for $\langle\bar{\psi}(z)\psi(y)\rangle$ into (2.10) leads to the following expression.

$$i\Delta S_1(p) = \frac{1}{4}g^2\int d^4x\, d^4y\, d^4z\; e^{ip\cdot x}\,\langle T\left(\psi(x)\bar{\psi}(y)\right)\rangle\,\gamma^\mu\lambda^a\,\frac{1}{288}\langle\bar{\psi}\sigma G\psi\rangle$$

$$\times\left(\left[-y^\epsilon z^\tau\sigma^{\epsilon\tau} - \frac{3}{2}i(y-z)^2\right] - \frac{m}{4}\gamma\cdot(y-z)(y-z)^2\right.$$

$$+\, m^2\left[\frac{1}{12}y^\epsilon z^\tau\sigma^{\epsilon\tau}(y-z)^2 + \frac{i}{8}(y-z)^4\right] + \frac{m^3}{64}\gamma\cdot(y-z)(y-z)^4$$

$$\left. +\, \ldots + m^{n-2}a_n\left(\gamma\cdot(y-z)\right)^n + m^{2n}b_{2n}y^\epsilon z^\tau\sigma^{\epsilon\tau}(y-z)^{2n}\right)$$

$$\times\ \gamma^{\nu}\lambda^{b} \langle T\left(\psi(z)\bar{\psi}(0)\right)\rangle \langle T\left(A^{a}_{\mu}(y)A^{b}_{\nu}(z)\right)\rangle \tag{3.140}$$

As demonstrated in Section 2-4, contributions to (3.140) from objects of the form $(\gamma\cdot(y-z))^n$ are zero for $n>1$, implying that (3.140) may be written as.

$$i\Delta S_1(p) = \frac{?}{4} g^2 \int d^4x\, d^4y\, d^4z\, e^{ip\cdot x} \langle T\left(\psi(x)\bar{\psi}(y)\right)\rangle \gamma^{\mu}\lambda^{a} \frac{1}{288} \langle\bar{\psi}\sigma G\psi\rangle$$

$$\times \left(-y^{\varepsilon}z^{\tau}\sigma_{\varepsilon\tau} + \frac{1}{12} m^2 y^{\varepsilon} z^{\tau}\sigma_{\varepsilon\tau}(y-z)^2 + \ldots + m^{2n} b_{2n} y^{\varepsilon} z^{\tau}\sigma_{\varepsilon\tau}(y-z)^{2n} \right)$$

$$\times\ \gamma^{\nu}\lambda^{b} \langle T\left(\psi(z)\bar{\psi}(0)\right)\rangle \langle T\left(A^{a}_{\mu}(y)A^{b}_{\nu}(z)\right)\rangle \tag{3.141}$$

Using the techniques of Section 2-4 the integrals in (3.141) can be reduced to delta functions by changing the variables of integration. For a general term in (3.141) the amplitude thus becomes

$$i\Delta S_1(p) = -\frac{4}{3} i g^2 \frac{1}{288} \langle\bar{\psi}\sigma G\psi\rangle \sum_{n=0}^{\infty} b_{2n} m^{2n} \frac{\not{p}+m_L}{p^2-m_L^2} \int d^4(y-z)\, d^4z\, e^{i\,p\cdot(y-z)}$$

$$\times\ e^{ip\cdot z} \gamma^{\mu} \left[\left((y-z)+z\right)^{\varepsilon} z^{\tau} (y-z)^{2n} \sigma_{\varepsilon\tau} \right] \gamma^{\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(y-z)}}{k^2+i\varepsilon}$$

$$\times \left(-g^{\mu\nu} + (1-a)\frac{k^{\mu}k^{\nu}}{k^2+i\varepsilon} \right) \int \frac{d^4T}{(2\pi)^4} e^{-iT\cdot z} \frac{\not{T}+m_L}{T^2-m_L^2+i\varepsilon} \tag{3.142}$$

The (y-z) and z integrals in (3.142) can now be performed through the use of (2.76).

$$i\Delta S_1(p) = \frac{4}{3} i g^2 \frac{1}{288} \langle\bar{\psi}\sigma G\psi\rangle \sum_{n=0}^{\infty} b_{2n} m^{2n} \frac{\not{p}+m_L}{p^2-m_L^2} \gamma^{\mu}\sigma_{\varepsilon\tau}\gamma^{\nu} (-1)^n$$

$$\times \frac{\partial}{\partial p_\varepsilon}\left[\left(\frac{\partial^2}{\partial p^2}\right)^n\left(-\frac{g^{\mu\nu}}{p^2} + (1-a)\frac{p^\mu p^\nu}{p^4}\right)\frac{\partial}{\partial p_\tau}\left(\frac{\not p+m_L}{p^2-m_L^2}\right)\right] \qquad (3.143)$$

Only the n=0 term in (3.143) will be non-zero after the equation is simplified. To see this, consider the following expression with n≥1.

$$\gamma^\mu\sigma_{\varepsilon\tau}\gamma^\nu\frac{\partial}{\partial p_\varepsilon}\left[\left(\frac{\partial^2}{\partial p^2}\right)^n\left(-\frac{g^{\mu\nu}}{p^2} + (1-a)\frac{p^\mu p^\nu}{p^4}\right)\frac{\partial}{\partial p_\tau}\left(\frac{\not p+m_L}{p^2-m_L^2}\right)\right]$$

$$= \gamma^\mu\sigma_{\varepsilon\tau}\gamma^\nu\frac{\partial}{\partial p_\varepsilon}\left(\frac{\partial^2}{\partial p^2}\right)^n\left(-\frac{g^{\mu\nu}}{p^2} + (1-a)\frac{p^\mu p^\nu}{p^4}\right)\left(\frac{\partial}{\partial p_\tau}\frac{\not p+m_L}{p^2-m_L^2}\right)$$

$$+ \gamma^\mu\sigma_{\varepsilon\tau}\gamma^\nu\left(\frac{\partial^2}{\partial p^2}\right)^n\left(-\frac{g^{\mu\nu}}{p^2} + (1-a)\frac{p^\mu p^\nu}{p^4}\right)\left(\frac{\partial^2}{\partial p_\varepsilon\partial p_\tau}\frac{\not p+m_L}{p^2-m_L^2}\right) \qquad (3.144)$$

The second term in the above equation is zero, since $\sigma_{\varepsilon\tau}$ is antisymmetric in ε,τ and the derivatives are symmetric. The Dirac identities

$$\gamma^\mu\sigma_{\mu\nu} = \sigma_{\nu\mu}\gamma^\mu = 3i\gamma_\nu \qquad (3.145a)$$

$$\gamma^\mu\gamma_\lambda\gamma_\varepsilon\gamma_\mu = 4g_{\lambda\varepsilon} \qquad (3.145b)$$

and the results contained in (2.80) and (2.81) can now be used to simplify (3.144).

$$\gamma^\mu\sigma_{\varepsilon\tau}\gamma^\nu\frac{\partial}{\partial p_\varepsilon}\left(\frac{\partial^2}{\partial p^2}\right)^n\left(-\frac{g^{\mu\nu}}{p^2} + (1-a)\frac{p^\mu p^\nu}{p^4}\right)\frac{\partial}{\partial p_\tau}\left(\frac{\not p+m_L}{p^2-m_L^2}\right)$$

$$= 2(1-a)\left(\frac{\partial^2}{\partial p^2}\right)^{n-1}\left[\gamma^\mu\sigma_{\varepsilon\tau}\gamma^\nu\frac{\partial}{\partial p_\varepsilon}\left(\frac{g^{\mu\nu}}{p^4} - 4\frac{p^\mu p^\nu}{p^6}\right)\right]\frac{\partial}{\partial p_\tau}\frac{\not p+m_L}{p^2-m_L^2}$$

$$= \frac{2}{p^6}(1-a)\left(\frac{\partial^2}{\partial p^2}\right)^{n-1}\gamma^\mu\sigma_{\varepsilon\tau}\gamma^\nu\left[-4\left(g_{\mu\nu}p_\varepsilon + g_{\mu\varepsilon}p_\nu + g_{\nu\varepsilon}p_\mu\right) + 24\frac{p^\mu p^\nu p^\varepsilon}{p^2}\right]$$

$$\times \frac{\partial}{\partial p_\tau}\frac{\not p + m_L}{p^2 - m_L^2} = 0 \quad (n \geq 1) \qquad (3.146)$$

The above result shows that the series of (3.143) truncates after the first term, leaving only the first term of (3.141) to provide a non-trivial contribution to the quark self-energy.[36,38] Evaluating this term leads to the following result[36-38,48] for the $O(g^3)$ mixed condensate contribution to the quark self-energy.

$$i\Delta S_1(p) = -\left(\not p + m_L\right)\frac{g^2\langle\bar\psi\sigma G\psi\rangle\not p(1-a)}{36p^4(p^2-m_L^2)^2} \qquad (3.147)$$

The procedure for evaluating the mixed condensate contribution from amplitudes (3.28) to (3.30) should now be evident. Results for the $\langle\bar\psi\sigma G\psi\rangle$ projection of $\langle\bar\psi(z)A^a_\mu(w)\psi(y)\rangle$ are substituted into the amplitudes, and the integrals are reduced to various combinations of derivatives acting upon propagators. This leads to a truly formidable task of algebraic manipulation, which for higher order terms in (3.139) quickly becomes unmanageable if the calculations are done manually. The symbolic manipulation program REDUCE[58] has therefore been used to evaluate the amplitudes of (3.28) to (3.30), with a manual check performed upon the lowest order (lead, $O(m)$ and $O(m^2)$) terms.

REDUCE has a built-in facility for handling tensor indices and Dirac algebra, but has no intrinsic ability to differentiate with respect to four-vectors. This is not a major difficulty since REDUCE can be taught additional rules for differentiation. However, REDUCE must be taught the differentiation rules in a certain order, a procedure which will now be outlined. After defining P, U, V to be VECTOR's in

REDUCE, the following rules are defined.

DF(P.U , P.V) = U.V $\left\{ \text{translation} \quad \frac{\partial p^\mu}{\partial p_\nu} = g^{\mu\nu} \right\}$

DF(P.P , P.U) = 2*P.U $\left\{ \text{translation} \quad \frac{\partial p^2}{\partial p_\mu} = 2p^\mu \right\}$ (3.148)

To begin the evaluation of the amplitudes in (3.28) to (3.30), the quark-gluon-quark vev of (3.139) is written in the form

$$\frac{ig}{2} \langle \bar{\psi}(z) A_\mu^a(w) \psi(y) \rangle = \frac{1}{1536} \lambda^a \langle \bar{\psi}\sigma G \psi \rangle C_\mu(y-z,w) \tag{3.149}$$

allowing a short-hand notation to be developed for the $\langle\bar{\psi}\sigma G\psi\rangle$ component of these amplitudes.

$$i\Delta S_4(p) = g^2 \frac{\lambda^a \lambda^b \lambda^a \lambda^c}{(4)1536} \langle \bar{\psi}\sigma G\psi \rangle \int d^4x\, d^4y\, d^4z\, d^4w\; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma^\mu$$

$$\times \langle T(\psi(y)\bar{\psi}(z))\rangle \gamma^\nu C_\mu(z-w,y) \gamma^\rho \langle T(\psi(w)\bar{\psi}(0))\rangle \langle T(A_\nu^b(z) A_\rho^c(w))\rangle \tag{3.150}$$

$$i\Delta S_5(p) = g^2 \frac{\lambda^a \lambda^c \lambda^b \lambda^c}{(4)1536} \langle \bar{\psi}\sigma G\psi \rangle \int d^4x\, d^4y\, d^4z\, d^4w\; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma^\mu$$

$$\times C_\rho(y-z,w) \gamma^\nu \langle T(\psi(z)\bar{\psi}(w))\rangle \gamma^\rho \langle T(\psi(w)\bar{\psi}(0))\rangle \langle T(A_\mu^a(y) A_\nu^b(z))\rangle \tag{3.151}$$

$$i\Delta S_6(p) = -g^2 \frac{\lambda^a \lambda^e \lambda^b}{(2)1536} f_{ehl} \langle \bar{\psi}\sigma G\psi \rangle \int d^4x\, d^4y\, d^4z\, d^4w\; e^{ip\cdot x} \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma^\mu$$

$$\times \langle T(A_\mu^a(y) A_\rho^h(w))\rangle \langle T(A_\nu^l(w) A_\lambda^b(z))\rangle$$

$$\times \left\{ \frac{\partial}{\partial w_\rho} C_\lambda(y-z,w) - \frac{\partial}{\partial w_\lambda} C_\rho(y-z,w) \right\} \gamma^\nu \langle T(\psi(z)\bar{\psi}(0))\rangle$$

$$- \frac{g^2 \lambda^a \lambda^l \lambda^b}{(2)1536} f_{ehl} \langle\bar{\psi}\sigma G\psi\rangle \int d^4x\, d^4y\, d^4z\, d^4w\; e^{ip\cdot x}$$

$$\times \langle T(\psi(x)\bar{\psi}(y))\rangle \gamma^\mu \langle T(A^a_\mu(y) A^h_\rho(w))\rangle\, 2C_\lambda(y-z,w) \qquad (3.152)$$

$$\times \left[\frac{\partial}{\partial w_\rho} \langle T(A^e_\lambda(w) A^b_\nu(z))\rangle - \frac{\partial}{\partial w_\lambda} \langle T(A^e_\rho(w) A^b_\nu(z))\rangle \right] \gamma^\nu \langle T(\psi(z)\bar{\psi}(0))\rangle$$

The above equations are simplified by shifting the integration variables to (x-y), (y-w), (w-z), z from x,y,z,w, permitting the integrations to be performed. Colour algebra results which are also needed are

$$\lambda^b\lambda^c\lambda^b\lambda^c = -\frac{32}{9} \qquad (3.153a)$$

and

$$\lambda^a\lambda^h\lambda^e f_{eah} = 16i \qquad (3.153b)$$

Performing the resulting integrals in (3.150) to (3.152) with the assistance of (2.76), using (3.153), and recalling the propagators from (2.12) and (2.13), allows (3.150) to (3.152) to be written as operator equations.

$$i\Delta S_4(p) = -\frac{8g^2}{(9)1536} \langle\bar{\psi}\sigma G\psi\rangle S_o(p,m_L)\, C_\mu\left(\frac{\partial}{i\partial k}, \frac{1}{i}\left(\frac{\partial}{\partial q} + \frac{\partial}{\partial T} + \frac{\partial}{\partial k} \right) \right)$$

$$\times \left[\gamma^\mu S_o(q,m_L)\, \gamma^\nu (\text{Dirac matrices from } C_\mu)\, \gamma^\rho S_o(T,m_L)\, D_{\nu\rho}(k) \right] \Big|_{k=q=T=p} \qquad (3.154a)$$

$$i\Delta S_5(p) = -\frac{8q^2}{1536}\langle\bar{\psi}\sigma G\psi\rangle S_o(p,m_L)\, C_\rho\left(\frac{\partial}{i\partial k},\frac{\partial}{i\partial T}\right)$$

$$\times\left[\gamma^\mu(\text{Dirac matrices from } C_\rho)\,\gamma^\nu S_o(q,m_L)\,\gamma^\rho S_o(T,m_L)\,D_{\mu\nu}(k)\right]\Bigg|_{k=q=T=p} \quad (3.154b)$$

$$i\Delta S_6(p) = \frac{8iq^2}{1536}\langle\bar{\psi}\sigma G\psi\rangle S_o(p,m_L)\,\gamma^\mu\left[\left(\frac{\partial}{\partial w_\rho}C_\lambda(y-z,w) - \frac{\partial}{\partial w_\lambda}C_\rho(y-z,w)\right)\right.$$

$$\left(\frac{\partial}{i\partial k}+\frac{\partial}{i\partial T},\ \frac{\partial}{i\partial T}+\frac{\partial}{i\partial q}\right)\times\left[\gamma^\nu S_o(q,m_L)\,D_{\mu\rho}(k)\,D_{\nu\lambda}(T)\right]$$

$$\left. + 2C_\lambda\left(\frac{\partial}{i\partial k}+\frac{\partial}{i\partial T},\ \frac{\partial}{i\partial T}+\frac{\partial}{i\partial q}\right)\left[\gamma^\nu S_o(q,m_L)\,D_{\mu\rho}(k)\,\frac{i(T^\lambda g^{\rho\nu} - T^\rho g^{\nu\lambda})}{T^2}\right]\right]\Bigg|_{T=k=q=p} \quad (3.155)$$

The expressions in (3.139) for the $\langle\bar{\psi}\sigma G\psi\rangle$ projection of $\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$ can now be substituted in the operator equations (3.154) and (3.155). REDUCE is then used to perform the algebra, leading to the results[36,38]

$$i\Delta S_4(p) = (\not{p}+m_L)\frac{g^2\langle\bar{\psi}\sigma G\psi\rangle}{288p^4(p^2-m_L^2)^3}\Big[-\not{p}(2p^2+m_L^2)(1-a)$$

$$+ m\,4m_L\not{p} + m^2\,3\not{p}(1-a) + m^3\,(0) + O(m^4)\Big] \quad (3.156)$$

$$i\Delta S_5(p) = (\not{p}+m_L)\frac{g^2\langle\bar{\psi}\sigma G\psi\rangle}{288p^4(p^2-m_L^2)^3}\Big[-\not{p}p^2(1-a) + m\,2\not{p}(\not{p}+m_L)$$

$$+ m^2\not{p}(1-a) + m^3\,(0) + O(m^4)\Big] \quad (3.157)$$

$$i\Delta S_6(p) = (\not{p}+m_L)\frac{g^2\langle\bar{\psi}\sigma G\psi\rangle}{288p^8(p^2-m_L^2)^2}\Big[(-54-18a)\not{p}p^4 + (-45-9a)m_Lp^4$$

$$+ \left(m-m_L\right)\left[-\frac{18a}{p^2-m_L^2}\left(p^6 + m_L p^4 \not{p}\right)\right] + m\,(45+9a)\left(p^4 + m_L \not{p} p^2\right)$$

$$+ m^2 \left[-(45+9a)\left(\not{p} p^2 + m_L p^2\right)\right]$$

$$+ m^3\,(45+9a)\left(p^2 + m_L \not{p}\right) + \mathcal{O}(m^4) \Bigg] \qquad (3.158)$$

The amplitudes of (3.156) and (3.157), which are generated by the abelian graphs of Figures Four and Five, are zero for the order m^3 term of the quark-gluon-quark vev. This behaviour is suggestive of the truncation observed in the amplitude (3.147) which is also generated by an abelian graph (Figure One). The apparent failure of the non-abelian graph of Figure Six to truncate is better understood when the self-energies are identified in (3.156) to (3.158). Using the definition of (2.74), the self-energies of (3.147) and (3.156) to (3.158) are given by[36,38]

$$\Sigma_1(p) = -\frac{ig^2\langle\bar{\psi}\sigma G\psi\rangle}{288p^4}\left[-\frac{8(1-a)\left(p^2 - m_L \not{p}\right)}{p^2 - m_L^2}\right] \qquad (3.159)$$

$$\Sigma_4(p) = -\frac{ig^2\langle\bar{\psi}\sigma G\psi\rangle}{288p^4}\left[-\frac{(1-a)\left(2p^2 + m_L^2\right)}{(p^2 - m_L^2)^2}\left(p^2 - m_L \not{p}\right) + m\,4m_L \frac{\left(p^2-m_L \not{p}\right)}{(p^2-m_L^2)^2} + m^2\,3(1-a)\frac{\left(p^2-m_L \not{p}\right)}{(p^2-m_L^2)^2} + \mathcal{O}(m^4)\right] \qquad (3.160)$$

$$\Sigma_5(p) = -\frac{ig^2\langle\bar{\psi}\sigma G\psi\rangle}{288p^4}\left[-(1-a)p^2\frac{\left(p^2-m_L \not{p}\right)}{(p^2-m_L^2)^2} + m\frac{2\not{p}}{p^2-m_L^2} + m^2(1-a)\frac{\left(p^2-m_L \not{p}\right)}{(p^2-m_L^2)^2} + \mathcal{O}(m^4)\right] \qquad (3.161)$$

$$\Sigma_6(p) = -\frac{ig^2\langle\bar{\psi}\sigma G\psi\rangle}{288p^4}\left\{-9(1+a)\frac{(p^2-m_L\not{p})}{p^2-m_L^2} - (m-m_L)\frac{18a\not{p}}{p^2-m_L^2}\right.$$

$$\left. - 9(5+a)\left[1 - \frac{m\not{p}}{p^2} + \frac{m^2}{p^2} - \frac{m^3\not{p}}{p^4}\right] + \mathcal{O}(m^4)\right\} \quad . \tag{3.162}$$

To proceed further, it is assumed that the truncation of the $\mathcal{O}(m^3)$ terms in (3.160) and (3.161) continues for all higher orders, consistent with the behaviour of (3.159). Secondly, the alternating character of the series in the bottom line of (3.162) is assumed to continue, accounting for all second and higher order contributions in m. This implies that the abelian contributions (Figures Four and Five) truncate after order m^2 while the non-abelian contribution (Figure Six) generates a geometric series in $m\not{p}/p^2$. Under these assumptions, the aggregate order g^3 mixed-condensate component of the quark self-energy becomes[36,38]

$$\Sigma(p) = -\frac{ig^2\langle\bar{\psi}\sigma G\psi\rangle}{288(p^2-m_L^2)^2(p^2-m^2)p^4}\left[E(p^2)\not{p} + F(p^2)\right] \tag{3.163a}$$

$$E(p^2) = \left[(20+16a)m_L + (47-9a)m\right]p^4$$

$$+ \left[(-16-20a)m_L^3 - 96m_L^2 m + (-24-12a)m_L m^2 + (-2+18a)m^3\right]p^2$$

$$+ \left[(45+9a)m_L^4 m + (16+20a)m_L^3 m^2 + (6-18a)m_L^2 m^3 + (4-4a)m_L m^4\right] \tag{3.163b}$$

$$F(p^2) = (-65-7a)p^6 + \left[(106+20a)m_L^2 + 4m_L m + (24+6a)m^2\right]p^4$$

$$+ \left[(-45-9a)m_L^4 + (-16-2a)m_L^2 m^2 - 4m_L m^3 + (-4+4a)m^4\right]p^2 \quad . \tag{3.163c}$$

To determine the effect of the $\langle\bar{\psi}\sigma G\psi\rangle$ condensate upon dynamical mass generation, the effective quark mass must be obtained. As in Section 2-4 the inverse propagator is

$$S_F^{-1}(p) = \not{p} - m_L - \Sigma(p) = \not{p} - m_L + \frac{ig^2\langle\bar{\psi}\sigma G\psi\rangle}{288(p^2-m_L^2)^2(p^2-m^2)p^4}\left[E(p^2)\not{p} + F(p^2)\right] . \tag{3.164}$$

Removing the total coefficient of $\not{p}$, which is then absorbed into a wave function renormalization,[54] identifies the effective quark mass.[36,38]

$$S_F^{-1}(p) = \left[1 + \frac{ig^2\langle\bar{\psi}\sigma G\psi\rangle E(p^2)}{288(p^2-m_L^2)^2(p^2-m^2)p^4}\right]\left[\not{p} - M(p^2)\right] \tag{3.165a}$$

$$M(p^2) = \frac{m_L - \dfrac{ig^2\langle\bar{\psi}\sigma G\psi\rangle F(p^2)}{288(p^2-m_L^2)^2(p^2-m^2)p^4}}{1 + \dfrac{ig^2\langle\bar{\psi}\sigma G\psi\rangle E(p^2)}{288(p^2-m_L^2)^2(p^2-m^2)p^4}} \tag{3.165b}$$

If the mixed condensate corrections to the quark propagator support the quark condensate result of Chapter Two, then the pole position of (3.165a) must remain at $\not{p}=m$, implying that

$$\lim_{p^2 \to m^2} M(p^2) = m \tag{3.166}$$

This limit is now taken in the effective mass of (3.165b), leading to the constraint

$$\lim_{p^2 \to m^2} M(p^2) = -\frac{F(m^2)}{E(m^2)} = m \tag{3.167}$$

This constraint is satisfied for arbitrary values of both m_L and the gauge parameter "a", since (3.163) implies that[36,38]

$$\lim_{p^2 \to m^2} mE(p^2) = (45+9a)m^2(m^2 - m_L^2)^2 = \lim_{p^2 \to m^2} -F(p^2) \quad . \tag{3.168}$$

Thus the mixed condensate contribution to the quark self-energy supports a gauge independent propagator pole at $\not{p}=m$, providing evidence for the stability of the dynamical mass under corrections from higher dimensional condensates.[36,38]

An important aspect of this result is the ability to support an arbitrary current mass in the presence of a stable, gauge independent dynamical mass. In earlier work considering only the first two orders in the expansion of the quark-quark-gluon vev, it was found that gauge independence required that m_L and m be equilibrated.[37,48] This conclusion was questioned when plane-wave methods were employed to calculate the mixed condensate component of the quark self-energy.[59] However, the analysis of this section (see references 36,38) should resolve the questions regarding the mixed condensate's effect upon dynamical mass generation.

The non-abelian graph (Figure Six) plays an important role in permitting m_L to remain arbitrary. In a "toy" theory which does not have non-abelian couplings; leaving only the contributions from Figures One, Four and Five, it is not possible to have a gauge independent propagator pole without equilibrating m_L and m. This suggests that the non-abelian nature of QCD allows distinct current and constituent mass scales to coexist without compromising gauge independence. It should be stressed that the assumptions made regarding the truncation of the abelian graphs beyond order m^3, and the hypothesized form of the order and higher-order terms from the non-abelian graph are crucial to this conclusion.

In summary, the lowest order mixed condensate contribution to the quark self-energy has been explicitly calculated to order m^3 in the expansions of $\langle\bar{\psi}(z)\psi(y)\rangle$ and $\langle\bar{\psi}(z)A^a_\mu(w)\psi(y)\rangle$. The results were then extended to all orders in m by assuming that the truncation of abelian graphs continued beyond order m^3, and that the alternating character of the non-abelian graph accounted for higher order terms. The resulting self-energy was found to support a gauge independent mass shell at $\not{p}=m$, independent of current mass value. The failure of the mixed condensate to shift the pole position from the $\not{p}=m$ value observed in Chapter Two suggests a stability of the dynamical mass under corrections from higher dimensional condensates. This behaviour indicates that a physical hierarchy of condensates may exist, based upon the order of perturbation theory at which a condensate may first enter the OPE.

CHAPTER FOUR

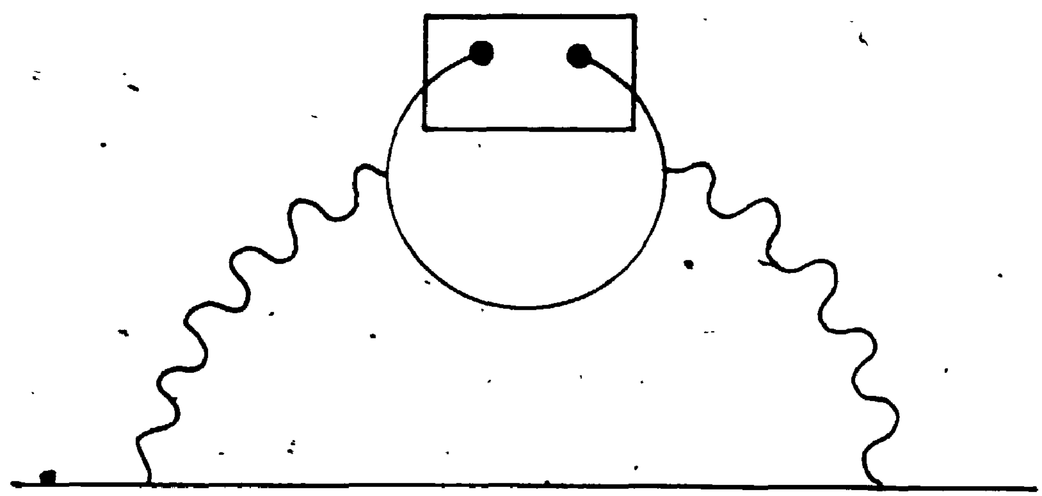
QUARK CONDENSATE COMPONENT OF THE GLUON SELF-ENERGY

4-1 Transversality of the Gluon Self-Energy

In this chapter the lowest order quark condensate contributions to the gluon propagator will be calculated. Motivation for the study of this problem was originally provided by consideration of order g^4 (one-loop) corrections to the quark propagator, as illustrated in Figure Seven.[60] The order g^2 correction to the gluon propagator, which is a sub-process of Figure Seven, will thus determine the gauge dependence of these higher order corrections to the quark propagator. In particular, transversality of the gluon self-energy will guarantee the gauge independence of the two-point amplitude of Figure Seven, since the gauge parameter appearing in the longitudinal portion of the gluon propagators vanishes when combined with a transverse self-energy.

Transversality of the gluon self-energy is also required by the symmetries of the QCD Lagrangian. After quantization, invariance of the effective Lagrangian for QCD is described by an extension of the gauge symmetry which is known as BRS (Becchi-Rouet-Stora) invariance.[61] This symmetry places many restrictions upon the QCD Green functions.[62] For the gluon propagator, BRS invariance demands that the gluon self-energy is transverse, a property known as the Slavnov-Taylor identity.[63] Thus if the SU(3) gauge symmetry of QCD is to be maintained, then the quark condensate component of the gluon self-energy must be transverse.

FIGURE SEVEN



One loop, order g^4 correction to the quark self-energy through the quark-quark vev.

Consider the OPE for the gluon propagator, which has the form

$$iD^{ab}_{\mu\nu}(p) = \int d^4x\, e^{ip\cdot x} \langle 0|T(A^a_\mu(x)A^b_\nu(0))|0\rangle$$

$$= \delta^{ab}C^I_{\mu\nu}(p) + C^{\bar\psi\psi}_{\mu\nu}(p)\,\delta^{ab}\langle\bar\psi\psi\rangle + C^{FF}_{\mu\nu}(p)\,\delta^{ab}\langle F^c_{\lambda\sigma}F^c_{\lambda\sigma}\rangle$$

$$+ \text{higher dimensional condensates} \quad . \qquad (4.1)$$

The lowest order contribution to the $C^{\bar\psi\psi}_{\mu\nu}(p)$ coefficient in (4.1) is represented by the vacuum polarization diagram in Figure Eight. The amplitude corresponding to this process is obtained by Wick expanding an expression similar to (2.8), leading to the equation[39]

$$i\Delta D^{ab}_{\alpha\beta}(p) = -\frac{1}{2}g^2\int d^4x\, d^4y\, d^4z\, e^{ip\cdot x} \langle T\left(A^a_\alpha(x)A^e_\mu(y)\right)\rangle$$

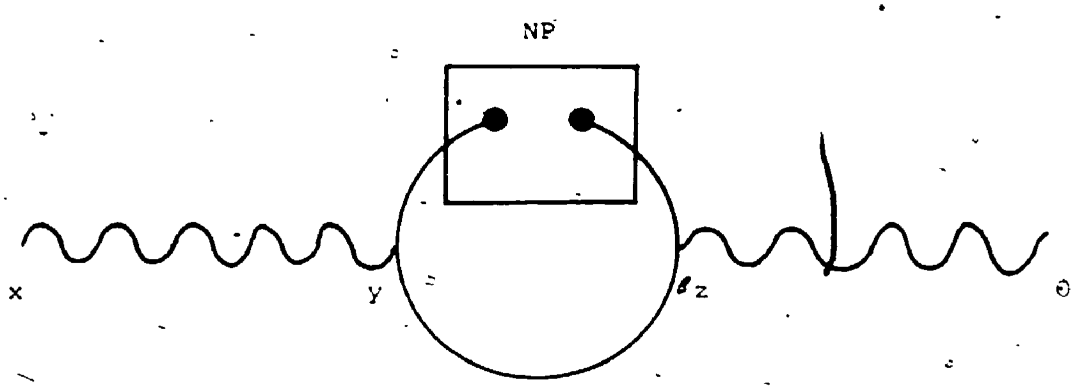
$$\times\ \mathrm{Tr}\left[\lambda^c\gamma^\mu\langle T\left(\psi(y)\bar\psi(z)\right)\rangle\gamma^\nu\lambda^e\langle\bar\psi(y)\psi(z)\rangle\right]\langle T\left(A^e_\nu(z)A^b_\beta(0)\right)\rangle \qquad (4.2)$$

where contributions from all quark flavours are included.

The non-perturbative content of (4.2) resides in the vev $\langle\bar\psi(y)\psi(z)\rangle$. The quark condensate component of this object is translation invariant, and in Section 2-3 was determined to all orders in its covariantized Taylor series. Shifting the variables of integration in (2.4) to (x-y),(y-z),z from x,y,z and using the basic form of $\langle\bar\psi(y)\psi(z)\rangle$ in (2.69) simplifies (4.2),

$$i\Delta D^{ab}_{\alpha\beta}(p) = -\frac{1}{2}g^2\mathrm{Tr}\left(\lambda^a\lambda^b\right) i\left[-\frac{g^{\alpha\mu}}{p^2} + (1-a)\frac{p^\alpha p^\mu}{p^4}\right]$$

FIGURE EIGHT



Order g^2 correction to the gluon self-energy from the quark-quark vev.

$$\times \int d^4(y-z)\, e^{ip\cdot(y-z)} \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(y-z)}$$

$$\times \mathrm{Tr}\left\{ \gamma^\mu \frac{\not{q}+m_L}{q^2-m_L^2} \gamma^\nu \langle\bar{\psi}(y)\psi(z)\rangle \right\} i\left\{ -\frac{g^{\nu\beta}}{p^2} + (1-a)\frac{p^\nu p^\beta}{p^4} \right\} \quad , \qquad (4.3)$$

where the free-field propagators from (2.12) and (2.13) have been used.

The gluon self-energy $\Pi^{ab}_{\mu\nu}(p)$ is defined by the following expression[47]

$$iD^{ab}_{\mu\nu}(p) = i\delta^{ab}D_{\mu\nu}(p) + iD_{\mu\alpha}(p)\, i\Pi^{ab}_{\alpha\beta}\, iD_{\beta\nu}(p) + \ldots \qquad (4.4)$$

where $D_{\mu\nu}\delta^{ab}$ represents the free gluon propagator. Using (4.4) and the colour algebra result $\mathrm{Tr}(\lambda^a\lambda^b)=2\delta^{ab}$ identifies the gluon self-energy in (4.3) as

$$\Pi^{ab}_{\mu\nu}(p) \equiv \delta^{ab}\Pi_{\mu\nu}(p) \qquad (4.5a)$$

$$\Pi^{ab}_{\mu\nu}(p) = -g^2\delta^{ab}\int d^4(y-z)\, e^{ip\cdot(y-z)} \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(y-z)}$$

$$\times \mathrm{Tr}\left[\gamma^\mu \frac{\not{q}+m_L}{q^2-m_L^2} \gamma^\nu \langle\bar{\psi}(y)\psi(z)\rangle \right] \qquad (4.5b)$$

where only the quark condensate component of (4.5b) will be extracted.

The $\langle\bar{\psi}\psi\rangle$ projection of the $\langle\bar{\psi}(y)\psi(z)\rangle$ vev is obtained from (2.69) by interchanging y and z.

$$\langle\bar{\psi}(y)\psi(z)\rangle = \frac{1}{3}\langle\bar{\psi}\psi\rangle \sum_{n=0}^{\infty} \frac{(-im)^{2n}}{(2n)!}\, a_{2n}(y-z)^{2n}$$

$$-\frac{1}{3}\langle\bar{\psi}\psi\rangle \sum_{n=0}^{\infty} \frac{(-im)^{2n+1}}{(2n+1)!}\, b_{2n+1}\, \gamma\cdot(y-z)(y-z)^{2n} \qquad (4.6a)$$

$$\frac{a_{2n}}{(2n)!} = \frac{1}{n!(n+1)!4^{n+1}} \tag{4.6b}$$

$$\frac{b_{2n+1}}{(2n+1)!} = \frac{1}{2(n+2)!n!4^{n+1}} \tag{4.6c}$$

The mass parameter m appearing in (4.6a) was determined to order g^2 in Section 3-2, with the result $m=m_L+O(g^2)$. Thus for the lowest order contribution to the gluon self-energy, the Lagrangian mass m_L and the mass parameter m must be equilibrated to the common value denoted by m. The order g^2 corrections to m in (4.5b) can only be consistently included with processes such as illustrated in Figure Nine.

Equation (4.5b) is now evaluated as a power series in m by substituting (4.6a) for the $\langle\bar{\psi}(y)\psi(z)\rangle$ vev, and by expanding the propagator (with m_L replaced by m) as*

$$\frac{1}{q^2-m^2} = \frac{1}{q^2}\sum_{j=0}^{\infty}\left(\frac{m^2}{q^2}\right)^j \tag{4.7}$$

leading to the following expression for $\Pi_{\mu\nu}(p)$.

$$\Pi_{\mu\nu}(p) = -g^2\int d^4(y-z)\; e^{ip\cdot(y-z)}\int\frac{d^4q}{(2\pi)^4}\; e^{-iq\cdot(y-z)}\;\frac{1}{3}\langle\bar{\psi}\psi\rangle$$
$$\times\Bigg[-\mathrm{Tr}\left(\gamma^\mu \not{q}\gamma^\nu\sum_{j=0}^{\infty}\left(\frac{m^2}{q^2}\right)^j\frac{1}{q^2}\sum_{n=0}^{\infty}\gamma\cdot(y-z)(y-z)^{2n}\frac{(-im)^{2n+1}}{(2n+1)!}b_{2n+1}\right)$$
$$+\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)\sum_{j=0}^{\infty}\left(\frac{m^2}{q^2}\right)^j\frac{m}{q^2}\sum_{n=0}^{\infty}(y-z)^{2n}\frac{(-im)^{2n}}{(2n)!}a_{2n}\Bigg] \tag{4.8}$$

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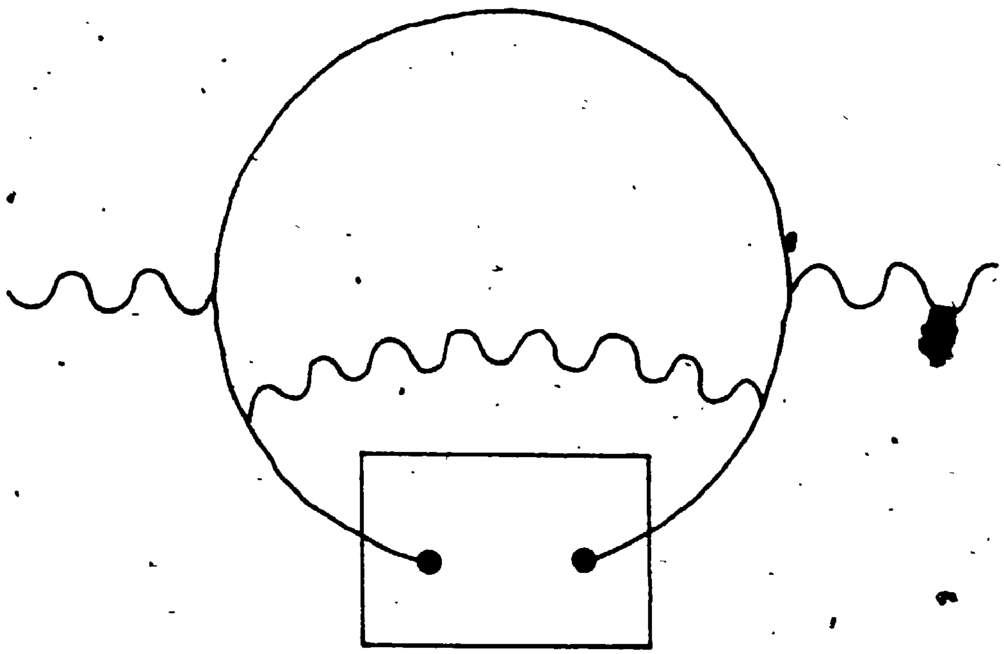
*Expansion for large p^2 has been carried out since this is the general range of validity for the OPE.

FIGURE NINE



One loop, order g^4 correction to the gluon self-energy from the quark-quark vev.

The aggregate coefficients of m are collected in (4.8), leading to a series representation for $\Pi_{\mu\nu}(p)$.[39]

$$\Pi_{\mu\nu}(p) \equiv -\frac{1}{3}g^2\langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} m^{2N+1}\Pi^{(N)}_{\mu\nu}(p) \tag{4.9a}$$

$$\Pi^{(N)}_{\mu\nu}(p) = \int d^4(y-z)\, e^{ip\cdot(y-z)} \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(y-z)}$$

$$\times \left[-\mathrm{Tr}\left(\gamma^\mu \not{q} \gamma^\nu \sum_{n=0}^{N} \frac{\gamma\cdot(y-z)}{(2n+1)!} \frac{(-i)^{2n+1}}{q^{2N-2n+2}} (y-z)^{2n} b_{2n+1}\right)\right.$$

$$\left. + \mathrm{Tr}(\gamma^\mu\gamma^\nu) \sum_{n=0}^{N} \frac{(-i)^{2n}(y-z)^{2n}}{(2n)!\,q^{2N-2n+2}} a_{2n} \right] . \tag{4.9b}$$

Consider $\Pi^{(N)}_{\mu\nu}(p)$ for the special cases of N=0,1, representing the coefficients of m, m³ in (4.9a). Using the values of a_{2n}, b_{2n+1} from (4.6), evaluating the traces, and performing the integrals with the assistance of (2.76) leads to the following expressions.[39]

$$\Pi^{(0)}_{\mu\nu}(p) = \int d^4(y-z)\, e^{ip\cdot(y-z)} \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(y-z)}$$

$$\times \left[\mathrm{Tr}\left(\gamma^\mu \not{q} \gamma^\nu \gamma\cdot(y-z)\right) \frac{i}{16q^2} + \mathrm{Tr}(\gamma^\mu\gamma^\nu) \frac{1}{4q^2} \right] \tag{4.10a}$$

$$\Pi^{(0)}_{\mu\nu}(p) = \frac{1}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \tag{4.10b}$$

$$\Pi^{(1)}_{\mu\nu}(p) = \int d^4(y-z)\, e^{ip\cdot(y-z)} \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(y-z)}$$

$$\times \left[\mathrm{Tr}\left(\gamma^\mu \not{q} \gamma^\nu \gamma\cdot(y-z) \frac{i}{16q^4} - \frac{(-i)^3}{192q^2} \gamma^\mu \not{q} \gamma^\nu \gamma\cdot(y-z)(y-z)^2\right)\right.$$

$$\left. + \mathrm{Tr}(\gamma^\mu\gamma^\nu) \left(\frac{1}{4q^4} + \frac{(-i)^2}{32q^2}(y-z)^2 \right) \right] , \tag{4.11a}$$

$$\Pi^{(1)}_{\mu\nu}(p) = \frac{4}{3p^4}\left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right] \tag{4.11b}$$

The values

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\lambda\right) = 4\left[g^{\mu\lambda}g^{\nu\alpha} - g^{\alpha\lambda}g^{\mu\nu} + g^{\mu\alpha}g^{\nu\lambda}\right] \tag{4.12a}$$

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4\,g^{\mu\nu} \tag{4.12b}$$

were used in the simplification of (4.10) and (4.11).

The results of (4.10b), (4.11b) show that the lowest order quark condensate component of the gluon self-energy is transverse up to order m^3. The general case of $N>1$ in (4.9b) can now be calculated. Performing integrals through use of (2.76), and substituting the traces from (4.12) leads to a general expression for $\Pi^{(N)}_{\mu\nu}(p)$.

$$\begin{aligned}\Pi^{(N)}_{\mu\nu}(p) &= 4\left[g^{\mu\lambda}g^{\nu\alpha} - g^{\alpha\lambda}g^{\mu\nu} + g^{\mu\alpha}g^{\nu\lambda}\right] \\ &\quad\times \sum_{n=0}^{N} \frac{b_{2n+1}}{(2n+1)!}\frac{\partial}{\partial p_\lambda}\left(\frac{\partial^2}{\partial p^2}\right)^n \frac{p^\alpha}{p^{2N-2n+2}} \\ &\quad + 4\,g^{\mu\nu}\sum_{n=0}^{N}\frac{a_{2n}}{(2n)!}\left(\frac{\partial^2}{\partial p^2}\right)^n \frac{1}{p^{2N-2n+2}}\end{aligned} \tag{4.13}$$

The derivatives appearing in (4.13) are easily evaluated for $N>1$.[39]

$$\left(\frac{\partial^2}{\partial p^2}\right)^n \frac{1}{p^{2N-2n+2}} = \begin{cases} \dfrac{4^n N!(N-1)!}{(N-n)!(N-n-1)!}\dfrac{1}{p^{2N+2}} & ;\ N-n>0 \\ 0 & ;\ N=n\neq 0 \end{cases} \tag{4.14}$$

$$\frac{\partial}{\partial p_\lambda}\left(\frac{\partial^2}{\partial p^2}\right)^n \frac{p^\alpha}{p^{2N-2n+2}} = \begin{cases} \frac{1}{p^{2N+2}}\left(g^{\alpha\lambda} - (2N+2)\frac{p^\alpha p^\lambda}{p^2}\right)\frac{4^n N!(N-2)!}{(N-n)!(N-n-2)!} & ; N-n>1 \\ 0 & ; N-n=1,\ n\neq 0 \\ 0 & ; N=n>1 \end{cases} \tag{4.15}$$

Substituting the above results into (4.13) gives $\Pi^{(N)}_{\mu\nu}$ for N>1.

$$\Pi^{(N)}_{\mu\nu}(p) = \frac{4g^{\mu\nu}}{p^{2N+2}}\left(\sum_{n=0}^{N-1}\frac{a_{2n}}{(2n)!}\frac{4^n N!(N-1)!}{(N-n)!(N-n-1)!} + \sum_{n=0}^{N-2} 2N\frac{b_{2n+1}}{(2n+1)!}\frac{4^n N!(N-2)!}{(N-n)!(N-n-2)!}\right)$$
$$- \frac{4p^\mu p^\nu}{p^{2N+4}}\sum_{n=0}^{N-2}\frac{b_{2n+1}}{(2n+1)!}\frac{4^{n+1}(N+1)!(N-1)!}{(N-n)!(N-n-2)!} \tag{4.16}$$

The values of the coefficients a_{2n}, b_{2n+1} are recalled from (4.6), simplifying (4.16).

$$\Pi^{(N)}_{\mu\nu}(p) = \frac{g^{\mu\nu}}{p^{2N+2}}\left(\sum_{n=0}^{N-1}\binom{N}{n}\binom{N}{n+1}\frac{1}{N} + \sum_{n=0}^{N-2}\binom{N}{n}\binom{N}{n+2}\frac{1}{N-1}\right)$$
$$- \frac{p^\mu p^\nu}{p^{2N+4}}\sum_{n=0}^{N-2}\frac{2(N+1)}{N(N-1)}\binom{N}{n}\binom{N}{n+2} \tag{4.17}$$

The sums over binomial coefficients in (4.17) are tabulated, having the value[64]

$$\sum_{n=0}^{N-k}\binom{N}{n}\binom{N}{n+k} = \frac{(2N)!}{(N-k)!(N+k)!} \tag{4.18}$$

A general expression for $\Pi^{(N)}_{\mu\nu}(p)$ can now be obtained by using (4.18) to simplify (4.17).

$$\Pi^{(N)}_{\mu\nu}(p) = \frac{(2N)!2(N+1)}{N!(N+2)!}\frac{1}{p^{2N+2}}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \quad ; N > 1 \tag{4.19}$$

Recalling the special cases of N=0,1 from (4.10b) and (4.11b) reveals that (4.19) is valid for all N. Thus the order g^2 quark condensate component of the gluon self-energy is transverse to all orders in m. The final result for $\Pi_{\mu\nu}(p)$ is obtained by substituting (4.19) into (4.9a).[39]

$$\Pi_{\mu\nu}(p) = \frac{2g^2}{3p^2}m\langle\bar{\psi}\psi\rangle\left(\frac{p^\mu p^\nu}{p^2} - g^{\mu\nu}\right)\sum_{N=0}^{\infty}\frac{(2N)!(N+1)}{N!(N+2)!}\left(\frac{m^2}{p^2}\right)^N \tag{4.20}$$

In summary, the order g^2 quark condensate component of the gluon self-energy has been calculated to all orders in the expansion of $\langle\bar{\psi}(y)\psi(z)\rangle$. The mass m which appears in the quark-quark vev was equilibrated with m_L, since the dynamical component of the quark mass will only contribute at higher orders in g. This leads to a transverse gluon self-energy, satisfying the Slavnov-Taylor identity which reflects the symmetries of the QCD Lagrangian. The transversality of the gluon self-energy also ensures that the order g^4 $\langle\bar{\psi}\psi\rangle$ correction to the quark self-energy in Figure Seven will be gauge independent. Physical consequences of (4.20) upon the gluon propagator will be investigated in the next section.

4-2 Physical Implications of the Gluon Self-Energy

The lowest order quark condensate contribution to the gluon self-energy was calculated in the previous section. The effect of this self-energy upon the gluon propagator will now be examined in the context of gluon mass generation. Others have suggested that the gluon condensate sets the scale for an effective gluon mass, which then provides a mass estimate for the lightest scalar glueball.[65] Since the quark condensate was ignored in these calculations, it is of interest to determine the magnitude of the quark condensate effects in comparison with the gluonic ones. This is an important consideration since for heavy quarks, a contribution to a gluon effective mass of the form $12m\langle\bar{\psi}\psi\rangle/p^2$ is of the same magnitude as a gluonic component of $\alpha_s\langle FF\rangle/\pi p^2$.

To determine the effect of the gluon self-energy on the structure of the gluon propagator, the functional behaviour of the infinite series in (4.20) must be determined. The object $\Pi(p^2)$ is defined as follows.

$$\Pi_{\mu\nu}(p) \equiv \left(\frac{p^\mu p^\nu}{p^2} - g^{\mu\nu}\right)\Pi(p^2) \qquad (4.21a)$$

$$\Pi(p^2) = \frac{2g^2}{3p^2}m\langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{(2N)!(N+1)}{N!(N+2)!}\left(\frac{m^2}{p^2}\right)^N \qquad (4.21b)$$

Investigation of (4.21b) shows that the series converges for $|m^2/p^2| < \frac{1}{4}$. The infinite sum can be evaluated with the assistance of tables,[66] leading to the expression[39]

$$\Pi(p^2) = \frac{2g^2}{3p^2}m\langle\bar{\psi}\psi\rangle\left(\frac{p^4}{12m^4}\left\{\left(1 - 4\frac{m^2}{p^2}\right)^{3/2} - 1 + 6\frac{m^2}{p^2}\right\} + \frac{8m^2}{3p^2}\left[1 + \left(1 - 4\frac{m^2}{p^2}\right)^{\frac{1}{2}}\right]^{-3}\right) \qquad (4.22)$$

where $p^2 > 4m^2$.

If only Euclidean momenta are considered, then (4.22) can be analytically continued to values of $|p^2| < 4m^2$, which implies that

$$\Pi(Q^2) \equiv \Pi(-p^2) = \frac{2g^2}{3m}\langle\bar{\psi}\psi\rangle \left[-\frac{Q^2}{12m^2}\left\{ \left[1 + 4\frac{m^2}{Q^2}\right]^{3/2} - 1 - 6\frac{m^2}{Q^2} \right\} + \frac{8m^4}{3Q^4}\left\{1 + \left[1 + 4\frac{m^2}{Q^2}\right]^{\frac{1}{2}} \right\}^{-3} \right] \quad (4.23)$$

The momentum regime which is of interest lies between 500 MeV and 750 MeV, the range of effective gluon mass estimates.[65] In this region m^2/Q^2 will be small for light quarks but relatively large for heavy quarks. Defining $x = Q^2/m^2$, (4.23) becomes

$$\Pi(x) = \frac{2g^2}{3m}\langle\bar{\psi}\psi\rangle x^{-\frac{1}{2}} \left[-\frac{1}{12}(x+4)^{\frac{3}{2}} + \frac{1}{12}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + \frac{8}{3}\left[x^{\frac{1}{2}} + (x+4)^{\frac{1}{2}}\right]^{-3} \right] \quad (4.24)$$

which has the following asymptotic forms for x large and for x small.

$$\Pi(Q^2) \sim -\frac{2g^2}{9Q}\langle\bar{\psi}\psi\rangle \quad ; \quad Q^2 << m^2 \quad (4.25a)$$

$$\Pi(Q^2) \sim -\frac{mg^2}{3Q^2}\langle\bar{\psi}\psi\rangle \quad ; \quad Q^2 >> m^2 \quad (4.25b)$$

Now return to (4.21a) and consider the effect of $\Pi(p^2)$ upon the gluon propagator. In Landau gauge, the self-energy modifies the gluon propagator, leading to the form

$$D^{\mu\nu}(p) = \left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right]\frac{1}{p^2 + \Pi(p^2)} \quad (4.26)$$

Equation (4.26) implies that $\Pi(p^2)$ can be interpreted as an effective mass.

$$M^2(p^2) \equiv -\Pi(p^2) \qquad (4.27)$$

In order to rigorously demonstrate that $\Pi(p^2)$ will lead to an effective gluon mass, it must be demonstrated that $\Pi(p^2)$ is non-zero at $p^2=0$. However, since OPE methods cannot be extended to such a low momentum value, only a comparison between the scales generated by the quark and gluon condensate can be made.

In the large Q^2 limit, the behaviour of (4.25b) implies that $M^2(Q^2)$ approaches zero, and the massless gluon of perturbative QCD is recovered. In Section 2-4, the assumption that mass scales generated by OPE effects are invariant upon reflection from space-like (Euclidean) to time-like momenta was made. Investigation of the bracketed expression in (4.24) reveals a function which is negative for all x, approaching zero as x becomes large. Combined with the negative values of the quark condensate, this implies that $M^2(Q^2)<0$, implying that a gluon mass cannot be generated by the quark condensate alone.

Since the effect of the quark condensate contribution to the gluon self-energy will tend to suppress any mechanism which generates a dynamical gluon mass, it is interesting to examine whether it is still possible to develop an effective gluon mass when gluon condensate effects are included. The gluon condensate must lead to a self-energy of the form

$$\Pi(Q^2) = -\frac{\Delta^4}{Q^2} \qquad (4.28)$$

where $\Delta \simeq 600$ MeV. At this energy scale the contributions from $\langle \bar{u}u \rangle$ and

$\langle \bar{d}d \rangle$ will be negligible in comparison to (4.28). For the s and c quarks, the contributions from (4.24) are*

$$\Pi_s((600\text{MeV})^2) \simeq \frac{(2\pi)(0.7)(250\text{MeV})^3(0.04)}{3(125\ \text{MeV})} \simeq 7000\ \text{MeV}^2 \qquad (4.29a)$$

$$\Pi_c((600\text{MeV})^2) \simeq \frac{(2\pi)(175\text{MeV})^3(0.95)}{3(1000\text{MeV})} \simeq 10000\ \text{MeV}^2\ . \qquad (4.29b)$$

Thus the total quark condensate effects on gluon mass generation are negligible in comparison with the gluon condensate contributions, confirming the accepted view that $\langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle$ is the dominant parameter for estimating the glueball mass.

To conclude this section, the self-energy in (4.23) will be analyzed for time-like momentum to determine whether unitarity is maintained. It is reasonable to assume that $\Pi(Q^2)$ can be analytically continued to $0<p^2<4m^2$, since in perturbation theory all integrals must be evaluated in Euclidean space via a Wick rotation and the final results are then continued to time-like momenta. For $0<p^2<4m^2$, $\Pi(p^2)$ develops an imaginary part which is equal to [39]

$$\text{Im}\,\Pi(p^2) = \frac{2g^2}{3m} \langle \bar{\psi}\psi \rangle \left(\frac{4-x}{x} \right)^{\frac{1}{2}} \left[\frac{(x-4)^2}{12} - \frac{2(x-1)}{3} \left\{ x(x-1)^2 + (4-x)(x-1)^2 \right\}^{-1} \right] . \qquad (4.30)$$

where $0<x=p^2/m^2<4$. To maintain unitarity, the imaginary part of $\Pi(p^2)$ must be positive, which is indeed the case in (4.30) since both the bracketed portion of (4.30) and the quark condensate $\langle \bar{\psi}\psi \rangle$ are negative.*

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*The behaviour of the bracketed part of (4.30) was determined numerically.

implying that $Im\Pi > 0$. This illustrates the importance of the quark condensate's sign for the existence of sensible field-theoretical results.

4-3 An Application to QCD Sum Rules

The calculation of the gluon self-energy in Section 4-1 is virtually identical to a specific application of the OPE to QCD sum rules. To illustrate this point consider the electromagnetic current for the ρ meson.*

$$J^{\mu}_{(\rho)}(x) = \frac{1}{2}\left(\bar{u}\gamma^{\mu}u - \bar{d}\gamma^{\mu}d\right) \tag{4.31}$$

The important quantity for applications to QCD sum rules is the two point spectral function, defined by

$$\Pi^{\mu\nu}_{(\rho)}(p) \equiv i\int d^4x\, e^{ip\cdot x} \langle 0|T\left(J^{\mu}_{(\rho)}(x)J^{\nu}_{(\rho)}(0)\right)|0\rangle \tag{4.32}$$

After substituting (4.31) into (4.32) and performing a Wick expansion, $\Pi^{\mu\nu}_{(\rho)}(p)$ is equal to

$$\Pi^{\mu\nu}_{(\rho)}(p) = \frac{i}{2}\int d^4x\, e^{ip\cdot x}\, \mathrm{Tr}\left(\gamma^{\mu}\langle T\left(u(x)\bar{u}(0)\right)\rangle\gamma^{\nu}\langle :\bar{u}(x)u(0):\rangle + \cdots\right) \tag{4.33}$$

The values of the up and down current masses and condensates are virtually equal, implying that (4.33) can be written in terms of $\psi(x)$, representing a generic light quark

$$\Pi^{\mu\nu}_{(\rho)}(p) = i\int d^4x\, e^{ip\cdot x}\, \mathrm{Tr}\left(\gamma^{\mu}\langle T\left(\psi(x)\bar{\psi}(0)\right)\rangle\gamma^{\nu}\langle :\bar{\psi}(x)\psi(0):\rangle\right) \tag{4.34}$$

---

*The introduction to the QCD sum rule for the ρ meson is based upon the treatment of reference 47.

where contributions from higher dimensional condensates and perturbation theory have been ignored. Comparing the above expression with (4.5b) reveals that (4.34) can be obtained from (4.20) simply by ignoring the factor of g^2 and multiplying by three to account for the colour trace in (4.34).

$$\Pi^{\mu\nu}_{(\rho)}(p) = \frac{2}{p^2}\, m\langle\bar{\psi}\psi\rangle \left(\frac{p^\mu p^\nu}{p^2} - g^{\mu\nu} \right) \sum_{N=0}^{\infty} \frac{(2N)!(N+1)}{N!(N+2)!} \left(\frac{m^2}{p^2} \right)^N$$

$$+ \text{ other dimension condensates} \tag{4.35}$$

The transversality of the spectral function in (4.35) is demanded by conservation of the electromagnetic charge. Thus transversality of (4.35) is more significant than in the application to the gluon self-energy, since even a small violation of charge conservation is not anticipated, whereas it is conceivable that $SU_c(3)$ may not be a perfect symmetry of nature.

The spectral function $\Pi^{\mu\nu}_{(\rho)}$ is now written as[25,28,47]

$$\Pi^{\mu\nu}_{(\rho)}(p) = \left(p^\mu p^\nu - p^2 g^{\mu\nu} \right) \Pi_{(\rho)}(p^2) \tag{4.36}$$

which upon comparison with (4.35) shows that the lowest order quark condensate component of $\Pi_{(\rho)}(p^2)$ is

$$\Pi_{(\rho)}(p^2) = \frac{2m}{p^4} \sum_{N=0}^{\infty} \frac{(2N)!(N+1)}{N!(N+2)!} \left(\frac{m^2}{p^2} \right)^N \langle\bar{\psi}\psi\rangle \; . \tag{4.37}$$

The N=1 and higher terms in (4.37) have not been calculated previously since m^2/p^2 is small when (4.37) is applied to QCD sum rules.

The spectral function is related to the ratio $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \text{muons})$ by the dispersion relation[25,28,47]

$$\frac{1}{12\pi^2}\int_0^\infty ds\, \frac{R^{T=1}(s)}{(s+Q^2)^2} = -\frac{d\Pi_{(\rho)}(Q^2)}{dQ^2} \qquad (4.38)$$

where $Q^2 = -p^2 > 0$. Using the result of (4.37), the quark condensate contribution to the dispersion relation becomes

$$\int_0^\infty ds\, \frac{R^{T=1}(s)}{(s+Q^2)^2} = \frac{24}{Q^6} m\langle\bar{\psi}\psi\rangle\pi^2 \sum_{N=0}^{\infty} (-1)^N \frac{(2N)!}{N!N!}\left(\frac{m^2}{Q^2}\right)^N \qquad (4.39)$$

To enhance the contribution of the ρ meson on both sides of (4.39), a Borel transformation is taken.[25] This transformation is defined by the operator $\hat{B}$, such that[25,28,47]

$$\hat{B} = \lim_{\substack{Q^2 \to \infty \\ N \to \infty \\ Q^2/N \equiv M^2}} \frac{1}{\Gamma(N)} (-Q^2)^N \left(\frac{d}{dQ^2}\right)^N \qquad (4.40)$$

For the function $(s+Q^2)^{-\beta}$, the Borel transform is equal to[47]

$$\hat{B}\,(s+Q^2)^{-\beta} = \frac{e^{-s/M^2}}{\Gamma(\beta)\, M^{2\beta}} \qquad (4.41)$$

which leads to the following expression for (4.39).

$$\int_0^\infty ds\, R^{T=1}(s)\, e^{-s/M^2} = \frac{24\pi^2}{M^2} m\langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{(-1)^N (2N)!}{N!N!(N+2)!}\left(\frac{m^2}{M^2}\right)^N \qquad (4.42)$$

For $M^2 \simeq (1\ \text{GeV})^2$ the LHS of (4.42) is dominated by the ρ meson, and R can be estimated by the narrow width approximation.[25,28,47]

$$R^{T=1}(s) = \frac{12\pi^2}{f_\rho^2} m_\rho^2 \delta(s - m_\rho^2) \quad ; \; m_\rho^2 = 0.6 \text{ GeV}^2 \quad f_\rho^2 = 9.6\,\pi \tag{4.43}$$

Under this approximation, the lowest order quark condensate contribution to the ρ meson QCD sum rule is

$$\frac{12\pi^2}{f_\rho^2} m_\rho^2 e^{-m_\rho^2/M^2} = \frac{24\pi^2}{M^2} m\langle\bar{\psi}\psi\rangle \sum_{N=0}^{\infty} \frac{(-1)^N (2N)!}{N!N!(N+2)!} \left(\frac{m^2}{M^2}\right)^N \tag{4.44}$$

Thus the quark condensate contribution to the ρ meson sum rule has been calculated to all orders in the expansion of $\langle\bar{\psi}(y)\psi(z)\rangle$. The spectral function was found to be transverse as required by charge conservation, a result intimately related to that of Section 4-2. It is hoped that knowledge of the higher order terms in QCD sum rules, as provided in this section, will prove useful for obtaining accurate estimates of the quark and gluon condensates.

SUMMARY AND CONCLUSIONS

Quantum Chromodynamics can be extended into the non-perturbative regime through use of the operator-product expansion. This introduces the gauge invariant, dimensionful objects known as condensates into the Green functions of QCD, leading to a power-law dependence which lies beyond the scope of perturbative methods. The OPE would not have any utility unless the contributions from the condensates can be calculated in a simple fashion. Fortunately, for condensates of small mass dimension, the OPE factors long distance QCD behaviour into the condensates allowing a perturbative evaluation of the OPE.

Condensates originate non-perturbatively, serving as a parametrization of the QCD vacuum. Hence the existence of chiral-violating condensates imply that quark masses can be dynamically generated by the OPE. The condensate of lowest mass dimension which violates chiral symmetry is the quark condensate. Since the condensates of lowest dimension tend to dominate the OPE, the quark condensate is expected to be foremost in determining the dynamical quark mass. Thus the order g^2 quark condensate component of the quark self-energy was calculated to determine the dominant non-perturbative portion of the quark mass.

To calculate a mass scale resulting from the quark self-energy, the existence of a mass m must be included in all areas of the calculation. The value of m can then be determined by self-consistently demanding that the self-energy generates this mass. This procedure leads to a power-series in m for the quark self-energy, caused by the corresponding expansion of the quark-quark vacuum expectation value.

The result of this calculation is a series which truncates at order m^2, leading to a gauge independent self-energy at the $p=m$ mass shell. Further confidence in this result was supplied by a calculation of the quark self-energy in the light-cone gauge. The on-shell value of the quark self-energy was found to be identical in both the covariant and non-covariant gauges, a condition which must be satisfied if a mass scale derived from the self-energy is physically meaningful. The gauge independence of the self-energy provides field-theoretical support for the concept of a running dynamical mass and the associated value of m. Using estimates for the quark condensate, m is found to be approximately 300 MeV, a value in excellent agreement with the phenomenology of constituent masses.

Higher dimension condensates can also contribute to the quark self-energy. The only other condensate which leads to an order g^2 correction to the quark propagator is the dimension-four gluon condensate. This condensate is associated with the quark string tension and is thus expected to be important in the context of confinement. However, the chiral invariance of the gluon condensate suggests that a dynamical quark mass will not be developed in the chiral limit. Explicit calculation of the order g^2 gluon condensate component of the quark self-energy, valid to all orders in the expansion of the gluon-gluon vev, revealed a self-energy which vanished in the chiral limit.

This behaviour suggests that for light quarks the parameters of confinement and dynamical mass generation are decoupled. A reflection of this is observed when the gluon and quark condensate effects are combined to obtain the entire set of order g^2 terms in the OPE. The contribution of the gluon condensate to the dynamical mass scale m was then found to be negligible for light quarks, shifting m from the chiral-limiting value by an amount equal to the current mass m_L. Conversely, for heavy quarks the quark condensate is unimportant, while the gluon condensate controls the small shift of m away from the current mass m_L. The strange quark lies between these two limits, and for certain values of the gluon condensate, the current and constituent masses of the s can be accommodated.

Apart from the quark condensate, the lowest dimension object which violates chiral symmetry is the dimension-five mixed condensate, which first enters the OPE at order g^3. If the terms in the OPE can be evaluated perturbatively, then it would be anticipated that dynamical mass generation depends weakly upon the mixed condensate. The mixed condensate component of the quark self-energy was calculated explicitly to order m^3, and extended to all orders through some plausible assumptions. The resulting self-energy is gauge independent on the $\not{p}=m$ mass shell for arbitrary values of the current mass, and the value of m is unshifted from that determined by the quark and gluon condensates. QCD's non-abelian nature appears to be responsible for the lack of constraint upon the current mass, since the non-abelian graph that generates the mixed condensate is crucial for maintaining an arbitrary value of m_L. Stability and gauge independence of the dynamical mass under mixed condensate corrections, provides evidence that dynamical mass generation is insensitive to higher dimensional condensates.

On-shell gauge independence of the self-energy is the field-theoretical restriction which has been investigated for quarks. A restriction upon Green functions which must be satisfied off-shell is the Slavnov-Taylor identity, which requires that the gluon self-energy be transverse. The order g^2 quark condensate component of the gluon propagator was thus calculated to all orders in the expansion of the quark-quark vev. A transverse self-energy was obtained, satisfying the Slavnov-Taylor identity which ultimately is a statement that SU(3) gauge invariance is preserved. The quark condensate component appears with a sign which would tend to suppress mechanisms for generating an effective gluon mass. Unfortunately, this effect is small in the region around 600 MeV where an effective gluon mass is expected.

An application to the QCD sum rule for the ρ meson can be obtained from the calculation of the gluon self-energy. Apart from a numerical factor, the two calculations are identical, allowing the quark condensate portion of the two point spectral function to be extracted from the gluon self-energy. Transversality of the spectral function is required for conservation of the electromagnetic current, a restriction of fundamental importance since even small deviations from charge conservation are not expected. As higher accuracy is demanded from QCD sum rules, it is hoped that the expansions of the ρ meson spectral function to all orders in the quark-quark vev will be required.

In conclusion, the OPE has been applied to the two point Green functions of QCD. Field-theoretical aspects of these amplitudes have been examined, including on-shell gauge independence and the Slavnov-Taylor identity. No violations of these properties were observed, supporting the validity of OPE methods. Thus the concept of a running dynamical quark mass, leading to a light quark mass scale on the order of 300 MeV, has been provided with field-theoretical support. The success of this program can be optimistically viewed as leading to future progress in the analytic solution of fundamental problems in QCD.

APPENDIX

Conventions and Notations of QCD and Special Relativity

This appendix will briefly review some of the notations and conventions employed in the thesis.

The units in which $\hbar=c=1$ will be used. In these units the Minkowski space metric of signature -2 is defined by the following line element.

$$ds^2 = +dt^2 - (d\underline{x})^2 = g^{\mu\nu} dx_\mu dx_\nu \qquad \text{(A.1)}$$

The Einstein summation convention of "sum over repeated indices" as illustrated in (A.1) will be followed, but rigorous distinction between raised and lowered indices will not be made.

Symmetrization and anti-symmetrization of tensors is denoted as follows.

$$V_{(\mu\nu)} = \frac{1}{2}\left(V_{\mu\nu} + V_{\nu\mu}\right) \qquad \text{(A.2a)}$$

$$V_{[\mu\nu]} = \frac{1}{2}\left(V_{\mu\nu} - V_{\nu\mu}\right) \qquad \text{(A.2b)}$$

This definition is easily extended to n indices. Commutators and anti-commutators will be represented by the standard notation.

$$[A,B] = AB - BA \qquad \text{(A.3a)}$$

$$\{A,B\} = AB + BA \qquad \text{(A.3b)}$$

The Lagrangian for free Dirac fermions of mass m is ($\hbar$=c=1)

$$L(x) = \bar{\psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) \tag{A.4}$$

where γ_{μ} are the 4x4 matrices which satisfy the algebra

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2\, g^{\mu\nu} I \tag{A.5}$$

The Bjorken and Drell representation for the Dirac matrices is followed,[67] and the notation $\not{a}=a^{\mu}\gamma_{\mu}$ will be used. The matrix $\sigma_{\mu\nu}$ which will occur in many applications is defined by

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu},\gamma_{\nu}] \tag{A.6}$$

The concept of mass dimension is important for categorizing condensates. In the units $\hbar$=c=1, length and time have mass dimension negative one denoted by [x]=-1. The action defined by

$$S = \int d^4x\ L(x) \tag{A.7}$$

must be dimensionless, implying that [L(x)]=+4. Thus in (A.4) $[\psi]=[\bar{\psi}]=3/2$, identifying the dimension of the fermion fields.

The QCD Lagrangian is obtained by assuming that the quark fields $\psi(x)$ transform as the fundamental representation of $SU_c(3)$

$$\psi'(x) = e^{-ig\lambda^a\theta^a}\psi(x) \tag{A.8}$$

where the eight λ matrices are the generators of $SU_c(3)$ in its fundamental representation. For a general representation of $SU_c(3)$, the eight generators T^a satisfy the following algebra.

$$[T^a, T^b] = if_{abc}T^c \quad ; \; Tr[T^a] = 0 \tag{A.9a}$$

$$f_{123} = 1 \tag{A.9b}$$

$$-f_{156} = f_{147} = f_{257} = -f_{367} = f_{246} = f_{345} = \tfrac{1}{2} \tag{A.9c}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2} \tag{A.9d}$$

In the fundamental representation, the 3x3 matrices which generate the algebra of (A.9) are known as the Gell-Mann matrices, and have the normalization $\lambda^a = \frac{1}{2}T^a$. Some important identities for the Gell-Mann matrices are

$$Tr(\lambda^a \lambda^b) = 2\delta^{ab} \tag{A.10a}$$

$$\lambda^a \lambda^a = \frac{16}{3} I \tag{A.10b}$$

$$\lambda^b \lambda^a \lambda^b = -\frac{2}{3}\lambda^a \tag{A.10c}$$

The QCD Langrangian is now obtained by gauging the global symmetry of (A.8), requiring (A.4) to be invariant under a local $SU_c(3)$ rotation. This introduces the need for the eight gauge bosons (gluons), which transform as the adjoint representation of $SU_c(3)$.

$$\psi'(x) = \psi(x) + \delta\psi(x) \quad ;$$

$$\delta\psi(x) = -igT^a\delta\theta^a(x)\,\psi(x) \tag{A.11}$$

$$A^{a\,\prime}_\mu = A^a_\mu + \delta A^a_\mu \quad ; \quad \delta A^a_\mu = g\,f_{abc}\delta\theta^b(x)\,A^c_\mu - \partial_\mu\delta\theta^a(x) \tag{A.12}$$

The transformation laws of (A.11) and (A.12) allow the introduction of the covariant derivative D_μ which permits a gauge invariant Lagrangian to be constructed.

$$D_\mu \equiv \partial_\mu - igT^aA^a_\mu(x) \tag{A.13a}$$

$$L(x) = \bar{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x) \tag{A.13b}$$

An additional term must be added to the Lagrangian in (A.13b) in order to describe the behaviour of gluons. The usual choice is to construct the field strength $G_{\mu\nu}$ identified by

$$G_{\mu\nu} = [D_\nu, D_\mu] \equiv igT^aF^a_{\mu\nu} \tag{A.14a}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu \tag{A.14b}$$

A gauge invariant, scalar quantity can be constructed from the field strength and then added to the QCD Lagrangian. Thus the total QCD Lagrangian for a single quark flavour is

$$L(x) = \bar{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x) + \frac{1}{2g^2}\,\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) \tag{A.15}$$

Chiral symmetry is an important concept for the considerations of dynamical quark mass. Projection operators can be defined by

$$L = \frac{1}{2}(I + \gamma_5) \qquad R = \frac{1}{2}(I - \gamma_5) \qquad \text{(A.16a)}$$

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad \gamma_5^2 = I \qquad \text{(A.16b)}$$

to project out the left and right-handed components of the quark field. Performing the projections of the quark portion of the Lagrangian in (A.15) leads to the result

$$\bar{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x) = \overline{\psi_R} i\gamma^\mu D_\mu \psi_R + \overline{\psi_L} i\gamma^\mu D_\mu \psi_L + m\left(\overline{\psi_R}\psi_L + \overline{\psi_L}\psi_R\right) \qquad \text{(A.17)}$$

Hence for m=0 the right and left-handed components of the quark fields decouple. This decoupling is intimately related to the invariance of the Lagrangian under a chiral transformation defined by

$$\psi'(x) = e^{i\gamma_5\alpha}\psi(x) \simeq \left(1 + i\alpha\gamma_5\right)\psi(x)$$

$$\bar{\psi}'(x) \simeq \bar{\psi}(x)\left(1 + i\alpha\gamma_5\right) \qquad \text{(A.18)}$$

Transforming the LHS of (A.17) under the chiral transformation (A.18) leads to the following expression

$$\delta\left[\bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\right] \simeq -2i\alpha m\bar{\psi}\gamma_5\psi \qquad \text{(A.19)}$$

Hence for m=0 the QCD Lagrangian is invariant under chiral symmetry.

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