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Sphericals And Primitives Classes In The Bordism Of Compact Lie Groups

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SPHERICALS AND PRIMITIVES CLASSES IN THE BORDISM OF COMPACT LIE GROUPS.

by

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ABSTRACT

The main purpose of this thesis is to study the following question: Do primitives and sphericals agree in $MU_*(X)/tor$ when X is a 1-connected compact Lie group. Our answer is no and the calculation of both subgroups appear in Part IV for the classical groups (stable cases) and in Part V for two exceptional cases namely G_2 and F_4 (ignoring the prime 2).

The main tool for our study is the rational MU operation $\varphi : MU_*(X)\otimes\mathbb{Q} \to MU_*(X)\otimes\mathbb{Q}$; $\varphi = \sum_E m_E S_E$ which detects primitives rationally. Our method is to find the least positive integer $k_\alpha \in \mathbb{Z}$ such that $k_\alpha \varphi(\alpha) \in MU_*(X)/tor \subset MU_*(X)\otimes\mathbb{Q}$ for $\alpha \in MU_*(X)/tor$. This is done in Part III §1. In Part III §2 and §3, we calculate the primitive elements in $MU_*(\mathbb{C}P^\infty)$ and $MU_*(HP^\infty)$ respectively. In §4 we establish a relation between φ and $ch^* : K^*(X) \to H^{**}(X,Q)$ the Chern character as well as between the integrality problem mentioned above and Chern character integrality condition. Also we disprove the Atiya-Mimura conjecture: For X a Lie group as above and $x \in H_*(X)/tor$ then x is spherical if and only if $\langle ch^*(y), x\rangle \in \mathbb{Z}$ for all $y \in K^*(X)$.

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Also I want to express my gratitude to Atzimba Villamar,my wife for her valuable support and companionship

. I dedicate this work to her.

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INTRODUCTION

In general if E is a suitable ring spectra v.g.: MU, BP, $H\mathbb{Z}$, K and $\iota_E: S^0 \longrightarrow E$ the unity map the (generalized) Hurewicz map is defined by applying $\pi_*(\_)$ to the composition.

$X \simeq X\wedge S^0 \xrightarrow{1\wedge\iota_E} X\wedge E$ we get $h_E: \pi_*(X) \longrightarrow \pi_*(X\wedge E) \cong E_*(X)$.

Consider the coaction map for X in E-theory

$\Psi_X: E_*(X) \longrightarrow E_*(E)\otimes E_*(X)$ Define the primitive elements in $E_*(X)$ by $Prim_E(X) := \{x \in E_*(X) \mid \Psi_X(x) = 1\otimes x\}$ Then

$Im\ h_E \subseteq Prim_E(X) \subset E_*(X)$ (See Lemma 4.12 of Part III §4)

One can ask the question: "for what spaces (or spectra) X and what spectra E do we have the equality $Im\ h_E = Prim_E(X)$?. The well known Hattori-Stong theorem gives a positive answer to that question when X = MU and E = K (See [Switzer 1] Chapter 20).

In the case E = MU and $X = B\mathbb{Z}/p$ p odd prime a positive answer has also been given while a negative one has been given for E = MU and $X = \mathbb{R}P^\infty$. [Hansen - Johnson 1] We are going to study this question when E = MU (or BP) and X is a compact, connected, 1-connected Lie group. We call a positive answer modulo torsion the Kane conjecture. (see: [Kane 1])

The coaction map description of MU primitives is equivalent to one in terms of the action of the Landweber-Novikov operation on $MU_*(X)$ or Quillen operations on $BP_*(X)$). Namely an element $x \in E_*(X)$ is primitive (recall E = MU or BP) if all the operations of positive degree act trivially on x.

From one viewpoint the Kane Conjecture is a question as to whether all the attaching maps in the cell structure of X are detected by primary $\mathcal{P}$-operation ? The Kane Conjecture is linked with another problem: the Atiyah-Mimura Conjecture.

If X is a complex, $ch^*: K^*(X) \longrightarrow H^{**}(X;\mathbf{Q})$ the Chern character and x is a spherical element, i.e. $x \in \mathrm{Im}\,(h:\pi_*(X) \longrightarrow H_*(X;\mathbf{Z}))$ then (See Corollary 4.3, Part III) $\langle ch^*(\xi),x\rangle \in \mathbf{Z}$ for all $\xi \in K^*(X)$

The Atiyah-Mimura Conjecture says that if X is a Lie group as above then a necessary and sufficient condition for $x \in PH_*(X,\mathbf{Z})/tor$ to be spherical is that $\langle ch^*(\xi),x\rangle \in \mathbf{Z}$ for all $\xi \in K^*(X)$.

In chapter 3, §4 we show that the Atiyah-Mimura Conjecture implies the Kane Conjecture. We also establish a relation between $\mathcal{P}$ and ch^* via the Todd genus. In chapter I and II we give an account of ring spectra and MU-theory which is necessary for our study of $\mathcal{P}$ in chapter III. We emphasize in chapter II the use of Landweber-Novikov operations.

In chapter III, §1 we define $\mathcal{P}$ and its basic properties while in §2 and §3 we calculate the primitive elements in $MU_*(\mathbf{C}P^\infty)$ and $MU_*(\mathbf{H}P^\infty)$.

In chapter IV we proceed to calculate sphericals and then primitives in $MU_*(X)$ when X is a (stable) classical group. Amazingly the calculation of the Hurewicz map for the simple Lie groups is not available in the literature except for the famous result for Bott about SU(n). We recollect all necessary results result for Bott about SU(n). We recollect all necessary results from [Kervaire 1], [Kervaire 2] and some other authors and calculate the Hurewicz map modulo torsion for the classical case. We give another proof of Bott's classical result about SU(n) and derive from that proof the Kane conjecture for SU(n). For Sp(n) and SO the Kane Conjecture does not hold and the failure is estimated.

In chapter V we use widely the idea of localization. We use BP theory and we localize $X = G_2$, F_4 in order to get the results.

We also consider the Harper Space [Harper 1].

§ 1. Definition of a (Co)homology theory by a Spectrum.

Definition 1.1: A CW spectrum $E = \{E_n, e_n\}$ is a sequence of CW spaces E_n and maps $e_n : SE_n \to E_{n-1}$, $n \geq 0$.

E is called a suspension spectrum if e_n is a weak homotopy equivalence

We can associate to each CW-space X. the suspension spectrum $\overline{X} = \{S^n X, \Sigma\}$ when $\Sigma_n : S(S^n X) \to S^{n+1} X$ is the canonical isomorphism. In particular we have the sphere spectrum $\underline{S}^i = \{S^{n+i}, \Sigma_n\}$ where $\Sigma_n : S(S^{n+i}) \cong S^{n+i-1}$ is the canonical isomorphism.

A spectrum map of degree $k \geq 0$ is a sequence of maps $\{f_n : E_n \to E_{n+k}\}$ which are compatible in the sense that the following diagram commutes for each $n \geq 0$:

$$\begin{array}{ccc} SE_n & \xrightarrow{e_n} & E_{n+1} \\ {\scriptstyle Sf_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ SF_{n+k} & \xrightarrow{e_{n+k}} & F_{n+k+1} \end{array}$$

We will denote the set of such graded maps by $[E,F]_k$. One can define a smash product $E \wedge F$ of spectrum. The general definition is quite complicated and is given in [Adams 1] part III §4 page 158. We will only note that we can think of it as being represented by any spectrum of the form $\{E_i \wedge F_j\}$ where $i, j \to \infty$. In particular, given a spectrum

$\{E_n, e_n\} = E$ a multiplication

$\mu : E \wedge E \to E$ consists of a compatible family of maps

$\{\mu_{m,n} : E_m \wedge E_n \to E_{m+n},\ m, n \geq 0\}$

Definition 1.2: E is a ring spectrum if there exist maps

$\mu : E \wedge E \longrightarrow E$ and $\iota : S^0 \longrightarrow E$ such that the following diagrams

commute:

$$\begin{array}{ccc} E_m \wedge E_n \wedge E_k & \xrightarrow{1 \wedge \mu_{n,k}} & E_m \wedge E_{n+k} \\ \downarrow \mu_{m,n} \wedge 1 & & \downarrow \mu_{m,n+k} \\ E_{m+n} \wedge E_k & \longrightarrow & E_{m+n+k} \end{array}$$

$$\begin{array}{ccc} E_m \wedge E_n & & \\ \downarrow T & \searrow \mu_{n,m} & \\ E_n \wedge E_m & \xrightarrow{\mu_{m,n}} & E_{m+n} \end{array} \qquad \begin{array}{ccc} E_m \wedge S^n & \xrightarrow{1 \wedge \iota_n} & E_m \wedge E_n \\ & \searrow e_{m,n} & \downarrow \mu_{m,n} \\ & & E_{m+n} \end{array}$$

Given a space X we define E-homology and cohomology as

$$E_*(X) = E_*(\bar{X}^+)$$
$$E^*(X) = E^*(\bar{X}^+)$$

where $\bar{X}^+$ is the suspension spectrum of $X^+ = X$ plus a disjoint basepoint.

Given a spectrum $F = \{F_n\}$ we can always define

$$E_*(F) = \varinjlim_n E_*(F_n)$$

Notation: $E_* = E_*(S) = \pi_*(E)$ $\quad E^* = E^*(S) = \pi_{-*}(E)$

Lemma 1.3 For any CW spectrum X $E_*(X)$ is an E_*-module and $E^*(X)$ is an E^*-module

Proof: To define the required multiplication we interpret all elements as maps and use composition.

Given $F \in E_i = \pi_i(E)$ and $g \in E_j(X) = \pi_j(E\, X)$ we define

$$f\cdot g: S^{i+j} = S^i \wedge S^j \xrightarrow{f\wedge g} E\wedge E\wedge X \xrightarrow{\mu\wedge 1} E\wedge X$$

Given $f \in E^i = \pi_{-i}(E)$ and $g \in E^j(X) = [S^{-j}\wedge X, E]$ we define

$$f\cdot g: S^{-i-j}\wedge X \longrightarrow S^{-i}\wedge S^{-j}\wedge X \xrightarrow{f\wedge g} E\wedge E \xrightarrow{\mu} E \qquad \text{Q.E.D.}$$

Proposition 1.4. The E-(Co)homology of X has the following properties:

(1) If $f \simeq g$ then $f_* = E_*(f) = E_*(g) = g_*$

$$\text{and } f^* = E^*(f) = E^*(g) = g^*$$

(2) There exist isomorphisms

$$\sigma_m : E_m(X) \xrightarrow{\cong} E_{m+1}(SX)$$

$$\sigma^n : E^n(X) \xleftarrow{\cong} E^{n+1}(SX)$$

called the suspension homomorphism in E-homology and E-cohomology respectively

(3) If $X \xrightarrow{f} Y$ is a map between CW-spaces and $Y\cup_f CX$ denotes the mapping cone of f then we have two exact triangles

$$\begin{array}{ccc} E_*(X) & \longrightarrow & E_*(Y) \\ & & \downarrow \\ & & E_*(Y\cup_f CX) \end{array} \qquad \begin{array}{ccc} E^*(X) & \longleftarrow & E^*(Y) \\ & & \uparrow \\ & & E^*(Y\cup_f CX) \end{array}$$

Proof: See [Gray 1] chapter 18

§2 Atiyah-Hirzebruch Spectral Sequence

Let X be a CW-spectrum and $X^0 \subset X^1 \subset X^2 \subset \ldots X$ be the skeleton decomposition of X. Let $\iota_q : X^q \longrightarrow X$ be the q-inclusion.

Definition 2.1. Given any two spaces $A \subset B$ we define

$E_*(B,A) = E_*(B/A)$ and $E^*(B,A) = E^*(B/A)$

In particular we define $E_*(X) = E_*(X,\emptyset)$ and $E^*(X) = E^*(X,\emptyset)$.

The modules $E_*(X)$ and $E^*(X)$ admit filtrations

(i) $0 \subset F_0 \subset \dots F_{q-1} \subset F_q \subset \dots F_*(E)$ where

$F_q = \mathrm{Im}(i_{q*}: E_*(X^q) \longrightarrow E_*(X))$

(ii) $E^*(X) \supset \dots \supset F^{q-1} \supset F^q \supset \dots \supset F^0 = 0$ where

$F_q = \mathrm{Ker}(E^*(X) \longrightarrow E^*(X))$ Since

$$\begin{array}{ccc} E_*(X^{q-1}) & \longrightarrow & E_*(X) \\ & & \uparrow \\ & & E_*(X^q) \end{array}$$

commutes

we have $F_{q-1} \subset F_q$

Since

$$\begin{array}{ccc} E^*(X) & \longrightarrow & E^*(X^{q-1}) \\ & \searrow & \uparrow \\ & & E^*(X^q) \end{array}$$

commutes.

we also have $F^q \subset F^{q-1}$.

By the method of exact couples [Adams 2] paper #4 of Massey we can construct a Spectral Sequence induced by the exact couple

$$\begin{array}{ccc} E^*(X^{p-1}) & \xrightarrow{i_*} & E_*(X^p) \\ \partial \nwarrow & & \downarrow j_* \\ & & E_*(X^p,X^{p-1}) \end{array}$$

We denote this spectral sequence by $\{E^s_*, d^s\}$ We have:

(2.1) $E^1_{p,q} = E_{p+q}(X^p,X^{p-1})$

(2.2) $E^\infty_{p,q} = F_{p+q} | F_{p+q-1}$ so $E^\infty_{p,q} = E_{p+q}(X)$

(2.3) $E^2_{p,q} = H_p(X,Eq)$

See [Switzer 1] chapter 15 Incise 15.6

Similarly we can construct the cohomology Atiyah-Hirzebruch spectral sequence $\{E_r^{p,q}, d_r\}$ where .

(2.4) $E_2^{p,q} = H^p(X, E^q)$

(2.5) $E_\infty^{p,q} = F^{p-q}/F^{p-q+1}$

this spectral sequence has the disadvantage that the filtration (F^q) is not always complete. So we cannot always speak of it as converging to $E^*(X)$.

However if X is finite that the problem disappears. The filtration is also complete for E = MU provided $H^*(X)$ is torsion free (see [Landweber 1]).

We have, as a consequence of the Atiyah-Hirzebruch Spectral Sequence, [Dold 1]

Lemma 2.2 If E and F are spectra and $\tau: E_*(\ ) \to F_*(\ )$ is a natural transformation such that $\tau(S^0): E_*(S^0) \to F_*(S^0)$ is an isomorphism then $\tau(X): E_*(X) \xrightarrow{\cong} F_*(X)$ is an isomorphism for all CW spectrum X.

Proof: Consider the A. H. Spectral sequences converging to $E_*(X)$ and $F_*(X)$ respectively.

By hypothesis $\alpha_*: \pi_*(E) \xrightarrow{\cong} \pi_*(F)$ is an isomorphism So $H_p(X, \pi_*(E)) \xrightarrow[\alpha_*]{} H_p(X, \pi_*(F))$ is an isomorphism

By naturality of the spectral sequences we are done. Q.E.D.

Lemma 2.3. Rationally the Atiyah-Hirzebruch spectral sequence collapses.

Proof: Consider the homology theories $H_*(\_\_, \pi_*(E)) \otimes \mathbb{Q}$ and $E_*(\_\_) \otimes \mathbb{Q}$. Then the edge map in the Atiyah-Hirzebruch spectral sequences (also called the Thom Map) defines a natural transformation

$\mu_X: E_*(X) \otimes \mathbb{Q} \longrightarrow H_*(X, \pi_*(E)) \otimes \mathbb{Q}$

When X = point μ_X $\pi_*(E)\otimes\mathbb{Q} \longrightarrow H_*(\text{point}, \pi_*(E))\otimes\mathbb{Q}$ is an isomorphism. So, by Lemma 2.1. $E_*(X)\otimes\mathbb{Q} \cong H_*(X, \pi_*(F))\otimes\mathbb{Q}$ But the Atiyah Hirzebruch spectral sequence collapses for ordinary homology. So it collapses for $E_*(X)\otimes\mathbb{Q}$ Q.E.D.

<u>Corollary 2.4</u> If $H_*(X)$ is torsion free and $\pi_*(F)$ is torsion free then the Atiyah Hirzebruch S.S converging to $E_*(X)$ collapses

<u>Proof</u>: We have an embedding $H_*(X, \pi_*(E)) \longrightarrow H_*(X, \pi_*(E)\otimes\mathbb{Q})$, so, by Lemma 2.2 the spectral sequence converging to $E_*(X)$ collapses. Q.E.D.

<u>§3: The Spaces BU(m) and $BU_@$</u>

<u>§3.1. $H^*(BU(n))$ As A Polynomial Algebra</u>

We recall that $U(n)$ is the group of nxn matrices with coefficient in $\mathbb{C}$ such that

$A \in U(n)$ if and only if A preserves the usual inner-product in $\mathbb{C}^n$. We have a fibre bundle

$$U(n) \xrightarrow{i} U(n+1) \xrightarrow{\varphi} S^{2n-1}$$

where $i(A) = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ and φ is the Gauss map $\varphi(A) = A\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in S^{2n+1}$

Using the Serre spectral sequence associated to this fibration one can prove, by induction on n, that

<u>Lemma 3.1.1</u>: $H^*(U(n)) = \Lambda(x_1, x_3, \ldots, x_{2n-1})$.

The Lie group multiplication $U(n)xU(n) \longrightarrow U(n)$ induces an associative product $H_*(U(n))\otimes H_*(U(n)) \longrightarrow H_*(U(n))$ It follows from Lemma 3.1.1 plus the Leray-Samelson theorem see: [Switzer 1] chapter to that.

<u>Lemma 3.1.2</u>: $H_*(U(n)) = \Lambda(x_1, x_2, \ldots, x_{2n-1})$.

We can use this result to calculate the cohomology of the classifying space $BU(n)$

<u>Proposition 3.1.3</u>: $H^*(BU(n)) = \mathbf{Z}[y_2, y_4, \ldots, y_{2n}]$ where $|y_i| = 2i$

<u>Proof</u>: It suffices to show that, for every prime p

$H^*(BU(n);\mathbf{Z}/p) = \mathbf{Z}/p[y_2, \ldots, y_{2n}]$. There exists an Eilenberg-Moore spectral sequence

$$E_2 = \mathrm{Ext}_{H_*(U(n);\mathbf{Z}/p)}(\mathbf{Z}/p;\mathbf{Z}/p) \Rightarrow H^*(BU(n);\mathbf{Z}/p)$$

By Lemma 3.1.2 $H_*(U(n);\mathbf{Z}/p) = \Lambda(x_1,$ [illegible] $_{2n+1})$.

Now $\mathrm{Ext}_{\Lambda(x)}(\mathbf{Z}/p,\mathbf{Z}/p) = \mathbf{Z}/p[s^{-1}x]$ where $s^{-1}x$ has bidegree$(-1,|x|)$

and $\mathrm{Ext}_{C\otimes D}(\mathbf{Z}/p,\mathbf{Z}/p) = \mathrm{Ext}_C(\mathbf{Z}/p,\mathbf{Z}/p)\otimes \mathrm{Ext}_D(\mathbf{Z}/p,\mathbf{Z}/p)$ so we have

$E_2 \cong \mathbf{Z}/p[s^{-1}x_1, s^{-1}x_3, \ldots, s^{-1}x_{2n-1}]$

Since all non-trivial elements have even total degree in the Eilenberg-Moore spectral sequence then $E_2 = E_\infty$. Also, there is no extension problem. Puting $s^{-1}x_{2i+1} = y_{2i}$ and $y_0 = 1$ we are done. Q.E.D

<u>§3-2 $H^*(BU(n))$ as a Ring of Invariants.</u>

Let $T(n) = S^1 x \ldots x S^1$ be the maximal torus of $U(n)$. It consists of the diagonal matrices in $U(n)$. The symmetric group Σ_n is also a subgroup of $U(n)$ It consists of the permutation matrices. They are the matrices where each row and column contains exactly one non-zero entry and that entry equals 1. Σ_n acts on $U(n)$ by conjugation and $T(n) \subset U(n)$ is invariant under this action.

Using Milnor's join construction [Husemoler 1]of a classifying space (namely $BG = G*G*\cdots/G$) we have induced actions of Σ_n on $BT(n)$ and $BU(n)$. Moreover the inclusion $i : T(n) \subset U(n)$ induces a map $B_i : BT(n) \to BU(n)$ which is Σ_n equivariant. We can pass to cohomology and obtain an equivariant map $(B_i)^* : H^*(BU(n)) \to H^*(BT(n))$. The action of Σ_n on $H^*(BU(n))$ is trivial. For any inner automorphism of G induces the identity map on $H^*(BG)$. So we have

<u>Lemma 3.2.1</u>: Image of $Bi_* : H^*(BU(n)) \to H^*(BT(n))$ is contained in $H^*(BT(n))^{\Sigma_n}$ = the invariants of the Σ_n action on $H^*(BT(n))$.

It is easy to calculate $H^*(BT(n))$. For Σ_n acts on $H^*(BT(n)) = \mathbf{Z}[t_1 \ldots t_n]$ ($|t_n| = 2$) by permuting the factors. So $H^*(BT(n))^{\Sigma_n} = \mathbf{Z}[\sigma_1, \sigma_2, \ldots, \sigma_n]$ where $\sigma_i(t_1, \ldots, t_n)$ is the i-th elementary symmetric polynomial.

<u>Proposition 3.2.2</u>: $H^*(BU(n)) = H^*(BT(n))^{\Sigma_n}$

<u>Proof</u>: Since we have shown that both $H^*(BU(n))$ and $H^*(BT(n))^{\Sigma_n}$ are polynomial algebras on generators of degrees 2,4,...,2n it suffices, by Lemma 3.2.1 to show that $Bi^* : H^*(BU(n);\mathbf{Z}/_p) \to H^*(BT(n);\mathbf{Z}/_p)$ is injective for each prime p. We will use the mod p Serre spectral sequence associated to the fibration

$$U(n)/_{T(n)} \to BT(n) \to BU(n) \qquad (*)$$

We claim that $H^{odd}(U(n)/_{T(n)}) = 0$ This follows by induction using the Serre spectral sequence associated to the fibration

$$U(n-1)/_{T(n-1)} \to U(n)/_{T(n)} \to U(n)/_{U(n-1)\times S^1} \simeq \mathbf{C}P^{\infty}$$

It follows from our claim that $H^*(U(n)_{T(n)};\mathbf{Z}_p)\otimes H^*(BU(n);\mathbf{Z}_p)$ is concentrated in even degree. So the spectral sequence associated to (*) collapses and we have

$$H^*(U(n)_{T(n)};\mathbf{Z}_p)\otimes H^*(BU(n);\mathbf{Z}_p) \Rightarrow E_0H^*(BU(n);\mathbf{Z}_p)$$

Under this identity $(Bi)^*$ becomes the obvious inclusion map. In particular it is injective. Q.E.D

Remark: The class $\sigma_k \in H^{2k}(BU(n))$ is called the k^{th} Chern class.

§3.3 Calculation of $H_*(BU(n))$

Definition. $H_*(BT(n))_{\Sigma_n}$ denotes the quotient of $H_*(BT(n);\mathbf{Z})$ by the subgroup generated by $\{\sigma(x) - x \mid \sigma \in \Sigma_n\}$.

$H^*(BT(n))^{\Sigma_n}$ and $H_*(BT(n))_{\Sigma_n}$ are dual. So if we dualize the discussion in §3.2 we obtain an isomorphism.

Proposition 3.3.1: $\otimes_{i=1}^n H_*(\mathbf{C}P^\infty)_{\Sigma_n} \xrightarrow{\cong} H_*(BU(n))$ and a monomorphism $H_*(\mathbf{C}P^\infty) \rightarrow H_*(BU(n))$.

Recall $H^*(\mathbf{C}P^\infty) = \mathbf{Z}[x]$ $|x| = 2$ Put $\beta_i \in H_*(\mathbf{C}P^\infty)$ as given by $\langle\beta_i, x^i\rangle = 1$. So $\{\beta_i\}$ is a basis of $H_*(\mathbf{C}P^\infty;\mathbf{Z})$ which "injects" into $H_*(BU(n))$. The above isomorphism tells us:

Proposition 3.3.2: $H_*(BU(n))$ has basis monomials of the form $\{\beta_{i_1}\cdot\beta_{i_2}\cdot\dots\cdot\beta_{i_n}\}$

§3.4 The Space BU

We define $BU = \lim_{n\to\infty} BU(n)$. For each $m,n\geq 1$ the inclusion $U(m)\times U(n) \rightarrow U(m+n)$ given by: $(A,B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}$ induces maps $\mu_{m,n}: BU(m)\times BU(n) \rightarrow BU(m+n)$ These maps fit together to give a multiplication $\mu: BU\times BU \rightarrow BU$. We can iterate μ to obtain maps

$\prod_{i=1}^{n} BU \to BU$. Since $BU(1) = \mathbb{C}P^\infty$ we have an inclusion $\mathbb{C}P^\infty \to BU$. We have a commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^{n} \mathbb{C}P^\infty & \longrightarrow \prod_{i=1}^{n} BU \longrightarrow & BU \\ \big\downarrow & & \big\uparrow \\ BT(n) & \xrightarrow{\quad Bi_n \quad} & B(n) \end{array}$$

So using the multiplication $\mu_*: H_*(BU)\otimes H_*(BU) \longrightarrow H_*(BU)$ we can deduce from §3.3 that

<u>Theorem 3.4.1</u> $H_*(BU) = \mathbb{Z}[\beta_1, \beta_2, \ldots]$ where $\beta_i \in H_{2i}(\mathbb{C}P^\infty) \subset H_{2i}(BU)$

<u>§4. The Space MU(n)</u>

Consider the universal bundle $U(n) \longrightarrow EU(n) \longrightarrow BU(n)$. We will denote it ξ_n. We can define subspace $D(\xi_n)$ and $S(\xi_n)$ of $EU(n)$ by the rule that for each $b \in BU(n)$

$D(\xi_n)_b = \{x\in EU(n)_b \mid |x| \le 1\}$ $S(\xi_n)_b = \{x\in EU(n)_b \mid |x| = 1\}$.

Clearly $S(\xi_n) \subset D(\xi_n)$ are subcomplexes of $EU(n)$ and $MU(n) = D(\xi_n)/S(\xi_n)$ is a CW-complex. It is called the unitary Thom complex. It is also possible to define $MU(n)$ as the cofibre of $Bi_{n-1}: BU(n-1) \longrightarrow BU(n)$ where $i_{n-1}: U(n-1) \longrightarrow U(n)$ is the inclusion. As a result we have

<u>Proposition 4.2</u>: There exist an exact sequence

$$0 \longrightarrow H^{2q}(MU(n)) \longrightarrow H^{2q}(BU(n)) \longrightarrow H^{2q}(BU(n-1)) \longrightarrow 0.$$

<u>Proof</u>: The Barrat-Puppe sequence $BU(n-1) \longrightarrow BU(n) \longrightarrow MU(n) \longrightarrow \Sigma BU(n-1) \longrightarrow$ induces an exact sequence

$$\to H^{2q}(\Sigma BU(n-1)) \longrightarrow H^{2q}(MU(n)) \longrightarrow H^{2q}BU(n)) \longrightarrow H^{2q}(BU(n-1)) \longrightarrow$$

Since $H^{even}(\Sigma BU(n-1)) = 0$ we are done. Q.E.D.

By the proposition $H^*(MU(n))$ lies in $H^*(BU(n))$ we have $H^*(MU(n)) = \mathrm{Ker}\, i^* =$ the ideal of $H^*(BU(n))$ generated by σ_n. the n-th symmetric polynomial as defined in §3.2. So we have

<u>Theorem 4.3</u>. (Thom Isomorphism Theorem). There exists an isomorphism $\phi : H^*(BU(n)) \to H^*(MU(n))$ of abelian groups of degree 2n.

<u>Proof</u>: Define $\phi(x) = \sigma_n \cdot x$. Q.E.D.

Dually we have $\phi_* : H_*(BU(n)) \to H_*(MU(n))$ an isomorphism of degee 2n.

<u>§5: The Speotrum MU</u>.

<u>§5.1 The Spectrum MU</u>.

If ξ_K denote the universal U(K)-bundle over BU(K) then the inclusion $\iota_K : BU(K) \to BU(K+1)$ define an isomorphism of bundles $\iota^*(\xi_{K+1}) \cong \xi_K \oplus 1$ so ι_K induce a map.

$\iota_k : MU(k) \wedge S^2 \cong M(\xi_k \oplus 1) \cong M(\iota_k^*(\xi_{k+1})) \longrightarrow M(\xi_{k+1}) \cong MU(k+1)$

We define the spectrum MU by the rule $MU_{2k} = MU(k)$ $MU_{2k+1} = SMU(k)$. The map $e_{2k} : SM_{2k} \longrightarrow M_{2k+1}$ is the identity map while $SM_{2k+1} \longrightarrow M_{2k+2}$ is ι_{k+1}.

The spectrum MU is a ring spectrum. The maps $\mu_{m,n} : BU(n) x BU(m) \longrightarrow BU(n+m)$ described in §3.4 satisy the identity $\mu^*_{m,n}(\xi_{n+m}) = \xi_n x \xi_m$. Therefore $\mu_{m,n}$ induce a map.

$\mu_{m,n} : MU(n) x MU(m) \longrightarrow M(\xi_n \wedge \xi_m) \xrightarrow{M(\mu_{m,n})} M(\xi_{n+m}) = MU(n+m)$

These maps give the multiplication $\mu : MU \wedge MU \longrightarrow MU$ Since $\mathbb{C}P^\infty \cong MU(1)$ we have a map $\iota : H_*(\mathbb{C}P^\infty) \to H_*(MU)$ of degree -2. Let

$b_k = \iota(\beta_k) \in H_{2i}(MU)$.

It follows from the fact that the following diagram commutes

$$\begin{array}{ccc} H_*(MU(m))\otimes H_*(MU(n)) & \xrightarrow{\mu_*} & H_*(MU(n+m)) \\ \downarrow{\scriptstyle \varphi_m\otimes\varphi_n} & & \downarrow{\scriptstyle \varphi_{m+n}} \\ H_*(BU(m))\otimes H_*(BU(n)) & \xrightarrow{\mu_*} & H_*(BU(m+n)) \end{array}$$

that we can mimic the calculation of $H_*(BU)$ in §3-4 and deduce

Proposition 5.1.1: $H_*(MU) = \mathbf{Z}[\ell_1, \ell_2, \ldots]$

The multiplication $\mu: MU\wedge MU \to MU$ also gives $\pi_*(MU)$ an algebra structure.

Proposition 5.1.2 $\pi_*(MU) = \mathbf{Z}[x_1, x_2, \ldots]$ where $|x_2| = 2i$.

So, in particular $\pi_*(MU)$ is torsion free. Unlike $H_*(MU)$ there is no easy way to give canonical generators for $\pi_*(MU)$. The above structure theorem follows from an Adams spectral sequence argument. Since $\pi_*(MU)\otimes\mathbf{Q} \cong H_*(MU)\otimes\mathbf{Q}$ it follows that the Hurewicz map $h: \pi_*(MU) \to H_*(MU)$ is injective.

§5.2 E Homology of MU

The following argument holds for E a ring spectra with $\pi_*(E)$ a $\mathbf{Z}$-free module and connected i.e., $E_k \cong *$ for $k < 0$ We already proved those things for $E = HZ$.

Proposition 5.2.1: $E^*(\mathbf{C}P^\infty) \cong \pi_*(E)[[x^E]]$ when $|x| = 2$ when $i^*(x^E) = e \in E^2(S^2) \cong \mathbf{Z}$

Proof: Since $H^*(\mathbf{C}P^\infty) \cong \mathbf{Z}[e]$ $|e| = 2$ we have that $H^*(\mathbf{C}P^\infty;\pi_*(E)) \cong \pi_*(E)[x]$ consider the Atiyah-Hirzebruch spectral sequence since $E_2^{**} \cong H^*(\mathbf{C}P^\infty;\pi_*(E))$ is torsion free then the spectral sequence collapses, i.e.,

$$E^0(E(\mathbb{C}P^\infty)) = H^*(\mathbb{C}P^\infty, \pi_*(E)) = \pi_*(E)[x]$$

We have to allow infinite sums in passing from $E^0(E^*(\mathbb{C}P^\infty))$ to $E^*(\mathbb{C}P^\infty)$ [illegible] $E^*(\mathbb{C}P^\infty) = \pi_*(E)[[x^E]]$. Q.E.D.

Now the Kronecker product in E-homology when (E, μ, ι) is a ring spectrum is given by:

If $f \in E^m(X)$ and $g \in E_n(X)$ then $f: X \wedge S^{-m} \to E$ and $g: S^n \to E \wedge X$ so $\langle g, f \rangle \in \pi_{n-m}(E)$ is given by :

$$S^n \wedge S^{-m} \xrightarrow{g \wedge 1} E \wedge X \wedge S^{-m} \xrightarrow{1 \wedge f} E \wedge E \xrightarrow{\mu} E$$

If we dualize Proposition 5.2.1 we obtain:

<u>Proposition 5.2.2</u> If $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ is defined by $\langle (x^E)^j ; \beta_i \rangle = \delta_{i,j}$ then $E_*(\mathbb{C}P^\infty)$ is generated as $\pi_*(E)$-module by $\{1, \beta_1, \beta_2, \ldots, \beta_n, \ldots\}$ Moreover the coproduct $\Psi : E_*(\mathbb{C}P^\infty) \longrightarrow E_*(\mathbb{C}P^\infty) \otimes_{\pi_*(E)} E_*(\mathbb{C}P^\infty)$ when $\Psi = \Delta_*$ and $\Delta : \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ is the diagonal map is given by

$$\Psi(\beta_i) = \sum_{j+k=1} \beta_j \otimes \beta_k .$$

As before we have a canonical map $\xi : E_*(\mathbb{C}P^\infty) \longrightarrow E_*(MU)$ of degree -2. Let $b_k^E = \xi(\beta_k^E)$

Since the Atiyah-Hirzebruch spectral sequence for $E_*(MU)$ collapses we can deduce from Proposition 5.1 that

<u>Proposition 5.2.3</u> $E_*(MU) \cong \pi_*(E)[b_1^E, b_2^E, \ldots.]$

<u>Remark</u>: Both β_i^E and b_i^E depend on the choice of x^E. In the case $E = MU$ we can make a canonical choice of x. Namely x is represented by the map $\mathbb{C}P^\infty = MU(1) \to MU$.

§5.3 The Hurewicz Map

The Hurewicz map $h:\pi_*(MU) \to H_*(MU)$ can be described by using formal group theory. We begin by working with spectrum E when

$\pi_*(E)$ is a free $\mathbb{Z}$ module. We have $E^*(\mathbb{C}P^\infty) = \pi_*(E)[[x^E]]$

Now if $m:\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the multiplication in the H-space $\mathbb{C}P^\infty$, i.e., the map induced by the tensor product of U(1)-bundle ($BU(1) \cong \mathbb{C}P^\infty$) it induces $m^*:E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ and we have $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty) \times_{\pi_*(E)} E^*(\mathbb{C}P^\infty) \cong \pi_*(E)[[x_1]] \otimes \pi_*(E)[[x_2]] \cong$ $\pi_*(E)[[x_1, x_2]]$ Now if $E^*(\mathbb{C}P^\infty) = \pi_*(E)[[x^E]]$ we have

$$m^*(x^E) = x_1^E + x_2^E + \Sigma\, a_{i,j} (x_1^E)^i (x_2^E)^j \qquad (*)$$

Clearly if $E = H\mathbb{Z}$ $a_{ij} = 0$, for all i, j i.e.: $m^*(x) = x_1^E + x_2^E$

We will study those $a_{ij} \in \pi_*(E)$ for E = MU. Let $\mu^E(x_1, x_2)$ denote the formal power series law associated to the spectrum E by (*)

Recall that $E_*(MU) \cong \pi_*(E)[b_1^E, b_2^E, \ldots]$ Let $g(x) = \Sigma\, b_i^E x^{i+1}$

The Hurewicz map $h:\pi_*(MU) \longrightarrow E_*(MU)$ relates the elements $\{a_{ij}^E\}$ in $\pi_*(MU)$ to $\{b_i^E\}$ in $E_*(MU)$

<u>Proposition 5.3.</u> $\mu^{MU}(x, y) = g(\mu^E(g^{-1}(x), g^{-1}(y)))$

<u>Proof</u>: If we imbed $MU^*(\mathbb{C}P^\infty) = \pi_*(MU)[[x^{MU}]]$ and $E^*(\mathbb{C}P^\infty) = \pi_*(E)[[x^E]]$ in $(E\wedge MU)^*(\mathbb{C}P^\infty)$ we have the identity $x^{MU} = g(x^E)$.

So, in $(E\wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ we have the identities $\mu^{MU}(x_1^{MU}, x_2^{MU}) =$ $m^*(x^{MU}) = m^*(g(x^E)) = g(m^* x^E)) = g(\mu^E(x_1^E, x_2)) =$ $g(\mu^E(g^{-1}(x_1^{MU}, g^{-1}(x_2^{MU}))))$ Q.E.D.

In the case $E = H\mathbb{Z}$ we have $\mu^H(x, y) = x + y$ and

$$g(x) = \log(x) = \Sigma\, m_i x^{i+1}$$

$$g^{-1}(x) = \exp(x) = \Sigma\, b_i x^{i+1}$$

The identity $\mu^{MU}(x,y) = \log(\exp(x)+\exp(y))$ tells us how to write $\{h(a_{ij})\}$ in terms of $\{b_i\}$ and $\{m_i\}$, hence just in terms of $\{b_i\}$. Notably one can show

Proposition 5.3.2 $h(a_{i,j}) = \frac{(i+j)!}{i!j!}$ mod decomposable in $H_*(MU)$.

One important fact which we will make use of in our arguments is:

Proposition 5.3.3 $(n+1)m_n \in \text{Image } h:\pi_*(MU) \to H_*(MU)$

The elements of $\pi_*(MU)$ can be interpreted as cobordism classes of manifolds. In particular, by a charateristic class argument, we can identify $(n+1)m_n$ with $[\mathbb{C}P^n]$ (see [Adams 1])

§6 MU Theory

The ring spectrum MU defines a homology theory $MU_*(X) = \pi_*(E\wedge X)$ and a cohomology theory $MU^*(X) = [X,MU]$. They are module over $\pi_*(MU) = \mathbb{Z}[x_1,x_2,\ldots]$. If we rationalize we can make canonical choices of the generators $\{x_i\}$. For $\pi_*(MU)\otimes\mathbb{Q} \cong H_*(MU)\otimes\mathbb{Q}$. So by the results in §5 we can write

$$\pi_*(MU)\otimes\mathbb{Q} = \mathbb{Q}[b_1,b_2,\ldots]$$

or

$$\pi_*(MU)\otimes\mathbb{Q} = \mathbb{Q}[m_1,m_2,\ldots]$$

where the formal power series $\exp(x) = \Sigma b_i x^{i+1}$ and $\log(x) = \Sigma m_i x^{i+1}$ are compositional inverses of each other. Because of these canonical choices we will often find it profitable to rationalize even if we only wish to describe phenomena in $\pi_*(MU)$, $MU_*(X)$ or $MU^*(X)$.

MU theory is related to both ordinary (co)homology and to K-theory. For ordinary (co)homology we have the Thom map

$$T: MU_*(X) \to H_*(MU)$$

$$T: MU^*(X) \to H^*(MU)$$

It is induced by a map $T: MU \to H\mathbf{Z}$ representing a generator of $H^0(MU) = \mathbf{Z}$. The Thom map is onto if and only if the Atiyah-Hirzebruch spectral sequence converging to $MU_*(X)$ or $MU^*(X)$ collapses. Indeed an element of $H_*(X) \subset H_*(X, \pi_*(MU))$ or $H^*(X) \subset H^*(X, \pi_*(MU))$ lies in the image of T precisely when it is a permanent cycle in the spectral sequence. T acts on $\pi_*(MU) \otimes Q$ by the rule $\quad T(b_i) = T(m_i) = 0 \quad$ for $i \geq 1$

The relation between MU theory and K-theory is determined by the Todd map [Hirzebruch 1]

$$Td: \pi_*(MU) \to \mathbf{Z}$$

It is the unique ring homomorphism such that $Td([\mathbf{C}P^n]) = 1$. If we rationalize then Td satisfies

$$Td(m_i) = \frac{1}{i+1}$$

$$Td(b_i) = \frac{1}{(i+1)!}$$

The first fact follows from the identity $m_i = (i+1)[\mathbf{C}P^i]$. The second fact follows from exp(x) and log(x) being inverse power series. The Conner-Floyd isomorphism asserts that (See:[Conner-Floyd 1])

$$K_*(X) = MU_*(X) \otimes_{\pi_*(MU)} \mathbf{Z}$$

$$K^*(X) = MU^*(X) \otimes_{\pi_*(MU)} \mathbf{Z}$$

We will use

$$\beta: MU^*(X) \to K^*(X)$$

$$\beta: MU_*(X) \to K_*(X)$$

to denote the resulting surjective map. We will call β the Conner-Floyd map.

Part II

§1. The Hopf Algebra $MU_*(MU)$

We have already shown that $MU_*(MU) = \pi_*(MU)[B_1, B_2, \ldots]$. In this section we demonstrate that MU_*MU has a Hopf algebra structure. The Hopf algebra structure is provided by maps

$$\varphi_{MU}: MU_*(MU) \underset{\pi_*(MU)}{\otimes} MU_*(MU) \longrightarrow MU_*(MU) \qquad \text{product}$$

$$\psi_{MU}: MU_*(MU) \longrightarrow MU_*(MU) \underset{\pi_*(MU)}{\otimes} MU_*(MU)$$

$$\chi: MU_*(MU) \longrightarrow MU_*(MU) \qquad \text{conjugation}$$

$$\varepsilon: MU_*(MU) \longrightarrow \pi_*(MU) \qquad \text{augmentation}$$

$$\eta_L: \pi_*(MU) \longrightarrow MU_*(MU) \qquad \text{left unit}$$

$$\eta_R: \pi_*(MU) \longrightarrow MU_*(MU) \qquad \text{right unit}$$

when φ_{MU} is induced by ring spectrum multiplication

$$\mu: MU \wedge MU \longrightarrow MU$$

i.e.

$$\varphi_{MU} = \mu_*: MU_*(MU \wedge MU) \cong MU_*(MU) \otimes MU_*(MU) \longrightarrow MU_*(MU).$$

χ is induced by the switching map $C: MU \wedge MU \longrightarrow MU \wedge MU$

i.e.

$$\chi = C_*: \pi_*(MU \wedge MU) \longrightarrow \pi_*(MU \wedge MU)$$

ε is also induced by μ in the following way

$$\varepsilon = \mu_*: \pi_*(MU \wedge MU) \cong MU_*(MU) \longrightarrow \pi_*(MU).$$

η_L and η_R are the left and right units which define two <u>different</u> $\pi_*(MU)$-module structures on $MU_*(MU)$ [See Adams 1 part II §11]. They are defined by application of the functor $\pi_*(-)$ on

$$MU \simeq MU \wedge S^0 \longrightarrow MU \wedge MU$$

and

$$MU \simeq S^0 \wedge MU \longrightarrow MU \wedge MU$$

respectively.

To define the coproduct we need the following

<u>Lemma 1.1</u>: If $E_*(-)$ is a reduced homology theory and $E_*(-)$ is a free $\pi_*(E)$-module then the exterior product gives an isomorphism

$$E_*(X) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow[\mu_*]{\cong} E_*(X \wedge Y)$$

<u>Proof</u>: Since $E_*(X) \otimes_{\pi_*} E_*(-)$ and $E_*(X \wedge -)$ are (reduced) cohomology theories and since they agree when $Y = S^0$ it follows that they agree for all Y. Q.E.D.

Since $MU_*(MU)$ is a free MU_* module we can define Ψ_{MU} as the composition

$$\Psi_{MU}: MU_*(MU) \xrightarrow{\iota_*} MU_*(MU \wedge MU) \xrightarrow{\mu_*^{-1}} MU_*(MU) \otimes_{MU_*} MU_*(MU).$$

These maps satisfy the following relations. Define $m_i = C(B_i)$. Let $B = 1 + B_1 + B_2 + \ldots$ and let B_j^i denote the graded component of degree $2j$ in B^j.

<u>Proposition 1.2</u>:

(a) $\epsilon(B_k) = 0$ for $k > 0$

(b) $n_L(b_k) = b_k$

$$n_R(b_k) = \sum_{i+j=k} B_j(b)_i^{j+1}$$

(c) $$\Psi(B_k) = \sum_{i+j=k} B_j^{i+1} \otimes B_i$$

In the above the elements $\{b_i\}$ do not belong to $\pi_*(MU)$ but rather to

$$\pi_*(MU)\otimes\mathbf{Q} \cong H_*(MU)\otimes\mathbf{Q} = \mathbf{Q}[b_1, b_2, \ldots].$$

So we are really giving a description of $MU_*(MU)\otimes\mathbf{Q}$. However $MU_*(MU)$ is torsion free. Sw we can recover a description of $MU_*(MU)$ from the above. We also have a dual description of $MU_*(MU)$. The conjugacy map satisfies

Proposition 1.3:

$$\chi^2 = I, \quad \chi\eta_L = \eta_R, \quad \chi\eta_R = \eta_L$$

If we define $M_k \in MU_{2k}(MU)$ by $M_k = \chi(B_k)$ and

$$m_k \in \pi_{2k}(MU)\otimes\mathbf{Q} = H_{2k}(MU)\otimes\mathbf{Q}$$

as in Part 1 §5 then

Proposition 1.4

(a) $\varepsilon(M_k) = 0$ for $k > 0$

(b) $\eta_L(m_k) = m_k$

$$\eta_R(m_k) = \sum_{i+j=k} m_i(M)_j^{i+1}$$

(c) $\Psi(M_k) = \sum_{i+j=k} M_i \otimes M_j^{i+1}$

§2: The Hopf Algebra $MU^*(MU)$

We can use §1 to obtain a description of $MU^*(MU)$, the algebra of MU cohomology operations

Proposition 2.1:

$MU^*(MU)$ is the dual $\pi_*(MU)$-module of $MU_*(MU)$

Proof:

We have seen that $H^*(MU)$ is the dual of $H_*(MU)$ as $\mathbf{Z}$-modules. Hence the Atiyah-Hirzebruch spectral sequence

$$E_*^2 \cong H^*(MU;\pi_*(MU)) \Rightarrow MU^*(MU)$$

collapses and $\pi_*(MU)$ is the dual of $\pi_*(MU)$. So E_*^2 is the $\pi_*(MU)$-dual of E_2^* and the collapsing gives the proposition. Q.E.D.

More generally, $MU^*(MU)$ is the dual Hopf algebra of $MU_*(MU)$ (over $\pi_*(MU)$). We can dualize our description of $MU_*(MU)$.

Given a sequence $E = (e_1, e_2, 0, 0, 0, \ldots)$ let

$$B^E = B_1^{e_1} B_2^{e_2} \ldots B_k^{e_k}$$

$$M^E = M_1^{e_1} M_2^{e_2} \ldots M_k^{e_k}$$

Both $\{B^E\}$ and $\{M^E\}$ are a $\pi_*(MU)$ basis of $MU_*(MU)$. So if we let

$$S_E = \text{the dual of } B^E$$

$$r_E = \text{the dual of } M^E$$

then both $\{S_E\}$ and $\{r_E\}$ are bases of $MU^*(MU)$ provided we allow infinite sums. The need for infinite sums arises from the fact that

$$MU^*(MU) = \varprojlim_n MU^*(MU^n).$$

The operation $\{S_E\}$ and $\{r_E\}$ satisfy the Thom formula

$$\sum_{E_1+E_2=E} s_{E_1} r_{E_2} = 0$$

§3: The Action and Coaction Maps

Given any spectrum X we have actions

(1) $$\Psi_*: MU^*(MU)\otimes MU_*(X) \longrightarrow MU_*(X)$$

(2) $$\Psi^*: MU^*(MU)\otimes MU^*(X) \longrightarrow MU^*(X)$$

as well as a coaction.

(3) $$\Psi_X: MU_*(X) \longrightarrow MU_*(MU)\otimes MU_*(X).$$

In this section we describe these maps and the relation between them.

Regarding the action map an element of $MU^*(MU)$ can be interpreted as a map $\theta: MU \longrightarrow MU$ of spectra. Since $MU^*(X) = [X,MU]_*$ and $MU_*X = \pi_*(MU\wedge X)$ it follows that θ induces maps

$$\theta: MU^*(X) \longrightarrow MU^*(X)$$

$$\theta: MU_*(X) \longrightarrow MU_*(X).$$

Regarding the coaction map it is the composite

$$\Psi: MU_*(X) \xrightarrow{\bar{\iota}_*} MU_*(MU\wedge X) \xrightarrow{\mu_*^{-1}} MU_*(MU)\otimes MU_*(X)$$

where $\bar{\iota}$ is the map $X\simeq = S^0\wedge X \longrightarrow MU\wedge X$ and μ_* is the isomorphism given by the exterior product i.e. $MU_*(MU\wedge\_)$ and $MU_*(MU)\otimes MU(\_)$ are homology theories which agree on X = point

The action Ψ_X and the coaction Ψ_X can be determined one from the other via the Adams formulas:

Proposition 3.1 (Formula #1)

Given $\theta \in MU^*(MU)$ and $x \in MU_*(X)$ suppose that $\Psi_*(x) = \sum e_i \otimes x_i$ where $e_i \in MU_*(MU)$ and $x_i \in MU_*(X)$ then

$$\theta(x) = \sum \langle\theta, \chi(e_i)\rangle x_i$$

where χ is the canonical anti-automorphism,

Proof: See [Switzer 1] prop. 17.10.

Proposition 3.2: (Formula #2)

Given $\theta \in MU^*(MU)$, $x \in MU_*(MU)$ and $y \in MU^*(X)$ and suppose

$$\Psi_X(x) = \sum e_i \otimes x_i \quad \text{where } e_i \in MU_*(MU) \quad x_i \in MU_*(X)$$

Then

$$\langle\theta(y), x\rangle = \sum (-1)^{|y||e_i|} \langle\theta, e_i\langle y, x_i\rangle\rangle$$

Proof: See [Switzer 1] Proposition 17.11.

Proposition 3.3:

The coaction map preserves external products i.e. if $x \in MU_*(X)$ and $y \in MU_*(Y)$ then

$$\Psi_{X\wedge Y}(x\times y) = \Psi_X(x)\times\Psi_Y(y).$$

Proof: See [Switzer 1] theorem 17.8.

Corollary 3.4: (Cartan formula)

$$S_E(x\times y) = \sum_{E = E_1+E_2} S_{E_1}(x)\times S_{E_2}(y)$$

Proof: We recall that $MU_*(MU) = \pi_*(MU)[B_1, \ldots B_k, \ldots]$. If $x \in MU_*(X)$ then $\Psi_X(x) = \sum \alpha_{E,i} B^E \otimes x_i$ and if $y \in MU_*(Y)$ then $\Psi_Y(y) = \sum \beta_{F,j} B^F \otimes y_j$ where $\alpha_{E,i}$ and $\beta_{E,i}$ are in $\pi_*(MU)$ while E and F are integers sequence. Then by Propostion 3.3

$$\Psi_{X\wedge Y}(x\times y) = \sum ((\alpha_{E_1,i})(\beta_{E_2,i})B^{E_1+E_2})$$

So by Adams formula #1.

$$S_E(x\times y) = \sum S_{E_1}(x)\times S_{E_2}(y).$$ Q.E.D.

The coaction map in the case $X = \mathbf{C}P^\infty$ has a simple description

Proposition 3.5.

$$\Psi: MU_*(\mathbf{C}P^\infty) \longrightarrow MU_*(MU)\otimes_{\pi_*(MU)} MU_*(\mathbf{C}P^\infty)$$

is given by

$$\Psi(B_k) = \sum_{i+j=k} (B)_i^j \otimes B_j.$$

Here $B = 1+B_1+B_2\ldots$ and B_i^j denotes the i^{th} component of B^j. This result is actually proved as a preliminary to determining the coproduct $\Psi: MU_*(MU) \longrightarrow MU_*(MU)\otimes_{\pi_* MU} MU_*(MU)$ as in §1-1.

Proof: See [Switzer 1] theorem 17.16.

§4: Action of $MU^*(MU)$ on $\pi_*(MU)$.

We begin by showing that the action of $MU^*(MU)$ on $MU_*(pt) \cong \pi_*(MU)$ is completely determined by

$$\eta_R : \pi_*(MU) \longrightarrow MU_*(MU).$$

Proposition 4.1:

Given $x \in \pi_*(MU)$ and $\theta \in MU^*(MU)$ then

$$\theta(x) = \langle \theta, \eta_R(x) \rangle$$

Proof: The diagram

$$\begin{array}{ccccc} S^i & \longrightarrow & MU & \xrightarrow{\theta} & MU \\ & & \downarrow{\scriptstyle \eta_R} & & \uparrow{\scriptstyle \mu} \\ & & MU \wedge MU & \xrightarrow{1\wedge\theta} & MU \wedge MU \end{array}$$

commutes. Q.E.D.

Proposition 4.2: In $\pi_*(MU)\otimes\mathbb{Q}$ we have

$$S_E(b_k) = \begin{cases} 0 & \text{if } |E| \geq 2k \text{ and } E \neq \Delta_k \\ 1 & \text{if } E = \Delta_k \end{cases}$$

$$r_E(m_k) = \begin{cases} 0 & \text{if } |E| \geq 2k \text{ and } E \neq \Delta_k \\ 1 & \text{if } E = \Delta_k \end{cases}$$

Here $|E| = \sum e_i$ and $\Delta_k = (0, \ldots \underset{k\text{-place}}{1} \ldots)$. Recall

$$\pi_*(MU)\otimes\mathbb{Q} \cong H_*(MU)\otimes\mathbb{Q} = \mathbb{Q}[b_1 \ldots b_k \ldots] = \mathbb{Q}[m_1, \ldots m_k \ldots]$$

Proof: By our dscription of η_R in §1 we have

$$\eta_R(b) \equiv B_k \text{ mod the ideal } (b_1, b_2, \ldots).$$

$$\eta_R(m_k) \equiv M_k \text{ mod the ideal } (m_1, m_2, \ldots).$$

Applying Proposition 4.1 and the definition of S_E and r_E we have the result. Q.E.D.

It follows from the above corollary and the Cartan Formula that:

Corollary 4.3: In $\pi_*(MU)\otimes\mathbb{Q}$

$$S_E(b^F) = \begin{cases} 0 & \text{if } |E| \geq |F| \text{ and } E \neq F \\ 1 & \text{if } E = F \end{cases}$$

$$r_E(m^F) = \begin{cases} 0 & \text{if } |E| \geq |F| \text{ and } E \neq F \\ 1 & \text{if } E = F. \end{cases}$$

Part III The Operation $\mathcal{P}$.

§1. The Operation $\mathcal{P}$ and Primitivity.

Definition 1.1: The total operation in MU-cohomolgy is

$$\mathcal{P}: MU^*(X)\otimes\mathbb{Q} \longrightarrow MU^*(X)\otimes\mathbb{Q}$$
$$\mathcal{P} = \sum_E m^E S_E$$

Definition 1.2: Given $\beta \in MU^*(X)$ we said that α is primitive if and only if all the operations in $MU^*(MU)$ with positive degree acts trivially on β i.e for all $E \neq (0,0,0\ldots)$ $S_E(x) = 0$.

Proposition 1.3: $x \in MU^*(X)\otimes\mathbb{Q}$ is primitive if and only if $x = \mathcal{P}(y)$ for some $y \in MU_*(X)\otimes\mathbb{Q}$

Proof: Suppose x is primitive. Then $S_E(x) = 0$ for $E \neq (0,0,\ldots)$ implies $\mathcal{P}(x) = x$. Conversely we will show that $S_E(\mathcal{P}(y)) = 0$ for all $E \neq (0,0,\ldots)$, and $y \in MU^*(X)\otimes\mathbb{Q}$. Define a filtration of $MU^*(X)\otimes\mathbb{Q}$.

$$\mathcal{F}_q = \{x \in MU^*(X)\otimes\mathbb{Q} \mid |x| \le q\}$$

If $F \neq (0,0,\ldots)$

$$S_F(\mathcal{P}(y)) = S_F(\sum_E m^E S_E(y))$$
$$= \sum_E (\sum_{F_1+F_2=F} S_{F_1}(m^E) S_{F_2}(S_E(y)).$$

But

$$S_{F_1}(m^E) = \begin{cases} 1 & \text{if } E = F_1 \\ 0 & \text{if } E \neq F_1. \end{cases}$$

So

$$S_F(\mathcal{P}(y)) \equiv \sum_{F_1+F_2=F} S_{F_2}(S_{F_1}(y)) \equiv 0 \mod \mathcal{F}.$$

Inductively suppose that $S_E(\varphi(y)) \equiv 0 \mod \mathcal{F}_q$ $q > 1$. If $S_E(\varphi(y)) \equiv \sum m^F x_F \mod \mathcal{F}_{q-1}$ then $S_F(S_E(\varphi(y))) \equiv x_F \mod \mathcal{F}_1$. But $S_F S_E = \sum \alpha_G S_G$. By case $q = 1$ we have $S_G(\varphi(y)) \equiv 0 \mod \mathcal{F}_1$. Hence $x_F \equiv 0 \mod \mathcal{F}_1$ and $m^F x_F \equiv 0 \mod \mathcal{F}_{q+1}$. Q.E.D.

<u>Proposition 1.4</u>: φ is multiplicative i.e. $\varphi(xy) = \varphi(x)\varphi(y)$ for any $x, y \in MU^*(X)\otimes\mathbb{Q}$.

<u>Proof</u>: The Cartan formula says that

$$S_E(xy) = \sum_{E_1+E_2=E} S_{E_1}(x) S_{E_2}(y)$$

and

$$m^{E_1+E_2} = m^{E_1} \cdot m^{E_2}$$. so we are done Q.E.D.

<u>Proposition 1.5</u>: There exists a monomorphism

$$\hat{\varphi} : H^*(X)\otimes\mathbb{Q} \longrightarrow MU^*(X)\otimes\mathbb{Q}$$

such that the following diagram commutes

$$\begin{array}{ccc} MU^*(X)\otimes\mathbb{Q} & \longrightarrow & MU^*(X)\otimes\mathbb{Q} \\ \downarrow T\otimes 1_{\mathbb{Q}} & \nearrow \hat{\varphi}_* & \\ H(X)\otimes\mathbb{Q} & & \end{array}$$

when $T: MU^*(X) \longrightarrow H^*(X)$ is the Thom map.

<u>Proof</u>: By our description of the operators $\{S_E\}$ on $MU^*\otimes\mathbb{Q} = \pi_*(MU)\otimes\mathbb{Q} = \mathbb{Q}[b_1, b_2, \ldots]$ (see §4 of II) only the elements of degree 0 are primitive. By Proposition 1-1 φ annihilates the elements from $MU^*\otimes\mathbb{Q}$ of degree $\neq 0$. In particular $\varphi(b_i) = 0$. But Ker $T\otimes 1_{\mathbb{Q}}$ is the ideal $(b_1, b_2, \ldots)$ and φ is multiplicative. So φ factors through $H^*(X;\mathbb{Q}) = MU^*(X)\otimes\mathbb{Q}/(b_1, b_2, \ldots)$. The resulting map $\hat{\varphi}$ is injective. For $T\cdot\hat{\varphi}$ is the identity map on $H^*(X,\mathbb{Q})$. Q.E.D.

In particular the proof of Proposition 1.5 shows that the primitive elements of $MU^*(X)\otimes\mathbb{Q}$ project to a basis of $H^*(X;\mathbb{Q})$.

Variations of the above discussion are also possible. If we replace $\{m_i\}$ by $\{b_i\}$ and $\{S_E\}$ by $\{r_E\}$ then the above discussion is still valid. So we can define

$$\mathcal{P} = \sum_E b^E r_E.$$

We can also define $\mathcal{P}$ in MU homology G by the same formula. We have

$$\mathcal{P}_*: MU_*(X)\otimes\mathbb{Q} \longrightarrow MU_*(X)\otimes\mathbb{Q}$$

$$\mathcal{P}_* = \sum_E m^E S_E \quad \sum_E b^E r_E.$$

In homology we can also characterize primitive elements via the coaction map. They are elements $\alpha \in MU_*(X)\otimes\mathbb{Q}$ with $\Psi_X(x) = 1\otimes\alpha$.

The map $\mathcal{P}$ and $\mathcal{P}_*$ are related by the rule

$$\langle \mathcal{P}(x), \mathcal{P}_*(y)\rangle = \langle x, y \rangle$$

for all $x \in MU^*(X)\otimes\mathbb{Q}$, $y \in MU_*(X)\otimes\mathbb{Q}$. Now our main concern is to calculate primitive elements in $MU_*(X)$. To do that we consider two problems:

1.- Calculate the image of $\mathcal{P}$ and $\mathcal{P}_*$ in $MU_*(X)\otimes\mathbb{Q}$. So we obtain primitive elements in $MU_*(X)\otimes\mathbb{Q}$.

2 - Integrality problem: Since $\mathcal{P}_*(\alpha) \in MU_*(X)\otimes\mathbb{Q}$ we want to calculate the (minimal) number $k_\alpha \in \mathbb{Z}$ such that $k_\alpha \mathcal{P}_*(\alpha) \in MU_*(X)$.

The first problem involves calculating the action of the Landweber-Novikov operations on $MU_*(X)$. This second problem is related to other integrality problems in algebraic topology e.g. Chern character integrality conditions, calculation of the Hurewicz map of X in ordinary and generalized homology theories, and others. We will discuss these topics in subsequent chapters.

§2: Primitives in $MU_*(\mathbb{C}P^\infty)$ We now solve for $X = \mathbb{C}P^\infty$ the two problems discussed at the end of §1. We know that

$$MU^*(\mathbb{C}P^\infty) = MU^*[[x]], \quad [x] = 2.$$

Here x is the canonical element given by the map $\mathbb{C}P^\infty = MU(1) \longrightarrow MU$.

Proposition 2.1: In $MU^*(\mathbb{C}P^\infty)$

$$S_E(x) = \begin{cases} 0 & \text{if } E \neq \Delta_n \quad n = 1, 2, \ldots \\ x^{n+1} & \text{if } E = \Delta_n \end{cases}$$

where

$$\Delta_q = (0, \ldots 0, 1, 0, \ldots)$$
$$\qquad\qquad\qquad \downarrow$$
$$\qquad\qquad\qquad \text{q-place}$$

Proof: From chapter II we know that

$$\Psi_{\mathbb{C}P^\infty}: MU_*(\mathbb{C}P^\infty) \longrightarrow MU_*(MU) \otimes MU_*(\mathbb{C}P^\infty)$$

is given by

$$\Psi_{\mathbb{C}P^\infty}(\beta_k) = \sum (b)^i_{k-i+1} \otimes \beta_i$$

where $\langle x, \beta_i \rangle = \delta_{ij}$ and $(b)^i_{k-1} = 2(k-i)$-component of $(b_1 + b_2 + \ldots)^i$. In other words $\Psi_{\mathbb{C}P^\infty}(\beta_k) = b_k \otimes \beta_1 + (b)^2_{k-1} \otimes \beta_2 + (b)^3_{k-2} \otimes \beta_3 + \ldots$ By Proposition 3.1 Chapter II $\langle \theta(z), y \rangle = \sum (-1)^i \langle \theta, e_i \rangle \cdot \langle z, y_i \rangle$ $\theta \in MU^*(MU)$ and $\Psi_{\mathbb{C}P^\infty}(y) = \sum e_i \otimes y_i$ y, g_i in $MU_*(\mathbb{C}P^\infty)$. By the definition of S_E:

$$\langle S_E, b^E \rangle = \begin{cases} 1 & E = \Delta_k \\ 0 & \text{otherwise.} \end{cases}$$

So

$$\langle S_E(x), \beta_k \rangle = \langle S_E, b_k \rangle \langle x, \beta_1 \rangle \Rightarrow S_E(x) = x^{n+1} \text{ only if } E = \Delta_n \quad \text{Q.E.D.}$$

Since $\varphi = \sum m^E s_E$ we have

Corollary 2.2: In $MU^*(\mathbb{C}P^\infty)$

$$\varphi(x) = \sum m_i x^{i+1} = \log(x) \in MU^*(\mathbb{C}P^\infty)\otimes\mathbb{Q}.$$

Corollary 2.3: The primitive elements in $MU^*(\mathbb{C}P^\infty)\otimes\mathbb{Q}$ are generated over $\mathbb{Q}$ by $\{\varphi(x), \varphi(x)^2, \varphi(x)^3, \ldots \varphi(x)^k, \ldots\}$.

Proof: Since φ is multiplicative $\varphi(x^k) = \varphi(x)^k$ and $\varphi(x)^k$ is primitive. Since $\{x, x^2, \ldots\}$ project under T to a basis of $H^*(\mathbb{C}P^\infty;\mathbb{Q})$ and since $\varphi(x) \equiv x \bmod \operatorname{Ker}T$ $(\ast\ (m_1, m_2, \ldots))$ it follows that $\{\varphi(x), \varphi(x)^2, \ldots\}$ also project to a basis of $H^*(\mathbb{C}P^\infty;\mathbb{Q})$. Proposition 1-5 then implies the corollary. Q.E.D.

Now we recall by Proposition 5.2 in Chapter I that $MU_*(\mathbb{C}P^\infty)$ is generated by $\{\beta_1, \beta_2, \ldots \beta_k, \ldots\}$ as an MU_*-algebra Since

$$\delta_{ij} = \langle x^i, \beta_j\rangle = \langle\varphi(x^i), \varphi_*(\beta_j)\rangle = \langle\varphi(x)^i, \varphi_*(\beta_j)\rangle,$$

Corollary 2.3 implies that. The primitive elements in $MU_*(\mathbb{C}P^\infty)\otimes\mathbb{Q}$ are generated by

$$\{\varphi_*(\beta_1),\ \varphi_*(\beta_2), \ldots \varphi_*(\beta_k), \ldots\} \text{ etc.}$$

We want an explicit calculation of $\varphi_*(\beta_k)$ $k = 1, 2, \ldots$ Write

$$\varphi(\beta_k) = a_{k,k}\beta_k + a_{k-1,k}\beta_{k-1} + \ldots + a_{1,k}\beta_1$$

when

$$a_{i,j} \in \pi_{2(j-i)}(MU)\otimes\mathbb{Q}.$$

So we want an expression (polynomial) of $a_{i,j}$ in terms of b_i i.e. $a_{i,j}$ are polynomials with $\mathbb{Q}$ coefficients. Since $\varphi(x) = \log(x)$ it follows that

$$x = \exp(\varphi(x)) \sum_{i \geq 0} b_i \varphi(x)^{i+1}$$

So

$$x^k = \sum (b)^k_i \varphi(x)^{k+i}$$

where $b = (1+b_1+b_2+\dots)$ and $(b)^k_i$ is the $2i^{th}$ graded component of $(b)^k$. Since $\{\beta_i\}$ is the dual basis of $\{x^i\}$ and $\{\varphi_*(\beta_i)\}$ is the dual basis of $\{\varphi(x^i)\}$ we deduce

<u>Theorem 2.4</u>: (Segal)

$$\varphi_*(\beta_m) = \sum_{i=1}^{m} (b)^i_{m-i}\beta_i$$

<u>Proposition 2.5</u>: $(k!)\varphi_*(\beta_k) \in MU_{2k}(\mathbb{C}P^\infty)$.

<u>Proof</u>: [Segal 1] Let $f(k)$ be the least positive integer such that $f(k)(b)^i_{m-i} \in MU_*$ for all $j \leq k$. We want to prove :

(i) $f(k)|k!$

(ii) $k!|f(k)$

By Hattori-Stong theorem [Switzer 1] chapter 20 we know that $x \in H_*(MU)$ lie in $Im(h:\pi_*(MU) \longrightarrow H_*(MU))$ if and only if $Td(S_E(x)) \in \mathbb{Z}$ for all sequence $E = (e_1, e_2, \dots)$ and S_E the corresponding Landweber-Novikov operation. Since the b_i's lie in $H_*(MU)\otimes\mathbb{Q}$ via the Hurewicz map we have $Td(f(k)(b)^1_{k-1}) = f(k)Td(b_{k-1}) = f(k)/k!$ then $k!|f(k)$ so (ii) is proved

TO prove (i) we need some other facts :

First we define on $H_*(MU)$ the following filtration :

$F^m = F^mH_*(MU) =$ the subgroup generated by monomials of the form $b_{a(1)} \cdot b_{a(2)} \cdots b_{a(n)}$ such that $(n+\Sigma\, a(i)) \leq m$.

<u>Claim</u> If $b_{a(1)} \cdots b_{a(n)} \in F^k$ then $(k!)Td(b_{a(1)} \cdots b_{a(n)}) \in \mathbf{Z}$

Directly we can see that $Td(b_{a(1)} \cdots b_{a(n)}) = 1/k!(\,_{a(1)+1,\ldots a(n)+1}^{+k})$

so $k!Td(b_{a(1)} \cdots b_{a(n)}) \in \mathbf{Z}$

By theorem 5.2 in [Adams 1] part I

$S_E(b) = (b)^{n+1}$ if $\Sigma e_i = n$ and

$S_E(b) = 0$ otherwise

and $S_E(xy) = \sum_{E = E_1 + E_2} (S_{E_1}(x))(S_{E_2}(y))$ for $x, y \in H_*(MU)$. Therefore F^k is closed under the action of Landweber-Novikov operations, since $(b)^j_{k-j} \in F^k$ then $Td(S_E((k!)(b)^j_{k-j})) \in \mathbf{Z}$ so we have (i) Q.E.D

§ 3 Primitives In $MU_*(\mathbb{H}P^\infty)$

Now we consider the case $X = \mathbb{H}P^\infty$, the quaternionic projective space.

Proposition 3.1. For $E = H\mathbb{Z}$, MU

If $i: \mathbb{C}P^\infty \longrightarrow \mathbb{H}P^\infty$ is the standard inclusion then

$$i^*: E^*(\mathbb{H}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty)$$

is a monomorphism

Proof Since

$$\begin{array}{ccccc} S^2 & \longrightarrow & \mathbb{C}P^\infty & \longrightarrow & \mathbb{H}P^\infty \\ \| & & \| & & \| \\ Sp(1)/U(1) & \longrightarrow & BU(1) & \longrightarrow & BSp(1) \end{array}$$

is a fibration, the collapsing Serre spectral sequence tell us that

$$i^*: H_*(BSp(1)) \longrightarrow H^*(BU(1))$$

is a monomorphism For $E = MU$ we observe that both spaces are torsion free and even graded so the Atiyah-Hirzebruch Spectral Sequence collpases and $i^*: MU^*(BSp(1)) \longrightarrow MU^*(BU(1))$ is a monomorphism. Q.E.D.

We can explicitly determine the image of $E^*(\mathbb{H}P^\infty)$ in $E^*(\mathbb{C}P^\infty)$ for the cases $E = H$ or MU. We can write

$$E^*(\mathbb{H}P^\infty) = E^*[[y]] \quad |y| = 4$$

$$E^*(\mathbb{C}P^\infty) = E^*[[x]] \quad |x| = 2$$

where we regard $y = p_1(\eta_H)$ and $x = c_1(\eta_C)$, the first Pontrjagin and Chern characteristic classes of η_H and η_C, the canonical line bundle over $\mathbb{H}P^\infty = BSp(1)$ and $\mathbb{C}P^\infty = BSU(1)$ respectively. Let $\bar{\eta}_C$ be the inverse line bundle of η_C In other words

$$\eta_C \otimes \bar{\eta}_C = 1.$$

Let $\bar{x} = c_1(\bar{\eta}_C)$.

<u>Proposition 3.2</u>: $i^*(y) = x\bar{x}$.

<u>Proof</u>: The map $i: \mathbb{C}P^\infty \longrightarrow \mathbb{H}P^\infty$ satisfies

$$i^*(\eta_H) = \eta_C \otimes \bar{\eta}_C$$

so

$$i^*(y) = i^*(c_2(\eta_H)) = c_2(i^*(\eta_H)) = c_2(\eta_C \otimes \bar{\eta}_C)$$
$$= c_1(\eta_C)c_1(\bar{\eta}_C) = x\bar{x}.$$ Q.E.D.

In particular, when $E = H_1$, $\bar{x} = -x$. We can also think of $E^*(\mathbb{H}P^\infty)$ as lying in a ring of invariants. Let

$$\tau: \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$$

be the classifying map of $\bar{\eta}_C = \eta_C^{-1}$. We have $\tau^2 = 1$. So $\bar{x} = c_1(\bar{\eta}_C) = c_1(\tau^*(\eta)) = \tau^* c_1(\eta_C) = \tau^*(x)$. Thus $x\bar{x} \in E^*(\mathbb{C}P^\infty)^\tau$. For $\tau^*(x\bar{x}) = \tau^*(x)\tau^*(\bar{x}) = \tau^*(x)\tau^*\tau^*(x) = \bar{x}\cdot x$.

As in the previous section the elements $\{\varphi(y), \varphi(y)^2, \ldots\}$ are a basis of the primitive elements in $MU^*(\mathbb{H}P^\infty)$. If we define $\gamma_n \in MU_{4n}(\mathbb{H}P^\infty)$ by the rule

$$\langle y^k, \gamma_n \rangle = \delta_{k,n}$$

then $\{\gamma_n\}$ is a $\pi_*(MU)$ basis of $MU_*(\mathbb{H}P^\infty)$ and the elements $\{\varphi_*(\gamma_n)\}$ are a basis of the primitives elements in $MU_*(\mathbb{H}P^\infty)$. So we have the pairing

$$\delta_{k,n} = \langle y^k, \gamma_n \rangle = \langle \varphi(y)^k, \varphi_*(\gamma_n) \rangle$$

If we expand $\varphi(\gamma_n)$ in terms of $\{\gamma_i\}$ then we obtain the following: Let

$$B_{2i} = 2\cdot\sum(-1)^{i+j+1} b_{i+j} b_{i-j} + (-1)^{i+1} b_i^2$$
$$B = 1 - B_2 + B_4 + \ldots + (-1)^i B_{2i} + \ldots$$

<u>Theorem 3.3:</u> (Segal) $\varphi(\gamma_n) = \sum_{i=1} (B)^i_{n-1}\gamma_i$ ...

<u>Proof</u>: Since $\overline{x}^H = -x^H$ it follows that $\varphi(\overline{x}^{MU}) = \varphi(-x^{MU})$ (see Proposition 1-3) From now on let $x = x^{MU}$ Write

$$x = \exp\varphi(x) = \sum_{i\geq 0} b_i \varphi(x)^{i-1}$$

Hence

$$\overline{x} = \sum_{i\geq 0} B_{2i}\varphi(\overline{x})^{i-1} = \sum_{i\geq 0} (-1)^{i-1} b_i \varphi(x)^{i-1}$$

and

$$x\overline{x} = \sum_{i\geq 1} B_{2i}\varphi(x)^2 = \sum_{i\geq 1} B_{2i}\varphi(x^2)^i.$$

If we let $y = x\overline{x}$ then we have

$$y = \sum (-1)^i B_{2i}\varphi(y)^i.$$

Here we use the fact that $\varphi(x^2) = \varphi(x)\varphi(x) = -\varphi(x)\varphi(\overline{x}) = -\varphi(y)$

So

$$y^k = \sum (B)^k_i \varphi(y)^{k-i}.$$

If we pass to the dual basis $\{\gamma_i\}$ and $\{\varphi_*(\gamma_i)\}$ then we obtain the theorem. Q.E.D

<u>Proposition 3.4</u>: (Segal) $((2k)!/2)\varphi_*(\gamma_k) \in MU_*(\mathbb{H}P^\infty)$.

<u>Proof</u>: [Segal 1] Similary as in Propositio 2.5 .Let g(k) be the least positive integer such that $g(k)(B)^j_{k-j} \in MU_*$ We want to prove

(i) $g(k) \mid (2k)!/2$

(ii) $(2k!)!/2 \mid g(k)$

Since

$$Td(g(k)(B)^1_{k-1}) = g(k)\Sigma_{j\leq 2k-2} (-1)^j Td(b_j b_{2k-2-j}) = (g(k)/(2k)!)\Sigma_{j\leq 2k-2} (-1)\binom{k}{j-1}$$

$= g(k)\ (2k)!(-(1-1)^{2k}-1-1) = 2g(k)/(2k)!$ so $(2k)!TdS_E(B)^j_{k-j}) \in 2\mathbb{Z}$ for all $j \leq k$ and (i) is done

If F^k is the filtration of $H_*(MU)$ as in Proposition 2.5 we have : if $b_{a(1)}\cdots b_{a(n)} \in F^k$ then $(2n)!Td(b_{a(1)}\cdots b_{a(n)}) \in 2\mathbb{Z}$ it follows by our Claim in Proposition 2.5 and that $(2n)!/((n)!)^2 \in 2\mathbb{Z}$

Let G^k be the subgroup of F^{2k} generated (additively) by $(F^k)^2$ and $2F^{2k}$ so if $x \in G^k$ then $(2k)!Td(x) \in 2\mathbb{Z}$ and G^k is closed under Landweber - Novikov because of theorem in [Adams 1] Part I. Since $(B)^j_{k-j} \in G^k$ we are done Q.E.D.

§4: The Chern Character

It is known (by [Dold 1]) that given two ring spectra, say F and E, and a homomorphism $\varphi: \pi_*(F) \longrightarrow \pi_*(E)$ one can choose a map $\phi: F \longrightarrow E$ between spectra such that $\varphi = \phi_*$ whenever $\pi_*(E)$ is a rational vector space. The choice of ϕ is unique. Consider the following diagram

$$\begin{array}{ccc} F^*(X) & \longrightarrow & \mathrm{Hom}(\pi_*^S(X), \tilde{F}^*(S^0)) \\ \downarrow \phi & & \downarrow \varphi \\ E^*(X) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_*^S(X), \tilde{E}^*(S^0)) \end{array}$$

Since $\tilde{E}^*(S^0)$ is $\mathbb{Q}$-vector space. $\mathrm{Hom}(\pi_*^S(-), \tilde{E}^*(S^0))$ defines a cohomology theory. So by the uniqueness of such functors and Serre's theorem that $\pi_q^S(S^0) = \begin{cases} \mathbb{Z} & q = 0 \\ G_q & q > 0 \end{cases}$ where G_q is a finite group. We can check that bottom homomorphism is in fact an isomorphism. So we get ϕ as assserted above.

If we let $F = MU$, $E = K\mathbb{Q}$, the <u>unitary</u> K-theory with rational coefficents, and $\phi: \pi_*(MU) \longrightarrow \pi_*(K)$ the Todd map then our map $\phi: MU \longrightarrow K\mathbb{Q}$ is known as the Conner-Floyd map (see [Conner-Floyd 1] §6). In the same fashion we can define the Chern character. Again let E and F be a ring spectra, X a finite complex. The following diagram commutes

$$\begin{array}{ccc} F^*(X) & \xrightarrow{B} & (E \wedge F)^*(X) \\ & \searrow \alpha & \downarrow \tau \\ & & \mathrm{Hom}^*(E_*(X), F_*(X)) \end{array}$$

see [Switzer 1] chapter 13 page 291

Here B is the Boardman map i.e. the map induced by

$F \simeq F\wedge S^0 \xrightarrow{1\wedge i} F\wedge E$.

τ is given by $\tau([f])([g]) = [\mu_E\circ(1\wedge f)\circ(g)]$ for $f: X \longrightarrow E\wedge F$ and

$g: S^0 \longrightarrow E\wedge X$ and $\alpha([z]) = (z_*: E_*(X) \longrightarrow E_*(F))$, $z:X \longrightarrow F$.

Moreover if $E_*(F)$ is flat as E_*-module then

$$\begin{array}{ccc} F^n(X)\otimes F(X) & \xrightarrow{\langle\ \rangle} & \pi_{m-n}(F) \\ \downarrow \alpha\otimes H & & H \downarrow \\ \mathrm{Hom}^{-n}_{E_*}(E_*(X),E_*(F))\otimes E_*(F)\otimes_{E_*} E_*(X) & \xrightarrow{K} & E_{m-n}(F) \end{array}$$

commutes, where H is the corresponding Hurewicz map and

$$K(\phi\otimes e\otimes x) = \phi(x)\cdot e \quad \text{in} \quad E_{m-n}(F) \text{ (Pontrjagin product)}$$

See [Switzer 1] chapter 13, pages 291-292.

Taking $E = H\mathbf{Z}$ and $F = K$, we define the universal Chern character as

$$\overline{ch}: K^*(X) \longrightarrow \mathrm{Hom}^*(H_*(X),H_*(K))$$

$$\overline{ch} = \tau\circ B = \alpha.$$

Since $H_*(K) \cong \mathbf{Q}[u,u^{-1}]$ when $\langle c_1,u\rangle = 1$ and since $\mathrm{Hom}_{\mathbf{Z}}(-;\mathbf{Q})$ is an exact functor for $\mathbf{Z}$-modules we have

$$\mathrm{Hom}^n(H_*(X),H_*(K)) = \begin{cases} \bigoplus_{q\geq 0} H^{2q}(X;\mathbf{Q}) & \text{for } n \text{ even} \\ \bigoplus_{q\geq 0} H^{2q+1}(X;\mathbf{Q}) & \text{for } n \text{ odd} \end{cases}$$

So we can regard $\overline{ch}$ as the map $\overline{ch}: K^*(X) \longrightarrow H^{**}(X;\mathbf{Q})$.

<u>Proposition 4.1</u>: If $ch^*: K^*(X) \longrightarrow H^{**}(X;\mathbf{Q})$ denotes the $\mathbf{Z}/2$-graded Chern character then $ch^* = \overline{ch}$.

Proof: By the naturality of ch^* and the splitting principle in K-theory it suffices to show the statement for $X = \mathbb{C}P^\infty$. Now $H^*(\mathbb{C}P^\infty)$ is a free finitely generated $\mathbb{Z}$-module. So
$Hom^*(H_*(\mathbb{C}P^\infty);H_*(K)) \cong H^*(\mathbb{C}P^\infty)\hat{\otimes}H_*(K)$.
For if $\{z_\alpha\}$ is a basis of $H^*(X;\mathbb{Z})$ and $\{x_\alpha\}$ denote the respective dual elements in $H_*(X;\mathbb{Z})$ then each $\phi: H_*(\mathbb{C}P^\infty) \longrightarrow H_*(K)$ defines an element

$$\sum_\alpha z_\alpha \otimes \phi(x_\alpha) \quad \text{in} \quad H^*(\mathbb{C}P^\infty)\otimes H_*(K).$$

This association: $\phi \longmapsto \sum z_\alpha \otimes \phi(x_\alpha)$ defines the above isomorphism.

Let γ be the Hopf bundle over $\mathbb{C}P^\infty$. Now $(\gamma-1) \in K^0(BU(1)) = K^0(\mathbb{C}P^\infty)$ is represented by

$$i: BU(1) \rightarrowtail BU \rightarrowtail K.$$

Here we regard the space BU as the 0-term of the spectrum K.

Let $g_i \in H_{2i}(BU(1))$ be given by $\langle y^k, x^k\rangle = \delta_{n,k}$, $x^n \in H^{2n}(BU(1))$, then

$$(\tau\circ B)(\gamma-1) = \alpha(\gamma-1): H_*(BU(1)) \longrightarrow H_*(K).$$

So

$$(\tau\circ B)(\gamma-1)(y_i) = \alpha(\gamma-1)(y_i) = y_1^i \in H_{2i}(K).$$

By construction $y_1^i = (1/(i!))U^i \in H_{2i}(K)$. So

$$\overline{ch}(\gamma)-1 = \overline{ch}(\gamma-1) = \sum x^i \otimes \tau B(\gamma-1)(y_i) = \sum_{i\geq 0}(1/i)x^i \otimes U^i.$$

But the even-stage of $H^*(\mathbb{C}P^\infty)\hat{\otimes}H_*(K)$ is $\bigoplus_{i\geq 0} H^{2i}(BU(1);\mathbb{Q}) \cong H^{even}(\mathbb{C}P^\infty;\mathbb{Q})$. So we have

$$\overline{ch}(\gamma) = 1 + y + y^2/2! + y^3/3! + \dots = e^y$$

where y is the dual of x. Therefore $\overline{ch}(\gamma) = ch^*(\gamma)$. Q.E.D.

<u>Remark</u>: For the definition of the ordinary Chern character, splitting principle, etc. we refer to [Bott 1].

Dually we can define a homomorphism <u>ch</u>: $K_*(X) \longrightarrow H_{**}(X;\mathbb{Q})$ as follows. Let $K_*(X) \cong \pi_*(K \wedge X) \xrightarrow{H} H_*(K \wedge X)$ be the Hurewicz map and $H_*(K) \otimes H_*(X) \xrightarrow{x} H_*(K \wedge X)$ be the exterior product. So we have a map

$$K_*(X) \longrightarrow H_*(K \wedge X) \xrightarrow{x^{-1}} H_*(K) \otimes H_*(X).$$

But

$$H_*(K) \otimes_{\mathbb{Z}} H_*(X) \cong H_{**}(X,\mathbb{Q}) = \begin{cases} \oplus H_{2q}(X,\mathbb{Q}) \\ \oplus H_{2q+1}(X,\mathbb{Q}). \end{cases}$$

Since $H_*(K) \cong \mathbb{Q}[u,u^{-1}]$ therefore the Chern character in homology is essentially the Hurewicz map $K_*(X) \xrightarrow{H} (H\mathbb{Z} \wedge K)_*(X)$. We are now able to prove:

<u>Lemma 4.2</u>: If X is a finite complex and $h: \pi_*(X) \longrightarrow H_*(X)$ denote the Hurewicz map, then the following square commutes.

$$\begin{array}{ccc} \pi_*(X)/\mathrm{tor} & \xrightarrow{\quad h \quad} & H_*(X)/\mathrm{tor} \\ \Big\downarrow h^k & & \Big\downarrow \\ K_*(X)/\mathrm{tor} & \xrightarrow[\quad ch_* \quad]{} & H_*(X;\mathbb{Q}) \end{array}$$

<u>Proof</u>: The following square is commutative.

$$\begin{array}{ccccccc} S^n \longrightarrow X \simeq X \wedge S^0 & \longrightarrow & X \wedge H\mathbb{Z} & \xrightarrow{inc} & X \wedge H\mathbb{Q} & \xrightarrow{\cong} & X \wedge H\mathbb{Q} \wedge S^0 \\ \Big\downarrow 1 \wedge \iota_k & & & & & & \Big\downarrow \tau \circ 1 \wedge \iota_k \\ X \wedge K \longrightarrow X \wedge K \wedge S^0 & \xrightarrow{1 \wedge \iota_H} & X \wedge K \wedge H\mathbb{Z} & \xrightarrow{inc} & & & X \wedge K \wedge H\mathbb{Q} \end{array}$$

Q.E.D

For each $x \in H_*(X)/\mathrm{tor}$ denote $G_x = \{z \in K^*(X) \mid \langle ch^*(z);x \rangle \in \mathbb{Z}\}$ Clearly G_x is a subgroup of $K^*(X)$. One of our main goals in this thesis is to study these subgroups when X is a Lie group.

The above lemma implies

Corollary 4.3: [Bott 1]

If $x \in \mathrm{Im}(h_X : \pi_*(X)/\mathrm{tor} \longrightarrow H_*(X_i)/\mathrm{tor})$ then

$$G_X = K^*(X).$$

Proof: If $x \in \mathrm{Im}\, h_X$ then there exists $\alpha \in \pi_*(X)/\mathrm{tor}$ with $x = h_X(\alpha) = ch_* h_1^k(\alpha)$ so for any $z \in K^*(X)$ we have $\langle ch^*(z), x\rangle = \langle ch^*(z), ch_* h^k(\alpha)\rangle = \langle z, h^k(\alpha)\rangle \in \mathbf{Z}$ because

$$\begin{array}{ccccc}
K^*(X)\otimes K_*(X) & \xrightarrow{\langle\ \rangle} & \pi_*(K) & \cong & \mathbf{Z} \\
\downarrow{\scriptstyle ch^*\otimes ch_*} & & \downarrow & & \downarrow \\
H^*(X;Q)\otimes H_*(X,Q) & \xrightarrow{\langle\ \rangle} & \pi_*(HQ) & \cong & \mathbf{Q}
\end{array}$$

commutes. Q.E.D.

The Chern Characters and MU Operations

Let $E = (e_1, e_2, \dots)$ a sequence of non negative intergers with $e_i = 0$ for all except a finite number of i's. Define $m(E) = m_1^{e_1} \dots m_1^{e_1} \dots \in \pi_*(MU)$. If r_F denotes the conjugate of S_F in $MU^*(MU)$ we know by Chapter II that

$$r_F(m(E)) = \begin{cases} 0 & \text{if } |E| \leq |F| \text{ and } E \neq F, \\ 1 & \text{if } E = F. \end{cases}$$

$$Tr_F(m(E)) = \begin{cases} 0 & E \neq F, \\ 1 & E = F. \end{cases}$$

We define the rational number $W(E)$ for each sequence as

$$W(E) = \Big(\prod_{n>1} (n+1)^{e_n}\Big)^{-1} = 1/2^{e_1} \cdot 1/3^{e_2} \cdot \dots$$

We define an operation $R\colon MU_*(X)\otimes Q \longrightarrow MU_*(X)\otimes Q$ as

$$R = \sum W(E) r_E .$$

Proposition 4.4: There exists a unique homomorphism

$$R\colon K_*(X)\otimes Q \longrightarrow MU_*(X)\otimes Q$$

such that the following diagram commutes

$$\begin{array}{ccc} MU_*(X)\otimes Q & \xrightarrow{R} & MU(X)\otimes Q \\ \big\downarrow \beta & & \big\downarrow T \\ K_*(X)\otimes Q) & \xrightarrow[R]{} & H_*(X)\otimes Q \end{array}$$

Here β is the Conner-Floyd map and T the Thom map.

Proof: We are going to prove the dual statement. Since we are dealing with Q-modules the dual statement is equivalent.

Suppose X = point. By Chapter II we know that r_E acts on MU^*(point) as

$$r_E(m(F)) = \begin{cases} 0 & |E| \geq F \quad E \neq F \\ 1 & \text{if} \quad E = F. \end{cases}$$

Then $T \circ R$ act on $MU^*(\text{point})$

$$\sum_{|E| \geq 0} W(E)T(r_E(m(F))) = W(F) \in \mathbb{Q}$$

So $T \circ R: \pi_*(MU) \longrightarrow \pi_*(H\mathbb{Q})$ factors through the Todd genus. By the definition of the Conner-Floyd map and by our discussion above there exists $\hat{R}: K^*(X)\otimes\mathbb{Q} \longrightarrow H^*(X;\mathbb{Q})$ with $\beta \circ \hat{R} = T \circ R$ (see: [Dold 1]). Since X is a finite complex, there exists $\hat{R}: K^*(X)\otimes\mathbb{Q} \longrightarrow H^*(X;\mathbb{Q})$ which is unique. Q.E.D.

In §1 we defined $\varphi_*: H_*(X)\otimes\mathbb{Q} \longrightarrow MU_*(X)\otimes\mathbb{Q}$ such that

$$\begin{array}{ccc} MU_*(X)\otimes\mathbb{Q} & \xrightarrow{\varphi_*} & MU_*(X)\otimes\mathbb{Q} \\ & \searrow \quad \nearrow \varphi_* & \\ & H_*(X)\otimes\mathbb{Q} & \end{array}$$

Define $\bar{\varphi}_*: H_*(X)\otimes\mathbb{Q} \longrightarrow K_*(X)\otimes\mathbb{Q}$ as $\beta \circ \hat{\varphi}_* = \bar{\varphi}$.

$$\begin{array}{ccc} H_*(X)\otimes\mathbb{Q} & \longrightarrow & K_*(X)\otimes\mathbb{Q} \\ \hat{\varphi}_* \searrow & & \nearrow \beta \\ & MU_*(X)\otimes\mathbb{Q} & \end{array}$$

<u>Proposition 4.5</u>: $\bar{\varphi}_*$ and $\hat{R}$ are inverses.

<u>Proof</u>: $\hat{R}\bar{\varphi}_* = \hat{R}\circ\beta\circ\hat{\varphi}_* = T\circ R\circ\hat{\varphi}_*$ but $\operatorname{Im}\hat{\varphi}_* = \operatorname{Im}\varphi_*$ and any element in the image of φ_* is primitive so $r_E(\varphi_*(y)) = 0$ for $|E| \neq 0$

Thus $T\circ R\circ\hat{\varphi}_*(y) \equiv y$ mod (Ker T). On the other hand

$$\bar{\varphi}_*\hat{R}(x) = \sum_E \sum_{E'+E''=E} W(E'')M(E')s_{E'}r_{E''}(x) \equiv \sum_{?} W(E) \sum_{E'+E''=E} x_{E'}r_{E''}(x)$$

$$= x \mod \operatorname{Ker} \beta$$

because $\beta(M(E')W(E'')) = W(E'')W(E') = W(E)$.

Since β and T are onto we are done. Q.E.D.

We are ready to prove

<u>Theorem 4.6</u>: The following square commutes for X a finite complex

$$\begin{array}{ccc} MU_*(X)\otimes\mathbb{Q} & \xrightarrow{\varphi} & MU_{**}(X)\otimes\mathbb{Q}' \\ \downarrow T & \overset{\varphi_*}{\nearrow} & \downarrow \beta \\ H_*(X)\otimes\mathbb{Q} & \xleftarrow[ch_*\otimes\mathbb{Q}]{} & K_*(X)\otimes\mathbb{Q} \end{array}$$

<u>Proof</u>: By Proposition 4.4 we have to prove that $(ch_*\otimes\mathbb{Q})\circ\beta = T\circ R$. If we suppose X = point we already have seen that $T\circ R$ is exactly the Todd genus.

Now the Chern character was defined (see the discussion after Proposition 4.1) as the Hurewicz map.

$$K_*(X) \longrightarrow (H\wedge K)_*(X)$$

When $X = pt$ this is the ordinary Hurewicz for K

$$h_K: \pi_*(K) \longrightarrow H_*(K)\ ,$$

which rationally is an isomorphism. The Conner-Floyd map β was defined in such a way that, for X = point, β coincides with the Todd genus so $(ch_*\otimes\mathbb{Q})\circ\beta = T\circ R$ when X is a point and we are done.

Q.E.D.

We have the following

<u>Corollary 4.7</u>: Let X be a finite complex. If $\alpha \in MU_*(X)$ is primitive then $G_{T(\alpha)} = K^*(X)$.

<u>Proof</u>: Let $z \in K^*(X)$. Since β is onto [Conner-Smith 1] there exist $\gamma \in MU^*(X)$ with $\beta(\gamma) = z$ and $\langle\gamma,\alpha\rangle \in \mathbb{Z}$. Now

$$\langle\gamma,\alpha\rangle = \langle ch^*\beta(\gamma), TR(\alpha)\rangle = \langle ch^*(z), T(\alpha)\rangle .$$

Regarding the last equality α primitive implies $R(\alpha) = \alpha$. So we are done. Q.E.D.

Consider the following commutative diagram

$$\begin{array}{ccc} X\wedge MU \simeq S^0\wedge X\wedge MU & \xrightarrow{\iota_{H\mathbf{Z}}\wedge 1\wedge 1} & H\mathbf{Z}\wedge X\wedge MU \\ \downarrow 1\wedge\beta & & \downarrow 1\wedge 1\wedge\beta \\ X\wedge K \simeq S^0\wedge X\wedge K & \xrightarrow{\iota_{H\mathbf{Z}}\wedge 1\wedge 1} & H\mathbf{Z}\wedge X\wedge K \end{array}$$

It induces the commutative diagram

$$\begin{array}{ccc} MU_*(X) & \xrightarrow{H^{MU}} & H_*(X\wedge MU) \cong H_*(X)\otimes H_*(MU) \\ \downarrow \beta & & \downarrow \beta \\ K_*(X) & \xrightarrow{H^{K}} & H_*(X\wedge K) \cong H_*(X)\otimes H_*(K) \end{array}$$

We have already seen that $H^K\circ\beta = (ch_*\otimes\mathbf{Q})\circ\beta = T\circ R$. So we have another expression for $T\circ R$, namely $T\circ R = \beta\circ H^{MU}$ that one is known by the Todd character and has been studied in see [Smith 1] [Smith 2]. Now $H^{MU}: MU_*(X) \longrightarrow H_*(X,\mathbf{Z})\otimes H_*(MU;\mathbf{Z})$ is given by

$$H^{MU}(\alpha) = \sum_E T(r_E(\alpha))\otimes b^E$$

where

$$H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_1,b_2,\dots] \quad \text{and} \quad b^E = b_1^{e_1}\cdot b_2^{e_2}\cdots$$

Therefore we will see how β acts on $H_*(MU;\mathbf{Z})$ and get an expression for $\beta\circ H^{MU} = T\circ R$.

Since $h\otimes\mathbf{Q}: \pi_*(MU)\otimes\mathbf{Q} \longrightarrow H_*(MU;Q)$ is an isomorphism and $h: \pi_*(MU) \longrightarrow H_*(MU;\mathbf{Z})$ is mono (see Chapter I §6) then β acts on $H_*(MU;\mathbf{Z})$ as it does in $\pi_*(MU)$ i.e. as the Todd genus.

Since $\beta: H_*(MU,\mathbf{Z}) \longrightarrow H_*(K;\mathbf{Z}) \equiv \mathbf{Q}[u,u^{-1}]$ then

Lemma 4.8: $\beta(b^E) = B_E$ where $B_E = (1/2)^{e_1} B_1^{e_2} 0^{e_3} B_2^{e_4} \ldots 0^{e_{2i-1}} B_2^{e_{2i-2}} \ldots$ when $0^0 = 1$ and $0^a = 0$ for $a \neq 0$ and $B_K = (-1)^k b_k\,(2k)!$ when b_k are the Bernoulli numbers as in [Hirzebruch 1] page 13

Proof Recall that $f(t) \in \mathbf{Q}[[t]]$ is the characteristic series for the Todd genus if $f(t) = \sum \beta(b_i)t^i \in \mathbf{Q}[[t]]$ and

$$f((t) = t/(1-e^{-t}) = 1 - (1/2)t - \sum_{k=1} (-1)^{k-1}(b_k/(2k)!)t^{2k}$$

[Hirzebruch 1] page 11. Recall $b_{2k-1} = 0$ for $k \geq 1$.

Since the following diagram commutes

$$\begin{array}{ccc} \pi_*(MU) & \xrightarrow{\text{Todd}} & \pi_*(K(\mathbf{Q}[[t]])) \cong \mathbf{Q}[[t]] \\ \downarrow h & & \cong \downarrow h \\ H_*(MU;\mathbf{Z}) & \xrightarrow{\beta} & H_*(K(\mathbf{Q}[[t]]) \cong \mathbf{Q}[[t]] \\ \downarrow & & \downarrow \cong \\ H_*(MU;Q) & \xrightarrow{\beta} & H_*(K(Q[[t]]);\mathbf{Q}) \cong [[t]]. \end{array}$$

we get that $\beta(b^E) = B_E t^{||E||} \in \mathbf{Q}[[t]]$. Q.E.D.

We have

Theorem 4.9 ([Smith 1]): For X finite complex and $\alpha \in MU_*(X)$.

$$T\circ R(\alpha) = \sum_{||E|| \geq 0} B_E T(r_E(\alpha)).$$

Proof: We have already seen that $T\circ R = \beta\circ H^{MU}$. In other words, $T\circ R$ is the composition

$$\begin{array}{ccc} MU_*(X) & \xrightarrow{H} & H_*(MU\wedge X) \cong H_*(MU;\mathbf{Z})\otimes H_*(X;\mathbf{Z}) \\ \downarrow T\circ R & & \downarrow \beta\otimes 1 \\ H_{**}(X;\mathbf{Q}) & \longleftarrow & \mathbf{Q}[u,u^{-1}]\otimes H_*(X;\mathbf{Z}) \cong H_*(K;\mathbf{Z})\otimes H_*(X;\mathbf{Z}). \end{array}$$

Also $H^{MU}(\alpha) = \sum_E T(r_E(\alpha))$. So by Lemma 4.8

$$T\circ R = \sum_E B_E T(r_E(\alpha)).$$

Q.E.D.

Now we want to calculate the denominator of B_E for each E and consequently give an upper bound number to make $T^{\circ}R(\alpha)$ and thus $R(\alpha)$ integral for each $\alpha \in MU_*(X)$.

First notice that

$$B_{E+F} = B_E \cdot B_F \quad \text{and} \quad B_{nE} = (B_E)^n \quad \text{for} \quad n \geq 0.$$

Definition $B_E = 0$ if $e_k \neq 0$ and K odd and

$$B_{\Delta_k} = \begin{cases} 1/2 & \text{if } k = 1 \\ B_l & \text{if } k = 2l \\ 0 & \text{if } k \text{ is odd} \end{cases} \qquad \Delta_k = (0,\ldots,\overset{k}{1},\ldots 0)$$

Since $B_E = (e_1, e_2, \ldots e_k, \ldots) = e_1\Delta_1 + e\Delta_2 + \cdots + e_k\Delta_k + \cdots$ then

$$B_E = B_{\Sigma e_k \Delta_k} = \prod_{k \geq 1} (B\Delta_k)^{e_k} = (1/2)^{e_1} \prod_{\substack{k>1 \\ k \text{ even}}} (B_k/(2k)!)^{e_k}.$$

<u>Definition</u>: Let $m(t)$ be the number theory funtion defined in [Adams 3] §1. Namely, for p odd

$$\nu_p(m(t)) = \begin{cases} 0 & \text{if } t \neq 0 \mod (p-1) \\ 1+\nu_p(t) & \text{if } t \neq 0 \mod (p-1) \end{cases}$$

and

$$\nu_2(m(t)) = \begin{cases} 1 & \text{if } t \neq 0 \mod 2 \\ 2+\nu_2(t) & \text{if } t = 0 \mod 2 \end{cases}$$

where $\nu_p(\ )$ is the p-adic valuation.

Now by Adams (op. cit. page 139) $m(2s)$ is the denominator of $(-1)^{-1} b_s/4s$. Thus we have

Corollary 4.10: If $E = (e_1, \ldots, e_k, \ldots)$ then

$$G(E) = 2^{e_1} \prod_{\substack{k>1 \\ k \text{ even}}} (((2k-1)!/2)m(2k))^{e_k}$$

is the denominator of B_E (up to sign). In particular B_{Δ_k} has denominator $((2k-1)!/2)m(2k)$, $k > 1$, B_{Δ_1} has denominator 2, $k = 1$.

Proof. $b_s/(2s)! = 2b_s/((4s)(2s-1)!) = (b_s/4s)/((2s-1)!/2)$. By the Adams theorem $(2s-1)!/2m(2s)$ is the denominator of $b_s/(2s)!$

Now $B_E = (1/2)^{e_1} \prod_{\substack{k>1 \\ k \text{ even}}} (\beta_k/(2k)!)^{e_k}$ and $G(E)$ is the denominator of B_E.

Q.E.D.

Corollary 4.11: The minimal number which made $R(\alpha)$ integral for $\alpha \in MU_*(X)$ is

$$g(\alpha) = \text{m.c.m.}\{G(E) \mid \|E\| \geq 0 \text{ and } r_E(\alpha) \neq 0\}.$$

Proof: Follows by Theorem 4.8 and last corollary.

Corollary 4.12: Suppose X is a finite complex. If $x \in H_*(X;\mathbf{Z})/\text{tor}$ and $G_x = K^*(X)$ then $g(\alpha)$ divides x for $\alpha \in MU_*(X)$ and $\bar{\varphi}(x) = \beta(\alpha)$.

Proof: By the diagram

$$\begin{array}{ccc} MU_*(X)\otimes\mathbf{Q} & \xleftarrow{\quad R \quad} & MU_*(X)\otimes\mathbf{Q} \\ \downarrow T & & \downarrow \beta \\ H_*(X)\otimes\mathbf{Q} & \underset{\bar{\varphi}}{\overset{ch_*\otimes\mathbf{Q}}{\rightleftarrows}} & K_*(X)\otimes\mathbf{Q} \end{array}$$

and since $G_x = K^*(X)$ if and only if $(ch_*\otimes\mathbf{Q})^{-1}(x) \in K_*(X) \subset K_*(X)\otimes\mathbf{Q}$, if $\beta(\alpha) = \bar{\varphi}(x) g(\alpha) T \circ R(\alpha) = ((ch_*\otimes\mathbf{Q})\beta(\alpha) \in \mathbf{Z}$ then $g(\alpha)$ divides x.

Q.E.D.

Let E be a ring spectrum and $\iota: S^0 \longrightarrow E$ the unit map. The Hurewicz homomorphism $h: \pi_*(X) \longrightarrow E_*(X)$ is defined by the rule that $f: S^m \longrightarrow X$ is sent to

$$S^n \xrightarrow{f} X \simeq X \wedge S^0 \xrightarrow{1\wedge i} X \wedge E.$$

__Definition__: If $x \in \operatorname{Im} h: \pi_*(X)/\mathrm{tor} \longrightarrow E_*(X)/\mathrm{tor}$ we say x is E-spherical.

__Lemma 4.12__: If $x \in MU_*(X)$ is spherical then x is primitive.

__Proof__: Since $MU^*(MU)$ acts trivially on $MU^*(S^n)$, then $MU^*(MU)$ acts trivially in $MU_*(S^n)$. If $f: S^n \longrightarrow X$ represents $\xi \in \pi_*(X)/\mathrm{tor}$ with $h^{MU}(\xi) = x$ then by naturality of the MU-operations $MU^*(MU)$ acts trivially on x. Q.E.D.

__Question__: Is the converse of this Lemma true? i.e. If $x \in MU_*(X)$ is primitive then is x MU-spherical?

This question has been the subject of much investigation recently. It has an affirmative solution for $X = B\mathbb{Z}/p$, p odd and negative for $X = \mathbb{R}P^\infty$ [Hansen-Johnson 1] or [Hansen 1].

We are going to devote the following chapters to study this question for the particular case of X as a Lie group. We will call an affirmative answer in that case the Kane conjecture. We can relate the Kane conjecture with another conjecture.

In Corollary 4.3 we saw that if $x \in H_*(X)/\mathrm{tor}$ is spherical then $G_x = K^*(X)$. Now we ask for the converse. Given $x \in H_*(X)/\mathrm{tor}$ such that $G_x = K^*(X)$ is x spherical? This question appears in [Bott 2] page 135 and in [Stasheff 1] Conjecture 42 under the name of the Atiyah-Mimura conjecture.

Proposition 4.13: [Kane 2]

If X is a finite H-space which satisfies the Atiyah-Mimura Conjecture then X satisfies the Kane Conjecture.

Proof: We assume that for $x \in H_*(X)/\mathrm{tor}$ x spherical if and only if $G_x = K^*(X)$. Suppose that $\alpha \in MU_*(X)$ is primitive. For any finite complex $\beta: MU_*(X) \longrightarrow K_*(X)$, the Conner-Floyd map, is onto. (See [Conner-Smith 1]). By Corollary 4.3 $G_{T(\alpha)} = K^*(X)$, by assumption $T(\alpha)$ is spherical and since

$$\begin{array}{ccc} & \overset{h^{MU}}{\nearrow} & MU_*(X)/\mathrm{tor} \\ \pi_*(X) & & \downarrow T \\ & \underset{h}{\searrow} & H_*(X;\mathbb{Z})/\mathrm{tor} \end{array}$$

commutes then α is MU-spherical. Q.E.D.

Remark: In particular if X is an H-space for which the Kane conjecture is not true then the Atiyah-Mimura Conjecture is not true as well.

Part IV: THE CLASSICAL GROUPS

§1: Definition of H-space

An H-space (X,μ) with base point $e \in X$ is a (finite) CW complex X together with a map $\mu\colon X\times X \longrightarrow X$ such that the diagram

$$\begin{array}{ccc} X\times X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle i}\nwarrow & & \nearrow{\scriptstyle \nabla} \\ & X\vee X & \end{array}$$

commutes up to homotopy. Here

$$X\vee X \cong (X\times e)\cup(e\times X) \xrightarrow{\ i\ } X\times X$$

and

$$\nabla(x,e) = \nabla(e,x) = x.$$

An H-space is finite if X has the homotopy type of a finite complex. If

$$\mu(x,\mu(y,z)) \simeq \mu(\mu(x,y),z)$$

we say (X,μ) is associative and if $\mu(x,y) = \mu(y,x)$ we say (X,μ) is commutative.

Examples: The main families of H-spaces are:

(i) Loop spaces

Given Y a CW-complex and $y_0 \in Y$ a base point

$\Omega(Y,y_0) = \{f\colon [0,1] \longrightarrow Y \mid f$ is continuous and $f(0) = f(1) = y_0\}$

with the compact open topology. It is known that

$$\mu\colon \Omega(Y,y_0)\times\Omega(Y,y_0) \longrightarrow \Omega(Y,y_0)$$

given by

$$\begin{aligned} \mu(f,g)(t) &= f(2t) && 0 \le t \le 1/2 \\ &= g(2t-1) && 1/2 \le t \le 1 \end{aligned}$$

defines a structure of H-space on $\Omega(Y,y_0)$. (See [Gray 1] §9). Most of the time loop spaces are not finite H-spaces e.g. ΩS^n.

(ii) Topological groups

A topological group (X,μ) is an associative H-space such that for each $a \in X$ there exist $L_a: X \longrightarrow X$ a homeomorphism with $L_a(x) = \mu(a,x)$.

Among the topological groups are the Lie groups. They are the ones which are analytic manifolds and μ is an analytic map while L_a is a diffeomorphism.

Since any manifold has the homotopy type of finite CW-complex, a compact Lie group is a finite H-space.

By the affirmative solution to Hilbert's fifth problem we know that every compact, simply connected manifold which is a topological group is actually a Lie group.

§2: Cartan classification of Lie groups

All Lie groups can be built up from the simple types, that is, those having no closed normal subgroups. In other words any compact Lie group is a product of the simple groups.

The compact simply connected types contain 4 infinite families: $n \geq 0$

A_n: SU(n) the group of unitary matrices with det = +1

B_n: The simply connected covering of SO(2n+1) denoted by Spin(2n+1) where SO(2n+1) is the matrices of (2n+1)×(2n+1) order with det = +1.

C_n: Sp(n) the group of n×n matrices with coefficients in $\mathbb{H}$ the quaternionic numbers, satisfying $X\bar{X}^t = 1$.

D_n: The simply connected covering of SO(2n) denoted by Spin (2n) where SO(2n) are the (2n×2n) orthogonal matrices of det = +1.

Thus A_n, B_n, C_n, D_n are known as the classical groups. The Cartan classification of the simply connected compact simple Lie groups is completed with the exceptional groups.

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8.$$

For a complete account about Lie groups and Cartan classifaction we refer to [Helgason 1] or to [Sophus Lie 1]. We will sketch a definition ofthe exceptional types in Chapter V §2.

We are interested in applying the algebraic topology developed in the last chapters to Lie groups.So we are looking at Lie groups as objects of the homotopy category. We have the following important theorem.

<u>Theorem</u>: (Scheerer) Two simply connected compact Lie groups are isomorphic if and only if they have the same homotopy type.

This theorem has been proved by different authors at different stages (Baum-Browder [1]) [Scheerer 1] [Toda 1]). A final short proof version is in [Hubbuck-Kane 1].

The hypothesis of simply connected is necessary: For instance, take $SO(3)\times SU(2)$ and $SO(4)$. These Lie groups are diffeomorphic: Let G a simply connected compact simply Lie group then $f: G\times G \longrightarrow G\times G$ given by $f(g,g') = (g,gg')$ induces an isomorphism $G/C\times G \xrightarrow{\cong} G\times G/f(G\times e)$ whenever C = center of $G \neq e$. Now let $G = SU(2)$. On the other hand $SO(3)XSU(2)$ and $SO(4)$ can not be isomorphic because the mod 2 cohomology group of their classifying spaces are different as Steenrod algebra modules. It should be noted that, in general, Steenrod operations do not suffice to distinguish homotopy type.

Consider $G_2\times Sp(2)$ and $Spin(7)\times SU(2)$. We will see that those groups have the same Steenrod algebra module structure in their mod p cohomology groups. But $\pi_6(G_2\times Sp(2)) \cong \mathbf{Z}/3$ and $\pi_6(Spin(7)\times SU(2)) \cong \mathbf{Z}/12$. See [Borel 1] page 428 and [Mimura-Toda 1] page 217.

So the theory of Steenrod operations in ordinary cohomology is far from deciding the homotopy type of the Lie groups. We wish to use MU operations in order to say something more about it.

§3: Homology and cohomolgoy of H-spaces

From now on by a Lie group we mean a compact, connected Lie group and by a simple Lie groups one of the Cartan classification and by a finite H-space we mean a finite complex with a H-space structure. (Finite in the sense of dimension: $H_n(X,\mathbf{Z}) = 0$ for $n \geq n_0$ and of finite type i.e. X has a finite number of cells in each dimension.) The first result about the cohomology of finite H-spaces is:

Theorem: [Hopf 1] Let (X,μ) be a finite H-space.

$$H^*(X;\mathbf{Q}) = \Lambda(x_1 \ldots x_r)$$

when $|x_i| = 2n_i - 1$.

The number r is known as the rank of X. In the case of X a Lie group agrees with the usual definition of rank (the dimension of the maximal torus). The numbers $\{n_i\}$ are called the exponents of X and the sum of the dimension of the x_i is the dimension of X.

If $H^*(X)$ has no torsion then $H^*(X;\mathbf{Q})$ determines it completely, i.e. $H^*(X)$ is an exterior algebra with generator as in $H^*(X;Q)$.

Regarding mod p cohomology we have:

Theorem: [Borel 2] Let (X,μ) be a finite H-space. For p any prime

$$H^*(X;\mathbf{Z}/p) = \Lambda(x_1,\ldots,x_r)\otimes\mathbf{Z}/p[y_1,\ldots,y_m]/(y_1^{r_1},\ldots,y_m^{r_m})$$

If $p > 2$ $|x_i|$ is odd and $|y_i|$ is even. If $p = 2$ all the generators have odd dimension.

Let $\Delta: X \longrightarrow X\times X$ the diagonal map defined by $\Delta(x) = (x,x)$. The diagram ,

$$\begin{array}{ccc} H^*(X\times X;\mathbf{Z}/p) & \xleftarrow{\Delta^*_{X\times X}} & H^*(X\times X\times X\times X;\mathbf{Z}/p) \\ \uparrow \mu^* & & \uparrow (\mu\times\mu)^* \\ H^*(X;\mathbf{Z}/p) & \xleftarrow{\Delta^*_X} & H^*(X\times X;\mathbf{Z}/p) \end{array}$$

commutes. As usual $(H^*(X;\mathbf{Z}/p),\Delta^*)$ is an algebra over k. Also $(H^*(X;\mathbf{Z}/p),\mu^*)$ is a coalgebra over $\mathbf{Z}/p$. By the diagram above Δ^* is a morphism of coalgebras over $\mathbf{Z}/p$. By definition ([Milnor-Moore 1] pages 226-227 etc.) $(H^*(X;\mathbf{Z}/p),\mu^*,\Delta^*)$ is a Hopf algebra which is connected. Write $A = H^*(X;\mathbf{Z}/p)$ and $A_i = H^i(X;\mathbf{Z}/p)$. By definition $\overline{A} = \mathrm{Ker}\ \varepsilon$ i.e. $\overline{A} = \bigoplus_{i>0} A_i$. Denote $\phi = \Delta^*$ and $\Psi = \mu^*$.

If $\overline{\phi}: \overline{A}\otimes\overline{A} \longrightarrow \overline{A}$ is the restriction of ϕ to $\overline{A}\otimes\overline{A}$ then QA = Coker of $\overline{\phi}$ = indecomposables of A. If $\overline{\Psi}: \overline{A} \longrightarrow \overline{A}\otimes\overline{A}$ is the restriction of Ψ to $\overline{A}$ then $PA = \mathrm{Ker}\overline{\Psi}$ = primitives of A. In order to avoid confusion with the primitives in the last chapter we will refer to these as Hopf algebra primitives if it is necessary.

A Hopf algebra is <u>primitively generated</u> if the composition

$$PA \xrightarrow{i} A \xrightarrow{q} QA$$

is surjective. Here i is standard inclusion and q the standard projection. The Borel theorem can be extended to say that if $H^*(X;\mathbf{Z}/p)$ is primitively generated then x_i and y_j can be chosen to be primitives.

In ([Milnor-Moore 1] §4) it is proved that A is primitively generated if and only if A^*, the dual, is a commutative and associative algebra and for $x \in \overline{A}^*$ $x^p = 0$. In particular one can show:

<u>Theorem</u>: For a connected associative finite H-space X $H^*(X;\mathbb{Q}) = \Lambda(x_1, \ldots x_r)$ when x_i is primitive.

If we dualize we obtain

<u>Corollary</u>: For a connected associative finite H-space

$$H_*(X;\mathbb{Q}) = \Lambda(x_1 \ldots x_r),$$

where x_i is primitive.

4: The Hurewicz map of a H-space modulo torsion

Lets look for a moment at the Hopf algebra over $\mathbb{Q}$, $A = H_*(X;\mathbb{Q})$, where X is a H-space not necessarily finite dimensional.

Let PA the $\mathbb{Q}$-subspace of primitive elements. In this case

$$\Delta^*: H_*(X;\mathbb{Q}) \longrightarrow H_*(X;\mathbb{Q})\otimes H_*(X;\mathbb{Q})$$

is the comultiplication and

$$\mu_*: H_*(X;\mathbb{Q})\otimes H_*(X;\mathbb{Q}) \longrightarrow H_*(X;\mathbb{Q})$$

is the multiplication. So

$$PA = \{x \in H_*(X,\mathbb{Q}) \mid \Delta_*(x) = x\otimes 1 + 1\otimes x\}.$$

Let $h: \pi_*(X)\otimes\mathbb{Q} \longrightarrow H_*(X;\mathbb{Q})$ be the Hurewicz map. i.e. if $\xi \in \pi_n(X)\otimes\mathbb{Q}$ is represented by $\xi = [f: S^n \longrightarrow X]$ then $H_*(\xi) = f_*(\alpha_n)$ where $\alpha_n \in H_n(S^n) \cong \mathbb{Z}$ is the generator and $f_*: H_n(S^n;\mathbb{Q}) \longrightarrow H_n(X;\mathbb{Q})$.

Proposition 4.1: $\operatorname{Im} h \subset P(H_*(X;\mathbb{Q}))$

Proof: First notice that each element of $H_n(S^n)$ is primitive for $n \geq 0$.

Secondly suppose that $f: Y \longrightarrow Z$ is any map and that for $y \in H_q(Y)$ $\Delta^Y_*(y) = y\otimes 1 + 1\otimes y$ then

$$\Delta^Z_* f_*(y) = (f\times f)_*\Delta^Y_*(y) = (f\times f)_*(y\otimes 1 + 1\otimes y)$$
$$= f_*(y)\otimes 1 + 1\otimes f_*(y).$$

Therefore, by the definition of h, $\operatorname{Im} h \subset P(H_*(X;\mathbb{Q}))$. Q.E.D.

In [Milnor-Moore 1] page 263 it is proved that $\operatorname{Im} h$ is in fact $PH_*(X,\mathbb{Q})$. Then looking to the diagram

$$\begin{array}{ccc} \pi_*(X)/\text{tor} & \xrightarrow{h} & PH_*(X)/\text{tor} \\ \downarrow & & \downarrow \\ \pi_*(X)\otimes\mathbb{Q} & \xrightarrow{\cong} & PH_*(X;\mathbb{Q}) \end{array}$$

we have

<u>Theorem</u>: (Cartan-Serre). For X an associative H-space

$$\pi_*(X)/\text{tor} \xrightarrow{h} PH_*(X)/\text{tor}$$

is a monomorphism. (See [Adams 2] paper #12).

<u>Definition</u>: For X an associative finite H-space $x \in PH_n(X)/\text{tor}$ is <u>spherical</u> if and only if $x \in \text{Im } h$.

To calculate the spherical elements in $PH_*(X)/\text{tor}$ we proceed in the following way. Write $H_*(X;\mathbb{Q}) = \Lambda(x_1, x_2, \ldots x_r)$ when $|x_i|$ = odd and x_i is primitive. Then there exists a minimal positive integer $N(i)$ which depends on i and X such that $N(i)x_i \in \text{Im } h$ for all the primitive generators of degree i.

Our main problem is to calculate such minimal number when X is a simply connected compact Lie group. Using the Hurewicz theorem [Whitehead 1] Chapter IV §7 we can deduce that for a simple connected finite H-space, any non-zero generator which occurs in dimension 3 is spherical. So $N(3) = 1$. By Browder's theorem [Browder 1] $\pi_2(X) = 0$ for any finite H-space. So if X is simply connected then X is 2-connected and $\pi_3(X)$ is the first non-vanishing homotopy group. Actually $\pi_3(X) \cong \mathbb{Z}$ if X is no trivial (See [Clark 1]) and X is an associative finite H-space.

Using the process called "killing" the homotopy group we can construct spaces [Whitehead 1] $X(n,\ldots\infty)$ for each $n \geq 1$ with the properties:

(1) $\pi_i(X(n,\ldots\infty)) = 0$ for $i < n$.

(2) $\pi(n-1,n)_*: \pi_j(X(n,\ldots,\infty)) \longrightarrow \pi_j(X(n-1,\ldots,\infty))$ is an isomorphism.

(3) Each $\pi(n-1,n): X(n,\ldots\infty) \longrightarrow X(n-1,\ldots\infty)$ is a principal $K(\pi_{n-1}(X), n-2)$ bundle.

We just have seen that $X(3,\ldots\infty) = X$. By Hurewicz's theorem

$$\pi_n(X(n,\ldots\infty)) \xrightarrow[\cong]{h} H_n(X(n,\ldots\infty;\mathbf{Z})).$$

So looking at the commutative diagram

$$\begin{array}{ccc} \pi_n(X(n,\ldots\infty)) & \longrightarrow & H_n(X(n,\ldots\infty);\mathbf{Z}) \\ \downarrow \pi(n,\ldots\infty)_* & & \downarrow \pi(n,\ldots 0)_* \\ \pi_n(X) & \longrightarrow & H_n(X;\mathbf{Z}) \end{array}$$

we can calculate $N(n)$ as the index of the image of $\pi(n,\ldots,0)_*$ in homology.

The first restriction on $N(n)$ is the following.

<u>Proposition</u> 4.2: Let X be a 1-connected associative H-space, $n = 2t+1 \geq 3$. If $H_n(X;\mathbf{Z})/tor \neq 0$, p a prime and $p|N(n)$, then $p \leq t$.

<u>Proof</u>: See [Smith 3] Corollary 2.4 or [Serre 1] Proposition IV 6.

Therefore we know which primes can occur in $N(n)$ for $n = 2t+1$, namely those satisfying $p \leq t$. For instance: If $n = 11$, $t = 5$ and $p = 2,3,5$.

Next we restrict the exponents of such primes. These restrictions are given by a result of Smith [Smith 3].

Proposition 4.3: If X is an associative, 1-connected H-space and $x \in PH_{2t+1}(X)/tor$ then $N(t) \mid M(t)$. where

$$M(t) = \prod_{p \leq t} p^{[t+1/p-1]}.$$

That means that if

$$N(t) = p_1^{e_{p_1}} \cdots p_S^{e_{p_S}} \qquad p_i \leq t \quad \text{for} \quad 1 \leq i \leq S$$

then

$$e_{p_i} \leq \sum t+1/(p_i-1).$$

For instance when $n = 11$, then, $t = 5$ and $e_2 = 6$ $e_3 = 3$ $e_5 = 1$.

Now our concern is to look at the lower bounds. Let

$$\rho\colon H_*(X;\mathbf{Z}) \longrightarrow H_*(X;\mathbf{Z}/p)$$

be the reduction mod p and $A^*(p)$ the mod p Steenrod algebra. Since $A^*(p)$ acts trivially on $H_*(S^n;\mathbf{Z}/p)$ then by naturality of Steenrod operations we have that if $A^*(p)$ acts nontrivially on

$$x \in PH_n(X;\mathbf{Z}/p)$$

then x is not spherical and moreover p must divide $N(n)$. [Kane 2] proved the following:

Theorem 4.4: Let Y connected CW-spectrum of finite type with $H_*(Y)$ torsion free. Given $y \in H_*(X)$ such that $\wp^s(\rho(x)) \neq 0$ then $\wp^q(x)$ is not spherical unless $q \geq s$.

So we have the corollary

<u>Corollary 4.5</u>: If X is a finite connected H-space and $\wp^{s}(\rho(y)) \neq 0$ for $y \in H_{n-1}(\Omega X;\mathbf{Z}/p)$ then $p^{s} | N(n)$.

<u>Proof</u>: By Loop theorem [Lin 1] we know that ΩX is torsion free. Now

$$\begin{array}{ccc} \pi_{n-1}(\Omega X) & \xrightarrow{\cong} & \pi_{n}(X)/\mathrm{tor} \\ \downarrow h & & \downarrow h \\ H_{n-1}(\Omega X) & \xrightarrow{\Omega_*} & H_{n}(X)/\mathrm{tor} \end{array}$$

commutes. So by the theorem we have $p^{s} | N(n)$. Q.E.D.

So the action of $A^{*}(p)$ on $H_{n-1}(\Omega X;\mathbf{Z}/p)$ gives us information about $N(n)$. The problem is that occasionally $A^{*}(p)$ acts trivially on $H_{*}(X;\mathbf{Z}/p)$. So the theorem is not applicable. What we mean is that there are finite H-space X and primes p with $A^{*}(p)$ acting trivially on $H_{n}(X;\mathbf{Z}/p)$ for $n \geq 3$ but $p | N(n)$. We will see that $X = G_2$, $p = 3$ is an example of this situation. We can describe this phenomenon by saying that ordinary homology does not "decide" G_2 at prime 3. (See [Ray 1] for definition of decideability and [Mimura-Toda 1] for specific treatment of the $(G_2,3)$).

5: The Hurewicz map for the classical groups

Case $X = SU(n)$

Since

$$\begin{array}{ccc} \pi_{2t+1}(SU(n)) & \xrightarrow{\iota_n^*} & \pi_{2t+1}(SU)/\mathrm{tor} \\ \downarrow h & & \downarrow h \\ PH_{2t+1}(SU(n);\mathbb{Z}) & \xrightarrow{\iota_n^*} & PH_{2t+1}(SU) \end{array}$$

commutes and ι_n^* is an isomorphism either in homotopy or homology, we can reduce to the study of the stable case.

The stable case is one of the consequences of the Bott periodicity theorem:

Theorem 5.1: [Bott 3] The image of

$$\pi_{2t+1}(SU) \longrightarrow PH_{2t+1}(SU)$$

has index in $PH_{2t+1}(SU)$ equal to $t!$ i.e. $N(2t+1) = t!$

Recall $PH_*(SU(k);\mathbb{Z}) = \Lambda(x_3, x_5, x_7, \ldots, x_{2k-1})$. We can give a different proof of the Bott theorem.

Lemma 5.1: There exists a map $\gamma: \Sigma\mathbb{C}P^\infty \longrightarrow SU$ such that

$$\gamma_*: H_*(\Sigma\mathbb{C}P^\infty;\mathbb{Z}) \longrightarrow PH_*(SU;\mathbb{Z})$$

is an isomorphism.

Proof: Let $i_1: \mathbb{C}P^\infty \simeq BU(1) \longrightarrow BU$ be the 1-th inclusion in the direct limit $\lim\limits_{k\to\infty} BU(k) \simeq BU$. We saw in Chapter [illegible] that i_1^* is an isomorphism in homology.

By Bott periodicity $BU \simeq \Omega SU$ so taking γ the adjoint map of i_1 we have the desired map. Q.E.D.

Lemma 5.2: There exists a map $g: \Omega S^3 \longrightarrow \mathbf{C}P^\infty$ such that $g_*: H_{2n}(\Omega S^3) \longrightarrow H_{2n}(\mathbf{C}P^\infty)$ is multiplication by $n!$.

Proof: By definition of $K(\mathbf{Z},3)$, $\pi_3(K(\mathbf{Z},3)) \cong \mathbf{Z}$. Let $f: S^3 \longrightarrow K(\mathbf{Z},3)$ be the generator and take

$$g = \Omega f. \Omega S^3 \longrightarrow \Omega K(\mathbf{Z},3) \cong K(\mathbf{Z},2) \cong \mathbf{C}P^\infty.$$

But $H_*(\Omega S^3) = Z[y]$ $|y| = 2$ by an easy argument of the Eilenberg-Moore spectral sequence ([Smith 4] Chapter II). For $H^*(S^3) = \Lambda(X)$ implies

$$E_2 = \mathrm{Tor}_{H_*(S^3)}(\mathbf{Z},\mathbf{Z}) = \Gamma[S^{-1}x_3].$$

Clearly the spectral sequence collapses. Therefore $g_*(\gamma_n(y))$ is divisible by $n!$ because $\gamma_n(y)$ is divisible by $n!$. So

$$g_*(\gamma_n(y)) = n!\beta_n \in H_{2n}(\mathbf{C}P^\infty;\mathbf{Z})$$

when β_n is the generator in $H_{2n}(\mathbf{C}P^\infty;\mathbf{Z})$. Q.E.D.

Now

$$\Sigma\Omega S^3 = \bigvee_{n\geq 1} S^{2n+1}$$

and the commutativity of the diagram

$$\begin{array}{ccccc}
\pi_{2n+1}(\Sigma\Omega S^3;\mathbf{Z}) & \xrightarrow{(\Sigma g)_*} & \pi_{2n+1}(\Sigma\mathbf{C}P^\infty;\mathbf{Z}) & \longrightarrow & \pi_{2n+1}(SU;\mathbf{Z}) \\
\downarrow t'' & & \downarrow & & \downarrow h \\
H_{2n+1}(\Sigma\Omega S^3;\mathbf{Z}) & \xrightarrow{(\Sigma g)_*} & H_{2n+1}(\Sigma\mathbf{C}P^\infty;\mathbf{Z}) & \xrightarrow{\cong} & PH_{2n+1}(SU;\mathbf{Z})
\end{array}$$

implies that $N(2n+1) \mid n!$. We want to prove that $n! \mid N(2n+1)$. To see this we recall that

$$\begin{array}{ccc} & \xrightarrow{h^{MU}} & MU_*(X) \\ \pi_*(X) & & \downarrow T \\ & \xrightarrow{h} & H_*(X) \end{array}$$

commutes and that for any $\hat{x} \in MU_*(X)$ with $\hat{x} \in \text{Im } h^{MU}$ then $\hat{x}$ is primitive (with repect to the MU operations). So if $x \in H_*(X;\mathbb{Z})$ is spherical there exists $\hat{x} \in MU_*(X)$ with $\hat{x} \in \text{Im } h^{MU}$ and $T(\hat{x}) = x$ and $\hat{x}$ is primitive. Thus $\varphi(x) = \hat{x}$. If $\xi_n \in PH_{2n+1}(SU)$ is the generator then there exists $\hat{\xi}_n \in MU_{2n+1}(SU)$ with $T(\hat{\xi}_n) = \xi_n$. For since $H_*(SU)$ is torsion free the Atiyah-Hirzebruch spectral sequence converging to $MU_*(SU)$ collapses. Thus $T: MU_*(SU) \longrightarrow H_*(SU)$ is surjective. We have

$$q\varphi(\xi_n) = \varphi(q\xi_n) \in MU_*(X)\otimes\mathbb{Q}.$$

<u>Lemma 5.3</u>: If $q\xi_n$ is spherical then

$$q\varphi(\xi_n) \in MU_*(SU) \subset MU_*(SU)\otimes\mathbb{Q}.$$

<u>Proof</u>: As observed above, if $n = q\xi_n$ is spherical then there exists $\hat{x} \in MU_*(SU)$ such that $T(\hat{x}) = x$ and

$$\varphi(x) = \hat{x} \in MU_*(SU).$$

Since

$$T(\hat{x}) = T(q\hat{\xi}_n) = q\xi_n$$

it follows that $\varphi(\hat{x}) = \varphi(q\hat{\xi}_n)$. So $\varphi(q\hat{\xi}_n) \in MU_*(SU)$ as well. Q.E.D.

We now show

<u>Lemma 5.4</u>: $q\varphi(\xi_n) \in MU_*(SU)$ only if $n!|q$.

<u>Proof</u>: The isomorphism $\tau_*: H_*(\mathbb{C}P^\infty) \cong PH_*(SU)$ (of degree +1) from

Lemma (5.1) also gives an isomorphism $\gamma_*\ MU_*(\mathbb{C}P^\infty) \cong PMU_*(SU)$. So we can pick $\xi_n = \gamma_*(\beta_n)$

But by [Proposition 2.5 chapter III] $n!$ is the minimal number such that $n!\varphi_*(\beta_n) \in MU_*(\mathbb{C}P^\infty)$. So $n!$ is the one such that

$$n!\varphi(\xi_n) \in MU_{2n+1}(SU). \qquad \text{Q.E.D.}$$

Therefore we have the Bott theorem $n! = N(2n+1)$ for $X = SU$. Even more we have proved

Corollary 5.5: For $X = SU(n)$ the spherical elements and primitive elements in $MU_*(X)$ agree. Q.E.D.

Case $X = Sp(n)$.

Recall $H_*(Sp(k) = \Lambda(x_3, x_7, \ldots x_{4k-1})$ and $PH_*(Sp(k))$ is spanned by the elements $\{x_3, x_7, \ldots x_{4k-1}\}$. The inclusion

$$Sp(n) \longrightarrow Sp = \lim_{n\to\infty} Sp(n)$$

induce a isomorphism in homotopy and homology in dimensions $\leq 4k-1$. So we can reduce to the case $X = Sp$. The inclusion $\mathbb{C} \hookrightarrow \mathbb{H}$ induces the inclusion $SU(n) \longrightarrow Sp(n)$ as well as a map $SU \xrightarrow{i} Sp$. Also $\mathbb{H} \subset \mathbb{C}\times\mathbb{C}$ induces an inclusion $Sp(n) \xrightarrow{s_n} SU(2n)$ and a map $Sp \xrightarrow{i} SU$. Similarly $\mathbb{R} \subset \mathbb{C}$, and $\mathbb{C} \subset \mathbb{R}^2$ define inclusions

$$SO(n) \xrightarrow{\alpha} SU(n) \xrightarrow{\beta_n} SO(2n)$$

and thus induce

$$SO \xrightarrow{\alpha} SU \xrightarrow{\beta} SO.$$

[Kervaire 1] page 281 proved that if

$$a_s = \begin{cases} 2 & \text{if } s \text{ is odd} \\ 1 & \text{if } s \text{ is even} \end{cases}$$

and

$$b_s = \begin{cases} 1 & \text{if } s \text{ is odd} \\ 2 & \text{if } s \text{ is even} \end{cases}$$

then

$$\pi_{4s-1}(SO)/tor \xrightarrow{\alpha_*} \pi_{4s-1}(SU)/tor \xrightarrow{\beta_*} \pi_{4s-1}(SO)/tor$$

is given by

$$\alpha_*(z_{4s-1}) = a_s x_{4s-1}$$

and

$$\beta_*(x_{4s-1}) = b_s z_{4s-1}$$

By Bott periodicity $\Omega^4 SO \simeq Sp$ and $\Omega^2 SU \simeq SU$.

So

$$\pi_{4n-1}(Sp) \xrightarrow{\Omega^4\alpha_*} \pi_{4n-1}(SU) \xrightarrow{\Omega^4\alpha_*} \pi_{4n-1}(Sp)$$

is given by $a_{n+1} = |\Omega^4\alpha_*|$ and $b_{n+1} = |\Omega^4\alpha_*|$ respectively. But $\Omega^4\alpha_* = 1$, $\Omega^4\alpha_* = \iota_*$, $a_{n+1} = b_n$ and $b_{n+1} = a_n$. Therefore

$$\begin{array}{ccccc} \pi_{4n-1}(Sp) & \xrightarrow{b_n} & \pi_{4n-1}(SU) & \xrightarrow{a_n} & \pi_{4n-1}(Sp) \\ \downarrow ? & & \downarrow (2n-1)! & & \downarrow ? \\ PH_{4n-1}(Sp) & \xrightarrow[\cong]{} & PH_{4n-1}(SU) & \longrightarrow & PH_{4n-1}(Sp) \end{array}$$

commutes. Now

$$Sp(k-1) \longrightarrow Sp(k) \longrightarrow S^{4k-1}$$

$$SU(2k-1) \longrightarrow SU(2k) \longrightarrow S^{4k-1}$$

are fibrations. By the obvious recursion

$$s_*: PH_{4n-1}(Sp(k)) \longrightarrow PH_{4n-1}(SU(2k))$$

is an monomorphism. So we have

Proposition 5.6: The Hurewicz map for $Sp(k)$ in dimension $4n-1$, $n \geq 1$ is given by

$$h_{Sp(k)}(4n-1) = \begin{cases} 2(2n-1)! & \text{if } n \text{ is even} \\ (2n-1)! & \text{if } n \text{ is odd,} \end{cases}$$

and the homomorphism

$$\iota_*: H_{4n-1}(SU(k)) \longrightarrow H_{4n-1}(Sp(k))$$

is given by

$$\iota_*(y_{4n-1}) = 2x_{4k-1}.$$

Remark: For the first assertion in the above proposition see [Harris 1] page 174 and for the second one see [Cartan 1] page 17-07.

The case Spin(n) or SO(n)

Since $Spin(n)$ is by definition the universal covering space of $SO(n)$ i.e.

$$\mathbb{Z}/2 \longrightarrow Spin(n) \longrightarrow SO(n)$$

is a fibration so after dimension equal to 3, the index of image of Hurewicz map of both spaces agree.

Passing to the stable case $SO = \lim_{n\to\infty} SO(n)$ presents difficulties this time. For $\pi_*(SO(n))$ and $H_*(SO(n))$ are not always direct summands of $\pi_*(SO)$ and $H_*(SO)$ in the desired range of dimensions. For example recall that $Spin(5) \cong Sp(2)$ and $Spin(6) \cong SU(4)$ [Mimura 1] page 172 and that

$$\begin{array}{ccc} Spin(5) & \longrightarrow & Spin(6) \\ \downarrow \iota'' & & \downarrow \iota'' \\ Sp(2) & \xrightarrow{s_2} & SU(4) \end{array}$$

commutes.

We have already seen that

$$\pi_7(Sp(2)) \xrightarrow{s_{2*}} \pi_7(SU(4))$$

is multiplication by $b_2 = 2$. So $\pi_7(Spin(5)) \longrightarrow \pi_7(Spin(6))$, is multiplication by 2. We split the case $X = SO(n)$ or $Spin(n)$ into two sub cases.

<u>Sub case 1: Stable case</u>:

We want to calculate $\pi_{4n-1}(SO(n))/tor \longrightarrow PH_{4s-1}(SO(n))/tor$ when $4s < n$.

Consider the diagram

$$\begin{array}{ccccc}
\pi_{4s-1}(SO)/tor & \xrightarrow{\alpha_*} & \pi_{4s-1}(SU) & \xrightarrow{\beta_*} & \pi_{4s-1}(SO)/tor \\
\downarrow h & & \downarrow h & & \downarrow \\
PH_{4s-1}(SO)/tor & \xrightarrow{\alpha_*} & PH_{4s-1}(SU) & \xrightarrow{\beta_*} & PH_{4s-1}(SO)/tor
\end{array}$$

<u>Lemma 5.7</u>: The realification map $SU(k) \xrightarrow{\beta} SO(2k)$ induces a split monomorphism

$$PH_{4s-1}(SU(k)) \longrightarrow PH_{4s-1}(SO(k))/tor$$

for $s \leq k$.

<u>Proof</u>: We can replace SO(k) by SO and SU(k) by SU. We only have to prove that β_* is an isomorphism in homology with $\mathbb{Z}/p$ coefficients for each prime p. See [Cartan 1] [page 17 13] or [Dyer-Lashof 1]. [page 242]. Q.E.D

Therefore

Proposition 5.8 The Hurewicz map for SO in dimension 4n-1, $n \geq 1$ is given by

$$h_{SO}(4n-1) = \begin{cases} (2n-1)!\,2 & n \text{ is even} \\ (2n-1)! & n \text{ is odd} \end{cases}$$

and the homomorphism $\alpha_*\colon PH_{4n-1}(SO)/\text{tor} \longrightarrow PH_{4n-1}(SU)$ is multiplication by 2

Proof Diagram chasing (before Lemma 5.7) using Theorem 5.1 and the [Kervaire 1] result we are done Q.E.D.

Sub case 2 Unstable

First we have the Barratt-Mahowald theorem (See [Barratt-Mahowald 1] theorem 2 or [Mahowald 1] theorem 2 page 639.)

Theorem 5.9 (Barratt-Mahowald) For $n \geq 13$ $\pi_{4s-1}(SO(n))$ is a direct summand of $\pi_{4s-1}(SO)$, when $2s \leq n$

The rest of the cases, namely $3 \leq n < 13$ and $2s > n$ have been studied by [Kervaire 2]

Theorem 5.10 [Kervaire] If $k \geq 2s+1$ then

$$\pi_{4s-1}(SO(k))/\text{tor} \longrightarrow \pi_{4s-1}(SO)/\text{tor}$$

is an isomorphism for $s \neq 1,2,4$ and $k \geq 13$ and it is multiplication by 2 if

$(s,k) = (1,3), (2,7), (4,9), (4,10), (4,11), (4,12)$.

It is multiplication by 4 if $(s,k) = (2,6)$. It is multiplication by 8 if $(s,k) = (2,5)$.

The exceptional cases $s = 1, 2, 4$ are summarized by the following pictures:

$s = 1$

$$\begin{array}{ccc} \pi_3(SO(3))/\text{tor} & \xrightarrow{\times 2} & \pi_3(SO) \\ \downarrow \times 2 & & \downarrow \cong \\ PH_3(SO(3))/\text{tor} & \xrightarrow{\cong} & PH_3(SO)/\text{tor} \end{array}$$

$s = 2$

$$\begin{array}{ccccccc} & & & \pi_7(SO) & & & \\ & 8 & & 4 & 2 & & \cong \\ \pi_7(SO(5)) & \xrightarrow{\times 2} & \pi_7(SO(6)) & \xrightarrow{\times 2} & \pi_7(SO(7)) & \xrightarrow{\times 2} & \pi_7(SO(8)) \\ \downarrow 24 & & \downarrow 12 & & \downarrow 6 & & \downarrow 3 \\ PH_7(SO(5)) & \xrightarrow{\cong} & PH_7(SO(6)) & \xrightarrow{\cong} & PH_7(SO(7)) & \xrightarrow{\cong} & PH_7(SO(8)) \end{array}$$

$s = 4$

$$\begin{array}{ccccccccc} & & & & \pi_{15}(SO) & & & & \\ & \times 2 & & \times 2 & & \times 2 & & \times 2 & \cong \\ \pi_{15}(SO(9)) & \xrightarrow{\cong} & \pi_{15}(SO(10)) & \xrightarrow{\cong} & \pi_{15}(SO(11)) & \longrightarrow & \pi_{15}(SO(12)) & \xrightarrow{\times 2} & \pi_{15}(SO(13)) \\ \downarrow 7! & & \downarrow 7! & & \downarrow 7! & & \downarrow 7! & & \downarrow 7!/2 \\ PH_{15}(SO(9)) & \xrightarrow{\cong} & PH_{15}(SO(10)) & \xrightarrow{\cong} & PH_{15}(SO(11)) & \xrightarrow{\cong} & PH_{15}(SO(12)) & \longrightarrow & PH_{15}(SO(13)) \end{array}$$

We use these diagrams in the following way. We know the Hurewicz map in the right extreme case (it is the stable case) and we know the homomorphisms on the top triangles (Kervaire result). So we proceed backwards filling the blank spacoes. We deduce

<u>Theorem 5.11</u>: The Hurewicz map for $SO(n)$ is

$$h_{SO(n)}: \pi_{4s-1}(SO(n))/tor \longrightarrow PH_{4s-1}(SO(n))/tor$$

$$h_{SO(n)}(4s-1) = \begin{cases} (2s-1)!/2 & \text{if } s \text{ is even and } s \neq 2,4 \\ (2s-1)! & \text{if } s \text{ is odd and } s \neq 1 \\ 7! & \text{if } s = 4 \text{ and } n = 9,10,11,12 \\ 24 & \text{if } s = 2\ n = 5 \\ 12 & \text{if } s = 2,\ n = 6 \\ 6 & \text{if } s = 2\ n = 7 \\ 2 & \text{if } s = 1\ n = 3 \end{cases}$$

<u>Remark</u>: There is an alternative approach to this unstable case using the fibration

$$SO(n-2) \longrightarrow SO(n) \xrightarrow{p} V_{n,2} = SO(n)/SO(n-2)$$

In ([James 1] exercise 7 page 153) it is shown that for $k = 4s-1$ n odd

$$\pi_k(V_{n,2}) \xrightarrow{h} H_k(V_{n,2})$$

is multiplication by $4a_s$. On the other hand, the map

$$p_*: \pi_k(SO(n)) \longrightarrow \pi_k(V_{n,2})$$

has been studied by [Lundell 1]. It is given by multiplication by M_s where

$$M_s = \begin{cases} (2s-1)!/8 & s \neq 1,2,4 \\ 7!/4 & \text{if } s = 4 \\ 6 & \text{if } s = 2 \\ 1 & \text{if } s = 1 \end{cases}$$

We have a commutative diagram

$$\begin{array}{ccc} \pi_k(SO(n)) & \xrightarrow{p_*} & \pi_k(V_{n,2}) \\ \downarrow & & \downarrow \\ H_k(SO(n)) & \xrightarrow{p_*} & H_k(V_{n,2}) \end{array}$$

Also p_* induces an monomorphism in homology. (Again we only have to check at the prime 2 and it is known [Whitehead 1] page 348. We use the above facts to derive all the cases except $s = 2, 4$. Looking at the Kervaire result we get this particular case.

But the Lundell result is based on the Barratt-Mahowald and Kervaire papers.

§6: MU Primitive Elements for the Classical Groups

If X is an H-space then $\mu: X \times X \longrightarrow X$ defines a product

$$\mu_*: MU_*(X) \otimes MU_*(X) \longrightarrow MU_*(X)$$

and the diagonal map $\Delta: X \longrightarrow X \times X$ defines a coproduct

$$\Delta_*: MU_*(X) \longrightarrow MU_*(X) \otimes MU_*(X).$$

We call $x \in MU_*(X)$ a diagonal primitive if

$$\Delta_*(x) = x \otimes 1 + 1 \otimes x.$$

We use $PMU_*(X)$ to denote the diagonal primitives of $MU_*(X)$.

Warning: We will reserve the term primitive for the coaction primitives i.e. the elements $x \in MU_*(X)$ such that $\Psi_X(x) = 1 \otimes x$.

As in ordinary homology the Hurewicz map

$$h^{MU}: \pi_*(X) \longrightarrow MU_*(X)$$

has its image lying in the subgroup of diagonal primitives. The following diagram commutes

$$\begin{array}{ccc} & \xrightarrow{h^{MU}} & PMU_*(X) \\ \pi_*(X) & & \downarrow T \\ & \xrightarrow{h} & PH_*(X;\mathbb{Z}) \end{array}$$

Therefore the image of h determines completely the image of h^{MU} modulo torsion.

In §5 we show that actually MU-sphericals and primitives agree when $X = SU(n)$ (Corollary 5.5). Now we want to compare these subgroups of $MU_*(X)$ when X is a classical group distinct from $SU(n)$.

First of all consider the Bott periodicity maps given by the homotopy equivalences.

$$\Omega^2 SU \simeq SU \qquad \Omega^4 Sp \simeq SO \quad \text{and} \quad \Omega^4 SO \simeq Sp$$

when Ω^i means looping i-times.

Consider the diagram

$$\begin{array}{ccccc}
\pi_{4k-1}(Sp) & \xrightarrow{\Omega^4_*} & \pi_{4k+3}(\Omega^4 Sp) & \cong & \pi_{4k+3}(SO) \\
\downarrow h_{sp} & & \downarrow & & \downarrow h_{SO} \\
PH_{4k-1}(Sp) & \xrightarrow{\Omega^4_*} & PH_{4k+3}(\Omega^4 SP) & \cong & PH_{4k+3}(SO) \\
\downarrow & & & & \downarrow \\
PH_{4k-1}(SU) & \xrightarrow{\Omega^4_*} & PH_{4k+3}(\Omega^4 SU) & \cong & PH_{4k+3}(SU) \\
\uparrow \sigma_* & & & & \uparrow \sigma_* \\
H_{4k-2}(BU) & \xrightarrow{\Omega^4_*} & H_{4k+2}(\Omega^4 BU) & \cong & H_{4k+2}(BU)
\end{array}$$

This diagram commutes when σ_* is the suspension map in homology.

In ([Switzer 1] page 390 corollary 16.23) it is proved that if

$$B_*: H_{2t}(BU) \longrightarrow H_{2t+2}(BU)$$

is the Bott periodicity map and $y_t \in H_{2t}(BU)$ is the generator in dimension 2t then $B_*(y_t) = (t+1)y_{t+1}$ + Decomposables. Therefore the bottom horozontal homomorphism in the diagram is $\Omega^4_* = B_* \circ B_*$ and is given by

$$\Omega^4_*(y_{2k-1}) = (2k+1)(2k)y_{2k+1} + \text{Decomposables}.$$

Also σ_* kills decomposables and $\operatorname{Im}\sigma_* \subset$ Primitives. So

$$\Omega^4_*: PH_{4k-1}(SU) \longrightarrow PH_{4k+3}(SU)$$

is given by:

$$\Omega^4_*:(x_{4k-1}) = (2k+1)(2k)x_{4k+3}$$

when

$$H_*(SU) = \Lambda(x_3, x_5, \ldots x_{2n-1} \ldots).$$

Now the inclusion $Sp \longrightarrow U$ induces a split monomorphism in homology (see before prop.5.6) while $SO \longrightarrow SU$ induces multiplication by 2 in homology (see Prop. 5.8). Thus

$$PH_{4k-1}(Sp) \xrightarrow{\Omega_*^4} PH_{4k+3}(SO)$$

is multiplication by $(2k+1)(2k)/2$. Now

$$\Omega_*^4 : \pi_{4k-1}(Sp) \longrightarrow \pi_{4k+3}(SO)$$

is an isomorphism. So

$$((2k+1)2k/2)h_{Sp}(4k-1) = h_{SO}(4k+3).$$

Similarly consider the diagram

$$\begin{array}{ccccc}
PH_{4k-1}(SO)/tor & \xrightarrow{\Omega_*^4} & PH_{4k+3}(\Omega^4 Sp) & \cong & PH_{4k+3}(Sp) \\
\downarrow \times 2 & & & & \downarrow \cong \\
PH_{4k-1}(SU) & \xrightarrow{\Omega_*^4} & PH_{4k+3}(\Omega^4 SU) & \cong & PH_{4k+3}(SU) \\
\uparrow \sigma_* & & & & \uparrow \sigma_* \\
H_{4k-2}(BU) & \xrightarrow{\Omega_*^4} & H_{4k+2}(\Omega^4 BU) & \cong & H_{4k+2}(BU)
\end{array}$$

Since $B_*(y_{2k-1}) = 2ky_k$ + decomposables, σ_* kills decomposables, $\mathrm{Im}\,\sigma_* \subsetneq P$, $SO \xrightarrow{\iota} SU$ is multiplication by 2 in homology, $Sp \longrightarrow SU$ is isomorphism in homology then

$$\Omega_*^4 : PH_{4k-1}(SO)/tor \longrightarrow PH_{4k+3}(Sp)$$

is multiplication by

$$2(2k)(2k+1).$$

These calculations are compatible with the Hurewicz map calculation in §5.

Pick $\gamma_k \in PMU_{4k-1}(Sp)$ so that $MU_*(Sp) = \Lambda(\gamma_3, \gamma_7, \ldots)$. The canonical inclusion $Sp \longrightarrow SO$ induces a map $MU_*(Sp) \longrightarrow MU_*(SO)$. Let $\alpha_k \in PMU_{4k-1}(SO)/Tor$ be the image of γ_k under this map. Let

$N_{Sp}(k)$ and $N_{SO}(k)$ be the least positive integers such that

$$N_{Sp}(k)\varphi(\gamma_{4k-1}) \in MU_{4k-1}(Sp)$$

$$N_{SO}(k)\varphi(\alpha_{4k-1}) \in MU_{4k-1}(SO)/tor$$

respectively. First of all $N_{Sp}(1) = N_{SO}(1) = 1$.: it suffices to show that $\pi_3(Sp) \cong PH_3(Sp)$ and $\pi_3(SO) \cong PH_3(SO)$. For Sp we use the Hurewicz isomorphism theorem and the fact that Sp is 2-connected. For SO we use the isomorphism

$$PH_3(SU) \cong PH_3(SO)/Tor$$

plus the fact that SU is 2-connected.

We now prove

<u>Theorem 6.1</u>:

$$N_{Sp}(k) = (2k-1)! \quad \text{for } k \geq 2$$

$$N_{SO}(k) = (2k-1)!/2 \quad \text{for } k \geq 2$$

First of all

<u>Proposition 6.2</u>:

$$N_{Sp}(k) \geq (2k-1)!$$

$$N_{SO}(k) \geq (2k-1)!/2$$

<u>Proof</u>: The inclusion $Sp \longrightarrow U$ induces isomorphisms

$$PH_{4k-3}(Sp) \cong PH_{4k-3}(U).$$

Thus γ_k is sent to a generator $\bar{\gamma}_k$ of $P_{4k-1}MU_*(U)$. However, by Lemma 5.4

$$q\varphi(\bar{\gamma}_k) \in MU_*(U) \subset MU_*(U)\otimes Q$$

only if $q \geq (2k-1)!$. Since $\varphi(\gamma_k)$ maps to $\varphi(\bar{\gamma}_k)$ it follows that

$$q\varphi(\gamma_k) \in MU_*(Sp) \subset MU_*(Sp)\otimes Q$$

only if $q \geq (2k-1)!$.

The case of SO is handled similarly using the fact the inclusion $SO \longrightarrow U$ induces a map

$$P_{4k-3}H_*(SO)/Tor \longrightarrow P_{4k-3}H_*(U)$$

which is multiplication by 2. Q.E.D.

<u>Proposition 6.3</u>:

$$N_{Sp}(k) \leq (2k-1)! \qquad \text{for } k \geq 2$$
$$N_{SO}(k) \leq (2k-1)!/2 \qquad \text{for } k \geq 2$$

<u>Proof</u>: We proceed by induction.

<u>k = 2</u>:

The proof of the fact that $3!\varphi(\gamma_2) \in MU_*(Sp)$ will be delayed until we discuss BP theory (see in Part V §1 Proposition 1.3.1 for $3|\varphi(\gamma_2)$ and Proposition 1.3.2 for $2|\varphi(\gamma_2)$). The fact that

$$3\varphi(\alpha_2) \in MU_*(SO)$$

follows from Lemma 5.7 of §5 plus the fact that $\pi_1 SO/Tor \longrightarrow PH_1 SO/Tor$ was calculated in §5 to be multiplication by 3.

<u>General k+1</u>:

It follows from our calculation of the maps

$$PH_{4k-1}(SO)/Tor \longrightarrow PH_{4k+3}(Sp)$$
$$PH_{4k-1}(Sp) \longrightarrow PH_{4k+3}(SO)/Tor$$

that $\varphi(\gamma_k)$ is sent to $2(2k)(2k+1)\varphi(\gamma_{k+1})$ and $\varphi(\gamma_k)$ is sent to $(2k)(2k+1)/2\varphi(\gamma_{k+1})$. By induction $2k-1!/2\varphi(\gamma_k) \in MU_*(SO)/Tor \subset MU_*(SO)\otimes Q$ and $2k-1!\varphi(\gamma_k) \in MU_*(Sp) \subset MU_*(Sp)\otimes Q$. So we must have $2k+1!\varphi(\gamma_k) \in MU_*(Sp)$ and $2k+1!/2\varphi(\alpha_k) \in MU_*(SO)/Tor$. Q.E.D.

<u>Corollary 6.4</u>: In $PMU_{4k-1}(Sp(n))$

Sphericals = 2·Primitives if k is even.

Sphericals = Primitives if k is odd.

<u>Proof</u>: Since

$$h_{Sp(n)}(4k-1) = b_k(2k-1)!$$

and

$$N_{Sp}(4k-1) = (2k-1)!.$$

Q.E.D.

<u>Proposition 6.5</u>: In $PMU_{4k-1}(SO)/Tor$

Sphericals = 2·Primitives if k is odd.

Sphericals = Primitives if k is even.

Q.E.D.

Part V: THE EXCEPTIONAL LIE GROUPS G_2 AND F_4

§1: Brown-Peterson (co)homology theory

§1.1: Definition of Brown-Peterson Spectrum

First of all we recall the definition of (co)homology theory with coefficients [Adams 1] page 200. Let G be an abelian group and

(*) $$0 \longrightarrow R \xrightarrow{i} F \longrightarrow G \longrightarrow 0.$$

a free presentation i.e. R and F are free and (*) is an exact sequence. Take $\vee_{\alpha\in A} S$, $\vee_{\beta\in B} S$ such that $\pi_0(\vee_{\alpha\in A} S) = R$ and $\pi_0(\vee_{\beta\in B} S) = F$. Take a map $\vee_{\alpha\in A} S \longrightarrow \vee_{\beta\in B} S$ such that $f_* = i$. Let

$$MG = (\vee_{\beta\in B} S)\cup_f C(\vee_{\alpha\in A} S).$$

This is a Moore spectrum of G and

$$H_r(MG) = 0 \quad \text{for} \quad r \supset 0$$

$$\pi_0(MG) = H_0(MG) = G.$$

Let E be a spectrum. By definition the spectrum $EG = E\wedge MG$ and

$$0 \longrightarrow \pi_n(E)\otimes G \longrightarrow \pi_n(EG) \longrightarrow \mathrm{Tor}_1^{\mathbf{Z}}(\pi_{n-1}(E),G) \longrightarrow 0$$

is exact.

We are interested in the case $E = MU$, $G = \mathbf{Q}_p$ p prime number. $\mathbf{Q}_p$ is a subring of $\mathbf{Q}$ with $a/b \in \mathbf{Q}_p$ if and only if $(p,b) = 1$ The following are known results of Quillen,[Adams] There exists a map $\varepsilon_p: MUQ_p \longrightarrow MUQ_p$ for each prime p such that

(i) $\varepsilon_p^2 = \varepsilon_p$.

(ii) ε_p is map of ring spectra.

(iii) ε_p acts on $\pi_*(MU)$ as follows:

$$\varepsilon_p[\mathbb{C}P^n] = \begin{cases} [\mathbb{C}P^n] & n = p^f-1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently: ε_p acts on $\pi_*(MU)\otimes\mathbb{Q}$ as follows:

$$\varepsilon_p(m_n) = \begin{cases} m_n & n = p^f-1 \\ 0 & \text{otherwise} \end{cases}$$

Given p a prime the image of $\varepsilon_{p*}: MU\mathbb{Q}_p^*(X) \longrightarrow MU\mathbb{Q}_p^*(X)$ is a natural direct sumand of $MU\mathbb{Q}_p^*(S)$ so it is a representable functor (in the sense of Brown).Let BP denote the respective spectrum (BP = Brown-Peterson). Therefore BP is a ring spectrum and $BP_*(\ )$ is a multiplicative (co)homology theory with canonical map.

$$BP^*(X) \xrightarrow[i]{\quad} BP^*(X) \quad \pi_p \quad MU^*(X)_{(p)}$$

We observe

$$\pi_*(BP)\otimes\mathbb{Q} = H_*(BP)\otimes\mathbb{Q} = \mathbb{Q}[m_{p-1}, m_{p^2-1}, \ldots]$$

and by [Hazewinkel 1].

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \quad |v_n| = 2(p^n-1)$$

where

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}$$

§1.2: BP operations

Similarly to MU-theory we identify $BP^*(BP)$ with the set of stable additive cohomology operations. $BP_*(BP)$ has a structure of Hopf algebra and we have homomorphism

$$\phi: BP_*(BP)\otimes BP_*(BP) \longrightarrow BP_*(BP) \qquad \text{, action}$$

$\Psi: (BP) \longrightarrow BP_*(BP) \otimes BP_*(BP)$ coaction

$\eta_L: \pi_*(BP) \longrightarrow BP_*(BP)$ left unity

$\eta_R: \pi_*(BP) \longrightarrow BP_*(BP)$ right unity

$\varepsilon: BP_*(BP) \longrightarrow \pi_*(BP)$ counit

$c: BP_*(BP) \longrightarrow BP_*(BP)$ conjugation

and for each X complex or spectra

$$\Psi_X: BP_*(X) \longrightarrow BP_*(BP) \otimes BP_*(X)$$

with the same properties as in MU-theory. (see [Switzer 1] chapter 17 theorem 17.8

<u>Theorem</u>: (See [Wilson 1] page 17)

(i) There are $t_i \in BP_{2(p^i-1)}(BP)$ $t_0 = 1$ such that

$$\eta_R(m_k) = \sum m_i t_{k-i}^{p^i}$$

(ii) $BP_*(BP) \cong BP_*[t_1, t_2, \ldots]$

(iii) c is given by

$$m_k = \sum m_i t_j^{p^i} c(t_{k-i-j})^{p^{i+j}} \quad \text{or} \quad 1 = \sum_{n,j \geq 0} t_n c(t_j)^{p^n}$$

(iv) $\varepsilon(1) = 1$ $\varepsilon(t_i) = 0$ $i > 0$

(v) The coproduct Ψ is computed by

$$\sum m_i (\Psi(t_{k-i}))^{p^i} = \sum_{h+i+j=k} m_h t_i^{p^h} \otimes t_j^{p^{h+i}}$$

and

$$\Psi_{CP^\infty}: BP_*(CP^\infty) \longrightarrow BP_*(BP) \otimes_{BP_*} BP_*(CP^\infty)$$

is

$$\Psi_{CP^\infty}(\beta^{BP}) = \sum (c(\sum_{n \geq 0}{}^F t_n))^i \otimes \beta_i^{BP}$$

where $BP_*(CP^\infty) = BP_*[\beta_1,\beta_2 \ldots]$.

In general the Quillen splitting theorem implies that

$$\pi_*(BP)\otimes_{\pi_*(MU)} MU_*(X) \cong BP_*(X)$$

and

$$MU_{*(p)} \otimes_{BP_*} BP_*(X) \cong MU_*(X)_{(p)}.$$

That means that if we define $x \in BP_*(X)$ to be primitive if $\Psi_X(x) = 1\otimes x$, we do not lose any information from considering BP-primitives.

Now $BP_*(BP)$ and $BP^*(BP)$ are dual Hopf algebras over BP_*. If $t^E = t_1^{e_1} \cdots t_n^{e_n} \in BP_*(BP)$ we define $r_E \in BP_*(BP)$ as the dual of t^E in $BP^*(BP)$. $\{r_E\}$ form a basis dual to the $\{t^E\}$ and

$$BP^*(BP) = \langle\{\sum \alpha_E r_E \mid \alpha_E \in BP^* = \pi_*(BP)\}\rangle$$

(See [Zahler 1] §3).

Now we want to say how $BP^*(BP)$ acts on $\pi_*(BP)$. As in MU-theory for $\theta \in BP^*(BP)$ we have

$$\theta(\lambda) = \langle\theta, \eta_R(\lambda)\rangle$$

so

$$r_E(m_{p^k-1}) = \begin{cases} m_{p^{k-1}-1} & \text{if } E = p^i\Delta_{k-1} \\ 0 & \text{otherwise} \end{cases}$$

If

$$m^E = m_{p-1}^{e_1} m_{p^2-1}^{e_2} \cdots m_{p^n-1}^{e_n}$$

we have

$$r_E(m^F) = \begin{cases} 1 & E = F \\ 0 & |E| \geq |F| \text{ and } E \neq F. \end{cases}$$

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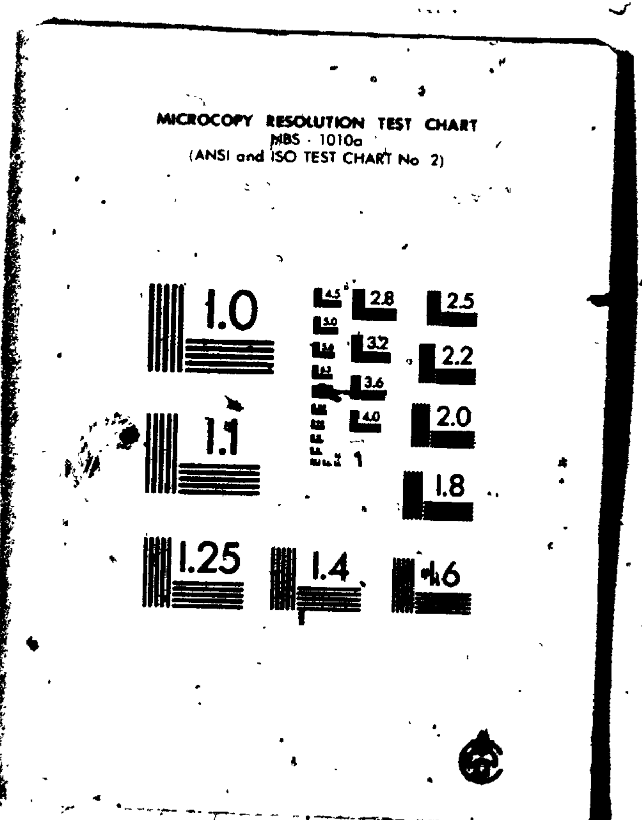
MICROCOPY RESOLUTION TEST CHART
NBS - 1010a
(ANSI and ISO TEST CHART No. 2)
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0

Now BP-operations and Steenrod operations are related by the following: See [Kane 3]. Define $T: BP \longrightarrow H\mathbf{Z}/p$ to be the Thom map such that

$$\begin{array}{ccccc} MU\mathbf{Q}_p & \xrightarrow{T} & H\mathbf{Q}_{(p)} & \xrightarrow{\rho} & H\mathbf{Z}/p \\ & {\scriptstyle i}\nwarrow & \uparrow{\scriptstyle T} & \nearrow{\scriptstyle T} & \\ & & BP & & \end{array}$$

ρ is the reduction mod p

commutes.

It can be chosen as the Thom class in $H^*(BP;\mathbf{Z}/p) = A^*(p)/\mathbf{Q}_0$. And by the Zahler op.cit. page 488

$$T_*: BP^*(BP) \longrightarrow H^*(BP;\mathbf{Z}/p)$$

is given by

$$T_*(\alpha_E r_E) = \begin{cases} \chi(\mathcal{P}^E) & \alpha_E = 1 \\ 0 & \text{otherwise} \end{cases}$$

and (See Kane op.cit.)

$$\begin{array}{ccc} BP^*(X) & \xrightarrow{r_E} & BP^*(X) \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ H^*(X;\mathbf{Z}/p) & \xrightarrow{\chi(\mathcal{P}^E)} & H^*(X;\mathbf{Z}/p) \end{array}$$

commutes where $\chi: A^*(p) \longrightarrow A^*(p)$ is the canonical automorphism. Therefore information about the action of Steenrod operations on $H^*(X;\mathbf{Z}/p)$ is included in information about the action of BP^*-operations on $BP^*(X)$.

We can define the operations $s_E \in BP^*(BP)$ by the recursive formula

$$\sum_{E_1+E_2=E} s_{E_1} r_{E_2} = 0$$

for each E. So s_E is the conjugate of r_E. Since $\{\wp^E\}$ and $\{\chi(\wp^E)\}$ are also related by the formula

$$\sum_{E_1+E_2=E} \wp^{E_1}\chi(\wp^{E_2}) = 0,$$

we can deduce that s_E covers $\wp^E$ i.e. the following diagram commutes

$$\begin{array}{ccc} BP^*(X) & \xrightarrow{s_E} & BP^*(X) \\ \downarrow T & & \downarrow T \\ H^*(X;\mathbf{Z}/p) & \xrightarrow{\wp^E} & H^*(X;\mathbf{Z}/p). \end{array}$$

§1.3: The Primitivity Operator

In analogue to the operator $\wp_*$ for MU theory we can define

$$\wp_*: BP_*(X)\otimes\mathbf{Q} \longrightarrow BP_*(X)\otimes\mathbf{Q}$$
$$\wp_* = \sum_E m_E s_E$$

where $m^E = m_{p-1}^{e_1}\cdots m_{p^k-1}^{e_k}$. The properties discussed for the MU $\wp_*$ in §1 of Part III also hold for this $\wp_*$. In particular $\wp_*$ factors through $H_*(X;\mathbf{Q})$. So the action $\wp_*$ in BP agrees with the image of $\wp_*$ in MU. In other words the following diagram commutes.

$$\begin{array}{ccc} MU_*(X)\otimes\mathbf{Q} & \xrightarrow{\wp_*} & MU_*(X)\otimes\mathbf{Q} \\ \uparrow & & \uparrow \\ BP_*(X)\otimes\mathbf{Q} & \xrightarrow{\wp_*} & BP_*(X)\otimes\mathbf{Q} \end{array}$$

Because the operations $\{s_E\}$ are related to Steenrod operations it is useful to use the BP version of $\wp_*$. Notably we have the following results. As before we will be interested in the minimal s such that $p^s\wp_*(x) \in BP_*(X) \subset BP_*(X)\otimes\mathbb{Q}$.

<u>Proposition 1.3.1</u>: Suppose $BP_*(X) = \Lambda(x_3, x_{2p+1})$. Then

(a) $$p\wp(x_{2p+1}) \in BP_*(X)$$

(b) $\wp(x_{2p+1}) \in BP_*(X)$ if and only if $\wp^1 T(x) = 0$ in $H_*(X;\mathbb{Z}/p)$.

<u>Proof</u>: We can write

$$\wp(x_{2p+1}) = x_{2p+1} + m_{p-1}s_1(x_{2p+1}).$$

Since

$$pm_{p-1} \in BP_* \subset BP_*\otimes\mathbb{Q}$$

we have part (a). Regarding (b) we know $s_1(x_{2p+1})$ is divisible by p if and only if $\wp^1(T(x_{2p+1})) = T(r_1(x_{2p+1})) = 0$. Q.E.D.

<u>Proposition 1.3.2</u>: Suppose $BP_*(X) = \Lambda(x_3, x_{4p-1})$. Then

$$p\wp(x_{4p-1}) \in BP_*(X).$$

<u>Proof</u>: Write

$$\wp(x_{4p-1}) = x_{4p-1} + m_1 s_1(x_{4p-1}) + m_1^2 s_2(x_{4p-1}).$$

As above we know $pm_1 \in BP_*$. Consider $pm_1^2 s_2(x_{4p-1})$. Since

$$\wp^2 : H^3(X;\mathbb{Z}/p) \longrightarrow H^{4p-1}(X;\mathbb{Z}/p)$$

is trivial ($A^*(p)$ acts unstably!). We also have

$$\wp^2 : H_{4p-1}(X;\mathbb{Z}/p) \longrightarrow H_3(X;\mathbb{Z}/p)$$

is trivial. So $T(s_2(x_{4p-1})) = \wp^2(T(x_{4p-1})) = 0$. Thus we can write $s_2(x_{4p-1}) = p\alpha x_3$ for some $\alpha \in \mathbb{Q}_{(p)}$. Thus $pm^2(s_2(x_{4p-1})) = p^2 m_1^2 x_3$ and $p^2 m_1^2 \in BP_*$. Q.E.D.

§2: Jacobson definition of exceptional Lie groups

Let K be the non associative algebra of Cayley numbers. i.e. K is the 8 dimensional algebra over the real numbers with generators 1, i, j, l and relations.

$$i^2 = j^2 = l^2 = -1, \; ij = -ji, \; il = -li \text{ and } jl = -lj$$

A real basis is $\{1, i, j, ij, l, il, jl, (ij)l\}$ and the subalgebra generated by $1, i$ and j is the quaternions algebra $\mathbb{H}$. Any $x \in K$ can be written as $x = a + bl$ where $a, b \in \mathbb{H}$ and the conjugate of x is $\overline{x} = \overline{a} - bl$. If $a \in \mathbb{H}$ is

$$a = a_0 + a_1 i + a_2 j + a_3 ij \qquad a_i \in \mathbb{R}$$

then

$$\overline{a} = a_0 - a_1 i - a_2 j - a_3 ij$$

the norm in K is $N(x) = x\overline{x} = \overline{x}x \in \mathbb{R}$ and the associated bilinear form $N(x_1 y) = (x\overline{y} + y\overline{x})/2$ is non-degenerate. Also

$$N(xy) = N(x)N(y).$$

Let M_3 be the exceptional algebra of 3×3 hermitian matirces with coefficients in K. i.e. $A \in M_3$ if A is a 3×3 matrix over K and $\overline{A}^{tr} = A$. The multiplication in M_3^8 is given by $A \cdot B = (AB+BA)/2$ (AB means the usual matrix multiplication.) Thus if $A \in M_3$ we can write

$$A = \begin{bmatrix} \alpha_1 & a & b \\ \overline{a} & \alpha_2 & c \\ \overline{b} & \overline{c} & \alpha_3 \end{bmatrix} \qquad \alpha_i \in \mathbb{R} \quad a, b, c \in K.$$

So M_3 is a 27 dimensional vector space.

We define

$$\varepsilon_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \varepsilon_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \varepsilon_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and for $a \in K$ and $i \neq j$ a_{ij} = the matrix with entries a and $\overline{a}$ in position (i,j) and (j,i) respectively and zero in all others. Then if $A \in M$ we can write

$$A = \sum_{i=1}^{3} \alpha_i \varepsilon_i + a_{12} + b_{13} + c_{23}.$$

The trace of $a \in K$ is $T(a) = a + \overline{a}$. So

$$T(A) = \sum_{i=1}^{3} \alpha_i.$$

The norm of $a \in K$ is

$$N(a) = \alpha_1\alpha_2\alpha_3 + T(a(c\overline{b})) + \alpha_1 N(c) - \alpha_2 N(\overline{b}) - \alpha_3 N(a).$$

The multiplication in M_3 is characterized by the formulas

$$\varepsilon_i \cdot a_{ij} = 1/2 a_{ij} = a_{ij} \cdot \varepsilon_j$$

$$a_{ij}^2 = N(a)(\varepsilon_i + \varepsilon_j)$$

$$2a_{ij} \cdot b_{jk} = (ab)_{ik} \qquad i \neq j,\ j \neq k,\ k \neq i$$

and by the fact that these are orthogonal idempotents.

<u>Definition</u>: $A \in M_3$ is a primitive idempotent if $A^2 = A$ and $A \neq 0$ and there do not exist non-zero idempotents E' and E'' with $A = E' + E''$ and $N < E', E'' > = 0$ or equivalently $T(A) = 1$.

Let P = the set of all primitive idempotents in M_3. If $A \in P$ $T(A) = \alpha_1 + \alpha_2 + \alpha_3 = 1$. Since $A^2 = A$ then writing

$$A = \begin{bmatrix} \alpha_1 & a & b \\ \overline{a} & \alpha_2 & c \\ \overline{b} & \overline{c} & \alpha_3 \end{bmatrix}$$

$$\alpha_1^2 + N(b)^2 + N(c)^2 = \alpha_1 \qquad (\alpha_2+\alpha_3)a + \overline{cb} = a$$

$$\alpha_2^2 + N(a)^2 + N(c)^2 = \alpha_2 \qquad (\alpha_1+\alpha_3)b + \overline{ac} = b$$

$$\alpha_3^2 + N(a)^2 + N(b)^2 = \alpha_3 \qquad (\alpha_1+\alpha_2)c + \overline{ba} = c$$

Then

$$bc = \alpha_1\overline{a} \qquad N(a)^2 = \alpha_2\alpha_3$$

$$ca = \alpha_2\overline{b} \qquad N(b)^2 = \alpha_3\alpha_1$$

$$ab = \alpha_3\overline{c} \qquad N(c)^2 = \alpha_1\alpha_2$$

and

$$\alpha_1 + \alpha_2 + \alpha_3 = 1.$$

In [Jacobson 1] it has been shown that

E_6 = the group of norm-preserving linear transforations of M_3,

F_4 = the group of automorphisms of M_3.

$\mathrm{Spin}(8) = \{t \in F_4 \mid t(\varepsilon_i) = \varepsilon_i,\ i = 1,2,3\}$

$\mathrm{Spin}(9) = \{t \in F_4 \mid t(\varepsilon) = \varepsilon$ when $\varepsilon^2 = \varepsilon$ and ε is not the sum of two orthogonal idempotents$\}$

$G_2 = \{t \in F_4 \mid t(1_{ij}) = 1_{ij} \quad i \neq j,\ i,j = 1,2,3,\ t(\varepsilon_i) = \varepsilon_i\}$,

$$1_{ij} = i\begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} \atop \qquad j$$

Therefore we have the following chain of inclusions

$$G_2 \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9) \subset F_4 \subset E_6$$

and clearly

$$Spin(7)/G_2 \cong S^7 = \{x \in K \mid N(x) = 1\}.$$

Let $P_0 = \{A \in P \mid \alpha_1 = 0\}$.

Lemma 2-1: $P_0 \cong S^8$

Proof: We see that S^8 as the set of all the pairs $(s,z) \in K \times R$ with $N(s)^2 + ||z||^2 = 1$. Define $\phi: S^8 \longrightarrow P_0$ by

$$\phi(s,z) = 1/2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+z & s \\ 0 & \bar{s} & 1-z \end{bmatrix}.$$

This is a continuous map with inverse $\Psi(X) = (2\alpha_2 - 1, 2a)$.

Theorem 2-2: The space P is homeomorphic to Π, the Cayley projective plane. (See [Whitehead 1] appendix A).

Theorem 2-3: $Spin(9) \longrightarrow F_4 \longrightarrow \Pi$ is a fibration.

§3: Quasi Regular Space

§3.1: Quasi-Regularity

Definitions: Let p a prime $f: X \longrightarrow Y$ a map between CW complexes. Then f is a p-equivalence if

$$f^*: H^*(X;\mathbf{Z}/p) \longrightarrow H^*(X;\mathbf{Z}/p)$$

is an isomorphism or equivalently

$$f_*: H_*(X;\mathbf{Z}/p) \longrightarrow H_*(Y,\mathbf{Z}/p)$$

is an isomorphism or

$$f_*: \pi_*(X)\otimes\mathbf{Z}/p \longrightarrow \pi_*(Y,\mathbf{Z}/p)$$

is an isomorphism.

f a is rational equivalence if it is a 0-equivalence. i.e.

$$f_*: H_*(X;\mathbf{Q}) \longrightarrow H_*(Y;Q)$$

is an isomorphism.

Notation: $X \underset{p}{\simeq} Y$ and $X \underset{0}{\simeq} Y$.

A prime number p is regular for a H-space X if X and

$$S_r^{n_i} = \prod_{1\leq i\leq r} S^{n_1}\times S^{n_2}\times\ldots\times S^{n_r} \quad (r = \text{rank of } X)$$

are p-equivalent i.e. There exists $f: X \longrightarrow S_r^{n_i}$ such that $f_*: H_*(X;\mathbf{Z}/p) \longrightarrow H_*(S_r^{n_i};\mathbf{Z}/p))$ is an isomorphism. The Hopf theorem says that any finite H-space is 0-regular.

Let $B_n(p)$ be the total space of a bundle with base $S^{2n+2p-1}$ and fiber S^{2n+1} such that

$$H^*(B_n(p);\mathbf{Z}/p) = \Lambda(x_{2n+1},\mathcal{P}^1(x_{2n+1})).$$

A finite H-space X is quasi-regular if there exists a p-equivalence between X and a space which is a product of spheres or bundles $B_n(p)$.

Notice that p can not be regular or quasi-regular for X unless $H^*(X)$ has no p-torsion. The most general theorem about quasi regularity is due to Harper [Harper 1] page 65.

Theorem 3.1: (Harper) Suppose X is 1-connected H-space with multiplication such that $H^*(X;\mathbb{Q})$ is primitvely generated. If p is a prime such that $4p-3 \geq \dim Q(H^*(X;\mathbb{Q}))$ then X is quasi regular at p.

A most precise result has been obtained for simple, 1-connected, compact, connected Lie groups.

Theorem 3.2: [Mimura-Toda 3]

p is quasi-regular at G if and only if

$p > n$	for	$G = Sp(n)$
$p > n/2$	for	$G = SU(n)$
$p > (n-1)/2$	for	$G = Spin(n)$
$p \geq 5$	for	$G = G_2, F_4, E_6$
$p \geq 11$	for	$G = E_7, E_8$.

Given a p quasi-regular space X, to distinguish S^n factors and $B_n(p)$ factors we only have to check the action of $\mathcal{P}^1$ on $H_*(X;\mathbf{Z}/p)$. [Mimura-Toda 3] (op. cit. page 318) have proved

Theorem 3.3: Let G be a Lie group which is p quasi regular. Write

$$H_*(X;\mathbf{Z}/p) = \Lambda(x_{s_1},\ldots,x_{s_r})$$

where $|x_{s_i}| = s_i$. Then $\mathcal{P}^1(x_s) = x_t$ if and only if $t-s = 2(p-1)$ and $s-1/2 \neq 0 \bmod p$.

Notably the above theorems give the following results for G_2 and F_4:

$$G_2 \underset{(5)}{\sim} B_1(5)$$

$$G_2 \underset{p}{\sim} S^3 \times S^{11} \quad p \geq 7$$

$$F_4 \underset{(5)}{\sim} B_1(5) \times B_7(5)$$

$$F_4 \underset{7}{\sim} B_1(7) \times B_5(7)$$

$$F_4 \underset{(11)}{\sim} B_1(11) \times S^{11} \times S^{15}$$

§3.2: Primitive and Spherical Classes for Quasi-Regular Spaces

Let X and Y be p equivalent spaces. Suppose

$$BP_*(X) = BP_*(Y) = \Lambda(x_{n_1}, \ldots x_{n_r})$$

where $\{x_{n_1} \ldots x_{n_r}\}$ are coalgebra primitives. We want to determine

$h(n_i)$ = the minimal integer such that $h(n_i)x_{n_i}$ is spherical

$N(n_i)$ = the minimal integer such that $N(n_i)x_{n_i}$ is primitive.

Since we are working localized at p we have $h(n_i) = p^s$ and $N(n_i) = p^t$ where $s \geq t$. If we consider p quasi regular spaces then we can reduce to spheres S^{2n-1} and the spaces $B_n(p)$.

(i) $\underline{X = S^{2n-1}}$

Then $h(2n-1) = N(2n-1) = 1$. In particular primitive elements agree with spherical elements in $BP_*(S^{2n-1})$.

(ii) $\underline{X = B_n(p)}$.

So $BP_*(X) = \Lambda(x_{2n+1}, x_{2n+2p-1})$. As above we have

$$h(2n+1) = N(2n+1) = 1.$$

We also have

$$h(2n+2p-1) = N(2n+2p-1)) = p.$$

Since $H_*(X;\mathbb{Z}/p) = \Lambda(T(x_{2n+1}), T(x_{2n+2p-1}))$ where $\mathcal{P}^1 T(s_{2n+sp-1}) = T(x_{2n+1})$ we have $N(2n+2p-1) = p$ by Proposition 1.3.2. We also have $p | h(2n+2p-1)$ by [Kane theorem 4.4. Chapter 1V of Part V]. Moreover, $h(2n+2(p-1)) \mid p$ by Proposition 4.3.

Again observe that primitives and sphericals agree in $BP_*(B_n(p))$.

§4: Harper's H-Space

For each odd prime p Harper constructeed a mod p finite H-space K(p) such that

$$H^*(K(p);\mathbf{Z}/p) = \Lambda(x_3, x_{2p+1}) \otimes \mathbf{Z}/p[x_{2p+2}]/(x_{2p+2}^p)$$

$$\mathcal{P}^1(x_3) = x_{2p-1}, \quad \beta(x_{2p+1}) = x_{2p+2}.$$

Our use of this space will come in the case of the exceptional Lie group F_4. For Harper has also shown

$$F_4 \underset{3}{\simeq} K(3) \times B_5(3)$$

(See [Harper 1] and [Harper 2].) In preparation for our study of F_4 we will study the Hurewicz map

$$h: \pi_*(K(p))(p)/\text{Tor} \longrightarrow H_*(K(p))(p)/\text{Tor}.$$

In all that follows we assume p is odd.

Proposition 4.1:

$$H^*(K(p))(p)/\text{Tor} = \Lambda(y_3, y_{2p^2+2p-1}).$$

Moreover, we can choose representatives y_3 and y_{2p^2+2p-1} such that

$$\rho(y_3) = x_3$$

and

$$\rho(y_{2p^2+2p-1}) = x_{2p+1}x_{2p+2}^{p-1}.$$

Proof: Let $\{\beta_r\}$ be the Bockstein spectal sequence analyzing p torsion in $H^*(K(p))_{(p)}$. Then

$$B_2 = H(H^*(K(p);\mathbf{Z}/p);\beta)$$

$$= \Lambda(y_3, y_{2p^2+2p-1})$$

where $y_3 = \{x_3\}$ and $y_{2p^2+2p-1} = \{x_{2p+1} \cdot x_{2p+2}^{p-1}\}$.

The spectral sequence must now collapse. So $\beta_2 = H^*(K(p))_{(p)}/\mathrm{Tor} \otimes \mathbf{Z}/p$ has the same description. Q.E.D.

Corollary 4.2:

$$H_*(K(p))_{(p)}/\mathrm{Tor} = \Lambda(z_3, z_{2p^2+2p-1})$$

If we consider the Hurewicz map then $\pi_3(K(p))_{(p)} \cong H_3(K(p))_{(3)}$. We now show

Proposition 4.3:

$$h: \pi_{2p^2+2p-1}(K(p))_{(p)} \longrightarrow PH_{2p^2+2p-1}(K(p))_{(p)}$$

is multiplication by p^2.

Consider the fibration

$$F \xrightarrow{f} K(p) \xrightarrow{g} K(\mathbf{Z}_{(p)}, 3)$$

where g represents $y_3 \in H^3(K(p))_{(p)}$.

Lemma 4.4: In degree $\leq 2p^2+2p$

$$H^*(F;\mathbf{Z}/p) = \Lambda(u_{2p^2+1}, u_{2p^2+2p-1}) \otimes \mathbf{Z}/p[u_{2p^2}]$$

where

$$\beta(u_{2p^2}) = u_{2p^2+1}$$

and

$$\mathcal{P}^1(u_{2p^2+1}) = u_{2p^2+2p-1}.$$

Proof: We use the Serre spectral sequence

$$E_2 = H^*(F;\mathbf{Z}/p) \otimes H^*(K_{(p)}, 3);\mathbf{Z}/p) \Rightarrow H^*(K(p);\mathbf{Z}/p).$$

By Cartan's calculation [Cartan 2] $H^*(K(\mathbf{Z}_{(p)}, 3))$ has no higher p torsion and

$H^*(K(\mathbf{Z}_{(p)};3);\mathbf{Z}/3) = \Lambda(z_3, \mathcal{P}^1(z_3), \mathcal{P}^p\mathcal{P}^1(z_3) \ldots) \otimes \mathbf{Z}/p[\beta\mathcal{P}^1(z_3), \beta\mathcal{P}^p\mathcal{P}^1((z_3)]$

In view of descrption of $H^*(K(p);\mathbf{Z}/(p)$ the elements $\mathcal{P}^p\mathcal{P}^1(z_3)$, $\beta\mathcal{P}^p\mathcal{P}^1(z_3)$ and $[\beta\mathcal{P}^1(z_3)]^p$ must be killed in the spectral sequence. This forces the existence of the elements u_{2p^2}, u_{2p^2+1} and u_{2p^2+2p-1}. We have

$$d_{2p^2}(u_{2p^2}) = \mathcal{P}^p\mathcal{P}^1(z_3)$$
$$d_{2p^2+1}(u_{2p^2+1}) = \beta\mathcal{P}^1\mathcal{P}^1(z_3)$$
$$d_{2p^2+2p-1}(u_{2p^2+2p-1}) = [\beta\mathcal{P}^1(z_3)]^p$$

Since differentials which act transgressively commute with the action of $A^*(p)$ the relations

$$\beta[\mathcal{P}^p\mathcal{P}^1(z_3)] = [\beta\mathcal{P}^p\mathcal{P}^1(z_3)]$$
$$\mathcal{P}^1[\beta\mathcal{P}^p\mathcal{P}^1(z_3)] = \mathcal{P}^1\mathcal{P}^p\beta\mathcal{P}^1(z_3)$$
$$= \mathcal{P}^{p+1}\beta\mathcal{P}^1(z_3)$$
$$= [\beta\mathcal{P}^1(z_3)]^2$$

force

$$\beta(u_{2p^2}) = u_{2p^2+1}$$
$$\mathcal{P}^1(u_{2p^2+1}) = u_{2p^2+2p-1}$$

Q.E.D.

<u>Corollary 4.5</u>: In degree $\leq 2p^2+2p$

(i) $H_*(F;\mathbf{Z}/p) = \Lambda(v_{2p^2+1}, v_{2p^2+2p-1}) \otimes \mathbf{Z}/p[v_{2p^2}]$

where

$$\mathcal{P}^1(v_{2p^2+sp-1}) = v_{2p^2+1}$$

(ii) . $H_*(F,\mathbf{Z}/p)/\mathrm{Tor} = \Lambda(w_{2p^2+1}, w_{2p^2+2p-1})/(pw_{2p^2+1})$

We have a commutative diagram

$$\begin{array}{ccc}
\mathbf{Z}_{(p)} = \pi_{2p^2+2p-1}(F)_{(p)} & \xrightarrow{h} & PH_{2p^2+2p-1}(F)_{(p)} = \mathbf{Z}_{(p)} \\
\wr\| & & \downarrow f_* \\
\mathbf{Z}_{(p)} = \pi_{2p^2+2p-1}(K(p))_{(p)} & \xrightarrow[h]{} & PH_{2p^2+2p-1}(K(p))_{(p)} = \mathbf{Z}_{(p)}
\end{array}$$

By the same reasoning as in §3-2 the top map is multiplication by p.

So to prove Proposition 4-3 it remains to show

Lemma 4-6: $f_*: PH_{2p^2+2p-1}(F)_{(p)} \longrightarrow PH_{2p^2+2p-1}(K(p))_{(p)}$ is multiplication by p.

Proof: Consider the Serre spectral sequence

$$E_2 = H^*(F;H^*(K(\mathbf{Z}_{(p)},3)) \Rightarrow H^*(K(p))_{(p)}.$$

Since $H^*(K(\mathbf{Z}_{(p)},3))_{(p)}$ has no higher p torsion we can reduce mod p without losing any information. In particular we can use our previous knowledge of the mod p Serre spectral sequence to obtain complete information in degree $\leq 2p^2+2p$ about the above spectral sequence. Notably we have $d_{2p^2+2p-1}(x) = b$ where a and b are integral representatives of u_{2p^2+2p-1} and $[\mathcal{P}^1(z_3)]^p$. Thus $d_{2p^2+2p-1}(pa) = pb = 0$. So pa is a permanent cycle in the spectral sequence. This tells us that

$$f^*: QH^{2p^2+2p-1}(K(p))_{(p)} \longrightarrow QH^{2p^2+2p-1}(F)_{(p)}$$

is multiplication by p. Dualizing, we have our result. Q.E.D.

Besides spherical elements we are also concerned with primitive BP elements. We can only obtain partial information in the case of the Harper H-spaces. First of all the BP structucure of $PBP_*(K(p))$ is more complicated than one might first suspect. Although $H_*(K(s))/Tor$ is an exterior algebra the same is not quite true of $BP_*(K(p))$.

Proposition 4.7:

$$BP_*(K(p)) = \wedge(x_3, x_{2p+1}, x_{2p^2+2p-1})/(px_{2p+1} - v_1x_3).$$

The argument extablishing this result is based on [Kane 1]. First of all, one uses an Eilenberg-Moore spectral sequence plus our knowledge of $H^*(K(p));\mathbf{Z}/p)$ to deduce

$$H_*(\Omega K(p);\mathbf{Z}/p) = \mathbf{Z}/p[a_2, a_{2p}, a_{2p^2+2p-1}]/(a_2^p)$$

$$\mathcal{P}^1(a_{2p}) = a_2.$$

One then uses the above facts to deduce that

$$BP_*(\Omega K(p)) = BP_*[A_2, A_{2p}, A_{2p^2+2p-2}]/(A_2^P - pA2p + v_1A_2)$$

One can then use a homology BP Eilenberg-Moore type spectral sequence to deduce the proposition from this fact.

It follows from the Proposition that $PBP_*(K(p))$ is spanned by $\{x_3, x_{2p+1}, x_{2p^2+2p-1}\}$ modulo the relation $px_{2p+1} = v_1x_3$. Now consider $\mathcal{P}(x_{2p^2+2p+1}) \in BP_*(K(p))\otimes\mathbf{Q}$. By Proposition 4.3 we know $p^2\mathcal{P}(x_{2p^2+2p+1}) \in BP_*(K(p)) \subset BP_*(K(p))\otimes\mathbf{Q}$. However we do not know that p^2 is the minimal number i.e. that $N(2p^2+2p-1) = p^2$. So we are left with an indeterminacy.

§5: The Exceptional Lie Groups G_2 and F_4

We now compare primitives and sphericals in $PBP_*(G_2)$ and $PBP_*(F_4)$ for p an odd prime.

§5.1: The Case $X = G_2$

First of all we determine the Hurewicz map

$$h: \pi_*(G_2)/\mathrm{Tor} \longrightarrow H_*(G_2)/\mathrm{Tor} = \Lambda(x_3, x_{11}).$$

We need only consider degree 11. We have fibrations

$$S^3 \longrightarrow G_2 \longrightarrow V_{7,2}$$

$$G_2 \longrightarrow \mathrm{Spin}(7) \longrightarrow S^7$$

where $V_{7,2} = SO(7)/SO(5)$ is the Stiefel manifold. We will rely on [James 1] for information about Stiefel manifolds. From the first fibration we have a diagram

$$\begin{array}{ccccccc}
\mathbf{Z}/2 & & \mathbf{Z}\oplus\mathbf{Z}/2 & & \mathbf{Z} & & \mathbf{Z}/15 \\
\| & & \| & & \| & & \| \\
\pi_{11}(S^3) & \longrightarrow & \pi_{11}(G_2) & \longrightarrow & \pi_{11}(V_{7,2}) & \longrightarrow & \pi_{10}(S^3) \\
 & & h\downarrow & & \downarrow h & & \\
 & & PH_{11}(G_2)/\mathrm{Tor} & \longrightarrow & PH_{11}(V_{7,2})/\mathrm{Tor} & &
\end{array}$$

By the exactness of the top row the map $\pi_{11}(G_2)/\mathrm{Tor} \longrightarrow \pi_{11}(V_{7,2})/\mathrm{Tor}$ is a map $\mathbf{Z} \xrightarrow{k} \mathbf{Z}$ where $k = 1,3,5,15$. By [James 1] page 153

$$h: \pi_{11}(V_{7,2})/\mathrm{Tor} \longrightarrow PH_{11}(V_{7,2})/\mathrm{Tor}$$

is multiplication by 8. Consequently

$$h_{11}(G_2) = 8, 3.8, 5.8 \text{ or } 15.8.$$

On the other hand, the second fibration gives a diagram

$$\begin{array}{ccc} \pi_{11}(G_2) & \longrightarrow & \pi_{11}(Spin(7)) \\ h \downarrow & & \downarrow h \\ PH_{11}(G_2)/Tor & \cong & H_{11}(Spin(7))/Tor \end{array}$$

By our previous calculations $h_{11}(Spin(7)) = 5!$. So

$$5! \mid h_{11}(G_2).$$

Thus $h_{11}(G_2) = 5!$

Now we consider primitive elements in $PBP_*(G_2)$ and compare them to spherical elements. For each odd prime we can write $BP_*(G_2) = \Lambda(x_3, x_{11})$. So $PBP_*(G_2)$ is spanned by x_3 and x_{11}. As in §3 $N(i)x_i$ and $h(n_i)$ are the minimal powers of p so that $N(i)x_i$ is primitive and $h(n_i)x_i$ is spherical. By the above plus the resolution §3 we have

$$N(3) = h(3) = 1 \quad \text{for } p \geq 3$$

$$N(11) = h(11) = 3 \quad \text{for } p = 3$$

$$N(11) = h(11) = 5 \quad \text{for } p = 5$$

$$N(11) = h(11) = 1 \quad \text{for } p \geq 7$$

In particular spherical and primitive elements agree in $PBP_*(G_2)$.

§5.2: The Case $X = F_4$

For $p \geq 5$ we can write $BP_*(F_4) = \Lambda(x_3, x_{11}, x_{15}, x_{23})$. By the results of §3 we have

$$N(3) = h(3) = 1 \quad \text{for } p \geq 5$$

$$N(11) = h(11) = \begin{cases} 5 & p = 5 \\ 1 & p \geq 7 \end{cases}$$

$$N(15) = h(15) = \begin{cases} 7 & \text{for } p = 7 \\ 1 & \text{for } p \geq 5 \;\; p \neq 7 \end{cases}$$

$$N(23) = h(23) = \begin{cases} 5 & \text{for } p = 5 \\ 7 & \text{for } p = 7 \\ 11 & \text{for } p = 7 \\ 1 & \text{for } p = 13 \end{cases}$$

For $p = 3$ we have Harper's splitting $F_{4(3)} \sim K(3) \times B_5(3)$. By the results of §3 we have

$$N(3) = h(3) = 1$$

$$N(11) = h(11) = 1$$

$$N(15) = h(15) = 3$$

We now show

$$N(23) = h(23) = 3^2$$

In §4 we demonstrated that $h(23) = 3^2$. Since $N(23) \mid h(23)$ it suffices to show

Lemma 5.2.1: $3^2 | N(23)$

Proof Consider the representation $\lambda : F_4 \longrightarrow SU(n)$ of maximal weight. (see [Bourbaki 1]). The map $\Omega\lambda : \Omega F_4 \longrightarrow \Omega SU(n)$ was studied in [Watanave 1]. One can write

$$H_*(F_4)_{(3)} = \mathbf{Z}_{(3)}[x_2, x_6, x_{10}, x_{14}, x_{22}]/(x_2^3 = 3x_6)$$

$$H_*(\Omega SU(n))_{(3)} = \mathbf{Z}_{(3)}[y_2, y_4, \ldots, y_{2n}]$$

It was proved in [Watanabe 1] that

$$\begin{array}{ccc} QH_{22}(\Omega F_4; \mathbf{Z}_{(3)}) & \xrightarrow{(\Omega\lambda)_*} & QH_{22}(SU(n); \mathbf{Z}_{(3)}) \\ \| & & \| \\ \mathbf{Z}_{(3)} & & \mathbf{Z}_{(3)} \end{array}$$

is multiplication by 3^3. Now consider the diagram

$$\begin{array}{ccc} \mathbf{Z}_{(3)} = PH_{23}(F_4)_{(3)}/tor & \xrightarrow{\lambda_*} & PH_{23}(SU(n))_{(3)} = \mathbf{Z}_{(3)} \\ \Omega_* \uparrow & & \uparrow \Omega_* \\ \mathbf{Z}_{(3)} = QH_{22}(\Omega F_4)_{(3)} & \xrightarrow{(\Omega\lambda)_*} & QH_{22}(\Omega SU(n))_{(3)} = \mathbf{Z}_{(3)} \end{array}$$

(i) $\Omega_*: QH_{22}(\Omega SU(n))_{(3)} \longrightarrow PH_{23}(SU(n))_{(3)}$ is an isomorphism. For we can write $H_*(SU(n))_{(3)} = \Lambda(z_3, z_5, \ldots, z_{2n+1})$ where $\Omega_*(y_{2i}) = z_{2i+1}$

(ii) $\Omega_*: QH_{22}(\Omega F_4)_{(3)} \longrightarrow PH_{23}(F_4)_{(3)}/tor$ is multiplication by 3^k where $k \geq 1$. To prove this result it suffices to show that $\Omega_*: QH_{22}(\Omega F_4; \mathbf{Z}/3) \longrightarrow PH_{23}(F_4; \mathbf{Z}/3)$ is trivial. In turn, it suffices to prove that the dual map $\Omega^*: QH^{23}(F_4; \mathbf{Z}/3) \longrightarrow PH^{22}(\Omega F_4; \mathbf{Z}/3)$ is trivial This is obvious. For $H^*(F_4; \mathbf{Z}/3) = \Lambda(x_3, x_7, x_{11}, x_{15}) \otimes \mathbf{Z}/3[x_8]/(x_8^3)$. So in particular, $QH^{23}(F_4; \mathbf{Z}/3) = 0$.

It follows for all above that $\lambda: PH_{23}(F_4)_{(3)}/tor \longrightarrow PH_{23}(SU(n))_{(3)}$ is multiplication by 3^k for $k \leq 2$. Pick a generator $x \in PH_{23}(F_4)_{(3)}/tor$. We have $\hat{\varphi}(x) \in BP_*(F_4) \otimes \mathbf{Q}$. We want to show that $3\hat{\varphi}(x) \notin BP_*(F_4) \subset BP_*(F_4) \otimes \mathbf{Q}$. We know that $\lambda_*(3\hat{\varphi}(x)) = 3\hat{\varphi}(\lambda_*(x)) = 3^{k+1}\hat{\varphi}(z_{23}) \notin BP_*(SU(n)) \subset BP_*(SU(n)) \otimes \mathbf{Q}$. It follows from Corollary 5.5 §5 Part IV that $N\hat{\varphi}(z_{23}) \notin MU_*(SU(n))$ unless $11! \mid N$ Now $11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. So, in BP theory, $N\hat{\varphi}(z_{23}) \notin BP_*(SU(n))$ unless $3^4 \mid N$. In particular $3^{k+1}\hat{\varphi}(z_{23}) \notin BP_*(SU(n))$. because $k \leq 2$.

Q.E.D

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