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Localized Algebraic K-theory

Felipe De Zaldivar-cruz

LOCALIZED ALGEBRAIC K-THEORY

by

Felipe de Jesus Zaldivar-Cruz

Department of Mathematics

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ABSTRACT

Let $K_*(A; \mathbf{Z}/\ell^n)$ denote the mod-ℓ^n algebraic K-theory of a $\mathbf{Z}[1/\ell]$-algebra A. V. Snaith has studied Bott-periodic algebraic K-theory $K_i(A; \mathbf{Z}/\ell^n)[1/\beta_n]$, the direct limit of iterated multiplications by β_n, the 'Bott element', using the K-theory product. For ℓ an <u>odd prime</u>, Snaith has given a description of $K_*(A; \mathbf{Z}/\ell^n)[1/\beta_n]$ using Adams maps between Moore spectra. These constructions are interesting, in particular, for their connections with the Lichtenbaum-Quillen conjecture.

In this thesis we obtain an analogous description of $K_*(A; \mathbf{Z}/2^n)[1/\beta_n]$, $n \geq 2$, for an algebra A with $1/2 \in A$ and such that A contains a fourth root of unity. We approach this problem using low dimensional computations of the stable homotopy groups of $BZ/4$, and transfer arguments to show that a power of the mod-4 'Bott element' is induced by an Adams map.

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TABLE OF CONTENTS

CHAPTER 1

ALGEBRAIC K-THEORY

The aim of this chapter is to recall Quillen's definition of the higher algebraic K-theory functors. In the first two sections we review the definition of the algebraic K-theory groups of a ring via the "plus" construction, and some of their properties. In the next two sections we summarize Quillen's generalization of the above groups to exact categories. Details of these constructions can be found in [Lo] and [Q_4].

§1.1: The "plus" construction.

Quillen's definition of the higher algebraic K-theory groups of a ring is based on the following result of homotopy theory:

1.1.1: Theorem:

Let X be a connected CW-complex with base point x_0 and let N be a normal subgroup of $\pi = \pi_1(X, x_0)$ which is perfect i.e. equal to its commutator subgroup. Then, there exists a CW-complex X^+ and a pointed map $i : X \to X^+$ such that:

a) $i_* = \pi_1(i)$ induces an isomorphism $\pi_1(X)/N \xrightarrow{\approx} \pi_1(X^+)$ and $i_* : \pi_1(X) \to \pi_1(X^+)$ corresponds to the canonical morphism $\pi \to \pi/N$.

b) For any system of local coefficients $\mathcal{L}$ on X^+, the map i induces an isomorphism: $i_* : H_*(X; i^*\mathcal{L}) \xrightarrow{\approx} H_*(X^+; \mathcal{L})$.

Recall that a system of local coefficients $\mathcal{L}$ on X^+ is given by a $\pi_1(X^+)$-module L. Thus, the map $i_* : \pi_1(X) \to \pi_1(X^+)$ allows us to consider L as a $\pi_1(X)$-module, and $i^*\mathcal{L}$ denotes this system of local coefficients on X.

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The construction (X^+, i) is universal up to homotopy in the following sense:

<u>1.1.2:</u> <u>Proposition</u>:

With the hypothesis and notation of (1.1.1). If $f : X \to Y$ is a pointed map, Y is connected and $\pi_1(f)(N) = 0$, then there exists a map $\hat{f} : X^+ \to Y$, unique up to homotopy, such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "i"] \arrow[dr, "f"'] & X^+ \arrow[d, dashed, "\hat{f}"] \\ & Y \end{tikzcd}$$

commutes up to homotopy.

///

<u>1.1.3</u>: <u>Corollary</u>:

If $f : X \to Y$ is a map such that $\pi_1(f)(M) \subseteq N$, where M and N are perfect normal subgroups of $\pi_1(X)$ and $\pi_1(Y)$ respectively, and if $i : X \to X^+$ and $j : Y \to Y^+$ are the maps of (1.1.1), then there exists a map $f^+ : X^+ \to Y^+$, unique up to homotopy, such that the following diagram homotopy commutes:

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "i"'] & Y \arrow[d, "i"] \\ X^+ \arrow[r, dashed, "f^+"'] & Y^+ \end{tikzcd}$$

///.

1.1.4: Remark:

Corollary (1.1.3) implies that any two choices of X^+ as in (1.1.1) are naturally homotopy equivalent.

1.1.5: Remark:

Corollary (1.1.3) also implies that the plus construction is functorial up to homotopy, i.e.:

a) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ satisfy the required hypothesis to construct $X^+ \xrightarrow{f^+} Y^+ \xrightarrow{g^+} Z^+$, then $(gf)^+$ exists and $(gf)^+ \simeq g^+f^+$.

b) The identity map $1_X : X \to X$ satisfies $(1_X)^+ \simeq 1_{X^+} : X^+ \to X^+$.

The following property is used for the definition of products in (§1.5).

1.1.6: Proposition:

Suppose X and Y satisfy the hypothesis of (1.1.1). Then, there exists a homotopy equivalence:

$$(X \times Y)^+ \xrightarrow{\simeq} X^+ \times Y^+ .$$

///.

We collect now some examples of the plus construction.

1.1.7: Example: [Barratt-Kahn-Priddy-Quillen]

Let Σ_n be the n-th symmetric group and let A_n be the alternating subgroup of Σ_n. Using the natural inclusions $\Sigma_n \to \Sigma_{n+1}$, define $\Sigma_\infty = \lim\limits_{\to} \Sigma_n$, and similarly $A_\infty = \lim\limits_{\to} A_n$. It is known that $A_\infty = [\Sigma_\infty, \Sigma_\infty]$ and A_∞ is perfect.

Let $B\Sigma_\infty$ be the classifying space of the discrete group Σ_∞. Thus, $N = A_\infty$ is a perfect subgroup of $\pi_1(B\Sigma_\infty) = \Sigma_\infty$ and we can form $B\Sigma_\infty^+$ as in (1.1.1).

Now, if X is a space, let $QX = \lim\limits_{\to} \Omega^n\Sigma^n X$, where ΣX is the suspension of X and ΩY is the loop space of Y.

Barratt, Kahn, Priddy and Quillen, see [P], have proved that $Z \times B\Sigma_\infty^+ \simeq QS^0$, where S^0 is the 0-th sphere.

1.1.8: Example:

Let Z/q be the group of integers mod q, and let $\Sigma_n \int Z/q$ denote the wreath product of the symmetric group Σ_n with Z/q, i.e. the semidirect product of Σ_n with $(Z/q)^n$. It is known that the commutator subgroup $N = [\Sigma_n \int Z/q, \Sigma_n \int Z/q]$ is perfect for $n \geq 5$.

Now, let $\Sigma_\infty \int Z/q = \lim\limits_{\to} \Sigma_n \int Z/q$, and let $B\Sigma_\infty \int Z/q$ be its classifying space. Then, we can form $(B\Sigma_\infty \int Z/q)^+$ as in (1.1.1).

Kahn and Priddy, see [H-S], have proved that

$$(B\Sigma_\infty \int Z/q)^+ \simeq Q_0(BZ/q)_+$$

$$= \text{base-point component of } Q(BZ/q)_+.$$

1.1.9: Example:

Let A be an associative ring with 1, and let GL_nA be the group of $n \times n$ non-singular matrices with coefficients in A. Let E_nA be the subgroup of GL_nA generated by the elementary matrices.

Let $GLA = \varinjlim GL_nA$, where the limit is induced by the natural inclusions $GL_nA \to GL_{n+1}A$. Similarly, let $EA = \varinjlim E_nA$.

By the Whitehead lemma, see for example [Mi;p.25], $EA = [GLA,GLA]$ and EA is perfect. Thus, we may form $BGLA^+$ and this is our most important example because this is the space that Quillen used to define the higher algebraic K-theory groups of the ring A.

§1.2: The higher algebraic K-theory groups of a ring.

Quillen's definition of the higher K-theory groups of a ring seems to have been motivated by certain computations in his work on the Adams conjecture [Q_1] and by the need of having a homotopic interpretation of Milnor's K_2.

1.2.1: Definition:

Let A be an associative ring with 1, and let $BGLA^+$ be the space of (1.1.9). Define $K_nA = \pi_n(BGLA^+)$ for $n \geqslant 1$.

1.2.2: Remarks:

1) From (1.1.1) it follows that $BGLA^+$ is connected and its fundamental group is $\pi_1(BGLA^+) \approx GLA/EA$. This abelian group is the classical algebraic K-theory group $K_1(A)$ of Bass [B].

2) Milnor's definition of $K_2(A)$ is isomorphic to $H_2(EA;Z)$, [Mi]. It can be shown that $\pi_2(BGLA^+) \approx H_2(EA;Z)$ so that Quillen's $K_2(A)$ agrees with Milnor's $K_2(A)$.

1.2.3: Remark:

If $f : A \to B$ is a ring morphism, then f induces a homomorphism $f : GLA \to GLB$ and hence maps $f : BGLA \to BGLB$ and $f^+ : BGLA^+ \to BGLB^+$ (1.1.3). Therefore, f induces a homomorphism $f_* = (f^+)_* : K_nA = \pi_nBGLA^+ \to \pi_nBGLB^+ = K_nB$. It follows that

$$K_n : \mathbf{Rings} \longrightarrow \mathbf{Abelian\ Groups}, \quad n \geqslant 1$$

are covariant functors (see 1.1.5).

The space $BGLA^+$ is a remarkable space. Quillen [Q_2] proved that it is an H-space and Gersten [G] and Wagoner [W] (see also Waldhausen [Wa] and May [Ma]) have proved that in fact it is an infinite loop space. Before stating this result we need some definitions:

<u>1.2.4</u>: <u>Definition</u>:

Let A be a ring with 1. The <u>cone of A</u>, denoted CA, is the ring of matrices over A generated by matrices of the form $P \cdot D$ where P is an infinite permutation matrix and D is an infinite diagonal matrix with entries in a finite subset of A.

Let $mA \subset CA$ be the ideal of finite matrices i.e. matrices with at most finitely many nonzero entries.

Karoubi defines the <u>suspension of A</u>, denoted SA, by $SA = CA/mA$.

<u>1.2.5</u>: <u>Theorem</u> [Gersten-Wagoner]:

The space $\Omega BGL(SA)^+$ is homotopy equivalent to $K_0A \times BGLA^+$. Consequently, $K_iA = K_{i+1}(SA)$ for all $i \geqslant 1$.

<u>1.2.6</u>: <u>Remark</u>:

It follows from the delooping (1.2.5) of $BGLA^+$ that the sequence of spaces:

$$\underline{KA}_n = K_0(S^nA) \times BGL(S^nA)^+ \;, \quad n \geqslant 0$$

is an Ω-spectrum. The Ω-spectrum $\underline{KA} = \{\underline{KA}_n\}$ is called the <u>algebraic</u> <u>K-theory spectrum</u> of the ring A, and its homotopy groups are the algebraic K-theory groups of A:

$$\pi_i(\underline{KA}) = \lim_{\rightarrow} \pi_{i+n}(\underline{KA}_n) = \pi_i BGLA^+ = K_iA$$

since $K_0A \times BGLA^+ \simeq \Omega^n BGL(S^n A)^+$.

§1.3: The Q-construction.

As observed by Quillen $[Q_3]$, $[Q_4]$, it is sometimes necessary for several reasons to work with the K-theory of categories more general than the category of rings. In this section and the next one, we recall Quillen's definition of the higher K-theory groups of an exact category.

1.3.1: Definition:

Let $\underline{C}$ be a small category. The nerve of $\underline{C}$, denoted $\underline{NC}$, is the simplicial set [Ma] given by:

1) Its p-simplices are the diagrams in $\underline{C}$ of the form :

$$(\underline{NC})_p = (X_o \to X_1 \to \ldots \to X_p), \; p \geq 0 .$$

2) The i-th face operator $d_i : (\underline{NC})_p \to (\underline{NC})_{p-1}$ is defined by deleting the object X_i, i.e. :

$$d_i(X_o \to \ldots \xrightarrow{f} X_i \xrightarrow{g} \ldots \to X_p) = (X_o \to \ldots \to X_{i-1} \xrightarrow{gf} X_{i+1} \to \ldots \to X_p)$$

3) The i-th degeneracy operator $s_i : (\underline{NC})_p \to (\underline{NC})_{p+1}$ is defined by replacing X_i by $1 : X_i \to X_i$, i.e.:

$$s_i(X_o \to \ldots \xrightarrow{f} X_i \xrightarrow{g} \ldots \to X_p) = (X_o \to \ldots \xrightarrow{f} X_i \xrightarrow{1} X_i \xrightarrow{g} \ldots \to X_p) .$$

1.3.2: Definition: [Segal]:

The classifiying space of a small category $\underline{C}$, denoted $\underline{BC}$, is the geometric realization [Ma] of the simplicial set $\underline{NC}$. Thus, $\underline{BC}$ is a CW-complex.

1.3.3: Remark:

If $f : \underline{C} \to \underline{C}'$ is a functor between small categories, then f induces a cellular map $Bf : B\underline{C} \to B\underline{C}'$.

With these definitions, we have a functor

B : *category of small categories* → *CW-complexes and cellular maps.*

called the classifying space functor.

Some properties of this functor are the following, see [Q_4]:

1.3.4: Proposition:

Let $\underline{C}$ and $\underline{C}'$ be small categories.

1) The canonical map $B(\underline{C} \times \underline{C}') \to B\underline{C} \times B\underline{C}'$ is a homeomorphism if either $B\underline{C}$ or $B\underline{C}'$ is a finite complex, and also if the product is given the compactly generated topology.

2) A natural transformation $\theta : f \to g$ of functors $f, g : \underline{C} \to \underline{C}'$ induces a homotopy $B\underline{C} \times I \to B\underline{C}'$ between Bf and Bg.

///

It follows from this proposition:

1.3.5: Corollary:

1) If a functor $f : \underline{C} \to \underline{C}'$ has a left adjoint, then $Bf : B\underline{C} \to B\underline{C}'$ is a homotopy equivalence.

2) In particular, a category $\underline{C}$ having either an initial or final object is contractible.

1.3.6: Definition:

An exact category is an additive category $\underline{M}$ provided with a family $\underline{E}$ of preferred short exact sequences $E' \overset{i}{\rightarrowtail} E \overset{j}{\twoheadrightarrow} E''$ such that:

1) Any sequence in $\underline{M}$ isomorphic to a sequence in $\underline{E}$ is in $\underline{E}$. Also, the split sequences of $\underline{M}$ are in $\underline{E}$.

2) The monomorphisms (epimorphisms) of the short exact sequences of $\underline{E}$, called admissible monomorphisms (epimorphisms), satisfy the following conditions:

a) The class of admissible epimorphisms is closed under composition and under base change (pullbacks) by arbitrary maps of $\underline{M}$, i.e. in the diagram

$$\begin{array}{ccccc} E' & \overset{\hat{i}}{\rightarrowtail} & M & \overset{\hat{j}}{\twoheadrightarrow} & M'' \\ \| & & \downarrow & & \downarrow f \\ E' & \underset{i}{\rightarrowtail} & E & \underset{j}{\twoheadrightarrow} & E'' \end{array}$$

if the bottom sequence is in $\underline{E}$, f is an arbitrary map in $\underline{M}$, and the right square is a pullback, then the top row is in $\underline{E}$, i.e. $\hat{j}$ is an admissible epimorphism.

b) Dually, the class of admissible monomorphisms is closed under composition and under cobase change (pushouts) by arbitrary maps of $\underline{M}$.

3) Let $M \to M''$ be a map which has a kernel in $\underline{M}$. If there exists a map $N \to M$ such that $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism.

Dually for admissible monomorphisms.

1.3.7: Examples:

1) Any abelian category $\underline{A}$ is an exact category with $\underline{E}$ the class of all short exact sequences.

2) If A is a Noetherian ring with 1, $\underline{P}_A$ the category of finitely generated projective A-modules is an exact category.

3) Similarly, the category $\underline{\text{Modf}}(A)$ of finitely generated left A-modules is exact.

4) If X is a scheme, $\underline{P}_X$ the category of (algebraic) vector bundles over X (i.e. locally free sheaves of $\mathcal{O}_X$-modules of finite rank), with the usual notion of exact sequence, is exact.

5) Similarly, the category $\underline{M}_X$ of coherent sheaves on a locally Noetherian scheme X is exact (in fact it is abelian).

Quillen's definition of the higher K-theory groups of an exact category is based on the following construction:

1.3.8: Definition: [The Q-construction]:

Let $\underline{M}$ be an exact category. The category $Q\underline{M}$ is formed as follows:

1) Objects:

$Q\underline{M}$ has the same objects as $\underline{M}$.

2) Morphisms:

A morphism $\alpha : M \to M'$ in $Q\underline{M}$ is an equivalence class of diagrams $M \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M'$ where j is an admissible epimorphism and i is an admissible monomorphism, under the equivalence relation given by:

$$(M \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M') \sim (M \overset{\hat{j}}{\twoheadleftarrow} \hat{N} \overset{\hat{i}}{\rightarrowtail} M')$$

iff there exists a commutative diagram:

$$\begin{array}{ccccc} M & \overset{j}{\twoheadleftarrow} & N & \overset{i}{\rightarrowtail} & M' \\ \| & & \approx\downarrow & & \| \\ M & \underset{\hat{j}}{\twoheadleftarrow} & \hat{N} & \underset{\hat{i}}{\rightarrowtail} & M' \end{array}$$

3) Composition:

If $\alpha : M \to M'$ and $\beta : M' \to M''$ are morphisms in $Q\underline{M}$, represented by the diagrams $\alpha : M \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M'$ and $\beta : M' \overset{\hat{j}}{\twoheadleftarrow} \hat{N} \overset{\hat{i}}{\rightarrowtail} M''$, consider the following diagram:

$$\begin{array}{ccccc} P & \overset{a}{\rightarrowtail} & \hat{N} & \overset{\hat{i}}{\rightarrowtail} & M'' \\ b\downarrow & & \downarrow\hat{j} & & \\ N & \underset{i}{\rightarrowtail} & M' & & \\ j\downarrow & & & & \\ M & & & & \end{array}$$

where the square is a pullback. Then, the composition $\beta\circ\alpha : M \to M''$ in $Q\underline{M}$ is the morphism represented by $M \overset{jb}{\twoheadleftarrow} P \overset{\hat{i}a}{\rightarrowtail} M''$.

1.3.9: Remark:

Let $(\underline{M}, \underline{E})$ and $(\underline{M}', \underline{E}')$ be exact categories. If $f : \underline{M} \to \underline{M}'$ is a functor that sends sequences in $\underline{E}$ to sequences in $\underline{E}'$, then f induces a functor $Qf : Q\underline{M} \to Q\underline{M}'$.

Also, if $\underline{M}^{op}$ is the opposite category of $\underline{M}$, then there is an isomorphism of categories $Q\underline{M}^{op} \approx Q\underline{M}$.

§1.4: The higher algebraic K-theory groups of an exact category.

Let $\underline{M}$ be a small category. Then $Q\underline{M}$ is also a small category and so $BQ\underline{M}$ its classifying space is defined (1.3.2). Let 0 be a given zero object of $\underline{M}$. Quillen [Q_4] proved:

1.4.1: Theorem:

The fundamental group $\pi_1(BQ\underline{M},0)$ is canonically isomorphic to the Grothendieck group $K_0\underline{M}$ of $\underline{M}$

///.

Using this theorem as motivation, Quillen [Q_4] proposed the following definition.

1.4.2: Definition:

Let $\underline{M}$ be a small exact category. Then define $K_n\underline{M} = \pi_{n+1}(BQ\underline{M},0)$ for $n \geq 0$.

1.4.3: Remarks:

1) These groups are independent of the choice of the zero object.

2) If $(\underline{M}, \underline{E})$ and $(\underline{M}', \underline{E}')$ are small exact categories and if $f : \underline{M} \to \underline{M}'$ is a functor that sends $\underline{E}$ to $\underline{E}'$, then f induces a map $BQf : BQ\underline{M} \to BQ\underline{M}'$ (1.3.9) and so a homomorphism of K-groups

$$f_* = (BQf)_* : K_i\underline{M} \to K_i\underline{M}' .$$

Thus, the K_i are covariant functors from the category of (small) exact categories and functors preserving (preferred) exact sequences, to the category of abelian groups.

3) Also, from (1.3.9), if $\underline{M}^{op}$ is the opposite category of $\underline{M}$, then $K_i(\underline{M}^{op}) = K_i(\underline{M})$.

4) If M and M' are exact categories, then:

$$K_i(\underline{M} \times \underline{M}') \approx K_i(\underline{M}) \oplus K_i(\underline{M}').$$

Now, let A be a ring with 1 and consider the category $\underline{P}_A$ of finitely generated projective A-modules. This is an exact category (1.3.7) so that we may consider its K-groups $K_i(\underline{P}_A)$. An important result of Quillen [G-Q] is the assertion that the groups $K_i(\underline{P}_A)$ agree with the groups $K_i(A)$ defined using $BGLA^+$ (1.2.1):

<u>1.4.4</u>: <u>Theorem</u>: [Quillen]:

There exists a homotopy equivalence $K_o(A) \times BGLA^+ \simeq \Omega BQ\underline{P}_A$.

Consequently : $K_i(A) = \pi_i(BGLA^+) \approx \pi_{i+1}(BQ\underline{P}_A) = K_i(\underline{P}_A)$ ///

<u>1.4.5</u>: <u>K-theory of schemes</u>:

Let X be a scheme and let $\underline{P}_X$ be the category of algebraic vector bundles over X. $\underline{P}_X$ is an exact category (1.3.7) and so we may consider its K-groups. Quillen [Q_4] defines $K_i(X) = K_i(\underline{P}_X)$.

1.4.6: Remark:

If $f : X \to Y$ is a morphism of schemes, then the inverse image functor $f^* : P_Y \to P_X$ sends exact sequences to exact sequences, and so it induces morphisms of K-groups $f^* : K_i(Y) \to K_i(X)$. Therefore, K_i : *Schemes* $\to$ *Abelian Groups* is a contravariant functor for all $i \geq 0$.

1.4.7: Remark:

If A is a ring with 1, we can consider the scheme $X = \mathrm{Spec}(A)$ the prime spectrum of A. It is well-known that $P_X \approx P_A$. It follows that $K_i(\mathrm{Spec}\ A) \approx K_i(A)$.

The following localization theorem for the K-theory of rings is proved using the general machinery of sections §1.3 and §1.4, and some other results, see $[Q_4]$.

1.4.8: Theorem (Localization):

If A is a Dedekind domain with quotient field F, there is a long exact sequence:

$$\cdots \to K_{n+1}(F) \xrightarrow{\partial} \bigoplus_M K_n(A/M) \xrightarrow{i_*} K_n(A) \xrightarrow{j_*} K_n(F) \to \cdots$$

where M runs over the maximal ideals of A.

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§1.5: Products in K-theory.

If A is a commutative ring with 1, Loday [Lo] has shown that the spectrum $\underline{KA}$ (1.2.6) has a product, so that the graded group $K_*(A) = \bigoplus_n K_n(A)$ is an anticommutative graded ring.

1.5.1: Let A and B be two rings. The tensor product of matrices induces a homomorphism $GL_m A \times GL_n B \longrightarrow GL_{mn}(A \otimes_Z B)$ and a continuous map $BGL_m A^+ \times BGL_n B^+ \longrightarrow BGL_{mn}(A \otimes_Z B)^+$ which is compatible with the stabilization maps $BGL_r(-)^+ \to BGL_{r+1}(-)^+$, so that it induces a map:

$$\hat{\mu} : BGLA^+ \times BGLB^+ \longrightarrow BGL(A \otimes_Z B)^+$$

which is well defined up to homotopy.

Moreover, $\hat{\mu}$ induces a map:

$$\mu : BGLA^+ \wedge BGLB^+ \longrightarrow BGL(A \otimes_Z B)^+$$

which is natural on A and B, bilinear, associative and commutative up to homotopy.

1.5.2: Definition:

If $A = B$, using the map μ of (1.5.1) and the map $\nabla^+ : BGL(A \otimes_Z A)^+ \longrightarrow BGLA^+$ induced by the codiagonal morphism $\nabla : A \otimes_Z A \to A$ given by $\nabla(a \otimes b) = a \cdot b$, Loday [Lo] defines a product

$$* : K_m A \times K_n A \to K_{m+n} A$$

so that $K_* A = \bigoplus_n K_n A$ becomes a graded anticommutative ring if A is commutative.

The product $* : K_1A \times K_1A \to K_2A$ agrees with the one defined by Milnor [Mi].

1.5.3: Remark:

Using the maps

$$\mu : BGL(S^nA)^+ \wedge BGL(S^mB)^+ \to BGL(S^{m+n}A \otimes_{\mathbb{Z}} B)^+$$

of (1.5.1), Loday [Lo;2:42] shows that these maps induce 'naive' (see [Ad_1]) pairing of spectra:

$$\mu : \underline{KA} \wedge \underline{KB} \to \underline{KA}$$

i.e. the algebraic K-theory spectrum $\underline{KA}$ is multiplicative.

CHAPTER 2

HOMOTOPY THEORY WITH COEFFICIENTS

In this chapter we recall the definition of (stable) homotopy groups with coefficients Z/m and some of their properties that we will need in the later chapters. For details we refer to Araki-Toda [A-T], Neisendorfer [N], Oka [O] and Browder [Br_1].

§2.1: Moore spaces and homotopy groups with coefficients.

2.1.1: Moore spaces:

Let $k \geq 2$ and let $m : S^{k-1} \to S^{k-1}$ be a map of degree m for the $k-1$ sphere S^{k-1}. Let $P^k(m) = S^{k-1} \cup_m e^k$ be the cofibre of m. Then, $P^k(m)$ is a Moore space of type $(Z/m, k)$, i.e. it has only one non-zero reduced integral cohomology group $H^k(P^k(m);Z) \approx Z/m$.

2.1.2: Definition: (Homotopy groups with coefficients Z/m):

Let X be a pointed space and let $k \geq 2$. Define:

$\pi_k(X; Z/m) = [\ P^k(m), X\ ]$ = set of based homotopy classes of base-point preserving maps from $P^k(m)$ to X.

2.1.3: Remarks:

1) Since for all $r \geq 0$, $\Sigma^r P^k(m) = P^{k+r}(m)$, where Σ^r is the suspension functor, then for $k \geq 3$, $P^k(m)$ is a co-H-space.

Therefore, if $k \geq 3$ $\pi_k(X;Z/m)$ is a group and if $k \geq 4$, it is abelian.

2) If X is an H-space then $\pi_2(X;Z/m)$ is a group and $\pi_3(X;Z/m)$ is abelian.

3) From the adjunction of the loop functor Ω^t and the suspension functor Σ^t, it follows that if $X = \Omega^t Y$ then $\pi_k(X;Z/m) \approx \pi_{k+t}(Y;Z/m)$.

2.1.4: Remark:

Let $f : X \to Y$ be a map. Then, there are induced functions $f_* : \pi_k(X;Z/m) \to \pi_k(Y;Z/m)$ defined by $f_*[\alpha] = [f\alpha]$ for $[\alpha] \in \pi_k(X;Z/m)$. These functions satisfy:

1) If $k \geq 3$, f_* is a homomorphism.

2) If $k = 2$, f_* is compatible with the actions of $\pi_2(X)$ on $\pi_2(X;Z/m)$ and of $\pi_2(Y)$ on $\pi_2(Y;Z/m)$. These actions are defined as follows:

From the cofibre sequence $S^1 \xrightarrow{m} S^1 \xrightarrow{i} P^2(m) \xrightarrow{j} \Sigma S^1 \approx S^2$ we obtain a map $\phi : P^2(m) \xrightarrow{j} S^2 \subset S^2 \vee P^2(m)$, and the action of $\pi_2(X) = [S^2, X]$ on $\pi_2(X;Z/m) = [P^2(m), X]$ is given by:

Let $[\alpha] \in \pi_2(X)$ and $[\beta] \in \pi_2(X;Z/m)$, then define:

$$[\alpha] * [\beta] = [P^2(m) \xrightarrow{\phi} S^2 \vee P^2(m) \xrightarrow{\alpha\vee\beta} X \vee X \xrightarrow{\nabla} X]$$

where $\nabla : X \vee X \longrightarrow X$ is the folding map $\nabla(x,*) = x = \nabla(*,x)$.

Similarly for Y.

One good sign that homotopy groups with coefficients as defined above are well-behaved is the following result of Peterson:

2.1.5: Proposition: [Universal Coefficient Theorem]:

If $k \geq 2$ and $m \geq 2$, there is a functorial exact sequence:

$$0 \to \pi_k(X)\otimes Z/m \xrightarrow{\rho} \pi_k(X;Z/m) \xrightarrow{\beta} \mathrm{Tor}_1^Z(\pi_{k-1}(X),\ Z/m) \to 0$$

///.

2.1.6: Remarks:

1) If $k \geq 3$ this is a sequence of groups and homomorphisms.

2) If $k = 2$ the sequence must be interpreted as a sequence of pointed sets and $\mathrm{Tor}_Z^1(\pi_1(X),\ Z/m)$ is interpreted as the kernel of $\pi_1(X) \xrightarrow{m} \pi_1(X)$, where $m(\alpha) = \alpha^m$.

3) The Universal Coefficient Sequence of (2.1.5) is obtained from the Barrat-Puppe sequence induced by the cofibre sequence:

$$(*) \qquad S^{k-1} \xrightarrow{m} S^{k-1} \xrightarrow{i} P^k(m) \xrightarrow{j} S^k \xrightarrow{m} S^k \longrightarrow \cdots$$

by applying the functor $[-,\ X]$, i.e. from the sequence:

$$(**) \qquad \cdots \longrightarrow \pi_k(X) \xrightarrow{m} \pi_k(X) \xrightarrow{\rho} \pi_k(X;Z/m) \xrightarrow{\beta} \pi_{k-1}(X) \xrightarrow{m} \cdots$$

where ρ, the map induced by j in (*) is called the mod-m reduction map, and β, the map induced by i in (*) is called the mod-m Bockstein map.

2.1.7: Proposition:

If $k \geq 3$ and $m = r\cdot s$ with r, s relatively prime, then there exists an isomorphism:

$$\pi_k(X;Z/m) \xrightarrow{\approx} \pi_k(X;Z/r) \times \pi_k(X;Z/s).$$

Hence, in order to compute $\pi_k(X;Z/m)$ it is enough to compute $\pi_k(X;Z/\ell^s)$ for all primes ℓ such that $\ell^s \mid m$ but $\ell^{s+1} \nmid m$.

///.

As for the classical long exact sequence of homotopy groups associated to a fibration of spaces, we have the corresponding result for homotopy groups with coefficients:

2.1.8: Proposition:

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Serre fibration. Then, there exists a functorial exact sequence:

$$\cdots \xrightarrow{p_*} \pi_{k+1}(B;Z/m) \xrightarrow{\partial} \pi_k(F;Z/m) \xrightarrow{i_*} \pi_k(E;Z/m) \xrightarrow{p_*} \pi_k(B;Z/m) \xrightarrow{\partial}$$
$$\cdots \longrightarrow \pi_2(B;Z/m) .$$

///.

2.1.9: Mod-m Hurewicz maps:

Consider the sequence of cofibrations $S^{k-1} \xrightarrow{m} S^{k-1} \xrightarrow{i} P^k(m) \xrightarrow{j} S^k$. Then:

$$j_* : H_k(P^k(m);Z/m) \longrightarrow H_k(S^k;Z/m) \approx Z/m$$

is an isomorphism. Let $e \in H_k(P^k(m);\, Z/m)$ be the generator such that $j_*(e) = \rho(\iota)$ where $\rho : H_k(S^k;\, Z) \longrightarrow H_k(S^k;\, Z/m)$ is the mod-m reduction map, and $\iota \in H_k(S^k;\, Z) \approx Z$ is the canonical generator.

2.1.10: Definition:

Let X be a space and $k \geq 2$. The mod-m Hurewicz map is the map: $h : \pi_k(X;\, Z/m) \longrightarrow H_k(X;\, Z/m)$ given by: $h[f] = f_*(e)$ for $f : P^k(m) \longrightarrow X$ a representative of $[f]$.

2.1.11: Remarks:

1) h is a well-defined map, i.e. it does not depend on the representative f of $[f]$.

2) $h : \pi_k(--;Z/m) \to H_k(--;Z/m)$ is a natural transformation.

3) The mod-m Hurewicz and the classical Hurewicz maps commute with the mod-m reduction and Bockstein maps, i.e. the following diagrams commute:

$$\begin{array}{ccc} \pi_k(X) & \xrightarrow{h} & H_k(X; Z) \\ \rho\downarrow & & \downarrow\rho \\ \pi_k(X; Z/m) & \xrightarrow[h]{} & H_k(X; Z/m) \end{array} \qquad \begin{array}{ccc} \pi_k(X; Z/m) & \xrightarrow{h} & H_k(X; Z/m) \\ \beta\downarrow & & \downarrow\beta \\ \pi_k(X) & \xrightarrow[h]{} & H_k(X; Z) \end{array}$$

4) If $k \geq 3$ $h : \pi_k(X; Z/m) \to H_k(X; Z/m)$ is a homomorphism.

5) If $k = 2$ h is compatible with the action of $\pi_2(X)$ on $\pi_2(X;Z/m)$ (see 2.1.4(2)).

§2.2: Moore spectra and stable homotopy groups with coefficients.

Much of what has been said about Moore spaces and homotopy groups with coefficients can be translated into the language of spectra and stable homotopy groups. This is done as follows:

2.2.1: The Moore spectrum:

Let $m \geq 2$ and consider the sphere spectrum $\underline{\Sigma^\infty S^0}$ i.e. the suspension spectrum of the 0-th sphere S^0. The mod-m Moore spectrum, denoted $\underline{P(m)}$, is the cofibre in the stable homotopy category [Ad_1] of the map $m : \underline{\Sigma^\infty S^0} \to \underline{\Sigma^\infty S^0}$ given by suspension of a map $S^1 \to S^1$ of degree m.

Thus, we have a cofibre (=fibre) sequence in the stable homotopy category: $\underline{\Sigma^\infty S^0} \xrightarrow{m} \underline{\Sigma^\infty S^0} \xrightarrow{i} \underline{P(m)}$,

Of course, the Moore spectrum $\underline{P(m)}$ is the suspension spectrum of the Moore space $P^2(m) = S^1 \cup_m e^2$.

2.2.2: Definition:

Let X be a spectrum. The (stable) homotopy groups with coefficients Z/m of X are:

$$\pi_k(X; Z/m) = \pi_k(X \wedge \underline{P(m)})$$

where $X \wedge \underline{P(m)}$ is the smash product of X with $\underline{P(m)}$ and for a spectrum $E = (E_n)$, $\pi_k(E) = \varinjlim \pi_{k+n}(E_n)$, see [$Ad_1$].

2.2.3: Remark:

Since Moore spectra are self S-dual, then for any spectrum X we have:

$\pi_k(X;\ Z/m) = \pi_k(X\quad \underline{P(m)}) \approx [\ \Sigma^k\underline{P(m)},\ X\ ]$ = group of (stable) homotopy classes from $\Sigma^k\underline{P(m)}$ to X.

2.2.4: Remark:

Most of the properties of $\pi_k(\text{—};\ Z/m)$ in §2.1 hold for the stable groups. Thus, these are functors from spectra to abelian groups; there is a (stable) mod-m Hurewicz map; the universal coefficient sequence:

$$0 \to \pi_k(X) \otimes Z/m \xrightarrow{\rho} \pi_k(X;\ Z/m) \xrightarrow{\beta} \mathrm{Tor}_Z^1(\pi_{k-1}(X),\ Z/m) \to 0$$

splits if m is odd or if $4 | m$; also, as in (2.1.7) since the Moore spectrum $\underline{P(m)}$ splits as a wedge of $\underline{P(\ell^n)}$ for the primes ℓ such that $\ell^n | m$ and $\ell^{n+1} \nmid m$, we may restrict m to a prime power ℓ^n .

§2.3: Copairings of Moore spectra and products.

In this section we recall the construction of products in the mod-m homotopy groups of a spectrum. Some of these results also hold for the unstable homotopy groups with a few restrictions, when necessary we shall remark these. Most of these constructions are due to, and can be found in, Araki-Toda [A-T], Toda [T_1], Oka [O], Neisendorfer [N] and Barratt [B].

2.3.1: Theorem:

Let $m \geq 2$ be an integer. The order of the identity map $1_{\underline{P}(m)}: \underline{P}(m) \to \underline{P}(m)$ is m when $m \neq 2 (\mathrm{mod}\ 4)$ and $2m$ when $m = 2(\mathrm{mod}\ 4)$.

Moreover, $m \cdot 1_{\underline{P}(m)} = 0$ is equivalent to the existence of a pairing:

$$\chi : \underline{P}(m) \wedge \underline{P}(m) \longrightarrow \underline{P}(m)$$

such that if $i : \underline{\Sigma}^\infty S^0 \to \underline{P}(m)$ is the inclusion into the bottom cell (see 2.2.1), then the following diagram commutes (in the stable homotopy category):

$$\begin{tikzcd} \underline{\Sigma}^\infty S^0 \wedge \underline{P}(m) \arrow[r, "i\wedge 1"] \arrow[dr, "\simeq"'] & \underline{P}(m) \wedge \underline{P}(m) \arrow[d, "\chi"] & \underline{P}(m) \wedge \underline{\Sigma}^\infty S^0 \arrow[l, "1\wedge i"'] \arrow[dl, "\simeq"] \\ & \underline{P}(m) & \end{tikzcd}$$

i.e. i is a unit for χ :

///.

2.3.2: Remark:

By an S-duality argument, the existence of the pairing $\chi : \underline{P}(m) \wedge \underline{P}(m) \to \underline{P}(m)$ is equivalent to the existence of a co-pairing

$\chi : \underline{P(m)} \to \underline{P(m)} \wedge \underline{P(m)}$ with analogous properties. This alternative approach will be useful sometimes (see also 2.2.3) in Chapters 3 and 5, and appears in, e.g., [N] and [Br_1].

2.3.3: Remark:

For the unstable case, Theorem (2.3.1) becomes (see [N]):

If $m \neq 2 \pmod 4$, then, there exists a copairing map $\chi_{i,j} : P^{i+j}(m) \to P^i(m) \wedge P^j(m)$ inducing an injection on $H_*(--;Z/m)$ so that if $i : P^{i+j-1}(m) \longrightarrow P^i(m) \wedge P^j(m)$ is any map which induces an isomorphism in integral homology in dimensions $\leq i+j-2$, then $\chi_{i,j} \vee i : P^{i+j}(m) \vee P^{i+j-1}(m) \longrightarrow P^i(m) \wedge P^j(m)$ is a homotopy equivalence.

2.3.4: Remark:

Let $m \neq 2 \pmod 4$. Then, for any spectrum X (or space X), $\pi_k(X; Z/m)$ is a Z/m-module (in the case of a space, if the group is abelian, see 2.1.3).

In general it is a $Z/2m$-module.

S.Oka [O] has obtained the following refinement of previous results of Araki-Toda [A-T], Barratt [B] and Neisendorfer [N]:

2.3.5: Theorem:

1) $\underline{P(m)}$ has pairing χ as in (2.3.1) if and only if $m \neq 2 \pmod 4$ and the number of (homotopy classes of) these pairings is 4 when

$m \equiv 0 \pmod 4$ and 1 when m is odd.

2) $\underline{P}(m)$ has a homotopy commutative pairing χ, i.e. the following diagram homotopy commutes

$$\begin{tikzcd}
\underline{P}(m) \wedge \underline{P}(m) \arrow[rr, "T"] \arrow[dr, "\chi"'] & & \underline{P}(m) \wedge \underline{P}(m) \arrow[dl, "\chi"] \\
& \underline{P}(m) &
\end{tikzcd}$$

where T denotes the twist map $T(x \wedge y) = y \wedge x$, if and only if m is odd or $m \equiv 0 \pmod 8$.

3) $\underline{P}(m)$ has a homotopy associative pairing χ, i.e. the following diagram homotopy commutes

$$\begin{tikzcd}
\underline{P}(m) \wedge \underline{P}(m) \wedge \underline{P}(m) \arrow[r, "\chi \wedge 1"] \arrow[d, "1 \wedge \chi"'] & \underline{P}(m) \wedge \underline{P}(m) \arrow[d, "\chi"] \\
\underline{P}(m) \wedge \underline{P}(m) \arrow[r, "\chi"'] & \underline{P}(m)
\end{tikzcd}$$

if and only if $m \not\equiv 2 \pmod 4$ and $m \not\equiv \pm 3 \pmod 9$.

///

2.3.6. Definition:

Let X be a spectrum. Let $\iota : \underline{P}(m) \to \underline{P}(m) \wedge \underline{P}(m)$ be the copairing of (2.3.2) for $m \not\equiv 2 \pmod 4$. Recall that $\pi_k(X; Z/m) = [\underline{P}(m), X]_k$ = group of stable homotopy classes of maps of degree k, see [Ad_1]. The (natural) product:

$$\mu : \pi_i(X; Z/m) \times \pi_j(X; Z/m) \to \pi_{i+j}(X; Z/m)$$

is defined as follows:

Let $[f] \in \pi_i(X;\mathbf{Z}/m) = [\underline{P}(m),X]_i$ and $[g] \in \pi_j(X;\mathbf{Z}/m) = [\underline{P}(m),X]_j$. Let $\mu : X \wedge X \longrightarrow X$ be the natural map. Then, the composite $\underline{P}(m) \xrightarrow{\chi} \underline{P}(m) \wedge \underline{P}(m) \xrightarrow{f \wedge g} X \wedge X \xrightarrow{\mu} X$, is a map of degree $i+j$, and so we define:

$$\mu([f],[g]) = [\mu \circ (f \wedge g) \circ \chi] \in \pi_{i+j}(X;\ \mathbf{Z}/m).$$

2.3.7: **Remarks:**

1) This product is well-defined, i.e. it does not depend on the choice of the representatives of [f] or [g].

2) The product is associative or commutative for the cases listed in (2.3.5)

§2.4: Homotopy Bockstein spectral sequence.

In this section we recall the construction of the mod-m Bockstein spectral sequence for homotopy groups. For details see Neisendorfer [N;§4,§5,§12] or Browder [Br_1;§5].

We also recall the definition of products in this spectral sequence and some of their properties [N;§12] and Oka [O].

2.4.1: Construction:

Let X be a spectrum. From the cofibre sequence

$$(*):\quad \underline{\Sigma^\infty S^0} \xrightarrow{m} \underline{\Sigma^\infty S^0} \xrightarrow{i} \underline{P(m)} \xrightarrow{j} \Sigma(\Sigma^\infty S^0) \longrightarrow \cdots$$

we obtain, as for the Universal Coefficient Theorem (2.2.4) an exact couple:

$$\begin{tikzcd} \pi_*(X) \arrow[rr, "m"] & & \pi_*(X) \arrow[dl, "\rho"] \\ & \pi_*(X;Z/m) \arrow[ul, "\beta"] & \end{tikzcd}$$

where m is the map induced by m in (*), ρ the mod-m reduction map is induced by j in (*), and β the mod-m Bockstein map is induced by i in (*).

The spectral sequence obtained by deriving this exact couple is called the mod-m (stable) homotopy Bockstein spectral sequence, and is denoted by $E_\pi^r(X)$ with differentials $\beta^r : E_\pi^r(X)_q \longrightarrow E_\pi^r(X)_{q-1}$ called Bockstein differentials.

Thus, $E_\pi^1(X)_q = \pi_q(X;\ Z/m)$ and $\beta^1 = \rho\circ\beta$.

For the unstable case, we assume X to be a loop space, i.e. $X = \Omega Y$ for some space Y, and we proceed as in the stable case above.

2.4.2: Proposition:

For either the stable and unstable cases, the Hurewicz maps (2.1.10, 2.2.4) induce a natural morphism of spectral sequences from the mod-m homotopy Bockstein spectral sequence defined above to the mod-m homology Bockstein spectral sequence of Browder $[Br_2]$.

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For the study of products in the homotopy Bockstein spectral sequence, the following result, see Oka [O], is important:

First, some notation: From the cofibre sequence:

$$(*): \quad \underline{\Sigma^\infty S^0} \xrightarrow{m} \underline{\Sigma^\infty S^0} \xrightarrow{i} \underline{P(m)} \xrightarrow{j} \Sigma(\underline{\Sigma^\infty S^0}) \xrightarrow{m} \Sigma(\underline{\Sigma^\infty S^0}) \xrightarrow{i} \Sigma(\underline{P(m)})$$

Let $\bar{\beta} = \underline{P(m)} \xrightarrow{j} \Sigma(\underline{\Sigma^\infty S^0}) \xrightarrow{i} \Sigma(\underline{P(m)})$

2.4.3: Proposition: [Oka, Theorem 2(b)]:

Suppose $m \neq 2 \pmod 4$. Then, $\underline{P(m)}$ has a multiplication χ (2.3.5) which satisfies:

$$\chi \circ (\bar{\beta} \wedge 1_{\underline{P(m)}}) + \chi \circ (1_{\underline{P(m)}} \wedge \bar{\beta}) = \bar{\beta} \circ \chi$$

in $[\underline{P(m)} \wedge \underline{P(m)}, \underline{P(m)}]$.

///.

This result means that $\bar{\beta}$ behaves as a derivation, i.e.:

2.4.4: Proposition:

Let X be a spectrum. For $m \neq 2 \pmod 4$, in the spectral sequence $(E^r_\pi(X), \beta^r)$ of (2.4.1), the Bockstein differentials are derivations, i.e. for the product $\mu : E_\pi(X) \otimes E_\pi(X) \longrightarrow E_\pi(X)$ induced by χ, we have:

$$\beta^r(\mu(x_1 \otimes x_2)) = \mu(\beta^r(x_1) \otimes x_2 + (-1)^{\deg(x_1)} \mu(x_1 \otimes \beta^r(x_2))).$$

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Compare this with Neisendorfer [N;§12].

CHAPTER 3

LOCALIZED ALGEBRAIC K-THEORY

The aim of this chapter is to review some results of Snaith [Sn_3] from his work on the Lichtenbaum-Quillen conjecture. These constructions and results form the general background and motivation for ours in Chapters 5.

§3.1: Localized Algebraic K-Theory and the Lichtenbaum-Quillen conjecture.

In this section we recall the construction of a localized version of algebraic K-theory due to Snaith [Sn_1;§9.1], [Sn_2], [Sn_3], which has many important properties, the most important being the fact that it satisfies the Lichtenbaum-Quillen conjecture by Thomason's main theorem [Th_1], see also [DFST]. This localization is performed by inverting a certain element of the mod ℓ^n algebraic K-theory of the given (suitable) ring (scheme). By analogy with the topological K-theory case and by the fact that this element maps to the generator of $\pi_*(\underline{BU};Z/\ell^n)$, where $\underline{BU}$ denotes the spectrum of topological (complex) K-theory [Sn_1;9.1.1], this element in mod ℓ^n algebraic K-theory is called a "Bott-element".

We now proceed to recall these constructions. For details we refer to [DFST].

3.1.1: Algebraic K-theory with coefficients:

Let ℓ be a prime integer. For A a ring (or scheme) let $\underline{K}A$ denote the algebraic K-theory spectrum of A (see May [Ma]). For $n \geq 1$, the mod ℓ^n algebraic K-theory groups of A are $K_*(A;Z/\ell^n) = \pi_*(\underline{K}A;Z/\ell^n)$.

3.1.2: Now, for ℓ^n a prime power, let $\zeta = \zeta_{\ell^n} = \exp(2\pi i/\ell^n)$ be a primitive ℓ^n-th root of unity . Assume $n \geq 2$ if $\ell = 2$. Let $R = Z[1/\ell]$ be the ring of rational integers localized away from ℓ, and let $A = Z[1/\ell,\zeta]$ be the ring obtained from R by adjoining ζ. Thus, the group of units of A , GL_1A , contains the group $\mu = \mu_{\ell^n}$ of ℓ^n-th roots of unity (generated by ζ).

3.1.3: Bott elements:

The inclusion $\mu \to GL_1A$ induces morphisms:

$$\Sigma_r \int \mu \to \Sigma_r \int GL_1A \to GL_rA$$

which induces maps of (topological) spectra

$$\gamma : \ \underline{\Sigma}^\infty(B\mu)_+ \to \underline{\Sigma}^\infty(BGL_1A) \to \underline{K}A$$

Now, let $\zeta \in \mu \approx \pi_1(B\mu)$. From the exact sequence (2.1.6):

$$0 \to \pi_2(B\mu;Z/\ell^n) \xrightarrow{\partial} \pi_1(B\mu) \xrightarrow{\ell^n} \pi_1(B\mu)$$

and since $\ell^n_*(\zeta) = (\zeta)^{\ell^n} = 1$, we see that there exists a unique element b in $\pi_2(B\mu;Z/\ell^n)$ which maps under the Bockstein map ∂ to ζ.

This element b goes under the natural stabilization map to an element, also denoted by b, in $\pi_2^s(B\mu;Z/\ell^n) = \pi_2(\underline{\Sigma}^\infty(B\mu);Z/\ell^n)$.

Under the map γ above, b goes to an element β in $\pi_2(\underline{KA};Z/\ell^n) = K_2(A;Z/\ell^n)$;this elements is called the <u>mod ℓ^n Bott element</u>.

<u>3.1.4</u>: <u>Remark</u>:

For ℓ <u>odd</u> take $n = 1$ and consider the $(\ell - 1)$ cup power $\beta_1 = \beta^{\ell-1} \in K_{2(\ell-1)}(A;Z/\ell)$ of the mod ℓ element β of (3.1.3). This element is also called a <u>Bott element</u>.

<u>3.1.5</u>: <u>Lemma</u> [DFST]:

Let ℓ be an <u>odd</u> prime. Let β_1 as in (3.1.4). For $n > 1$, the ℓ^{n-1} cup power of β_1 in $K_*(A;Z/\ell)$ is the reduction mod ℓ of an element β_n in $K_*(A;Z/\ell^n)$.

Moreover, β_n can be chosen to be invariant under the action of the Galois group $G = Gal(Q(\zeta)/Q)$ on $K_*(A;Z/\ell^n)$.

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The following lemma shows how these Bott elements β_n can be traced back to $K_*(Z[1/\ell];Z/\ell^n)$:

<u>3.1.6</u>:<u>Lemma</u>[DFST]:

For each $n \geqslant 1$, the element β_n in $K_*(A;Z/\ell^n)$ is the image, under the natural map induced by the inclusion $j : Z[1/\ell] \to Z[1/\ell,\zeta] = A$, of a unique element, denoted by $\bar{\beta}_n$, in $K_*(Z[1/\ell];Z/\ell^n)$.

///.

3.1.7: Summarizing the contents of the previous two lemmas, the Bott elements $\bar{\beta}_n$ in $K_*(Z[1/\ell];Z/\ell^n)$ are characterized as follows:

i) $\bar{\beta}_1$ in $K_{2(\ell-1)}(Z[1/\ell];Z/\ell)$ is such that $j_\#(\bar{\beta}_1) = \beta_1 = \beta^{\ell-1}$.

ii) For $n > 1$, $\bar{\beta}_n$ in $K_{2(\ell-1)\ell^{n-1}}(Z[1/\ell];Z/\ell^n)$ is such that

$r_\#(\bar{\beta}_n) = (\bar{\beta}_1)^{\ell^{n-1}}$, where $r_\# : K_*(\text{—};Z/\ell^n) \to K_*(\text{—};Z/\ell)$ is the mod ℓ reduction map.

3.1.8: Remark:

Although until now we have only been concerned with the particular rings $Z[1/\ell]$ and $Z[1/\ell,\zeta]$, the previous constructions can be generalized to algebras (or schemes) over these rings, as follows:

If X is an algebra over $Z[1/\ell]$, the action $Z[1/\ell] \times X \to X$ induces a map $BGLZ[1/\ell]^+ \times BGLX^+ \to BGLX^+$, which yields a natural pairing:

$$K_p(Z[1/\ell];Z/\ell^n) \otimes K_q(X;Z/\ell^n) \to K_{p+q}(X;Z/\ell^n)$$

in particular, multiplication by $\beta_n \in K_d(Z[1/\ell];Z/\ell^n)$ induces morphisms

$$K_i(X;Z/\ell^n) \xrightarrow{\beta_n} K_{i+d}(X;Z/\ell^n)$$

where $d = \deg(\beta_n) = 2(\ell-1)\ell^{n-1}$, and so we define

$$K_i(X;Z/\ell^n)[1/\beta_n] = \varinjlim \left(K_i(X;Z/\ell^n) \xrightarrow{\beta_n} K_{i+d}(X;Z/\ell^n) \right)$$

the limit of iterated multiplications by β_n.

If X is a scheme over $Z[1/\ell]$, then the structure map $X \to \operatorname{Spec} Z[1/\ell]$ induces $K_*(Z[1/\ell];Z/\ell^n) \to K_*(X;Z/\ell^n)$ so that.

$K_*(X;Z/\ell^n)$ is an algebra over $K_*(Z[1/\ell];Z/\ell^n)$ and we proceed as above to obtain the localized algebraic K-theory of X.

Notice that for $d = \deg(\beta_n)$, $K_i(X;Z/\ell^n)[1/\beta_n] \approx K_{i+d}(X;Z/\ell^n)[1/\beta_n]$ i.e. $K_*(X;Z/\ell^n)[1/\beta_n]$ is periodic of period d. This is called the <u>Bott-periodic algebraic K-theory</u> of X and is denoted $\mathcal{K}_*(X;Z/\ell^n)$ by Snaith [Sn_5].

3.1.9: <u>The Lichtenbaum-Quillen conjecture</u>:

Consider the canonical localization map $\rho : K_i(X; Z/\ell^n) \to K_i(X; Z/\ell^n)[1/\beta_n]$ for X a scheme over $Z[1/\ell]$.

From the work of Friedlander [F_1], [F_2], Dwyer-Friedlander [D-F], Dwyer-Friedlander-Snaith-Thomason [D-F-S-T] and Thomason [Th_1], [Th_2], the Lichtenbaum-Quillen conjecture [Q_3], [L], [So], for a scheme (ring) X over $Z[1/\ell]$ having suitable étale cohomological properties [D-F-S-T], can be reformulated as the assertion that the localization map

$$\rho : K_i(X; Z/\ell^n) \to K_i(X; Z/\ell^n)[1/\beta_n]$$

is an isomorphism for large i.

This conjecture is true for $X = F_q$ a finite field (or its algebraic closure) by Quillen's computation [Q_2] of the algebraic K-theory of finite fields.

Suslin [Su_1], [Su_2] has proved the conjecture for algebraically closed fields, local fields and the real numbers.

Jardine [J] has also, independently, obtained an elegant proof of this conjecture for algebraically closed fields.

Snaith [Sn3], [Sn4] has reduced the conjecture to the study of the kernel of $\rho : K_i(X; Z/\ell^n) \to K_i(X; Z/\ell^n)[1/\beta_n]$ when $i = 2$, giving a characterization of $\mathrm{Ker}(\rho)$ for $i = 2$ when $n = 1$ in [Sn3] and for all $n \geq 1$ in [Sn4]. We will elaborate on this matter on the subsequent sections.

§3.2: Odd-primary Bott elements and Adams maps.

In this section we review Snaith's main results in [Sn_3] which essentially provide a canonical way to localize mod ℓ^n algebraic K-theory. Snaith proves in [Sn_3] that a certain power of the Bott element β_n in $K_*(X; \mathbb{Z}/\ell^n)$, for X an algebra (scheme) over $Z[1/\ell]$, is induced by an Adams map $A_n : \underline{P(\ell^n)} \to \underline{P(\ell^n)}$ of degree $\deg(A_n) = 2(\ell-1)\ell^{n-1}$ between mod-ℓ^n Moore spectra. Recall that by definition an Adams map between Moore spectra is a map which induces an isomorphism on topological K-theory [Ad_2], [C-K].

We begin this section by recalling the definition and properties of the Adams maps. See [Ad_2] and [C-K] for more details.

3.2.1: Adams maps betwen Moore spectra: Let ℓ be an odd prime and $n \geq 1$.

Let $\underline{P(\ell^n)}$ be the mod-ℓ^n Moore spectrum (2.2.1). We can consider $\underline{P(\ell^n)}$ as the suspension spectrum of the Moore space $P^{2q}(\ell^n) = S^{2q-1} \cup_{\ell^n} e^{2q}$ (2.1.1). We know:

$$\overline{KU}(P^{2q}(\ell^n)) \approx Z/\ell^n$$

(and if q is even we also have $\overline{KO}(P^{2q}(\ell^n)) \approx Z/\ell^n$). Here, $\overline{KU}$ (respectively, $\overline{KO}$) denotes the reduced complex (real) topological K-theory functor.

Adams [Ad_1;1.7 & 12.1] has constructed stable (for q large enough) maps:

$$A_n : \Sigma^d P^{2q}(\ell^n) \longrightarrow P^{2q}(\ell^n)$$

for $d = 2(\ell-1)\ell^{n-1}$, which induce isomorphisms in KU-theory (respectively, KO-theory). We call these maps, Adams maps.

In addition, [Ad$_1$;12.5; 12.4], the Adams maps A_n, are such that, in the homotopy commutative diagram (for q large enough , and $d = 2(\ell-1)\ell^{n-1}$):

$$\begin{array}{ccc} \Sigma^d P^{2q}(\ell^n) & \xrightarrow{A_n} & P^{2q}(\ell^n) \\ i\uparrow & & \downarrow j \\ \Sigma^d S^{2q-1} = S^{d+2q-1} & \xrightarrow[\alpha_n]{} & S^{2q} \end{array}$$

where i is the inclusion into the bottom cell, j is projection onto the top cell, and $\alpha_n \simeq j \circ A_n \circ i$, we have for $\alpha_n \in \pi^s_{d-1}(S^0)$:

i) $\ell^n \alpha_n = 0$.

ii) The e-invariant [Ad$_1$;§3] $e_C(\alpha_n) = -1/\ell^n$ mod 1 .

iii) The Toda bracket [Ad$_1$;§5], [To], $\{\ell^n, \alpha_n, \ell^n\} = 0$ mod $\ell^n \pi^s_d(S^0)$.

Moreover, Adams [Ad$_1$;12.5] proved the following :

3.2.2:Proposition:

Suppose given $\alpha \in \pi^s_{2r-1}(S^0)$ and $m \in Z$ such that:

i) $m\cdot\alpha = 0$

ii) The e-invariant $e_C(\alpha) = -1/m$ mod 1

iii) The Toda bracket $\{m , \alpha , m\} = 0$ mod $m\, \pi^s_{2r}(S^0)$.

Then, for suitable large q , there exist maps A_n making the following diagram homotopy commutative:

$$\begin{array}{ccc} \Sigma^{2r}P^{2q}(m) & \xrightarrow{A_n} & P^{2q}(m) \\ i\uparrow & & \downarrow j \\ \Sigma^{2r}S^{2q-1} = S^{2r+2q-1} & \xrightarrow[\alpha]{} & S^{2q} \end{array}$$

and for any such A_n, $\overline{KU}^*(A_n)$ is an isomorphism.

///.

Then, for the case $m = \ell^n$, and for $r = 2(\ell-1)\ell^{n-1}$, ℓ an odd prime, Adams [Ad_1;12.4] showed that there exists an element $\alpha \in \pi^s_{2r-1}(S^o)$ satisfying (i), (ii), (iii), obtaining in this way the maps A_n of (3.2.1).

3.2.3: Remark:

It follows from the properties of these Adams maps (3.2.1) and (3.2.2), that for ℓ odd, the composite:

$$\underline{\Sigma^d S^o} \xrightarrow{i} \underline{\Sigma^d P(\ell^n)} \xrightarrow{A_n} \underline{P(\ell^n)} \xrightarrow{j} \underline{\Sigma S^o}$$

generates the ℓ-component of the (classical) image of J in $\pi^s_{d-1}(S^o)$. Here $d = 2(\ell-1)\ell^{n-1}$ and for a spectrum $\underline{E} = (E_n)$, $\underline{\Sigma^d E}$ is the spectrum obtained by shifting the spaces of $\underline{E}$ d places, i.e. $(\underline{\Sigma^d E})_n = E_{n+d}$.

In the remaining part of this section we describe Snaith's approach to Bott-periodic algebraic K-theory.

First, we need some definitions:

3.2.4: Let ℓ be an odd prime and let $\zeta_\ell = \exp(2\pi i/\ell)$ be a primitive ℓ-th root of unity.

Consider the following maps:

i) The inclusion $\mathbf{Z}/\ell \approx \langle\zeta_\ell\rangle \to \mathbf{Z}[\zeta_\ell]^* = GL_1\mathbf{Z}[\zeta_\ell]$ (where the isomorphism is given by sending a generator of $\mathbf{Z}/\ell$ to ζ_ℓ) induces morphisms:

$$d_1 : \Sigma_n\int\mathbf{Z}/\ell \to \Sigma_n\int GL_1\mathbf{Z}[\zeta_\ell] \to GL_n\mathbf{Z}[\zeta_\ell] \qquad (1 \leq n \leq \infty)$$

given by sending a permutation $\sigma \in \Sigma_n$ to the corresponding permutation matrix $[\sigma] \in GL_n\mathbf{Z}[\zeta_\ell]$.

ii) Similarly, inclusion of permutation matrices induces

$$d_2 : \Sigma_n \to GL_n\mathbf{Z}$$

iii) The natural map $\mathbf{Z} \to \mathbf{Z}[1/\ell]$ induces a morphism

$$\epsilon : GL_m\mathbf{Z} \to GL_m\mathbf{Z}[1/\ell]$$

iv) By considering $\mathbf{Z}[\zeta_\ell]$ as the free abelian group $\mathbf{Z}^{\ell-1}$ on generators $1, \zeta, \zeta^2, \ldots, \zeta^{\ell-2}$, we obtain a morphism (transfer map)

$$\tau : GL_n\mathbf{Z}[\zeta_\ell] \to GL_{n(\ell-1)}\mathbf{Z}$$

v) The morphism $\mathbf{Z}/\ell \to \Sigma_\ell$ given by sending a generator of $\mathbf{Z}/\ell$ to the ℓ-cycle (permutation) $\sigma = (1,\ldots,\ell) \in \Sigma_\ell$ induces a morphism

$$i : \Sigma_n\int\mathbf{Z}/\ell \to \Sigma_n\int\Sigma_\ell \to \Sigma_{n\ell}$$

vi) Finally, the morphism $\mathbf{Z}/\ell \to 1$ induces

$$\eta : \Sigma_n\int\mathbf{Z}/\ell \to \Sigma_n\int 1 \to \Sigma_n$$

With this notation, Snaith [Sn_3;Corrigendum] proves:

3.2.5: Proposition:

The following diagram of groups and homomorphisms commutes up to an inner automorphism:

$$\begin{tikzcd}
\Sigma_n \int Z/\ell \arrow[rr, "\tau d_1 \times d_2\eta"] \arrow[d, "d_2 i"'] & & GL_{n(\ell-1)}Z \times GL_n Z \arrow[d, "\oplus"] \\
GL_{n\ell}Z \arrow[dr, "\varepsilon"'] & & GL_{n\ell}Z \arrow[dl, "\varepsilon"] \\
& GL_{n\ell}Z[1/\ell] &
\end{tikzcd}$$

///.

3.2.6: Definition:

Let $d_1 : (B\Sigma_\infty \int Z/\ell)^+ \to BGLZ[\zeta_\ell]^+$, $d_2 : B\Sigma_\infty^+ \to BGLZ[1/\ell]^+$, $i : (B\Sigma_\infty \int Z/\ell)^+ \to B\Sigma_\infty^+$, $\tau : BGLZ[\zeta_\ell]^+ \to BGLZ^+$ and $\eta : (B\Sigma_\infty \int Z/\ell)^+ \to B\Sigma_\infty^+$ be the maps induced by : d_1 , $\epsilon \cdot d_2$, i, τ and η of (3.2.4) respectively.

As a corollary of (3.2.5) Snaith [Sn_3;Corrigendum] obtains:

3.2.7: Corollary:

With the notation of (3.2.6), there exists a commutative diagram:

$$\begin{tikzcd}
\pi_i^S(BZ/\ell;Z/\ell) \approx \pi_i(B\Sigma_\infty \int Z/\ell^+;Z/\ell) \arrow[r, "(d_1)_\# \times \eta_\#"] \arrow[d, "i_\#"'] & K_i(Z[1/\ell,\zeta_\ell];Z/\ell) \times \pi_i^S(S^0;Z/\ell) \arrow[d, "\tau_\# + (d_2)_\#"] \\
\pi_i^S(S^0; Z/\ell) \approx \pi_i(B\Sigma_\infty^+; Z/\ell) \arrow[r, "(d_2)_\#"'] & K_i(Z[1/\ell]; Z/\ell)
\end{tikzcd}$$

This corollary is proven taking $n \to \infty$ in (3.2.5), applying the classifying space functor $B(-)$, the plus construction (§1.1) and the fact that inner automorphisms on a group G induce the identity map on BG. Also, it uses the homotopy equivalences $B\Sigma_\infty^+ \simeq Q_0(S^0)$ and $B\Sigma_\infty \int Z/\ell^+ \simeq Q_0(BZ/\ell_+)$ of (1.1.7) and (1.1.8).

///

3.2.8: Remark : (Snaith; [Sn$_3$])

Using the diagram of (3.2.7) and the fact that since $b \in \pi_2(B\Sigma_\infty \int Z/\ell^+;\ Z/\ell)$ originates in $\pi_2^S(BZ/\ell;\ Z/\ell)$ and so $\eta_\#(b) = 0$ and consequently $\eta_\#(b^{\ell-1}) = 0$ also, we obtain:

$$\begin{aligned} (d_2)_\# i_\#(b^{\ell-1}) &= (\tau_\# + (d_2)_\#)\circ((d_1)_\# \times \eta_\#)(b^{\ell-1}) \quad \text{by (3.2.7)} \\ &= \tau_\#(d_1)_\#(b^{\ell-1}) + (d_2)_\#\eta_\#(b^{\ell-1}) \\ &= \tau_\#(d_1)_\#(b^{\ell-1}) \end{aligned}$$

but, since the map $d_1 : B\Sigma_\infty \int Z/\ell^+ \to BGLZ[1/\ell;\zeta_\ell]^+$ corresponds to the map of spectra $\gamma : \underline{\Sigma}^\infty(BZ/\ell) \to \underline{KZ}[1/\ell,\zeta_\ell]$ of (3.1.3), then by the definition (3.1.3) $\beta = \gamma_\#(b) = (d_1)_\#(b)$, therefore:

$$(*) \qquad (d_2)_\# i_\#(b)^{\ell-1} = \tau_\#(\beta)^{\ell-1}$$

Now, define:

$$\hat{\beta}_1 = (d_2)_\# i_\#(b)^{\ell-1} \in K_{2(\ell-1)}(Z[1/\ell];\ Z/\ell)$$

for the maps

$$d_2 : B\Sigma_\infty^+ \to BGLZ[1/\ell]^+ \quad \text{and} \quad i : B\Sigma_\infty \int Z/\ell^+ \to B\Sigma_\infty^+$$

of (3.2.6).

With this notation, using the formula (*) above and a transfer argument [Br_1; 2.8], using the fact that $(d_1)_{\#}(b)^{\ell-1}$ is invariant under the action of the Galois group $G = Gal(Q(\zeta_\ell)/Q) \approx (Z/\ell)^*$ Snaith [Sn_3] proves:

3.2.9: Lemma (Snaith, [Sn_3; 3.13]):

Let $j : BGLZ[1/\ell]^+ \to BGLZ[1/\ell;\zeta_\ell]^+$ be the map induced by the natural inclusion. Then :

$$j_{\#}(\hat{\beta}_1) = (\ell-1)(d_1)_{\#}(b)^{\ell-1} = (\ell-1)\beta^{\ell-1} \in K_{2(\ell-1)}(Z[1/\ell,\zeta_\ell]; Z/\ell)$$

///.

3.2.10: Remarks:

i) By (3.1.7)(i) the Bott element $\bar{\beta}_1 \in K_{2(\ell-1)}(Z[1/\ell]; Z/\ell)$ is such that $j_{\#}(\bar{\beta}_1) = \beta^{\ell-1}$, thus (3.2.9) implies that a choice for $\bar{\beta}_1$ in (3.1.6) is given (up to an ℓ-adic unit) by:

$$\bar{\beta}_1 = \hat{\beta}_1 = (d_2)_{\#} i_{\#}(b)^{\ell-1}$$

ii) Now, since $Z \times B\Sigma_\infty^+ \simeq QS^o$, the map $d_2 : B\Sigma_\infty^+ \to BGLZ[1/\ell]^+$ is the base-point component of the infinite loop-map $d : QS^o \to BGLZ[1/\ell]^+$ and the map d corresponds to the unit of the algebraic K-theory spectrum of $Z[1/\ell]$, i.e. to the map of spectra:

$$D : \Sigma^\infty S^o \to \underline{KZ[1/\ell]}$$

Thus, the previous discussion has shown that we may choose

$$\bar{\beta}_1 = \hat{\beta}_1 = D_{\#}(i_{\#}(b)^{\ell-1})$$

for $i_{\#}(b)^{\ell-1} \in \pi^s_{2(\ell-1)}(S^o; Z/\ell)$.

iii) Now, Snaith [Sn_3] observes that $i_{\#}(b)^{\ell-1}$ is the generator of the mod-ℓ image of J, and its Bockstein

$$\partial(i_{\#}(b)^{\ell-1}) \in \pi^s_{2(\ell-1)-1}(S^0)$$

generates the ℓ-primary image of J in $\pi^s_{2\ell-3}(S^0)$.

Now, from Adams [Ad_2;§12] the generator $a_1 = i_{\#}(b)^{\ell-1}$ of the mod-ℓ image of J can be factored as:

$$\begin{tikzcd} P^{q+2(\ell-1)}(\ell) \arrow[r, dashed, "A_1"] \arrow[dr, "a_1"] & P^q(\ell) \arrow[d, "j"] \\ S^{q+2\ell-3} \arrow[u, "i"] \arrow[r, "\alpha_1"'] & S^q \end{tikzcd}$$

where i and j are the canonical inclusion and projection maps respectively and A_1 is an Adams map (3.2.1).

iv) Therefore, (ii) and (iii) say that a choice for the Bott element $\bar{\beta}_1$ of (3.1.6) is given by:

$$\bar{\beta}_1 = D_{\#}(a_1)$$

with $a_1 = jA_1$ and $A_1 : \Sigma^{2(\ell-1)}\underline{P(\ell)} \to P(\ell)$ an Adams map between Moore spectra.

<u>3.2.11</u>: Now, in order to have a similar description for the Bott elements $\bar{\beta}_n$ of (3.1.6) for $n > 1$, Snaith [Sn_3] proceeds as follows:

i) First, for $a_1 = jA_1 \in \pi^s_{2(\ell-1)}(S^0; Z/\ell)$ form the product $a_1^{\ell^{n-1}} \in \pi^s_{2(\ell-1)\ell^{n-1}}(S^0; Z/\ell)$. By definition of the product (2.3.6) a representative of $a_1^{\ell^{n-1}}$ is given as a composite of the form:

$$\begin{tikzcd} P^{q+2(\ell-1)\ell^{n-1}}(\ell) \arrow[r, "\chi"] \arrow[dr, "a_1^{\ell^{n-1}}"'] & P^{q+2(\ell-1)}(\ell) \wedge \cdots \wedge P^{q+2(\ell-1)}(\ell) \arrow[d, "a_1 \wedge \cdots \wedge a_1"] & (\ell^{n-1}\text{-factors}) \\ & S^{q\ell^{n-1}} = S^q \wedge \cdots \wedge S^q & \end{tikzcd}$$

where χ is the copairing of (2.3.3) for q sufficiently large.

Taking $q = 2s\ell^{n-1}$ for convenience of exposition we see that since χ_* is injective on $H_*(-;Z/\ell)$ by [Br_1;1.4] then (using the Atiyah-Hirzebruch spectral sequence) χ_* sends a generator of $KU_0(P^{2\ell^{n-1}(s+\ell-1)}(\ell);\ Z/\ell)$ to the ℓ^{n-1}-fold tensor product of the generator of $KU_0(P^{2s+2(\ell-1)}(\ell);Z/\ell)$. Thus, $a_1^{\ell^{n-1}} = (a_1 \wedge \cdots \wedge a_1)\circ\chi$ is an isomorphism on $KU_0(-;\ Z/\ell)$.

ii) Now, if s is large enough, there exist maps A_n such that we have a homotopy commutative diagram:

$$\begin{tikzcd} P^{(2s+2(\ell-1))\ell^{n-1}}(\ell^n) \arrow[r, dashed, "A_n"] \arrow[ddr, "a_n"] & P^{2s\ell^{n-1}}(\ell^n) \arrow[dd, "j"] \\ P^{(2s+2(\ell-1))\ell^{n-1}}(\ell) \arrow[u, "\tilde{i}"] \arrow[dr, "a_1^{\ell^{n-1}}"] & \\ S^{(2s+2(\ell-1))\ell^{n-1}-1} \arrow[u, "i'"] \arrow[uu, bend left, "i"] \arrow[r, "\alpha_n"'] & S^{2s\ell^{n-1}} \end{tikzcd}$$

where $\tilde{i}$ is the map induced by $Z/\ell \to Z/\ell^n$, and i and i' are the canonical inclusions into the respective bottom cells, j is the canonical projection onto the top cell and $\alpha_n \simeq a_1^{\ell^{n-1}} \circ i'$.

In this diagram, $\tilde{i}_*$ is a $KU_0(-;\ Z/\ell)$ isomorphism and hence so is $(A_n)_*$. Thus, A_n is an Adams map and so $\alpha_n \simeq j \circ A_n \circ i$. The map a_n is defined by $a_n = j \circ A_n$.

Therefore,

$$a_n \tilde{i} = \tilde{i}^{\#}(a_n) = a_1^{\ell^{n-1}} \in \pi^s_{2(\ell-1)\ell^{n-1}}(S^o; Z/\ell)$$

i.e. the mod-ℓ reduction of a_n is $a_1^{\ell^{n-1}}$.

3.2.12: Remark:

Write $r_\# = \tilde{i}^{\#}(\ )$ for the mod-ℓ reduction map (induced by precomposition with $\tilde{i}$). Then, in (3.2.11) we have constructed an element $a_n \in \pi^s_{2(\ell-1)\ell^{n-1}}(S^o; Z/\ell^n)$ whose mod-ℓ reduction is:

$$r_\#(a_n) = \tilde{i}^{\#}(a_n) = a_1^{\ell^{n-1}} \in \pi^s_{2(\ell-1)\ell^{n-1}}(S^o; Z/\ell)$$

Thus, using the fact that the mod-ℓ reduction map $r_\#$ is natural, and that $D : \underline{\Sigma^\infty S^o} \to \underline{KZ[1/\ell]}$ is a map of ring spectra we obtain:

$$r_\# D_\#(a_n) = D_\# r_\#(a_n) = D_\#(a_1^{\ell^{n-1}}) = \left(D_\#(a_1)\right)^{\ell^{n-1}} = \left(\bar{\beta}_1\right)^{\ell^{n-1}}$$

(the last equality by (3.2.10)(iv)).

Therefore, a choice for $\bar{\beta}_n$ in (3.1.7)(ii) is given by:

$$\bar{\beta}_n = D_\#(a_n) \in K_{2(\ell-1)\ell^{n-1}}(Z[1/\ell]; Z/\ell^n)$$

for all $n \geqslant 1$.

3.2.13: Now, let X be a commutative $Z[1/\ell]$-algebra and let $\underline{S}$ denote the sphere spectrum $\underline{\Sigma^\infty S^o}$.

Let $[g] \in K_i(X; Z/\ell^n) = \pi_i(\underline{KX}; Z/\ell^n)$ be represented by a map of spectra:

$$g : \underline{P(\ell^n)} \to \underline{KX} \quad \text{of degree } i.$$

Consider the Bott element $\beta_n \in K_{2(\ell-1)\ell^{n-1}}(Z[1/\ell];\ Z/\ell^n)$ of (3.1.6) and consider a representative of it:

$$\beta_n : \underline{P(\ell^n)} \longrightarrow \underline{KZ[1/\ell]}$$

of degree $\deg(\beta_n) = d = 2(\ell-1)\ell^{n-1}$

Consider the following diagram of spectra where by definition (2.3.6) the composite of the top row represents the product

$$\beta_n \cdot [g] \in K_{i+d}(X;\ Z/\ell^n)$$

$$\begin{tikzcd}
\underline{P(\ell^n)} \arrow[r, "\chi"] \arrow[dd, "A'_n"'] & \underline{P(\ell^n)} \wedge \underline{P(\ell^n)} \arrow[rr, "\beta_n \wedge g"] \arrow[d, "A_n \wedge 1"] \arrow[drr, "a_n \wedge g"] & & \underline{KZ[1/\ell]} \wedge \underline{KX} \arrow[r, "\mu"] & \underline{KX} \\
\arrow[r, phantom, "1"] & \underline{P(\ell^n)} \wedge \underline{P(\ell^n)} \arrow[rr, "j \wedge g"] \arrow[d, "j \wedge 1"] \arrow[ur, phantom, "3"] & \arrow[d, phantom, "2"] & \underline{S} \wedge \underline{KX} \arrow[u, "D \wedge 1"] \arrow[ur, "\simeq"] \arrow[u, phantom, "4"', shift right=4ex] \arrow[ur, phantom, "5", shift left=3ex] & \\
\underline{P(\ell^n)} \arrow[r, "\simeq"] & \underline{S} \wedge \underline{P(\ell^n)} \arrow[urr, "1 \wedge g"'] & & &
\end{tikzcd}$$

where χ is the copairing of Moore spectra (2.3.2), μ is the "multiplication" induced by the action of $Z[1/\ell]$ on X, D is the unit of $\underline{KZ[1/\ell]}$ (3.2.10) and A_n, a_n, j are the maps of spectra (3.2.1).

In this diagram:

(5) commutes because $\underline{KX}$ is a $\underline{KZ[1/\ell]}$-module spectrum and D is the unit of $\underline{KX}$ $[Ad_1]$.

(4) commutes by definition (3.2.12) of β_n as $\beta_n = \tilde{D}_{\#}(a_n) = D \cdot a_n$

(3) commutes since $j \circ A_n \simeq a_n$ by (3.2.1).

(2) commutes trivially.

(1) defines the map A'_n. Note that since A_n, j and χ are $KU_0(-;\ Z/\ell^n)$-isomorphisms then A'_n is also a $KU_0(-;Z/\ell^n)$-isomorphism

i.e. A'_n is an Adams map too.

From the commutativity of this diagram it follows that:

$$\chi\circ(\beta_n \wedge g)\circ\mu \simeq (1\wedge g)\circ A'_n$$

and so:

3.2.14: Remark:

$$\beta_n\cdot[g] = [g\circ A'_n] = A'^{*}_n[g] \in K_{i+d}(X;\ Z/\ell^n)$$

i.e. multiplication by the Bott element β_n is precomposition with an Adams map A'_n.

From this remark, Snaith [Sn_3;3.22] obtains the following description of $K_*(X;\ Z/\ell^n)[1/\beta_n]$:

3.2.15: Theorem:

Let ℓ be an odd prime and let X be a $Z[1/\ell]$-algebra. Suppose there exists a map of Moore spaces

$$A_n : P^{s+2(\ell-1)\ell^{n-1}}(\ell^n) \to P^s(\ell^n)$$

such that its stable homotopy class is A'_n : $\underline{P(\ell^n)} \to \underline{P(\ell^n)}$ of (3.2.13). Write $d = \deg(\beta_n) = 2(\ell-1)\ell^{n-1}$ and suppose $i \geq s$. Then:

$$K_i(X;Z/\ell^n)[1/\beta_n] \approx \varinjlim \left(K_i(X;Z/\ell^n) \xrightarrow{(\Sigma^{i-s}A_n)^*} K_{i+d}(X;Z/\ell^n) \to \cdots \right)$$

///.

§3.3: J-theory.

In this section we recall the definition of J-theory and also the construction of a diagram [Sn_3;3.24] that factors the localization map $\rho : K_*(X; Z/\ell^n) \to K_*(X; Z/\ell^n)[1/\beta_n]$ using the Hurewicz map for J-homology and using the description of Bott-periodic K-theory given in (3.2.15).

Finally, we review the characterization of the kernel of ρ given by Snaith in [Sn_3;4.1] and [Sn_4] for dimension i = 2.

3.3.1: J-theory:

Let $\psi^t : KU_*(-; Z/\ell^n) \to KU_*(-; Z/\ell^n)$ be an Adams operation for t a prime in the sequence $\{1+\ell^a(n\ell+1) \mid n \in \mathbf{N}\}$ where $1 \leq n \leq a < \infty$ are integers.

ψ^t is induced by a map of spectra $\psi^t : \underline{BUZ/\ell^n} \to \underline{BUZ/\ell^n}$ where $\underline{BUZ/\ell^n}$ denotes the spectrum of mod-ℓ^n complex topological K-theory [Ad_1].

Let $\underline{JZ/\ell^n}$ be the fibre of (=cofibre) of ψ^t-1 in the stable homotopy category. Thus we have a fibration:

$$\underline{JZ/\ell^n} \xrightarrow{\lambda} \underline{BUZ/\ell^n} \xrightarrow{\psi^t-1} \underline{BUZ/\ell^n}$$

Let $J_*(-; Z/\ell^n)$ denote the homology theory defined by the spectrum $\underline{JZ/\ell^n}$. Thus, $J_*(-; Z/\ell^n)$ fits into a long exact sequence:

$$\cdots \to J_\alpha(X; Z/\ell^n) \xrightarrow{\lambda} KU_\alpha(X; Z/\ell^n) \xrightarrow{\psi^t-1} KU_\alpha(X; Z/\ell^n) \to \cdots$$

Note that $J_*(-; Z/\ell^n)$ as well as $KU_*(-; Z/\ell^n)$ are Z/2-graded theories.

3.3.2: Remark:

Sometimes we can compute $J_*(-;\ Z/\ell^n)$ using the long exact sequence of (3.3.1).

Example: Let $X = P^2(\ell^n)$. From the cofibration:

$$S^1 \xrightarrow{\ell^n} S^1 \xrightarrow{i} P^2(\ell^n) \xrightarrow{j} S^2$$

we obtain: $KU_0(P^2(\ell^n);\ Z/\ell^n) \underset{\approx}{\xrightarrow{j_*}} KU_0(S^2;\ Z/\ell^n) \approx Z/\ell^n$

and we know that the action of Ψ^t on $KU_0(S^2;\ Z/\ell^n)$ is multiplication by t.

Thus, from the commutative diagram:

$$\begin{array}{ccc} KU_0(P^2(\ell^n);\ Z/\ell^n) & \xrightarrow{\Psi^t} & KU_0(P^2(\ell^n);\ Z/\ell^n) \\ j_*\downarrow\approx & & \approx\downarrow j_* \\ KU_0(S^2;\ Z/\ell^n) & \underset{\Psi^t = t\cdot}{\longrightarrow} & KU_0(S^2;\ Z/\ell^n) \end{array}$$

it follows that $\Psi^t : KU_0(P^2(\ell^n);\ Z/\ell^n) \to KU_0(P^2(\ell^n);\ Z/\ell^n)$ is also multiplication by t.

So, the long exact sequence of (3.3.1) for $X = P^2(\ell^n)$ looks like:

$$\begin{array}{ccccccc} \cdots\longrightarrow J_0(P^2(\ell^n);Z/\ell^n) & \xrightarrow{\lambda} & KU_0(P^2(\ell^n);Z/\ell^n) & \xrightarrow{\Psi^t-1} & KU_0(P^2(\ell^n);Z/\ell^n) & \longrightarrow\cdots \\ & & \| & & \| & \\ & & Z/\ell^n & \underset{t-1}{\longrightarrow} & Z/\ell^n & \end{array}$$

Now, since $t-1 = \ell^a u$ with $u = n\ell+1$ (3.3.1) and $(n,\ell) = 1$ and $a \geq n$, then:

$$\lambda : J_0(P^2(\ell^n);\ Z/\ell^n) \to KU_0(P^2(\ell^n);\ Z/\ell^n) \approx Z/\ell^n$$

is an isomorphism.

Thus, given a generator $e \in KU_0(P^2(\ell^n);\ Z/\ell^n) \approx Z/\ell^n$ we can choose a generator $e' \in J_0(P^2(\ell^n);Z/\ell^n) \approx Z/\ell^n$ such that $\lambda(e') = e$.

3.3.3: Definition:

The Hurewicz map for J-homology is the map $h_J : \pi_2(X; Z/\ell^n) \to J_o(X; Z/\ell^n)$ defined by: $h_J[f] = f_*(e')$.

3.3.4: Remark:

Recall that the KU-homology Hurewicz map $h_K : \pi_2(X; Z/\ell^n) \to KU_o(X; Z/\ell^n)$ is defined as $h_K[f] = f_*(e)$.

Thus, since $\lambda(e') = e$, then $f_*(e) = f_*(\lambda e')$, i.e. we have a commutative diagram:

$$\begin{tikzcd} \pi_2(X; Z/\ell^n) \arrow[rr, "h_J"] \arrow[dr, "h_K"'] & & J_o(X; Z/\ell^n) \arrow[dl, "\lambda"] \\ & KU_o(X; Z/\ell^n) & \end{tikzcd}$$

3.3.4: Now, let $h_*(-;Z/\ell^n)$ denote either $KU_*(-;Z/\ell^n)$ or $J_*(-;Z/\ell^n)$.

Thus, for $n \geq 2$ $h_i(P^2(\ell^n); Z/\ell^n) \approx Z/\ell^n$ for each $i \in Z/2$.

Since for an Adams map $A_n : P^{s+d}(\ell^n) \to P^s(\ell^n)$ with $d = 2(\ell-1)\ell^{n-1}$, $(A_n)_*$ induces an isomorphism on KU_*-theory, from the five lemma it follows that $(A_n)_*$ is also an isomorphism for h_*-homology.

Snaith [Sn_3;3.24] obtains the following:

3.3.5: Theorem:

Let ℓ be an odd prime and let X be $Z[1/\ell]$-algebra. Let s be as in (3.2.15). Then, there exists a commutative diagram:

$$\begin{tikzcd} K_i(X;\ Z/\ell^n) \arrow[rr, "\rho"] \arrow[dr, "H"'] & & K_i(X;\ Z/\ell^n)[1/\beta_n] \arrow[dl, "I"] \\ & h_i(BGLX^+;Z/\ell^n) & \end{tikzcd}$$

where ρ is the localization map and H is the Hurewicz map .

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For dimension $i = 2$, Snaith [Sn_3;§4] has obtained the following analogous of (3.3.5):

3.3.6: Theorem:

Let ℓ be an odd prime and let X be a commutative $Z[1/\ell]$ algebra. Let H denote the Hurewicz map for h_*-theory, and set $\hat{H}(x) = H(x) - H(x)^\ell$. Then, there is a commutative diagram:

$$\begin{tikzcd} K_2(X;\ Z/\ell) \arrow[rr, "\rho"] \arrow[dr, "\hat{H}"'] & & K_2(X;\ Z/\ell)[1/\beta_1] \arrow[dl, "I"] \\ & h_0(BGLX^+;\ Z/\ell) & \end{tikzcd}$$

///.

In [Sn_4;3.12] generalizes this result for all $n \geq 2$, and in this case the map $\hat{H}$ is more complicated.

CHAPTER 4

THE STABLE HOMOTOPY OF BZ/4

In this chapter we compute the stable homotopy groups of the classifying space $BZ/4$ in low dimensions. This computation will be used in the next chapter to detect a certain power of the Bott element for the prime $\ell = 2$.

To do this computation we use results of Liulevicius [Li] on the stable homotopy groups of the complex projective space CP^{∞} to obtain a description of the E_2-term of the Adams spectral sequence for BZ/4 in the desired range. Next, we use a transfer argument to decide the differentials in this spectral sequence.

§4.1: The mod-2 cohomology of BZ/4.

In this section we establish some results about the mod-2 cohomology algebra of BZ/4.

4.1.1: For the Eilenberg-MacLane space $BZ/4 = K(Z/4, 1)$, it is known, see e.g. Serre [Se], that its mod-2 cohomology algebra is:

$$H^*(BZ/4; Z/2) = P[u] \otimes E(v)$$

the tensor product of a polynomial algebra $P[u] = Z/2[u]$ on a generator u of degree $\deg(u) = 2$, and an exterior algebra $E(v)$ on a generator v of degree $\deg(v) = 1$.

These generators are given as follows:

Let $v \in H^1(BZ/4; Z/2)$ be the canonical generator. Let

$d_2 : H^1(BZ/4;\ Z/4) \to H^2(BZ/4;\ Z/2)$ be the connecting (Bockstein) morphism associated to the coefficient sequence $Z/2 \rightarrowtail Z/8 \twoheadrightarrow Z/4$. Then, $u = d_2(v')$. Note that d_2 is the second Bockstein differential for the cohomology of BZ/4.

Now, let $r : H^1(BZ/4;\ Z/4) \to H^1(BZ/4;\ Z/2)$ be the reduction morphism associated to the sequence : $Z/2 \rightarrowtail Z/4 \xrightarrow{r} Z/2$. Then, $v = r(v')$.

Note that if d_1 is the Bockstein morphism associated to the coefficient sequence $Z/2 \rightarrowtail Z/4 \xrightarrow{r} Z/2$, then $d_1(v) = d_1(r(v')) = 0$ by exactness. Thus, since $d_1 = Sq^1$, the first Steenrod square, then $Sq^1(v) = 0$. But, since $\deg(v) = 1$, then $Sq^i(v) = 0$ for all $i \geqslant 1$.

4.1.2: Now, recall that for the complex projective space CP^∞ we have:

$$H^*(CP^\infty;\ Z/2) = P[u]$$

a polynomial algebra, over Z/2, on one generator u of degree $\deg(u) = 2$.

Also, the map $BZ/4 \to BS^1 = CP^\infty$ induced by the inclusion of Z/4 into the circle S^1, gives rise to a monomorphism

$$H^*(CP^\infty;\ Z/2) \xrightarrow{f} H^*(BZ/4;\ Z/2)$$

that sends the generator u of the first algebra (4.1.1) to the generator u of the second algebra.

Moreover:

4.1.3: Remark:

The following sequence of Steenrod modules splits:

$$0 \longrightarrow \underset{\underset{H^*(CP^\infty;Z/2)}{\parallel}}{P[u]} \xrightarrow{f} \underset{\underset{H^*(BZ/4;Z/2)}{\parallel}}{P[u] \otimes E(v)} \xrightarrow{g} P[u]\langle v\rangle \longrightarrow 0$$

where $P[u]\langle v\rangle$ = group generated by the elements of the form $x \cdot v$ with $x \in P[u]$, and the action of the Steenrod algebra $A(2)$ on $P[u]\langle v\rangle$ is given by: $Sq^1(x \cdot v) = Sq^1(x) \cdot v$.

Proof:

First observe that the action of the Steenrod algebra on $P[u]\langle v\rangle$ is obtained from the Cartan formula and the fact that $Sq^i(v) = 0$ for all $i \geq 1$ (4.1.1), since:

$$Sq^i(x \cdot v) = \sum_j Sq^j(x) \cdot Sq^{i-j}(v) = Sq^i(x) \cdot Sq^0(v)$$

because $Sq^{i-j}(v) = 0$ for all $i-j > 0$.

The morphism g is the canonical one and the splitting is given by the obvious map from $P[u] \otimes E(v)$ to $P[u]$.

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4.1.4: From the previous discussion, it follows that, as $A(2)$-modules,

$$H^*(BZ/4;\ Z/2) \approx H^*(CP^\infty;\ Z/2) \oplus \Sigma H^*(CP^\infty;\ Z/2)$$

where $\Sigma H^*(CP^\infty;\ Z/2) = H^*(CP^\infty;\ Z/2)\langle v\rangle = P[u]\langle v\rangle$, i.e. $H^*(CP^\infty;\ Z/2)$ with its elements lifted, by v, one degree.

§4.2: The Adams spectral sequence for $BZ/4$ in low degrees.

In this section we determine the E_2-term of the Adams spectral sequence of $BZ/4$ in total degrees $\leqslant 9$.

4.2.1: The Adams spectral sequence: Recall, see e.g. Adams [Ad_1], that for a nice suitable space X, the (mod-2) Adams spectral sequence (E_r, d_r) for X satisfies, among others, the following properties:

i) $E_2^{s,t} \approx \mathrm{Ext}_{A(2)}^{s,t}\left(H^*(X; Z/2), Z/2\right)$.

ii) The spectral sequence (E_r, d_r) converges to the 2-primary component of the stable homotopy groups of X, ${}_2\pi_*^S(X)$.

iii) The differentials d_r are derivations.

4.2.2: Let M denote the elements of positive degree in

$$H^*(CP^\infty; Z/2) = P[u].$$

Observe that $P[u]\langle v\rangle \approx M\langle v\rangle \oplus Z/2\langle v\rangle$ as Steenrod modules. Here, as in (4.1.3) $M\langle v\rangle$ = group generated by the elements $x \cdot v$ with $x \in M$ and with the Steenrod module structure as in (4.1.3). Similarly for $Z/2\langle v\rangle$.

We use the notation $\Sigma M = M\langle v\rangle$ and $\Sigma Z/2 = Z/2\langle v\rangle$.

From (4.1.4) it follows that:

$$H^*(BZ/4; Z/2) \approx Z/2 \oplus M \oplus \Sigma M \oplus \Sigma Z/2$$

as A(2)-modules.

Therefore, since $\mathrm{Ext}_{A(2)}(-,\ )$ commutes with direct sums, we have:

$$\mathrm{Ext}^{s,t}_{A(2)}(\tilde{H}^*(BZ/4;Z/2),Z/2) \approx \mathrm{Ext}^{s,t}_{A(2)}(M,Z/2) \oplus \mathrm{Ext}^{s,t}_{A(2)}(\Sigma M,Z/2) \oplus \mathrm{Ext}^{s,t}_{A(2)}(\Sigma Z/2,Z/2)$$

$$\approx \mathrm{Ext}^{s,t}_{A(2)}(M,Z/2) \oplus \Sigma\mathrm{Ext}^{s,t}_{A(2)}(M,Z/2) \oplus \Sigma\mathrm{Ext}^{s,t}_{A(2)}(Z/2,Z/2)$$

Now, $\mathrm{Ext}^{s,t}_{A(2)}(Z/2,Z/2)$ is well-known in low dimensions, see e.g. [M-T], and from Liulevicius [Li] we know $\mathrm{Ext}^{s,t}_{A(2)}(M,Z/2)$ explicitly in total dimensions ≤ 11:

4.2.3: Proposition [Li; II.3 & II.8]:

1) $\mathrm{Ext}^{s,t}_{A(2)}(M,Z/2)$ has the following Z/2-basis, for $t-s \leq 11$:

$e_{0,2}\cdot h_0^n$, $e_{0,6}\cdot h_0^n$, $e_{1,5}\cdot h_0^n$, $e_{2,12}\cdot h_0^n$, $e_{3,11}\cdot h_0^n$, $n = 0,1,\ldots$

$e_{0,2}\cdot h_2$, $e_{0,6}\cdot h_0h_2$, $e_{0,2}\cdot h_3h_0^k$, $k = 0,1,2,3.$

$e_{0,6}\cdot h_1$, $e_{0,6}\cdot h_r^2$, $e_{2,13}\cdot h_0^k$, $k = 0,1,2,3.$

$e_{3,14}$

Here $e_{s,t}$ denotes a class in $\mathrm{Ext}^{s,t}_{A(2)}(M,Z/2)$ and $h_i \in \mathrm{Ext}^{1,2^i}_{A(2)}(Z/2,Z/2)$ $i \geq 0$, denotes the Hopf generator [M-T] .

2) Moreover, the only nontrivial differential in total degrees ≤ 9 is : $d_2(e_{0,6}) = e_{0,2}h_0h_2$.

///.

4.2.4: Now, the generators for $\Sigma\mathrm{Ext}_{A(2)}(Z/2,Z/2)$ are obtained by lifting the generators of $\mathrm{Ext}_{A(2)}(Z/2,Z/2)$ one degree. Thus, using the standard notation [M-T] for the generators of $Ext_{A(2)}(Z/2,Z/2)$ in degrees ≤ 10, for example if $h_1 \in Ext_{A(2)}(Z/2,Z/2)$ with bidegree $(2^1-1,1)$, then, the corresponding generator $\bar{h}_1 \in \Sigma Ext_{A(2)}(Z/2,Z/2)$ has bidegree $(2^i,1)$.

Similarly for the generators of $\Sigma Ext_{A(2)}(M,Z/2)$ using (4.2.3).

With these notations and from (4.2.2) it follows that:

4.2.5: Proposition:

$Ext^{s,t}_{A(2)}(\tilde{H}^*(BZ/4;Z/2),Z/2)$ has, for $t-s \leq 9$, the Z/2-basis given by Table 1:

Notes:

1) In Table 1, the dots "$\cdots$" indicate that the column continues in the indicated way by powers of h_0.

2) As usual, we have arranged the table so that the bigrading of $Ext^{s,t}_{A(2)}(-,-)$ is:

s = filtration degree

t-s = stem-degree, i.e. the (t-s)-column corresponds to the (t-s)-stem of the stable homotopy groups of BZ/4.

s									
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	h_0^6v	$e_{0,2}h_0^6$	$e_{0,2}h_0^6v$	$e_{1,5}h_0^5$	$e_{1,5}h_0^5v$	$e_{0,6}h_0^6$	$e_{0,6}h_0^6v$	$e_{3,11}h_0^3$	$e_{3,11}h_0^3v$
5	h_0^5v	$e_{0,2}h_0^5$	$e_{0,2}h_0^5v$	$e_{1,5}h_0^4$	$e_{1,5}h_0^4v$	$e_{0,6}h_0^5$	$e_{0,6}h_0^5v$	$e_{3,11}h_0^2$	$e_{3,11}h_0^2v$
4	h_0^4v	$e_{0,2}h_0^4$	$e_{0,2}h_0^4v$	$e_{1,5}h_0^3$	$e_{1,5}h_0^3v$	$e_{0,6}h_0^4$	$e_{0,6}h_0^4v$	$h_0^3h_3v$ $e_{3,11}h_0$	$e_{0,2}h_3h_0^3$ $e_{3,11}h_0v$
3	h_0^3v	$e_{0,2}h_0^3$	$e_{0,2}h_0^3v$	$h_0^2h_2v$ $e_{1,5}h_0^2$	$e_{1,5}h_0^2v$	$e_{0,6}h_0^3$	$e_{0,6}h_0^3v$	$h_0^2h_3v$ $e_{3,11}$	c_0v $e_{0,2}h_3h_0^2$ $e_{3,11}v$
2	h_0^2v	$e_{0,2}h_0^2$	h_1^2v $e_{0,2}h_0^2v$	h_0h_2v $e_{1,5}h_0$	$e_{0,2}h_2h_0$ $e_{1,5}h_0v$	$e_{0,2}h_2h_0v$ $e_{0,6}h_0^2$	h_2^2v $e_{0,6}h_0^2v$	h_0h_3v $e_{0,6}h_1^2$	h_1h_3v $e_{0,2}h_3h_0$ $e_{0,6}h_1^2v$
1	h_0v	h_1v $e_{0,2}h_0$	$e_{0,2}h_0v$	h_2v $e_{1,5}$	$e_{0,2}h_2$ $e_{1,5}v$	$e_{0,2}h_2v$ $e_{0,6}h_0$	$e_{0,6}h_1$ $e_{0,6}h_0v$	h_3v $e_{0,6}h_1v$	$e_{0,2}h_3$
0	v	$e_{0,2}$	$e_{0,2}v$			$e_{0,6}$	$e_{0,6}v$		
$t-s \to$	1	2	3	4	5	6	7	8	9

TABLE 1: $\mathrm{Ext}_{A(2)}(\underline{H}^*(BZ/4;Z/2),Z/2)$ for $t-s \leqslant 9$.

§4.3: The stable homotopy groups of BZ/4 in degrees ≤ 8.

In this section we compute the differentials of the Adams spectral sequence of $BZ/4$ in dimensions ≤ 8, and simultaneously we obtain the stable homotopy groups of $BZ/4$ in dimensions ≤ 8.

To do this computations, the following lemma will be very useful:

4.3.1: Lemma:

Let F_5 denote the field with 5 elements. Then, the 2-primary component of the algebraic K-theory groups of F_5, ${}_2K_{2n-1}(F_5)$ is a direct summand of ${}_2\pi^S_{2n-1}(BZ/4)$.

Proof:

Since $Z/4 = GL_1(F_5)$, we have an inclusion:

$$\Sigma_n \int Z/4 \to GL_n F_5$$

Here, $\Sigma_n \int Z/4$ is the wreath product of the symmetric group Σ_n with $Z/4$ (see 1.1.8).

Now, since $\Sigma_n \int Z/4$ has odd index in $GL_n F_5$ by Quillen [Q_2; p.574], then the map

$$i : (B\Sigma_n \int Z/4)^+ \to (BGL_n F_5)^+$$

induced by the inclusion above, is such that the morphism

$$i_\# : {}_2\pi_*(B\Sigma_n \int Z/4)^+ \to {}_2\pi_*(BGL_n F_5)^+$$

is split surjective by Harris-Segal [H-S;3.1].

Now, since the spaces $GL_n F_5$ and $B\Sigma_n \int Z/4$ have homological stability by Quillen [Q_4] and Snaith splitting [Sn_7] respectively, then, passing to the limit, the morphism

$$1_{\#} : {}_2\pi_*(B\Sigma_\infty \int Z/4)^+ \to {}_2\pi_*(BGLF_5)^+$$

is split surjective.

But, $(B\Sigma_\infty \int Z/4)^+ \simeq Q_0(BZ/4_+)$ by (1.1.18) and so the lemma follows since ${}_2\pi_*(B\Sigma_\infty \int Z/4)^+ \approx {}_2\pi_*(Q_0(BZ/4)) \approx {}_2\pi_*^S(BZ/4)$ and ${}_2\pi_*(BGLF_5)^+ \simeq {}_2K_*(F_5)$.

///.

4.3.2: Remark:

From Quillen [Q_2] we know that $K_{2n-1}(F_5) \simeq Z/(5^n-1)$ (and $K_{2n}(F_5) = 0$) . Therefore, from (4.3.1) we obtain:

${}_2K_1(F_5) = Z/4$ is a direct summand of ${}_2\pi_1^S(BZ/4)$

${}_2K_3(F_5) = Z/8$ is a direct summand of ${}_2\pi_3^S(BZ/4)$

${}_2K_5(F_5) = Z/4$ is a direct summand of ${}_2\pi_5^S(BZ/4)$

${}_2K_7(F_5) = Z/8$ is a direct summand of ${}_2\pi_7^S(BZ/4)$.

All this information will be used for the calculation of the differentials of the Adams spectral sequence for BZ/4 in total dimensions $\leq$ 9.

4.3.3: Remark:

Before we start these computations, recall that since $BZ/4 = K(Z/4, 1)$ we do not have to consider the spectral sequences for odd primes; they are zero. Thus, the stable homotopy groups of

$BZ/4$ are equal to their 2-primary components.

Also, recall that multiplication by h_0 in the Adams spectral sequence for $BZ/4$ corresponds to multiplication by 2 in the stable homotopy groups, and also $h_i h_{i+1} = 0$. This will help us to decide certain extensions and differentials.

4.3.4: Differentials in the Adams spectral sequence of BZ/4 :

i) For $t-s = 1$, obviously $d_r(h_0^n v) = 0$ for all $r \geqslant 2$ and $n \geqslant 0$.

ii) For $t-s = 2$, since by (4.3.2) $Z/4$ is a direct summand of $\pi_1^S(BZ/4)$, we must then have that $\pi_1^S(BZ/4) = Z/4$.

Thus, $d_2(e_{0,2}) = h_0^2 v$ and consequently $d_2(e_{0,2}h_0^i) \doteq h_0^{i+1}v$ for all, $i \geqslant 0$.

Also, $d_2(h_1 v) = 0$ since $d_2(v) = 0$ by (i) and $d_2(h_1) = 0$.

Thus, $h_1 v$ is a d_r-cycle for all $r \geqslant 2$ and since it can not be a d_r-boundary by dimensional reasons, then it survives to E_∞ and so $\pi_2^S(BZ/4) \doteq Z/2$.

iii) For $t-s = 3$: All the generators in this column are d_2-cycles (and hence d_r-cycles for all $r \geqslant 2$) because if there were one that hits an element $e_{0,2}h_0^n$ in the second column, then $d_2(e_{0,2}h_0^n)$ will be zero, but this contradicts (ii).

iv) For $t-s = 4$: First, $d_2(h_2v) = 0$ since $d_2(h_2) = 0 = d_2(v)$. Similarly, $d_2(h_0h_2v) = 0$ and $d_2(h_0^2h_2v) = 0$.

Now, by (4.3.2) $Z/8$ is a direct summand of $\pi_3^s(BZ/4)$, therefore we must have that $d_2(e_{1,5}) = e_{0,2}h_0^3v$, and consequently $d_2(e_{1,5}h_0^i) = e_{0,2}h_0^{i+3}v$ for all $i \geq 0$.

v) For $t-s = 5$: Since $d_2(h_2) = 0$ and $d_2(e_{0,2}) = h_0^2v$ by (ii), then, $d_2(e_{0,2}h_2) = h_0^2h_2v$. Therefore:

$$d_2(e_{0,2}h_0h_2) = d_2(e_{0,2}h_2)h_0 + e_{0,2}h_2d_2(h_0) = h_0^2h_2v\, h_0 + 0$$

$$= h_0^3h_2v = \begin{cases} 0 \\ \text{or} \\ e_{1,5}h_0^3 \end{cases}$$

but since $d_2(e_{1,5}h_0^3) = e_{0,2}h_0^6v \neq 0$ by (iv), we must then have that $d_2(e_{0,2}h_0h_2) = h_0^3h_2v = 0$.

Finally, the generators $e_{1,5}h_0^nv$, $n \geq 0$, are all d_2-cycles since

$$\begin{aligned} d_2(e_{1,5}h_0^nv) &= d_2(e_{1,5}h_0^n)v + e_{1,5}h_0^n\, d_2(v) \\ &= (e_{0,2}h_0^{n+3}v)v + 0 \\ &= e_{0,2}h_0^{n+3}v^2 = 0 \quad \text{since } v^2 = 0. \end{aligned}$$

vi) For $t-s = 6$: By (4.2.3)(2) we know that $d_2(e_{0,6}) = e_{0,2}h_0h_2$.

Now, $$\begin{aligned} d_2(e_{0,2}h_2v) &= d_2(e_{0,2}h_2)v + e_{0,2}h_2d_2(v) \\ &= (h_0^2h_2v)v + 0 \quad \text{since } d_2(v) = 0 \text{ and (v).} \\ &= h_0^2h_2v^2 = 0 \quad \text{since } v^2 = 0. \end{aligned}$$

Hence, $e_{0,2}h_2v$ is a d_2-cycle and thus an infinite cycle for dimensional reasons.

Similarly $d_2(e_{0,2}h_2h_0v) = 0$.

Now, by (4.3.20) Z/4 is a direct summand of $\pi_5^s(BZ/4)$, so, from (v) we must have that $d_2(e_{0,6}h_0) = e_{1,5}h_0^2v$ and consequently $d_2(e_{0,6}h_0^n) = e_{1,5}h_0^{n+1}v$ for all $n \geqslant 1$.

vii) For $t-s = 7$: Since $d_2(e_{0,6}) = e_{0,2}h_0h_2$ by (vi) , it follows that $d_2(e_{0,6}v) = e_{0,2}h_0h_2v$. This implies that for $t-s = 6$ there is only one infinite cycle: $e_{0,6}h_2v$.

Now, for $n \geqslant 1$: $d_2(e_{0,6}h_0^nv) = d_2(e_{0,6}h_0^n)v + 0$

$$= e_{1,5}h_0^{n+1}vv = 0 .$$

Also, $d_2(e_{0,6}h_1) = d_2(e_{0,6})h_1 + e_{0,6}d_2(h_1)$

$= e_{0,2}h_2h_0h_1 + 0$ since $d_2(h_1) = 0$ and by (vi)

$= 0$ since $h_0h_1 = 0$ (see 4.3.3) .

Finally, $d_2(h_2^2v) = 0$ since it can not be $e_{0,6}h_0^5$ because $d_2(e_{0,6}h_0^5) = e_{1,5}h_0^6v \neq 0$ by (vi) .

Therefore, with the exception of $e_{0,6}v$ all the other generators in the column $t-s = 7$ are infinite cycles.

viii) For $t-s = 8$: First, $d_r(h_3v) = 0$ for all $r \geqslant 2$ since $d_r(h_3) = 0 = d_r(v)$.

Also, $d_r(e_{0,6}h_1v) = d_r(e_{0,6}h_1)v + e_{0,6}h_1 d_r(v)$

$= 0$ since $d_r(v) = 0$ and by (vii)

$$d_r(e_{0,6}h_1) = 0 .$$

Similarly, $d_r(h_0^nh_3v) = 0$ for $n = 1,2,3$, since by (i) $d_r(h_0^nv) = 0$ and $d_r(h_3) = 0$.

Now, from (4.3.2) $Z/8$ is a direct summand of $\pi_7^s(BZ/4)$, therefore we must have:

$$d_2(e_{0,6}h_1^2) = e_{0,6}h_0^4 v$$

and $$d_2(e_{3,11}h_0^n) = e_{0,6}h_0^{n+5}v \quad \text{for all } n \geq 0$$

ix) For $t-s = 9$: First, $d_2(e_{0,2}h_3) = d_2(e_{0,2})h_3 + e_{0,2}d_2(h_3)$

$$= h_0^2 v h_3 + 0 \quad \text{by (i)}.$$

$$= h_0^2 h_3 v$$

Similarly, $d_2(e_{0,2}h_3h_0) = h_0^3 h_3 v$.

These are all the differentials we have to consider, because by (viii) the generators $e_{3,11}h_0^n$ for $t-s = 8$ are not cycles, and so by dimensional reasons the remaining infinite cycles, for $t-s = 8$: $e_{0,6}h_1 v$, $h_3 v$ and $h_0 h_3 v$ are not d_r-boundaries for all $r \geq 2$.

///.

Summarizing the information gathered in (4.3.4) we obtain:

4.3.5: Proposition:

The E_∞-term of the Adams spectral sequence of $BZ/4$ in total dimensions ≤ 8 is given by the following table:

3								$e_{0,6}h_0^3v$	
2				h_1^2v $e_{0,2}h_0^2v$	h_0h_2v	$e_{1,5}h_0v$		h_2^2v $e_{0,6}h_0^2v$	h_0h_3v
1		h_0v	h_1v	$e_{0,2}h_0v$	h_2v	$e_{1,5}v$	$e_{0,2}h_2v$	$e_{0,6}h_1$ $e_{0,6}h_0v$	h_3v $e_{0,6}h_1v$
$s = 0$		v		$e_{0,2}v$					
$t-s =$	0	1	2	3	4	5	6	7	8

TABLE 2: E_∞-term of the Adams spectral sequence of BZ/4 for $t-s \leq 8$.

Now, since $h_0h_1 = 0$ and since multiplication by h_0 corresponds to multiplication by 2, we can easily obtain the group extensions in (4.3.5):

4.3.6: Corollary:

In dimensions ≤ 8, the stable homotopy groups of BZ/4 are as follows:

$\pi_*^s(BZ/4) =$	Z/4	Z/2	Z/8 ⊕ Z/2	Z/4	Z/4	Z/2	Z/8 ⊕ Z/2 ⊕ Z/2	Z/4 ⊕ Z/2
$* =$	1	2	3	4	5	6	7	8

///.

CHAPTER 5

2-PRIMARY BOTT-PERIODIC ALGEBRAIC K-THEORY

In this chapter we prove that for the algebraic K-theory with mod-4^n coefficients of $Z[1/2, \zeta_4]$-algebras, ζ_4 a fourth root of unity, localizing by inverting a Bott element gives the same result as localizing by inverting an appropiate Adams map.

This description of $K_*(X; Z/4^n)[1/\beta_n]$ is the 2-primary analogous of Snaith's theorem (3.2.15) for the odd-primary case.

We also obtain the 2-primary analogous of the J-theory diagrams of §3.3 .

§5.1: 2-Primary Bott elements.

We begin this chapter by recalling some properties of the mod-4^n Bott elements and adapting the results of §3.2 to the mod-4^n case.

5.5.1: Definition:

Let $\zeta_4 = \exp(2\pi i/4)$ be a fourth root of unity. Let $A = Z[1/2, \zeta_4]$ be the ring obtained by adjoining ζ_4 to the ring of integers localized away from 2.

where a_1 is a map that represents $\hat{\sigma}$, i and j are inclusion into the bottom cell and projection onto the top cell respectively, and $\alpha_1 \simeq a_1 \circ i = i^{\#}(a_1)$ represents $\partial(\hat{\sigma}) = 4\sigma$ (recall that the Bockstein morphism ∂ is given by $i^{\#}$).

Then:

i) $\alpha_1 \in \pi_7^S(S^0)$ has order 4 since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and σ is of order 16 .

ii) The Toda bracket $\{4 , \alpha_1 , 4\} = 0$ by [To; 3.7];

iii) The Adams e-invariant [Ad$_2$;§3] of α_1 is:

$$e(\alpha_1) = 1/4 \pmod 1$$

since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and $e(\sigma) = 1/16 \pmod 1$ by [Ad$_1$].

It follows, from [Ad$_2$; 12.5] (see 3.2.2), that there exists a map A_1 making the previous diagram homotopy commutative, and moreover, A_1 is an Adams map.

5.2.6: Transfer maps: Let $H \subseteq G$ be finite groups, and let $n = [G : H]$ = index of H in G . As usual, let Σ_r denote the r-th symmetric group for $1 \leq r \leq \infty$.

Consider the following construction:

The natural morphisms:

$$G \longrightarrow \Sigma_n \int H \longrightarrow \Sigma_\infty \int H$$

induce, upon applying the classifying space functor $B(-)$ and the plus construction $(-)^+$ (§1.1) , maps:

5.1.3: Lemma:

Let β_1 be as in (5.1.2). For $n > 1$, the 4^{n-1} cup power of β_1 in $K_*(Z[1/2, \zeta_4]; Z/4)$ is the reduction mod-4 of an element β_n in $K_*(Z[1/2, \zeta_4]; Z/4^n)$.

Proof:

As in (3.1.5) [DFST;Lemma 2], the proof is by induction on $n > 1$ using the fact (2.4.4) that the differentials in the mod-4 stable homotopy Bockstein spectral sequence are derivations, and the definition $K_*(A; Z/4^m) = \pi_*(\underline{KA}; Z/4^m)$.

1) $n = 2$: Since $\beta_1 = \beta^4$ and since $d_1 : K_*(A; Z/4) \longrightarrow K_{*-1}(A; Z/4)$ is a derivation, then:

$$d_1(\beta_1) = d_1(\beta^4) = 4 \cdot \beta^3 d_1(\beta) = 0$$

since $K_*(A; Z/4)$ is a $Z/4$-module.

Thus, β_1 is a d_1-cycle and so it survives to E_2.

Now, by the description of E_r, see [Br'_1; §5:p.75], $\beta_1 = \beta^4 \in E_2$ is represented by the class of a map $\beta_1 : P^8(4) \longrightarrow \underline{KA}$ such that there exists a factorization:

$$\begin{tikzcd} P^8(4) \arrow[rr, dashed, "\beta_1"] \arrow[dr, "i"'] & & \underline{KA} \\ & P^8(4^2) \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

i.e. $\beta_1 = \beta_2 \circ i = i^{\#}(\bar{\beta}_2)$, i.e. β_1 is the mod-4 reduction of $\beta_2 \in K_8(A; Z/4^2) = \pi_8(\underline{KA}; Z/4^2)$ (the mod-4 reduction map is $r_{\#} = i^{\#}(\_)$).

ii) Now, for $n > 2$, inductively we see that the cup powers:

$$\beta_1 = \beta^4 \ , \ \beta_1^4 = \beta^{4^2} \ , \ \ldots \ , \ \beta_1^{4^{n-1}} = \beta^{4^n}$$

are d_r-cycles for $1 \leqslant r \leqslant n-1$, and so in particular

$$\beta_1^{4^{n-1}} = \beta^{4^n} \in E_n^{8\cdot 4^{n-1}}$$

can be represented as:

$$\begin{tikzcd}
P^{8\cdot 4^{n-1}}(4) \arrow[rr, dashed, "\beta_1^{4^{n-1}}"] \arrow[dr, "i"'] & & \underline{KA} \\
& P^{8\cdot 4^{n-1}}(4^n) \arrow[ur, "\beta_n"'] &
\end{tikzcd}$$

by the description of $E_n^{8\cdot 4^{n-1}}$ [Br_1;§5].

Thus, for $\beta_n \in K_{8\cdot 4^{n-1}}\left(A\,;\,Z/4^n\right)$ we have: $\beta_1^{4^{n-1}} = i^{\#}(\beta_n)$ i.e. the mod-4 reduction of β_n is $\beta_1^{4^{n-1}}$.

///.

5.1.4: Definition:

Let X be an algebra over $A = Z[1/2\,,\,\zeta_4]$, define (as in 3.1.8):

$$K_i(X\,;\,Z/4^n)[1/\beta_n] = \varinjlim \left(K_i(X\,;\,Z/4^n) \xrightarrow{\beta_n} K_{i+d}(X\,;\,Z/4^n) \longrightarrow \cdots \right)$$

where $d = \deg(\beta_n) = 8\cdot 4^{n-1}$.

Notice that $K_i(X\,;\,Z/4^n)[1/\beta_n] \approx K_{i+d}(X\,;\,Z/4^n)[1/\beta_n]$ i.e $K_*(X\,;\,Z/4^n)[1/\beta_n]$ is periodic of period d.

These groups are called the mod-4^n <u>Bott-periodic algebraic K-theory groups of X</u>.

§5.2: 2-Primary Bott elements and Adams maps.

In this section we prove that an appropiate choice for the 2-primary Bott elements is given by an Adams map beetwen mod-4^n Moore spectra.

First, we recall some properties of these 2-primary Adams maps, see [C-K] for details on these maps.

§.2.1: 2-Primary Adams maps:

Let $u \in KU_0(S^0) = \pi_2(BU) = Z$ be a (Bott) generator. Then, $\bar{u} = u^{2r} \in KU_{2r}(S^0) = \pi_{2r}(BU) = Z$ is independent of the choice of u. This $\bar{u}$ will be called a Bott class.

Now, consider real K-theory KO_* and the complexification map

$$c : KO_*(-) \longrightarrow KU_*(-)$$

Choose a generator $v \in KO_{8r}(S^0) = \pi_{8r}(BO) = Z$ such that $c(v) = \bar{u}$ = Bott class in $KU_{8r}(S^0)$.

Now, let $n \geq 1$ and consider the Moore spectrum $\underline{P(2^n)} = S^0 \cup_{2^n} e^1$ (see 2.2.1).

Using this spectrum to introduce coefficients in KO-theory, write:

$$KO_*(X ; Z/2^n) = [\underline{P(2^n)} , X \wedge KO]_*$$

for X any spectrum and KO the spectrum representing KO_*-theory (see [Ad_1;Part 3]).

Now, for $v \in KO_{8r}(S^0) = [S^0 ; KO]_{8r}$ we have that:

$$\bar{v} = 1 \wedge v \in [\underline{P(2^n)} \wedge S^0 , \underline{P(2^n)} \wedge KO]_{8r}$$

$$= [\underline{P(2^n)}, \underline{P(2^n)} \wedge \underline{KO}]_{8r}$$
$$= KO_{8r}(\underline{P(2^n)}; Z/2^n)$$

is a generator, called the <u>mod-2^n Bott class</u>.

Now, let $h_{KO} : \pi_*^s(X; Z/2^n) \longrightarrow KO_*(X; Z/2^n)$ be the KO-Hurewicz map defined as follows:

If $[f] \in \pi_r^s(X; Z/2^n) = [\underline{P(2^n)}, X]_r$ is represented by a map $f : \underline{P(2^n)} \longrightarrow X$ of degree r, then f induces

$$f_* : KO_*(P(2^n); Z/2^n) \longrightarrow KO_*(X; Z/2^n)$$

and we define: $h_{KO}[f] = f_*(e) \in KO_r(X; Z/2^n)$ where $e \in KO_o(\underline{P(2^n)}; Z/2^n) = Z/2^n$ is a generator.

<u>5.2.2</u>: <u>Definition</u>:

A map $A_n : \Sigma^d \underline{P(2^n)} \longrightarrow \underline{P(2^n)}$, representing an element $A_n \in \pi_d^s(\underline{P(2^n)}; Z/2^n)$, is called an <u>Adams map</u> iff

$$h_{KO}[A_n] = \bar{v} = \text{Bott class} \in KO_d(P(2^n); Z/2^n)$$

<u>5.2.3</u>: <u>Remark</u>:

Observe that if A_n is an Adams map then $(A_n)_*$ is a KO_*-isomorphism.

M.C. Crabb and K. Knapp, [C-K], have proved the following:

<u>5.2.4</u>: <u>Proposition</u> [C-K; 3.2]:

Let $d = d(n) = \max(8, 2^{n-1})$, $n \geq 1$. Then, there exists a family of maps $A_n \in \pi_d^s(\underline{P(2^n)}; Z/2^n) = [\Sigma^d \underline{P(2^n)}, \underline{P(2^n)}]_o$ such that :

i) A_n is an Adams map.

ii) In the homotopy commutative diagram:

$$\begin{tikzcd} \Sigma^d \underline{P(2^n)} \arrow[r, "A_n"] & \underline{P(2^n)} \arrow[d, "j"] \\ \Sigma^d S^0 \arrow[u, "i"] \arrow[r, "\alpha_n"'] & \Sigma S^0 \end{tikzcd}$$

where i and j are inclusion into the bottom cell and projection onto the top cell respectively, and α_n is defined by the composite $\alpha_n = j \circ A_n \circ i$, we have that $[\alpha_n] \in \pi^s_{d-1}(S^0)$ generates the 2-primary component of the image of J if $n \geq 4$ (a subgroup of order 2^n if $1 \leq n < 4$).

5.2.5: Remark:

Recall, see e.g. [To], that ${}_2\pi^s_7(S^0) = Z/16$ generated by the Hopf map σ.

From the coefficient sequence (2.1.6):

$$\cdots \to {}_2\pi^s_8(S^0) \xrightarrow{4} {}_2\pi^s_8(S^0) \xrightarrow{r} {}_2\pi^s_8(S^0;Z/4) \xrightarrow{\partial} {}_2\pi^s_7(S^0) \xrightarrow{4} {}_2\pi^s_7(S^0) \to \cdots$$

and since ${}_2\pi^s_8(S^0) = Z/2 \oplus Z/2$, it follows that r is injective.

Also, from the Universal Coefficient Sequence (2.1.5) we see that

$${}_2\pi^s_8(S^0;Z/4) = Z/4 \oplus Z/2 \oplus Z/2 .$$

Let $\hat{\sigma}$ = generator of order 4 in ${}_2\pi^s_8(S^0;Z/4)$. Observe that $\partial(\hat{\sigma}) = 4\sigma$.

Consider now the following diagram for q sufficiently large:

$$\begin{tikzcd} P^{q+8}(4) \arrow[r, dashed, "A_1"] \arrow[dr, "a_1"] & P^q(4) \arrow[d, "j"] \\ S^{q+7} \arrow[u, "i"] \arrow[r, "\alpha_1"'] & S^0 \end{tikzcd}$$

where a_1 is a map that represents $\hat{\sigma}$, i and j are inclusion into the bottom cell and projection onto the top cell respectively, and $\alpha_1 \simeq a_1 \circ i = i^{\#}(a_1)$ represents $\partial(\hat{\sigma}) = 4\sigma$ (recall that the Bockstein morphism ∂ is given by $i^{\#}$).

Then:

i) $\alpha_1 \in \pi_7^S(S^0)$ has order 4 since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and σ is of order 16.

ii) The Toda bracket $\{4, \alpha_1, 4\} = 0$ by [To; 3.7].

iii) The Adams e-invariant [Ad_2; §3] of α_1 is:

$$e(\alpha_1) = 1/4 \pmod 1$$

since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and $e(\sigma) = 1/16 \pmod 1$ by [Ad_1].

It follows, from [Ad_2; 12.5] (see 3.2.2), that there exists a map A_1 making the previous diagram homotopy commutative, and moreover, A_1 is an Adams map.

5.2.6: Transfer maps:

Let $H \subseteq G$ be finite groups, and let $n = [G:H]$ = index of H in G. As usual, let Σ_r denote the r-th symmetric group for $1 \leq r \leq \infty$.

Consider the following construction:

The natural morphisms:

$$G \longrightarrow \Sigma_n \int H \longrightarrow \Sigma_\infty \int H$$

induce, upon applying the classifying space functor $B(\_)$ and the plus construction $(\_)^+$ (§1.1), maps:

$$BG \dashrightarrow B\Sigma_n \int H \longrightarrow \left(B\Sigma_\infty \int H\right)^+ \simeq Q_0\left(BH_+\right)$$

where the equivalence is that of (1.1.8).

Now, the natural extension of the map $BG \dashrightarrow Q_0\left(BH_+\right)$ to $Q_0\left(BG_+\right)$ is called the (stable) transfer map, and we will denote it by:

$$t : Q_0\left(BG_+\right) \longrightarrow Q_0\left(BH_+\right)$$

5.2.7: Theorem:

Let $b \in \pi_2^s(BZ/4\,;\,Z/4) = Z/4 \oplus Z/2$ be the generator of order 4 (see 5.1.1 and 4.3.6). Let $t : Q_0(BZ/4)_+ \longrightarrow Q_0(S^0)$ be the transfer map associated to the inclusion $1 \hookrightarrow Z/4$.

Consider $b^4 \in \pi_8^s(BZ/4\,;\,Z/4)$ and let $\hat{\sigma} \in \pi_8^s(S^0;\,Z/4)$ be as in (5.2.5). Then:

$$t_\# : \pi_8^s(BZ/4\,;\,Z/4) \longrightarrow \pi_8^s(S^0;Z/4)$$

sends b^4 to $\hat{\sigma}$.

Proof:

1) The transfer map $t_\#$ can be factored as :

$$t_\# : \pi_8^s(BZ/4\,;\,Z/4) \xrightarrow{t_2} \pi_8^s(\mathbb{R}P^\infty\,;\,Z/4) \xrightarrow{t_1} \pi_8^s(S^0\,;\,Z/4)$$

where $\mathbb{R}P^\infty = BZ/2$, t_1 is the transfer map associated to $1 \hookrightarrow Z/2$ and t_2 is the transfer associated to $Z/2 \hookrightarrow Z/4$.

2) Consider now the following commutative diagram:

$$\begin{array}{ccccc}
\pi_8^S(BZ/4\,;\,Z/4) & \overset{t_2}{\underset{f_2}{\rightleftarrows}} & \pi_8^S(RP^\infty\,;\,Z/4) & \overset{t_1}{\underset{f_1}{\rightleftarrows}} & \pi_8^S(S^0\,;\,Z/4) \\
\partial\downarrow & & \downarrow\partial & & \downarrow\partial \\
\pi_7^S(BZ/4) & \overset{t_2}{\underset{f_2}{\rightleftarrows}} & \pi_7^S(RP^\infty) & \overset{t_1}{\underset{f_1}{\rightleftarrows}} & \pi_7^S(S^0)
\end{array}$$

where f_i, $i = 1,2$ are the morphisms induced by the group inclusions, t_i are the corresponding transfer maps, and ∂ the Bockstein morphisms.

3): i) We know that $\partial(\hat{\sigma}) = 4\cdot\sigma$.

ii) Similarly, if $\tilde{a}$ is a generator of order 4 of $\pi_8^S(RP^\infty;Z/4) = Z/4 \oplus Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2$, see [Li], and a is a generator of order 16 of $\pi_7^S(RP^\infty) = Z/16 \oplus Z/2$, then:

$$\partial(\tilde{a}) = 4\cdot a$$

iii) Also, if $\tilde{b}$ is a generator of order 8 of $\pi_7^S(BZ/4) = Z/8 \oplus Z/2 \oplus Z/2$, then:

$$\partial(b^4) = 2\cdot\tilde{b}$$

4): i) By Kahn-Priddy [K-P], see also [H-S; Remark 4, p.26], t_1 is split surjective on the 2-primary components. Thus:

$$t_1(a) = \sigma \in {}_2\pi_7^S(S^0)$$

and so, by the commutativity of the right-hand side square of the diagram in (2), we have:

$$t_1(\tilde{a}) = \hat{\sigma}$$

ii) Now, observe that $f_2\cdot t_2$ = multiplication by 2 on $\pi_*^S(BZ/4)$ so that

$$t_2(\tilde{b}) - 2\cdot a \in \pi_7^S(RP^\infty)$$

Therefore:

$$t_2\,\partial(b^4) - 4\cdot a = \partial(\tilde{a})$$

Hence :

$$t_2(b^4) = \tilde{a} \; .$$

///.

5.2.8: Remarks:

i) Recall that for $A = Z[1/2, \zeta_4]$ we defined (5.1.1)

$$\beta = (d_1)_\#(b) \in K_2(A; Z/4)$$

where $d_1 : (B\Sigma_\infty \int Z/4)^+ \longrightarrow BGLA^+$

ii) We also defined (5.1.2)

$$\beta_1 = \beta^4 = \left((d_1)_\#(b)\right)^4 = (d_1)_\#(b^4) \in K_8(A; Z/4)$$

iii) Now, we have a commutative (up to an inner automorphism) diagram [Sn$_3$;Corrigendum 2.6]:

$$\begin{array}{ccccc}
\Sigma_n \int Z/4 & \xrightarrow{d_1 \times d_2\eta} & & & GL_n Z[1/2, \zeta_4] \times GL_n Z[1/2, \zeta_4] \\
t \downarrow & & & & \downarrow \oplus \\
\Sigma_{4n} & \xrightarrow{d_2} & GL_{4n} Z[1/2, \zeta_4] & \xleftarrow{s} & GL_{2n} Z[1/2, \zeta_4]
\end{array}$$

where:

s is the stabilization map

$d_1 : \Sigma_r \int Z/4 \longrightarrow \Sigma_r \int GL_1 Z[1/2, \zeta_4] \longrightarrow GL_r Z[1/2, \zeta_4]$

is induced by the inclusion $Z/4 \approx \mu_4 \longrightarrow GL_1 Z[1/2, \zeta_4]$

$d_2 : \Sigma_m \longrightarrow GL_m Z \longrightarrow GL_m Z[1/2, \zeta_4]$

is induced by inclusion of permutation matrices

$$\eta : \Sigma_n \int Z/4 \dashrightarrow Gl_n Z[1/2, \zeta_4]$$

is induced by the morphism $Z/4 \to 1$

$$t : \Sigma_m \int Z/4 \longrightarrow \Sigma_{4m}$$

is the transfer morphism induced by sending a generator of $Z/4$ to the cycle $(1,2,3,4) \in \Sigma_4$.

iv) From this diagram, applying the classifying space functor, the plus construction and taking $n \to \infty$; we obtain a commutative diagram:

$$\begin{array}{ccc}
\pi_*\left(B\Sigma_\infty \int Z/4^+; Z/4\right) & \xrightarrow{(d_1)_\# \times (d_2\eta)_\#} & K_*\left(Z[1/2, \zeta_4]; Z/4\right) \times K_*\left(Z[1/2, \zeta_4]; Z/4\right) \\
\downarrow t_\# & & \downarrow + \\
\pi_*\left(B\Sigma_\infty^+; Z/4\right) & \xrightarrow{(d_2)_\#} & K_*\left(Z[1/2, \zeta_4]; Z/4\right)
\end{array}$$

and since $B\Sigma_\infty \int Z/4^+ \simeq Q_0(BZ/4_+)$ and $b \in \pi_2^S(BZ/4_+; Z/4)$ originates in $\pi_2^S(BZ/4; Z/4)$ then $\eta_\#(b) = 0$ and hence $\eta_\#(b^4) = 0$ also.

Therefore, we have the formula:

$$(d_2)_\# t_\#(b^4) = (d_1)_\#(b^4)$$

v) Consequently, we have:

$$\begin{aligned}
\beta_1 = (d_1)_\#(b^4) &= (d_2)_\# t_\#(b^4) \\
&= (d_2)_\#(\hat{\sigma}) \quad \text{by (5.2.7)}
\end{aligned}$$

where $\hat{\sigma} = j \circ A_1$, A_1 an Adams map (5.2.5).

vi) Now, $d_2 : B\Sigma_\infty^+ \longrightarrow BGLZ[1/2, \zeta_4]^+$ is the base-point component of the 0-th spaces of the unit

$$D : \underline{S}^0 \longrightarrow \underline{KZ}[1/2, \zeta_4]$$

of the algebraic K-theory spectrum of $A = Z[1/2, \zeta_4]$.

Therefore,

$$\beta_1 = (d_2)_{\#}(\hat{\sigma}) = D_{\#}(\hat{\sigma}) .$$

5.2.9: Now, in order to have a similar description for the higher Bott elements $\beta_n \in K_*(A ; Z/4^n)$ of (5.1.3) for $n > 1$, we proceed as in [Sn_3; §3] as follows:

We want $\beta_n \in D_{\#}\left(\pi^s_{8 \cdot 4^{n-1}}(S^0; Z/4^n)\right)$ where

$$D_{\#} : \pi^s_*(S^0;Z/4^n) \longrightarrow K_*(A ; Z/4^n)$$

By induction on n suppose $\beta_n \in D_{\#}\left(\pi^s_{8 \cdot 4^{n-1}}(S^0;Z/4^n)\right)$ and consider $\beta_{n+1} \in K_{8 \cdot 4^n}(A ; Z/4^{n+1})$.

Let $r_{\#} : \pi_*(\; ; Z/4^{n+1}) \longrightarrow \pi_*(\; ; Z/4^n)$, be the reduction map.

Let $x_n \in \pi^s_{8 \cdot 4^{n-1}}(S^0;Z/4^n)$ such that $D_{\#}(x_n) = \beta_n$, and consider $x_n^4 \in \pi^s_{8 \cdot 4^n}(S^0 ; Z/4^n)$. Since the differentials in the homotopy Bockstein spectral sequence are derivations (2.4.4) then $d(x_n^4) = 0$ since $\pi^s_*(S^0;Z/4)$ is a $Z/4$-module.

Thus, there exists $x_{n+1} \in \pi^s_{8 \cdot 4^n}(S^0;Z/4^{n+1})$ such that $r_{\#}(x_{n+1}) = x_n^4$.

Now, since $D_{\#}$ is a ring map we have $D_{\#}(x_n^4) = \beta_n^4$.

Therefore, by naturality we have:

$$\begin{tikzcd}
x_{n+1} \arrow[r, mapsto, "r_{\#}"] \arrow[d, mapsto, "D_{\#}"'] & x_n^4 \arrow[d, mapsto, "D_{\#}"] \\
D_{\#}(x_{n+1}) \arrow[r, mapsto, dashed, "r_{\#}"'] & \beta_n^4
\end{tikzcd}$$

i.e. $D_{\#}(x_{n+1})$ is an element of $K_{8 \cdot 4^n}(A ; Z/4^{n+1})$ that reduces mod-4 to β_n^4 .

Therefore, we may choose $\beta_{n+1} = D_\#(x_{n+1})$ since this element reduces to β_n^4 which itself reduces to $\left(\beta_1^{4^{n-1}}\right)^4 = \beta_1^{4^n}$ by (5.1.3).

5.2.10: Remark:

Analogously to [Sn$_3$;§3], see 3.2.11, we can see that for $n \geq 1$, a suitable choice for $x_n \in \pi_*^s(S^0;Z/4^n)$ is given by an Adams map, i.e. by $a_n = j \circ A_n$ where j and A_n are maps in the diagram:

$$\begin{tikzcd}
P^{(sd_n+8)4^{n-1}}(4^n) \arrow[r, dashed, "A_n"] & P^{sd_n \cdot 4^{n-1}}(4^n) \arrow[d, "j"] \\
S^{(sd_n+8)4^{n-1}-1} \arrow[u, "i"] \arrow[r, "\alpha_n"] & S^{sd_n \cdot 4^{n-1}}
\end{tikzcd}$$

where $d_n = \max\{8, 4^{n-1}\} = \deg(A_n)$, and A_n an Adams map.

5.2.11: Now, let X be a commutative A-algebra, $A = Z[1/2, \zeta_4]$.

Then, $\underline{KX}$ is a $\underline{KA}$-module. We denote this action by

$$\mu : \underline{KA} \wedge \underline{KX} \longrightarrow \underline{KX}$$

Let $[g] \in K_i(X; Z/4^n) = \pi_i(\underline{KX}; Z/4^n)$ be represented by a map of spectra

$$g : \underline{P(4^n)} \longrightarrow \underline{KX} \quad \text{of degree } i.$$

Consider a representative of the Bott element

$$\beta_n \in K_{8\cdot 4^{n-1}}(A ; Z/4^n)$$

of (5.1.3):

$$\beta_n : \underline{P(4^n)} \longrightarrow \underline{KA}$$

We have a commutative diagram of spectra:

$$\begin{tikzcd}
P(4^n) \arrow[r, "\chi"] \arrow[dd, "A'_n"'] & P(4^n)\wedge P(4^n) \arrow[rr, "\beta_n\wedge g"] \arrow[d, "A_n\wedge 1"] \arrow[drr, "a_n\wedge g"] & & KA\wedge KX \arrow[r, "\mu"] & KX \\
 & P(4^n)\wedge P(4^n) \arrow[rr, "j\wedge g"] \arrow[d, "j\wedge 1"] & & S^0\wedge KX \arrow[u, "D\wedge 1"'] \arrow[ur, "\simeq"'] & \\
P(4^n) \arrow[r, "\simeq"] & S^0\wedge P(4^n) \arrow[urr, "1\wedge g"'] & & &
\end{tikzcd}$$

where the composite of the top row represents the product

$$\beta_n \cdot [g] \in K_{i+d}(X\,;\,Z/4^n)$$

$\underline{S}^0$ is the sphere spectrum, χ is the copairing of Moore spectra of (2.3.2); μ is the multiplication induced by the action of A on X, A_n and j are the maps of spectra of (5.2.4) and $a_n = j_* A_n$ in (5.2.4), and D is the unit of $\underline{KA}$.

It follows that A'_n is also an Adams map betwen Moore spectra.

From the commutativity of this diagram it follows that:

$$\beta_n \cdot [g] = [g \circ A'_n] = A'^{*}_n[g] \in K_{i+d}(X\,;\,Z/4^n)$$

i.e. multiplication by β_n is precomposition with an Adams map A'_n.

From this remark, we obtain the analogue of Snaith's theorem [Sn_3;3.22]:

5.2.12: Theorem:

Let X be a commutative $Z[1/2\,,\,\zeta_4]$-algebra. Suppose that there exists a map of Moore spaces

$$A_n : P^{q+d}(4^n) \longrightarrow P^q(4^n)$$

with $d = 8\cdot 4^{n-1}$, such that its stable homotopy class is

$$A'_n : \underline{P(4^n)} \longrightarrow \underline{P(4^n)}$$

an Adams map of Moore spectra as in (5.2.11).

Suppose $i \geq q$. Then:

$$K_i(X;Z/4^n)[1/\beta_n] \approx \lim_{\overrightarrow{k}} \left(K_{i+kd}(X;Z/4^n) \xrightarrow{(\Sigma^{i+kd-q}A_n)^*} K_{i+(k+1)d}(X;Z/4^n) \right)$$

Proof:

First, recall that there exist Adams maps

$$A_n : P^{q+d}(4^n) \longrightarrow P^q(4^n)$$

for $d = \max\left(8, 2^{2n-1}\right)$ and q large enough (5.2.4).

Now, by choosing appropiate compositions of suspensions of these Adams maps we get maps

$$A''_n : P^{q+8\cdot 4^{n-1}}(4^n) \longrightarrow P^q(4^n)$$

that still induce isomorphisms in K-theory, i.e. they are Adams maps.

Now, by the remark (5.2.11)

$$\beta_n \cdot [g] = A'^*_n[g] = [g \cdot A'_n]$$

and since the isomorphisms

$$K_i(X; Z/4^n) = [P^i(4^n), BGLX^+] \approx [\Sigma^i \underline{P(4^n)}, \underline{KX}]$$

are such that the following diagram commutes

$$\begin{array}{ccc} [P^i(4^n), BGLX^+] & \approx & [\Sigma^i \underline{P(4^n)}, \underline{KX}] \\ \downarrow{\scriptstyle (\Sigma^{i-q}A_n)^*} & & \downarrow{\scriptstyle (A'_n)^*} \\ [P^{i+d}(4^n), BGLX^+] & \approx & [\Sigma^{i+d} \underline{P(4^n)}, \underline{KX}] \end{array}$$

provided $i \geq q$, since we are assuming that the stable homotopy class of the map A_n is A'_n. Therefore the result follows.

///.

§5.3: 2-primary J-theory diagrams:

In this section we will obtain certain diagrams that give a factorization of the localization map for 2-primary Bott-periodic algebraic K-theory using the Hurewicz morphism for J-homology and a variant of it. These are the 2-primary analogous of the results of Snaith [Sn_3;§4] and [Sn_5].

5.3.1: J-theory:

Let $\Psi^t : KU_*(\ ;\ Z/4^n) \longrightarrow KU_*(\ ;\ Z/4^n)$ be an Adams operation for t a prime (or prime power) in the sequence $\{1 + 2^a(4n+1) : n \in \mathbf{N}\}$ where $2 \leqslant n \leqslant a < \infty$, are integers.

Since $(t, 4) = 1$ the Ψ^t are stable operations i.e. they are represented by maps of spectra

$$\Psi^t : \underline{BUZ/4^n} \longrightarrow \underline{BUZ/4^n}$$

where $\underline{BUZ/4^n}$ denotes the spectrum of mod-4^n KU_*-homology, i.e. $\underline{BUZ/4^n} = \underline{BU} \wedge P^2(4^n)$ where $\underline{BU}$ is the spectrum representing complex topological K-theory.

Now, let $\underline{JZ/4^n}$ be the fibre (cofibre) of

$$\Psi^t - 1 : \underline{BUZ/4^n} \longrightarrow \underline{BUZ/4^n}$$

in the stable homotopy category.

Thus, we have a fibre (cofibre) sequence:

$$\underline{JZ/4^n} \xrightarrow{\lambda} \underline{BUZ/4^n} \xrightarrow{\Psi^t - 1} \underline{BUZ/4^n}$$

Consider the homology theory defined by the spectrum $\underline{JZ/4^n}$:

$$J_1(X\,;\,Z/4^n) = (\underline{JZ/4^n})_1(X) = [\underline{S}^1,\, X_+ \wedge \underline{JZ/4^n}]\,.$$

Thus, $J_*(X\,;\,Z/4^n)$ fits into a long exact sequence:

$$\cdots \to J_1(X\,;\,Z/4^n) \xrightarrow{\lambda} KU_1(X\,;\,Z/4^n) \xrightarrow{\Psi^t-1} KU_1(X\,;\,Z/4^n) \to \cdots$$

and we also note that $J_*(\ ;\,Z/4^n)$ is a $Z/2$-graded homology theory as $KU_*(\ ;\,Z/4^n)$.

5.3.2: Remark:

As in example (3.3.2) we can compute $J_*(P^2(4^n);Z/4^n)$ using the exact sequence of 5.3.1, and the known fact that Ψ^t is multiplication by t on $KU_0(S^2;\,Z/4^n) \approx Z/4^n$. The result is:

$$J_0(P^2(4^n);\,Z/4^n) \approx Z/4^n\,.$$

5.3.3: Remark:

Recall that the Hurewicz morphisms

$h_K : \pi_2(X;Z/4^n) \longrightarrow KU_0(X;Z/4^n)$ and $h_J : \pi_2(X;Z/4^n) \longrightarrow J_0(X;Z/4^n)$

are defined as follows:

i) Given a generator $e \in KU_0(P^2(4^n);Z/4^n) \approx Z/4^n$ we choose a generator $e' \in J_0(P^2(4^n);\,Z/4^n) \approx Z/4^n$ such that $\lambda(e') = e$.

ii) Then:

For $[f] \in \pi_2(X;Z/4^n) = [P^2(4^n),\,X]$

$h_K[f] = f_*(e) \in KU_0(X;Z/4^n)$ and $h_J[f] = f_*(e') \in J_0(X;Z/4^n)$

iii) Thus, we have:

$$h_K[f] = \lambda \cdot h_J[f]$$

for any $[f] \in \pi_2(X;Z/4^n)$.

5.3.4: Remark:

For an Adams map $A_n : P^{q+d}(4^n) \longrightarrow P^q(4^n)$ $d = \max(8, 2^{2n-1})$, from the 5-lemma and from the commutative diagram with exact rows (5.3.1):

$$\begin{array}{ccccccc}
\cdots \to & J_0(P^{q+d}(4^n);Z/4^n) & \xrightarrow{\lambda} & KU_0(P^{q+d}(4^n);Z/4^n) & \to \cdots \\
& (A_n)_* \downarrow \approx & & \approx \downarrow (A_n)_* & \\
\cdots \to & J_0(P^q(4^n);Z/4^n) & \xrightarrow{\lambda} & KU_0(P^q(4^n);Z/4^n) & \to \cdots
\end{array}$$

it follows that $(A_n)_*$ is also an isomorphism on J-theory.

Thus, inverting iterated compositions with A_n in J-theory is an isomorphism, i.e.:

$$J_*(X;Z/4^n)[1/A_n] \approx J_*(X;Z/4^n)$$

5.3.5: Theorem:

Let X be a commutative $Z[1/2, \zeta_4]$-algebra. There exists a commutative diagram :

$$\begin{tikzcd}
K_i(X;Z/4^{2n}) \arrow[rr, "\rho"] \arrow[dr, "h_J"'] & & K_i(X;Z/2^{2n})[1/\beta_n] \arrow[dl, "I"] \\
& J_i(BGLX^+;Z/2^{2n}) &
\end{tikzcd}$$

where ρ is the localization map and h_J is the Hurewicz morphism for J-theory.

Moreover, I is injective.

Proof:

By Theorem (5.2.1) we have:

$$K_*(X;Z/4^n)[1/\beta_n] \approx K_*(X;Z/4^n)[1/A_n]$$

$$= \lim_{\rightarrow} \left(K_{i+kd}(X;Z/4^n) \xrightarrow{(\Sigma^{i+kd-q} A_n)^*} K_{i+(k+1)d}(X;Z/4^n) \right)$$

if $i \geq q$.

Now, we may choose generators

$$e_{m,i} \in KU_i(P^m(4^n);Z/4^n) \approx Z/4^n$$

such that for

$$\Sigma^{m-q}A_n : \Sigma^{m-q}P^{q+d}(4^n) \longrightarrow \Sigma^{m-q}P^q(4^n)$$
$$\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$
$$\qquad\qquad P^{m+d}(4^n) \qquad\qquad P^m(4^n)$$

we have:

$$(\Sigma^{m-q}A_n)^*(e_{m,i}) = e_{m-d,i}$$

Similarly, for J-homology, we can choose generators

$$e'_{m,i} \in J_i(P^m(4^n);Z/4^n) \approx Z/4^n$$

with analogous properties.

Now, observe that in this situation, we have commutative diagrams:

$$\begin{array}{ccc}
K_{i+kd}(X;Z/4^n) & \xrightarrow{(\Sigma^{i+kd-q}A_n)^*} & K_{i+(k+1)d}(X;Z/4^n) \\
\Big\downarrow h_J & & \Big\downarrow h_J \\
J_{i+kd}(X;Z/4^n) & \xrightarrow[(\Sigma^{i+kd-q}A_n)^*]{\approx} & J_{i+(k+1)d}(X;Z/4^n)
\end{array}$$

so that we can define:

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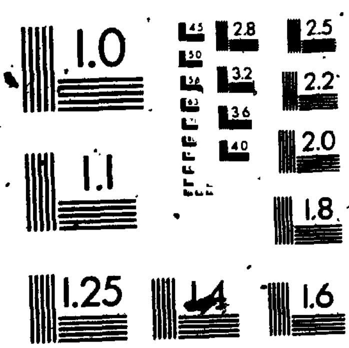

$$I = \varinjlim h_J : \varinjlim_k \left(K_{1+kd}(X;Z/4^n)\right) \longrightarrow \varinjlim_k \left(J_1(X;Z/4^n)\right)$$

i.e.

$$I = \varinjlim h_J : K_1(X;Z/4^n)[1/A_n] \longrightarrow J_1(X;Z/4^n)[1/A_n] \approx J_1(X;Z/4^n)$$

by sending a representative

$$x \in K_{1+kd}(X;Z/4^n) = [P^{1+kd}(4^n), BGLX^+]$$

of $[x] \in K_1(X;Z/4^n)[1/A_n]$, to

$$h_J(x) = x_*(e'_{1+kd,1+kd}) \in J_{1+kd}(X;Z/4^n)$$

Clearly I makes the diagram in the statement of the theorem commute.

Now, to show that I is injective, recall that by [Sn_3:3.4]

$$\operatorname{Ker}\rho = \operatorname{Ker} h_J \quad \text{on} \quad K_j(X;Z/4^n) \quad \text{if} \quad j \geq 3 .$$

Now, since the groups $\{\rho\left(K_{1+kd}(X;Z/4^n)\right) : k \geq 3\}$ generate $K_1(X;Z/4^n)[1/A_n]$, then if $[x] \in \operatorname{Ker} I \subset K_1(X;Z/4^n)[1/A_n]$ is represented say by:

$$x \in K_{1+kd}(X;Z/4^n) \quad \text{for some} \quad k \geq 0 , \quad \text{then:}$$

$$0 = I[x] = I\rho(x) = h_J(x) \qquad \text{since} \quad I \cdot \rho = h_J$$

Therefore, $x \in \operatorname{Ker} h_J = \operatorname{Ker}\rho$ on $K_{1+kd}(X;Z/4^n)$ and so $0 = \rho(x) = [x]$.

///.

5.3.6: Remark:

We have an obvious analogous of the diagram of (5.3.5) when we replace $J_*(X;Z/4^n)$ by $KU_*(X;Z/4^n)$.

5.3.7. Now, we are going to construct a diagram analogous to (5.3.5) in dimension 2 for $n = 1$. To do this, we take the Moore space: $P^2(4) = S^1 \cup_4 e^2$.

Consider now:

$$QP^2(4) = \lim \Omega^m \Sigma^m P^2(4)$$

We shall consider the primitives of $KU_0(QP^2(4);Z/4)$ and $KU_1(QP^2(4);Z/4)$.

In particular, for a generator $[f] \in \pi_2(BU;Z/4) \approx Z/4$ represented by a map $f : P^2(4) \to BU$, let $\bar{f} : QP^2(4) \to BU$ denote the (infinite loop map) which extends f to $QP^2(4)$, i.e. $\bar{f} = r\,Q(f)$ in the diagram:

$$\begin{tikzcd}
P^2(4) \arrow[r, "f"] \arrow[d, "i"'] & BU \\
QP^2(4) \arrow[r, "Q(f)"'] & QBU \arrow[u, "r"']
\end{tikzcd}$$

where r is the canonical retraction (since BU is an infinite loop space).

We will show that:

$$\bar{f}_* : PKU_0(QP^2(4);Z/4) \to PKU_0(BU;Z/4)$$

is injective.

Next, for an Adams map:

$$A_1 : P^{q+8}(4) \to P^q(4)$$

we consider its s-iterate

$$A_1^s : \Sigma^{q-2}P^{8s+2}(4) \longrightarrow \Sigma^{q-2}P^2(4)$$

and the adjoint

$$\tilde{A}_1^s : P^{8s+2}(4) \longrightarrow QP^2(4)$$

We shall consider the action of $(\tilde{A}_1^s)_*$ on $KU_o(\ ;Z/4)$ (or $J_o(\ ;Z/4)$).

First, we recall the definition of primitives:

5.3.8. Primitives.

Recall that the inclusion $i : Y \vee Y \to Y \times Y$ induces an injective morphism (for any Y):

$$i_* : \left(KU_*(Y;Z/\ell^n) \otimes 1\right) + \left(1 \otimes KU_*(Y;Z/\ell^n)\right) \rightarrowtail KU_*(Y;Z/\ell^n)$$

Now, let $d : Y \to Y \times Y$ be the diagonal map. The primitives of $KU_*(Y;Z/\ell^n)$ are defined by:

$$PKU_*(Y:Z/\ell^n) = \{z \in KU_*(Y:Z/\ell^n) : d_*(z) = i_*(z \otimes 1 + 1 \otimes z)\}$$

5.3.9: Recall now that the mod-4 reduced K-homology groups of $P^2(4)$ are given by:

$$KU_o(P^2(4):Z/4) \approx Z/4 \approx KU_1(P^2(4):Z/4)$$

(see 5.3.2).

Let $u \in KU_a(P^2(4):Z/4)$, $a = 0 \bmod(2)$, be a generator. Then, the Bockstein of u, $v = \partial(u) \in KU_{a-1}(P^2(4);Z/4)$, $a-1 = 1 \bmod(2)$, is also a generator.

Let $Q^s : KU_*(Y;Z/2^r) \to KU_*(Y:Z/2^{r-s})$, for $1 \leq s < r$, and for Y a suitable infinite loop space, be the Dyer-Lashof operations of Snaith-McClure [Mc], [Sn$_6$].

Let $\rho : KU_*(\ ;Z/2^r) \to KU_*(\ ;Z/2^t)$, $1 \leq t \leq r$, be the reduction map.

5.3.10: By [Mc: Theorem 5]:

$$KU_*(QP^2(4);Z/2) \cong Z/2[u_1, u_2] \otimes E(v_1, v_2)$$

where:

i) For $u \in KU_0(P^2(4);Z/4)$ and $v = \partial(u) \in KU_1(P^2(4);Z/4)$ generators as in (5.3.9), denote with the same symbols the corresponding elements in $KU_*(QP^2(4);Z/4)$. Then:

$$u_1 = \rho(u) \in KU_0(QP^2(4);Z/2)$$

$$v_1 = \rho(v) \in KU_1(QP^2(4);Z/2)$$

$$u_2 = Q^1u \in KU_0(QP^2(4);Z/2)$$

$$v_2 = \partial Q^1u = \partial u_2 \in KU_1(QP^2(4);Z/2)$$

where $Q^1 : KU_*(QP^2(4);Z/4) \to KU_*(QP^2(4);Z/2)$ and $\partial : KU_0(QP^2(4);Z/2) \to KU_1(QP^2(4);Z/2)$ is the Bockstein differential.

ii) Observe that all the generators v_i have zero Bocksteins and that their squares are zero. Thus, they generate an exterior algebra.

5.3.11: Primitives in $KU_*(QP^2(4);Z/2)$:

We now look a the primitive elements of $KU_*(QP^2(4);Z/2)$.

i) $v_1 = \rho(v) = \rho\partial(u) \in KU_1(QP^2(4);Z/2)$, is clearly primitive.

ii) $v_2 = \partial Q^1u = Q^1\partial(u) = Q^1v$, and so:

$$\begin{aligned} d_*(v_2) = d_*(Q^1v) &= Q^1(d_*(v)) \\ &= Q^1(1\otimes v + v\otimes 1) \\ &= 1\otimes Q^1v + Q^1v\otimes 1 \\ &= 1\otimes v_2 + v_2\otimes 1 \end{aligned}$$

since Q^1 is additive in odd degrees by [Mc; Theorem 1(ii)].

Thus, v_2 is primitive.

iii) Clearly u_1 is primitive.

iv) $u_2 = Q^1 u$, and we have:

$$d_*(u_2) = d_*(Q^1 u) = Q^1(d_*(u)) \quad \text{since } d_* \text{ is an infinite loop map}$$

$$= Q^1(u \otimes 1 + 1 \otimes u)$$

$$= Q^1(u \otimes 1) + Q^1(1 \otimes u) - \rho\big((u \otimes 1)(1 \otimes u)\big)$$

by [Mc; Theorem 1(ii)]

$$= Q^1(u) \otimes 1 + 1 \otimes Q^1(u) - u_1 \otimes u_1$$

$$= u_2 \otimes 1 + 1 \otimes u_2 - u_1 \otimes u_1$$

v) Now, since $KU_*(QP^2(4);Z/2)$ is finitely generated, we have an exact Milnor-Moore sequence:

$$0 \to P\left(KU_*(QP^2(4);Z/2)^2\right) \to PKU_*(QP^2(4);Z/2) \xrightarrow{\lambda} QKU_*(QP^2(4);Z/2)$$

where $KU_*(QP^2(4);Z/2)^2$ denotes the subalgebra of 2-th powers. $P(\ )$ denotes the primitives and $Q(\ )$ the indecomposables.

vi) Observe that $\operatorname{Im}\lambda$ is generated by $\{u_1, v_1, v_2\}$, and also $P\left(KU_*(QP^2(4);Z/2)^2\right) = \left(PKU_*(QP^2(4);Z/2)\right)^2$.

5.3.12: Proposition:

i) A basis for $PKU_0(QP^2(4);Z/2)$ is given by : $\{ u_1^{2^\alpha} : \alpha \geq 0 \}$

ii) A basis for $PKU_1(QP^2(4);Z/2)$ is given by : $\{ v_1, v_2 \}$.

Proof:

This follows from the previous remarks. ///.

5.3.13: Now, we consider the primitives in $KU_*(QP^2(4);Z/4)$.

We will need the following notation:

Let $u \in KU_0(Y;Z/2^n)$ and define $X(u) \in KU_0(Y;Z/2^n)$ by the formula:

$$X(u) = u^2 + 2_*Q(u)$$

where $Q(u) \in KU_0(Y;Z/2^{n-1})$ and $2_* : KU_0(Y;Z/2^{n-1}) \to KU_0(Y;Z/2^n)$.

5.3.14: Lemma:

If $u \in PKU_0(Y;Z/2^n)$ then $X(u) \in PKU_0(Y;Z/2^n)$.

Proof:

As in [Sn_4:1.6]:

i) $d_*(u^2) = d_*(u)^2 = (u \otimes 1 + 1 \otimes u)^2$ since u is primitive

$$= u^2 \otimes 1 + 1 \otimes u^2 + 2\cdot(u \otimes u)$$

ii) $d_*(Q^1u) = Q^1 d_*(u) = Q^1(u \otimes 1 + 1 \otimes u)$

$$= Q^1(u) \otimes 1 + 1 \otimes Q^1(u) - \rho(u \otimes u)$$ by [Mc;Theorem 1(ii)]

Therefore:

$$\begin{aligned} d_*(2_*Q(u)) &= 2_*(d_*Q(u)) \\ &= 2_*Q(u) \otimes 1 + 1 \otimes 2_*Q(u) - 2_*\rho(u \otimes u) \\ &= 2_*Q(u) \otimes 1 + 1 \otimes 2_*Q(u) - 2\cdot(u \otimes u) \end{aligned}$$

since $2_* \rho$ = multiplication by 2.

iii) Hence, adding (i) and (ii) we see that:

$$d_*\big(X(u)\big) = X(u) \otimes 1 + 1 \otimes X(u).$$

///.

5.3.15: Proposition:

$PKU_0(QP^2(4);Z/4)$ is generated by $\{\rho X^\alpha(u) : \alpha \geq 0\}$ where $u \in KU_0(P^2(4);Z/4)$ is a generator, and $X^\alpha(u) = X(+X\cdots(X(u))\cdots)$ is the α-iterate of $X(\ )$, and $\rho : KU_*(\ ;Z/4) \to KU_*(\ ;Z/2)$ is the reduction map.

Proof:

To prove this proposition we need some results about the Bockstein spectral sequence $\{E_r^*, d_r : r \geq 1\}$ for the K-homology of $QP^2(4)$. This result comes from [Mc]:

i) $E_1^* = KU_*(QP^2(4);Z/2) = Z/2[u_1,u_2] \otimes E(v_1,v_2)$

with $d_1(u_2) = v_2$ and d_1 zero on the other generators.

ii) Hence:

$$E_2^* = Z/2[u_1,u_2^2] \otimes E(v_1,u_2v_2)$$

with $d_2(u_2^2) = u_2v_2$ and $d_2(u_1) = v_1$ and d_2 zero on the other generators.

iii) We also know, (5.3.12), that:

$$PE_1^0 = PKU_0(QP^2(4);Z/2) = \langle u_1^{2^\alpha} : \alpha \geq 0\rangle$$
$$PE_1^1 = PKU_1(QP^2(4);Z/2) = \langle v_1,v_2\rangle$$

iv) Now, let $w \in PKU_0(QP^2(4);Z/4)$ and consider the exact sequence:

$$KU_1(QP^2(4);Z/2) \xrightarrow{\partial} KU_0(QP^2(4);Z/2) \xrightarrow{2_*} KU_0(QP^2(4);Z/4) \xrightarrow{\rho} KU_0(QP^2(4);Z/2)$$

Since $w \in PKU_0(QP^2(4);Z/4)$ then $\rho(w) \in PKU_0(QP^2(4);Z/2)$ and since $PKU_0(QP^2(4);Z/2) = \langle u_1^{2^\alpha} : \alpha \geq 0\rangle$, then:

$$\rho(w) = \sum_\alpha \lambda_\alpha \cdot u_1^{2^\alpha}$$

$$= \sum_\alpha \lambda_\alpha \cdot \rho(u^{2^\alpha}) \qquad \text{since} \quad u_1 = \rho(u)$$

$$= \sum_\alpha \lambda_\alpha \cdot \rho(X^\alpha(u)) \qquad \text{since } X^\alpha(u) = u^{2^\alpha}$$

Thus : $\rho\left(w - \sum_\alpha \lambda_\alpha \cdot X^\alpha(u)\right) = 0$,

i.e. $v = w - \sum_\alpha \lambda_\alpha \cdot X^\alpha(u) \in \operatorname{Ker}\rho = \operatorname{Im} 2_* \subset KU_0(QP^2(4);Z/4)$. i.e. there exists $r \in KU_0(QP^2(4);Z/2)$ such that $2_*(r) = v$. Moreover, v is primitive since w and $X^\alpha(u)$ are primitive elements.

Now, let $d : QP^2(4) \longrightarrow QP^2(4) \times QP^2(4)$ be the diagonal map, and consider $d_*(r)$.

From the commutative diagram , for $Y = QP^2(4)$, with exact rows:

$$\begin{array}{ccccc}
KU_1(Y;Z/2) & \xrightarrow{\partial} & KU_0(Y;Z/2) & \xrightarrow{2_*} & KU_0(Y;Z/4) \\
\downarrow d_* & & \downarrow d_* & & \downarrow d_* \\
KU_1(Y \times Y;Z/2) & \xrightarrow[\partial]{} & KU_0(Y \times Y;Z/2) & \xrightarrow[2_*]{} & KU_0(Y \times Y;Z/4)
\end{array}$$

it follows that, if we write

$$d_*(r) = 1 \otimes r + r \otimes 1 + z$$

then:

$$2_* d_*(r) = 1 \otimes 2_* r + 2_* r \otimes 1 + 2_*(z) = d_*(2_*(r))$$

and since $2_*(r)$ is primitive , then $2_*(z) = 0$. i.e. there exists $t \in KU_1(QP^2(4) \times QP^2(4);Z/2)$ such that $z = \partial(t)$, i.e.

$$d_*(r) = r \otimes 1 + 1 \otimes r + \partial(t)$$

Now, since $d_1(\partial(t)) = 0$, where d_1 is the first Bockstein differential, then:

$$d_1(r) \in PE_1^1 \cap \operatorname{Im}(d_1)$$

Now, since $PE_1^1 \cong \langle v_1, v_2\rangle$ and since $d_1(u_2) = v_2$, then:

$$d_1(r) = \lambda\cdot v_2 = \lambda\cdot d_1(u_2) = d_1(\lambda\cdot v_2)$$

Thus:

$x = r - \lambda\cdot u_2 \in KU_0(QP^2(4);Z/2)$ is a d_1-cycle, with diagonal given by:

$$d_*(x) = x\otimes 1 + 1\otimes x + \partial(t) + \lambda\cdot(u_1\otimes u_1)$$

Now, for $[x] \in E_2^* = \mathrm{Ker}(d_1)/\mathrm{Im}(d_1)$, and since $\partial(t)$ is a d_1-boundary, we have:

$$d_*[x] = [x]\otimes 1 + 1\otimes[x] + \lambda\cdot(u_1\otimes u_1)$$

Now, since the reduced diagonal of any canonical generator of E_2^* is a polynomial in u_1, u_2^2, it follows that $\lambda = 0$. Therefore $x = r$ and $[x]$ is primitive in E_2^*.

Therefore $\lambda = 0$, implies that

$$d_*(x) = x\otimes 1 + 1\otimes x \ \mathrm{mod}(\mathrm{Im}(\partial))$$

i.e. x is a primitive element mod(Im(∂)), and so it can be written as

$$x = \sum_\alpha \mu_\alpha\cdot u_1^{2^\alpha} \ \mathrm{mod}(\mathrm{Im}(\partial))$$

Thus, since $2_*\partial = 0$, and since $x = r$, we have:

$$\begin{aligned} 2_*(r) &= 2_*(x) = \sum_\alpha \mu_\alpha\cdot 2_*(u_1^{2^\alpha}) = \sum_\alpha \mu_\alpha\cdot 2_*(\rho u^{2^\alpha}) \\ &= \sum_\alpha \mu_\alpha\cdot 2\cdot X^\alpha(u) \quad \text{since } 2_*\rho = 2 \text{ and } 2\cdot Q(u) = 0 \end{aligned}$$

Thus, $y = w - \Sigma\lambda_\alpha\cdot X^\alpha(u) = 2_*(r) = \Sigma\mu_\alpha\cdot X^\alpha(u)$, i.e. w has the required form.

///.

Now, let $[f] \in \pi_2(BU;Z/4) = Z/4$ be a generator, represented by a map $f : P^7(4) \to BU$.

Let $\bar{f} : QP^2(4) \to BU$ be the natural extension of f to $QP^2(4)$.

5.3.16: Corollary:

$$\bar{f}_* : PKU_0(QP^2(4);Z/4) \to PKU_0(BU;Z/4)$$

is injective.

Proof:

By the previous Theorem $PKU_0(QP^2(4);Z/4) = \langle \rho(X^\alpha(u)) : \alpha \geq 0 \rangle$.

Write $KU_0(BU;Z/4) = Z/4[v_1, v_2, \ldots]$.

The operation Q induces an endomorphism of $QKU_0(BU;Z/4)$, and this has been computed by Snaith [Sn_6;§6]. From this computation it follows that

$$\bar{f}_*\left(X^\alpha(u)\right) \quad Q^\alpha(v_1) \in QKU_0(BU;Z/4)$$

But, by [Sn_6;§6], v_1, $Q(v_1)$, $Q^2(v_1)$, ..., are linearly independent mod-4, and so the result follows.

///.

5.3.17: Now, let $A_1 : P^{q+8}(4) \to P^q(4)$ be an Adams map (5.2.4) and consider the composite:

$$A_1^s = A_1 \circ \Sigma^8 A_1 \ \ldots \ \circ \Sigma^{8(s-1)} A_1 : P^{q+8s}(4) \to P^q(4)$$

Thus:

$$A_1^s : \Sigma^{q-2} P^{2+8s}(4) \to \Sigma^{q-2} P^2(4)$$

Consider the adjoint of A_1^s:

$$\bar{A}_1^s : P^{2+8s}(4) \to QP^2(4)$$

Let $\tilde{A}^s_{1*} : KU_o(P^{2+8s}(4);Z/4) \to KU_o(QP^2(4);Z/4)$ and let $A = Z[1/2, \zeta_4]$, and $[g] \in K_2(A;Z/4)$ be represented by a map

$$g : P^2(4) \to BGLA^+$$

Form the composite

$$P^{2+8s}(4) \xrightarrow{\tilde{A}^s_1} QP^2(4) \xrightarrow{Q(g)} QKA \xrightarrow{D} BKA$$

where D is the structure map of the infinite loop space $KA = K_o(A) \times BGLA^+$.

5.3.18: Definition:

With the notation of (5.3.17), if $q < 2 + 8s$, define

$$\rho'_s : K_2(A;Z/4) \to K_{2+8s}(A;Z/4) \xrightarrow{\rho} K_{2+8s}(A;Z/4)[1/A_1] \approx K_2(A;Z/4)[1/A_1]$$

by the rule

$$\rho'_s[g] = \rho\left([D \circ Q(g) \circ \tilde{A}^s_1]\right)$$

where ρ is the localization map.

5.3.19: Definition:

With the same notation. Define

$$\hat{H} : K_2(A;Z/4) \to KU_o(BGLA^+;Z/4)$$

by :

$$\hat{H}[g] = D_*Q(g)_*\tilde{A}^s_{1*}(u_{2+8s,0})$$

where $u_{2+8s,0} \in KU_o(P^{2+8s}(4);Z/4)$ is a generator chosen as in the proof of (5.3.5).

5.3.20: Remark:

Recall the definition of the morphism

$$I : K_2(A;\mathbf{Z}/4)[1/A_1] \rightarrowtail KU_0(BGLA^+;\mathbf{Z}/4) \qquad (5.3.5)$$

i.e. for $\bar{g} \in K_2(A;\mathbf{Z}/4)[1/A_1]$ represented by $[g] \in K_{2+kd}(A;\mathbf{Z}/4)$ we have: $I(\bar{g}) = g_*(u_{2+kd,0})$.

Thus, for $[g] \in K_2(A;\mathbf{Z}/4)$, we have:

$$I \circ \rho'_s[g] = I(\rho(D\cdot Q(g);\tilde{A}_1^s)) = (D\,Q(g)\,\tilde{A}_1^s)_*(u_{2+8s,0}) = \hat{H}[g]$$

i.e. the following diagram commutes:

$$\begin{tikzcd} K_2(A;\mathbf{Z}/4) \arrow[rr, dashed, "\rho'_s"] \arrow[dr, "\hat{H}"'] & & K_2(A;\mathbf{Z}/4)[1/A_1] \arrow[dl, "I"] \\ & KU_0(BGLA^+;\mathbf{Z}/4) & \end{tikzcd}$$

5.3.21: Lemma [Sn$_5$;3.10]:

For $q < 2+8s$, the element $\tilde{A}^s_{1*}(u_{2+8s,0})$ in $KU_0(QP^2(4);\mathbf{Z}/4)$ is independent of s, up to multiplication by a 2-adic unit.

Proof:

Since $u_{2+8s,0}$ is primitive and since by (5.3.16)

$$PKU_0(QP^2(4);\mathbf{Z}/4) \overset{\bar{f}_*}{\rightarrowtail} PKU_0(BU;\mathbf{Z}/4)$$

is injective, then it suffices to see that

$\bar{f}_*\,\tilde{A}_1^s\,(u_{2+8s,0}) \in KU_0(BU;\mathbf{Z}/4)$ is independent of s.

Now, since $\bar{f}_*\cdot\tilde{A}^s_{1*} = (\bar{f}\,\tilde{A}_1^s)_*$ and since $[\bar{f}\cdot\tilde{A}_1^s]$ generates $\pi_{2+8s}(BU;\mathbf{Z}/4)$ because $[f]$ is a generator and A_1 is an Adams map, then up to a 2-adic unit

$$\bar{f}\,\tilde{A}_1^s = \bar{f}\,\tilde{A}_1^s\,(\Sigma^{2+8s-q}A_1)$$

Now, since

$$(\Sigma^{2+8s-q}A_1)_* : KU_0(P^{2+8(s+1)}(4);Z/4) \longrightarrow KU_0(P^{2+8s}(4);Z/4)$$

is an isomorphism, then up to a 2-adic unit

$$(\bar{f}\ \tilde{A}_1^{s+1})_*(u_{2+8(s+1),0}) = (\bar{f}\ \tilde{A}_1^{s})_*(\Sigma^{2+8s-q}A_1)_*(u_{2+8(s+1),0})$$
$$= (\bar{f}\ \tilde{A}_1^{s})_*(u_{2+8s,0})$$

i.e. this value is the same for s and $s+1$.

///.

5.3.22: Remark:

Since by the previous lemma $\tilde{A}^s_{1*} = \tilde{A}^{s+1}_{1\ *}$ on the generator $u_{2+8s,0}$ and since by definition (5.3.18)

$$\rho'_s\ [g] = \rho[D\cdot Q(g)\cdot\tilde{A}_1^s]$$

then, ρ'_s is independent of s when $2+8s > q$.

Thus, we may define:

$$\rho' = \rho'_s : K_2(A;Z/4) \longrightarrow K_2(A;Z/4)[1/A_1]$$

for some value of s such that $2+8s > q$.

5.3.23: Remark:

It follows from the definition above and from (5.3.20), that the following diagram is commutative:

$$\begin{tikzcd} K_2(A;Z/4) \arrow[rr, "\rho'"] \arrow[dr, "\hat{H}"'] & & K_2(A;Z/4)[1/A_1] \arrow[dl, "I"] \\ & KU_0(BGLA^+;Z/4) & \end{tikzcd}$$

The following proposition gives a formula for $\hat{H}$ in terms of the KU-homology Hurewicz map and the K-theory Dyer-Lashof operations.

5.3.24: Proposition [Sn_5;3.11]:

Up to a 2-adic unit

$$\hat{H} : K_2(A;Z/4) \longrightarrow KU_0(BGLA^+;Z/4)$$

is given by:

$$\hat{H}(y) = h_{KU}(y) + \sum_{j=1}^{N} a_j \cdot X^j\left(h_{KU}(h)\right)$$

where $a_j \in Z/4$ and X^j is the j-iterate of $X(y) = y^2 + 2_*Q(y)$.

Proof:

Let $b_s \in \pi_{2+8s}(BU;Z/4) \approx Z/4$ be a generator represented by

$$b_s : P^{2+8s}(4) \to BU$$

This b_s can be factored as:

$$b_s : P^{2+8s}(4) \xrightarrow{\tilde{A}_1^s} QP^2(4) \xrightarrow{\bar{f}} BU$$

where $\bar{f}$ is the extension of $f : P^2(4) \to BU$ for $[f]$ a generator of $\pi_2(BU;Z/4)$.

Now, since $\bar{f}$ is an infinite loop map, then

$$\bar{f}_*(X(y)) = X(\bar{f}_*(y))$$

Also, we know, (5.3.16), that $\bar{f}_*$ is injective, and that for $u_{2,0} \in PKU_0(P^2(4);Z/4)$ the $X^j(u_{2,0})$ generate $PKU_0(QP^2(4);Z/4)$ by (5.3.15).

Thus, since $\tilde{A}^s_{1*}(u_{2+8s,0})$ is a primitive element, then it is a linear combination of the $X^j(u_{2,0})$:

$$\tilde{A}^s{}_*(u_{2+8s,0}) = \sum_{j=0}^{N} a_j X^j(u_{2,0}) = u_{2,0} + \sum_{j=1}^{N} a_j X^j(u_{2,0})$$

with $a_j \in Z/4$.

Thus:

$$(b_s)_*(u_{2+8s,0}) = \bar{f}_* \tilde{A}^s_{1*}(u_{2+8s,0}) = \omega\cdot\left(\beta_1 + \sum_{j=1}^{N} a_j X^j(\beta_1)\right)$$

where ω is a 2-adic unit and $KU_o(BU;Z/4) = Z/4[\beta_1, \beta_2, \ldots]$ and $\bar{f}_*(u_{2,0}) = \beta_1$.

Finally, for $y = [g] \in K_2(A;Z/4)$ we have:

$$\begin{aligned}\hat{H}[g] &= D_*Q(g)_*\tilde{A}^s_{1*}(u_{2+8s,0}) \qquad \text{by definition of } \hat{H} \\ &= D_*Q(g)_*\left(u_{2,0} + \sum a_j X^j(u_{2,0})\right) \\ &= D_*Q(g)_*(u_{2,0}) + \sum a_j X^j\left(D_*Q(g)_*(u_{2,0})\right) \\ &= h_{KU}[g] + \sum a_j X^j\left(h_{KU}[g]\right)\end{aligned}$$

since $D_*Q(g)_*(u_{2,0}) = g_*(u_{2,0}) = h_{KU}[g]$, and since D and $Q(g)$ are infinite loop maps.

///

Finally, we just remark that using this proposition, we can show as in [Sn_5;p.88,89], that $\rho' = \rho$, so that the diagram in (5.3.23) is a factorization of the localization map ρ in dimension 2, using the map $\hat{H}$.

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