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A STUDY OF OSEEN FLOW USING INTEGRAL CONDITIONS

by

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of the requirements for the degree of
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ABSTRACT

The primary objective of this work is to analyze the steady two-dimensional flow of a viscous incompressible fluid past a conformally-mappable cylinder in an unbounded field. The main concern is the correct satisfaction of the boundary conditions at large distances from the cylinder. It is shown that the steady-state asymmetrical flow problem needs careful examination, particularly with regard to the satisfaction of these conditions. A general method is developed to solve this class of problem, under the assumption that flow is governed by the Oseen linearized equations of motion. The method is based on satisfaction of the proper conditioning for the vorticity of integral type. This is considered as a very important part of the solution procedure since the integral conditions ensure both the correct decay of the vorticity at large distances from the cylinder and satisfaction of the physically essential results for the existence of the flow. For Oseen flow the method enables one to obtain the vorticity separately from the stream function. Once the vorticity can be approximated on the surface of the cylinder many properties of the flow can be determined for low Reynolds number.

As an example of the application of the method, the uniform flow past an elliptic cylinder at an arbitrary angle of incidence at low Reynolds number R_E is considered. An analytical expression for the vorticity on the surface of the elliptic cylinder is obtained correct to the order of $[R_E][\ln R_E]^{-1}$, the lowest order term being $O([\ln R_E]^{-1})$.

The leading terms for the asymptotic expansions for the lift, drag coefficients and the circulation round a large contour surrounding the elliptic cylinder are determined in terms of Rr. These approximations are based on Oseen linearized theory and are valid in the case of extremely small Reynolds number Rr as indicated by consideration of the pressure. In this case the Reynolds number Rr is based on the length of the major axis of the ellipse. The method is also applied to the cases of symmetrical and asymmetrical flows past circular cylinders. Results are also given for symmetrical flow in terms of low Reynolds number. A paradox is obtained for asymmetrical flow generated by a rotating circular cylinder. The paradox may be stated as

" *No steady two-dimensional asymmetrical Oseen flow of a viscous incompressible fluid past a rotating circular cylinder is possible* ". It is found to be impossible to obtain a solution in this case in which the circulation is non-zero. Comparisons with existing analytical results have been made, wherever possible, the overall agreement in these comparisons is quite satisfactory.

ACKNOWLEDGEMENTS

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1.2 BASIC EQUATIONS GOVERNING THE FLOW

Consider a cylinder of infinite length and constant cross-section in a viscous incompressible fluid in steady motion. We consider a Cartesian co-ordinate system (x,y,z) with origin inside the contour C of the cylinder and the z-axis being the axis of the cylinder. The fluid moves with uniform velocity U inclined at an angle α to the positive x-direction of the Cartesian co-ordinates. A typical situation is shown in figure 1. The fluid at large enough distances from the cylinder is assumed to remain undisturbed with uniform velocity ($U\cos\alpha$, $U\sin\alpha$, 0). The Navier-Stokes equation together with the equation of continuity, for the steady case, and appropriate boundary conditions are assumed to govern the motion of the fluid. The flow is also assumed to remain two-dimensional within the xy-plane.

The Navier-Stokes equation describing the steady motion of a viscous, incompressible, fluid is, in terms of the dimensionless pressure p, the dimensionless velocity vector $\underset{\sim}{q}$ and the dimensionless vorticity vector $\underset{\sim}{\omega} = \text{curl}\ \underset{\sim}{q}$,

$$\text{grad}\left(\frac{1}{2}\underset{\sim}{q}^2\right) - \underset{\sim}{q} \times \underset{\sim}{\omega} = -\,\text{grad}\ p - \frac{2}{R}\,\text{curl curl}\ \underset{\sim}{q}\ , \qquad (1.2.1)$$

where R is a Reynolds number based on some typical length and typical velocity in the flow field. The equation of continuity is

$$\text{div}\ \underset{\sim}{q} = 0\ . \qquad (1.2.2)$$

Dimensional variables would be given by

$$x' = dx, \quad y' = dy, \quad z' = dz;$$
$$\underset{\sim}{q}' = U\underset{\sim}{q}, \qquad p' = \rho\, U^2 p. \qquad (1.2.3)$$

LIST OF FIGURES

CHAPTER I

FORMULATION OF THE GENERAL PROBLEM

1.1 INTRODUCTION

The problem to be considered is that of the steady two-dimensional flow of a viscous incompressible fluid past a cylinder in an unbounded field. The Navier-Stokes equations, which mathematically describe these and other flows in terms of non-linear differential equations, were formulated in the middle of the last century, and they have been investigated ever since with varying degrees of success. In the present work the basic governing equations for the stream function and vorticity are given in a curvilinear co-ordinate system. Most previous work has employed this type of formulation.

One of the major difficulties encountered in dealing with viscous incompressible flows in two dimensions by means of the equations of the vorticity and stream function is to specify proper vorticity boundary conditions to solve the vorticity transport equation. In fact, the velocity boundary condition provides two boundary conditions on the stream function and its normal derivative whereas none on the vorticity. This formulation makes any uncoupling of the equations difficult. If the solution of the two second order problems is preferred, as in the subsequent analysis, an attempt should be made to provide each equation with its own conditions. It is shown that the boundary conditions for two second order partial differential equations can be separated in the sense that the boundary conditions given for the stream function can be used to provide boundary conditions on the vorticity where nonexisted in an explicit form. As it turns out, the proper conditioning for the vorticity is of an integral (global) type instead of the usual boundary (local) type. The integral conditions for the vorticity are obtained by making an asymmetric generalization of a method first proposed by Dennis & Chang [9] for symmetric flows, and used by Collins [7], Badr & Dennis [1], Staniforth [43]. The problem of establishing appropriate conditions for the vorticity transport equation is considered by Quartapelle & Valz-Gris [36] from a conceptual point of view.

There are numerous investigations in existence of two dimensional symmetrical flows, both steady and unsteady, about various types of cylinders in an unbounded field but relatively few in the case of asymmetrical flows. Only the steady-state problem is considered in this study. The steady-state problem is in some ways less complicated but in the case of asymmetrical flows needs careful examination, particularly with regard to satisfaction of the boundary conditions at large distances from the cylinder. Several authors have realized the importance of enforcing an accurate boundary condition at large distances from the cylinder, e.g., Chang [5], Dennis [8], Fornberg [13], Ingham [23]. In the paper by Dennis [8] he looks at the steady asymmetrical flow past an elliptic cylinder using the method of series truncation to solve the Navier-Stokes equations with the Oseen approximation throughout the flow. He found, by considering the asymptotic nature of the decay of the vorticity at large distances, that for asymmetrical flows it is not sufficient merely that the vorticity shall vanish far from the cylinder but it must decay rapidly enough. This is achieved by suitable adjustment of the leading term in the asymptotic expansion for the vorticity. The problem does not arise in behaviour of both ψ and ζ is known as infinity is approached. However, the important point here is that two conditions are prescribed for the stream function ψ on C and none for the vorticity ζ. The appropriate boundary conditions for the dimensionless stream function ψ in the case of steady two-dimensional flow of a viscous incompressible fluid past a fixed arbitrary cylinder and a rotating circular cylinder in an unbounded field are

(i) for flow past an arbitrary fixed cylinder

$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad C, \tag{1.2.10}$$

(ii) for flow past a rotating circular cylinder

$$\psi = 0, \quad \frac{\partial\psi}{\partial n} = -\Omega \quad \text{on} \quad C, \tag{1.2.11}$$

and

$$\frac{\partial\psi}{\partial y} \to \cos\alpha, \quad \frac{\partial\psi}{\partial x} \to -\sin\alpha \quad \text{as} \quad x^2 + y^2 \to \infty . \tag{1.2.12}$$

Here the parameter Ω which gives a measure of the rate of rotation of the cylinder relative to the velocity of the undisturbed stream is defined by $\Omega = a\omega_o/U$, where ω_o is the constant angular velocity with which the cylinder rotates about its centre in a counter-clockwise sense and a is the radius of the cylinder. Conditions (1.2.10) and (1.2.12) are the mathematical statements of no-slip on the boundary C and uniform flow at infinity. Implicit in condition (1.2.12) is that

$$\zeta(x,y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \tag{1.2.13}$$

which is obtained from the asymptotic form of the stream function at infinity i.e., $\psi = y\cos\alpha - x\sin\alpha$.

using complex variables. He showed that in principle no essential difficulty is involved in this procedure, and obtained explicitly a large number of terms. However, his procedure is unduly complicated. Williams [50] in 1965 reduced the problem of Oseen flow past an arbitrary obstacle to a set of integral equations and the approximate solution of these equations for low Reynolds number is obtained without use of special function theory. The only other technique which seems to have been effective for arbitrary bodies is Kaplun's method of matched asymptotic expansions which has been used, e.g., by Chang [5] in 1961, to treat related problems for any fixed Reynolds number. It is a characteristic, however, of the expansions devised in these latter works, that the successive approximations used soon become very complex and the development must be terminated due to the analytical difficulties involved.

There is, however, some interest in carrying out the calculations for elliptic cylinders as these approximate the problems of the circular cylinder and flat plate as limiting cases. Expansion formulas in terms of Reynolds number were first obtained using Oseen approximation for a fixed circular cylinder by Lamb [27] in 1911, later by Bairstow et al. [2], Harrison [18], Filon [12], Faxèn [11], Griffith [17], Sidrak [40], Tomotika & Aoi [46] & [47] and Yamada [51]; an elliptic cylinder by Berry & Swain [3] in 1922, later by Harrison [18], Meksyn [33], Lewis [28], Sidrak [41], Tomotika & Aoi [47], Hasimoto [19], Imai [22] and Williams [50].

As an application of the method proposed in this study, the uniform flow past an elliptic cylinder at an arbitrary angle of incidence is considered on the basis of Oseen approximation. This problem was first treated by Meksyn [33] in 1937 using Mathieu functions. Later Lewis [28] pointed out that Meksyn's solution is physically unacceptable on account of an infinite circulation and gave a formal method to give a correct solution on the basis of modified Oseen approximation due to Southwell & Squire [42]. Hasimoto [19] treated the same problem using elliptic co-ordinates and Mathieu functions. Finally, it may be added that the same problem has been investigated by Imai [22] and Williams [50] by means of the methods which were pointed out earlier. Their results coincide perfectly with those of Hasimoto's analysis. The uniform flow past an elliptic cylinder at an arbitrary angle of incidence, where the flow is assumed to be governed by the linearized equations of motion of Oseen, has been left untouched except in these latter works. However, it should be mentioned that all these investigations were mainly concerned with discussions on the drag experienced by the elliptic cylinder at an arbitrary angle of attack.

The main motivation for this work is to develop a general method involving integral conditions for solving Oseen linearized equations for a two dimensional steady flow of a viscous fluid past a cylinder. This method is based on the fact that the vorticity for the Oseen flow problems can be obtained separately from the stream function, the determination of which from the integral boundary conditions can be effected by successive approximations in terms of Reynolds number provided that the Reynolds number is sufficiently small. In this sense our method is quite different than the other methods of its kind. The method enables us to obtain many properties of the flow in terms of Reynolds number by using an approximated vorticity on the surface of the cylinder. Satisfaction of the integral conditions is considered as a very important part of the solution procedure because these conditions ensure the correct asymptotic decay of the vorticity at large distances from the cylinder and satisfaction of physically essential results for the existence of the flow. This method is applied to the case of uniform flow past an elliptic cylinder at an arbitrary angle of incidence and also, to the cases of symmetrical and asymmetrical flows past circular cylinders. The method is in effect applicable to flow around any cylinder whose boundary can be mapped conformally onto a straight line, although the details may be more complicated.

The analytical investigations of the steady two-dimensional flow of a viscous fluid past a cylinder at a small Reynolds number have hitherto been restricted to the case of a fixed cylinder where the flow is assumed to be governed by the linearized equations of motion of Oseen. In this study an analytical treatment of the asymmetrical flow which is generated by a rotating circular cylinder in a uniform viscous fluid for a small Reynolds number is first carried out on the basis of Oseen approximation by means of the same technique. The numerical solution of this problem is obtained for small rotation rates and moderate values of Reynolds number by Ingham [23] and Loc [29]. Shkadova [39] calculated the drag and lift in the problem of flow of a viscous fluid past a rotating circular cylinder, by solving the Navier-Stokes equations numerically, for Reynolds numbers $10 \leq R \leq 100$. Glauert [14] investigated the same problem for large values of the Reynolds number and Moore [34] for a rapidly rotating circular cylinder. Both of these investigators used analytical methods. The significance of the results obtained in the present study will be commented on later.

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The Navier-Stokes equation describing the steady motion of a viscous, incompressible, fluid is, in terms of the dimensionless pressure p, the dimensionless velocity vector $\underset{\sim}{q}$ and the dimensionless vorticity vector $\underset{\sim}{\omega} = \text{curl}\ \underset{\sim}{q}$,

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where R is a Reynolds number based on some typical length and typical velocity in the flow field. The equation of continuity is

$$\text{div}\ \underset{\sim}{q} = 0\ . \qquad (1.2.2)$$

Dimensional variables would be given by

$$x' = dx, \quad y' = dy, \quad z' = dz;$$

$$\underset{\sim}{q}' = U\underset{\sim}{q}, \qquad p' = \rho\, U^2 p. \qquad (1.2.3)$$

In these equations, d is a typical dimension, U is a representative velocity and ρ is the constant fluid density. The Reynolds number can be defined as $R = 2Ud/\nu$, where ν is the coefficient of kinematic viscosity of the fluid. Since the flow is assumed two dimensional in the xy-plane, the equations (1.2.1) and (1.2.2) in Cartesian co-ordinates are therefore,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + 2R^{-1}\nabla^2 u,$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + 2R^{-1}\nabla^2 v; \qquad (1.2.4)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (1.2.5)$$

where $\underline{q} = (u, v, 0)$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

It may be noted at this stage that the set of equations (1.2.4) and (1.2.5) for two-dimensional, incompressible flow has three unknown dependent variables, namely two velocity components and pressure. The disadvantage of this formulation is that the presence of the pressure terms introduces significant difficulty because there are no simple specified boundary conditions for the pressure. In addition, a method of solution of the equation (1.2.5) is difficult to formulate. The most widely used method of solution of two-dimensional incompressible flow problems such as (1.2.4) and (1.2.5) is by means of the stream function. In this case equation (1.2.5) can be satisfied by introducing the dimensionless stream function $\psi(x,y)$ which is related to the velocity components by the equations

$$u(x,y) = \frac{\partial\psi}{\partial y}, \quad v(x,y) = -\frac{\partial\psi}{\partial x}. \tag{1.2.6}$$

As a consequence of two-dimensional flow, the dimensionless vorticity vector is given by

$$\underline{\omega} = \text{curl}\ \underline{q} = (0,0,-\zeta),$$

where $\zeta(x,y)$ is the dimensionless (negative) scalar vorticity defined by

$$\zeta(x,y) = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{1.2.7}$$

Also, if we eliminate the pressure from equations (1.2.4), the equation which results can be expressed in terms of ψ and ζ. Thus from (1.2.4) and (1.2.5) it is found that the equations governing ψ and ζ are

$$\nabla^2\zeta = \frac{R}{2}\left(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\right), \tag{1.2.8}$$

$$\nabla^2\psi = \zeta. \tag{1.2.9}$$

The formulation (1.2.8) and (1.2.9) for the stream function ψ and the scalar vorticity ζ is now regarded as equivalent to using the momentum equation (1.2.1) and the continuity equation (1.2.2). The advantage of using a coupled pair of second-order partial differential equations will be noticeable in the analytical method of solution to be adopted.

The necessary boundary conditions are as follows.

ψ and $\partial\psi/\partial n$ are known on the contour C of the cylinder, where $\partial/\partial n$ is differentiation in the normal direction to the cylinder. Since the flow field is unbounded, conditions at infinity must be imposed. Generally these reduce to the fact that the asymptotic behaviour of both ψ and ζ is known as infinity is approached. However, the important point here is that two conditions are prescribed for the stream function ψ on C and none for the vorticity ζ. The appropriate boundary conditions for the dimensionless stream function ψ in the case of steady two-dimensional flow of a viscous incompressible fluid past a fixed arbitrary cylinder and a rotating circular cylinder in an unbounded field are

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and

$$\frac{\partial\psi}{\partial y} \to \cos\alpha, \quad \frac{\partial\psi}{\partial x} \to -\sin\alpha \quad \text{as} \quad x^2 + y^2 \to \infty . \tag{1.2.12}$$

Here the parameter Ω which gives a measure of the rate of rotation of the cylinder relative to the velocity of the undisturbed stream is defined by $\Omega = a\omega_0/U$, where ω_0 is the constant angular velocity with which the cylinder rotates about its centre in a counter-clockwise sense and a is the radius of the cylinder. Conditions (1.2.10) and (1.2.12) are the mathematical statements of no-slip on the boundary C and uniform flow at infinity. Implicit in condition (1.2.12) is that

$$\zeta(x,y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \tag{1.2.13}$$

which is obtained from the asymptotic form of the stream function at infinity i.e., $\psi = y\cos\alpha - x\sin\alpha$.

For future purposes it is advantageous at this point to consider a conformal transformation of the form

$$x + iy = F(\xi + i\eta) \qquad (1.2.14)$$

which maps the region outside the cylinder onto the semi-infinite strip shown in figure 2. In the present case it is supposed that the surface of the cylinder (oabcd in figure 1) maps to a curve of constant ξ, say $\xi = \xi_o$ (oabcd in figure 2), the region outside the cylinder corresponds to $\xi > \xi_o$, and that η is an angular co-ordinate which varies from $\eta = 0$ to $\eta = 2\pi$ as the whole flow field outside the cylinder is described, with x and y as periodic functions of η of period 2π. It is also assumed that as $\xi \to \infty$ the coincident curves $\eta = 0$, $\eta = 2\pi$ ultimately approach the direction of the undisturbed stream, and that the mapping (1.2.14) has the asymptotic form given by the equations

$$x \sim ke^{\xi}\cos(\eta + \alpha), \quad y \sim ke^{\xi}\sin(\eta + \alpha) \quad \text{as} \quad \xi \to \infty, \qquad (1.2.15)$$

where k is a constant depending on the transformation, i.e., the particular cylinder shape. A typical situation is illustrated in figure 2 by the elliptic co-ordinate system, but there are a number of transformations of related type having these properties which can deal with different cylinders. Transformations having the above mentioned properties have been used extensively in the literature. Dennis [8] has used the transformations

$$x + iy = \exp(\xi + i\eta) \qquad (1.2.16)$$

and

$$x + iy = \cosh(\xi + i(\eta + \alpha)) \qquad (1.2.17)$$

in his computations of the steady flow around a circular cylinder and an elliptic cylinder, respectively. Tomotika & Aoi [45] & [47] have considered variations of transformations (1.2.16) and (1.2.17) in their analytic investigations.

The question of a proper choice of co-ordinate system is important. From a study of the low Reynolds number Oseen approximations, it is seen that an appropriate system of curvilinear co-ordinates enables one to obtain the solution by separation of variables.

Another purpose of choosing such a transformation (1.2.14) is to create a domain which enables us to derive appropriate global conditions for the vorticity, i.e., to provide boundary conditions on the vorticity where none existed in an explicit form.

If (u_ξ, v_η) generally denote the dimensionless velocity component in the directions of increase of (ξ, η) then

$$u_\xi = \frac{1}{M}\frac{\partial\psi}{\partial\eta}, \qquad v_\eta = -\frac{1}{M}\frac{\partial\psi}{\partial\xi} \qquad (1.2.18)$$

and

$$\zeta = \frac{1}{M^2}\left[\frac{\partial}{\partial\eta}(M u_\xi) - \frac{\partial}{\partial\xi}(M v_\eta)\right], \qquad (1.2.19)$$

where

$$M = \left[\left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial x}{\partial\eta}\right)^2\right]^{\frac{1}{2}} = \left[\left(\frac{\partial y}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2\right]^{\frac{1}{2}}. \qquad (1.2.20)$$

The component equations of momentum are

$$\frac{\partial}{\partial\xi}\left(p + \frac{1}{2}q^2\right) = 2R^{-1}\frac{\partial\zeta}{\partial\eta} + \zeta\frac{\partial\psi}{\partial\xi}, \qquad (1.2.21)$$

$$\frac{\partial}{\partial\eta}\left(p + \frac{1}{2}q^2\right) = -2R^{-1}\frac{\partial\zeta}{\partial\xi} + \zeta\frac{\partial\psi}{\partial\eta}, \qquad (1.2.22)$$

where q is the magnitude of the dimensionless velocity vector $\underset{\sim}{q}$. Under the transformation (1.2.14) equations (1.2.8) and (1.2.9) become respectively

$$\nabla^2 \zeta = \frac{R}{2} \left(\frac{\partial\psi}{\partial\eta}\frac{\partial\zeta}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\eta} \right), \tag{1.2.23}$$

$$\nabla^2 \psi = M^2 \zeta , \tag{1.2.24}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} .$$

To complete the formulation of the problem in (ξ,η) space it remains to state the appropriate boundary conditions in the transformed co-ordinate system. We use the relation between the derivatives

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\xi} ,$$

$$\frac{\partial\psi}{\partial\eta} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\eta} . \tag{1.2.25}$$

Provided that the derivatives of x and y with respect to ξ and η remain finite at $\xi = \xi_o$, we can take

(i) for flow past a fixed cylinder

$$\psi = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{when} \quad \xi = \xi_o , \tag{1.2.26}$$

(ii) for flow past a rotating circular cylinder

$$\psi = 0 , \ \frac{\partial\psi}{\partial\xi} = \Omega \quad \text{when} \quad \xi = \xi_o . \tag{1.2.27}$$

Using (1.2.12) and (1.2.15) it may be deduced that

$$e^{-\xi}\frac{\partial\psi}{\partial\xi} \sim k\sin\eta , \quad e^{-\xi}\frac{\partial\psi}{\partial\eta} \sim k\cos\eta \quad \text{as} \quad \xi \to \infty . \tag{1.2.28}$$

As a consequence of (1.2.28) it follows that

$$\zeta(\xi,\eta) \to 0 \quad \text{as} \quad \xi \to \infty . \qquad (1.2.29)$$

The sets of conditions (1.2.26)-(1.2.29) must be satisfied for all η such that $0 \le \eta \le 2\pi$ and moreover, because of the choice of the co-ordinate system all the flow variables must be periodic functions of η with period 2π. Thus, in particular,

$$\psi(\xi,\eta) = \psi(\xi,\eta + 2n\pi) , \quad \zeta(\xi,\eta) = \zeta(\xi,\eta + 2n\pi) \qquad (1.2.30)$$

for $n = 0, \mp 1, \mp 2, \ldots\ldots.$

On account of the periodicity of ψ and the fact that the transformation (1.2.14) must obviously be chosen so that $M(\xi,\eta)$ is of period 2π in η it follows from (1.2.18) that the velocity components (u_ξ, v_η) will be of period 2π in η. In the case of flow which is symmetrical about the x-axis, $(\alpha = 0)$, both of the functions ψ and ζ are odd functions of η and the solutions of (1.2.23) and (1.2.24) are required only in the region $0 \le \eta \le \pi$ with the conditions

$$\psi(\xi,\eta) = \zeta(\xi,\eta) = 0 \quad \text{when} \quad \eta = 0, \pi . \qquad (1.2.31)$$

Here the problem is more complicated because, although the equations (1.2.18) ensure that the velocity components u_ξ and v_η are periodic functions of η with period 2π, care must exercised to ensure that the pressure in the fluid is likewise periodic in η. The pressure at large distances from the cylinder is taken to be uniform, and then integration of (1.2.21) along a path of constant η from a given station ξ to $\xi = \infty$ will determine a pressure $p(\xi,\eta)$ which is of period 2π in η. However, it is also necessary for consistency that integration of (1.2.22) along a path of constant ξ over a complete period in η shall give zero pressure difference. In view of the subsequent analysis we shall consider this point by assuming the expressions

$$\psi(\xi,\eta) = \frac{1}{2} F_o(\xi) + \sum_{n=1}^{\infty} [F_n(\xi)\cos n\eta + f_n(\xi)\sin n\eta] , \qquad (1.2.32)$$

$$\zeta(\xi,\eta) = \frac{1}{2} G_o(\xi) + \sum_{n=1}^{\infty} [G_n(\xi)\cos n\eta + g_n(\xi)\sin n\eta] , \qquad (1.2.33)$$

where it will be supposed that the first and second partial derivatives with respect to ξ and η of the functions on the left can be represented by the corresponding differentiated series on the right. We wish to point out that the particular assumptions (1.2.32) and (1.2.33) are valid because of the boundary conditions (1.2.30) of this problem. If we substitute (1.2.32) and (1.2.33) in the right side of (1.2.22) and integrate both sides with respect to η from (ξ,η) to $(\xi, \eta + 2\pi)$ we obtain

$$p(\xi,\eta + 2\pi) - p(\xi,\eta) = \pi [2R^{-1} G_o'(\xi) - S(\xi)] , \qquad (1.2.34)$$

where the prime denotes differentiation with respect to ξ and

$$S(\xi) = \sum_{n=1}^{\infty} n [f_n(\xi) G_n(\xi) - F_n(\xi) g_n(\xi)] . \qquad (1.2.35)$$

We can also substitute (1.2.32) and (1.2.33) into (1.2.23) and integrate both sides with respect to η from (ξ,η) to $(\xi, \eta + 2\pi)$ which gives

$$G_o''(\xi) = \frac{1}{2} R\, S'(\xi) . \qquad (1.2.36)$$

Since ζ must vanish as $\xi \to \infty$ for all η then $G_o'(\infty) = 0$. Integration of (1.2.36) and substitution in (1.2.34) then gives

$$p(\xi,\eta + 2\pi) - p(\xi,\eta) = -\pi\, S(\infty) , \qquad (1.2.37)$$

and hence the pressure difference on the left vanishes only if

$$S(\infty) = 0 .$$

Dennis [8] showed that this is satisfied only if vorticity decays rapidly enough as $\xi \to \infty$. His analysis is detailed in Section 2.2.

1.3 THE INTEGRAL BOUNDARY CONDITIONS FOR THE VORTICITY

The velocity boundary condition provides two boundary conditions on the stream function ψ and its normal derivative whereas none on the vorticity ζ. This occurrence poses no problem, if one considers the single fourth-order equation for the stream function, since conditions on the vorticity are contained implicitly in the non-linear coupling of the two equations (1.2.23) and (1.2.24). This formulation makes any uncoupling of the equations difficult. If the solution of the two second-order problems is preferred, as in the subsequent analysis, an attempt should be made of providing each equation with its own conditions. In this section we will show that the boundary conditions for the two second-order partial differential equations (1.2.23) and (1.2.24) can be *separated* in the sense that extra conditions given for the stream function ψ can be used to derive *global* conditions for the vorticity ζ, i.e., to provide boundary conditions on the vorticity where none existed in an explicit form.

We will proceed to derive mathematical conditions on the vorticity ζ, to be subsequently referred to as *the integral conditions*, by making an asymmetric generalization of a method first proposed by Dennis & Chang [9] for symmetric flows, and also used by Collins [7] and Staniforth [43]. In this method, the stream function is assumed to be represented by a Fourier sine series in the variable η. As was pointed out earlier, the generalization to the present investigation is to assume the following Fourier expansion for the stream function

$$\psi(\xi,\eta) = \frac{1}{2} F_c(\xi) + \sum_{n=1}^{\infty} [\, F_n(\xi)\cos n\eta + f_n(\xi)\sin n\eta \,] \,. \qquad (1.3.1)$$

Term by term differentiation of this series with respect to η is justified (see e.g. Jeffreys & Jeffreys [24]). Now if we substitute the series (1.3.1) for ψ into the equation (1.2.24) and multiply by $\sin n\eta$, $n \geq 1$ and integrate from 0 to 2π with respect to η we obtain, from the orthogonality of the trigonometric functions,

$$f_n'' - n^2 f_n = r_n , \qquad n \geq 1 . \tag{1.3.2}$$

Next the expansion (1.3.1) for ψ is substituted into (1.2.24) and a similar procedure of multiplying by $\cos n\eta$, $n \geq 0$ and integration from $\eta = 0$ to $\eta = 2\pi$ yields

$$F_n'' - n^2 F_n = s_n , \qquad n \geq 0 . \tag{1.3.3}$$

In these equations the prime denotes differentiation with respect to ξ, and

$$r_n(\xi) = \frac{1}{\pi} \int_0^{2\pi} M^2 \zeta \sin n\eta \, d\eta , \tag{1.3.4}$$

$$s_n(\xi) = \frac{1}{\pi} \int_0^{2\pi} M^2 \zeta \cos n\eta \, d\eta . \tag{1.3.5}$$

Boundary conditions for (1.3.2) and (1.3.3) follow from (1.3.1), (1.2.26), (1.2.27) and (1.2.28). They are that at the cylinder surface

(i) for flow past a fixed cylinder

$$\left.\begin{aligned} f_n(\xi_o) &= f_n'(\xi_o) = 0 , \quad n \geq 1 ; \\ F_n(\xi_o) &= F_n'(\xi_o) = 0 , \quad n \geq 0 , \end{aligned}\right\} \tag{1.3.6}$$

(ii) for flow past a rotating circular cylinder

$$\left.\begin{aligned} f_n(\xi_o) &= F_n(\xi_o) = 0 , \quad n \geq 1 ; \quad F_o(\xi_o) = 0 , \\ f_n'(\xi_o) &= F_n'(\xi_o) = 0 , \quad n \geq 1 ; \quad F_o'(\xi_o) = -2\Omega , \end{aligned}\right\} \tag{1.3.7}$$

and as $\xi \to \infty$

$$e^{-\xi} f_n \to k\delta_{n,1} , \quad e^{-\xi} F_n \to 0 , \quad n \geq 1 ; \quad e^{-\xi} F_o \to 0 , \tag{1.3.8}$$

$$e^{-\xi} f_n' \to k\delta_{n,1} , \quad e^{-\xi} F_n' \to 0 , \quad n \geq 1 ; \quad e^{-\xi} F_o' \to 0 , \tag{1.3.9}$$

where $\delta_{n,m}$ is the Kronecker delta symbol defined by

$$\delta_{n,m} = 1 \quad \text{if} \quad n = m, \qquad \delta_{n,m} = 0 \quad \text{if} \quad n \neq m.$$

The second condition (1.3.9) in effect follows from (1.3.8), but (1.3.8) and (1.3.9) together express the condition that the velocity components shall reduce to the components of the stream velocity as $\xi \to \infty$. We can now obtain from equations (1.3.2) and (1.3.3)

$$\frac{d}{d\xi}\left[e^{-n\xi}(f_n' + n f_n) \right] = e^{-n\xi} r_n, \quad n \geq 1;$$

$$\frac{d}{d\xi}\left[e^{-n\xi}(F_n' + n F_n) \right] = e^{-n\xi} s_n, \quad n \geq 1. \tag{1.3.10}$$

If we integrate both of the equations (1.3.10) with respect to ξ from $\xi = \xi_o$ to $\xi = \infty$ we find, after use of the conditions given in (1.3.6)-(1.3.9), that

$$\int_{\xi_o}^{\infty} e^{-n\xi} r_n(\xi)\, d\xi = 2k\delta_{n,1}, \quad n \geq 1;$$

$$\int_{\xi_o}^{\infty} e^{-n\xi} s_n(\xi)\, d\xi = 0, \quad n \geq 1. \tag{1.3.11}$$

A further condition is necessary to determine $s_o(\xi)$. We can deduce a suitable condition from (1.3.3) with $n = 0$. In this case integration of (1.3.3) with respect to ξ from $\xi = \xi_o$ to $\xi = \infty$ gives

$$F_o'(\infty) - F_o'(\xi_o) = \int_{\xi_o}^{\infty} s_o(\xi)\, d\xi. \tag{1.3.12}$$

It follows from the conditions (1.3.6) and (1.3.7) that

(i) for flow past a fixed cylinder

$$F_o'(\infty) = \int_{\xi_o}^{\infty} s_o(\xi)\, d\xi, \tag{1.3.13}$$

(ii) for a rotating circular cylinder

$$F_c'(\infty) + 2\Omega = \int_{\xi_o}^{\infty} s_o(\xi)\, d\xi \; . \qquad (1.3.14)$$

In these equations $F_c'(\infty)$ is an unknown constant and it is given, in terms of the circulation round a large contour surrounding the cylinder, by the equation (I-4) in Appendix I. As is well known in the case of steady-state asymmetrical flow problems, the circulation round a circular contour centred at the cylinder and of large enough radius is non-zero. We may then summarize the re-specification of the boundary conditions as being the following

$$\int_{\xi_o}^{\infty} s_o(\xi)\, d\xi = \beta \, , \qquad (1.3.15)$$

$$\int_{\xi_o}^{\infty} e^{-n\xi}\, s_n(\xi)\, d\xi = 0 \, , \quad n \geq 1 \, ; \qquad (1.3.16)$$

$$\int_{\xi_o}^{\infty} e^{-n\xi}\, r_n(\xi)\, d\xi = 2k\delta_{n,1} \, , \quad n \geq 1 \, , \qquad (1.3.17)$$

where

$$r_n(\xi) = \frac{1}{\pi} \int_0^{2\pi} M^2 \zeta \sin n\eta \, d\eta \, , \qquad (1.3.18)$$

$$s_n(\xi) = \frac{1}{\pi} \int_0^{2\pi} M^2 \zeta \cos n\eta \, d\eta \, , \qquad (1.3.19)$$

and β is an unknown constant. The definition of β in terms of the circulation round a large enough contour surrounding the cylinder for a fixed cylinder and a rotating cylinder is given by the equations (I-5) and (I-6) in Appendix I. The convergence of the integrals in the equations (1.3.15)-(1.3.17) will be discussed in Section 2.3.

The sets of conditions (1.3.15)-(1.3.17) effectively give conditions which must be satisfied by the vorticity ζ. These conditions have the peculiarity of being integral (global) type instead of the usual boundary value (local) type. The condition (1.3.15) is merely an expression of the fact that the circulation round a large enough contour surrounding the cylinder must be non-zero. Moreover, it will subsequently be shown that the satisfaction of (1.3.15) is the required condition to ensure that the pressure in the fluid is periodic with period 2π in η. The conditions (1.3.15) and (1.3.17) ensure that the velocity is approached the uniform stream far from the cylinder, just as the satisfaction of (1.3.18) gave this assurance in the symmetrical case, in which the functions $s_n(\xi)$ are identically zero, considered by Dennis & Chang [9]. All the conditions of the problem are therefore satisfied. In this sense we consider the satisfaction of the integral conditions (1.3.16)-(1.3.18) as a very important part of the solution procedure. Providing boundary conditions of global type to the vorticity equation (1.2.23) can be considered as an equivalent substitute for the boundary conditions on the stream function and the equation (1.2.24) which are given in the previous section. The problem of finding such vorticity conditions is considered from a conceptual point of view by Quartapelle and Valz-Gris [36]. They showed that the correct conditions on the vorticity are a consequence of the theorem stated in Appendix I.

It will subsequently be shown that for asymmetrical flows it is not sufficient to impose the necessary condition (1.2.29) as a boundary condition in obtaining solutions of (1.2.23) and it is necessary to which is periodic with period 2π in η and vanishes as $\xi \to \infty$ can be given as

$$\zeta(\xi,\eta) = e^{z\cos\eta}\left[A_o K_o(z) + \sum_{n=1}^{\infty}[A_n \cos n\eta + B_n \sin n\eta]\, K_n(z)\right], \qquad (2.1.23)$$

where the A_n and B_n are arbitrary constants. This expression gives the approximation which is the so-called Oseen solution of equation (1.2.23). The arbitrary constants A_n, $n \geq 0$ and B_n, $n \geq 1$ cannot be determined explicitly without further information, e.g. knowledge of the complete solution of (1.2.23) in a domain which overlaps the domain of validity of (2.1.10). In the subsequent analysis, these constants will be determined to satisfy the integral conditions (1.3.15)-(1.3.17) in terms of very small values of the Reynolds number so that the final result will hold throughout the whole flow field. When the flow is symmetrical about $\eta = 0$ the constants A_n, $n \geq 0$ are identically zero.

As was pointed out earlier, the basic structure of the solution for the vorticity is

$$\zeta(\xi,\eta) = \frac{1}{2}G_o(\xi) + \sum_{n=1}^{\infty}[\, G_n(\xi)\cos n\eta + g_n(\xi)\sin n\eta\,]\,. \qquad (2.1.24)$$

We can express the functions $G_n(\xi)$ in the series (2.1.24) in terms of the coefficients of the trigonometric terms in the series (2.1.23). Now if we equate the series (2.1.24) to the series (2.1.23), multiply each side by $\cos n\eta$ and integrate with respect to η from 0 to 2π we find, after use of a well-known result in connection with the theory of Bessel functions

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\eta}\cos n\eta\, d\eta \qquad (2.1.25)$$

and also,

CHAPTER II

OSEEN THEORY

2.1 THE OSEEN EQUATIONS

As is well known the solution of the Navier-Stokes equation for viscous flow cannot be obtained excatly owing to its nonlinearity; however its form at large distances may be obtained analytically. The flow at large enough distances from the cylindrical body is governed by the Oseen linearized equations in which the Navier-Stokes equations are linearized with the velocity of the uniform stream. To particularize the problem, it is assumed that the fluid velocity is a perturbation from the uniform stream. Thus the dimensionless velocity vector may be given as

$$\underset{\sim}{q} = (\cos\alpha\, i + \sin\alpha\, j) + \underset{\sim}{q}' , \qquad (2.1.1)$$

where $\underset{\sim}{q}'$ is the perturbational velocity vector with magnitude $q' \ll 1$ such that

$$\underset{\sim}{q}' \to 0 \quad \text{at infinity} \qquad (2.1.2)$$

and as stated before, the fluid at large enough distances from the cylinder is assumed to move with uniform velocity ($U\cos\alpha$, $U\sin\alpha$, 0). Then the Navier-Stokes equations may be linearized by substituting the equation (2.1.1) into (1.2.1) and (1.2.2) and neglecting quadratic terms in $\underset{\sim}{q}'$. The resulting equations in Cartesian co-ordinates are

$$\left(\cos\alpha \frac{\partial}{\partial x} , \sin\alpha \frac{\partial}{\partial y} \right) \left(\underset{\sim}{q} \right) = - \operatorname{grad} p - \frac{2}{R} \operatorname{curl} \operatorname{curl} \underset{\sim}{q} \qquad (2.1.3)$$

$$\operatorname{div} \underset{\sim}{q} = 0 . \qquad (2.1.4)$$

These equations are called the Oseen equations. The same equations hold for the perturbation $\underset{\sim}{q}'$. An immediate consequence of these equations, which follows by taking the divergence of the Oseen equation (2.1.3), is that p is a harmonic function

$$\nabla^2 p = 0 . \tag{2.1.5}$$

The corresponding equations for the vorticity ζ and the stream function ψ in Cartesian co-ordinates are

$$\nabla^2 \zeta = \frac{R}{2} \left(\cos\alpha \frac{\partial \zeta}{\partial x} + \sin\alpha \frac{\partial \zeta}{\partial y} \right) , \tag{2.1.6}$$

$$\nabla^2 \psi = \zeta . \tag{2.1.7}$$

If we substitute the conditions (1.2.28) into the equation (1.2.23) the Oseen equations take the forms in the (ξ,η) co-ordinate system

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} = \frac{1}{2} Rke^{\xi} \left(\cos\eta \frac{\partial \zeta}{\partial \xi} - \sin\eta \frac{\partial \zeta}{\partial \eta} \right) , \tag{2.1.8}$$

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = M^2 \zeta . \tag{2.1.9}$$

It may be noted that these equations are the same equations as (2.1.6) and (2.1.7). We wish to point out that the solution of (2.1.8) is, of course, only asymptotically equivalent to the solution of the equation (1.2.23) but the same symbol is used.

The validity of the Oseen equations may be discussed from two different points of view. For any fixed Reynolds number, it is to be expected that the linearized equations are approximately valid near infinity because at large distances from the cylinder the perturbational velocity $\underset{\sim}{q}'$ will be small compared to the free stream velocity. Near the cylinder, the linearization is not justified for any arbitrary Reynolds number. However, as the Reynolds number tends to zero the Oseen's equations have uniform validity. An explanation of this was found by Kaplun who pointed out that as Reynolds number tends to zero the whole flow field may be regarded as a perturbation of uniform flow (details are given in [25] & [26]). In this work we are concerned with flow at low Reynolds numbers. Hence in the present investigation the validity of the Oseen equations throughout the whole flow field is assumed to be relevant. The advantage of Oseen approximation lies essentially in that it becomes more and more accurate as the distances from the cylinder increases, so that the approximation is very good over an infinitely extended region of the flow field. Thus, the Oseen approximation is especially suitable for the discussion of the asymptotic behaviour of the flow field at large distances from the cylinder. This is useful in numerical work.

For future purposes it is advantageous at this point to obtain a fundamental solution of the linearized vorticity equation

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} = \frac{1}{2} Rke^{\xi} \left(\cos\eta \frac{\partial \zeta}{\partial \xi} - \sin\eta \frac{\partial \zeta}{\partial \eta} \right) \tag{2.1.10}$$

subject to the boundary condition

$$\zeta \to 0 \quad \text{as} \quad \xi \to \infty \tag{2.1.11}$$

and also,

$$\zeta(\xi,\eta) = \zeta(\xi, \eta + 2\pi) \tag{2.1.12}$$

by making an asymmetric generalization of a method used by Cole & Roshko [6], Dennis & Chang [9] and Illingworth [20] for symmetric flows. In this method the vorticity is assumed to be represented by

$$\zeta(\xi,\eta) = e^{\lambda(\xi,\eta)} \Phi(\xi,\eta), \tag{2.1.13}$$

where the function $\lambda(\xi,\eta)$ is chosen so that

$$\frac{\partial\lambda}{\partial\xi} = \frac{R}{4}\frac{\partial\psi}{\partial\eta}. \qquad (2.1.14)$$

The object of introducing the function $\lambda(\xi,\eta)$ in (2.1.13) is that when (2.1.13) is substituted into the equation (2.1.10) the resulting partial differential equation in $\Phi(\xi,\eta)$ does not contain terms involving the first derivatives of this function, which makes it easier to solve analytically. Also, it follows from the equations (1.2.14) and (1.2.28) that

$$\lambda(\xi,\eta) \sim \frac{1}{4} Rke^{\xi}\cos\eta \quad \text{as} \quad \xi \to \infty. \qquad (2.1.15)$$

If we substitute (2.1.13) into the equation (2.1.10) we may deduce the partial differential equation for the function $\Phi(\xi,\eta)$ given by

$$\frac{\partial^2\Phi}{\partial\xi^2} - \frac{\partial^2\Phi}{\partial\eta^2} - \frac{1}{16} R^2k^2e^{2\xi}\Phi = 0. \qquad (2.1.16)$$

For the symmetric flows, the function $\Phi(\xi,\eta)$ is assumed to be represented by a Fourier sine series in the variable η. The generalization to the present investigation is to assume the following Fourier expansion for the function $\Phi(\xi,\eta)$

$$\Phi(\xi,\eta) = \frac{1}{2} H_0(\xi) + \sum_{n=1}^{\infty} [\, H_n(\xi)\cos n\eta + h_n(\xi)\sin n\eta \,]. \qquad (2.1.17)$$

We wish to point out that this particular assumption is valid because of the boundary condition (2.1.12). Substitution of this series into the equation (2.1.16) yields the sets of ordinary differential equations for the functions $H_n(\xi)$ and $h_n(\xi)$

$$\frac{d^2H_n}{d\xi^2} - \left(n^2 + \frac{1}{16} R^2k^2e^{2\xi} \right) H_n = 0, \quad n \geq 0; \qquad (2.1.18)$$

$$\frac{d^2 h_n}{d\xi^2} - \left(n^2 + \frac{1}{16} R^2 k^2 e^{2\xi} \right) h_n = 0 , \quad n \geq 1 . \tag{2.1.19}$$

Let z be the variable defined by

$$z = \frac{1}{4} Rke^{\xi} \tag{2.1.20}$$

We then obtain from the equations (2.2.9) and (2.2.10)

$$z^2 \frac{d^2 H_n}{dz^2} + z \frac{d H_n}{dz} - (n^2 + z^2) H_n = 0 , \quad n \geq 0 ; \tag{2.1.21}$$

$$z^2 \frac{d^2 h_n}{dz^2} + z \frac{d h_n}{dz} - (n^2 + z^2) h_n = 0 , \quad n \geq 1 . \tag{2.1.22}$$

These differential equations are satisfied by the modified Bessel. functions of integer order n. Hence the fundamental solutions of (2.1.21) and (2.2.22) are

$$I_n(z) , \quad K_n(z) \quad \text{where} \quad z = \frac{1}{4} Rke^{\xi} ,$$

where $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of first and second kinds of order n respectively. Definitions of these functions and their properties are given in Appendix IV. By virtue of the asymptotic properties of the modified Bessel functions for large z, i.e.,

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} [1 + O(z^{-1})] ,$$

$$K_n(z) \sim \frac{\sqrt{\pi}}{\sqrt{2z}} e^{-z} [1 + O(z^{-1})]$$

as $z \to \infty$, only the functions $K_n(z)$, $n \geq 0$ are admissible solutions since from the boundary condition (2.1.11), ζ is to remain finite as $\xi \to \infty$. Thus it follows from the basic structure of the function $\Phi(\xi,\eta)$ and the equations (2.1.13) and (2.1.15) that the complete solution of (2.1.10) which is periodic with period 2π in η and vanishes as $\xi \to \infty$ can be given as

$$\zeta(\xi,\eta) = e^{z\cos\eta}\left[A_o K_o(z) + \sum_{n=1}^{\infty}[A_n \cos n\eta + B_n \sin n\eta]\, K_n(z)\right], \qquad (2.1.23)$$

where the A_n and B_n are arbitrary constants. This expression gives the approximation which is the so-called Oseen solution of equation (1.2.23). The arbitrary constants A_n, $n \geq 0$ and B_n, $n \geq 1$ cannot be determined explicitly without further information, e.g. knowledge of the complete solution of (1.2.23) in a domain which overlaps the domain of validity of (2.1.10). In the subsequent analysis, these constants will be determined to satisfy the integral conditions (1.3.15)-(1.3.17) in terms of very small values of the Reynolds number so that the final result will hold throughout the whole flow field. When the flow is symmetrical about $\eta = 0$ the constants A_n, $n \geq 0$ are identically zero.

As was pointed out earlier, the basic structure of the solution for the vorticity is

$$\zeta(\xi,\eta) = \frac{1}{2}G_o(\xi) + \sum_{n=1}^{\infty}[\, G_n(\xi)\cos n\eta + g_n(\xi)\sin n\eta\,]\,. \qquad (2.1.24)$$

We can express the functions $G_n(\xi)$ in the series (2.1.24) in terms of the coefficients of the trigonometric terms in the series (2.1.23). Now if we equate the series (2.1.24) to the series (2.1.23), multiply each side by $\cos n\eta$ and integrate with respect to η from 0 to 2π we find, after use of a well-known result in connection with the theory of Bessel functions

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\eta}\cos n\eta\, d\eta \qquad (2.1.25)$$

and also,

$$\int_0^{2\pi} e^{z\cos\eta} \sin n\eta \, d\eta = 0 \,, \tag{2.1.26}$$

that the series for the functions $G_n(\xi)$, $n \geq 0$ may be given as

$$G_n(\xi) = \sum_{m=0}^{\infty} A_m \left[I_{m-n}(z) + I_{m+n}(z) \right] K_m(z) \,, \quad n \geq 0 \,. \tag{2.1.27}$$

Next expansion (2.1.24) is equated to (2.1.23) and a similar procedure of multiplying each side by $\sin n\eta$ and integration from $\eta = 0$ to $\eta = 2\pi$ yields

$$g_n(\xi) = \sum_{m=0}^{\infty} B_m \left[I_{m-n}(z) - I_{m+n}(z) \right] K_m(z) \,, \quad n \geq 1 \,. \tag{2.1.28}$$

We propose to solve the Oseen equations (2.1.8) and (2.1.9) subject to the boundary conditions (1.2.26)-(1.2.28) *separately* for the vorticity. In this case one needs to consider the solution of the linearized vorticity equation (2.1.8) subject to its integral conditions (1.3.15)-(1.3.17) since the vorticity equation is linear and independent of the stream function and the boundary conditions (1.2.26)-(1.2.28) are equivalent to the integral conditions on the vorticity. The analytical solution of the Oseen problem can be carried out for various cylinder shapes (e.g. circular, elliptic) *separately* for the vorticity. In Chapter III and Chapter IV, it will be shown that all the information about the Oseen problems for an elliptic cylinder and a circular cylinder can be obtained from the vorticity.

2.2 THE PRINCIPLE OF RAPID DECAY OF THE VORTICITY

The correct satisfaction of the boundary conditions at large distances from the cylinder is a particularly crucial matter in the case of asymmetrical flows. Unless conditions are satisfied properly an unacceptable solution throughout the whole domain can result. In this section it will be shown that it is not sufficient to impose the necessary condition that the vorticity must vanish at large distances from the cylinder where the flow is a uniform stream, as a boundary condition in obtaining the solutions of (1.2.23). Dennis [9] found that by considering the asymptotic nature of the decay of vorticity at large distances that for *asymmetrical flows* it is not sufficient merely that the vorticity shall vanish far from the cylinder but that the *vorticity must decay rapidly enough*. Since a number of important results may be extracted from his analysis, it is detailed in this section.

As previously noted, equations (1.3.15) through (1.3.19) provide the integral conditions for the vorticity. Equations (1.3.15)-(1.3.19) may be re-arranged to obtain

$$\int_{\xi_o}^{\infty} e^{-n\xi}\, r_n(\xi)\, d\xi = 2k\delta_{n,1}, \quad n \geq 1; \tag{2.2.1}$$

$$\int_{\xi_o}^{\infty} e^{-n\xi}\, s_n(\xi)\, d\xi = \beta\delta_{n,o}, \quad n \geq 0, \tag{2.2.2}$$

where

$$r_n(\xi) = \frac{1}{\pi}\int_0^{2\pi} M^2 \zeta \sin n\eta\, d\eta, \tag{2.2.3}$$

$$s_n(\xi) = \frac{1}{\pi}\int_0^{2\pi} M^2 \zeta \cos n\eta\, d\eta. \tag{2.2.4}$$

In order to investigate the convergence of the integrals in equations (2.2.1) and (2.2.2) it is necessary to determine the character of the steady-state vorticity distribution for large ξ. As stated before, the solution of the Oseen linearized equation for the vorticity equation (2.1.10) which is valid for large ξ is

$$\zeta(\xi,\eta) = \frac{1}{2} G_0(\xi) + \sum_{n=1}^{\infty} \left[G_n(\xi) \cos n\eta + g_n(\xi) \sin n\eta \right] . \tag{2.2.5}$$

where

$$G_n(\xi) = \sum_{m=0}^{\infty} A_m \left[I_{m-n}(z) + I_{m+n}(z) \right] K_m , \quad n \geq 0 ; \tag{2.2.6}$$

$$g_n(\xi) = \sum_{m=1}^{\infty} B_m \left[I_{m-n}(z) - I_{m+n}(z) \right] K_m , \quad n \geq 1 , \tag{2.2.7}$$

with $z = \frac{1}{4} Rke^{\xi}$. This solution satisfies the necessary conditions

$$\zeta(\xi,\eta) = \zeta(\xi,\eta+2\pi) \quad \text{and} \quad \zeta \to 0 \quad \text{as} \quad \xi \to \infty .$$

From the asymptotic properties of the modified Bessel functions for large z which are given in Appendix IV, we have

$$I_m(z)K_m(z) = \frac{1}{2} z^{-1} + O(z^{-3}) , \tag{2.2.8}$$

$$\left[I_{m-n}(z) + I_{m+n}(z) \right] K_m(z) = z^{-1} + O(z^{-2}) , \tag{2.2.9}$$

$$\left[I_{m-n}(z) - I_{m+n}(z) \right] K_m(z) = mn z^{-2} + O(z^{-3}) , \tag{2.2.10}$$

as $z \to \infty$. Since ξ is large when z is large, the limiting behaviours of the functions $G_n(\xi)$, $n \geq 0$ and $g_n(\xi)$, $n \geq 1$ in the series (2.2.5) as $\xi \to \infty$ are

$$G_0(\xi) = \frac{4}{Rk} \left[\sum_{m=0}^{\infty} A_m \right] e^{-\xi} + O\left(e^{-3\xi}\right) , \tag{2.2.11}$$

$$G_n(\xi) = \frac{4}{Rk} \left[\sum_{m=0}^{\infty} A_m \right] e^{-\xi} + O\left(e^{-2\xi}\right) , \quad n \geq 1 ; \tag{2.2.12}$$

$$g_n(\xi) = \frac{16n}{R^2k^2} \left[\sum_{m=1}^{\infty} m B_m \right] e^{-2\xi} + O\left(e^{-3\xi}\right) , \quad n \geq 1 \tag{2.2.13}$$

then for large ξ equation (2.2.5) can be replaced by

$$\zeta(\xi,\eta) = \left[\mathfrak{F}(\eta)e^{-\xi} + \mathcal{X}(\eta)e^{-2\xi} \right]\left[1 + O\left(e^{-\xi}\right) \right] \qquad (2.2.14)$$

where $\mathfrak{F}(\eta)$ and $\mathcal{X}(\eta)$ are functions of η alone which are given by

$$\mathfrak{F}(\eta) = \frac{4}{Rk}\left[\sum_{m=0}^{\infty} A_m \right]\left[\frac{1}{2} + \cos\eta + (\cos 2\eta) \right], \qquad (2.2.15)$$

$$\mathcal{X}(\eta) = \frac{16}{R^2k^2}\left[\sum_{m=1}^{\infty} m\, B_m \right]\left[\sin\eta + 2(\sin 2\eta) \right]. \qquad (2.2.16)$$

In these equations the existence of the sums

$$S_1 = \sum_{m=0}^{\infty} A_m \quad \text{and} \quad S_2 = \sum_{m=1}^{\infty} m\, B_m \qquad (2.2.17)$$

are assumed. In practice they are approximated by a finite number of terms. The functions $\mathfrak{F}(\eta)$ and $\mathcal{X}(\eta)$ are known only precisely when the constants A_m, $m \geq 0$ and B_m, $m \geq 1$ are known.

We will proceed to determine the functions $r_n(\xi)$, $n \geq 1$ and $s_n(\xi)$, $n \geq 0$ which are given by the equations (2.2.3) and (2.2.4) when ξ is large. It follows from (1.2.15) and (1.2.20) that the quantity M^2 in (2.2.3) and (2.2.4) has the asymptotic form

$$M^2 \sim k^2 e^{2\xi} \quad \text{as} \quad \xi \to \infty. \qquad (2.2.18)$$

If we replace ζ and M^2 in equation (2.2.3) and (2.2.4) by their asymptotic forms for large ξ which are given by the equations (2.2.14) and (2.2.18), we obtain

$$r_n(\xi) - k^2e^{2\xi} g_n(\xi) = \frac{16n}{R^2} S_2 + O\left(e^{-\xi}\right), \quad n \geq 1; \qquad (2.2.19)$$

$$s_0(\xi) - k^2e^{2\xi} G_0(\xi) = \frac{4k}{R} S_1 e^{\xi} + O\left(e^{-\xi}\right), \qquad (2.2.20)$$

$$s_n(\xi) - k^2e^{2\xi} G_n(\xi) = \frac{4k}{R} S_1 e^{\xi} + O(1), \quad n \geq 1. \qquad (2.2.21)$$

It follows from (2.2.20) and (2.2.21) that the flow must be adjusted so that $S_1 = 0$. Then the functions

$$s_n(\xi) \to c_n \quad \text{as} \quad \xi \to \infty, \qquad (2.2.22)$$

where the constants c_n depend upon a sum involving the constants A_n and the coefficients of the next order terms in equations (2.2.11) and (2.2.12). Only under this circumstance will the unknown constant β determined from (2.2.2) be such that the conditions (1.3.8) and (1.3.9) are satisfied. Further, the infinite integral in (2.2.2) will then converge at the upper limit for $n \geq 0$. The necessary condition

$$\sum_{m=0}^{\infty} A_m = 0,$$

indicates the *rapid decay of the vorticity* at large distances from the cylinder since its application eliminates the slowly decaying term in $G_n(\xi)$ as $\xi \to \infty$. It also follows from (2.2.19) that

$$r_n(\xi) \to nc \text{ as } \xi \to \infty, \qquad (2.2.23)$$

where $c = \frac{16n}{R^2} S_2$ and is a definite constant for a given Reynolds number. It follows from (2.2.23) that the integral in the condition (2.2.1) always converges at the upper limit.

As previously noted, because of the choice of the co-ordinate system all the flow variables must be periodic functions of the angular co-ordinate η with period 2π. It follows from the equation (1.2.37) that the pressure in the fluid will only turn out to be periodic provided that

$$\sum_{n=1}^{\infty} n\,[\, f_n(\xi)G_n(\xi) - F_n(\xi)g_n(\xi)\,] \to 0 \quad \text{as} \quad \xi \to \infty. \qquad (2.2.24)$$

Thus we find, after use of (1.3.8), (2.2.12) and (2.2.13), that

$$\sum_{n=1}^{\infty} n\,[\, f_n(\xi)G_n(\xi) - F_n(\xi)g_n(\xi)\,] \to \frac{4k}{R}\sum_{m=0}^{\infty} A_m \quad \text{as} \quad \xi \to \infty. \qquad (2.2.25)$$

The condition (2.2.24) required for the periodicity is then satisfied only if

$$S_1 = \sum_{s=0}^{\infty} \lambda_s = 0 . \qquad (2.2.26)$$

This condition must be enforced on the flow in some manner. The integral conditions for the vorticity at least in theory allows this to be done, although the practical difficulties are great. The problem does not arise in flows which are symmetrical about $\eta = 0$ since ζ is then an odd function of η. The functions $G_n(\xi)$ in the series (2.2.5) are then identically zero and the condition (2.2.26) is satisfied since by the equation (2.2.13) the functions $g_n(\xi)$ always vanish rapidly enough.

Thus even if the correct conditions of periodicity are enforced upon the governing equations for the stream function and the vorticity (1.2.23) and (1.2.24), the pressure in the fluid will only turn out to be periodic provided that the integral conditions on the vorticity have been satisfied. It is not yet completely clear how the necessary adjustment of the flow so that $S_1 = 0$ is to be made in the general case of the Navier-Stokes equations. The central problem is the elimination of the slowly decaying term in $G_n(\xi)$ as $\xi \to \infty$. It is not sufficient to impose the necessary condition (1.2.29) as a boundary condition in obtaining the solutions of (1.2.23). The problem has been solved in the case when the whole flow field is assumed to be governed by the Oseen equations. The technique will be described in the next section and some analytical illustrations will be given in subsequent chapters.

2.3 ANALYSIS AND METHOD OF SOLUTION

The object of this method is to reduce the partial differential equations (2.1.8) and (2.1.9) to sets of second order linear ordinary differential equations in one space variable which are integrable. This is done by standard Fourier analysis and then the integration constants are determined in terms of the Reynolds number R, treating R as small, provided that certain necessary conditions involving integrals of the vorticity ζ evaluated throughout the flow field are satisfied. In theory each set of equations is an infinite set which is then truncated by setting to zero all terms after a certain stage. A further object of this method is to deal with the solution of the Oseen problem separately for the vorticity. We consider the Oseen linearized equation for the vorticity

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} = \frac{1}{2} Rke^{\xi} \left(\cos\eta \frac{\partial \zeta}{\partial \xi} - \sin\eta \frac{\partial \zeta}{\partial \eta} \right), \qquad (2.3.1)$$

where $\xi_0 \le \xi < \infty$, $0 \le \eta \le 2\pi$, subject to its integral conditions

$$\int_{\xi_0}^{\infty} \int_0^{2\pi} e^{-n\xi} M^2 \zeta \cos n\eta \, d\eta \, d\xi = \pi\beta\delta_{n,0}, \quad n \ge 0 ; \qquad (2.3.2)$$

$$\int_{\xi_0}^{\infty} \int_0^{2\pi} e^{-n\xi} M^2 \zeta \sin n\eta \, d\eta \, d\xi = 2\pi k\delta_{n,1}, \quad n \ge 1 . \qquad (2.3.3)$$

In addition, the vorticity must satisfy the condition at infinity

$$\zeta(\xi,\eta) \to 0 \quad \text{as} \quad \xi \to \infty \qquad (2.3.4)$$

and also,

$$\zeta(\xi,\eta) = \zeta(\xi, \eta + 2\pi). \qquad (2.3.5)$$

Equations (2.3.1) through (2.3.5) provide the mathematical formulation of the Oseen problem for the vorticity in the (ξ,η) co-ordinate system. In this method, we assume

$$\zeta(\xi,\eta) = \frac{1}{2} G_0(\xi) + \sum_{n=1}^{\infty} \left[G_n(\xi)\cos n\eta + g_n(\xi)\sin n\eta \right]. \qquad (2.3.6)$$

Substitution of the series (2.3.6) into (2.3.1) yields two infinite sets of linear second order ordinary differential equations for the functions $G_n(\xi)$ and $g_n(\xi)$

$$G_0'' = \frac{1}{2} Rke^{\xi} [G_1' + G_1] , \qquad (2.3.7)$$

$$G_n'' - n^2 G_n = \frac{1}{4} Rke^{\xi} [G_{n-1}' - (n-1) G_{n-1} + G_{n+1}' + (n+1) G_{n+1}] ; \quad n \geq 1; \qquad (2.3.8)$$

$$g_n'' - n^2 g_n = \frac{1}{4} Rke^{\xi} [g_{n-1}' - (n-1) g_{n-1} + g_{n+1}' + (n+1) g_{n+1}] , \quad n \geq 1, \qquad (2.3.9)$$

where $g_0 = 0$, and the prime denotes differentiation with respect to ξ.

We note that the dependence of the problem on the parameter R can be considered to be isolated in the variable

$$z = \frac{1}{4} Rke^{\xi} . \qquad (2.3.10)$$

Thus if we consider flows at low Reynolds numbers, z will be small. Under these conditions we can remove the R-dependence from the differential equations (2.3.7)-(2.3.9). By means of (2.3.10), equations (2.3.7)-(2.3.9) become

$$z G_0'' + G_0' = \frac{1}{2} [z G_1' + G_1] , \qquad (2.3.11)$$

$$z G_{n+1}' + (n+1) G_{n+1} = z G_n'' + G_n' - n^2 z^{-1} G_n - z G_{n-1}' + (n-1) G_{n-1} , \quad n \geq 1; \qquad (2.3.12)$$

$$z g_{n+1}' + (n+1) g_{n+1} = z g_n'' + g_n' - n^2 z^{-1} g_n - z g_{n-1}' + (n-1) g_{n-1} , \quad n \geq 1. \qquad (2.3.13)$$

The prime now denotes differentiation with respect to z. The integral conditions for the vorticity (2.3.2) and (2.3.3) become

$$\int_{\frac{Rk}{4} e^{\xi_0}}^{\infty} z^{-n-1} G_0(z) \int_0^{2\pi} M^2(z,\eta) \cos n\eta \, d\eta \, dz + 2 \sum_{m=1}^{\infty} \Big[\int_{\frac{Rk}{4} e^{\xi_0}}^{\infty} z^{-n-1} G_m(z) \int_0^{2\pi} M^2(z,\eta) \cos m\eta \cos n\eta \, d\eta \, dz$$

$$+ \int_{\frac{Rk}{4} e^{\xi_0}}^{\infty} z^{-n-1} g_m(z) \int_0^{2\pi} M^2(z,\eta) \sin m\eta \cos n\eta \, d\eta \, dz \Big] = \Big[\frac{4}{Rk} \Big]^n 2\pi\beta\delta_{n,0} , \quad n \geq 0 ; \qquad (2.3.14)$$

$$\int_{\frac{Rk}{4}e^{\xi_0}}^{\infty} z^{-n-1} G_0(z) \int_0^{2\pi} M^2(z,\eta) \sin n\eta \, d\eta \, dz + 2\sum_{m=1}^{\infty} \Big[\int_{\frac{Rk}{4}e^{\xi_0}}^{\infty} z^{-n-1} G_m(z) \int_0^{2\pi} M^2(z,\eta) \cos m\eta \sin n\eta \, d\eta \, dz$$

$$+ \int_{\frac{Rk}{4}e^{\xi_0}}^{\infty} z^{-n-1} g_m(z) \int_0^{2\pi} M^2(z,\eta) \sin m\eta \sin n\eta \, dz \, d\eta \Big] = \Big[\frac{4}{Rk} \Big]^n 4\pi k \delta_{n,1} , \quad n \geq 1, \quad (2.3.15)$$

and also, from the equation (2.3.4) we have

$$G_n(z) \to 0 \quad \text{as} \quad z \to \infty , \quad n \geq 0 ; \qquad (2.3.16)$$

$$g_n(z) \to 0 \quad \text{as} \quad z \to \infty , \quad n \geq 1 . \qquad (2.3.17)$$

Equations (2.3.11) through (2.3.17) provide the complete formulation of the Oseen problem for the steady-state vorticity.

If we integrate the equations (2.3.11) through (2.3.13) with respect to z we find, after re-arranging these equations and multiplying each side of the equations (2.3.12) and (2.3.13) by z^n, $n \geq 1$, that

$$G_1(z) = \frac{1}{2} G_0'(z) + c_1 z^{-1} , \qquad (2.3.18)$$

$$G_{n+1}(z) = z^n \frac{d}{dz} [z^{-n} G_n(z)] - G_{n-1}(z) + 2nz^{-n-1} [\mathcal{X}_{n+1}(z) + c_{n+1}] , \quad n \geq 1 ; \quad (2.3.19)$$

$$g_{n+1}(z) = z^n \frac{d}{dz} [z^{-n} g_n(z)] - g_{n-1}(z) + 2nz^{-n-1} [\mathcal{L}_{n+1}(z) + d_{n+1}] , \quad n \geq 1 , \quad (2.3.20)$$

where $\mathcal{X}_{n+1}(z)$ and $\mathcal{L}_{n+1}(z)$ are the functions given by

$$\mathcal{X}_{n+1}(z) + c_{n+1} = \int z^n G_{n-1}(z) \, dz , \quad n \geq 1 ; \qquad (2.3.21)$$

$$\mathcal{L}_{n+1}(z) + d_{n+1} = \int z^n g_{n-1}(z) \, dz , \quad n \geq 1 . \qquad (2.3.22)$$

Here c_{n+1}, $n \geq 0$ and d_{n+1}, $n \geq 1$ are the constants of integration. The functions $G_{n+1}(z)$, $n \geq 0$ and $g_{n+1}(z)$, $n \geq 1$ are then only known precisely when the functions $G_0(z)$, $g_1(z)$ and the integration constants are known.

In order to determine the functions $G_0(z)$ and $g_1(z)$, we can use basic structures of the functions $G_n(z)$, $n \geq 0$ and $g_n(z)$, $n \geq 1$

$$G_n(z) = \sum_{m=0}^{\infty} A_m \left[I_{m-n}(z) + I_{m+n}(z) \right] K_m(z), \quad n \geq 0; \tag{2.3.23}$$

$$g_n(z) = \sum_{m=1}^{\infty} B_m \left[I_{m-n}(z) - I_{m+n}(z) \right] K_m(z), \quad n \geq 1. \tag{2.3.24}$$

which are obtained by standard processes of analysis in Section 2.1. The constants A_m, $m \geq 0$ and B_m, $m \geq 1$ can be determined in terms of the Reynolds number R, treating R as small, by evaluating the integral conditions (2.3.14) and (2.3.15). Substitution of the equations (2.3.23) and (2.3.24) into both of the conditions (2.3.14) and (2.3.15) yields an infinite set of algebraic equations for the unknown constants A_m, $m \geq 0$; B_m, $m \geq 1$ and β if $S_1 = 0$. These constants are to be found by solving the resulting system of equations provided that the necessary condition $S_1 = 0$ is satisfied. Theoretically, the solution of the resulting infinite set of equations is to be achieved by means of infinite determinants; practically to find the approximate solution we solve a finite number of equations. A special technique is used for solving these equations. It is explained and illustrated for various cylinder shapes in subsequent chapters.

Assuming the constants A_m, $m \geq 0$ and B_m, $m \geq 1$ are known, the functions $G_0(z)$ and $g_1(z)$ can be determined completely from the equations (2.3.23) for $n = 0$ and (2.3.24) for $n = 1$ and then the functions $G_{n+1}(z)$, $n \geq 0$ and $g_{n+1}(z)$, $n \geq 1$ can be determined in an alternating manner from the equations (2.3.18)-(2.3.22). The constants of integration c_{n+1}, $n \geq 0$ and d_{n+1}, $n \geq 1$ have to be determined in terms of low Reynolds numbers provided that the integral conditions on the vorticity (2.3.14) and (2.3.15) are satisfied. The complexity of the evaluation of the integral conditions (2.3.14) and (2.3.15) depends upon the form of $M(z,\eta)$ which is determined by the cylinder shape under consideration. The method is particularly suited for flow past a circular cylinder in which case $M(z,\eta)$ is a function of z alone. In this case the integral conditions are of a fairly simple nature and may be evaluated by straight-forward analytical procedures.

Determination of the functions $G_n(z)$, $n \geq 0$ and $g_n(z)$, $n \geq 1$ and the constants of integration in the equations (2.3.18) through (2.3.22) will enable us to find an asymptotic expansion of the steady-state vorticity for $R \ll 1$ when the whole flow field is assumed to be governed by the Oseen equations. The final result is of the form (2.3.6) and assumed to hold throughout the whole flow field since all the unknown functions and constants appearing in equations (2.3.18) through (2.3.22) are determined by evaluating the necessary conditions involving integrals of the vorticity throughout the flow region. The satisfaction of these conditions allows the vorticity ζ to be determined on the contour C of the cylinder without using any information about the stream function ψ. The solution for the vorticity is then complete. Since the vorticity is now known on the contour C of the cylinder as well as in the infinite domain outside the cylinder, it is now possible to integrate the equation (1.2.24) subject to the conditions (1.2.26)-(1.2.28). The procedure is based on the reduction of the equation (1.2.24) to a sets of ordinary differential equations in one space variable. This is done by standard Fourier analysis in Section 1.3. The applicability of the method to various cylinder shapes will be discussed in subsequent chapters.

CHAPTER III

APPLICATION OF THE NEW METHOD OF SOLVING OSEEN EQUATIONS TO THE ASYMMETRICAL FLOW PAST AN ELLIPTIC CYLINDER

3.1 THE OSEEN EQUATIONS FOR ASYMMETRICAL FLOW PAST AN ELLIPTIC CYLINDER

In this chapter, as an example of the application of the new method described in Section 2.3, the uniform flow past an elliptic cylinder at an arbitrary angle of incidence will be dealt with. The x and y axes are along the major and minor axes of the dimensionless elliptic cylinder, the angle of incidence between the uniform flow and the major axis of the dimensionless ellipse is α. A typical situation is shown in figure 2. For flow past an elliptic cylinder the appropriate transformation corresponding to (1.2.14) is

$$x + iy = \cosh[\xi + i(\eta + \alpha)], \tag{3.1.1}$$

$$x = \cosh\xi \cos(\eta + \alpha), \quad y = \sinh\xi \sin(\eta + \alpha). \tag{3.1.2}$$

The boundary $\xi = \xi_o$ corresponds to a dimensionless ellipse with semi-major and semi-minor axes of lengths

$$a = \cosh\xi_o, \qquad b = \sinh\xi_o \tag{3.1.3}$$

respectively. The dimensionless ellipse itself is then given by

$$\xi = \xi_o = \tanh^{-1}\left(\frac{b}{a}\right) \tag{3.1.4}$$

where $a > b$. The corresponding value of k in (1.2.15) is $k = \frac{1}{2}$ and the function $M(\xi, \eta)$ in (1.1.20) is

$$M^2(\xi, \eta) = \frac{1}{2}[\cosh 2\xi - \cos 2(\eta + \alpha)] \tag{3.1.5}$$

in this case. Since the dimensionless length of the major axis is $2\cosh\xi_o$ we define a Reynolds number $R_E = 2Ud\cosh\xi_o/\upsilon$ based on the dimensional length of the major axis. The number R_E can be defined in terms of the generalized Reynolds number R as

$$R = \frac{R\varepsilon}{\cosh\xi_o} . \tag{3.1.6}$$

The Oseen equations (2.1.8) and (2.1.9) take the forms in the (ξ,η) co-ordinate system

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2} = \frac{R\varepsilon}{4\cosh\xi_o} e^{\xi}\left(\cos\eta \frac{\partial\zeta}{\partial\xi} - \sin\eta \frac{\partial\zeta}{\partial\eta} \right), \tag{3.1.7}$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = \frac{1}{2}[\cosh 2\xi - \cos 2(\eta+\alpha)]\ \zeta , \tag{3.1.8}$$

where $\xi_o \le \xi < \infty$, $0 \le \eta \le 2\pi$. These are to be solved subject to the following conditions

$$\int_{\xi_o}^{\infty}\int_0^{2\pi} e^{-n\xi}[\cosh 2\xi - \cos 2(\eta+\alpha)]\ \zeta \cos n\eta \, d\eta \, d\xi = 2\pi\beta\delta_{n,o} , \quad n \ge 0 ; \tag{3.1.9}$$

$$\int_{\xi_o}^{\infty}\int_0^{2\pi} e^{-n\xi}[\cosh 2\xi - \cos 2(\eta+\alpha)]\ \zeta \sin n\eta \, d\eta \, d\xi = 2\pi\delta_{n,1} , \quad n \ge 1, \tag{3.1.10}$$

$$\zeta(\xi,\eta) \to 0 \quad \text{as} \quad \xi \to \infty, \tag{3.1.11}$$

$$\psi(\xi,\eta) = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{when} \quad \xi = \xi_o , \tag{3.1.12}$$

$$e^{-\xi}\frac{\partial\psi}{\partial\xi} \to \frac{1}{2}\sin\eta , \qquad e^{-\xi}\frac{\partial\psi}{\partial\eta} \to \frac{1}{2}\cos\eta \quad \text{as} \quad \xi \to \infty, \tag{3.1.13}$$

$$\psi(\xi,\eta) = \psi(\xi,\eta+2\pi), \qquad \zeta(\xi,\eta) = \zeta(\xi,\eta+2\pi). \tag{3.1.14}$$

If we substitute the assumed expansion for the vorticity $\zeta(\xi,\eta)$ given by

$$\zeta(\xi,\eta) = \frac{1}{2} G_o(\xi) + \sum_{n=1}^{\infty} [G_n(\xi)\cos n\eta + g_n(\xi)\sin n\eta] \tag{3.1.15}$$

in equation (3.1.7) we find, after the standard use of orthogonal functions, that the equations for arbitrary components $G_n(\xi)$, $n \ge 0$ and $g_n(\xi)$, $n \ge 1$ of the series (3.1.15) are

$$G_o'' = \frac{R\varepsilon}{4\cosh\xi_o} e^{\xi} [\, G_1' + G_1 \,] \,, \qquad (3.1.16)$$

$$G_n'' - n^2 G_n = \frac{R\varepsilon}{8\cosh\xi_o} e^{\xi} [\, G_{n-1}' - (n-1) G_{n-1} + G_{n+1}' + (n+1) G_{n+1} \,] \,, \; n \ge 1 ; \qquad (3.1.17)$$

$$g_n'' - n^2 g_n = \frac{R\varepsilon}{8\cosh\xi_o} e^{\xi} [\, g_{n-1}' - (n-1) g_{n-1} + g_{n+1}' + (n+1) g_{n+1} \,] \,, \; n \ge 1, \qquad (3.1.18)$$

where $g_o = 0$, and the prime denotes differentiation with respect to the variable ξ. In terms of the functions $G_n(\xi)$, $n \ge 0$ and $g_n(\xi)$, $n \ge 1$ associated with the expansion (3.1.15) for $\zeta(\xi,\eta)$ the integral conditions (3.1.9) and (3.1.10) become

$$\int_{\xi_o}^{\infty} e^{-n\xi} \Big[2\cosh 2\xi \, G_n(\xi) - \cos 2\alpha [\, G_{n+2}(\xi) + G_{n-2}(\xi)] + \sin 2\alpha [\, g_{n+2}(\xi) - g_{n-2}(\xi)] \Big] d\xi = 4\beta\delta_{n,o} \,, \; n \ge 0 \,; \qquad (3.1.19)$$

$$\int_{\xi_o}^{\infty} e^{-n\xi} \Big[2\cosh 2\xi \, g_n(\xi) - \cos 2\alpha [\, g_{n+2}(\xi) + g_{n-2}(\xi)] + \sin 2\alpha [\, G_{n-2}(\xi) - G_{n+2}(\xi)] \Big] d\xi = 4\delta_{n,1} \,, \; n \ge 1 \,, \qquad (3.1.20)$$

where

$$G_{-k}(\xi) = G_k(\xi) \,, \qquad g_{-k}(\xi) = -\, g_k(\xi) \qquad \text{for} \;\; k = 1,2 \,. \qquad (3.1.21)$$

Also, from the condition (3.1.11) we have

$$G_n(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \quad n \ge 0 \,; \qquad (3.1.22)$$

$$g_n(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \quad n \ge 1 \,. \qquad (3.1.23)$$

The two sets of linear second order ordinary differential equations (3.1.16)-(3.1.18) for the functions $G_n(\xi)$, $n \ge 0$ and $g_n(\xi)$, $n \ge 1$, subject to the conditions (3.1.19) through (3.1.23) are to be solved in a manner similar to that outlined in Section 2.3. Now let us make a Fourier expansion of $\psi(\xi,\eta)$

$$\psi(\xi,\eta) = \frac{1}{2} F_o(\xi) + \sum_{n=1}^{\infty} [\, F_n(\xi) \cos n\eta + f_n(\xi) \sin n\eta \,] \,. \qquad (3.1.24)$$

Substitution of this series in equation (3.1.8) and the standard use of orthogonal functions yields the pair of equations for arbitrary components $F_n(\xi)$, $n \geq 0$ and $f_n(\xi)$, $n \geq 1$ of the series (3.1.24)

$$F_n''(\xi) - n^2 F_n(\xi) = \frac{1}{2}\Big[\cosh 2\xi\, G_n(\xi) - \frac{1}{2}\cos 2\alpha\,[\,G_{n+2}(\xi) + G_{n-2}(\xi)\,] + \frac{1}{2}\sin 2\alpha\,[\,g_{n+2}(\xi) - g_{n-2}(\xi)\,]\Big]; \quad (3.1.25)$$

$$f_n''(\xi) - n^2 f_n(\xi) = \frac{1}{2}\Big[\cosh 2\xi\, g_n(\xi) - \frac{1}{2}\cos 2\alpha\,[\,g_{n+2}(\xi) + g_{n-2}(\xi)\,] + \frac{1}{2}\sin 2\alpha\,[\,G_{n-2}(\xi) - G_{n+2}(\xi)\,]\Big]. \quad (3.1.26)$$

The boundary conditions on the functions $F_n(\xi)$, $n \geq 0$ and $f_n(\xi)$, $n \geq 1$ follow from equations (3.1.24), (3.1.12) and the uniform flow at infinity condition (3.1.13). They are at the cylinder surface

$$F_n(\xi_o) = F_n'(\xi_o) = 0, \quad n \geq 0; \quad (3.1.27)$$

$$f_n(\xi_o) = f_n'(\xi_o) = 0, \quad n \geq 1, \quad (3.1.28)$$

and as $\xi \to \infty$

$$e^{-\xi} f_n \to \frac{1}{2}\delta_{n,1}, \quad e^{-\xi} F_n \to 0, \quad n \geq 1; \quad e^{-\xi} F_o \to 0, \quad (3.1.29)$$

$$e^{-\xi} f_n' \to \frac{1}{2}\delta_{n,1}, \quad e^{-\xi} F_n' \to 0, \quad n \geq 1; \quad e^{-\xi} F_o' \to 0. \quad (3.1.30)$$

Assuming the functions $G_n(\xi)$, $n \geq 0$ and $g_n(\xi)$, $n \geq 1$ are known, the linear second order ordinary differential equations (3.1.25) and (3.1.26) for the functions $F_n(\xi)$, $n \geq 0$ and $f_n(\xi)$, $n \geq 1$ are then to be solved subject to the conditions (3.1.27) through (3.1.30).

As is to be expected, the reduction of equations (3.1.7) and (3.1.8) into component form yields a complicated system of equations, this essentially being due to the fact that M is a function of both of the variables ξ and η.

3.2 THE APPROXIMATE SOLUTION OF THE OSEEN FLOW PROBLEM FOR THE VORTICITY IN TERMS OF LOW REYNOLDS NUMBER

In this section the approximation to the vorticity distribution over the surface of the elliptic cylinder will be obtained correct to the order of $[R\varepsilon][\ln R\varepsilon]^{-1}$, the lowest order term being $O([\ln R\varepsilon]^{-1})$.

The mathematical formulation of the Oseen flow problem for the vorticity components $G_n(z)$, $n \geq 0$ and $g_n(z)$, $n \geq 1$ in the case of the elliptic cylinder which is described in the previous section may be summarized as being the following

$$z G_0'' + G_0' = \frac{1}{2}[\, z G_1' + G_1 \,], \qquad (3.2.1)$$

$$z G_{n+1}' + (n+1)G_{n+1} = z G_n'' + G_n' - n^2 z^{-1} G_n - z G_{n-1}' + (n-1)G_{n-1}, \quad n \geq 1; \qquad (3.2.2)$$

$$z g_{n+1}' + (n+1)g_{n+1} = z g_n'' + g_n' - n^2 z^{-1} g_n - z g_{n-1}' + (n-1)g_{n-1}, \quad n \geq 1, \qquad (3.2.3)$$

where $z = \dfrac{R\varepsilon\, e^{\xi}}{8\cosh\xi_0}$ and $g_0 = 0$;

$$G_n(z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad n \geq 0, \qquad (3.2.4)$$

$$g_n(z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad n \geq 1, \qquad (3.2.5)$$

$$\int_{\frac{R\varepsilon e^{\xi_0}}{8\cosh\xi_0}}^{\infty} \Big[\Big(z^{1-n} + \Big(\frac{R\varepsilon}{8\cosh\xi_0}\Big)^4 z^{-3-n}\Big)G_n(z) - \Big(\frac{R\varepsilon}{8\cosh\xi_0}\Big)^2 z^{-1-n}\Big(\cos 2\alpha[\, G_{n+2}(z) + G_{n-2}(z)] - \sin 2\alpha[\, g_{n+2}(z) - g_{n-2}(z)]\Big)\Big]\, dz = \Big(\frac{R\varepsilon}{8\cosh\xi_0}\Big)^{2-n} 4\beta\delta_{n,0}, \quad n \geq 0;$$

$$(3.2.6)$$

$$\int_{\frac{R\varepsilon e^{\xi_0}}{8\cosh\xi_0}}^{\infty} \Big[\Big(z^{1-n} + \Big(\frac{R\varepsilon}{8\cosh\xi_0}\Big)^4 z^{-3-n}\Big)g_n(z) - \Big(\frac{R\varepsilon}{8\cosh\xi_0}\Big)^2 z^{-1-n}\Big(\cos 2\alpha[\, g_{n+2}(z) + g_{n-2}(z)] - \sin 2\alpha[\, G_{n-2}(z) - G_{n+2}(z)]\Big)\Big]\, dz = \Big(\frac{R\varepsilon}{8\cosh\xi_0}\Big)^{2-n} 4\delta_{n,1}, \quad n \geq 1,$$

where $g_{-k}(z) = -g_k(z)$, $G_{-k}(z) = G_k(z)$ for $k = 1, 2$.

As stated before, the two sets of linear second order ordinary differential equations (3.2.1)-(3.2.3) are integrable and the functions

$G_n(z)$, $n \geq 1$ and $g_n(z)$, $n \geq 2$ are only known precisely when the functions $G_o(z)$ and $g_1(z)$ are known. If we re-arrange the equations (2.3.19) for $n = 2, 3$, after re-writing the equation (2.3.18), we find

$$G_1(z) = \frac{1}{2}\frac{d}{dz}[\,G_o(z)\,] + C_1 z^{-1}, \tag{3.2.7}$$

$$G_2(z) = z\frac{d}{dz}[\,z^{-1}G_1(z)\,] - G_o(z) + 2z^{-2}\int^z z\,G_o(z)\,dz + C_2 z^{-2}, \tag{3.2.8}$$

$$G_3(z) = z^2\frac{d}{dz}\left[z^{-1}\frac{d}{dz}[\,z^{-1}G_1(z)\,]\right] - G_1(z) + 4z^{-3}\int^z z^2\left(G_1(z) + \frac{d}{dz}[\,G_o(z)\,]\right)dz - \frac{d}{dz}[\,G_o(z)\,] + C_3 z^{-3}. \tag{3.2.9}$$

Also, the equations (2.3.20) for $n = 2, 3$ may be re-arranged to obtain

$$g_2(z) = z\frac{d}{dz}[\,z^{-1}g_1(z)\,] + D_2 z^{-2}, \tag{3.2.10}$$

$$g_3(z) = z^2\frac{d}{dz}\left[z^{-1}\frac{d}{dz}[\,z^{-1}g_1(z)\,]\right] - g_1(z) + 4z^{-3}\int^z z^2 g_1(z)\,dz + D_3 z^{-3}. \tag{3.2.11}$$

where the C_1, C_k and D_k for $k = 2, 3$ are arbitrary constants.

We now proceed to determine the functions $G_o(z)$ and $g_1(z)$. In order to determine these functions, we use basic structures of the vorticity components

$$G_n(z) = \sum_{m=o}^{\infty} A_m\,[\,I_{m-n}(z) + I_{m+n}(z)\,]\,K_m(z), \quad n \geq 0\,; \tag{3.2.12}$$

$$g_n(z) = \sum_{m=1}^{\infty} B_m\,[\,I_{m-n}(z) - I_{m+n}(z)\,]\,K_m(z), \quad n \geq 1, \tag{3.2.13}$$

which are obtained by standard processes of analysis in section 2.1. To find the constants A_m, $m \geq 0$ and B_m, $m \geq 1$ we substitute the expressions (3.2.12) and (3.2.13) for the functions $G_n(z)$, $n \geq 0$ and $g_n(z)$, $n \geq 1$ into both of the integral conditions (3.2.6). This gives the two infinite sets of linear simultaneous equations

$$\left[\sum_{m=0}^{\infty} \Gamma_{nm} A_m + \sum_{m=1}^{\infty} \Lambda_{nm} B_m \right] = 4\beta\delta_{n,0}, \quad n \geq 0; \tag{3.2.14}$$

$$\left[\sum_{m=1}^{\infty} \chi_{nm} B_m + \sum_{m=0}^{\infty} \Omega_{nm} A_m \right] = 4\delta_{n,1}, \quad n \geq 1, \tag{3.2.15}$$

where

$$\Gamma_{nm} = \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^n \left[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-n} \left(\left[\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{-2} z + \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{2} z^{-3} \right] \left[I_{m-n}(z) + I_{m+n}(z) \right] - \cos2\alpha\, z^{-1} \left[I_{m-n-2}(z) + I_{m+n+2}(z) + I_{m-n+2}(z) + I_{m+n-2}(z) \right] \right) K_m(z)\, dz \right], \quad n \geq 0,\ m \geq 0, \tag{3.2.16}$$

$$\chi_{nm} = \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^n \left[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-n} \left(\left[\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{-2} z + \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{2} z^{-3} \right] \left[I_{m-n}(z) - I_{m+n}(z) \right] - \cos2\alpha\, z^{-1} \left[I_{m-n-2}(z) - I_{m+n+2}(z) + I_{m-n+2}(z) - I_{m+n-2}(z) \right] \right) K_m(z)\, dz \right], \quad n \geq 1,\ m \geq 1, \tag{3.2.17}$$

$$\Lambda_{nm} = \sin2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^n \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1-n} \left[I_{m-n-2}(z) - I_{m+n+2}(z) - I_{m-n+2}(z) + I_{m+n-2}(z) \right] K_m(z)\, dz, \quad n \geq 0,\ m \geq 1, \tag{3.2.18}$$

$$\Omega_{nm} = \sin2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^n \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1-n} \left[-I_{m-n+2}(z) + I_{m+n-2}(z) + I_{m-n-2}(z) - I_{m+n+2}(z) \right] K_m(z)\, dz, \quad n \geq 1,\ m \geq 0. \tag{3.2.19}$$

We note that the property (IV-5) is used in obtaining the expressions (3.2.16)-(3.2.19). In Appendix VII, it is shown that the necessary condition for the existence of the sums

$$\sum_{m=0}^{\infty} \Gamma_{om} A_m \quad \text{and} \quad \sum_{m=0}^{\infty} \Gamma_{1m} A_m \tag{3.2.20}$$

in (3.2.14) is

$$\sum_{m=0}^{\infty} A_m = 0 . \tag{3.2.21}$$

The unknown constants β, A_m, $m \geq 0$ and B_m, $m \geq 1$ are then to be found by solving the system of equations (3.2.14)-(3.2.15) provided that the necessary condition (3.2.21) is satisfied. In practice, the number of equations that can be obtained from (3.2.14)-(3.2.15) together with the condition (3.2.21) is assumed to be finite, say $\{2N+2\}$ which is identical to the number of unknown constants. Theoretically, the solution of the resulting infinite set of equations is to be achieved by means of infinite determinants; practically, to find the approximate solution in terms of the low Reynolds number $R\varepsilon$, we solve a finite number of equations.

We will proceed to approximate the unknown constants in terms of $R\varepsilon$ as $R\varepsilon \to 0$ by making a generalization of a method, used by Tomotika & Aoi [45] & [47] and Sidrak [40] & [41], for solving a coupled system of equations. The two infinite sets of linear simultaneous equations (3.2.14) and (3.2.15) involving the constants β, A_m ; $m \geq 0$ and B_m , $m \geq 1$ may be re-written as

$$\sum_{m=1}^{\infty} \chi_{nm} B_m = 4\delta_{n,1} - \sum_{m=0}^{\infty} \Omega_{nm} A_m , \quad n \geq 1 ; \tag{3.2.22}$$

$$\sum_{m=0}^{\infty} \Gamma_{nm} A_m = 4\beta\delta_{n,0} - \sum_{m=1}^{\infty} \Lambda_{nm} B_m , \quad n \geq 0 \tag{3.2.23}$$

where the coefficients of the constants A_m and B_m are defined by the equations (3.2.16) through (3.2.19). The final equation necessary to complete these sets is given by the condition

$$\sum_{m=0}^{\infty} A_m = 0 . \tag{3.2.24}$$

By this means we do in effect adopt a global procedure of adjusting the solution to the correct conditions at large distances. For the first approximation, we put B_m , $m \geq 2$ and A_m , $m \geq 0$ equal to zero in (3.2.22), and solve the first equation in (3.2.22) for B_1 ; and then we put A_m , $m \geq 2$ and B_m , $m \geq 2$ equal to zero in (3.2.23) and (3.2.24), and solve the first two equations in (3.2.23) together with the condition (3.2.24) for A_0, A_1 and β by using the first approximation to B_1 . For the second approximation, we put B_m , $m \geq 3$ and A_m , $m \geq 2$ equal to zero in (3.2.22), and solve the first two equations in (3.2.22) for B_1, B_2 by using the first approximation to A_0, A_1 ; and then we put A_m , $m \geq 3$ and B_m , $m \geq 3$ equal to zero in (3.2.23) and (3.2.24), and solve the first three equations in (3.2.23) together with the condition (3.2.24) for A_0, A_1, A_2 and β by using the second approximations to B_1, B_2, and so on.

In this approximation, it is assumed that as $R\varepsilon \to 0$ the set of constants A_m , $m \geq 0$ and B_m , $m \geq 1$ are decreasing when m increases. Only under this circumstance we can determine the approximations to the vorticity components on the surface of the cylinder in terms of the low Reynolds number $R\varepsilon$ from the expansions (3.2.12) and (3.2.13). Also, in this approximation all the terms involving powers of $R\varepsilon$ above a certain order are neglected, and all below that order are taken into account.

To obtain approximate solutions to the constants B_m , $m \geq 1$ and A_m, $m \geq 0$ we must determine the coefficients of these constants, namely,

$$\begin{aligned} &\chi_{nm}\ , \ n \geq 1,\ m \geq 1\ ;\ \ \Lambda_{nm}\ ,\ n \geq 0,\ m \geq 1\ , \\ &\Omega_{nm}\ ,\ n \geq 1,\ m \geq 0\ ;\ \ \Gamma_{nm}\ ,\ n \geq 0,\ m \geq 0\ , \end{aligned} \qquad (3.2.25)$$

by evaluating the integrals involving the product of the two modified Bessel functions in equations (3.2.16) through (3.2.19). For our present purpose, it would not be much use to give a complicated general expression for these coefficients. We can obtain an approximation to a finite number of constants β, A_s, B_s by replacing the infinite sums in (3.2.12) and (3.2.13) by finite sums over N terms satisfying (3.2.22) and (3.2.23) for all integer values of m, n up to m = n = N and finally finding the approximate solution to the set of (2N + 1) simultaneous algebraic equations which arise from (3.2.22) and (3.2.23) together with the condition (3.2.24). In Appendix VII, by taking N = 3, forty-nine coefficients have been determined in terms of $R\varepsilon$, treating $R\varepsilon$ as small. They are sufficient to enable us to proceed to the third approximation.

The first approximation gives

$$B_1 = -\frac{R\varepsilon}{2\cosh\xi_0}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_0} - e^{-2\xi_0}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^2\right]\right], \quad (3.2.26)$$

$$\beta = 4e^{-2\xi_0}\sin 2\alpha[\cosh\xi_0][R\varepsilon]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma + \frac{1}{2}\left(1 - e^{-2\xi_0}\cos 2\alpha - \frac{1}{2}e^{-4\xi_0}\right)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma - \frac{1}{2}\left(1 - e^{-4\xi_0} + e^{-2\xi_0}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^2\right)\right], \quad (3.2.27)$$

$$A_0 = \frac{1}{4}e^{-2\xi_0}\sin 2\alpha[\cosh\xi_0]^{-1}[R\varepsilon]\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma - \frac{1}{2}e^{-2\xi_0}\left(\cos 2\alpha + \frac{1}{2}e^{-2\xi_0}\right)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma - \frac{1}{2}\left(1 - e^{-4\xi_0} + e^{-2\xi_0}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([\ln R\varepsilon]^{-1}\right)\right], \quad (3.2.28)$$

$$A_1 = -\frac{1}{4} e^{-2\xi_o} \sin 2\alpha [\cosh\xi_o]^{-1} [R\varepsilon] \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma + \frac{1}{2}\left(1 - e^{-2\xi_o}\cos 2\alpha - \frac{1}{2} e^{-4\xi_o}\right)\right]^{-1} \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 - e^{-4\xi_o} + e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([R\varepsilon]^2 [\ln R\varepsilon]^2\right)\right], \quad (3.2.29)$$

The second approximation gives

$$B_1 = -\frac{R\varepsilon}{2\cosh\xi_o} \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([\ln R\varepsilon]^{-2}\right)\right], \quad (3.2.30)$$

$$B_2 = -3.2^{-8} e^{2\xi_o} \left(3 + e^{-4\xi_o}\right)^{-1} [\cosh\xi_o]^{-3} [R\varepsilon]^3 \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2} \cdot \left(1 + \frac{1}{2} e^{-4\xi_o} + e^{-4\xi_o}[\sin 2\alpha]^2\right)\right] \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1} \left[1 + O\left([\ln R\varepsilon]^{-2}\right)\right], \quad (3.2.31)$$

$$\beta = 4e^{-2\xi_o} \sin 2\alpha [\cosh\xi_o] [R\varepsilon]^{-1} \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma + \frac{1}{2}\left(1 - e^{-2\xi_o}\cos 2\alpha - \frac{1}{2} e^{-4\xi_o}\right)\right]^{-1} \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right)\right], \quad (3.2.32)$$

$$A_o = \frac{1}{4} e^{-2\xi_o} \sin 2\alpha [\cosh\xi_o]^{-1} [R\varepsilon] \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2} e^{-2\xi_o}\left(\cos 2\alpha + \frac{1}{2} e^{-2\xi_o}\right)\right]^{-1} \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([\ln R\varepsilon]^{-1}\right)\right], \quad (3.2.33)$$

$$A_1 = -\frac{1}{4}e^{-2\xi_o}\sin 2\alpha[\cosh\xi_o]^{-1}[R\varepsilon]\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}e^{-2\xi_o}\left(\cos 2\alpha + \frac{1}{2}e^{-2\xi_o}\right)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^2\right)\right], \quad (3.2.34)$$

$$A_2 = -3.2^{-9}\left(3 + e^{-4\xi_o}\right)^{-1}\sin 2\alpha[\cosh\xi_o]^{-3}[R\varepsilon]^3\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2} \cdot \left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1}\left[1 + O\left([\ln R\varepsilon]^{-1}\right)\right]. \quad (3.2.35)$$

It follows from the equations (3.2.33) through (3.2.35) that

$$A_o = O\left([R\varepsilon][\ln R\varepsilon]^{-2}\right), \quad A_1 = O\left([R\varepsilon][\ln R\varepsilon]^{-2}\right),$$

$$A_2 = O\left([R\varepsilon]^3[\ln R\varepsilon]^{-1}\right).$$

Also, it follows from the equations (3.2.30) and (3.2.31)

$$B_1 = O\left([R\varepsilon][\ln R\varepsilon]^{-1}\right), \quad B_2 = O\left([R\varepsilon]^3\right).$$

Without going into detail it may be shown that

$$A_3 = O\left([R\varepsilon]^4[\ln R\varepsilon]^{-1}\right), \quad A_4 = O\left([R\varepsilon]^5[\ln R\varepsilon]^{-1}\right)$$

$$A_5 = O\left([R\varepsilon]^6[\ln R\varepsilon]^{-1}\right), \ldots\ldots \quad (3.2.36)$$

and also,

$$B_3 = O\left([R\varepsilon]^4[\ln R\varepsilon]^{-1}\right), \quad B_4 = O\left([R\varepsilon]^5[\ln R\varepsilon]^{-1}\right),$$

$$B_5 = O\left([R\varepsilon]^7[\ln R\varepsilon]^{-1}\right), \ldots\ldots \quad (3.2.37)$$

As is to be expected, the solution sets of the constants A_m, $m \geq 0$ and

B_m, $m \geq 1$ yield decreasing sequences as $R\varepsilon \to 0$ provided that

$$A_{m+1} = O[A_m] \quad \text{as} \quad R\varepsilon \to 0, \quad m \geq 1$$

$$A_{m+1} = O\big([R\varepsilon][A_m]\big) \quad \text{as} \quad R\varepsilon \to 0, \quad m \geq 2$$

and also,

$$B_{m+1} = O[B_m] \quad \text{as} \quad R\varepsilon \to 0, \quad m \geq 1$$

$$B_{m+1} = O\big([R\varepsilon]^2[B_m]\big) \quad \text{as} \quad R\varepsilon \to 0, \quad m \geq 4.$$

We can now approximate the vorticity components $G_o(\xi)$ and $g_1(\xi)$ of the Oseen flow problem on the elliptic cylinder surface in terms of the low Reynolds number $R\varepsilon$. As stated before, the equation of the surface of the elliptic cylinder under consideration is $\xi = \xi_o$. If we write down the equations (3.2.12) for $n = 0$, (3.2.13) for $n = 1$ and then substitute the result (IV-18) into (3.2.13) we can immediately arrive at

$$G_o(\xi) = \sum_{m=0}^{\infty} 2A_m I_m(z)K_m(z), \tag{3.2.38}$$

$$g_1(\xi) = \sum_{m=1}^{\infty} 2m B_m z^{-1} I_m(z)K_m(z). \tag{3.2.39}$$

If we put ξ equal to ξ_o in these equations and then expand the modified Bessel functions in (3.2.38) and (3.2.39) by using (IV-2) and (IV-7) we find, after use of (3.2.30), (3.2.33) and (3.2.34), that as $R\varepsilon \to 0$ the approximations to the vorticity components G_o, g_1 of the Oseen flow problem at $\xi = \xi_o$ are

$$G_o(\xi_o) = -\frac{1}{2}e^{-2\xi_o}\sin 2\alpha[\cosh\xi_o]^{-1}[R\varepsilon]\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\Big(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\Big)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}e^{-2\xi_o}\left(\cos 2\alpha + \frac{1}{2}e^{-2\xi_o}\right)\right]^{-1} \cdot \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma\right]\left[1 + O\big([\ln R\varepsilon]^{-1}\big)\right], \tag{3.2.40}$$

$$g_1(\xi_o) = -4e^{-\xi_o}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - \cos2\alpha\, e^{-2\xi_o}\right)\right]^{-1}$$

$$\cdot\left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^2\right]\right. \quad (3.2.41)$$

We will proceed to determine the vorticity components $G_1(\xi)$, $G_2(\xi)$ and $g_2(\xi)$ by using the results that are obtained for the vorticity components $G_o(\xi)$ and $g_1(\xi)$. In order to determine these vorticity components, we must obtain the approximations to the constants C_1, C_2 and D_2 in equations (3.2.7), (3.2.8) and (3.2.10) as $R\varepsilon \to 0$. This can be done by the satisfaction of the two sets of the integral conditions (3.2.6) for the vorticity components of the Oseen flow problem. Without going into detail, it may be shown, by using the expressions (3.2.7)-(3.2.8), (3.2.10) and the results (3.2.38), (3.2.39) in (3.2.6), that satisfaction of the integral conditions gives

$$C_1 = 0, \quad (3.2.42)$$

$$C_2 = \sum_{m=o}^{\infty} 4m^2 A_m I_m K_m, \quad (3.2.43)$$

$$D_2 = \left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)^2 \sum_{m=o}^{\infty} B_m\left[I_{m-1}K_{m+1} - I_{m+1}K_{m-1}\right], \quad (3.2.44)$$

where the arguments of the modified Bessel functions are $\dfrac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}$. It can be deduced from these equations that

$$C_2 = -e^{-2\xi_o}\sin2\alpha[\cosh\xi_o]^{-1}[R\varepsilon]\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(e^{-2\xi_o}\cos2\alpha\right.\right.$$

$$\left.\left.+\frac{1}{2}e^{-4\xi_o}\right)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\ _o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos2\alpha\right)\right]^{-1}$$

$$\cdot\left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^2\right)\right], \quad (3.2.45)$$

$$D_2 = -\frac{R\varepsilon}{\cosh\xi_o}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1} \cdot \left[1 + O\left([\ln R\varepsilon]^{-2}\right)\right]. \quad (3.2.46)$$

In order to find the approximations to the vorticity components G_1, G_2 and g_2 on the cylinder surface as $R\varepsilon \to 0$, we put ξ equal to ξ_o in the equations (3.2.7) and (3.2.8) and also, use the results (3.2.26), (3.2.28), (3.2.29), (3.2.36), (3.2.37) (3.2.38), (3.2.39), (3.2.40)-(3.2.42), (3.2.45), (3.2.46) in these equations and then expand the modified Bessel functions in the resulting equations by using (IV-2) and (IV-7) in terms of $R\varepsilon$. This gives

$$G_1(\xi_o) = -2e^{-3\xi_o}\sin 2\alpha\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}e^{-2\xi_o}\left(\cos 2\alpha + \frac{1}{2}\cdot e^{-4\xi_o}\right)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 - e^{-2\xi_o}\cos 2\alpha + e^{-4\xi_o}\right)\right]^{-1} \cdot \left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^{-2}\right)\right], \quad (3.2.47)$$

$$G_2(\xi_o) = -\frac{3}{4}e^{-2\xi_o}\left(3 + e^{-4\xi_o}\right)^{-1}[\cosh\xi_o]^{-1}[R\varepsilon]\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\cdot\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1}\left[1 + O\left([\ln R\varepsilon]^{-1}\right)\right], \quad (3.2.48)$$

$$g_2(\xi_o) = -\frac{3}{2}\left(3 + e^{-4\xi_o}\right)^{-1}[\cosh\xi_o]^{-1}[R\varepsilon]\left[1 + O\left([\ln R\varepsilon]^{-1}\right]\right. . \quad (3.2.49)$$

It follows from (3.2.40), (3.2.47) and (3.2.48) that

$$G_0(\xi_0) = O\left([\,R\varepsilon\,][\,\ln R\varepsilon\,]^{-1}\right), \quad G_1(\xi_0) = O\left([\,\ln R\varepsilon\,]^{-2}\right),$$

$$G_2(\xi_0) = O\left([\,R\varepsilon\,][\,\ln R\varepsilon\,]^{-1}\right).$$

Also, it follows from (3.2.41) and (3.2.42) that

$$g_1(\xi_0) = O\left([\,\ln R\varepsilon\,]^{-1}\right), \quad g_2(\xi_0) = O\left([\,R\varepsilon\,]\right).$$

Without going into detail, it may be shown that

$$G_3(\xi_0) = O\left([\,R\varepsilon\,][\,\ln R\varepsilon\,]^{-1}\right), \quad G_4(\xi_0) = O\left([\,R\varepsilon\,][\,\ln R\varepsilon\,]^{-1}\right),$$

$$G_5(\xi_0) = O\left([\,R\varepsilon\,]^2[\,\ln R\varepsilon\,]^{-1}\right), \ldots\ldots \qquad (3.2.50)$$

and also,

$$g_3(\xi_0) = O\left([\,R\varepsilon\,][\,\ln R\varepsilon\,]^{-1}\right), \quad g_4(\xi_0) = O\left([\,R\varepsilon\,][\,\ln R\varepsilon\,]^{-1}\right),$$

$$g_5(\xi_0) = O\left([\,R\varepsilon\,]^2[\,\ln R\varepsilon\,]^{-1}\right), \ldots\ldots \qquad (3.2.51)$$

It may be deduced from these equations that as $R\varepsilon \to 0$ the significant vorticity component at $\xi = \xi_0$ is $g_1(\xi_0)$. The exact form of the dominant terms in the expansions of the limited number of vorticity components g_1, g_2, G_0, G_1 and G_2 at $\xi = \xi_0$ are determined in terms of $R\varepsilon$ for $R\varepsilon \ll 1$. They are sufficient to enable us to obtain an approximation to the vorticity distribution over the surface of the elliptic cylinder correct to order of $[\,R\varepsilon\,][\,\ln R\varepsilon\,]^{-1}$, the lowest order term being $O\left([\,\ln R\varepsilon\,]^{-1}\right)$. Also, from these approximated vorticity components many properties of the Oseen flow problem may be obtained. If we substitute the results (3.2.40), (3.2.47)-(3.2.51) in the expansion (3.1.15), we obtain, after putting $\xi = \xi_0$ in (3.1.15), the following approximation to the vorticity distribution over the surface of the cylinder as $R\varepsilon \to 0$

$$\zeta(\xi_o,\eta) = -2e^{-\xi_o}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\right)\right]^{-1}$$

$$\cdot\left(2\sin 2\eta + e^{-2\xi_o}[\sin 2\alpha][\cos\eta]\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}e^{-2\xi_o}\left(\cos 2\alpha\right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{2}e^{-2\xi_o}\right)\right]^{-1}\right)\left[1 + O\left([R\varepsilon]\right)\right], \quad (3.2.52)$$

where $\eta = O(1)$.

Alternatively, the approximation (3.2.52) to the vorticity distribution over the surface of the elliptic cylinder can be obtained from (2.1.23) as $R\varepsilon \to 0$.

The components of the stream function $F_n(\xi)$, $n \geq 0$ and $f_n(\xi)$, $n \geq 1$ can now be obtained by integrating the sets of differential equations (3.1.25) and (3.5.26) subject to the initial conditions (3.1.27) and (3.1.28). These functions can be expressed in terms of integrals involving the products of two modified Bessel functions which cannot be evaluated by analytical methods. The form of these functions will not be stated here due to their complexity.

3.3 APPROXIMATIONS IN TERMS OF LOW REYNOLDS NUMBER

In this section, many properties of the Oseen flow problem under consideration will be obtained by using the approximated vorticity components on the surface of the cylinder.

We will proceed to obtain an approximation to the circulation at a great distance from the elliptic cylinder. The circulation round a large contour surrounding the cylinder is given by

$$K_\infty = -\pi \lim_{\xi\to\infty}\left[F_o'(\xi)\right]. \quad (3.3.1)$$

Definition of circulation round a close circuit is given in Appendix I.

If we write down the equation (3.1.25) for $n = 0$ and integrate with respect to ξ from ξ_o to ξ we find, after use of the second of (3.1.27) for $n = 0$, (3.2.38), (3.2.8), (3.2.10), (3.2.45) and (3.2.46), that

$$
\begin{aligned}
F_o'(\xi) = \sum_{m=o}^{\infty} A_m \Big[& 2^4 [\cosh\xi_o]^2 [R\varepsilon]^{-2} z^2 \big(2 I_m K_m + I_{m-1} K_{m+1} + I_{m+1} K_{m-1} \big) \\
& - \tfrac{1}{4} \cos 2\alpha \big(I_{m-2} K_m + I_{m+2} K_m + I_{m-1} K_{m-1} + I_{m+1} K_{m+1} \big) \Big] + 2^{-8} \\
& \cdot [\cosh\xi_o]^{-2} [R\varepsilon]^2 A_o \Big[z^{-2} \big(I_o K_o + I_1 K_1 \big) - \tfrac{1}{2} \big(I_2 K_o + I_1 K_1 + I_o K_2 \big) \\
& + \tfrac{1}{4} \big(I_o K_o + I_2 K_2 \big) - \int_{z_o}^{z} z^{-1} I_o(z) K_o(z)\, dz \Big] - 2^{-10} [\cosh\xi_o]^{-2} [R\varepsilon]^2 \\
& \cdot A_1 \Big[I_o K_2 + I_2 K_o + I_1 K_1 \Big] - \tfrac{1}{2} \big(I_o K_o + I_2 K_2 \big) + 2 \int_{z_o}^{z} z^{-1} I_o(z) K_o(z)\, dz \Big] \\
& - 2^{-7} [\cosh\xi_o]^{-2} [R\varepsilon]^2 \sum_{m=2}^{\infty} A_m \Big[\frac{z^{-2}}{m[m^2 - 1]} \Big(I_1 K_1 + \sum_{n=2}^{m-1} n^2 [I_n K_n] \\
& + \tfrac{1}{2} m[m-1] I_m K_m \Big) \Big] - \tfrac{1}{4} \sin 2\alpha \sum_{m=1}^{\infty} B_m \Big[I_{m-2} K_m + I_{m-1} K_{m-1} - I_{m+2} K_m \\
& - I_{m+1} K_{m+1} \Big] - \tfrac{1}{8} A_1 \lim_{\xi \to \xi_o} \big[e^{2\xi} I_o K_2 \big] + O(R\varepsilon), \qquad (3.3.2)
\end{aligned}
$$

where the arguments of the modified Bessel functions are $z = \dfrac{R\varepsilon\, e^{\xi}}{8\cosh\xi_o}$ and z_o denotes its value at $\xi = \xi_o$. The limit of $F_o'(\xi)$ as $\xi \to \infty$ can now be determined by using the necessary condition (3.2.24) and the asymptotic properties (IV-12), (IV-13) and (IV-14) of the modified Bessel functions for large argument in (3.3.2) as $\xi \to \infty$. Thus

$$
F_o'(\infty) = -\frac{1}{8} A_1 \lim_{\xi \to \xi_o} \big[e^{2\xi} I_o K_2 \big] + O[R\varepsilon] . \qquad (3.3.3)
$$

If we substitute the first approximation (3.2.29) to the constant A_1 as $R\varepsilon \to 0$ and the expansions (IV-2) for $n = 0$ and (IV-7) for $n = 2$ in (3.3.3) we obtain

$$F_0'(\infty) = 4e^{-2\xi_0}\sin 2\alpha[\cosh\xi_0][R\varepsilon]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma + \frac{1}{2}\left(1 - e^{-2\xi_0}\right.\right.$$

$$\left.\left. \cdot\cos 2\alpha - \frac{1}{2}e^{-4\xi_0}\right)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma - \frac{1}{2}\left(1 - e^{-4\xi_0} + e^{-2\xi_0}\cos 2\alpha\right)\right]^{-1}$$

$$\cdot\left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^2\right)\right]. \quad (3.3.4)$$

It follows from (1.3.13), (1.3.15) and the first of (3.2.6) that

$$\beta = F_0'(\infty) = \frac{1}{2}\int_{\xi_0}^{\infty}\left(\cosh 2\xi\, G_0(\xi) - \cos 2\alpha\, G_2(\xi) + \sin 2\alpha\, g_2(\xi)\right)d\xi\ . \quad (3.3.5)$$

We note that the approximation (3.3.4) to $F_0'(\infty)$ as $R\varepsilon \to 0$ coincides perfectly with the approximation (3.2.27) to the constant β that is obtained in the previous section. We can now obtain an approximation to the circulation at a great distance from the elliptic cylinder by substituting (3.3.4) in (3.3.1).

$$K_- = -4\pi e^{-2\xi_0}\sin 2\alpha[\cosh\xi_0][R\varepsilon]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma + \frac{1}{2}\left(1 - e^{-2\xi_0}\cos 2\alpha\right.\right.$$

$$\left.\left. - \frac{1}{2}e^{-4\xi_0}\right)\right]^{-1}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \gamma - \frac{1}{2}\left(1 - e^{-4\xi_0} + e^{-2\xi_0}\cos 2\alpha\right)\right]^{-1}$$

$$\cdot\left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]\right)\right]. \quad (3.3.6)$$

We now turn our attention to providing a check on the periodicity of pressure distribution over the surface of the elliptic cylinder. Because of the choice of the co-ordinate system (3.1.2), all of the dependent variables in the flow domain must be periodic functions of η with period 2π. Thus, in particular, the pressure in the fluid must be periodic in η with period 2π. As stated in Section 2.2, the pressure in the fluid will only turn out to be periodic provided that the correct conditions at large distances from the cylinder have been satisfied.

The pressure coefficient which describes the pressure variation round the surface of the cylinder can be used to check the periodicity. The pressure coefficient $p^*_{\xi_o}$ is defined in Appendix III and is given by the expression (III-3). For the present case of an ellipse this reduces to

$$p^*_{\xi_o} = p(\xi_o,\eta) - p(\xi_o,\eta_o) = -\frac{1}{R\varepsilon}[\,2\cosh\xi_o] \int_c^{\eta} \left(\frac{\partial\zeta}{\partial\xi}\right)_{\xi=\xi_o} d\eta \,, \tag{3.3.7}$$

where η_o, the point taken for the base pressure, has been arbitrarily set to zero.

If we differentiate the series (3.1.15) for the vorticity ζ and integrate with respect to η from $\eta = 0$ to $\eta = 2\pi$ we find that

$$\int_0^{2\pi} \left(\frac{\partial\zeta}{\partial\xi}\right)_{\xi=\xi_o} d\eta = G_o'(\xi_o) \,, \tag{3.3.8}$$

where the prime denotes differentiation with respect to ξ. We now replace the derivative of G_o with respect to z by using $z = \frac{R\varepsilon\, e^{\xi}}{8\cosh\xi_o}$ and then substitute (3.2.7) and (3.2.42) in (3.3.8). It is then found that the expression (3.3.7) for $\eta = 2\pi$, is replaced by

$$p^*_{\xi_o} = p(\xi_o,\eta) - p(\xi_o,\eta_o) = -\frac{\pi}{2} e^{-\xi_o} G_1(\xi_o) \,. \tag{3.3.9}$$

It may be shown, by using the approximation (3.2.47) in (3.3.9), that

$$p(\xi_o,2\pi) - p(\xi_o,0) = \pi e^{-4\xi_o} \sin 2\alpha \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}e^{-2\xi_o}\left(\cos 2\alpha + \frac{1}{2}e^{-4\xi_o}\right)\right]^{-1} \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 - e^{-2\xi_o}\cos 2\alpha + e^{-4\xi_o}\right)\right]^{-1} \cdot \left[1 + O\left([R\varepsilon]^2[\ln R\varepsilon]^2\right)\right] \tag{3.3.10}$$

This gives the pressure difference on the left being of order $[\ln R\varepsilon]^{-2}$. The pressure distribution on the surface of the cylinder will turn out to be periodic only if the left side of (3.3.10) is zero. In this sense the result (3.3.10) indicates a slight inconsistency. However, the pressure difference on the left side of (3.3.10) approaches to zero as $R\varepsilon \searrow 0$. We may then conclude that the Oseen approximations obtained for this problem are valid only for very small Reynolds number $R\varepsilon$.

We now proceed to describe various properties of the flow obtainable from the approximated vorticity components. The non-dimensional coefficients of the drag and lift, C_D and C_L, are defined by the equations (III-9) and (III-10) in Appendix III. We can obtain a simple formula for the lift and drag coefficients for $R\varepsilon \to 0$ by making use of formula which may be found to approximate these quantities as $R\varepsilon \to 0$ from the approximated vorticity components. For the present case of an ellipse, the non-dimensional coefficients of drag and lift can be defined by

$$C_D = \frac{\pi}{R\varepsilon}\Big[2[(\cos\alpha)\cosh\xi_o]^2 + (\sin\alpha)^2 \sinh 2\xi_o\Big] g_1(\xi_o) - \frac{\pi}{R\varepsilon}\Big[2[(\sin\alpha)\cosh\xi_o]^2 + (\cos\alpha)^2 \sinh 2\xi_o\Big] g_1'(\xi_o) + \frac{\pi}{2R\varepsilon}\sin 2\alpha\Big[2(\cosh\xi_o)^2 - \sinh 2\xi_o\Big]\Big[G_1'(\xi_o) + G_1(\xi_o)\Big] + \frac{1}{2}\sin 2\alpha\Big[\cosh\xi_o - \sinh\xi_o\Big] p^*_{\xi_o}(2\pi), \quad (3.3.11)$$

$$C_L = -\frac{\pi}{R\varepsilon}\Big[2[(\sin\alpha)\cosh\xi_o]^2 + (\cos\alpha)^2 \sinh 2\xi_o\Big] G_1(\xi_o) + \frac{\pi}{R\varepsilon}\Big[2[(\cos\alpha)\cosh\xi_o]^2 + (\sin\alpha)^2 \sinh 2\xi_o\Big] G_1'(\xi_o) - \frac{\pi}{2R\varepsilon}\sin 2\alpha\Big[2(\cosh\xi_o)^2 - \sinh 2\xi_o\Big]\Big[g_1(\xi_o) + g_1'(\xi_o)\Big] + \Big[(\sin\alpha)^2 \sinh\xi_o + (\cos\alpha)^2 \cosh\xi_o\Big] p^*_{\xi_o}(2\pi), \quad (3.3.12)$$

where $p^*_{\xi_o}(2\pi) = p(\xi_o, 2\pi) - p(\xi_o, 0)$ and the prime denotes differentiation with respect to ξ. It may be noted that the lift and drag coefficients each consist of the friction coefficient which depends on G_1 and g_1 and also, that the pressure coefficient which depends on G_1', g_1' and $p^*_{\xi_o}$.

Without going into detail, it may be shown that

$$C_D = \frac{\pi}{R_\varepsilon}\Big[2[(\cos\alpha)\cosh\xi_o]^2 + (\sin\alpha)^2\sinh 2\xi_o\Big]\, g_1(\xi_o) - \frac{\pi}{R_\varepsilon}\Big[2[(\sin\alpha)\cosh\xi_o]^2 + (\cos\alpha)^2\sinh 2\xi_o\Big]\, g_1'(\xi_o) + O\Big([\ln R_\varepsilon]^{-2}\Big), \quad (3.3.13)$$

$$C_L = -\frac{\pi}{R_\varepsilon}\Big[2[(\sin\alpha)\cosh\xi_o]^2 + (\cos\alpha)^2\sinh 2\xi_o\Big]\, G_1(\xi_o) + \frac{\pi}{R_\varepsilon}\Big[2[(\cos\alpha)\cosh\xi_o]^2 + (\sin\alpha)^2\sinh 2\xi_o\Big]\, G_1'(\xi_o) + O\Big([\ln R_\varepsilon]^{-2}\Big). \quad (3.3.14)$$

These expressions give the following results as $R_\varepsilon \to 0$

$$C_D = -\frac{4\pi}{R_\varepsilon} e^{-\xi_o}\Big[2(\cosh\xi_o)^2 + \sinh 2\xi_o\Big]\Big[\ln\Big(\frac{R_\varepsilon e^{\xi_o}}{16\cosh\xi_o}\Big) + \gamma - \frac{1}{2}e^{-4\xi_o} - \cos 2\alpha\, e^{-2\xi_o}\Big)\Big]^{-1}\Big[1 + O\Big([R_\varepsilon][\ln R_\varepsilon]^{-1}\Big]\Big], \quad (3.3.15)$$

$$C_L = \frac{2\pi}{R_\varepsilon} e^{-3\xi_o}\sin 2\alpha\Big[2(\cosh\xi_o)^2 + \sinh 2\xi_o\Big]\Big[\ln\Big(\frac{R_\varepsilon e^{\xi_o}}{16\cosh\xi_o}\Big) + \gamma - \frac{1}{2}e^{-2\xi_o}\cdot\Big(\cos 2\alpha + \frac{1}{2}e^{-4\xi_o}\Big)\Big]^{-1}\Big[\ln\Big(\frac{R_\varepsilon e^{\xi_o}}{16\cosh\xi_o}\Big) + \gamma - \frac{1}{2}\Big(1 - e^{-2\xi_o}\cos 2\alpha + e^{-4\xi_o}\Big)\Big]^{-1}\Big[1 + O\Big([R_\varepsilon]^2[\ln R_\varepsilon]^2\Big)\Big]. \quad (3.3.16)$$

We note that the leading terms deduced from the expressions (3.3.6), (3.3.15), (3.3.16) for the circulation in the counter-clockwise sense round a large contour surrounding the elliptic cylinder, K_∞, and the non-dimensional coefficients of the drag and lift, C_D and C_L, are in perfect agreement with the analytical results obtained by Hasimoto [19]. His analysis is summarized and also, the comparison of his results with those of the present analysis has been made in Appendix VIII.

CHAPTER IV

APPROXIMATIONS TO OSEEN FLOW PAST A CIRCULAR CYLINDER USING THE NEW METHOD

4.1 THE OSEEN EQUATIONS FOR SYMMETRIC FLOW PAST A CIRCULAR CYLINDER

In this chapter, the method outlined in Section 2.3 is applied to the symmetrical flow past a circular cylinder. For flow past a circular cylinder the details of the transformation (1.2.14) with $\alpha = 0$ are given by

$$x + iy = \exp(\xi + i\eta), \tag{4.1.1}$$

$$x = e^{\xi}\cos\eta, \qquad y = e^{\xi}\sin\eta. \tag{4.1.2}$$

Thus the (ξ, η) co-ordinate system corresponds to polar co-ordinates with $r = e^{\xi}$. The surface of the cylinder is at $\xi = 0$. The value of k in (1.2.15) is $k = 1$ and the function M in (1.2.20) is

$$M = e^{\xi}. \tag{4.1.3}$$

The Reynolds number R is based on the radius a of the circular cylinder and is given by

$$R = \frac{2aU}{\upsilon}. \tag{4.1.4}$$

The Oseen equations (2.1.8) and (2.1.9) take the forms in the (ξ, η) co-ordinate system

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2} = \frac{R}{2}e^{\xi}\left(\cos\eta\,\frac{\partial\zeta}{\partial\xi} - \sin\eta\,\frac{\partial\zeta}{\partial\eta}\right), \tag{4.1.5}$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = e^{2\xi}\zeta, \tag{4.1.6}$$

where $0 \le \xi < \infty$, $0 \le \eta \le \pi$. We note that in the present case the flow is symmetrical about $\eta = 0$ and both of the functions ψ and ζ are odd functions of η. The solutions of (4.1.5) and (4.1.6) are then required only in the region $0 \le \eta \le \pi$. The equations (4.1.5) and (4.1.6) are to be solved subject to the conditions

$$\int_0^\infty \int_0^\pi e^{(2-n)\xi} \zeta \sin n\eta \, d\eta \, d\xi = \pi\delta_{n,1} , \quad n \ge 1 , \tag{4.1.7}$$

$$\zeta(\xi,\eta) \to 0 \quad \text{as} \quad \xi \to \infty, \tag{4.1.8}$$

$$\psi(\xi,\eta) = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{when} \quad \xi = 0 , \tag{4.1.9}$$

$$e^{-\xi} \frac{\partial\psi}{\partial\xi} \sim \sin\eta , \qquad e^{-\xi} \frac{\partial\psi}{\partial\eta} \sim \cos\eta \quad \text{as} \quad \xi \to \infty, \tag{4.1.10}$$

$$\psi(\xi,0) = \psi(\xi,\pi) = \zeta(\xi,0) = \zeta(\xi,\pi) = 0. \tag{4.1.11}$$

Under the assumption of symmetrical flow it follows that the functions $F_n(\xi)$, $G_n(\xi)$ appearing in the expansions for ψ and ζ take the simplified forms

$$\zeta(\xi,\eta) = \sum_{n=1}^{\infty} g_n(\xi) \sin n\eta , \tag{4.1.12}$$

$$\psi(\xi,\eta) = \sum_{n=1}^{\infty} f_n(\xi) \sin n\eta . \tag{4.1.13}$$

In terms of the functions $g_n(\xi)$, $n \ge 1$ associated with the expansion (4.1.12) for the vorticity ζ, the differential equation (4.1.5) and the boundary conditions (4.1.7) and (4.1.8) become

$$g_n'' - n^2 g_n = \frac{R}{4} e^{\xi} [\, g_{n-1}' - (n-1)g_{n-1} + g_{n+1}' + (n+1)g_{n+1} \} , \quad n \ge 1 , \tag{4.1.14}$$

$$\int_0^\infty e^{(2-n)\xi} g_n(\xi) \, d\xi = 2\delta_{n,1} , \quad n \ge 1 , \tag{4.1.15}$$

$$\zeta(\xi,\eta) \to 0 \quad \text{as} \quad \xi \to \infty . \tag{4.1.16}$$

Here $g_0 = 0$, and the prime denotes differentiation with respect to ξ. The substitution of (4.1.12) and (4.1.13) in (4.1.6) yields

$$f_n''(\xi) - n^2 f_n(\xi) = e^{2\xi} g_n(\xi), \quad n \geq 1 . \tag{4.1.17}$$

The boundary conditions (4.1.9) and (4.1.10) can be written in terms of the functions $f_n(\xi)$, $n \geq 1$ by substituting the series (4.1.13) for ψ in these conditions. It is found from (4.1.9) and (4.1.10) that

$$f_n(0) = f_n'(0) = 0 , \quad n \geq 1 , \tag{4.1.18}$$

and as $\xi \to \infty$

$$e^{-\xi} f_n(\xi) \to \delta_{n,1} , \quad e^{-\xi} f_n'(\xi) \to \delta_{n,1} , \quad n \geq 1 . \tag{4.1.19}$$

The equations (4.1.14) for $g_n(\xi)$, $n \geq 1$ are to be solved subject to the conditions (4.1.15) and (4.1.16), and then the functions $f_n(\xi)$, $n \geq 1$ can now be determined by solving (4.1.17) so that the conditions (4.1.18) and (4.1.19) are satisfied.

4.2 APPROXIMATIONS FOR SYMMETRICAL FLOW PAST A CIRCULAR CYLINDER IN TERMS OF LOW REYNOLDS NUMBER

In this section approximations to the Oseen flow problem for the fixed circular cylinder described in the previous section will be presented in terms of the low Reynolds number R.

We will proceed to find an approximation to the vorticity distribution over the surface of the cylinder under consideration. The mathematical formulation of the Oseen flow problem for the vorticity components $g_n(z)$, $n \geq 1$ in the case of the fixed circular cylinder may be given as follows

$$z g_{n+1}' + (n+1) g_{n+1} = z g_n'' + g_n' - n^2 z^{-1} g_n - z g_{n-1}' + (n-1) g_{n-1} , \quad n \geq 1 , \tag{4.2.1}$$

where $z = \frac{R}{4} e^{\xi}$ and $g_0 = 0$;

$$\int_{\frac{R}{4}}^{\infty} z^{(1-n)} g_n(z)\,dz = \left(\frac{R}{4}\right)^{2-n} 2\delta_{n,1}, \quad n \geq 1, \tag{4.2.2}$$

$$g_n(z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad n \geq 1. \tag{4.2.3}$$

As previously noted, the set of differential equations (4.2.1) are integrable and the functions $g_n(z)$, $n \geq 2$ are only known precisely when the function $g_1(z)$ is known. It follows from equation (2.3.2) when $n = 2, 3$ that

$$g_2(z) = z \frac{d}{dz}[z^{-1} g_1(z)] + d_2 z^{-2}, \tag{4.2.4}$$

$$g_3(z) = z^2 \frac{d}{dz}\left[z^{-1} \frac{d}{dz}[z^{-1} g_1(z)]\right] - g_1(z) + 4z^{-3} \int^z z^2 g_1(z)\,dz + d_3 z^{-3}. \tag{4.2.5}$$

where d_2 and d_3 are arbitrary constants. In order to determine the function $g_1(z)$ we use the basic structure of the vorticity components

$$g_n(z) = \sum_{m=1}^{\infty} B_m [I_{m-n}(z) - I_{m+n}(z)] K_m(z), \quad n \geq 1 \tag{4.2.6}$$

once again. Now the B_m are to be found in terms of the low Reynolds number R provided that the integral conditions (4.2.2) on the vorticity components $g_n(z)$, $n \geq 1$ are satisfied. Thus, as in Section 3.2, to find the constants B_m, $m \geq 1$ we substitute the expression (4.2.6) into the integral conditions (4.2.2). This gives the infinite set of simultaneous equations

$$\sum_{m=1}^{\infty} \lambda_{nm} B_m = 2\delta_{n,1}, \quad n \geq 1 \tag{4.2.7}$$

where

$$\lambda_{nm} = \left(\frac{R}{4}\right)^{n-2} \int_{\frac{R}{4}}^{\infty} z^{1-n} [I_{m-n}(z) - I_{m+n}(z)] K_m(z)\,dz, \quad n \geq 1, \quad m \geq 1. \tag{4.2.8}$$

The conditions (4.2.7) may be recognized as equivalent to the conditions which are used to find a similar set of constants in the various applications of Oseen linearized theory, e.g. Tomotika & Aoi [45] & [47] and Sidrak [40] & [41], although the process is not exactly the same. To evaluate the integral in (4.2.8) we use the result (V-14) in (4.2.8) for $n = 1$ and the results (V-6), (V-7) in (4.2.8) for $n \geq 2$ and then if we expand all the modified Bessel functions in the resulting equation as $R \to 0$ by using (IV-2) and (IV-7) we find, after use of (IV-12) and (IV-13), that the coefficients λ_{nm} are functions of the Reynolds R and expressed as

$$\lambda_{1m} = \frac{4}{R} \sum_{n=1}^{m} \left[I_{n-1} K_{n-1} + I_n K_n \right], \quad m \geq 1,$$

$$\lambda_{nm} = \frac{1}{2(n-1)} \left[I_{n-m-1} K_{m-1} + I_{m-n} K_m - I_{n+m-1} K_{m+1} - I_{n+m} K_m \right], \quad n \geq 2, \ m \geq 1, \tag{4.2.9}$$

the arguments of all the modified Bessel functions being $\frac{\bar{R}}{4}$. We note that the properties (IV-5) and (IV-9) are used in obtaining the expressions (4.2.9).

The constants B_m, $m \geq 1$ are then to be found by solving the system of simultaneous linear algebraic equations (4.2.7). Theoretically, the solution of (4.2.8) is to be achieved by means of infinite determinants; in practice, to find the approximate solution in terms of the low Reynolds number $\bar{R}$ we solve a finite number of equations. For the first approximation we put B_2, B_3, B_4 and so on, equal to zero, and solve the first equation for B_1. For the second approximation, we put B_3, B_4, and so on, equal to zero, and solve the first two equations for B_1 and B_2, and so on. We note that the same method is used by Tomotika & Aoi [45] & [47] and Sidrak [40] to approximate a set of constants in their investigations of the drag on a circular cylinder. In this method of approximation all the terms involving powers of R above a certain order are neglected, and below that are taken into account. Also, in this approximation we assume that as $R \to 0$ the set of constants B_m, $m \geq 1$ are decreasing when m increases. Only under this circumstance can we determine the approximations to the vorticity components on the surface of the cylinder in terms of low Reynolds number R from the expansion (4.1.12).

To obtain approximate solutions to the constants B_m, $m \geq 1$ we must determine the coefficients (4.2.9) of B_m, $m \geq 1$ by using the expansions (IV-2) and (IV-7) of modified Bessel functions as $R \to 0$, and then these coefficients can be expressed in an infinite number of terms. For our present purpose, however, which is to obtain approximate solutions, it would not be of much use to give a complicated general expression for (4.2.9). The first sixteen coefficients have been calculated which are sufficient to enable us to proceed to the fourth approximation. Without going into detail it may be shown that the diagonal terms of the matrix of coefficients which result from the (4.2.7) has the following properties

$$\lambda_{nm} / \lambda_{nn} \to 0 \quad \text{as} \quad R \to 0, \quad 1 \leq m \leq n,$$

$$\lambda_{nm} = O(\lambda_{nn}) \quad \text{as} \quad R \to 0, \quad 1 \leq n \leq m,$$

and also,

$$\lambda_{(n+1)(n+1)} / \lambda_{nn} \to 0 \quad \text{as} \quad R \to 0, \quad n \geq 1.$$

The first approximation gives

$$B_1 = \frac{R}{2(I_0K_0 + I_1K_1)}, \tag{4.2.10}$$

where the arguments of the modified Bessel functions are again $\frac{R}{4}$. Expanding all the modified Bessel functions in (4.2.10) as $R \to 0$ yields

$$B_1 = -\frac{R}{2}\left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}\left[1 + O\left(R^2\right)\right]. \tag{4.2.11}$$

The second approximation gives

$$B_1 = -\frac{R}{2}\left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}\left[1 + O\left([R^2][\ln R]\right)\right], \tag{4.2.12}$$

$$B_2 = -2^{-6}R^3\left[1 + O\left([\ln R]^{-1}\right)\right]. \tag{4.2.13}$$

The third approximation gives

$$B_1 = -\frac{R}{2}\left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}\left[1 + O\left([R^2][\ln R]\right)\right], \tag{4.2.14}$$

$$B_2 = -2^{-6}R^3\left[1 + O\left([\ln R]^{-1}\right)\right], \tag{4.2.15}$$

$$B_3 = 2^{-13}R^5\left[1 + O\left([\ln R]^{-1}\right)\right]. \tag{4.2.16}$$

The fourth approximation gives

$$B_1 = -\frac{R}{2}\left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}\left[1 + O\left([R^2\ln R]\right)\right], \tag{4.2.17}$$

$$B_2 = -2^{-6}R^3\left[1 + O\left([\ln R]^{-1}\right)\right], \tag{4.2.18}$$

$$B_3 = 2^{-13} R^5 \left[1 + O\left([\ln R]^{-1} \right) \right], \tag{4.2.19}$$

$$B_4 = -\frac{1}{3} \cdot 2^{-20} R^7 \left[1 + O\left([\ln R]^{-1} \right) \right]. \tag{4.2.20}$$

It may then be shown that

$$B_{m+1} = O[R^{2m+1}] \quad \text{as} \quad R \to 0, \quad m \geq 1. \tag{4.2.21}$$

Thus the solution set of the constants yields a rapidly decreasing sequence as $R \to 0$ provided that

$$B_{m+1} = O[B_m] \quad \text{as} \quad R \to 0, \quad m \geq 1, \tag{4.2.22}$$

$$B_{m+1} = O\left([R^2][B_m] \right) \quad \text{as} \quad R \to 0, \quad m \geq 2. \tag{4.2.23}$$

We can now find an approximation to the vorticity component $g_1(\xi)$ of the Oseen flow problem on the fixed circular cylinder surface, $\xi = 0$, in terms of the low Reynolds number R. As previously noted, the vorticity component $g_1(\xi)$ in (4.2.6) may be written as

$$g_1(\xi) = \sum_{m=1}^{\infty} 2m B_m z^{-1} I_m(z) K_m(z). \tag{4.2.24}$$

If we put ξ equal to zero in this expansion and then expand the modified Bessel functions in (4.2.24) by using (IV-2) and (IV-7), we find, after use of (4.2.12) and (4.2.13), that as $R \to 0$ the approximation to the vorticity component g_1 of the Oseen flow problem under consideration at $\xi = 0$ is

$$g_1(0) = -2\left[\ln\left(\frac{R}{8} \right) + \gamma - \frac{1}{2} \right]^{-1} \left[1 + O\left([R]^2 [\ln R] \right) \right]. \tag{4.2.25}$$

The vorticity components $g_2(\xi)$ and $g_3(\xi)$ can now be determined by using (4.2.4) and (4.2.5). In order to do this, we need to approximate the constants d_2 and d_3 in (4.2.4) and (4.2.5) as $R \to 0$. It may be shown, by using the expressions (4.2.4), (4.2.5) and result (4.2.24) in (4.2.2) for $n = 2$ and $n = 3$ that satisfaction of the integral conditions gives

$$d_2 = \frac{R}{2} g_1(0), \tag{4.2.26}$$

$$d_3 = -2R\Big[g_1(0) - \sum_{m=1}^{\infty} m B_m \big(I_{m+1} K_m - I_m K_{m-1} \big) \Big] - \frac{R^3}{64}\Big[g_1(0) + \sum_{m=1}^{\infty} B_m \big(I_{m-2} K_{m+1} - I_{m+2} K_{m-1} \big) \Big], \tag{4.2.27}$$

where the arguments of the modified Bessel functions are $\frac{R}{4}$. It may be deduced from (4.2.4) and (4.2.5), by using the results (4.2.25), (4.2.26) and (4.2.27), that as $R \to 0$ the approximations to the vorticity components g_2 and g_3 at $\xi = 0$ are

$$g_2(0) = -\frac{R}{2}\Big[1 + O\big([\ln R]^{-1}\big) \Big], \tag{4.2.28}$$

$$g_3(0) = -2^{-6} R^2 \Big[\ln\Big(\frac{R}{8}\Big) + \gamma - \frac{1}{2} \Big]^{-1} \Big[1 + O\big([\ln R]^{-1}\big) \Big]. \tag{4.2.29}$$

Without going into detail it may be shown that

$$g_{n+1}(0) = O[R^n] \quad \text{as} \quad R \to 0, \quad n \geq 1. \tag{4.2.30}$$

Thus the approximations $g_n(0)$, $m \geq 1$ to the vorticity components yield a decreasing sequence as $R \to 0$ provided that

$$g_{n+1}(0) = O[g_n(0)] \quad \text{as} \quad R \to 0, \quad n \geq 1, \tag{4.2.31}$$

$$g_{n+1}(0) = O\left([R]\, g_n(0)\right) \quad \text{as} \quad R \to 0, \quad n \geq 1 . \tag{4.2.32}$$

The significant vorticity component at $\xi = 0$ is then $g_1(0)$ as $R \to 0$. We can now obtain an approximation to the vorticity distribution over the surface of the fixed circular cylinder, by using approximated vorticity components g_1, g_2 and g_3 at $\xi = 0$, correct to the order of $[R][\ln R]^{-1}$, the lowest order term being $O\left([\ln R]^{-1}\right)$. It follows from (4.1.11), (4.2.25), (4.2.28) and (4.2.29) that as $R \to 0$ the approximation to the vorticity distribution over the surface of the fixed cylinder is

$$\zeta(0,\eta) = -\left[2\left(\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right)^{-1} \sin\eta + \frac{R}{2}\sin 2\eta \right]\left[1 + O(R)\right], \tag{4.2.33}$$

where $\eta = O(1)$.

Alternatively, an approximation to the vorticity distribution over the surface of the cylinder under consideration may be obtained from

$$\zeta(\xi,\eta) = e^{z\cos\eta} \left[\sum_{n=1}^{\infty} B_n K_n(z) \sin n\eta \right] \tag{4.2.34}$$

as $R \to 0$. The result (4.2.33) is the same as that obtained from (4.2.34).

We can now obtain an approximation to the non-dimensional drag coefficient C_D as $R \to 0$. For the present case of a fixed circular cylinder the non-dimensional drag coefficient is defined by

$$C_D = \frac{2\pi}{R} \left[g_1(0) - g_1'(0) \right], \tag{4.2.35}$$

where the prime denotes differentiation with respect to ξ. We note that this formula is given by the equation (III-21) in Appendix III. It may be shown, by differentiating the series (4.2.6) for $n = 1$ with respect to z and noticing $z = \frac{R}{4} e^{\xi}$, that

$$g_1'(\xi) = -g_1(\xi) + 2\sum_{m=1}^{\infty} m B_m \left[I_{m+1} K_m - I_m K_{m-1} \right] . \qquad (4.2.36)$$

Using the results (4.2.25), (4.2.35) and (4.2.36) we obtain the result for the drag coefficient

$$C_D = -\frac{8\pi}{R}\left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2} \right]^{-1} \left[1 + O\left([R^2] \right) \right] . \qquad (4.2.37)$$

This is the well-known formula obtained first by Lamb in 1911 (see e.g. Rosenhead [37] p.180).

4.3 CALCULATED RESULTS FOR THE DETERMINATION OF CRITICAL VALUE OF THE REYNOLDS NUMBER R

Experimental observations on the flow past circular cylinders indicate that for very low Reynolds numbers it is possible to obtain flow patterns in which no separation of the fluid from the cylinder takes place before it reaches the downstream generator. As the Reynolds number is increased, a critical value is reached at which separation does occur and a pair of attached eddies forms behind the cylinder. Dennis & Chang [10] made a very careful numerical calculation on the basis of the full Navier-Stokes equations. In their work the critical value of R is estimated at R = 6.2 which is in good agreement with the experimental results. On the other hand, in a theoretical study of Oseen's linearization of the Navier-Stokes equations of motion, Tomotika & Aoi [45] found that the attached eddies are formed even for Reynolds numbers as small as R = 0.05, in direct contradiction of experimental fact. Such a conclusion is worthy of comment, for it is generally accepted that the Oseen equations give an adequate representation of the theoretical flow for small enough values of R. This certainly appears to be so far the drag coefficient. As stated before, a value for the drag based on these equations was given as early as 1911 by Lamb.

On account of this discrepancy between theory and observation and also, to be able to provide a useful check on the validity of the approximations, obtained in the previous section, based on low Reynolds number expansions, we will now proceed to present the results of a further investigation of the Oseen equations for Reynolds numbers $R = 0.05, 0.5, 1, 2, 3$. In this investigation the finite sums in (4.2.6) and (4.2.7) are replaced by finite sums over four terms. As was pointed out before the Oseen equations are only a low Reynolds number approximations to the full Navier-Stokes equations so that the extra terms retained in the full Oseen equations are probably not justified to the order of approximation of the Navier-Stokes equations. The first sixteen coefficients of the constants B_m, in (4.2.7), given by the expressions (4.2.9), have been determined using the calculated values of the modified Bessel functions in these expressions at $z = \frac{R}{4}$ for $R = 0.05, 0.5, 1, 2, 3$.

The calculated results based on both expanded and non-expanded forms of the Oseen equations may be summarized as follows.

(i) for $R = 0.05$

m	B_m (analytical)	$f''_m(0)$ (analytical)	B_m (full Oseen)	$f''_m(0)$ (full Oseen)
1	0.5002×10^{-2}	0.4002	0.5004×10^{-2}	0.4000
2	-0.1953×10^{-5}	-0.0250	-0.1854×10^{-5}	-0.0213
3	0.3815×10^{-10}	0.7816×10^{-5}	0.3630×10^{-10}	0.8023×10^{-5}
4	-0.2484×10^{-15}	0.[illegible]28 $\times 10^{-7}$	-0.2311×10^{-15}	0.2727×10^{-7}

(ii) for $R = 0.5$

m	B_m (analytical)	$f_m''(0)$ (analytical)	B_m (full Oseen)	$f_m''(0)$ (full Oseen)
1	0.9275×10^{-1}	0.7420	0.9467×10^{-1}	0.7283
2	-0.1953×10^{-2}	-0.2500	-0.1820×10^{-2}	-0.1856
3	0.3815×10^{-5}	0.1449×10^{-2}	0.3596×10^{-5}	0.1647×10^{-2}
4	-0.2484×10^{-8}	0.3019×10^{-4}	0.2361×10^{-8}	0.3160×10^{-4}

(iii) for $R = 1$

m	B_m (analytical)	$f_m''(0)$ (analytical)	B_m (full Oseen)	$f_m''(0)$ (full Oseen)
1	0.2497	0.9989	0.2654	0.9436
2	-0.1563×10^{-1}	-0.5000	-0.1492×10^{-1}	-0.3453
3	0.1221×10^{-3}	0.0078	0.1141×10^{-3}	0.0070
4	-0.3179×10^{-6}	0.3252×10^{-3}	-0.3014×10^{-6}	0.1908×10^{-3}

(iv) for $R = 2$

m	B_m (analytical)	$f_m''(0)$ (analytical)	B_m (full Oseen)	$f_m''(0)$ (full Oseen)
1	0.7639	1.5278	0.8992	1.2866
2	-0.1250	-1.000	-0.1333	-0.6314
3	0.3906×10^{-2}	0.4774×10^{-1}	0.3636×10^{-2}	0.2111×10^{-1}
4	-0.4070×10^{-4}	0.3979×10^{-2}	-0.3398×10^{-4}	0.2825×10^{-2}

(v) for $R = 3$

m	B_m (analytical)	$f''_m(0)$ (analytical)	B_m (full Oseen)	$f''_m(0)$ (full Oseen)
1	1.6600	2.2133	2.1654	1.5842
2	− 0.4219	− 1.5000	− 0.5338	− 0.8991
3	0.2966×10^{-1}	0.1556	0.3102×10^{-1}	0.6416×10^{-1}
4	-0.6952×10^{-3}	0.1945×10^{-2}	-0.6994×10^{-3}	− 0.1[illegible]93 × 10^{-3}

We note that the approximations obtained in the previous section for the low Reynolds number R are used in obtaining the analytical results which are presented in the tables, by substituting the indicated values of R into the approximations. The approximations to the constants B_m in (4.2.6) and the vorticity components g_m at $\xi = 0$ for $m = 1, 2, 3, 4$, based on low Reynolds number expansions, may be summarized as follows.

$$B_1 \sim -\frac{R}{2}\left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}, \tag{4.3.1}$$

$$B_2 \sim -2^{-6} R^3, \tag{4.3.2}$$

$$B_3 \sim 2^{-15} R^5, \tag{4.3.3}$$

$$B_4 \sim -\frac{1}{3} \cdot 2^{-20} R^7, \tag{4.3.4}$$

and also,

$$g_1(0) \sim -2\left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}, \tag{4.3.5}$$

$$g_2(0) \sim -\frac{R}{2} \tag{4.3.6}$$

$$g_3(0) \sim -2^{-6} R^2 \left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}, \tag{4.3.7}$$

$$g_4(0) \sim -\frac{1}{3} \cdot 2^{[illegible]} R^3 \left[\ln\left(\frac{R}{8}\right) + \gamma - \frac{1}{2}\right]^{-1}. \tag{4.3.8}$$

It may be also noted that the estimated values of the second derivatives of the stream function components $f''_m(\xi)$ at the cylinder surface for the indicated values of Reynolds number which are presented in the tables are determined by using the result

$$f''_m(0) = g_m(0)\,, \quad m \geq 1\,. \tag{4.3.9}$$

This result follows from (4.1.17) and (4.1.18). It may then be deduced from the calculated results for flow past the circular cylinder at $R = 0.05$, 0.5, 1, 2, 3, contrary to the view put forward by Tomotika & Aoi [45], that the standing vortex-pair is not formed for very small Reynolds numbers. In fact we estimate the critical value of R at which separation does occur to be around $R = 2$ for the approximations to the Oseen flow problem under consideration based on the low Reynolds number expansions and around $R = 3$ according to the investigation of the same problem based on the non-expanded form of the full Oseen equations, at the same time admitting that it is debatable whether Oseen equations can be applied at Reynolds number as high as these. We note that the separation first occurs when R is such that $\sum_{m=1}^{\infty} m f''_m(0) = 0$. These estimated critical values of the Reynolds number R are quite reasonable. The improvement that is obtained for the critical value of R in the latter investigation is due to the fact that the first 4 terms of both series (4.2.9) for $n = 1, 2, 3, 4$ and (4.2.6) are taken into account for $m = 1, 2, 3, 4$, in this investigation, whereas only the dominant terms of these series are considered in obtaining the corresponding analytical results. We also note that Dennis [8] exhibits the streamlines of the motion under the consideration according to Oseen theory for $R = 1$ and $R = 5$ in his numerical investigation. The diagrams verify that separation does not start to take place between $R = 1$ and $R = 5$ (see e.g., figure 4).

CHAPTER V

NON-EXISTENCE OF OSEEN FLOW IN THE CASE OF A ROTATING CYLINDER

5.1 THE OSEEN EQUATIONS FOR FLOW PAST A ROTATING CYLINDER

In this chapter, the asymmetrical flow which is generated by a rotating circular cylinder in a uniform viscous fluid will be discussed on the basis of Oseen approximation for low Reynolds number using the new method described in Section 2.3.

The problem considered is that of a circular cylinder of radius a rotating about its axis with constant angular velocity ω_0 in a uniform stream velocity U. It is assumed that the rotation is in a counter-clockwise sense. We use the modified polar co-ordinates (ξ,η) defined by (4.1.2), where $\xi = \ln\left(\frac{r}{a}\right)$. Thus the cylinder is situated at $\xi = 0$ and the domain of the solution is $0 \le \xi < \infty$, $0 \le \eta \le 2\pi$, with $\eta = 0$ the downstream direction.

The Oseen equations of the motion in the (ξ,η) co-ordinate system can be expressed as the two equations

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\eta^2} = \frac{R}{2}e^{\xi}\left(\cos\eta\,\frac{\partial\zeta}{\partial\xi} - \sin\eta\,\frac{\partial\zeta}{\partial\eta}\right) \tag{5.1.1}$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = e^{2\xi}\zeta . \tag{5.1.2}$$

Here R is the Reynolds number defined by $R = 2aU/\nu$. These equations are to be solved subject to the following conditions

$$\int_0^\infty \int_0^{2\pi} e^{(2-n)\xi} \zeta \cos n\eta \, d\eta \, d\xi = \pi\beta\delta_{n,0} , \quad n \geq 0 ; \tag{5.1.3}$$

$$\int_0^\infty \int_0^{2\pi} e^{(2-n)\xi} \zeta \sin n\eta \, d\eta \, d\xi = \pi\delta_{n,1} , \quad n \geq 1 , \tag{5.1.4}$$

$$\zeta(\xi,\eta) \to 0 \quad \text{as} \quad \xi \to \infty, \tag{5.1.5}$$

$$\psi(\xi,\eta) = 0 , \quad \frac{\partial\psi}{\partial\xi} = -\Omega \quad \text{when} \quad \xi = 0 , \tag{5.1.6}$$

$$e^{-\xi} \frac{\partial\psi}{\partial\xi} \sim \sin\eta , \quad e^{-\xi} \frac{\partial\psi}{\partial\eta} \sim \cos\eta \quad \text{as} \quad \xi \to \infty , \tag{5.1.7}$$

$$\psi(\xi,\eta) = \psi(\xi,\eta+2\pi) , \quad \zeta(\xi,\eta) = \zeta(\xi,\eta+2\pi) . \tag{5.1.8}$$

The rotation of the cylinder enters through the parameter Ω which gives a measure of the rate of rotation of the cylinder relative to the undisturbed stream, in the boundary conditions (5.1.3) and (5.1.6), and is defined by $\Omega = a\omega_o / U$. We note that the parameter β in (5.1.3) depends on the circulation round a large enough contour surrounding the rotating cylinder and the parameter Ω and is given by the equation (I-5) in Appendix I.

In the present case, the Fourier series for ψ and ζ are of the form

$$\zeta(\xi,\eta) = \frac{1}{2} G_o(\xi) + \sum_{n=1}^{\infty} [G_n(\xi)\cos n\eta + g_n(\xi)\sin n\eta] , \tag{5.1.9}$$

$$\psi(\xi,\eta) = \frac{1}{2} F_o(\xi) + \sum_{n=1}^{\infty} [F_n(\xi)\cos n\eta + f_n(\xi)\sin n\eta] . \tag{5.1.10}$$

The equations for the functions $G_n(\xi)$, $n \geq 0$ and $g_n(\xi)$, $n \geq 1$ in (5.1.9) are obtained by substituting the series (5.1.9) in the equation (5.1.1). It is found that

$$G_0''(\xi) = \frac{R}{2} e^{\xi} [\, G_1' + G_1 \,] , \tag{5.1.11}$$

$$G_n''(\xi) - n^2 G_n = \frac{R}{4} e^{\xi} [\, G_{n-1}' - (n-1) G_{n-1} + G_{n+1}' + (n+1) G_{n+1} \,] , \ n \geq 1 ; \tag{5.1.12}$$

$$g_n''(\xi) - n^2 g_n = \frac{R}{4} e^{\xi} [\, g_{n-1}' - (n-1) g_{n-1} + g_{n+1}' + (n+1) g_{n+1} \,] , \ n \geq 1 , \tag{5.1.13}$$

where $g_0 = 0$, and the prime denotes differentiation with respect to the variable ξ. Boundary conditions for (5.1.11)-(5.1.13) follow from (5.1.3)-(5.1.5) and are given by

$$\int_0^{\infty} e^{(2-n)\xi} G_n(\xi)\, d\xi = \beta \delta_{n,0} , \quad n \geq 0 ; \tag{5.1.14}$$

$$\int_0^{\infty} e^{(2-n)\xi} g_n(\xi)\, d\xi = 2 \delta_{n,1} , \quad n \geq 1 , \tag{5.1.15}$$

and

$$G_n(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \quad n \geq 0 ; \tag{5.1.16}$$

$$g_n(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \quad n \geq 0 . \tag{5.1.17}$$

If we substitute the assumed expansions (5.1.9) and (5.1.10) in equation (5.1.2) we find, after the standard use of orthogonal functions, that

$$F_n''(\xi) - n^2 F_n(\xi) = e^{2\xi} G_n(\xi) , \quad n \geq 0 , \tag{5.1.18}$$

$$f_n''(\xi) - n^2 f_n(\xi) = e^{2\xi} g_n(\xi) , \quad n \geq 0 . \tag{5.1.19}$$

The boundary conditions for these equations follow from (5.1.6) and (5.1.7). They are at the cylinder surface

$$F_n(0) = F_n'(0) = 0 , \quad n \geq 1 ; \quad F_0(0) = 0 , \tag{5.1.20}$$

$$f_n(0) = f_n(0) = 0 , \quad n \geq 1 ; \quad f_0(0) = -2\Omega , \tag{5.1.21}$$

and as $\xi \to \infty$

$$e^{-\xi} f_n \to \delta_{n,1} , \quad e^{-\xi} F_n \to 0 , \quad n \geq 1 ; \quad e^{-\xi} F_0 \to 0 , \tag{5.1.22}$$

$$e^{-\xi} f_n' \to \delta_{n,1} , \quad e^{-\xi} F_n' \to 0 , \quad n \geq 1 ; \quad e^{-\xi} F_0' \to 0 . \tag{5.1.23}$$

The functions $F_n(\xi)$ and $n \geq 0$, $f_n(\xi)$, $n \geq 0$ can now be determined by solving (5.1.18) and (5.1.19) subject to (5.1.20)-(5.1.23). We note that the reduction of equations (5.1.1) and (5.1.2) into component form yields a system of equations simpler than those obtained in Chapter III for flow past an elliptic cylinder because of the fact that M is a function of only the variable ξ.

5.2 A PARADOX : NON-EXISTENCE OF OSEEN FLOW

As stated before, the correct satisfaction of the boundary conditions at large distances from the cylinder is a particularly crucial matter in the case of asymmetrical flows and unless conditions are satisfied properly an unacceptable solution throughout the whole domain can result. In this section, a *paradox* that is obtained as a result of correct satisfaction of the boundary conditions at large distances from the cylinder under consideration will be presented and discussed. By a paradox we mean a plausible argument which yields conclusions at variance with physical observation, as it is in this sense of an apparent inconsistency that the term is most used in hydrodynamics. The importance of hydrodynamical paradoxes are discussed by Birkhoff [4] from the practical, the mathematical and the philosophical points of view.

In the present case formulation of the Oseen flow problem for the vorticity components associated with the symmetrical and asymmetrical part of the problem can be separated as follows .

(i) The set of differential equations for the vorticity components $g_n(z)$, $n \geq 1$ associated with the symmetrical part of the problem and the boundary conditions on these components are

$$z g'_{n+1} + (n+1) g_{n+1} = z g''_n + g'_n - n^2 z^{-1} g_n - z g'_{n-1} + (n-1) g_{n-1}, \quad n \geq 1, \quad (5.2.1)$$

where $z = \frac{R}{4} e^{\xi}$ and $g_o = 0$;

$$\int_{\frac{R}{4}}^{\infty} z^{(1-n)} g_n(z)\, dz = \left(\frac{R}{4}\right)^{2-n} 2\delta_{n,1}, \quad n \geq 1, \quad (5.2.2)$$

$$g_n(z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad n \geq 1. \quad (5.2.3)$$

(ii) The set of differential equations for the vorticity components $G_n(z)$, $n \geq 0$ associated with the asymmetrical part of the Oseen flow problem and the boundary conditions on these functions are

$$z G''_o + G'_o = \frac{1}{2}[\, z G'_1 + G_1 \,], \quad (5.2.4)$$

$$z G'_{n+1} + (n+1) G_{n+1} = z G''_n + G'_n - n^2 z^{-1} G_n - z G'_{n-1} + (n-1) G_{n-1}, \quad n \geq 1, \quad (5.2.5)$$

where $z = \frac{R}{4} e^{\xi}$;

$$\int_{\frac{R}{4}}^{\infty} z^{(1-n)} G_n(z)\, dz = \left(\frac{R}{4}\right)^{2-n} \beta\delta_{n,o}, \quad n \geq 0, \quad (5.2.6)$$

$$G_n(z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad n \geq 1. \quad (5.2.7)$$

The separation is essentially due to the fact that the function M in (4.1.3) depends only on the variable ξ. We note that the set of differential equations (5.2.1) for the vorticity components $g_n(\xi)$, $n \geq 1$ associated with the symmetrical part of the Oseen flow problem under consideration, and the conditions (5.2.2) and (5.2.3) on these functions, are identical to the set of equations (4.2.1) and the conditions (4.2.2) and (4.2.3) that were given, in the previous chapter, for the vorticity components $g_n(\xi)$, $n \geq 1$ for the case of symmetrical flow past a circular cylinder. Thus we only need to consider the solution of the set of equations (5.2.4) and (5.2.5) for the vorticity components $G_n(\xi)$, $n \geq 0$ associated with the asymmetrical part of the Oseen flow problem subject to the conditions (5.2.6) and (5.2.7). As previously noted, the function $G_o(\xi)$ may be determined by using the basic structures of the vorticity components

$$G_n(\xi) = \sum_{m=o}^{\infty} A_m [\, I_{m-n}(z) + I_{m+n}(z)] \, K_m(z) \, . \qquad (5.2.8)$$

The constants A_m, $m \geq 0$ in this expression have to be determined provided that the integral conditions (5.2.6) on the vorticity components $G_n(\xi)$, $n \geq 0$ are satisfied. If we substitute the expression (5.2.8) for the functions $G_n(\xi)$, $n \geq 0$ into the integral conditions (5.2.6) we obtain the infinite set of linear simultaneous equations

$$\sum_{m=o}^{\infty} \mu_{nm} A_m = \beta \delta_{n,o} \, , \quad n \geq 0 \qquad (5.2.9)$$

where

$$\mu_{nm} = \left(\frac{R}{4} \right)^{n-2} \int_{\frac{R}{4}}^{\infty} z^{1-n} [\, I_{m-n}(z) + I_{m+n}(z) \,] \, K_m(z) \, dz \, , \; n \geq 0 \, , \; m \geq 0 \, . \qquad (5.2.10)$$

If we write the equation (5.2.9) for $n = 0$ we immediately arrive at the result

$$\sum_{m=o}^{\infty} A_m \int_{\frac{R}{4}}^{\infty} 2z \, I_m(z) K_m(z) \, dz = \left(\frac{R}{4} \right)^2 \beta \, . \qquad (5.2.11)$$

It may be shown, by substituting (V-5) in (5.2.11) and then using the result (IV-2), (IV-7) and (IV-14) in the resulting equation, that if

$$\sum_{m=0}^{\infty} A_m = 0$$

the consequence of (5.2.11) in terms of low Reynolds number R is

$$\sum_{m=0}^{\infty} A_m \left[I_m K_m + \frac{1}{2}[I_{m-1} K_{m+1} + I_{m+1} K_{m-1}] \right] = -\beta , \qquad (5.2.12)$$

where the arguments of the Bessel functions are $\frac{R}{4}$. The expression (5.2.10) for $n = 1$ may be substituted in (5.2.9) by using (IV-18) to obtain

$$\frac{4}{R} \sum_{m=0}^{\infty} A_m \int_{\frac{R}{4}}^{\infty} 2\, I_m'(z) K_m(z)\, dz = 0 . \qquad (5.2.13)$$

If we substitute the result (V-3) in (5.2.13) and then using the results (IV-2), (IV-7) and (IV-14) in the resulting equation we find that if

$$\sum_{m=0}^{\infty} A_m = 0$$

the consequence of (5.2.13) is

$$\frac{4}{R} \sum_{m=0}^{\infty} A_m \; I_m K_m = 0 . \qquad (5.2.14)$$

We now consider the integrals

$$\mu_{nm} = \left(\frac{R}{4} \right)^{n-2} \int_{\frac{R}{4}}^{\infty} z^{1-n} [I_{m-n}(z) + I_{m+n}(z)] K_m(z)\, dz , \qquad (5.2.15)$$

in (5.2.10) for $n \geq 2$ and $m \geq 0$. These integrals may be evaluated exactly by using the results (V-6), (V-7) in Appendix V. It may be shown by substituting these results in (5.2.15) and then using the asymptotic properties (IV-2), (IV-7), (IV-12) and (IV-13) of the modified Bessel functions in the resulting equation, that

$$\mu_{nm} = \frac{1}{2(n-1)}\left[I_{n-m-1}K_{m-1} + I_{n-m}K_m + I_{n+m-1}K_{m+1} + I_{n+m}K_m \right], \; n \geq 2$$

where $m \geq 0$ and the arguments of the Bessel functions are $\frac{R}{4}$. We note that the properties (IV-5) and (IV-9) of the modified Bessel functions are used in obtaining the above expressions. The necessary condition for the existence of the sums

$$\sum_{m=0}^{\infty} \mu_{om} A_m \quad \text{and} \quad \sum_{m=0}^{\infty} \mu_{1m} A_m \tag{5.2.16}$$

in (5.2.9) is then

$$\sum_{m=0}^{\infty} A_m = 0 . \tag{5.2.17}$$

The system of equations (5.2.9) can now be written as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \ell_{nm} A_m = -\beta\delta_{n,o} \tag{5.2.18}$$

where the coefficients ℓ_{nm} are functions of the Reynolds number R and are expressed as

$$\ell_{om} = I_m K_m + \frac{1}{2}\left[I_{m-1}K_{m+1} + I_{m+1}K_{m-1} \right], \tag{5.2.19}$$

$$\ell_{1m} = \frac{4}{R} I_m K_m , \tag{5.2.20}$$

$$\ell_{nm} = \frac{1}{2(n-1)}\left[I_{n-m-1}K_{m-1} + I_{n-m}K_m + I_{n+m-1}K_{m+1} + I_{n+m}K_m \right], \; n \geq 2, \tag{5.2.21}$$

the arguments of all the modified Bessel functions being $\frac{R}{4}$. In these formulas $m \geq 0$. We note that in practice the number of equations that can be obtained from (5.2.18) together with the condition (5.2.17) is finite, say $(2N+2)$ which is identical to the number of unknown constants. The unknown constants A_m, $m \geq 0$ are then to be found by solving the simultaneous linear algebraic equations (5.2.18) provided that the necessary condition (5.2.17) is satisfied.

We will proceed to approximate the unknown constants β, A_m, $m \geq 0$ in terms of the low Reynolds number R, by solving (5.2.18) together with the necessary condition (4.3.12). Thus, as in Section 4.2, to find the first approximation we put A_m, $m \geq 2$ equal to zero in (5.2.18), and solve the first two equations in (5.2.18), together with the condition (5.2.17), for A_o, A_1 and β. For the second approximation, we put A_m, $m \geq 3$ equal to zero in (5.2.18), and solve the first three equations in (5.2.19), together with the condition (5.2.17), for A_o, A_1, A_2 and β, and so on.

To find the first approximation we equate A_2, A_3, ... to zero in (5.2.18) and then solve the first two resulting equations in (5.2.18) for A_o and A_1. This gives

$$A_o = \frac{2\beta(I_1 K_1)}{I_o K_o (2I_1 K_1 + I_o K_2 + I_2 K_o) - 2I_1 K_1 (I_o K_o + I_1 K_1)}, \quad (5.2.22)$$

$$A_1 = -\frac{2\beta(I_o K_o)}{I_o K_o (2I_1 K_1 + I_o K_2 + I_2 K_o) - 2I_1 K_1 (I_o K_o + I_1 K_1)}, \quad (5.2.23)$$

where the arguments of the modified Bessel functions are again $\frac{R}{4}$. The constant β in (5.2.22) and (5.2.23) has to be determined by applying the necessary condition (5.2.17). By this means we do in effect adopt a global procedure of adjusting the solution to the correct conditions at large distances. Equating the constants A_m, $m \geq 2$ to zero in (5.2.17), the necessary condition reduces to

$$A_o + A_1 = 0 \quad (5.2.24)$$

for the first approximation. It may be shown, by using (5.2.22), (5.2.23) and (5.2.24), satisfaction of the necessary condition (5.2.24) gives

$$\beta = 0 . \tag{5.2.25}$$

Thus, the first approximation gives the trivial solution to the constants

$$A_o = A_1 = \beta = 0 . \tag{5.2.26}$$

If we proceed to find the second, third, and so on, approximations to the constants it can be shown that β appears as a common factor in all the constants A_m, $m \geq 0$ and then application of the necessary condition (5.2.17) in each approximation gives

$$A_m = \beta = 0 \quad \text{for all } m . \tag{5.2.27}$$

It follows from this result and (5.2.8) that the vorticity components $G_n(\xi)$, $n \geq 0$ associated with the asymmetrical part of the Oseen problem under consideration are identically equal to zero. Due to this the expansion (5.1.9) for the vorticity ζ takes the simplified form

$$\zeta(\xi,\eta) = \sum_{n=1}^{\infty} g_n(\xi) \sin n\eta \tag{5.2.28}$$

where the vorticity components are identical to the case of symmetrical flow past a circular cylinder. Thus we get the symmetrical solution of the form (5.2.28) for the vorticity ζ which contradicts the assumed form of the expansion (5.1.9) for the steady-state vorticity. The result (5.2.27) in this sense is thus an apparent inconsistency and indicates *non-existence of the Oseen flow in the case of the rotating cylinder.*

From a mathematical point of view, the reason for not being able to approximate the constants associated with the asymmetrical part of the Oseen flow problem under consideration in terms of the low Reynolds number R may be analyzed as follows. As previously noted, determination of the unknown constants depends on satisfaction of the integral conditions. In the present case, these conditions are simpler than those obtained in Chapter III for flow past an elliptic cylinder. This is due to the fact that the function M defined by the equation (4.1.3) depends only on the single variable ξ. Because of the single variable ξ dependency, satisfaction of the integral conditions (5.1.15) and (5.1.16) enables us to obtain an uncoupled set of equations for the constants associated with the symmetrical and asymmetrical part of the Oseen flow problem in the case of the rotating cylinder. As a result of the uncoupling of two sets of equations, the unknown constant β defined by equation (I.6) in Appendix I appears as a common factor in all the approximations to the constants A_m, $m \geq 0$ related with the asymmetrical part of the problem under consideration. Finally, satisfaction of the necessary condition (5.2.17) in each approximation determines the common factor β as being zero. This gives the trivial solution (5.2.27) for all the constants associated with the asymmetric part of the Oseen flow problem. As a result of the application of the new method to the Oseen flow problem under consideration, the *paradox* that is obtained may be stated as follows

" *No steady two-dimensional asymmetrical Oseen flow of a viscous incompressible fluid past a rotating cylinder is possible* ".

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CHAPTER VI

SUMMARY AND CONCLUSION

The object of this thesis is to show how the two-dimensional Navier-Stokes equations, describing the steady flow of a viscous incompressible fluid past a cylinder in an unbounded field, can be analyzed and conveniently solved using methods involving integral conditions, where the flow is assumed to be governed by the linearized equations of Oseen. It is shown that satisfaction of correct conditions is a particularly crucial matter in the case of asymmetrical flows and that unless conditions are satisfied properly an unacceptable solution throughout the whole domain can result.

We give the basic governing equations for the stream function and vorticity in a curvilinear co-ordinate system. A necessary boundary condition is then that the vorticity must vanish at large distances from the cylinder where the flow is a uniform stream. It is shown that for asymmetrical flows it is not sufficient merely that the vorticity shall vanish far from the cylinder but it must decay rapidly enough. As it turns out, the proper conditioning for the vorticity has the peculiarity of being integral (global) type instead of the usual boundary value (local) type.

A general method of solving the problem of two-dimensional steady flow of viscous liquid past a cylinder using Oseen's method of approximation is developed. The method is based on satisfaction of the proper conditioning for the vorticity of integral type. We consider this is a very important part of the solution procedure since the integral conditions ensure both the correct decay of the vorticity at large distances from the cylinder and satisfaction of the physically essential results for the existence of the flow. It can be pointed out that in numerical work nobody seems to realize the importance of satisfaction of these correct conditions because they take the boundaries at finite distances.

As an application of the new method, the uniform flow past an elliptic cylinder at an arbitrary angle of incidence is considered on the basis of Oseen approximation. The correct condition of periodicity of the pressure in the fluid gives a useful check on the validity of the Oseen approximations. The fully-analytical method enables one to obtain many properties of the flow past an elliptic cylinder in a straightforward and systematic manner. The results obtained in terms of low Reynolds number are in excellent agreement with those of Hasimoto's analysis. The new method is also, tested by applying it to the case of a fixed circular cylinder. Good agreement is found between these results and the results obtained for the same problem previously. Also, according to our solution for the Oseen flow problem in the case of a fixed circular cylinder we found, contrary to the view put forward by Tomotika & Aoi [45], that the standing vortex-pair is not formed for very small Reynolds number which is in a good agreement with the experimental findings.

An analytical treatment of the asymmetrical flow which is generated by a rotating cylinder is first carried out on the basis of Oseen approximation by means of the same technique. As a result of this the first paradox is obtained for Oseen flow

" No steady two-dimensional asymmetrical Oseen flow of a viscous incompressible fluid past a rotating cylinder is possible."

We note that the Navier-Stokes equations in the case of the rotating circular cylinder are investigated for small rotation rates and moderate values of Reynolds number by Ingham [23] and Loc [29]. Both of these investigators used numerical methods, but there are wide discrepancies between the results of these works for the lift and drag coefficients. These differences seem to depend upon the conditions assumed at finite distances in these works. One of the object of considering the Oseen flow for the rotating circular cylinder in our work was to attempt to resolve the differences in these workers. This is done by means of fully-analytical method and as a result of the application of this method it is shown that there is no satisfactory Oseen solution for the rotating circular cylinder.

The ultimate aim of this investigation was to indicate an effective method of obtaining approximate solutions of the Navier-Stokes equations for steady asymmetrical flow past cylinders in terms of Reynolds number expansions, but this far the work has not been advanced to this stage. It may be noted, however, that only step which remains to be completed is to find a satisfactory technique of solving the equation (1.2.23) so that the correct decay of the vorticity as $\xi \to \infty$ is enforced. The method of solution of (1.2.24) by means of the solution of the ordinary differential equations (1.3.2) and (1.3.3) once the functions $r_n(\xi)$, $n \geq 1$ and $s_n(\xi)$, $n \geq 0$ have been determined and the conditions (1.3.15)-(1.3.17) are satisfied, together with the additional condition on $s_0(\xi)$, remains valid in the general Navier-Stokes case.

APPENDIX I

The concept of circulation was first introduced by Thompson (Lord Kelvin) [44] in 1869. It is defined (see e.g. McCormack & Crane [30] p.198) as the line integral of the tangential component of the velocity round a closed contour C. Thus

$$\text{circulation} = K = \oint_C \underset{\sim}{q} \cdot d\underset{\sim}{r} \,, \tag{I-1}$$

where $\underset{\sim}{q}$ is the velocity vector and $\underset{\sim}{r}$ is the position vector. The circulation is a scalar, but has directional properties in so far its sign depends on the sense of the circulation.

If we choose the closed path C in (I-1) as a circle C_R centered at the cylinder and of very large radius $R - ke^{\xi}$ as $\xi \to \infty$ and substitute the second of (1.2.18) in (I-1) we find, after the integration around C_R in the anti-clockwise sense and after use of the asymptotic property (2.2.18), that the circulation round a large contour surrounding the cylinder is given by

$$K_\infty = -\lim_{\xi \to \infty} \int_0^{2\pi} \left(\frac{\partial \psi}{\partial \xi} \right) d\eta \,. \tag{I-2}$$

It follows from the expression (1.2.32) for the stream function that

$$\frac{\partial \psi}{\partial \xi} = \frac{1}{2} F_o'(\xi) + \sum_{n=1}^{\infty} [\, F_n'(\xi) \cos n\eta + f_n(\xi) \sin n\eta \,] \,. \tag{I-3}$$

If we substitute the result (I-3) in (I-2) we immediately arrive at

$$K_\infty = -\pi \lim_{\xi \to \infty} [\, F_o'(\xi) \,] \,. \tag{I-4}$$

The constant β in (1.3.15) can now be defined in terms of the circulation round a large contour surrounding the cylinder by using

(1.3.13), (1.3.14) and (I-4). Thus

(i) for a fixed cylinder

$$\beta = F_o'(\infty) = -\frac{1}{\pi} K_\infty . \qquad \text{(I-5)}$$

(ii) for a rotating circular cylinder

$$\beta = F_o'(\infty) + 2\Omega = -\frac{1}{\pi} K_\infty + 2\Omega . \qquad \text{(I-6)}$$

APPENDIX II.

The problem of establishing appropriate conditions for the vorticity transport equation is considered by Quartapelle & Valz-Gris [36] from a conceptual point of view. They showed that the correct conditions of an integral type on the vorticity are a consequence of the following theorem.

THEOREM A function ζ, in R, exists such that $\zeta = \nabla^2\psi$, with $\psi|_S = a$ and $\frac{\partial\psi}{\partial n}\Big|_S = b$, if and only if

$$\iint_R \phi\,\zeta\,ds = \oint_S \left(b\,\phi - a\,\frac{\partial\phi}{\partial n} \right) ds \qquad \text{(II-1)}$$

for any harmonic function ϕ in R, i.e., such that $\nabla^2\phi = 0$ in R. Here R is a simply connected domain of the plane with boundary S; a and b are two functions defined on S, and ds is the line element of S. $\frac{\partial}{\partial n}$ is taken in the normal direction to the boundary S. The proof of this theorem follows by a trivial application of Green's formula.

As a simple application of the above remark we consider two-dimensional, steady, symmetrical flow of a viscous incompressible fluid past a circular cylinder in an unbounded field. Let us consider a large circle, surrounding the cylinder, drawn in the fluid, say S_1. The domain R can be taken as the region between the contour S_2 of the cylinder and S_1. The boundary of R may then be given as $S_1 + S_2$. For the symmetrical flow past a circular cylinder we have used the following expansions and the conditions

$$\zeta = \sum_{n=1}^{\infty} g_n(\xi) \sin n\eta , \qquad \psi = \sum_{n=1}^{\infty} f_n(\xi) \sin n\eta , \tag{II-2}$$

$$f_n(0) = f_n'(0) = 0 , \quad n \geq 1 \tag{II-3}$$

$$e^{-\xi} f_n(\xi) \to \delta_{n,1} , \quad e^{-\xi} f_n'(\xi) \to \delta_{n,1} \quad \text{as} \quad \xi \to \infty . \tag{II-4}$$

If we choose the harmonic function ϕ in (II-1) as

$$\phi = e^{-n\xi} \sin n\eta \tag{II-5}$$

and use the expansions (II-2) and the conditions (II-3) and (II-4) in (II-1) we find, after use of $\psi_{\xi\xi} + \psi_{\eta\eta} = e^{2\xi} \zeta$, that

$$\int_0^{\infty} e^{(2-n)\xi} g_n(\xi) \, d\xi = 2\delta_{n,1} , \quad n \geq 1 . \tag{II-6}$$

This condition is identical to the condition (4.1.15) which is obtained for the case of symmetrical flow past a circular cylinder in Section 4.1.

APPENDIX III

The dimensionless pressure coefficient $p^*_{\xi_o}(\eta)$ on the surface of the cylinder is defined to be

$$p^*_{\xi_o}(\eta) = p(\xi_o,\eta) - p(\xi_o,\eta_o)\,, \qquad \text{(III-1)}$$

where $p(\xi_o,\eta)$ is the non-dimensional pressure on the surface of the cylinder and η_o is some base point on the surface.

The pressure distribution over the cylinder may be obtained from the component equations (1.2.21) and (1.2.22) of momentum. If we take the component equation (1.2.22) in the η-direction on the surface of the cylinder, $\xi = \xi_o$ where $q = 0$, we can show that

$$\left(\frac{\partial p}{\partial \eta}\right)_{\xi=\xi_o} = -\frac{2}{R}\left(\frac{\partial \zeta}{\partial \xi}\right)_{\xi=\xi_o}. \qquad \text{(III-2)}$$

The dimensionless pressure distribution over the surface of the cylinder may be obtained by integrating (III-2) round $\xi = \xi_o$ from $\eta = \eta_o$ to a given station η, which gives

$$p^*_{\xi_o}(\eta) = p(\xi_o,\eta) - p(\xi_o,\eta_o) = -\frac{2}{R}\int_{\eta_o}^{\eta}\left(\frac{\partial \zeta}{\partial \xi}\right)_{\xi=\xi_o} d\eta\,. \qquad \text{(III-3)}$$

Alternatively $p^*_{\xi_o}(\eta)$ may be obtained in terms of the uniform free-stream pressure $p(\infty,\eta_o)$ by integrating the component equation of momentum (1.2.21) in the ξ-direction along $\eta = \eta_o$ from $\xi = \xi_o$ to $\xi = \infty$. Thus

$$p(\xi_o,\eta_o) - p(\infty,\eta_o) = \frac{1}{2} - \int_{\xi_o}^{\infty}\left(\frac{2}{R}\frac{\partial \zeta}{\partial \eta} + \zeta\,\frac{\partial \psi}{\partial \xi}\right) d\xi\,. \qquad \text{(III-4)}$$

Pressure variations may then be expressed in terms of the pressure coefficient

$$P(\eta) = p(\xi_o,\eta) - p(\infty,\eta_o).$$

which is obtained by addition of (III-3) to (III-4) giving

$$P(\eta) = \frac{1}{2} - \frac{2}{R}\int_{\eta_o}^{\eta}\left(\frac{\partial\zeta}{\partial\xi}\right)_{\xi=\xi_o} d\eta - \int_{\xi_o}^{\infty}\left(\frac{2}{R}\frac{\partial\zeta}{\partial\eta} + \zeta\frac{\partial\psi}{\partial\xi}\right) d\xi . \qquad \text{(III-5)}$$

When employing either (III-3) or (III-5) to express pressure variations for symmetrical flow past a circular cylinder and also, for asymmetrical flow past an elliptic cylinder, the value $\eta_o = 0$ was chosen in which for the case of symmetric flow the second integral in (III-5) is zero.

We will proceed to calculate the coefficients of drag and lift experienced by the cylinders considered in Chapter III and Chapter IV. Let C be the contour of the cylinder corresponding to $\xi = \xi_o$ in the transformation (1.2.14) and let X and Y be the resultant non-dimensional forces applied to the cylinder by the fluid in the x and y directions respectively. The non-dimensional forces X and Y exerted by the fluid on the cylinder are defined (see e.g. Schlicting [38], p.58) by

$$X = -\oint_C \left(p^*_{\xi_o} dy + \frac{2}{R}[\zeta]_{\xi=\xi_o} dx \right), \qquad \text{(III-6)}$$

$$Y = \oint_C \left(p^*_{\xi_o} dx - \frac{2}{R}[\zeta]_{\xi=\xi_o} dy \right), \qquad \text{(III-7)}$$

where the first integral in each gives the pressure contribution to the force and the second gives the frictional contribution. The equations which relate the dimensionless forces X and Y to the corresponding (primed) dimensional forces are

$$X' = (\rho U^2 d) X, \qquad Y' = (\rho U^2 d) Y . \qquad \text{(III-8)}$$

In terms of X , Y and the angle of incidence α, the non-dimensional drag coefficient C_D and lift coefficient C_L on the cylinder in the directions parallel and normal to the free stream, respectively, are given by

$$C_D = X\cos\alpha + Y\sin\alpha\,, \tag{III-9}$$

$$C_L = Y\cos\alpha - X\sin\alpha\,. \tag{III-10}$$

In the case of an elliptic cylinder we have used the transformation

$$x = \cosh\xi\cos(\eta+\alpha)\,, \qquad y = \sinh\xi\sin(\eta+\alpha)\,. \tag{III-11}$$

For the elliptic cylinder, equations (III-6) and (III-7) become in the (ξ,η) co-ordinate system

$$X = -\sinh\xi_o\int_0^{2\pi} p^*_{\xi_o}\cos(\eta+\alpha)\,d\eta + \frac{2}{R_E}(\cosh\xi_o)^2\int_0^{2\pi}[\,\zeta\,]_{\xi=\xi_o}\sin(\eta+\alpha)\,d\eta\,,$$

$$Y = -\cosh\xi_o\int_0^{2\pi} p^*_{\xi_o}\sin(\eta+\alpha)\,d\eta - \frac{1}{R_E}\sinh2\xi_o\int_0^{2\pi}[\,\zeta\,]_{\xi=\xi_o}\cos(\eta+\alpha)\,d\eta\,.$$

We can evaluate the integrals in these equations using the series (3.1.15) for the vorticity ζ. The first integral in each of these equations is evaluated directly while the second of each is evaluated by integration by parts and then using the result (III-3). Thus we obtain

$$X = \frac{2\pi}{R_E}(\cosh\xi_o)^2\Big[\cos\alpha\, g_1(\xi_o) + \sin\alpha\, G_1(\xi_o)\Big] - \frac{\pi}{R_E}\sinh2\xi_o\Big[\cos\alpha\, g_1'(\xi_o) + \sin\alpha\, G_1'(\xi_o)\Big] - (\sin\alpha)\sinh\xi_o\,[\,p^*_{\xi_o}(2\pi)]\,, \tag{III-12}$$

$$Y = -\frac{\pi}{R_E}\sinh2\xi_o\Big[\cos\alpha\, G_1(\xi_o) - \sin\alpha\, g_1(\xi_o)\Big] + \frac{2\pi}{R_E}(\cosh\xi_o)^2\Big[\cos\alpha\, G_1'(\xi_o) - \sin\alpha\, g_1'(\xi_o)\Big] + (\cos\alpha)\cosh\xi_o\,[\,p^*_{\xi_o}(2\pi)]\,. \tag{III-13}$$

The non-dimensional coefficients of drag and lift, C_D and C_L, can now be determined using both of the equations (III-12) and (III-13) in (III-9) and (III-10).

$$C_D = \frac{\pi}{R\varepsilon}\left[2[(\cos\alpha)\cosh\xi_o]^2 + (\sin\alpha)^2 \sinh 2\xi_o \right] g_1(\xi_o) - \frac{\pi}{R\varepsilon}\left[2[(\sin\alpha)\cosh\xi_o]^2 + (\cos\alpha)^2 \sinh 2\xi_o \right] g_1'(\xi_o) + \frac{\pi}{2R\varepsilon}\sin 2\alpha \left[2(\cosh\xi_o)^2 - \sinh 2\xi_o \right] \cdot \left[G_1(\xi_o) + G_1'(\xi_o) \right] + \frac{1}{2}\sin 2\alpha \left[\cosh\xi_o - \sinh\xi_o \right] p^*_{\xi_o}(2\pi), \qquad \text{(III-14)}$$

$$C_L = -\frac{\pi}{R\varepsilon}\left[2[(\sin\alpha)\cosh\xi_o]^2 + (\cos\alpha)^2 \sinh 2\xi_o \right] G_1(\xi_o) + \frac{\pi}{R\varepsilon}\left[2[(\cos\alpha)\cosh\xi_o]^2 + (\sin\alpha)^2 \sinh 2\xi_o \right] G_1'(\xi_o) - \frac{\pi}{2R\varepsilon}\sin 2\alpha \left[2(\cosh\xi_o)^2 - \sinh 2\xi_o \right]\left[g_1(\xi_o) + g_1'(\xi_o) \right] + \left[(\sin\alpha)^2 \sinh\xi_o + (\cos\alpha)^2 \cosh\xi_o \right] p^*_{\xi_o}(2\pi) . \qquad \text{(III-15)}$$

For symmetrical flow about the x-axis, we have $\alpha = 0$, $Y = 0$ in which case

$$C_D = X , \qquad \text{(III-16)}$$

$$C_L = 0 . \qquad \text{(III-17)}$$

In the case of a circular cylinder we have used the transformation

$$x = e^{\xi} \cos\eta , \qquad y = e^{\xi} \sin\eta . \qquad \text{(III-18)}$$

Thus equation (III-6) and (III-18) give

$$C_D = \frac{2}{R}\int_0^{\pi} \zeta_o \sin\eta \, d\eta - \int_0^{\pi} p^*_o \cos\eta \, d\eta . \qquad \text{(III-19)}$$

Integrating the second integral by parts and using the result (III-3), C_D may be written as

$$C_D = \frac{\pi}{R}\left[g_1(0) - g_1'(0) \right] . \qquad \text{(III-20)}$$

The first term on the right hand side of (III-20) gives the friction drag coefficient and the second term gives the pressure drag coefficient. Making use of the symmetry property C_D can be written as

$$C_D = \frac{2\pi}{R}\left[g_1(0) - g_1'(0) \right] . \qquad \text{(III-21)}$$

APPENDIX IV

Bessel's modified differential equation is given by

$$z^2 \frac{d^2y}{dz^2} + \frac{dy}{dz} - [\, n^2 + z^2 \,]\, y = 0 , \quad n \geq 0 . \tag{IV-1}$$

One solution of this equation (see e.g. McLachlan [32]) is

$$I_n(z) = \left[\frac{z}{2}\right]^n \sum_{k=0}^{\infty} \frac{4^{-k} z^{2k}}{k!\, \Gamma[\, n+k+1 \,]} \tag{IV-2}$$

which is known as the modified Bessel's function of the first kind of order n. Here the Gamma Function $\Gamma[\, n+k+1 \,]$ is introduced instead of $(n+k)!$, with which it is identical when n is a positive integer. Throughout this appendix it will be supposed, unless the contrary is expressed, that the parameter n is a positive integer or zero. The integral representations of the function $I_n(z)$ are given by the following equations

$$I_0(z) = \frac{1}{\pi} \int_0^{\pi} e^{\pm z\cos\theta}\, d\theta = \frac{1}{\pi} \int_0^{\pi} \cosh(\cos\theta)\, d\theta , \tag{IV-3}$$

$$I_n(z) = \frac{1}{\pi} \int_0^{\pi} e^{z\cos\theta} \cos n\theta \, d\theta . \tag{IV-4}$$

An obvious deduction from these equations is the following property

$$I_{-n}(z) = I_n(z), \quad n = 0, 1, 2, \ldots \tag{IV-5}$$

The function defined by

$$K_n(z) = \lim_{p \to n} \frac{\pi}{2\sin p\pi} \left[\, I_{-p}(z) - I_p(z) \,\right], \quad n = 0, 1, 2, \ldots . \tag{IV-6}$$

which is a solution of the differential equation (IV-1), is known as the modified Bessel function of the second kind of order n. Taking the limit in (IV-6), using L'Hopitals rule, yields

$$K_n(z) = [-1]^{n+1} \ln\left[\frac{z}{2}\right] I_n(z) + \frac{1}{2}\left[\frac{z}{2}\right]^{-n} \sum_{k=0}^{n-1} [-1]^k \frac{(n-k-1)!}{k!} \left[\frac{z}{2}\right]^{2k}$$

$$+ [-1]^n \frac{1}{2}\left[\frac{z}{2}\right]^n \sum_{k=0}^{\infty} \left[\Psi(k+1) + \Psi(n+k+1)\right] \frac{4^{-k} z^{2k}}{k!\,(n+k)!}, \qquad \text{(IV-7)}$$

$$\text{where} \quad \Psi(1) = -\gamma, \qquad \Psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}, \quad n \geq 2. \qquad \text{(IV-8)}$$

Here $\gamma = 0.5772156....$ is Euler's constant. The function $K_n(z)$ then satisfies

$$K_{-n}(z) = K_n(z). \qquad \text{(IV-9)}$$

The limiting forms of the functions $I_n(z)$ and $K_n(z)$ when n is fixed and $z \to 0$ are—

$$I_n(z) \sim \frac{2^{-n} z^n}{\Gamma[n+1]}, \quad n \geq 0; \qquad \text{(IV-10)}$$

$$K_0(z) \sim -\ln z, \qquad K_n(z) \sim \frac{1}{2}\Gamma[n]\left[\frac{z}{2}\right]^{-n}, \quad n \geq 1. \qquad \text{(IV-11)}$$

The asymptotic expansions of the modified Bessel functions $I_n(z)$ and $K_n(z)$ for large argument z when n is fixed and $\gamma = 4n^2$ are given by

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{\gamma - 1}{8z} + \frac{(\gamma-1)(\gamma-9)}{2![8z]^2} - \frac{(\gamma-1)(\gamma-9)(\gamma-25)}{3![8z]^3} + \ldots\ldots. \right], \qquad \text{(IV-12)}$$

$$K_n(z) \sim \frac{\sqrt{\pi}}{\sqrt{2z}} e^{-z} \left[1 + \frac{\gamma - 1}{8z} + \frac{(\gamma-1)(\gamma-9)}{2![8z]^2} + \frac{(\gamma-1)(\gamma-9)(\gamma-25)}{3![8z]^3} + \ldots\ldots. \right] \qquad \text{(IV-13)}$$

and also, it follows from these expansions that—

$$I_n(z)K_n'(z) \sim \frac{1}{2z}\left[1 - \frac{1}{2}\frac{\gamma-1}{[2z]^2} + \frac{1.3}{2.4}\frac{(\gamma-1)(\gamma-9)}{[2z]^4} - \ldots\right]. \quad \text{(IV-14)}$$

The recurrence formulas for the modified Bessel functions are given by

$$z\mathcal{F}_n'(z) = n\mathcal{F}_n(z) + z\mathcal{F}_{n+1}(z), \quad \text{(IV-15)}$$

$$z\mathcal{F}_n'(z) = -n\mathcal{F}_n(z) + z\mathcal{F}_{n-1}(z) \quad \text{(IV-16)}$$

where $\mathcal{F}_n(z) = I_n(z)$ or $\mathcal{F}_n(z) = [-1]^n K_n(z)$.

An obvious deduction from these equations by addition is

$$2\mathcal{F}_n'(z) = \mathcal{F}_{n-1}(z) + \mathcal{F}_{n+1}(z). \quad \text{(IV-17)}$$

Also, it follows from the equations (IV-15) and (IV-16) by subtraction that

$$\frac{2n}{z}\mathcal{F}_n(z) = \mathcal{F}_{n-1}(z) - \mathcal{F}_{n+1}(z). \quad \text{(IV-18)}$$

The formulas connected with the derivatives of modified Bessel functions are

$$\left[\frac{1}{z}\frac{d}{dz}\right]^k [z^n\mathcal{F}_n] = z^{n-k}\mathcal{F}_{n-k}(z), \quad k = 0,1,2\ldots. \quad \text{(IV-19)}$$

$$\left[\frac{1}{z}\frac{d}{dz}\right]^k [z^{-n}\mathcal{F}_n] = z^{-n-k}\mathcal{F}_{n+k}(z), \quad k = 0,1,2\ldots. \quad \text{(IV-20)}$$

and also; we note that

$$I_o'(z) = I_1(z), \quad K_o'(z) = -K_1(z). \quad \text{(IV-21)}$$

The Wronskian of $I_n(z)$ and $K_n(z)$ is given by

$$I_n(z)K_n'(z) - I_n'(z)K_n(z) = -z^{-1}, \quad n \geq 0 \text{ and } z \neq 0. \quad \text{(IV-22)}$$

This may be expressed in another form by using the recurrence formulas (IV-15) and (IV-16)

$$I_n(z)K_{n+1}(z) + I_{n+1}(z)K_n(z) = z^{-1}. \quad \text{(IV-23)}$$

APPENDIX-V

In this appendix indefinite integrals involving the product of two modified Bessel functions of the integer order and the same argument are evaluated. The results are referenced in Appendix VI.

We now consider the integrals

$$\int^{z} I_m'(z)K_m(z)\,dz\,, \quad m \geq 0\,, \tag{V-1}$$

where the prime denotes differentiation with respect to z. Applying integration by parts formula to the integral (V-1) yields

$$\int^{z} I_m'(z)K_m(z)\,dz = I_m(z)K_m(z) - \int^{z} I_m(z)K_m'(z)\,dz\,. \tag{V-2}$$

It follows from the Wronskian relation (IV-22) of the modified Bessel functions and (V-2) that

$$\int^{z} I_m'(z)K_m(z)\,dz = \frac{1}{2}\left[I_m(z)K_m(z) + \ln z\right]. \tag{V-3}$$

To evaluate the integrals involving the two modified Bessel functions of the same integer order of the form

$$\int^{z} z\,I_m(z)K_m(z)\,dz\,, \quad m \geq 0$$

we use the result

$$\frac{d}{dz}\left[z^p\left(I_q(z)K_r(z) + I_{q+1}(z)K_{r+1}(z)\right)\right] = z^{p-1}\Big[\left[p+q+r\right]I_q(z)K_r(z) + \left[p-q-r\right]I_{q+1}(z)K_{r+1}(z)\Big], \tag{V-4}$$

where p, q and r are any integers. This result is given by Watson [48] for the cylinder functions but it is equally valid for the modified Bessel functions. By suitable choice of p, q and r in (V-4) it follows that

$$\int^z z\, I_m(z)K_m(z)\, dz = \frac{1}{2} z^2 \left[I_m(z)K_m(z) + \frac{1}{2}\left(I_{m-1}(z)K_{m+1}(z) + I_{m+1}(z)K_{m-1}(z) \right) \right]. \quad \text{(V-5)}$$

It follows from (V-4), after suitable choice of p, q and r, that

$$\int^z z^{1-n}\, I_{m-n}(z)K_m(z)\, dz = \frac{z^{2-n}}{2[1-n]} \left[I_{m-n}(z)K_m(z) + I_{m-n+1}(z)K_{m-1}(z) \right], \quad \text{(V-6)}$$

$$\int^z z^{1-n}\, I_{m+n}(z)K_m(z)\, dz = \frac{z^{2-n}}{2[1-n]} \left[I_{m+n}(z)K_m(z) + I_{m+n-1}(z)K_{m+1}(z) \right]; \quad \text{(V-7)}$$

$$\int^z z^{-1-n}\, I_{m-n-2}(z)K_m(z)\, dz = -\frac{z^{-n}}{2[1+n]} \left[I_{m-n-2}(z)K_m(z) + I_{m-n-1}(z)K_{m-1}(z) \right], \quad \text{(V-8)}$$

$$\int^z z^{-1-n}\, I_{m+n+2}(z)K_m(z)\, dz = -\frac{z^{-n}}{2[1+n]} \left[I_{m+n+2}(z)K_m(z) + I_{m+n+1}(z)K_{m+1}(z) \right]; \quad \text{(V-9)}$$

$$\int^z z^{-1}\, I_{m-1}(z)K_{m+1}(z)\, dz = -\frac{1}{2} \left[I_{m-1}(z)K_{m+1}(z) + I_m(z)K_m(z) \right], \quad \text{(V-10)}$$

$$\int^z z^{-1}\, I_{m+1}(z)K_{m-1}(z)\, dz = -\frac{1}{2} \left[I_{m+1}(z)K_{m-1}(z) + I_m(z)K_m(z) \right]; \quad \text{(V-11)}$$

$$\int^z z^{-1} I_{m-2}(z) K_m(z)\, dz = -\frac{1}{2}\left[I_{m-2}(z)K_m(z) + I_{m-1}(z)K_{m-1}(z) \right], \qquad \text{(V-12)}$$

$$\int^z z^{-1} I_{m+2}(z) K_m(z)\, dz = -\frac{1}{2}\left[I_{m+2}(z)K_m(z) + I_{m+1}(z)K_{m+1}(z) \right]. \qquad \text{(V-13)}$$

We now consider the indefinite integrals involving the product of two modified Bessel functions of the form

$$\int^z z^{-1} I_m(z) K_m(z)\, dz, \quad m \geq 1.$$

In order to evaluate these integrals we need to consider particular cases for $m = 1, 2, 3, \ldots$ after suitable choice of p, q and r in the formula (V-4) for each case, we obtain

$$\int^z z^{-1} I_m(z) K_m(z)\, dz = -\frac{1}{2m}\left[I_0(z)K_0(z) + 2\sum_{n=1}^{m-1} \left[I_n(z)K_n(z) \right] + I_m(z)K_m(z) \right], \quad m \geq 1. \qquad \text{(V-14)}$$

We note that there is no simple formula for $\int^z z^{-1} I_0(z) K_0(z)\, dz$.

We will proceed to evaluate the integrals involving the modified Bessel function of the same integer order and the same argument of the form

$$\int^z z^{-3} I_m(z) K_m(z)\, dz, \quad m \geq 0. \qquad \text{(V-15)}$$

It follows from (V-4) by suitable choice of p, q and r that

$$\int^z z^{-3} I_0(z) K_0(z)\, dz = -\frac{1}{2} z^{-2}\left[I_0(z)K_0(z) + I_1(z)K_1(z) \right] - 2\int^z z^{-3} I_1(z) K_1(z)\, dz. \qquad \text{(V-16)}$$

The integral on the right can be written by using the formula (IV-18) for $n = 1$, as

$$\int^z z^{-3} I_1(z)K_1(z)\,dz = \frac{1}{4}\int^z z^{-1}\Big[\,I_0(z)K_2(z) + I_2(z)K_0(z) - I_0(z)K_0(z) - I_2(z)K_2(z)\,\Big]dz\,. \quad \text{(V-17)}$$

The integrals on the right can be evaluated by using the results (V-12) for $m = 2$, (V-13) for $m = 0$ and also, the result (V-14) for $m = 2$. Thus we obtain

$$\int^z z^{-3} I_1(z)K_1(z)\,dz = -\frac{1}{8}\Big[\,I_0(z)K_2(z) + I_2(z)K_0(z) + I_1(z)K_1(z)\,\Big] + \frac{1}{16}\Big[\,I_0(z)K_0(z) + I_2(z)K_2(z)\,\Big] - \frac{1}{4}\int^z z^{-1} I_0(z)K_0(z)\,dz\,. \quad \text{(V-18)}$$

Using this result in (V-16) yields

$$\int^z z^{-3} I_0(z)K_0(z)\,dz = -\frac{1}{2}z^{-2}\Big[\,I_0(z)K_0(z) + I_1(z)K_1(z)\,\Big] + \frac{1}{4}\Big[\,I_2(z)K_0(z) + I_1(z)K_1(z) + I_0(z)K_2(z)\,\Big] - \frac{1}{8}\Big[\,I_0(z)K_0(z) + I_2(z)K_2(z)\,\Big] + \frac{1}{2}\int^z z^{-1} I_0(z)K_0(z)\,dz\,. \quad \text{(V-19)}$$

If we consider (V-15) for $m \geq 2$ and by suitable choice of p, q and r it follows from (V-4) that

$$\int^z z^{-3} I_m(z)K_m(z)\,dz = -\frac{z^{-2}}{m[\,m^2-1]}\Big[\,I_1(z)K_1(z) + \sum_{n=2}^{m-1} n^2[\,I_n(z)K_n(z)\,] + \frac{1}{2}m[\,m-1]\,I_m(z)K_m(z)\,\Big]\,,\quad m \geq 2\,. \quad \text{(V-20)}$$

If we apply integration by parts formula to the integrals (V-15), we find after use of (IV-22), that

$$\int^{z} z^{-2} I_m'(z)K_m(z) = \frac{1}{2} z^{-2} I_m(z)K_m(z) - \frac{1}{4} z^{-2} + \int^{z} z^{-3} I_m(z)K_m(z)\,dz\,, \quad m \geq 0 \qquad \text{(V-21)}$$

where the prime denotes differentiation with respect to z.

We will proceed to evaluate the integrals given by

$$\int^{z} z^{-5} I_m(z)K_m(z)\,dz\,, \quad m \geq 0\,. \qquad \text{(V-22)}$$

It follows from (V-4), after suitable choices of p, q and r, that

$$\int^{z} z^{-5} I_0(z)K_0(z)\,dz = -\frac{1}{4} z^{-4} \left[I_0(z)K_0(z) + I_1(z)K_1(z) \right] - \frac{3}{2} \int^{z} z^{-5} I_1(z)K_1(z)\,dz\,, \qquad \text{(V-23)}$$

$$\int^{z} z^{-5} I_1(z)K_1(z)\,dz = -\frac{1}{2} z^{-4} \left[I_2(z)K_2(z) + I_1(z)K_1(z) \right] - \int^{z} z^{-5} I_2(z)K_2(z)\,dz\,. \qquad \text{(V-24)}$$

Using the result (IV-18) for $n = 2$ yields

$$\int^{z} z^{-5} I_2(z)K_2(z)\,dz = \frac{1}{16} \int^{z} z^{-3} \left[I_1(z)K_3(z) + I_3(z)K_1(z) - I_1(z)K_1(z) - I_3(z)K_3(z) \right]. \qquad \text{(V-25)}$$

If we evaluate the integrals on the right by using the result (V-6) for $n = 4$, $m = 3$, the result (V-7) for $n = 4$, $m = -1$ and also, the result (V-20) both for $m = 1$ and $m = 3$, we may then obtain

$$\int^z z^{-5}\, I_2(z)K_2(z)\,dz = \frac{z^{-2}}{3.2^7}\Big[\, I_1(z)K_1(z) + 4\,I_2(z)K_2(z) + 3\,I_3(z)K_3(z)$$

$$- 4\Big(\, I_1(z)K_3(z) + I_3(z)K_1(z) + I_0(z)K_2(z) + I_2(z)K_0(z) \Big)\Big]$$

$$- \frac{1}{2^8}\Big[\, I_0(z)K_0(z) + I_2(z)K_2(z) - 2\Big(\, I_2(z)K_0(z) + I_1(z)K_1(z)$$

$$+ I_0(z)K_2(z) \Big)\Big] + \frac{1}{2^6}\int^z z^{-1}\, I_0(z)K_0(z)\,dz\ . \qquad \text{(V-26)}$$

Using this result in (V-23) yields

$$\int^z z^{-5}\, I_0(z)K_0(z)\,dz = -\frac{1}{4}z^{-4}\Big[\, I_0(z)K_0(z) - 2\,I_1(z)K_1(z) - 3\,I_2(z)K_2(z)\Big]$$

$$+ \frac{z^{-2}}{2^6}\Big[\, I_1(z)K_1(z) + 4\,I_2(z)K_2(z) + 3\,I_3(z)K_3(z) - 4\big[\, I_1(z)K_3(z)$$

$$+ I_3(z)K_1(z) + I_0(z)K_2(z) + I_2(z)K_0(z)\big]\Big] - \frac{3}{2^7}\Big[\, I_0(z)K_0(z)$$

$$+ I_2(z)K_2(z) - 2\big[\, I_2(z)K_0(z) + I_1(z)K_1(z) + I_0(z)K_2(z)\big]\Big]$$

$$+ \frac{3}{2^5}\int^z z^{-1}\, I_0(z)K_0(z)\,dz\ , \qquad \text{(V-27)}$$

and also, using the result (V-26) in (V-24) gives

$$\int^z z^{-5}\, I_1(z)K_1(z)\,dz = -\frac{1}{2}z^{-4}\Big[\, I_2(z)K_2(z) + I_1(z)K_1(z)\Big] - \frac{z^{-2}}{3.2^5}\Big[\, I_1(z)K_1(z)$$

$$+ 4\,I_2(z)K_2(z) + 3\,I_3(z)K_3(z) - 4\big[\, I_1(z)K_3(z) + I_3(z)K_1(z) + I_0(z)K_2(z)$$

$$+ I_2(z)K_0(z)\big]\Big] + \frac{1}{2^6}\Big[\, I_0(z)K_0(z) + I_2(z)K_2(z) - 2\big[\, I_2(z)K_0(z)$$

$$+ I_1(z)K_1(z) + I_0(z)K_2(z)\big]\Big] - \frac{1}{2^4}\int^z z^{-1}\, I_0(z)K_0(z)\,dz\ . \qquad \text{(V-28)}$$

Also, it follows from (V-4) for suitable values of p, q and r that

$$\int^{z} z^{-5}\, I_m(z)K_m(z)\, dz = -\frac{4!\, z^{-4}}{2m[\,m^2-4][\,m^2-1]}\Big[\, I_2(z)K_2(z) + \sum_{n=3}^{m-1} a_n\, I_n(z)K_n(z)$$

$$+ \frac{1}{4!}\, m[\,m-2\,][\,m^2-1\,]\; I_m(z)K_m(z)\,\Big], \quad m \ge 3 \qquad \text{(V-29)}$$

where $a_3 = 6$, $a_4 = 20$, $a_5 = 50$, $a_6 = 105$, $a_7 = 196$, $a_8 = 336$ . . . . . .

If we apply integration by parts formula to the integrals (V-22), we find after use of (IV-22), that

$$\int^{z} z^{-4}\, I_m'(z)K_m(z)\, dz = \frac{1}{2} z^{-4}\, I_m(z)K_m(z) - \frac{1}{8} z^{-4}$$

$$+ 2\int^{z} z^{-5}\, I_m(z)K_m(z)\, dz, \qquad m \ge 0 \qquad \text{(V-30)}$$

where the prime denotes differentiation with respect to z.

APPENDIX VI

In this appendix infinite integrals involving the product of two modified Bessel functions are evaluated. These integrals are referenced in Chapter III and Chapter IV. Throughout this appendix it will be supposed that the parameter $R\varepsilon$ is small i.e., $R\varepsilon \ll 1$.

To evaluate the integral

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1}\, I_o(z) K_o(z)\, dz ,$$

we expand $I_o(z)$ and $K_o(z)$ in powers of their argument by using both of the formulas (IV-2) and (IV-7) and integrate term by term. Thus we find, after use of the asymptotic properties of the modified Bessel functions (IV-12) and (IV-13) for large z, which gives

$$z^{-1}\, I_m(z) K_n(z) \sim \frac{1}{2} z^{-2} \qquad \text{as} \qquad z \to \infty, \tag{VI-1}$$

for $m \geq 0$ and $n \geq 0$, that

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1}\, I_o(z) K_o(z)\, dz = \frac{1}{2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right]^2 + [\gamma - \ln 2]\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right] + O\Big([R\varepsilon]^2[\ln R\varepsilon]\Big) . \tag{VI-2}$$

Using the results (V-14), (VI-1) and also, both of the formulas (IV-2) and (IV-7), yields

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1}\, I_1(z) K_1(z)\, dz = -\frac{1}{2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{2}\right)\right] + O\Big([R\varepsilon]^2[\ln R\varepsilon]\Big) , \tag{VI-3}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} I_2(z) K_2(z)\, dz = -\frac{1}{4}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{5}{4}\right)\right] + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right), \qquad \text{(VI-4)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} I_3(z) K_3(z)\, dz = -\frac{1}{6}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{5}{3}\right)\right] + O\left([R\varepsilon]^2\right). \qquad \text{(VI-5)}$$

If we use the result (V-12) for $m = 0, 2, 3$ and also, the formulas (IV-2) and (IV-7) we find, after use of (VI-1), that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} I_2(z) K_o(z)\, dz = \frac{1}{2^{10}} \cdot \frac{e^{2\xi_o}}{[\cosh\xi_o]^2} [R\varepsilon]^2 \left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{2}\right)\right] + O\left([R\varepsilon]^4 [\ln R\varepsilon]\right), \qquad \text{(VI-6)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} I_o(z) K_2(z)\, dz = 2^6 e^{-2\xi_o} [\cosh\xi_o]^2 [R\varepsilon]^{-2} + 2^{-2} + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right), \qquad \text{(VI-7)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} I_1(z) K_3(z)\, dz = 2^7 e^{-2\xi_o} [\cosh\xi_o]^2 [R\varepsilon]^{-2} + 2^{-3} + O\left([R\varepsilon]^2\right). \qquad \text{(VI-8)}$$

Using the results (V-13) for $m = 2$, (VI-1) and also, both of the formulas (IV-2) and (IV-7), gives

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} I_4(z) K_2(z)\, dz = -\frac{1}{3.2^{13}} \cdot \frac{e^{2\xi_o}}{[\cosh\xi_o]^2} [R\varepsilon]^2 + \frac{1}{15.2^{20}} \cdot \frac{e^{4\xi_o}}{[\cosh\xi_o]^4} . [R\varepsilon]^4 + O\left([R\varepsilon]^6 [\ln R\varepsilon]\right). \qquad \text{(VI-9)}$$

If we use the result (V-7) for $n = 3$, $m = -1$ and also, both of the formulas (IV-2) and (IV-7) we find, after use of the asymptotic properties of the modified Bessel functions for large z, which gives

$$z^{-2} I_m(z) K_n(z) \sim \frac{1}{2} z^{-3} \quad \text{as} \quad z \to \infty, \tag{VI-10}$$

for $m \geq 0$ and $n \geq 0$, that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2} I_2(z) K_{[illegible]}(z)\, dz = -\frac{1}{8}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{4}\right)\right] + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right). \tag{VI-11}$$

To evaluate the integrals

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2} I_m(z) K_n(z)\, dz$$

for $m = 1$, $n = 0$ and $m = 0$, $n = 1$ we expand the functions $I_m(z)$ and $K_n(z)$ in powers of their argument by using both of the formulas (IV-2) and (IV-7) and integrate term by term. Thus we find, after using the asymptotic property (VI-10) for large z, that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2} I_1(z) K_0(z)\, dz = \frac{1}{4}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right]^2 + \frac{1}{2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\left(\gamma - \ln 2\right)\right] + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right), \tag{VI-12}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2} I_0(z) K_1(z)\, dz = 2^5 e^{-2\xi_o} [\cosh\xi_o]^2 [R\varepsilon]^{-2} - \frac{1}{4}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right]^2 - \frac{1}{2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\left(\gamma - \ln 2\right)\right] + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right). \tag{VI-13}$$

If we use the results (V-19), (VI-2) and both of the formulas (IV-2) and (IV-7), we find, after use of the asymptotic properties of the modified Bessel functions for large argument, which gives

$$z^{-3} I_m(z) K_n(z) \sim \frac{1}{2} z^{-4} \quad \text{as} \quad z \to \infty, \qquad \text{(VI-14)}$$

for $m \geq 0$ and $n \geq 0$, that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3} I_o(z) K_o(z)\, dz = -2^5 e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2}\left[\ln\left(\frac{R\varepsilon e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{2}\right)\right] + \frac{1}{4}\left[\ln\left(\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}\right)\right]^2 + O[\ln R\varepsilon] . \qquad \text{(VI-15)}$$

Using the results (V-18), (VI-2), (VI-14) and also, both of the formulas (IV-2) and (IV-7), yields

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3} I_1(z) K_1(z)\, dz = 2^4 e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} - \frac{1}{8}\left[\ln\left(\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}\right)\right]^2 + O[\ln R\varepsilon] . \qquad \text{(VI-16)}$$

If we use the results (V-20) for $m = 2, 3$, (VI-14), and both of the formulas (IV-2) and (IV-7), we obtain

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3} I_2(z) K_2(z)\, dz = 2^3 e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} + \frac{1}{3.2^3}\left[\ln\left(\frac{R\varepsilon e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{5}{12}\right)\right] + O\left([R\varepsilon]^2[\ln R\varepsilon]\right). \qquad \text{(VI-17)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3} I_3(z) K_3(z)\, dz = \frac{2^4}{3} e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} + \frac{1}{3.2^5}\left[\ln\left(\frac{R\varepsilon e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{25}{24}\right)\right] + O\left([R\varepsilon]^2[\ln R\varepsilon]\right). \qquad \text{(VI-18)}$$

Using the results (V-9) for $n = 2$, $m = -1$, (IV-14), and also, both of the formulas (IV-2) and (IV-7), yields

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\, I_3(z)K_1(z)\,dz = -\frac{1}{3.2^4}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{6}\right)\right] + O\left([R\varepsilon]^2[\ln R\varepsilon]\right). \quad \text{(VI-19)}$$

If we use the result (V-7) for $n = 4$, $m = -3$ and the formulas (IV-2) and (IV-7), we find, after use of (VI-14), that

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\, I_1(z)K_3(z)\,dz = 2^{12}\, e^{-4\xi_o}[\cosh\xi_o]^4[R\varepsilon]^{-4} - \frac{1}{3.2^4}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{4}\right)\right] + O\left([R\varepsilon]^2[\ln R\varepsilon]\right). \quad \text{(VI-20)}$$

To evaluate the integrals

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-4}\, I_m(z)K_n(z)\,dz\ ,$$

for $m = 0$, $n = 1$; $m = 1$, $n = 0$; $m = 1$, $n = 2$; $m = 2$, $n = 3$; and $m = 4$, $n = 3$ we expand the functions $I_m(z)$ and $K_n(z)$ in powers of their argument by using the formulas (IV-2) and (IV-7) and integrate term by term. Thus we find, after use of the asymptotic properties of the modified Bessel functions (IV-12) and (IV-13) for large z, which gives

$$z^{-4}\, I_m(z)K_n(z) \sim \frac{1}{2} z^{-5} \text{ as } z \to \infty, \quad \text{(VI-21)}$$

for $m \geq 0$ and $n \geq 0$, that

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-4}\, I_0(z)K_1(z)\,dz = 2^{10}\, e^{-4\xi_o}[\cosh\xi_o]^4[R\varepsilon]^{-4} + 2^4\, e^{-2\xi_o}[\cosh\xi_o]^2 \cdot [R\varepsilon]^{-2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma + \frac{1}{2}\right)\right] + O\left([\ln R\varepsilon]^2\right). \quad \text{(VI-22)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-4}\, I_1(z)K_o(z)\, dz = -\, 2^{4}\, e^{-2\xi_o}[\cosh\xi_o]^{2}[R\varepsilon]^{-2}\Big[\ln\Big(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\Big) + \Big(\gamma + \frac{1}{2}\Big)\Big] + O\Big([\ln R\varepsilon]^{2}\Big). \quad \text{(VI-23)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-4}\, I_1(z)K_2(z)\, dz = 2^{10}\, e^{-4\xi_o}[\cosh\xi_o]^{4}[R\varepsilon]^{-4} - 4\, e^{-2\xi_o}[\cosh\xi_o]^{2} \cdot [R\varepsilon]^{-2} + O\Big([\ln R\varepsilon]^{2}\Big), \quad \text{(VI-24)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-4}\, I_2(z)K_3(z)\, dz = 2^{10}\, e^{-4\xi_o}[\cosh\xi_o]^{4}[R\varepsilon]^{-4} - \frac{4}{3}\, e^{-2\xi_o}[\cosh\xi_o]^{2} \cdot [R\varepsilon]^{-2} + O\Big([\ln R\varepsilon]\Big), \quad \text{(VI-25)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-4}\, I_4(z)K_3(z)\, dz = \frac{2}{3}\, e^{-2\xi_o}[\cosh\xi_o]^{2}[R\varepsilon]^{-2} + O\Big([\ln R\varepsilon]\Big). \quad \text{(VI-26)}$$

If we use the result (V-17) and (VI-2) and also, both of the formulas (IV-2) and (IV-7), we find, after use of the asymptotic properties of the modified Bessel functions (IV-12) and (IV-13) for large z, which gives

$$z^{-5}\, I_m(z)K_n(z) \sim \frac{1}{2} z^{-6} \quad \text{as} \quad z \to \infty, \quad \text{(VI-27)}$$

for $m \geq 0$ and $n \geq 0$, that

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_o(z)K_o(z)\, dz = -\, 2^{10} e^{-4\xi_o}[\cosh\xi_o]^{4}[R\varepsilon]^{-4}\Big[\ln\Big(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\Big) + \Big(\gamma + \frac{1}{4}\Big)\Big] - 2^{4} e^{-2\xi_o}[\cosh\xi_o]^{2}[R\varepsilon]^{-2}\Big[\ln\Big(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\Big) + \Big(\gamma - \frac{1}{2}\Big)\Big] + O\Big([R\varepsilon]^{-2}\Big). \quad \text{(VI-28)}$$

Using the results (V-28), (VI-2) and (VI-27) and also, both of the formulas (IV-2) and (IV-7), yields

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_1(z)K_1(z)\, dz = 2^9 e^{-4\xi_o}[\cosh\xi_o]^4[R\varepsilon]^{-4} + 2^3 e^{-2\xi_o}[\cosh\xi_o]^2 \cdot [R\varepsilon]^{-2}\left[\ln\left(\frac{R\varepsilon e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma\right] + O\left([R\varepsilon]^{-2}\right). \quad \text{(VI-29)}$$

If we use the results (V-25), (VI-2) and both of the formulas (IV-2) and (IV-7), we find, after use of the result (VI-27), that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_2(z)K_2(z)\, dz = 2^8 e^{-4\xi_o}[\cosh\xi_o]^4[R\varepsilon]^{-4} - \frac{4}{3}e^{-2\xi_o}[\cosh\xi_o]^2 \cdot [R\varepsilon]^{-2} + O\left([\ln R\varepsilon]^2\right). \quad \text{(VI-30)}$$

Using the results (V-29), (VI-27) and both of the formulas (IV-2) and (IV-7) yields

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_3(z)K_3(z)\, dz = \frac{2^9}{3} e^{-4\xi_o}[\cosh\xi_o]^4[R\varepsilon]^{-4} - \frac{1}{3}e^{-2\xi_o}[\cosh\xi_o]^2 \cdot [R\varepsilon]^{-2} + O\left([\ln R\varepsilon]\right). \quad \text{(VI-31)}$$

We now use the result (V-7) for $n = 6$, $m = -4$ and also, both of the formulas (IV-2) and (IV-7). Thus we find, after use of the asymptotic property (VI-27) for large z, that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_2(z)K_4(z)\, dz = 2^3\left[2^{15} e^{-6\xi_o}[\cosh\xi_o]^6[R\varepsilon]^{-6} + \frac{1}{15}e^{-2\xi_o} \cdot [\cosh\xi_o]^2[R\varepsilon]^{-2} + O\left([\ln R\varepsilon]\right)\right]. \quad \text{(VI-32)}$$

To evaluate the integrals

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_m(z) K_n(z)\, dz ,$$

for m = 0, n = 2; m = 1, n = 3; m = 2, n = 0; m = 3, n = 1; m = 4, n = 2; m = 5, n = 3 and m = 6, n = 4, we expand the functions $I_m(z)$ and $K_n(z)$ in powers of their arguments by using the formulas (IV-12) and (IV-13) and integrate term by term. Thus we find, after use of (VI-27), that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_0(z) K_2(z)\, dz = \frac{1}{3}\left[2^{18} e^{-6\xi_o}[\cosh\xi_o]^6[R\varepsilon]^{-6} \right] + O\Big([R\varepsilon]^{-2}[\ln R\varepsilon]\Big), \quad \text{(VI-33)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_1(z) K_3(z)\, dz = \frac{1}{3}\cdot 2^{19} e^{-6\xi_o}[\cosh\xi_o]^6[R\varepsilon]^{-6} + 2e^{-2\xi_o}[\cosh\xi_o]^2 \cdot [R\varepsilon]^{-2} \Big] + O\Big([\ln R\varepsilon]^2\Big), \quad \text{(VI-34)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_2(z) K_0(z)\, dz = -2^2 e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma + \frac{1}{2}\right)\right] + O\Big([\ln R\varepsilon]^2\Big), \quad \text{(VI-35)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_3(z) K_1(z)\, dz = \frac{e^{2\xi_o}}{3.2^{10}}[\cosh\xi_o]^{-2}[R\varepsilon]^2 - \frac{1}{3.2^6}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right]^2 + O\Big([\ln R\varepsilon]\Big), \quad \text{(VI-36)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_4(z) K_2(z)\, dz = \frac{1}{6} e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} + \frac{1}{15.2^6}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right] + O\Big([R\varepsilon]^2[\ln R\varepsilon]\Big), \quad \text{(VI-37)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_5(z)K_3(z)\, dz = \frac{1}{15}\left[\, e^{-2\xi_o}[\,\cosh\xi_o]^2[\,R\varepsilon\,]^{-2} + \frac{1}{3.2^7}\ \ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right] + O\Big([\,R\varepsilon\,]^2\Big), \quad \text{(VI-38)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\, I_6(z)K_4(z)\, dz = 4\,e^{-2\xi_o}[\,\cosh\xi_o]^2[\,R\varepsilon\,]^{-2} + \frac{837}{2^4.7!}\left[\,\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right)\right] + O\Big([\,R\varepsilon\,]^2\Big). \quad \text{(VI-39)}$$

To evaluate the integrals

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-6}\, I_m(z)K_n(z)\, dz\ ,$$

for $m = 0$, $n = 3$; $m = 1$, $n = 2$; $m = 1$, $n = 4$; $m = 2$, $n = 1$; $m = 3$, $n = 0$; $m = 4$, $n = 1$; $m = 5$, $n = 2$; $m = 6$, $n = 3$ and $m = 7$, $n = 4$, we expand the functions $I_m(z)$ and $K_n(z)$ in powers of their argument by using (IV-2) and (IV-7) and integrate term by term. Thus we find, after use of the asymptotic properties of the modified Bessel functions (IV-12) and (IV-13) for large z, which gives

$$z^{-6}\, I_m(z)K_n(z) \sim \frac{1}{2}\, z^{-7} \qquad \text{as} \qquad z \to \infty, \qquad \text{(VI-40)}$$

for $m \geq 0$ and $n \geq 0$, that

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-6}\, I_0(z)K_3(z)\, dz = 2^{24}\, e^{-8\xi_o}[\,\cosh\xi_o]^8[\,R\varepsilon\,]^{-8} + \frac{1}{3}\, 2^{17}\, e^{-6\xi_o}[\,\cosh\xi_o]^6 \cdot [\,R\varepsilon\,]^{-6} + O\Big([\,R\varepsilon\,]^{-4}\Big), \quad \text{(VI-41)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-6}\, I_1(z)K_2(z)\, dz = \frac{1}{3}\, 2^{17}\, e^{-6\xi_o}[\,\cosh\xi_o]^6[\,R\varepsilon\,]^{-6} - 2^7\, e^{-4\xi_o}[\,\cosh\xi_o]^4 \cdot [\,R\varepsilon\,]^{-4} + O\Big([\,R\varepsilon\,]^{-2}[\,\ln R\varepsilon\,]\Big), \quad \text{(VI-42)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}}^{\infty} z^{-6}\, I_1(z) K_4(z)\, dz = 3.2^{24}\, e^{-8\xi_0}[\cosh\xi_0]^{8}[R\varepsilon]^{-8} + \frac{1}{3}\cdot 2^{17}\, e^{-6\xi_0}\cdot[\cosh\xi_0]^{6}[R\varepsilon]^{-6} + O\left([R\varepsilon]^{-4}\right), \quad \text{(VI-43)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}}^{\infty} z^{-6}\, I_2(z) K_1(z)\, dz = 2^{7}\, e^{-4\xi_0}[\cosh\xi_0]^{4}[R\varepsilon]^{-4} + 2\, e^{-2\xi_0}[\cosh\xi_0]^{2}\cdot[R\varepsilon]^{-2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \left(\gamma + \frac{1}{6}\right)\right] + O\left([\ln R\varepsilon]^{2}\right), \quad \text{(VI-44)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}}^{\infty} z^{-6}\, I_3(z) K_0(z)\, dz = -\frac{2}{3} e^{-2\xi_0}[\cosh\xi_0]^{2}[R\varepsilon]^{-2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) + \left(\gamma + \frac{1}{2}\right)\right] + O\left([\ln R\varepsilon]^{2}\right), \quad \text{(VI-45)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}}^{\infty} z^{-6}\, I_4(z) K_1(z)\, dz = \frac{1}{12} e^{-2\xi_0}[\cosh\xi_0]^{2}[R\varepsilon]^{-2} + O\left([\ln R\varepsilon]^{2}\right), \quad \text{(VI-46)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}}^{\infty} z^{-6}\, I_5(z) K_2(z)\, dz = \frac{2}{5!} e^{-2\xi_0}[\cosh\xi_0]^{2}[R\varepsilon]^{-2} + \frac{1}{2^{7}.3.4!}\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi}\right) + O\left([R\varepsilon]^{2}[\ln R\varepsilon]\right), \quad \text{(VI-47)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}}^{\infty} z^{-6}\, I_6(z) K_3(z)\, dz = \frac{4}{6!} e^{-2\xi_0}[\cosh\xi_0]^{2}[R\varepsilon]^{-2} + \frac{5}{2^{6}.7!}\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}\right) + O\left([R\varepsilon]^{2}\right), \quad \text{(VI-48)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_0}}{8\cosh\xi_0}}^{\infty} z^{-6}\, I_7(z) K_4(z)\, dz = \frac{12}{7!} e^{-2\xi_0}[\cosh\xi_0]^{2}[R\varepsilon]^{-2} + O\left([\ln R\varepsilon]\right). \quad \text{(VI-49)}$$

To evaluate the integrals

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_m(z) K_n(z)\, dz\,,$$

for m = 0, n = 0; m = 0, n = 4; m = 1, n = 1; m = 1, n = 3; m = 2, n = 2; m = 3, n = 1; m = 5, n = 1; m = 6, n = 2; m = 7, n = 3 and m = 8, n = 4, we expand the functions $I_m(z)$ and $K_n(z)$ in powers of their argument by using (IV-2) and (IV-7) and integrate term by term. Thus we find, after use of the asymptotic properties of the modified Bessel functions, which gives

$$z^{-7}\, I_m(z) K_n(z) \sim \frac{1}{2} z^{-8} \qquad \text{as} \qquad z \to \infty, \qquad \text{(VI-50)}$$

for $m \geq 0$ and $n \geq 0$, that

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_0(z) K_0(z)\, dz = -\frac{1}{3} \cdot 2^{17}\, e^{-6\xi_o} [\cosh\xi_o]^6 [R\varepsilon]^{-6} \Big[\ln\Big(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\Big) + \Big(\gamma + \frac{1}{6}\Big) \Big] + O\Big([R\varepsilon]^{-4}[\ln R\varepsilon]\Big), \qquad \text{(VI-51)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_0(z) K_4(z)\, dz = \frac{3}{5} \cdot 2^{33}\, e^{-10\xi_o} [\cosh\xi_o]^{10} [R\varepsilon]^{-10} + 2^{24}\, e^{-8\xi_o} \cdot [\cosh\xi_o]^8 [R\varepsilon]^{-8} + O\Big([R\varepsilon]^{-6}\Big), \qquad \text{(VI-52)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_1(z) K_1(z)\, dz = \frac{1}{3} \cdot 2^{16}\, e^{-6\xi_o} [\cosh\xi_o]^6 [R\varepsilon]^{-6} + O\Big([R\varepsilon]^{-4}[\ln R\varepsilon]\Big), \qquad \text{(VI-53)}$$

$$\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_1(z) K_3(z)\, dz = 2^{23}\, e^{-8\xi_o} [\cosh\xi_o]^8 [R\varepsilon]^{-6} + \frac{1}{3} \cdot 2^{8}\, e^{-4\xi_o} [\cosh\xi_o]^4 \cdot [R\varepsilon]^{-4} + O\Big([R\varepsilon]^{-2}[\ln R\varepsilon]\Big), \qquad \text{(VI-54)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_2(z)K_2(z)\,dz = \frac{2^{15}}{3} e^{-6\xi_o}[\cosh\xi_o]^6[R\varepsilon]^{-6} - \frac{2^7}{3} e^{-4\xi_o}[\cosh\xi_o]^4 \cdot [R\varepsilon]^{-4} + O\left([R\varepsilon]^{-2}[\ln R\varepsilon]\right), \quad \text{(VI-55)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_3(z)K_1(z)\,dz = \frac{1}{3}\cdot 2^6 e^{-4\xi_o}[\cosh\xi_o]^4[R\varepsilon]^{-4} + \frac{1}{3} e^{-2\xi_o}[\cosh\xi_o]^2 \cdot [R\varepsilon]^{-2}\left[\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma + \frac{1}{8}\right)\right] + O\left([\ln R\varepsilon]^2\right), \quad \text{(VI-56)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_5(z)K_1(z)\,dz = \frac{1}{5!} e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} + O\left([\ln R\varepsilon]^2\right), \quad \text{(VI-57)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_6(z)K_2(z)\,dz = \frac{1}{6!} e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} + O\left([\ln R\varepsilon]\right), \quad \text{(VI-58)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_7(z)K_3(z)\,dz = \frac{2}{7!} e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} + \frac{3}{2^9\cdot 7!}\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}\right) + O\left([R\varepsilon]^2\right), \quad \text{(VI-59)}$$

$$\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-7}\, I_8(z)K_4(z)\,dz = \frac{6}{8!} e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2} + O\left([\ln R\varepsilon]\right). \quad \text{(VI-60)}$$

APPENDIX VII

In this appendix, the infinite set of equations which result from the system of equations (3.2.14) and (3.2.15) for the unknown constants β, A_m, $m \geq 0$ and B_m, $m \geq 1$ will be presented in terms of the Reynolds number $R\varepsilon$ as $R\varepsilon \to 0$.

The system of equations (3.2.14) and (3.2.15) can be re-written as

$$\sum_{m=1}^{\infty} \chi_{nm} B_m = 4\delta_{n,1} - \sum_{m=0}^{\infty} \Omega_{nm} A_m \,, \quad n \geq 1 \,; \tag{VII-1}$$

$$\sum_{m=0}^{\infty} \Gamma_{nm} A_m = 4\beta\delta_{n,0} - \sum_{m=1}^{\infty} \Lambda_{nm} B_m \,, \quad n \geq 0 \,. \tag{VII-2}$$

If we substitute both (3.2.17) and (3.2.19) for $n = 1$ in (VII-1), we find, after use of the formulas (IV-17) and (IV-18), that

$$\sum_{m=1}^{\infty} B_m \left[2m \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} I_m(z) K_m(z)\,dz - \cos 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o} \right)^2 \left[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2} \Big(I_{m-3}(z) - I_{m+3}(z) \Big) K_m(z)\,dz \right. \right.$$

$$\left. - 2m \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3} I_m(z) K_m(z)\,dz \right] + 2m \left(\frac{R\varepsilon}{8\cosh\xi_o} \right)^4 \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5} I_m(z) K_m(z)\,dz \Bigg] = \frac{R\varepsilon}{2\cosh\xi_o} - \sin 2\alpha$$

$$\cdot \left(\frac{R\varepsilon}{8\cosh\xi_o} \right)^2 \sum_{m=0}^{\infty} A_m \left[2 \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2} I_m'(z) K_m(z)\,dz + \int_{\frac{\varepsilon e^{\xi_o}}{8\operatorname{osh}\xi_o}}^{\infty} z^{-2} \Big(I_{m-3}(z) + I_{m+3}(z)\,dz \Big) \right] . \tag{VII-3}$$

Without going into detail it may be shown, by substituting both of the formulas (V-6) and (V-7) for n = 3, the result (V-21) and also, the expressions (VI-3)-(VI-5), (VI-16)-(VI-18) and (VI-29)-(VI-31) for the integrals in (VII-3) and then using (IV-2), (IV-7), (IV-12), and (IV-13), that the consequence of (VII-3) in terms of the Reynolds number R_E, treating R_E as small, is

$$B_1\left[-\ln\left(\frac{R_E e^{\xi_o}}{16\cosh\xi_o}\right)-\left(\gamma-\frac{1}{2}\right)+\frac{1}{2}e^{-2\xi_o}\left[e^{-2\xi_o}-\cos 2\alpha\right]+O\left([R_E]^2[\ln R_E]^2\right)\right]$$

$$+B_2\left[-\ln\left(\frac{R_E e^{\xi_o}}{16\cosh\xi_o}\right)-\left(\gamma-\frac{5}{4}\right)+\frac{1}{2}e^{-4\xi_o}+O\left([R_E]^2[\ln R_E]\right)\right]$$

$$+B_3\left[-2^7 e^{-4\xi_o}\cos 2\alpha[\cosh\xi_o]^2[R_E]^{-2}-\ln\left(\frac{R_E e^{\xi_o}}{16\cosh\xi_o}\right)-\left(\gamma-\frac{5}{3}\right)+\frac{1}{4}e^{-2\xi_o}\right.$$

$$\left.\cdot\left(2e^{-2\xi_o}+\cos 2\alpha\right)+O\left([R_E]^2[\ln R_E]\right)\right]+O\left(B_4[R_E]^{-2}\right)$$

$$+O\left(B_5[R_E]^{-2}\right)+O\left(B_6[R_E]^{-2}\right)+\cdots\cdots$$

$$=\frac{R_E}{2\cosh\xi_o}+A_o\left[-\frac{1}{2}e^{-2\xi_o}\sin 2\alpha+O\left([R_E]^2[\ln R_E]^2\right)\right]+A_1\left[2^{-8}\sin 2\alpha\right.$$

$$\left.\cdot[\cosh\xi_o]^{-1}[R_E]\left(\ln\left(\frac{R_E e^{\xi_o}}{8\cosh\xi_o}\right)\right)^2+O\left([R_E]^2[\ln R_E]\right)\right]+A_2\left[\frac{1}{2}e^{-2\xi_o}\right.$$

$$\left.\cdot\sin 2\alpha+O\left([R_E]^2[\ln R_E]\right)\right]+A_3\left[2^7 e^{-4\xi_o}\sin 2\alpha[\cosh\xi_o]^2[R_E]^{-2}\right.$$

$$\left.+\frac{1}{4}e^{-2\xi_o}+O\left([R_E]^2[\ln R_E]\right)\right]-\frac{1}{2}e^{-2\xi_o}\sin 2\alpha\left[\sum_{m=0}^{\infty}A_m\right].$$

$$+O\left(A_4[R_E]^{-2}\right)+O\left(A_5[R_E]^{-2}\right)+O\left(A_6[R_E]^{-2}\right)+\cdots\cdots \qquad \text{(VII-4)}$$

Substituting both of (3.2.17) and (3.2.19) for $n = 2$ in (VII-1) gives

$$\sum_{m=1}^{\infty} B_m \Bigg[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1}\left(I_{m-2}(z) - I_{m+2}(z)\right)K_m(z)\,dz + \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^4 \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\left(I_{m-2}(z) - I_{m+2}(z)\right)$$

$$\cdot K_m(z)\,dz - \cos 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^2 \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\left(I_{m-4}(z) - I_{m+4}(z)\right)K_m(z)\,dz \Bigg]$$

$$= \sin 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^2 \sum_{m=0}^{\infty} A_m \Bigg[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\left(I_{m-4}(z) + I_{m+4}(z) - 2\,I_m(z)\right)K_m(z)\,dz \Bigg] . \qquad \text{(VII-5)}$$

Inserting the expressions (VI-3), (VI-7), (VI-8), (VI-15)-(VI-18), (VI-20), (VI-29), (VI-33) and (VI-34) in (VII-5) yields

$$B_1 \left[-\frac{1}{2} \left(\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{2}\right)\right) + \frac{1}{8} e^{-4\xi_o} + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right)\right]$$

$$+ B_2 \left[2^6 e^{-2\xi_o} [\cosh\xi_o]^2 [R\varepsilon]^{-2} \left(1 + \frac{1}{3} e^{-4\xi_o}\right) + \frac{1}{4} - \frac{3}{8} e^{-2\xi_o} \cos 2\alpha \right.$$

$$\left. + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right)\right] + B_3 \left[2^7 e^{-2\xi_o} [\cosh\xi_o]^2 [R\varepsilon]^{-2} \left(1 + \frac{1}{3} e^{-4\xi_o} \right.\right.$$

$$\left.\left. - \frac{1}{2} e^{-2\xi_o} \cos 2\alpha\right) + \frac{1}{8} + O\left([R\varepsilon]^2 [\ln R\varepsilon]\right)\right] + O\left(B_4 [R\varepsilon]^{-4}\right)$$

$$+ O\left(B_5 [R\varepsilon]^{-4}\right) + O\left(B_6 [R\varepsilon]^{-4}\right) + \cdots\cdots$$

$$= A_0 \left[e^{-2\xi_o} \sin 2\alpha \left(\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{2}\right)\right) + O\left([R\varepsilon]^2 [\ln R\varepsilon]^2\right)\right]$$

$$- A_1 \left[\frac{1}{2} e^{-2\xi_o} \sin 2\alpha + O\left([R\varepsilon]^2 [\ln R\varepsilon]^2\right)\right] - A_2 \left[\frac{1}{8} e^{-2\xi_o} \sin 2\alpha \right.$$

$$+ \; O\left([\,R\varepsilon\,]^2[\,\ln R\varepsilon\,]\right)\Big] + A_3\Big[\sin 2\alpha\Big(2^6 e^{-4\xi_o}[\cosh\xi_o]^2[\,R\varepsilon\,]^{-2}$$

$$- \frac{1}{6}e^{-2\xi_o}\Big) + O\left([\,R\varepsilon\,]^2[\,\ln R\varepsilon\,]\right)\Big] + O\left(A_4[\,R\varepsilon\,]^{-4}\right) + O\left(A_5[\,R\varepsilon\,]^{-4}\right)$$

$$+ \; O\left(A_6[\,R\varepsilon\,]^{-4}\right) + O\left(A_7[\,R\varepsilon\,]^{-4}\right) + \cdots\cdots \qquad \text{(VII-6)}$$

Both of the expansions (3.2.17) and (3.2.19) for $n \geq 3$ may be substituted in (VII-1) to obtain

$$\sum_{m=1}^{\infty} B_m \Bigg[\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{n-2} \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{1-n}\Big(I_{m-n}(z) - I_{m+n}(z)\Big)K_m(z)\,dz + \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{2+n} \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3-n}$$

$$\cdot \Big(I_{m-n}(z) - I_{m+n}(z)\Big)K_m(z)\,dz - \cos 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{n} \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1-n}\Big(I_{m-n-2}(z) - I_{m+n+2}(z)$$

$$+ I_{m-n+2}(z) - I_{m+n-2}(z)\Big)K_m(z)\,dz \Bigg] = \sin 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{n} \sum_{m=0}^{\infty} A_m \Bigg[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1-n}$$

$$\cdot \Big(I_{m-n-2}(z) + I_{m+n+2}(z) - I_{m-n+2}(z) - I_{m+n-2}(z)\Big)K_m(z)\,dz \Bigg]. \qquad \text{(VII-7)}$$

It may be shown, by substituting the results (V-6) through (V-9) in (VII-7) and then inserting the expressions (VI-22)-(VI-26), (VI-41) and (VI-42) for the integrals in the resulting equation for $n = 3$ and also, using (IV-2), (IV-7), (IV-12) and (IV-13), that the consequence of (VII-7) for $n = 3$ in terms of the Reynolds number $R\varepsilon$, treating $R\varepsilon$ as small, is

$$B_1\left[-2e^{-4\xi_o}\cos 2\alpha[\cosh\xi_o][R\varepsilon]^{-1}+O([R\varepsilon][\ln R\varepsilon])\right]+B_2\left[2e^{-2\xi_o}\right.$$

$$\left.\cdot\cosh\xi_o[R\varepsilon]^{-1}\left(2+\frac{2}{3}e^{-4\xi_o}-\cos 2\alpha\, e^{-2\xi_o}\right)+O([R\varepsilon][\ln R\varepsilon])\right]$$

$$+B_3\left[2^9 e^{-4\xi_o}\left(2+e^{-4\xi_o}\right)[\cosh\xi_o]^3[R\varepsilon]^{-3}+O([R\varepsilon]^{-1})\right]$$

$$+O(B_4[R\varepsilon]^{-3})+O(B_5[R\varepsilon]^{-5})+O(B_6[R\varepsilon]^{-5})+\cdots\cdots$$

$$=A_o\left[2^{-4}e^{-2\xi_o}\sin 2\alpha[\cosh\xi_o]^{-1}[R\varepsilon]\left(\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right)+\left(\gamma+\frac{1}{2}\right)\right)\right.$$

$$\left.+O([R\varepsilon]^3[\ln R\varepsilon]^2)\right]-A_1\left[2e^{-4\xi_o}\sin 2\alpha[\cosh\xi_o][R\varepsilon]^{-1}\right.$$

$$\left.+O([R\varepsilon][\ln R\varepsilon])\right]-A_2\left[2e^{-4\xi_o}\sin 2\alpha[\cosh\xi_o][R\varepsilon]^{-1}\right.$$

$$\left.+O([R\varepsilon][\ln R\varepsilon]^2)\right]-A_3\left[\frac{1}{3}\cdot 2^{-8}e^{-2\xi_o}\sin 2\alpha[\cosh\xi_o]^{-1}[R\varepsilon]\right.$$

$$\left.+O([R\varepsilon]^3[\ln R\varepsilon])\right]+O(A_4[R\varepsilon]^{-3})\pm O(A_5[R\varepsilon]^{-5})$$

$$+O(A_6[R\varepsilon]^{-5})+\cdots\cdots \qquad \text{(VII-8)}$$

Without going into detail it may be shown that the consequence of (VII-7) for $n=4$ and $n=5$, are respectively

$$O([B_1])+O(B_2[R\varepsilon]^{-2})+O(B_3[R\varepsilon]^{-2})+O(B_4[R\varepsilon]^{-4})+O(B_5[R\varepsilon]^{-4})$$

$$+\cdots\cdots=O(A_o[R\varepsilon]^2[\ln R\varepsilon])+O([A_1])+O(A_2[R\varepsilon]^{-2})$$

$$+ O\left(A_3[R\varepsilon]^{-2}\right) + O\left(A_4[R\varepsilon]^{-2}\right) + O\left(A_5[R\varepsilon]^{-4}\right) + \cdots\cdots\cdots \quad \text{(VII-9)}$$

and

$$O\left(B_1[R\varepsilon]\right) + O\left(B_2[R\varepsilon]^{-1}\right) + O\left(B_3[R\varepsilon]^{-3}\right) + O\left(B_4[R\varepsilon]^{-3}\right)$$

$$+ O\left(B_5[R\varepsilon]^{-5}\right) + O\left(B_6[R\varepsilon]^{-5}\right) + \cdots\cdots\cdots = O\left(A_0[R\varepsilon]^3[\ln R\varepsilon]\right)$$

$$+ O\left(A_1[R\varepsilon]\right) + O\left(A_2[R\varepsilon]^{-1}\right) + O\left(A_3[R\varepsilon]^{-3}\right) + O\left(A_4[R\varepsilon]^{-3}\right)$$

$$+ O\left(A_5[R\varepsilon]^{-3}\right) + O\left(A_6[R\varepsilon]^{-5}\right) + \cdots\cdots\cdots \quad \text{(VII-10)}$$

Both of the expressions (3.2.16) and (3.2.18) for $n = 0$ may be substituted in (VII-2) to obtain

$$\sum_{m=0}^{\infty} A_m \left[\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{-2} \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z\, I_m(z) K_m(z)\, dz + \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{2} \cdot \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\, I_m(z) K_m(z)\, dz \right.$$

$$- \cos 2\alpha \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} \left(I_{m-2}(z) + I_{m+2}(z)\right) K_m(z)\, dz = 2\beta - \sin 2\alpha \sum_{m=1}^{\infty} B_m \left[- \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1} \left(I_{m-2}(z)\right.\right.$$

$$\left.\left. - I_{m+2}(z)\right) K_m(z)\, dz \right] . \quad \text{(VII-11)}$$

If we substitute the expressions (V-5), (V-12) and (V-13) for the integrals in (VII-11) and then insert the expansions (IV-2) and (IV-7) for the modified Bessel functions, the expressions (VI-15) through

(VI-18) for the integrals in the resulting equation, we find, after use of (IV-12) and (IV-13), that if

$$\sum_{m=0}^{\infty} A_m = 0 \qquad \text{(VII-12)}$$

the consequence of (VII-11), in terms of the small Reynolds number R_ε, is

$$A_0\left[\sinh 2\xi_0\left(\ln\left(\frac{R_\varepsilon e^{\xi_0}}{16\cosh\xi_0}\right)+\left(\gamma-\frac{1}{2}\right)\right)-\frac{1}{2}\cos 2\alpha+O\left([R_\varepsilon]^2[\ln R_\varepsilon]^2\right)\right]$$

$$+A_1\left[-2^5[\cosh\xi_0]^2[R_\varepsilon]^{-2}+\frac{1}{2}\cos 2\alpha\left(\ln\left(\frac{R_\varepsilon e^{\xi_0}}{16\cosh\xi_0}\right)+\left(\gamma-\frac{3}{4}\right)\right)-\frac{1}{2}\sinh 2\xi_0\right.$$

$$\left.+\;O\left([R_\varepsilon]^2[\ln R_\varepsilon]^2\right)\right]+A_2\left[-2^6[\cosh\xi_0]^2[R_\varepsilon]^{-2}\left(1+e^{-2\xi_0}\cos 2\alpha\right)\right.$$

$$\left.-\frac{1}{4}\sinh 2\xi_0-\frac{1}{3}\cos 2\alpha+O\left([R_\varepsilon]^2[\ln R_\varepsilon]\right)\right]+A_3\left[-2^5[\cosh\xi_0]^2[R_\varepsilon]^{-2}\right.$$

$$\left.\cdot\left(3+4e^{-2\xi_0}\cos 2\alpha\right)-\frac{3}{16}\cos 2\alpha-\frac{1}{6}\sinh 2\xi_0+O\left([R_\varepsilon]^2[\ln R_\varepsilon]\right)\right]$$

$$+\;O\left(A_4[R_\varepsilon]^{-2}\right)+O\left(A_5[R_\varepsilon]^{-2}\right)+O\left(A_6[R_\varepsilon]^{-2}\right)+\cdots\cdots$$

$$=2\beta+B_1\left[\frac{1}{2}\sin 2\alpha\left(\ln\left(\frac{R_\varepsilon e^{\xi_0}}{16\cosh\xi_0}\right)+\left(\gamma-\frac{1}{4}\right)\right)+O\left([R_\varepsilon]^2[\ln R_\varepsilon]\right)\right]$$

$$-\;B_2\left[\sin 2\alpha\left(2^6e^{-2\xi_0}[\cosh\xi_0]^2[R_\varepsilon]^{-2}+\frac{1}{6}\right)+O\left([R_\varepsilon]^2[\ln R_\varepsilon]\right)\right]$$

$$-\;B_3\left[\sin 2\alpha\left(2^7e^{-2\xi_0}[\cosh\xi_0]^2[R_\varepsilon]^{-2}+\frac{1}{16}\right)+O\left([R_\varepsilon]^2\right)\right]$$

$$+\;O\left(B_5[R_\varepsilon]^{-2}\right)+O\left(B_6[R_\varepsilon]^{-2}\right)+\cdots\cdots \qquad \text{(VII-13)}$$

We note that (VII-12) is the necessary condition for the existence of the summation

$$\sum_{m=0}^{\infty} A_m \left[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z\, I_m(z) K_m(z)\, dz \right] \qquad \text{(VII-14)}$$

in (VII-11). It may be shown, by substituting (V-5) into (VII-14) and also, using (IV-2), (IV-7) and (IV-14), that the integrals in (VII-14) diverge to infinity at the upper limit for all m. Thus, the summation (VII-14) can be approximated in terms of $R\varepsilon$ only if (VII-12) is satisfied.

If we substitute both of the expressions (3.2.16) and (3.2.18) for n = 1 in (VII-2) we find, after use of (V-17) and (IV-18), that

$$\sum_{m=0}^{\infty} A_m \left[\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{-1} \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} 2\, I_m'(z) K_m(z)\, dz - 2\cos 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right) \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2}\, I_m'(z) K_m(z)\, dz \right.$$

$$+ \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{3} \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} 2z^{-4}\, I_m'(z) K_m(z)\, dz - \cos 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right) \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-2} \Big(I_{m-1}(z) + I_{m+1}(z) \Big)$$

$$\left. \cdot\, K_m(z)\, dz \right] = -\sin 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right) \sum_{m=1}^{\infty} B_m \left[2m \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\, I_m(z) K_m(z)\, dz + \int z^{-2} \Big(I_{m-1}(z) \right.$$

$$\left. - I_{m+1}(z) \Big) K_m(z)\, dz \right] . \qquad \text{(VII-15)}$$

It may be shown, by substituting both of (V-6) and (V-7) for n = 3, (V-3) (V-21), (V-30) and also, the expressions (VI-16) through (VI-18) in (VII-15) and then inserting the expansions (IV-2), (IV-7) for the modified Bessel functions, the expressions (VI-28) through (VI-31) for the integrals, and using (IV-12) and (IV-13) in the resulting equation, that if

$$\sum_{m=0}^{\infty} A_m = 0$$

the consequence of the equation (VII-15) in terms of the small Reynolds number R_E, is

$$A_0\Big[2^3[\cosh\xi_0][R_E]^{-1}\Big(\ln\Big(\frac{R_E e^{\xi_0}}{16\cosh\xi_0}\Big) + \Big(\gamma - \frac{1}{2}e^{-2\xi_0}\cos 2\alpha - \frac{1}{4}e^{-4\xi_0}\Big)\Big)$$

$$+ O\big([R_E][\ln R_E]\big)\Big] - A_1\Big[4[\cosh\xi_0][R_E]^{-1} - 2^{-5}\cos 2\alpha[\cosh\xi_0]^{-1}[R_E]$$

$$\cdot\Big[\ln\Big(\frac{R_E}{8\cosh\xi_0}\Big)\Big]^2 + O\big([R_E][\ln R_E]\big)\Big] - A_2\Big[4[\cosh\xi_0][R_E]^{-1}\Big(e^{-2\xi_0}$$

$$\cdot\cos 2\alpha + \frac{1}{2}\Big) + \frac{1}{3}\cdot 2^{-5}\cos 2\alpha[\cosh\xi_0]^{-1}[R_E]\ln\Big(\frac{R_E e^{\xi_0}}{16\cosh\xi_0}\Big) + O\big([R_E]\big)\Big]$$

$$- A_3\Big[2^{10}e^{-4\xi_0}\cos 2\alpha[\cosh\xi_0]^2[R_E]^{-2} + 4[\cosh\xi_0][R_E]^{-1}\Big(\frac{1}{3} + \frac{1}{2}e^{-2\xi_0}$$

$$\cdot\cos 2\alpha\Big) + O(1)\Big] + O\big(A_4[R_E]^{-3}\big) + O\big(A_5[R_E]^{-3}\big) + \cdots\cdots$$

$$= -B_1\Big[4\sin 2\alpha\big(e^{-2\xi_0}[\cosh\xi_0][R_E]^{-1} - 2^{-7}[\cosh\xi_0]^{-1}[R_E]\Big(\ln\Big(\frac{R_E e^{\xi_0}}{8\cosh\xi_0}\Big)\Big)^2\big)$$

$$+ O\big([R_E][\ln R_E]\big)\Big] - B_2\Big[8e^{-2\xi_0}\sin 2\alpha[\cosh\xi_0][R_E]^{-1} + O\big([R_E]$$

$$\cdot[\ln R_E]\big)\Big] - B_3\Big[8\sin 2\alpha\Big(2^7 e^{-4\xi_0}[\cosh\xi_0]^3[R_E]^{-3} + \frac{3}{4}e^{-2\xi_0}$$

$$\cdot[\cosh\xi_0][R_E]^{-1} + O(1)\Big] + O\big(B_4[R_E]^{-3}\big) + O\big(B_5[R_E]^{-3}\big)$$

$$+ O\big(B_6[R_E]^{-3}\big) + \cdots\cdots \qquad \text{(VII-16)}$$

We note that the necessary condition for the existence of the summation

$$\sum_{m=0}^{\infty} A_m \left[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} I_m'(z) K_m(z)\, dz \right] \qquad \text{(VII-17)}$$

in (VII-15) is

$$\sum_{m=0}^{\infty} A_m = 0 \,. \qquad \text{(VII-18)}$$

It may be shown, by substituting (V-3) into (VII-17) and also, using (IV-2), (IV-7) and (IV-14), that the integrals in (VII-17) diverge to infinity at the upper limit for all m. Thus (VII-17) can be approximated in terms of $R\varepsilon$ only if (VII-18) is satisfied.

If we substitute (3.2.16) and (3.2.18) in (VII-2) for $n = 2$ we immediately arrive at the result

$$\sum_{m=0}^{\infty} A_m \left[\int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1}\left(I_{m-2}(z) + I_{m+2}(z)\right) K_m(z)\, dz + \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^4 \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-5}\left(I_{m-2}(z) + I_{m+2}(z)\right) K_m(z)\, dz - \cos 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^2 \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\left(I_{m-4}(z) + I_{m+4}(z) + 2I_m(z)\right) K_m(z)\, dz \right]$$

$$= - \sum_{m=1}^{\infty} B_m \left[\sin 2\alpha \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^2 \int_{\frac{R\varepsilon e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3}\left(I_{m-4}(z) - I_{m+4}(z)\right) K_m(z)\, dz \right] . \qquad \text{(VII-19)}$$

Without going into detail it may be shown, by substituting the expressions (VI-3), (VI-7), (VI-8) and (VI-15)-(VI-20) for the integrals in (VII-19), that the consequence of (VII-19) in terms of the small Reynolds number $R\varepsilon$, is

$$A_o\left[\cos 2\alpha\, e^{-2\xi_o}\left(\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{2}\right)\right) + O\left([R\varepsilon]^2[\ln R\varepsilon]\right)\right]$$

$$+ A_1\left[-\frac{1}{2}\left(\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \gamma - \frac{1}{2}\left(1 + \frac{1}{2}e^{-4\xi_o}\right) + \cos 2\alpha\, e^{-2\xi_o}\right) + O\left([R\varepsilon]^2\right.\right.$$

$$\left.\left. .[\ln R\varepsilon]^2\right)\right] + A_2\left[2^6 e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2}\left(1 + \frac{1}{3}e^{-4\xi_o}\right) + \frac{1}{4} - \frac{3}{8}\right.$$

$$\left. .\, e^{-2\xi_o}\cos 2\alpha + O\left([R\varepsilon]^2[\ln R\varepsilon]\right)\right] + A_3\left[2^7 e^{-2\xi_o}[\cosh\xi_o]^2[R\varepsilon]^{-2}\right.$$

$$\left. .\left(1 + \frac{1}{3}e^{-4\xi_o} - \frac{1}{2}e^{-2\xi_o}\cos 2\alpha\right) + \frac{1}{8} - \frac{1}{6}e^{-2\xi_o}\cos 2\alpha + O\left([R\varepsilon]^2[\ln R\varepsilon]\right)\right]$$

$$+ O\left(A_4[R\varepsilon]^{-4}\right) + O\left(A_5[R\varepsilon]^{-4}\right) + O\left(A_6[R\varepsilon]^{-4}\right) + \cdots\cdots$$

$$= B_1\left[\frac{1}{3}\cdot 2^{-10}\sin 2\alpha[\cosh\xi_o]^{-2}[R\varepsilon]^2\left(\ln\left(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\right) + \left(\gamma - \frac{1}{16}\right)\right) + O\left([R\varepsilon]^4\right.\right.$$

$$\left.\left. .[\ln R\varepsilon]\right)\right] - B_2\left[\frac{1}{8}e^{-2\xi_o}\sin 2\alpha + O\left([R\varepsilon]^2[\ln R\varepsilon]\right)\right] - B_3\left[2^6 e^{-4\xi_o}\right.$$

$$\left. .\sin 2\alpha[\cosh\xi_o]^2[R\varepsilon]^{-2} + O\left([R\varepsilon]^2[\ln R\varepsilon]\right)\right]\Big\} + O\left(B_4[R\varepsilon]^{-4}\right)$$

$$+ O\left(B_5[R\varepsilon]^{-4}\right) + O\left(B_6[R\varepsilon]^{-4}\right) + \cdots\cdots \qquad \text{(VII-20)}$$

Both of the expansions (3.2.16) and (3.2.18) for $n \geq 3$ may be substituted in (VII-2) to obtain

$$\sum_{m=0}^{\infty} A_m\left[\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{n-2}\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{1-n}\left(I_{m-n}(z) + I_{m+n}(z)\right)K_m(z)\,dz + \left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^{2+n}\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-3-n}\right.$$

$$\cdot \left(I_{m-n}(z) + I_{m+n}(z)\right)K_m(z)\,dz - \cos 2\alpha\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^n \int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1-n}\left(I_{m-n-2}(z) - I_{m+n+2}(z)\right.$$

$$\left. + I_{m-n+2}(z) + I_{m+n-2}(z)\right)K_m(z)\,dz\ \Big] = -\sin 2\alpha\left(\frac{R\varepsilon}{8\cosh\xi_o}\right)^n \sum_{m=1}^{\infty} B_m \Big[\int_{\frac{R\varepsilon\, e^{\xi_o}}{8\cosh\xi_o}}^{\infty} z^{-1-n}$$

$$\cdot \left(I_{m-n-2}(z) - I_{m+n+2}(z) - I_{m-n+2}(z) + I_{m+n-2}(z)\right)K_m(z)\,dz \Big] . \qquad \text{(VII-21)}$$

It may be shown, by substituting the results (V-6) through (V-9) in (VII-21) and then inserting the expressions (VI-22)-(VI-26), (VI-41) and (VI-42) for the integrals in the resulting equation for $n = 3$ and also, using (IV-2), (IV-7), (IV-12) and (IV-13), that the consequence of (VII-7) for $n = 3$ in terms of the Reynolds number $R\varepsilon$, treating $R\varepsilon$ as small, is

$$A_o\Big[2^{-4}[\cosh\xi_o]^{-1}[R\varepsilon]\Big(e^{-2\xi_o}\cos 2\alpha\Big(\ln\Big(\frac{R\varepsilon\, e^{\xi_o}}{16\cosh\xi_o}\Big) + \Big(\gamma + \frac{1}{2}\Big)\Big) + \frac{1}{8}\Big)$$

$$+ O\big([R\varepsilon]^3[\ln R\varepsilon]^2\big)\Big] - A_1\Big[2\,e^{-4\xi_o}\cos 2\alpha[\cosh\xi_o][R\varepsilon]^{-1} + O\big([R\varepsilon]$$

$$\cdot[\ln R\varepsilon]\big)\Big] + A_2\Big[2\,e^{-2\xi_o}[\cosh\xi_o][R\varepsilon]^{-1}\Big(2 + \frac{2}{3}e^{-4\xi_o} - e^{-2\xi_o}\cos 2\alpha\Big)$$

$$+ O\big([R\varepsilon][\ln R\varepsilon]\big)\Big] + A_3\Big[2^9\,e^{-4\xi_o}\big(2 + e^{-4\xi_o}\big)[\cosh\xi_o]^3[R\varepsilon]^{-3}$$

$$+ O\big([R\varepsilon]^{-1}\big)\Big] + O\big(A_4[R\varepsilon]^{-3}\big) + O\big(A_5[R\varepsilon]^{-5}\big) + O\big(A_6[R\varepsilon]^{-5}\big)$$

$$+ \cdots\cdots = -B_1\Big[2\,e^{-4\xi_o}\sin 2\alpha[\cosh\xi_o][R\varepsilon]^{-1} + O\big([R\varepsilon][\ln R\varepsilon]\big)\Big]$$

$$- B_2\left[2e^{-4\xi_o}\sin 2\alpha[\cosh\xi_o][R\varepsilon]^{-1} + O([R\varepsilon][\ln R\varepsilon]^2)\right] - B_3\left[\frac{1}{3}\cdot 2^{-8}e^{-2\xi_o}\right.$$

$$\left.\cdot\sin 2\alpha[\cosh\xi_o]^{-1}[R\varepsilon] + O([R\varepsilon]^3[\ln R\varepsilon])\right] + O(B_4[R\varepsilon]^{-3})$$

$$+ O(B_5[R\varepsilon]^{-5}) + O(B_6[R\varepsilon]^{-5}) + \cdots\cdots \quad \text{(VII-22)}$$

Without going into detail it may be shown that the consequence of (VII-21) for $n = 4$ and $n = 5$, respectively, are

$$O(A_o[R\varepsilon]^2[\ln R\varepsilon]) + O([A_1]) + O(A_2[R\varepsilon]^{-2}) + O(A_3[R\varepsilon]^{-2})$$

$$+ O(A_4[R\varepsilon]^{-4}) + O(A_5[R\varepsilon]^{-4}) + \cdots\cdots = O([B_1]) + O(B_2[R\varepsilon]^{-2})$$

$$+ O(B_3[R\varepsilon]^{-2}) + O(B_4[R\varepsilon]^{-2}) + O(B_5[R\varepsilon]^{-4}) + \cdots\cdots \quad \text{(VII-23)},$$

and

$$O(A_o[R\varepsilon]^3[\ln R\varepsilon]) + O(A_1[R\varepsilon]) + O(A_2[R\varepsilon]^{-1}) + O(A_3[R\varepsilon]^{-3})$$

$$+ O(A_4[R\varepsilon]^{-3}) + O(A_5[R\varepsilon]^{-5}) + O(A_6[R\varepsilon]^{-5}) + \cdots\cdots$$

$$= O(B_1[R\varepsilon]) + O(B_2[R\varepsilon]^{-1}) + O(B_3[R\varepsilon]^{-3}) + O(B_4[R\varepsilon]^{-3})$$

$$+ O(B_5[R\varepsilon]^{-3}) + O(B_6[R\varepsilon]^{-5}) + \cdots\cdots \quad \text{(VII-24)}$$

APPENDIX VIII

In the paper by Hasimoto [19], the steady flow of a viscous fluid past an elliptic cylinder at an arbitrary angle of incidence is discussed on the basis of Oseen linearized equation of motion using elliptic co-ordinates and Mathieu functions. He obtained the expansion formula for the lift and drag in powers of Reynolds number by using the fundamental solutions of Oseen equations due to himself. In this Appendix, we will summarize Hasimoto's analysis to be able to compare his results with those of the present analysis.

Fundamental Equations and Its Solutions

Using a stream function Ψ, the Navier-Stokes equation for the two dimensional steady motion of an incompressible viscous fluid is expressed as follows

$$\Delta\Delta\Psi = -\frac{1}{\upsilon}\frac{\partial(\Psi,\Delta\Psi)}{\partial(x,y)}, \qquad \text{(VIII-1)}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and υ is the kinematic the viscosity.

Let us assume that a cylindrical obstacle is placed in a uniform flow of velocity U streaming parallel to the x-axis. Writing

$$\Psi = Uy + \phi, \qquad \text{(VIII-2)}$$

and using the Oseen approximation which neglects terms of the second order in ϕ, i.e., $-\frac{1}{\upsilon}\frac{\partial(\phi,\Delta\phi)}{\partial(x,y)}$, we obtain

$$\Delta\left(\Delta - 2k\frac{\partial}{\partial x}\right)\phi = 0, \qquad \text{(VIII-3)}$$

$$2k = U/\upsilon. \qquad \text{(VIII-4)}$$

The solution of (VIII-3) can be written in the form

$$\phi = \phi_1 + \phi_2 \,, \tag{VIII-5}$$

$$\Delta \phi_1 = 0 \,, \tag{VIII-6}$$

$$(\Delta - k^2)(e^{-kx} \phi_2) = 0 \,, \tag{VIII-7}$$

According to Filon [12], the solutions satisfying (VIII-6), (VIII-7) and the condition at infinity are

$$\phi_1 = A_o \ln r + B_o \theta + \sum_{n=1}^{\infty} r^{-n} [\, A_n \cos n\theta + B_n \sin n\theta \,] \,, \tag{VIII-8}$$

$$\phi_2 = a_o e^{kx} K_o(kr) + b_o \int_0^{\theta} kr[\, K_1(kr) + K_o(kr)\,] e^{kr\cos\theta} d\theta + e^{kx} \sum_{n=1}^{\infty} K_n(kr)[\, a_n \cos n\theta + b_n \sin n\theta \,] \,, \tag{VIII-9}$$

where

$$b_o = -B_o \,, \qquad x = r\cos\theta \,, \qquad y = r\sin\theta \,, \tag{VIII-10}$$

and $K_n(kr)$ is the modified Bessel function of order n which tends to zero when $kr \to \infty$. Then the lift L and drag D experienced by the cylinder are, respectively, given by

$$L = (2\pi\rho U) A_o = \rho U \Gamma \,, \tag{VIII-11}$$

$$D = (2\pi\rho U) B_o = \rho U m \,, \tag{VIII-12}$$

where ρ is the density of the fluid, m and Γ are, respectively, the inward flow in the wake and the circulation in the clockwise sense round a large circle surrounding the cylinder. (VIII-10) is the condition for the stream function Ψ to be continuous in the field of flow.

Introducing elliptic co-ordinates ξ, η, related with x, y by the equation

$$x + iy = c e^{-i\alpha} \cosh(\xi + i\eta) \,, \tag{VIII-13}$$

$$\begin{aligned} x &= c[\, (\cos\alpha)(\cosh\xi)(\cos\eta) + (\sin\alpha)(\sinh\xi)(\sin\eta)\,] \,, \\ y &= c[\, (\cos\alpha)(\sinh\xi)(\sin\eta) - (\sin\alpha)(\cosh\xi)(\cos\eta)\,] \,, \end{aligned} \tag{VIII-14}$$

we obtain from (VIII-6) and (VIII-7),

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \phi_1 = 0 , \qquad \text{(VIII-15)}$$

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - k^2 c^2 (\cosh^2 \xi - \cos^2 \alpha) \right) \left(e^{-kx} \phi_2 \right) = 0 . \qquad \text{(VIII-16)}$$

The surface of the elliptic cylinder is given by $\xi = \xi_o$, the angle of incidence between the uniform flow and the major axis of the ellipse is α.

If we take

$$\sigma = e^{-\xi_o} , \quad c = 2\sigma , \qquad \text{(VIII-17)}$$

the half lengths of the major and minor axes are, respectively

$$a = c(\cosh \xi_o) = 1 + \sigma^2 , \qquad \text{(VIII-18)}$$

$$b = c(\sinh \xi_o) = 1 - \sigma^2 . \qquad \text{(VIII-19)}$$

Making $\xi \to \infty$ in (3.1) we see

$$\xi \sim \ln r \quad \text{or} \quad r^{-n} \sim e^{-n\xi} , \qquad \text{(VIII-20)}$$

$$\eta \sim \theta + \alpha . \qquad \text{(VIII-21)}$$

Therefore, taking account of (VIII-8) and (VIII-9), we may take as the general solutions satisfying (VIII-15) and (VIII-16) and the condition at infinity,

$$\phi_1 = A_o \xi + B_o \eta + \sum_{n=1}^{\infty} \frac{1}{n} e^{-n(\xi - \xi_o)} [A_n \cos n\eta - B_n \sin n\eta] , \qquad \text{(VIII-22)}$$

$$\phi_2 = \sum_{n=o}^{\infty} [a_n F_n (\xi, \eta) + b_n G_n (\xi, \eta)] , \qquad \text{(VIII-23)}$$

where

$$F_n (\xi, \eta) = e^{kx} \, FEK_n (\xi) \, ce_n (\eta) ,$$

$$G_n (\xi, \eta) = \int_{\eta_o}^{\eta} \left(\frac{\partial}{\partial \eta} - 2k \frac{\partial x}{\partial \xi} \right) F_n (\xi, \eta) \, d\eta - \int_{\xi}^{\infty} \left(\frac{\partial}{\partial \eta_o} - 2k \frac{\partial x(\xi, \eta_o)}{\partial \eta_o} \right) F_n (\xi, \eta_o) \, d\xi . \qquad \text{(VIII-24)}$$

$ce_n(\eta)$ are proper solutions of the Mathieu equation (see e.g. McLachlan [31] or Whittaker & Watson [49])

$$\left(\frac{d^2}{d\eta^2} + (\lambda_n + k^2c^2\cos^2\eta) \right) ce_n(\eta) = 0 , \qquad \text{(VIII-25)}$$

which are periodic and even in η

$$ce_{2n}(\eta) = (-1)^n \sum_{p=0}^{\infty} a_{2p}^{2n} \cos 2p\eta , \qquad \text{(VIII-26)}$$

$$ce_{2n+1}(\eta) = (-1)^n \sum_{p=0}^{\infty} a_{2p+1}^{2n+1} \cos(2p+1)\eta . \qquad \text{(VIII-27)}$$

$FEK_n(\xi)$ are solutions of the associated Mathieu equation, corresponding to the characteristic values λ_n's of (VIII-25)

$$\left(\frac{d^2}{d\xi^2} - (\lambda_n + k^2c^2\cosh^2\xi) \right) FEK_n(\xi) = 0 , \qquad \text{(VIII-28)}$$

and tend to zero as ξ tends to infinity. According to McLachlan [31]

$$FEK_{2n}(\xi) = \frac{1}{A_0^{(2n)}} \sum_{p=0}^{\infty} A_{2p}^{(2n)} I_p(v_1)K_p(v_2) , \qquad \text{(VIII-29)}$$

$$FEK_{2n+1}(\xi) = \frac{1}{B_1^{(2n+1)}} \sum_{p=0}^{\infty} B_{2p+1}^{(2n+1)} [I_p(v_1)K_{p+1}(v_2) - I_{p+1}(v_1)K_p(v_2)] , \qquad \text{(VIII-30)}$$

where $A_{2p}^{(2n)} = (-1)^{n+p} a_{2p}^{(2n)}$, $B_{2p+1}^{(2n+1)} = (-1)^{n+p} a_{2p+1}^{(2n+1)}$, $v_1 = \frac{1}{2}kce^{-\xi}$ and $v_2 = \frac{1}{2}kce^{\xi}$.

Taking into account (VIII-8), (VIII-9), (VIII-20) and (VIII-21) we see that the lift L and drag D experienced by the elliptic cylinder are expressed by (VIII-11) and (VIII-12) using the same A_o and B_o. Also, circulation at $\xi = \infty$ takes a finite value $2\pi A_o$.

The constants a_n, b_n, A_n and B_n in the stream function Ψ have to be determined from the following boundary conditions

(i) Ψ is one-valued and continuous in the field of flow.

(ii) $u = v = 0$ on the surface of the cylinder, or in terms of normal and tangential velocities

$$u_n = h\frac{\partial \Psi}{\partial \eta} = 0\,, \quad u_s = -h\frac{\partial \Psi}{\partial \xi} = 0 \quad \text{at } \xi = \xi_o\,, \quad h^{-2} = c^2(\cosh^2\xi - \cos^2\eta)$$

(iii) $u = \dfrac{\partial \Psi}{\partial y} \to U$ and $v = -\dfrac{\partial \Psi}{\partial x} \to 0$ as $x, y \to \infty$.

These coefficients are determined, by Hasimoto [19] (p.656 & 657), in powers of Reynolds number

$$R = 2aU/\nu = 2(1+\sigma^2)U/\nu = 4ka = 4k(1+\sigma^2)\,, \tag{VIII-31}$$

treating R as small. For our present purpose, however, which is to compare his results with those of the present analysis, it would not be of much use to write down the laborious and long calculations in obtaining complicated expansions for these constants. We will only present results for A_o and B_o which are sufficient to enable us to determine the leading terms in the expansions for the circulation at a great distance from the cylinder, Γ_∞, and the non-dimensional coefficients of the drag and lift, C_D and C_L. The expressions for A_o and B_o are

$$A_o = k^{-1}(\bar{\alpha})\left[1 + O(k^2)\right], \tag{VIII-32}$$

$$B_o = k^{-1}(\beta)\left[1 + O(k^2)\right]. \tag{VIII-33}$$

In these expressions $\bar{\alpha}$ and β are given by

$$\bar{\alpha} = (2\sigma^2 U \sin 2\alpha) / D_1 , \quad \beta = 2U[2s - (1 + \sigma^2 \cos 2\alpha)] / D_1 , \tag{VIII-34}$$

where

$$D_1 = 4s^2 - (1 + 2\sigma^2 \cos 2\alpha + \sigma^4) . \tag{VIII-35}$$

Hasimoto's Results

It may be shown, by using (VIII-11), (VIII-12), (VIII-30), (VIII-32) and (VIII-33) that the lift and drag coefficients can be expressed in terms of the constants A_o and B_o, as

$$C_L = 2\pi A_o / Ua , \quad C_D = 2\pi B_o / Ua . \tag{VIII-36}$$

We note that the formulas

$$C_L = L / (\rho U^2 a) , \quad C_D = D / (\rho U^2 a), \tag{VIII-37}$$

are used in obtaining the above expressions. Hasimoto [19] shows that the circulation, Γ_∞, in the clockwise sense, round a large circle surrounding the cylinder takes a finite value

$$\Gamma_\infty = 2\pi A_o , \tag{VIII-38}$$

by using the well-known formula

$$\Gamma_\infty = \lim_{\xi \to \infty} \int_0^{2\pi} \left(\frac{\partial \Psi}{\partial \xi} \right) d\eta . \tag{VIII-39}$$

Here Ψ is the dimensional stream function related to the non-dimensional stream function $\bar{\psi}$ by the equation

$$\bar{\psi} = \Psi / Ua . \tag{VIII-40}$$

It may be shown, by using (VIII-32)-(VIII-36) and (VIII-40), that Hasimoto's results for Γ_∞, C_L and C_D are

$$C_L \sim \frac{4\pi}{R} e^{-2\xi_o} (\sin 2\alpha) \left[\ln\left(\frac{R e^{\xi_o}}{16 \cosh \xi_o} \right) \right]^{-2}, \tag{VIII-41}$$

$$C_D \sim - \frac{8\pi}{R} \left[\ln\left(\frac{R e^{\xi_o}}{16 \cosh \xi_o} \right) \right]^{-2}, \tag{VIII-42}$$

$$\Gamma_\infty \sim \frac{8\pi}{R} e^{-3\xi_0} U \sin 2\alpha \left[\ln\left(\frac{R e^{\xi_0}}{16\cosh\xi_0}\right) \right]^{-2} . \qquad \text{(VIII-43)}$$

Comparison of the Results

It follows from equations (3.3.6), (3.3.15) and (3.3.16), in Chapter III, that our results for the circulation, in the counter-clockwise sense, at $\xi = \infty$, K_∞, and the non-dimensional lift and drag coefficients, C_L and C_D, are

$$C_L \sim \frac{2\pi}{R\varepsilon} e^{-3\xi_0} \left[2(\cosh\xi_0)^2 + (\sinh 2\xi_0) \right] \left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) \right]^{-2} , \qquad \text{(VIII-44)}$$

$$C_D \sim -\frac{4\pi}{R\varepsilon} e^{-\xi_0} \sin 2\alpha \left[2(\cosh\xi_0)^2 + (\sinh 2\xi_0) \right] \left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) \right]^{-2} , \qquad \text{(VIII-45)}$$

$$K_\infty \sim -4\pi e^{-2\xi_0} \sin 2\alpha (\cosh\xi_0) [R\varepsilon]^{-1} \left[\ln\left(\frac{R\varepsilon\, e^{\xi_0}}{16\cosh\xi_0}\right) \right]^{-2} , \qquad \text{(VIII-46)}$$

where $R\varepsilon = 2Ud(\cosh\xi_0)/\nu$. (VIII-47)

We note that the formulas

$$C_L = L/(\rho U d) , \qquad C_D = D/(\rho U d) , \qquad \text{(VIII-48)}$$

and also,

$$K_\infty = \lim_{\xi \to \infty} \int_0^{2\pi} \left(\frac{\partial\psi}{\partial\xi} \right) d\eta , \qquad \text{(VIII-49)}$$

are used in obtaining the expressions (VIII-44)-(VIII-46). In (VIII-49) the non-dimensional stream function ψ related to the dimensional stream function ψ' by the equation

$$\psi = \psi'/Ud . \qquad \text{(VIII-50)}$$

Noticing the differences between the formulas (VIII-37)&(VIII-48), (VIII-39)&(VIII-49) and (VIII-40)&(VIII-50) and taking $d = 2e^{-\xi_0}$, it may be shown that Hasimoto's results are in perfect agreement with those of the present analysis as expected. We note that only the leading terms are taken into account for comparison of the results. This is due to the fact that, as stated by Hasimoto, his method is very laborious, and comparison with other results is not easy.

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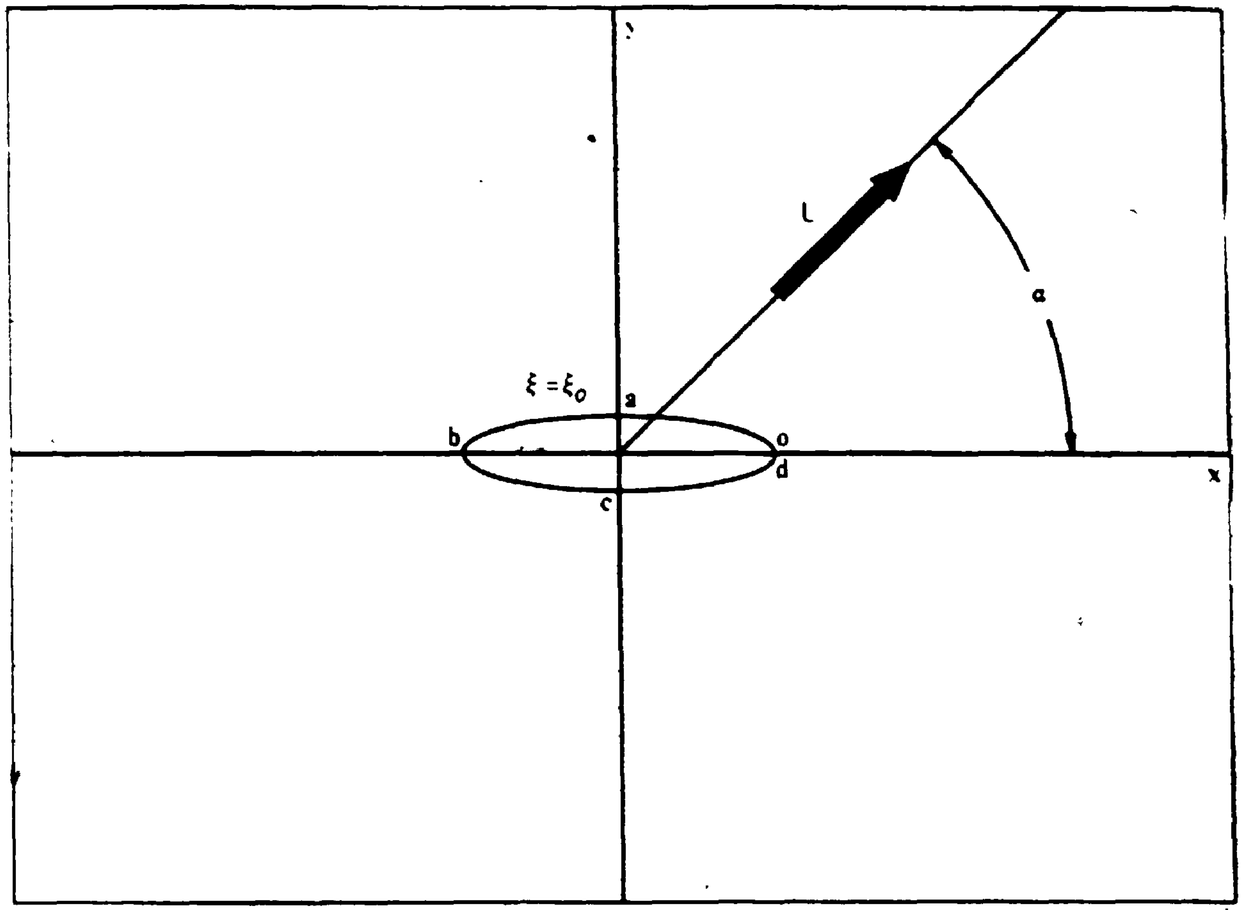


Fig. 1 Orientation of Cartesian axes.

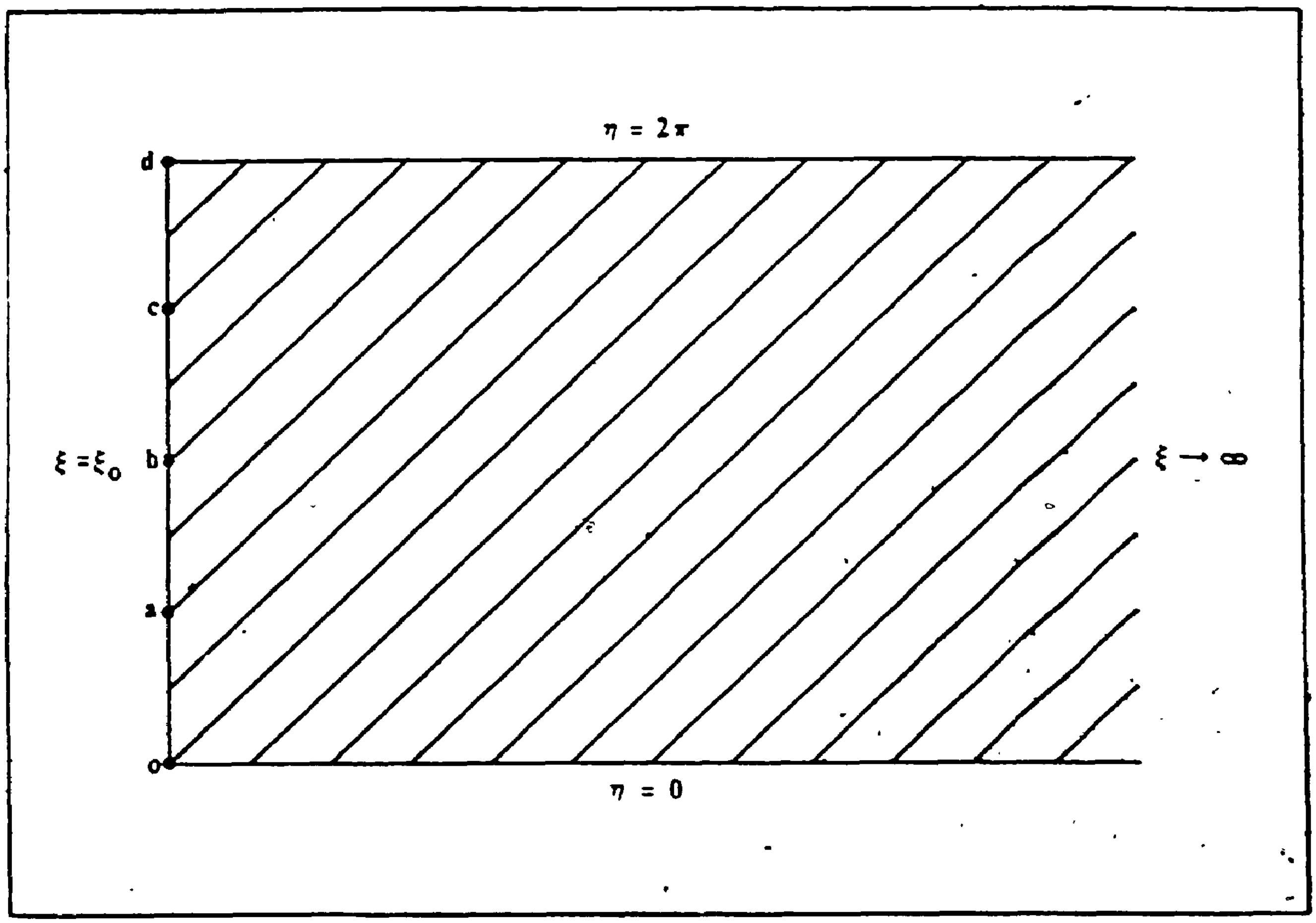


Fig. 2 Transformed domain.

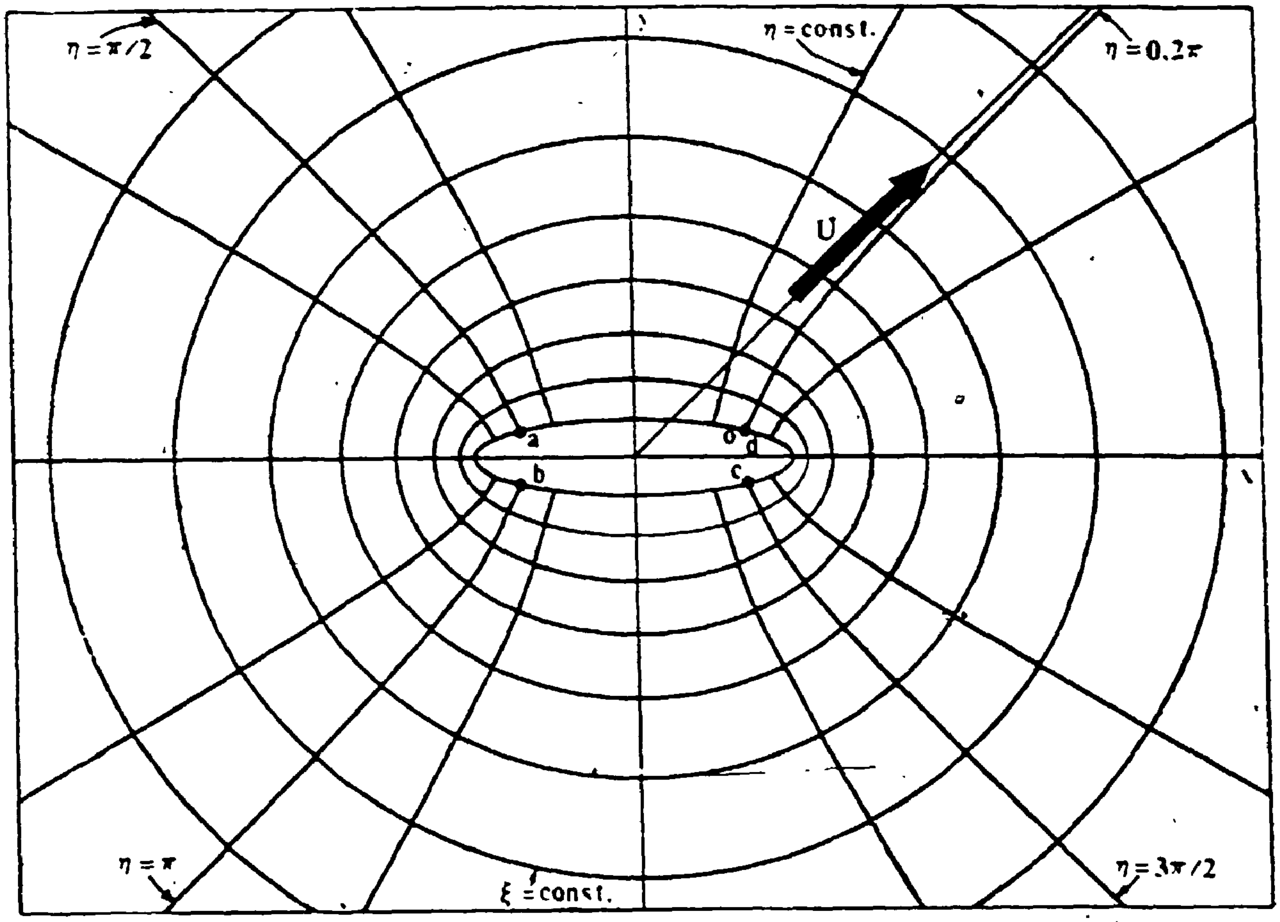
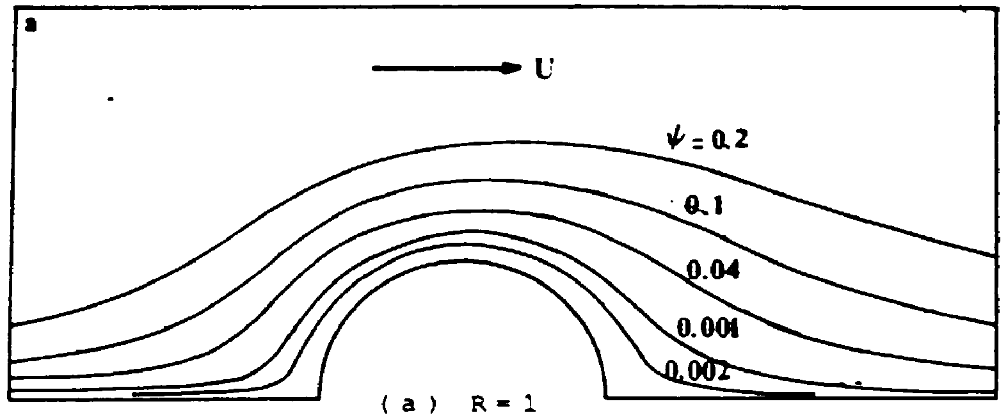
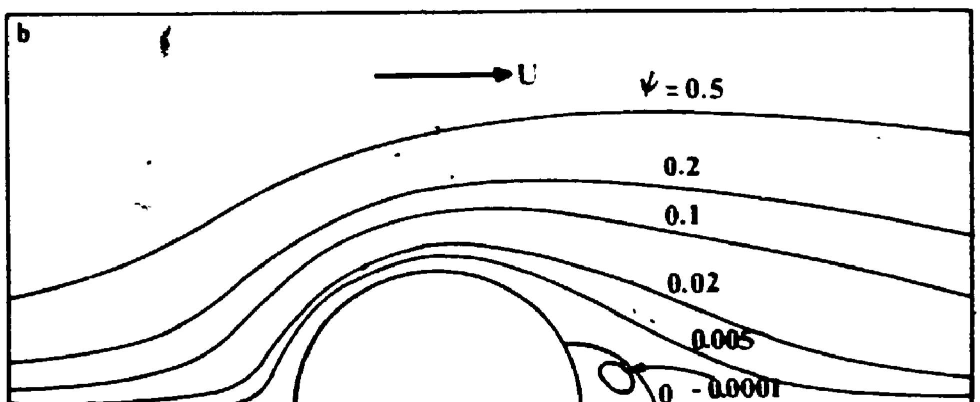


Fig. 3 Elliptic co-ordinates.



(a) R = 1



(b) R = 5

Fig. 4 Streamlines for Oseen flow past a circular cylinder.

(from the paper by Dennis [8] : p.168)