1985

Homology And K-theory Of Loop Spaces

Jesus-manuel Mayorquin-garcia

HOMOLOGY AND K-THEORY OF LOOP SPACES

by

Jesus-Manuel Mayorquin-Garcia

Submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Faculty of Graduate Studies

The University of Western Ontario

London, Ontario

August 1985

ABSTRACT

Part 1 of this work is an application of the Dyer-Lashof Algebra modulo p, an odd prime, to the determination of the Steenrod annihilated indecomposables in the $\mathbb{Z}/p$ homology of the infinite loop space associated to a CW-complex.

Part 2 is concerned with the determination of the algebra structure of $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$ where $K_*(\_; \mathbb{Z}/2)$ stands for mod 2, periodic, reduced, complex K-homology theory. Moreover the Atiyah-Hirzebruch spectral sequence for $K_*(\Omega^m S^n X; \mathbb{Z}/2)$ is studied. The main tools in Part 2 are the mod 2 Dyer-Lashof operations acting on finite loop spaces, as well as the Atiyah-Hirzebruch spectral sequence for K-homology.

ACKNOWLEDGEMENTS

I am grateful to Professors S. O. Kochman and V. P. Snaith for their help during the preparation of this work.

I would also like to thank the PSPA, OGS, and the University of Western Ontario for making possible my studies.

Thanks also to J. Williams for her fast and excellent typing.

TABLE OF CONTENTS

	Page
CERTIFICATE OF EXAMINATION	ii
ABSTRACT	iii
ACKNOWLEDGEMENTS	iv
TABLE OF CONTENTS	v
PART 0. INTRODUCTION	1
PART 1	6
CHAPTER 0. INTRODUCTION	7
CHAPTER 1. H*-SPACES	10
CHAPTER 2. DYER-LASHOF OPERATIONS	14
CHAPTER 3. DYER-LASHOF OPERATIONS AND $H_*(QX; \mathbb{Z}/p)$	18
CHAPTER 4. PROOF OF THE MAIN RESULT	22
CHAPTER 5. A RELATED RESULT	54
BIBLIOGRAPHY TO PART 1	70
PART 2	72
CHAPTER 0. INTRODUCTION	73
CHAPTER 1. MOD 2 HOMOLOGY OPERATIONS	76
A. H_n-Spaces	76
B. λ_n	77
C. Q_m	78

D. Properties of the Operations 79

E. $H_*(\Omega^n S^n X; \mathbb{Z}/2)$ 82

F. The Bockstein Spectral Sequence for mod 2 Homology 83

CHAPTER 2. K-THEORIES 86

A. Generalities on G-vector Bundles 86

B. $K_G(X)$ 87

C. $K_*(X; \mathbb{Z}/2)$ 91

D. Multiplications in K-Theory 93

E. The Bockstein Spectral Sequence 97

F. The Transfer Homomorphism 98

G. $R(\pi_2)$ 99

H. The Atiyah-Hirzebruch Spectral Sequence 99

I. $K^*(RP^n; \mathbb{Z}/2)$ 101

CHAPTER 3. THE ROTHENBERG STEENROD SPECTRAL SEQUENCE 103

A. Preliminaries 103

B. $E_2^{**}(pt,\phi); \pi_2; \mathbb{Z}/2)$ 108

C. $E_2^{**}((X,Y)^2; \pi_2; \mathbb{Z}/2)$ 109

D. $K^*_{\pi_2}(S^v_\pi \times X^2; \mathbb{Z}/2)$ and $K_*^{\pi_2}(S^v_\pi \times X^2; \mathbb{Z}/2)$, $v = 1,2$ 110

E. q_1 and $\overline{Q}_1$ 113

F. q_2 and $\overline{Q}_2$ 115

CHAPTER 4. E^4 OF SOME ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCES 126

CHAPTER 5. E^∞ OF THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE FOR $K_*(\Omega^3 S^3 X; \mathbb{Z}/2)$ 140

CHAPTER 6. THE ALGEBRA $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$, $n \geq 1$ 143

A. $D_{n,q}$ 143

B. $K_*(QS^r; \mathbb{Z}/2)$, r Odd 146

C. Determination of $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$ 146

BIBLIOGRAPHY TO PART 2 152

VITA 156

PART 0

INTRODUCTION

This work consists of two parts, 1 and 2, in each of which different objectives are pursued, the common underlying subject being the iterated loop spaces.

Part 1 deals with homology mod p, an odd prime, and only infinite loop spaces are considered. This part arose from a problem proposed by Dr. S. O. Kochman.

In Part 2 the mod 2 K-homology of finite loop spaces is studied, and it answers a problem proposed by Dr. V. P. Snaith. Although there is an introduction to each of the parts, we want to consider here some common features to both parts.

0.1. Let us start with the notion of iterated loop space, in the manner of [Ma 1].

An n-th loop sequence, $n \leq \infty$, is a sequence $B = \{B_i \mid 0 \leq i \leq n\}$ of based spaces such that $B_i = \Omega B_{i+1}$. A map $g : B \to C$ of n-th loop sequences is a sequence of base point preserving maps $g_i : B_i \to C_i$ which satisfy $g_i = \Omega g_{i+1}$. B_0 is called a perfect n-th loop space, and g_0 is called a perfect n-th loop map. If $n = \infty$, one speaks of "infinite loop spaces", and "infinite loop maps".

If $n < \infty$, the n-th loop sequence whose first space is $\Omega^n S^n X$ is $Q_n = \{\Omega^{n-i} S^n X \mid 0 \leq i \leq n\}$. If $n = \infty$, the adjunction

$$[S^{n+1}X, S^{n+1}X] \overset{\phi}{\cong} [S^nX; \Omega S^{n+1}X]$$

provides the map $\phi(1_{S^{n+1}X}) : S^nX \to \Omega S^{n+1}X$. Applying Ω^n to $\phi(1_{S^{n+1}X})$ one obtains a map

$$\phi : \Omega^n S^n X \to \Omega^{n+1} S^{n+1} X.$$

Define $QX = \lim_{\overrightarrow{n}} \Omega^n S^n X$, with the topology of the union. The infinite loop sequence whose initial object is QX is $\{QS^iX \mid i \geq 0\}$. For a discussion of the notions introduced above see [Ma 1] and [Ma 3].

The Dyer-Lashof operations acting on the mod p homology of QX generate this group, and in Part 1 of this work we study a particular feature of $H_*(QX; \mathbb{Z}/p)$, p odd prime. (See the introduction to Part 1, as well as Chapters 2 to 5 there.) For finite loop spaces the Dyer-Lashof operations are insufficient to generate $H_*(\Omega^n S^n X; \mathbb{Z}/p)$, p a prime, and more operations must be used to compute this group. If $p = 2$, W. Browder [Br] defined the extra operations required to calculate $H_*(\Omega^n S^n X; \mathbb{Z}/2)$, and this group will be of use in our analysis of $K_*(\Omega^n S^n X; \mathbb{Z}/2)$ in Part 2.

A complete treatment of the geometry and homology of $\Omega^n S^n X$ is carried out by F. R. Cohen in Part III of [Co-La-Ma], based on the foundational work by J. P. May in [Ma 4]. We will come back to $\Omega^n S^n X$, $n < \infty$, in Part 2.

0.2. The other topic we want to mention in this introduction is the Steenrod algebra, whose properties will play a fundamental role in Parts 1 and 2.

Proposition 0.3. [St] There are homomorphisms

$$Sq^i : H^q(X; \mathbb{Z}/2) \to H^{q+i}(X; \mathbb{Z}/2),\ i \geq 0,$$

which are natural, and satisfy the following properties

1) Sq^0 = identity

2) Sq^1 is the Bockstein operator β of the coefficient sequence

$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$

3) If dim x = n, then $Sq^n x = x^2$.

4) If dim x = n, then $Sq^i x = 0$ for all $i > n$.

5) (Adem relations) If $a < 2b$, then

$$Sq^a Sq^b = \binom{b-1}{a} Sq^{a+b} + \sum_{j=1}^{[a/2]} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j_q$$

6) (Cartan formula) If $x,y \in H^*(X; \mathbb{Z}/2)$ then

$$Sq^i (xy) = \sum_{j=0}^{i} (Sq^j x)(Sq^{i-j} y).$$

Similarly, if p is an odd prime:

Proposition 0.4. [Ibid.] There are homomorphisms

$$P^i : H^q(X; \mathbb{Z}/p) \to H^{q+2i(p-1)}(X; \mathbb{Z}/p),\ i \geq 0,$$

and the Bockstein homomorphism β associated with the exact sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$, such that

1) P^0 = identity.

2) If dim x = 2n, then $P^n x = x^p$.

3) If $2i > \dim x$, then $P^i x = 0$.

4) (Adem relations). If $a < pb$, then

$$P^aP^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-ip} P^{a+b-i}P^i.$$

If $a \leq pb + 1$, then

$$P^a\beta P^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)}{a-ip} \beta P^{a+b-i}P^i$$

$$+ \sum_{i=0}^{[a-1/p]} (-1)^{a-1+i} \binom{(p-1)(b-i)-1}{a-ip-1} P^{a+b-i} \beta P^i.$$

5) (Cartan formula). If $x,y \in H^*(X; \mathbb{Z}/p)$, then

$$P^i(xy) = \sum_{j=0}^{i} (P^jx)(P^{i-j}y);$$

$$\beta(xy) = (\beta x)y + (-1)^{\dim x} x(\beta y).$$

The Steenrod algebra mod 2, A_2, is defined to be the graded associative algebra over $\mathbb{Z}/2$ generated by the Sq^i, $i = 0,1,2,\ldots$, subject to the relations 1) and 5) of 0.3. Similarly, if $p \geq 2$, A_p is the graded associative algebra over $\mathbb{Z}/p$ generated by β and the P^i subject to the relations 1) and 4), and $\beta^2 = 0$. Sq^i has degree i, degree of $\beta = 1$, degree of $P^i = 2i(p-1)$.

With this terminology, the cohomology of a space X, $H^*(X; \mathbb{Z}/p)$ for each p, is a graded A_p-module.

The following result of Adem [A] is important to study the Steenrod algebra.

0.5. The algebra A_p is generated by β and the P^{p^i} for $i = 0,1,\ldots$; and A_2 is generated by the Sq^{2^i}, $i = 0,1,\ldots$.

0.6. The duals of the Steenrod operations are defined as usual and they act on the homology group $H_*(X; \mathbb{Z}/p)$ of a space X. They are denoted Sq^i_*, P^i_*, $i \geq 0$, and β.

The properties listed in 0.3 and 0.4 are valid for the Sq^i_*, P^i_* and β if one reverses the order of any composite of operations appearing there.

In particular, 0.5 has the same form for the dual A^*_p of the Steenrod algebra:

0.7. A^*_p is generated by β, $P^{p^i}_*$, $i = 0,1,\ldots$; and A^*_2 is generated by $Sq^{2^i}_*$, $i = 0,1,\ldots$. We will denote A^*_p simply by A_p.

PART 1

CHAPTER 0

INTRODUCTION

This work is an attempt to extend some of the techniques developed by Snaith and Tornehave in [SN-T], from the prime two to odd primes. Snaith and Tornehave reduced the problem of the existence of a framed manifold M^{2^n-2} with non-trivial Arf-Kervaire invariant to that of showing that a certain element $\underline{b}$ of $H_{2^n-2}(BO; \mathbb{Z}/2)$ is stably spherical.

There are two main features involved in this reduction:

1. A suitable description of the Arf-Kervaire invariant is required.
2. An effective method is needed to identify the primitive elements annihilated by the (dual) mod 2 Steenrod Algebra in $H_{2^n-\varepsilon}(QBO, \mathbb{Z}/2)$, $\varepsilon = 1,2$.

The solution of the second problem involves long computations with the mod 2 Dyer-Lashof operations which act on the homology of infinite loop spaces. Specifically, Snaith and Tornehave proved:

[Sn-T, Theorem 1.1] Let X be any connected CW-complex. Then

$$\underline{Q}\, H_{2^n-2}(QX; \mathbb{Z}/2)_A = H_{2^n-2}(X; \mathbb{Z}/2)_A$$

$$\underline{Q}\, H_{2^n-2}(QX; \mathbb{Z}/2)_A = H_{2^n-1}(X; \mathbb{Z}/2) \oplus Q^{2^{n-1}}(H_{2^{n-1}-1}(X, \mathbb{Z}/2)_A)$$

Here $\underline{Q}$ denotes indecomposables in homology, QX is the infinite loop space associated with X, $Q^{2^{n-1}}$ is the Dyer-Lashof operation, and the subscript a denotes elements annihilated by the mod 2 Steenrod Algebra. Our main results consist of versions for odd primes of the theorem above. Specifically we prove in Chapter 4 the following theorem.

Theorem 4.1. Let X be any connected CW-complex and let p be any odd prime. Then

$$\underline{Q}\, H_m(QX;\ \mathbb{Z}/p)_A = H_m(X;\ \mathbb{Z}/p)_A$$

whenever the p-adic expansion of m satisfies the following conditions:

1. If $m = 2k$, then $m = a_0 + a_1 p + \dots + a_n p^n$ with either

   a) $a_0 \neq p-2$; $a_0 \neq p-1$ or $a_1 \neq p-1$, while $a_i \neq p-1$ and $a_i \neq p-2$ for $i \geq 2$; or

   b) $a_i = p-1$ for all i.

2. If $m = 2k+1$, then $m = a_0 + a_1 p + \dots + a_n p^n$ with either

   a) $a_0 \neq p-3$ or $a_1 \neq p-1$, while $a_i \neq p-1$ and $a_i \neq p-2$ for $i \geq 2$; or

   b) $a_i = p-1$ for all i.

3. Either a) $m \leq p^n + (p-2)p^{n-1}$; or

   b) $m \geq p^n + (p-1)(p^{n-1} + \dots + 1)$.

Here $\underline{Q}$ denotes indecomposables in $\mathbb{Z}/p$-homology, QX is the infinite loopspace associated to X, and the subscript a denotes elements annihilated by the mod p Steenrod Algebra.

By slightly modifying the arguments of the proof of Theorem 4.1 we prove the following:

Theorem 5.1. Let X be any connected CW-complex, p any odd prime, and $t \geq 0$. Then

$$\underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX;\mathbb{Z}/p)_A =$$

$$H_{2p^{t+1}(p-1)-2}(X;\mathbb{Z}/p)_A$$

$$\oplus\ \beta Q^{p^t(p-1)}[H_{2(p-1)^2 p^t-1}(X;\mathbb{Z}/p)_A].$$

β is the mod p Bockstein homomorphism, $Q^{p^t(p-1)}$ is the mod p Dyer-Lashof operation, and the rest of the symbols are as in Theorem 4.1.

We emphasize that no homotopical application of our results is attempted in this work.

CHAPTER 1

H*-SPACES

We give here a brief description of the properties of the spaces on which Dyer-Lashof operations are defined. The general reference is [D-L]. Let Σ_p be the group of permutations of p symbols, and form the n-th iterated join of Σ_p with itself, $J^n\Sigma_p$. A point of $J^n\Sigma_p$ is determined by a sequence $t_0, t_1 \ldots t_{n-1}$ of real numbers such that $t_i \geq 0$ and $t_0 + \ldots + t_{n-1} = 1$, and by an element $\sigma_i \in \Sigma_p$ for each $t_i \neq 0$. The points of $J^n\Sigma_p$ are denoted $t_0\sigma_0 \oplus t_1\sigma_1 \oplus \ldots \oplus t_{n-1}\sigma_{n-1}$.

Definition 1.1. A space X is an H_p^n-space, $n \geq 0$, if it is an associative H-space with unit e and in addition there is a map

$$\theta_p^n : J^{n+1}\Sigma_p \times X^p \to X$$

which has the following properties:

a) (Σ_p-equivariant) for each $\sigma \in \Sigma_p$,

$$\theta_p^n (t_0\sigma_0 \oplus \ldots \oplus t_n\sigma_n; x_1, \ldots, x_p) =$$

$$\theta_p^n (t_0\sigma_0\sigma^{-1} \oplus \ldots \oplus t_n\sigma_n\sigma^{-1}; x_{\sigma(1)}, \ldots, x_{\sigma(p)}),$$

b) (Normalized) for each $\sigma \in \Sigma_p$

$$\theta_p^n (0 \oplus \ldots \oplus 0 \oplus 1 \cdot \sigma; x_1, \ldots, x_p) = x_{\sigma(1)} \ldots x_{\sigma(p)}.$$

Define the inclusion $J^n\Sigma_p \subset J^{n+1}\Sigma_p$ by

$$(t_0\sigma_0 \oplus \ldots \oplus t_{n-1}\sigma_{n-1}) \to (0 \oplus t_0\sigma_0 \oplus \ldots \oplus t_{n-1}\sigma_{n-1}) .$$

Let $J^\infty\Sigma_p = \varinjlim_n J^n\Sigma_p$.

Definition 1.2. A space X is an H_p^∞-space if there is a map $\theta_p^\infty : J^\infty\Sigma_p \times X^p \to X$ such that, for each n, $\theta_p^\infty \mid J^{n+1}\Sigma_p \times X^p$ makes X an H_p^n-space.

Araki and Kudo proved that an (n+1) - st loop space is an H_2^n-space [AR-K].

Definition 1.3. A space X is a special H_p^n-space if it is an H_p^n-space, and in addition θ_p^n is projective, i.e.,

$$\theta_p^n(w,e, \ldots, e, \overset{i}{x}, e, \ldots, e) = x, \text{ for } 1 \le i \le p,$$

$$w \in J^{n+1}\Sigma_p,\ x \in X.$$

The next result is fundamental in the theory of H_p^n-spaces.

Theorem 1.4. [D-L, Th. 1.1] An H-space is a special H_p^0-space for all p. If X is a special H_p^n-space, then ΩX, the loop space of X, is a special H_p^{n+1}-space. So an (n+1)-st-loop space is an H_p^n-space for all p.

(1.5) A map $f : X \to \overline{X}$ of H_p^n-spaces is called an H_p^n-map if the following diagram commutes up to a Σ_p-equivariant homotopy:

$$\begin{array}{ccc} J^{n+1}\Sigma_p \times X^p & \xrightarrow{\theta^n_p} & X \\ \downarrow 1 \times f^p & & \downarrow \\ J^{n+1}\Sigma_p \times \overline{X}^p & \xrightarrow{\overline{\theta}^n_p} & \overline{X} \end{array}$$

In particular if $f : \Omega^{n+1}Y \to \Omega^{n+1}\overline{Y}$ and if f is the n-th loops of a map $g : Y \to \overline{Y}$, then f is an H^n_p-map.

We mention an additional property satisfied by the iterated loop spaces.

(1.6) Let $\Sigma_m \int \Sigma_\ell$ be the wreath product of Σ_m by Σ_ℓ.

There is a $\Sigma_m \int \Sigma_\ell$ equivariant map $\psi : J^{s+1}\Sigma_\ell \times (J^{r+1}\Sigma_m)^\ell \to J^{t+1}\Sigma_{m\ell}$, $t = s + r\ell$. Moreover ψ is a simplicial map when the spaces above are suitably triangulated. (See [D-L] p. 38).

Definition 1.7. An H^t-space is an H-space X which is an H^t_p-space for all positive integers p, and if $t = s + r\ell$, then the following diagram commutes up to a $\Sigma_m \int \Sigma_\ell$-equivariant homotopy:

$$(1.9)\qquad \begin{array}{ccc} J^{s+1}\Sigma_\ell \times (J^{r+1}\Sigma_m \times X^m)^\ell & \xrightarrow{1 \times (\theta^r_m)^\ell} & J^{s+1}\Sigma_\ell \times X^\ell \\ \downarrow \overline{\psi} & & \downarrow \theta^s_\ell \\ J^{t+1}\Sigma_{m\ell} \times X^{m\ell} & \xrightarrow{\theta^t_{m\ell}} & X \end{array}$$

Here $\overline{\psi} : J^{s+1}\Sigma_\ell \times (J^{r+1}\Sigma_m \times X^{r+1})^\ell \to J^{s+1}\Sigma_\ell \times (J^{r+1}\Sigma_m)^\ell \times X^{m\ell}$

$$\xrightarrow{\psi \times 1} J^{t+1}\Sigma_{m\ell} \times X^{m\ell} .$$

Theorem 1.8. ([D-L], Th. 1.2) Every $(t+1)$-st loop space is an H^t-space.

Remark 1.10. The existence of diagrams like 1.9 led to the formulation and proof of Adem relations for Dyer-Lashof operations by P. May [Ma 2].

Remark 1.11. The notion of H^t-space has been generalized to that of C_t-space, in the sense of May, [Ma 4]. In May's approach the join $J^n\Sigma_p$ is replaced by the little cubes operad ([B-V], [Ma 4]), and the resulting theory has an enormous richness which however we will not use in Part 1 of this work.

CHAPTER 2

DYER-LASHOF OPERATIONS

We give an outline of the definition of the Dyer-Lashof operations in the manner of [D-L].

Let A be a space and $C(A)$ the normalized cubical chains of A. If Σ_p acts on A then it acts on $C(A)$. Let X be an H_p^n-space and $\theta = \theta_p^n : J^{n+1}\Sigma_p \times X^p \to X$ be the map of Def. 1.1. Then $\theta : C(J^{n+1}\Sigma_p \times X^p) \to C(X)$ is Σ_p-equivariant chain map. Let $h : C(J^{n+1}\Sigma_p) \otimes C(X^p) \to C(J^{n+1}\Sigma_p \times X^p)$ be the Eilenberg-Zilber chain equivalence. Then $\theta \cdot h : C(J^{n+1}\Sigma_p) \otimes C(X^p) \to C(X)$ is equivariant. Moreover, if $\pi \subset \Sigma_p$ and $J^{n+1}\pi \subset J^{n+1}\Sigma_p$ is the canonical inclusion we have:

(2.1) For π a subgroup of Σ_p, the equivariant map $\theta : J^{n+1}\pi \times X^p \to X$ induces an equivariant chain group homomorphism $\theta \cdot h : C(J^{n+1}\pi) \otimes C(X^p) \to C(X)$ on normalized cubical chains ([D-L], p. 41).

The following result is the basic fact for the definition of the Dyer-Lashof operations, (see [D-L], p. 43):

Proposition 2.2. Let W be any π-free chain complex, where $\pi \subset \Sigma_p$. Given a π-equivariant chain map $t : W \to C(J^{n+1}\pi)$, there exists a natural π-equivariant chain map $F : W \otimes C(X)^p \to C(J_\pi^{n+1}) \otimes C(X^p)$ such that

$F(w \otimes x_1 \otimes \ldots \otimes x_p) = t(w) \otimes (x_1, \ldots, x_p)$, where $\dim(x_j) = 0$, $j = 1, \ldots, p$. Moreover $F(w^{(i)} \otimes C(X)^p) \subset C(J^{n+1}\pi)^{(i)} \otimes C(X^p)$, where the super-index denotes i-skeleton. Also, if $H_0(W) = \mathbb{Z}$, $H_i(W) = 0$ for $0 < i < n$, then there is a map

$$G : C(J^{n+1}\pi)^{(n-1)} \otimes C(X^p) \to W \otimes C(X)^p \text{ so that}$$

$$GF : W^{(n-1)} \otimes C(X)^p \to W \otimes C(X)^p \text{ and}$$

$$FG : C(J^{n+1}\pi)^{(n-1)} \otimes C(X^p) \to C(J^{n+1}\pi) \otimes C(X^p)$$ are equivariantly homotopic to the respective inclusions. Furthermore, if F and F^1 are obtained as above from equivariantly chain homotopic maps t and t^1, then F and F^1 are equivariantly chain homotopic. If t is an equivariant chain equivalence, so is F.

Proposition 2.3. (See Lemma 2.4 of [D-L].) Suppose p is a prime, then

$$H(W^{(n)} \otimes_\pi C(X)^p; \mathbb{Z}/p) \cong H(W^{(n)} \otimes_\pi H(X; \mathbb{Z}/p)^p)$$

(2.4) Let p be an odd prime, π the cyclic group of order p, generated by α. Take W to be the complex

$$\ldots \to \mathbb{Z} \xrightarrow{\Gamma} \mathbb{Z}\pi \xrightarrow{\Delta} \mathbb{Z}\pi \to \ldots \to \mathbb{Z}\pi \xrightarrow{\Gamma} \mathbb{Z}\pi \xrightarrow{\Delta} \mathbb{Z}\pi \xrightarrow{E} \mathbb{Z}.$$

So W_i has a single generator e_i, $i \geq 0$, and $\partial e_{2i+1} = \Delta e_{2i}$, $\partial e_{2i+2} = \Gamma e_{2i+1}$, $\varepsilon e_1 = 1$, where $\Delta = \alpha - 1$, $\Gamma = 1 + \partial + \ldots + \partial^{p-1}$.

(2.5) Let π be the cyclic group on p elements, for p an odd prime, and let W be the π-free resolution of $\mathbb{Z}$ above. Suppose $K = H(X; \mathbb{Z}/p)$ is a mod p chain complex with trivial boundary. Then if $x_1, x_2, \ldots$ is a vector space basis of homogeneous elements for K, the homology classes of the following cycles form a vector space basis for $H(W \otimes_\pi K^p)$:

$e_i \otimes x_j^p$, for $i > 0$, all j.

$e_0 \otimes x_{j_1} \otimes \ldots \otimes x_{j_p}$, modulo cyclic permutation of the indices $j_1, \ldots, j_p$.

Definition 2.6. Let X be an H_p^n-space, and $x \in H_j(X; \mathbb{Z}/p)$. For $0 \le i \le n$ define $Q_i(x) \in H_{pj+i}(X; \mathbb{Z}/p)$ by $Q_i(x) = \theta h F(e_i \otimes_\pi x^p)$. Here $e_i \otimes x^p$ is the homology class in $H(W^{(n)} \otimes_\pi C(X)^p; \mathbb{Z}/p)$ represented by the cycle $e_i \otimes x^p$.

Since the definition of Q_n involves the choice of the equivariant map $t : W \to C(J^{n+1}\pi)$ we remark that a canonical equivariant chain map $t : W \to J^\infty \mathbb{Z}/p$ exists, which is defined in Lemma 2.5 of [D-L], and that if the restrictions $t^{(n)} : W^{(n)} \to C(J^{n+1}\mathbb{Z}/p)$ of t are used to define the maps F of 2.2, then we have uniquely defined operations Q_n in an H_p^∞-space.

Assume from now on that X is an H_p^∞-space, p odd prime.

Remark 2.7. The definition of Q_n is induced by the top row in the following commutative diagram

$$\begin{array}{ccccc} W^{(n)} \otimes C(X)^p & \xrightarrow{t \otimes 1^p} & C(J^{n+1}\mathbb{Z}/p) \otimes C(X)^p & \xrightarrow{\theta \cdot F} & C(X) \\ & & \downarrow{\scriptstyle j \otimes 1^p} & \nearrow{\scriptstyle \theta \cdot \overline{F}} & \\ & & C(J^{n+1}\Sigma_p) \otimes C(X)^p & & \end{array}$$

Thus we see that the homology of Σ_p plays a role in the definition of Q_n. Specifically, in [Ma 2, Def. 2.2], it is proved that $Q_i(x) = 0$ unless either:

1) deg (x) is even and $i = 2t(p-1)$ or $i = 2t(p-1) - 1$, or

2) deg(x) is odd and $i = 2(t+1)(p-1)$ or $i = 2(t+1)(p-1) - 1$.

Also, $\beta Q_{2i}(x) = Q_{2i-1}(x)$, β the Bockstein homomorphism induced by the sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$.

Definition 2.8. (See [Ma 2], Def. 2.2). $Q^s : H_q(X; \mathbb{Z}/p) \to H_{q+2s(p-1)}(X; \mathbb{Z}/p)$ is given by

i) $Q^s(x) = 0$ if $2s < q$, and

ii) $Q^s(x) = (-1)^s v(q)\, Q_{(2s-q)(p-1)}(x)$, if $2s \geq q$.

Moreover $\beta Q^s : H_q(X; \mathbb{Z}/p) \to H_{q+2s(p-1)-1}(X; \mathbb{Z}/p)$ is such that

iii) $\beta Q^s(x) = 0$ if $2s \leq q$, and

iv) $\beta Q^s(x) = (-1)^s v(q)\, Q_{(2s-q)(p-1)-1}(x)$, if $2s > q$.

Here $v(q) = (-1)^{q(q-1)m/2}(m!)^q$, with $m = \frac{p-1}{2}$.

CHAPTER 3

DYER-LASHOF OPERATIONS FOR AN ODD PRIME p, AND $H_*(QX; \mathbb{Z}/p)$

The proofs of Theorems 4.1 and 5.1 consist essentially of an extensive use of the Nishida relations satisfied by the Dyer-Lashof operations in homology mod p, for p an odd prime. An extensive treatment of the theory and applications of the Dyer-Lashof operations is given in [Co-La-Ma].

The following theorem collects the properties of the Dyer-Lashof operations which we will make use of. Homology is taken with odd prime coefficients, $\mathbb{Z}/p$.

Theorem 3.1. (See [Co-La, Ma], Part I, Theorem 1.1.) Let B be an infinite loop space. There exist natural homomorphisms $Q^i : H_*(B) \to H_*(B)$, $i \geq 0$, of degree $2i(p-1)$. These homomorphisms satisfy the following properties.

1) The Q^s are natural with respect to infinite loop maps.

2) $Q^0(e) = e$, and $Q^i(e) = 0$ for $i > 0$, where $e \in H_0(B)$ is the identity of the loop product of $H_*(B)$.

3) $Q^i x = 0$ if $2i < \deg x$.

4) $Q^i x = x^p$ if $2i = \deg x$.

5) (Cartan formula)

$$Q^s(xy) = \sum_{i=0}^{s} Q^i(x)Q^{s-i}(y).$$

If $\psi(x) = \Sigma\, x' \otimes x''$, $\psi(Q^s(x)) = \sum_{i=0}^{s} Q^i(x') \otimes Q^{s-i}(x'')$.

6) The Q^s are stable: $Q^s\sigma_* = \sigma_* Q^s$, where $\sigma_* : \tilde{H}_*(\Omega X, \mathbb{Z}/p) \to H_*(X)$ is the homology suspension.

7) (Adem relations) If $a > pb$, then

$$Q^a Q^b = \sum_i (-1)^{a+i} \binom{(p-1)i - (p-1)b - 1}{pi - a} Q^{a+b-i} Q^i \;,$$

and if $a \geq pb$, and β is the mod p Bockstein, then

$$Q^a \beta Q^b = \sum_i (1)^{a+i} \binom{(i-b)(p-1)}{pi-a} \beta Q^{a+b-i} Q^i$$

$$- \sum_i (-1)^{a+i} \binom{(i-b)(p-1)-1}{pi-a-1} Q^{a+b-i} \beta Q^i$$

8) (Nishida relations) Let $P^s_* : H_*(B) \to H_*(B)$ of degree $-2s(p-1)$, be dual to the Steenrod power P^s. Then

$$P^s_* Q^r = \sum_i (-1)^{i+s} \binom{(r-s)(p-1)}{s-pi} Q^{r-s+i} P^i_* \;.$$

$$P^s_* \beta Q^r = \sum_i (-1)^{i+s} \binom{(r-s)(p-1)-1}{s-pi} \beta Q^{r-s+i} P^i_*$$

$$+ \sum_i (-1)^{i+s} \binom{(r-s)(p-1)-1}{s-pi-1} Q^{r-s+i} P^i_* \beta \;.$$

Definition 3.2. (See [Co-La-Ma], Part 1, Def. 2.2.) Let F be the free associative algebra generated by $\{Q^s, \beta Q^{s+1} \mid s \geq 0\}$, and let J be the two sided ideal consisting of all elements of F which annihilate every homology class of every infinite loop space. Then the Dyer-Lashof algebra R is defined as F/J.

The following notions are useful in describing the Dyer-Lashof algebra mod p, p odd prime.

Definition 3.3. (See [Co-La-Ma], Part 1, Def. 2.1.) Let I be a sequence $(\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k)$ where $\varepsilon_j = 0,1$ and $s_j \geq \varepsilon_j$. Define

degree of I = $d(I) = \sum_{j=1}^{k} [2s_j(p-1) - \varepsilon_j]$

length of I = $\ell(I) = k$.

$$\text{excess of I} = e(I) = 2s_k - \varepsilon_1 - \sum_{j=2}^{k} [2ps_j - \varepsilon_j - 2s_{j-1}]$$

$$= 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} [2s_j(p-1) - \varepsilon_j]$$

Each I as above determines the composite $Q^I = \beta^{\varepsilon_1} Q^{s_1} \ldots \beta^{\varepsilon_k} Q^{s_k} \in R$. I is admissible if $ps_j - \varepsilon_j \geq s_{j-1}$, for $2 \leq j \leq k$. The empty sequence I is admissible by convention and it satisfies $d(I) = 0$, $\ell(I) = 0$ and $e(I) = \infty$. It determines $Q^I = 1 \in R$.

Theorem 3.4. (See [Co-La-Ma], Part 1, Theorem 2.3.)

a) In the odd primary Dyer-Lashof algebra the ideal J of F is generated by

i) the Adem relations;

ii) the relations obtained by applying β to the Adem relations

iii) the relations $Q^I = 0$ if $e(I) < 0$.

b) R has $\mathbb{Z}/p$ basis $\{Q^I \mid I \text{ is admissible}, e(I) \geq 0\}$.

c) R admits a Hopf algebra structure with coproduct given on generators by

$$\psi(Q^s) = \sum_{i+j=s} Q^i \otimes Q^j,$$

$$\psi(\beta Q^{s+1}) = \sum_{i+j=s} [\beta Q^{i+1} \otimes Q^j + Q^i \otimes \beta Q^{j+1}]$$

$$\psi(\beta) = \beta \otimes 1 + 1 \otimes \beta.$$

d) R admits a structure of left coalgebra over A^0, the opposite Hopf algebra of the Steenrod algebra, and the action of A^0 on R is given by the Nishida relations (3.1.8).

Next we quote the theorem of [D-L] which determines the $\mathbb{Z}/p$-homology of QX in terms of the Dyer-Lashof operations. In this theorem X is assumed to be connected, which is the case we will be dealing with in Chapters 4 and 5.

Theorem 3.5. (See Th. 5.1 of [D-L].) If X is connected, and p is an odd prime, then $H_*(QX; \mathbb{Z}/p)$ is isomorphic to the free commutative associative graded algebra generated by the monomials $\beta^{\varepsilon_1} Q^{r_1} \ldots \beta^{\varepsilon_s} Q^{r_s} x_\alpha$, where the sequence $(\varepsilon_1, r_1, \ldots, \varepsilon_s, r_s)$ is admissible (see 3.3), and $\{x_\alpha\}$ is a vector space basis of $H_*(X; \mathbb{Z}/p) \subset H_*(QX; \mathbb{Z}/p)$.

Remark 3.6. This theorem has been generalized by May ([Ma 3], [Ma 5] and [Co-La-Ma]) to the case when X is not connected, and to a wider class of spaces than that of infinite loop spaces. However, we will have no need of the advantages of May's approach to the theory of infinite loop spaces in Part I of this work.

CHAPTER 4

PROOF OF THE MAIN RESULT

In this chapter we prove the main theorem of Part I of this work, which was stated in the Introduction. Later, by slightly modifying the arguments in the proof of Theorem 4.1 we will establish Theorem 5.1 which deals with the degrees of some well known elements in the stable homotopy of the spheres.

Theorem 4.1. Let X be any connected CW-complex and let p be any odd prime. Then

$$\underline{Q}\, H_m(QX; \mathbb{Z}/p)_A = H_m(X; \mathbb{Z}/p)_A$$

whenever the p-adic expansion of m satisfies the following conditions:

1) If $m = 2k$, then $m = a_0 + a_1 p + \ldots + a_n p^n$ with either
   a) $a_0 \neq p-2$; $a_0 \neq p-1$ or $a_1 \neq p-1$, while $a_i \neq p-1$ and $a_i \neq p-2$ for $i \geq 2$; or
   b) $a_i = p-1$ for all i.
2) If $m = 2k+1$, then $m = a_0 + a_1 p + \ldots + a_n p^n$ with either
   a) $a_0 \neq p-3$ or $a_1 \neq p-1$; while $a_i \neq p-1$ and $a_i \neq p-2$ for $i \geq 2$; or
   b) $a_i = p-1$ for all i.
3) Either a) $m \leq p^n + (p-2)p^{n-1}$ or
   b) $m \geq p^n + (p-1)(p^{n-1} + \ldots + 1)$

Since the proof of Theorem 4.1 is quite long we will first give some indications of the procedure we follow.

(4.2) Given $y \in \underline{Q}\, H_m(QX; \mathbb{Z}/p)_A$ express y as a sum of non-trivial admissible, indecomposable monomials:

$$(4.3) \qquad y = \sum_{\underline{r}} \beta Q^{r_1} \ldots \beta^{\varepsilon_s} Q^{r_s} (x(\underline{r}))$$

(See Def. 3.3 and Theorems 3.1 and 3.4.)

By assumption y is A-annihilated, so that $P_*^t(y) = 0$, $t \geq 1$. On the other side of (4.3), the Nishida relations (3.1.(8)) provide an expansion for $P_*^t(\beta Q^{r_1} \ldots \beta^{\varepsilon_s} Q^{r_s}(x(\underline{r})))$, and in this expansion the monomials are all of the same length s (see 3.3). Consequently we can assume that the monomials in the expansion (4.3) of y are all of the same length.

Notice that by the Adem relations of A, y is A-annihilated if and only if $P_*^{p^j}(y) = 0$, $j \geq 0$, so that we will center our analysis on this condition of y. As a result of this analysis we will conclude that for each monomial $\beta Q^{r_2} \ldots \beta^{\varepsilon_s} Q^{r_s}(x(\underline{r}))$ the admissible sequence $(1, r_1, \ldots, \varepsilon_s, r_s)$ has $\varepsilon_k = 1$, $1 \leq k \leq s$, and it also satisfies, simultaneously, certain properties H(j), $j \geq 0$.

The simultaneous validity of the properties H(j), $j \geq 0$, for each monomial $\beta Q^{r_1} \ldots \beta^{\varepsilon_s} Q^{r_s} x(\underline{r})$ in (4.3) will be established by induction on s, the length of the monomials.

Properties H(j), $j \geq 0$, will force the leading operation Q^{r_1} to be of too large a degree to be present in the monomial considered, thus implying that the monomials in (4.3) have length smaller than s. Another induction on s will imply that there cannot be operations in (4.3), proving Theorem 4.1. We now formulate the conditions H(j), $j \geq 0$, on the monomial $\beta Q^{r_1} \ldots \beta^{\varepsilon_s} Q^{r_s}(x(\underline{r}))$.

Definition 4.4. The monomial $\beta Q^{r_1} \ldots \beta^{\varepsilon_s} Q^{r_s}(x(\underline{r}))$ satisfies $H(j)$, $j \geq 0$, if $\varepsilon_k = 1$, $1 \leq k \leq s$, and

a) If $j < s$, $r_t \equiv 0$, mod p^{j-t+2}, $t = 1, \ldots, j+1$.

b) If $s \leq j$, $r_t \equiv 0$, mod p^{j-t+2}, $t = 1, \ldots, s$, $P_*^{p^m}(x(\underline{r})) = 0$, $m = 0, \ldots, j-s$.

c) $\beta(x(\underline{r})) = 0$.

Conditions $H(j)$ are suggested by the following observation. Consider $y = \beta Q^{r_1}(x(\underline{r})) \in Q H_m(QX, \mathbb{Z}/p)_A$, and look at the Nishida expansion

$$P_*^{p^j}(y) = \sum_i (-1)^{i+j} \binom{(r_1-p^j)(p-1)-1}{p^j - pi} \beta Q^{r_1-p^j+i} P_*^i(x(\underline{r}))$$

$$+ \sum (-1)^{i+j} \binom{(r_1-p^j)(p-1)-1}{p^j-pi-1} Q^{r_1-p^j+i} P_*^i \beta(x(\underline{r}))$$

Suppose we can show that the non-trivial monomials in this expansion (if there are any) are indecomposable. Then equation $P_*^{p^j}(y) = 0$ will force the binomial coefficients to be zero modulo p. In particular the first binomial coefficient $(-1)^j \binom{(r_1-p^j)(p-1)-1}{p^j}$ must be zero modulo p. This in turn imposes certain conditions on the p-adic expansion of r_1. Clearly, if $j = 0$, and if $\binom{(r_1-1)(p-1)-1}{1} \equiv 0$, modulo p, then necessarily $r_1 \equiv 0$, modulo p. On the other hand, for $Q^{r_1-1}\beta(x(\underline{r}))$ to be zero, $\beta x(\underline{r}) = 0$. These two facts are precisely condition $H(0)$ for $\beta Q^{r_1} x(\underline{r})$. Properties $H(j)$, $j \geq 0$ will be shown to hold for monomials of arbitrary length in Lemma 4.41. The following lemma will considerably shorten our analysis of the equation $P^t_*(y) = 0$.

Lemma 4.5. If $\beta Q^{r_1} \ldots \beta Q^{r_s}(x(\underline{r}))$ satisfies $H(j)$, $s > 0$, then $\beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r}))$ satisfies $H(j-1)$.

Proof. Replace j by j-1, t by t-1 and s by s-1 in 4.4. Since r_t becomes r_{t-1} when we suppress βQ^{r_1}, the congruences of Def. 4.4, a), b) remain valid after that substitution and give $H(j-1)$ for $\beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r}))$.

We next show how the properties $H(j)$ decrease the number of terms in the Nishida expansion of $P^t_*(\beta Q^{r_1} \ldots \beta Q^{r_s}(x(\underline{r}))$.

Lemma 4.6. Suppose $y = \beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r}))$ has property $H(j)$; then $P^t_*(y) = 0$ for $0 < t < p^{j+1}$.

Proof. By Adem relations in A, it suffices to prove the statement for $t = p^\ell$, $0 \le \ell \le p^j$. We induct on $s \ge 0$, the length of y. If $s = 0$, then 4.4, b) gives the assertion for $j \ge 0$.

Assume $s > 0$ and that the lemma is true for $q < s$. Then for $0 \le \ell \le j$, the Nishida relations (3.1.8) give

$$P^{p^\ell}_*(y) = (-1)^{p^\ell} \begin{pmatrix} (r_1-p\ )(p-1)-1 \\ p^\ell \end{pmatrix} \beta Q^{r_1-p^\ell} \beta Q^{r_2} \ldots Q^{r_s}(x(\underline{r}))$$

$$+ (-1)^{p^\ell} \begin{pmatrix} (r_1-p\ )(p-1)-1 \\ p^\ell-1 \end{pmatrix} Q^{r_1-p^\ell} \beta\beta \ldots Q^{r_s}(x(\underline{r}))$$

$$+\sum_{m=1}^{p^\ell-1}(-1)^{m+p^\ell}\begin{pmatrix}(r_1-p^\ell)(p-1)-1\\ p^\ell-pm\end{pmatrix}\beta Q^{r_1-p^\ell+m}P_*^m\beta Q^{r_2}\ldots Q^{r_s}x(\underline{r})$$

$$+\sum_{m=1}^{p^\ell-1}(-1)^{m+p^\ell}\begin{pmatrix}(r_1-p^\ell)(p-1)-1\\ p^\ell-pm-1\end{pmatrix}Q^{r_1-p^\ell+m}P_*^m\beta\beta\ldots Q^{r_s}x(\underline{r})$$

Notice that each summand in the second and fourth lines of the Nishida expansion above involve $\beta\beta = 0$ and so they do not contribute at all. The third line is also trivial since $\beta Q^{r_2}\ldots\beta Q^{r_s}(x(\underline{r}))$ satisfies $H(j-1)$, by 4.5, and so the induction hypothesis for $s-1$ and $0 \le \ell - 1 \le j-1$ applies. It remains only to consider the first term

$$(-1)^{p^\ell}\begin{pmatrix}(r_2-p^\ell)(p-1)-1\\ p^\ell\end{pmatrix}\beta Q^{r_1-p^\ell}\beta Q^{r_2}\ldots\beta Q^{r_s}(x(\underline{r})).$$

We have, by $H(j)$, that $r_1 \equiv 0, \pmod{p^{j+1}}$; so $r_1 = a_hp^h + a_{h+1}p^{h+1} + \ldots +$, $a_h > 0$, is the p-adic expansion of r_1, with $h \ge j+1$. Then

$$(4.7)\qquad (r_1-p^\ell)(p-1)-1 = [(a_hp^h + a_{h+1}p^{h+1}+\ldots) - p^\ell](p-1)-1$$

$$= [p^\ell(a_hp^{h-\ell}-1) + a_{h+1}p^{h+1}+\ldots](p-1)-1,$$

and

$$a_hp^{h-\ell}-1 = (a_h-1)p^{h-\ell} + (p^{h-\ell}-1) = (a_h-1)p^{h-\ell}$$

$$+ (p^{h-\ell-1}+\ldots+1)(p-1).$$

Substituting this in (4.7) we get

$$(p^\ell[(a_h-1)p^{h-\ell}+(p^{h-\ell-1}+\ldots+1)(p-1)] + a_{h+1}p^{h+1}+\ldots)(p-1)-1$$

$$= Mp^k + p^\ell - 1 = Mp^k + (p-1)(p^{\ell-1}+\ldots+1),\ k > \ell.$$

Thus p^ℓ has coefficient zero in the p-adic expansion of $(r_1 - p^\ell)(p - 1) - 1$, proving

$$\binom{(r_1 - p^\ell)(p - 1) - 1}{p^\ell} \equiv 0, \text{ mod } p.$$

This completes the proof of the lemma.

Next we exhibit the possibly non-trivial terms in the Nishida expansion of $P_*^{p^j}(y)$, for suitable y.

Lemma 4.8. Let $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ satisfy $H(j-1)$. Then

$$P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^j} \beta Q^{r_1-p^j} \ldots \beta Q^{r_s}(x(\underline{r}))$$

$$+ \beta Q^{r_1-p^j+p^{j-1}} P_*^{p^{j-1}} \beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r}))$$

Proof. By the Nishida relations (3.1.8),

$$P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^j} \beta Q^{r_1-p^j} \beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r}))$$

$$+ \sum_{0<t}^{p^{j-1}} (-1)^{p^j+t} \binom{(r_1-p^j)(p-1)-1}{p^j-pt} \beta Q^{r_1-p^j+t} P_*^t \beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r}))$$

$$+ \sum_{t=0}^{p^{j-1}} (-1)^{p^j+t} \binom{(r_1-p^j)(p-1)-1}{p^j-pt-1} \beta Q^{r_1-p^j+t} P_*^t \beta\beta \ldots (x(\underline{r})),$$

The last sum is clearly zero, since $\beta^2 = 0$.

By 4.5 $\beta Q^{r_s} \ldots \beta Q^{r_s} x(\underline{r})$ satisfies $H(j-2)$ and so, by 4.6, $P_*^t(\beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r}))) = 0$, $0 < t < p^{j-1}$. We see that only the last term of the second sum survives, together with the first term.

We now explicitly determine $P_*^{p^j}(y)$, y a monomial as in 4.8, by the iterated application of the Nishida relations.

Lemma 4.9. Suppose $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ has property $H(j-1)$, with $s > j$. Then

$$P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^j} \beta Q^{r_1-p^j} \beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r})$$

$$+ \sum_{u=2}^{j+1} (-1) \binom{(r_u-p^{j-u+1})(p-1)-1}{p^{j-u+1}} \beta Q^{r_1-p^j+p^{j-1}} \beta Q^{r_2-p^{j-1}+p^{j-2}} \ldots \beta Q^{r_{u-1}-p^{j-u+2}+p^{j-u+1}} \beta Q^{r_u-p^{j-u+1}} \beta Q^{r_{u+1}} \ldots \beta Q^{r_s} x(\underline{r}).$$

Proof. We use induction on the lengths of y, $s \geq 1$.

If $s = 1$, then $j = 0$ and only the first term is present. This is immediately seen from the coefficients in Nishida expansion and by the use of $H(0)$, (part c) of 4.4). Suppose now that $s > 1$, and that the lemma is true for monomials of type $\beta Q^{r_1} \ldots \beta Q^{r_q} x(\underline{r})$, $q < s$, which satisfy $H(j-1)$, $j < q$. Consider now $y = \beta Q^{r_1} \ldots \beta Q^{r_s}(x(\underline{r}))$, satisfying $H(j-1)$, with $j < s$. Applying 4.8 we get

29

$$P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^j}\, \beta Q^{r_1-p^j} \cdots \beta Q^{r_s} x(\underline{r})$$

$$+\ \beta Q^{r_1-p^j+p^{j-1}}\, P_*^{p^{j-1}}\, \beta Q^{r_2} \cdots \beta Q^{r_s} x(\underline{r})$$

We will use the induction hypothesis on the second summand, since $\beta Q^{r_2} \cdots \beta Q^{r_s} x(\underline{r})$ has length s-1 and satisfies H(j-2), by 4.2.

Let $\eta \in \{1,\ldots, s-1\}$ and define $r'_\eta = r_{\eta+1}$, so that $\beta Q^{r_2} \cdots \beta Q^{r_s} x(\underline{r}) = \beta Q^{r'_1} \cdots \beta Q^{r'_{s-1}} x(\underline{r})$. The induction assumption gives

$$P_*^{p^{j-1}}(\beta Q^{r'_1} \cdots \beta Q^{r'_{s-1}} x(\underline{r}))$$

$$= (-1)\binom{(r'_1-p^{j-1})(p-1)-1}{p^{j-1}}\, \beta Q^{r'_1-p^{j-1}}\, \beta Q^{r'_2} \cdots x(\underline{r})$$

$$+ \sum_{u=2}^{j} (-1)\binom{(r'_u-p^{j-1-u+1})(p-1)-1}{p^{j-1-u+1}}\, \beta Q^{r'_1-p^{j-1}+p^{j-2}} \cdots \beta Q^{r'_{u-1}-p^{j-1-u+2}+p^{j-1-u+1}}\, \beta Q^{r'_u-p^{j-1-u+1}}\, \beta Q^{r'_{u+1}} \cdots \beta Q^{r'_{s-1}} x(\underline{r})$$

Replacing r'_u by r_{u+1} and u by u+1 in this formula we obtain

$$P_*^{p^{j-1}}(\beta Q^{r_2}\ldots\beta Q^{r_s}x(\underline{r}))$$

$$= (-1)\binom{(r_2 - p^{j-1})(p-1)-1}{p^{j-1}} \beta Q^{r_2-p^{j-1}} \beta Q^{r_3}\ldots x(\underline{r})$$

$$+ \sum_{u+1=3}^{j+1} (-1)\binom{(r_{u+1} - p^{j-(u+1)+1})(p-1)-1}{p^{j-(u+1)+1}} \beta Q^{r_2-p^{j-1}+p^{j-2}} \beta Q^{r_u-p^{j-(u+1)+2}+p^{j-(u+1)+1}} \beta Q^{r_{u+1}-p^{j-(u+1)+1}} \beta Q^{r_{u+2}} \ldots \beta Q^{r_s}x(\underline{r})$$

Substitution of this expression for $P_*^{p^{j-1}}(\beta Q^{r_2}\ldots\beta Q^{r_s}x(\underline{r}))$ in $\beta Q^{r_1-p^j+p^{j-1}} P_*^{p^{j-1}} \beta Q^{r_2}\ldots\beta Q^{r_s}(\underline{x}(r))$ gives the lemma.

The next lemma is entirely analogous to the previous one and covers the case when $s \leq j$.

Lemma 4.10. Let $y = \beta Q^{r_1}\ldots\beta Q^{r_s}(x(\underline{r}))$ satisfy property $H(j-1)$, and let $j \geq s \geq 0$. Then

$$P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^{j-u+1}} \beta Q^{r_1-p^j} \beta Q^{r_2}\ldots\beta Q^{r_s}x(\underline{r})$$

$$+ \sum_{u=2}^{s} (-1)\binom{(r_u-p^{j-u+1})(p-1)-1}{p^{j-u+1}} \beta Q^{r_1-p^j+p^{j-1}} \beta Q^{r_2-p^{j-1}+p^{j-2}} \beta Q^{r_{u-1}-p^{j-u+2}+p^{j-u+1}} \beta Q^{r_u-p^{j-u+1}} \beta Q^{r_{u+1}}\ldots\beta Q^{r_s}(x(\underline{r}))$$

$$+ \beta Q^{r_1-p^j+p^{j-1}} \ldots \beta Q^{r_s-p^{j-s+1}+p^{j-s}} P_*^{p^{j-s}}(x(\underline{r})).$$

Proof. We induct on $s \geq 0$.

If $s = 0$, only the last term appears, and (4.4, b) clearly gives the assertion. Let $s > 0$ and suppose the lemma true for monomials $\beta Q^{r_1} \ldots \beta Q^{r_q} x(\underline{r})$, $q < s$, which satisfy property $H(j-1)$, $j > q$. Consider $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ satisfying $H(j-1)$, $j > s$, and apply 4.8 to obtain

$$P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^j} \beta Q^{r_1-p^j} \beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r})$$

$$+ \beta Q^{r_1-p^j+p^{j-1}} P_*^{p^{j-1}} \beta Q^{r_2} \ldots x(\underline{r}).$$

With the notation of 4.9 we have by the induction hypothesis

$$P_*^{p^{j-1}}(\beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r}))$$

$$= (-1)\binom{(r'_1-p^{j-1})(p-1)-1}{p^{j-1}} \beta Q^{r'_1-p^{j-1}} \beta Q^{r'_2} \ldots \beta Q^{r'_{s-1}} x(\underline{r})$$

$$+ \sum_{u=2}^{s-1} (-1)\binom{(r'_u-p^{j-1-u+1})(p-1)-1}{p^{j-1-u+1}} \beta Q^{r'_1-p^{j-1}+p^{j-2}} \ldots \beta Q^{r'_{u-1}-p^{j-1-u+2}+p^{j-1-u+1}}$$

$$\beta Q^{r'_u-p^{j-1-u+1}} \beta Q^{r'_{u+1}} \ldots \beta Q^{r'_{s-1}}(x(\underline{r}))$$

$$+ \beta Q^{r'_1-p^{j-1}+p^{j-2}} \ldots \beta Q^{r'_{s-1}-p^{j-1-(s-1)+1}+p^{j-1-(s-1)}} P_*^{p^{j-1-(s-1)}}(x(\underline{r})).$$

Replacing r'_n by r'_{n+1} and u by u+1 we obtain

$$P_*^{p^{j-1}}(\beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r}))$$

$$= (-1)\binom{(r_2-p^{j-1})(p-1)-1}{p^{j-1}} \beta Q^{r_2-p^{j-1}} \beta Q^{r_3} \ldots \beta Q^{r_s} x(\underline{r})$$

$$+ \sum_{u+1=3}^{s} (-1)\binom{(r_{u+1}-p^{j-(u+1)+1})(p-1)-1}{p^{j-(u+1)+1}} \beta Q^{r_2-p^{j-1}+p^{j-2}} \ldots \beta Q^{r_u-p^{j-(u+1)+2}+p^{j-(u+1)+1}} \beta Q^{r_{u+1}-p^{j-(u+1)+1}} \beta Q^{r_{u+2}} \ldots \beta Q^{r_s}(x(\underline{r}))$$

$$+ \beta Q^{r_2-p^{j-1}+p^{j-2}} \ldots \beta Q^{r_s-p^{j-s+1}+p^{j-s}} P_*^{p^{j-s}}(x(\underline{r})).$$

If we replace $P_*^{j-1}(\beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r}))$ by this expression in $\beta Q^{r_1-p^j+p^{j-1}} P_*^{p^{j-1}}(\beta Q^{r_2} \ldots x(\underline{r}))$ we get the desired formula for $P_*^{p^j}(y)$.

As pointed out in section 4.2, our aim is to prove that conditions H(j) hold for monomials whose degree satisfy either 1) or 2) in the statement of Theorem 4.1, and to achieve this, we analyze the Nishida expansion of $P_*^{p^j}(y)$, y as above. Furthermore, Lemmas 4.9 and 4.10 provide us with a very simple expression for $P_*^{p^j}(y)$ in the case that y satisfies H(j-1).

Our next task is to show that Lemmas 4.9 and 4.10 are the best possible in the sense that, if y satisfies H(j-1), then the monomials appearing in 4.9 and 4.10 are admissible.

Lemma 4.11. If $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ is admissible and satisfies H(j-1), then each term in 4.9 and 4.10 is admissible.

Proof. We have by assumption that the sequence $(1, r_1, \ldots, 1, r_s)$ is admissible, i.e., $r_m \le pr_{m+1} - 1$, $m = 1, \ldots, s-1$. (See def. 3.3.) Then the sequence

$$(1, r_1 - p^j + p^{j-1}, \ldots, 1, r_{u-1} - p^{j-u+2} + p^{j-u+1}, r_u - p^{j-u+1}, 1,$$

$r_{u+1}, \ldots, 1, r_s)$ is admissible, since $r_m \le pr_{m+1} - 1$ implies that for $k \ge 1$, $r_1 - p^k + p^{k-1} \le pr_{k+1} - p^k + p^{k-1} - 1 = (p(r_{k+1} - p^{k-1} + p^{k-2}) - 1$.

It also implies that $r_k - p^k \le pr_{k+1} - 1$ for $k \ge 1$. It remains to verify that

$$(4.12) \quad r_k - p^{j-k+1} + p^{j-k} \le p(r_{k+1} - p^{j-k}) - 1, \ 1 \le k \le j.$$

We will prove it by contradiction. Suppose

$$(4.13) \quad r_k - p^{j-k+1} + p^{j-k} > p(r_{k+1} - p^{j-k}) - 1.$$

By H(j-1), $r_{k+1} \equiv 0 \pmod{p^{j-k}}$, and so $pr_{k+1} \equiv 0 \pmod{p^{j-k+1}}$. H(j-1) also implies that $r_k \equiv 0$, mod p^{j-k+1}. Notice that $r_k - p^{j-k+1} + p^{j-k} \le (pr_{k+1} - 1) - p^{j-k+1} + p^{j-k}$, and consequently, by use of 4.13, that $pr_{k+1} - p^{j-k+1} - 1 < r_k - p^{j-k+1} + p^{j-k} \le pr_{k+1} - p^{j-k+1} + p^{j-k} - 1$. These inequalities clearly imply that $pr_{k+1} - p^{j-k} - 1 + K = r_k$, K an integer, $0 < K < p^{j-k}$. Then H(j-1) gives $p^{j-k} - K + 1 \equiv 0$, mod p^{j-k+1}, which is impossible since $K < p^{j-k}$.

We have proved inequality 4.12 and thus completed the proof of the Lemma.

We will determine the maximal possible length of any indecomposable monomial $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ whose degree satisfies the conditions in Theorem 4.1.

Lemma 4.14. Let $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ be an indecomposable monomial, and let $\deg(y) = a_n p^n + \ldots + a_1 p + a_0$ satisfy either condition 1) or 2) of Theorem 4.1. Then length $(y) = s \leq n$.

Proof. If $s > 0$, notice that by connectedness $\deg(x(\underline{r})) \geq 1$. Also $\deg x(\underline{r}) < 2r_s$ since y is indecomposable. Then $\deg \beta Q^{r_s}(x(\underline{r})) \geq 1 + 2(p-1) - 1 = 2(p-1)$, and so $2r_{s-1} > 2(p-1)$ by the indecomposability of y. Thus $r_{s-1} \geq p$, which gives

$$\deg \beta Q^{r_{s-1}} \beta Q^{r_s}(x(\underline{r})) \geq 2p(p-1) + 2(p-1) - 1.$$

Indecomposability of y implies $2r_{s-2} \geq 2(p-1)p + 2(p-1) - 1$, and then $r_{s-2} \geq p^2 - 1$. Assume inductively that for $3 < s$, $m < k \leq s - 3$

$$(4.15) \qquad r_k \geq p^{s-k} - p^{s-k-2} - p^{s-k-4} - \ldots - p^{\varepsilon},$$

where $\varepsilon = 0$ if $s-k$ is even and $\varepsilon = 1$ if $s-k$ is odd. Then, since y is indecomposable,

$$(4.16) \qquad 2r_m > 2(p-1)(p^{s-(m+1)} - \ldots - p^{\varepsilon_0})$$
$$+ 2(p-1)(p^{s-(m+2)} - \ldots - p^{\varepsilon_1}) + \ldots$$
$$+ 2(p-1)(p^2-1) + 2(p-1)p + 2(p-1) - (s - m - 1)$$

where the last term accounts for the Bocksteins in

$y = \beta Q^{r_1} \ldots \beta Q^{r_m} \ldots \beta Q^{r_s} x(\underline{r})$, and it is $s - m - 1$ because

$2(p-1) \le \deg(\beta Q^{r_s} x(\underline{r}))$ as we saw above. Notice that if we exclude the last three terms in the right hand side of (4.16), the remaining sum satisfies the following condition:

$2p(p^{s-(m+t)} - \ldots - p^{\varepsilon_t})$, $t > 1$, cancels with a summand of the immediately preceding $2(p^{s-[m+(t-1)]} - \ldots - p^{\varepsilon_{t-1}})$. If $\varepsilon_0 = 0$ then inequality (4.16) becomes

$$2r_m > 2p(p^{s-m-1} - p^{s-m-3} - \ldots - p^{\varepsilon_0}) - 2p^2 + 2(p-1)p + 2(p-1) - (s-m-1) + 2\frac{(s-m-1)}{2}$$

$$= 2p(p^{s-m-1} - p^{s-m-3} - \ldots - p^{\varepsilon_0}) - 2,$$

i.e.,

$$r_m > p^{s-m} - p^{s-m-2} - \ldots - p - 1, \text{ and so}$$

$$r_m \ge p^{s-m} - p^{s-m-2} - \ldots - p.$$

On the other hand, if $\varepsilon_0 = 1$, then (4.16) becomes

$$2r_m > 2p(p^{s-m-1} - p^{s-m-3} - \ldots - p^{\varepsilon_0}) - 2p^2 + 2(p-1)p + 2(p-1) - (s-m-1) + s-m-2$$

$$= 2(p^{s-m} - \ldots - p^2) - 3, \text{ and then}$$

$$r_m \ge p^{s-m} - \ldots - p^2 - 1.$$

We have completed the downwards induction establishing (4.15).

We go back to determine an upper bound for s, the length of y. Consider

$$p^{s-m} - p^{s-m-2} - \cdots - p^{\varepsilon} = p^{s-m} + (p^{s-m-1} - p^{s-m-1}) - p^{s-m-2} - \cdots - p^{\varepsilon},$$

and write

$$r_m \geq (p-1)p^{s-m-1} + (p-1)p^{s-m-2} - p^{s-m-4} - \cdots - p^{\varepsilon}.$$

If $s = n + 1$ we take $m = 1$ to get $r_1 \geq (p-1)p^{n-1} + (p-1)p^{n-2} - p^{n-4} - \cdots - p^{\varepsilon}$. Then

$$(4.17) \quad \deg(y) \geq 2(p-1)r_1 \geq 2(p-1)^2(p^{n-1} + p^{n-2}) - 2(p-1)(p^{n-4} + \cdots + p^{\varepsilon})$$

$$= (p^2 + p(p-4) + 2)(p^{n-1} + p^{n-2}) - 2(p-1)(p^{n-4} + \cdots + p^{\varepsilon})$$

$$= p^{n+1} + (p-3)p^n + (p-2)p^{n-1} + 2p^{n-2} - 2(p-1)(p^{n-4} + \cdots + p^{\varepsilon}) > p^{n+1}.$$

Now $\deg(y) > p^{n+1}$ contradicts the fact that the p-adic expansion of $\deg(y)$ is $a_n p^n + \cdots + a_0$. Thus we have established that $s < n + 1$.

The following lemma is the key to establishing properties H(j), $j \geq 0$, for monomials $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ whose degree satisfy either condition 1) or 2) in the statement of Theorem 4.1. The need for condition 3) in Theorem 4.1 will become evident later when we prove the validity of conditions H(j), $j \geq 0$.

Recall that in 4.9 and 4.10 we proved that a monomial $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ with property H(j-1) $s > j$, satisfies the following formula:

$$P_*^{p^j}(y) = (-1)^{\binom{(r_1-p^j)(p-1)-1}{p^j}} \beta Q^{r_1-p^j} \beta Q^{r_2} \cdots \beta Q^{r_s} x(\underline{r})$$

(4.18)

$$+ \sum_{u=2}^{j+1} (-1)^{\binom{(r_u-p^{j-u+1})(p-1)-1}{p^{j-u+1}}} \beta Q^{r_1-p^j+p^{j-1}} \beta Q^{r_2-p^{j-1}+p^{j-2}} \cdots \beta Q^{r_{u-1}-p^{j-u+2}+p^{j-u+1}} \beta Q^{r_u-p^{j-u+1}} \beta Q^{r_{u+1}} \cdots \beta Q^{r_s} x(\underline{r})$$

If $s \le j$ there is a similar formula in which the sum runs from $u = 2$ to $u = s$ and there is an extra summand

$$\beta Q^{r_1-p^j+p^{j-1}} \cdots \beta Q^{r_s-p^{j-s+1}+p^{j-s}} P_*^{p^{j-s}} x(\underline{r}).$$

Lemma 4.19. Let $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ be an indecomposable monomial in $H_*(QX; \mathbb{Z}/p)_A$ whose degree satisfies either condition 1) or 2) of Theorem 4.1 as well as condition 3) of that Theorem. Assume that y has property $H(j-1)$, $j \ge 1$, (see def. 4.4). Then the monomials in the Nishida expansion of $P_*^{p^j}(y)$, $j \ge 1$, given in (4.18) are indecomposable.

Proof. By assumption $\deg(y) = a_n p^n + \ldots + a_0$, $a_n \ne 0$, $0 \le a_i \le p-1$, satisfies the following conditions:

1) If $\deg(y)$ is even, then either a) $a_0 \ne p-1$ or $a_1 \ne p-1$, while $a_i \ne p-1$ and $a_i \ne p-2$ for $i \ge 2$; moreover $a_0 \ne p-2$, or b) $a_i = p-1$ for all i.

2) If $\deg(y)$ is odd then either a) $a_0 \ne p-3$ or $a_1 \ne p-1$, while $a_i \ne p-1$ and $a_i \ne p-2$ for $i \ge 2$, or b) $a_i = p-1$ for all i.

Recall that a monomial $\beta^{\varepsilon_1}Q^{r_1}\ldots\beta^{\varepsilon_s}Q^{r_s}x(\underline{r})$ is indecomposable if the inequality $2r_u > \deg\left(\beta^{\varepsilon_{u+1}}Q^{r_{u+1}}\ldots\beta^{\varepsilon_s}Q^{r_s}x(\underline{r})\right)$ holds for all $1 \le u \le s$. We will check these inequalities for each monomial in the expansion (4.18) of $P_*^{p^j}(y)$.

It will turn out that property H(j-1) and the assumption 1) or 2) on deg (y) are of use only when it comes to verify that

(4.20) $\quad 2(r_u - p^{j-u+1}) > \deg\ (\beta\ Q^{r_{u+1}}\ldots\beta\ Q^{r_s}\ x(\underline{r})),\ 1 \le u \le j+1.$

We first prove inequalities (4.20) using induction on u.

Consider u = 1. It turns out that only if j = 1 the parity of deg (y) is important. Suppose that, contrary to (4.20), the following inequality takes place.

(4.21) $\quad 2(r_1 - 1) \le \deg(y) + 1 - 2(p-1)r_1 < 2r_1.$

The right inequality expresses the fact that the monomial of y is indecomposable. (4.21) is equivalent to:

$$2pr_1 - 2 \le \deg(y) + 1 < 2pr_1.$$

By parity we get:

(4.22) $\quad 2pr_1 - 2 = \deg(y) \quad$ if deg(y) is even,

(4.23) $\quad 2pr_1 - 1 = \deg(y) \quad$ if deg(y) is odd.

Condition H(0) means that $r_1 \equiv 0 \pmod p$, i.e., $2pr_1 \equiv 0 \pmod{p^2}$, so that (4.22) and (4.23) imply:

(4.24) $2pr_1 = \deg(y) + 2 \equiv 0 \pmod{p^2}$, if deg(y) even,

(4.25) $2pr_1 \equiv \deg(y) + 3 \equiv 0 \pmod{p^2}$, if deg(y) odd.

On the other hand, if deg(y) is even, assumption 1), a) on deg(y) implies $\deg(y) + 2 = (a_0 + 2) + a_1 p + \ldots + a_n p^n$

(4.26)
$$= \begin{cases} (a_1+1)p + a_2p^2 + \ldots + a_np^n & \text{if } a_0 = p-2,\ a_1 \neq p-1 \\ (a_0+2) + a_1p + a_2p^2 + \ldots + a_np^n & \text{if } a_0 < p-2 \\ 1 + (a_1+1)p + a_2p^2 + \ldots + a_np^n & \text{if } a_0 = p-1 \end{cases}$$

In all cases one observes that

(4.27) $\deg(y) + 2 \not\equiv 0 \pmod{p^2}$.

However (4.27) contradicts (4.24). Assumption 1), b) also contradicts (4.24) as it is the last case in (4.26).

(4.28) We conclude that if deg(y) is even and satisfies 1) then

$$\beta Q^{r_1-1} \beta Q^{r_2} \ldots \beta Q^{r_s} x(r) \text{ is indecomposable.}$$

Assumption 2) implies, for deg(y) odd, that $\deg(y) + 3 \not\equiv 0 \pmod{p^2}$.

(4.29) Thus if deg(y) is odd and satisfies 2) then

$$\beta Q^{r_1-1} \beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r})) \text{ is indecomposable.}$$

We will prove next the inequality

$$2(r_1 - p^{j-1+1}) > \deg(\beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r})),\ j \geq 1.$$

Suppose, on the contrary, that

(4.30) $2(r_1-p^j) \leq \deg(\beta Q^{r_2} \ldots \beta Q^{r_s}x(\underline{r})) = \deg(y) + 1 - 2(p-1)r_1 < 2r_1.$

The inequality on the right holds by indecomposability of the original monomial y. Then $2pr_1 - 2p^j \leq \deg(y) + 1 < 2p^{r_1}$. Thus

(4.31) $2pr_1 - 2p^j + k = \deg(y) + 1$, for $k \in \mathbb{Z}$, $0 \leq k < 2p^j$.

Condition $H(j-1)$ implies, on the other hand, that $2pr_1 \equiv 0 \pmod{p^{j+1}}$, so that, by (4.31)

(4.32) $k \equiv (a_0+1) + a_1p + \ldots + a_{j-1}p^{j-1} + (a_j+2)p^j \pmod{p^{j+1}}.$

However, assumption 1), a) on deg (y) gives

(4.33) $(a_0+1) + a_1p + \ldots + a_{j-1}p^{j-1} + (a_j+2)p^j < p^{j+1}.$

Thus (4.32) is an equality.

Notice that 1) a) also implies that $k = (a_0+1) + a_1p + \ldots + a_{j-1}p^{j-1} + (a_j+2)p^j > 2p^j$. This contradicts the condition $0 \leq k < 2p^j$. Similarly 1), b) gives $k = (a_0+1) + a_1p + \ldots + a_{j-1}p^{j-1} + (a_j+2)p^j = 2p^j$, contradicting the condition $0 \leq k < 2p^j$. Clearly, if deg(y) is odd and satisfies either 2), a) or 2), b), one can proceed as in the case deg (y) even to establish the required inequality

$$2(r_1-p^j) > \deg(\beta Q^{r_2} \ldots \beta Q^{r_s}x(\underline{r})), \ j \geq 1.$$

So far we have proved the first step of the induction on u announced in (4.20).

Assume inductively that the monomials

$$\beta Q^{r_1-p^j+p^{j-1}} \cdots \beta Q^{r_{v-1}-p^{j-v+2}+p^{j-v+1}} \beta Q^{r_v-p^{j-v+1}} \beta Q^{r_{v+1}} \cdots \beta Q^{r_s} x(\underline{r})$$

are indecomposable at r_v-p^{j-v+1}, where $a \leq v < u < s$. That is to say, assume

$$(4.34) \quad 2(r_v - p^{j-v+1}) > \deg(\beta Q^{r_{v+1}} \cdots \beta Q^{r_s}(x(\underline{r}))) \text{ for } 2 \leq v < u < s.$$

Consider now the monomial

$$\beta Q^{r_1-p^j+p^{j-1}} \cdots \beta Q^{r_u-p^{j-u+1}} \beta Q^{r_{u+1}} \cdots \beta Q^{r_s} x(\underline{r}).$$

Assume that, contrary to (4.34)

$$2(r_u - p^{j-u+1}) \leq \deg(y) - 2(p-1)(r_1 + \ldots + r_u) + u.$$

Equivalently,

$$(4.35) \quad 2pr_u - 2p^{j-u+1} \leq \deg(y) - 2(p-1)(r_1 + \ldots + r_{u-1}) + u.$$

Since $y = \beta Q^{r_1} \ldots \beta Q^{r_s}$ is indecomposable,

$$(4.36) \quad \deg(y) - 2(p-1)(r_1 + \ldots + r_{u-1}) + u-1 < 2r_{u-1}.$$

(4.35) and (4.36) imply

$$(4.37) \quad 2pr_u - 2p^{j-u+1} \leq \deg(y) - 2(p-1)(r_1 + \ldots + r_{u-1}) + u \leq 2r_{u-1}.$$

Now $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ is admissible, so in particular, $2r_{u-1} \leq 2pr_u - 2$. Thus

(4.38) $\quad 2r_{u-1} - 2p^{j-u+2} < 2r_{u-1} - 2p^{j-u+1} < 2pr_u - 2p^{j-u+1}$

Together (4.37) and (4.36) imply

$$2r_{u-1} - 2p^{j-u+2} \leq \deg(y) - 2(p-1)(r_1 + \ldots + r_{u-1}) + u - 1.$$

This contradicts the induction hypothesis (4.34). The induction establishing

$$2(r_u - p^{j-u+1}) > \deg(\beta Q^{r_{u+1}} \ldots \beta Q^{r_s} x(\underline{r})), \ 1 \leq u \leq s,$$

is thus complete.

To complete the proof of the lemma, consider the following inequality which holds by the indecomposability of y:

(4.39) $\quad \deg(y) - 2(p-1)(r_1 + \ldots + r_k) + k < 2r_k, \ 1 \leq k \leq s.$

Add $2(-p^{j-k+1} + p^{j-k})$ to both sides of (4.39) to get

(4.40) $\quad \deg(y) - 2(p-1)p^j + k - 2(p-1)[r_1 - p^j + p^{j-1} + r_2 - p^{j-1} + p^{j-2} + \ldots$

$$+ r_k - p^{j-k+1} + p^{j-k}] < 2(r_k - p^{j-k+1} + p^{j-k}).$$

Notice that (4.40) is the set of inequality required to show that the monomial

$$\beta Q^{r_1 - p^j + p^{j-1}} \ldots \beta Q^{r_k - p^{j-k+1} + p^{j-k}} \beta Q^{r_{k+1} - p^{j-k} + p^{j-k-1}} \ldots \beta Q^{r_s}(x(\underline{r})),$$

$$1 \leq k < j < s$$

is indecomposable at each step. Proceeding as in the derivation of

(4.40) gives us, finally, that for $s \le j$,

$$\beta Q^{r_1-p^j+p^{j-1}} \ldots \beta Q^{r_s-p^{j-s+1}+p^{j-s}} P_*^{p^{j-s}} (x(\underline{r}))$$

is indecomposable, provided that $P_*^{p^{j-s}} x(\underline{r}) \neq 0$.

We are now in a position to prove the validity of properties H(j), $0 \le j$ for monomials $y = \beta Q^{r_1} \ldots \beta Q^{r_1} x(\underline{r})$ satisfying the conditions stated in Theorem 4.1. Properties H(j), $j \ge 0$ are defined in (4.4), and as noted in (4.2), they constitute the main ingredient in the proof of Theorem 4.1.

Lemma 4.41. Suppose that $y = \beta Q^{r_1} \ldots \beta Q^{r_s}(x(\underline{r})) \in \underline{Q}H_*(QX; \mathbb{Z}/p)_A$, with y homogeneous, and suppose furthermore that deg(y) satisfies the conditions 1), 2) and 3) of Theorem 4.1. Then y satisfies H(j), $0 \le j \le h$. Here $h = n-2$ if $\deg(y) \le p^n + (p-2)p^{n-1}$ while $h = n-1$ if $\deg(y) \ge p^n + (p-1)(p^{n-1} + \ldots + 1)$.

Proof. We use the induction on s = length (y). We need the following remarks concerning the sequence $(1, r_1, 1, r_2, \ldots, 1, r_s)$.

First of all, since $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ is indecomposable, the following inequality must hold:

$$2r_1 > \deg(\beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r})) = \deg(y) - 2(p-1)r_1 + 1.$$

Equivalently

$$2pr_1 > \deg(y) + 1 \quad \text{and}$$

$$2pr_1 - 1 > \deg(y).$$

If $r_1 \leq p^{n-1}$ then

$$2p^n - 1 > \deg(y), \text{ i.e., } p^n + (p-1)(p^{n-1} + \dots + 1) > \deg(y).$$

We conclude that if $\deg(y) \geq p^n + (p-1)(p^{n-1} + \dots + 1)$, then $r_1 > p^{n-1}$. That is to say, assumption 3), b) in Theorem 4.1 implies that $r_1 > p^{n-1}$.

We now extract some conclusions on the sequence $(1, r_1, \dots, 1, r_s)$ from assumption 3), a) in Theorem 4.1. Since the space X is connected, then

$$(4.43) \quad r_1 < p^{n-1}, \text{ for otherwise } 2(p-1)r_1 \geq p^n + (p-2)p^{n-1} \geq \deg(y),$$

which is impossible (see Theorem 3.1.2). The upper bounds for r_1 obtained in (4.42) and (4.43), namely:

$$r_1 < p^n \qquad \text{if } \deg(y) \geq p^n + (p-1)(p^{n-1} + \dots + 1),$$

$$r_1 < p^{n-1} \qquad \text{if } \deg(y) \leq p^n + (p-2)p^{n-1},$$

account for the top condition H(h) considered in the statement of the lemma.

We now proceed to prove the lemma. If $s = 0$, i.e., $y \in H_*(X; \mathbb{Z}/2)_A$, then properties 4.4, b), c) are clearly satisfied, since y is A-annihilated. Suppose the lemma true for monomials of length q, where $0 \leq q < s$. That is to say, suppose that if

$$\beta Q^{r_1} \dots \beta Q^{r_q}(x(\underline{r})) \in \underline{Q}\, H_*(QX; \mathbb{Z}/p)_A, \; 0 \leq q < s,$$

is a homogeneous monomial of degree as in the statement of Theorem 4.1, then properties H(j), $j \geq 0$, are simultaneously satisfied by $r_1, \dots, r_q$ and $x(\underline{r})$. We will prove that, for y as in Theorem 4.1, $y = \beta Q^{r_1} \dots \beta Q^{r_s} x(\underline{r}) \in \underline{Q}\, H_*(QX; \mathbb{Z}/p)_A$ satisfies simultaneously properties

H(j), $0 \le j$. The proof itself is an induction on $j \ge 0$. Let $j = 0$. The Nishida relations (3.1.8) imply

$$(4.44) \quad 0 = P_*^1(y) = (-1)\binom{(r_1-1)(p-1)-1}{1} \beta Q^{r_1-1} \beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r})$$

$$+ \binom{(r_1-1)(p-1)-1}{1-1} Q^{r_1-1} \beta \cdot \beta Q^{r_2} \ldots \beta Q^{r_s}(x(\underline{r})).$$

The second summand of (4.44) is zero, since $\beta^2 = 0$. The monomial in the first summand of (4.44) is indecomposable, as proved in Lemma 4.19. Then $P_*^1(y) = 0$ forces

$$\binom{(r_1-1)(p-1)-1}{1} \equiv 0, \ (\mathrm{mod}\ p),$$

This clearly implies that $(r_1-1)(p-1) \equiv 1$, (mod p) and thus $r_1 \equiv 0$, (mod p). We conclude that y satisfies H(0).

Suppose that y satisfies H(k), for $0 \le k < j$. Consider now $P_*^{p^j}(y) = 0$. As proved in 4.19 the monomials in the Nishida expansion of $P_*^{p^j}(y)$ are indecomposable, since we are assuming property H(j-1) on y. Then $P_*^{p^j}(y) = 0$ forces the coefficients of the monomials of (4.18) to be zero, i.e.,

$$(4.45) \quad \binom{(r_u-p^{j-u+1})(p-1)-1}{p^{j-u+1}} \equiv 0, \ (\mathrm{mod}\ P), \ 1 \le u \le j+1, \ j < s.$$

Property H(j-1) on y implies that $r_u = h_{j-u+1}p^{j-u+1} + Mp^{j-u+2}$, where

$M \in \mathbb{Z}$; $M \geq 0$, $0 \leq h_{j-u+1} < p$. Then

$$(r_u - p^{j-u+1})(p-1)-1 = ((p-1)(h_{j-u+1} - 1)(p-1)p^{j-u+1} + (p-1)Mp^{j-u+2} - 1$$

$$= (h_{j-u+1}-1)(p-1)p^{j-u+1} + ((p-1)M-1)p^{j-u+2} + p^{j-u+2} - 1$$

$$(4.46) \quad = (h_{j-u+1}-1)(p-1)p^{j-u+1} + ((p-1)M-1)p^{j-u+2} + (p-1)(p^{j-u+1} + \dots + 1).$$

Observe that for the binomial coefficient (4.45) to be zero it suffices that $(h_{j-u+1}-1)(p-1) \equiv 1$, (mod p), i.e.,

$$(4.47) \quad h_{j-u+1} \equiv 0, \text{ (mod p)}. \quad \text{Thus } h_{j-u+1} = 0.$$

We must prove that (4.47) is necessarily the case. Notice that $M = 0$ and $h_{j-u+1} > 1$ would imply, proceeding as in (4.46) that $\binom{(r_u - p^{j-u+1})(p-1)-1}{p^{j-u+1}}$ is not congruent to zero mod p, which contradicts that $P_*^{p^j}(y) = 0$. Moreover, since y is admissible,

$$(4.48) \quad r_1 \leq pr_2 - 1 < pr_2 \leq p(pr_3-1) < p^2 r_3 \leq \dots < p^{u-1} r_u.$$

So if $r_u = p^{j-u+1}$, then $r_1 < p^{u-1}p^{j-u+1} = p^j$, which is impossible, since $r_1 \equiv 0$, (mod P^j). From (4.46), (4.47), (4.48) we conclude that

$$r_u \equiv 0, \text{ (mod } P^{j-u+2}), \quad 1 \leq u \leq j+1.$$

This establishes the property H(j) when $j < s$. Consider the monomials

$$(4.49) \quad \binom{(r_s - p^{j-s+1})(p-1)-1}{p^{j-s+1}-pt} \beta Q^{r_1 - p^j + p^{j-1}} \dots \beta Q^{r_s - p^{j-s+1}+t} P_*^t x(r),$$

$$1 \leq t \leq p^{j-s}.$$

They appear in $P_*^{p^j}(y)$ when $j \geq s$, as seen in 4.10. Observe that in (4.49), $(r_s - p^{j-s+1})(p-1)-1$ has p-adic expansion of form

(4.50) $\quad M\, p^{j-s+1} + (p-1)(p^{j-s} + \dots + 1)$, by $H(j-1)$.

On the other hand, the p-adic expansion of $p^{j-s+1} - pt$, for t as in (4.49), is of form

(4.51) $\quad b_0 + \dots + b_{j-s}p^{j-s}$

We conclude from (4.50) and (4.51) that the binomial coefficient in (4.49) is not congruent to zero mod p. However $P_*^{p^j}(y) = 0$ by assumption on y, which can occur only if $P_*^t x(\underline{r}) = 0$ for $1 \leq t \leq p^{j-s}$. For we recall from (4.19) that the monomial in (4.49) is indecomposable.

We have established properties $H(j)$, $j \geq 0$, for

$y = \beta Q^{r_1} \dots \beta Q^{r_s} x(\underline{r}) \in \underline{Q}H_*(QX; \mathbb{Z}/p)_A$ thus proving the lemma.

So far in our results we have assumed that the monomial summands of an element of $\underline{Q}H_*(QX; \mathbb{Z}/p)_A$ are of type

(4.52) $\quad \beta Q^{r_1} \dots \beta Q^{r_s} x(\underline{r})$.

In order to characterize $\underline{Q}\, H_*(QX; \mathbb{Z}/p)_A$ in the degrees of Theorem 4.1, we need to know that any $y \in \underline{Q}\, H_*(QX; \mathbb{Z}/p)_A$ is of type (4.52).

Lemma 4.53. Let $y = \beta^{\varepsilon_1} Q^{r_1} \dots \beta^{\varepsilon_s} Q^{r_s} x(\underline{r})$, $\varepsilon_i \in \{0,1\}$ be an A-annihilated indecomposable monomial whose degree satisfies the conditions of Theorem 4.1. Then $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_s = 1$, and $\beta x(\underline{r}) = 0$.

Proof. We use induction on s = length $(y) \geq 1$, (see 3.3).

If $\beta x = 1$, clearly y has to be of the form $\beta Q^{r_1}x(\underline{r})$, as it must be annihilated by β.

Consider the Nishida expansion

(4.54) $$P_*^1(\beta Q^{r_1}x(\underline{r})) = (-1)\binom{(r_1-1)(p-1)-1}{1}\beta Q^{r_1-1}x(\underline{r}) + (-1)\binom{(r_1-1)(p-1)-1}{1-1}Q^{r_1-1}\beta x(\underline{r}).$$

As proved in Lemma 4.41, the first summand is trivial only if r_1 satisfies H(0). Notice furthermore that the second summand in (4.54) is trivial only if $Q^{r_1-1}\beta x(\underline{r}) = 0$. One checks that the monomial $Q^{r_1-1}\beta x(\underline{r})$ is indecomposable, for, otherwise,

(4.55) $$2(r_1-1) \leq \deg(y) - 2(p-1)r_1 < 2r_1-1.$$

If deg(y) is even, clearly (4.55) implies

(4.56) $$2pr_1 - 2 = \deg(y).$$

However condition 1,a) of Theorem 4.1 prevents the occurrence of (4.56). Thus $Q^{r_1-1}\beta x(\underline{r})$ is indecomposable, when deg(y) is even. Similarly, one proves that $Q^{r_1-1}\beta x(\underline{r})$ is indecomposable if deg(y) is odd. Therefore (4.55) is impossible. Since $P_*^1(y) = 0$, by assumption, the previous conclusions imply that $\beta x(\underline{r}) = 0$.

We have established the first step of the induction. Notice also that y satisfies H(0), since the first term of (4.54) must be zero. We need some remarks concerning the lemmas numbered from 4.5 to 4.41.

First of all, notice that in the Nishida expansion (3.1.8) of

$$P_*^{p^t}(\beta^{\varepsilon_1}Q^{r_1} \ldots \beta^{\varepsilon_s}Q^{r_s}x(\underline{r})), \quad s \geq 2,$$

only the operations Q^{r_q} are affected, where $q \leq t + 1$. This is clear from the form of the coefficients in

$$(4.57) \quad P_*^s\beta^\varepsilon Q^r = \sum_i (-1)^{i+s}\binom{(r-s)(p-1)-\varepsilon}{s-pi}\beta^\varepsilon Q^{r-s+i}P_*^i$$

$$+ \sum_i (-1)^{i+s}\binom{(r-s)(p-1)-\varepsilon}{s-pi-1}Q^{r-s+i}P_*^i\beta^\varepsilon, \quad \varepsilon = 0;1.$$

Observe that only the first sum is considered when $\varepsilon = 0$. Furthermore, notice that condition H(j) on $\beta Q^{r_1} \ldots \beta Q^{r_s}x(\underline{r})$, $j < s$, concerns only the first j+1 operations in the monomial. Thus one can define properties H(j), $j < s$, for monomials $y = \beta Q^{r_1} \ldots \beta Q^{r_{j+1}}\beta^{\varepsilon_{j+2}} \ldots \beta^{\varepsilon_s}Q^{r_s}x(\underline{r})$, and one proves that such properties are satisfied by y, by looking at the Nishida expansion of $P_*^{p^t}(y)$, $t \leq j$. This allows us to see that the Nishida expansion of $P_*^{p^j}(\beta Q^{r_1} \ldots \beta Q^{r_{j+1}}\beta^{\varepsilon_{j+2}}Q^{r_{j+2}} \ldots \beta^{\varepsilon_s}Q^{r_s}x(\underline{r}))$, $j < s$ looks like the one in (4.9), i.e., if we assume H(j-1) for y,

$$(4.58) \quad P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^j}\beta Q^{r_1-p^j}\beta Q^{r_2} \ldots \beta Q^{r_{j+1}}\beta^{\varepsilon_{j+2}} \ldots x(\underline{r})$$

$$+\sum_{u=2}^{j+1}(-1)\binom{(r_u-p^{j-u+1})(p-1)-1}{p^{j-u+1}}\,\beta Q^{r_1-p^j+p^{j-1}}\,\beta Q^{r_2-p^{j-1}+p^{j-2}}\cdots\beta Q^{r_{u-1}-p^{j-u+2}+p^{j-u+1}}\,\beta Q^{r_u-p^{j-u+1}}\,\beta Q^{r_{u+1}}\cdots\beta Q^{r_{j+1}}\beta^{\varepsilon_{j+2}}\cdots x(\underline{r})$$

$$+(-1)\binom{(r_{j+1}-1)(p-1)-1}{1-1}\,\beta Q^{r_1-p^j+p^{j-1}}\,\beta Q^{r_2-p^{j-1}+p^{j-2}}\cdots\beta Q^{r_{j+1}-1}\,\beta\beta^{\varepsilon_{j+2}}\,Q^{r_{j+2}}\cdots x(\underline{r}).$$

The last term of (4.58) appears only if $\beta^{\varepsilon_{j+2}} = 0$, or if $s = j+1$ and $\beta x(\underline{r}) \neq 0$. If we can prove that the last term in (4.58) is an admissible indecomposable monomial then its presence will contradict $P_*^{p^j}(y) = 0$.

We are now ready to continue the induction which will prove the lemma. Suppose the lemma is true for monomials of length q, $q < s$, i.e., suppose that

$$\beta^{\varepsilon_1}Q^{r_1}\cdots\beta^{\varepsilon_q}Q^{r_q}x(\underline{r}) \in \underline{Q}\,H_*(QX;\mathbb{Z}/p)_A \text{ implies}$$

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_q = 1 \text{ and } \beta x(\underline{r}) = 0.$$

Consider, with deg(y) as in Theorem 4.1,

$$y = \beta Q^{r_1}\beta^{\varepsilon_2}\cdots\beta^{\varepsilon_s}Q^{r_s}x(\underline{r}) \in \underline{Q}\,H_*(QX;\mathbb{Z}/p)_A;$$

Suppose for a moment that $\varepsilon_2 = \cdots = \varepsilon_s = 1$ and either $\varepsilon_{j+2} = 0$ if $1 \le j < s-1$, or $j = s-1$ and $\beta x(\underline{r}) \neq 0$. Thus

$$y = \beta Q^{r_2}\cdots\beta Q^{r_{j+1}}Q^{r_{j+2}}\beta^{\varepsilon_{j+3}}\cdots x(\underline{r}).$$

y satisfies H(j-1), as pointed out above. Consider

$$(4.59)\quad P_*^{p^j}(y) = (-1)\binom{(r_1-p^j)(p-1)-1}{p^j} \beta Q^{r_1-p^j} \beta Q^{r_2} \ldots \beta Q^{r_{j+1}} Q^{r_{j+2}} \ldots x(\underline{r})$$

$$+ \sum_{u=2}^{j+1}(-1)\binom{(r_u-p^{j-u+1})(p-1)-1}{p^{j-u+1}} \beta Q^{r_1-p^j+p^{j-1}} \beta Q^{r_2-p^{j-1}+p^{j-2}} \ldots \beta Q^{r_{u-1}-p^{j-u+2}+p^{j-u+1}} \beta Q^{r_u-p^{j-u+1}} \beta Q^{r_{u+1}} \ldots \beta Q^{r_{j+1}} Q^{r_{j+2}} \beta^{\varepsilon_{j+3}} \ldots x(\underline{r})$$

$$+ (-1)\binom{(r_{j+1}-1)(p-1)-1}{1-1} \beta Q^{r_1-p^j+p^{j-1}} \beta Q^{r_2-p^{j-1}+p^{j-2}} \ldots \beta Q^{r_j-p+1} Q^{r_{j+1}-1} \beta Q^{r_{j+2}} \ldots x(\underline{r}).$$

Except for the last term, we know that (4.59) consists of admissible, indecomposable monomials, and that, for these terms to be 0, $(1, r_1, \ldots, 1, r_{j+1}, 0, r_{j+2}, \ldots)$ must satisfy H(j). We will prove that the last term is not trivial, which will contradict $P_*^{p^j}(y) = 0$. Notice that the last two terms of (4.59) are determined, respectively, by the sequences

$$(4.60)\quad (1,\ r_1-p^j+p^{j-1},\ 1,\ r_2-p^{j-1}+p^{j-2}, \ldots, 1,\ r_j-p+1,\ 1,\ r_{j+1}-1,\ 0,\ r_{j+2},\ \varepsilon_{j+3}, \ldots)$$

and

$$(4.61)\quad (1,\ r_1-p^j+p^{j-1},\ 1,\ r_2-p^{j-1}+p^{j-2}, \ldots, 1,\ r_j-p+1,\ 0,\ r_{j+1}-1,\ 1,\ r_{j+2},\ \varepsilon_{j+3}, \ldots).$$

Sequence (4.60) was proven to be admissible in Lemma 4.11. So to prove that (4.61) is admissible we must only check that

(4.62) $r_j - p + 1 \le p(r_{j+1} - 1)$ and that

(4.63) $r_{j+1} - 1 \le pr_{j+2} - 1$.

Observe that $r_j - p + 1 \le p(r_{j+1} - 1) - 1$ by Lemma 4.11, which clearly implies (4.62). (4.63) is immediate from the admissibility of y. Thus the last summand of (4.59) is admissible. To prove that the last monomial in (4.59) is indecomposable one argues exactly as in Lemma 4.19, since H(j-1) holds in our present situation. Clearly the coefficient of the last monomial of (4.59) is non-zero. From these facts we conclude that if $\varepsilon_{j+2} = 0$ or $\beta(x(\underline{r})) \neq 0$, then the last monomial of $P_*^{p^j}(y)$ is non-zero, and thus we arrive to the desired contradiction. Thus the lemma is proved for monomials $y \in \underline{Q}\, H_*(QX; \mathbb{Z}/p)_A$ of arbitrary length, as long as they are in the required degrees.

Theorem 4.1 is now an easy consequence of the sequence of Lemmas from (4.5) to (4.53).

(4.64) Proof of Theorem 4.1. Let $z \in \underline{Q}\, H_*(QX; \mathbb{Z}/p)_A$, where deg(z) satisfies conditions either 1) or 2) of Theorem 4.1, plus condition 3) of this Theorem. Then

(4.65) $$z = \sum_{\underline{r}} \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r}),$$

where each summand is an admissible, indecomposable monomial. Each monomial in (4.65) is of the asserted type, by Lemma 4.53. Moreover, each monomial of (4.65) satisfies properties H(j), j ≥ 0, by Lemma 4.41.

Let h be the largest integer such that $P_*^{p^h}(z) = 0$ not just by degree reasons, i.e., h is the largest integer such that

(4.66) $2(p-1)p^h \leq \deg(z)$.

It follows from Lemma 4.41 that each monomial in (4.65) satisfies property H(h). This, in particular, implies that for each monomial of (4.65), $r_1 \equiv 0, \pmod{p^{h-1+2}}$. Notice that this, in turn, implies $\deg(z) = \deg(\beta Q^{r_1} \ldots x(\underline{r})) \geq 2(p-1)p^{h+1}$. The last inequality holds since $\deg x(\underline{r}) > 0$, and it contradicts our assumption (4.66) on h.

We conclude that no operations Q^t, $t \geq 0$, can be present in the monomials of (4.65). Thus $z \in \underline{Q} H_*(QX; \mathbb{Z}/p)_A$ implies $z \in H_*(QX; \mathbb{Z}/p)_{A*}$ for z homogeneous of degree as prescribed in Theorem 4.1. This establishes Theorem 4.1.

CHAPTER 5

A RELATED RESULT

This chapter is devoted to the proof of Theorem 5.1. It turns out that most of the auxiliary lemmas of Chapter 4 are of use here, though some essential arguments in their proofs cannot be carried out in these new degrees without changes. Recall that the main ingredients in the proof of Theorem 4.1 are Lemmas 4.19 and 4.41. Lemma 4.19 states that if $\beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r}) \in \underline{Q}\, H_*(QX;\mathbb{Z}/2)_A$ has the required degree then $P_*^{p^j}(\beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r}))$ consists of sums of indecomposable monomials, provided that $\beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ satisfies property $H(j-1)$.

Lemma 4.19 and $\beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ A-annihilated imply Lemma 4.41, which establishes property $H(j)$, $j \geq 0$, for $\beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ of the degree required in Theorem 4.1.

Theorem 5.1. Let X be any connected CW-complex, p an odd prime and $t \geq 0$. Then

$$\underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX;\mathbb{Z}/2)_A = H_{2p^{t+1}(p-1)-2}(X;\mathbb{Z}/p)_A \oplus \beta Q^{p^t(p-1)}[H_{2(p-1)^2p^t-1}(X;\mathbb{Z}/p)_A]$$

$\underline{Q}$ denotes indecomposable elements,

QX the infinite loop space associate to X,

$Q^{p^t(p-1)}$ is the Dyer-Lashof operation, and the subindex A denotes Steenrod annihilated elements.

We will establish in this section results analogous to Lemmas 4.19 and 4.41 though the present ones will hold only for a proper subset of $\underline{Q}\,H_m(QX;\mathbb{Z}/2)_A$, m as in Theorem 5.1.

First, notice that

$$(5.2)\quad 2p^{t+1}(p-1)-2 = (p-1)p^{t+1} + (p-2)p^{t+1} + p^{t+1} - 1 - 1$$

$$= p^{t+2} + (p-3)p^{t+1} + (p-1)p^t + \ldots + (p-1)p + p - 2.$$

So our present degrees are not among those of Theorem 4.1. Let

$$y = \sum_{\underline{r}} \beta^{\varepsilon_1}Q^{r_1}\ldots\beta^{\varepsilon_s}Q^{r_s}x(\underline{r}) \in \underline{Q}\,H_m(QX;\mathbb{Z}/2)_A .$$

As done in Chapter 4, we may assume in analyzing $P^q_*(y)$, $q \geq 1$, that the monomials all have the same length $s \geq 1$. Moreover $\varepsilon_1 = 1$, since $\beta y = 0$ by assumption. Let $s = 1$, and consider the Nishida expansion (3.1.8):

$$(5.3)\quad 0 = P^1_*(\beta Q^r x(\underline{r})) = (-1)\binom{(r-1)(p-1)-1}{1}\beta Q^{r-1}x(\underline{r})$$

$$+ (-1)\binom{(r-1)(p-1)-1}{1-1}Q^{r-1}\beta x(\underline{r}) .$$

The monomial in the first summand of (5.3) is indecomposable, which is proved exactly as in Lemmas 4.19-4.29. Thus r satisfies H(0), as seen in 4.41. We now analyze the second summand in (5.3). For it to be decomposable one must have

$$(5.4)\quad 2(r-1) \leq \deg(\beta Q^r x(\underline{r})) - 2(p-1)r < 2r-1 .$$

This is seen as in Lemma 4.19. (5.4) implies $2pr-2 = \deg(\beta Q^r x(\underline{r}))$, in which case $2pr = \deg(\beta Q^r x(\underline{r})) + 2 = 2p^{t+1}(p-1)$. Then $r = p^t(p-1)$.

(5.5) So $Q^{p^t(p-1)-1}\beta x \doteq (\beta x)^2$, which is trivial only if $\beta x = 0$.

It is clear that inequality (5.4) cannot hold if $r > (p-1)p^t$. We put aside for a while the case $r = (p-1)p^t$ in monomials $\beta Q^r x(\underline{r}) \in \underline{Q}\, H_m(QX; \mathbb{Z}/p)_A$, $m = 2p^{t+1}(p-1) - 2$.

Assuming $r > (p-1)p^t$ and property $H(j-1)$ on $y = \beta Q^r x(\underline{r}) \in \underline{Q}\, H_m(QX; \mathbb{Z}/p)_A$, consider

$$(5.6) \quad P_*^{p^j}(\beta Q^r x(\underline{r})) = \begin{pmatrix}(r-p^j)(p-1)-1 \\ 1\end{pmatrix} \beta Q^{r-p^j} x(\underline{r}).$$

Notice that by (5.5) only this one term appears on the right. Suppose for a moment that the monomial on the right of (5.6) is decomposable. Then $2pr-2p^j \leq \deg(y)+1 < 2pr$, so that

$$(5.7) \quad 2pr - 2p^j + k = \deg(y) + 1, \; 0 \leq k < 2p^j.$$

Applying $H(j-1)$, this implies $-2p^j + k \equiv \deg(y) + 1, \pmod{p^{j+1}}$. From the p-adic expansion of $\deg(y)$ we get, for $j \leq t$:

$$(5.8) \quad k \equiv (p-1) + (p-1)p + \ldots + (p-1)p^{j-1} + p^j = 2p^j - 1, \pmod{p^{j+1}}.$$

(5.8) is indeed an equality, since $0 \leq k < 2p^j$, so that (5.7) becomes

$$2pr - 1 = \deg(y) + 1, \text{ i.e., } 2pr = 2p^{t+1}(p-1),$$

and so $r = (p-1)p^t$, contrary to our assumption on r.

This contradiction allows us to conclude that if $r > p^t(p-1)$ then the monomial $\beta Q^{r-p^j}x(\underline{r})$ in $P_*^{p^j}(\beta Q^r x(\underline{r}))$, $j \le t$, is indecomposable.

Notice that now we can prove properties $H(j)$, $0 \le j \le t$, exactly as in 4.41, for the monomial $\beta Q^r x(\underline{r})$, $r > (p-1)p^t$. This is a consequence of the previous remarks.

In order to prove properties $H(j)$, $0 \le j \le t$ for any monomial of the specified degree we will make use of the sequence of Lemmas 4.5 to 4.53. We are still considering monomials $y = \beta Q^{r_1}\beta^{\varepsilon_2} \ldots \beta^{\varepsilon_s}Q^{r_s}x(\underline{r})$ where $r_1 > (p-1)p^t$.

Notice that in the proof of the Lemmas 4.5 to 4.11 and 4.41 only the assumption of $H(j-1)$ and the fact that $y \in \underline{Q}\, H(QX; \mathbb{Z}/p)_A$ play a role.

The specific degrees of Theorem 4.1 are used in the proofs of Lemmas 4.19 and 4.53. These lemmas establish, respectively, that the monomials in $P_*^{p^j}(y)$ are indecomposable, and that y is of form $y = \beta Q^{r_1} \ldots \beta Q^{r_s}x(\underline{r})$ with $\beta(x(\underline{r})) = 0$. Moreover the degrees of Theorem 4.1 are used only when analyzing the first monomial $\beta Q^{r_1-p^j}\beta \ldots \beta Q^{r_s}(x(\underline{r}))$ of $P_*^{p^j}(y)$, as can be checked in 4.19-4.33. The rest of the proof of 4.19 depends only on this first monomial of $P_*^{p^j}(y)$.

For monomials of degree $2p^{t+1}(p-1)-2$, which we are now analyzing, the first monomial of $P_*^{p^j}(\beta Q^r x(\underline{r}))$, $0 \le j \le t$ has already been studied in (5.2) to (5.8) when $r > p^t(p-1)$. There we proved that $\beta Q^{r-p^j}x(\underline{r})$ is indecomposable, and that $\beta x(\underline{r}) = 0$. Then properties $H(j)$, $0 \le j \le t$ can be established for monomials $\beta Q^r x(\underline{r}) \in \underline{Q}\, H(QX; \mathbb{Z}/p)_A$, $r > p^t(p-1)$.

As pointed out in our remarks above, the fact that the monomial summands of $P_*^{p^j}(y)$, for $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(\underline{r})$ of degree $2p^{t+1}(p-1)-2$, $s > 1$, $r_1 > p^t(p-1)$, are indecomposable is proven exactly as in Lemma 4.19. Similarly, to prove that $y \in \underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$ must be of the form $\beta Q^{r_2} \ldots \beta Q^{r_s} x(\underline{r})$ when $r > p^t(p-1)$ we can proceed exactly as in Lemma 4.53. We have then proved the following lemma which is analogous to Lemma 4.41.

Lemma 5.9. Let $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(r) \in \underline{Q}\, H_m(QX; \mathbb{Z}/p)_A$, $m = 2p^{t+1}(p-1)-2$, $r_1 > (p-1)p^t$. Then y satisfies properties H(j), $0 \le j \le t$.

As a consequence of Lemma 5.9 we see that if $y = \beta Q^{r_1} \ldots \beta Q^{r_s} x(r) \in \underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$, $r_1 > p^t(p-1)$, then y satisfies H(t). This in turn implies that $r_1 \equiv 0$, (mod p^{t+1}). In particular $r_1 \ge p^{t+1}$, which gives

$$\deg(y) \ge 2(p-1)p^{t+1} > p^{t+2} + (p-2)p^{t+1}.$$

This is impossible, by the p-adic expansion of deg(y) given in (5.2). We have established the following partial conclusion on $\underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$.

Lemma 5.10. There are no monomials $\beta Q^{r_1} \beta^{\varepsilon_2} \ldots \beta^{\varepsilon_s} Q^{r_s} x)\underline{r})$ in $\underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$ with $r_1 > p^t(p-1)$ and $s \ge 1$.

Remark 5.11. If $y \in \underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$, $y = \beta Q^{r_1}\beta^{\varepsilon_2}Q^{r_2}\ldots x(\underline{r})$, then $2r_1 > \deg(y) + 1 - 2(p-1)r_1$. This implies that $r_1 > [\deg(y)+1]/2p = [2p^{t+1}(p-1)-1]/2p$. Thus $r_1 \geq p^t(p-1)$. In view of Lemma 5.10 it only remains to analyze $\beta Q^{p^t(p-1)}\beta^{\varepsilon_2}Q^{r_2}\ldots x(\underline{r})$ in $\underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$. Moreover the argument in (5.3)-(5.5) applies to show that $P^1_*\beta Q^{p^t(p-1)}\beta^{\varepsilon_2}Q^{r_2}\ldots x(\underline{r})$, $r_2 > 0$, is zero only if $\varepsilon_2 = 1$.

Our next goal is to prove the following:

Lemma 5.12. If $y = \beta Q^{p^t(p-1)}\beta Q^{r_2}\ldots x(\underline{r})$ is in $\underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$, then $y = \beta Q^{p^t(p-1)}x(\underline{r})$, where $\beta x(\underline{r}) = 0$.

Proof. We need the following observations.

$$\deg(\beta Q^{r_2}\beta^{\varepsilon_3}\ldots x(\underline{r})) = 2p^{t+1}(p-1) - 2 + 1 - 2(p-1)^2p^t$$
$$= 2(p-1)p^t - 1.$$

Also, y indecomposable implies $2r_2 > 2(p-1)p^t - 1 + 1 - 2(p-1)r_2$. Then $2(p-1)p^t < 2pr_2$, and so $r_2 \geq (p-1)p^{t-1} + 1$. This and the indecomposability of y imply that

(5.13) $\quad 2(p-1)p^t + 2p - 2 \leq 2pr_2 - 2 \leq 2(p-1)p^t < 2pr_2$.

The right inequality is obtained from $2(p-1)p^t - 2(p-1)r_2 < 2r_2$. However (5.13) is clearly impossible. We will use this fact in proving the lemma.

In order to proceed with the proof, consider the following Nishida expansion (3.1.8):

$$(5.14)\quad P_*^p(\beta Q^{p^t(p-1)}\beta Q^{r_2}\beta^{\varepsilon_3}\ldots x(\underline{r}))$$

$$= (-1)\binom{[p^t(p-1)-p](p-1)-1}{1}\beta Q^{p^t(p-1)-p}\beta Q^{r_2}\beta^{\varepsilon_3}\ldots x(\underline{r})$$

$$+ (-1)\binom{(r_2-1)(p-1)-1}{1}\beta Q^{p^t(p-1)-p+1}\beta Q^{r_2-1}\beta^{\varepsilon_3}\ldots x(\underline{r}).$$

Notice that the first summand of (5.14) is 0, since $\deg(\beta Q^{r_2}\ldots x(\underline{r})) = 2(p-1)p^t-1$. Thus $2(p^t(p-1)-p) < \deg(\beta Q^{r_2}\ldots x(\underline{r}))$, i.e., the monomial $\beta Q^{p^t(p-1)-p}\beta Q^{r_2}\ldots x(\underline{r})$ is trivial.

If we prove that the second summand of (5.14) is non-trivial, then its presence will contradict that y is A-annihilated, and so we will establish our claim: there is no $r_2 > 0$ in the monomial $y \in \underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$. To prove that the second summand of (5.14) is non-trivial, we analyze next the monomial

$$(5.15)\quad \beta Q^{p^t(p-1)-p+1}\beta Q^{r_2-1}\beta^{\varepsilon_3}\ldots x(\underline{r})$$

and its coefficient $\binom{(r_2-1)(p-1)-1}{1}$. Consider the inequality

$$(5.16)\quad p^t(p-1) - p + 1 < p[(p-1)p^{t-1} + 1 - 1] - 1 = (p-1)p^t - 1.$$

(5.16) clearly implies that for $r_2 \geq (p-1)p^{t-1} + 1$ the following inequality holds:

(5.17) $p^t(p-1) - p + 1 < p(r_2 - 1) - 1.$

Moreover, $r_2 \le pr_3 - \varepsilon_3$ by the admissibility of y, so that

(5.18) $r_2 - 1 < pr_3 - \varepsilon_3.$

Both (5.17) and (5.18) combine to prove that

$\beta Q^{p^t(p-1)-p+1} \beta Q^{r_2-1} \beta^{\varepsilon_3} \ldots x(\underline{r})$ is admissible.

(5.19) Now we prove that the monomial (5.15) is indecomposable. Suppose on the contrary that

$$2(r_2-1) \le \deg(y) - 2(p-1)(r_1+r_2)+2 < 2r_2 .$$

Then the lower bound $(p-1)p^{t-1} + 1$ for r_2 obtained above gives the inequalities of (5.13). As already observed, the inequalities of (5.13) are contradictory. We must also check that the inequality

$$2(p^t(p-1) - p + 1) \le \deg(y) - 2(p-1)r_1 + 1 - 2(p-1)p$$

is impossible. However this is a consequence of the indecomposability of y, i.e., of $2(p^t(p-1)) > \deg(y) - 2(p-1)r_1 + 1$. Thus we have proved that $\beta Q^{p^t(p-1)-p+1} \beta Q^{r_2-1} \beta^{\varepsilon_3} \ldots x(\underline{r})$ is indecomposable. We analyze next the coefficient $\begin{pmatrix} (r_2-1)(p-1)-1 \\ 1 \end{pmatrix}$. First, recall that $\deg (Q^{r_2} \beta^{\varepsilon_3} \ldots x(\underline{r})) = 2(p-1)p^t$. This and connectedness of X imply that $r_2 < p^t$, which together with the lower bound $(p-1)p^{t-1} + 1$ for r_2 give

(5.20) $(p-1)p^{t-1}+1 \le r_2 < p^t.$

Thus

(5.21) $(p-1)p^{t-1} + 1 \leq r_2 \leq (p-1)(p^{t-1} + \ldots + 1)$.

Next, it is clear that $\binom{(r_2-1)(p-1)-1}{1}$ is zero (mod p) if and only if

$(r_2-1) \equiv (p-1) \pmod{p}$, i.e., $r_2 \equiv 0, \pmod{p}$.

We have established the following: if $r_2 \not\equiv 0 \pmod{p}$ then the second summand of (5.14) is non-trivial. This contradicts that $y = \beta Q^{r_1}\beta Q^{r_2} \ldots x(\underline{r})$ is A-annihilated. Thus no $r_2 \not\equiv 0 \pmod{p}$ can appear in y.

We deal now with the case $r_2 \equiv 0, \pmod{p}$. Let $r_2 = (p-1)p^{t-1} + h_{t-2}p^{t-2} + \ldots + h_q p^q$ be the p-adic expansion of r_2. That it has this form is a consequence of (5.21). Here $h_q > 0$ and $q > 0$ by assumption.

We will exhibit a non-trivial summand in $P_*^{p^{q+1}}(y)$, a fact that will contradict the assumption that y is A-annihilated. Consider the following Nishida expansion

$$(5.22) \quad P_*^{p^{q+1}}(y) = \sum_{i=0}^{p^q} (-1)^{i+1} \binom{[p^t(p-1)-p^{q+1}](p-1)-1}{p^{q+1}-pi} \beta Q^{p^t(p-1)-p^{q+1}+i} P_*^i \beta Q^{r_2} \beta^{\varepsilon_3} \ldots (\underline{r})$$

There is the corresponding Nishida expansion for each term of (5.22), so that the i-th term equals

$$(5.23)\quad (-1)^{i+1}\binom{[p^t(p-1)-p^{q+1}](p-1)-1}{p^{q+1}-pi}\Bigg[\sum_{j=0}^{[\frac{i}{p}]}(-1)^{i+j}\binom{(r_2-i)(p-1)-1}{i-pj}$$

$$\beta Q^{p^t(p-1)-p^{q+1}+i}\beta Q^{r_2-i+j}P_*^j\beta^{\varepsilon_3}\ldots x(\underline{r})$$

$$+\sum_{j=0}^{[\frac{i-1}{p}]}(-1)^{i+j}\binom{(r_2-i)(p-1)-1}{i-pj-1}\beta Q^{p^t(p-1)-p^{q+1}+i}Q^{r_2-i+j}P_*^j\beta\varepsilon^{\varepsilon_3}Q^{r_3}\ldots x(\underline{r})\Bigg].$$

The second sum of (5.23) survives only if $\varepsilon_3 = 1$. The monomial $\beta Q^{p^t(p-1)-p^{q+1}+p^q}\beta Q^{r_2-p^q}\beta^{\varepsilon_3}\ldots x(\underline{r})$, which is the one corresponding to $i = q$, $j = 0$, is admissible. To see this, we must check:

$$(5.24)\quad p^t(p-1) - p^{q+1} + p^q \le p(r_2-p^q) - 1 = p((p-1)p^{t-1}+h_{t-2}p^{t-2}+\ldots+h_qp^t) - p^{q+1} - 1.$$

(5.24) is clearly equivalent to

$$(5.25)\quad p^t(p-1) + p^q \le (p-1)p^t + h_{t-2}p^{t-1} + \ldots + h_qp^{q+1} - 1 .$$

(5.25) is true, since $h_q \ge 1$.

From (5.24) we can deduce the admissibility of all the monomials determined by $j = 0$, $i \le p^q$, in (5.22). This is proven by the following inequalities:

$$(5.26)\quad p^t(p-1) - p^{q+1} + i \le p^t(p-1) - p^{q+1} + p^q$$
$$\le p(r_2-p^q) - 1 \le p(r_2-i) - 1 < p(r_2-i).$$

The second inequality of (5.26) is that in (5.24). The last two terms of (5.26) give the admissibility of the monomials, in both sums of

(5.23), determined by $j = 0$, $i \le p^q$.

Granted this, our next goal is to prove that when we fully develop the Nishida expansion in (5.22), no monomial in this expansion equals the monomial $\beta Q^{p^t(p-1)-p^{q+1}+p^q}\beta Q^{r_2-p^q}\ldots$. It suffices to prove that the monomial $\beta Q^{p^t(p-1)p^{q+1}+p^q}\beta Q^{r_2-p^q}\ldots x(\underline{r})$ is indecomposable, a fact we will establish later, and to show that no Adem relation (3.1.7) in the monomials of (5.22) produces the monomial $\beta Q^{p^t(p-1)-p^{q+1}+p^q}\beta Q^{r_2-p^q}\ldots$. We now prove the second statement, making use of the following observation concerning the Adem relations. Consider, with $r \ge ps$,

$$Q^r\beta Q^s = \sum_i (-1)^{r+i}\binom{(i-s)(p-1)}{pi-r}\beta Q^{r+s-i}Q^i$$
$$- \sum_i (-1)^{r+i}\binom{(i-s)(p-1)}{pi-r-1}Q^{r+s-i}\beta Q^i .$$

We see from the coefficients above that for the terms to be non-zero it must be $i \ge s$. In particular $r + s - i \le r$, so that the effect of an Adem relation on $Q^r\beta Q^s$ is to lower r, or to leave r and s unchanged but moving β. We apply this observation to a typical monomial in (5.23), say

$$\beta Q^{p^t(p-1)-p^{q+1}+i}\beta Q^{r_2-i+j}\ldots x(r).$$

Suppose that an Adem relation occurs involving $Q^{p^t(p-1)-p^{q+1}+i}\beta Q^{r_2-i+j}\ldots x(\underline{r})$. By the previous observation we obtain a sum of monomials $\beta Q^m\beta Q^n\ldots x(\underline{r})$ where $m < p^t(p-1) - p^{q+1} + i$. In particular $m < p^t(p-1) - p^{q+1} + p^q$, since $i = p^q$ is the maximal value of i

in the monomials of (5.22). Then the monomials $\beta Q^m \beta Q^n \ldots x(\underline{r})$ cannot equal $\beta Q^{p^t(p-1)-p^{q+1}+p^q} \beta Q^{r_2-p^q} \beta^{\varepsilon_3} Q^{r_3} \ldots x(\underline{r})$. We proved our claim: No Adem relation in (5.22) produces the monomial

$$\beta Q^{p^t(p-1)-p^{q+1}+p^q} \beta Q^{r_2-p^q} \beta^{\varepsilon_3} Q^{r_3} \ldots x(\underline{r}).$$

The monomial $\beta Q^{p^t(p-1)-p^{q+1}+p^q} \beta Q^{r_2-p^q} \beta^{\varepsilon_3} \ldots x(\underline{r})$ is indecomposable. To prove this it suffices to show that

$$(5.27) \qquad 2(r_2 - p^q) > \deg(Q^{r_2} \beta^{\varepsilon_3} \ldots \beta Q^{r_s} x(\underline{r})) - 2(p-1)r_2.$$

For, as seen in (4.17), the other cases follow directly from the indecomposability of y. (5.27) is equivalent to

$$(5.28) \qquad 2pr_2 - 2p^q > 2(p-1)p^t .$$

Now,

$$(5.29) \qquad 2pr_2 - 2p^q = 2p[(p-1)p^{t-1} + h_{t-2}p^{t-2} + \ldots + h_q p^q] - 2p^q$$

$$= 2(p-1)p^t + 2h_{t-2}p^{t-1} + \ldots + 2h_q p^{q+1} - 2p^q > 2(p-1)p^t .$$

The last inequality holds since $h_q \geq 1$. (5.24)-(5.26) and (5.27) together imply that the monomial $\beta Q^{p^t(p-1)-p^{q+1}+p^q} \beta Q^{r_2-p^q} \beta^{\varepsilon_3} \ldots x(\underline{r})$ is non-trivial, and indecomposable.

To prove that the summand in (5.22) with $i = p^q$ is non-trivial, it remains to show that its binomial coefficient is not zero, modulo p. We proceed to prove this. Write

(5.30) $(r_2-p^q)(p-1)-1 = [(p-1)p^{t-1} + h_{t-2}p^{t-2} + \ldots + (h_q-1)p^q](p-1) - 1.$

If $h_q > 1$, then $(h_q-1)(p-1) \equiv u$, (mod p), $1 < u \leq p-1$. Then, if $h_q > 1$, (5.30) has the following form

(5.31) $(r_2-p^q)(p-1)-1 = Mp^{q+1} + (u-1)p^q + p^q-1$

$$= Mp^{q+1} + (u-1)p^q + (p-1)(p^{q-1} + \ldots + 1).$$

But (5.31) implies that $\binom{(r_2-p^q)(p-1)-1}{p^q}$ is non-zero modulo p.

In case $h_q = 1$,

(5.32) $(r_2-p^q) = (p-1)p^t + h_{t-2}p^{t-2} + \ldots + h_v p^v$, $h_v \neq 0$, $q < v \leq t$.

In this case the term $h_v p^v - 1$ in $(r_2-p^q)(p-1)-1$ produces a non-trivial summand $(p-1)p^q$, which again implies that (5.30) is non-zero modulo p.

(5.33) Thus $\binom{(r_2-p^q)(p-1)-1}{p^q}$ is non-zero, modulo p.

This ends the proof of Lemma 5.12.

We are now prepared to prove Theorem 5.1.

(5.34) Proof of Theorem 5.1: All that is left to do is to show that if a monomial $\beta Q^{(p-1)p^t} x(\underline{r})$ of degree $2p^{t+2}(p-1)-2$ is A-annihilated indecomposable, then $x(\underline{r})$ is A-annihilated.

To see this suppose on the contrary that there exists a natural number k such that $P_*^{p^k} x(\underline{r})$ is not zero. Assume that k is the smallest natural number with this property.

Consider the following Nishida expansion (see 3.1.8):

$$(5.35)\quad P_*^{p^{k+1}}\beta Q^{(p-1)p^t}x(\underline{r}) = \sum_{i=0}^{p^k}(-1)\binom{[p^t(p-1)-p^{k+1}](p-1)-1}{p^{k+1}-pi}\beta Q^{p^t(p-1)-p^{k+1}+i}P_*^i\,x(\underline{r})$$

$$+\sum_{i=0}^{p^k-1}(-1)\binom{[p^t(p-1)-p^{k+1}](p-1)-1}{p^{k+1}-pi-1}Q^{p^t(p-1)-p^{k+1}+i}\beta P_*^i\,x(\underline{r})$$

$$=(-1)\binom{[p^t(p-1)-p^{k+1}](p-1)-1}{p^{k+1}}\beta Q^{p^t(p-1)-p^{k+1}}x(\underline{r})$$

$$+(-1)\beta Q^{p^t(p-1)-p^{k+1}+p^k}P_*^{p^k}x(\underline{r})$$

$$+(-1)\binom{[p^t(p-1)-p^{k+1}](p-1)-1}{p^{k+1}-1}Q^{p^t(p-1)-p^{k+1}}\beta x$$

$$+(-1)\binom{[p^t(p-1)-p^{k+1}](p-1)-1}{p-1}Q^{p^t(p-1)-p^{k+1}+p^k-1}\beta P_*^{p^k-1}\,x(\underline{r})\,.$$

The simplification is due to the assumption on k plus the Adem relations for the dual of the Steenrod Algebra, A. Notice that the first and third summands of (5.35) are zero since $[\deg x(\underline{r})] - 1 > 2(p^t(p-1) - p^{k+1})$. Moreover, the fourth summand is zero by assumption on k and by the Adem relations of A.

The second summand is non-trivial, by assumption on k. These facts imply that $P_*^{p^{k+1}}(\beta Q^{p^t(p-1)}x(\underline{r})) \neq 0$, which contradicts that $\beta Q^{p^t(p-1)}x(\underline{r})$ is A-annihilated.

Thus $P_*^{p^j}(x(\underline{r})) = 0$, for all $j \geq 0$. Moreover $\beta x(r) = 0$, as shown in (5.5). The proof of Theorem 5.1 is complete. For if a monomial y is in $\underline{Q}\, H_{2p^{t+1}(p-1)-2}(QX; \mathbb{Z}/p)_A$, then either Lemma 5.10 applies to show that $y \in H_{2p^{t+1}(p-1)-2}(X; \mathbb{Z}/p)_A$ or $y = \beta Q^{p^t(p-1)} x(\underline{r})$, with $x(r) \in H_{2(p-1)^2 p^t - 1}(X; \mathbb{Z}/p)_A$ as was just proved.

5.36. An Application. We have the following simple application of Theorem 5.1.

(5.37) Let Σ_p be the symmetric group, p an odd prime, and $B\Sigma_p$ the classifying space of Σ_p (see [A]).

The homology of $B\Sigma_p$ with $\mathbb{Z}/p$ coefficients is given by

$$(5.38)\quad H_m(B\Sigma_p; \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p, & m = 2t(p-1) \text{ or } m = 2t(p-1)-1 \\ 0, & \text{otherwise.} \end{cases}$$

Moreover the action of the dual of the Steenrod Algebra on $H_*(B\Sigma_p; \mathbb{Z}/p)$ is as follows:

$$(5.39)\quad P_*^i e_j = \binom{[j/2] - (p-1)i}{i} e_{j-2(p-1)i} \quad \text{(see [A], [Ma 2]).}$$

We determine $\underline{Q}\, H_{*\,2(p-1)p^{t+1}-2}(QB\Sigma_p; \mathbb{Z}/p)_A$. By Theorem 5.1,

$$(5.40)\quad \underline{Q}\, H_{2p^{t+1}(p-1)-2}(QB\Sigma_p; \mathbb{Z}/p)_A$$

$$= H_{2p^{t+1}(p-1)-2}(B\Sigma_p; \mathbb{Z}/p)_A \oplus \beta Q^{p^t(p-1)}[H_{2(p-1)^2 p^t - 1}(B\Sigma_p; \mathbb{Z}/p)_A].$$

From (5.38) we see that the first direct summand in (5.40) is trivial. To determine $\underline{Q}\,H_{2p^{t+1}(p-1)-2}(QB\Sigma_p;\mathbb{Z}/p)_A$ we are thus left to compute $H_{2(p-1)^2p^t-1}(B\Sigma_p;\mathbb{Z}/p)_A$. In degree $2(p-1)^2p^t-1$ formula (5.39) is

$$(5.41)\qquad P_*^i\, e_{2(p-1)^2p^t-1} = \binom{(p-1)^2p^t-1-(p-1)i}{i} e_{2(p-1)^2p^t-1-2(p-1)i}$$

If $i = p^n$, $0 \le n \le t-1$, the p-adic expansion of $(p-1)p^t-1-(p-1)i$ equals

$$(5.42)\qquad (p-2)p^{t+1}+(p-1)(p^{t-1}+\dots+1)-(p-1)p^n.$$

We see that (5.42) has no p^n term, so that the binomial coefficient of (5.41) is zero. The Adem relations on the dual of the Steenrod Algebra [Ma 2] imply that $e_{2(p-1)^2p^t-1}$ is Steenrod-annihilated, so we can conclude that

$$\underline{Q}\,H_{2p^{t+1}(p-1)-2}(QB\Sigma_p;\mathbb{Z}/p) = \langle \beta\, Q^{p^t(p-1)} e_{2(p-1)^2p^t-1} \rangle$$

BIBLIOGRAPHY TO PART 1

[A] J. Adem. The relations on Steenrod powers of cohomology classes. Algebraic Geometry and Topology. (A Symposium in Honor of S. Lefschetz), Princeton Univ. Press, Princeton, N.J. (1957), pp. 191-238.

[Br] W. Browder. Homology operations and loop spaces. Illinois J. Math. 4(1960), pp. 347-357.

[Co-La-Ma] F. R. Cohen, Thomas J. Lada and J. Peter May. The homology of iterated loop spaces. Lecture Notes in Math., Vol. 533. Springer-Verlag.

[D-L] E. Dyer and R. K. Lashof. Homology of iterated loop spaces. Amer. J. Math. 84(1962), pp. 35-88.

[Ma 1] J. Peter May. Categories of spectra and infinite loop spaces. Lecture Notes in Math., Vol. 99. Springer-Verlag.

[Ma 2] J. Peter May. A general algebraic approach to Steenrod operations. Lecture Notes in Math., Vol. 168. Springer-Verlag, pp. 153-231.

[Ma 3] J. Peter May. Homology operations on infinite loop spaces. Proc. Symp. Pure Math., Vol. 22. Amer. Math. Soc., 1971, pp. 171-185.

[Ma 4] J. Peter May. The geometry of iterated loop spaces. Lecture Notes in Math., Vol. 271. Springer-Verlag.

[Ma 5] The homology of E_∞ spaces and infinite loop spaces. Lecture Notes in Math. Vol.

[Mi] J. Milnor. The Steenrod algebra and its dual. Annals of Math. 67(1958), pp. 150-171

[N] G. Nishida. Cohomology operations in iterated loop spaces. Proc. Japan Acad. 44, 1968.

[P] S. Priddy. On $\Omega^{\infty}S^{\infty}$ and the infinite symmetric group. Proc. Symp. Pure Math., Vol. 22. Amer. Math. Soc. 1971, pp. 217-220.

[Sn-T] V. Snaith and J. Tornehave. On $\pi_*^S(BO)$ and the Arf invariant of framed manifolds. Contemporary Math., Vol. 12, 1982, pp. 299-313.

[St-E] N. E. Steenrod and D. B. A. Epstein. Cohomology operations. Annals of Math. Study No. 50, Princeton University Press, 1962.

PART 2

CHAPTER 0

INTRODUCTION

This work arose as an attempt to solve the problem proposed by Dr. V. P. Snaith concerning the determination of the algebra structure of the mod 2 K-homology theory, complex, periodic, of the spaces $\Omega^2 S^{2n+1}$. An antecedent in this direction was the knowledge of the Atiyah-Hirzebruch spectral sequence $H_*(\Omega^2 S^3 X; \mathbb{Z}/2) \Rightarrow K_*(\Omega^2 S^3 X; \mathbb{Z}/2)$, computed by Snaith in [Sn 6], where X is a finite CW-complex with $H_*(X; \mathbb{Z})$ torsion free (see Theorem 3.6 of the reference above). Snaith achieved the computation of the Atiyah-Hirzebruch spectral sequence by means of a secondary operation of Dyer-Lashof type he defined (and whose properties he studied) on $\frac{\ker \beta}{\operatorname{im} \beta} \subset \frac{K(X; \mathbb{Z}/2)}{\operatorname{im} \beta}$ when X is an H_1-space. This operation, denoted Q_α in [Sn 6] is defined as the composite

$$\frac{\ker \beta}{\operatorname{im} \beta} \xrightarrow{q} K_1^{\pi_2}(S^1_\pi \times X^2; \mathbb{Z}/2) \xrightarrow{\theta} K_1(X; \mathbb{Z}/2).$$

Here θ is the H_1-structure map, the middle group is the equivariant, mod 2, periodic K-homology of [At 2], [Se], and q is a function (without inderterminacy) defined using the techniques developed by Snaith in [Sn 5].

The link of the operation Q_α with the Atiyah-Hirzebruch spectral sequence was exhibited by Snaith [Sn 6]. He proved that the chain determining q(x) in the Atiyah-Hirzebruch spectral sequence also determines the homology Dyer-Lashof class $Q_1(\bar{x})$, for $\bar{x} \xmapsto[A-H]{} x$, (see Chapter 3). With this operation at hand, and with the knowledge of the first differential d_3 in the Atiyah-Hirzebruch spectral sequence, plus homotopical properties of loop spaces, the spectral sequence $H_*(\Omega^2S^3X; \mathbb{Z}/2) \underset{\overline{AH}}{\Rightarrow} K_*(\Omega^2S^3X; \mathbb{Z}/2)$ is reduced to a homological algebra problem. The differential $d_3 = Sq^3_* + Sq^1_*Sq^2_*$ was determined by Snaith in [Sn 5].

It turns out that a similar analysis applies to $K_*(\Omega^3S^3X; \mathbb{Z}/2)$, which allows us to determine the Atiyah-Hirzebruch spectral sequence converging to this group. We will deal with this problem in Chapter 5.

Returning to the analysis of the algebra structure of $K_*(\Omega^2S^{2n+1}; \mathbb{Z}/2)$, we attack it by means of a mixture of techniques, namely by:

1) The basic properties of $K_*(\ ; \mathbb{Z}/2)$ as a generalized homology theory endowed with an external multiplication nicely related to the product in $K_*(\ ; \mathbb{Z})$. The foundational paper by Araki and Toda [Ar-T] is the source of this theory. We will pursue on this point in Chapter 3.

2) The Atiyah-Hirzebruch spectral sequences for $K_*(\ ; \mathbb{Z})$ and $K_*(\ ; \mathbb{Z}/2)$, related by the reduction homomorphism.

An essential point here is the result of F. Cohen which states that the 2-primary torsion in $H(\Omega^2S^m; \mathbb{Z})$ is all of order 2 [Co-La-Ma, Part III].

3) The stable splitting of Ω^nS^nX due to Snaith [Sn 1], and especially the rich information on this splitting when $X = S^m$, a sphere, as dealt with in [B-P].

4) The knowledge of $K_*(QS^{2n+1}; \mathbb{Z}/2)$, computed by Snaith and Miller in [M-Sn].

5) Finally, and essentially whenever the Atiyah-Hirzebruch spectral sequence $H_*(\_; \mathbb{Z}/2) \Rightarrow K_*(\_; \mathbb{Z}/2)$ is involved, the Dyer-Lashof operations on the mod 2 homology of finite loop spaces as treated by W. Browder in [Br] and by P. May and F. Cohen in [Co-La-Ma]. The properties of the Dyer-Lashof operations on loop spaces are listed in Chapter 1.

CHAPTER 1

MOD 2 HOMOLOGY OPERATIONS

We give here the elements of the definition of the Arak-Kudo and Browder operations defined on H_n-spaces, following [Br]. The Araki-Kudo operations are called Dyer-Lashof operations in May's terminology [Ma 3].

A. H_n-spaces

Let π be the group of two elements, and let π act on $X \times X$ by permuting factors, for any space X, while π acts on S^n by the antipodal map.

Definition 1.1. [Br] A space X is called an H_n-space if there exists an equivariant map

$$\phi : S^n \times (X \times X) \to X$$

where π acts on the factors S^n and $X \times X$ as above and it acts trivially on X, such that there exists $e \in X$ for which, for all $t \in S^n$ and $x \in X$, $\phi(t, (e,x)) = \phi(t, (x,e)) = x$. ϕ is called the structure map of X.

Examples 1.2. An H-space X is an H_0-space with $\phi : S^0 \times X \times X \to X$ such that $\phi(1, (x,y)) = xy$ and $\phi(-1, (x,y)) = yx$.

b) Homotopy-commutative H-spaces are H_1-spaces.

c) Commutative H-spaces are H_n-spaces for all n.

d) Let P be the space of paths of a space X, E the space of paths which begin at $x_0 \in X$, and $p : E \to X$ the map sending a path of E to its end point. p is a fibre map, and the space of loops of X is defined as the fibre of p on x_0, i.e. $\Omega = p^{-1}(x_0)$.

There is the following theorem due to Araki and Kudo.

Theorem 1.3. [Br] Let X be an H_n-space. Then the space E of paths over X beginning a e is also an H_n space such that $p : E \to X$ is a map of H_n-spaces. Moreover Ω is an H_{n+1}-space.

B. $\underline{\lambda_n}$

For X an H_n-space W. Browder [Br] defined a function of two variables for any coefficients

$$\lambda_n : H_*(X;A) \otimes H_*(X,A) \to H_*(X;A)$$

in the following way.

Let ∇ be the natural map of normalized singular chains

$$\nabla : C(S^n) \otimes C(X) \otimes C(X) \to C(S^n \times X \times X).$$

Let $\phi_\#$ be the chain map induced by ϕ, and let $\phi = \phi_\# \cdot \nabla$. ϕ is an equivariant chain map and so it induces a map

$$\phi_* : H_*(S^n) \otimes H_*(X;A) \otimes H_*(X;A) \to H_*(X;A)$$

for any coefficients A. Choose a generator γ of $H_n(S^n)$ and define λ_n as follows:

Definition 1.4. [Br] Let $x \in H_p(X;A)$, $y \in H_q(X;A)$. Then $\lambda_n(x,y) \in H_{p+q+n}(X)$ is given by $\phi_*(\gamma \otimes x \otimes y)$. If X is an H_0-space then, for one choice of γ,

$$\lambda_0(x,y) = x * y - (-1)^{pq} y * x,$$

where * is the Pontrjagin product.

C. $\underline{Q_m}$.

The following definition of the Araki-Kudo operations was given by W. Browder in [Br].

The map ϕ factors through the collapsed module, as the action of π on X is trivial.

$$C(S^n) \otimes C(X) \otimes C(X) \xrightarrow{n} C(S^n) \underset{\pi}{\otimes} (C(X) \otimes C(X)) \xrightarrow{\phi_\#} C(X).$$

For a class $\bar{u} \in H_q(X; \mathbb{Z}/2)$, define an elementary chain complex $M(q)$ by: $C_r(M(q)) = 0$ if $r \neq q$ or $r \neq q-1$, $C_q(M(q))$ is infinite cyclic with generator u, and $C_{q-1}(M(q))$ is infinite cyclic with generator v. Define $\partial u = 2v$. Every chain map $f: M(q) \to C(X)$ defines a homology class $\bar{u} = \{f(u)\} \in H_q(X; \mathbb{Z}/2)$ and for every $\bar{u} \in H_q(X; \mathbb{Z}/2)$ one finds a chain representative of $\bar{u}$ which gives rise to a map $f: M(q) \to C(X)$. Thus there is an equivariant map

$$f : C(S^n) \underset{\pi}{\otimes} (M \otimes M) \to C(S^n) \otimes C(X) \otimes C(X),$$

which induces

$$f_\# : C(S^n) \underset{\pi}{\otimes} (M \otimes M) \to C(S^n) \underset{\pi}{\otimes} (C(X) \otimes C(X))$$

and the composite

$$\phi_{\#} \cdot f_{\#} : C(S^n) \underset{\pi}{\otimes} (M \otimes M) \to C(X)$$

induces the homomorphism

$$\Phi : H_*(C(S^n) \underset{\pi}{\otimes} (M \otimes M)) \to H_*(X).$$

Moreover any two chain representatives of $\overline{u}$ define the same homomorphism Φ. Giving S^n the antipodal triangulation so that $Te_i = \overline{e}_i$, the antipodal cell to e_i, $i \leq n$, we have

Definition 1.5. [Br] The m-th operation of Araki and Kudo is $Q_m(\overline{u}) = \phi_* f_*(e_m \otimes u \otimes u) = \Phi(\xi_m)$, where ξ_m is the generator of $H_m(RP^n; A)$, with RP^n the n-real projective space and $A = u \otimes u \otimes \mathbb{Z}_2$.

Definition 1.6. ([Ma 2, def. 2.2]) For $0 \leq s \leq n$ and $x \in H_q(X; \mathbb{Z}/2)$, define $Q^s(x) \in H_{q+s}(X)$ by $Q^s(x) = 0$ if $s < q$, and $Q^s(x) = Q_{s-q}(x)$ if $s \geq q$.

D. <u>Properties of the Operations</u>

The next theorem deals with the properties of the Araki-Kudo operations defined on an H_n-space (def. 1.1), except for the top one, Q_n. Moreover, as in most of the literature, the upper notation for the operations is used (see def. 1.6).

We also remark that a proof of the theorem requires the theory of operads of P. May, a topic that we will not treat here (see [Ma 4], [co-La-Ma]).

Theorem 1.7. [Co-La-Ma, Part III, Thm. 1.1] For $s - q < n$, the operations $Q^s : H_q(X) \to H_{q+s}(X)$ are homomorphisms, and are natural with respect to maps of H_n-spaces. These homomorphisms satisfy

(1) $Q^s x = 0$ if $s < \deg x$, $x \in H_*X$.

(2) $Q^s = x^2$ if $s = \deg x$, $x \in H_*X$.

(3) $Q^s \phi = 0$ if $s > 0$, where $\phi \in H_0X$ is the identity element.

(4) The ~~cartan for~~mula holds

$$Q^s(xy) = \sum_{i+j=s} (Q^i(x))(Q^j(y)), \; x,y \in H_*X.$$

$$\psi Q^s(x) = \sum_{i+j=s} (Q^i(x')) \otimes (Q^j(x'')), \text{ if } \psi x = \sum x' \otimes x'', \; x \in H_*X.$$

(5) (Adem relations) If $r > ps$, then

$$Q^r Q^s = \sum (-1)^{r+i} \begin{pmatrix} i-s-1 \\ 2i-r \end{pmatrix} Q^{r+s-i} Q^i.$$

(6) (Nishida relations) Let $Sq^r_* : H_*X \to H_*X$ be dual to Sq^r. Then

$$Sq^r_* Q^s = \sum_i (-1)^{r+i} \begin{pmatrix} s-r \\ r-2i \end{pmatrix} Q^{s-r+i} Sq^i_* .$$

Next, the properties of the Browder operation λ_n are recorded.

Theorem 1.8. ([Co-La-Ma, Part III, Thm. 1.2]. The Browder operations $\lambda_n : H_qX \otimes H_rX \to H_{q+r+n}$ are natural with respect to maps of H_n-spaces. They satisfy:

(1) If X is an H_{n+1}-space, $\lambda_n(x,y) = 0$ for $x,y \in H_*X$.

(2) $\lambda_0(x,y) = xy + yx$, $x,y \in H_*(X)$.

(3) $\lambda_n(x,y) = \lambda_n(y,x)$, $x,y \in H_*(X)$

$\lambda_n(x,x) = 0$,

(4) $\lambda_n(\phi,x) = 0 = \lambda\,(x,\phi)$, where ϕ is the identity element of H_*X, and $x \in H_*X$.

(5) (Nishida relations)

$$Sq_*^s \lambda_n(x,y) = \sum_{i+j=s} \lambda_n[Sq_*^i x, Sq_*^j y]$$

$$\beta \lambda_n(x,y) = \lambda_n(\beta x,y) + \lambda_n(x,\beta y), \ x,y \in H_*X.$$

(6) $\lambda_n(x,Q^s y) = 0 = \lambda_n(Q^s x,y)$, $x,y \in H_*X$.

The top Araki-Kudo operation on an H_n-space enjoys the properties stated in the following theorem. Following F. Cohen, Q_n is denoted ξ_n.

Theorem 1.9. (Ibid, Thm. 1.3) The function $\xi_n : H_qX \to H_{2q+n}X$ is natural with respect to maps of H_n-spaces. It satisfies the following properties, where $ad_n(x)(y) = \lambda_n(y,x)$, $ad_n^i(x)(y) = ad_n(x)(ad^{i-1}(x)(y))$.

(1) If X is an H_{n+1}-space, then $\xi_n(x) = Q^{n+q}(x)$, for $x \in H_qX$.

(2) If we let $Q^{n+q}x$ denote $\xi_n(x)$, then $\xi_n(X)$ satisfies formulas (1)-(3), (5) of Theorem 1.7. It also satisfies

$$\xi_n(xy) = \sum_{i+j=n+|xy|} (Q^i x)(Q^j y) + x\,\lambda_n(x,y)y,$$

where $|\ |$ denotes degree.

(3) (Nishida relations)

$$Sq_*^r \xi_n(x) = \Sigma\ (-1)^{r+i} \binom{n+q-r}{r-2i} Q^{m-r+i} Sq_*^i x + \Sigma\ ad_n\left(Sq_*^{i_{\sigma(1)}} x\right)\left(Sq_*^{i_1} x\right).$$

Here $m = n+|x|$, and the second sum runs over all sequences (i_1,i_2) such that $i_1+i_2 = r$, $i_1 < i_2$

$$\beta\ \xi_n(x) = (|x|+n-1)\ Q^{|x|+n-1}x + \lambda_n(\beta x,x).$$

(4) $\lambda_n(x,\xi_n y) = ad_n^2(y)(x)$, $x,y \in H_*X$.

(5) $\xi_n(x+y) = \xi_n(x) + \xi_n(y) + \lambda_n(x,y)$, $x,y \in H_*X$.

Theorem 1.10. (Ibid. Thm. 1.4) If $X = \Omega^{n+1}Y$, then

(1) $\sigma_* Q^s(x) = Q^s \sigma_*(x)$, $x \in H_*X$

(2) $\sigma_* \xi_n(x) = \xi_{n-1}\sigma_* x$, $x \in H_*X$

(3) $\sigma_* \lambda_n(x,y) = \lambda_{n-1}(\sigma_* x, \sigma_* y)$, $x,y \in H_*X$.

Here $\sigma_* : \tilde{H}_*\Omega^{n+1}Y \to H_*\Omega^n Y$ is the homology suspension.

E. $H_*(\Omega^n S^n X; \mathbb{Z}/2)$

We describe the procedure used by W. Browder [Br] to compute $H_*(\Omega^n S^n X; \mathbb{Z}/2)$, when this group is finitely generated (in the graded sense). There are canonical maps

$$\Sigma_n : X \to \Omega^n S^n X$$

with the property that $\sigma^n \Sigma_{n_*} = S^n$, where σ is the homology suspension and S is the suspension isomorphism. Thus Σ_{n_*} is a monomorphism.

If X is connected, it is a classical result that $H_*(\Omega S^n X; \mathbb{Z}/2) \cong T(H_*(S^{n-1}X; \mathbb{Z}/2))$, $n \geq 1$, where T(M) is the tensor algebra over $\mathbb{Z}/2$ generated by the graded module M. Moreover $T(M) = \otimes_i P(x_i)$ as $\mathbb{Z}/2$-modules, where $\{x_i\}$ is a basis of the graded Lie algebra generated by M in T(M). If $M = H_*(S^{n-1}X; \mathbb{Z}/2)$, $n > 1$, each x_i above is transgressive, for if $x_i[a_1, [a_2, [a_3, \ldots], \ldots]]$ with $a_i \in H_*(S^{n-1}X; \mathbb{Z}/2)$, and if $\sigma^n \alpha_i = a_i$, $\alpha_i \in H_*(X; \mathbb{Z}/2) \subseteq H_*(\Omega^n S^n X; \mathbb{Z}/2)$, then the class $\xi_i = \lambda_n(\alpha_1, \lambda_n(\alpha_2, \lambda_n(\alpha_3, \ldots), \ldots))$ is such that $\sigma^n \xi_i = x_i$.

The set $\{x_i^{2^r}\}$ forms a simple system of transgressive generators for $H_*(\Omega S^n X; \mathbb{Z}/2)$. Then by use of the comparison theorem for spectral sequences W. Browder proved

Theorem 1.11. [Br] $H_*(\Omega^n S^n X; \mathbb{Z}/2) = P(QH_*(X; \mathbb{Z}/2))$, $n \geq 2$, where $P(M)$ is the graded polynomial ring over $\mathbb{Z}/2$ generated by M, and $Q(H_*(X; \mathbb{Z}/2))$ is the submodule of $H_*(\Omega^n S^n X; \mathbb{Z}/2)$ generated by all elements $Q_1^{i_1} \ldots Q_{n-1}^{i_{n-1}}(\xi_j)$, for ξ_j as defined above, Q_m are the operations of Araki-Kudo and $(i_1, \ldots, i_{n-1})$ is any sequence of nonnegative integers, with $Q_m^{i_m}$ denoting iteration of Q_m (in particular, for all m, Q_m^0 = identity).

F. The Bockstein Sprectral Sequence for mod 2 Homology

1.12. The exact sequence

$$\ldots \xrightarrow{\partial} H_*(X; \mathbb{Z}) \xrightarrow{2\cdot -} H_*(X; \mathbb{Z}) \xrightarrow{\rho} H_*(X; \mathbb{Z}/2) \xrightarrow{\partial} H_*(X; \mathbb{Z}) \xrightarrow{2\cdot -} \ldots$$

derived from the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2\cdot -} \mathbb{Z} \xrightarrow{r} \mathbb{Z}/2 \to 0$ gives rise to an exact couple [Mas] which can be briefly described as a triangle of graded groups and graded maps

(1.13)
$$\begin{array}{ccc} H_*(X; \mathbb{Z}) & \xrightarrow{(2\cdot -)} & H_*(X; \mathbb{Z}) \\ {\scriptstyle\partial}\nwarrow & & \swarrow{\scriptstyle\rho} \\ & H_*(X; \mathbb{Z}/2) & \end{array}$$

with $\deg(2\cdot -) = \deg \rho = 0$ and $\deg(\partial) = -1$, and which is exact at each corner.

Let E_*^1 denote $H_*(X; \mathbb{Z}/2)$ and V denote $H_*(X; \mathbb{Z})$. The mod 2 Bockstein homomorphism $\beta = \rho\partial$ satisfies $\beta^2 = 0$, and allows to define $E_*^2 = \frac{\ker \beta}{\operatorname{im} \beta}$, which fits in an exact triangle

(1.14)
$$\begin{array}{ccc} 2V & \xrightarrow{2\cdot -} & 2V \\ {\scriptstyle \partial_2} \nwarrow & & \swarrow {\scriptstyle \rho_2 = \rho \cdot 2^{-1}} \\ & E^2 & \end{array}$$

called the derived couple of the exact couple (1.13). The maps of (1.14) are induced by those of (1.13) in the obvious manner.

Setting $B_2 = \rho_2 \cdot 2^{-1} \cdot \rho_2$ one defines $E^3_* = \frac{\ker B_2}{\operatorname{im} B_2}$.

The procedure can be iterated and the r-step gives the r-th derived couple

(1.15)
$$\begin{array}{ccc} 2^r V & \xrightarrow{2\cdot -} & 2^r V \\ {\scriptstyle \partial_r} \nwarrow & & \swarrow {\scriptstyle \rho_r = \rho \cdot 2^{-r}} \\ & E^r & \end{array}$$

$$B_r = \rho \cdot 2^{-r} \cdot \partial_r$$

Definition 1.16. The spectral sequence of the exact couple (1.13) is $\{E^r_*, B_r\}$, as defined above, and is called the Bockstein spectral sequence.

As is customary, we denote by Z^r the group $\ker B_r$, and B^r the group $\operatorname{im} B_r$. The following properties of $\{E^r_*, B_r\}$ will be useful to us.

Proposition 1.17. [Ar-T, II, sec. 11].

a) $\partial(Z^r(E^1)) = (2^r V) \cap (\ker 2\cdot -)$, $r \geq 1$

b) $\rho^{-1}(B^r(E^1)) = \ker(2^r : V \to V) + 2V$, $r \geq 1$

The Bockstein spectral sequence provides information on the $\mathbb{Z}$-homology of X.

Theorem 1.18. [Ibid.]

$$E_i^{\infty}(X; \mathbb{Z}/2) \cong \left[\frac{\widetilde{H}_i(X; \mathbb{Z})}{\text{tors}\,\widetilde{H}_i(X; \mathbb{Z})}\right] \otimes \mathbb{Z}/2$$

where tors $\widetilde{H}_i(X; \mathbb{Z})$ denotes the torsion subgroup of $\widetilde{H}_i(X; \mathbb{Z})$.

Analysis of the effect of higher Bocksteins on the operations Q_m, λ_n and ξ_n allowed F. Cohen to prove the following result.

Theorem 1.19. ([Co-La-Ma], Part III, Cor. 3.13) If X has no 2-torsion, then the 2-torsion of $H_*(\Omega^2S^2X; \mathbb{Z})$ is all of order 2.

This theorem will be crucial for our computations in Chapter 6.

CHAPTER 2

K-THEORIES

A. Generalities on G-vector Bundles

2.1. [Se] 1) Fix a topological group G. Then a G-space is a topological space X together with a (continuous) map $G \times X \to X$, denoted $(g,x) \to g \cdot x$, satisfying $g \cdot (g' \cdot x) = (g \cdot g')(x)$ and $1 \cdot x = x$, 1 the identity element of G.

2) Let X be a G-space. A G-vector bundle on X is a G-space E together with a G-map $p : E \to X$, i.e., a map such that $p(g \cdot x) = g \cdot p(x)$, which satisfies

a) $p : E \to X$ is a complex vector bundle on X.

b) For any $g \in G$, $x \in X$, the group action $g : E_x \to E_{gx}$ is a homomorphism of vector spaces.

3) The direct sum $\oplus$ and tensor product $\otimes$ of two G-vector bundles are defined fibrewise by $(E \oplus F)_x = E_x \oplus F_x$ and $(E \otimes F)_x = E_x \otimes F_x$, respectively. There is also the G-vector bundle $(\mathrm{Hom}(E,F))_x = \mathrm{Hom}(E_x,F_x)$ associated to the E-bundles E and F on X.

4) The sections of a G-vector bundle $E \xrightarrow{p} X$ are the maps $s : X \to E$ such that $ps = \mathrm{Id}$. If a section is a G-map, it is called equivariant.

5) A homomorphism $f : E \to F$ of G-vector bundles on X is a continuous G-map which induces a homomorphism of vector spaces $f_x : E_x \to F_x$ for each $x \in X$.

5) If M is a finite dimensional complex representation space of G (or simply: M is a G-module), the obvious G-vector bundle $X \times M$ is denoted $\mathbf{M}$.

6) Let $\phi : Y \to X$ be a G-map of G-spaces, and E a vector bundle on X. The induced vector bundle on Y, $\phi^*(E)$ defined as usual, is a G-vector bundle on Y.

Suppose that Y is an H-space, X is an G-space, $\alpha : H \to G$ a homomorphism, and $\phi : Y \to X$ satisfies $\phi(h \cdot y) = \alpha(h) \cdot \phi(y)$. Then ϕ^*E is an H-vector bundle on Y.

If $i : Y \subset X$ is the inclusion of a subspace i^*E is also denoted $E|_Y$.

Suppose from now on that G is a compact group. The following propositions of [Se] are the basic facts in the definition of equivariant K-theory.

Proposition 2.2. If E is a G-vector bundle on a compact G-space X, and A is a closed G-subspace of X, then an equivariant section of $E|_A$ can be extended to an equivariant section of E.

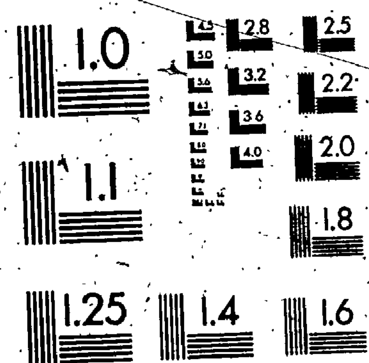
Proposition 2.3. Under the assumptions of 2.2, if F is another G-vector bundle on X and $f : E|_A \to F|_A$ is an isomorphism then there is a G-neighbourhood U of A in X and an isomorphism $f : E|_U \to F|_U$ extending f.

Proposition 2.4. If $\phi_0, \phi_1 : Y \to X$ are G-homotopic G-maps, and Y is compact, and E is a G-vector bundle on X, then $\phi_0^*E \simeq \phi_1^*E$.

B. $\underline{K_G(X)}$

2.5. 1) Let X be a compact G-space. The set of isomorphism classes of G-vector bundles on X forms an abelian semigroup under $\oplus$. The

2 OF / DE 2



Grothendieck construction on this semigroup is called $K_G(X)$, and it consists of formal differences $E_0 - E_1$ of G-vector bundles on X, modulo the equivalence relation: $E_0 - E_1 \doteq E'_0 - E'_1$ if and only if $E_0 \oplus E'_1 \oplus F \cong E'_0 \oplus E_1 \oplus F$ for some G-vector bundle F on X.

2) The tensor product of G-vector bundles induces a commutative ring structure in $K_G(X)$. K_G is a contravariant functor from compact G-spaces to commutative rings, via the induced G-vector bundle construction of 2.1.6.

3) A homomorphism $\alpha : H \to G$ induces a morphism of restriction via 2.1.6, and more generally if $\phi : Y \to X$ is a map from an H-space to a G-space compatible with α, (see 2.1.6), then $\phi^* : K_G(X) \to K_H(Y)$.

4) If G = 1, write K(X) for $K_G(X)$.

Example 2.6. If X = pt, then $K_G(X) \cong R(G)$, the representation ring of G. $K_G(X)$ is an algebra over R(G) via the map $X \to pt$, which assigns $M \mapsto [\mathbf{M}] = [X \times \hat{M}]$, (see 1 above).

2.7. If X is a compact H-space, the following G-space is constructed: $G \times X/H = G \times_H X$. There is an embedding $\phi : X \to G \times_H X$ which identifies X with the H-subspace $H \times_H X$ of $G \times_H X$. The induced vector-bundle construction (2.5.3) is an equivalence between G-vector bundles on $G \times_H X$ and H-vector bundles on X.

Let X be a compact G-space. The projection $pr : X \to X/G$ induces $pr^* : K(X/G) \to K_G(X)$.

Suppose that G acts freely on X; i.e., that $gx = x$ if and only if $g = 1$. The following proposition is proved in [Se].

Proposition 2.8. If G acts freely on X then $pr^* : K(X/G) \xrightarrow{\cong} K_G(X)$.

G acts trivially on X if $g \cdot x = x$ for all $g \in G$ and $x \in X$. There is the homomorphism $K(X) \to K_G(X)$ which gives a vector bundle the trivial G-action. There is also the natural map $R(G) \to K_G(X)$. With both these homomorphisms define $R(G) \otimes K(X) \to K_G(X)$.

Proposition 2.9. If X is a trivial G-space the natural map $\mu : R(G) \otimes K(X) \to K_G(X)$ is an isomorphism of rings.

The following property is deduced from Proposition 2.4.

Proposition 2.10. If $\phi_0, \phi_1 : Y \to X$ are G-homotopic G-maps then $\phi_0^* = \phi_1^* : K_G(X) \to K_G(Y)$.

The next result plays an important role in equivariant K-theory, as does the corresponding result for the non-equivariant case [Se].

Proposition 2.11. If E is a G-vector bundle on X, then there is a G-module M and a G-vector bundle $E^{\perp}$ such that $E \oplus E^{\perp} \cong \mathbf{M}$.

Definition 2.12. Two G-vector bundles E, E' on X are called stably equivalent if there exist G-modules M, M' such that $E \oplus \mathbf{M} \cong E' \oplus \mathbf{M}'$. From 2.11 the stable equivalence classes of G-vector bundles on X form an abelian group under $\oplus$. This group is called $\tilde{K}_G(X)$, and can be naturally identified with a quotient of $K_G(X)$.

Let X be a compact G-space with base point x_0. The reduced cone of X, CX, is obtained from the quotient of $X \times [0,1]$ by $(X \times 0) \cup (x_0 \times [0,1])$. For two based maps of compact G-spaces, $i_1 : X \to Y_1$, $i_2 : X \to Y_2$, $Y_1 \cup_X Y_2$ denotes the identification space obtained from the topological sum $Y_1 \cup Y_2$ by setting $i_1(x) = i_2(x)$ for all $x \in X$. The map from X to CX such that $x \mapsto [x,0]$ is an embedding, and $CX \cup_X CX$ is called the reduced suspension of X, denoted SX [Se].

Proposition 2.13: If X is a compact G-space with base point x_0, and A is a closed G-subspace (base point at x_0), then the sequence

$$\tilde{K}_G(X \cup_A CA) \to \tilde{K}_G(X) \to \tilde{K}_G(A)$$

is exact.

Via the obvious inclusion $X \subset X \cup_A CA$ construct $CX \cup_A CA$, and similarly from $X \subset X \cup_A CA \subseteq CX \cup_A CA$ construct $CX \cup_X C(X \cup_A (A) \simeq CX \cup_X CX \cup_{CA} C(CA)$. The maps in the following diagram are the obvious ones, and the diagram commutes up to G-homotopy, with rows which are G-homotopy equivalences

$$\begin{array}{ccc} SA & \longrightarrow & CX \cup_A CA \\ \downarrow & & \downarrow \\ SX & \longrightarrow & CX \cup_X (CX \cup_{CA} CA) \end{array}.$$

This fact produces the exact sequence

$$\tilde{K}_G(SX) \to \tilde{K}_G(SA) \to \tilde{K}_G(X \cup_A CA) \to \tilde{K}_G(X) \to \tilde{K}_G(A).$$

Definition 2.14. For a compact G-space with base points x_0 and A a closed G-subspace, and $q \in \mathbb{N}$, define

$$\tilde{K}_G^{-q}(X) = \tilde{K}_G(S^q X), \qquad S^q X = S(\ldots(SX)),$$

$$\tilde{K}_G^{-q}(X,A) = \tilde{K}_G(S^q(X \cup_A CA)),$$

$$\tilde{K}_G^{-q}(X,x_0) = \tilde{K}_G^{-q}(X).$$

Due to the identity $S^q(X \cup_A CA) = S^qX \cup_{S^qA} CS^qA$, iteration of the sequence 2.13 give the following exact sequence, infinite to the left

$$\to \tilde{K}_G^{-q}(X,A) \to \tilde{K}_G^{-q}(X) \to \tilde{K}^{-q}(A) \to \tilde{K}^{-q+1}(X,A) \to \cdots$$

$$\to \tilde{K}_G(X,A) \to \tilde{K}_G(X) \to \tilde{K}_G(A) \ .$$

For a locally compact G-space X which is not compact, X^+ denotes its one point compactification, which is a compact, based space. If X is compact, $X^+ = X \cup x_0$, disjoint union with a base point.

Definition 2.15. If X is a locally compact G-space, and A is a closed G-subspace, define

$$K_G^{-q}(X) = \tilde{K}_G^{-q}(X^+), \text{ and } K_G^{-q}(X,A) = \tilde{K}_G^{-q}(X^+,A^+).$$

Thus $K_G^{-q}(X,\phi) = K_G^{-q}(X)$.

C. $K_*(X;\, \mathbb{Z}/2)$

2.16. Let M_2 be the space $S^1 \cup_2 \subset S^1$, and let $U_n = \begin{cases} \mathbb{Z}\times BU, & (n \text{ even}) \\ U, & (n \text{ odd}) \end{cases}$ be the spaces of the unitary spectrum.

Definition 2.17. [Sn 5] The based maps from M_2 into the spaces U_n of the unitary spectrum constitute the spaces of a $\mathbb{Z}/2$-graded Ω-spectrum. The maps of this spectrum are induced by the Bott maps, $\{SU_n \xrightarrow{\alpha_n} U_{n+1}\}$.

The spectrum $\{U_n^{M_2}, \alpha_n\}$ defined above represents the generalized cohomology theory denoted $\tilde{K}^*(\_;\, \mathbb{Z}/2)$, which is $\mathbb{Z}/2$-graded. Thus $\tilde{K}^0(X;\, \mathbb{Z}/2) = [X, U_{2n}^{M_2}]$, $\tilde{K}^1(X;\, \mathbb{Z}/2) = [X, U_{2n+1}^{M_2}]$. The associated homology theory, denoted as usual $\tilde{K}_*(\_;\, \mathbb{Z}/2)$, is thus $\mathbb{Z}/2$-graded, and

$$\tilde{K}_0(X;\mathbb{Z}/2) = \lim_{\substack{\to \\ n}} [S^n, X \wedge U_n^{M_2}]$$

$$\tilde{K}_1(X;\mathbb{Z}/2) = \lim_{\substack{\to \\ n}} [S^{n+1}, X \wedge U_n^{M_2}]$$

Consider the cofibration $S^1 \xrightarrow{i} M_2 \xrightarrow{\pi} S^2$, where i is the inclusion and π shrinks S^1 to a point. From this one gets, by Bott periodicity, the exact sequence

$$(2.18) \quad [S^n, X \wedge U_n^{S^2}] \to [S^n, X \wedge U_n^{M_2}] \to [S^n, X \wedge U_n^{S^1}]$$
$$[S^n, X \wedge U_n^{S^2}] \cong [S^n, X \wedge U_{n-2}], \qquad [S^n, X \wedge U_n^{S^1}] \cong [S^n, X \wedge U_{n-1}]$$

Taking direct limits everywhere in (2.18) yields the exact sequence

$$\tilde{K}_*(X) \xrightarrow{\rho} \tilde{K}_*(X;\mathbb{Z}/2) \xrightarrow{\delta} \tilde{K}_{*-1}(X) .$$

ρ is called the reduction mod 2 and δ is the Bockstein homomorphism. The mod 2 Bockstein homomorphism is $\beta = \rho\cdot\delta$. From the sequence above follows the exact sequence

$$(2.19) \quad \cdots \xrightarrow{2\cdot} \tilde{K}_*(X) \xrightarrow{\rho} \tilde{K}_*(X;\mathbb{Z}/2) \xrightarrow{\delta} \tilde{K}_{*-1}(X) \xrightarrow{2\cdot} \tilde{K}_{*-1}(X) \xrightarrow{\rho} \cdots$$

and then the universal coefficient exact sequence

$$(2.20) \quad 0 \to \tilde{K}_*(X) \otimes \mathbb{Z}/2 \xrightarrow{\bar{\rho}} \tilde{K}_*(X;\mathbb{Z}/2) \xrightarrow{\bar{\delta}} \mathrm{Tor}(\tilde{K}_{*-1}(X), \mathbb{Z}/2) \to 0$$

where $\bar{\rho}$ and $\bar{\delta}$ are induced by ρ and δ respectively. Moreover, $\tilde{K}_*(X;\mathbb{Z}/2)$ is a $\mathbb{Z}/2$-vector space, a fact proved in [Ar-T 1] for $K^*(\_;\mathbb{Z}/2)$ and which holds for $K_*(\_;\mathbb{Z}/2)$ by the duality

$\tilde{K}_*(\_; \mathbb{Z}/2) = \mathrm{Hom}(\tilde{K}^*(\_; \mathbb{Z}/2), \mathbb{Z}/2)$, (see [An]).

Due to this duality a non-singular pairing is defined:

$$K_\alpha(X; \mathbb{Z}/2) \otimes K^\alpha(X; \mathbb{Z}/2) \to \mathbb{Z}/2 .$$

D. Multiplications in K-theory

2.21. The external product of complex vector bundles defines the multiplication in periodic, reduced K-theory

$$\nu : \tilde{K}^i(X) \otimes \tilde{K}^j(Y) \to \tilde{K}^{i+j}(X \wedge Y), \quad [\text{At-H}].$$

The definition of $K^*(X; \mathbb{Z}/2)$ given in 2.17 can be stated as $K^i(X, \mathbb{Z}/2) = \tilde{K}^{i+2}(X \wedge M_2)$, and the cofibration $X \wedge S^1 \to X \wedge M_2 \to X \wedge S^2$ provides us with the following notions [An-T I].

1) Suspension $\sigma_2 : K^j(X; \mathbb{Z}/2) \to K^{j+1}(SX; \mathbb{Z}/2)$, which is defined by the composite

$$\sigma_2 : \tilde{K}^{j+2}(X \wedge M_2) \xrightarrow[\cong]{\sigma} K^{j+3}(X \wedge M_2 \wedge S^1) \xrightarrow{(1\wedge T)^*} K^{j+3}(X \wedge S^1 \wedge M_2) = K^{j+1}(SX, \mathbb{Z}/2).$$

2) Reduction $\rho : \tilde{K}^j(X) \to \tilde{K}^j(X; \mathbb{Z}/2)$, given by

$$(1 \wedge \pi)\sigma^2 : \tilde{K}^j(X) \to \tilde{K}^{j+2}(X \wedge S^2) \to \tilde{K}^{j+1}(X \wedge M_2).$$

3) Bockstein homomorphism (coboundary), $\delta : \tilde{K}^j(X; \mathbb{Z}/2) \to \tilde{K}^{j+1}(X)$, defined as

$$\sigma^{-1}(1 \wedge i)^* : \tilde{K}^{j+2}(X \wedge M_2) \to \tilde{K}^{j+2}(X \wedge S) \xrightarrow{\cong} \tilde{K}^{j+1}(X) .$$

4) Mod 2 Bockstein homomorphism

$\beta : \tilde{K}^i(X; \mathbb{Z}/2) \to \tilde{K}^{i+1}(X; \mathbb{Z}/2)$, given by

$\beta = \rho \cdot \delta$.

5) The cofibration above yields the following exact sequence

$$\dots \to \tilde{K}(X) \xrightarrow{2\cdot} \tilde{K}^i(X) \xrightarrow{\rho} \tilde{K}^i(X; \mathbb{Z}/2) \xrightarrow{\delta} \tilde{K}^{i+1}(X) \to \dots$$

2.22. The product v induces the following homomorphisms v_R and v_L [Ar-T I].

$$\begin{array}{ccccc} \tilde{K}^i(X; \mathbb{Z}/2) \otimes \tilde{K}^j(Y) & & \xrightarrow{\quad v_R \quad} & & \tilde{K}^{i+j}(X \wedge Y; \mathbb{Z}/2) \\ \| & & & & \| \\ \tilde{K}^{i+2}(X \wedge M_2) \otimes \tilde{K}^j(Y) & \xrightarrow[v]{} & \tilde{K}^{i+j+2}(X \wedge M_2 \wedge Y) & \xrightarrow[(1\wedge T)^*]{} & \tilde{K}^{i+j+2}(X \wedge Y \wedge M_2) \end{array}$$

$$\begin{array}{ccc} \tilde{K}^i(X) \otimes \tilde{K}^j(Y; \mathbb{Z}/2) & \xrightarrow{v_L} & \tilde{K}(X \wedge Y; \mathbb{Z}/2) \\ \| & & \| \\ \tilde{K}^i(X) \otimes \tilde{K}^{j+2}(Y \wedge M_2) & \xrightarrow[v]{} & \tilde{K}^{i+j+2}(X \wedge Y \wedge M_2) \end{array}$$

The following formulas are stated in [Ar-T I].

2.23. 1) $v_R(\rho \otimes 1) = \rho v = v_L(1 \otimes \rho)$

2) $\delta v_R(x \otimes y) = v(\delta x \otimes y)$

3) $\delta v_L(x \otimes y) = v(x \otimes \delta y)$

4) $\beta v_R(x \otimes y) = v_R(\beta x \otimes y)$

5) $\beta v_L(x \otimes y) = v_L(x \otimes \beta y)$,

with the notation of 2.21

2.24. There is a mulitplication

$$v_2 : \tilde{K}^i(X;\mathbb{Z}/2) \otimes \tilde{K}^j(Y;\mathbb{Z}/2) \to \tilde{K}^{i+j}(X\wedge Y;\mathbb{Z}/2)$$

defined in [Ar-T II, sec. 6], and which is described as follows:

For $x \in \tilde{K}^i(X\wedge M_2)$ and $y \in \tilde{K}^j(Y\wedge M_2)$, the external product gives

$$x\cdot y \in \tilde{K}^{i+j}(X\wedge Y\wedge M_2\wedge M_2).$$

In [Ar-T I, sec. 4] is defined a complex $N_2 = S^2 \cup_g C(SM_2)$ and a map $\alpha : N_2 \to M_2\wedge M_2$ with the property that the cofibration sequence $0 \to K^*(X\wedge S^2M_2) \to K^*(X\wedge N_2) \to K^*(X\wedge S^2) \to 0$ is naturally split exact for all X. $v_2(x\otimes y) \in \tilde{K}^{i+j}(X\wedge Y;\mathbb{Z}/2) \cong \tilde{K}^{i+j}(X\wedge Y\wedge S^2M_2)$ is then defined as the component of $\alpha^*(x\cdot y)$ in this group.

Proposition 2.25. (Künneth isomorphism) v_2 is an isomorphism

$$v_2 : \tilde{K}^*(X;\mathbb{Z}/2)\otimes\tilde{K}^*(Y;\mathbb{Z}/2) \xrightarrow{\cong} \tilde{K}^*(X\wedge Y;\mathbb{Z}/2).$$ [Ar-T II].

2.26. The multiplication v_2 satisfies the following formulae [Ar-T I]:

1) $v_R = v_2(1\otimes\rho)$,

2) $v_L = v_2(\rho\otimes 1)$,

3) $\beta v_2(x\otimes y) = v_2(\beta x\otimes y) + v_2(x\otimes\beta y)$,

4) $v_2(v_2(x\otimes y)\otimes z] = v_2(x\otimes v_2(y\otimes z))$,

5) $v_2(\rho\otimes\rho) = \rho v$,

6) $\tau^* v_2(x\otimes y) = v_2(y\otimes x) + v_2(\beta x\otimes\beta y)$, where $\tau : X\wedge Y \to Y\wedge X$ is the switching map.

2.27. We notice that all the definitions above can be carried into K_*-theory, when the spaces involved are finite CW-complexes. This is due to the fact that $\widetilde{K}_*(X) \simeq \widetilde{K}^{-*}(DX)$, where DX denotes the Spanier-Whitehead dual of X (see [coh]).

We denote the homomorphisms in K-homology with the same symbols, while the duals of the products v, v_R, v_L and v_2 are denoted, respectively, by u, u_R, u_L and u_2.

Thus we have the K-homology counterparts of the formulae in 2.23 and 2.26 if we replace v_* by u_*, $* = R$, L, 2, or no symbol at all. Dual to 2.25 we have

Proposition 2.28. u_2 is an isomorphism,

$$u_2 : \widetilde{K}_*(X; \mathbb{Z}/2) \otimes \widetilde{K}_*(Y; \mathbb{Z}/2) \to \widetilde{K}_*(X \wedge Y; \mathbb{Z}/2)$$

2.29. Let X be a finite CW-complex which is a homotopy associative H-space with unit e. Let $i : \{e\} \to X$, $p : X \to e$ denote the inclusion and constant maps, and $h : X \times X \to X$, $\Delta : X \to X \times X$ the multiplication and diagonal maps of X. Define

$$\phi = \Delta^* \cdot v_2 : \widetilde{K}^*(X; \mathbb{Z}/2) \otimes \widetilde{K}^*(X; \mathbb{Z}/2) \to \widetilde{K}^*(X \wedge X; \mathbb{Z}/2) \to \widetilde{K}^*(X; \mathbb{Z}/2),$$

$$\psi = v_2^{-1} \cdot h^* : \widetilde{K}^*(X; \mathbb{Z}/2) \to \widetilde{K}^*(X \wedge X; \mathbb{Z}/2) \to \widetilde{K}^*(X; \mathbb{Z}/2) \otimes \widetilde{K}^*(X; \mathbb{Z}/2).$$

From [Ar] we have the following result:

Proposition 2.30. Suppose $v_2 : \widetilde{K}^*(X; \mathbb{Z}/2) \otimes \widetilde{K}^*(X; \mathbb{Z}/2) \to \widetilde{K}^*(X \wedge X; \mathbb{Z}/2)$ is commutative. Then $\widetilde{K}^*(X; \mathbb{Z}/2)$ is a Hopf Algebra with multiplication ϕ, comultiplication ψ, unit $\eta = p^*$, and counit $\varepsilon = i^*$.

With the obvious definitions of multiplication and comultiplication, unit and counit we have, dual to 2.30:

Proposition 2.31. If $u_2 : \widetilde{K}_*(X; \mathbb{Z}/2) \otimes \widetilde{K}_*(X; \mathbb{Z}/2) \to \widetilde{K}_*(X \wedge X; \mathbb{Z}/2)$ is commutative, then $\widetilde{K}_*(X; \mathbb{Z}/2)$ is a Hopf algebra.

Definition 2.32. For a compact G-space X define

$$\widetilde{K}^*_G(X; \mathbb{Z}/2) = \widetilde{K}^*_G(X \wedge M_2). \quad [\text{Sn } 5]$$

E. The Bockstein Spectral Sequence

2.33. Recall the Bockstein spectral sequence for $\mathbb{Z}/2$-homology described in 1.12-1.18. If we replace $H_*(X; \mathbb{Z})$ and $H_*(X; \mathbb{Z}/2)$ by $K_*(X; \mathbb{Z})$ and $K_*(X; \mathbb{Z}/2)$, respectively, and if we use the homomorphisms of 2.21 in their K-homology versions, then we obtain the exact couple for $K_*(\_; \mathbb{Z}/2)$-theory:

(2.34)

$$\begin{array}{ccc} K_*(X; \mathbb{Z}) & \xrightarrow{2\cdot -} & K_*(X; \mathbb{Z}) \\ {\scriptstyle \partial} \nwarrow & & \swarrow {\scriptstyle \rho} \\ & K_*(X; \mathbb{Z}/2) & \end{array}$$

Properties 1.17 are valid in this situation, except that, as we are in periodic $K_*\mathbb{Z}$-theory, degree must be replaced by filtration. The analogous of theorem 1.18 also holds in $K_*\mathbb{Z}$-theory.

We will exploit the following fact concerning the duality (see 2.20)

$$K_*(X; \mathbb{Z}/2) \simeq \mathrm{Hom}(K^*(X; \mathbb{Z}/2), \mathbb{Z}/2).$$

(2.35). The quotient $\dfrac{\ker \beta}{\mathrm{im}\ \beta}$ is self-dual for the duality above.

F. The Transfer Homomorphism

2.36. Let X and Y be compact spaces, and $f : X \to Y$ a finite covering. If E is a vector bundle on X, the direct image bundle $f_!(E)$ over Y is the bundle whose fibre at $y \in Y$ is $\bigoplus_{f(x)=y} E_x$, E_x the fibre of E over $x \in X$ ([Se], [Sn 5]). The function $E \mapsto f_!(E)$ is functorial on vector bundles, commutes with direct sums, so as to give rise to a homomorphism $f_! : K^*(X) \to K^*(Y)$. $f_!$ is called the transfer homomorphism associated with f ([At 1], [Sn 5]). The transfer homomorphism can be defined for the reduced theory, $f_! : \tilde{K}(X) \to \tilde{K}(Y)$ [Se].

The following property of the transfer $f_!$ is proved in [At 1].

2.36. If F is a vector bundle over Y and E is a vector bundle over X, then

$$f_!(E \otimes f^*(F)) \cong f_!(E) \otimes F.$$

Here $f^*(F)$ is the pullback of E along f.

2.38. Let X be a compact G-space, Y a closed G-subspace of X, and $j : H \subset G$ be the inclusion of a finite index subgroup. Define the map $f : (G \times X)/H \to X$ by $f[g,x] = x \cdot g^{-1}$, where $x \in G$ and $[\ ,\ ]$ an H-orbit equivalence class. Then f and its restriction to $G \times Y/H$ are finite coverings. G acts on $G \times X/H$ by multiplication on the G factor, and with this action, f is a G-map, [Se]. There is the isomorphism

$$\phi : K_H^*(X,Y) \cong K_G^*(G \times X/H, G \times Y/H)$$

and $f_!$ composed with ϕ gives a homomorphism $K_H^*(X;Y) \to K_G^*(X,Y)$, which happes to coincide with the homomorphism $j_! : K_H^*(X,Y) \to K_G^*(X,Y)$ induced by j, ([At 1], [Sn 5]).

G. $R(\pi_2)$

2.39. Consider the representation ring $R(\pi_2)$ of the group π_2, [At 1]. Let $y \in R(\pi_2)$ be the class of the one-dimensional complex representation of π_2 whose character is $e^{\frac{i \cdot 2\pi}{2}}$ on the canonical p-cycle.

Then $R(\pi_2) = \dfrac{\mathbb{Z}[y]}{(y^2 - 1)}$.

Let $G = 1 + y \in R(\pi_2)$. Thus in $R(\pi_2) \otimes \mathbb{Z}/2$, $\sigma^2 = 0$. Moreover $\{1,\sigma\}$ is a $\mathbb{Z}/2$-basis for $R(\pi_2) \otimes \mathbb{Z}/2$. With the notation of (2.38) for $X = \{pt\}$ $i_!(1) = \sigma \in R(\pi_2) \otimes \mathbb{Z}/2$, [Sn 5].

Let $i^* : K^*_{\pi_2}(X,Y; \mathbb{Z}/2) \to K^*(X,Y; \mathbb{Z}/2)$ be the forgetful homomorphism. For general X in (2.38), the composite $i_! i^*$ equal multiplcation by σ, and $i^* i_! = 1 + \tau^*$, [Sn 5].

We will use these facts in our computations in Chapter 3.

H. The Atiyah-Hirzebrich Spectral Sequence

2.40. The Atiyah-Hirzebrich spectral sequence for K^*-theory was set up by M. F. Atiyah and F. Hirzebruch in their foundational paper [At-H]. They stated it for finite simplicial complexes, but its validity for finite CW-complexes is proven in [Ad, Part III], where also a discussion of the spectral sequence for general, filtered, CW-complexes is given.

Notation. Let X be a finite CW-complex and X^n the n-th eleton of X. Filter $\tilde{K}^n(X)$ by $\tilde{K}^n_p(X) = \text{kernel } [\tilde{K}^n(X) \to \tilde{K}^n(X^{p-1})]$.

Theorem 2.41. Let X be a finite CW-complex, and {pt} the one point space, so that $K^*(pt) \simeq \mathbb{Z}$ if * is even and $K^*(pt) \simeq 0$ if * is odd. There exists a spectral sequence $E_r^{p,q}$, $(r \geq 1, -\infty < p,q < \infty)$ with

i) $E_1^{p,q} \cong C^p(X; K^q(pt))$, d_1 the ordinary coboundary,

ii) $E_2^{p,q} \cong H^p(X; K^q(pt))$,

iii) $E_\infty^{p,q} \cong G_p\tilde{K}^{p+q}(X) = \tilde{K}^{p+q}(X)/\tilde{K}^{p+q}_{p+1}(X)$,

iv) $dr : E_r^{p,q} \to E_r^{p+r,q-r+1}$ vanishes for r even, as $E_r^{p,q} = 0$ for all odd values of q.

The spectral sequence above is compatible with the Bott periodicity, so that the grading can be disregarded and we have

Theorem 2.42. [Ad] Let X be a finite CW-complex. Let $\tilde{K}^*_p(X)$ be the kernel of $\tilde{K}^*(X) \to \tilde{K}^*(X^{p-1})$. There exists a spectral sequence $E_r^p(X)$, $r \geq 1$, with

i) $E_1^p(X) \cong C^p(X; \mathbb{Z})$

ii) $E_2^p(X) \cong H^p(X, \mathbb{Z})$

iii) $E_\infty^p(X) \cong G_pK^*(X) = \tilde{K}^*_p(X)/\tilde{K}^*_{p+1}(X)$

iv) The differentials d_r vanish for even r.

Concerning the multiplicative structure of $K^*(X)$, the following holds.

Theorem 2.43. [At-H] Let X be as in 2.42, and consider the spectral sequence $E_r^p(X)$, $r \geq 2$, with the differentials d_r. Then the cup-product $E_2^p(X) \otimes E_2^q(X) \to E_2^{p+q}(X)$ induces pairings $E_r^p(X) \otimes E_r^q(X) \to E_r^{p+q}(X)$ which are maps of spectral sequences if $E_r^p(X) \otimes E_r^q(X)$ is given the differential $d_r \otimes 1 + (-1)^{p+q} 1 \otimes d_r$, (i.e., the differentials d_r are derivations). Moreover the pairing induced in $GK^*(X)$ coincides with the product induced by the ring structure of $\tilde{K}^*(X)$.

2.44. $\widetilde{K}^*$ and $\widetilde{K}_*$ satisfy the wedge axiom [Mi] so that, if $X_1 \subset X_2 \subset \ldots \subset X$ is a filtration of X by CW sub-complexes, then

Theorem 2.45. The group $\widetilde{K}_m(X)$ is canonically isomorphic to $\varinjlim_n \widetilde{K}_m(X_n)$.

Remark. We must make clear that 2.40-2.45 are valid if we introduce coefficients, and that the pairings of 2.43 are natural with respect to based maps in the category.

I. $\underline{K^*(RP^n; \mathbb{Z}/2)}$.

The K* theory of the real projective spaces was computed by Atiyah.

Proposition 2.46. [At 2] The structure of $K^*(RP^m)$ is as follows

$$K^1(RP^{2n+1}) = \mathbb{Z}$$

$$K^1(RP^{2n}) = 0$$

$$\widetilde{K}^0(RP^{2n+1}) = \widetilde{K}^0(RP^{2n}) = \mathbb{Z}_{2^n}.$$

2.46 and the universal coefficient theorem imply the following exact sequences:

2.47.

a) $$0 \to \widetilde{K}^0(RP^{2n}) \otimes \mathbb{Z}/2 \to \widetilde{K}^0(RP^{2n}; \mathbb{Z}/2) \to \mathrm{Tor}(\widetilde{K}^1(RP^{2n}), \mathbb{Z}/2) \to 0$$

where $\widetilde{K}^0(RP^{2n}) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2^n \otimes \mathbb{Z}/2 = \mathbb{Z}/2$ and $\mathrm{Tor}(\widetilde{K}^1(RP^{2n}), \mathbb{Z}/2) = 0$

for all $n \geq 0$.

Then $\widetilde{K}^0(RP^{2n}; \mathbb{Z}/2) = \mathbb{Z}/2$.

b) $$0 \to \widetilde{K}^1(RP^{2n}) \otimes \mathbb{Z}/2 \to \widetilde{K}^1(RP^{2n};\mathbb{Z}/2) \to \mathrm{Tor}(\widetilde{K}^0(RP^{2n}),\mathbb{Z}/2) \to 0$$

$$\widetilde{K}^1(RP^{2n}) \otimes \mathbb{Z}/2 \cong 0, \qquad \mathrm{Tor}(\widetilde{K}^0(RP^{2n}),\mathbb{Z}/2) = \mathrm{Tor}(\mathbb{Z}/2^n,\mathbb{Z}/2)$$

for all $n \geq 0$.

Then $K^1(RP^{2n};\mathbb{Z}/2) \cong \mathbb{Z}/2$.

c) $$0 \to \widetilde{K}^0(RP^{2n+1}) \otimes \mathbb{Z}/2 \to \widetilde{K}^0(RP^{2n+1};\mathbb{Z}/2) \to \mathrm{Tor}(\widetilde{K}^1(RP^{2n+1});\mathbb{Z}/2) \to 0$$

$$\widetilde{K}^0(RP^{2n+1}) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2^n \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2, \qquad \mathrm{Tor}(\widetilde{K}^1(RP^{2n+1});\mathbb{Z}/2) = \mathrm{Tor}(\mathbb{Z},\mathbb{Z}/2) = 0$$

for $n \geq 1$.

Then $\widetilde{K}^0(RP^{2n+1};\mathbb{Z}/2) \cong \mathbb{Z}/2$.

d) $$0 \to \widetilde{K}^1(RP^{2n+1}) \otimes \mathbb{Z}/2 \to \widetilde{K}^1(RP^{2n+1};\mathbb{Z}/2) \to \mathrm{Tor}(\widetilde{K}^0(RP^{2n+1}),\mathbb{Z}/2) \to 0$$

$$\widetilde{K}^1(RP^{2n+1}) \otimes \mathbb{Z}/2 \cong \mathbb{Z} \otimes \mathbb{Z}/2, \qquad \mathrm{Tor}(\widetilde{K}^0(RP^{2n+1}),\mathbb{Z}/2) = \mathrm{Tor}(\mathbb{Z}/2^n,\mathbb{Z}/2)$$

$$\text{Thus } \widetilde{K}^1(RP^{2n+1};\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \geq 1 \\ \mathbb{Z}/2 & n = 0. \end{cases}$$

Dualizing gives the same groups in $K_*\mathbb{Z}/2$.

2.48. [Sn-M] Let

$\xi_{2n} \in K_0(RP^{2n};\mathbb{Z}/2)$, $n \geq 1$

$\xi_{2n+1} \in K_0(RP^{2n+1};\mathbb{Z}/2)$, $n \geq 1$

$v_{2n} \in K_1(RP^{2n};\mathbb{Z}/2)$, $n \geq 1$

$v_{2n+1}, \alpha_{2n+1} \in K_1(RP^{2n+1};\mathbb{Z}/2)$, $n \geq 1$

be the generators. They are related by the Bocksteins (see 2.33) as follows:

$$v_{2n} = B_n\xi_{2n}, \quad v_{2n+1} = B_n\xi_{2n+1}$$

CHAPTER 3

THE ROTHENBERG-STEENROD SPECTRAL SEQUENCE

A. Preliminary

We briefly give the elements of the construction of spectral sequences of the Rothenberg-Steenrod type, which we record from [Sn 2].

3.1. Let H be a compact Lie group, with $K^*(H; k)$ a flat k-module. Let EH be a (free) right H-space, filtered by closed subspaces: $(pt) = D_0 \subset E_0 \subset D_1 \subset E_1 \subset \ldots$, where the H-invariant subspaces E_i satisfy:

i) $EH = \cup_n E_n$, with the topology of the union.

ii) E_n is contractible in E_{n+1}, for each n.

iii) For each n the action map $\psi_n : E_n \times H \to E_n$ restricts to a relative homeomorphism $\phi_n : (D_n, E_{n-1}) \times H \to (E_n, E_{n-1})$ and $\phi_0 : H \to E_0$ is a homeomorphism.

iv) Let I denote the unit interval. Then for each n there exists $u : D_n \to I$ and $h : I \times D_n \to D_n$ representing E_{n-1} as a neighbourhood deformation retract in D_n, so that $E_{n-1} \subset D_n$ is a cofibration and the unique maps v_n, h_n defined by the diagrams

(3.2)
$$\begin{array}{ccc} D_n \times H & \xrightarrow{p_1} & D_n \\ \uparrow & & \downarrow u_n \\ E_n & \xrightarrow[v_n]{} & I \end{array} \qquad \begin{array}{ccc} I \times D_n \times H & \xrightarrow{h_n \times 1} & D_n \times H \\ I \times \phi_n \downarrow & & \downarrow \phi_n \\ I \times E_n & \xrightarrow[h_n]{} & E_n \end{array}$$

are continuous. Properties (ii) and (iii) above ensure that the complex

$$0 \to k \xrightarrow{\eta} K^*(E_{0;k}) \xrightarrow{\delta} K^*(E_1, E_{0;k}) \xrightarrow{\delta} \cdots$$

is an exact sequence of right $K^*(H;k)$-comodules. Defining an H-action on $(D_n, E_{n-1}) \times H$ by $(D_n, E_{n-1}) \times H \times H \to (D_n, E_{n-1}) \times H$, $(z,h,h_0) \mapsto (z, h\cdot h_0)$, the following isomorphisms represent $K^*(E_n, E_{n-1};k)$ as an extended right $K^*(H;k)$-comodule:

$$K^*(D_n, E_{n-1}; k) \otimes K^*(H;k) \cong K^*((D_n, E_{n-1}) \times H; k) \cong K(E_n, E_{n-1}; k).$$

3.3. An example of a space EH is the Milnor H-resolution: Let $D_0 = \{1\} \subset E_0 = H$. Define, inductively, $D_n = C\,E_{n-1}$, the cone on E_{n-1}, and $E_n = (D_n \times H) \cup_{\psi_{n-1}} E_{n-1}$, $\psi_n : E_n \times H \to E_n$ such that $\psi_n | E_{n-1} \times H = \psi_{n-1}$, $\psi_{n-1} = \psi_n | \{t\} \times E_{n-1} \times H \to \{t\} \times E_{n-1}$ for $t \in I$.

For $H = \pi_2$, the Milnor resolution is the filtration given by $\cdots \subset S^n = E_n \subset S^{n+1} = E_{n+1} \subset \cdots S^\infty = B\pi_2$, with the π_2-intopodal action on each sphere.

3.4. In what follows k is taken to be $\mathbb{Z}/2$. Following [Ho] let Q_H denote the category of compact, locally contractible H-spaces of finite covering dimensions. On Q_H the groups $K_H^*(\ ;\mathbb{Z}/2)$ are defined as in 2.32. For spaces $X = \cup_n X_n$, $X_n \in Q_H$, $X_n \subset X_{n+1}$, $K_H^*(X;\mathbb{Z}/2)$ can be defined as

$$K_H^*(X;\mathbb{Z}/2) = \varprojlim_n K^*(X_n;\mathbb{Z}/2).$$

$K_H^*(X;\mathbb{Z}/2)$ is a module over $R(H;\mathbb{Z}/2) = K_H^*(pt;\mathbb{Z}/2)$.

3.5. Let $X \in Q_H$ be an H-space. Define

$$X_H = X \wedge \frac{EH^+}{G}, \qquad X_{H,n} = X_n \wedge \frac{E_n^+}{E}.$$

The filtration $X_{H,n}$ of X_H gives, upon application of $K^*(\_;\mathbb{Z}/2)$, a spectral sequence, called the Rothenberg-Steenrod spectral sequence for X. For an H-pair (X,Y) with Y a closed H-subspace of X, the filtration $(X,Y)_{H,n} = \frac{X}{Y} \wedge E_n^+$ of $(X,Y)_H = \frac{X}{Y} \wedge EA^+$ induces, upon applying $K^*(\ ;\mathbb{Z}/2)$, a corresponding relativized spectral sequence called the Rothenberg-Steenrod spectral sequence for (X,Y).

Theorem 3.6. The Rothenberg-Steenrod spectral sequence for X, $\{E_s, d_s\}$, $(s \geq 2)$, is a strongly convergent spectral sequence of $\mathbb{Z} \times \mathbb{Z}_2$ bigraded $\mathbb{Z}/2$-algebras and $E_s^{**}(pt,\phi\ \mathbb{Z}/2)$-modules satisfying:

1) $E_2^{p,\alpha} = \mathrm{Cotor}^{p,\alpha}_{K^*(H,\mathbb{Z}/2)}(\mathbb{Z}/2, K^*(X;\mathbb{Z}/2)) \Rightarrow K^*(X_H;\mathbb{Z}/2) \cong K_H^*(X;\mathbb{Z}/2)^\wedge$

2) $d_s : E_s^{p,\alpha} \to E_s^{p+s,\alpha-s+1}$ is a derivation with respect to the $\mathbb{Z}/2$ and $E_s^{**}(pt,\phi;\mathbb{Z}/2)$ actions mentioned above.

3) The spectral sequence is multiplicative.

4) If H is a subgroup of Σ_n and X is a finite complex, the spectral sequence converges strongly to $K_H^*(X;Y;\mathbb{Z}/2)$. This is so as, for such an H, $K_H^*(X,Y;\mathbb{Z}/2) \cong K_H^*(X,Y;\mathbb{Z}/2)^\wedge$ [Sn 5].

Due to the isomorphism $K_H^*(X,Y;\mathbb{Z}/2) \cong K_H^*((X,Y)_H;\mathbb{Z}/2)^\wedge$ for $H \subset \Sigma_n$, we are entitled to define

$$(3.7)\qquad K_*^H(X;Y;\mathbb{Z}/2) = K_*((X,Y)_H;\mathbb{Z}/2).$$

Dual to 3.5 we have

Theorem 3.7. Let $H \subset \Sigma_n$. There is a spectral sequence $E^s_{**}(X,Y,\mathbb{Z}/p), d^s)$, $s \geq 2$, natural in the H-pair (X,Y), satisfying

1) The terms E^S_{**} are $\mathbb{Z}\times\mathbb{Z}_2$-bigraded coalgebras, and $E^S_{**}(pt,\phi;\mathbb{Z}/2)$ comodule.

2) d^S is a derivation with respect to the coactions above.

3) $E^2_{p,\alpha}(X,Y;\mathbb{Z}/2) \cong \mathrm{Tor}^{\mathbb{Z}/2[H]}_{p,\alpha}(K_*(X,Y),\mathbb{Z}/2)$.

4) The spectral sequence is multiplicative.

5) The spectral sequence converges strongly to $K^H_*(X,Y;\mathbb{Z}/2)$.

The properties (ii) and (iii) of the H-resolution imply

$$(3.9)\qquad \mathbb{Z}/2 \xrightarrow{\varepsilon} K^*(E_0;\mathbb{Z}/2) \xrightarrow{\delta} K^*(E_1,E_0;\mathbb{Z}/2) \xrightarrow{\delta} K^*(E_2,E_1;\mathbb{Z}/2) \xrightarrow{\delta} \cdots$$

$$(3.10)\qquad \mathbb{Z}/2 \xleftarrow{\eta} K_*(E_0;\mathbb{Z}/2) \xleftarrow{\delta} K_*(E_1,E_0;\mathbb{Z}/2) \xleftarrow{\delta} K_*(E_2,E_1;\mathbb{Z}/2) \xleftarrow{\delta} \cdots$$

are, respectively, free $K^*(H;\mathbb{Z}/2)$-comodule and $K_*(H;\mathbb{Z}/2) = \mathbb{Z}/2[H]$-module resolutions of $\mathbb{Z}/2$. 3.8 and 3.9 are dual chain complexes, by the natural isomorphism $K^*(\_;\mathbb{Z}/2) \cong \mathrm{Hom}(K_*(\_;\mathbb{Z}/2),\mathbb{Z}/2)$. We have from [Sn 5, sec 1]:

Condition (iii) on the resolution EH implies

$$(3.11)\qquad \tilde{K}^*((\tfrac{X}{Y}\wedge E_n^+)/H,\ (\tfrac{X}{Y}\wedge E_{n-1})/H;\mathbb{Z}/2) \cong$$

$$\tilde{K}^*([(X,Y)\wedge(E_n,E_{n-1})]/H;\mathbb{Z}/2) \cong$$

$$K^*(X,Y;\mathbb{Z}/2)\ \square_{K^*(H;\mathbb{Z}/2)}\ K^*(E_n,E_{n-1};\mathbb{Z}/2)$$

and dually

$$(3.12)\qquad \tilde{K}_*((\tfrac{X}{Y}\wedge E_n)/H,\ (\tfrac{X}{Y}\wedge E_{n-1})/H;\mathbb{Z}/2) \cong$$

$$K_*(X,Y;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2[H]} K_*(E_n,E_{n-1};\mathbb{Z}/2).$$

Under the identifications (3.11) and (3.12) the E_1-differentials are $1\,\square\,\delta$ and $1\otimes\delta$.

Recall the Milnor π_2-resolution described in 3.3 where $D^n_\pi = CS^{n-1}_\pi$, the cone on S^{n-1}_π, with the obvious action of π_2 on D^n_π. The following result is proved in [Sn 5] as Proposition 2.3.

Lemma 3.13. Let X be a compact π_2-space. For m > 0, there are isomorphisms

$$K_{\pi_2}(X,Y;\mathbb{Z}/2) \cong K^*_{\pi_2}((X,Y) \times (D^{2m}_\pi, S^{2m-1}_\pi; \mathbb{Z}/2) \cong$$

$$K^*_{\pi_2}((X,Y) \times (E\pi_2, S^{2m-1}_\pi); \mathbb{Z}/2).$$

We also record the following result which is proved in [Sn 5] as Proposition 2.4.

Lemma 3.14. Let $i : \{1\} \to \pi_2$ be the inclusion of the identity, and let X be a compact π_2-space. Under the isomorphism of 3.13 and the isomorphism

$$K^\alpha_{\pi_2}((X,Y) \times (S^1_\pi, S^0_\pi); \mathbb{Z}/2) \to K^{\alpha+1}_{\pi_2}((X,Y); \mathbb{Z}/2)$$

the coboundary

$$\delta : K^\alpha_{\pi_2}((X,Y) \times (S^1_\pi, S^0_\pi); \mathbb{Z}/2) \to K^{\alpha+1}_{\pi_2}((X,Y) \times (E\pi_2, S^1_\pi); \mathbb{Z}/2)$$

corresponds to the transfer $(i)_!$, (2.36).

Lemma 3.15. Let i and (X,Y) be as in 3.14. Under the isomorphisms of 3.13 and the isomorphism $K^\alpha_{\pi_2}((X,Y) \times (S^2_\pi, S^1_\pi); \mathbb{Z}/2) \cong K^\alpha(X,Y;\mathbb{Z}/2)$ the restriction

$$j : K^\alpha_{\pi_2}((X,Y) \times (E\pi_2, S^1_\pi); \mathbb{Z}/2) \to K^\alpha_{\pi_2}((X,Y) \times (S^2_\pi, S^1_\pi); \mathbb{Z}/2)$$

corresponds to the homomorphism $(i)^*$.

B. $E_2^{**}((pt,\phi);\ \pi_2;\ \mathbb{Z}/2)$

Snaith [Sn 3] proved that the spectral sequence obtained by applying integral K*-theory to the $B\pi_2$ filtration collapses, and has non-zero groups

$$E_2^{2q,0} = \mathrm{Cotor}^{2q,0}_{K^*(\pi_2)}(\mathbb{Z},\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/2, & q > 0. \end{cases}$$

From this he deduced the spectral sequence obtained by applying $K^*\ \mathbb{Z}/2$ to $B\pi_2$.

Theorem 3.16. [Sn 5, Prop. 2.6]

(a) i) $E_2^{q,t}((pt,\phi);\ \pi_2;\ \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/p, & t \equiv 0 \mod 2 \\ 0, & t \equiv 1 \mod 2 \end{cases}$

ii) $E_2^{2q,0}((pt,\phi);\ \pi_2;\ \mathbb{Z}/2)$ consists of permanent cycles.

iii) Any element of the form $\sigma^q + \sum_{j>q} \sigma^j \otimes a_j \in R(\pi_2) \otimes \mathbb{Z}/2$ is represented by the canonical generator of $E_2^{2q,0} \cong E_\alpha^{2q,0}$.

iv) The only non-zero differential is

$$d_3 : E_2^{2q+1,0} \cong E_3^{2q+1,0} \xrightarrow{\cong} E_3^{2(q+2),0} \cong E_2^{2(q+2),0}.$$

Dually

(b) i) $E^2_{q,t}((pt,\phi);\pi_2;\ \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & t \equiv 0 \pmod 2, \\ 0, & t \equiv 1 \pmod 2. \end{cases}$

ii) If j is odd or j < 4, $E^2_{j,0}((pt,\phi);\ \pi_2;\ \mathbb{Z}/2)$ consists of permanent cycles.

iii) The only non-zero differential is

$$d^3 : E^2_{2(q+2),0} \cong E^3_{2(q+2),0} \xrightarrow{\cong} E^3_{2q+1,0} \cong E^2_{2q+1,0}$$

C. $\underline{E_2^{**}((X,Y)^2;\ \pi_2;\ \mathbb{Z}/2)}$

Recall from 3.6 and 3.7 the E_2^{**} and E^2_{**} terms of the Rothenberg-Steenrod spectral sequences for $K_H^*(\_;\mathbb{Z}/2)_*$ and $K_*^H(\_;\mathbb{Z}/2)$, respectively, where $H \subset \Sigma_n$. In [Sn 5, sec. 1] the isomorphisms $K_*((X,Y)^2;\mathbb{Z}/2) \cong K_*(X,Y;\mathbb{Z}/2)^{\otimes 2} \cong K_*(X,Y;\mathbb{Z}/2)^{\otimes 2} \otimes_{\mathbb{Z}/2[\pi_2]} e_i$, (see 3.3), are shown to render $\mathrm{Tor}_{**}^{\mathbb{Z}/p[\pi_2]}(K_*((X,Y)^2;\mathbb{Z}/2),\mathbb{Z}/2)$ computable as the homology of the complex

$$0 \leftarrow K_*(X,Y;\mathbb{Z}/2)^{\otimes 2} \xleftarrow{1+\tau_*} K_*(X,Y;\mathbb{Z}/2)^{\otimes 2} \xleftarrow{1+\tau_*} K_*(X,Y;\mathbb{Z}/2)^{\otimes 2} \longleftarrow;$$

π_2 acts on $(K_*(X,Y;\mathbb{Z}/2))^{\otimes 2}$ by $\tau_*(x\otimes y) = y\otimes x + \beta x\otimes \beta y$. Moreover, for a suitable basis for $K_*(X,Y;\mathbb{Z}/2)$, [Ar-Y, 6.14], $(K_*(X,Y;\mathbb{Z}/2))^{\otimes 2}$ is, as a π_2-module, the direct sum of π_2-submodules of type $\{u_i\otimes u_j,\ u_i\otimes u_j + \beta u_i\otimes \beta u_j\}$ and $\{u_i^{\otimes 2}\}$ where $\beta u_i = 0$ and $u_i \notin \mathrm{im}\,\beta$.

We state the following result from [Sn 5, Thm. 3.8]:

Theorem 3.17. (a) In the spectral sequence $\{E^r((X,Y)^2;\pi_2;\mathbb{Z}/2)\}$ the only non-trivial differential is d_3, and if $x \in K_\alpha(X;Y;\mathbb{Z}/2)$, then

i) $d_3(x^{\otimes 2}\otimes e_{2q}) = x^{\otimes 2}\otimes e_{2q-3}$ if $\alpha \equiv 0 \mod 2$

ii) $d_3(x^{\otimes 2}\otimes e_{2q+1}) = x^{\otimes 2}\otimes e_{2(q-1)}$ if $\alpha \equiv 1 \mod 2$

where, in (i) and (ii), $x \in \ker\beta - \mathrm{im}\,\beta$.

iii) d_3 is zero otherwise.

The dual result is

(b) In the spectral sequence $\{E_r((X,Y)^2;\pi_2;\mathbb{Z}/2)\}$ the only differential is d_3, whose action is

i) $d_3(x^{\otimes 2} \otimes e_{2q+1}) = x^{\otimes 2} \otimes e_{2(q+2)}$ if $\alpha \equiv 0$, mod 2

ii) $d_3(x^{\otimes 2} \otimes e_{2q}) = x^{\otimes 2} \otimes e_{2q+3}$ if $\alpha \equiv 1$, mod 2,

where in both i) and ii), $x \in \ker \beta - \operatorname{im} \beta \subset K^{\alpha}(X,Y; \mathbb{Z}/2)$;

iii) d_3 is zero otherwise.

3.18. The pairing $\langle , \rangle : K_{\alpha}(; \mathbb{Z}/2) \otimes K^{\alpha}(; \mathbb{Z}/2) \to \mathbb{Z}/2$ of 2.21 passes to $K_{\alpha}^{\pi_2}(; \mathbb{Z}/2) \otimes K^{\alpha}_{\pi_2}(; \mathbb{Z}/2)$ by the spectral sequences, a fact which is exploited in the sequel.

In [Sn 5], using bundle theoretic constructions, the following theorem is proved.

Theorem 3.19. [Sn 5, Prop. 4.10]

i) Let $Z_1^{\otimes 2} \otimes e_1 \in K^1_{\pi_2}((X,Y)^2; \mathbb{Z}/2)$ determine a class in $E_{\alpha}^{1,0}((X,Y)^2; \pi_2; \mathbb{Z}/2)$. Then $\beta_2(Z_1^{\otimes 2} \otimes e_1) = i_!(B_2(Z_1)^{\otimes 2})$
$\in K^0_{\pi_2}((X,Y)^2; \mathbb{Z}/2)$.

ii) Let $Z_0^{\otimes 2} \otimes e_0 \in (i^*)^{-1}(Z_0^{\otimes 2}) \subset K^0_{\pi_2}((X,Y)^2; \mathbb{Z}/2)$. Then

$$\beta_2(Z_0^{\otimes 2} \otimes e_0) = B_2(Z_0)^{\otimes 2} \otimes e_1 \in K^1_{\pi_2}((X,Y)^2; \mathbb{Z}/2).$$

D. $K^*_{\pi_2}(S^v_{\pi} \times X^2; \mathbb{Z}/2)$ and $K_*^{\pi_2}(S^v_{\pi} \times X^2; \mathbb{Z}/2)$, $v = 1,2$

3.20. Denote S^v_{π} the v-sphere with the antipodal action of π_2, the group of order 2, where $v = 1,2$. Let π_2 act on X^2 permuting the factors and consider the π_2-pair $(S^v_{\pi} \times X^2; \phi)$. From 3.6 we distill the following facts:

(a) For the π_2-pair (X,Y), $Y \subset X$ exists a multiplicative spectral sequence such that

i) $E_2^{q,\alpha}(S_\pi^V \times X^2, \phi) = \mathrm{Cotor}^{q,\alpha}_{K^*(\pi_2;\, \mathbb{Z}/2)}(K^*(S_\pi^V \times X^2;\, \mathbb{Z}/2)$

$$\Rightarrow K^*_{\pi_2}(S_\pi^V \times X^2;\, \mathbb{Z}/2)$$

bigraded by $\mathbb{Z} \times \mathbb{Z}/2$.

Let A_V denote the $\mathbb{Z}/2$-vector space with basics 1, ξ_V such that $A_V = K^*(S^V;\, \mathbb{Z}/2)$;

Snaith's computation of $E_2^{q,}\ (S_\pi^V \times X^2;\, \phi)$, [Sn 6, sec. 1] takes the following form

ii) $\mathrm{Cotor}^{q,\alpha}_{K^*(\pi_2;\, \mathbb{Z}/2)}(K^*(S_\pi^V \times X^2;\, \mathbb{Z}/2, \mathbb{Z}/2) \cong$

$$\begin{cases} \pi_2\text{-invariants of } A_V \otimes K^*(X;\, \mathbb{Z}/2) & \text{if } q = 0 \\ A_V \otimes \dfrac{\ker \beta}{\mathrm{im}\ \beta} \subset K^*(S_\pi^V;\, \mathbb{Z}/2) \otimes \dfrac{K^*(X;\, \mathbb{Z}/2)}{\mathrm{im}\ \beta} & \text{if } q > 0 \end{cases}$$

iii) $E_1^{q,*} \cong A_V \otimes (K^*(X;\, \mathbb{Z}/2))^{\otimes 2} \otimes \{e_q\}$, $E_2^{0,*} \subset E_1^{0,*}$ as the inclusion of the π_2-invariants.

iv) If $q > 0$, $E_2^{q,*} \xleftarrow[\delta]{\cong} A_V \otimes \dfrac{\ker \beta}{\mathrm{im}\ \beta} \subset A_V \otimes \dfrac{K^*(X;\, \mathbb{Z}/2)}{\mathrm{im}\ \beta}$

with $\delta(a \otimes [x + \mathrm{im}\ \beta]) = a \otimes x^{\otimes 2} \otimes e_q$.

Dual to (a) we have, with $(q,\alpha) \in \mathbb{Z} \times \mathbb{Z}/2$,

(b) i) $E^2_{q,\alpha}(S_\pi^V \times X^2, \phi) = \mathrm{Tor}^{\mathbb{Z}/2[\pi_2]}_{q,\alpha}(K_*(S_\pi^V \times X^2;\, \mathbb{Z}/2);\, \mathbb{Z}/2)$

$$\Rightarrow K_*^{\pi_2}(S_\pi^V \times X^2;\, \mathbb{Z}/2);$$

If $A_V = K_*(S^V;\, \mathbb{Z}/2) = \{1, \xi_V\}$, as $\mathbb{Z}/2$-vector space, then

ii) $\mathrm{Tor}^{\mathbb{Z}/2[\pi_2]}_{q,\alpha}(K_*(S^V_\pi \times X^2; \mathbb{Z}/2); \mathbb{Z}/2) \cong$

$$\begin{cases} \pi_2\text{-coinvariants of } A_V \otimes K_*(X, \mathbb{Z}/2) & \text{if } q = 0 \\ A_V \otimes \dfrac{\ker \beta}{\operatorname{im} \beta} \subset K_*(S^V; \mathbb{Z}/2) \otimes \dfrac{K_*(X; \mathbb{Z}/2)}{\operatorname{im} \beta} & \text{if } q > 0 \end{cases}$$

iii) $E_1^{q,\alpha} \cong A_V \otimes (K_*(X; \mathbb{Z}/2))^{\otimes 2} \otimes \{e_q\}$.

iv) For $q > 0$, $E^2_{q,\alpha} \xrightarrow[\delta]{\cong} A_V \otimes \dfrac{\ker \beta}{\operatorname{im} \beta}$ is given by

$$\delta(a \otimes x^{\otimes 2} \otimes e_q) = a \otimes [x + \operatorname{im} \beta].$$

3.21. In [Sn 6, sec. 1] Snaith computed the Rothenberg-Steenrod spectral sequences $E_2^{**}(S^V \times X^2; \phi) \Rightarrow K^*_{\pi_2}(S^V_\pi \times X^2; \mathbb{Z}/2)$ $V = 1,2$. In the above spectral sequence the only non-zero differential is d_{V+1} whose action is given by

a) $d_2(\delta_1 \otimes x^{\otimes 2} \otimes e_j) = 1 \otimes x^{\otimes 2} \otimes e_{j+2}$

b) $$d_3((\lambda + \mu\xi_2) \otimes x^{\otimes 2} \otimes e_j) = \begin{cases} \mu(1 + \xi_2) \otimes x^{\otimes 2} \otimes e_{j+3}, & j \equiv \deg(x) \pmod 2 \\ (\lambda + \mu) \otimes x^{\otimes 2} \otimes e_{j+3}, & \text{otherwise} \end{cases}$$

where $\beta x = 0$.

Dually, in $E^2_{**}(S^V_\pi \times X^2, \phi) \Rightarrow K_*^{\pi_2}(S^V_\pi \times X^2; \mathbb{Z}/2)$ the only non-zero differential is d_{V+1}, with

a) $d_2(1 \otimes x^{\otimes 2} \otimes e_{j+2}) = \xi_1 \otimes x^{\otimes 2} \otimes e_j$

b) $$d_3((\lambda + \mu\xi_2) \otimes x^{\otimes 2} \otimes e_{j+3}) = \begin{cases} (\lambda + \mu\xi_2) \otimes x^{\otimes 2} \otimes e_j, & j \not\equiv \deg(x) \pmod 2 \\ \lambda(1 + \xi_2) \otimes x^{\otimes 2} \otimes e_j, & \text{otherwise} \end{cases}$$

where $\beta x = 0$.

c) Moreover $E^{\infty}_{j,*}(S^{v}_{\pi} \times X^2; \mathbb{Z}/2) = 0$ if $j \geq v+1$. For $1 \leq j \leq v$, $\frac{\ker \beta}{\text{im } \beta} \xrightarrow[\delta]{\cong} E^{\infty}_{j,0}$, $\delta(x) = 1 \otimes x^{\otimes 2} \otimes e_q$. $\rho_* : K^{\pi_2}_*(S^2_{\pi} \times X^2; \mathbb{Z}/2) \to K^{\pi_2}_*(X^2; \mathbb{Z}/2)$ is onto; ρ collapses S^2.

E. $\underline{q_1 \text{ and } \overline{Q}_1}$

Using the Rothenberg-Steenrod spectral sequence

$$E^*_2((S^1_{\pi} \times_{\pi_2} X^2, \phi), \pi_2, \mathbb{Z}/2) \Rightarrow K^*(S^1_{\pi} \times_{\pi_2} X^2; \mathbb{Z}/2)$$

Snaith [Sn 6, sec. 2] defined a function

$$q_1 : \frac{\ker \beta}{\text{im } \beta} \to K_1 \ (S^1_{\pi} \times_{\pi_2} X^2; \mathbb{Z}/2)$$

for compact spaces X, which extends to arbitrary X by taking direct limits. We give an outline of the definition of q_1.

Definition 3.22. [Sn 6, sec. 2] For X a compact space, let

$$\phi : \frac{\ker \beta}{\text{im } \beta} \to K^1(S^1_{\pi} \times_{\pi_2} X^2; \mathbb{Z}/2) \quad \text{be defined by}$$

$$\phi(x + \text{im } \beta) = 1 \otimes x^{\otimes 2} \otimes e_1 \in E^{\infty}_{1,0} \subset K^1 \ (S^1_{\pi} \times_{\pi_2} X^2; \mathbb{Z}/2).$$

Properties of the transfer and forgetful maps (2.36, 2.39, 3.14, 3.15) show that the homomorphism

$$\Phi : (K^1(S^1 \times X^2; \mathbb{Z}/2)/J) \otimes \frac{\ker \beta}{\text{im } \beta} \to K^1(S^1_{\pi} \times_{\pi_2} X^2; \mathbb{Z}/2),$$

where $\Phi(w + j, (x + \text{im } \beta)) = i_!(w) + \phi(x)$,

is an isomorphism. Here J is generated by $\{(1 + \tau^*)(w), \xi_1 \otimes x^{\otimes 2}$ with $\beta x = 0\} \subset [K^*(S^1; \mathbb{Z}/2) \otimes K^*(X; \mathbb{Z}/2)^{\otimes 2}]^1$, the superscript 1 meaning odd filtration.

Definition 3.23. [Sn 6, 2.1] For $f \in \frac{\ker \beta}{\operatorname{im} \beta} \subset K_*(X; \mathbb{Z}/2)/\operatorname{im} \beta$, with dual $\overline{f} \in \operatorname{Hom}(\frac{\ker \beta}{\operatorname{im} \beta}, \mathbb{Z}/2)$, let

$$q_1(f) = (0 \oplus \overline{f})\,\Phi^{-1} \in \operatorname{Hom}(K^1(S^1_\pi \times_{\pi_2} X^2; \mathbb{Z}/2), \mathbb{Z}/2)$$

(see 2.35).

Definition 3.24. [Sn 6, 2.13] Let X be a two-fold loop space and let $\theta : S^1_\pi \times_{\pi_2} X^2 \to X$ be the H_1-space structure map (see 1.1). Define

$$\overline{Q}_1 : \frac{\ker \beta}{\operatorname{im} \beta} \to K_1(X; \mathbb{Z}/2) \text{ by } \overline{Q}_1(x) = \theta_* q_{1_*} x.$$

Here $\frac{\ker \beta}{\operatorname{im} \beta} \subset K_*(X; \mathbb{Z}/2)\ \operatorname{im} \beta$.

The properties of q_1 discussed in [Sn 6, sec. 2] imply the following result:

Theorem 3.25. [Sn 6, Thm. 2.15] (i) $\overline{Q}_1$ is a homomorphism, natural for H_1-maps.

(ii) For $x, y \in \frac{\ker \beta}{\operatorname{im} \beta}$, $\overline{Q}_1(x \cdot y) = \overline{Q}_1(x)y^2 + x^2\overline{Q}_1(y)$.

(iii) If $(\psi^k)_*$ is the dual of the Adams operation ψ^k for k odd, then $(\psi^k)_*\overline{Q}_1(x) = \overline{Q}_1(\psi^k_*(x))$.

(iv) If β and B are the first and second Bocksteins and $x \in \ker \beta \subset K_\alpha(X; \mathbb{Z}/2)$, then

$$\beta\,\overline{Q}_1(x) = \begin{cases} B(x)^2, & \text{if } \alpha \equiv 0 \pmod 2, \\ B(x)^2 + x^2, & \text{if } \alpha \equiv 1 \pmod 2. \end{cases}$$

Here $B(x)^2$ means z^2 for any $z \in B(x)$.

(v) Let $\sigma : K_\gamma(\Omega X; \mathbb{Z}/2) \to K_{\gamma-1}(X; \mathbb{Z}/2)$ be the suspension homomorphism and let $x \in \ker \beta \subset K_\alpha(\Omega X; \mathbb{Z}/2)$. Then $\sigma\overline{Q}_1(x) = \sigma(x)^2 \in K_0(X; \mathbb{Z}/2)$.

F. q_2 and $\overline{Q}_2$

We distill some consequences from the determination of the Rothenberg-Steenrod spectral sequences 3.21.b and its dual.

3.26. (a) Let

$$B = K_1 \oplus K_2 \subset \left(A_2 \otimes \frac{K^*(X;\ \mathbb{Z}/2)}{\text{im } \beta}\right) \oplus \left(A_2 \otimes \frac{K^*(X;\ \mathbb{Z}/2)}{\text{im } \beta}\right),$$

where

$$K_1 = \langle\{a \otimes [x + \text{im } \beta] \mid a = 1 \text{ if } \deg(x) \equiv 1,\ a = (1+\xi_2) \text{ other}\}\rangle$$

$$K_2 = \langle\{a \otimes [x + \text{im } \beta] \mid a = 1 \text{ if } \deg(x) \equiv 0,\ a = (1+\xi_2) \text{ other}\}\rangle$$

$$M \subset \{K(S^2 \times X^2;\ \mathbb{Z}/2)\}^{\pi_2},$$

$$M = \langle\{\pi_2\text{-invariants which are not permanent cycles}\}\rangle$$

$$= \left\langle \left\{\begin{matrix} (1+\xi_2) \otimes x^{\otimes 2} \\ \xi_2 \otimes x^{\otimes 2} \end{matrix} \middle| \begin{matrix} \deg(x) \equiv 0 \\ \beta_2\ x = 0 \end{matrix}\right\} \left\{\begin{matrix} \xi_2 \otimes x^{\otimes 2} \\ 1 \otimes x^{\otimes 2} \end{matrix} \middle| \begin{matrix} \deg\ x \equiv 1 \\ \beta\ x = 0 \end{matrix}\right\} \right\rangle$$

(b) Define $\Delta = \delta_1 \oplus \delta_2 : K_1 \oplus K_2 \to K^*_{\pi_2}(S^2 \times X^2;\ \mathbb{Z}/2)$ by

$$\delta_i(a \otimes \text{im } \beta]) = a \otimes x^{\otimes 2} \otimes e_i.$$

(c) Then the following sequence is exact

$$0 \longleftarrow \frac{\{K^*(S^2 \times X^2;\ \mathbb{Z}/2)\}^{\pi_2}}{M} \xleftarrow{\ i^*\ } K^*_{\pi_2}(S^2_\pi \times X^2;\ \mathbb{Z}/2) \xleftarrow{\ \Delta\ } B \longleftarrow 0$$

i^* is the forgetful map (2.39, 3.15).

Proof. The sequence expresses the determination of $K^*_{\pi_2}(S^2_\pi \times X^2;\ \mathbb{Z}/2)$ in 3.21.b.

(d) Dual to (c) there is the following exact sequence

$$0 \to \frac{\{K_*(S^2 \times X^2; \mathbb{Z}/2)\}_{\pi_2}}{M'} \xrightarrow{i_*} K_*^{\pi_2}(S^2 \times X^2; \mathbb{Z}/2) \xrightarrow{\Delta'} B' \to 0.$$

Here $M' = \langle\{\pi_2\text{-coinvariants which are not permanent cycles}\}\rangle$

$$= \langle\{\xi_2 \otimes x^{\otimes 2} \mid \deg x \equiv 1,\ \beta x = 0\} \cup \{(1+\xi_2) \otimes x^{\otimes 2} \mid \deg x \equiv 0,\ \beta x = 0\}\rangle$$

$B' = K_1^! \oplus K_2^!$, $k_i^!$ dual to K_i, and from 3.21.c, $K_i^! \cong \frac{\ker \beta}{\operatorname{im} \beta} \subset \frac{K_*(X; \mathbb{Z}/2)}{\operatorname{im} \beta}$;

the maps are dual to those in (c).

For later use, we determine the image of the transfer

$$i_! : K^0(S^2 \times X^2; \mathbb{Z}/2) \to K^0_{\pi_2}(S^2_\pi \times X^2; \mathbb{Z}/2) \quad \text{(see 2.36)}.$$

The procedure mimics that of Proposition 3.10.ii of [Sn 5].

Lemma 3.27. Let $Q \subset K^0(S^2_\pi \times X^2; \mathbb{Z}/2)$ be the subspace generated by $\{(1+\tau^*)(w)\}$,

$$\{(1+\xi_2) \otimes x^{\otimes 2} \mid \deg(x) \equiv 0,\ \beta x = 0\},\ \{1 \otimes x^{\otimes 2} \mid \deg(x) \equiv 1,\ \beta x = 0\}.$$

Then

$$K^0(S^2_\pi \times X^2; \mathbb{Z}/2)/Q \cong \operatorname{im} i_! = \ker(\sigma\cdot -) \subset K^0_{\pi_2}(S^2_\pi \times X^2; \mathbb{Z}/2).$$

Proof. (a) By 2.39 $i_! i^*(z) = z \cdot \sigma$, $z \in K^*_{\pi_2}(S^2_\pi \times X^2; \mathbb{Z}/2)$. If $x \in \ker \beta_2$, then $i^* i_!(u \otimes x^{\otimes 2}) = (1+\tau^*)(u \otimes x^{\otimes 2}) = 0$ by 3.17, so

$i_!(a \otimes x^{\otimes 2}) = \Delta(a \otimes [x + \operatorname{im} \beta_2])$. Moreover

$\sigma(i_!(a \otimes x^{\otimes 2})) = i_! i^* i_!(a \otimes x^{\otimes 2}) = i_! i^*(\Delta(a \otimes [x + \operatorname{im} \beta_2])) = 0$, (3.26.c).

For general $i_!(a \otimes x_1 \otimes x_2)$:

$$\sigma \cdot i_!(a \otimes x_1 \otimes x_2) = i_!(a \otimes x_1 \otimes x_2)i^*(\sigma)) = 0$$

which shows that $\operatorname{im} i_! \subset \ker(\sigma\cdot -)$.

(b) Let $i^*(z) = \Sigma\, a_i \otimes x_i^{\otimes 2} + (1+\tau^*)(w)$, with $a_i \otimes x_i^{\otimes 2}$ a permanent cycle, i.e., $x \in \ker \beta$. As above, $\sigma\cdot z = i_! i^*(z) = \Sigma\, i_!(a_i \otimes x_i^{\otimes 2})$. Since $i^* i_!(a_i \otimes x^{\otimes 2}) = (1+\tau^*)(a_i \otimes x^{\otimes 2}) = 0$, $i_!(\Sigma(a_i \otimes x_i^{\otimes 2})) = \Delta(\Sigma\, a_i \otimes [x_i + \operatorname{im}\beta])$. Suppose $\sigma\cdot z = 0$. Then $\Delta(a_i \otimes [x_i + \operatorname{im}\beta]) = 0$, so that $x_i \in \operatorname{im}\beta$, by the spectral sequence. Say $x_i = \beta y_i$, so that $a_i \otimes x_i^{\otimes 2} = a_i \otimes (\beta y_i)^{\otimes 2}$ $= i^* i_!(a_i \otimes y_i^{\otimes 2})$. Then $i^*(z) = (1+\tau^*)(w) = i^* i_!(w)$, $i^*(z + i_!(w)) = 0$, and so $z = i_!(w) + \Sigma_i\, b_i \otimes w_i \otimes e_2 \in \operatorname{im} i_!$.

Thus we proved $\operatorname{im} i_! = \ker(\sigma\cdot -)$.

(c) Consider now $(1+\xi_2) \otimes x^2$ if $\deg(x) \equiv 0$, and $1 \otimes x^{\otimes 2}$ if $\deg(x) \equiv 1$. From 3.21.b we have that $(1 \otimes \xi_2) \otimes x^{\otimes 2} \otimes e_1$ and $1 \otimes x^{\otimes 2} \otimes e_1$, $\deg(x) \equiv 0$ and $\deg(x) \equiv 1$, respectively, are permanent cycles in the Rothenberg-Steenrod spectral sequence. By 3.14 we have the exact sequence with δ determining the transfer $i_!$ (see 3.14).

$$K^1_{\pi_2}((S^2_\pi \times X^2) \times (E\pi_2, S^0_\pi); \mathbb{Z}/2) \xrightarrow{j} K^1_{\pi_2}((S^2_\pi \times X^2) \times (S^1_\pi, S^0_\pi); \mathbb{Z}/2)$$
$$\xrightarrow{\delta} K^0_{\pi_2}((S^2_\pi \times X^2) \times (E\pi_2, S^1_\pi); \mathbb{Z}/2).$$

Via the isomorphism $K^1((S^2_\pi \times X^2) \times (S^1_\pi, S^0_\pi; \mathbb{Z}/2) \simeq K^0(S^2_\pi \times X^2; \mathbb{Z}/2)$ stated in 3.14, both $(1+\xi_2) \otimes x^{\otimes 2} \otimes e_1$ $\deg(x) \equiv 0$ and $1 \otimes x^{\otimes 2} \otimes e_1$, $\deg(x) \equiv 1$ are in $\operatorname{im}(j)$, so that

(1) $\delta((1+\xi_2)\otimes x^{\otimes 2}\otimes e_1) = 0$ if $\deg(x) \equiv 0$, and

(2) $\delta(1\otimes x^{\otimes 2}\otimes e_1) = 0$ if $\deg(x) \equiv 1$.

Then $i_!((1+\xi_2)\otimes x^{\otimes 2}) = 0$, $\deg(x) \equiv 0$, and $i_!(1\otimes x^{\otimes 2}) = 0$, $\deg(x) \equiv 1$. Similarly, since $\xi_2 \otimes x^{\otimes 2}\otimes e_1$ is not in $\mathrm{im}(j)$, for both $\deg(x) \equiv 0$ and $\deg(x) \equiv 1$, (see 3.21.b), we have that $i_!(\xi_2\otimes x^{\otimes 2}) \neq 0$, $\deg x \equiv 0$ or 1. From (1) and (2)

$$\langle\{(1+\xi_2)\otimes x^{\otimes 2} \mid \deg(x) \equiv 0\} \cup \{1\otimes x^{\otimes 2} \mid \deg(x) = 1\}\rangle \subset \ker i_! .$$

Moreover, $i_!((1+z^*)(w)) = i_! i^* i_!(w) = \sigma(i_!(w)) = 0$, as we proved above. Then $Q \subset \ker i_!$.

(d) Suppose now $i_!(\Sigma\, a_i\otimes x_i\otimes x_i') = 0$. Then $0 = i^* i_!(\Sigma\, a_i\otimes x_i'\otimes x_i'') = (1+\tau^*)(\Sigma\, a_i\otimes x_i'\otimes x_i'')$, so that $\Sigma\, a_i\otimes x_i'\otimes x_i''$ is π_2-invariant, i.e.,

$$\Sigma\, a_i\otimes x_i'\otimes x_i'' = (1+\tau^*)(w) + \Sigma\, a_i\otimes x_i^{\otimes 2}, \quad x_i \in \ker \beta_2 .$$

So $0 = i_!(\Sigma\, a_i\otimes x_i'\otimes x_i'') = i_!(\Sigma\, a_i\otimes x_i^{\otimes 2})$, which, from the computations in (c), is possible only if $a_i = 1+\xi_2$ if $\deg x_i \equiv 0$, and $a_i = 1$ if $\deg x_i \equiv 1$. Thus $Q = \ker i_!$, as claimed.

The next Lemma will be of use in the analysis of the operation q_2 of 3.31. The statement and result are analogous to those of [Sn 5, 3.11].

Lemma 3.28. Let X be a compact space. Then $\mathrm{im}\, i_! \subset K^0_{\pi_2}(S^2\times X^2; \mathbb{Z}/2)$ is dual to $K_0^{\pi_2}(S^2\times X^2; \mathbb{Z}/2)/\mathrm{Ind}$, where Ind is generated by

$$\{i_*(a\otimes x^{\otimes 2}) \mid \beta x = 0,\ a = 1 \text{ if } \deg(x) \equiv 0,\ a = 1+\xi_2 \text{ if } \deg(x) \equiv 1\}.$$

Proof. As usual, $<\ ,\ >\ : K_0(\_;\mathbb{Z}/2)\otimes K^0(\_;\mathbb{Z}/2)\to \mathbb{Z}/2$ denotes the pairing of 2.20, which passes to $K_{\pi_2}(\_;\mathbb{Z}/2)$ as mentioned in 3.18.

(a) Suppose $z\ \ K_0^{\pi_2}(S^2\times X^2;\mathbb{Z}/2)$ is such that $0 = <z, i_!(w)>$ for all $w\in K^0_{\pi_2}(S^2\times X^2;\mathbb{Z}/2)$. In particular $0 = <z, i_!(a\otimes x^{\otimes 2})>$, $x\in \ker\beta$. By 3.23 $i_!(a\otimes x^{\otimes 2}) = \Delta(y)$, and so $0 = <z, i_!(a\otimes x^{\otimes 2})> = <z, \Delta(y)>$ $= <\Delta' z, y>$, all y. 3.26 implies $z = i_*(z')$.

(b) Moreover

$$0 = <z, i_!(a\otimes x_1\otimes x_2)> = <i_* z', i_!(a\otimes x_1\otimes x_2)>$$

$$= <z', i^* i_!(a\otimes x_1\otimes x_2)> = <z', (1+\tau^*)(a\otimes x_1\otimes x_2)>.$$

Thus z' does not pair with any $(1+\tau^*)(w)$, forcing $z' = \Sigma\, a_i\otimes z_i''^{\otimes 2}$.

(c) By the assumption on z, for any $b\otimes x_1\otimes x_2$:

$$0 = <i_* a_i\otimes z''^{\otimes 2}, b\otimes x_1\otimes x_2> = \Sigma <a_i\otimes z_i''^{\otimes 2}, (1+\tau^*)(b\otimes x_1\otimes x_2)>$$

$$= \Sigma <a_i\otimes z_i''^{\otimes 2}, b\otimes\beta x_1\otimes\beta x_2> = \Sigma <a_i, b><z_i'', \beta x_1><z_i'', \beta x_2>$$

$$= \Sigma <a_i, b><\beta z_i'', x_1><\beta z_i'', x_2>.$$

Fixing i, and taking $b = a_i$, $x_1 = x_2 = \beta z_i''$ we conclude that, for each i, $\beta z_i'' = 0$. Then $\operatorname{im} i_! \subset K^0_{\pi_2}(S^2_\pi\times X^2;\mathbb{Z}/2)$ has the dual in the statement of the lemma since, from the spectral sequence 3.21.b', the classes $i_*(a\otimes x^{\otimes 2})$, $\beta x = 0$, with $a = 1$ if $\deg(x)\equiv 0$ and $a = 1+\xi_2$ if $\deg(x)\equiv 1$ determine non-trivial elements in $K_0^{\pi_2}(S^2_\pi\times X^2;\mathbb{Z}/2)$. Also, $1\otimes x^{\otimes 2}$ for $\deg x\equiv 0$, and $(1+\xi_2)\otimes x^{\otimes 2}$ for $\deg(x)\equiv 1$, are equivalent to $(1\otimes x^{\otimes 2})$, $\deg(x)\equiv 0$ and 1, respectively, in the spectral sequence 3.21.b' and we can write $\operatorname{Ind} = <\{i_*(1\otimes x^{\otimes 2}\ \ \beta x = 0\}>$.

3.29. We define an operation on $\frac{\ker\beta}{\operatorname{im}\beta} \subset \frac{K_*(X;\mathbb{Z}/2)}{\operatorname{im}\beta}$, where X is an H_2-space (see 1.1), valued in $\frac{K_0(X;\mathbb{Z}/2)}{\text{Ind}}$. This operation, denoted $\overline{Q}_2$, will be nicely related to the Dyer-Lashof operation Q_2, through the Atiyah-Hirzebruch spectral sequence.

Through 3.30-3.35 we make use of the notation of 3.20-3.28, as will be made explicit.

Definition 3.30. Define the function

$$\eta_2: \frac{\ker\beta}{\operatorname{im}\beta} \longrightarrow K^0_{\pi_2}(S^2_\pi \times X^2; \mathbb{Z}/2)$$

$$\cap$$

$$\frac{K^*(X;\mathbb{Z}/2)}{\operatorname{im}\beta}$$

by $\eta_2(x) = a \otimes x^{\otimes 2} \otimes e_2$, where $a = 1$ if $\deg(x) \equiv 0$, $a = 1 + \xi_2$ if $\deg(x) \equiv 1$, both (mod 2). See 3.21 for the notation. η_2 is natural on $\ker\beta$. By 3.14

$$\eta_2(x) \in (\operatorname{im} i_!) \subset K^0_{\pi_2}(S^2_\pi \times X^2; \mathbb{Z}/2).$$

See (2.36) for the meaning of $i_!$.

Definition 3.31. Let

$$q_2 : \frac{\ker \beta}{\operatorname{im} \beta} \to \frac{K_0^{\pi_2}(S^2_\pi \times X^2; \mathbb{Z}/2)}{\text{Ind}}$$

be given by $q_2(x) = (\eta_2(x^0))^0$. Here $(\ )^0$ stands for dual (2.20, 2.35) and $\text{Ind} = \langle\{1 \otimes x^{\otimes 2} \mid \beta(x) = 0\}\rangle$, the subspace dual to im $i_!$ determined in 3.27.

Next we use the discussion of 3.28 to prove that $\beta\, q_2(x) = 0$ if $K^*(X; \mathbb{Z})$ is torsion free.

Lemma 3.32. $\beta\, q_2(x) = 0$ if $K^*(X; \mathbb{Z})$ is torsion free.

Proof. Take $z \in K^1_{\pi_2}(S^2_\pi \times X^2; \mathbb{Z}/2)$. Using the pairing of (2.21, 3.18) we have

$$\langle \beta\, q_2(x), z\rangle = \langle q_2(x), \beta\,(z)\rangle.$$

Notice from 3.27 that the only odd classes not in (im $i_!$) are of form $z = a \otimes y^{\otimes 2} \otimes e_1$, a depending on deg(y). So, to prove our claim, it suffices to show that $\beta\,(a \otimes y^{\otimes 2} \otimes e_1)$ does not pair with $q_2(x)$, for all $y \in \ker \beta$.

(a) We treat the parities of y separately. Let $\deg(y) \equiv 1$, (mod 2). By 3.20.b we must only study $\beta\,(1 \otimes y^{\otimes 2} \otimes e_1)$. Now

$$\beta(1 \otimes y^{\otimes 2} \otimes e_1) = \beta\, \rho^*(y^{\otimes 2} \otimes e_1) =$$

$$\rho^*\beta\,(y^{\otimes 2} \otimes e_1) = \rho^*(i_!(B_2(y)^{\otimes 2})),$$

where ρ is the map $S^2_\pi \times X^2 \to X^2$ in 3.21.c, and B_2 is the second Bockstein introduced in 2.33. The first two equations hold by the naturality of the Bockstein β. The last equation follows from 3.19.i. Since X is torsion free, $B_2(y) = 0$, thus giving the claim for deg(y) odd.

(b) Suppose now that deg(y) $\equiv 0$, (mod 2). From 3.21.b we must analyze $\beta((1+\xi_2) \otimes y^{\otimes 2} \otimes e_1)$. First observe that we can factorize $(1+\xi_2) \otimes y^{\otimes 2} \otimes e_1 = ((1+\xi_2) \otimes 1 \otimes e_1) \cdot (1 \otimes y^{\otimes 2} \otimes e_0)$ in equivariant $K^*_{\pi_2}$ $\mathbb{Z}/2$-theory (see 3.6.3). This is so since $(1 \otimes \xi_2) \otimes e_1$ is a generator of $K^1(RP^2; \mathbb{Z}/2)$, as seen from 3.21.b for X = point, and since $1 \otimes y^{\otimes 2} \otimes e_0$ is a generator of $K^0_{\pi_2}(X^2; \mathbb{Z}/2)$, by 3.17.b. Then

$$\beta((1+\xi_2) \otimes y^{\otimes 2} \otimes e_1)$$

$$= (\beta[(1+\xi_2) \otimes e_1])(1 \otimes y^{\otimes 2} \otimes e_0) + [(1+\xi_2) \otimes e_1][\beta(1 \otimes y^{\otimes 2} \otimes e_0)]$$

$$= \beta[(1+\xi_2) \otimes e_1](1 \otimes y^{\otimes 2} \otimes e_0) + [(1+\xi_2) \otimes e_1][B_2(y)^{\otimes 2} \otimes e_1]$$

$$= \beta[(1+\xi_2) \otimes e_1](y^{\otimes 2} \otimes e_0).$$

The first equality is the derivation property of β, the second by 3.19.b, and the third since X is torsion free. Now, $K_1(RP^2; \mathbb{Z}/2)$ has trivial first Bockstein β, (2.33), and dualizing, $K^1(RP^2; \mathbb{Z}/2)$ has trivial Bockstein β (see 2.35). This proves the claim for deg(y) $\equiv 0$, (mod 2).

3.33. In order to exhibit the relationship between the function q_2 defined above and the Dyer-Lashof operation Q_2, (see 1.7 and 1.9) we make use of the properties of the Atiyah-Hirzebruch spectral sequence. Let X be a torsion free CW-complex, X_k, $k \geq 0$, its k-skeleton, and the maps below the canonical ones:

$$i_k : X_k \to X, \; j_k : X_k \to \frac{X_k}{X_{k-1}}.$$

Given $y \in K_\alpha(X; \mathbb{Z}/2)$, let $[y] \in H_k(X; \mathbb{Z}/2)$ represent y in the Atiyah-Hirzebruch spectral sequence. There is a class $x \in K_\alpha(Y_k, \mathbb{Z}/2)$ with $(i_k)_*(x) = y$, and $(j_k)_*(x) \in K_\alpha(Y_k, Y_{k-1}; \mathbb{Z}/2) \cong C_k(Y, K_{\alpha-k}(*, \mathbb{Z}/2))$ is a cycle in the chains on Y which represents [y].

Suppose that y is integral, so that x is also integral in $K_*(Y_k; \mathbb{Z}/2)$. Then

$$q_2(x) \in \frac{K_0(S^2_\pi \times_{\pi_2} (Y_k)^2; \mathbb{Z}/2)}{\text{Ind}}$$

is defined, and $q_2(x) = 1 \otimes x^{\otimes 2} \otimes e_2 + i_*(x_1^{\otimes 2})$; $x, x_1 \in \ker \beta$ (see 3.28). The component $1 \otimes x^{\otimes 2} \otimes e_2$ is in the $2k+2$ skeleton of $[S^2_\pi \times (Y_k)^2] \times_{\pi_2} S^2_\pi$, which is the second space in the π_2- resolution of $S^2_\pi \times (Y_k)^2$, (see 3.1-3.5). And indeed $1 \otimes x^{\otimes 2} \otimes e_2$ lies in $* \times (Y_k)^2 \times_{\pi_2} S^2_\pi$, assuming that Y_k is the minimal filtration where x is. The following canonical procedure gives us the cycle determined by $1 \otimes x^{\otimes 2} \otimes e_2$ in the chains on $(Y_k)^2 \times_{\pi_2} S^2_\pi$:

$$(j_{2k+2})_* : K_1(S^2_\pi \times_{\pi_2} (Y_k)^2; \mathbb{Z}/2) \to K_1(* \times (Y_k, Y_{k-1})^2 \times_{\pi_2} (S^2_\pi, S^1_\pi); \mathbb{Z}/2)$$

$$\cong C_{2k+2}(Y^2 \times_{\pi_2} S^2_\pi; \mathbb{Z}/2).$$

Clearly the image of $1 \otimes x^{\otimes 2} \otimes e_2$ in the group $C_{2k+2}(Y^2 \times_{\pi_2} S^2_\pi; \mathbb{Z}/2)$ is a chain representing the Dyer-Lashof operation $Q_2([y])$.

A similar procedure exhibits the class $x_1^{\otimes 2}$ as a square in some even filtration degree lower than or equal to 2k.

3.34. We proved in 3.32 that $\beta\, q_2(x) = 0$, $x \in \ker\beta - \operatorname{im}\beta$, which implies that we can iterate q_2 at least twice, and that also $q_1 q_2$ is defined, (see def. 3.23). We only pursue on $q_1 q_2$.

Proceeding as in 3.33 one sees that $q_1 q_2(x)$ determines a chain in

$$C_{4k+5}([S^2_\pi \times_{\pi_2} (Y_k)^2]^2 \times_{\pi_2} S^1_\pi; \mathbb{Z}/2),$$

and that this chain represents the Dyer-Lashof composite $Q_1 Q_2([y])$. There is also the class $q_1(x_1^{\otimes 2}) \in K_1(S^1_\pi \times_{\pi_2} (Y_\ell)^4; \mathbb{Z}/2)$, $\ell \le k$. However from the properties of q_1 and $\overline{Q}_1$ we conclude that $\overline{Q}_1(x_1^{\otimes 2}) = 0$, (see 3.25.ii).

Definition 3.35. Let X be an H_2-space, and let $\theta_2 : S^2_\pi \times_{\pi_2} X^2 \to X$ be the H_2-space structure map. On $\frac{\ker\beta}{\operatorname{im}\beta} \subset \frac{K_*(X; \mathbb{Z}/2)}{\operatorname{im}\beta}$ define

$$\overline{Q}_2 : \frac{\ker\beta}{\operatorname{im}\beta} \to \frac{K_*(X; \mathbb{Z}/2)}{\text{Ind}} \quad \text{by} \quad \overline{Q}_2(x) = \theta_{2*} q_2(x).$$

From the discussion in 3.34 we see that if x is integral, then $q_1 q_2(x)$ is defined.

3.36. We carry out a similar analysis to that of 3.34 for the Browder operation λ_2 of 1.4 (which is denoted ψ_2 in [Br]),

$$\lambda_2 : H_*(S^2_\pi \times X^2; \mathbb{Z}/2) \to H_*(X; \mathbb{Z}/2).$$

Let $\overline{\lambda}_2 : K_*(S^2_\pi \times X^2; \mathbb{Z}/2) \to K_*(X; \mathbb{Z}/2)$ be induced by ϕ (see 1.4) in $K_*\,\mathbb{Z}$-theory. Take $z = \overline{\lambda}_2(\alpha_1, \alpha_2) \in K_*(X; \mathbb{Z}/2)$, where $\alpha_1, \alpha_2 \in K_*(X; \mathbb{Z}/2)$. Represent α_1 and α_2 by $[\alpha_1]$ and $[\alpha_2]$, respectively, in the Atiyah-

Hirzebruch spectral sequence, say $[\alpha_1] \in H_k(Y; \mathbb{Z}/2)$, $[\alpha_2] \in H_\ell(Y; \mathbb{Z}/2)$. Let $\gamma_1 \in K_\alpha(Y_k; \mathbb{Z}/2)$, $\gamma_2 \in K_\beta(Y_\ell; \mathbb{Z}/2)$ be such that $(i_k)_*(\gamma_1) = \alpha_1$, $(i_\ell)_*(\gamma_2) = \alpha_2$. The commutativity of the next diagram is clear.

$$\begin{array}{ccc}
K_{\alpha+\beta}((S^2_\pi \times (X\times X))_{2+k+\ell}; \mathbb{Z}/2) & \xrightarrow{\overline{\lambda}_2} & K_{\alpha+\beta}(X_{2+k+\ell}; \mathbb{Z}/2) \\
\downarrow & & \downarrow \\
K_{\alpha+\beta}((S^2_\pi\times(X\times X))_{2+k+\ell}, (S^2_\pi\times(X\times X))_{1+k+\ell}; \mathbb{Z}/2) & \rightarrow & K_{\alpha+\beta}(X_{2+k+\ell}, X_{1+k+\ell}; \mathbb{Z}/2) \\
\| \wr & & \| \wr \\
C_{2+k+\ell}(S^2_\pi \times X^2; \mathbb{Z}/2) & \longrightarrow & C_{2+k+\ell}(X; \mathbb{Z}/2)
\end{array} \tag{3.37}$$

We see that $\overline{\lambda}_2\ (\gamma_1, \gamma_2)$ determines in the chains a cycle which represents $\lambda_2([\alpha_1], [\alpha_2])$ in homology. In the top square of (3.37) $\overline{\lambda}_2$ is the obvious restriction.

CHAPTER 4.

E^4_* OF THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE

Let Y be an H_1-space (see 1.1). We record the action of the first non-trivial differential $d_3 = Sq^3_* + Sq^1_*Sq^2_*$ in the Atiyah-Hirzebruch spectral sequence $H_*(Y; \mathbb{Z}/2) \Rightarrow K_*(Y; \mathbb{Z}/2)$, as computed in [Sn 6].

Proposition 4.1. [Sn 6, Lemma 3.4] Let $x \in H_s(\Omega^2 X; \mathbb{Z}/2)$. Then, for $p \geq 0$,

$$[Sq^3_* + Sq^1_*Sq^2_*](Q_1^{p+2}(x)) = \begin{cases} (Q_1^p(x))^4, & \text{if } p > 0 \text{ or } s \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

The proof of this proposition is an application of the Nishida relations (see 1.7-1.9). We will make use of this result in the rest of this chapter.

We determine now the action of $d_3 = Sq^3_* + Sq^1_*Sq^2_*$ in the Atiyah-Hirzebruch spectral sequence

$$H_*(\Omega^s S^t X; \mathbb{Z}/2) \Rightarrow K_*(\Omega^s S^t X; \mathbb{Z}/2), \ s \leq t.$$

Here X is a torsion-free, finite CW-complex. As the differentials are derivations, we centre our analysis on the effect of d_3 on the algebra generators of $H_*(\Omega^s S^t X; \mathbb{Z}/2)$, (see 1.11 for the notation).

We first consider $s \geq 4$, so that in the top operation, Q_{s-1}, $s-1 \geq 3$. Under these conditions we have

Theorem 4.2. E_*^4 of the Atiyah-Hirzebruch spectral sequence $H_*(\Omega^s S^t X; \mathbb{Z}/2) \Rightarrow K_*(\Omega^s S^t X; \mathbb{Z}/2)$ $s \geq 4$, has the following multiplcative generators:

(1) $Q_j(\xi)$ if $j \geq s-3$, and $j + \deg(\xi) \equiv 1$, (mod 2)

(2) $Q_{s-2}Q_{s-1}(\xi)$ if $s-1+\deg(\xi) \equiv 1$, (mod 2)

(3) $Q_k Q_{s-1}(\xi)$ if $1 \leq k \leq 2$, and $s-1+\deg(\xi) \equiv 1$, (mod 2)

(4) $Q_{j_1} Q_{s-1}(\xi)$ if $3 \leq j_1$ odd, $s-1+\deg(\xi) \equiv 1$, (mod 2),
$s-1+j_1 \equiv 1$, (mod 2). So s-1 is even.

(5) $Q_2 Q_{j_2} Q_{j_3} \ldots (\xi)$ if j_2 even, and
$\deg(Q_{j_3} \ldots (\xi)) + j_2 \equiv 1$, (mod 2)

(6) $(Q_{2k} Q_{2\ell} \ldots (\xi))^2$ if $k \geq 1$

(7) $(Q_{2k+1} Q_{2\ell+1} \ldots (\xi))^2$ if $k \geq 1$, or if $k = 0$ and $\ell = 0$.

(8) $Q_1(\xi_{2n+1})$, $Q_2(\xi_{2n})$, $Q_2^2(\xi_{2n+1})$

(9) $Q_1(\xi_{2n})$ if $s = 4$, and $Q_2(\xi_{2n+1})$ if $s \leq 5$;
ξ_{2n}, ξ_{2n+1}

Remark. If $4 \leq s \leq 6$, we see that the classes in (2), (3) and (4) intersect.

Proof. The procedure is a systematic use of the Nishida relations (see 1.7-1.9). See 1.6 for the lower notation in the homology operations. We denote $m = \deg(\xi)$.

(A) We analyse first $Q_j(\xi)$. The Nishida relations give, for $j < s-1$,

(a) $$Sq_*^2(Q_j(\xi)) = \binom{m+j-2}{2} Q^{m+j-2}(\xi) + \binom{m+j-1}{0} Q^{m+j-1} Sq_*^1(\xi)$$

(b) $$Sq_*^3(Q_j(\xi)) = \binom{m+j-3}{3} Q^{m+j-3}(\xi) + \binom{m+j-3}{1} Q^{m+j-2} Sq_*^1(\xi)$$

From (a) and (b) we deduce

(I)

(1) The second summands of (a) and (b) are trivial since ξ is annihilated by Sq^1_*.

(2) Suppose deg $(\xi) = 2k+1$, $j = 2\ell + 1 \geq 3$. Then in (a) $\binom{m+j-2}{2} = \binom{2(k+\ell)}{2}$, which equals

(i) 0 if $k \equiv \ell$, (mod 2)

(ii) 1 if $k + \ell \equiv 1$, (mod 2).

(3) In (b), $\binom{m+j-3}{3} = \binom{2(k+\ell)-1}{3}$, which is

(i) 0 if $k + \ell \equiv 1$, (mod 2)

(ii) 1 if $k \equiv \ell \equiv 0$, (mod 2) and $k + \ell \geq 2$ or $k \equiv \ell \equiv 1$, (mod 2).

(4) In case (2.ii) above we have $Sq^1_* Sq^2_*(\xi) = Sq^1_* Q^{m+j-2}(\xi)$ $= Q^{m+j-3}(\xi) \neq 0$, as $j \geq 3$ by assumption, and m+j even. By (2), (3) and (4) we conclude, for $j \geq 3$, that

$$(Sq^3_* + Sq^1_* Sq^2_*)\, Q_j(\xi) \neq 0, \quad \text{if } \deg(\xi) \equiv j \equiv 1, \text{ (mod 2)}.$$

(II) Suppose now $m = 2k$, $j = 2\ell$, $m+j \geq 6$. Then

(1) In (a), $\binom{m+j-2}{2} = \binom{2(k+\ell)-2}{2} = \begin{cases} 0 & k+\ell \text{ odd} \\ 1 & k+\ell \text{ even} \end{cases}$

(2) Moreover in (b) $\binom{m+j-3}{3} = \binom{2(k+\ell)-3}{3} =$

(i) 1 if $k+\ell$ odd

(ii) 0 if $k+\ell$ even.

(III) By (I) and (II) $Q_j(\xi)$, $j + \deg(\xi) \equiv 1$, (mod 2) is a d_3-cycle, $m+j \geq 1$. Also from (I) and (II) $Q_j(\xi)$ is not a d_3-cycle, $j+m \equiv 0$, (mod 2) $m+j \geq 6$.

(IV) In case $j = s-1$, Q_{s-1} is the top operation, and $Sq_*^r Q_{s-1}(\xi)$ involves an extra sum (see 1.9.3). However, if $r \leq 3$, each extra summand involves $Sq_*^1(\xi)$, which is 0, as noted above. Then the extra sum does not contribute to the Nishida expansions of $Sq_*^3 Q_{s-1}(\xi)$ and $Sq_*^1 Sq_*^2 Q_{s-1}(\xi)$, and the analysis (I-III) applies for the top operation.

(B) Consider now the monomials $Q_{j_1} Q_{j_2} \ldots (\xi)$, with $s-1 > j_1 \geq 3$, $j_2 > 0$. Let $m = \deg(Q_{j_2} \ldots)$.

The Nishida relations imply

(a) $$Sq_*^2(Q_{j_1} Q_{j_2} \ldots (\xi)) = \binom{m+j_1-2}{2} Q^{m+j_1-2} Q_{j_2} \ldots (\xi)$$

$$+ \binom{m+j_1-2}{0} \binom{\frac{m+j_2}{2} - 1}{1} Q^{m+j_1-1} Q^{\frac{m+j_2}{2} - 1} Q_{j_3} \ldots (\xi)$$

(b) $$Sq_*^3(Q_{j_1} Q_{j_2} \ldots (\xi)) = \binom{m+j_1-3}{3} Q^{m+j_1-3} Q_{j_2} \ldots (\xi)$$

$$+ \binom{m+j_1-3}{1} \binom{\frac{m+j_2}{2} - 1}{1} Q^{m+j_1-2} Q^{\frac{m+j_2}{2} - 1} Q_{j_3} \ldots (\xi)$$

(1) Notice that $\binom{\frac{m+j_2}{2} - 1}{1} \equiv 0$ if and only if $m + j_2 \equiv 2 \pmod 4$, which is equivalent to

(i) $m \equiv j_2 \equiv 1$, or 3, (mod 4); i.e., either

(ii) $m \equiv 0$, $j_2 \equiv 2$, both (mod 4), or

(iii) $m \equiv 2$, $j_2 \equiv 0$, both (mod 4).

(iv) Moreover $m + j_2 \equiv 2$, (mod 4) is easily seen to be equivalent to $j_2 + \deg(Q_{j_3} \ldots (\xi)) \equiv 1$, (mod 2) which, in case $j_3 > 0$, in turn is equivalent to $j_2 + j_3 \equiv 1$, (mod 2).

Condition (1) will be used over and over, together with the binomial coefficients $\binom{m+j_1-3}{3}$, $\binom{m+j_1-3}{1}$, $\binom{m+j_1-2}{2}$ of (a) and (b), to determine d_3-cycles and d_3-boundaries. Assume through (2) and (4), that condition (1) is satisfied by $Q_{j_1}Q_{j_2} \ldots (\xi)$, $s-1 > j_1 \geq 3$, $s-1 > j_2 > 0$.

(2) Let $m \equiv j_1 \equiv 1$ (mod 2), so that $m + j_1 = 4k + 2$. Then $\binom{m+j_1-2}{2} \equiv 0$, while $\binom{m+j_1-3}{3} = \binom{4k-1}{3} \neq 0$.

(3) Let $m + j_1 \equiv 1$, (mod 2). By condition (1)

(i) $Sq_*^1 Sq_*^2 = \left(Q^{m+j_1-2} Q_{j_2} \ldots (\xi)\right) = \binom{m+j_1-3}{1} Q^{m+j_1-3} \ldots (\xi)$.

Also by (1),

(ii) $Sq_*^3(Q_{j_1}Q_{j_2} \ldots (\xi)) = \binom{m+j_1-3}{3} Q^{m+j_1-3} Q_{j_2} \ldots (\xi)$.

(iii) By parity $\binom{m+j_1-3}{1} = \binom{m+j_1-3}{3} = 0$.

(4) Assume now $m \equiv j_1 \equiv 0$, (mod 2). Then either

(i) $\binom{m+j_1-3}{3} \neq 0$ if $m \not\equiv j_1$, (mod 4), or

(ii) $\binom{m+j_1-2}{2} \neq 0$ if $m \equiv j_1$, (mod 4).

(iii) By (2), (i) and (ii) $(Sq^1_*Sq^2_* + Sq^3_*)(Q_{j_1} \ldots (\xi))$

$$= Q^{m+j_1-3} Q_{j_2} \ldots (\xi) \neq 0$$

in case $m \equiv j_1$, (mod 2).

Let us assume through (5) - (7) that condition (1) is not satisfied, and, as above, $s-1 > j_1 \geq 3$; $s-1 > j_2 > 0$.

(5) Suppose $m + j_1 \equiv 1$, (mod 2). As (1) does not hold, $\binom{\frac{m+j_2}{2} - 1}{1} \neq 0$. Moreover $\binom{m+j_1-3}{3} = \binom{m+j_1-3}{1} = 0$ which gives $Sq^3_*(Q_{j_1} \ldots (\xi)) = 0$. Also $Sq^1_*(Q^{m+j_1-2} Q_{j_2} \ldots (\xi)) = 0$, as, by parity, $\binom{m+j_1-3}{1} = 0$. We then have

$$d_3(Q_{j_1} \ldots (\xi)) = Sq^1_*(Q^{m+j_1-1} Q^{\frac{m+j_2}{2}-1} Q_{j_3} \ldots (\xi))$$

$$= \binom{m+j_1-1}{1} Q^{m+j_1-2} Q^{\frac{m+j_2}{2}-1} Q_{j_3} \ldots (\xi) \neq 0.$$

(6) Let $m \equiv j_1$, (mod 4), m even. Then $\binom{m+j_1-2}{2} \neq 0$. Now, $\binom{m+j_1-3}{3} = 0$, since the term 2 is not in the 2-adic expansion of $m + j_1 - 3$. We obtain

$$Sq^1_*Sq^2_*(Q_{j_1} \ldots (\xi)) = Q^{m+j_1-3} Q_{j_2} \ldots (\xi),$$

$$Sq^3_*(Q_{j_1} \ldots (\xi)) = Q^{m+j_1-2} Q^{\frac{m+j_2}{2}-1} Q_{j_3} \ldots (\xi).$$

(7) Suppose $m \equiv j_1$, (mod 4), m odd. Then only

$$\binom{m+j_1-3}{1} \equiv \binom{m+j_1-3}{3} \equiv 1$$ is non zero. Then

$$d_3(Q_{j_1} \ldots (\xi)) = Sq^3_*(Q_{j_1} \ldots (\xi))$$

$$= Q^{m+j_1-3} Q_{j_2} \ldots (\xi) + Q^{m+j_1-2} Q^{\frac{m+j_2}{2}-1} Q_{j_3} \ldots (\xi) .$$

(C) In (A) and (B) we have analyzed $d_3Q_{j_1}(\xi)$ for $j_1 \geq 3$, and $d_3(Q_{j_1}Q_{j_2} \ldots (\xi))$ when $3 \leq j_1, j_2 < s-1$. We deal presently with $j_1 = 1,2$, when the rest of the conditions in (A) and (B) on j_1, j_2 are kept. In particular the analysis of the binomial coefficients made in (A) and (B) applies. However a difference between the conclusions in (A) and (B) and the present ones arises, as $j_1 = 1,2$ is too small. Specifically, we have $d_3(Q_2(\xi)) = d_3(Q_1(\xi)) = 0$ as seen easily from 1.7.6 and 1.9.3. In case $Q_2Q_{j_2} \ldots (\xi)$, B(1) gives $d_3(Q_2Q_{j_2} \ldots (\xi)) = Q_{2-3}Q_{j_2} \ldots (\xi) = 0$. This is different from (B.3.i) and B.4.iii). Similarly, if B(1) is assumed, then $d_3(Q_1Q_{j_2} \ldots (\xi)) = 0$, as opposed to B(2) and B(3), where, depending on the parity of j_2, the differential d_3 is non-trivial.

(D) Through (A)-(C) we assumed both $j_1, j_2 < s-1$, the reason being the different behaviour of the top operation Q_{s-1}, compared with the lower ones, when Sq^r_* acts on it. Nevertheless, observe that in case of $Q_{s-1}(\xi)$, (A.IV) applies, thus making the extra sum in $Sq^3(Q_{s-1}(\xi)$

trivial. (See Thm. 1.9.3). If we now allow $j_1 = s-1$ and $j_2 = s-1$ in the analysis (B)-(C), we see that the extra sum appearing in $Sq^r(Q_{j_1}Q_{j_2}\ldots(\xi))$, $r = 1,2,3$, is trivial, as comes from property (1.8.6) of the Browder operation λ_{s-1}. Then the whole analysis of (B)-(C) applies if $j_1 = s-1$ or $j_2 = s-1$.

(E) Notice that in (A) and (B) we made the assumption $m + j_1 \geq 6$. The reason is that we wanted to deal with the binomial coefficients involved by a general procedure, as we indeed did. However, the case $m + j_1 < 6$ produces only d_3 cycles $Q_{j_1}\ldots(\xi)$, since, $m + j_1$ being small, all the relevant binomial coefficients in (A)-(D) are trivial.

(F) From (A)-(E) we distill the following information.

(1) If $j_1 \geq 3$, $j_2 > 0$, $Q_{j_1}Q_{j_2}\ldots(\xi)$ is d_3 cycle only if $j_1 + \deg(Q_{j_2}\ldots(\xi)) \equiv 1$ (mod 2), and the condition B(1) is satisfied.

(2) $Q_j(\xi)$, $j + \deg(\xi) \equiv 1$ is a d_3-cycle, $j \geq 3$.

(3) If $j_1 = 1,2, j_2 > 0$, and condition B.(1) is satisfied, then $Q_{j_1}Q_{j_2}\ldots(\xi)$ is a d_3-cycle.

(4) $Q_1(\xi)$, $Q_2(\xi)$ are d_3-cycles for all ξ.

(5) Clearly all squares $(Q_{j_1}\ldots(\xi))^2$ are d_3-cycles.

(G) Our next task is to determine which of the classes listed in (F) are d_3-boundaries. From this the proof of the theorem will be accomplished.

(1) Consider the d_3-cycle $Q_{j_1}Q_{j_2}\ldots(\xi)$, which satisfies B.(1) by F.(1), and assume that both $0 \leq j_1, j_2 < s-1$, so that $Q_{j_1+1}Q_{j_2+1}Q_{j_3}\ldots(\xi)$ is defined. By condition B.(1), $j_2 + \deg(Q_{j_3}\ldots(\xi)) \equiv 1$, (mod 2), and as pointed out in F.(1), $j_1 + \deg(Q_{j_2}\ldots(\xi)) \equiv 1$, (mod 2). Then $Q_{j_1+1}Q_{j_2+1}Q_{j_3}\ldots(\xi)$ does not satisfy B.(1) since $j_2+1+\deg(Q_{j_3}\ldots(\xi)) \equiv 0$, (mod 2), and moreover, from B.(5)-B.(7), we see that $d_3(Q_{j_1+1}Q_{j_2+1}Q_{j_3}\ldots(\xi)) = Q_{j_1}Q_{j_2}Q_{j_3}\ldots(\xi)$. Notice that if $j_3 > 0$, then condition B.(1) forces $j_3 > j_2$.

(2) Consider now $Q_j(\xi)$, $j + \deg(\xi) \equiv 1$, $j < s-3$. We see from (A.III) that $d_3 Q_{j+3}(\xi) = Q_j(\xi)$.

(3) If $j_1 = s-1$, which forces $j_2 = s-1$ if $j_2 > 0$, the monomial $Q_{s-1}Q_{s-1}\ldots(\xi)$ is not a d_3-cycle and so we don't deal with it anymore. Here $s-1+\deg(Q_{s-1}\ldots(\xi)) \equiv 0$, (mod 2), and so, if condition B.(1) holds, from B.(2) and B.(4) we see that

$$d_3(Q_{s-1}Q_{s-1}\ldots(\xi)) = Q_{s-4}Q_{s-1}\ldots(\xi).$$

On the other hand, if condition B.(1) does not hold, B.(6) and B.(7) give:

$$d_3(Q_{s-1}Q_{s-1}\ldots(\xi)) = Q_{s-4}Q_{s-1}\ldots(\xi) + Q_{s-3}Q_{s-2}\ldots(\xi).$$

(4) Consider now the monomial $Q_{s-2}Q_{s-1}(\xi)$. If $s-1+\deg(\xi) \equiv 1$ (mod 2), this is a d_3-cycle, as pointed out in F.(1). Moreover it is clear that $Q_{s-2}Q_{s-1}(\xi)$ is not a d_3-boundary, since there are not the

necessary higher operations. This accounts for part (2) of the theorem.

(5) Similarly $Q_{s-3}(\xi)$ or $Q_{s-2}(\xi)$ or $Q_{s-1}(\xi)$ are d_3-cycles if $j + \deg(\xi) \equiv 1$, (mod 2), and they are not d_3-boundaries by the argument in (4). This accounts for (1) of the theorem.

(6) Consider next d_3-cycles $Q_jQ_{s-1}(\xi)$, $0 < j < s-3$, i.e., $s-1 + \deg(\xi) \equiv 1$, (mod 2), by F.(1). By (B), Q_jQ_{s-1} can only be a d_3-boundary of $Q_{j+3}Q_{s-1}(\xi)$, which is of the type analyzed in B.(2) and B.(4). There we saw that $d_3(Q_{j+3}Q_{s-1}(\xi)) = Q_j \; Q_{s-1}(\xi)$ if $j+3$ is odd, i.e., if j is even, while $d_3(Q_{j+3}Q_{s-1}(\xi)) = 0$ if $j+3$ is even, i.e., if j is odd. The last sentence proves (4) of the theorem.

(7) As for d_3-cycles $Q_j(\xi)$, $1 \le j < s-3$, satisfying $j + \deg(\xi) \equiv 1$, (mod 2), we see from (A.III) that $d_3Q_{j+3}(\xi) = Q_j(\xi)$.

(8) Some classes in (3) and (5) of the theorem fail to satisfy the condition B.(1). They are nevertheless d_3-cycles, as 1, and 2 are small. They are not d_3-boundaries, since d_3-boundaries satisfy B.(1).

(9) All classes ξ are d_3-cycles, by their definition and by 1.9.5 and 1.9.6, as $H_*(X; \mathbb{Z})$ is torsion free. It follows easily from the Nishida relations that they cannot be hit by d_3.

(10) If $s > 4$, Q_4 is defined and $d_3 \, Q_4(\xi_{2n}) = Q_1(\xi_{2n})$, and similarly for $Q_2(\xi_{2n+1})$ if $s \ge 6$, completing (9) of the theorem.

This completes the proof of the theorem. The determination of the $d_3$cycles in Theorem 4.2 clearly applies to the case $s = 3$. A difference between the E^4_* terms of the Atyah-Hirzebruch spectral sequences for Ω^sS^tX and Ω^3S^tX could only arise when deciding the d_3-boundaries. We can then state:

Theorem 4.3. The E^4_* of the Atiyah-Hirzebruch spectral sequence $H_*(\Omega^3 S^t X; \mathbb{Z}/2) \Rightarrow K_*(\Omega^3 S^t X; \mathbb{Z}/2)$, (X as in 4.2) has the following multiplicative generators

$$\xi_{2n},\ \xi_{2n+1},\ Q_1(\xi_{2n+1}),\ Q_2(\xi_{2n+1}),\ Q_2^2(\xi_{2n+1}),$$

$$Q_1Q_2(\xi_{2n+1}),\ Q_1(\xi_{2n}),\ Q_2(\xi_{2n}),\ (Q_{\alpha_1}Q_{\alpha_2}\cdots(\xi))^2$$

for $\alpha_i = 1,2$; $\alpha_i \leq \alpha_{i+1}$ (excluding from these $Q_1Q_2(\xi_{2n+1})$ and $Q_2^2(\xi_{2n+1})$).

Proof. The only class listed above not mentioned in 4.2 is $Q_1Q_2(\xi_{2n+1})$. If Q_3 is defined, $d_3(Q_2Q_3(\xi_{2n+1}) = Q_1Q_2(\xi_{2n+1})$. At present we do not dispose of Q_3 and the theorem follows.

We give here the explicit form of the E^4_* term of the Atiyah-Hirzebruch spectral sequence $H_*(\Omega^s S^t X; \mathbb{Z}/2) \Rightarrow K_*(\Omega^s S^t X; \mathbb{Z}/2)$, for $s = 3$ and for $s = 2n \geq 4$. X satisfies the conditions in 4.2. We only write down the E^4_* term for such values of s, since only about them we will be able to obtain more results on the spectral sequence. The statement is analogous to that of [Sn 6, Thm. 3.6], without the E^∞_* assertion.

Theorem 4.4. a) $E^4_* \cong A \otimes B$ in the Atiyah-Hirzebruch spectral sequence

$$H_*(\Omega^3 S^3 X; \mathbb{Z}/2) \Rightarrow K_*(\Omega^3 S^3 X; \mathbb{Z}/2),$$

for X finite, torsion free CW complex; here A and B are given by the following expressions, where ξ is as in 1.11.

$$A = \bigotimes_{\xi_i} \frac{P(\xi_i, Q_1(\xi_i), Q_1Q_2(\xi_i), Q_2(\xi_i), Q_2^2(\xi_i))}{(\xi_i^4, (Q_1(\xi_i))^4, (Q_1Q_2(\xi_i))^2, (Q_2(\xi_i))^4, (Q_2^2(\xi_i))^4)}$$

$$\otimes \left[\bigotimes_{\alpha_1 = \alpha_2} E\left((Q_{\alpha_1} Q_{\alpha_2} \cdots (\xi_i))^2 \right) \right].$$

Here $\deg(\xi_i) \equiv 1$, (mod 2).

$$B = \bigotimes_{\xi_i} \frac{P(\xi_i, Q_1(\xi_i), Q_2(\xi_i))}{((Q_1(\xi_i))^2, (Q_2(\xi_i))^4)}$$

$$\otimes \left[\bigotimes_{\alpha_1 = \alpha_2} E\left((Q_{\alpha_1} Q_{\alpha_2} \cdots (\xi_i))^2 \right) \right]$$

Here $\deg(\xi_i) \equiv 0$, (mod 2).

Both in A and B, P stands for polynomial and E for Exterior.

b) In the Atiyah-Hirzebruch spectral sequence

$H_*(\Omega^s S^t X; \mathbb{Z}/2) \Rightarrow K_*(\Omega^s S^t X; \mathbb{Z}/2)$, X as above, s even, $s \geq 6$, $E_*^4 \cong A \otimes B$, where

$$A = \bigotimes_{\xi_i} \frac{P(\xi_i, Q_1(\xi_i), Q_2^2(\xi_i), Q_{s-2}(\xi_i))}{(\xi_i^2, (Q_1(\xi_i))^4, (Q_2^2(\xi_i))^4, (Q_{s-2}(\xi_i))^4)}$$

$$\otimes \left[\bigotimes_{j_1 \equiv j_2, \ (\mathrm{mod}\ 2)} E\left((Q_{j_1} Q_{j_2} Q_{j_3} \cdots (\xi_i))^2 \right) \right]$$

Here $\deg(\xi_i) \equiv 1$, (mod 2).

$$B = \bigotimes_{\xi_i} \frac{P(\xi_i, Q_2(\xi_i), Q_{s-3}(\xi), Q_{s-1}(\xi_i), Q_1Q_{s-1}(\xi_i),}{((Q_2(\xi_i))^4, (Q_{s-3}(\xi_i))^2, (Q_{s-1}(\xi_i))^2, (Q_1Q_{s-1}(\xi_i))^2,}$$

$$\frac{Q_2Q_{s-1}(\xi_i), Q_{s-2}Q_{s-1}(\xi_i))}{(Q_2Q_{s-1}(\xi_i))^2, (Q_{s-2}Q_{s-1}(\xi_i))^2)}$$

$$\otimes \left[\bigotimes_{\substack{j_1 \equiv j_2 \pmod 2 \\ (j_1, j_2) \neq (1, s-1)}} \left(E\ (Q_{j_1} Q_{j_2} \cdots (\xi_i))^2 \right) \right]$$

Here $\deg(\xi_i) \equiv 0, \pmod 2$.

(c) For $s = 4$, A and B are the same as in (b), when the overlapping of classes in the polynomial part P is simplified.

Proof. (1) Let $Q_j(\xi)$ be a d_3-cycle, $j \geq 3$. Suppose that j is odd. Then $Q_3Q_j(\xi)$ is of the type analyzed in 4.2.B.(2) and B.(7) so that $d_3(Q_3Q_j(\xi)) = (Q_j(\xi))^2$.

(2) If $\deg(\xi) \equiv 0$, mod 2, we saw in 4.2.B.(5) that

$$d_3(Q_1Q_2(\xi)) = (Q_1(\xi))^2.$$

(3) From 4.1 we have

$$d_3Q_1^3Q_{s-1}(\xi) = (Q_1Q_{s-1}(\xi))^4.$$

(4) If $\deg(\xi) \equiv 1$, (mod 2), 4.2.A.I gives

$$d_3(Q_3(\xi)) = \xi^2.$$

(5) Consider $(Q_{j_1}Q_{j_2}\ldots(\xi))^2$, $j_1 \geq 1$, $j_1 < j_2$. If $Q_1Q_{j_1+1}Q_{j_2}\ldots(\xi)$ does not satisfy condition 4.2.B.(1), then

$$d_3(Q_1Q_{j_1+1}Q_{j_2}\ldots(\xi)) = (Q_{j_1}Q_{j_2}\ldots(\xi))^2,$$

with $Q_{j_1}Q_{j_2}\ldots(\xi)$ such that $j_1 \neq j_2$, mod 2.

(6) If $j_2 \geq 2$, by 4.2.B.(6) we get

$$d_3(Q_1Q_2Q_{j_2}\ldots(\xi)) = (Q_1Q_{j_2}\ldots(\xi))^2$$

if 4.2.B.(1) does not hold for $Q_1Q_2Q_{j_2}\ldots(\xi)$, i.e., if j_2 even.

(7) The last two paragraphs have as a consequence that if $j_1 \geq 1$, $j_1 \neq j_2$, (mod 2), the class $(Q_{j_1}Q_{j_2}\ldots(\xi))^2$ is a d_3-boundary.

(8) For classes $Q_{j_1}Q_{j_2}\ldots(\xi)$, $j_1 \equiv j_2$, (mod 2), $j_1 \geq 1$, we have by 4.1 that $d_3(Q_1^2Q_{j_1}Q_{j_2}\ldots(\xi)) = (Q_{j_1}Q_{j_2}\ldots(\xi))^4$. (1)-(8) give b), and c) of the theorem.

(9) In case $s = 3$, ξ^2 is not a d_3-boundary, as there is not Q_3 operation. The situation is similar for some other d_3-cycles, which gives a). The discussion above gives the form of E_*^4 stated in a) - c).

CHAPTER 5

E_*^∞ OF THE ATIYAH-HIRZEBRUCH SPEGTRAL SEQUENCE FOR $K_*(\Omega^3S^3X;\ \mathbb{Z}/2)$

We state the result of [Sn 6] on the Atiyah-Hirzebruch spectral sequence for Ω^2S^3X, where X is a torsion-free, finite CW-complex.

Theorem 5.1. [Sn 6, Thm. 3.6] Let X be a finite CW-complex which is torsion-free. Then, in the Atiyah-Hirzebruch spectral sequence

$$H_*(\Omega^2S^3X;\ \mathbb{Z}/2) \Rightarrow K_*(\Omega^2S^3X;\ \mathbb{Z}/2),$$

$E_*^4 = E_*^\infty \cong A \otimes B$, where A and B are

$$A = \bigotimes_{\xi_i} \frac{P(\xi_i,\ Q_1(\xi_i))}{(\xi_i^4,\ (Q_1(\xi_i))^4)} \otimes \left[\bigotimes_{t\geq 2} E((Q_1^t(\xi_i))^2)\right],$$

if $\deg(\xi_i) \equiv 1$, mod 2.

$$B = \bigotimes_{\xi_i} \frac{P(\xi_i,\ Q_1(\xi_i),\ Q_1^2(\xi_i))}{((Q_1(\xi_i))^4,\ (Q_1^2(\xi_i))^4)} \otimes \left[\bigotimes_{t\geq 3} E((Q_1^t(\xi_i))^2)\right]$$

if $\deg(\xi_i) \equiv 0$, mod 2.

Here P stands for polynomial, and E for exterior.

We are now in a position to determine the Atiyah-Hirzebruch spectral sequence

$$H_*(\Omega^3S^3X;\mathbb{Z}/2) \Rightarrow K_*(\Omega^3S^3X;\mathbb{Z}/2),$$

for X a finite, torsion free CW-complex. For the rest of this section let ξ denote the classes on which the group $H_*(\Omega^3S^3X;\mathbb{Z}/2)$ is generated by applying the Dyer-Lashof operations.

Theorem 5.2. $d_3 = Sq^3_* + Sq^1_*Sq^2_*$ is the only non-trivial differential in the Atiyah-Hirzebruch spectral sequence

$$H_*(\Omega^3S^3X;\mathbb{Z}/2) \Rightarrow K_*(\Omega^3S^3X;\mathbb{Z}/2),$$

X as in 5.1.

Proof. The multiplicative generators of E^4_* were computed in 4.3. They are

(1) ξ_{2n}, ξ_{2n+1}, $Q_1(\xi_{2n+1})$, $Q_2(\xi_{2n+1})$, $Q_2^2(\xi_{2n+1})$,

$Q_1Q_2(\xi_{2n+1})$, $Q_1(\xi_{2n})$, $Q_2(\xi_{2n})$, $(Q_{\alpha_1}Q_{\alpha_2}\dots(\xi))^2$

for $\alpha_i = 1,2$, $\alpha_i \le \alpha_{i+1}$.

Notice that the odd classes in (1) are all infinite cycles. That $Q_1(\xi)$ is so is the content of Thm. 3.6 of [Sn 6], (see 3.22-3.25). $Q_1Q_2(\xi_{2n+1})$ is an infinite cycle. This fact is clear when $X = S^n$, and the general case may be reduced to his one by a result of J. McClure.

The proof of the theorem will be complete if we prove that the odd classes above cannot be hit by any differential d_s, $s > 3$. Now, if either $Q_1(\xi)$ or $Q_1Q_2(\xi_{2n+1})$ is the image of some differential, then so is $\sigma_*Q_1(\xi)$ or $\sigma_*(Q_1Q_2(\xi_{2n+1}))$, respectively, by the naturality of the Atiyah-Hirzebruch spectral sequence. Here

$$\sigma_* : H_*(\Omega^3 S^3 X; \mathbb{Z}/2) \to H_*(\Omega^2 S^3 X; \mathbb{Z}/2)$$

is the homology suspension.

But $\sigma_* Q_1(\xi) = \xi^2$ and $\sigma_* Q_1 Q_2(\xi_{2n+1}) = Q_1(\sigma\xi_{2n+1})^2$ and we know from 5.1 that neither ξ^2 nor $(Q_1(\sigma\xi_{2n+1}))^2$ are targets of any differential in the spectral sequence for $\Omega^2 S^3 X$.

CHAPTER 6

THE ALGEBRA $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$, $n \geq 1$

In this chapter we determine $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$ as an algebra. We give briefly the notions which will appear in the proofs of our results.

A. $D_{k,q}$

6.1. Let $C_k(q)$ denote the space of ordered q-tuples of little cubes disjointly embedded in I^k [Bo-V], on which the symmetric group acts freely. For a based space X, C_kX is defined by

$$C_kX = \bigcup_{q\geq 1} C_k(q) \times_{\Sigma_q} X^q/\sim$$

with $[(c_1,\ldots, c_q), (x_1,\ldots, x_{q-1}, *)] \sim [(c_1,\ldots, c_{q-1}), (x_1,\ldots, x_{q-1})]$ determining the equivalence relation [Ma 4]. The importance of the spaces C_kX is that they are approximations to Ω^kS^kX, i.e.: there are natural maps $\alpha_k: C_kX \to \Omega^kS^kX$ which preserve the additive structure and which are homotopy equivalences when X is connected [Ma 4]. The space C_kX is filtered by closed subspaces, $F_nC_kX \subset F_{n+1}C_kX$, where

$$F_nC_kX = \bigcup_{q=0}^{n} C_k(q) \times_{\Sigma_q} X^q/\sim \qquad \text{[ibid.]}$$

The quotients $F_nC_kX/F_{n-1}C_kX = D_{k,q}X$ play an important role in the rest of this chapter.

$D_{k,q}X$ are called the reduced extended power spaces, and $D_{k,q} = C_k(q)^+ \wedge_{\Sigma_q} X^{[q]}$, where Y^+ is the union of Y with a disjoint base point, and $X^{[q]}$ is the q-fold smash product.

Theorem 6.2. [Sn 1] Let $\Sigma^\infty Y$ denote the suspension spectrum of a space Y. Then

$$\Sigma^\infty \Omega^k S^k X \simeq \bigvee_{q\geq 1} \Sigma^\infty D_{k,q}X.$$

There are maps $F_n C_k X \times F_m C_k X \to F_{n+m} C_k X$ such that the composite

$$F_n C_k X \times F_m C_k X \to F_{n+m} C_k X \to D_{k,n+m} X$$

factors through the projection

$$F_n C_k X \times F_m C_k \to D_{k,n} X \wedge D_{k,m} X,$$

thus giving maps

$$D_{k,n} X \wedge D_{k,m} X \to D_{k,m+n} X, \quad [\text{Ma } 4].$$

Projection for each q of $\bigvee_{q\geq 1} D_{k,q}X$, and then adjunction in 6.2 provides maps

$$j_q : \Omega^k S^k X \to QD_{k,q}X$$

called the James-Hopf maps. Q denotes the associated infinite loop space [Co-Ma-Ta]. Consider the weak infinite product $\prod_{q\geq 1} QD_{k,q}X$, where all but a finite number of coordinates are the base point; then the pairings described above give $\prod_{q\geq 1} QD_{k,q}X$ the structure of an E_∞-ring space ([Ma 5], [C-C-M-T]).

With the notation above, in [C-C-M-T] is proved a refined version of Snaith's theorem 6.2, namely

Theorem 6.3. [C-C-M-T, Thm. 1.1] For $n \geq 1$ and connected spaces X, the following is a natural commutative diagram in the stable category in which the horizontal arrows are equivalences

$$\begin{array}{ccc} \Sigma^\infty(\Omega^k S^k X \times \Omega^k S^k X) & \xrightarrow{\sum_{r\geq 1}\sum_{p+q=r} j_p \wedge j_q} & \bigvee_{r\geq 1}\bigvee_{p+q=r} \Sigma^\infty(D_{k,p}X \wedge D_{k,q}X) \\ \downarrow & & \downarrow \\ \Sigma^\infty \Omega^k S^k X & \xrightarrow{\sum_{r\geq 1} j_r} & \bigvee_{r\geq 1} \Sigma^\infty D_{k,r}X \end{array}$$

Here the map on the left is loop addition and that on the right is given by the pairings mentioned in 6.1. Thus the stable equivalence $\sum_{r\geq 1} j_r$ is exponential, that is to say, it sends sums in $\Sigma^\infty(\Omega^k S^k X)$ to products in $\bigvee_{r\geq 1} \Sigma^\infty(D_{k,r}X)$. [ibid.]

6.4. We specialize to $X = S^r$, r odd, $k = 2$, in the previous discussion, so that we are dealing with $\Omega^2 S^2 S^r$, r odd. We suppress the index 2 in the symbols $D_{2,q}$, and denote the components of the splitting of $\Omega^2 S^{r+2}$, r odd, by D_q^r. With these conventions we state the result of [Co-Mah-Mi] on D_q^r, $q > 1$.

Theorem 6.5. $D_q^r = S^{q(r-1)} D_q^1$.

Thus Snaith's splitting 6.2 becomes

Proposition 6.6. $\Sigma^{\infty}\Omega^2 S^{r+2} \simeq \bigvee_{q=1} \Sigma^{\infty} S^{q(r-1)} D_q^1$.

The following result is due to P. May and F. Cohen [Co-La-Ma]. Let $a_j = Q_1^{j-1}(\iota_1) \in H_*(\Omega^2 S^3; \mathbb{Z}/2)$, and define $wt(a_j) = 2^{j-1}$; extend wt by $wt(x \cdot y) = wt(x) + wt(y)$.

Theorem 6.7. [B-P] $H_*(D_q^1; \mathbb{Z}/2) \subset H_*(\Omega^2 S^3; \mathbb{Z}/2)$ is generated by all monomials of wt $\leq$ q. Due to 6.5 the theorem holds for $H_*(D_q^r; \mathbb{Z}/2) \subset H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$ if $a_j = Q_1^{j-1}(\iota_{2n-1})$ and $wt(a_j) = 2^{j-1}$, extended to decomposables as above.

B. $K_*(QS^r; \mathbb{Z}/2)$, r odd

In [M-Sn] Snaith and Miller computed the algebra $K_*(QS^r; \mathbb{Z}/2)$, where QS^r is as usual the infinite loop space on S^r. We will only need the case r odd, which we state.

Theorem 6.8. $K_*(QS^r; \mathbb{Z}/2) = E(x_1, x_2, \ldots)$, if r odd.

C. Determination of $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$

We now have at hand the necessary notations to start with the study of the algebra $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$, $n \geq 1$.

6.9. Consider the set

$$G = \{(Q_1^t)(\iota))^2 \mid t \geq 2\} \subset H_{even}(\Omega^2 S^{2n+1}; \mathbb{Z}/2).$$

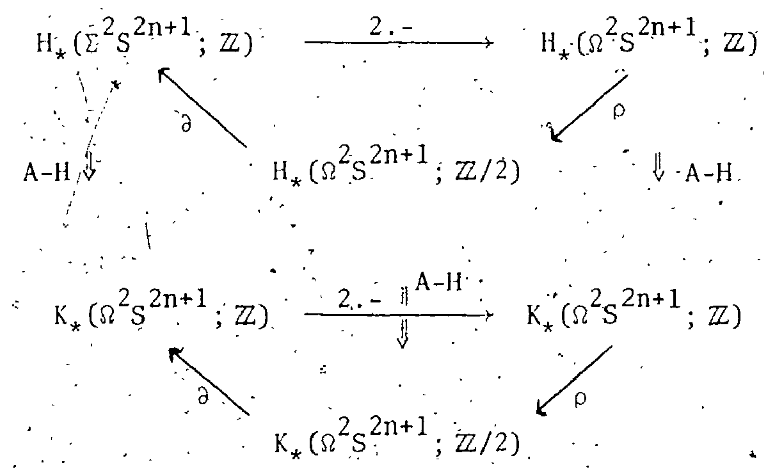
Observe that $\beta\, Q_1^{t+1}(\iota) = (Q_1^t(\iota))^2$, so that $g \in$ im ρ for all $g \in G$, and from the exact couple (1.13) we conclude that $G \subset \ker \partial$.

We will use these observations on G, together with the properties of the Bockstein spectral sequences for $H_*\mathbb{Z}/2$ and $K_*\mathbb{Z}/2$ and the splitting 6.2 of $\Sigma^\infty\Omega^2S^{2n+1}$ to prove the following.

Theorem 6.10. Let $\overline{G} = \{K_*\mathbb{Z}/2$ classes determined by $G \subset H_*(\Omega^2S^{2n+1};\mathbb{Z}/2)\}$. Then $\overline{G} \subset \operatorname{im}\rho$, (in $K_*\mathbb{Z}/2$).

Proof. The naturality of the Atiyah-Hirzebruch spectral sequence implies the following "diagram convergence" (modulo lower filtration)

(6.11)



The top triangle is the first one of the exact couple for $\mathbb{Z}/2$-homology, and the bottom triangle is the corresponding one for $K\mathbb{Z}/2$-homology, (see 1.12 and 2.33). Notice that from the definition of $K_*(\_;\mathbb{Z}/2)$ given in 2.27 (see also 2.21) it is naturality of $\mathbb{Z}$-homology and $\mathbb{Z}$-K homology which we are considering.

We proceed to prove the theorem.

From 1.17, $\rho^{-1}(\operatorname{im}\beta) = \{\text{order 2 elements}\} + \{\text{2-divisible elements}\}$

$$\subset H_*(\Omega^2S^{2n+1};\mathbb{Z}).$$

Choose for each $g \in G$ a 2-torsion element y such that $\rho(y) = g$. We claim that y is an infinite cycle in the spectral sequence

$H_*(\Omega^2S^{2n+1}; \mathbb{Z}) \Rightarrow K_*(\Omega^2S^{2n+1}; \mathbb{Z})$. For suppose there exists a differential d_{2r+1} in this spectral sequence for which $d_{2r+1}(y) \neq 0$, and assume furthermore that 2r+1 is the smallest integer, bigger than 3, with this property. Then the naturality of the A-H spectral sequence implies that

$$\rho(d_{2r+1}(y)) = d_{2r+1}(\rho(y)) + \{\text{terms in } (\text{im } d_3)\} \subset H_*(\Omega^2S^{2n+1}; \mathbb{Z}/2).$$

This is so by Theorem 5.1, which also implies that $d_{2r+1}(\rho(y)) = 0$, $r > 1$, thus giving $\rho(d_{2r+1}(y)) \in \text{im } d_3$. We next show that this forces $\rho(d_{2r+1}(y)) = 0$. For if $d_3(z) = \rho\, d_{2r+1}(y) \neq 0$, then Theorem 5.1 implies that $z = \Sigma[\otimes_{j_k} Q_1^{j_k}(\iota)] \otimes w_k$, where the j_k's are distinct and bigger than 1, and w is a square. Moreover, notice that there must be an even number of factors $Q_1^{j_k}(\iota)$, $j_k > 1$, at least 2 of them: this is so since deg(z) is even and $d_3 z \in D^{2n-1}_{2^{t+1}}$, the component of g.

Using the Nishida relations (1.9.3) one checks that this form of z implies that $\beta\, d_3(z) \neq 0$, which contradicts that $d_3(z) \in \text{im } \rho$. Thus $\rho(d_{2r+1}(y)) = 0$, as claimed. This implies that $d_{2r+1}(y)$ is 2-divisible, say 2x, and it is 2-torsion by the linearity of d_{2r+1}, (recall the choice of y). Then $2d_{2r+1}(y) = 2(2x) = 0$. However this contradicts 1.19 on the torsion of $H_*(\Omega^2S^{2n+1}; \mathbb{Z}/2)$. Thus $d_{2r+1}(y) = 0$ for all $r \geq 1$, so that y is a permanent cycle in

$$H_*(\Omega^2S^{2n+1}; \mathbb{Z}) \xRightarrow[\text{A-H}]{} K_*(\Omega^2S^{2n+1}; \mathbb{Z}).$$

Looking at diagram (6.11) we see that the naturality of the spectral sequence implies that $\overline{G} \subset \text{im } \rho$, which proves the theorem.

We are now prepared to prove the result which, together with 6.10 constitutes the main ingredient for the determination of the algebra $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$.

Let $X = S^{2n-2}$ in Theorem 5.1; then the $E_*^4 = E_*^\infty$ term is

$$G_r K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2) = \frac{\mathbb{Z}/2[\iota, Q_1\iota]}{[\iota^4, (Q_1\iota)^4]} \otimes (\bigotimes_{t\geq 2} E(Q_1^t(\iota))^2),$$

$$\deg(\iota) = 2n-1, \ \deg[(Q_1^t(\iota))^2] = 2^{t+2}n-2.$$

Theorem 6.12. The classes ι, $Q_1(\iota)$ have height 4, while the classes $(Q_1^t(\iota))^2$, $t \geq 2$ have height 2 in the algebra $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$.

Proof. Inspection of filtration shows that $\iota^4 = \lambda\,\iota\,Q(\iota) + \mu\,\iota^2$, $\lambda, \mu \in \{0,1\}$. However the right member of this equality does not fall in the component D_4^{2n-1}, where ι^4 lies (see 6.4 for the notation).

Similarly, using that $\iota^4 = 0$, we have

$$(Q\iota)^4 = \lambda_1\,\iota(Q_1(\iota))^3 + \lambda_2\,\iota^2(Q_1(\iota))^2 + \lambda_3\,\iota^3 Q_1(\iota) + \lambda_4(Q(\iota))^2$$
$$+ \lambda_5(Q_1(\iota) + \lambda_6\iota^2, \ \lambda_j \in \{0,1\}.$$

However none of the terms in the right member lies in the component D_8^{2n-1} of $(Q(\iota))^4$, which proves that $(Q_1(\iota))^4 = 0$. Consider now $(Q_1^t(\iota))^4$, $t \geq 2$. If this class is non-trivial in $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$, then

$$(6.13) \quad (Q_1^t(\iota))^4 = \sum_i \iota^{k_i} \otimes (Q_1(\iota))^{\ell_i} \otimes (\otimes_{j_i} (Q^{t_{j_i}}(\iota))^2)$$

where $0 \leq k_i, \ell_i < 4$, $k_i + \ell_i \equiv 0$, mod 2, $t_{j_i} \geq 2$, and with each term of the right member of filtration lower than that of $(Q_1^t(\iota))^4$.

Thus $(Q_1^t(\iota))^4 = 0$ in $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$, proving the theorem.

We prove by induction on $t \geq 2$ that (6.13) is impossible.

For $t = 2$, $(Q_1^2(\iota))^4 \in D_{2^4}$, and one checks that no values of k_i, ℓ_i and t_{j_i} in the required range are such that any summand at the right of (6.13) is in D_{2^4}.

Suppose $(Q_1^s(\iota))^4 = 0$ for $s \leq t - 1$. Then in the expression 6.13 for $(Q^t(\iota))^4$ we have that if $2 \leq t_{j_i} < t$ then t_{j_i} appears at most once in the right summands. Also, by filtration, $(Q^t(\iota))^2$ appears at most once in the right terms. Again use of 6.7 allows us to see that no values of k_i, ℓ_i and t_{j_i} are such that a monomial

$$\iota^{k_i} \otimes (Q_1(\iota))^{\ell_i} \otimes (\otimes_{j_i} (Q^{t_{j_i}}(\iota))^2)$$

lies in $D^{2^{n-1}}_{2^{t+2}}$, which is the component of $(Q_1^t(\iota))^4$. Thus $(Q_1^t(\iota))^4 = 0$ in $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$ proving the theorem.

We state our result on $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$.

Theorem 6.14. As an algebra,

$$K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2) \cong \frac{\mathbb{Z}/2[\iota, Q_1(\iota)]}{(\iota^4, [Q_1(\iota)]^4)} \otimes (\bigotimes_{t\geq 2} E(Q_1^t(\iota))^2),$$

where ι is a $K_*\mathbb{Z}/2$ representative of the fundamental class of $H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$.

Proof. In $E_*^4 \cong E_*^\infty$ of the Atiyah-Hirzebruch spectral sequence $H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2) \underset{A-H}{\Longrightarrow} K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$, $(n \geq 1)$ stated in 5.1, there are two possible sources of algebra relations, namely:

a) those arising from the identity $x \cdot y + y \cdot x = \beta x \cdot \beta y$, (2.26.6, 2.27), and

b) d_3-boundaries which are non-trivial in $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$.

From 6.10 we have that if x is a basic generator, $x \neq Q_1(\iota)$ then $\beta(x) = 0$, while by 6.12 $\iota^4 = (Q_1(\iota))^4 = 0$, so that $x \cdot y + y \cdot x = 0$ for all x and y. Moreover, 6.12 also shows that $(Q_1^t(\iota))^4 = 0$, for $t \geq 2$.

Thus neither a) nor b) produce new relations among the multiplcative generators of $K_*(\Omega^2 S^{2n+1}; \mathbb{Z}/2)$, other than those derived from the spectral sequence, which proves the theorem.

BIBLIOGRAPHY TO PART 2

[Ad] J. F. Adams. Lectures on generalized homology. Mathematics Lecture Notes, University of Chicago (1970).

[A] J. Adem. The relations on Steenrod powers of cohomology classes. Algebraic Geometry and Topology. (A symposium in honor of S. Lefshetz.) Princeton University Press (1957), pp. 191-238.

[An] D. W. Anderson. Universal Coefficient Theorems for K-theory (preprint).

[Ar] S. Araki. Hodgkin's Theorem. Ann. Math. 85 (1957), pp. 508-525.

[Ar-K] S. Araki and T. Kudo. Topology of H_n-spaces and H-squaring operations. Mem. Fac. Kyusyu Univ., Ser. A, 10(1956), pp. 85-120.

[Ar-T] S. Araki and H. Toda. Multiplicative structures in mod q cohomology theories I, II. Osaka J. Math. 2(1965), 71-115 and 3(1966), pp. 81-120.

[Ar-Y] S. Araki and Z. Yosimuya. Differential Hopf algebras modelled on K-theory mod p I. Osaka J. Math. 8(1971), pp. 151-206.

[At 1] M. F. Atiyah. Characters and cohomology of finite groups. Pub. Math., No. 9, IHES Paris.

[At 2] M. F. Atiyah. K-theory. Benjamin Press (1968).

[At-Se] M. F. Atiyah and G. B. Segal. Equivariant K-theory and completion. J. Diff. Geom. 3(1969), pp. 1-18.

[Bo-V] J. M. Boardman and R. M. Vogt. Homotopy-everything H-spaces. Bull. Amer. Math. Soc. 74(1968), pp. 1117-1122.

[Br] W. Browder. Homology operations and loop spaces. Illinois J. Math. 4(1960), pp. 347-357.

[B-P] E. H. Brown and F. P. Peterson. On the stable decomposition of $\Omega^2 S^{r+2}$. Trans. Am. Math. Soc., Vol. 243 (1978), pp. 287-298.

[Coh] J. M. Cohen. Stable Homotopy. Lecture Notes in Math. 165. Springer-Verlag.

[C-C-M-T] J. Caruso, F. R. Cohen, J. P. May, L. R. Taylor. James maps, Segal maps and the Kahn-Priddy theorem. Trans. AMS 1, 281.

[Co-La-Ma] F. R. Cohen, T. J. Láda, J. P. May. The homology of iterated loop spaces. Lecture Notes in Math., No. 533 (1976).

[Co-Mah-Mi] F. R. Cohen, M. Mahowald, R. J. Milgram. The stable decomposition of the double loop space of a sphere. AMS Proc. Symp. Pure Math. 32(1978), part 2 (1978), pp. 225-228.

[Co-Ma-Ta] F. R. Cohen, J. P. May, L. R. Taylor. Splitting of certain spaces CX. Math. Proc. Camb. Phil. Soc. 84(1978), pp. 465-496.

[D-L] E. Dyer and R. K. Lashof. Homology of iterated loop spaces. Amer. J. Math. 84(1962), pp. 35-88.

[E-M] S. Eilenberg and John C. Moore. Homology and fibrations I. Com. Mat. Helv., Vol. 40, (1966), pp. 199-236.

[Ho] L. Hodgkin. A Künneth formula in equivariant K-theory. Warwick University preprint, 1968.

[Ma 1] J. P. May. Categories of spectra and infinite loop spaces. Lecture Notes in Math., Vol. 99. Springer-Verlag.

[Ma 2] J. P. May. A general algebraic approach to Steenrod operations. Lecture Notes in Math., Vol. 168. Springer-Verlag.

[Ma 3] J. P. May. Homology operations on infinite loop spaces. Proc. Symp. Pure Math., Vol. 22. Amer. Math. Soc. 1971.

[Ma 4] J. P. May. The geometry of iterated loop spaces. Lecture Notes in Math., Vol. 271. Springer-Verlag.

[Ma 5] J. P. May. E_∞ ring spaces and E_∞ ring spectra. Lecture Notes in Math. 577. Springer-Verlag.

[Mas] W. S. Massey. Exact couples in algebraic topology I, II. Ann. Math. 56 and 57.

[R-S] M. Rothenberg and N. E. Steenrod. The cohomology of classifying spaces of H-spaces. Bull. Amer. Math. Soc. 71(1961).

[Se] G. B. Segal. Equivariant K-theory. Pub. Math., No. 34 (1968) IHES Paris.

[Sn 1] V. P. Snaith. A stable decomposition of $E_n S_n X$. J. London Math. Soc. 7, 1974, pp. 577-583.

[Sn 2] V. P. Snaith. On the K-theory of homogeneous spaces and conjugate bundles of Lie groups. Proc. L. M. Soc. (3) 22, 1971, pp. 562-584.

[Sn 3] V. P. Snaith. On cyclic maps. Proc. Camb. Phil. Soc. (1972) 71, pp. 449-456.

[Sn 4] V. P. Snaith. Massey products in K-theory I, II. Proc. Camb. Phil. Soc. (1970), 68, 303-320 and (1971) 69, 259-289.

[Sn 5] V. P. Snaith. Dyer-Lashof operations in K-theory. Lecture Notes in Math. 496 (1975).

[Sn 6] V. P. Snaith. On $K_*(\Omega^2 X; \mathbb{Z}/2)$. Quart. J. Math. Oxford (3), 26 (1975), pp. 421-436.

[M-Sn] Hanes Miller and V. P. Snaith. On $K_*(QRP^n; \mathbb{Z}/2)$. Canadian Math. Soc. Conference Proc., Vol. 2, Part 1 (1982).

[St] N. E. Steenrod. The cohomology algebra of a space. L'énseignement Math. II Serie Tome VII (1961), pp. 153-178.

[St-E] N. E. Steenrod and D. B. A. Epstein. Cohomology operations. Annals of Math. Study, No. 50. Princeton University Press, 1962.

[Mi] J. W. Milnor. On axiomatic homology theory. Pacific J. Math. 12 (1962), pp. 337-341.

END

11·03·86

FIN