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Contemporaneous Carma Modelling With Applications

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CONTEMPORANEOUS CARMA MODELLING WITH APPLICATIONS

by

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ABSTRACT

This thesis presents a comprehensive study of the statistical properties of the contemporaneous Autoregressive Moving-Average (CARMA) model. The research results constitute a more general framework than previously available for the analysis of many actual sets of time series data. It is shown in the thesis that the joint estimation is asymptotically efficient. For the case of the CAR(1) model, asymptotic theory and small sample simulation show that the gain in efficiency over univariate estimation can be in excess of 50%. A computationally efficient procedure to obtain the joint estimation of the parameters together with a useful estimation procedure for the case of unequal sample sizes is also given in the thesis. Applications in hydrology are presented, where the physical restrictions of the system often suggest that a CARMA model would be appropriate. Test statistics for two important hypotheses are also considered: -(a) whether a joint set of univariate models will suffice and (b) whether $\underset{\sim}{\beta}_h = \underset{\sim}{\beta}$, or otherwise, where $\underset{\sim}{\beta}_h$ is the vector of parameters for the series h.

To Gloria and Ximena

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CHAPTER 1

INTRODUCTION

Univariate Autoregressive Moving Average models, i.e., Univariate ARMA models, popularized by Box and Jenkins (1976) are widely used today to fit time series data in engineering, economics and many other fields of application. These models describe the dynamics of the series Z_t in the following form:

$$\phi(B)Z_t = \theta(B)\, a_t$$

where

$$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$$

$$\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$$

$$a_t \sim NID(0, \sigma^2).$$

and B is the lag operator such that $BY_t = Y_{t-1}$.

If the zeros of the polynomial $\phi(B)$ lie outside the unit circles the model is stationary. If the zeros of $\theta(B)$ lie outside the unit circle the model is invertible.

In many situations, however, not only one series but several series need to be jointly considered. This leads to the extension of the univariate ARMA model to the k-dimensional multivariate ARMA (p, q) model of the form:

$$\Phi(B)\underset{\sim}{Z}_t = \Theta(B)\underset{\sim}{a}_t$$

where $\Phi(B)$ and $\Theta(B)$ are matrix polynomials given by

$$\Phi(B) = I_{k\times k} - \Phi_1 B - \ldots - \Phi_p B^p$$

$$\Theta(B) = I_{k\times k} - \Theta_1 B - \ldots - \Theta_q B^q$$

and the vectors $\underset{\sim}{a}_t$ are $NID_k\ (0,\Delta)$

This model is said to be stationary if the zeros of the determinant equation $|\Phi(B)| = 0$ lie outside the unit circle and invertible if the zeros of the determinant equation $|\Theta(B)| = 0$ lie outside the unit circle. These models have been studied by, among others, Tiao and Box (1981), Jenkins and Alavi (1981), Hillmer and Tiao (1979), Nicholls and Hall (1979), Wilson (1973) and Hannan (1970). The multivariate ARMA model is very useful in studying the dynamic relationships among different series. Such relationships may usefully be categorized in the following way, each of which is closely related to the concept of Granger Causality (Granger, 1969) and has a representation which characterizes it. (a) When the Φ and Θ matrices are all diagonal the model is said to be contemporaneous only ARMA or CARMA; in this case only current values of one series affect current values of the other series. (b) When the Φ and Θ matrices are all lower (or upper) triangular the model is said to be a transfer function model; in this case one series is a leading indicator for the other. (c) When at least one of the Φ or Θ matrices is a full matrix the model is said to be a feedback model; in this case past values of one series affect future values of the other series and vice versa.

It is of interest to consider the class of Contemporaneous only ARMA models. This class has proved useful in the modelling of many actual time series. Indeed, Pierce (1977) with economic time series and Hipel et al. (1984) with geophysical time series have provided evidence of the adequacy of the CARMA model in many situations. Risager (1980) has fitted a CAR model to series of measurements of relative content of oxygen isotope $0^{18}/0^{16}$ of two ice cores in central Greenland, while Salas et al. (1979) have suggested the use of CARMA models in fitting multisite hydrologic time series. The CARMA model also corresponds to the case when only Granger instantaneous causality is present in a system (Pierce and Haugh, 1977 and 1979) and, as is pointed out by Granger and Newbold (1977), this may arise when some time aggregation is present in the data, a situation which occurs frequently in many fields. These considerations show that the class of CARMA models is in fact a very rich class of models and that a detailed analysis would be desirable. In this thesis, a detailed study of the statistical properties of the CARMA models is given. The nice diagonal structure of the model allows the derivation of many results which are either very complicated in the general multivariate ARMA model or intractable.

The CARMA model can also be considered as a collection of K-univariate ARMA models with contemporaneously correlated innovations. In fact, the CARMA (p, q) model can be written as:

$$\phi_h(B)Z_{h,t} = \theta_h(B)a_{ht} \qquad h = 1, \ldots ,k \qquad (1.1)$$
$$\underset{\sim}{a}_t = (a_{1t}, \ldots ,a_{kt})' \sim \text{NID } (0,\Delta) .$$
$$p = \max\{p_1, \ldots ,p_k\} \qquad q - \max\{q_1, \ldots ,q_k\}$$

This representation raises many questions regarding the performance of statistics obtained from the univariate models compared with those obtained from the multivariate model. In particular, questions regarding the efficiency and consistency of parameter estimators and the distribution of the residual cross correlations are of interest. These questions are dealt with in Chapter 2.

The basic idea of considering a set of univariate models with contemporaneously correlated innovations is not new. In fact, Zellner (1962) introduced the seemingly unrelated regression equations (SURE) model of the form:

$$Y_{ht} = X_{ht}\underset{\sim}{\beta}_h + a_{ht} \qquad h = 1, \ldots ,k; \qquad t = 1, \ldots ,T$$
$$\underset{\sim}{a}_t = (a_{1t}, \ldots ,a_{kt}) \sim \text{IID } (0,\Delta)$$

where X_{ht} is a $\ell \times 1$ vector of explanatory variables and $\underset{\sim}{\beta}_h = (\beta_{h1}, \ldots ,\beta_{h\ell_h})$ is the vector of parameters for the h^{th} model. He pointed out many potential applications of the model and observed that such contemporaneous correlation is a common feature of sets of regression equations. The SURE model has been studied by many authors: Maeshiro (1980), Revankar (1974), Kmenta and Gilbert (1970 and 1968), Kakwani (1967), Parks (1967), Zellner (1963, 1962) and Zellner and Huanq (1962). These authors have shown that the joint estimation of the parameters leads, in general, to a gain in efficiency compared with the univariate estimators even for small sample size. Parks (1967), Kmenta and Gilbert (1970) and Maeshiro (1980) have considered the case of autocorrelated disturbance and proposed and compared several estimation techniques for the case of AR(1) disturbances. The consideration of autocorrelated disturbances is, as pointed out by Kmenta and Gilbert (1970), very important because many of the actual data sets encountered by researchers are time series data sets. The class of contemporaneous transfer functions of the form:

$$Y_{ht} = \sum_{r=1}^{\ell_h} \frac{\omega_{rh}(B)}{\delta_{rh}(B)} X_{rht} + \frac{\theta_h(B)}{\phi_h(B)} a_{ht} \qquad h = 1, \ldots, h$$

$$\underset{\sim}{a}_t = (a_{1t}, \ldots, a_{kt})' \sim NID\ (0, \Delta)$$

with the roots of the polynomials $\theta_h(B)$, $\phi_h(B)$ and $\delta_{rh}(B)$ outside the unit circle, provide a more general framework to the class of SURE models. The asymptotic properties of the estimators of the parameters are considered in Chapter 2.

One of the constraints of the general multivariate ARMA model is that an equal number of observations for each one of the series is required for the estimation of the parameters of the model. However, in many applications, series with differenct sample sizes are available (see Hipel et al., 1984; Risager, 1980) and naturally the researcher would like to make use of as much information as possible in the estimation of the parameters. In Chapter 3, the likelihood function for the parameters of the CARMA model when the series have different sample sizes is obtained and an adequate algorithm is developed for the estimation of the parameters. Large sample properties of these estimators are also given.

It is important to consider empirical applications of the CARMA model. These are considered in Chapter 4. In particular, attention is focussed on the use of the CARMA model in hydrology.

Many models have been proposed in the literature to model multisite streamflow time series (see for example Fiering, 1964; Matalas, 1967; Young and Pisano, 1968; Matalas and Wallis, 1971; Bernier, 1971, Pegram and James, 1972; Valencia and Shaake, 1973; Mejia et al., 1974; Kahan, 1974; O'Connell, 1974; Yevjevich, 1975; Lawrence, 1976; Mejia and Rouselle, 1976; Salas and Pegram, 1977; Ledolter, 1978; Salas et al., 1979; Cooper and Wood, 1982 a,b; and Deutsch and Ramos, 1984). In general the proposed models belong to the class of multivariate ARMA models. The full multivariate ARMA model is very complicated and the number of parameters increase rapidly with the size of the model. Salas et al. (1979) suggested reducing the number of parameters of the multivariate model by considering diagonal parameter matrices.

There are important physical constraints in the multisite system that impose specific structures on the model. For example, feedback relationships would never be expected so that either a triangular model (transfer function) or a diagonal model (CARMA) would always be entertained to model multisite hydrologies.

Specifically, the CARMA model may arise in several situations. An important case is the modelling of two-station riverflows. When the stations are located at separate rivers, then physical restrictions of the system imply that the CARMA model is adequate to model the system. Another case is when temporal aggregation is present in the data. In this case it is likely that the dynamic relationships between series collapse, simplyfying the model to a CARMA (see Grange and Newbold, 1977).

There are two important tests of hypotheses associated with the CARMA model which require special attention. Consider a bivariate CARMA model. The first hypothesis is concerned with the significance of ρ, the correlation between the innovations of the series. This hypothesis is of greater relevance in the CARMA model because, to quote Pierce and Haugh (1979), "in many situations the important consideration is whether a bivariate model is necessary or a single-equation model suffices". The other hypothesis compares the parameters of the two series. In particular, if $\beta_h = (\phi_{h1}, \ldots, \phi_{hp}, \theta_{h1}, \ldots, \theta_{hq})$, $h = 1, 2$, it is desired to test the hypothesis $H_o : \underset{\sim}{\beta}_1 = \underset{\sim}{\beta}_2$. Zellner (1962) considered a similar test for the SURE model and stated its relevance. On the other hand, Risager (1980), by considering the nature of the process at hand, concluded that it was reasonable to expect the hypothesis to be true. These two test statistics are considered in detail in Chapter 5. Monte Carlo simulation experiments comparing the power of the alternative test statistics are also reported.

CHAPTER 2

DISTRIBUTION OF ESTIMATORS AND RESIDUAL AUTOCORRELATIONS IN CARMA MODELS

2.1 INTRODUCTION

The contemporaneous ARMA (p, q) model, CARMA (p, q), is defined as:

$$\phi_h(B)(Z_{h,t} - \mu_h) = \theta_h(B)a_{h,t} \qquad h = 1,\dots,k \qquad (2.1.1)$$

$$\underset{\sim}{a}_t = (a_{1t}, \dots, a_{kt})' \sim \text{NID}(0,\Delta)$$

where

$$\phi_h(B) = 1 - \phi_{h1}B - \dots - \phi_{hp_h}B^{p_h}$$

$$\theta_h(B) = 1 - \theta_{h1}B - \dots - \theta_{hq_h}B^{q_h}$$

$\Delta = (\sigma_{gh})$ the variance covariance matrix of $\underset{\sim}{a}_t$

$\mu_h =$ the mean of series Z_h.

$p = \max(p_1, \dots, p_k)$ and $q = \max(q_1, \dots, q_k)$.

It is assumed that the zeros of the polynomial equations $\phi_h(B) = 0$ and $\theta_h(B) = 0$, $h = 1, \dots, k$, lie outside the unit circle so that the model is stationary and invertible. In the case that $\sigma_{gh} = 0$ for $g \neq h$ the model collapses to a set of k independent univariate ARMA (p, q) models as defined by Box and Jenkins (1976). The CARMA model describes the case when only contemporaneous Granger causality is present among the series (see Granger, 1969; Pierce and Haugh, 1979 and 1977). Pierce (1977) and Hipel et al. (1983) provide empirical evidence that many economic and geophysical time series possess in fact only Granger instantaneous causality, so that they can be adequately fitted with the CARMA model.

In 1962 Zellner proposed a similar model using a set of regression equations of the form:

$$Y_{ht} = \underset{\sim}{X}_{ht} \, \underset{\sim}{\beta}_h + U_{ht} \qquad (2.1.2)$$

where $\underset{\sim}{X}_{ht}$ is a vector of ℓ_h non-stochastic explanatory variables, $\underset{\sim}{\beta}_h$ is a vector of ℓ_h unknown regression coefficients and $\underset{\sim}{U}_t = (U_{1t}, \ldots, U_{kt})$ is a vector of random disturbances with mean zero and covariance matrix Δ. He called this model the seemingly unrelated regression equation (SURE) model. One of the main problems associated with this model is the efficient estimation of the parameters of the model. In particular, how do the estimators obtained from the univariate models (i.e., using only data for series h) behave compared with the estimators obtained from the multivariate model (i.e., joint estimation using all the data)? Zellner and other researchers have found that in general, even in small samples, joint estimation is more efficient than univariate estimation (see Zellner 1962 and 1963; Zellner and Huong 1962; Parks, 1967; Kmenta and Gilbert, 1968 and 1970; Revankar, 1974; Maeshiro, 1980). A similar situation holds for the CARMA model. In Section 2.2 the asymptotic distribution of the univariate estimators is compared with the asymptotic distribution of the multivariate estimator. It will be shown that in some critical cases the loss in efficiency of unvariate estimators may be well over 50%. A simulation experiment carried out to compare the efficiency of the estimators for small sample sizes using the CAR(1) model, is reported in Section 2.3. In Section 2.4, the results of Section 2.2 are extended to the case of contemporaneous transfer function models which provide a more general framework to analyze the SURE model of equation (2.1.2). In Section 2.5, the distribution of the residual autocorrelation matrices is derived. Finally, in Section 2.6 an algorithm is developed to simulate the CARMA models.

Risager (1980 and 1981) proposed the CARMA (p,0) process, which he called the simple correlated autoregressive process, to model climate variations and derived some of the results presented in this Chapter for this particular model. The results presented here are, however, more general. Also, the techniques used in the proofs of the results are different from those given by Risager.

2.2 ESTIMATION OF PARAMETERS

The estimation of the parameters of the CARMA(p,q) model given by equation (2.1.1), is considered in this section. To fix ideas the following notation is introduced. Let $\{\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_N\}$, where $\underset{\sim}{Z}_t = (Z_{1t}, \ldots, Z_{kt})'$, $t = 1, \ldots, N$, be a sample of N consecutive observations from a CARMA(p,q) process. Let $\underset{\sim}{\beta}_h = (\phi_{h1}, \ldots, \phi_{hp}, \theta_{h1}, \ldots, \theta_{hq})$ denote the parameters of series Z_{ht}, $h = 1, \ldots, k$, and let $\underset{\sim}{\beta} = (\underset{\sim}{\beta}_1', \ldots, \underset{\sim}{\beta}_k')'$ denote the vector of parameters of the CARMA model. It is assumed, without loss of generality, that the order of the univariate models are the same, i.e., $p_h = p$, $q_h = q$, $h = 1, \ldots, k$. It is also assumed that the process is (i) stationary, (ii) invertible (as was pointed out in Section 2.1 a necessary and sufficient condition is that the zeros of the polynomial equations $\phi_h(B) = 0$ and $\theta_h(B) = 0$ lie outside the unit circle) (iii) that $\phi_h(B)$ and $\theta_h(B)$ do not have common factors and (iv) that the innovations are Gaussian. Let $\bar{\underset{\sim}{\beta}}_h$ denote the univariate maximum likelihood estimator of $\underset{\sim}{\beta}_h$ obtained using the data $\{Z_{h1}, \ldots, Z_{hN}\}$. Algorithms to obtain these estimators are given elsewhere (see for example McLeod, 1977; Ansley 1979; McLeod and Sales, 1983). Let $\bar{\underset{\sim}{\beta}} = (\bar{\underset{\sim}{\beta}}_1', \ldots, \bar{\underset{\sim}{\beta}}_k')'$ denote the vector of univariate estimators. The first Lemma gives the asymptotic distribution of $\bar{\underset{\sim}{\beta}}$.

<u>Lemma 2.2.1</u> The asymptotic distribution of $\sqrt{N}(\bar{\beta} - \beta)$ is normal with mean vector zero and covariance matrix $V_{\bar{\beta}}$:

$$V_{\bar{\beta}} = \begin{bmatrix} \sigma_{11} I_{11}^{-1} & \cdots & \sigma_{1k} I_{11}^{-1} I_{1k} I_{kk}^{-1} \\ \\ \sigma_{k1} I_{kk}^{-1} I_{k1} I_{11}^{-1} & \cdots & \sigma_{kk} I_{kk}^{-1} \end{bmatrix} \qquad (2.2.1)$$

where

$$I_{gh} = \begin{bmatrix} \gamma_{V_g V_h}(i-j) & \gamma_{V_g U_h}(i-j) \\ \gamma_{U_g V_h}(i-j) & \gamma_{U_g U_h}(i-j) \end{bmatrix} \begin{matrix} p_g \\ q_g \end{matrix}$$

$$\qquad\qquad\quad p_h \qquad\qquad\qquad q_h$$

$$\gamma_{cd}(i-j) = \langle c_{t-i} \cdot d_{t-j} \rangle$$

c,d standing for V_g, U_g, V_h, U_h. $\langle . \rangle$ denotes expectation and the auxiliary time series are defined by:

$$\begin{aligned} \phi_h(B) V_{ht} &= -a_{ht} & h &= 1, \ldots, k \\ \theta_h(B) U_{ht} &= a_{ht} & h &= 1, \ldots, k \end{aligned} \qquad (2.2.2)$$

<u>Proof:</u> It is well known that under normality, identifiability, stationarity and invertibility conditions the univariate ARMA model meets the usual regularity conditions for the maximum likelihood estimator to be asymptotically normal and efficient. Therefore, the MLE $\bar{\beta}_h$ can be expanded as:

$$\bar{\beta}_h - \beta_h = \sigma_{hh} I_{hh}^{-1} S_h + O_p(1/N) \qquad (2.2.3)$$

where

$S_h = (S_{h1}, \ldots, S_{hp+q})'$ is the score function.

$$S_{hi} = \begin{cases} -(N\sigma_{hh})^{-1} \sum_{t=1}^{N} a_{ht} V_{ht-i} & i = 1, \ldots, p \\ -(N\sigma_{hh})^{-1} \sum_{t=1}^{N} a_{ht} U_{ht-i} & i = p+1, \ldots, p+q \end{cases} \qquad (2.2.4)$$

From equations (2.2.3) and (2.2.4) it is straightforward to show that $N\langle(\bar{\beta}_g - \beta_g)\cdot(\bar{\beta}_h - \beta_h)'\rangle = \sigma_{gh} I_{gg}^{-1} I_{gh} I_{hh}^{-1}$ which gives equation (2.2.1). It is easy to see that linear combinations of the S 's are the average of Martingale differences with convergent finite variance. Therefore, normality follows from the Martingale Central limit theorem (Billingsley, 1961).

Let $\hat{\beta} = (\hat{\beta}_1', \ldots \hat{\beta}_k')'$ denote the MLE of β using joint estimation. The following Lemma gives the asymptotic distribution of $\hat{\beta}$.

Lemma 2.2.2 The asymptotic distribution of $\sqrt{N}(\hat{\beta} - \beta)$ is normal with zero mean and variance covariance $V_{\hat{\beta}}$ given by:

$$V_{\hat{\beta}} = \begin{pmatrix} \sigma^{11} I_{11} & \cdots & \sigma^{1k} I_{1k} \\ & & \\ \sigma^{k1} I_{k1} & \cdots & \sigma^{kk} I_{kk} \end{pmatrix}^{-1} \qquad (2.2.5)$$

where the I_{gh} submatrices are defined in Lemma 2.2.1 and $\Delta^{-1} = (\sigma^{gh})$ is the inverse of the innovation variance covariance matrix.

<u>Proof</u>. It is very easy to see that assumption (i) through (iii) of the CARMA model imply stationary, invertibility and triangular identifiability of the model when it is considered as a multivariate ARMA model (Dunsmuir and Hannan, 1976). Wilson (1973) and later Dunsmuir and Hannan, (1976) showed that under such conditions $\sqrt{N}(\hat{\beta} - \beta)$ is asymptotically normal with zero mean and covariance matrix Γ^{-1} where

$$\Gamma = \lim_{N\to\infty} \langle \partial^2 S / \partial\beta\,\partial\beta' \rangle$$

with $S = \sum_{t=1}^{N} a_t' \Delta^{-1} a_t / 2N$

Now from equation (2.1.1) it follows that

$$\partial a_t / \partial\phi_{h\ell} = (0,\ldots,V_{ht-\ell},\ldots,0)' \quad h=1,\ldots,k; \quad \ell=1,\ldots,p$$
$$\partial a_t / \partial\theta_{h\ell} = (0,\ldots,U_{ht-\ell},\ldots,0)' \quad h=1,\ldots,k; \quad \ell=1,\ldots,q$$

where V_{ht} and U_{ht} are the auxiliary series defined by equations (2.2.2). The second derivatives of S are given by:

$$\partial^2 S/\partial\beta_{gi}\,\partial\beta_{hj} = \frac{1}{N}\sum_{t=1}^{N}(\partial a_t'/\partial\beta_{gi}\,\Delta^{-1}\,\partial a_t/\partial\beta_{hj} + a_t'\Delta^{-1}\,\partial^2 a_t/\partial\beta_{gi}\,\partial\beta_{hj})$$

Taking expectations, the first term of this equation becomes:

$$\sum_{t=1}^{N} \sigma^{gh} \langle W_{gt-i} W_{ht-j} \rangle / N = \sigma^{gh}\,\gamma_{W_g W_h}(i-j)$$

where W stands for V if $\beta = \phi$ or for U if $\beta = \theta$.

The second term of this equation has zero expectation. The Theorem follows comparing this result with I_{gh}.

In the following Theorem the asymptotic distributions of $\bar{\beta}$ and $\hat{\beta}$ are compared. Although both estimators are asymptotically unbiased and asymptotically consistent, $\bar{\beta}$ is not as efficient as $\hat{\beta}$.

Theorem 2.2.1 $V_{\bar{\beta}} - V_{\hat{\beta}}$ is a positive semidefinite matrix, so that $\bar{\beta}$ is not an asymptotically efficient estimator if Δ is not a diagonal matrix.

Proof: Consider the vector $\alpha = \sqrt{N}\,([\bar{\beta} - \beta]', \partial S/\partial\beta')'$

where $S = \sum_{t=1}^{N} a_t' \Delta^{-1} a_t /2N$. Then,

$$\partial S/\partial\beta_{hj} = \sum_{r=1}^{K} \sigma^{rh} \sum_{t=1}^{N} a_{rt} W_{ht-j}/N$$

where W stands for V if $\beta = \phi$ and for U if $\beta = \theta$. Now because of the normality assumption, it follows from a well known result of Isserlis (1918) that:

$$\begin{aligned} \langle a_{gt} W_{gt-i}\, a_{rt'} W_{ht'-j} \rangle &= \langle a_{gt} W_{gt-i} \rangle \langle a_{rt'} W_{ht'-j} \rangle + \\ &\quad \langle a_{gt} a_{rt'} \rangle \langle W_{gt-i} W_{ht'-j} \rangle + \\ &\quad \langle a_{gt} W_{ht'-j} \rangle \langle W_{gt-i}\, a_{rt'} \rangle \\ &= \sigma_{gr} \gamma_{W_g W_h} (i - j) \,.\, \delta(t - t') \end{aligned} \qquad (2.2.6)$$

where $\delta(t) = 1$ for $t = 0$ and $\delta(t) = 0$ for $t \neq 0$.

From equation (2.2.4), it follows that:

$$N \langle S_{gi} \partial S / \partial \beta_{hj} \rangle = \gamma_{W_g W_h}(i - j) \cdot \sum_{r=1}^{K} \sigma^{rh} \sigma_{gr} / \sigma_{gg}$$

$$= \gamma_{W_g W_h}(i - j) / \sigma_{gg} \quad \text{if } g = h$$

$$= 0 \quad \text{otherwise.}$$

This result and equation (2.2.3) imply that

$$N \langle (\bar{\beta} - \beta) \cdot \partial S / \partial \beta' \rangle = 1_{k(p+q) \times k(p+q)}$$

where 1_m is the m-dimensional identity matrix.

Now, from equation (2.2.6) it is very easy to show that

$$N \cdot \langle \partial S / \partial \beta \cdot \partial S / \partial \beta' \rangle = V_{\hat{\beta}}^{-1} .$$

Therefore, the variance covariance matrix of α, which is positive semidefinite, is given by:

$$\begin{bmatrix} V_{\bar{\beta}} & 1 \\ 1 & V_{\hat{\beta}}^{-1} \end{bmatrix}$$

It follows from a result of matrix algebra that

$$V_{\bar{\beta}} - 1.(V_{\hat{\beta}}^{-1})^{-1}1 = V_{\bar{\beta}} - V_{\hat{\beta}}$$

is a positive semidefinite matrix, which is the desired result. In the case that Δ is a diagonal matrix, it is very easy to see that $V_{\bar{\beta}} = V_{\hat{\beta}}$.

The following Lemma provides a computationally and statistically efficient algorithm to estimate the parameters of the CARMA model.

Lemma 2.2.3 Let $\underset{\sim}{\beta}^* = \underset{\sim}{\bar{\beta}} - V_{\hat{\beta}}(\partial S/\partial \underset{\sim}{\beta})_{\underset{\sim}{\beta} = \underset{\sim}{\bar{\beta}}}$ where $S = \sum_{t=1}^{N} \underset{\sim}{a}'_t \Delta^{-1} \underset{\sim}{a}_t/2N$. Then $\underset{\sim}{\beta}^*$ is an asymptotically efficient estimator.

Proof. From Lemma 2.2.1, $\underset{\sim}{\bar{\beta}}$ is an asymptotically consistent estimator of $\underset{\sim}{\beta}$. Therefore, $\underset{\sim}{\beta}^*$, which corresponds to one iteration of the method of scores, has the same asymptotic properties as the MLE of $\underset{\sim}{\beta}$ (Cox and Hinkley, 1974; Harvey, 1981).

The main idea in the above procedure is to estimate the parameters of the series h, $\underset{\sim}{\beta}_h$ h = 1, ..., k, using univariate ARMA estimation algorithm and then to calculate one iteration of the Gauss Newton optimization scheme. Of course, iterations may be continued until convergence is obtained to give the MLE $\hat{\underset{\sim}{\beta}}$.

The following Theorem gives the distribution of the estimators of the mean vector $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_k)$ and the variance convariance matrix Δ in the CARMA model. As before, $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_k)$ denotes the vector of univariate estimators for $\underset{\sim}{\mu}$ and $\hat{\underset{\sim}{\mu}}$ the joint estimator. Similar notation is used for Δ.

<u>Theorem 2.2.2</u> The asymptotic distributions of $\sqrt{N}(\bar{\underset{\sim}{\mu}} - \mu, \bar{\Delta} - \Delta)$ and $\sqrt{N}(\hat{\underset{\sim}{\mu}} - \underset{\sim}{\mu}, \hat{\Delta} - \Delta)$ are identical. Both are normal with zero mean and variance covariance given by:

$$V = \begin{pmatrix} I_\mu^{-1} & 0 \\ 0 & I_\Delta^{-1} \end{pmatrix}$$

where

$$I_\mu = (\sigma^{gh}\phi_g(1)\phi_h(1)/\theta_g(1)\theta_h(1))$$

$$I_\Delta = (\iota(\sigma_{ij}, \sigma_{rs}))/2$$

$$\iota(\sigma_{ij}, \sigma_{rs}) = (\sigma^{si}\sigma^{jr} + \sigma^{sj}\sigma^{ir})/2$$

Furthermore, this distribution is statistically independent of $\underset{\sim}{\beta}$ and $\bar{\underset{\sim}{\beta}}$.

<u>Proof:</u> Consider first the distribution of $\sqrt{N}(\hat{\underset{\sim}{\mu}} - \underset{\sim}{\mu}, \hat{\Delta} - \Delta)$. As in Lemma 2.2.2, the normality, identifiablity, stationarity and

invertibility conditions ensure that the regularity conditions for the asymptotic results of the MLE are satisfied. Moreover, the likelihood can be approximated by (Hillmer and Tiao, 1979).

$$l(\underset{\sim}{\beta}, \underset{\sim}{\mu}, \Delta) = C - N \log|\Delta|/2 - \sum_{t=1}^{N} \underset{\sim}{a}'_t \Delta^{-1} \underset{\sim}{a}_t/2 \qquad (2.2.7)$$

It follows that the asymptotic distribution of $\sqrt{N}(\hat{\mu} - \mu, \hat{\Delta} - \Delta)$ is normal with mean zero and variance covariance I^{-1} where $I = \lim_{N\to\infty} \langle -\partial^2 l/\partial^2(\mu,\Delta)\rangle /N$ is the large sample Fisher information matrix per observation.

Now

$$\partial l/\partial\mu_h = \sum_{t=1}^{N} a_t' \Delta^{-1} \begin{pmatrix} 0 \\ \vdots \\ C_h \\ \vdots \\ 0 \end{pmatrix}$$

where $C_h = -\partial a_h/\partial\mu_h = \phi_h(1)/\theta_h(1)$ (see equation 2.1.1)

So
$$I_\mu = \langle -\partial^2 l/\partial^2 \underset{\sim}{\mu}\rangle/N = (\sigma^{gh} C_g C_h)$$
$$= \text{dig}(C_1, \ldots, C_k)\, \Delta^{-1}\, \text{dig}(C_1, \ldots, C_k) \qquad (2.2.8)$$

Also,

$$\partial^2 l/\partial\sigma_{ij}\partial\mu_h = -\sum_{t=1}^{N} a_t' \Delta^{-1} \underset{\sim}{k}_{ij} \Delta^{-1} \begin{pmatrix} 0 \\ \vdots \\ C_h \\ \vdots \\ 0 \end{pmatrix}$$

$$= O + O_p(\sqrt{N}) \qquad (2.2.9)$$

where $\underset{\sim}{K}_{ij} = (K_{ij} + K_{ji})/2$ and K_{ij} is the matrix with zero entries everywhere except 1 in position (i,j). The last equality follows because $\partial^2 l/\partial\sigma_{ij}\partial\mu_h$ has zero expectation and variance 0(N). The derivatives with respect to Δ are given by:

$$\partial l/\partial\sigma_{ij} = -N\sigma^{ij}/2 + \mathrm{tr}(\Delta^{-1}\underset{\sim}{K}_{ij}\Delta^{-1}\sum_{t=1}^{N} \underset{\sim}{a}_t\underset{\sim}{a}_t')/2$$

and

$$\partial^2 l/\partial\sigma_{ij}\partial\sigma_{rs} = N(\sigma^{ri}\sigma^{js} + \sigma^{rj}\sigma^{js})/4 - \mathrm{tr}\{(\Delta^{-1}\underset{\sim}{K}_{ij}\Delta^{-1}\underset{\sim}{K}_{rs}\Delta^{-1} + \Delta^{-1}\underset{\sim}{K}_{rs}\Delta^{-1}\underset{\sim}{K}_{ij}\Delta^{-1}).(\sum_{t=1}^{N}\underset{\sim}{a}_t\underset{\sim}{a}_t')\}/2$$

Taking expectations this becomes:

$$\langle -\partial^2 l/\partial\sigma_{ij}\partial\sigma_{rs}\rangle = N(\sigma^{si}\sigma^{jr} + \sigma^{ri}\sigma^{js})/4$$

In general I_Δ can be expressed as:

$$I_\Delta = \langle -\partial^2 l/\partial^2\Delta\rangle/N = (\Delta^{-1}\otimes\Delta^{-1})(1 + P)/4$$

where P is a permutation matrix such that $P^2 = 1_{k^2 x k^2}$ the identity matrix and $P(\Delta^{-1}\otimes\Delta^{-1}) = (\Delta^{-1}\otimes\Delta^{-1})P$. (Given that Δ is a symmetric matrix it is only necessary to consider the k(k + 1)/2 elements of the upper (or lower) triangular part of the matrix to obtain the Fisher information and the correlation matrices of Δ. When the k^2 elements of the matrix Δ are considered in the calculation of the Fisher information matrix of Δ, the resulting matrix I_Δ is singular because some rows of the matrix are repeated. This representation is, however, somewhat easier to work with.) A generalized inverse for I_Δ can be easily obtained. In fact, I_Δ^{-1} can be expressed as

$$I_\Delta^{-1} = (1 + P)(\Delta \otimes \Delta) = (\Delta \otimes \Delta)(1 + P) \qquad (2.2.10)$$

The result for $(\hat{\underset{\sim}{\mu}} - \underset{\sim}{\mu}, \hat{\Delta} - \Delta)$ follows from equations (2.2.8) to (2.2.10).

Consider now the distribution of $\bar{\underset{\sim}{\mu}}$ and $\bar{\Delta}$. As in Lemma 2.2.1 the univariate MLE $\bar{\mu}_h$ can be expanded as:

$$\bar{\mu}_h - \mu = I_{\mu_h}^{-1} \partial l_h / \partial \mu_h + O_p(1/N) \qquad (2.2.11)$$

where

$$l_h(\underset{\sim}{\beta}_h, \underset{\sim}{\mu}_h, \sigma_{hh}) = C - N \log(\sigma_{11})/2 - \sum_{t=1}^{N} a_{ht}^2/2\sigma_{hh}$$

and

$$I_{\mu_h} = \lim_{N \to \infty} \langle \partial^2 l_h / \partial^2 \mu_h \rangle = N\, C_h^2/\sigma_{hh}$$

Now N Cov($\bar{\mu}_g, \bar{\mu}_h$) is given by:

$$N < I_{\mu_g}^{-1} \partial l_g/\partial\mu_g \cdot I_{\mu_h}^{-1} \partial l_h / \partial\mu_h > = (N^2 C_g^2 C_h^2/\sigma_{gg}\sigma_{hh})^{-1} \cdot \sum_{t=1}^{N} \sum_{t'=1}^{N} < a_{gt}\, a_{ht'} >$$
$$= \sigma_{gh}/C_g C_h$$

The variance covariance of $\sqrt{N}\,(\bar{\mu} - \mu)$ is given by

$$\text{diag}\left(\frac{1}{c_1},\ldots,\frac{1}{c_k}\right) \Delta^{-1}\, \text{diag}\left(\frac{1}{c_1},\ldots,\frac{1}{c_k}\right)$$

which is the inverse of I_μ given by equation (2.2.8). Now the estimators for $\bar{\Delta}$ are given by:

$$\bar{\sigma}_{gh} = \sum_{t=1}^{N} \bar{a}_{gt}\, \bar{a}_{ht}/N$$

where $\bar{a}_{ht}$ are the residuals obtained from (2.1.1) using $\bar{\beta}_h$ instead of β, the true value. Taking a Taylor expansion around the true parameters β, evaluating at $\bar{\beta}$ and observing that $(\bar{\beta} - \beta)$ and $\partial\bar{\sigma}_{gh}/\partial\beta_{hj}$ are both $O_p(1/\sqrt{N})$, it follows that:

$$\bar{\sigma}_{gh} = \sum_{t=1}^{N} a_{gt}\, a_{ht}/N + O_p(1/N) \qquad (2.2.12)$$

The expectation of $\bar{\sigma}_{gh}$ is σ_{gh} and the variance covariance of $\bar{\sigma}_{gh}$ and $\bar{\sigma}_{ij}$, neglecting terms of $O(1/N^2)$ is given by

$$<\bar{\sigma}_{ij} \cdot \bar{\sigma}_{gh}> - \sigma_{gh}\sigma_{ij} = (1/N^2) \cdot \sum_{t=1}^{N} \sum_{t'=1}^{N} <a_{it'}\, a_{jt'}\, a_{gt}\, a_{ht}> - \sigma_{gh}\sigma_{ij}$$
$$= (\sigma_{gi}\sigma_{jh} + \sigma_{gj}\sigma_{ih})/N$$

so that the variance covariance matrix of $\sqrt{N}(\bar{\Delta}-\Delta)$ can be written as $(\Delta \otimes \Delta)\cdot(1_{k^2+k^2} + P)$ which is equal to equation (2.2.10). Normality is obtained from the Martingale central limit theorem as in Lemma 2.2.1.

The last statement of the theorem can be easily proved considering the Taylor expansions of the form (see equations (2.2.3), (2.2.11), (2.2.12))

$$\sqrt{N}(\hat{\beta} - \beta) = [I_{\hat{\beta}}]^{-1} \, \partial l/\partial \beta + 0_p(1/\sqrt{N})$$

where $l(\cdot)$ is given by equation (2.2.7) and observing that

$$<\partial l/\partial \beta \cdot \partial l/\partial(\mu', \Delta)> = 0$$

$$<\partial l/\partial \beta \cdot \partial l_h/\partial \mu_h> = 0$$

$$<\partial l/\partial \beta \cdot \bar{\sigma}_{gh}> = 0$$

$$<\partial l/\partial \beta \cdot \partial l/\partial(\mu', \Delta)> = 0.$$

It also can be proved that the joint distributions are normal and then because they are uncorrelated the independence result is obtained.

2.3 THE CAR(1) MODEL

2.3.1 Theoretical Results

It is interesting to study the relative efficiency of the estimator $\bar{\beta}$ with respect to $\hat{\beta}$. The 2-dimensional CAR(1) model can be used to illustrate some of the results. The model is given by:

$$Z_{1t} = \phi_1 Z_{1t-1} + a_{1t} \qquad (2.3.1)$$
$$Z_{2t} = \phi_2 Z_{2t-1} + a_{2t}$$
$$a_t = (a_{1t}, a_{2t})' \sim NID(0, \Delta)$$

with

$$\Delta = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

In view of Lemma 2.2.1 it is necessary to consider the auxiliary series given by $\phi_h(B)V_{ht} = -a_{ht}$, $h = 1,2$; Multiplying by V_{gt} and taking expectations the covariance between V_{ht} and V_{gt} is found to be

$$\gamma_{V_h V_g}(0) = \sigma_{hg}/(1 - \phi_g\phi_h)$$

Therefore, the asymptotic variance of $(\bar{\phi}_1, \bar{\phi}_2)'$ is given by:

$$V_{\bar{\phi}} = \frac{1}{N} \begin{pmatrix} (1 - \phi_1^2) & \rho^2(1-\phi_1^2)(1-\phi_2^2)/(1-\phi_1\phi_2) \\ \text{symm.} & (1 - \phi_2^2) \end{pmatrix}$$

and from Lemma 2.2.2 the asymptotic variance of $(\hat{\phi}_1, \hat{\phi}_2)$ is given by:

$$V_{\hat{\phi}} = \frac{(1-\rho^2)}{N} \begin{pmatrix} 1/(1-\phi_1^2) & -\rho^2/(1-\phi_1\phi_2) \\ \text{symm.} & 1/(1-\phi_2^2) \end{pmatrix}^{-1}$$

$$= \frac{(1-\rho^2)}{(1-\rho_\phi^2)} V_{\bar{\phi}}$$

where $\rho_\phi^2 = \rho^4(1-\phi_1^2)(1-\phi_2^2)/(1-\phi_1\phi_2)^2$ the relative efficiency of $\bar{\phi}_i$ with respect to $\hat{\phi}_i$ is then given by:

$$\text{eff} = V(\hat{\phi}_i)/V(\bar{\phi}_i) = (1-\rho^2)/(1-\rho_\phi^2) \tag{2.3.2}$$

which is a complicated function of ρ and the parameter values ϕ_1 and ϕ_2. Table 2.1 provides some idea of the efficiency values that could be expected for different values of ρ and $a = (1-\phi_1^2)(1-\phi_2^2)/(1-\phi_1\phi_2)^2$. For example, when $\phi_1 = \phi_2$ then $a = 1$ and the efficiency becomes $\frac{1}{2} < \text{eff} = 1/(1+\rho^2) < 1$. On the other hand, when $a \to 0$ (for example if one of the parameters approaches ± 1 while the other stays away) then $\text{eff} \to (1-\rho^2)$ and for large values of ρ the value of eff can be very small. It is interesting to observe that the effect of ρ on the variance of the estimator of ϕ_1 say, is equivalent to an increase in the sample size of Z_{1t} by a factor $(1-\rho_\phi^2)/(1-\rho^2)$. It is also clear from equation (2.3.2) that for $\rho = 0$ joint estimation does not result in any gain in efficiency.

TABLE 2.1

THEORETICAL RELATIVE EFFICIENCY OF $\bar{\phi}$ WITH RESPECT TO $\hat{\phi}$

FOR A BIVARIATE CAR(1) MODEL

A → / ρ	.0	.2	.4	.6	.8	1.0
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.10	0.990	0.990	0.990	0.990	0.990	0.990
0.20	0.960	0.960	0.961	0.961	0.961	0.962
0.30	0.910	0.911	0.913	0.914	0.916	0.917
0.40	0.840	0.844	0.849	0.853	0.858	0.862
0.50	0.750	0.759	0.769	0.779	0.789	0.800
0.60	0.640	0.657	0.675	0.654	0.714	0.735
0.70	0.510	0.536	0.564	0.556	0.631	0.671
0.80	0.360	0.352	0.431	0.477	0.535	0.610
0.90	0.190	0.219	0.258	0.313	0.400	0.552

$$A = (1-\phi_1^2)(1-\phi_2^2) / (1-\phi_1\phi_2)^2$$

In the case of a K-dimensional CAR(1) model with $\phi_h = \phi$, h=1, ..., k and $\sigma_{hh} = 1$, $\sigma_{hg} = \rho$, $g \neq h$, it can be shown that the efficiency of $\bar{\phi}_h$ relative to $\hat{\phi}_h$ is given by

$$\text{eff} = \{1 + (k - 2)\rho\ (1-\rho^2)\}/\{1 + \rho^2 + (K - 2)\rho\}$$

This shows that in general the efficiency of $\bar{\beta}$ depends upon several factors including the number of equations, the correlation values of the innovations and the parameter values.

2.3.2 Simulation Study

Using the bivariate CAR(1) model of equation (2.3.2), a simulation experiment was carried out to compare how the values of efficiency of $\bar{\phi}_i$ obtained using small sample sizes compare with the theoretical asymptotic results of section 2.3.1. 19 pairs of values for (ϕ_1, ϕ_2), namely (.0, .0), (.3, .3), (.6, .6), (.9, .9), (.0, .3), (.3, .6), (.3, - .3), (.0, .6), (.6, .9), (.6, - .3), (.3, .9), (.6, -.6), (.0, .9), (.9 -.3), (.9, -.6), (.9, -.9), (.0, .1), (.3, .4), (.6, .7), and 11 values for ρ, namely 0, $\pm$ .1, $\pm$ .2, $\pm$ .3, $\pm$ .5, $\pm$ .9, were included in the simulation. All the 209 models obtained from combinations of the values of (ϕ_1, ϕ_2) and ρ, were simulated using sample sizes of N = 50 and N = 200. The number of replications for each of the models was 1000. The random number generator Superduper (Marsaglia, 1976) together with the method of Beasley and Springer (1977) to obtain the percentile value of the normal distribution were used to generate pseudo-random normal variates. The random number generator Superduper, shown in Appendix 1, was coded in CDC Fortran 5 using the technique given in McLeod (1982). To simulate the models, the algorithm given in section 2.6 was used. The seed used for the random number generator were:

for N = 50, ISEED = 81310 JSEED = 10024

for N= 200, ISEED = 84671 JSEED = 29296

The calculations were done on a CYBER-8935 (NOS) computer.

The conditional maximum likelihood estimator was used to calculate the univariate MLE $\bar{\phi}_1$ and $\bar{\phi}_2$. The score algorithm given in Lemma 2.2.3 was used to obtain the joint MLE $\hat{\phi}_1$ and $\hat{\phi}_2$. A maximum of 10 iterations were allowed and the iterations were stopped when

$$|| \tilde{\underset{\sim}{\phi}}_k - \hat{\underset{\sim}{\phi}}_{k+1} || < 10^{-3}$$

The efficiency values were calculated as:

$$\text{eff} \doteq \text{Var}\,(\hat{\phi})/\text{Var}\,(\bar{\phi})$$

where VAR (ϕ) is the sample variance of ϕ_r, r = 1, ... ,1000 and ϕ_r is the observed value of $\hat{\phi}$ or $\bar{\phi}$ in the r^{th} replication. The variance of eff was calculated as:

$$\text{Var(eff)} = S^2/\{1000 \cdot \text{Var}^2(\bar{\phi})\}$$

where

$$S^2 = S_{11}^2 - 2 \cdot \text{eff} \cdot S_{12}^2 + (\text{eff})^2 \cdot S_{22}^2$$

S_{11}^2 is the sample variance of $(\hat{\phi} - \bar{\hat{\phi}})^2$ $r = 1, \ldots, 1000$.

S_{22}^2 is the sample variance of $(\phi_r - \bar{\bar{\phi}})^2$ $r = 1, \ldots, 1000$

S_{22}^2 is the sample covariance of $(\hat{\phi}_r - \bar{\hat{\phi}})$ and $(\bar{\phi}_r - \bar{\bar{\phi}})$,

$r = 1, \ldots, 1000$

The efficiency values of the estimator $\bar{\phi}$ relative to the estimator $\hat{\phi}$ as well as their standard errors are given in Table 2.2 for $N = 50$ and in Table 2.3 for $N = 200$. It can be seen from these tables that the observed values of eff agree quite well with the theoretical results even for a sample size of 50. For small values of ρ i.e. $|\rho| < .3$, some of the estimated efficiency values are greater than one, particularly for a sample size of 50. In general, these values are not significantly different from one. As the sample size increases, this problem is overcome and in fact the observed efficiency values for small values of ρ do not differ significantly from the asymptotic results.

The simulation study was also used to compare the behaviour of the score estimators $\tilde{\phi}$ given by Lemma 2.2.3 with the MLE $\hat{\phi}$. The variances of the score estimators $\tilde{\phi}$ were compared with the variances of the MLE $\hat{\phi}$. Tables A.2.1 and A.2.2 of Appendix 2 list the observed values of the efficiency of $\tilde{\phi}$ relative to $\hat{\phi}$. It is observed that for a sample size of 50 some gain efficiency is

TABLE 2.2

EFFICIENCY VALUES OF THE UNIVARIATE ESTIMATORS
RELATIVE TO THE JOINT ESTIMATORS
NUMBER OF OBSERVATIONS PER SERIES: 50
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.586	.852	.995	1.047	1.050	1.069	1.061	1.015	.984	.853	.611
			(.025)	(.025)	(.018)	(.016)	(.013)	(.010)	(.011)	(.015)	(.020)	(.026)	(.026)
		ϕ_2	.606	.846	1.019	1.029	1.063	1.052	1.091	1.037	1.023	.889	.619
			(.025)	(.024)	(.021)	(.017)	(.012)	(.010)	(.014)	(.015)	(.019)	(.024)	(.026)
(.3, .3)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.580	.847	.977	1.030	1.080	1.062	1.031	1.025	1.003	.860	.609
			(.025)	(.025)	(.020)	(.017)	(.013)	(.011)	(.013)	(.015)	(.022)	(.025)	(.028)
		ϕ_2	.566	.856	.978	1.046	1.072	1.072	1.020	1.016	.981	.843	.590
			(.025)	(.027)	(.019)	(.018)	(.013)	(.011)	(.013)	(.015)	(.018)	(.024)	(.028)
(.6, .6)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.570	.863	.977	1.014	1.035	1.048	1.049	1.014	.946	.850	.564
			(.028)	(.024)	(.020)	(.019)	(.014)	(.016)	(.015)	(.017)	(.020)	(.025)	(.025)
		ϕ_2	.559	.829	.943	1.022	1.071	1.046	1.030	1.041	.988	.835	.575
			(.028)	(.025)	(.020)	(.020)	(.014)	(.013)	(.014)	(.019)	(.022)	(.024)	(.027)
(.9, .9)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.483	.837	.934	.951	1.019	1.021	1.007	.951	.949	.794	.491
			(.031)	(.037)	(.027)	(.023)	(.023)	(.023)	(.025)	(.025)	(.025)	(.028)	(.041)
		ϕ_2	.555	.762	.890	.937	.942	1.006	1.018	.938	.967	.792	.541
			(.033)	(.030)	(.028)	(.030)	(.025)	(.024)	(.019)	(.025)	(.025)	(.041)	(.038)
(0.0, .1)	.99	TEFF	.542	.799	.917	.962	.990	1.000	.990	.962	.917	.799	.542
		ϕ_1	.570	.837	1.002	1.011	1.053	1.063	1.042	1.029	.978	.884	.579
			(.023)	(.023)	(.019)	(.016)	(.012)	(.011)	(.012)	(.015)	(.018)	(.024)	(.025)
		ϕ_2	.590	.880	.979	1.027	1.055	1.084	1.051	1.025	1.013	.872	.600
			(.023)	(.026)	(.021)	(.015)	(.013)	(.012)	(.012)	(.016)	(.020)	(.024)	(.025)

TABLE 2.2 (Continued)

(.3, .4)	.99	TEFF	.539	.799	.917	.962	.990	1.000	.990	.962	.917	.799	.539
		ϕ_1	.605	.888	1.006	1.033	1.048	1.059	1.063	1.024	.957	.845	.583
			(.026)	(.025)	(.019)	(.017)	(.012)	(.012)	(.013)	(.015)	(.018)	(.024)	(.025)
		ϕ_2	.593	.888	.985	1.028	1.045	1.061	1.053	1.051	.968	.856	.583
			(.025)	(.025)	(.021)	(.016)	(.012)	(.011)	(.012)	(.018)	(.021)	(.026)	(.024)
(.6, .7)	.97	TEFF	.523	.798	.917	.961	.990	1.000	.990	.961	.917	.798	.523
		ϕ_1	.531	.872	1.039	1.013	1.069	1.027	1.075	1.004	.943	.858	.572
			(.026)	(.027)	(.021)	(.017)	(.015)	(.011)	(.017)	(.019)	(.021)	(.026)	(.026)
		ϕ_2	.540	.859	.953	1.015	1.054	1.035	1.034	.986	.936	.874	.561
			(.026)	(.027)	(.021)	(.020)	(.017)	(.015)	(.014)	(.018)	(.021)	(.027)	(.030)
(0.0, .3)	.91	TEFF	.472	.795	.917	.961	.990	1.000	.990	.961	.917	.795	.472
		ϕ_1	.496	.838	.987	1.013	1.036	1.067	1.055	1.016	.958	.856	.515
			(.022)	(.024)	(.021)	(.016)	(.012)	(.010)	(.013)	(.014)	(.018)	(.024)	(.024)
		ϕ_2	.491	.822	.996	1.024	1.040	1.050	1.061	1.049	.966	.864	.493
			(.022)	(.026)	(.020)	(.018)	(.013)	(.010)	(.012)	(.019)	(.022)	(.026)	(.024)
(.3, .6)	.87	TEFF	.440	.793	.916	.961	.990	1.000	.990	.961	.916	.793	.440
		ϕ_1	.503	.842	.958	1.055	1.039	1.057	1.050	1.046	.990	.849	.501
			(.025)	(.022)	(.019)	(.018)	(.012)	(.011)	(.013)	(.017)	(.020)	(.024)	(.025)
		ϕ_2	.492	.850	.973	.971	1.043	1.054	1.045	1.034	.983	.798	.543
			(.024)	(.027)	(.022)	(.016)	(.015)	(.013)	(.014)	(.019)	(.022)	(.022)	(.029)
(.3,-.3)	.70	TEFF	.350	.784	.915	.961	.990	1.000	.990	.961	.915	.784	.350
		ϕ_1	.362	.816	.990	1.020	1.042	1.066	1.052	1.034	.952	.850	.423
			(.018)	(.023)	(.020)	(.016)	(.013)	(.011)	(.012)	(.017)	(.021)	(.025)	(.021)
		ϕ_2	.372	.811	1.005	1.043	1.061	1.074	1.071	1.006	.960	.839	.412
			(.018)	(.026)	(.021)	(.015)	(.011)	(.012)	(.012)	(.015)	(.018)	(.025)	(.020)
(0.0, .6)	.64	TEFF	.328	.781	.915	.961	.990	1.000	.990	.961	.915	.781	.328
		ϕ_1	.368	.908	.977	1.019	1.043	1.074	1.053	1.036	.994	.881	.378
			(.018)	(.027)	(.018)	(.015)	(.012)	(.011)	(.011)	(.015)	(.020)	(.026)	(.018)
		ϕ_2	.352	.815	.986	1.021	1.018	1.044	1.025	1.013	.938	.873	.365
			(.018)	(.024)	(.018)	(.018)	(.016)	(.012)	(.013)	(.017)	(.022)	(.027)	(.019)

TABLE 2.2 (Continued)

(.6, .9)	.58	TEFF	.305	.778	.914	.961	.990	1.000	.990	.961	.914	.778	.305
		ϕ_1	.358	.878	.987	1.002	1.035	1.052	1.053	1.011	.962	.793	.332
			(.019)	(.026)	(.022)	(.019)	(.015)	(.014)	(.014)	(.016)	(.018)	(.026)	(.021)
		ϕ_2	.328	.738	.882	.958	1.008	1.010	1.001	1.001	.936	.802	.327
			(.019)	(.026)	(.025)	(.026)	(.023)	(.020)	(.025)	(.027)	(.030)	(.037)	(.020)
(.6,-.3)	.42	TEFF	.262	.770	.913	.961	.990	1.000	.990	.961	.913	.770	.262
		ϕ_1	.318	.874	.974	1.009	1.048	1.028	1.048	1.006	.957	.871	.312
			(.018)	(.028)	(.023)	(.017)	(.015)	(.013)	(.015)	(.018)	(.025)	(.029)	(.017)
		ϕ_2	.327	.851	.923	1.000	1.048	1.074	1.045	.998	.970	.794	.300
			(.017)	(.025)	(.020)	(.016)	(.014)	(.013)	(.012)	(.015)	(.020)	(.023)	(.016)
(.3, .9)	.32	TEFF	.241	.766	.912	.960	.990	1.000	.990	.960	.912	.766	.241
		ϕ_1	.281	.848	.938	1.032	1.059	1.065	1.064	1.018	1.001	.848	.286
			(.015)	(.027)	(.023)	(.018)	(.014)	(.011)	(.014)	(.016)	(.021)	(.025)	(.015)
		ϕ_2	.212	.768	.931	.950	1.027	.999	.988	.962	.918	.719	.264
			(.016)	(.028)	(.027)	(.025)	(.019)	(.022)	(.023)	(.023)	(.028)	(.030)	(.021)
(.6,-.6)	.22	TEFF	.222	.761	.912	.960	.990	1.000	.990	.960	.912	.761	.222
		ϕ_1	.254	.787	.933	.999	1.058	1.024	1.021	1.012	.957	.813	.247
			(.015)	(.027)	(.021)	(.018)	(.014)	(.014)	(.015)	(.021)	(.020)	(.027)	(.014)
		ϕ_2	.252	.768	.967	.990	1.021	1.078	1.021	1.029	.939	.782	.203
			(.013)	(.023)	(.021)	(.017)	(.015)	(.014)	(.013)	(.016)	(.019)	(.025)	(.012)
(0.0, .9)	.19	TEFF	.217	.759	.911	.960	.990	1.000	.990	.960	.911	.759	.217
		ϕ_1	.267	.780	.984	1.034	1.046	1.072	1.034	1.022	.990	.865	.260
			(.015)	(.026)	(.021)	(.015)	(.012)	(.011)	(.012)	(.015)	(.022)	(.028)	(.014)
		ϕ_2	.247	.792	.910	.979	1.019	.982	1.025	.947	.930	.788	.214
			(.018)	(.030)	(.028)	(.032)	(.022)	(.022)	(.021)	(.022)	(.030)	(.030)	(.013)
(.9,-.3)	.11	TEFF	.204	.755	.911	.960	.990	1.000	.990	.960	.911	.755	.204
		ϕ_1	.225	.744	.917	.897	1.039	.993	.983	.939	.924	.779	.217
			(.015)	(.027)	(.025)	(.025)	(.024)	(.021)	(.023)	(.025)	(.026)	(.032)	(.015)
		ϕ_2	.218	.766	.958	1.029	1.046	1.070	1.069	1.012	.933	.813	.243
			(.012)	(.023)	(.019)	(.016)	(.013)	(.011)	(.014)	(.016)	(.019)	(.025)	(.014)

TABLE 2.2 (Continued)

(.9,-.6)	.05	TEFF	.197	.752	.910	.960	.990	1.000	.990	.960	.910	.752	.197
		ϕ_1	.193	.699	.874	.962	.991	.965	1.017	.932	.902	.758	.188
			(.012)	(.028)	(.024)	(.028)	(.025)	(.029)	(.021)	(.026)	(.030)	(.032)	(.012)
		ϕ_2	.229	.871	.918	.995	1.045	1.031	1.047	1.007	.940	.809	.185
			(.013)	(.028)	(.020)	(.017)	(.013)	(.013)	(.014)	(.017)	(.022)	(.024)	(.010)
(.9,-.9)	.01	TEFF	.191	.751	.910	.960	.990	1.000	.990	.960	.910	.751	.191
		ϕ_1	.184	.734	.890	.962	.971	1.036	.961	1.001	.854	.749	.164
			(.012)	(.027)	(.028)	(.024)	(.022)	(.022)	(.023)	(.026)	(.028)	(.030)	(.012)
		ϕ_2	.197	.776	.858	.973	.960	.986	.998	.914	.864	.733	.192
			(.015)	(.027)	(.032)	(.022)	(.021)	(.021)	(.022)	(.026)	(.026)	(.030)	(.012)

NOTE : $A = (1. - \phi_1^2)(1. - \phi_2^2)/(1. - \phi_1\phi_2)^2$

Values in parentheses indicate the Standard Errors

TABLE 2.3

EFFICIENCY VALUES OF THE UNIVARIATE ESTIMATORS
RELATIVE TO THE JOINT ESTIMATORS
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.543	.799	.958	.996	1.002	1.008	.999	.969	.923	.823	.520
			(.023)	(.024)	(.017)	(.013)	(.008)	(.004)	(.007)	(.012)	(.017)	(.022)	(.022)
		ϕ_2	.550	.845	.932	.965	1.001	1.010	1.008	.992	.918	.825	.543
			(.022)	(.024)	(.017)	(.012)	(.008)	(.005)	(.007)	(.013)	(.018)	(.024)	(.022)
(.3, .3)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.559	.777	.909	.979	.986	1.016	1.008	.990	.924	.803	.550
			(.023)	(.020)	(.017)	(.012)	(.008)	(.005)	(.007)	(.014)	(.016)	(.022)	(.024)
		ϕ_2	.536	.815	.959	.970	1.013	1.022	1.009	.996	.958	.776	.576
			(.022)	(.023)	(.016)	(.013)	(.009)	(.006)	(.008)	(.012)	(.017)	(.021)	(.025)
(.6, .6)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.599	.827	.928	.961	1.010	1.006	.994	.953	.920	.822	.552
			(.027)	(.024)	(.016)	(.013)	(.008)	(.007)	(.009)	(.012)	(.018)	(.024)	(.024)
		ϕ_2	.612	.812	.929	.947	1.001	1.013	.991	.987	.903	.797	.564
			(.027)	(.025)	(.018)	(.013)	(.008)	(.006)	(.010)	(.014)	(.017)	(.023)	(.024)
(.9, .9)	1.00	TEFF	.552	.800	.917	.962	.990	1.000	.990	.962	.917	.800	.552
		ϕ_1	.543	.792	.903	.947	.981	.987	.998	.942	.906	.822	.502
			(.028)	(.028)	(.020)	(.017)	(.014)	(.016)	(.014)	(.019)	(.020)	(.028)	(.029)
		ϕ_2	.528	.779	.911	.925	.962	.999	.980	.976	.878	.784	.505
			(.029)	(.022)	(.020)	(.021)	(.014)	(.015)	(.013)	(.021)	(.023)	(.027)	(.026)
(0.0, .1)	.99	TEFF	.542	.799	.917	.962	.990	1.000	.990	.962	.917	.799	.542
		ϕ_1	.583	.830	.964	.999	1.017	1.016	1.016	.972	.947	.834	.573
			(.024)	(.023)	(.019)	(.013)	(.008)	(.005)	(.007)	(.011)	(.019)	(.023)	(.022)
		ϕ_2	.565	.861	.948	.997	.994	1.014	1.007	.966	.949	.814	.567
			(.024)	(.025)	(.019)	(.014)	(.009)	(.004)	(.008)	(.013)	(.017)	(.024)	(.026)

TABLE 2.3 (Continued)

(.3, .4)	.99	TEFF	.539	.799	.917	.962	.990	1.000	.990	.962	.917	.799	.539
		ϕ_1	.602	.799	.919	.994	1.003	1.018	1.003	.984	.937	.795	.572
			(.026)	(.023)	(.017)	(.013)	(.008)	(.005)	(.008)	(.013)	(.016)	(.023)	(.024)
		ϕ_2	.566	.791	.936	.979	1.008	1.027	1.007	.974	.935	.803	.571
			(.024)	(.023)	(.017)	(.012)	(.008)	(.005)	(.008)	(.013)	(.018)	(.024)	(.025)
(.6, .7)	.97	TEFF	.523	.798	.917	.961	.990	1.000	.990	.961	.917	.798	.523
		ϕ_1	.547	.820	.922	.987	1.008	1.011	.997	.938	.933	.813	.526
			(.025)	(.024)	(.018)	(.014)	(.009)	(.006)	(.008)	(.012)	(.017)	(.024)	(.023)
		ϕ_2	.519	.848	.935	.969	1.005	1.013	1.000	.963	.910	.813	.491
			(.023)	(.025)	(.018)	(.013)	(.011)	(.009)	(.009)	(.013)	(.018)	(.025)	(.024)
(0.0, .3)	.91	TEFF	.472	.795	.917	.961	.990	1.000	.990	.961	.917	.795	.472
		ϕ_1	.465	.781	.955	.979	1.007	1.008	1.005	.958	.983	.792	.518
			(.022)	(.024)	(.017)	(.012)	(.007)	(.005)	(.008)	(.013)	(.019)	(.022)	(.023)
		ϕ_2	.477	.834	.948	.962	1.014	1.019	.997	.991	.941	.788	.484
			(.021)	(.024)	(.017)	(.012)	(.007)	(.006)	(.008)	(.013)	(.017)	(.021)	(.022)
(.3, .6)	.87	TEFF	.440	.793	.916	.961	.990	1.000	.990	.961	.916	.793	.440
		ϕ_1	.484	.825	.921	.970	.991	1.017	.986	1.001	.924	.855	.499
			(.025)	(.022)	(.017)	(.013)	(.008)	(.005)	(.007)	(.013)	(.016)	(.024)	(.024)
		ϕ_2	.499	.797	.928	.973	.997	1.005	1.004	.953	.913	.826	.472
			(.024)	(.023)	(.017)	(.013)	(.008)	(.007)	(.009)	(.013)	(.019)	(.023)	(.023)
(.3, -.3)	.70	TEFF	.350	.784	.915	.961	.990	1.000	.990	.961	.915	.784	.350
		ϕ_1	.337	.805	.930	.982	.998	1.007	1.004	.985	.935	.828	.358
			(.017)	(.023)	(.017)	(.013)	(.008)	(.005)	(.007)	(.013)	(.019)	(.025)	(.018)
		ϕ_2	.334	.787	.953	.965	1.009	1.026	1.008	.981	.941	.760	.379
			(.018)	(.022)	(.017)	(.012)	(.008)	(.005)	(.009)	(.012)	(.017)	(.023)	(.019)
(0.0, .6)	.64	TEFF	.328	.781	.915	.961	.990	1.000	.990	.961	.915	.781	.328
		ϕ_1	.322	.753	.944	.970	1.008	1.017	1.022	.985	.936	.826	.343
			(.016)	(.021)	(.017)	(.013)	(.007)	(.005)	(.007)	(.013)	(.018)	(.024)	(.017)
		ϕ_2	.365	.766	.942	.981	.982	1.013	.996	.972	.940	.773	.327
			(.017)	(.024)	(.018)	(.014)	(.009)	(.006)	(.009)	(.013)	(.018)	(.024)	(.017)

TABLE 2.3 (Continued)

(.6, .9)	.58	TEFF	.305	.778	.914	.961	.990	1.000	.990	.961	.914	.778	.305
		ϕ_1	.315	.759	.902	.964	.997	1.012	.993	.957	.921	.805	.345
			(.017)	(.023)	(.018)	(.012)	(.009)	(.007)	(.009)	(.013)	(.018)	(.024)	(.017)
		ϕ_2	.294	.756	.901	.952	.988	.980	.972	.929	.921	.772	.328
			(.020)	(.029)	(.025)	(.016)	(.015)	(.013)	(.017)	(.020)	(.019)	(.028)	(.017)
(.6,-.3)	.42	TEFF	.262	.770	.913	.961	.990	1.000	.990	.961	.913	.770	.262
		ϕ_1	.283	.782	.919	.948	1.008	1.004	1.015	.955	.910	.779	.275
			(.016)	(.024)	(.018)	(.014)	(.009)	(.007)	(.010)	(.014)	(.017)	(.024)	(.016)
		ϕ_2	.271	.753	.922	.969	1.004	1.009	1.003	.969	.948	.783	.289
			(.015)	(.024)	(.017)	(.014)	(.008)	(.005)	(.008)	(.013)	(.017)	(.023)	(.017)
(.3, .9)	.32	TEFF	.241	.766	.912	.960	.990	1.000	.990	.960	.912	.766	.241
		ϕ_1	.247	.753	.944	.971	1.023	1.013	1.009	.985	.902	.786	.252
			(.013)	(.023)	(.018)	(.013)	(.008)	(.005)	(.008)	(.013)	(.018)	(.024)	(.014)
		ϕ_2	.229	.757	.852	.942	.971	1.008	.973	.932	.859	.743	.252
			(.014)	(.026)	(.020)	(.017)	(.014)	(.011)	(.014)	(.016)	(.021)	(.026)	(.015)
(.6,-.6)	.22	TEFF	.222	.761	.912	.960	.990	1.000	.990	.960	.912	.761	.222
		ϕ_1	.233	.789	.941	.964	.994	1.015	1.000	.998	.919	.774	.223
			(.013)	(.025)	(.020)	(.014)	(.010)	(.008)	(.009)	(.014)	(.018)	(.024)	(.013)
		ϕ_2	.216	.750	.941	.959	.007	1.010	.993	.958	.923	.819	.236
			(.012)	(.024)	(.019)	(.013)	(.009)	(.006)	(.010)	(.014)	(.020)	(.025)	(.014)
(0.0, .9)	.19	TEFF	.217	.759	.911	.960	.990	1.000	.990	.960	.911	.759	.217
		ϕ_1	.227	.749	.931	.976	1.009	1.019	1.011	.964	.917	.732	.234
			(.013)	(.023)	(.016)	(.012)	(.008)	(.004)	(.008)	(.012)	(.018)	(.022)	(.012)
		ϕ_2	.208	.774	.882	.919	.995	.975	.947	.949	.879	.686	.195
			(.012)	(.028)	(.019)	(.017)	(.015)	(.013)	(.015)	(.018)	(.021)	(.025)	(.011)
(.9,-.3)	.11	TEFF	.204	.755	.911	.960	.990	1.000	.990	.960	.911	.755	.204
		ϕ_1	.191	.740	.879	.952	1.006	.993	1.007	.934	.916	.731	.193
			(.012)	(.027)	(.020)	(.016)	(.014)	(.014)	(.012)	(.017)	(.022)	(.024)	(.013)
		ϕ_2	.195	.789	.931	.963	1.000	1.013	1.012	.976	.945	.783	.214
			(.011)	(.023)	(.017)	(.013)	(.007)	(.005)	(.008)	(.013)	(.019)	(.023)	(.012)

TABLE 2.3 (Continued)

(.9,-.6)	.05	TEFF	.197	.752	.910	.960	.990	1.000	.990	.960	.910	.752	.197
		ϕ_1	.170	.718	.932	.954	.987	.991	1.015	.948	.902	.744	.197
			(.011)	(.024)	(.021)	(.018)	(.015)	(.012)	(.015)	(.017)	(.022)	(.030)	(.012)
		ϕ_2	.208	.732	.956	.976	.993	1.011	1.018	.978	.913	.737	.200
			(.012)	(.023)	(.019)	(.014)	(.009)	(.006)	(.008)	(.014)	(.018)	(.023)	(.012)
(.9,-.9)	.01	TEFF	.191	.751	.910	.960	.990	1.000	.990	.960	.910	.751	.191
		ϕ_1	.186	.739	.884	.936	.942	.979	.964	.904	.876	.726	.167
			(.012)	(.029)	(.021)	(.016)	(.021)	(.015)	(.014)	(.016)	(.020)	(.023)	(.010)
		ϕ_2	.174	.711	.909	.968	.966	.991	.993	.942	.896	.714	.187
			(.011)	(.025)	(.023)	(.018)	(.014)	(.013)	(.014)	(.017)	(.024)	(.024)	(.011)

NOTE : $A = (1. - \phi_1^2)(1. - \phi_2^2)/(1. - \phi_1\phi_2)^2$

Values in parentheses indicate the Standard Errors obtained when the MLE is used. Such a gain in efficiency is more marked in models where the value of

$$(1-\rho^2)/(1-\rho^4 \{(1-\phi_1^2)(1-\phi_2^2)/(1-\phi_1\phi_2)^2\})$$

is small. When the sample size is increased to 200 the efficiency values tend to 1 as should be expected.

The number of iterations required to obtain convergence with the score algorithm were also recorded. Tables A.2.3 and A.2.4 of Appendix 2 list the average number of iterations together with their standard errors. Two features of the simulation with respect to the number of iterations required should be noted: (a) In general, more iterations were required for models where a large gain in efficiency was observed. (b) As the sample size increases to 200, fewer iterations were required and on average only one or two iterations were necessary to obtain convergence.

2.4 CONTEMPORANEOUS TRANSFER FUNCTION MODELS

In this section a model given by the following equation is considered:

$$Z_{ht} = \frac{\omega_h(B)}{\delta_h(B)} X_{ht} + \frac{\theta_h(B)}{\phi_h(B)} a_{ht} \qquad h = 1, \ldots ,k$$

where $\underset{\sim}{a}_t = (a_{1t}, \ldots ,a_{kt})' - \text{NID}(0,\Delta)$ and

$$\omega_h(B) = \omega_{ho} - \omega_{h1}B - \ldots -\omega_{h\nu}B^{\nu}$$

$$\delta_h(B) = 1 - \delta_{h1}B - \ldots -\delta_{hu}B^u$$

$$\theta_h(B) = 1 - \theta_{h1}B - \ldots -\theta_{hq}B^q$$

$$\phi_h(B) = 1 - \phi_{h1}B - \ldots -\phi_{hp}B^p$$

It is assumed (i) That the roots of the polynomials $\delta_h(B)$, $\theta_h(B)$ and $\theta_h(B)$, $h = 1, \ldots ,k$, lie outside the unit circle (ii) The numerator and denominator of the polynomial ratios $\omega_h(B)/\delta_h(B)$ and $\theta_h(B)/\phi_h(B)$ and $\theta_h(B)/\phi_h(B)$ have no root in common (iii) $\underset{\sim}{a}_t$ is statistically independent of $X_{ht'}$ for all t and t' and $h = 1, \ldots ,k$ (iv) For given S the matrices

$$\text{plim} \left\{ \sum_{t=1}^{N} X_{ht-i}X_{ht-j}/N \right\} i,j = 1, \ldots, s, \; h = 1, \ldots ,k$$

are positive definite.

The model given in equation (2.4.1) for k = 1 is referred to by Box and Jenkins (1976) as a transfer function model. This model extends the CARMA model in the sense that it allows for "explanatory variables" or using Box and Jenkins terminology, input variables in in the model. On the other hand, it also extends the SURE model proposed by Zellner (1962) in the sense that it allows autocorrelated innovations to be present in the model and as is pointed out by Kmenta and Gilbert (1970), this is the case with many actual series fitted using the SURE model. In this section, the asymptotic distribution of the parameter estimators is given.

To fix ideas, the following notation is introduced. Let

$$\underset{\sim}{\beta}_h = (\phi_{h1}, \ldots, \phi_{hp}, \theta_{h1}, \ldots, \theta_{hq})'$$
$$\underset{\sim}{\alpha}_h = (\omega_{h0}, \ldots, \omega_{hv}, \delta_{h1}, \ldots, \delta_{hu})'$$
$$\underset{\sim}{\tau}_h = (\underset{\sim}{\beta}_h', \underset{\sim}{\alpha}_h')'$$
$$\underset{\sim}{\beta} = (\underset{\sim}{\beta}_1', \ldots, \underset{\sim}{\beta}_k')' \text{ and } \underset{\sim}{\alpha} = (\underset{\sim}{\alpha}_1', \ldots, \underset{\sim}{\alpha}_k')'$$
$$\underset{\sim}{\tau} = (\underset{\sim}{\beta}', \underset{\sim}{\alpha}')'$$

Given a sample of N consecutive observations from a model given by equation (2.4.1), $\{(Z_{ht}, X_{ht})\}$ $t = 1, \ldots, N$; $h = 1, \ldots, k$, it is desired to estimate the parameters of the model $\underset{\sim}{\tau}$. One possible estimator is $\bar{\underset{\sim}{\tau}} = (\bar{\underset{\sim}{\beta}}, \bar{\underset{\sim}{\alpha}})$ where $\bar{\underset{\sim}{\tau}}_h = (\bar{\underset{\sim}{\beta}}_h, \bar{\underset{\sim}{\alpha}}_h)$ is the MLE of $\underset{\sim}{\tau}_h$ calculated using only the data for series h, i.e., $\{(Z_{ht}, X_{ht})\}$ $t = 1, \ldots, N$. Another estimator is $\hat{\underset{\sim}{\tau}}$ the joint MLE of $\underset{\sim}{\tau}$ calculated using all the available data. In order to obtain the asymptotic distribution, the following notation is introduced: Let $\dot{\underset{\sim}{\tau}}$ be a vector of parameters which satisfy conditions (i) and (ii) and let

$$\dot{a}_{ht} = \frac{\dot{\phi}_h(B)}{\dot{\theta}_h(B)} Z_{ht} - \frac{\dot{\phi}_h(B)\dot{\omega}_h(B)}{\dot{\theta}_h(B)\dot{\delta}_h(B)} X_{ht} \qquad (2.4.2.a)$$

The auxiliary series U_{ht}, V_{ht}, b_{ht} and d_{ht} are defined by:

$$\dot{\phi}_h(B)\,\dot{V}_{ht} = -\dot{a}_{ht} \qquad (2.4.2.b)$$

$$\dot{\theta}_h(B)\,\dot{U}_{ht} = \dot{a}_{ht} \qquad (2.4.2.c)$$

$$\dot{b}_{ht} = -\{\dot{\phi}_h(B)/\dot{\theta}_h(B)\,\dot{\delta}_h(B)\}\,X_{ht} \qquad (2.4.2.d)$$

$$\dot{d}_{ht} = \{\dot{\omega}_h(B)/\dot{\delta}_h(B)\}\,\dot{b}_{ht} \qquad (2.4.2.e)$$

It is very easy to see that

$$\begin{aligned} \partial \dot{a}_{ht} / \partial \dot{\eta}_{jl} &= f_{ht-l} && \text{if } j = h \\ &= 0 && \text{otherwise} \end{aligned} \qquad (2.4.3)$$

where η stands for ϕ, θ, ω or δ and f stands for V, U, b or d.

The following Lemma gives the asymptotic distribution of $\bar{\tau}$.

Lemma 2.4.1. The distribution of $\sqrt{N}(\bar{\tau} - \tau)$ is multivariate normal with mean zero and variance covariance given by:

$$V_{\bar{\tau}} = \begin{pmatrix} V_{\bar{\beta}} & 0 \\ & \\ 0 & V_{\bar{\alpha}} \end{pmatrix}$$

where $V_{\bar{\beta}}$ is given in Lemma 2.2.1 and $V_{\bar{\alpha}}$ by:

$$V_{\bar{\alpha}} = \begin{bmatrix} \alpha_{11} I^{-1}_{\alpha_{11}} & \cdots & \alpha_{1k} I^{-1}_{\alpha_{11}} I_{\alpha_{1k}} I^{-1}_{\alpha_{kk}} \\ & & \\ \text{symm} & & \alpha_{kk} I^{-1}_{\alpha_{kk}} \end{bmatrix}$$

$$I_{\alpha_{gh}} = \left[\begin{array}{c|c} \gamma_{b_g b_h}(i-j) & \gamma_{b_g d_h}(i-j) \\ \hline \gamma_{d_g b_h}(i-j) & \gamma_{d_g d_h}(i-j) \end{array}\right] \qquad (2.4.4)$$

$$\gamma_{ef}(i-j) = \text{plim} \left\{ \sum_{t=1}^{N} e_{t-i} \cdot f_{t-j} / N \right\}$$

where e and f stand for b or d and these auxiliary series are defined by equation (2.4.2.d) and (2.4.2.e).

Proof: The asymptotic distribution of $\bar{\tau}_h$ was given by Pierce (1972). It follows from his result that:

$$\bar{\tau} - \tau = I^{-1}_{(\beta_h, \alpha_h)} S^{C}_{h} + O_p(1/N) \qquad (2.4.5)$$

where

$$I_{(\beta_h, \alpha_h)} = \frac{1}{\sigma_{hh}} \begin{bmatrix} I_{\beta_{hh}} & 0 \\ 0 & I_{\alpha_{hh}} \end{bmatrix}$$

$$\underset{\sim}{S}_h^c = (\partial S_h/\partial \underset{\sim}{\beta}'_h, \ \partial S_h/\partial \alpha'_h)'$$

$$S_h = \sum_{t=1}^{N} a^2_{ht}/2N\sigma_{hh}$$

and a_{ht} is defined by equation (2.4.2.a.) when $\dot{\underset{\sim}{\tau}} = \underset{\sim}{\tau}$, the true parameters values.

Now

$$N < S^c_{\eta_{gi}} \cdot S^c_{\eta_{hj}} > = < \sum_{t=1}^{N} \sum_{t'=1}^{N} a_{gt} e_{gt-i} a_{ht'} f_{ht'-j} > / N\sigma_{gg}\sigma_{hh}$$

where η stands for ϕ, θ, ω or δ and e and f stand for V, U, b or d respectively, and the auxilary series V, U, b, d are defined by equations (2.4.2).

The expected value $\langle a_{gt}\ e_{gt-i}\ a_{ht'}\ f_{ht'-j}\rangle$ is given by

(i) e = U,V; f = U,V

$$\langle a_{gt}\ e_{gt-1}\ a_{ht'}\ f_{ht'-j}\rangle = \sigma_{gh}\ \gamma_{e_g f_\eta}(i-j)\cdot\xi(t-t')$$

where $\xi(t) = 0$ if $t=0$ and $\xi(0) = 1$

(ii) e = u,v; f = b, d

$$\langle a_{gt}\ e_{gt-i} a_{ht'}\ f_{ht'-j}\rangle = \langle a_{gt}\ e_{gt-i}\ a_{ht'}\rangle\ \langle f_{ht'-j}\rangle = 0$$

(The first equality follows because X_{ht}' is independent of $\underset{\sim}{a}_t$ and the second because the third moments of the normal distribution are zero.).

(iii) $e = b,d; \quad f = b,d$

$$<a_{gt}\, e_{gt-i}\, a_{ht'}\, f_{ht'-j}> = \sigma_{gh} < e_{gt-i} f_{ht-j}> \cdot \quad \varepsilon \quad (t' - t')$$

Therefore

$$N < S^{c}_{\eta_{gi}} S_{\eta_{hj}} > \; = \; \sigma_{gh} \gamma_{e_g f_h}(i - j)/\sigma_{gg}\sigma_{hh} \quad e = u,v; \; f = u,v$$

$$= \; 0 \quad e = u,v; \; f = b,d$$

$$= \; \sigma_{gh} \gamma_{e_g f_h}(i - j)/\sigma_{gg}\sigma_{hh} \quad e = b,d; \; f = b,d$$

The Lemma follows from the above results and equations (2.4.5). Normality is obtained using the Martingale central limit theorem as in Lemma 2.2.1.

The following Lemma gives the asymptotic distribution of $\hat{\underset{\sim}{\tau}}$, the MLE of $\underset{\sim}{\tau}$.

<u>Lemma 2.4.2</u> The asymptotic distribution of $\sqrt{N}(\hat{\underset{\sim}{\tau}} - \underset{\sim}{\tau})$ is multivariate normal with mean zero and covariance matrix given by:

$$V_{\hat{\tau}} = \begin{bmatrix} V_{\hat{\beta}} & 0 \\ 0 & V_{\hat{\alpha}} \end{bmatrix}$$

where $V_{\hat{\alpha}}$ is given by Lemma 2.2.2 and

$$V_{\hat{\alpha}} = \begin{pmatrix} \sigma^{11} I_{\alpha_{11}} & \sigma^{1k} I_{\alpha_{1k}} \\ \sigma^{k1} I_{\alpha_{k1}} & \sigma^{kk} I_{\alpha_{kk}} \end{pmatrix}^{-1}$$

$I_{\alpha_{gh}}$ given by equation (2.4.4).

<u>Proof</u>: Conditional on the values of $\{X_{ht}\}$ h = 1, ... ,k; t = 1, ... ,N the log likelihood function can be expressed as:

$$l(\underset{\sim}{\tau}) = C - \frac{1}{2}\sum_{t=1}^{N} \underset{\sim}{a}_t' \Delta^{-1} \underset{\sim}{a}_t - \frac{N}{2}\log|\Delta| + m_N + O(r^n)$$

where m_N is bounded for all n (Hillmer and Tiao, 1979). So, from equation (2.4.3)

$$\partial l(\dot{\tau})/\partial\dot{\eta}_{hj} = -\sum_{t=1}^{N} \dot{\underset{\sim}{a}}_t' \, \dot{\Delta}^{-1} \begin{pmatrix} 0 \\ f_{ht-j} \\ 0 \end{pmatrix}$$

$$= -\sum_{s=1}^{k} \dot{\sigma}^{sh} \left(\sum_{t=1}^{N} \dot{a}_{st} \, \dot{f}_{ht-j}\right)$$

Taking a Taylor expansion of $\partial l/\partial\dot{\underset{\sim}{\tau}}$ around $\underset{\sim}{\tau}$, the true value, and evaluating at $\dot{\underset{\sim}{\tau}} = \hat{\underset{\sim}{\tau}}$ gives

$$0 = \partial l/\partial\underset{\sim}{\tau} + [\partial^2 l/\partial\underset{\sim}{\tau}\partial\underset{\sim}{\tau}'](\hat{\underset{\sim}{\tau}} - \underset{\sim}{\tau}) + O_p(1/N) \qquad (2.4.6)$$

The second derivatives are given by:

$$\partial^2 l/\partial\eta_{gi}\partial\eta_{hj} = -\sum_{s=1}^{k} \sigma^{sh} \sum_{t=1}^{N} (\partial a_{st}/\partial\eta_{gi} \cdot f_{ht-j} + a_{st}\,\partial f_{ht-j}/\partial\eta_{gi})$$

$$= -\sigma^{gh} \sum_{t=1}^{N} e_{gt-i} f_{ht-j} - \sum_{s=1}^{k} \sigma^{sh} \sum_{t=1}^{N} a_{st}\,\partial f_{ht-j}/\partial\eta_{gi}$$

It can be shown that the second term converges in probability to zero for all values of η_{gi} and η_{hj}, whereas the first term converges in probability to:

$$- N \sigma^{gh} \gamma_{e_g f_h} (i,j)$$

From equation 2.4.5 it follows that:

$$\sqrt{N}(\hat{\tau} - \tau) = \begin{pmatrix} V_{\hat{\beta}} & 0 \\ 0 & V_{\hat{\beta}} \end{pmatrix}^{-1} \frac{1}{\sqrt{N}} \cdot \frac{\partial l}{\partial \underset{\sim}{\tau}} + 0_p(1/\sqrt{N})$$

Noting that $\langle \partial l / \partial \underset{\sim}{\tau} \cdot \partial l / \partial \underset{\sim}{\tau}' \rangle / N = \begin{pmatrix} V_{\hat{\beta}} & 0 \\ 0 & V_{\hat{\alpha}} \end{pmatrix}$

which can be proved along the lines of the proof of Lemma 2.4.1 and that normality can be obtained as in Lemma 2.2.1, the statements of the Lemma are demonstrated.

Lemma 2.4.3 $V_{\bar{\tau}} - V_{\hat{\tau}}$ is a positive semidefinite matrix so that $\bar{\tau}$ is not asymptotically efficient.

Proof: This follows as a corollary of Theorem 2.2.1.

The above results can be easily generalized to a more general model of the form:

$$Z_{ht} = \sum_{\ell=1}^{s_h} \frac{\omega_{\ell h}(B)}{\delta_{\ell h}(B)} X_{\ell ht} + \frac{\theta_h(B)}{\phi_h(B)} a_{ht} \qquad h = 1, \ldots, k$$

In this case, all the distributions given by Lemmas 2.4.1 and 2.4.2 remain basically the same if appropriate changes are made in notation. In the case s = 2, for example equation (2.4.4) should be changed to (see also Pierce, 1972) the following matrix:

$$I_{\alpha_{gh}} = \begin{pmatrix} \gamma_{b_g^1 b_h^1}(i-j) & \gamma_{b_g^1 d_h^1}(i-j) & \gamma_{b_g^1 b_h^2}(i-j) & \gamma_{b_g^1 d_h^2}(i-j) \\ \gamma_{d_g^1 b_h^1}(i-j) & \gamma_{d_g^1 d_h^1}(i-j) & \gamma_{d_g^1 b_h^2}(i-j) & \gamma_{d_g^1 d_h^2}(i-j) \\ \gamma_{b_g^2 b_h^1}(i-j) & \gamma_{b_g^2 d_h^1}(i-j) & \gamma_{b_g^2 b_h^2}(i-j) & \gamma_{b_g^2 d_h^2}(i-j) \\ \gamma_{d_g^2 b_h^1}(i-j) & \gamma_{d_g^2 d_h^1}(i-j) & \gamma_{d_g^2 b_h^2}(i-j) & \gamma_{d_g^2 d_h^2}(i-j) \end{pmatrix}$$

with the obvious notation for b_h^ℓ and d_h^ℓ. (see equations 2.4.2.d. and 2.4.2.e.). For the case where there is no dynamic relationship in the system, i.e.: $\omega(B) = \omega_0$ and all the other polynomials are equal to 1, the results of Lemmas 2.4.1 and 2.4.2 collapse to those given by Zellner (1962) for the SURE model.

It is also interesting to observe that the method of scores of Lemma 2.2.3 can easily be extended to estimate the parameters of the contemporaneous transfer function model where the block diagonality of $V_{\hat{\tau}}$ can be exploited to obtain computationally efficient algorithms.

2.5 DISTRIBUTION OF THE RESIDUAL AUTOCORRELATIONS

In this section, the large sample distribution of the residual autocorrelations for the CARMA model and an adequate Portmanteau test for the independence of the residuals is given.

Li and McLeod (1981) derived the large sample distribution of the residual autocorrelations for the general multivariate ARMA model. The result for the general model is rather too complicated to be of direct applicability. For the CARMA model, a significant amount of simplification may be obtained which gives more easily applicable results.

To fix ideas let $\dot{\beta}$ be a vector of parameter values satisfying conditions (i) through (iii) of Section 2.2. For $p + 1 \leq t \leq N$ let

$$\dot{a}_{ht} = z_{ht} - \dot{\phi}_{h1} z_{ht-1} - \cdots - \dot{\phi}_{hp} z_{ht-p} + \dot{\theta}_{h1} \dot{a}_{ht-1} + \cdots + \dot{\theta}_{hq} \dot{a}_{ht-q} ,$$

$$\dot{a}_{ht} = 0 \text{ for } t \leq p, \qquad h = 1, \ldots ,k$$

The corresponding autocorrelations are defined by:

$$\dot{r}_{gh}(\ell) = \dot{c}_{gh}(\ell)/\sqrt{\dot{c}_{gg}(\ell)\, \dot{c}_{hh}(\ell)}$$

$$\dot{C}_{gh}(\ell) = \sum_{t=1}^{N-\ell} \dot{a}_{gt}\dot{a}_{ht+\ell}/N$$

Let also $\dot{\underset{\sim}{r}} = (\dot{\underset{\sim}{r}}_{11}', \dot{\underset{\sim}{r}}_{21}', \ldots, \dot{\underset{\sim}{r}}_{12}', \ldots, \dot{\underset{\sim}{r}}_{kk}')'$ where $\dot{\underset{\sim}{r}}_{ij} = (\dot{r}_{ij}(1), \ldots, \dot{r}_{ij}(M))$. For $\dot{\underset{\sim}{\beta}} = \bar{\underset{\sim}{\beta}}$ the vector of univariate estimator (see section 2.1), let $\bar{a}_{ht}$ and $\bar{r}_{ij}(\ )$ denote the corresponding residuals and residual autocorrelations. Similarly let $a_{ht}, r_{ij}(1)$ and $\hat{a}_{ht}, \hat{r}_{ij}(1)$ be the residuals and the autocorrelations corresponding to $\dot{\underset{\sim}{\beta}} = \underset{\sim}{\beta}$, the true parameter values, and to $\dot{\underset{\sim}{\beta}} = \hat{\underset{\sim}{\beta}}$, the MLE of $\underset{\sim}{\beta}$, respectively. It is also assumed throughout this section that $\Delta = \langle a_t \cdot a_t' \rangle$ is in correlation form.

McLeod (1979) derived the distribution of the residual cross-correlation in univariate ARMA time series models. His results can be particularized to obtain the distribution of $\bar{r}$. The main results for the CARMA model are summarized in the following Lemma.

Lemma 2.5.1

(i) The asymptotic joint distribution of $\sqrt{N}(\bar{\underset{\sim}{\beta}} - \underset{\sim}{\beta}\ \underset{\sim}{r}')$ is normal with mean zero and variance covariance

$$\begin{pmatrix} V_{\bar{\beta}} & -\,\mathrm{Diag}\,(I_{hh}^{-1})\,A' \\ -A\ \mathrm{Diag}\,(I_{hh}^{-1}) & Y \end{pmatrix}$$

where $V_{\bar{\beta}}$ and I_{hh} are given in Lemma 2.2.1,

$$Y = \Delta \otimes \Delta \otimes 1_M$$

$$A = (\sigma_{gh} \cdot X_{hh})$$

$$X_{hh} = \begin{pmatrix} \sigma_{1h} \\ \vdots \\ \cdot \\ \sigma_{kh} \end{pmatrix}_{k\times 1} \otimes (-\pi_{h,i-j} \mid \psi_{h,i-j})_{M\times(p+q)} = \sigma_{\cdot h} \otimes X_h$$

$$\phi(B)^{-1} = \sum_{r=0}^{\infty} \pi_{hr} B^r \quad , \qquad \theta_h^{-1}(B) = \sum_{r=0}^{\infty} \psi_{hr} B^r$$

and $A \otimes B$ denotes the kronecker product of matrices.

(ii) The asymptotic distribution of $\sqrt{N}\bar{r}$ is normal with mean zero and variance covariance

$$Y + X V_{\bar{\beta}} X' - X \text{ Diag } (I_{hh}^{-1}) A' - A \text{ Diag } (I_{hh}^{-1}) X'$$

where $X = \text{Diag } (X_{II}, \ldots, X_{KK})_{k^2 M \times K^2 M}$.

In particular the variance of $\bar{r}_{gh} = (\bar{r}_{gh}(1), \ldots, \bar{r}_{gh}(M))$ is given by:

$$N \text{ Var } (\bar{r}_{gh}) = 1_M - {}_{gh} X_h I_{hh}^{-1} X_h' \qquad (2.5.1)$$

The following Lemmas give the asymptotic distribution of $\hat{r}$.

Lemma 2.5.2

(i) The asymptotic joint distribution of $\sqrt{N}(\hat{\beta}-\beta, r')$ is normal with mean zero and variance covariance given by

$$\begin{pmatrix} V_{\hat{\beta}} & -V_{\hat{\beta}}X' \\ -X.V_{\hat{\beta}} & Y \end{pmatrix}$$

where $V_{\hat{\beta}}$ is given by Lemma 2.2.2, X and Y by Lemma 2.5.1.

(ii) The asymptotic distribution of $\sqrt{N}\,\hat{r}$ is asymptotically normal with zero mean and covariance matrix

$$Y - X \cdot V_{\hat{\beta}} \cdot X'$$

In particular the variance of $\hat{r}_{gh} = (\hat{r}_{gh}(1),\ldots,\hat{r}_{gh}(M))$ is given by

$$N \,.\, \mathrm{Var}\,(\hat{r}_{gh}) = 1_M - \sigma_{gh}^2 X_h \cdot \mathrm{Var}(\hat{\beta}_h)\, X_h \qquad (2.5.2)$$

Proof: Expanding $\partial l/\partial\beta$, where l is given by equation (2.2.7), using a Taylor series expansion and evaluating at $\hat{\beta}$ gives:

$$0 = \partial l/\partial\beta + \partial^2 l/\partial\beta\partial\beta' \cdot (\hat{\beta}-\beta) + o_p(1)$$

From this equation and Lemma 2.2.2 it is very easy to see that

$$\hat{\beta} - \beta = V_{\hat{\beta}}\,(\partial l/\partial\beta')/N + o_p(1/N).$$

Now if W_{ht} stands for V_{ht} or U_{ht}, the auxiliary series given by equation (2.2.2), it follows from the well known fourth moment result that:

$$< a_{gt}\, a_{ft+i}\, a_{st'}\, W_{ht-j} > = \sigma_{ps} < a_{gt} W_{ht'-j} > \quad \text{for } t = t' - i$$
$$= 0 \quad \text{otherwise}$$

and

$$< a_{gt}\, W_{ht+r} > = -\sigma_{gh}\, \Pi_{hr} \quad \text{if} \quad W = V$$
$$= \sigma_{gh}\, \psi_{hr} \quad \text{if} \quad W = U$$

Now because $r_{gf}(i) = \sigma_{gf}(i) + O_p(1/N)$ it follows that

$$- < r_{gf}(i)\, \partial l / \partial \beta_{hj} > = < (\sum_{t=1}^{N} a_{gt}\, a_{ft+j} / N) \cdot (\sum_{t=1}^{N} \sum_{s=1}^{K} \sigma^{sh} a_{st'} W_{ht-j}) >$$
$$= \eta_{h(i-j)}\, \sigma_{gh} \sum_{s=1}^{K} \sigma^{sh}\, \sigma_{fs}$$

where η stands for π if $\beta = \phi$ or for ψ if $\beta = \theta$

$$= \sigma_{gh}\, \eta_{h(i-j)} \quad \text{if} \quad f = h$$
$$= 0 \quad \text{otherwise.}$$

This result implies that $N \cdot < \underset{\sim}{r} (\underset{\sim}{\hat{\beta}} - \underset{\sim}{\beta})' > = X\, V_{\hat{\beta}}$. Normality is obtained from the Martingale central limit theorem as in Lemma 2.2.1. This proves statement (i) of the Lemma.

To prove (ii) it is observed that a Taylor expansion of $\underset{\sim}{\dot{r}}$ around (β , Δ) and evaluated at $(\hat{\beta} , \hat{\Delta})$ gives:

$$\hat{\underset{\sim}{r}} = r + X\,(\hat{\beta} - \beta) + O_p(1/N)$$

The result follows immediately.

Let $\underline{q}$ be such that $\underline{q}\ \Delta\ \underline{q}' = 1_K$ and $\underline{qq}' = \Delta^{-1}$ and let $G = \underline{q} \otimes \underline{q} \otimes 1_M$. It follows as a corollary of theorem 5 in Li and McLeod (1981) that for M sufficiently large

$$\pi_M \simeq 0 \text{ and } \phi_m \simeq 0,\ X'(GG')\,X \simeq V_{\hat{\beta}}^{-1}.$$

This implies that $Q = 1_{Mk^2} - G'XV_{\hat{\beta}}X'G$ is almost idempotent with rank rank $k^2M - k(p+q)$. Now $\sqrt{N}\underset{\sim}{r}$ where

$$\tilde{\underset{\sim}{r}} = G'\hat{\underset{\sim}{r}}$$

is N(0, Q). This suggests the following modified portmanteau test statistics for testing the independence of the residuals (see Li and McLeod, 1981):

$$Q^*_m = N\ \tilde{\underset{\sim}{r}}'\ \tilde{\underset{\sim}{r}} + k^2M(M+1)/2$$

$$= N \sum_{\ell=1}^{M} \hat{r}(\ell)'\ (\hat{\Delta}^{-1} \otimes \hat{\Delta}^{-1})\hat{r}(\ell) + K^2M(M+1)/2 \quad (2.5.3)$$

where

$$\hat{r}(\ell) = (\hat{r}_{11}(\ell), \hat{r}_{21}(\ell), \ldots, \hat{r}_{K1}(\ell), \hat{r}_{12}(\ell) \ldots, \hat{r}_{K2}(\ell), \ldots, \hat{r}_{KK}(\ell))'$$

which is approximately χ^2-distributed with $k^2\ M - k(p + q)$ d.f. for large N and M. As shown by Li and McLeod this modified test provides a better approximation to the null distribution than $Q_m = N \cdot \underset{\sim}{\tilde{r}}' \underset{\sim}{\tilde{r}}$.

Expressions (2.5.1) and (2.5.2) also provide a method for testing the independence of the residuals comparing the observed values of $\hat{r}_{ij}(\ell)$ or $\bar{r}_{ij}(\ell)$ with the respective asymptotic standard deviations which are easily calculated. Large values or $\hat{r}_{ij}$ or $\bar{r}_{ij}$ should detect misspecification of the model.

2.6 SIMULATION OF CARMA MODELS

2.6.1 General Method

Simulation of stochastic models is today a widely used technique in industry and academia. Applications vary from the design and operation of complex systems to the assessment and development of new statistical and other theoretical methods. Therefore, it is very important to use appropriate methods to simulate the model at hand to avoid making wrong decisions when the techniques are used in industry and bias when they are used in theoretical studies. In this section the simulation of the CARMA model is considered and an appropriate algorithm is given. It will be assumed, without loss of generality, that $\underset{\sim}{\mu} = 0$.

McLeod and Hipel (1978) have developed adequate techniques to simulate univariate ARMA models. The techniques were designed so that random realizations of the underlying model were used for starting values, avoiding the introduction of systematic bias in the simulated series by employing fixed starting values. The method given in this section is an extension of the techniques of McLeod and Hipel (1978) to the CARMA model (equation 2.1.1).

Let $\underset{\sim}{Z}_{h,p} = (Z_{h1}, \ldots, Z_{hp})'$ $h = 1, \ldots, k$ and let $\underset{\sim}{A}_{h,q} = (a_{h\,p-q+1}, \ldots, a_{hp})$ $h = 1, \ldots, k$. Then $\underset{\sim}{Z}_{h,p}$ and $\underset{\sim}{A}_{h,q}$ represent the initial values for the series h, h = 1, ... ,k.

Suppose it is necessary to generate N synthetic observations for the CARMA model of equation 2.1.1. The following algorithm is used to obtain $\tilde{Z}_1, \ldots, \tilde{Z}_N$, where $\tilde{Z}_t = (Z_{1t}, \ldots, Z_{kt})$. This algorithm is exact in the sense that it is not subject to inaccuracies associated with fixed initial values.

1. Determine the lower triangular matrix M by Cholesky decomposition such that (Ralston, 1965)

$$\Delta = MM'$$

2. Obtain the vectors of initial values $\tilde{Z}_{h,p}$, $\tilde{A}_{h,q}$, $h = 1, \ldots, k$. (See next section for the method used to calculate the initial values).

3. Generate $\tilde{a}_{p+1}, \ldots, \tilde{a}_N$, a sequence of N-p k-dimensional vector NID(0,Δ) in the following way:
   - Generate $e_{1t}, \ldots, e_{kt}$, $t = p+1, \ldots, N$ a sequence of k(N-p) NID (0,1) random variables
   - Calculate $a_{ht} = \sum_{j=1}^{h} m_{hj} e_{jt}$, $h = 1, \ldots, k$; $t = p+1, \ldots, N$

4. Obtain $\tilde{Z}_{p+1}, \ldots, \tilde{Z}_N$ by using

$$Z_{ht} = \phi_{h1} Z_{ht-1} + \ldots + \phi_{hp} Z_{ht-p}$$

$$+ a_{ht} - \theta_{h1} a_{ht-1} - \ldots - \theta_{hq} a_{ht-q}$$

$$h = 1, \ldots, k; \; t = p+1, \ldots N$$

5. If another series of length N is required return to step 2.

It is possible that in some application the researcher may be interested not in simulating $\tilde{Z}_t$ but in simulating $\tilde{W}_t$ where $\tilde{Z}_t$ follows a CARMA model and $Z_{ht} = \nabla_{s_h}^{D_h} \cdot \nabla^{d_h} W_{ht}$

$\nabla^d = (I - B)^d$ and $\nabla_s^D = (I - B^s)^D$ are the regular and seasonal differencing operators. McLeod and Hipel (1978) have given a detailed algorithm to obtain the integrated series W_{ht} from the simulated series Z_{ht}. Models with power transformations of the form (see Box and Cox, 1964)

$$Z_{ht} = \{(W_{ht} + \text{const})^{\lambda_h} - 1\}/\lambda_h \qquad \lambda_h \neq 0$$
$$= \ln(W_{ht} + \text{const}) \qquad \lambda_h = 0$$

where $\underset{\sim}{Z}_t$ follows a CARMA model can be easily simulated from the synthethic values Z_{ht}. The generated values of W_{ht} are obtained by the inverse transformation.

$$W_{ht} = (\lambda_h Z_{ht} + 1)^{1/\lambda_h} \qquad \lambda_h \neq 0$$
$$= \exp Z_{ht} = \text{const} \qquad \lambda_h = 0$$

2.6.2 Calculation of the Initial Values

The joint distribution of $Z_{h,p}$ and $A_{h,p}$, $h = 1, \ldots, k$ is used to generate the initial values for the simulation of the CARMA model. The following Lemma gives this joint distribution.

Lemma 2.6.2.1 The joint distribution of $\underset{\sim}{U} = (\underset{\sim}{Z}_{1,p}, \ldots, \underset{\sim}{Z}_{k,p}, \underset{\sim}{A}_{1q}, \ldots, \underset{\sim}{A}_{kq})$ is multivariate normal with zero mean and variance covariance given by:

$$V = \left(\begin{array}{c|c} \gamma_{gh}(i-j) & \sigma_{gh} \quad \psi_g(i-j) \\ \hline \text{Symm} & \Delta \otimes 1_{qxq} \end{array} \right) \qquad (2.6.2.1)$$

where

$$\gamma_{gh}(r) = \langle Z_{gt} \; Z_{ht+r} \rangle \qquad g, h = 1, \ldots, k$$

$$\phi_h(B) \cdot \psi_h(B) = \phi_h(B) \qquad h = 1, \ldots, k \quad (2.6.2.2)$$

Proof. It is easy to see that linear combinations of the elements of $\underset{\sim}{U}$ are normally distributed, so that the joint distribution of $\underset{\sim}{U}$ is normal. The only term in the variance covariance matrix which requires special consideration is:

$$\langle Z_{gt} e_{ht'} \rangle = \langle \Sigma \, \psi_{gr} \; Z_{gt-r} e_{ht'} \rangle = \psi_g(t - t') \sigma_{gh}$$

The other terms of V are easy to obtain. This completes the proof of the Lemma.

Ansley (1980) and Kohn and Ansley (1982) have provided an algorithm to obtain the theoretical autocovariance function of the general multivariate ARMA model. This algorithm could be employed to calculate the terms $\gamma_{gh}(i-j)$ of equation (2.6.2.1). However, due to the diagonal structure of the CARMA model, it is possible to develop a computationally efficient algorithm for the calculation of the theoretical autocovariance function of the CARMA model.

$\gamma_{qh}(\cdot)$, $q \neq h$, can be calculated by solving a linear system of 2p - 1 equations in the 2p - 1 unknowns

$$\gamma_{qh}(r),\ r = 1 - p, \ldots ,0, 1, \ldots ,p - 1$$

The system is formed by the following equations:

$$\gamma_{qh(o)} - \sum_{r=0}^{p-1} \Big(\sum_{i=1}^{p-r} \phi_{qi} \phi_{hr+i} \Big) \gamma_{qh(r)} - \sum_{r=1}^{p-1} \Big(\sum_{i=1}^{p} \phi_{qh} \phi_{hr+i} \Big) \gamma_{qh}(-r) = b_o \qquad (2.6.2.3)$$

where

$$b_0 = \sigma_{qh} \Big\{ \sum_{j=o}^{q} \theta_{gj} \theta_{hj} - \sum_{i=1}^{p} \sum_{j=1}^{q} (\phi_{hi} \phi_{qj} \Psi_h(j-i) + \phi_{qi} \theta_{hj} \Psi_q(j-i) \Big\}$$

$$\gamma_{qh}(r) - \sum_{i=1}^{p} \phi_{hi} \gamma_{qh}(r-i) = - \sum_{j=r}^{q} \theta_{hj} \Psi_q(j-r) \sigma_{qh} \qquad (2.6.2.4)$$

$$r = 1, \ldots ,p-1$$

$$\gamma_{qh}(-r) - \sum_{i=1}^{p} \phi_{qi} \gamma_{qh}(i-r) = -\sum_{j=r}^{q} \theta_{qj} \Psi_h(j-r) \sigma_{qh} \qquad (2.6.2.5)$$

The values of $\Psi_q(\cdot)$ are obtained by solving equation (2.6.2.2). It is easy to see that equations (2.6.2.3) through (2.6.2.5) can be written in the form $A\phi = b$. This system of equations can be easily solved.

To calculate $\gamma_{gg}(\cdot)$ it is only necessary to consider equations (2.6.2.3) and (2.6.2.4) to form a system of p linear equations in $\gamma_{gg}(r)$, $r = 0,\ldots,p - 1$. This corresponds to the algorithm given by McLeod (1975, 1977) to obtain the theoretical autocovariance function of univariate ARMA models.

The following algorithm can be used to obtain the initial values for the simulation of the CARMA model (See step 2 of the algorithm given in Section 2.6.1).

1. Calculate $\psi_g(s)$, $g = 1, \ldots, k$; $s = 0, \ldots, \max\{p, q\}$ from equation (2.6.2.2).
2. Calculate the theoretical autocovariance functions $\gamma_{gh}(r)$,

$$r = 1 - p, \ldots, 0, \ldots, p - 1, \quad 1 < g < h < k$$

   solving the system of linear equations obtained from equations (2.6.2.3.) through (2.6.2.5).
3. Form the variance covariance matrix V of $\underset{\sim}{U}'$

$$\underset{\sim}{U} = (\underset{\sim}{Z}'_{1,p}, \ldots, \underset{\sim}{Z}'_{1,q}, \underset{\sim}{A}'_{1,q}, \ldots, \underset{\sim}{A}'_{k,q})'$$

   given by equation (2.6.2.1) and obtain the lower triangular matrix L by Cholesky decomposition such that

$$V = LL'$$

4. Generate $e_1, \ldots, e_{k(p+q)}$, a sequence of $k(p+q)$ NID(0,1) random variables and determine the vector of initial values by:

$$U_j = \sum_{i=1}^{j} \ell_{ji} e_i \qquad j = 1, \ldots, k(p+g)$$

Note that if another series is required only step 4 is needed.

2.7 CONCLUSIONS

It is interesting to observe how the results obtained in this chapter can be efficiently employed in the model building procedure. Given the data set $\{\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_N\}$, $\underset{\sim}{Z}_t = (Z_{1t}, \ldots, Z_{kt})'$ the set $\{Z_{ht}\}$ $t = 1, \ldots, N$ can be used to fit an adequate univariate ARMA model for Z_{ht}. This produces a series of residuals and estimated parameters for each model, $\bar{a}_{ht}$ and $\bar{\underset{\sim}{\beta}}_h$ say, which can be used to test whether or not a CARMA model could be usefully employed. For example, the hypothesis of lagged relationships of the form:

$$H_o^{(i,j)}: \quad \rho_{ij}(1) = \ldots \rho_{ij}(M) = 0 \qquad i \neq j$$

against the simple negation can be tested using the Portmanteau test statistic:

$$\bar{Q}_M(i,j) = N \; \bar{\underset{\sim}{r}}_{ij}' \; \mathrm{Var}(\bar{\underset{\sim}{r}}_{ij})^{-1} \; \bar{\underset{\sim}{r}}_{ij}$$

where Var $(\bar{\underset{\sim}{r}}_{ij})$ is given by equation (2.5.1) calculated using $\bar{\underset{\sim}{\beta}}_h$ rather than $\underset{\sim}{\beta}_h$. $\bar{Q}_M(i,j)$ has under the null hypothesis a χ^2 distribution with M degrees of freedom. (See also McLeod, 1979; Haugh, 1976). If the hypothesis is not rejected, one iteration (or more if desired) of the score algorithm (Lemma 2.2.3) produces asymptotically efficient estimators for the parameters of the model. With these new parameters, the residual cross correlations

$\hat{\underset{\sim}{r}}$ can be obtained and the Portmanteau statistic (2.5.3) can be employed to assess the adequacy of the overall model. If the null hypothesis is rejected, then a more general multivariate ARMA model should be considered (see Haugh and Box, 1977).

The results can also be extended to the seasonal multiplicative CARMA models of the form:

$$\Phi_h (B^s)\ \phi_h\ (B)\ Z_{ht} = \Theta_h\ (B^s)\ \theta_h(B)\ a_{ht} \qquad (2.7.1)$$

$$\underset{\sim}{a}_t = (a_{1t}, \ldots, a_{kt})' \sim \text{NID}\ (O, \Delta)$$

$$\Theta_h(B^s) = 1 - \Theta_{h1} B^s - \ldots - \Phi_{hp_s} B^{sp_s}$$

$$\Theta_h(B^s) = 1 - \Theta_{h1} B^s - \ldots - \Phi_{hq_s} B^{sq_s}$$

where the polynomials $\Phi_h(B^s)$, $\Theta_h(B^s)$, $\phi_h(B)$ and $\theta_h(B)$ satisfy the conditions of stationarity, invertibility and non-redundancy. This model provides an adequate parsimonious repesentation for many seasonal economic time series (Box and Jenkins, 1976).

The results of section 2.2 extend throughout if the submatrices $[I_{cd}\ (i-j)]$ of equation (2.2.1) are replaced by:

$$\begin{pmatrix} \gamma_{cd}(i-j) & \gamma_{CD}(i-js) \\ \gamma_{cd}(is-j) & \gamma_{CD}(is-js) \end{pmatrix} \begin{matrix} r \\ r_s \end{matrix}$$

$$\qquad r' \qquad\qquad r'_s$$

where c,d stand for u or v the auxiliary series of equation (2.2.2), and C, D stand for U or V defined by:

$$\Phi_h(B^s)\, V_{ht} = -a_{ht}$$

and

$$\Theta_h(B^s)\, U_{ht} = a_{ht}$$

and r, r_s denote the order of polynomials, i.e.: p, q, p_s or q_s, $\underset{\sim}{\beta}_h$ is now defined to be

$$\underset{\sim}{\beta}_h = (\phi_{h1}, \ldots, \phi_{hp}, \Phi_{h1}, \ldots, \Phi_{hp_s}, \theta_{h1}, \ldots, \theta_{hq}, \Theta_{h1}, \ldots, \Theta_{hq_s})$$

The results of the distribution of the residual autocorrelations of section 2.5 can also be extended to the multiplicative model (2.7.1). In this case the matrix $X = [-\pi_{h,i-j} \mid \psi_{h,i-j}]_{Mx(p,\, q)}$ should be changed to

$$X_h = \left[-\pi_{hi-j} \mid -\Pi_{h,i-js} \mid \psi_{h,i-j} \mid \Psi_{h,i-js}\right]_{Mx(p+p_s+q+q_s)}.$$

(See also McLeod, 1978).

CHAPTER 3

ESTIMATION OF PARAMETERS FOR CARMA MODELS WITH UNEQUAL SAMPLE SIZE

3.1 INTRODUCTION

In this chapter, the likelihood function of the parameters of the bivariate CARMA (p,q) model when m + N observations $\{Z_{1t}\}$ $t = 1-m, \ldots 0, 1, \ldots, N$, say, and N observations $\{Z_{2t}\}$ $t = 1, \ldots, N$, say, are available, is discussed. This situation often arises in practice. For example, Risager (1980) fitted a bivariate CAR involving two third order autoregressions to mean annual ice core measurements for which data were available for the years 1861-1974 and 1969-1975, respectively. In Risager's (1980) analysis, only data for the common period 1861-1974 could be used. A better technique, based on maximum likelihood estimation, for utilizing all of the available data is presented in this Chapter. The gain in efficiency is also discussed.

The likelihood function for a general multivariate ARMA model has been given by Hillmer and Tiao (1979) and Nicholls and Hall (1979). However, this likelihood function was derived under the assumption of equal sample sizes for all the series and hence is not applicable here. One possible approach is to consider the observations $\{Z_{2t}\}$ $t = 1-m, \ldots, 0$ as missing and treat them as additional parameters to be estimated from the data. Missing observations in time series have been studied by several authors. Ansley and Kohn (1980) proposed the use of the Kalman filter to obtain the likelihood of vector moving average processes with missing data. The procedure of Ansley and Kohn is basically an extension of the approach given by Jones (1980) and Gardner et al. (1980) for the univariate case. This approach, although sensible, is not computationally efficient for large samples (see Gardner et al., 1980). Ljung (1982) also considered missing observations in univariate time series and gave expressions for the likelihood function and for the estimation of the missing observations. An equivalent and more straightforward approach is to set the missing observations to zero and then use intervention analysis (Box and Tiao, 1975) to estimate the missing values. (See Baracos et al., 1983.) One problem in considering the observations $\{Z_{2t}\}$, $t = 1-m, \ldots ,0$ as missing is the introduction of too many parameters into the estimation process. This problem can fortunately be avoided when the likelihood function is considered directly.

In the next section, expressions for the likelihood function are given. In Section 3.3, some simplifications are considered for a computationally efficient implementation of the likelihood function. Lastly, in Section 3.4 the asymptotic distribution of the parameter estimators is given and the possible gain in efficiency in the estimation of the parameters is considered.

3.2 THE LIKELIHOOD FUNCTION

Let $\{Z_{1t}\}$ $t = 1 - m, \ldots, 0, 1, \ldots, N, N + m$ observations of Z_{1t} and $\{Z_{2t}\}$ $t = 1, \ldots, N$, N observations of Z_{2t} where $\underset{\sim}{Z}_t = (Z_{1t}, Z_{2t})'$ follows a bivariate CARMA(p,q) model of the form:

$$\phi_h(B)(Z_{ht} - \mu_h) = \theta_h(B)a_{ht} \qquad h = 1,2 \qquad (3.1.2)$$

$$a_t = (a_{1t}, a_{2t})' \sim \text{NID}(0, \Delta)$$

$$\mu_h : \text{ mean of series } h \qquad h = 1,2$$

$$\phi_h(B) = 1 - \phi_{h1} B - \ldots - \phi_{hp} B^p \qquad h = 1,2$$

$$\theta_h(B) = 1 - \theta_{h1} B - \ldots - \theta_{hq} B^q \qquad h = 1,2$$

It is assumed that the polynomials $\phi_h(B)$ and $\theta_h(B)$, $h = 1,2$, have their roots outside the unit circle and the pair $(\phi_h(B), \theta_h(B))$ does not have common factors, $h = 1,2$. These assumptions assure stationarity, invertibility and identifiability of the CARMA model. The purpose of this section is to obtain the likelihood function for the parameters of the model $(\beta, \mu', \Delta)'$ where $\underset{\sim}{\beta} = (\underset{\sim}{\beta}_1', \underset{\sim}{\beta}_2')'$, $\beta_h = (\phi_{h1}, \ldots, \phi_{hp}, \theta_{h1}, \ldots, \theta_{hq})'$ and $\underset{\sim}{\mu} = (\mu_1, \mu_2)'$. It will be assumed in the following discussion, without loss of generality, that $\underset{\sim}{\mu} = 0$. It is very straightforward to modify the results to include the case $\underset{\sim}{\mu} \neq 0$.

The following notation is introduced in order to find an expression for the likelihood. Let:

$$\underset{\sim}{a}_{10} = (a_{11-m}, \ldots, a_{10})' \qquad \underset{\sim}{a}_{11} = (a_{11}, \ldots, a_{1N})'$$

$$\underset{\sim}{a}_{1} = (\underset{\sim}{a}_{10}', \underset{\sim}{a}_{11}')'$$

$$\underset{\sim}{a}_{2} = (a_{21}, \ldots, a_{2N})' \qquad \underset{\sim}{a} = (\underset{\sim}{a}_1', \underset{\sim}{a}_2')'$$

so $\underset{\sim}{a}$ represents the innovations of the process. Let also

$$\underset{\sim}{Z}_{10} = (Z_{11-m}, \ldots, Z_{10})' \qquad \underset{\sim}{Z}_{11} = (Z_{11}, \ldots, Z_{1N})'$$

$$\underset{\sim}{Z}_{1} = (\underset{\sim}{Z}_{10}, \underset{\sim}{Z}_{11})'$$

$$\underset{\sim}{Z}_{2} = (Z_{21}, \ldots, Z_{2N})' \qquad \underset{\sim}{Z} = (\underset{\sim}{Z}_1, \underset{\sim}{Z}_2)'$$

so $\underset{\sim}{Z}$ represents the observations of the process. Finally, let

$$\underset{\sim}{e}_1 = (a_{1-(m+q)}, \ldots, a_{1-m}, Z_{1-(m+p)}, \ldots, Z_{1-m})'$$

$$\underset{\sim}{e}_2 = (a_{2,1-q}, \ldots, a_{2,0}, Z_{2,1-p}, \ldots, Z_{2,0})'$$

$$\underset{\sim}{e} = (\underset{\sim}{e}_1, \underset{\sim}{e}_2)'$$

so $\underset{\sim}{e}$ represents the initial values of the process. It is easy to see that for suitable matrices H_h (which depend on $\underset{\sim}{\phi}_h$), F_h (which depend on $\underset{\sim}{\theta}_h$) and G_h (which depend on $\underset{\sim}{\beta}_h$) h = 1,2.

$$\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \cdot \begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{a}_1 \\ \underset{\sim}{a}_2 \end{pmatrix} + \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \end{pmatrix}$$

or in short form:

$$H \underset{\sim}{Z} = F\underset{\sim}{a} + G\underset{\sim}{e}$$

Therefore the vector (e', a) can be written as

$$\begin{pmatrix} \underset{\sim}{e} \\ \\ \underset{\sim}{a} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ K \end{pmatrix} \underset{\sim}{z} + \begin{pmatrix} I \\ \\ L \end{pmatrix} \underset{\sim}{e}$$

$$= \psi \underset{\sim}{z} + \Lambda \underset{\sim}{e} \qquad (3.2.2)$$

where

$$K = \left(\begin{array}{c|c} F_1^{-1} H_1 & 0 \\ \hline 0 & F_2^{-1} H_2 \end{array}\right) \begin{array}{l} N+1 \\ \\ N \end{array}$$
$$\qquad\quad N+m \qquad\quad N$$

and

$$L = \left(\begin{array}{c|c} -F_1^{-1} G_1 & 0 \\ \hline 0 & -F_2^{-1} G_2 \end{array}\right) \begin{array}{l} N+M \\ \\ N \end{array}$$
$$\qquad\quad p+q \qquad\quad p+q$$

Now, because of the assumptions of normality, stationarity and invertibility, it follows that the joint distribution of (e', a') is multivariate normal with mean zero and variance covariance Ω, say. From equation (3.2.2), it can be seen that the Jacobian of the linear transformation from (e', a')' to (e', z')' is one, so that the joint distribution of (e', z')' is multivariate normal with probability density function given by:

$$L(\underset{\sim}{e}, \underset{\sim}{z}) = (2\pi)^{-\frac{1}{2}(2N+m+2(p+q))} \cdot |\Omega|^{-\frac{1}{2}} \cdot \exp\{-\tfrac{1}{2}S(\underset{\sim}{z}, \underset{\sim}{e})\} \quad (3.2.3)$$

where

$$S(\underset{\sim}{z}, \underset{\sim}{e}) = (\psi \underset{\sim}{Z} + \Lambda \underset{\sim}{e})' \cdot \Omega^{-1} \cdot (\psi \underset{\sim}{Z} + \Lambda \underset{\sim}{e})$$

$$= S(\underset{\sim}{z}, \hat{\underset{\sim}{e}}) + (\underset{\sim}{e} - \hat{\underset{\sim}{e}})' \Lambda' \Omega^{-1} \Lambda (\underset{\sim}{e} - \hat{\underset{\sim}{e}})$$

where

$$\hat{\underset{\sim}{e}} = -(\Lambda' \Omega^{-1} \Lambda)^{-1} \cdot \Lambda' \Omega^{-1} \psi \underset{\sim}{Z}$$

which corresponds to the maximum likelihood estimate for $\underset{\sim}{e}$ given the data $\underset{\sim}{Z}$. Integrating out $\underset{\sim}{e}$ from equation (3.2.3), the distribution of $\underset{\sim}{Z}$ is obtained and is given by:

$$L(\underset{\sim}{Z}) = (2\pi)^{-\frac{(2n+m)}{2}} \cdot |\Omega|^{-\frac{1}{2}} \cdot |\Lambda' \Omega^{-1} \Lambda|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}S(\underset{\sim}{z}, \underset{\sim}{e})\} \quad (3.2.4)$$

which corresponds to the likelihood of the parameters of the CARMA model $L(\underset{\sim}{\beta}, \Delta)$ say.

3.3 CALCULATION OF THE LIKELIHOOD

It is interesting to observe that the form of the likelihood function $L(\beta, \Delta)$ given by equation (3.2.4) is similar to the likelihood of a multivariate ARMA model as given by Hillmer and Tia (1979) or Nicholls and Hall (1979). Hall and Nicholls (1980) gave an algorithm to evaluate the likelihood function of the general multivariate model. Although their approach could be employed in the CARMA case, it would not be computationally efficient because some simplifications can be made, due to the structure of the model, which gives a more efficient procedure. In this Section, some explicit expressions for the terms of the likelihood function as well as some simplifications are given.

3.3.1 Calculation of the Sum of Squares $S(Z,\hat{e})$

In this section a method to calculate $S(Z, \hat{e})$ which corresponds to the (unconditional) sum of squares is presented. $S(Z, \hat{e})$ is given by:

$$S(Z, \hat{e}) = (\psi Z + \Lambda \hat{e})'\Omega^{-1} (\psi Z + \Lambda \hat{e}) \qquad (3.3.1.1)$$

where $\hat{e}$ is the vector of estimated initial values and Ω the variance covariance of $(e', a')' = (e_1', e_2', a_{10}', a_{11}', a_2')'$ (see section 3.2). Now Ω is given by:

$$\Omega = \begin{pmatrix} \Omega_{e_1e_1} & \Omega_{e_1e_2} & 0 & 0 & 0 \\ \Omega_{e_2e_1} & \Omega_{e_2e_2} & \Omega_{e_2a_{10}} & 0 & 0 \\ 0 & \Omega_{a_{10}e_2} & \sigma_{11}1_{mxm} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{11}1_{NxN} & \sigma_{12}1_{NxN} \\ 0 & 0 & 0 & \sigma_{21}1_{NxN} & \sigma_{22}1_{NxN} \end{pmatrix}$$

where 1_{RxR} is the identity matrix

$$\Omega_{e_he_h} = \left(\begin{array}{c|c} \sigma_{hh}1_{gxg} & \gamma_{e_hZ_h}(j-i) \\ \hline \gamma_{Z_he_h}(j-i) & \gamma_{Z_hZ_h}(j-i) \end{array}\right) \begin{array}{c} q \\ \\ p \end{array}$$

$$= \sigma_{hh} \left(\begin{array}{c|c} 1_{mxm} & \Psi_h(j-i) \\ \hline \Psi_{h(j-i)} & \gamma_h(j-i) \end{array}\right) \qquad h = 1,2$$

where $\phi_h(B)\,\Psi(B) = \theta_h(B)$ and $\gamma_{gh}(k) = \langle Z_{gt}Z_{ht+k}\rangle/\sigma_{gh}$

$$\Omega_{e_1e_2} = \left(\begin{array}{c|c} 0 & 0 \\ \hline \begin{array}{c} \Psi_{2m+q-p+1}\cdots\Psi_{2m-p+2} \\ \Psi_{2m+q}\cdots\Psi_{2m+1} \end{array} & \begin{array}{c} \gamma_{21}(m+1)\ \ldots\ \gamma_{21}(m-p+2) \\ \gamma_{21}(m+p)\ \ldots\ \gamma_{21}(m+1) \end{array} \end{array}\right)\sigma_{12}$$

$$\qquad\qquad q \qquad\qquad\qquad\qquad p$$

$$\Omega_{a_{10}e_2} = \sigma_{12} \left[\begin{array}{c|ccc} 0 & \psi_{2m-p} & \cdots & \psi_{2m-1} \\ & \psi_{21} & \cdots & \psi_{2p} \\ & \psi_{20} & \cdots & \psi_{2p-1} \\ 1_{q\times q} & & \ddots & \\ & & & \psi_{20} \end{array} \right] m$$

$$\qquad\qquad q \qquad\qquad p$$

The inverse of Ω can be expressed as

$$\Omega^{-1} = \left[\begin{array}{c|c} \Gamma^{-1} & 0 \\ \hline 0 & \Delta^{-1} \otimes I \end{array} \right] \begin{array}{l} 2(p+q)+m \\ 2N \end{array}$$

where

$$\Gamma^{-1} = \begin{bmatrix} 1_{2(p+q)\times 2(p+q)} \\ -\dfrac{\Omega_{a_{10}e}}{\sigma_{11}} \end{bmatrix} P^{-1} \left(1_{2(p+q)\times(2(p+q)} \Big| -\frac{\Omega_{ea_{10}}}{\sigma_{11}} \right) + \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1_{m\times m}}{\sigma_{11}} \end{pmatrix} \tag{3.3.1.2}$$

$$P = \begin{bmatrix} \Omega_{e_1e_1} & \Omega_{e_1e_2} \\ \Omega_{e_2e_1} & \Omega_{e_1e_2} - \dfrac{\Omega_{e_2a_{10}}\Omega_{a_{10}e_2}}{\sigma_{11}} \end{bmatrix}$$

It can be shown that for MA processes, or in general when m is large so that $\gamma_{22}(k) \doteq \sum_{r=0}^{m} \psi_{2r}\psi_{2r+k}$, the matrix $\Omega_{e_2 a_{10}} \cdot \Omega_{a_{10} e_2}$ can be approximated by $\sigma_{11}\rho^2\Omega_{e_2 e_2}$, where ρ is the correlation coefficient between a_{1t} and a_{2t} given by $\rho = \sigma_{12} / \sqrt{\sigma_{11}\sigma_{22}}$. Therefore P can be approximated by:

$$P \doteq \begin{pmatrix} \Omega_{e_1 e_1} & \Omega_{e_1 e_2} \\ \Omega_{e_2 e_1} & (1-\rho^2)\Omega_{e_2 e_2} \end{pmatrix}$$

Now let $\underset{\sim}{U} = (\underset{\sim}{\psi Z} + \Lambda \hat{\underset{\sim}{e}}) = (\underset{\sim}{\hat{e}}', \hat{\underset{\sim}{a}}_{10}', \hat{\underset{\sim}{a}}_{11}', \hat{\underset{\sim}{a}}_{2}')$ where $\hat{a}_{ht}$ corresponds to the estimated value of the innovations of series Z_{ht}, h = 1,2, using the data $\{Z_{1t}\}$ t = 1 - m, ... ,N and $\{Z_{2t}\}$ t=1,...,N and the vector of starting values $\hat{\underset{\sim}{e}}$. The values of $\hat{a}_{ht}$ can be obtained recursively using equation (3.2.1). For example $\hat{a}_{1t}$ can be obtained as:

$$\hat{a}_{1t} = Z_{1t} - \phi_{11}Z_{1t-1} - \dots - \phi_{1p}Z_{1t-p} + \theta_{11}\hat{a}_{1t-1} + \dots + \theta_{1q}\hat{a}_{1t-q} \quad t=1-m,\dots,N$$

with starting values given by $\hat{\underset{\sim}{e}}_1$.

From equations (3.3.1.1.) and (3.3.1.2.) $S(\underset{\sim}{z}, \underset{\sim}{e})$ can then be calculated as:

$$S(\underset{\sim}{z}, \hat{\underset{\sim}{e}}) = \underset{\sim}{U}' \, \Omega^{-1} \, \underset{\sim}{U}$$

$$= (\hat{e}' \;\; \hat{a}'_{10}) \begin{pmatrix} 1 \\ -\dfrac{\Omega_{a_{10}\hat{e}}}{\sigma_{11}} \end{pmatrix} P^{-1} \left(1 \;\; -\frac{\Omega_{a_{10}e}}{\sigma}\right) \begin{pmatrix} \hat{e} \\ a_{10} \end{pmatrix}$$

$$+ \frac{1}{\sigma_{11}} \sum_{t=1-m}^{0} \hat{a}^2_{1t} + \sum_{t=1}^{N} (\hat{a}_{1t}, \hat{a}_{2t}) \, \Delta^{-1} \begin{pmatrix} \hat{a}_{1t} \\ \hat{a}_{2t} \end{pmatrix} \quad (3.3.1.3)$$

Now for large values of m the first term of this expression can be expressed as $\zeta' P^{-1} \zeta$ where

$$\zeta = \hat{e}_1 - \frac{\Omega_{ea_{10}}}{\sigma_{11}} \hat{a}_{10}$$

$$= \left\{ e_1', \; e_2 - \frac{\sigma_{21}}{\sigma_{11}} (\hat{a}_{11-q}, \ldots, \hat{a}_{10}, \Psi_2(B)\hat{a}_{11-p}, \ldots, \Psi_2(B)\hat{a}_{10})' \right\}' \quad (3.3.1.4)$$

3.3.2 Calculation of the Starting Values.

Even though the vector of initial values $\hat{e}$ can be calculated as:

$$\hat{e} = -(\Lambda'\Omega^{-1}\Lambda)^{-1}\Lambda'\Omega^{-1} \psi Z$$

it would be useful to have an alternative algorithm to obtain the vector $\hat{e}$ which could be more efficient, in particular, when dealing with seasonal or large order models. In this section, the back forecasting method of Box and Jenkins (1976, Ch. 7) for univariate

ARMA models is extended to obtain estimates of the initial values of CARMA models.

It should be observed that the backward and forward representations of the CARMA models do not have in general the same parameters as is the case for univariate ARMA models. Futhermore, for the CARMA model the parameter matrices of the forward representation are in general non-diagonal. Consider, for example, the bivariate CAR(1) model. Following Whittle (1963), the parameter matrices of the backward model ϕ_B and of the forward model ϕ_F can be obtained from:

$$\Gamma_1 = \phi_B \Gamma_o$$
$$\Gamma_{\cdot 1} = \Gamma_o \phi_F$$

where $\Gamma_k = \langle \underset{\sim}{Z}_t \; \underset{\sim}{Z}'_{t+k} \rangle$. After solving for ϕ_F, the following expression is obtained:

$$\phi_F = \Gamma_o^{-1} \; \phi_B \; \Gamma_o$$

For the bivariate CAR (1) model, ϕ_B = Diag (ϕ_1, ϕ_2) and $\Delta = (\sigma_{ij})$, so that ϕ_F is given by:

$$\phi_F = \phi_B + \frac{(\phi_1-\phi_2)}{(1-\rho_\phi^{\,2})}\begin{pmatrix} \rho_\phi^{\,2} & \rho_\phi\sqrt{\dfrac{\sigma_{22}(1-\phi_1^{\,2})}{\sigma_{11}(1-\phi_2^{\,2})}} \\ -\rho_\phi\sqrt{\dfrac{\sigma_{11}(1-\phi_2^{\,2})}{\sigma_{22}(1-\phi_1^{\,2})}} & \rho_\phi^{\,2} \end{pmatrix}$$

where

$$\rho_\phi^{\,2} = \rho^4\,\frac{(1-\phi_1^{\,2})(1-\phi_2^{\,2})}{(1-\phi_1\phi_2)^2} \qquad \text{and} \qquad \rho^2 = \frac{\sigma_{12}^{\,2}}{\sigma_{11}\sigma_{22}}$$

Therefore, the backforecasting technique can not be applied directly to the CARMA model.

In order to apply the backforecasting technique to the CARMA model consider the modified Cholesky decomposition of the variance covariance matrix Δ i.e. let $\Delta = LVL'$ where L is the lower triangular matrix given by:

$$L = \begin{pmatrix} 1 & 0 \\ \sigma_{12}/\sigma_{11} & 1 \end{pmatrix}$$

and V is the diagonal matrix given by $V = \text{Diag}\,(\sigma_{11}, \sigma_{22} - \sigma_{12}^{\,2}/\sigma_{11}) = \text{Diag}\,(\sigma_{11}, \sigma_{2:1})$. The CARMA model can be expressed as:

$$\begin{pmatrix} \phi_1(B)Z_{1t} \\ \\ \phi_2(B)Z_{2t} \end{pmatrix} = \begin{pmatrix} \theta_1(B) & 0 \\ \\ 0 & \theta_2(B) \end{pmatrix} L \cdot L^{-1} \begin{pmatrix} a_{1t} \\ \\ a_{2t} \end{pmatrix}$$

$$= \begin{pmatrix} \theta_1(B) & \\ \\ \sigma_{12}\theta_2(B)/\sigma_{11} & \theta_2(B) \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \\ \varepsilon_{2t} \end{pmatrix}$$

where

$$\underset{\sim}{\varepsilon}_t = L^{-1}\underset{\sim}{a}_t = (a_{1t}, a_{2t} - \sigma_{12}a_{1t}/\sigma_{11})'$$

Let
$$X_t = Z_{2t} - \frac{\sigma_{12}}{\sigma_{11}} \frac{\theta_2(B)}{\phi_2(B)} a_{1t}$$

$$= Z_{2t} - \sigma_{12}\psi_2(B)\, a_{1t}/\sigma_{11}$$

The CARMA model can then be written as two independent series:

$$\phi_1(B)Z_{1t} = \theta_1(B)\, a_{1t}$$
$$\phi_2(B)X_t = \theta_2(B)\, \varepsilon_{2t}$$

where $\varepsilon_{2t} = a_{2t} - \sigma_{12}\, a_{1t}/\sigma_{11}$ and $\mathrm{Var}(\varepsilon_{2t}) = \sigma_{2:1}$.

The iterative backforecasting algorithm of McLeod and Sales (1983) can be applied to each one of these models to obtain the estimated innovations $\hat{a}_1$, $\hat{\varepsilon}_2$ and the initial values $\hat{e}_1$ and $\hat{\xi}$, say, of the series Z_{1t} and χ_t. The initial values for Z_{2t} are easily obtained from X_t as

$$Z_{2t} = \chi_t + \sigma_{12}\,\psi_2(B)\,\hat{a}_{1t}/\sigma_{11}$$

and the innovations for the series Z_{Zt} are obtained as:

$$\hat{a}_{zt} = \hat{\varepsilon}_{zt} + \sigma_{12}\,\hat{a}_{1t}/\sigma_{11}$$

Although χ_t is not directly observable, it can be readily obtained as:

$$\chi_t = Z_t - \sigma_{12}\,Y_t/\sigma_{11}$$

where

$$\phi_2(B)Y_t = \theta_2'(B)\hat{a}_t$$

The values of Y_t can be calculated using for example the algorithm of McLeod and Sales. Another possibility is to use a finite series approximation for $\psi_2(B) = \sum_{k=0}^{m} \psi_{2K} B^K$ say to obtain the initial values of Y_t as

$$Y_t = \sum_{k=0}^{m} \psi_{2K}\hat{a}_{1t-k} \qquad t = 1 - p, \ldots, 0$$

and then obtain the values of Y_t $t = 1, \ldots, N$ recursively. It is interesting to observe that the series of innovations $\hat{a}_{2t}$ are not required to calculate the likelihood function. In fact, from equations (3.3.1.3) and (3.3.1.4)

$$S(\underset{\sim}{Z}_1\underset{\sim}{\hat{e}}) = \zeta' P^{-1}\zeta + \frac{1}{\sigma_{11}} \sum_{t=1-m}^{0} \hat{a}_{1t}^2 + \frac{1}{\sigma_{11}} \sum_{t=1}^{N} \underset{\sim}{\hat{a}}_t' L'^{-1} (L'\Delta^{-1}L) L^{-1} \underset{\sim}{\hat{a}}_t$$

$$= \zeta' P^{-1}\zeta + \frac{1}{\sigma_{11}} \sum_{t=1-m}^{0} \hat{a}_{1t}^2 + \frac{1}{\sigma_{11}} \sum_{t=1}^{N} \hat{a}_{1t}^2 + \frac{1}{\sigma_{2:1}} \sum_{t=1}^{N} \hat{\varepsilon}_t^2$$

(3.3.2.1)

where

$$\zeta' = (\hat{a}_{1-m-q}, \ldots, \hat{a}_{1-m}, \hat{Z}_{1-m-p}, \ldots, \hat{Z}_{1-m}, \hat{\varepsilon}_{2\,1-q}, \ldots, \hat{\varepsilon}_{20}, \hat{X}_{1-p}, \ldots, \hat{X}_o)$$

When the process is MA the expression $\zeta' P^{-1}\zeta$ is given by:

$$\zeta' P^{-1}\zeta = \frac{1}{\sigma_{11}} \sum_{t=-(m+q)}^{-m} a_{1t}^2 + \frac{1}{\sigma_{2:1}} \sum_{t=1-q}^{0} \varepsilon_{2t}^2 \qquad (3.3.2.2)$$

Because the general CARMA(p,q) model can be approximated by a CMA process of order Q, say; the above result suggests an approximation of the term $\zeta' P^{-1}\zeta$ by

$$\zeta' P^{-1}\zeta = \frac{1}{\sigma_{11}} \sum_{t=-(m+Q)}^{-m} \hat{a}_{1t}^2 + \frac{1}{\sigma_{2:1}} \sum_{t=1-Q}^{0} \hat{\varepsilon}_{2t}^2$$

This approximation avoids the inversion of P which in some cases may be quite laborious; the additional values for $\hat{a}_{1t}$ and $\hat{\varepsilon}_t$ are readily available from lthe backforecasting algorithm (see McLeod and Sales, 1983).

3.3.3 Calculation of the Covariance Determinant

The calculation of the term $|\Omega| \cdot |\Lambda'\Omega^{-1}\Lambda|$ of equation (3.2.4) is now considered. The inclusion of this term in the likelihood function improves the small sample properties of the parameter estimators, particularly in models with moving average operators having roots close to the unit circle (Hillmer and Tiao, 1979).

Because of the normality, stationarity and invertibility conditions, the joint distribution of $\{Z_{1t}\}$ t=1-m,...,N and $\{Z_{zt}\}$ t=1,...N is normal so that the likelihood function can also be expressed as

$$L(\underset{\sim}{\beta},\Delta|\underset{\sim}{Z}) = (2\pi)^{-\frac{2N+m}{2}} |\Gamma|^{-\frac{1}{2}} \{\exp -\tfrac{1}{2}(\underset{\sim}{Z}'\Gamma^{-1}\underset{\sim}{Z})\}$$

where Γ is the variance covariance matrix of $\underset{\sim}{Z}$. Comparing this expression with equation (3.2.4) it follows that

$$|\Omega| \; |\Lambda'\Omega^{-1}\Lambda| \doteq |\Gamma|$$

the determinant of the covariance matrix. The calculation of this determinant may be quite laborious so that it would be desirable to obtain an adequate approximation which is computationally attractive.

The variance covariance is given by:

$$\Gamma = \begin{pmatrix} \sigma_{11}\Gamma_{11}(i-j) & \sigma_{12}\Gamma_{12}(i-j) \\ \sigma_{21}\Gamma_{21}(i-j) & \sigma_{22}\Gamma_{22}(i-j) \end{pmatrix} \begin{matrix} m+N \\ N \end{matrix}$$

$$\qquad\qquad m+N \qquad\qquad N$$

where

$$\sigma_{gh}\Gamma_{gh}(i-j) = \langle Z_{gt-1} \; Z_{ht-j} \rangle$$

$$= \sigma_{gh} \sum_{k=o}^{\infty} \psi gk \quad \psi hk+(i-j)$$

$$\phi_h(B)\,\psi_h(B) = \theta_h(B) \qquad h = 1,2$$

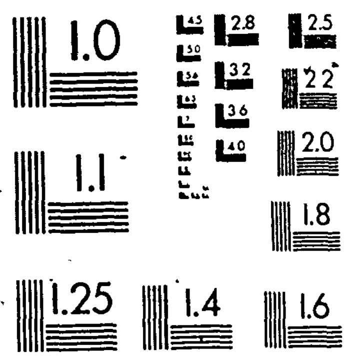
and it is understood that $\psi_{h-k} = 0$ for k>0.

Define the matrices $A_{(m+N)xR}$ and B_{NxR} with R>N+m as

$$A_{(m+N)XR} = \begin{pmatrix} \psi_{10} & \psi_{11} \cdots & \psi_{1R} \\ & \psi_{10} \cdots & \psi_{1R-1} \\ & \ddots & \vdots \\ & \psi_{10} & \psi_{1R-(m+N)} \end{pmatrix}$$

$$B_{NXR} = \begin{pmatrix} \psi_{20} & \psi_{21} \cdots & \psi_{2R} \\ & \psi_{20} & \psi_{2R-1} \\ & \ddots & \vdots \\ & \psi_{20} & \psi_{2R-N} \end{pmatrix}$$

2



It is easy to see that Γ can be approximated by

$$\Gamma = \begin{bmatrix} \sigma_{11}AA' & \sigma_{12}AB' \\ \sigma_{21}BA' & \sigma_{22}BB' \end{bmatrix} , \qquad (3.3.3.1)$$

The error order of this approximation is $0(\lambda^{R-N-m})$ where $|\lambda|<1$ and corresponds to the largest root of the polynomial $\phi_h(B) = 0$, h = 1,2. Now using a well known result for the determinant of a partitioned matrix it follows that

$$|\Gamma| = |\sigma_{11}\Gamma_{11}|\cdot|\sigma_{22}\Gamma_{22} - (\sigma_{12}^2/\sigma_{11})\cdot\Gamma_{21}\Gamma_{11}\ \Gamma_{12}| \qquad (3.3.3.2)$$

Using the approximation of equation (3.3.3.1)

$$\sigma_{22}\Gamma_{22} - (\sigma_{12}/\sigma_{11})\Gamma_{21}\Gamma_{11}^{-1}\Gamma_{12}$$
$$= \sigma_{22}B(1_{RXR} - \rho^2 A'(AA')^{-1}A)B' + 0(\lambda^{R-N-M})$$

where

$$\rho^2 = \sigma_{21}^2/\sigma_{11}\sigma_{22}$$

It can be shown that the determinant of this last expression is given by:

$$|\sigma_{22}B(1_{RXR}-\rho^2 A'(AA')^{-1}A)\ B'| = \sigma_{22}^N\cdot(1-\rho^2)^{N-1}|\ BB'|\cdot(1-\rho^2 a)$$
$$= \sigma_{22}^N(1-\rho^2)^N\cdot|\Gamma_{22}|$$

where $0 < a < 1$ and depends on m, N and R.

The determinant of Γ can then be approximated by

$$|\Gamma| = \sigma_{11}^{N+m} \sigma_{22}^{N} (1 - \rho^2)^N \cdot |\Gamma_{11}| \cdot |\Gamma_{1}| \qquad (3.3.3.3)$$
$$= |\Delta|^N \sigma_{11}^{m} |\Gamma_{11}| \cdot |\Gamma_{22}|$$

The error introduced in the above approximation is negligible for moderate to large values of $N + m$. It can be shown that the exact expression of $|\Gamma|$ for the bivariate CAR(1) is given by

$$|\Gamma| = |\Gamma_{11}| \cdot |\Gamma_{22}| \cdot \sigma^{N+m} \sigma_{22}^{N} (1 - \rho^2)^{N-1} \cdot \left(1 - \frac{\rho^2 \phi^2 (\phi_1 - \phi_2) \cdot (\phi_1 - \phi_2^{2m-1})}{(1-\rho^2) \cdot (1 - \phi_1 \phi_2)^2}\right)$$

Taking logarithms it is readily seen that the logarithm of the last factor is $0(1)$ whereas the logarithm of the rest of the expression is $0(N + m)$.

3.3.4 Algorithm to Calculate the Likelihood Function

The algorithm given in this section calculates the approximate likelihood function of the CARMA model when the series have different sample sizes, using the simplifications discussed in the the previous sections. It can be shown from equations (3.3.2.1), (3.3.2.2) and (3.3.3.3) that apart from an arbitrary constant the logarithm of the likelihood function (equation 3.2.4) maximized over Δ can be written as:

$$\log L(\underset{\sim}{\beta}) = -\frac{(N+m)}{2}\log(S_{m1}/(N+m)) - \frac{N}{2}\log(S_{m2}/N) \qquad (3.3.4.1)$$

where S_{m1} and S_{m2} are the modified sums of squares defined by:

$$S_{m1} = S_2[M_{1(N+m)}(p,q)]^{-1/N+m} \qquad (3.3.4.2)$$

$$S_{m2} = S_1[M_{2N}(p,q)]^{-1/N} \qquad (3.3.4.3)$$

S_1 and S_2 representing the unconditional sum of squares approximated by

$$S_1 = \sum_{t=-(M+Q)}^{N} \hat{a}_{1t}^2$$

$$S_2 = \sum_{t=1-Q}^{N} \hat{\varepsilon}_{zt}^2$$

where ε_{zt} is the auxiliary series defined by

$$\phi_2(B)\{Z_{ht} - \frac{\sigma_{12}}{\sigma_{11}} \frac{\theta_2(B)}{\phi_2(B)} a_{1t}\} = \theta_2(B)\,\varepsilon_{2t}$$

The terms S_{m1}, and S_{m2} can easily be calculated using the subroutine SARMA given by McLeod and Sales (1983). In order to obtain the auxiliary series ε_{2t}, initial estimated values for σ_{12} and σ_{22} are required. These can be obtained using equations (3.4.4.2) and (3.4.4.3) and the "univariate" residuals $[a_{ht}]$ calculated from:

$$\phi_h(B)\, Z_{ht} = \theta_h(B)[a_{ht}]$$

log L($\underset{\sim}{\beta}$) of equation (3.3.4.1) can be calculated using the following algorithm:

1. Using $\{Z_{1t}\}$ t = 1 - m, ... ,N and $\underset{\sim}{\beta}_1 = (\phi_{11},\ldots,\phi_{1p},\theta_{11},\ldots,\theta_{1q})$ obtain the residuals a_{1t} and S_{m1} (equation 3.3.4.2) using the algorithm SARMAs given by McLeod and Sales (1983).

2. Using$\{Z_{zt}\}$ t=1,...,N and $\underset{\sim}{\beta}_2 = (\phi_{21},\ldots,\phi_{2p},\theta_{21},\ldots\theta_{2q})$ obtain the residuals $[a_{2t}]$. These can be calculated using the subroutine SARMAS.

3. Calculate initial estimated values for σ_{11} and σ_{12} using equations (3.4.4.2) and (3.4.4.3) and the residuals $\hat{a}_{1t}$ and $[a_{zt}]$.

4. Calculate the auxiliary series Y_t given by

$$\theta_2(B)\, \tilde{a}_{1t} = \phi_2(B) Y_t$$

The series can be obtained using subroutine SARMA.

5. Using$\{Z_{zt} - (\sigma_{21}/\sigma_{11})Y_t\}$ t = 1, ... ,N and $\underset{\sim}{\beta}_2$ obtain the auxiliary residuals ε_{2t} and S_{m2} (equation 3.3.4.3). These can be calculated using subroutine SARMA.

6. Calculate log L($\underset{\sim}{\beta}$) using equation (3.3.4.1).

3.4 LARGE SAMPLE PROPERTIES OF THE ESTIMATORS

In this section the asymptotic properties of the estimators ,
obtained by maximizing the likelihood of equation (3.2.4) are considered. It should be noted that for large N+m the likelihood can be approximated by (see Hillmer and Tiao, 1979):

$$L(\dot{\underset{\sim}{\beta}},\dot{\Delta}) = (2\pi)^{-(N+\frac{m}{2})}\dot{\sigma}_{11}^{-m/2}|\dot{\Delta}|^{-\frac{m}{2}}.\exp\{-\frac{1}{2\sigma_{11}}\sum_{t=1-m}^{0}\dot{a}_{1t}^{2} - \tfrac{1}{2}\sum_{t=1}^{N}\dot{\underset{\sim}{a}}_t'\dot{\Delta}^{-1}\dot{\underset{\sim}{a}}_t\} \qquad (3.4.1)$$

3.4.1 Distribution of $\tilde{\underset{\sim}{\beta}}$

From equation 3.4.1, the log-likelihood is given for large N+m by:

$$\ell(\dot{\underset{\sim}{\beta}},\dot{\Delta}) = -(\frac{N+m}{2})\log 2\pi-\frac{m}{2}\log\dot{\sigma}_{11}-\frac{N}{2}\log|\dot{\Delta}|-\frac{1}{2\sigma_{11}}\sum_{t=1-m}^{0}\dot{a}_{1t}^{2} - \frac{1}{2}\sum_{t=1}^{N}\dot{\underset{\sim}{a}}_{1t}'\dot{\Delta}^{-1}\dot{\underset{\sim}{a}}_t \qquad (3.4.1.1)$$

The first derivatives of $\ell(\dot{\underset{\sim}{\beta}},\dot{\Delta})$ with respect to $\underset{\sim}{\beta}$ are given by:

$$\frac{\partial\ell}{\partial\beta_{1j}} = -\frac{1}{\sigma_{11}}\sum_{t=1-m}^{0} a_{1t}W_{1t-j} - \sum_{t=1}^{N}\underset{\sim}{a}_t'\cdot\Delta^{-1}\cdot\begin{pmatrix} W_{1t-j} \\ 0 \end{pmatrix}$$

$$\frac{\partial\ell}{\partial\beta_{2j}} = -\sum_{t=1}^{N}\underset{\sim}{a}_t'\,\Delta^{-1}\begin{pmatrix} 0 \\ W_{2t-j} \end{pmatrix}$$

where W stands for V or U depending on whether β is ϕ or θ and the auxiliary series V and U are defined by:

$$\phi_h(B)V_{ht} = -a_{ht} \qquad h = 1,2$$

$$\theta_h(B)U_{ht} = a_{ht} \qquad h = 1,2$$

The second derivatives of $\ell(\beta,\Delta)$ with respect to β are given by:

$$-\frac{\partial^2 \ell}{\partial \beta_{1i}\partial \beta_{1j}} = \frac{1}{\sigma_{11}} \sum_{t=1-m}^{0} \left(a_{1t} \frac{\partial W_{1t-j}}{\partial \beta_{1i}} + W_{1t-i}^{(i)} W_{1t-j}^{(j)} \right)$$

$$+ \sum_{t=1}^{N} \left\{ a_t' \Delta^{-1} \begin{bmatrix} \frac{\partial W_{1t-j}}{\partial \beta 1j} \\ 0 \end{bmatrix} + \sigma^{11} W_{1t-i}^{(i)} W_{1t-j}^{(j)} \right\}$$

$$= \left(\frac{m}{\sigma_{11}} + N\sigma^{11}\right) \gamma_{W_1^{(i)} W_1^{(j)}}(i-j) + O_p(\{m+N\}^{\frac{1}{2}})$$

$$-\frac{\partial^2 \ell}{\partial \beta_{2i}\partial \beta_{1j}} = \sum_{t=1}^{N} \sigma^{21} W_{2t-i} W_{1t-j}$$

$$= N \sigma^{12} \gamma_{W_2 W_1}(i-j) + O_p(\sqrt{N})$$

$$\frac{\partial^2 \ell}{\partial \beta_{2i}\partial \beta_{2j}} = \sum_{t=1}^{N} \left\{ a_t' \Delta^{-1} \begin{bmatrix} 0 \\ \frac{\partial W_{2t-j}^{(j)}}{\partial \beta_{2i}} \end{bmatrix} + W_{2t-i}^{(i)} W_{2t-j}^{(j)} \right\}$$

$$= N \sigma^{22} \gamma_{W_2^{(i)} W_2^{(j)}}(i-j) + O_p(\sqrt{N})$$

It is easy to see that $\mathrm{Var}(\partial \ell/\partial\beta_{1i}\partial\beta_{2j}) = 0(m + N)$ and $\mathrm{Var}(\partial\ell/\partial\beta_{gi}\partial\beta_{hj}) = 0(N)$ for $g = 2$, $h = 1.2$, justifying the second expression for each term.

The following Lemma gives the asymptotic distribution of $\underset{\sim}{\beta}$.

Lemma 3.4.1.1 The asymptotic distribution of $\sqrt{N}(\tilde{\underset{\sim}{\beta}} - \underset{\sim}{\beta})$ is multivariate normal with mean zero and variance covariance.

$$V_{\tilde{\beta}} = \begin{pmatrix} (\sigma^{11} + \frac{m_N}{\sigma_{11}}) I_{11} & \sigma^{12} I_{12} \\ \text{symm.} & \sigma^{22} I_{22} \end{pmatrix}^{-1}$$

where the I_{gh} are defined by Lemma 2.2.1.
and $m_N = \lim_{N\to\infty} m/N$. In practical applications it can be assumed that $m_N = m/N$.

Proof: Under the assumptions of normality, stationarity and invertibility the log-likelihood satisfies the usual regularity conditions. It follows from taking a Taylor expansion of $\partial\ell/\partial\underset{\sim}{\beta}$ about $\underset{\sim}{\beta}$, the true value, and evaluating at $\tilde{\underset{\sim}{\beta}}$ that

$$0 = \frac{\partial \ell}{\partial \beta} + \frac{\partial^2 \ell}{\partial\beta\partial\beta'} (\tilde{\underset{\sim}{\beta}} - \underset{\sim}{\beta}) + o_p(1)$$

Further, from the derivations given before the Lemma it follows that

$$\sqrt{N}(\tilde{\beta} - \beta) = V_{\tilde{\beta}} \cdot \frac{1}{N} \frac{\partial \ell}{\partial \beta} + O_p(1/N)$$

Now it is easy to see that apart from terms $O_p(1/N)$, linear combinations of $(\tilde{\beta} - \beta)$ are an average of martingale differences with bounded variances. Normality then follows from the Martingale central limit theorem (Billingsley, 1961).

Lemma 3.4.1.2 The matrix $V_{\hat{\beta}} - V_{\tilde{\beta}}$ is positive semidefinite where $V_{\hat{\beta}}$ is the variance covariance of the estimator of β using only the N pairs of observations $\{Z_{ht}\}$ t = 1, ... ,N; h = 1,2; and given by Lemma 2.2.2.

Proof It follows immediately since $V_{\tilde{\beta}} = V_{\hat{\beta}}^{-1} + m_N \frac{(1-\rho^2)}{\sigma_{11}} \begin{pmatrix} I_{11} & 0 \\ 0 & 0 \end{pmatrix}$

where I_{11} is a positive semidefinite matrix.

3.4.2 The CAR(1) Model

The bivariate CAR(1) is used as an example to study the possible gain in efficiency of the estimators obtained using all the available information, $\tilde{\beta}$, compared to the estimators obtained using only part of the data, $\hat{\beta}$ or $\bar{\beta}$ (see Chapter 2).

The CAR(1) model is given by

$$Z_{ht} = \phi_h Z_{ht-1} + a_{ht} \qquad h = 1,2$$

$$a_t = (a_{1t}, a_{2t})' \sim NID(0, \quad)$$

$$\Delta = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \text{symm.} & \sigma_2^2 \end{bmatrix}$$

From the results of Section 2.3 it follows that

$$V_{\hat{\beta}} = \frac{(1-\rho^2)}{N(1-\rho_\phi^2)} \begin{bmatrix} (1-\phi_1^2) & \rho^2(1-\phi_1^2)(1-\phi_2^2)/(1-\phi_1\phi_2) \\ \text{symm.} & (1-\phi_2^2) \end{bmatrix}$$

where $\rho_\phi^2 = \rho^4(1-\phi_1^2)(1-\phi_2^2)/(1-\phi_1\phi_2) = a\rho^4$ say, and

$$V_{\tilde{\beta}} = \frac{(1-\rho^2)}{N} \begin{bmatrix} (1+m_N(1-\rho^2))/(1-\phi_1^2) & -\rho^2/(1-\phi_1\phi_2) \\ \text{symm.} & 1/(1-\phi_2^2) \end{bmatrix}^{-1}$$

$$= \frac{(1-\rho^2)}{N(1+m_N(1-\rho^2)-\rho_\phi^2)} \begin{bmatrix} (1-\phi_1^2) & \rho^2(1-\phi^2)(1-\phi^2)/(1-\phi_1\phi_2) \\ \text{symm.} & (1+m_N(1-\rho^2))(1-\phi_2^2) \end{bmatrix}$$

The efficiency of $\hat{\phi}_2$ relative to $\tilde{\phi}_2$ is given by:

$$\text{eff}(\phi_2) = \frac{V(\tilde{\phi}_2)}{V(\hat{\phi}_2)} = \frac{(1-a\rho^4)(1+m_N(1-\rho^2))}{(1+m_N(1-\rho^2) - a\rho^4)}$$

which decreases asymptotically to $(1 - a\rho^4)$ as m_N increases. If $\phi_1 = \phi_2$, for example, then $a = 1$ and $\text{eff}(\phi_2) \to 1 - \rho^4$. As a decreases to zero the values of $\text{eff}(\phi_2) \to 1$ so that no gain in efficiency is obtained in the estimation of ϕ_2.

To study the possible gain in efficiency in the estimation of ϕ_1 two cases are considered. In the first case, it is assumed that $m_N < \rho^2(1 - a\rho^2)/(1 - \rho^2)$ so that the joint estimator of ϕ_1 and ϕ_2, $\hat{\phi}_1$, say, using the N common pairs of observations of the series has smaller (asymptotic) variance than the univariate estimator, $\tilde{\phi}_1$, say, obtained using only the m + N observations of the Z_{1t}. The relative efficiency of $\hat{\phi}_1$ respect to $\tilde{\phi}_1$ in this case is given by:

$$\text{eff}(\phi_1) = \frac{V(\tilde{\phi}_1)}{V(\hat{\phi}_1)} = \frac{(1 - a\rho^4)}{1 + m_N (1 - \rho^2) - a\rho^4}$$

In the second case, i.e., $m_N > \rho^2(1 - a\rho^2)/(1 - \rho^2)$, the asymptotic variances of $\bar{\phi}_1$ and $\tilde{\phi}_1$ are compared. The relative efficiency of $\bar{\phi}_1$ respect to $\tilde{\phi}_1$ is given by

$$\text{eff}(\phi_1) = \frac{V(\tilde{\phi}_1)}{V(\bar{\phi}_1)} = \frac{1 + m_N (1 - \rho^2) - \rho^2}{1 + m_N (1 - \rho^2) - a\rho^4}$$

As can be expected, the value of $\text{eff}(\phi_1)$ tends to 1 as m_N increases. Table 3.1 gives some numerical results of the efficiency values that can be obtained when all the information is used during the estimation. As observed in the Table, the gain in efficiency may be quite substantial.

TABLE 3.1

EFFICIENCY VALUES FOR $\bar{\phi}$ RELATIVE TO $\tilde{\phi}$ FOR A CAR(1) MODEL

with Unequal Sample Sizes Relative Efficiency for ϕ_1

ρ = 0.30

MN	A : .0	.2	.4	.6	.8	1.0
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.08	0.931	0.930	0.930	0.930	0.930	0.930
0.20	0.924	0.925	0.926	0.928	0.929	0.930
0.40	0.934	0.935	0.936	0.937	0.938	0.940
0.60	0.942	0.943	0.944	0.945	0.946	0.947
0.80	0.948	0.949	0.950	0.951	0.951	0.952
1.00	0.953	0.954	0.954	0.955	0.956	0.957
2.00	0.968	0.969	0.969	0.970	0.970	0.971
3.00	0.976	0.976	0.977	0.977	0.978	0.978
4.00	0.981	0.981	0.981	0.982	0.982	0.982

ρ = 0.60

MN						
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.20	0.887	0.884	0.881	0.878	0.875	0.872
0.26	0.871	0.865	0.865	0.852	0.845	0.837
0.40	0.713	0.728	0.744	0.778	0.778	0.795
0.60	0.740	0.754	0.769	0.800	0.800	0.816
0.80	0.762	0.775	0.789	0.813	0.818	0.833
1.00	0.780	0.793	0.806	0.833	0.833	0.847
2.00	0.842	0.852	0.862	0.882	0.882	0.893
3.00	0.877	0.885	0.893	0.909	0.909	0.917
4.00	0.899	0.905	0.912	0.926	0.926	0.933

ρ = 0.90

MN						
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.20	0.972	0.963	0.951	0.941	0.926	0.900
0.40	0.220	0.251	0.949	0.926	0.885	0.819
0.45	0.247	0.282	0.327	0.926	0.483	0.802
0.60	0.273	0.309	0.357	0.422	0.516	0.664
0.80	0.297	0.335	0.384	0.451	0.545	0.690
1.00	0.319	0.359	0.410	0.477	0.571	0.712
2.00	0.413	0.456	0.510	0.578	0.667	0.787
3.00	0.484	0.528	0.581	0.646	0.727	0.832
4.00	0.540	0.583	0.634	0.695	0.769	0.861

TABLE 3.1 (continued)

Relative Efficiency for $\phi 2$

$\rho = 0.30$

MN	A: .0	.2	.4	.6	.8	1.0
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.08	1.000	1.000	1.000	1.000	1.000	0.999
0.20	1.000	1.000	0.999	0.999	0.999	0.999
0.40	1.000	1.000	0.999	0.999	0.998	0.998
0.60	1.000	0.999	0.999	0.998	0.998	0.997
0.80	1.000	0.999	0.998	0.998	0.997	0.997
1.00	1.000	0.999	0.998	0.998	0.997	0.996
2.00	1.000	0.999	0.998	0.997	0.996	0.995
3.00	1.000	0.999	0.998	0.996	0.995	0.994
4.00	1.000	0.999	0.997	0.996	0.995	0.994

$\rho = 0.60$

MN						
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.20	1.000	0.997	0.994	0.991	0.987	0.983
0.26	1.000	0.997	0.993	0.989	0.984	0.979
0.40	1.000	0.995	0.989	0.983	0.977	0.971
0.60	1.000	0.993	0.985	0.977	0.969	0.960
0.80	1.000	0.991	0.982	0.972	0.962	0.952
1.00	1.000	0.990	0.979	0.968	0.957	0.945
2.00	1.000	0.985	0.970	0.955	0.939	0.923
3.00	1.000	0.983	0.965	0.947	0.929	0.911
4.00	1.000	0.981	0.962	0.943	0.923	0.903

$\rho = 0.90$

MN						
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.20	1.000	0.995	0.987	0.977	0.961	0.935
0.40	1.000	0.995	0.987	0.971	0.940	0.881
0.45	1.000	0.989	0.975	0.956	0.928	0.870
0.60	1.000	0.985	0.965	0.938	0.898	0.837
0.80	1.000	0.980	0.955	0.921	0.873	0.799
1.00	1.000	0.976	0.946	0.906	0.850	0.767
2.00	1.000	0.960	0.911	0.848	0.767	0.656
3.00	1.000	0.948	0.886	0.809	0.714	0.591
4.00	1.000	0.939	0.867	0.781	0.677	0.548

$A = (1-\phi_1^2)(1-\phi_2^2) / (1-\phi_1 \phi_2)^2$

$MN = \lim_{N \to \infty} M/N$

3.4.3 Distribution of $\tilde{\Delta}$ and $\tilde{\mu}$

In this section, the asymptotic distribution of $\tilde{\Delta}$ and $\tilde{\mu}$ obtained by maximizing equation (3.2.4) is given. Throughout this section $\tilde{\Delta}$ denotes the vector $\tilde{\Delta} = (\tilde{\sigma}_{11}, \tilde{\sigma}_{21}, \tilde{\sigma}_{12}, \tilde{\sigma}_{22})$. The following lemma gives the distribution of $\tilde{\mu}$.

Lemma 3.4.3.1 The asymptotic distribution of $\sqrt{N}(\tilde{\mu}-\mu)$ is multivariate normal with mean zero and variance covariance matrix

$$V_{\beta} = 1/(1+m_N)\begin{bmatrix} \sigma_{11}/c_1^2 & \sigma_{11}/c_1c_2 \\ \text{symm.} & (1+m_N(1-\rho^2))\sigma_{22}/c_2^2 \end{bmatrix}$$

where $m_N = \lim m/N$ and $C_h = \phi_h(1)/\theta_h(1) \quad h = 1,2$

Furthermore, it is independent of the asymptotic distribution of $\sqrt{N}(\tilde{\beta}-\beta)$.

Proof The proof is similar to the one given for theorem 2.2.2. From equations (3.2.1) and (3.4.1) the second derivatives of $\ell(\hat{\beta}, \hat{\Delta}, \hat{\mu})$ with respect to μ are given by

$$\frac{\partial^2 \ell}{\partial \mu_1^2} = - Nc_1^2\,(\sigma^{11} + m_N/\sigma_{11})$$

$$\frac{\partial^2 \ell}{\partial \mu_2 \partial \mu_h} = - N\,\sigma^{2h} c_2 c_h \qquad h = 1,2$$

As in theorem 2.2.2 V_{μ} is given by:

$$V_{\hat{\mu}} = \left\{ \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} \sigma^{11} + m_N/\sigma_{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{pmatrix} \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \right\}^{-1}$$

$$= \frac{1}{(1+m_N)} \begin{bmatrix} \sigma_{11}/c_1^2 & \sigma_{12}/c_1c_2 \\ \text{symm.} & (1+m_N(1-\rho^2))\sigma_{22}/c_2^2 \end{bmatrix}$$

The rest of the proof follows the same lines as the proof for theorem 2.2.2., so that it can be omitted for brevity.

It is interesting to compare the asymptotic variance of $\tilde{\mu}_2$ with that of $\hat{\mu}_2$ or $\bar{\mu}_2$ given by Lemma 2.2.2 and obtained using only the N pairs of common observations. The relative efficiency of $\hat{u}_2$ with respect to $\tilde{\mu}_2$ is given by:

$$\mathrm{eff}(\hat{\mu}_2) = V(\tilde{\mu}_2)/V(\hat{\mu}_2) \doteq 1 - m\rho^2/(1+m_N)$$

This shows that, in general, there is a gain in efficiency in the estimation of $\hat{\mu}_2$ when all the available information is used. On the other hand, the variance of $\tilde{\mu}_1$ is the same as the asymptotic variance of $\bar{\mu}_1$ obtained using the m+N observations of Z_{1t}, so that no gain in efficiency is expected in the estimation of μ_1. The following Lemma gives the distribution of $\tilde{\Delta}$.

Lemma 3.4.3.2 The asymptotic distribution of $\sqrt{N}(\tilde{\Delta}-\Delta)$ is normal with mean zero and variance covariance $V_{\tilde{\Delta}}$ given by

$$V_{\tilde{\Delta}} = \Delta\otimes\Delta(1+P)\begin{pmatrix} \frac{1}{\{1+m_N\}} & -\frac{m_N}{\{1+m_N\}}\left(\frac{\sigma_{12}}{\sigma_{11}}\right) & -\frac{m_N}{\{1+m_N\}}\left(\frac{\sigma_{12}}{\sigma_{11}}\right) & -\frac{m_N}{\{1+m_N\}}\left(\frac{\sigma_{12}^2}{\sigma_{11}^2}\right) \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

where P is a permutation matrix given in Theorem 2.2.2 and 1_{4x4} is the identity matrix. Furthermore the distribution is independent of $\tilde{\beta}$ and $\tilde{\mu}$.

Proof: The proof is similar to the proof of theorem 2.2.2. In particular the variance matrix $V_{\tilde{\Delta}}$ is given by the inverse of the information matrix $I_\Delta = \lim_{N\to\infty} < -\frac{1}{N} \partial^2 \ell / \partial\Delta\partial\Delta >$ where ℓ is given by equation 3.4.1 and < > denotes expectation. Now

$$-\frac{1}{N} < \frac{\partial^2 \ell}{\partial \sigma_{11}^2} > = \frac{\sigma^{11}\sigma^{11}}{2} (1 + m_N (1-\rho^2)^2)$$

$$-\frac{1}{N} < \frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{11}} > = \frac{\sigma^{1j}\sigma^{1j}}{2} \qquad \text{for } (i,j) \neq (1,1)$$

$$-\frac{1}{N} < \frac{\partial^2 \ell}{\partial \sigma_{ij} \partial_{rs}} > = \frac{1}{2} \left(\frac{\sigma^{si}\sigma^{jr} + \sigma^{hi}\sigma^{js}}{2}\right) \quad (i,j)\neq(1,1) \text{ and } (r,s)\neq(1,1).$$

Therefore, the information matrix can be written as:

$$I_\Delta = \Delta^{-1} \otimes \Delta^{-1} \frac{(1+P)}{4} + \frac{m_N}{2} \begin{pmatrix} \sigma^{11}(1-\rho^2) \\ 0 \\ 0 \\ 0 \end{pmatrix} (\sigma^{11}(1-\rho^2), 0, 0, 0)$$

where P is an adequate permutation matrix.

From here it can be shown that $V_{\tilde{\Delta}}$ is then given by

$$V_{\tilde{\Delta}} = \Delta \otimes \Delta \, (1+p) \begin{pmatrix} \frac{1}{\{1+m_N\}} & -\frac{m_N}{\{1+m_N\}} \left(\frac{\sigma_{12}}{\sigma_{11}}\right) & -\frac{m_N}{\{1+m_N\}} \left(\frac{\sigma_{12}}{\sigma_{11}}\right) & -\frac{m_N}{\{1+m_n\}} \left(\frac{\sigma_{12}^2}{\sigma_{11}^2}\right) \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

The explicit expressions for the variances are:

$$N \cdot Var(\tilde{\sigma}_{11}) = 2\sigma_{11}^2/\{1+m_N\} \qquad (3.4.3.1)$$

$$N \, Var(\tilde{\sigma}_{22}) = 2\sigma_{22}^2 \, (1-m_N\rho^4/(1+m_N)) \qquad (3.4.3.1)$$

$$N \, Var(\tilde{\sigma}_{21}) = N \, Var(\tilde{\sigma}_{12}) = \sigma_{11}\sigma_{22}(1+\rho^2\{1-m_N\}/\{1+m_N\}) \qquad (3.4.3.1)$$

Normality and the independence of $\tilde{\mu}$ and $\tilde{\beta}$ is obtained as in theorem 2.2.2.

It is interesting to observe that the asymptotic variance of $\tilde{\sigma}_{11}$ (equation 3.4.3.1) is the same as the asymptotic variance of $\bar{\sigma}_{11}$ obtained using the m + N observations of Z_{1t}. On the other hand, it can be seen from equations 3.4.3.2 and 3.4.3.3 that there is a gain in efficiency of the estimators $\tilde{\sigma}_{22}$ and $\tilde{\sigma}_{21}$ compared with the estimators $\hat{\sigma}_{22}$ and $\hat{\sigma}_{21}$ obtained using the N pairs of common observations.

3.4.4. <u>On the Estimation of $\tilde{\beta}$ and $\tilde{\Delta}$</u>

As was mentioned before, for moderate or large sample size, i.e. moderate to large values of m+N, the estimators $\tilde{\beta}$ and $\tilde{\Delta}$ can be obtained by maximizing equation (3.4.1). To obtain the estimator for $\underset{\sim}{\beta}$ the following non-linear system of equations needs to be solved:

$$\partial \ell / \partial \underset{\sim}{\beta} = 0 .$$

An iterative procedure like that of Newton-Raphson is required to obtain $\tilde{\beta}$. In particular, $\bar{\beta}_1$ and $\bar{\beta}_2$, the estimators of β_1 and β_2 obtained using the N+m observations of Z_{1t} and the N observations of Z_{2t} respectively, can be used as initial values for the following algorithm:

$$\beta_{k+1} = \beta_k - V_{\hat{\beta}} \cdot \frac{1}{N} \frac{\partial \ell}{\partial \beta} \qquad (3.4.4.1)$$

where the last term is evaluated at $\beta = \beta_k$. It can be shown that just one iteration of equation (3.4.4.1) with $\bar{\beta}$ as initial values produces asymptotically efficient estimators.

The estimator for Δ is obtained by solving the equation:

$$0 = \frac{\partial \ell}{\partial \Delta} = -\frac{N}{2}\Delta^{-1} - \frac{m}{2\sigma_{11}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{t=1-m}^{0} \frac{a_{1t}}{2\sigma_{11}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\Delta^{-1} \left(\sum_{t=1}^{N} a_t a_t' \right) \Delta^{-1}$$

This system of equations can be solved explicitly as follows:

$$\tilde{\sigma}_{11} = (SS_{11} + S_{10}) / (N + m) \qquad (3.4.4.2)$$

$$\tilde{\sigma}_{j1} = \tilde{\sigma}_{1j} = SS_{1j} / (N + m - S_{10}/\tilde{\sigma}_{11}) \qquad j > 1 \qquad (3.4.4.3)$$

$$\tilde{\sigma}_{ij} = \frac{1}{N} \left(SS_{22} + \frac{\tilde{\sigma}_{i1}\tilde{\sigma}_{1j}}{\tilde{\sigma}_{11}} \left(\frac{S_{10}}{\tilde{\sigma}_{11}} - m \right) \right) \qquad i,j > 1 \qquad (3.4.4.4)$$

where $S_{10} = \sum_{t=1-m}^{0} a_{1t}^2$ and $SS = [SS_{ij}] = \sum_{t=1}^{N} a_t a_t'$

So, given $\bar{\underset{\sim}{\beta}} = (\bar{\underset{\sim}{\beta}}_1' \bar{\underset{\sim}{\beta}}_2')'$, initial estimators for $\tilde{\Delta}$ can be obtained using equations (3.4.4.2) to (3.4.4.4), replacing a_{ht} for $\bar{a}_{ht}$, the residuals being obtained from the univariate estimation.

3.5 DISTRIBUTION OF THE RESIDUAL AUTOCORRELATIONS

In this section the distribution of the residual autocorrelations when the series have unequal sample size is derived. The autocovariances $\dot{C}_{ij}$ and autocorrelations $\dot{r}_{ij}$ when $\dot{\beta}$ is used are defined as follows:

$$\dot{C}_{11}(\ell) = \sum_{t=1-m}^{N-\ell} \dot{a}_{1t}\,\dot{a}_{1t+\ell}/(N+m) \qquad l = 1, \ldots, M$$

$$\dot{C}_{gh}(\ell) = \sum_{t=1}^{N-\ell} \dot{a}_{gt}\,\dot{a}_{ht+\ell}/N \qquad (g, h) \neq (1, 1)$$

$$\dot{r}_{gh}(\ell) = \dot{C}_{gh}(\ell)/\sqrt{\tilde{\sigma}_{gg}\cdot\tilde{\sigma}_{hh}(o)} \qquad l = 1, \ldots, M$$

where σ_{hh} is given by equation (3.4.4.2) or (3.4.4.4)

$$\dot{\underset{\sim}{r}}_{ij}' = (\dot{r}_{ij}(1), \ldots, \dot{r}_{ij}(M))$$
$$\dot{\underset{\sim}{r}}' = (\dot{\underset{\sim}{r}}_{11}', \dot{\underset{\sim}{r}}_{21}', \dot{\underset{\sim}{r}}_{12}', \dot{\underset{\sim}{r}}_{22}')$$

It can be assumed, without loss of generality, that Δ is in correlation form. The following Lemma gives the distribution of $\underset{\sim}{r}$ when $\dot{\underset{\sim}{\beta}} = \underset{\sim}{\beta}$ the true parameter values.

Lemma 3.5.1 The joint distribution of $\sqrt{N}\underset{\sim}{r}$ is multivariate normal with mean zero and variance covariance Y given by:

$$
Y = \begin{bmatrix}
\sigma_{11}\sigma_{11}/\{1+m_N\} & \sigma_{11}\sigma_{12}/\{1+m_N\} & \sigma_{12}\sigma_{11}/\{1+m_N\} & \sigma_{12}\sigma_{12}\{1+m_N\} \\
\sigma_{11}\sigma_{12}/\{1+m_N\} & \sigma_{11}\sigma_{22} & \sigma_{12}\sigma_{21} & \sigma_{12}\sigma_{22} \\
\sigma_{21}\sigma_{11}/\{1+m_N\} & \sigma_{21}\sigma_{12} & \sigma_{22}\sigma_{11} & \sigma_{22}\sigma_{12} \\
\sigma_{21}\sigma_{21}/\{1+m_N\} & \sigma_{21}\sigma_{22} & \sigma_{22}\sigma_{21} & \sigma_{22}\sigma_{22}
\end{bmatrix} \otimes 1_M
$$

where 1_M is the identity matrix and $m_N = \lim m/N$.

Proof: It can be shown by taking a Taylor expansion of $r_{ij}(\ell)$ around $(\gamma_{ij}(\ell), \gamma_{ii}(0), \gamma_{jj}(0))$, where $\gamma_{ij}(\ell) = \langle a_{it}\, a_{jt+\ell} \rangle$ and evaluating at $(\dot{C}_{ij}(\ell), \dot{C}_{ii}(0), \dot{C}_{jj}(0))$ that

$$r_{ij}(\ell) = C_{ij}(\ell) + O_p(1/N)$$

Also, neglecting quantities of $O(1/N)$.

$$N \cdot \mathrm{Cov}\,(r_{11}(\ell) \cdot r_{11}(k)) = \delta_{\ell k}/\{1+m_N\}$$

$$N \cdot \mathrm{Cov}\,(r_{11}(\ell) \cdot r_{gh}(k)) = \sigma_{1g}\sigma_{1h} \cdot \delta_{\ell k}/\{1+m_N\} \qquad (g,h) \neq (1,1)$$

$$N \cdot \mathrm{Cov}\,(r_{ij}(\ell) \cdot r_{gh}(k)) = \sigma_{ig}\sigma_{jh} \cdot \delta_{\ell k} \qquad (i,j)\,,\ (g,h) \neq (1,1)$$

where $\delta_{\ell k} = 1$ for $\ell = k$ and $\delta_{\ell k} = 0$ $\ell \neq k$.

Normality is obtained from the Martingale Central limit theorem (Billingsley, 1961) as in Lemma 2.2.1.

The following Lemma gives the distribution of $\sqrt{N}\tilde{r}$, the autocorrelations obtained when $\hat{\beta} = \tilde{\beta}$ the estimator obtained maximizing equation (3.4.1). The Lemma parallels Lemma 2.5.2 so that the proof will be omitted.

Lemma 3.5.2

(1) The asymptotic joint distribution of $\sqrt{N}(\tilde{\beta}-\beta', r')'$ is normal with mean zero and variance covariance given by:

$$\begin{pmatrix} V_{\tilde{\beta}} & -V_{\tilde{\beta}} X' \\ -XV_{\tilde{\beta}} & Y \end{pmatrix}$$

where

$$X = \text{Diag}\,(X_{11}, X_{22})\quad 4M \times 4M$$

$$X_{hh} = \begin{bmatrix} \sigma_{1h} \\ \sigma_{2h} \end{bmatrix} \otimes \left[-\pi_{h,i-j} \,\middle|\, \psi_{hi-j}\right] \quad M \times (p+q)$$

$$\phi_h^{-1}(B) = \sum_{r=0}^{\infty} \pi_{hr}, \quad \theta_h^{-1}(B) \quad \sum_{r=0}^{\infty} \psi_{hr} B^r$$

$V_{\tilde{\beta}}$ is given by Lemma 3.4.1.1. and Y given by Lemma 3.5.1.

(ii) The asymptotic distribution of $\sqrt{N}\tilde{r}$ is multivariate normal with mean zero and variance covariance $Y - XV_{\tilde{\beta}}X'$.

The following Lemma gives the distribution of $\tilde{r}_{12}(0)$ the estimator for the correlation coefficient of the innovations.

Lemma 3.5.3 The asymptotic distribution of $\tilde{r}_{12}(0)$ is normal with mean $\rho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$ and variance given by

$$N \ \mathrm{Var}\ (\tilde{r}_{12}(0)) = (1-\rho^2)^2 - \frac{m_N \rho^2}{2(1+m_N)}$$

Proof: Taking a Taylor expansion of $\tilde{r}_{12}(0)$ as a function of $(\dot{\sigma}_{12}, \dot{\sigma}_{11}, \dot{\sigma}_{12})$ around $(\sigma_{12}, \sigma_{11}, \sigma_{22}) = (\rho, 1, 1)$ and evaluating at $(\tilde{\sigma}_{12}, \tilde{\sigma}_{11}, \tilde{\sigma}_{22})$, it follows that

$$\tilde{r}_{12}(0) = \rho + \tilde{\sigma}_{12} - \frac{\sigma_{12}}{2}(\tilde{\sigma}_{11} + \tilde{\sigma}_{22}) + O_p(1/N)$$

From Lemma 3.4.3.2, it follows that $\tilde{r}_{12}(0)$ is normal with mean ρ and variance given by

$$N \ \mathrm{Var}\ (\tilde{r}_{12}(o)) = b' V_{\tilde{\Delta}} b$$

where $b = \frac{1}{2} \cdot (-\sigma_{12}, 1, 1, -\sigma_{12})$

after some algebra it is seen that $\mathrm{Var}(\tilde{r}_{12}(0))$ is given by

$$N \cdot \mathrm{Var}\ (\tilde{r}_{12}(0)) = (1-\rho^2)^2 - m_N\rho^2/2\ (1+m_N)$$

3.6 CONCLUSIONS

A complete procedure was developed to handle unequal sample sizes in the estimation of the parameters of the bivariate CARMA model. The results can be easily extended to the bivariate seasonal multiplicative CARMA model of equation (2.6.1). It is also possible to extend the results to the case of three or more series, one of which has m additional observations. In particular, when m+N is large, the likelihood can be closely approximated by equation (3.4.1), and equations (3.4.4.1) through (3.4.4.4) provide an algorithm for the estimation of the parameters of the model.

The more general case of k-series, each one of them having a different sample size is more difficult to handle. In particular, the exact likelihood function can be very complicated. For large values of N, the number of common observations, it is proposed to approximate the log-likelihood by:

$$\ell(\dot{\underset{\sim}{\beta}}, \dot{\underset{\sim}{\Delta}}) = -\frac{N}{2}\log\dot{\Delta} - \sum_{h=1}^{k}\frac{m_h}{2}\log\sigma_{hh} - \frac{SS}{2} - \frac{1}{2}\sum_{h=1}^{k}\frac{S_{h0}}{\sigma_{hh}}.$$

where

$$SS = \sum_{t=1}^{N} \dot{\underset{\sim}{a}}_t' \dot{\underset{\sim}{a}}_t \quad , \quad S_{h0} = \sum_{t=1-m_h}^{0} \dot{a}_{ht}^2$$

and $\{Z_{ht}\}$ $t = 1 - m_h, \ldots, 0, 1, \ldots N$ is the set of observations available for series Z_{ht}.

With this approximation equation (3.4.4.1) can be used to estimate $\underset{\sim}{\beta}$, but $V_{\underset{\sim}{\beta}}$ is now given by

$$V_{\underset{\sim}{\beta}}^{-1} = [\sigma^{gh} I_{gh}] + \text{Diag}\left(\frac{m_1}{N\sigma_{11}} I_{11}, \ldots, \frac{m_k}{N\sigma_{kk}} I_{kk}\right)$$

where I_{gh} is defined in Lemma 2.2.1.

Unfortunately, in this case Δ can not be solved explicitly, so that an iterative procedure is also required to estimate Δ.

CHAPTER 4

APPLICATIONS OF CARMA MODELLING IN HYDROLOGY

4.1- INTRODUCTION

The planning and operation of water resources, which necessitates the simultaneous consideration of several river flows at different sites and the interdependencies which might be expected between them, motivated the application of multivariate models in hydrology.

Since the pioneering work of Fiering (1964), several multivariate models have appeared in the literature. These include at least the following: Matalas (1967), Young and Pisano (1968), Matalas and Wallis (1971), Bernier (1971), Pegram and James (1972), Valencia and Shaake (1973), Mejia et al. (1974), Kahan (1974), O'Connell (1974), Yevjevich (1975), Lawrance (1976), Mejia and Rouselle (1976), Salas and Pegram (1977), Ledolter (1978), Cooper and Wood (1982) and Deutsch and Ramos (1984) (For a summary and critical review of these models, see Camacho et al. 1983). The most important use of these models has been the generation of synthetic hydrology series where the major concern has been to preserve the statistical characteristics of the historical data set.

Prior to 1977, multivariate AR(1) and ARMA(1,1) models were formulated to reproduce in the synthetic flows the first two moments of the observed data. The exact form of the model was specified before the actual data was even seen. Such a procedure clearly leads to the possibility, if not the probability, that the model will not fit the data very well. This, not surprisingly, led to the finding that synthetic hydrologies generated from these models were inadequate (Finzi et al. 1975). Another particular danger of this approach is discussed below, Section 4.3.

To overcome this problem, Ledolter (1978) suggested the use of multivariate ARMA models, so that following the application of the standard model building methodology of identification, estimation and diagnostic checking the best model from the class could be selected. Another proposal was the application of multivariate Input-Output models, however these may be shown to be equivalent to the multivariate ARMA models (Cooper and Wood 1982).

There are two big disadvantages to the use of the general multivariate ARMA models in hydrology: (a) they are very complicated and, in particular, the number of parameters increases exponentially with the dimensionality of the model and (b) an important feature is still being omitted, namely that the physical structure of the system imposes restrictions on the model (Terasvirta 1982).

In response to these disadvantages of the multivariate ARMA model, Salas et al. (1979) proposed the use of a multivariate ARMA model which was restricted to have diagonal parameter matrices i.e. the CARMA model, arguing that this would reduce the number of parameters to be estimated. The CARMA model can, however, cope with the second disadvantage of the multivariate ARMA model as well. Consider the modelling of multistation riverflow time series. Feedback relationships are not present in the system (unless some kind of external interventions stipulate otherwise), so that the full multivariate ARMA model is not required. On the other hand, the network structure of the river flow basin implies that a transfer function (triangular) model would always be adequate. In the case where the stations are not physically connected, the CARMA model would suffice. The CARMA model is also applicable in many situations where temporal aggregation is present in the data, because it is likely that some of the lagged relationships in the model will collapse, thus simplifying the model to be CARMA (see Granger and Newbold, 1977). A further advantage of the CARMA model is that it can handle the case of unequal sample sizes, commonly encountered in practice (Hipel et al. 1983; Risager 1980). Various applications of the CARMA model to hydrology will be presented in the next section where there are three examples of modelling two station riverflows, two of which have unequal sample sizes and an example of how the CARMA model can be efficiently employed to model water quality.

Before moving on, however, it would be instructive to consider a recently proposed alternative to the CARMA model, which can also overcome the disadvantages of the multivariate ARMA model in this situation, namely the Space-Time Autoregressive Integrated Moving

Average (STARMA) model (Deutsch and Ramos 1984; Pfeifer and Deutsch 1980; Deutsch and Pfeifer 1981).

The STARMA ($p_{\lambda_1}, \ldots, p_{\lambda_p}, q_{m_1}, \ldots, q_{m_q}$) model is defined as:

$$\underset{\sim}{z}_t = \sum_{k=1}^{p} \sum_{\ell=0}^{\lambda_k} \phi_{k\ell} W_\ell \underset{\sim}{z}_{t-\ell} - \sum_{k=1}^{q} \sum_{\ell=0}^{m_k} \theta_{k\ell} W_\ell \underset{\sim}{a}_{t-k} + \underset{\sim}{a}_t \qquad (4.1.1)$$

where:

p is the autoregressive order

q is the moving average order

λ_k is the spatial order of the k^{th} autoregressive term

m_k is the spatial order of the k^{th} moving-average term

$\left.\begin{matrix}\phi_{k\ell} \\ \theta_{k\ell}\end{matrix}\right\}$ are parameters

W_ℓ is the NxN matrix for spatial order $(W_0=I)$ and

$\underset{\sim}{a}_t$ is the random normally distributed innovation or disturbance vector at time t with

$E[\underset{\sim}{a}_t] = \underset{\sim}{0}$

$$E[\underset{\sim}{a}_t \underset{\sim}{a}_{t+s}{}'] = \begin{cases} G & s=0 \\ 0 & s\neq 0 \end{cases}$$

$E[\underset{\sim}{z}_t \underset{\sim}{a}_{t+s}{}'] = 0$ for $s > 0$.

The spatial order matrices W_ℓ are weighting matrices specified by the researcher in order to capture the physical properties of the particular spatial system of interest. For a multisite riverflow network, Deutsch and Ramos (1984) have suggested that the elements of the W matrices should be specified in the following way:

$$W_{\ell}(i,j) = \begin{cases} 1 & \text{if site i and j are } \ell \text{ order neighbors} \\ 0 & \text{otherwise} \end{cases}$$

As can be seen from equation (4.1.1), the STARMA model reduces the number of parameters of the general multivariate model by imposing several restrictions on the parameters. These restrictions are, however, sometimes too severe to allow wide applicability of the model. Consider, for example, the modelling of a two-station river system, where it is assumed for simplicity, that the rivers in the system are not connected. The appropriate STARMA model for the riverflow series $\underset{\sim}{Z}_t = (Z_{1t}, Z_{2t})'$, would be chosen from the subclass of STARMA $(p_{o,\ldots o}, q_{o,\ldots,o})$ models of the form:

$$\underset{\sim}{Z}_t = \phi_1 \underset{\sim}{Z}_{t-1} + \ldots + \phi_p \underset{\sim}{Z}_{t-p} - \theta_1 \underset{\sim}{a}_{t-1} - \ldots - \theta_q \underset{\sim}{a}_{t-q} + \underset{\sim}{\varepsilon}_t$$

$$\underset{\sim}{\varepsilon}_t \;\ddot{\sim}\; \text{NID}\ (0,\Delta)$$

If only the physical restrictions of the system were imposed in modelling $\underset{\sim}{Z}_t$, the appropriate model would be selected from the class of CARMA(p, q) models of the form:

$$Z_{ht} = \phi_{h1} Z_{ht-1} + \ldots + \phi_{hp_h} Z_{ht-p_h} - \theta_{h1} a_{ht-1} - \ldots \theta_{hq_h} a_{ht-q_h} \qquad h=1,2$$

$$\underset{\sim}{a}_t = (a_{1t}, a_{2t})' \quad \text{NID } (0,\Delta) \qquad (4.1.2)$$

$$p = \max (p_1, p_2) \qquad q = \max (q_1, q_2)$$

From this equation, it can be seen that the STARMA model imposes two strong restrictions on the system: i) The orders of the marginal models are equal ie: $p_1 = p_2$; $q_1 = q_2$. ii) The parameters of the two models are equal, ie: $\phi_{1i} = \phi_{2i}$ $i = 1,\ldots,p$ and $\theta_{1j} = \theta_{2j}$ $j = 1,\ldots,q$. These two restrictions may be too restrictive in some applications.

4.2 APPLICATION

This section presents four applications of the CARMA model to actual time series data. The first three applications are examples of modelling the time series of two-station annual riverflows. The series considered are:

(1) Wolf River near London, Wisconsin (1899-1965) and
Fox River near London, Wisconsin (1899-1965)

(2) French Broad River at Asheville, N.C. (1896-1965)
French Broad River near Newport, Tenn. (1965)

(3) Saint Lawrence River near Ogdensburg, N.Y. (1860-1957)
McKenzie River at McKenzie Bridge, Oregon (1911-1957).

The data for these series was taken from Yevjevich (1963) and from the hydrological data tapes of Colorado State University. In the fourth application, two time series corresponding to different measurements of the concentration of Nitrogen in the Middle Fork Creek near Seebe, located in the province of Alberta, are presented. The series represent monthly measurements of Total Nitrogen and Nitrogen Kjeldahl from 1972 to 1979 and are taken from McLeod and Hipel (1981). A more precise definition of these water quality variables is given by McNeely et al. (1979) A listing of the data used in this chapter is given in Appendix A.

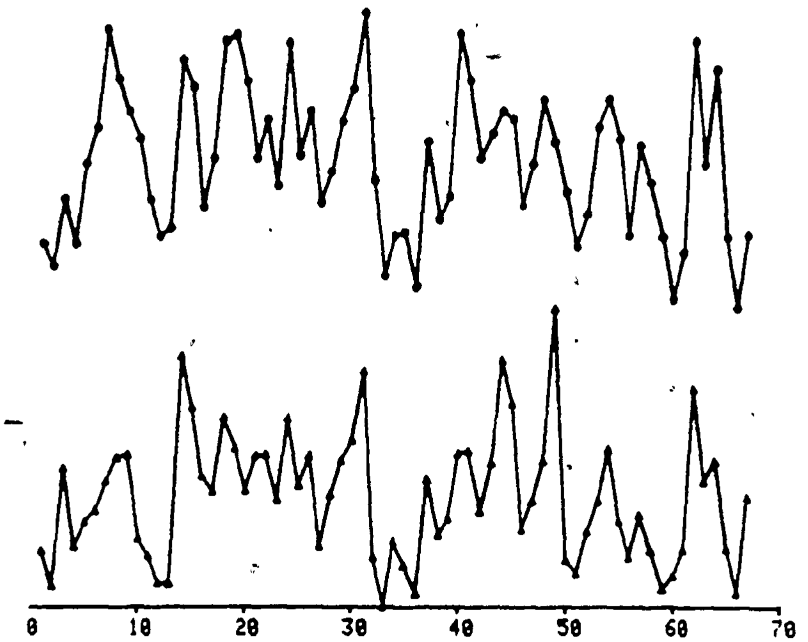
Fox and Wolf Rivers

The data for this case are plotted in Figure 4.1. The first step in identifying the model is to fit univariate ARMA models to each of the component series. This was done using the identification techniques of Box and Jenkins (1976) and Hipel et al (1977). MA(1) models were found to be adequate for both of the series.

A plot of the residual cross correlations is also given in Figure 4.1. From this it can be seen that only the cross correlation of lag zero is significant, implying that a CMA(1) model is adequate to model the bivariate series. The different parameter estimates are given in Table 4.1. As can be seen from this Table there is a significant reduction in the variance of the parameter estimates when joint estimation is used.

French River at Asheville and Near Newport

A plot of the 70 observations of the flows at Asheville and the 45 observations of the flows near Newport is given in Figure 4.2. Univariate MA(1) models were found to be adequate to fit the logarithms of the series. A plot of the residual cross correlations, obtained using the residuals of the 45 common observations, is also given in Figure 42. Although the flows near Newport are measured downstream from the flows measured at Asheville



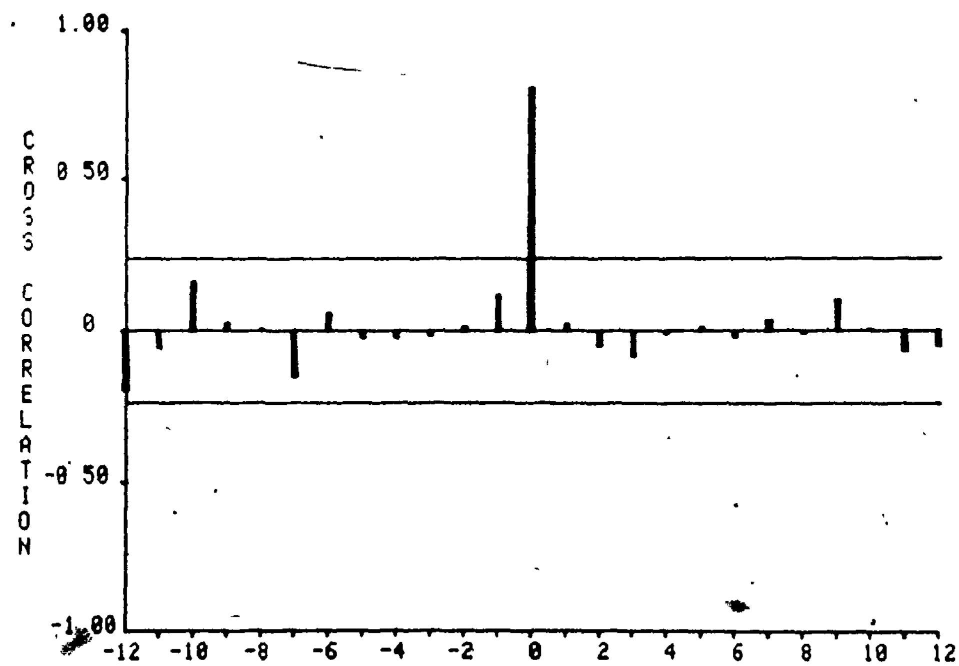


FIGURE 4.1

Plot of the Series and the Residual Cross-Correlation for the Fox and Wolf Rivers.

TABLE 4.1

PARAMETER ESTIMATES FOR THE CAM(1) MODEL
FOR THE FOX AND WOLF RIVERS

	FOX RIVER	WOLF RIVER
UNIVARIATE ESTIMATION OF θ_h	- .483 (.110)	- .411 (.111)
JOINT ESTIMATION ESTIMATION OF θ_h	- .170 (.088)	- .470 (.091)
EFFICIENCY OF UNIVARIATE ESTIMATOR	.640	.532
MEAN OF $\text{Log } Z_{ht}$	6.96 (.037)	7.41 (.042)
RESIDUAL VARIANCE	5.30×10^{-2}	5.22×10^{-2} $\rho = .82$

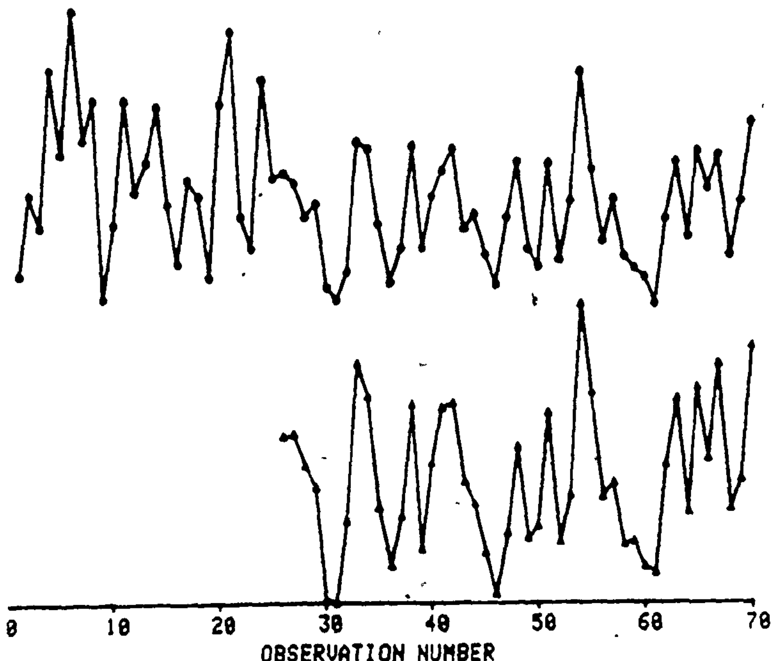
MODEL $\log Z_{ht} = \mu_h + (1 - \theta_h B)_{ht}$ $h = 1,2$

implying that a transfer function would be required to model the riverflows, it is observed from the plot of the residual cross correlations that a CARMA model would suffice (only the cross correlation at lag zero is significantly different from zero). This is due to the fact that here annual river flows are considered and this temporal aggregation of the data, by its very nature, incorporates some of the lagged relationships, which would be expected to hold in the model of the system (see Granger and Newbold, 1977). If monthly data or less temporally aggregated data were considered, the transfer function would probably be required to model the data.

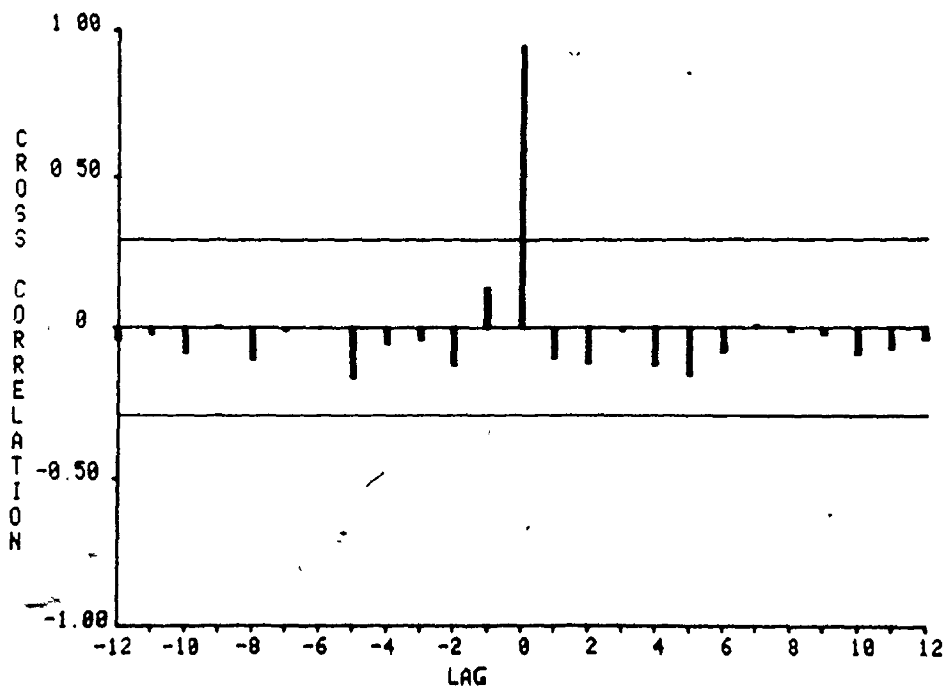
The method outlined in Chapter 3 was used to jointly estimate the parameters of the series. The different estimated parameters are listed in Table 4.2. It can be seen from this table that the variance of the estimators is reduced by almost 50% when a joint estimation is used.

Saint Lawrence and McKenzie Rivers

A plot of the 97 observations for the Saint Lawrence River located in New York and the 47 observations for the McKenzie river located in Oregon is given in Figure 4.3. Univariate AR(1) models were found to be adequate to model the two component series. Because one of the rivers is located in the east of the United States, which the other is located in the west, it would be expected that two



TOP SERIES   FRENCH BROAD RIVER AT ASHEVILLE, N.C. ANNUAL FLOWS CFS.
BOTTOM SERIES: FRENCH BROAD RIVER NEAR NEWPORT, TENN. ANNUAL FLOWS CFS.



1ST SERIES: 45 RESIDUALS(0, 0, 1)(0, 0, 0) 0-FRENCH BROAD RIVER A
2ND SERIES: 45 RESIDUALS(0, 0, 1)(0, 0, 0) 0-FRENCH BROAD RIVER N

FIGURE 4.2

Plot of the Series and the Residual Cross-Correlation for the French River at Asheville and near Newport

TABLE 4.2

PARAMETER ESTIMATES FOR THE CAM(1) MODEL
FOR FRENCH RIVER AT ASHEVILLE AND NEAR NEWPORT

	AT ASHEVILLE N = 70	NEAR NEWPORT N = 45
UNIVARIATE ESTIMATION OF θ_h	- .283 (.115)	- .467 (.131)
JOINT ESTIMATION ESTIMATION OF θ_h	- .170 (.087)	- .470 (.081)
EFFICIENCY OF UNIVARIATE ESTIMATOR	.572	.382
MEAN OF $\log Z_{ht}$	7.59 (.040)	7.94 (.048)
RESIDUAL VARIANCE	6.72×10^{-2}	5.79×10^{-2} $\rho = .91$

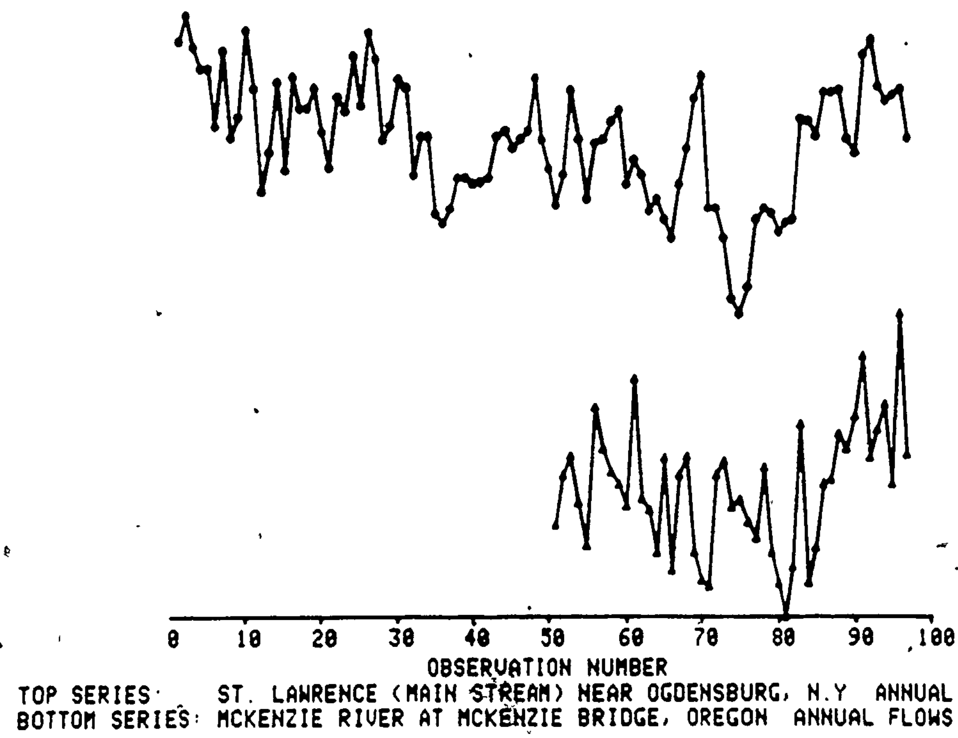
MODEL $\log Z_{ht} = \mu_h + (1-\theta_h B) a_{ht}$

independent series would be sufficient to fit the data. However, a perusal of the residual cross correlations (Figure 4.3) reveals that a small, but significantly different from zero, value of the cross correlation is observed at lag zero. This correlation may be due to weather patterns which affect the two rivers. The parameter estimates of the model are given in Table 4.3. It can be seen from the table that there is an improvement in the efficiency of the estimators when the models are joint estimated.

Total Nitrogen and Nitrogen Kjeldahl Series for The Middle Fork Creek

A plot of these series is given in Figure 4.4. Seasonal ARMA $(1, 0)_6 \times (0, 0)$ model was needed to fit the Total Nitrogen series whereas an AR(1) model was required to fit the Nitrogen Kjeldahl series. A plot of the residual cross correlation is also shown in Figure 4.4. As can be seen from this plot, only the cross correlation at lag zero is significantly different from zero, implying that the CARMA model is adequate to jointly fit the two series. Values of the parameter estimates are given in Table 4.4. As can be seen from this table, there is a reduction in the variance of the estimators of almost 75%. The same reduction would be obtained by increasing the sample size of the series by a factor of four.

The above applications have illustrated how the CARMA model can be efficiently employed to model hydrological and water quality time



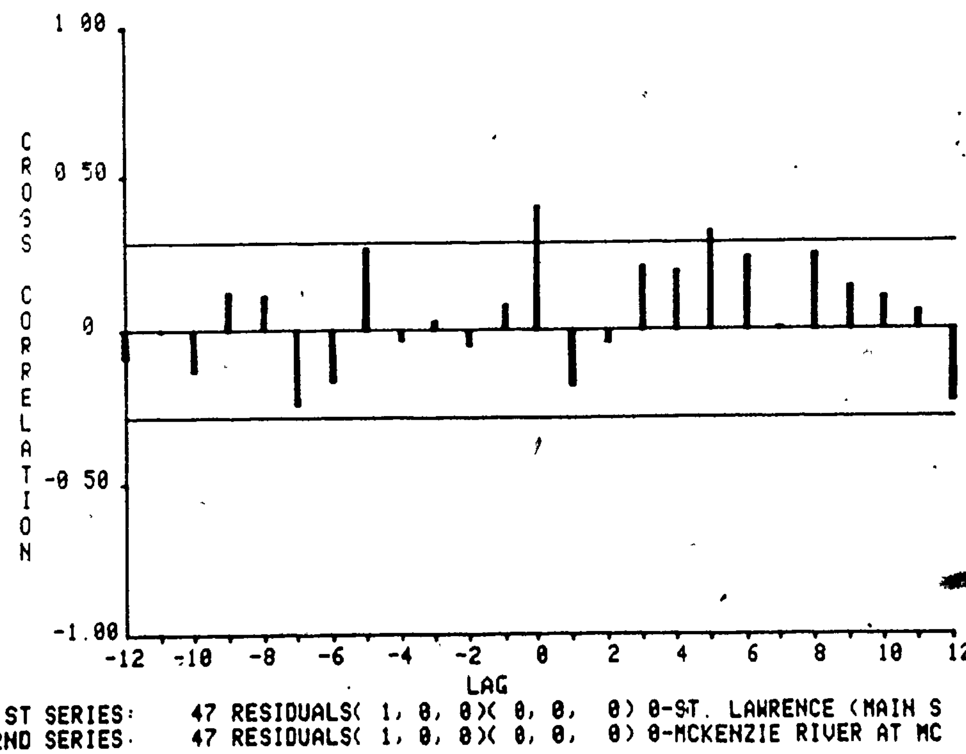


FIGURE 4.3

Plot of the Series and the Residual Cross-Correlation for the Saint Lawrence and McKenzie Rivers

TABLE 4.3

PARAMETER ESTIMATES FOR THE CAR(1) MODEL
FOR THE SAINT LAWRENCE AND MCKENZIE RIVERS

	SAINT LAWRENCE N = 97	McKENZIE N = 47	
UNIVARIATE ESTIMATION OF ϕ_h	.723 (.071)	.347 (.136)	
JOINT ESTIMATION ESTIMATION OF ϕ_h	.7651 (.062)	.372 (.125)	
EFFICIENCY OF UNIVARIATE ESTIMATOR	.762	.844	
MEAN OF $\log Z_{ht}$	12.38 (.023)	7.39 (3.47)	
RESIDUAL VARIANCE	3.2×10^{-3}	2.41×10^{-3}	$\rho = .39$

MODEL $(1 - \phi_h B)(\log Z_{ht} - \mu_h) = a_{ht}$

series. They have also illustrated how the CARMA model deals with time series having unequal sample sizes.

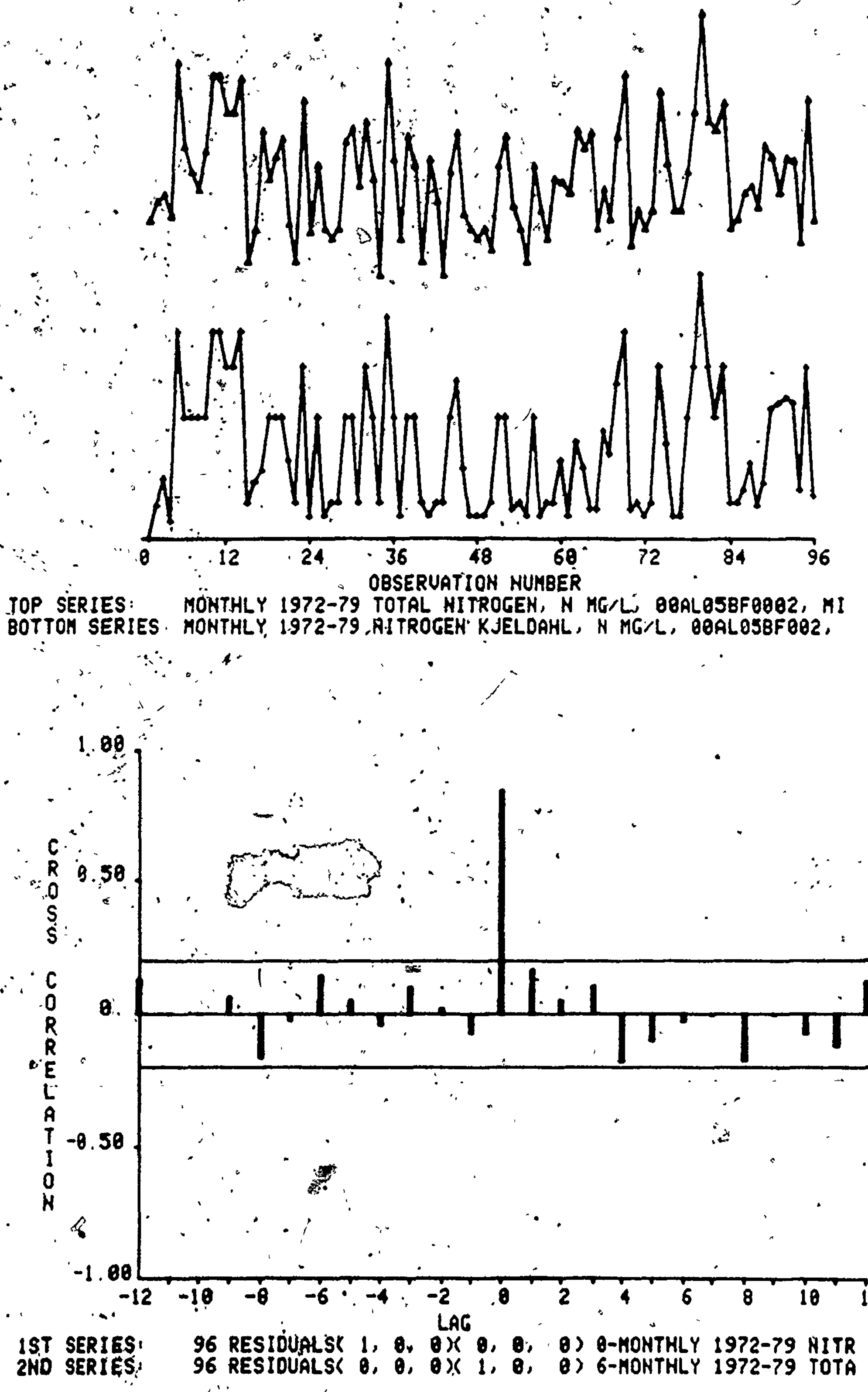


FIGURE 4.4

Plot of the Series and the Residual Cross-Correlation for the Total Nitrogen and Nitrogen Kjeldahl

TABLE 4.4

PARAMETER ESTIMATES FOR THE CARMA MODEL
FOR THE TOTAL NITROGEN AND NITROGEN KJELDAHL SERIES FOR
THE MIDDLE FORK CABIN CREEK

	TOTAL NITROGEN	NITROGEN KJELDAHL	
UNIVARIATE ESTIMATION OF ϕ_h	.310 (.097)	.294 (.097)	
JOINT ESTIMATION ESTIMATION OF ϕ_h	.141 (.049)	.141 (.049)	
EFFICIENCY OF UNIVARIATE ESTIMATOR	.255	.255	
MEAN OF Log Z_{ht}	- 1.33 (.084)	- 1.59 (.104)	
RESIDUAL VARIANCE	.131	.152	ρ = .88

MODEL

$$(1 - \theta_1 B^6)(\log T - \mu_1) = a_{1t}$$
$$(1 - \theta_2 B)(\log K - \mu_2) = a_{2t}$$

4.3 COMPARISON OF THE CAR(1) MODEL AND THE MATALAS AR(1) MODEL

As was pointed out in the introduction of this Chapter, the multivariate AR(1) model proposed by Matalas (1967) has been widely used to generate multivariate synthetic hydrologies.

In this section, the effect on the estimation of the parameters of incorrectly modelling a bivariate series $\underset{\sim}{X}_t = (X_{1t}, X_{2t})'$ as AR(1) when CAR(1) would suffice is considered. The more general AR(1) model is given by:-

$$\underset{\sim}{X}_t = \Gamma \cdot \underset{\sim}{X}_{t-1} + B \cdot \varepsilon_t$$

$$\varepsilon_t \sim NID_2 \quad (0,I)$$

where I is the identity matrix and Γ and B are parameter matrices estimated as:

$$\bar{\Gamma} = M(1) \cdot M(0)^{-1} \tag{4.3.1}$$

$$\bar{B}\bar{B}' = M(0) - M(1)\, M(0)^{-1}\, M(1)'$$

where M(0) and M(1) are the lag zero and lag one sample cross correlation matrices of the series $\underset{\sim}{X}_t$. It can be shown that $\bar{\Gamma}$ is asymptotically equivalent to the MLE of Γ. Therefore the asymptotic distribution of $\bar{\Gamma}$ is multivariate normal with variance covariance matrix given by:

$$N \ \mathrm{Var}(\bar{\Gamma}) = \Sigma \otimes \Gamma_0^{-1}$$

where

$$\Gamma_o = \langle X_t \cdot X_t' \rangle$$

$$\Sigma = BB' = (\sigma_{ij})$$

If the parameter of this AR(1) model are estimated, when the bivariate series X_t is really a CAR(1), then the asmptotic variance for the diagonal elements of $\bar{\Gamma}, \bar{\phi}_1$ and $\bar{\phi}_2$ say, is given by

$$\mathrm{Var}(\bar{\phi}_i) = (1 - \phi_i^2)/(1 - \rho^2 A)$$

$$A = (1 - \phi_1^2) \cdot (1 - \phi_2^2)/(1 - \phi_1\phi_2)^2$$

where ϕ_1 and ϕ_2 are the true parameters of the model and $\rho = \sigma_{12}/\sigma_{11}\sigma_{22}$.

If, on the other hand, the CAR(1) bivariate series X_t is estimated by imposing the CAR(1) model restriction on the more general AR(1) model, using for example, the score algorithm given in Lemma 2.2.3, then the asymptotic variance of the obtained estimators for ϕ_1 and ϕ_2, $\hat{\phi}_1, \hat{\phi}_2$ say, are (see section 2.3.1)

$$\mathrm{Var}(\hat{\phi}_i) = (1 - \rho^2)(1 - \phi_i^2)/(1 - \rho^4 A)$$

so that the asymptotic efficiency of $\bar{\phi}_i$ relative to $\hat{\phi}_i$ is given by

$$\text{eff} = (1 - \rho^2)(1 - \rho^2 A)/(1 - \rho^4 A) \qquad (4.3.2)$$

A simulation experiment was carried out using the technique of section 2.6 in order to compare the efficiency values obtained for small sample sizes with the theoretical asymptotic efficiency value of equation (4.3.2). A total of 45 models corresponding to the parameter settings (ϕ_1, ϕ_2) = (.3,.3), (.3,.6), (.3,.9), (.6,.6), (.9,.9), ρ = .3,.6,.9 and N = 50, 100, 200 were included and for each model 1000 replications were done. For each model, the multivariate moment estimators were obtained using equations (4.3.1) and the score algorithm of Lemma 2.2.3 was used to obtain the restricted estimators. The efficiency values and their standard errors were obtained using the methodology described in section 2.3.2 and are listed in Table 4.5 From this table, it can be seen that the observed efficiency values are of the same order as the theoretical values, even for a sample size of 50. It is also seen that the loss in efficiency of the estimators obtained using the full multivariate model can be very substantial and in many cases can be well over 50%.

TABLE 4.5

EFFICIENCY VALUES OF THE MULTIVARIATE MOMENT ESTIMATORS
RELATIVE TO THE SCORE ESTIMATORS
NUMBER OF OBSERVATIONS PER SERIES : 50, 100, 200
NUMBER OF REPLICATIONS : 1000

$\rho = .3$

MODEL :	(.3,.3)		(.3,.6)		(.3,.9)		(.6,.6)		(.9,.9)	
50	.815	.873	.888	.855	.892	.905	.819	.810	.663	.674
	(.024)	(.024)	(.025)	(.024)	(.024)	(.040)	(.024)	(.025)	(.031)	(.033)
100	.833	.817	.891	.880	.873	.867	.802	.790	.719	.752
	(.022)	(.022)	(.022)	(.031)	(.021)	(.022)	(.023)	(.023)	(.032)	(.026)
200	.891	.831	.821	.852	.910	.884	.797	.826	.796	.756
	(.023)	(.023)	(.022)	(.024)	(.019)	(.021)	(.020)	(.025)	(.025)	(.027)
THEO	.835		.845		.886		.835		.835	

$\rho = .6$

50	.465	.441	.477	.451	.604	.624	.611	.521	.355	.366
	(.022)	(.020)	(.023)	(.023)	(.024)	(.030)	(.038)	(.028)	(.021)	(.026)
100	.465	.460	.505	.505	.562	.615	.439	.461	.400	.422
	(.024)	(.021)	(.023)	(.025)	(.023)	(.030)	(.022)	(.024)	(.025)	(.024)
200	.501	.505	.468	.463	.594	.664	.448	.495	.450	.444
	(.022)	(.021)	(.021)	(.022)	(.025)	(.032)	(.021)	(.022)	(.021)	(.023)
THEO	.471		.496		.590		.471		.471	

$\rho = .9$

50	.102	.110	.131	.146	.300	.316	.100	.110	.083	.090
	(.006)	(.007)	(.008)	(.010)	(.017)	(.027)	(.007)	(.007)	(.006)	(.007)
100	.108	.115	.129	.124	.236	.237	.171	.202	.102	.108
	(.006)	(.007)	(.007)	(.008)	(.013)	(.017)	(.013)	(.018)	(.008)	(.008)
200	.097	.105	.140	.142	.200	.201	.099	.095	.092	.097
	(.005)	(.006)	(.008)	(.008)	(.011)	(.015)	(.006)	(.006)	(.007)	(.006)
THEO	.105		.131		.178		.105		.105	

CHAPTER 5

ON TESTING TWO IMPORTANT HYPOTHESES CONCERNING THE CARMA MODEL

5.1 INTRODUCTION

Even though a variety of hypotheses could be formulated concerning the parameters of the CARMA model, there are two important hypotheses which require special attention. The first hypothesis is concerned with the statistical independence of the equations of the overall model. The null hypothesis is $H_o : \rho_{ij} = 0, i \neq 0$. It is important to test this hypothesis because in many cases the relevant consideration is to see whether a joint model is required to fit the data or if a set of univariate models will suffice (Pierce and Haugh, 1979). In particular, the rejection of the null hypothesis implies the existence of contemporaneous causality in the system (Granger, 1969; Pierce and Haugh, 1977, 1979). In Section 5.2 a test statistic for this hypothesis is given.

The second hypothesis is concerned with the homogeneity of the parameters of the models of the different series. The null hypothesis is $H_0 : \beta_h = \beta_1$ $h = 2, \ldots, k$, where $\beta_h = (\phi_{h1}, \ldots, \phi_{hp}, \theta_{h1}, \ldots, \theta_{hq})$ are the parameters of the model of series h. Zellner (1962) considered a similar hypothesis for the regression parameters of the SURE model. Zellner pointed out that when the null hypothesis is true "there will be no bias involved in the simple linear aggregation of the data", a situation which occurs frequently with microeconomic data. Risager (1980) assumed that the null hypothesis was true for the bivariate CAR model fitted to two series of mean annual ice core measurements. Risager argued that "the nature of the two processes made it reasonable" to assume that the null hypothesis was true. No statistical tests were reported concerning the validity or otherwise of the hypothesis. In section 5.3 the test of this hypothesis is considered in more detail.

5.2 TESTING FOR THE SIGNIFICANCE OF THE CORRELATION

Suppose that the series $\{Z_{ht}\}$ $h = 1, \ldots, k$, $t = 1, \ldots, N$, satisfy the CARMA model given by

$$\phi_h(B)\, Z_{ht} = \theta_h(B)\, a_{ht} \qquad h = 1, \ldots, k \quad (5.2.1)$$

$$\underset{\sim}{a}_t = (a_{1t}, \ldots, a_{kt}) \sim \text{NID}(0, \Delta)$$

where

$$\phi_h(B) = 1 - \phi_{h1}B - \ldots - \phi_{hp}B^p$$

$$\phi_h(B) = 1 - \theta_{h1}B - \ldots - \theta_{hq}B^q$$

$$\Delta = (\sigma_{gh})$$

It is assumed that the zeros of the polynomial equations $\phi_h(B) = 0$ and $\theta_h(B) = 0$, $h = 1, \ldots, k$, lie outside the unit circle, so that the model is stationary and invertible. The likelihood test for testing for the significance of the correlation among the series Z_{ht}, $h = 1, \ldots, k$, which under normality assumptions is equivalent to a test for independence, is given in the following lemma.

<u>Lemma 5.2.1</u>. The likelihood ratio test statistic for testing the null hypothesis H_o : $\rho_{ij} = 0$ $i \neq j$ ($\rho_{ij} = \sigma_{ij} / \sqrt{\sigma_{jj}\, \sigma_{jj}}$) against H_o : $\rho_{ij} \neq 0$ for some pair (i, j) $i \neq j$, i.e., the simple negation of H_o, is given by:

$$\lambda = N \log |\bar{R}| + O_p(N^{-\frac{1}{2}}) \qquad (5.2.2)$$

where $|\bar{R}|$ denotes the determinant of the matrix $\bar{R} = (\bar{r}_{ij})$

$$\bar{r}_{ij} = \bar{\sigma}_{ij} / \sqrt{\bar{\sigma}_{ii}\,\bar{\sigma}_{jj}}$$

$$\bar{\sigma}_{ij} = \sum_{t=1}^{N} \bar{a}_{it}\;\bar{a}_{jt}/N \qquad (5.2.3)$$

The $\bar{a}_{it}$ are the estimated residuals for the series h obtained using the univariate MLE of $\underset{\sim}{\beta}_h$. Under the null hypothesis, λ is asymptotically distributed as a χ^2 variable with K(K-1)/2 degrees of freedom.

<u>Proof</u>: The maximized likelihood function under H_o is given, apart from terms $0_p(N^{-\frac{1}{2}})$, by

$$\bar{L}_o = \prod_{h=1}^{k} |\bar{\sigma}_{hh}|^{-N/2} \exp\{-N/2\}$$

The maximized likelihood function under H_1 is given, apart from terms $0_p(N^{-\frac{1}{2}})$, by

$$\hat{L} = |\hat{\Delta}|^{-N/2} \cdot \exp\{-Nk/2\}$$

$$= |\hat{R}|^{-N/2} \prod_{h=1}^{K} |\hat{\sigma}_{KK}| \exp\{-N/2\}$$

where $\hat{\Delta}$ is the estimated variance covariance matrix under H_1, i.e., using joint estimation.

Now, it follows from the results of theorem 2.2.2 that

$$\hat{\sigma}_{ij} = \bar{\sigma}_{ij} \{ 1 + O_p(N^{-\frac{1}{2}}) \}$$

so that $\hat{L}$ can be written as:

$$\hat{L} = |\bar{R}|^{-N/2} \cdot \bar{L}_o \cdot \{1 + O_p(N^{-\frac{1}{2}})\}$$

therefore, the likelihood ratio test is given by:

$$\begin{aligned} \lambda &= 2 \log (\bar{L}_o/\hat{L}) \\ &= - N \log |\bar{R}| + O_p(N^{-\frac{1}{2}}) \end{aligned}$$

The last statement of the Lemma follows because there are $k(k-1)/2$ independent restrictions in the null hypothesis (see Cox and Hinkley, 1974).

It is instructive to consider the case $k = 2$. In this situation the likelihood ratio test statistic of equation 5.2.2 for testing H_o P ≠ 0 vs. $H_i: \rho \neq 0$ simplifies to

$$\lambda = - N \log (1 - \bar{\rho}^2) \qquad (5.2.4)$$

where

$$\bar{\rho} = \bar{r}_{12} = \sum_{t=1}^{N} \bar{a}_{it} \bar{a}_{zt} / \sqrt{\sum_{t=1}^{N} a_{1t} \sum_{t=1}^{N} a_{zt}^2} \qquad (5.2.5)$$

Equation (5.2.4) shows that the test statistic based on $\bar{\rho}$, the residual cross correlation of the univariate series, is asymptotically equivalent to the likelihood ratio test for testing the independence of the series Z_{1t} and Z_{2t}. This result gives an asymptotic justification to the intuitive idea of considereing the univariate residual cross correlation for testing the independence of two series which has been discussed by several authors (Jenkins and Alavi, 1968; Haugh, 1976; Haugh and Box, 1977; Pierce, 1977; Pierce and Haugh, 1977; McLeod, 1979; Li, 1981). McLeod (1979) has shown that the distribution of is asymptotically distributed as $N(\rho,(1-\rho^2)^2/N)$. This distribution can be used to obtain the power of the test statistic.

An empirical comparison of the small sample properties of the likelihood ratio test and the test based on $\bar{\rho}$ was carried out making use of the simulation study for the bivariate CAR(1) model, described in Section 2.3.2. For each simulated model, the likelihood ratio test was calculated as

$$LR = N \log(|\bar{\sigma}_{11} \cdot \bar{\sigma}_{22}|/|\hat{\Delta}|)$$

where $\bar{\sigma}_{hh}$ is given by equation 5.2.3 and $|\hat{\Delta}|$ is the determinant of the estimated variance covariance matrix under H_1. The null hypothesis $H_0 : \rho = 0$ was rejected whenever $LR > \chi_1^2(1-\alpha)$ where $\chi_1^2(1-\alpha)$ denotes the 100 (1-α)% quartile of the χ^2 distribution with one degree of freedom. $\bar{\rho}$ was calculated using equation (5.2.5) and H_0 was rejected whenever $\sqrt{N}\,\bar{\rho} > Z(1-\alpha/2)$ where $Z(1-\alpha)$ denotes the 100(1-α)% quartile of the standard normal distribution.

The number of rejections of the null hypothesis using the LR test and the test based on $\bar{\rho}$ were recorded. Two significance levels, 5% and 1%, were used. The results are given in Table 5.1 for N=50 and in Table 5.2 for N=200. Several points deserve to be singled out concerning the results of the simulation:

(a) The results of the simulation do not seem to depend on the values of (ϕ_1, ϕ_2).

(b) The power of the LR test slightly dominates the power of the test based on $\bar{\rho}$. This dominance is more marked for the sample size of 50. In this case, however, the observed significance level for the LR test is greater than 5%. Such an increase in the significance level may at least partly account for the dominance of the LR test over the residual correlation test with respect to power.

(c) With a sample size of 200 the two tests perform almost equally as would be expected from the asymptotic theory.

TABLE 5.1

COMPARISON OF THE LIKELIHOOD RATIO TEST
AND THE TEST BASED ON THE VALUE OF ρ
NUMBER OF REJECTIONS AT 5% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES : 50
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
		THEO	1000	982	570	285	106	50	106	285	570	982	1000
(0.0,0.0)	1.00	ρ	1000	955	545	313	109	48	105	291	545	957	1000
		LR	1000	961	578	332	131	58	120	316	570	966	1000
(.3, .3)	1.00	ρ	1000	958	570	269	94	49	112	291	580	965	1000
		LR	1000	962	589	297	112	63	126	311	600	968	1000
(.6, .6)	1.00	ρ	1000	966	552	291	100	57	92	292	579	966	1000
		LR	1000	971	577	333	117	66	108	324	613	969	1000
(.9, .9)	1.00	ρ	1000	961	570	300	99	56	101	271	554	960	1000
		LR	1000	979	642	370	157	112	168	348	625	977	1000
(0.0, .1)	.99	ρ	1000	963	554	317	125	45	97	293	551	968	1000
		LR	1000	971	583	352	140	50	107	321	572	976	1000
(.3, .4)	.99	ρ	1000	962	556	302	112	62	119	281	576	957	1000
		LR	1000	967	595	332	125	73	140	322	608	964	1000
(.6, .7)	.97	ρ	1000	954	550	272	96	59	119	266	575	964	1000
		LR	1000	963	588	312	119	72	139	303	614	970	1000
(0.0, .3)	.91	ρ	1000	958	588	295	101	54	108	297	563	964	1000
		LR	1000	962	625	327	122	63	129	316	598	969	1000
(.3, .6)	.87	ρ	1000	955	532	285	115	51	108	287	568	967	1000
		LR	1000	958	562	317	135	65	133	323	599	976	1000

TABLE 5.1 (Continued)

(.3, -.3)	.70	ρ	1000	960	551	280	105	67	98	291	537	962	1000
		LR	1000	967	591	308	121	75	118	314	576	971	1000
(0.0, .6)	.64	ρ	1000	965	556	283	99	45	109	260	545	962	1000
		LR	1000	971	585	317	124	61	120	288	582	967	1000
(.6, .9)	.58	ρ	1000	962	571	292	99	46	88	270	529	954	1000
		LR	1000	976	625	351	134	72	134	324	594	967	1000
(.6, -.3)	.42	ρ	1000	952	541	293	96	44	105	302	553	962	1000
		LR	1000	960	586	329	112	54	123	337	595	967	1000
(.3, .9)	.32	ρ	1000	969	548	274	116	53	95	266	562	963	1000
		LR	1000	978	602	313	144	76	144	311	615	975	1000
(.6, -.6)	.22	ρ	1000	975	545	287	103	58	103	289	536	961	1000
		LR	1000	977	576	311	123	74	118	320	573	970	1000
(0.0, .9)	.19	ρ	1000	954	598	299	113	61	97	294	543	966	1000
		LR	1000	967	639	351	149	89	139	341	608	973	1000
(.9, -.3)	.11	ρ	1000	954	541	267	93	45	100	268	565	965	1000
		LR	1000	966	605	322	130	79	137	326	621	972	1000
(.9, -.6)	.05	ρ	1000	955	566	313	134	53	109	281	541	961	1000
		LR	1000	969	616	370	172	79	150	338	587	970	1000
(.9, -.9)	.01	ρ	1000	969	571	306	100	46	106	273	562	966	1000
		LR	1000	979	654	376	156	79	174	350	641	977	1000

TABLE 5.1 (Continued)

COMPARISON OF THE LIKELIHOOD RATIO TEST
AND THE TEST BASED ON THE VALUE OF ρ
NUMBER OF REJECTIONS AT 1% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES : 50
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
		THEO	1000	899	309	113	30	10	30	113	309	899	1000
(0.0,0.0)	1.00	ρ	1000	869	280	120	37	9	32	102	302	858	1000
		LR	1000	890	325	154	43	12	40	138	347	893	1000
(.3, .3)	1.00	ρ	1000	869	305	115	22	11	32	126	321	860	1000
		LR	1000	899	357	141	32	15	42	149	367	906	1000
(.6, .6)	1.00	ρ	1000	867	293	110	30	10	25	101	298	873	1000
		LR	1000	890	349	150	37	15	37	127	357	895	1000
(.9, .9)	1.00	ρ	1000	849	296	124	26	9	28	106	293	869	1000
		LR	1000	897	394	178	52	22	55	158	377	904	1000
(0.0, .1)	.99	ρ	1000	861	283	118	29	8	24	109	312	875	1000
		LR	1000	887	343	148	46	11	32	135	362	903	1000
(.3, .4)	.99	ρ	1000	859	323	102	20	9	29	95	292	869	1000
		LR	1000	890	363	126	26	12	41	132	341	899	1000
(.6, .7)	.97	ρ	1000	855	292	113	19	7	38	107	313	869	1000
		LR	1000	888	344	144	26	13	55	141	369	902	1000
(0.0, .3)	.91	ρ	1000	872	316	113	22	8	31	119	300	884	1000
		LR	1000	904	374	150	41	11	43	143	351	904	1000
(.3, .6)	.87	ρ	1000	860	285	110	31	13	27	119	315	865	1000
		LR	1000	884	340	145	40	19	38	152	374	895	1000

TABLE 5.1 (Continued)

(.3,-.3)	.70	ρ	1000	866	311	119	26	15	25	107	277	869	1000
		LR	1000	899	363	138	35	18	33	136	328	900	1000
(0.0, .6)	.64	ρ	1000	854	270	104	39	3	29	89	297	862	1000
		LR	1000	898	329	134	46	5	42	114	355	899	1000
(.6, .9)	.58	ρ	1000	877	291	124	29	8	26	105	289	864	1000
		LR	1000	915	369	160	50	12	47	137	350	898	1000
(.6,-.3)	.42	ρ	1000	864	314	109	28	6	29	124	286	871	1000
		LR	1000	889	372	136	40	11	45	158	334	907	1000
(.3, .9)	.32	ρ	1000	883	310	112	35	10	26	96	313	860	1000
		LR	1000	923	388	141	55	19	44	143	389	890	1000
(.6,-.6)	.22	ρ	1000	889	292	108	21	14	30	102	283	872	1000
		LR	1000	913	351	139	34	21	38	140	342	905	1000
(0.0, .9)	.19	ρ	1000	855	308	116	36	13	26	96	302	855	1000
		LR	1000	905	399	149	55	18	39	134	374	893	1000
(.9,-.3)	.11	ρ	1000	862	288	104	18	6	26	112	288	868	1000
		LR	1000	898	368	149	29	10	38	153	363	916	1000
(.9,-.6)	.05	ρ	1000	857	300	119	37	6	29	115	297	858	1000
		LR	1000	896	370	162	61	12	50	167	379	902	1000
(.9,-.9)	.01	ρ	1000	875	283	119	30	10	30	100	303	871	1000
		LR	1000	914	391	185	60	31	59	150	389	918	1000

NOTE : $A = (1. - \phi_1^2)(1. - \phi_2^2)/(1. - \phi_1\phi_2)^2$

Theoretical values are based on the asymptotic distribution of $\bar{\rho} \sim N(\rho,(1-\rho^2)^2/N)$

TABLE 5.2

COMPARISON OF LIKELIHOOD RATIO TEST
AND THE THE TEST BASED ON THE VALUE OF ρ
NUMBER OF REJECTIONS AT 5% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
		THEO	1000	999	993	817	291	50	291	817	993	999	1000
(0.0,0.0)	1.00	ρ	1000	1000	989	815	296	49	274	815	993	1000	1000
		LR	1000	1000	989	821	303	52	285	818	994	1000	1000
(.3, .3)	1.00	ρ	1000	1000	990	801	261	52	278	812	990	1000	1000
		LR	1000	1000	991	805	268	54	281	817	991	1000	1000
(.6, .6)	1.00	ρ	1000	1000	991	818	287	58	306	812	989	1000	1000
		LR	1000	1000	991	824	297	62	313	817	991	1000	1000
(.9, .9)	1.00	ρ	1000	1000	992	818	305	44	289	802	993	1000	1000
		LR	1000	1000	992	826	320	56	311	819	995	1000	1000
(0.0, .1)	.99	ρ	1000	1000	992	822	289	49	300	811	992	1000	1000
		LR	1000	1000	992	825	300	51	307	814	992	1000	1000
(.3, .4)	.99	ρ	1000	1000	990	805	298	42	302	814	996	1000	1000
		LR	1000	1000	991	807	305	46	310	816	996	1000	1000
(.6, .7)	.97	ρ	1000	1000	995	829	262	45	276	839	992	1000	1000
		LR	1000	1000	995	834	271	49	283	846	992	1000	1000
(0.0, .3)	.91	ρ	1000	1000	990	792	273	48	281	823	989	1000	1000
		LR	1000	1000	990	804	280	51	290	829	990	1000	1000
(.3, .6)	.87	ρ	1000	1000	988	799	300	49	287	813	993	1000	1000
		LR	1000	1000	989	805	304	50	294	820	993	1000	1000

TABLE 5.2 (Continued)

(.3,-.3)	.70	ρ	1000	1000	992	813	299	48	291	801	992	1000	1000
		LR	1000	1000	992	818	303	52	299	809	994	1000	1000
(0.0, .6)	.64	ρ	1000	1000	990	825	291	50	306	822	989	1000	1000
		LR	1000	1000	990	830	301	53	313	834	989	1000	1000
(.6, .9)	.58	ρ	1000	1000	996	799	289	44	285	801	996	1000	1000
		LR	1000	1000	996	809	309	49	299	816	996	1000	1000
(.6,-.3)	.42	ρ	1000	1000	987	791	296	58	297	809	990	1000	1000
		LR	1000	1000	988	802	303	59	306	816	990	1000	1000
(.3, .9)	.32	ρ	1000	1000	993	824	300	51	295	811	994	1000	1000
		LR	1000	1000	993	833	319	52	304	817	994	1000	1000
(.6,-.6)	.22	ρ	1000	1000	994	797	295	47	275	805	990	1000	1000
		LR	1000	1000	995	805	301	48	287	809	990	1000	1000
(0.0, .9)	.19	ρ	1000	1000	993	806	287	43	313	803	991	1000	1000
		LR	1000	1000	994	813	300	47	323	810	991	1000	1000
(.9,-.3)	.11	ρ	1000	1000	992	788	277	49	300	806	994	1000	1000
		LR	1000	1000	992	796	289	52	311	816	995	1000	1000
(.9,-.6)	.05	ρ	1000	1000	993	813	281	45	292	801	985	1000	1000
		LR	1000	1000	993	822	294	53	311	812	986	1000	1000
(.9,-.9)	.01	ρ	1000	1000	989	815	268	48	288	826	989	1000	1000
		LR	1000	1000	992	826	294	59	312	839	992	1000	1000

TABLE 5.2 (Continued)

COMPARISON OF THE LIKELIHOOD RATIO TEST
AND THE TEST BASED ON THE VALUE OF ρ
NUMBER OF REJECTIONS AT 1% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
		THEO	1000	999	966	604	120	10	120	604	966	999	1000
(0.0,0.0)	1.00	ρ	1000	1000	958	607	130	9	120	595	956	1000	1000
		LR	1000	1000	960	622	134	11	129	604	960	1000	1000
(.3, .3)	1.00	ρ	1000	1000	947	606	106	10	131	587	958	1000	1000
		LR	1000	1000	949	616	112	10	139	603	961	1000	1000
(.6, .6)	1.00	ρ	1000	1000	968	600	119	10	117	605	964	1000	1000
		LR	1000	1000	968	617	127	10	124	617	965	1000	1000
(.9, .9)	1.00	ρ	1000	1000	965	617	117	4	108	580	956	1000	1000
		LR	1000	1000	968	645	128	7	117	604	962	1000	1000
(0.0, .1)	.99	ρ	1000	1000	962	599	124	6	131	593	966	1000	1000
		LR	1000	1000	963	614	129	6	136	607	967	1000	1000
(.3, .4)	.99	ρ	1000	1000	955	596	126	10	122	590	957	1000	1000
		LR	1000	1000	960	612	134	10	125	609	960	1000	1000
(.6, .7)	.97	ρ	1000	1000	960	617	117	9	121	641	964	1000	1000
		LR	1000	1000	965	635	122	10	130	657	965	1000	1000
(0.0, .3)	.91	ρ	1000	1000	958	586	103	14	127	622	964	1000	1000
		LR	1000	1000	962	599	109	14	132	634	965	1000	1000
(.3, .6)	.87	ρ	1000	1000	956	606	120	9	105	594	962	1000	1000
		LR	1000	1000	961	617	131	10	113	607	965	1000	1000

TABLE 5.2 (Continued)

(.3,-.3)	.70	ρ	1000	1000	958	621	138	8	117	608	947	1000	1000
		LR	1000	1000	959	636	145	9	124	623	952	1000	1000
(0.0, .6)	.64	ρ	1000	1000	963	587	128	14	121	599	961	1000	1000
		LR	1000	1000	967	609	134	14	127	614	964	1000	1000
(.6, .9)	.58	ρ	1000	1000	962	600	116	9	110	581	971	1000	1000
		LR	1000	1000	967	623	125	10	118	597	973	1000	1000
(.6,-.3)	.42	ρ	1000	1000	962	606	127	14	127	605	966	1000	1000
		LR	1000	1000	962	618	134	14	134	620	969	1000	1000
(.3, .9)	.32	ρ	1000	1000	964	589	132	9	114	604	960	1000	1000
		LR	1000	1000	968	616	146	10	125	618	962	1000	1000
(.6,-.6)	.22	ρ	1000	1000	959	585	128	8	116	587	957	1000	1000
		LR	1000	1000	963	600	133	9	129	595	962	1000	1000
(0.0, .9)	.19	ρ	1000	1000	959	584	118	7	125	597	966	1000	1000
		LR	1000	1000	964	601	124	7	137	616	969	1000	1000
(.9,-.3)	.11	ρ	1000	1000	966	583	109	13	113	583	969	1000	1000
		LR	1000	1000	971	595	120	15	134	599	973	1000	1000
(.9,-.6)	.05	ρ	1000	1000	954	605	122	10	101	587	947	1000	1000
		LR	1000	1000	958	623	131	12	111	600	955	1000	1000
(.9,-.9)	.01	ρ	1000	1000	958	611	112	9	129	621	961	1000	1000
		LR	1000	1000	965	631	121	12	136	636	964	1000	1000

NOTE : $A = (1. - \phi_1^2)(1. - \phi_2^2)/(1. - \phi_1\phi_2)^2$

Theoretical values are based on the asymptotic distribution of $\bar{\rho} \sim N(\rho,(1-\rho^2)^2/N)$

5.3 TESTING FOR EQUALITY OF THE PARAMETERS

In this section, the test of the hypothesis H_o: $\underset{\sim}{\beta}_h = \underset{\sim}{\beta}_1$, h=2, ... ,k against the alternative H_1: $\underset{\sim}{\beta}_h = \underset{\sim}{\beta}_1$ for at least one h is considered The distribution of the parameter estimates under the null and the alternative hypotheses is considered in the following two lemmas:

Lemma 5.3.1 Let $\hat{\underset{\sim}{\beta}}$ denote the MLE of $\underset{\sim}{\beta} = (\underset{\sim}{\beta}_1', \ldots ,\underset{\sim}{\beta}_k')'$ under H_1. If the null hypothesis is true, the asymptotic distribution of $\sqrt{N}(\hat{\underset{\sim}{\beta}} - \underset{\sim}{\beta})$ is normal with zero and variance covariance given by:

$$V_{\hat{\beta}} = [\ (\sigma^{gh} \ \sigma_{gh}) \otimes I \ ]^{-1}$$

$$\Delta^{-1} = (\sigma^{gh})$$

where

$$I = \begin{pmatrix} \gamma_{VV}(i - j) & \gamma_{VU}(i - j) \\ \hline \gamma_{UV}(i - j) & \gamma_{UU}(i - j) \end{pmatrix},$$

were U and V are the auxiliary series given by:

$$\theta_1(B) \ V_t = a_t$$

$$\phi_1(B) \ U_t = -a_t$$

with a_t NID (0, 1),

$$\gamma_{cd} = \langle \ C_{t-1} \ \ d_{t-j} \ \rangle$$

and c, d standing for U or V. $\langle \cdot \rangle$ denotes expectation and $A \otimes B$

denotes the Kronecker product of matrices.

Proof: It follows as a corollary of Lemma 2.2.2 observing that under H_o, $I_{gh} = \sigma_{gh} I$.

Lemma 5.3.2 The distribution of $\sqrt{N}(\hat{\underset{\sim}{\beta}}_o - \underset{\sim}{\beta}_1)$, where $\hat{\underset{\sim}{\beta}}_o$ denotes the estimator of $\underset{\sim}{\beta}_h$ imposing the restriction $\underset{\sim}{\beta} = \underset{\sim}{B}_1$, is asymptotically normal with zero mean and variance covariance matrix given by:

$$V_{\hat{\beta}_o} = \frac{1}{k} I^{-1}$$

where I is given in lemma 5.3.1.

Proof: The proof follows the lines of the proof of lemma 2.2.2. It is only observed here that

$$\begin{aligned}\left\langle -\frac{\partial^2 S}{\partial\beta_i\,\partial\beta_j}\right\rangle/N &= \text{trace}\left\{\Delta^{-1}\cdot\left\langle \sum_{t=1}^{N} W^{(i)} W^{(j)}_{t-j}\right\rangle/_N\right\}\\ &= \text{trace}\left\{\Delta^{-1}\gamma_W(i)_W(j)(i-j)\right\}\\ &= K\,\gamma_W(i)_W(j)\,(i-j)\end{aligned}$$

so that $V_{\hat{\beta}}^{-1} = k.I.$

The result of lemma 5.3.2 shows that if indeed the null hypothesis were true, the asymptotic variance of the estimated parameters of $\underset{\sim}{\beta}_1$ obtained imposing the restrictions of the null hypothesis, is the same as the asymptotic variance of the univariate estimated parameters same for $\underset{\sim}{\beta}_1$ when the sample size of Z_{1t} is increased by a factor of k.

Three asymptotically equivalent test statistics can be used to test the null hypothesis $H_o : \underset{\sim}{\beta}_h = \underset{\sim}{\beta}_1 \quad h = 2, \ldots, k$, against $H_o : \underset{\sim}{\beta}_h \neq \underset{\sim}{\beta}_1$ for at least one h. These statistics are the likelihood ratio test, the Wald test and the Lagrange multiplier test (see Harvey, 1981; Cox and Hinkley, 1974). The likelihood ratio test is given by:

$$LR = N \log (|\hat{\Delta}_o| / |\hat{\Delta}|)$$

where $|\hat{\Delta}_o|$ is the determinant of the estimated variance covariance matrix when $\hat{\beta}_o$ is used and $|\hat{\Delta}|$ is the determinant of the estimated variance when $\hat{\beta}$ is used. The Wald test is given by

$$W = N(R\hat{\underset{\sim}{\beta}})' (R V_{\hat{\beta}} R)^{-1} (R\hat{\underset{\sim}{\beta}})$$

where

$$\underset{-}{R} = \begin{pmatrix} 1 & -1 & \cdots & 0 \\ \vdots & & & \\ 1 & 0 & \cdots & -1 \end{pmatrix}_{(K-1)x(p+q) / kx(p+q)}$$

1 is the $(p + q)x(p + q)$ identity matrix and $V_{\hat{\beta}}$ is given by lemma 2.2.2, evaluated at $\underset{\sim}{\beta} = \hat{\underset{\sim}{\beta}}$.

The Lagrange multiplier test can be calculated as:

$$LM = \frac{1}{N} \left(\frac{\partial S}{\partial \underset{\sim}{\beta}} \right)' V_{\underset{\sim}{\beta}} \left(\frac{\partial S}{\partial \underset{\sim}{\beta}} \right) \Bigg|_{\underset{\sim}{\beta} = \hat{\underset{\sim}{\beta}}_o}$$

where

$$S = \sum_{t=1}^{N} \underset{\sim}{a}_t' \Delta^{-1} \underset{\sim}{a}_t / 2$$

The three tests are asymptotically distributed χ^2 with (p+q)(k-1) degrees of freedom. However, the small sample properties of the test statistics are not known and it is very likely that they behave quite differently for small sample sizes.

Using the bivariate CAR(1) model and the simulation set-up described in Section 2.3.2, small sample properties of the three tests were compared. The number of rejections of the null hypothesis were recorded in each case. For a given test, the null hypothesis was rejected whenever the observed value of the test statistic exceeded the value $\chi^2 (1 - \alpha)$, where $\chi^2_1 (1 - \alpha)$ denotes the 100(1 - α)% quartile of the χ^2 distribution with one degree of freedom. Two values for the significance level, α = 5% and α = 1% were used. Table 5.3 gives the number of rejections for a sample size of 50 and Table 5.4 for a sample size of 200. The following points should be noted concerning the results of the simulation:

(a) In general the following relationship is observed among the power functions of the three tests: W > LR > LM. The last inequality is more marked. Berndt and Savin (1977) have found a similar relation among the three tests when testing for

TABLE 5.3

EMPIRICAL COMPARISON OF TEST STATISTICS FOR THE
HYPOTHESIS $\phi_1 = \phi_2$
NUMBER OF REJECTIONS AT 5% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES : 50
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	LR	64	56	53	53	63	58	45	40	42	55	39
		LM	55	46	47	44	51	50	41	33	38	46	33
		WALD	66	57	56	59	68	66	51	43	47	55	39
(.3, .3)	1.00	LR	55	57	64	62	58	55	54	64	55	56	45
		LM	47	48	53	48	46	43	44	57	47	49	39
		WALD	57	58	67	66	61	60	59	70	61	57	47
(.6, .6)	1.00	LR	65	55	39	51	52	58	59	57	45	53	65
		LM	50	41	26	36	41	41	39	40	28	39	41
		WALD	65	56	45	57	57	66	62	60	54	50	62
(.9, .9)	1.00	LR	51	61	53	59	56	47	57	44	70	60	52
		LM	19	18	14	15	12	13	12	12	22	18	22
		WALD	59	74	61	63	64	57	64	53	67	62	72
(0.0, .1)	.99	LR	345	109	80	82	83	66	91	86	93	100	367
		LM	319	97	67	73	74	58	76	80	76	82	346
		WALD	343	106	82	88	88	70	96	91	95	100	359
(.3, .4)	.99	LR	388	100	95	100	77	90	90	97	94	111	372
		LM	354	85	81	83	66	76	79	77	76	97	345
		WALD	375	107	101	103	84	102	93	106	97	117	356
(.6, .7)	.97	LR	467	134	124	97	93	105	100	106	108	137	508
		LM	393	101	92	62	58	63	63	67	76	93	419
		WALD	463	142	125	104	100	110	105	116	111	136	488
(0.0, .3)	.91	LR	991	504	367	346	354	315	313	340	388	491	992
		LM	988	471	336	310	329	295	292	311	366	469	991
		WALD	991	503	375	368	372	325	324	355	393	489	991

TABLE 5.3 (Continued)

(.3, .6)	.87	LR	992	574	457	420	397	386	388	419	444	546	996
		LM	985	518	392	360	349	347	325	373	395	495	993
		WALD	992	576	462	432	416	401	402	433	457	543	995
(.6, .9)	.58	LR	1000	787	674	648	598	614	604	641	669	814	1000
		LM	728	440	331	267	271	258	256	288	341	467	759
		WALD	1000	810	683	661	613	637	626	659	681	821	1000
(.3,−.3)	.70	LR	1000	959	890	860	846	848	841	867	891	953	1000
		LM	1000	947	878	841	835	835	832	860	883	948	1000
		WALD	1000	959	897	866	856	860	850	866	896	955	1000
(0.0, .6)	.64	LR	1000	967	921	888	888	864	864	889	902	964	1000
		LM	1000	955	902	868	869	849	848	877	889	955	1000
		WALD	1000	966	922	896	895	873	867	893	911	967	1000
(.3, .9)	.32	LR	1000	995	977	978	961	953	964	975	974	996	1000
		LM	837	870	854	859	849	842	843	845	843	878	846
		WALD	1000	995	979	976	963	958	966	976	978	995	1000
(.6,−.3)	.42	LR	1000	1000	998	997	998	990	993	992	997	1000	1000
		LM	1000	1000	998	998	994	988	992	991	995	1000	1000
		WALD	1000	1000	997	997	998	992	996	992	999	1000	1000
(0.0, .9)	.19	LR	1000	1000	1000	999	999	999	998	999	1000	1000	1000
		LM	862	927	919	942	937	937	941	936	947	916	873
		WALD	1000	1000	1000	999	1000	999	998	999	1000	1000	1000
(.6,−.6)	.22	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.9,−.3)	.11	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	904	932	944	952	947	962	949	958	964	947	918
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.9,−.6)	.05	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	944	956	961	953	967	964	959	958	953	960	946
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

TABLE 5.3 (Continued)

EMPIRICAL COMPARISON OF TEST STATISTICS FOR THE HYPOTHESIS $\phi_1 = \phi_2$
NUMBER OF REJECTIONS AT 1% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES: 50
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	LR	12	7	14	8	11	11	7	11	9	14	13
		LM	8	4	12	6	8	8	5	9	7	9	10
		WALD	11	7	16	14	11	13	11	13	8	12	11
(.3, .3)	1.00	LR	13	10	18	8	18	14	15	13	15	6	11
		LM	10	4	8	5	13	11	7	11	8	4	8
		WALD	11	11	17	10	18	16	18	18	16	6	7
(.6, .6)	1.00	LR	14	9	8	14	12	12	9	15	8	4	11
		LM	9	5	3	6	3	5	4	5	4	2	4
		WALD	16	11	7	18	12	13	13	17	11	7	11
(.9, .9)	1.00	LR	6	11	12	19	17	13	11	7	18	12	17
		LM	1	1	1	1	1	0	1	0	0	0	4
		WALD	18	14	15	19	22	10	19	9	20	15	22
(0.0, .1)	.99	LR	167	32	27	30	26	13	23	32	22	27	164
		LM	137	28	23	23	24	12	16	27	19	19	135
		WALD	150	28	32	32	29	17	23	36	23	26	151
(.3, .4)	.99	LR	184	30	38	27	25	28	23	20	23	29	186
		LM	146	20	29	19	14	19	18	13	15	24	146
		WALD	166	31	38	28	28	34	24	21	33	31	161
(.6, .7)	.97	LR	236	47	39	24	22	23	33	27	30	33	261
		LM	139	18	11	8	3	6	6	8	9	13	140
		WALD	220	37	45	30	28	25	35	31	30	34	236
(0.0, .3)	.91	LR	955	284	182	155	164	134	141	157	189	288	968
		LM	943	244	148	125	130	117	107	131	155	237	948
		WALD	948	276	194	162	173	150	160	172	194	280	955

TABLE 5.3 (Continued)

(.3, .6)	.87	LR	971	306	226	191	176	183	171	207	236	323	982
		LM	938	227	154	131	120	123	123	126	165	228	939
		WALD	970	311	239	211	185	201	195	216	241	323	979
(.6, .9)	.58	LR	998	560	413	370	343	317	340	369	415	560	998
		LM	407	87	[illegible]	23	19	16	16	18	34	56	404
		WALD	1000	566	438	407	377	362	372	401	433	571	1000
(.3,-.3)	.70	LR	1000	864	740	697	681	657	663	722	740	880	1000
		LM	1000	841	715	657	633	624	627	675	689	847	1000
		WALD	1000	864	746	710	696	676	686	736	753	877	1000
(0.0, .6)	.64	LR	1000	892	781	740	732	704	712	721	774	907	1000
		LM	1000	855	727	688	668	637	647	669	727	869	1000
		WALD	1000	896	795	760	759	727	734	744	789	914	1000
(.3, .9)	.32	LR	1000	977	923	922	888	853	885	909	928	975	1000
		LM	567	443	369	362	322	331	337	336	379	469	558
		WALD	1000	981	935	940	896	872	900	922	936	985	1000
(.6,-.3)	.42	LR	1000	1000	990	985	980	965	976	976	991	998	1000
		LM	1000	999	986	979	973	962	967	971	987	998	1000
		WALD	1000	1000	990	986	982	967	980	978	990	998	1000
(0.0, .9)	.19	LR	1000	1000	999	997	996	992	993	998	998	999	1000
		LM	595	678	683	692	707	692	701	691	696	667	624
		WALD	1000	1000	999	996	997	994	993	998	999	1000	1000
(.6,-.6)	.22	LR	1000	1000	1000	1000	1000	1000	1000	998	1000	1000	1000
		LM	1000	1000	1000	1000	999	1000	1000	998	999	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	998	1000	1000	1000
(.9,-.3)	.11	LR	1000	1000	1000	1000	1000	1000	999	1000	1000	1000	1000
		LM	716	746	770	761	805	772	778	773	832	779	718
		WALD	1000	1000	1000	1000	1000	1000	999	1000	1000	1000	1000
(.9,-.6)	.05	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	788	805	825	797	807	824	817	832	816	810	775
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

TABLE 5.4

EMPIRICAL COMPARISON OF TEST STATISTICS FOR THE HYPOTHESIS $\phi_1 = \phi_2$
NUMBER OF REJECTIONS AT 5% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	LR	44	66	44	50	57	51	56	53	52	54	63
		LM	43	63	43	49	54	51	56	52	52	53	61
		WALD	44	66	44	51	58	50	58	54	52	56	62
(.3, .3)	1.00	LR	49	42	59	52	41	44	51	53	45	54	66
		LM	43	40	56	51	38	43	49	49	44	53	64
		WALD	49	41	57	53	43	45	52	55	46	55	66
(.6, .6)	1.00	LR	49	59	51	54	54	48	57	38	50	43	42
		LM	46	60	49	48	51	41	54	36	46	37	40
		WALD	46	57	53	53	56	49	56	38	51	45	39
(.9, .9)	1.00	LR	54	50	49	44	55	52	52	44	64	40	52
		LM	42	36	30	25	34	35	35	30	39	30	40
		WALD	47	56	51	48	57	58	55	46	61	48	55
(0.0, .1)	.99	LR	868	256	189	173	173	169	166	176	190	253	872
		LM	865	249	185	171	169	16[illegible]	161	173	187	244	870
		WALD	870	255	190	175	172	172	170	178	194	257	871
(.3, .4)	.99	LR	896	279	214	183	197	201	200	223	212	264	910
		LM	893	270	208	180	190	196	192	216	202	260	908
		WALD	890	282	214	185	198	208	205	222	214	267	913
(.6, .7)	.97	LR	977	374	310	303	256	260	235	278	296	396	982
		LM	973	358	294	286	235	244	224	261	287	383	973
		WALD	979	377	313	306	260	267	241	279	296	396	981
(0.0, .3)	.91	LR	1000	974	903	879	860	864	859	882	919	965	1000
		LM	1000	973	902	878	858	862	858	879	913	963	1000
		WALD	1000	974	904	881	862	863	862	884	922	966	1000

TABLE 5.4 (Continued)

(.3, .6)	.87	LR	1000	986	949	943	918	926	926	937	951	988	1000
		LM	1000	985	945	939	913	922	921	933	948	988	1000
		WALD	1000	986	950	945	919	927	926	938	951	987	1000
(.6, .9)	.58	LR	1000	1000	997	994	991	996	997	996	999	999	1000
		LM	1000	1000	997	994	989	994	995	996	999	999	1000
		WALD	1000	1000	998	994	991	996	997	997	998	999	1000
(.3,-.3)	.70	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(0.0, .6)	.64	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.3, .9)	.32	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.6,-.3)	.42	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(0.0, .9)	.19	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.6,-.6)	.22	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.9,-.3)	.11	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.9,-.6)	.05	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

TABLE 5.4 (Continued)

EMPIRICAL COMPARISON OF TEST STATISTICS FOR THE HYPOTHESIS $\phi_1 = \phi_2$
NUMBER OF REJECTIONS AT 1% SIGNIFICANCE LEVEL
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	LR	6	15	6	6	14	7	11	12	12	12	11
		LM	4	14	6	6	12	7	10	10	12	11	11
		WALD	6	15	6	6	15	7	11	15	12	12	11
(.3, .3)	1.00	LR	10	11	10	15	10	10	7	10	8	12	15
		LM	8	9	10	12	9	9	5	9	7	10	15
		WALD	9	11	10	14	10	10	9	10	8	10	14
(.6, .6)	1.00	LR	9	10	18	11	11	6	10	12	9	12	10
		LM	8	6	17	9	9	5	7	7	9	10	9
		WALD	11	10	17	12	15	6	11	11	10	12	10
(.9, .9)	1.00	LR	6	11	10	4	10	2	12	8	9	6	7
		LM	1	2	6	1	3	0	2	1	2	2	1
		WALD	8	9	11	5	14	2	12	10	10	9	8
(0.0, .1)	.99	LR	708	90	73	62	56	56	55	62	61	90	668
		LM	694	86	72	58	54	51	51	57	57	88	655
		WALD	698	89	74	63	57	61	56	62	62	92	662
(.3, .4)	.99	LR	743	122	65	57	74	80	74	78	86	99	765
		LM	727	113	61	53	69	70	70	73	80	93	745
		WALD	737	118	66	58	79	83	75	80	86	98	758
(.6, .7)	.97	LR	924	177	127	122	91	91	88	117	141	189	913
		LM	906	151	113	109	76	80	75	100	114	169	892
		WALD	919	176	132	124	94	96	92	119	139	186	914
(0.0, .3)	.91	LR	1000	911	775	719	678	674	692	714	771	902	1000
		LM	1000	908	766	705	672	663	680	710	762	896	1000
		WALD	1000	911	778	724	687	680	700	716	768	902	1000

TABLE 5.4 (Continued)

(.3, .6)	.87	LR	1000	944	858	843	784	786	808	811	843	958	1000
		LM	1000	935	848	825	775	767	793	801	833	953	1000
		WALD	1000	946	860	847	790	793	813	817	847	957	1000
(.3,-.3)	.70	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.6, .9)	.58	LR	1000	998	990	980	968	979	974	988	991	998	1000
		LM	1000	996	986	962	962	965	962	978	987	997	998
		WALD	1000	998	988	980	968	978	972	988	991	998	1000
(0.0, .6)	.64	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.3, .9)	.32	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.6,-.3)	.42	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(0.0, .9)	.19	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.6,-.6)	.22	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.9,-.3)	.11	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
(.9,-.6)	.05	LR	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		LM	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
		WALD	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

linear restrictions in the multiple regression model.

(b) The power of the test strongly depends on the value of ρ, particularly for small departures from the null hypothesis, i.e. for models with small values of $|\phi_1 - \phi_2|$. The power increases with the value of $|\rho|$.

(c) As the sample size increases, the differences among the three tests diminish.

Although the scope of the simulation is very limited, some conclusions may be drawn for the general case:

(a) Even though the tests are asymptotically equivalent, they may give conflicting results for small sample sizes.

(b) The power of the tests depend not only on the degree of departure from the null hypothesis, but also on the correlation structure of the model innovations.

(c) If a specific test statistic needs to be chosen, computational convenience should also be taken into account. In this regard, the Wald test may be preferable because it is very easy to estimate the unrestricted model.

The restricted estimator was iteratively calculated in the simulation as:

$$\hat{\phi}_o = \text{trace } \Delta^{-1}\tau_1 / \text{trace } \Delta^{-1}\tau_o \tag{5.3.1}$$

where

$$\tau_o = \sum_{t=2}^{N} \underset{\sim}{Z}_{t-1} \underset{\sim}{Z}'_{t-1}$$

$$\tau_1 = \sum_{t=2}^{N} \underset{\sim}{Z}_{t-1} \underset{\sim}{Z}_t'$$

A maximum of 10 iterations were allowed. The initial value was set to $\hat{\phi}_o = (\bar{\phi}_1 + \bar{\phi}_2)/2$ where $\bar{\phi}_i$ was the univariate MLE of ϕ_i. Appendix 3 lists the summary statistics of the restricted estimators. In particular, the mean, the standard deviation, the MSE, the relative efficiency of $\hat{\phi}_i$ with respect to $\hat{\phi}_o$ given by:

$$\text{eff} = V(\hat{\phi}_o)/V(\hat{\phi}_i)$$

with their respective standard errors and the number of iterations required to obtain convergence are listed. It is observed from the tables of Appendix 3 that, in general, there is a gain in efficiency when the restricted estimator is used. It can be shown that the expected efficiency value is given by

$$\text{eff} = (1 + \rho^2)/2$$

which agrees with the observed efficiency values.

CHAPTER 6

SUMMARY AND CONCLUSIONS

A comprehensive study of the statistical properties of the contemporaneously only (CARMA) model has been made. This CARMA model, or more generally the set of contemporaneously correlated transfer function models (discussed in section 2.4), provide a more general framework for the analysis of many actual time series than the SURE model (Zellner, 1962).

It has been shown that while both the univariate and the joint estimation procedures are consistent, the joint estimation procedure is asymptotically efficient. The gain in efficiency from using joint estimation has been considered in detail for the bivariate CAR(1) model using both asymptotic theory and small sample simulation.

A computationally efficient procedure has been proposed for calculating the joint estimates. In Chapter 3 a new useful procedure has been given to include the case of the CARMA model with unequal sample sizes, a situation which occurs frequently in practice, thus avoiding the waste of valuable additional information.

Application of the CARMA model to four sets of hydrological time series has been presented in Chapter 4, where it was shown how often the physical restrictions of the system suggest that the CARMA model wold be appropriate.

Test statistics for two important hypothesis have been considered in Chapter 5: - (a) whether a joint model is required or whether a set of univariate models will suffice and (b) whether $\underset{\sim}{\beta}_h = \underset{\sim}{\beta}$ or otherwise, where $\underset{\sim}{\beta}_h$ is a vector of parameters for the series h. For the first of these hypothesis it has been shown that both the likelihood ratio test and the test based on the significance of the correlation of the pre-whitened series are asymptotically equivalent.

APPENDIX 1

CYBER-FTN5 VERSION OF THE RSUPER RANDOM NUMBER GENERATOR

```
C RSUPER
C CYBER-FTN5 VERSION OF THE RANDOM NUMBER GENERATOR SUPER-DUPER
C
      FUNCTION RSUPER(JUNK)
      COMMON /RNDM/ ISEED, JSEED
C
      BOOLEAN MASK, I1, I2, I3 , IS1, ISE, JSE
C                   2.0**-32
      DATA TPM32/2.3283064365386 9E-10/
C
C MASK - HAS 1'S IN THE 32 RIGHTMOST BIT POSITIONS AND 0'S ELSEWHERE
C
      DATA MASK/O'00000000037777777777'/
C
C GENERATE RANDOM INTEGER USING CONGRUENTIAL GENERATOR
C
      ISEED = ISEED*69069
      ISE = AND(ISEED, MASK)
      ISEED = ISE
C
C GENERATE RANDOM INTEGER USING SHIFT REGISTER GENERATOR
C
      I1 = SHIFT(JSEED, -15)
      I2 = XOR(I1, JSEED)
      I3 = SHIFT(I2, 17)
      I3 = AND(I3 , MASK)
      JSE = XOR(I2, I3 )
      JSEED = JSE
C
C COMBINE AND CONVERT TO UNIFORM(0,1) VARIABLE
C
      IS1 = XOR(ISE, JSE)
      IF (IS1 .EQ. 0) RSUPER = 0.5
      RSUPER = FLOAT(IS1)*TPM32
      RETURN
      END
```

APPENDIX 2.

SIMULATION RESULTS FOR THE SCORE ALGORITHM

This Appendix reports the results of the simulation study of Section 2.3.2 concerning the efficiency of the Score Algorithm of Lemma 2.2.3 for the estimation of the parameters of the CARMA model. In the Tables

$$A = (1. - \phi_1^2)(1. - \phi_2^2)/(1. - \phi_1\phi_2)^2$$

and the values in parentheses indicate the Standard Errors.

TABLE A.2.1

EFFICIENCY VALUES OF THE SCORE ESTIMATORS
RELATIVE TO THE JOINT ESTIMATORS
NUMBER OF OBSERVATIONS PER SERIES : 50
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	ϕ_1	.975	.997	1.005	1.006	1.007	1.005	1.006	1.004	1.004	.999	.985
			(.007)	(.003)	(.002)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.003)	(.008)
		ϕ_2	.978	1.001	1.008	1.004	1.006	1.004	1.010	1.004	1.009	1.001	.991
			(.007)	(.004)	(.002)	(.002)	(.001)	(.001)	(.002)	(.001)	(.002)	(.003)	(.007)
(.3, .3)	1.00	ϕ_1	.967	.997	1.004	1.004	1.006	1.005	1.005	1.004	1.004	1.007	.997
			(.008)	(.003)	(.003)	(.002)	(.001)	(.001)	(.002)	(.002)	(.002)	(.004)	(.009)
		ϕ_2	.970	1.000	1.005	1.006	1.006	1.007	1.003	1.003	1.004	1.005	.981
			(.007)	(.004)	(.002)	(.002)	(.001)	(.002)	(.001)	(.001)	(.002)	(.004)	(.008)
(.6, .6)	1.00	ϕ_1	.975	.997	1.000	1.000	1.002	1.003	1.003	.996	1.000	.996	.980
			(.008)	(.003)	(.002)	(.002)	(.002)	(.002)	(.002)	(.002)	(.002)	(.004)	(.008)
		ϕ_2	.973	.994	1.000	1.000	1.001	.998	1.000	1.001	.998	.996	.984
			(.008)	(.004)	(.002)	(.003)	(.001)	(.001)	(.001)	(.002)	(.002)	(.004)	(.009)
(.9, .9)	1.00	ϕ_1	.979	.979	.977	.968	.969	.977	.971	.974	.981	.986	.926
			(.018)	(.010)	(.007)	(.007)	(.007)	(.006)	(.007)	(.007)	(.007)	(.011)	(.025)
		ϕ_2	1.000	.971	.989	.977	.970	.974	.975	.975	.988	.946	.961
			(.019)	(.010)	(.008)	(.008)	(.007)	(.006)	(.006)	(.007)	(.007)	(.012)	(.017)
(0.0, .1)	.99	ϕ_1	.970	1.001	1.004	1.006	1.007	1.006	1.005	1.005	1.006	1.000	.966
			(.009)	(.003)	(.002)	(.002)	(.001)	(.001)	(.001)	(.001)	(.002)	(.003)	(.008)
		ϕ_2	.966	1.003	1.003	1.005	1.006	1.008	1.005	1.008	1.007	1.009	.970
			(.008)	(.003)	(.002)	(.002)	(.001)	(.002)	(.002)	(.002)	(.002)	(.004)	(.008)
(.3, .4)	.99	ϕ_1	.984	1.001	1.005	1.007	1.004	1.005	1.006	1.005	1.000	.996	.975
			(.008)	(.003)	(.002)	(.002)	(.001)	(.001)	(.001)	(.001)	(.002)	(.003)	(.009)
		ϕ_2	.987	.998	1.007	1.004	1.003	1.004	1.003	1.007	1.003	.998	.978
			(.008)	(.003)	(.003)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.004)	(.009)

TABLE A.2.1 (Continued)

(.6, .7)	.97	ϕ_1	.966	1.001	1.003	.998	1.002	1.001	1.003	.996	1.001	.995	.969
			(.009)	(.004)	(.002)	(.001)	(.002)	(.001)	(.002)	(.002)	(.003)	(.004)	(.010)
		ϕ_2	.945	.996	.994	.994	.996	.998	.997	.995	.998	.994	.960
			(.009)	(.003)	(.003)	(.002)	(.002)	(.002)	(.001)	(.002)	(.003)	(.004)	(.010)
(0.0, .3)	.91	ϕ_1	.955	1.003	1.007	1.006	1.004	1.005	1.007	1.006	1.003	.999	.965
			(.010)	(.003)	(.002)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.003)	(.010)
		ϕ_2	.961	.998	1.005	1.005	1.004	1.003	1.006	1.007	1.003	1.006	.971
			(.010)	(.004)	(.002)	(.002)	(.002)	(.001)	(.001)	(.002)	(.003)	(.003)	(.009)
(.3, .6)	.87	ϕ_1	.963	.996	1.001	1.004	1.003	1.004	1.006	1.006	1.004	.998	.959
			(.010)	(.003)	(.002)	(.002)	(.001)	(.001)	(.002)	(.002)	(.002)	(.004)	(.010)
		ϕ_2	.965	.995	1.002	.997	1.002	1.001	.999	.999	1.004	.991	.957
			(.009)	(.004)	(.002)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.003)	(.009)
(.3, -.3)	.70	ϕ_1	.945	.998	1.004	1.004	1.004	1.003	1.004	1.005	1.000	.998	.968
			(.012)	(.003)	(.002)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.004)	(.010)
		ϕ_2	.945	.999	1.005	1.004	1.003	1.008	1.005	1.002	1.000	1.001	.962
			(.012)	(.004)	(.002)	(.002)	(.001)	(.002)	(.001)	(.002)	(.002)	(.003)	(.010)
(0.0, .6)	.64	ϕ_1	.940	1.005	1.006	1.004	1.004	1.005	1.004	1.003	1.005	1.005	.939
			(.013)	(.003)	(.002)	(.002)	(.001)	(.001)	(.001)	(.001)	(.002)	(.004)	(.011)
		ϕ_2	.959	.991	1.000	1.000	.996	1.000	.999	.998	.995	1.002	.948
			(.011)	(.003)	(.002)	(.002)	(.002)	(.001)	(.001)	(.002)	(.003)	(.003)	(.011)
(.6, .9)	.58	ϕ_1	.866	.996	.997	.998	1.000	1.004	1.002	1.002	.996	.995	.838
			(.019)	(.006)	(.003)	(.003)	(.002)	(.003)	(.002)	(.002)	(.003)	(.007)	(.019)
		ϕ_2	.903	.967	.966	.969	.966	.978	.975	.965	.962	.957	.877
			(.026)	(.007)	(.006)	(.006)	(.005)	(.006)	(.006)	(.007)	(.006)	(.008)	(.022)
(.6, -.3)	.42	ϕ_1	.953	1.000	1.001	.999	1.003	.999	1.002	.995	1.003	.998	.946
			(.015)	(.003)	(.003)	(.002)	(.002)	(.001)	(.001)	(.002)	(.004)	(.004)	(.014)
		ϕ_2	.961	1.000	1.000	1.003	1.004	1.008	1.004	1.001	1.005	.999	.938
			(.014)	(.003)	(.003)	(.002)	(.001)	(.003)	(.002)	(.002)	(.002)	(.004)	(.014)

TABLE A.2.1 (Continued)

(.3, .9)	.32	ϕ_1	.887	1.002	1.002	1.003	1.004	1.005	1.005	1.002	1.003	1.003	.869
			(.021)	(.007)	(.005)	(.003)	(.002)	(.002)	(.002)	(.002)	(.003)	(.006)	(.017)
		ϕ_2	.859	.959	.976	.969	.978	.977	.971	.974	.961	.949	.911
			(.024)	(.009)	(.007)	(.007)	(.005)	(.006)	(.006)	(.006)	(.007)	(.009)	(.030)
(.6,-.6)	.22	ϕ_1	.939	.994	.996	.998	.999	.998	.998	1.000	.998	.989	.946
			(.013)	(.004)	(.002)	(.002)	(.001)	(.001)	(.001)	(.002)	(.003)	(.005)	(.014)
		ϕ_2	.968	.994	.999	.998	.999	1.002	1.000	1.000	.999	.992	.943
			(.013)	(.004)	(.002)	(.002)	(.002)	(.002)	(.001)	(.002)	(.002)	(.004)	(.014)
(0.0, .9)	.19	ϕ_1	.861	.990	1.004	1.002	1.005	1.005	1.005	1.002	1.005	.995	.878
			(.026)	(.006)	(.003)	(.002)	(.002)	(.002)	(.002)	(.002)	(.003)	(.006)	(.019)
		ϕ_2	.867	.974	.958	.959	.974	.972	.978	.967	.968	.963	.906
			(.023)	(.009)	(.008)	(.008)	(.005)	(.005)	(.006)	(.006)	(.007)	(.009)	(.024)
(.9,-.3)	.11	ϕ_1	.890	.964	.972	.982	.960	.972	.966	.973	.963	.963	.849
			(.022)	(.007)	(.007)	(.006)	(.006)	(.005)	(.006)	(.005)	(.006)	(.008)	(.019)
		ϕ_2	.899	.987	1.002	1.001	1.005	1.005	1.007	1.004	1.001	.991	.913
			(.018)	(.005)	(.003)	(.002)	(.002)	(.002)	(.002)	(.003)	(.003)	(.005)	(.017)
(.9,-.6)	.05	ϕ_1	.858	.952	.957	.956	.967	.959	.970	.966	.961	.961	.844
			(.021)	(.011)	(.006)	(.006)	(.006)	(.008)	(.006)	(.007)	(.006)	(.009)	(.021)
		ϕ_2	.895	.988	.994	.995	1.000	1.000	1.003	1.000	.994	.990	.884
			(.021)	(.005)	(.003)	(.002)	(.002)	(.001)	(.002)	(.003)	(.003)	(.004)	(.020)
(.9,-.9)	.01	ϕ_1	.850	.960	.959	.964	.976	.964	.964	.963	.960	.950	.849
			(.023)	(.009)	(.007)	(.007)	(.006)	(.006)	(.007)	(.006)	(.007)	(.009)	(.023)
		ϕ_2	.804	.954	.953	.963	.967	.968	.964	.952	.972	.956	.876
			(.036)	(.009)	(.008)	(.006)	(.005)	(.006)	(.006)	(.006)	(.007)	(.009)	(.022)

TABLE A.2.2

EFFICIENCY VALUES OF THE SCORE ESTIMATORS
RELATIVE TO THE JOINT ESTIMATORS
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	A	ρ :	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	ϕ_1	.991	1.000	1.001	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.990
			(.002)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.002)
		ϕ_2	.991	1.002	1.000	1.000	1.000	1.000	1.000	1.001	1.000	1.000	.993
			(.002)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.002)
(.3, .3)	1.00	ϕ_1	.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.999	.989
			(.004)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.003)
		ϕ_2	.993	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.000	.999	.991
			(.003)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.002)
(.6, .6)	1.00	ϕ_1	.987	.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.988
			(.004)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.004)
		ϕ_2	.988	.998	1.001	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.992
			(.004)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.004)
(.9, .9)	1.00	ϕ_1	.940	.988	.998	.992	.994	.991	.993	.996	.997	.993	.922
			(.014)	(.005)	(.002)	(.002)	(.001)	(.002)	(.002)	(.002)	(.002)	(.005)	(.019)
		ϕ_2	.947	.995	.995	.993	.995	.991	.994	.993	.997	.994	.923
			(.015)	(.004)	(.002)	(.002)	(.001)	(.002)	(.001)	(.002)	(.003)	(.005)	(.019)
(0.0, .1)	.99	ϕ_1	.991	1.001	1.000	1.001	1.001	1.000	1.001	1.000	1.000	1.000	.993
			(.002)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.002)
		ϕ_2	.989	1.001	1.001	1.001	1.000	1.000	1.000	1.000	1.001	1.000	.993
			(.002)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.003)
(.3, .4)	.99	ϕ_1	.995	.999	1.000	1.001	1.000	1.001	1.000	1.000	1.001	.999	.990
			(.003)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.003)
		ϕ_2	.993	1.000	1.001	1.000	1.000	1.001	1.000	1.001	1.000	1.000	.990
			(.003)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.003)

TABLE A.2.2 (Continued)

(.6, .7)	.97	ϕ_1	.997	.999	.999	1.000	1.000	1.000	1.000	1.000	1.000	1.003	.981
			(.005)	(.002)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.002)	(.006)
		ϕ_2	.990	.998	1.000	1.000	1.000	.999	1.000	.999	.999	1.001	.986
			(.005)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.002)	(.006)
(0.0, .3)	.91	ϕ_1	.988	.997	1.001	1.000	1.000	1.000	1.000	1.000	1.001	1.000	.992
			(.004)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.004)
		ϕ_2	.985	1.000	1.000	1.000	1.001	1.001	1.000	1.000	1.000	1.000	.992
			(.003)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.003)
(.3, .6)	.87	ϕ_1	.986	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.000	1.002	.989
			(.006)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.006)
		ϕ_2	.993	.999	1.000	1.000	1.000	1.000	1.000	1.000	.998	1.000	.986
			(.005)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.005)
(.3, -.3)	.70	ϕ_1	.977	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.001	.999	.989
			(.006)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.006)
		ϕ_2	.983	1.000	1.000	1.000	1.001	1.001	1.001	1.000	1.001	.999	.983
			(.006)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.006)
(0.0, .6)	.64	ϕ_1	.977	.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.992
			(.006)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.007)
		ϕ_2	.977	.999	1.000	1.000	1.000	1.000	1.000	1.000	.999	1.000	.994
			(.005)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.006)
(.6, .9)	.58	ϕ_1	.916	.998	.998	1.000	1.000	1.000	1.000	1.000	1.000	.999	.942
			(.017)	(.003)	(.001)	(.001)	(.000)	(.000)	(.000)	(.001)	(.001)	(.003)	(.015)
		ϕ_2	.904	.995	.994	.997	.992	.994	.993	.995	.989	.997	.912
			(.021)	(.004)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.002)	(.003)	(.018)
(.6, -.3)	.42	ϕ_1	.988	1.000	1.000	.999	1.000	1.000	1.000	1.000	.999	1.000	.974
			(.005)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.006)
		ϕ_2	.980	.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.971
			(.006)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.007)

TABLE A.2.2 (Continued)

(.3 , .9)	.32	ϕ_1	.959	.995	1.000	1.000	1.000	1.000	1.000	.999	.999	.997	.946
			(.012)	(.002)	(.001)	(.001)	(.000)	(.000)	(.000)	(.001)	(.001)	(.003)	(.014)
		ϕ_2	.966	.995	.993	.994	.993	.992	.995	.997	.997	.996	.938
			(.014)	(.003)	(.002)	(.003)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.015)
(.6,-.6)	.22	ϕ_1	.988	.999	1.000	.999	1.000	1.000	1.000	1.001	.999	1.000	.981
			(.005)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.007)
		ϕ_2	.975	.996	.999	1.000	1.000	1.000	1.000	1.000	.999	.998	.983
			(.006)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.006)
(0.0, .9)	.19	ϕ_1	.967	.997	.999	1.000	1.000	1.000	1.000	1.000	.998	.994	.969
			(.010)	(.002)	(.001)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.002)	(.011)
		ϕ_2	.959	.985	.992	.996	.991	.994	.994	.992	.992	.992	.969
			(.012)	(.003)	(.002)	(.002)	(.002)	(.002)	(.001)	(.002)	(.002)	(.002)	(.015)
(.9,-.3)	.11	ϕ_1	.982	.993	.994	.991	.993	.993	.993	.995	.992	.991	.966
			(.009)	(.002)	(.002)	(.002)	(.001)	(.001)	(.001)	(.002)	(.002)	(.003)	(.009)
		ϕ_2	.973	.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.984
			(.008)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.002)	(.009)
(.9,-.6)	.05	ϕ_1	.964	.990	.991	.990	.990	.992	.992	.996	.988	.991	.976
			(.011)	(.002)	(.002)	(.002)	(.002)	(.002)	(.002)	(.001)	(.003)	(.002)	(.007)
		ϕ_2	.987	.998	.999	.999	1.000	1.000	1.000	1.000	1.000	.998	.984
			(.007)	(.001)	(.001)	(.000)	(.000)	(.000)	(.000)	(.000)	(.001)	(.001)	(.006)
(.9,-.9)	.01	ϕ_1	.958	.986	.991	.994	.994	.993	.994	.994	.994	.992	.971
			(.008)	(.002)	(.002)	(.001)	(.001)	(.001)	(.002)	(.001)	(.002)	(.002)	(.008)
		ϕ_2	.953	.990	.991	.990	.993	.989	.992	.995	.991	.990	.980
			(.008)	(.002)	(.002)	(.002)	(.002)	(.002)	(.002)	(.001)	(.003)	(.002)	(.011)

TABLE A.2.3

NUMBER OF ITERATIONS TO OBTAIN THE MLE
NUMBER OF OBSERVATIONS PER SERIES : 50
NUMBER OF REPLICATIONS : 1000

MODEL	ρ : A	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	2.04	1.92	1.69	1.57	1.42	1.36	1.46	1.57	1.69	1.87	2.04
		(.67)	(.63)	(.65)	(.62)	(.61)	(.58)	(.62)	(.63)	(.63)	(.63)	(.62)
(.3, .3)	1.00	2.09	1.93	1.70	1.59	1.45	1.41	1.49	1.61	1.74	1.91	2.10
		(.69)	(.66)	(.65)	(.65)	(.60)	(.59)	(.61)	(.62)	(.65)	(.66)	(.68)
(.6, .6)	1.00	2.27	2.03	1.82	1.73	1.56	1.52	1.53	1.68	1.82	2.04	2.21
		(.78)	(.70)	(.67)	(.69)	(.65)	(.65)	(.64)	(.65)	(.69)	(.73)	(.74)
(.9, .9)	1.00	3.82	3.44	3.09	2.86	2.71	2.70	2.72	2.86	3.02	3.58	3.87
		(2.24)	(2.11)	(1.97)	(1.79)	(1.72)	(1.83)	(1.73)	(1.89)	(1.84)	(2.18)	(2.30)
(0.0, .1)	.99	2.02	1.86	1.70	1.56	1.44	1.38	1.48	1.61	1.71	1.89	2.06
		(.65)	(.63)	(.63)	(.62)	(.61)	(.60)	(.63)	(.65)	(.62)	(.63)	(.71)
(.3, .4)	.99	2.14	1.92	1.73	1.62	1.49	1.42	1.50	1.63	1.72	1.94	2.11
		(.68)	(.63)	(.63)	(.64)	(.62)	(.58)	(.59)	(.67)	(.63)	(.66)	(.70)
(.6, .7)	.97	2.44	2.09	1.85	1.69	1.60	1.57	1.57	1.68	1.85	2.12	2.41
		(1.00)	(.73)	(.69)	(.69)	(.66)	(.63)	(.70)	(.68)	(.70)	(.82)	(.94)
(0.0, .3)	.91	2.17	1.91	1.73	1.61	1.47	1.38	1.49	1.59	1.76	1.89	2.17
		(.72)	(.60)	(.63)	(.64)	(.62)	(.60)	(.62)	(.64)	(.65)	(.63)	(.71)
(.3, .6)	.87	2.31	1.99	1.78	1.68	1.50	1.48	1.52	1.65	1.80	1.95	2.38
		(.78)	(.68)	(.67)	(.65)	(.61)	(.67)	(.59)	(.66)	(.66)	(.62)	(.92)
(.3,-.3)	.70	2.34	1.96	1.75	1.61	1.49	1.44	1.47	1.59	1.72	2.00	2.29
		(.78)	(.67)	(.67)	(.63)	(.60)	(.63)	(.59)	(.63)	(.63)	(.67)	(.77)
(0.0, .6)	.64	2.44	1.97	1.75	1.62	1.50	1.46	1.49	1.58	1.78	1.99	2.40
		(.87)	(.69)	(.64)	(.67)	(.63)	(.57)	(.64)	(.63)	(.73)	(.67)	(.89)

TABLE A.2.3 (Continued)

(.6, .9)	.58	4.47 (2.71)	2.86 (1.71)	2.51 (1.57)	2.36 (1.45)	2.22 (1.52)	2.16 (1.39)	2.31 (1.55)	2.33 (1.46)	2.54 (1.61)	2.95 (1.76)	4.36 (2.55)
(.6,-.3)	.42	2.40 (.94)	2.00 (.71)	1.82 (.73)	1.63 (.65)	1.49 (.62)	1.45 (.60)	1.53 (.64)	1.67 (.71)	1.80 (.75)	2.08 (.77)	2.47 (.91)
(.3, .9)	.32	4.29 (2.60)	2.91 (1.78)	2.43 (1.57)	2.25 (1.43)	2.09 (1.39)	2.05 (1.35)	2.19 (1.59)	2.36 (1.56)	2.45 (1.51)	2.81 (1.61)	4.14 (2.54)
(.6,-.6)	.22	2.66 (1.04)	2.06 (.80)	1.77 (.74)	1.70 (.68)	1.52 (.65)	1.48 (.65)	1.53 (.65)	1.66 (.68)	1.79 (.75)	2.09 (.83)	2.73 (1.12)
(0.0, .9)	.19	3.90 (2.50)	2.85 (1.69)	2.51 (1.56)	2.27 (1.51)	2.07 (1.38)	2.06 (1.40)	2.13 (1.46)	2.21 (1.43)	2.40 (1.40)	2.87 (1.76)	3.85 (2.42)
(.9,-.3)	.11	3.72 (2.35)	2.78 (1.75)	2.45 (1.51)	2.30 (1.49)	2.15 (1.37)	2.13 (1.54)	2.12 (1.36)	2.33 (1.52)	2.38 (1.45)	2.77 (1.66)	3.79 (2.40)
(.9,-.6)	.05	3.77 (2.29)	2.68 (1.61)	2.40 (1.60)	2.23 (1.49)	2.22 (1.42)	2.12 (1.50)	2.14 (1.46)	2.29 (1.51)	2.37 (1.48)	2.66 (1.66)	3.84 (2.43)
(.9,-.9)	.01	4.50 (2.89)	3.25 (2.03)	2.89 (2.03)	2.69 (1.80)	2.61 (1.80)	2.66 (1.85)	2.65 (1.78)	2.73 (1.83)	2.81 (1.81)	3.16 (1.91)	4.38 (2.78)

TABLE A.2.4

NUMBER OF ITERATIONS TO OBTAIN THE MLE
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	ρ : A	-.9	-.5	-.3	-.2	-.1	0.0	.1	.2	.3	.5	.9
(0.0,0.0)	1.00	1.36 (.48)	1.22 (.42)	1.09 (.30)	1.05 (.24)	.95 (.29)	.87 (.35)	.95 (.29)	1.06 (.24)	1.10 (.30)	1.21 (.41)	1.38 (.49)
(.3, .3)	1.00	1.43 (.50)	1.23 (.43)	1.10 (.30)	1.05 (.23)	.99 (.25)	.93 (.27)	.99 (.20)	1.06 (.25)	1.10 (.31)	1.22 (.41)	1.45 (.50)
(.6, .6)	1.00	1.61 (.52)	1.38 (.49)	1.15 (.36)	1.05 (.24)	.99 (.21)	.96 (.23)	.99 (.21)	1.05 (.23)	1.13 (.35)	1.34 (.48)	1.57 (.52)
(.9, .9)	1.00	1.90 (.73)	1.63 (.59)	1.42 (.53)	1.31 (.51)	1.17 (.44)	1.12 (.51)	1.17 (.47)	1.27 (.49)	1.43 (.53)	1.65 (.57)	1.91 (.79)
(0.0, .1)	.99	1.35 (.48)	1.21 (.41)	1.10 (.30)	1.04 (.21)	.98 (.25)	.86 (.36)	.95 (.29)	1.04 (.23)	1.10 (.31)	1.20 (.40)	1.34 (.48)
(.3, .4)	.99	1.50 (.51)	1.26 (.44)	1.09 (.29)	1.05 (.25)	.99 (.23)	.94 (.27)	.98 (.23)	1.05 (.23)	1.10 (.30)	1.25 (.43)	1.50 (.51)
(.6, .7)	.97	1.66 (.52)	1.40 (.49)	1.17 (.37)	1.06 (.24)	.99 (.23)	.95 (.25)	.99 (.19)	1.06 (.27)	1.17 (.38)	1.41 (.50)	1.68 (.53)
(0.0, .3)	.91	1.50 (.51)	1.23 (.42)	1.10 (.30)	1.05 (.25)	.97 (.25)	.89 (.34)	.97 (.26)	1.05 (.23)	1.10 (.31)	1.22 (.41)	1.53 (.51)
(.3, .6)	.87	1.62 (.51)	1.33 (.47)	1.12 (.33)	1.06 (.25)	.99 (.23)	.94 (.25)	.98 (.20)	1.06 (.25)	1.12 (.32)	1.32 (.47)	1.61 (.50)
(.3,-.3)	.70	1.65 (.49)	1.25 (.43)	1.11 (.32)	1.04 (.24)	.98 (.23)	.92 (.29)	.99 (.23)	1.05 (.26)	1.11 (.32)	1.26 (.44)	1.66 (.48)
(0.0, .6)	.64	1.68 (.48)	1.33 (.47)	1.11 (.32)	1.07 (.27)	.96 (.26)	.93 (.27)	.97 (.24)	1.06 (.26)	1.13 (.34)	1.34 (.47)	1.69 (.48)

TABLE A.2.4 (Continued)

(.6, .9)	.58	2.18	1.58	1.33	1.18	1.08	1.04	1.08	1.16	1.34	1.55	2.14
		(1.20)	(.55)	(.49)	(.42)	(.40)	(.40)	(.37)	(.41)	(.50)	(.55)	(1.05)
(.6,-.3)	.42	1.72	1.32	1.11	1.05	.99	.95	.99	1.07	1.14	1.32	1.71
		(.50)	(.47)	(.32)	(.26)	(.21)	(.24)	(.23)	(.26)	(.35)	(.47)	(.50)
(.3, .9)	.32	2.10	1.55	1.30	1.17	1.08	1.02	1.08	1.15	1.31	1.54	2.10
		(1.00)	(.54)	(.48)	(.41)	(.37)	(.41)	(.39)	(.37)	(.48)	(.55)	(1.14)
(.6,-.6)	.22	1.67	1.30	1.11	1.04	.99	.96	.99	1.04	1.10	1.27	1.69
		(.54)	(.46)	(.32)	(.23)	(.21)	(.20)	(.18)	(.22)	(.31)	(.45)	(.53)
(0.0, .9)	.19	1.96	1.53	1.28	1.14	1.07	1.00	1.08	1.15	1.29	1.52	1.93
		(.86)	(.55)	(.48)	(.39)	(.40)	(.43)	(.38)	(.39)	(.48)	(.55)	(.90)
(.9,-.3)	.11	1.87	1.45	1.22	1.14	1.05	1.04	1.05	1.12	1.22	1.43	1.88
		(.84)	(.52)	(.43)	(.37)	(.39)	(.38)	(.31)	(.36)	(.44)	(.54)	(.99)
(.9,-.6)	.05	1.80	1.30	1.17	1.12	1.06	1.03	1.07	1.10	1.18	1.35	1.78
		(1.02)	(.51)	(.41)	(.37)	(.38)	(.41)	(.36)	(.35)	(.43)	(.52)	(.95)
(.9,-.9)	.01	1.69	1.33	[illegible]	1.17	1.14	1.11	1.12	1.16	1.23	1.31	1.79
		(.91)	(.59)	(.45)	(.40)	(.44)	(.49)	(.45)	(.40)	(.46)	(.51)	(1.23)

APPENDIX 3

SIMULATION RESULTS FOR THE RESTRICTED ESTIMATORS

This Appendix reports the results of the simulation study correspondent to the restricted estimator of equation (5.3.1).

TABLE A.3.1

SUMMARY STATISTICS FOR THE RESTRICTED PARAMETER ESTIMATOR
NUMBER OF OBSERVATIONS PER SERIES : 50
NUMBER OF REPLICATIONS : 1000

MODEL	ρ	MEAN	STD	EF1	STD1	EF2	STD2	ITR	STDI
(0.0,0.0)	-.9	.001	.099	.922	.0193	.898	.0190	1.645	.649
	-.5	-.003	.101	.603	.0227	.660	.0261	1.506	.647
	-.3	.002	.102	.566	.0256	.522	.0210	1.341	.603
	-.2	.003	.101	.508	.0203	.570	.0249	1.227	.569
	-.1	.005	.102	.523	.0229	.478	.0200	1.126	.528
	0.0	.002	.104	.519	.0227	.546	.0224	1.074	.539
	.1	.003	.102	.554	.0245	.521	.0210	1.126	.546
	.2	-.000	.103	.566	.0234	.546	.0226	1.222	.563
	.3	-.006	.104	.576	.0241	.581	.0227	1.279	.590
	.5	.001	.104	.681	.0260	.630	.0225	1.469	.616
	.9	-.005	.101	.911	.0159	.934	.0184	1.638	.646
(.3, .3)	-.9	.291	.098	.877	.0165	.965	.0199	1.651	.688
	-.5	.293	.097	.626	.0254	.659	.0254	1.503	.636
	-.3	.290	.097	.532	.0223	.562	.0242	1.352	.590
	-.2	.299	.097	.542	.0238	.542	.0232	1.267	.568
	-.1	.294	.097	.511	.0237	.491	.0220	1.251	.582
	0.0	.294	.099	.535	.0230	.529	.0228	1.176	.523
	.1	.291	.095	.494	.0206	.551	.0259	1.228	.550
	.2	.293	.097	.530	.0224	.523	.0244	1.267	.568
	.3	.289	.098	.566	.0241	.540	.0228	1.334	.587
	.5	.290	.096	.645	.0223	.636	.0246	1.471	.616
	.9	.291	.093	.893	.0179	.913	.0187	1.671	.673
(.6, .6)	-.9	.591	.083	.893	.0202	.912	.0201	1.620	.651
	-.5	.587	.082	.603	.0248	.632	.0284	1.456	.614
	-.3	.589	.082	.534	.0237	.556	.0263	1.378	.574
	-.2	.593	.084	.504	.0239	.548	.0265	1.367	.599
	-.1	.586	.085	.553	.0268	.484	.0222	1.324	.558

TABLE A.3.1 (Continued)

	0.0	.591	.086	.507	.0264	.524	.0251	1.330	.563
	.1	.589	.084	.488	.0239	.530	.0256	1.330	.567
	.2	.585	.086	.497	.0235	.574	.0273	1.356	.583
	.3	.588	.081	.524	.0253	.573	.0270	1.362	.567
	.5	.589	.084	.671	.0278	.615	.0247	1.460	.592
	.9	.586	.084	.896	.0201	.917	.0211	1.580	.631
(.9, .9)	-.9	.886	.049	.889	.0267	.810	.0240	1.440	.573
	-.5	.882	.051	.512	.0291	.538	.0330	1.412	.573
	-.3	.884	.052	.453	.0366	.495	.0331	1.397	.555
	-.2	.884	.048	.384	.0292	.432	.0309	1.393	.559
	-.1	.884	.049	.374	.0279	.424	.0335	1.412	.563
	0.0	.884	.050	.415	.0323	.435	.0255	1.404	.543
	.1	.885	.050	.447	.0320	.381	.0413	1.387	.558
	.2	.886	.048	.448	.0281	.428	.0278	1.390	.559
	.3	.885	.049	.432	.0332	.411	.0313	1.393	.568
	.5	.882	.051	.521	.0324	.564	.0314	1.413	.573
	.9	.886	.049	.851	.0258	.785	.0272	1.432	.598

TABLE A.3.2

SUMMARY STATISTICS FOR THE RESTRICTED PARAMETER ESTIMATOR
NUMBER OF OBSERVATIONS PER SERIES : 200
NUMBER OF REPLICATIONS : 1000

MODEL	ρ	MEAN	STD	EF1	STD1	EF2	STD2	ITR	STDI
(0.0,0.0)	-.9	.001	.049	.907	.0183	.912	.0176	1.132	.380
	-.5	.002	.049	.626	.0257	.581	.0229	1.062	.332
	-.3	-.002	.051	.567	.0234	.565	.0239	.959	.282
	-.2	-.000	.053	.543	.0223	.589	.0245	.900	.329
	-.1	-.003	.050	.506	.0225	.499	.0226	.774	.423
	0.0	.001	.051	.471	.0199	.561	.0247	.616	.493
	.1	.002	.050	.513	.0235	.490	.0215	.760	.427
	.2	.000	.051	.527	.0230	.543	.0236	.921	.311
	.3	.001	.050	.525	.0215	.574	.0244	.968	.274
	.5	.002	.050	.611	.0237	.631	.0242	1.046	.303
	.9	.003	.049	.926	.0194	.866	.0174	1.139	.395
(.3, .3)	-.9	.297	.049	.897	.0170	.927	.0175	1.131	.377
	-.5	.299	.048	.602	.0228	.664	.0264	1.025	.332
	-.3	.298	.050	.585	.0270	.525	.0205	.966	.298
	-.2	.302	.049	.517	.0218	.547	.0244	.930	.288
	-.1	.298	.048	.502	.0222	.538	.0235	.846	.364
	0.0	.301	.046	.498	.0225	.492	.0212	.827	.381
	.1	.298	.048	.496	.0207	.513	.0232	.847	.374
	.2	.296	.050	.540	.0232	.529	.0233	.932	.306
	.3	.298	.050	.609	.0243	.565	.0238	.973	.284
	.5	.300	.046	.635	.0253	.597	.0237	1.036	.311
	.9	.297	.046	.917	.0201	.866	.0180	1.131	.377
(.6, .6)	-.9	.597	.042	.918	.0165	.907	.0182	1.102	.363
	-.5	.595	.040	.601	.0246	.623	.0257	1.021	.294
	-.3	.596	.042	.536	.0247	.554	.0230	.955	.288
	-.2	.600	.040	.509	.0236	.533	.0228	.945	.313
	-.1	.598	.040	.488	.0227	.507	.0230	.916	.315

TABLE A.3.2 (Continued)

	0.0	.597	.040	.509	.0233	.487	.0218	.913	.299
	.1	.596	.042	.484	.0218	.548	.0240	.907	.326
	.2	.598	.041	.503	.0221	.550	.0258	.966	.259
	.3	.598	.039	.520	.0229	.555	.0253	.962	.288
	.5	.596	.040	.604	.0236	.656	.0270	1.009	.302
	.9	.596	.040	.917	.0186	.896	.0188	1.088	.353
(.9, .9)	-.9	.897	.022	.891	.0216	.845	.0222	1.008	.279
	-.5	.895	.024	.637	.0287	.610	.0255	.971	.276
	-.3	.895	.023	.530	.0288	.520	.0252	.951	.281
	-.2	.895	.024	.495	.0238	.556	.0291	.941	.289
	-.1	.895	.022	.462	.0239	.459	.0257	.934	.293
	0.0	.896	.023	.482	.0231	.491	.0256	.946	.267
	.1	.895	.023	.455	.0261	.479	.0257	.941	.289
	.2	.896	.023	.526	.0312	.499	.0264	.933	.298
	.3	.895	.022	.478	.0257	.480	.0256	.950	.296
	.5	.895	.023	.585	.0268	.608	.0265	.972	.282
	.9	.895	.023	.877	.0227	.884	.0219	1.025	.297

APPENDIX 4.

DATA USED IN CHAPTER 4.

FOX RIVER, NEAR BERLIN, WISC. ANNUAL FLOWS CFS. 1899-1965

786	708	946	787	1071
1200	1561	1380	1261	1160
943	813	842	1451	1350
915	1091	1520	1541	1370
1091	1231	991	1510	1101
1261	932	1040	1221	1340
1621	1010	672	810	825
634	1147	871	954	1543
1373	1091	1177	1260	1230
923	1070	1302	1147	970
778	890	1201	1302	1159
816	1132	1001	810	590
755	1513	1062	1407	808
559	813			

WOLF RIVER NEAR NEW LONDON, WISC. ANNUAL FLOWS CFS. 1899-1965

1320	1035	1962	1349	1547
1635	1862	2050	2072	1411
1278	1057	1054	2842	2439
1901	1780	2360	2120	1790
2060	2070	1720	2350	1831
2060	1349	1761	2021	2179
2721	1260	864	1390	1201
967	1877	1447	1575	2080
2101	1633	2005	2810	2468
1486	1711	2024	3216	1237
1138	1469	1711	2111	1553
1266	1597	1322	1008	1115
1329	2580	1859	2018	1334
974	1736			

FRENCH BROAD RIVER AT ASHEVILLE, N.C. ANNUAL FLOWS CFS. 1896-1965

1420	2090	1820	3160	2430
3670	2550	2890	1230	1840
2890	2120	2360	2840	2010
1520	2220	2080	1400	2870
3490	1910	1640	3070	2230
2270	2190	1900	2010	1310
1210	1440	2530	2470	1840
1350	1640	2490	1640	2071

2280	2469	1797	1913	1582
1329	1892	2353	1628	1486
2340	1538	2024	3122	2286
1688	2039	1560	1457	1383
1166	1867	2343	1718	2430
2125	2398	1568	2009	2673

FRENCH BROAD RIVER NEAR NEWPORT, TENN. ANNUAL FLOWS CFS. 1921-1965

3300	3320	3020	2820	1750
1720	2510	3980	3670	2640
2080	2550	3589	2243	3051
3561	3612	2874	2667	2203
1801	2394	3195	2346	2458
3520	2309	2753	4530	3704
2732	2863	2284	2312	2071
2014	3032	3636	2592	3755
3088	3964	2614	2893	4136

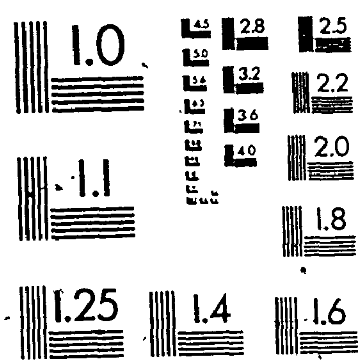
MCKENZIE RIVER AT MCKENZIE BRIDGE, OREGON. ANNUAL FLOWS CFS. 1911-1957

1500	1710	1790	1590	1410
2000	1820	1720	1670	1580
2120	1610	1560	1380	1780
1300	1710	1790	1380	1260
1230	1710	1770	1576	1602
1511	1441	1742	1381	1246
1098	1317	1925	1249	1402
1671	1690	1887	1819	1959
2213	1786	1904	2013	1672
2397	1799			

ST. LAWRENCE (MAIN STREAM) NEAR OGDENSBURG, N.Y. ANNUAL FLOWS CFS. 1860-1957

275016	283927	273090	265865	265865
245877	271886	242024	249008	278870
258881	223963	236967	261049	230946
262975	251898	251898	258881	243951
231910	255992	250934	269959	252861
277906	268996	241061	245877	262012
258881	229020	242024	242024	216016
212885	217942	228057	228057	225889
226852	228057	242024	243951	237930
241061	243951	262012	240820	230946
218905	229020	257918	241061	221073
239857	241061	247081	250934	225889
234077	229020	216979	221073	214089
208068	225889	237930	255028	262975
217942	217942	208068	188080	183023
191934	214089	217942	216016	209995
212885	214089	248045	247081	242024
256955	256955	257918	241061	236004
269959	275016	258881	254065	255992
257918	241061			

3 OF/DE 3



MONTHLY 1972-79 TOTAL NITROGEN, N MG/L, MIDDLE FORK CREEK NEAR SEEBE

1.5873E-01	1.7983E-01	1.8847E-01	1.6372E-01	4.7000E-01	2.6000E-01
2.2000E-01	1.9442E-01	2.5298E-01	4.3000E-01	4.3000E-01	3.3000E-01
3.3000E-01	4.2000E-01	1.2000E-01	1.4866E-01	2.8983E-01	2.1000E-01
2.4372E-01	2.7495E-01	1.5556E-01	1.2000E-01	3.6000E-01	1.4638E-01
2.3000E-01	1.5000E-01	1.4000E-01	1.5000E-01	2.7000E-01	2.9496E-01
2.0000E-01	3.1000E-01	2.1000E-01	1.1000E-01	4.7286E-01	2.4000E-01
1.4000E-01	2.8000E-01	2.3000E-01	1.2000E-01	2.4000E-01	1.8000E-01
1.1000E-01	2.2000E-01	2.8650E-01	1.6577E-01	1.5000E-01	1.4000E-01
1.5000E-01	1.3000E-01	2.3000E-01	2.8000E-01	1.7493E-01	1.5000E-01
1.2000E-01	2.3000E-01	1.7000E-01	1.4000E-01	2.1000E-01	2.0396E-01
1.9000E-01	2.8983E-01	2.5690E-01	2.8460E-01	1.5000E-01	1.9442E-01
1.6000E-01	2.7770E-01	4.3000E-01	1.3416E-01	1.7000E-01	1.4967E-01
1.7000E-01	3.8000E-01	2.3213E-01	1.7000E-01	1.7000E-01	2.2000E-01
3.3000E-01	6.5521E-01	3.1000E-01	2.9000E-01	3.5000E-01	1.5000E-01
1.6000E-01	1.9005E-01	1.9918E-01	1.7303E-01	2.6181E-01	2.4257E-01
1.8959E-01	2.3886E-01	2.3590E-01	1.3649E-01	3.5970E-01	1.5824E-01

MONTHLY 1972-79 NITROGEN KJELDAHL, N MG/L, MIDDLE FORK CREEK NEAR SEEBE

7.4353E-02	9.7739E-02	1.2241E-01	8.5950E-02	4.0000E-01	2.0000E-01
2.0000E-01	2.0000E-01	2.0000E-01	4.0000E-01	4.0000E-01	3.0000E-01
3.0000E-01	4.0000E-01	1.0000E-01	1.1832E-01	1.3038E-01	2.0000E-01
2.0000E-01	2.0000E-01	1.4142E-01	1.0000E-01	3.0000E-01	8.9280E-02
2.0000E-01	9.0000E-02	1.0000E-01	1.0000E-01	2.0000E-01	2.0000E-01
1.0000E-01	3.0000E-01	2.0000E-01	1.0000E-01	4.4721E-01	2.0000E-01
9.0000E-02	2.0000E-01	2.0000E-01	1.0000E-01	9.0000E-02	1.0000E-01
1.0000E-01	2.0000E-01	2.6823E-01	1.3277E-01	9.0000E-02	9.0000E-02
9.0000E-02	1.0000E-01	2.0000E-01	2.0000E-01	9.4868E-02	1.0000E-01
9.0000E-02	2.0000E-01	9.0000E-02	1.0000E-01	1.0000E-01	1.4142E-01
9.0000E-02	1.6432E-01	1.3416E-01	9.4868E-02	9.4868E-02	1.7889E-01
1.5000E-01	2.6132E-01	4.0000E-01	9.4868E-02	1.0000E-01	9.0000E-02
1.0000E-01	3.0000E-01	1.6255E-01	9.0000E-02	9.0000E-02	2.0000E-01
3.0000E-01	6.3246E-01	3.0000E-01	2.0000E-01	3.0000E-01	1.0000E-01
1.0000E-01	1.1139E-01	1.3950E-01	9.7950E-02	1.1748E-01	2.1616E-01
2.2503E-01	2.3495E-01	2.2503E-01	1.1139E-01	2.9931E-01	1.0623E-01

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