1984

The Exploitation Of A Nonrenewable Resource Under Imperfect Competition

Nguyen Van Quyen

THE EXPLOITATION OF A NONRENEWABLE RESOURCE UNDER IMPERFECT COMPETITION

by

Nguyen Van Quyen

Department of Economics

Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
London, Ontario

July 1984

ABSTRACT

In this thesis, we investigate some problems concerning the exploitation of a nonrenewable resource under imperfect competition. The approach adopted here is that of partial equilibrium. The thesis consists of five chapters, with each chapter dealing with a specific topic.

In Chapter One, we study the pattern of resource extraction expected of a dominant firm facing competition from a naive competitive fringe. To model the behavior of the fringe, we assume that only the current price affects its output decision. More specifically, at any time when the instantaneous output of the dominant firm alone is not sufficient to keep the market price below $\bar{c}_2$, the fringe's cost, then entry will occur until either the market price is driven down to $\bar{c}_2$ or the fringe reaches its extraction capacity, whichever happens first. Our model should be viewed as an abstraction of the world oil market in which OPEC is assigned a two part structure - the "core" on the one hand and the "fringe" on the other. It is assumed further that the absence of a perfect capital market reduces the fringe in particular to the apparently naive behavior described above.

In Chapter Two, we build a model of monopoly resource extraction under stochastic entry with consumers as potential entrants. We assume that at any instant the monopolist might lose a block of customers who decide to leave the monopolist and make heavy investments in an alternative source to provide for their own needs. The loss in customers depends probabilistically on the monopolist's pricing strategy and is irreversible once it occurs.

In Chapter Three, we build a model of simultaneous resource extraction and exploration under monopoly. Our model of exploration is spatially oriented and deals directly with location uncertainty. To deal with location uncertainty, we model the exploration process in the spirit of the Theory of Optimal Search.

As is well known, the discovery of oil and gas on a piece of land provides information about the potential of adjacent lots. Because it is almost impossible for a prospector to conceal information about his exploratory activities and also because markets for information can never be expected to exist, there is a strong incentive for each leaseholder just to sit back and wait for the outcomes of the exploratory efforts of other leaseholders to obtain free information about his own tract. The game-theoretical model we build in Chapter Four represents an attempt to study this phenomenon.

In Chapter Five, we formulate a differential game model of resource extraction under oligopoly. Here, we present a rigorous proof of the existence of a Cournot-Nash equilibrium. When the market demand curve satisfies a suitable assumption, we show that this equilibrium is unique and provide an algorithm to find it.

ACKNOWLEDGEMENTS

First, I would like to thank Professor A. Robson, my chief thesis supervisor, from whom I have learned a lot in writing this thesis. Second, I would also like to thank the other two members in my thesis committee, Professor M. Hoy and Professor J. McMillan, for their many comments and suggestions in improving this thesis.

TABLE OF CONTENTS

Page

CHAPTER ONE

THE MONOPOLY PRICE OF A NONRENEWABLE RESOURCE UNDER THREAT OF ENTRY

1. INTRODUCTION

The pricing strategy of a monopolist who owns a stock of a nonrenewable resource has been extensively studied in the economic literature (see Solow [1974], Weinstein and Zeckhauser [1975]). When the resource in question has a close substitute, there arises the problem of potential competition which might dissipate monopoly power.

Hoel [1978] studied the problem of monopoly resource extraction when there exists a substitute produced by a backstop technology. This author considered both the case when the monopolist owns the substitute and the case when the substitute is supplied competitively.

Gilbert and Goldman [1978] studied the pricing behavior of a monopolist under threat of entry by rival firms which supply the same kind of nonrenewable resource. These authors presented two formulations. In the first formulation, entry by rival firms is assumed to occur whenever the monopolist's remaining stock is reduced to a certain threshold level. In the second formulation, entry occurs when the monopolist's price exceeds a certain critical level. By making some suitable assumptions, these authors reduced the second formulation to the first formulation. The limit pricing problem that the monopolist must solve is thus reduced to finding a time horizon τ and an extraction strategy which brings the remaining stock down to the threshold level at the exact time τ to maximize the sum of discounted pre-entry and post-entry profits. The post-entry profit is assumed to be completely determined by the remaining stock at the exact time entry occurs. The main results asserted by these authors are: (1) until entry actually occurs, the price is higher with potential competition than with complete monopoly when entry is prohibited; (2) social welfare under complete monopoly might exceed social welfare when the monopolist has to face potential competition.

Gilbert [1978] investigated the market for a nonrenewable resource characterized by the existence of a dominant firm and a large number of small firms collectively called the competitive fringe. This author assumed that the dominant firm is the price leader and the competitive fringe are price takers. Furthermore, a constraint on the extraction capacity of the fringe was also imposed.

Lewis and Schmalensee [1979] also studied a model similar to that of Gilbert [1978]. They assumed zero extraction costs for both the dominant firm and the fringe, but they put no limit on the extraction capacity of the fringe.

In this chapter, we build a model of limit pricing for a nonrenewable resource market characterized by the existence of a dominant firm and a large number of nameless small firms, collectively called the fringe. Compared with the fringe, the dominant firm has a lower extraction cost, a higher extraction capacity, and a higher reserve level. The first two factors combined might allow the dominant firm to drive the market price below the extraction cost of the fringe and completely shut it out of the market, while still making a profit. The third factor, i.e., the dominant firm's initial reserve, will determine how long it can stay in business.

Concerning the behavior of the fringe, we shall assume that only the current price affects its behavior. More specifically, at any time t when the output of the dominant firm alone is sufficient to keep the market price below the fringe's extraction cost, the fringe will be completely shut out. Otherwise, entry will occur until instantaneous market equilibrium is reached when either the fringe's output at t reaches its extraction capacity or the combined output of the dominant firm and the fringe has driven the market price down to the level of the fringe's extraction cost, whichever happens first. Our model should be viewed as an abstraction of the world oil market in which OPEC is assigned a two part structure - the "core" on the one hand and the "fringe" on the other. It is assumed further that the absence of a perfect capital market reduces the fringe in particular to the apparently naive behavior described above.

Our main contributions in this chapter are Theorem 1 and Theorem 2 in section 3, which can be summarized as follows:

(i) If the dominant firm's initial stock is sufficiently large, then the industry extraction pattern consists of three phases. During phase one, the dominant firm is the sole supplier in the market. During phase two, the dominant firm and the fringe share the market for some time, with the fringe operating at full capacity without any interruption until its stock is exhausted at the end of this phase. Phase three begins immediately after the fringe's stock has been exhausted. During this phase, the dominant firm becomes a monopolist. Phase three might or might not exist. Its existence is guaranteed if either the choke price is allowed to become infinite or the fringe's initial stock is small. On the other hand, if the fringe's initial stock is large, phase three will not exist; this is clearly the case if the fringe possesses a backstop technology.

As for the market price, we can describe it as follows: (1) the market price begins below the fringe's extraction cost $\bar{c}_2$, rises steadily, reaches $\bar{c}_2$ and stays at $\bar{c}_2$ until phase one ends; (2) at the exact time phase two begins, the market price might take an upward jump, then increases steadily after that; (3) if phase three exists, then the market price will take another upward jump when this phase begins and, after this jump, will rise steadily until reaching the choke price.

(ii) If the dominant firm's initial stock is sufficiently large, then the dominant firm exhausts its stock sooner under potential competition than under complete monopoly. Furthermore, the time that the dominant firm operates in the region of the demand curve which precludes entry is longer, and the dominant firm's cumulative extraction during this time is larger under potential competition than under complete monopoly. Loosely speaking, this last result means that the threat of potential competition induces the dominant firm to concentrate its production activity more in the pre-entry period.

The plan of this chapter is as follows. In section 2, we present our model. The main results are given in section 3. Section 4 contains the proofs of the results presented in section 3. In section 5, we have some concluding remarks.

2. THE MODEL

2.1 The Market Demand Curve

Let $f: Q \to f(Q)$ denote the market demand curve, as a function of output Q. The following assumptions are imposed upon f:

(f1) $0 < f(0) < +\infty$

$f(Q) > 0$, $f'(Q) < 0$ for all $Q \geqq 0$

(f2) the industry total revenue curve $\pi: Q \to \pi(Q) = Qf(Q)$ is strictly concave and achieves a maximum at $Q = Q_{max} > 0$.

(f3) the function $h: (Q_1, Q_2) \to h(Q_1, Q_2) = Q_1 f(Q_1+Q_2)$, which is defined for all $Q_1 \geqq 0$, $Q_2 \geqq 0$, is assumed to satisfy

$$h_1(Q_1, Q_2) \leqq \pi'(Q_1)$$

with strict inequality holding whenever $Q_2 > 0$.

Here h_1 denotes the partial derivative of h, taken with respect to Q_1.

(f4) $\bar{Q}_2 < f^{-1}(\bar{c}_2) < Q_{max}$. Here $\bar{Q}_2$ and $\bar{c}_2$ are the extraction capacity and unit extraction cost of the fringe, respectively.

Assumptions (f1) and (f2) are standard. If we interpret Q_1 and Q_2 as the outputs of the dominant firm and the fringe, respectively, then $h(Q_1,Q_2)$ is the dominant firm's total revenue as a function of Q_1 and Q_2. Assumption (f3) then means that for any given $Q_2 > 0$, the dominant firm's marginal revenue curve corresponding to Q_2 lies strictly below the industry marginal revenue curve. If the dominant firm's marginal revenue at each output level Q_1 is decreasing in Q_2, i.e., if $h_{12} < 0$, then (f3) is clearly satisfied. However, assumption (f3) might not imply that $h_{12} < 0$. Many demand curves satisfy (f3). For example, if f is linear, then we can readily verify that (f3) is fulfilled. As for assumption (f4), the assumed strict inequality $\bar{Q}_2 < f^{-1}(\bar{c}_2)$ is not so restrictive as it seems to be. Indeed, if $\bar{Q}_2 \geqq f^{-1}(\bar{c}_2)$, then the fringe's extraction capacity constraint $\bar{Q}_2$ will cease to be binding because only the fringe's extraction cost $\bar{c}_2$ operates to put an upper bound on the fringe's output. In this case, the fringe's output can vary between 0 and $f^{-1}(\bar{c}_2)$, i.e., for all practical purposes, we can also accept $f^{-1}(\bar{c}_2)$ as the fringe's extraction capacity if $\bar{Q}_2 \geq f^{-1}(\bar{c}_2)$, and all the lemmas and theorems are still valid if $\bar{Q}_2$ is replaced by the effective constraint $f^{-1}(\bar{c}_2)$.

2.2 Specification of the Dominant Firm and the Fringe

Let $\bar{x}_1 > 0$ denote the dominant firm's initial fixed stock. We assume that its extraction cost is zero and its extraction capacity is at least as great as Q_{max}.

As for the fringe, we assume that all its individual members have the same extraction cost $\bar{c}_2 > 0$. The combined stock and combined extraction capacity of all the members of the fringe are denoted by $\bar{x}_2 > 0$ and $0 < \bar{Q}_2 < +\infty$, respectively.

We shall now describe the entry behavior of the fringe. At each time t, let Q_1 be the dominant firm's output and x_2 be the fringe's remaining stock. The fringe's output at this time t, say Q_2, is then determined as follows

$$u: \quad (Q_1, x_2) \to u(Q_1, x_2) = Q_2$$

with $\quad Q_2 = 0$ if $x_2 = 0$

$$= \max\{Q \mid 0 \leqq Q \leqq \bar{Q}_2,\ \bar{c}_2 \leqq f(Q_1+Q)\} \text{ if } x_2 > 0$$

As just defined, u is the fringe's reaction function. It describes the fringe's instantaneous output as a function of the fringe's remaining stock x_2 and the dominant firm's instantaneous output Q_1. As specified, $Q_2 = 0$ if the fringe's remaining stock is zero. If the fringe's stock has not yet been exhausted, then the fringe's output will increase until either it reaches the extraction capacity $\overline{Q}_2$ or the market price is driven down to $\overline{c}_2$.

Given any feasible extraction strategy $q_1: \ t \to q_1(t)$ chosen by the dominant firm, the extraction path of the fringe is completely and jointly determined by q_1 and the reaction function. More specifically, this extraction path is described by the following differential equation

$$
\begin{aligned}
(1) \qquad & dx_2/dt = -q_2(t) \\
& q_2(t) = u(q_1(t), x_2(t)) \\
& x_2(0) = \overline{x}_2
\end{aligned}
$$

with $x_2(t)$, $q_2(t)$ representing the fringe's remaining stock at time t and extraction rate at time t, respectively.

2.3 Statement of the Limit Pricing Problem

The problem that the dominant firm has to solve is to find an extraction strategy $q_1: \ t \to q_1(t)$ to maximize the following discounted profit

$$\int_0^\infty e^{-rt} q_1(t) f(q(t))dt$$

subject to (1) and

(2) $dx_1/dt = -q_1(t)$, $x_1(0) = \overline{x}_1$

$x_1(t) \geqq 0$, $q_1(t) \geqq 0$ for all t

and $q(t) = q_1(t) + q_2(t)$

Here r is the market rate of interest.

3. THE OPTIMAL SOLUTION

We shall assume that the preceding limit pricing problem has a solution q_1^*: $t \rightarrow q_1^*(t)$ among the class of non-negative piecewise and right continuous controls satisfying the stock constraint. The dominant firm's optimal remaining stock at each instant is described by the following differential equation

$$dx_1^*/dt = -q_1^*(t)$$

$$x_1^*(0) = \overline{x}_1$$

The exact time $\overline{x}_1$ is exhausted under q_1^* will be denoted by T.

The fringe's remaining stock $x_2^*(t)$ and extraction rate $q_2^*(t)$, as induced by q_1^*, are described by the following differential equation

$$dx_2^*/dt = -\ q_2^*(t)$$

$$x_2^*(0) = \bar{x}_2$$

$$q_2^*(t) = u(q_1^*(t), x_2^*(t)).$$

Our major results in this chapter are summarized by the following two theorems. Their proofs are given in the next section.

<u>Theorem 1</u>

Let τ denote the exact time entry first occurs, i.e.,

$$\tau = \inf\ \{t | q_2^*(t) > 0\}$$

and $\Delta\tau = \bar{x}_2/\bar{Q}_2$. If the dominant firm's initial stock is sufficiently large, then

(a) The extraction pattern of the industry consists of three phases:

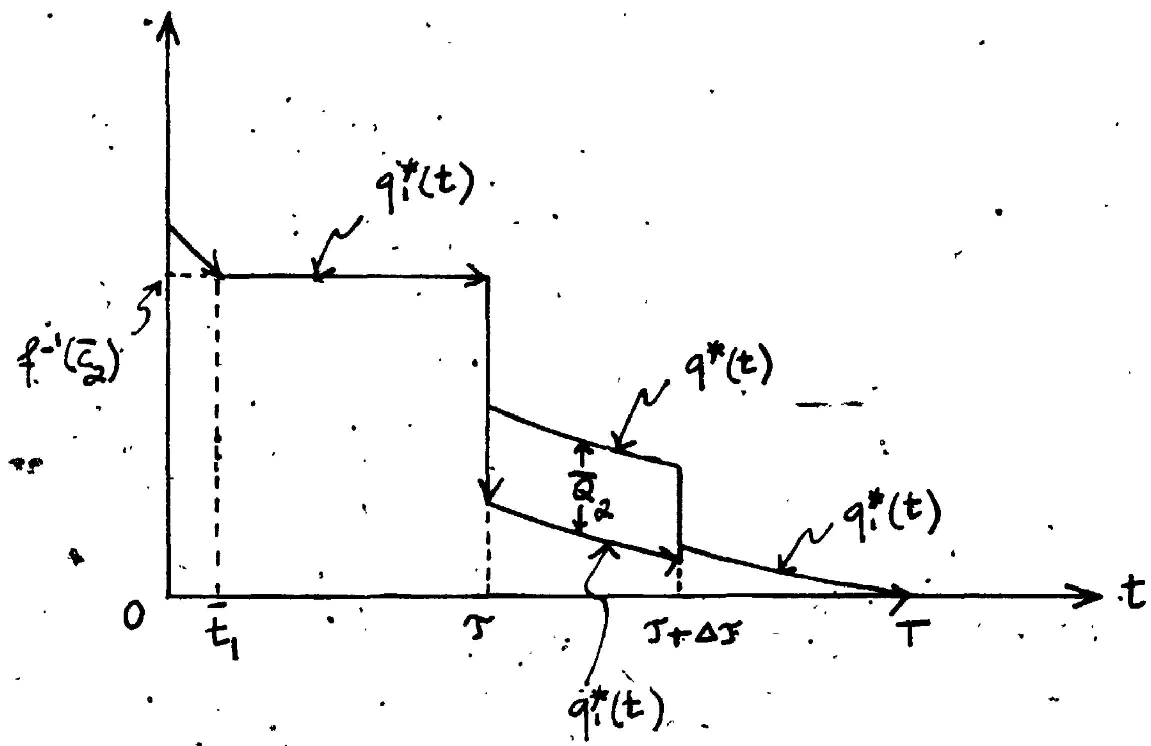
(a1) During phase one, $0 \leqq t < \tau$, the dominant firm is the sole supplier in the market

(a2) During phase two, $\tau \leqq t < \tau+\Delta\tau$, the dominant firm and the fringe share the market for at least some time. Furthermore, throughout phase two, the fringe operates at full capacity without any interruption since time τ, when entry first occurs, until time $\tau+\Delta\tau$, when its stock is exhausted.

(a3) During phase three, $\tau+\Delta\tau \leq t < +\infty$, the dominant firm enjoys complete monopoly power. Phase three might not always exist. Its existence can be guaranteed if $f(\bar{Q}_2)e^{r\Delta\tau} \leq f(0)$. On the other hand, if $\pi'(f^{-1}(\bar{c}_2))e^{r\Delta\tau} \ngeq f(0)$, then phase three will not exist.

(b) The market price begins below $\bar{c}_2$, rises steadily, reaches $\bar{c}_2$ at time $\bar{t}_1 < \tau$, remains constant at $\bar{c}_2$ for the rest of phase one. At time τ when phase two begins, the market price might take an upward jump (this jump certainly occurs if phase three does not exist). After τ, the market price is monotone increasing until the end of phase two. If phase three exists, then the market price will take an upward jump at $\tau+\Delta\tau$, when this phase begins, then increases steadily until reaching the choke price at T.

The following figure gives a graphical representation of Theorem 1. In this figure, we define $q^*(t) = q_1^*(t)+q_2^*(t)$.



Theorem 2

Suppose that the dominant firm's initial stock is sufficiently large. Let q^m: $t \to q^m(t)$ be the dominant firm's optimal extraction strategy under complete monopoly, i.e., q^m is the solution of the following optimal control problem

$$\max \int_0^\infty e^{-rt}\,\pi(q^m(t))\,dt$$

subject to $\int_0^\infty q^m(t)dt = \bar{x}_1$, $q^m(t) \geq 0$ for all t

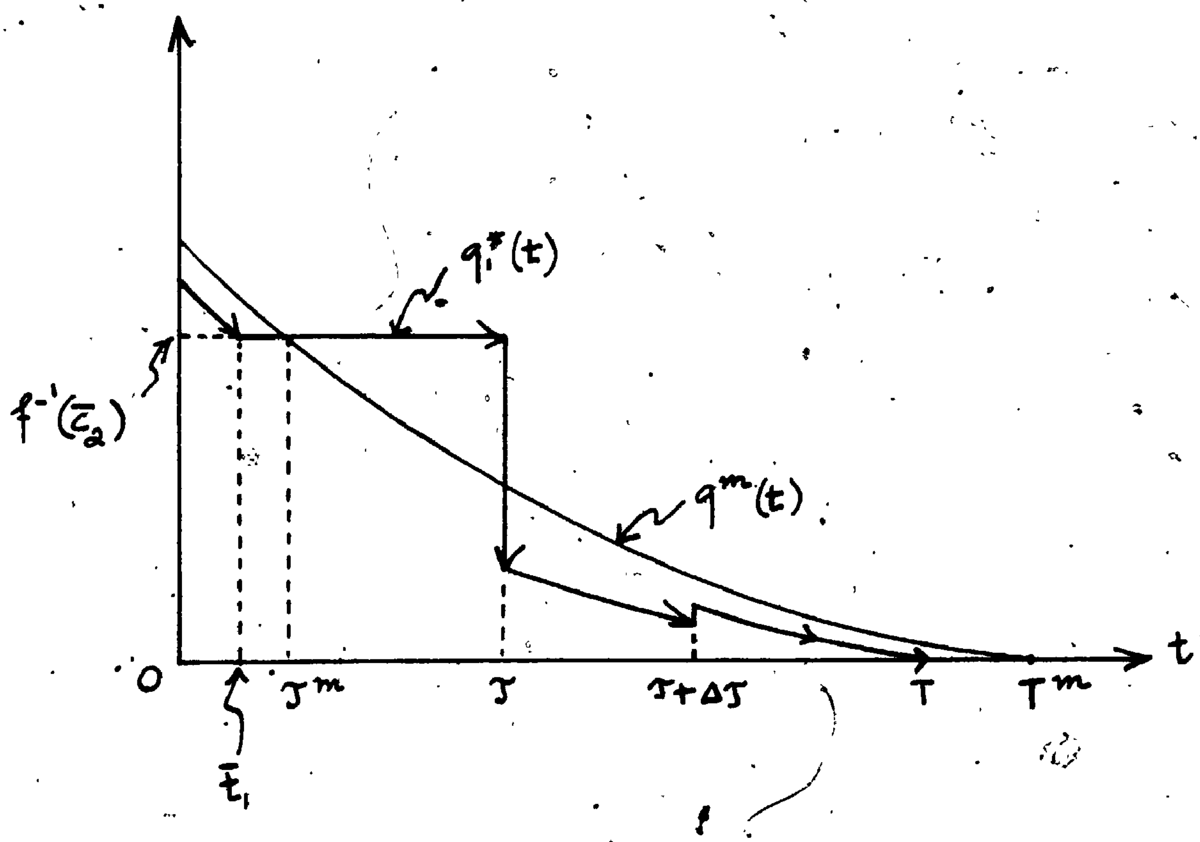
If T^m is the exact time q^m exhausts $\overline{x}_1$ and τ^m is the exact time the market price, under q^m, reaches $\overline{c}_2$, i.e., $f(q^m(\tau^m)) = \overline{c}_2$, then

(a) $T < T^m$, i.e., the dominant firm exhausts its stock sooner under potential competition than under complete monopoly.

(b) $\tau^m < \tau$ and $\int_0^{\tau^m} q^m(t)dt < \int_0^{\tau} q_1^*(t)dt$.

The first strict inequality means that the threat of potential competition causes the dominant firm to operate longer in the region of the demand curve which makes entry unprofitable; the comparison is made with q^m. The second strict inequality reinforces the first strict inequality in the sense that cumulative extraction in the region of the demand curve which makes entry unprofitable is larger under potential competition than under complete monopoly.

The following figure gives a graphical representation of Theorem 2.



4. THE PROOFS

The proofs of Theorems 1 and 2 will be given step by step.

Lemma 1

For any given $Q_2 \geqq 0$, the function of $Q_1 \geqq 0$

$$h(.,Q_2): \quad Q_1 \rightarrow h(Q_1,Q_2) = Q_1 f(Q_1+Q_2)$$

is strictly concave.

Proof

The second partial derivative of h with respect to Q_1 is given by

$$h_{11}(Q_1,Q_2) = 2\,f'(Q_1+Q_2) + Q_1 f''(Q_1+Q_2)$$

This second derivative is clearly negative if $f''(Q_1+Q_2) < 0$. If $f''(Q_1+Q_2) \geqq 0$, then we rewrite $h_{11}(Q_1,Q_2)$ as

$$\begin{aligned} h_{11}(Q_1,Q_2) &= 2f'(Q_1+Q_2) + (Q_1+Q_2)f''(Q_1+Q_2) \\ &\quad - Q_2 f''(Q_1+Q_2) \\ &= \pi''(Q_1+Q_2) - Q_2 f''(Q_1+Q_2) < 0 \end{aligned}$$

due to the strict concavity of π.

q.e.d.

Lemma 2

The fringe's output at each instant is either zero or equal to its maximum extraction capacity. That is, at any time t, either $q_2^*(t) = 0$ or $q_2^*(t) = \overline{Q}_2$.

Proof

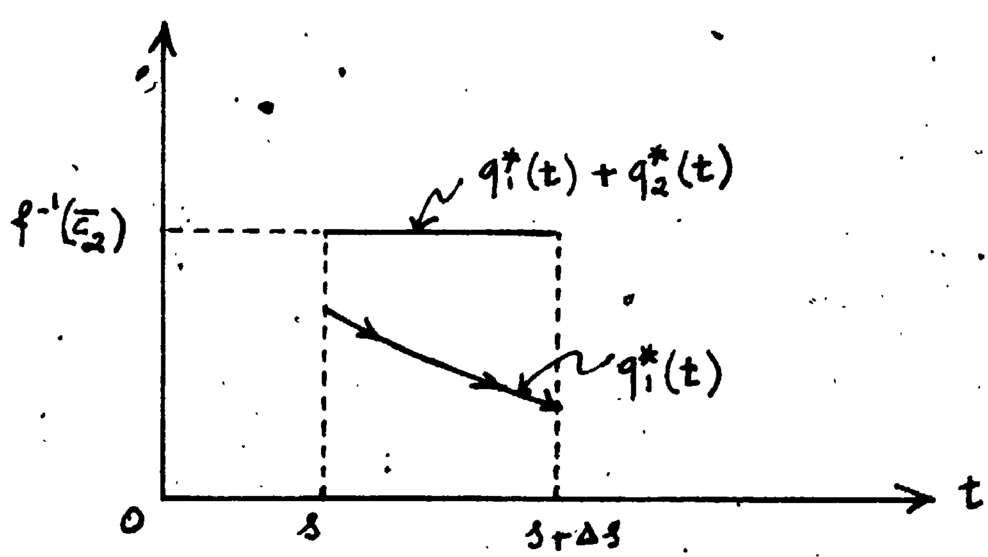
To prove the lemma, suppose the contrary, say, $0 < q_2^*(s) < \overline{Q}_2$ at some time s. According to the entry mechanism, as represented by the fringe's reaction function u, we must have

(3) $$f(q_1^*(s) + q_2^*(s)) = \bar{c}_2$$

$$0 < q_1^*(s),\ 0 < q_2^*(s) < \bar{Q}_2,\ x_2^*(s) > 0$$

In (3), the strict inequality $0 < q_1^*(s)$ is obtained by using the supposition $q_2^*(s) < \bar{Q}_2$ and assumption (f4), which asserts $f(\bar{Q}_2) > \bar{c}_2$.

Now because q_1^* is piecewise and right continuous, it must also be true that q_2^* is piecewise and right continuous. This fact allows us to extend (3) to a closed right neighborhood N(s) of s, i.e.,



(4) $f(q_1^*(t) + q_2^*(t) = \bar{c}_2$

$0 < q_1^*(t),\ 0 < q_2^*(t) < \bar{Q}_2,\ x_2^*(t) > 0$

for all t in N(s). Clearly (4) contradicts the optimality of q_1^* because the dominant firm's discounted revenue at time s is $\bar{c}_2 e^{-rs}$, which is strictly greater than $\bar{c}_2 e^{-r(s+\Delta s)}$, the discounted revenue at time $s+\Delta s$ for this same firm.

q.e.d.

Lemma 3

(a) Once entry has occurred, the fringe will operate at full capacity until its stock is exhausted, i.e.,

$$q_2^*(t) = \bar{Q}_2,\ \tau \leqq t < \tau+\Delta\tau$$

Here τ is the exact time entry first occurs, i.e.,

$$\tau = \inf\{t|q_2^*(t) > 0\}$$

and $\Delta\tau = \bar{x}_2/\bar{Q}_2$.

(b) q_1^* is continuous and monotone decreasing in $[\tau,\tau+\Delta\tau]$.

Proof

To prove part (a) of the lemma, suppose the contrary, i.e., $q_2^*(s) = 0$ at some time s, $\tau < s < \tau+\Delta\tau$. Here we recall from Lemma 2 that either $q_2^*(s) = 0$ or $q_2^*(s) = \overline{Q}_2$. If $q_2^*(s) = 0$, then the dominant firm's output at s alone must be sufficiently large to keep the market price below $\overline{c}_2$. Furthermore, by assumption (f4), it is never profitable for the dominant firm to extract more than Q_{max} at each instant. Hence we must have

$$(5) \qquad f^{-1}(\overline{c}_2) \leqq q_1^*(s) < Q_{max}$$

Now, using the definition of τ, we can find s', $\tau < s' < s$, such that $q_2^*(s') > 0$. Furthermore, by Lemma 2 we must also have $q_2^*(s') = \overline{Q}_2$. This last result together with the entry mechanism then imply $f(q_1^*(s')+\overline{Q}_2) \geqq \overline{c}_2$, or, equivalently,

$$(6) \qquad q_1^*(s') \leqq f^{-1}(\overline{c}_2) - \overline{Q}_2$$

If we let $\pi_1^*(t)$ denote the dominant firm's optimal undiscounted profit at time t, i.e.,

$$\pi_1^*(t) = q_1^*(t)f(q_1^*(t) + q_2^*(t))$$

then (5) and (6), taken together, imply

$$(7) \qquad \pi_1^*(s) > \pi_1^*(s')$$

Hence (7) implies

$$(8) \qquad \pi_1^*(s')e^{-rs'}+\pi_1^*(s)e^{-rs} < \pi_1^*(s)e^{-rs'}+\pi_1^*(s')e^{-rs}$$

The left side of (8) is the sum of the discounted profits at s' and s, enjoyed by the dominant firm under q_1^*. The right side of (8) is the sum of the discounted profits at s' and s, enjoyed by the dominant firm if it adopts the extraction strategy $\hat{q}_1$: $t \to \hat{q}_1(t)$ such that $\hat{q}_1(s') = q_1^*(s)$ and $\hat{q}_1(s) = q_1^*(s')$ while $\hat{q}_1(t) = q_1^*(t)$ elsewhere. Therefore, (8) contradicts the optimality of q_1^*, and we are then forced to accept that $q_2^*(t) = \overline{Q}_2$ for all t in $(\tau,\tau+\Delta\tau)$. Furthermore, because q_2^* is right continuous, we must also have $q_2^*(t) = \overline{Q}_2$ for all t in $[\tau,\tau+\Delta\tau)$, proving part (a) of the lemma.

To prove part (b) of the lemma, we note that the restriction of q_1^* to the time interval $I_2 = [\tau,\tau+\Delta\tau)$ is the solution of the following optimal control problem

$$\max \int_{I_2} h(q_1(t),\overline{Q}_2)e^{-rt}\,dt$$

subject to

$$q_1(t) \geqq 0 \ , \ f(q_1(t) + \overline{Q}_2) \geqq \overline{c}_2 \text{ for all t in } I_2$$

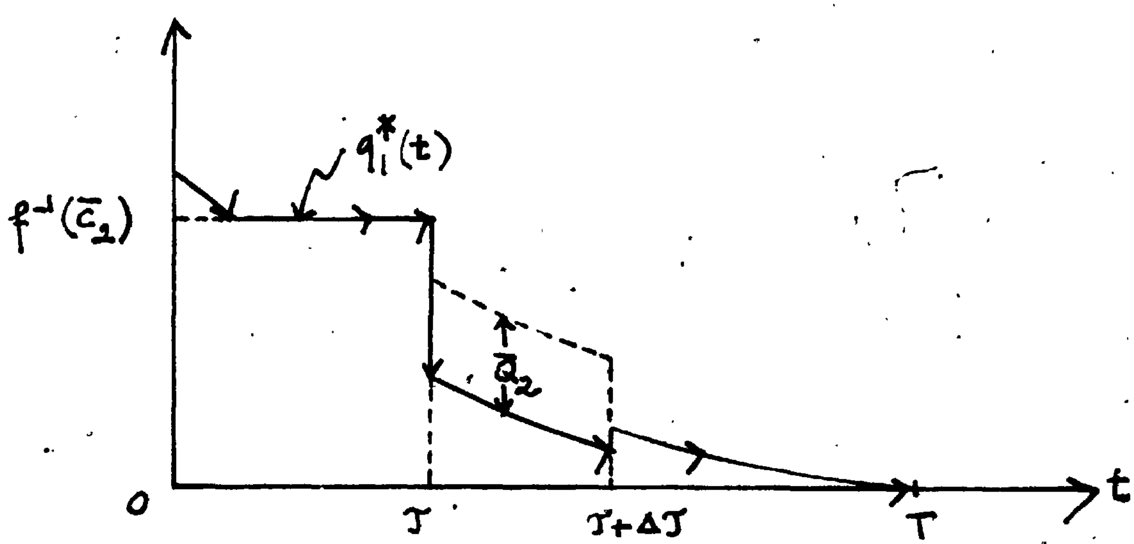
$$\int_{I_2} q_1(t)dt \leqq \int_{I_2} q_1^*(t)dt$$

By Lemma 1, $h_1(Q_1,\overline{Q}_2)$ is strictly monotone decreasing in Q_1. Furthermore, if we also assume that $h_1(Q_1,\overline{Q}_2)$ is continuous in Q_1, then it is clear that q_1^* is continuous and monotone decreasing in I_2.

Lemma 4

If the dominant firm's initial stock is sufficiently large then $\tau > 0$, i.e., entry only occurs after some time has elapsed, and the dominant firm shares the market with the fringe for some time.

Proof



By Lemmas 2 and 3, $q_1^*(t) < f^{-1}(\bar{c}_2)$ for all $t \geq \tau$. Hence the dominant firm's total discounted profit yielded by q_1^* from time τ on is strictly less than $\pi(f^{-1}(\bar{c}_2))/r$.

Now let $\bar{x}_1$ be the dominant firm's initial stock and

$\hat{q}_1: t \to \hat{q}_1(t)$ be the extraction strategy defined by

$$\hat{q}_1(t) = Q_{max}, \quad 0 \leqq t \leqq \bar{x}_1/Q_{max}$$
$$= 0 \quad , \text{elsewhere}$$

Then, for $\bar{x}_1$ sufficiently large, the dominant firm's discounted profit under $\hat{q}_1$ approaches $\pi(Q_{max})/r > \pi(f^{-1}(\bar{c}_2))/r$. This last strict inequality means that $\tau > 0$ if $\bar{x}_1$ is large, i.e., entry will not occur until some time has elapsed.

To show that the dominant firm and the fringe share the market for some time, we prove $q_1^*(\tau) > 0$. To this end, we shall show that $q_1^*(\tau) = 0$ leads to a contradiction. First, let $s < \tau$ be sufficiently close to τ. Then $f(q_1^*(s)) \leqq \bar{c}_2$. Now, a small reduction, say $\epsilon > 0$, in the output at s from $q_1^*(s)$ to $q_1^*(s) - \epsilon$ will cause a reduction in the dominant firm's discounted profit no larger than $\epsilon\bar{c}_2 e^{-rs}$. If $q_1^*(\tau) = 0$, then a small increase in its output, say ϵ, at τ will increase the dominant firm discounted profit by the amount $\epsilon h_1(0,\bar{Q}_2)e^{-r\tau} = \epsilon f(\bar{Q}_2)e^{-r\tau} > \epsilon\bar{c}_2 e^{-r\tau} \cong \epsilon\bar{c}_2 e^{-rs}$: contradicting the optimality of q_1^*.

q.e.d.

<u>Lemma 5</u>

If the dominant firm's initial stock is sufficiently large, then

$q_1^*(t) \leqq q^m(t)$ for all $t \geqq \tau$.

with strict inequality holding whenever $q_1^*(t) > 0$. Here, q^m, as defined in the statement of Theorem 2, is the dominant firm's optimal extraction strategy if it enjoys complete monopoly power.

Proof

Let λ and λ^m be the shadow prices of the dominant firm's initial stock under potential competition and under complete monopoly, respectively. If the dominant firm's initial stock is sufficiently large, then it can be shown that $\lambda > \lambda^m$; for a proof, see Gilbert and Goldman [1978].

It is sufficient to prove the lemma for t inside $I_2 = [\tau, \tau+\Delta\tau)$. Let t be any time in I_2 such that $q_1^*(t) > 0$. We claim that

(9) $$e^{-rt}h_1(q_1^*(t),\overline{Q}_2) \geqq \lambda$$

Indeed, if $e^{-rt}h_1(q_1^*(t),\overline{Q}_2) < \lambda$, then the dominant firm's discounted marginal revenue at t is strictly less than $\lambda = \pi'(q_1^*(0))$, its marginal revenue at time zero. This means that q_1^* can certainly be improved by shifting an infinitesimal amount of output at t back to time zero: a contradiction to the optimality of q_1^*. Therefore, (9) must be true and

(10) $$e^{-rt}h_1(q_1^*(t),\overline{Q}_2) \geqq \lambda > \lambda^m = e^{-rt}\pi'(q^m(t))\,.$$

The strict inequality between the first term and the last term of (10) together with assumption (f3) then imply $q^m(t) > q_1^*(t)$.

q.e.d.

Lemma 6

If the dominant firm initial stock is sufficiently large, then

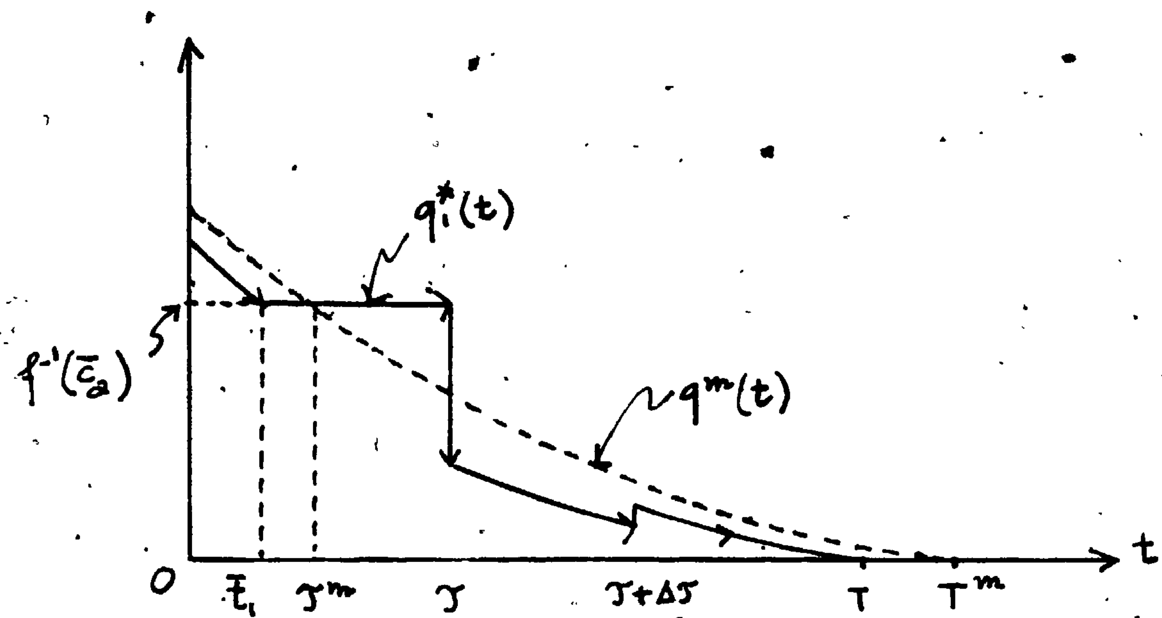
(a) $q_1^*(t) > f^{-1}(\overline{c}_2)$ for all t in $(0,\overline{t}_1)$

$= f^{-1}(\overline{c}_2)$ for all t in $[\overline{t}_1,\tau)$

(b) $0 < \overline{t}_1 < \tau^m < \tau$

Here $\overline{t}_1$ is the first time the market price reaches $\overline{c}_2$ under potential competition and τ^m is the first time the market price reaches $\overline{c}_2$ under the assumption that the dominant firm enjoys complete monopoly power.

Proof



If the dominant firm's initial stock is sufficiently large, then $\tau > 0$ by Lemma 4. That is, $q_1^*(t) \geqq f^{-1}(\bar{c}_2)$ for all $t < \tau$. We claim that $q_1^*(s) = f^{-1}(\bar{c}_2)$ for some time $s < \tau$. Indeed, if this is not the case, then we must have

(11) $\quad q_1^*(t) < q^m(t)$ for all $t < \tau$

because $\lambda > \lambda^m$. Furthermore, by Lemma 5,

(12) $\quad q_1^*(t) \leqq q^m(t)$ for all $t \geqq \tau$

Hence, taken together, (11) and (12) violate the stock constraint for the dominant firm. If we let $\bar{t}_1$ be the first time q_1^* reaches $f^{-1}(\bar{c}_2)$, then because q_1^* is monotone decreasing in phase one, we must have

$$q_1^*(t) > f^{-1}(\bar{c}_2) \text{ for all } t \text{ in } (0,\bar{t}_1)$$

$$= f^{-1}(\bar{c}_2) \text{ for all } t \text{ in } [\bar{t}_1,\tau)$$

$$0 < \bar{t}_1 < \tau$$

Finally, because (12) is true and (11) holds for all $t < \bar{t}_1$, the extraction strategies q_1^* and q^m must intersect at some time τ^m inside $(\bar{t}_1,\tau)$, due to the stock constraint.

q.e.d.

<u>Lemma 7</u>

Suppose that the dominant firm's initial stock is sufficiently large. Then

(a) Phase three will not exist if $\pi'(f^{-1}(\bar{c}_2))e^{r\Delta\tau} \geqq f(0)$

(b) Phase three will exist if $f(\bar{Q}_2)e^{r\Delta\tau} \leqq f(0)$.

(c) If phase three exists, then

$$a < q_1^*(\tau+\Delta\tau) < a + \bar{Q}_2$$

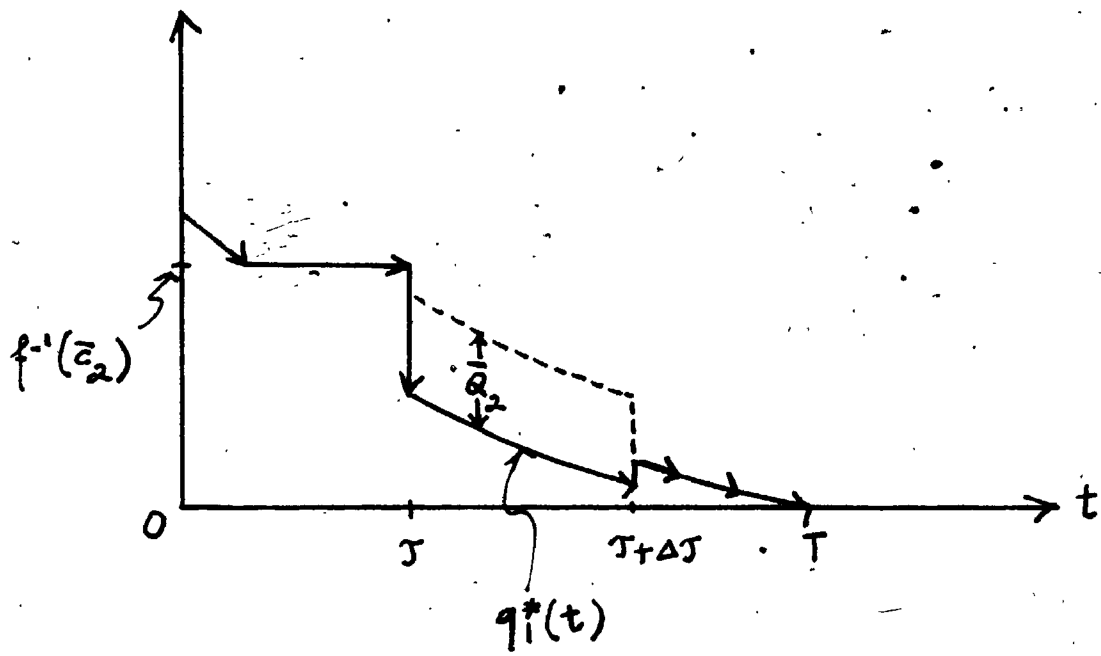
where a is the left limit of q_1^* at $\tau+\Delta\tau$, i.e.,

$$a = \lim_{t<\tau+\Delta\tau} q_1^*(t)$$

(d) If phase three does not exist, then

$$q_1^*(\tau) + \bar{Q}_2 < f^{-1}(\bar{c}_2)$$

Proof



(a) To prove (a), suppose the contrary, i.e., $q_1^*(\tau+\Delta\tau) > 0$. Then we must have

$$\pi'(q_1^*(\tau+\Delta\tau)) = \lambda e^{r(\tau+\Delta\tau)}$$

$$> \lambda e^{rt_1} e^{r\Delta\tau} = \pi'(f^{-1}(\bar{c}_2))e^{r\Delta\tau}$$

$$\geqq f(0) \doteq \pi'(0)$$

a contradiction.

(b) To prove (b), we first note that if the dominant firm's initial stock is large, then $q_1^*(\tau) > 0$ by Lemma 4. This last strict inequality means that

$$\lambda e^{r\tau} \leqq h_1(q_1^*(\tau),\overline{Q}_2) < h_1(0,\overline{Q}_2) = f(\overline{Q}_2) \tag{13}$$

If $f(\overline{Q}_2)e^{r\Delta\tau} \leqq f(0)$, then from (13) we must have

$$\lambda e^{r(\tau+\Delta\tau)} < f(0) = \pi'(0) \tag{14}$$

Clearly, the strict inequality (14) means that for q_1^* to be optimal we must have $q_1^*(\tau+\Delta\tau) > 0$.

(c) Now suppose that phase three exists. If a is the left limit of q_1^* at $\tau+\Delta\tau$, there are three possibilities to consider: $a = 0$, $0 < a < f^{-1}(\overline{c}_2) - \overline{Q}_2$, and $a = f^{-1}(\overline{c}_2) - \overline{Q}_2$.

First, if $a = 0$, then we must have

$$f(\overline{Q}_2) = h_1(0,\overline{Q}_2) \leqq \pi'(q_1^*(\tau+\Delta\tau))$$

i.e., $0 = a < q_1^*(\tau+\Delta\tau) < \overline{Q}_2$.

Second, if $0 < a < f^{-1}(\overline{c}_2) - \overline{Q}_2$, then we must have

$$\begin{aligned}\pi'(q_1^*(\tau+\Delta\tau)) = h_1(a,\overline{Q}_2) &= f(a + \overline{Q}_2) + af'(a + \overline{Q}_2)\\ &> \pi'(a + \overline{Q}_2)\end{aligned}$$

i.e., $a < q_1^*(\tau+\Delta\tau) < a + \overline{Q}_2$, by assumption (f4) and the assumed concavity of π.

Third, if $a = f^{-1}(\overline{c}_2) - \overline{Q}_2$, i.e., $a + \overline{Q}_2 = f^{-1}(\overline{c}_2)$, then because

$$\lambda = e^{-r(\tau+\Delta\tau)}\pi'(q_1^*(\tau+\Delta\tau)) = e^{-r\overline{t}_1}\pi'(f^{-1}(\overline{c}_2)) ,$$

we must have $q_1^*(\tau+\Delta\tau) < f^{-1}(\overline{c}_2)$. Furthermore, because $h_1(a,\overline{Q}_2) \geqq \pi'(q_1^*(\tau+\Delta\tau))$, we must also have $a < q_1^*(\tau+\Delta\tau)$ by assumption (f4).

(d) If phase three does not exist, we claim that $q_1^*(\tau) + \overline{Q}_2 < f^{-1}(\overline{c}_2)$. To prove this, suppose the contrary, i.e., $q_1^*(\tau) + \overline{Q}_2 = f^{-1}(\overline{c}_2)$. The equality means that

$$(15) \qquad \lambda e^{r\tau} \leqq h_1(q_1^*(\tau),\overline{Q}_2) = f(q_1^*(\tau) + \overline{Q}_2) + q_1^*(\tau)f'(q_1^*(\tau) + \overline{Q}_2) < \overline{c}_2$$

If the dominant firm transfers one unit of output from time zero to time τ, its net gain in discounted profit will be $\overline{c}_2 e^{-r\tau} - \lambda > 0$ by (15): a contradiction to the optimality of q_1^*.

q.e.d.

We are finally ready to prove Theorems 1 and 2. First, we note that Lemmas 3, 4, 6, and 7 constitute Theorem 1.

As for the proof of Theorem 2, we proceed as follows.

(a) Because $\lambda > \lambda^m$, we must have

$$f(0) = \pi'(0) = \lambda^m e^{rT^m} < \lambda e^{rT^m}$$

Furthermore, because $\lambda e^{rT} \leqq f(0)$, it then must be true that $T < T^m$, establishing part (a) of Theorem 2.

(b) To prove part (b) of Theorem 2, we note that

$$(16) \qquad \int_0^{\tau} q_1^*(t)dt > \int_0^{\tau} q^m(t)dt > \int_0^{\tau^m} q^m(t)dt$$

The first strict inequality in (16) is due to Lemma 5; the second strict inequality in (16) is due to part (b) of Lemma 6.

5. CONCLUSION

The model presented in this chapter deals with the phenomenon of limit pricing in a market of a certain non-renewable resource. If the dominant firm's initial stock is large while the fringe's stock is small, then we show that the extraction pattern of the whole market consists of three phases. During phase one, the dominant firm is the sole supplier in the market. During the second phase, the dominant firm and the fringe share the market at least for some time. Phase three begins after the fringe has exhausted its stock, and during this last phase, the dominant firm becomes a monopolist. Furthermore, Theorem 2, in some sense, represents an extension of the limit pricing theory into the context of a nonrenewable resource: facing potential entry at price $\geq \bar{c}_2$, a sensible solution for the dominant form is to operate longer below $\bar{c}_2$. Finally, we note that if the fringe possesses a backstop technology, i.e., there are no constraints on its extraction capacity and its stock, then our model is identical to that of Hoel [1978] and that of Gilbert and Goldman [1978].

CHAPTER TWO

THE MONOPOLY PRICE OF A NONRENEWABLE RESOURCE UNDER STOCHASTIC ENTRY

1. INTRODUCTION

In Chapter One, we studied the pricing strategy, in a market for a certain nonrenewable resource, of a dominant firm facing potential competition from a competitive fringe. The fringe was assumed to exist right from the beginning, and its entry behavior was postulated to be as follows: in any period such that the remaining stock of the fringe is still positive and the instantaneous output of the dominant firm alone is not sufficient to keep the market price below $\bar{c}_2$, the fringe's extraction cost, entry will occur until either the market price is driven down to $\bar{c}_2$ or the fringe's combined extraction capacity $\bar{Q}_2$ is reached, whichever happens first.

In this chapter, we build a model of monopoly resource extraction under stochastic entry. Unlike the model of Chapter One, entry cannot be controlled with absolute certainty in the present model. Here, we make the entry process depend probabilistically on the pricing strategy of the monopolist. Furthermore, in the present model, we assume that the monopolist is the only firm in the market and the consumers are the potential entrants; this assumption contrasts sharply with the model of Chapter One in which entry comes from the fringe in the form of intra-industry competition.

The subject of consumers as potential entrants has been investigated by Newbery [1978], who studied the pricing strategy that one would expect from OPEC. According to this author, a high price charged by OPEC might induce the consuming countries to switch to alternative sources of energy such as shale oil or large scale nuclear generating plants. The switch, of course, involves heavy investments and, once made, makes the opportunity cost of operating the plants so low that it will never pay to scrap the plants and return to the dependence on foreign imports. We can also visualize consumers switching from oil to solar energy or natural gas in heating their homes. Finally, we should not forget that a higher price for oil also intensifies the research conducted in breeder technology or hydrogen fusion.

To give more structure to our model, we postulate the possibility of the occurrence of a single event: the decision of a block of countries to leave the monopolist and provide for their own needs by making the necessary heavy investments in an alternative source. As in Newbery [1978], we also assume that the monopolist's loss in customers, once it occurs, is permanent. If we interpret the market demand curve as the sum of the individual demand curves of the consumers, then the monopolist's post-entry demand curve is the sum of the individual demand curves of those who did not make the switch. As for the exact time the switch occurs, we shall assume that its probability distribution, to be precisely described in the following section, depends only on the pricing strategy pursued by the monopolist.

A model somewhat similar to our present model was formulated by Dasgupta and Stiglitz [1981]. These two authors investigated the behavior of a monopolist who possesses a finite stock of a certain nonrenewable resource, and who faces the threat of entry by a competitive fringe which can produce a perfect substitute from a backstop technology. However, while our model allows the monopolist to influence the probability of the time entry occurs, these two authors assumed that this probability is exogenous, i.e., outside the control of the monopolist. Furthermore, the questions asked by these two authors are different from ours. Mathematically speaking, the model of Dasgupta and Stiglitz [1981] is almost the same as the one formulated by Long

[1975], who investigated the problem of resource extraction under the uncertainty of possible nationalization. Under the drastic assumption of no compensation after nationalization, this author demonstrated that the optimal extraction strategy requires extracting more of the resource at the beginning and ceasing production earlier under uncertainty about nationalization than under no threat of nationalization.

Our main contributions in this chapter include the following results:

(i) The monopolist's optimal extraction strategy under stochastic entry, $\dot{q}^*: t \to q^*(t)$, is strictly monotone decreasing and exhausts his initial stock $\bar{x}$ in finite time, say at $T < \infty$. This result is Theorem 1.

(ii) $T < T^m$. Here T^m is the exact time the monopolist would exhaust his stock $\bar{x}$ if he enjoyed complete monopoly, i.e., if entry were not allowed. That is, facing possible entry, the monopolist plans to exhaust his stock $\bar{x}$ sooner than if entry were prohibited. This result is the substance of Theorem 2.

(iii) If the instantaneous probability of entry is linear, then, at time zero, price is lower under stochastic entry than under complete monopoly. Furthermore, we also show that the uncertain future under stochastic entry has induced the monopolist to attach a risk premium on top of the market rate of interest in the sense that the discounted marginal revenue $e^{-rt}\pi'(q^*(t))$ is rising along the optimal trajectory. These results are given by Theorem 3.

(iv) In the case the instantaneous probability of entry is linear in the monopolist's output Q, say

$$m(Q) = -aQ + b \ , \quad a > 0, \ b > 0$$

we show in Theorem 4 that a decrease in the value of the parameter a, ceteris paribus, will cause an increase in the monopolist's price at time zero. That is, an upward shift in the instantaneous probability of entry function, engendered by a decrease in a, will induce the monopolist to raise his price in the beginning.

The plan of this chapter is as follows. In section 2, we present our model. In section 3, the optimal solution is investigated. Section 4 contains some concluding remarks. Some more technical results are proved in the appendix.

2. THE MODEL

2.1 The Pre-Entry Market Demand Curve

Let $f: Q \to f(Q)$ be the market demand curve, as enjoyed by the monopolist before entry occurs. Here Q denotes output. We impose the following assumptions on f:

(f1) $0 < f(0) < +\infty$

$f(Q) > 0 \ , \ f'(Q) < 0 \quad \text{for all } Q \geqq 0 \ .$

(f2) the total revenue curve $\pi: Q \to \pi(Q) = Qf(Q)$ is strictly concave and achieves a maximum at $Q = Q_{max} > 0$.

2.2 The Post-Entry Demand Curve Enjoyed By the Monopolist

To model the entry process, we visualize the possibility of the occurrence of a single event: the decision of a block of consumers to leave the monopolist and make the necessary heavy investments to provide for their own needs; we shall say that entry occurs whenever this event happens. Furthermore, we shall assume that this loss in customers, once it occurs, is irreversible. The post-entry demand curve enjoyed by the monopolist is then given by the sum of the individual demand curves of the customers who remain loyal to the monopolist. We shall let

$\hat{f}$: $Q \rightarrow \hat{f}(Q)$ denote this post-entry demand curve and impose the following assumptions on $\hat{f}$

($\hat{f}1$) $\quad 0 < \hat{f}(0) < +\infty$

$\hat{f}(Q) > 0$, $\hat{f}'(Q) < 0$ for all $Q \geqq 0$

($\hat{f}2$) $\quad$ the total revenue curve $\hat{\pi}$: $Q \rightarrow \hat{\pi}(Q) = Q\hat{f}(Q)$ is strictly concave

($\hat{f}3$) $\quad \hat{\pi}'(Q) < \pi'(Q)$ for all $Q > 0$

($\hat{f}4$) $\quad \hat{f}(0) = f(0)$

$\hat{f}(Q) < f(Q)$ for all $Q > 0$

Assumptions ($\hat{f}1$), ($\hat{f}2$), ($\hat{f}3$) are standard. Assumption ($\hat{f}3$), which is fundamental to our model, is needed to derive strong results. We can readily verify that if $\hat{f}^{-1} = \alpha f^{-1}$, $0 < \alpha < 1$, then $\hat{f}$ certainly satisfies all the assumptions ($\hat{f}1$) through ($\hat{f}4$). In particular, if both f and $\hat{f}$ are linear and ($\hat{f}4$) also holds, then we can also show that ($\hat{f}3$) is satisfied.

2.3 Stochastic Specification of the Entry Process

Because it is more convenient to work with quantity than with price, we shall describe the random time of entry in terms of the monopolist's extraction strategy. To this end, let q: $t \to q(t)$ be any feasible extraction strategy for the monopolist, i.e.,

$$q(t) \geqq 0 \text{ for all } t$$

$$\int_0^\infty q(t)dt \leqq \bar{x}$$

Here q(t) is the extraction rate at time t, given that entry has not yet occurred, and $\bar{x} > 0$ is the monopolist's initial stock. If we let x(t) denote the remaining stock under q, then

$$dx/dt = -q(t) \quad , \; x(0) = \bar{x}$$

Pick any $t \geqq 0$ such that $x(t) > 0$. If entry has not occurred by time t, we shall assume that in an infinitesimal time interval (t,t+h] entry will occur with probability

$$hm(q(t)) + O(h)$$

and the probability that entry will not occur in (t,t+h] is equal to

$$1 - hm(q(t)) + O(h)$$

Here $h > 0$ is sufficiently small and $\lim O(h)/h = 0$ as $h \to 0$. Also,

$$m: \; Q \to m(Q)$$

is a nonnegative function defined for all $Q \geqq 0$. As just defined, the instantaneous probability of entry $m(Q)$ depends only on the current extraction rate, or, equivalently, the current price. We shall assume that

$$m'(Q) \leqq 0 \ , \ m''(Q) \geqq 0 \quad \text{for all } Q \geqq 0 \ .$$

The assumption $m'(Q) \leqq 0$ means that the monopolist can reduce the probability of entry by lowering his price. The assumption $m''(Q) \geqq 0$, a convexity assumption, implies diminishing return in the probability of entry as the monopolist expands his output.

Once m is given, each feasible extraction strategy q induces on the time axis a unique probability distribution characterizing the random time of entry. To determine this probability distribution, we proceed as follows.

Let $G(t,q)$ denote the probability, under q, that entry has not occurred by time t. For any given $h > 0$ sufficiently small, the probability that entry has not occurred by time t+h is

$$G(t+h,q) = G(t,q)[1-hm(q(t)) + O(h)]$$

Subtracting $G(t,q)$ from both sides of the preceding expression, dividing the result by h, then letting $h \to 0$, we obtain the following differential equation

$$\frac{dG}{dt}(t,q) = -m(q(t))G(t,q)$$
$$G(0,q) = 1$$

the solution of which is

$$G(t,q) = \exp\left[-\int_0^t m(q(s))ds\right]$$

For our purpose, we shall let $F(t,q)$ denote the cumulative probability, under q, that entry occurs inside the time interval $[0,t]$. That is,

$$(1) \qquad F(t,q) = 1 - G(t,q)$$

2.4 Statement of the Problem of Monopoly Resource Extraction Under Stochastic Entry

Find an extraction strategy $q: t \rightarrow q(t)$ to maximize the following expected discounted profit

$$(2) \qquad \int_0^\infty \left[\int_0^t e^{-rs}\pi(q(s))ds + V(x(t),t)\right]dF(t,q)$$

subject to

$$dx/dt = -q(t)$$
$$x(0) = \bar{x}$$
$$q(t) \geqq 0, \; x(t) \geqq 0 \text{ for all } t$$

where r is the market rate of interest, and

$$V(z,t) = \max \int_t^\infty e^{-rs} \hat{\pi}(\hat{q}(s))ds$$

subject to

$$\hat{q}(s) \geqq 0 \quad \text{for all } s \geqq t$$

$$\int_t^\infty \hat{q}(s)ds \leqq z$$

i.e., V(z,t) represents the monopolist's post-entry discounted profit, given that entry occurs at t and his remaining stock at t is z.

We note that in the model we have just formulated, there is no need for revision of the conditional plan at any future time.

3. THE OPTIMAL SOLUTION

We shall solve the problem of limit pricing under stochastic entry by the technique of Dynamic Programming. To this end, let

$$W: \quad (x,t) \rightarrow W(x,t)$$

be a function defined for all $x \geqq 0$, $t \geqq 0$. Here we shall interpret W(x,t) as the monopolist's optimal expected discounted profit, given that he begins at time $t \geqq 0$ with initial stock $x \geqq 0$ and that entry has not occurred yet. In particular, $W(\overline{x},0)$ gives the expected discounted profit yielded by the optimal solution to the original problem stated in section 2.3.

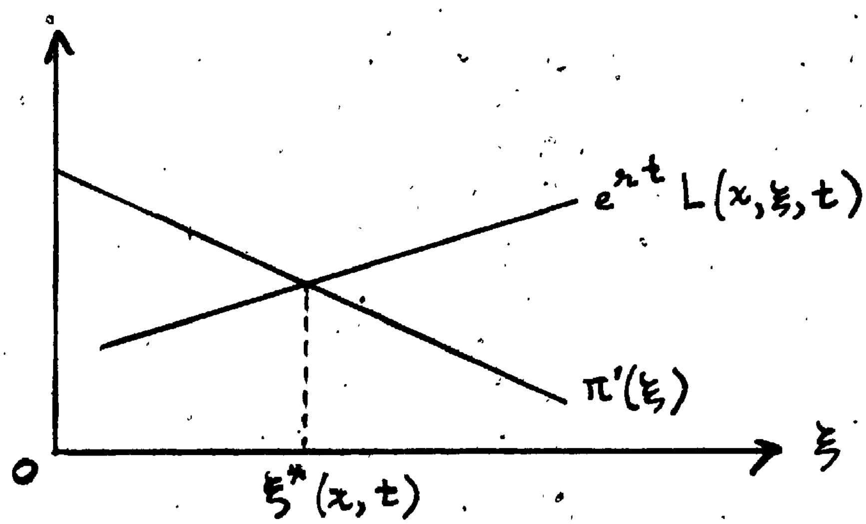
Assuming that W is sufficiently smooth, we can use a standard argument of Dynamics Programming to derive the following Hamilton-Jacobi-Bellman equation for the problem of monopoly resource extraction under stochastic entry (the derivation is given in the appendix to this chapter)

$$(3) \qquad \max_{\xi \geq 0} \begin{bmatrix} W_t(x,t) - \xi W_x(x,t) + e^{-rt}\pi(\xi) \\ \\ + m(\xi)(V(x,t) - W(x,t) \end{bmatrix} = 0$$

Maximizing the expression in the left side of (3) with respect to ξ, we obtain the following first order necessary condition for an interior solution

$$(4) \qquad e^{-rt}\pi'(\xi) = W_x(x,t) - m'(\xi)(V(x,t) - W(x,t))$$

$$= L(x,\xi,t)$$



Due to the assumed convexity of $m(\xi)$, the right side of (3) must be an increasing function of ξ. Hence (4) has a unique solution, say $\xi^*(x,t)$.

To get an economic interpretation of (3), we first note that the term $m(\xi)(W(x,t) - V(x,t))$ represents the expected loss, due to entry, as a function of output ξ at time t (recall that $m(\xi)$ is the instantaneous probability of entry). Equation (4) then means that along the optimal trajectory, the monopolist's output is increased until the marginal gain, which is the sum of the marginal revenue $e^{-rt}\pi'(\xi)$ and the marginal reduction in instantaneous expected loss $m'(\xi)(W(x,t)-V(x,t))$, is equal to the shadow price $W_x(x,t)$.

To learn more about the optimal extraction strategy of the monopolist, let's consider the following differential equation

$$(5) \qquad dx^*/dt = -\xi^*(x^*,t)$$
$$x^*(0) = \bar{x}$$

That is, $x^*(t)$ represents the monopolist's optimal remaining stock at time t before entry occurs.

Next, let $\lambda(t) = W_x(x^*(t),t)$ be the shadow price of the monopolist's remaining stock along the optimal trajectory, given that entry has not yet occurred. In the appendix to this chapter, we show that $\lambda(t)$ satisfies the following differential equation

(6) $$d\lambda/dt = m(q^*(t))[\lambda(t) - V_x(x^*(t),t)]$$

where

(7) $$q^*: \ t \to q^*(t) = \xi^*(x^*(t),t)$$

is the optimal solution of the problem of monopoly resource extraction under stochastic entry.

Before presenting our major results, we state the following lemma, which plays a fundamental role in characterizing the optimal solution. The proof of this lemma is long and technical, hence we delegate it to the appendix of this chapter.

<u>Lemma</u>

For any $x > 0$ and any $t \geq 0$, we have $W_x(x,t) > V_x(x,t)$.

We are now ready for our first main result.

<u>Theorem 1</u>

The optimal extraction strategy $q^*: \ t \to q^*(t)$, as defined by (6), is strictly monotone decreasing and exhausts the initial stock at time $T < \infty$. Furthermore, $\lambda(T)e^{rT} = f(0)$.

Proof

Using the lemma just stated and (6), we can assert immediately that the shadow price $\lambda(t) = W_x(x^*(t),t)$ is strictly monotone increasing along the optimal trajectory. Furthermore, the preceding lemma also implies that $W(x^*(t),0) - V(x^*(t),0)$ is monotone decreasing in t.

Now we note that $q^*(t)$ is the unique solution of (4), i.e., the equation in one unknown ξ

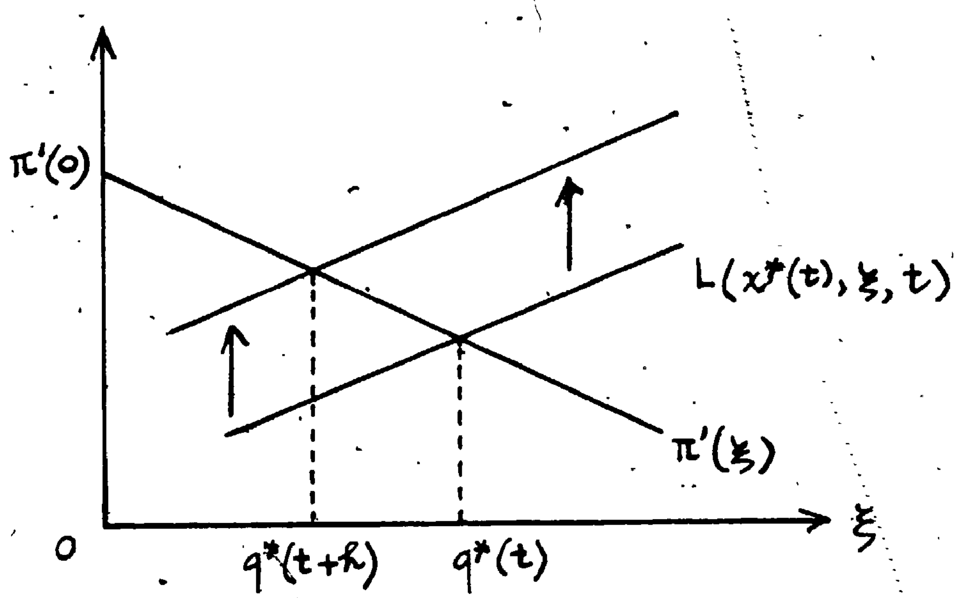
$$(8) \qquad \pi'(\xi) = e^{rt} L(x^*(t),\xi,t)$$

has $q^*(t)$ as the solution. Next, we note that for each given ξ

$$L(x^*(t),\xi,t)e^{rt} = e^{rt} W_x(x^*(t),t) + m'(\xi)(W(x^*(t),0)-V(x^*(t),0))$$

increases through time by the results of the preceding paragraph.

Therefore q^* must be strictly monotone decreasing through time.



To show that q^* depletes $\bar{x}$ in finite time, suppose the contrary, i.e., $x^*(t) > 0$ for all t. Then because $\lambda(t)$ is strictly monotone increasing, we must have

$$\lambda(t)e^{rt} = W_x(x^*(t),t)e^{rt}$$

$$= W_x(x^*(t),0) > f(0)$$

when t gets sufficiently large: a clear contradiction because the undiscounted shadow price $\lambda(t)e^{rt}$ can never exceed the choke price. Therefore, q^* exhausts $\bar{x}$ in finite time, say at $T < +\infty$.

Finally, because $W_x(x,0) > V_x(x,0)$ for all $x > 0$, as asserted by the lemma, we must have

$$W_x(0,0) \geqq V_x(0,0) = f(0)$$

This inequality together with the fact $W_x(0,0) \leqq f(0)$ then mean that $W_x(0,0) = f(0)$. Hence at the exact time $\overline{x}$ is exhausted we must have

$$\lambda(T)e^{rT} = W_x(0,0) = f(0)$$

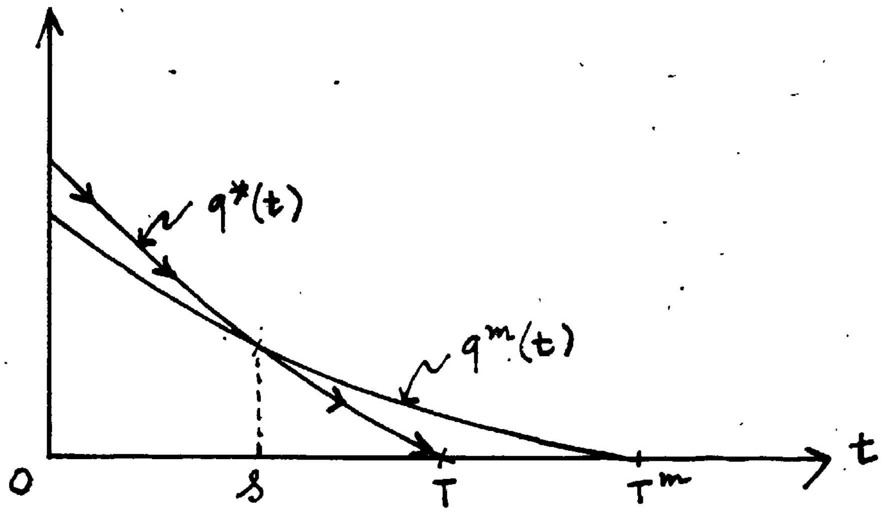
as desired.

q.e.d.

Theorem 2

Let q^m: $t \to q^m(t)$ be the monopolist's optimal extraction strategy under the assumption that entry by consumers is prohibited, and T^m be the exact time $\overline{x}$ is depleted under q^m. Then $T < T^m$, i.e., the possibility of entry has induced the monopolist to exhaust his stock faster under stochastic entry than under absolute monopoly, given that q^* is successfully carried out before entry might occur.

Proof



Let λ^m be the shadow price under complete monopoly. There are two possibilities to consider.

First, if $\lambda(0) < \lambda^m$, then $q^*(0) > q^m(0)$ by (4). Hence by the stock constraint q^* and q^m must intersect at some time $s < T^m$. At time s, we must have

$$e^{-rs}\,\pi(q^m(s)) = \lambda^m = e^{-rs}\,\pi(q^*(s))$$

$$= \lambda(s) - m'(q^*(s))(V(x^*(s),s) - W(x^*(s),s))$$

i.e., $\lambda(s) > \lambda^m$. Furthermore, because λ is strictly monotone increasing, we also have

$$\lambda(s)e^{rT} < \lambda(T)e^{rT} = f(0)$$

Therefore $f(0) = \lambda(T)e^{rT} > \lambda^m e^{rT}$, implying $T < T^m$.

Second, if $\lambda(0) \geqq \lambda^m$, then again we must have $f(0) = \lambda(T)e^{rT} > \lambda^m e^{rT}$ because λ is strictly monotone increasing, proving $T < T^m$.

q.e.d.

Intuitively speaking, we expect that the threat of entry will induce the monopolist to shift production to the present and put less emphasis on the uncertain future. Indeed, if the instantaneous probability of entry is linear, then we can show that the monopolist's output at the beginning is higher under stochastic entry than under complete monopoly. Furthermore, the uncertain future under stochastic entry also induces the monopolist to attach a risk premium on top of the market rate of interest in the sense that discounted marginal revenue is rising along the optimal trajectory. These results are given by the following theorem.

Theorem 3

Suppose that

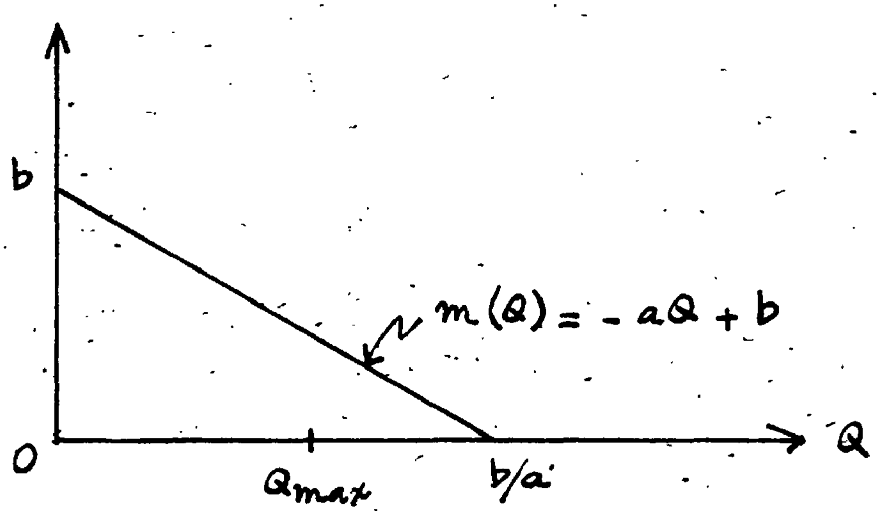
$$m(Q) = -aQ + b, \quad 0 \leqq Q \leqq b/a$$
$$= 0 \quad , \quad Q > b/a$$

where a and b are two positive constants satisfying $Q_{max} < b/a$. Then

(a) $e^{-rt} \pi'(q^*(t))$ is rising through time

(b) $q^*(0) > q^m(0)$

<u>Proof</u>



Using the assumption $Q_{max} < b/a$, we see immediately that $q^*(t) < b/a$ for all t. Next, by (4), we must have

$$(9) \qquad e^{-rt} \pi'(q^*(t)) = W_x(x^*(t),t) + a(V(x^*(t),t) - W(x^*(t),t))$$

The first term on the right side of (8) is rising; the second term on the right side of (8) is negative and its absolute value is decreasing. Hence, the right side of (9) must be rising through time, proving (a).

To prove (b), suppose the contrary, i.e., $q^*(0) \leq q^m(0)$. Then (a) implies

$$e^{-rt} \pi'(q^*(t)) > e^{-rt} \pi'(q^m(t)) = \lambda^m$$

for all t in (0,T], i.e., $q^*(t) < q^m(t)$ for all t in (0,T], violating the stock constraint.

q.e.d.

The final result of this chapter describes the effect, on the monopolist's optimal extraction strategy, of an upward shift in the instantaneous probability of entry function.

Theorem 4

Suppose that the instantaneous probability of entry is as given in Theorem 3, i.e.,

$$\begin{aligned} m(Q) &= -aQ + b \quad , \quad 0 \leq Q \leq b/a \\ &= 0 \qquad\qquad\quad Q > b/a \end{aligned}$$

where a and b are two positive constants satisfying $Q_{max} < b/a$. For each given value of a, let $q^*(\cdot,a): t \to q^*(t,a)$ be the monopolist's corresponding optimal extraction strategy under stochastic entry. Then

$$q^*(0,a_1) > q^*(0,a_2)$$

if $a_1 > a_2 > 0$, $Q_{max} < b/a_1$. That is, an increase in the instantaneous probability of entry engendered by a decrease of the value of a from a_1 to a_2 induces the monopolist to decrease the extraction rate in the beginning.

Proof

For any given value of a, let $W(x,t,a)$ be the corresponding optimal expected discounted profit for the monopolist from the initial condition (x,t).

Now, at time $t = 0$, the Hamilton-Jacobi-Bellman equation (2) can be written as follows

$$
\begin{aligned}
(10)\qquad -W_t(\bar{x},0,a) &= rW(\bar{x},0,a) \\
&= -q^*(0,a)W_x(\bar{x},0,a) + \pi(q^*(0,a)) \\
&\quad (-aq^*(0,a)+b)(V(\bar{x},0)-W(\bar{x},0,a)) \\
&= \pi(q^*(0,a))-q^*(0,a)\pi'(q^*(0,a)) \\
&\quad +b(V(\bar{x},0)-W(\bar{x},0,a))
\end{aligned}
$$

with the help of (3). A rearrangement of (10) gives

$$
\begin{aligned}
(11)\qquad & \pi(q^*(0,a))-q^*(0,a)\pi'(q^*(0,a)) \\
&= rW(\bar{x},0,a) + b(W(\bar{x},0,a)-V(\bar{x},0))
\end{aligned}
$$

To prove Theorem 4, we claim that it is sufficient to prove that $W(\bar{x},0,a)$ decreases when a decreases. Indeed, if this is the case then $a_1 > a_2$ together with (11) imply

(12) $\pi(q^*(0,a_1)) - q^*(0,a_1)\pi'(q^*(0,a_1)) >$

$\pi(q^*(0,a_2)) - q^*(0,a_2)\pi'(q^*(0,a_2))$

i.e., $q^*(0,a_1) > q^*(0,a_2)$ because $\pi(Q) - Q\pi'(Q)$ is strictly increasing in Q.

Therefore, all the following effort is devoted to showing that $W(\bar{x},0,a_1) > W(\bar{x},0,a_2)$.

First, let $q: t \to q(t)$ be any feasible extraction strategy for the monopolist. If $a_1 > a_2$, then using (1), we have

(13) $F(t,q,a_2) > F(t,q,a_1)$ for all $t > 0$

Here $F(t,q,a_i)$, $i = 1,2$, is the cumulative probability, under q, that entry occurs inside $[0,t]$ when $a = a_i$.

The expected discounted profit, under q, is given by (2), or, equivalently, by

(14) $\int_0^\infty A(t,q)dF(t,q,a_i)$

when $a = a_i$. Here

$$A(t,q) = \int_0^t e^{-rs}\pi(q(s))ds + V(x(t),t)$$

We claim that if $q^*(\cdot,a_i)$ maximizes (2) or (14), then it is necessary that $A(t,q^*(\cdot,a_i))$ is monotone increasing in t. Indeed, if we let $A'(t,q^*)$ be the time derivative of $A(t,q^*)$, then (suppressing a_i)

$$(15)\qquad \begin{aligned} A'(t,q^*) &= e^{-rt}\,\pi(q^*(t)) - q^*(t)\,V_x(x^*(t),t) \\ &\quad -rV(x^*(t),t) \\ &\geqq e^{-rt}\,\pi(q^*(t)) - q^*(t)W_x(x^*(t),t) \\ &\quad -rW(x^*(t),t) \\ &= m(q^*(t))(W(x^*(t),t) - V(x^*(t),t)) \\ &\geqq 0 \end{aligned}$$

The first inequality in (15) is due to the fact $W_x(x,t) > V_x(x,t)$ for all $x > 0$ and all $t \geqq 0$, as guaranteed by the lemma, and the obvious fact $W(x,t) > V(x,t)$ for all $x > 0$ and all $t \geqq 0$. The second equality is due to the Hamilton-Jacobi-Bellman equation (3). We also note that all the inequalities in (15) are strict whenever $x^*(t) > 0$. Therefore, to maximize (14), we only need to consider feasible extraction strategies q's such that $A(t,q)$ is monotone increasing in t. This means that we can give an alternative expression for (14) in terms of the Lebesque-Stieltjes integral as follows

$$(16)\qquad \begin{aligned} \int_0^\infty A(t,q)dF(t,q,a_i) &= [A(t,q)F(t,q,a_i]_0^\infty \\ &\quad - \int_0^\infty F(t,q,a_i)dA(t,q) \\ &= \int_0^{T(q)} e^{-rt}\pi(q(t))dt - \int_0^\infty F(t,q,a_i)dA(t,q) \end{aligned}$$

Here $T(q)$ is the exact time of exhaustion. Due to Theorem 1, we can assume $T(q) < \infty$, because the optimal extraction strategy q^* exhausts the monopolist in finite time. Now, using (13) and (16), we must have

$$(17) \qquad \int_0^\infty A(t,q)\,dF(t,q,a_1) > \int_0^\infty A(t,q)\,dF(t,q,a_2)$$

proving $W(\bar{x},0,a_1) > W(\bar{x},0,a_2)$, as desired.

q.e.d.

4. CONCLUSION

The main results of this chapter, Theorems 2 and 3, demonstrate that the threat of stochastic entry has induced the monopolist to extract more at the beginning and cease production earlier than if entry is prohibited. Furthermore, as shown in Theorem 3, discounted marginal revenue is rising along the optimal trajectory. Economically speaking, this last result means that to compensate for the possible loss of customers that he can control probabilistically, the monopolist has added a risk premium, which is increasing through time, on top of the market rate of interest.

The last result of this chapter, Theorem 4, deals with the effect, on the monopolist's optimal extraction strategy, of an upward shift in the schedule of the instantaneous probability of entry. Intuitively, we expect that this will result in increasing the monopolist's output at the beginning. However, Theorem 4 asserts the opposite result: the monopolist's price is higher after the shift. This seemingly counterintuitive result arises because the monopolist's optimal expected discounted profit is lower after the upward shift in the schedule of the instantaneous probability of entry. Gilbert and Goldman [1978] proved that the monopoly price at time zero is higher under potential competition than under complete monopoly; the reason for their result is because the monopolist's optimal discounted profit is lower under potential competition than under absolute monopoly. The result of Gilbert and Goldman [1978], just mentioned, and Theorem 4 of this chapter are thus due to an external circumstance which lowers the optimal value of the monopolist's objective function.

APPENDIX

A1 THE HAMILTON-JACOBI-BELLMAN EQUATION

Let (x,t) be an arbitrary initial condition and $W(x,t)$ be the monopolist's optimal expected discounted profit, given that he starts at time t with remaining stock x and entry has not yet occurred.

Let $q: s \rightarrow q(s)$ be a nonnegative continuous function defined in $t \leq s \leq t+h$, with $h > 0$ a sufficiently small number. If the monopolist follows the control q inside $[t,t+h]$ then continue optimally afterward, then his expected discounted profit will be

$$(A1.1) \qquad \int_t^{t+h} e^{-rs}\pi(q(s))ds + [hm(q(s))+O(h)]V(x+\Delta x,t+h)$$
$$+ [1-hm(q(s))+O(h)]W(x+\Delta x,t+h)$$
$$\leq W(x,t)$$

In the preceding inequality $\Delta x = -\int_t^{t+h} q(s)ds$. We note that (A1.1) will become an equality if q is the optimal extraction strategy for the monopolist inside $[t,t+h]$. We can rearrange (A1.1) as follows

$$\text{(A1.2)} \qquad W(x+\Delta x,t+h)-W(x,t) + \int_t^{t+h} e^{-rs}\,\pi(q(s))ds$$

$$+ [hm(q(s))+O(h)][V(x+\Delta x,t+h)-W(x+\Delta x,t+h)]$$

$$\leqq 0$$

If W is sufficiently smooth, we can divide (A1.2) by h, then let $h \to 0$ to obtain

$$W_t(x,t) - q(t)W_x(x,t) + e^{-rt}\pi(q(t))$$

$$+ m(q(t))[V(x,t)-W(x,t)] \leqq 0$$

with equality holding if q(t) is equal to the optimal extraction rate at the initial condition (x,t). These results can be compactly expressed by the following equation, known as the Hamilton-Jacobi-Bellman equation of Dynamic Programming

$$\text{(A1.3)} \qquad \max_{\xi \geqq 0} \begin{bmatrix} W_t(x,t) - \xi W_x(x,t) + e^{-rt}\pi(\xi) \\ + m(\xi)(V(x,t)-W(x,t)) \end{bmatrix} = 0$$

which is exactly equation (3) of section 3.

A2 THE ADJOINT EQUATION

To derive the differential equation (5), which describes the movement of the shadow price along the optimal trajectory, we proceed as follows.

First, let $\xi^*(x,t)$ be the value of ξ which maximizes the left side of (A1.3). Next, let H: $z \to H(z)$ be the function of $z \geqq 0$, defined as follows

$$H(z) = W_t(z,t) - \xi^*(x,t)W_z(z,t) + e^{-rt}\pi(\xi^*(x,t))$$

$$+ m(\xi^*(x,t))[V(z,t)-W(z,t)]$$

i.e., H(z) is obtained from the left side of (A1.3) by replacing ξ with $\xi^*(x,t)$ and x with z.

By its definition, $H(z) \leqq 0$ for all z and achieved a maximum at $z = x$. Therefore

$$\text{(A2.1)} \qquad H'(z{=}x) = W_{tx}(x,t)-\xi^*(x,t)W_{xx}(x,t)$$

$$+ m(\xi^*(x,t))[V_x(x,t)-W_x(x,t)] = 0$$

Equation (A2.1) holds for any initial condition (x,t). In particular, for $(x^*(t),t)$ we must also have

$$\text{(A2.2)} \qquad W_{tx}(x^*(t),t)-\xi^*(x^*(t),t)W_{xx}(x^*(t),t)$$

$$= m(\xi^*(x^*(t),t))[W_x(x^*(t),t)-V_x(x^*(t),t)]$$

Here $x^*(t)$ is the solution of the differential equation

$$dx^*/dt = -\xi^*(x^*(t),t)$$

$$x^*(0) = \bar{x}$$

i.e., $x^*(t)$ is the monopolist's optimal remaining stock at time t, given that entry has not yet occurred.

Finally, if we define $\lambda(t) = W_x(x^*(t),t)$, then $d\lambda/dt$ is given by the left side of (A2.2), i.e.,

$$d\lambda/dt = m(q^*(t))[\lambda(t)-V_x(x^*(t),t)] \tag{A2.3}$$

In (A2.3), we have used $q^*(t)$ to denote $\xi^*(x^*(t),t)$. Equation (A2.3) is exactly (6) of section 3.

A3 PROOF OF THE LEMMA

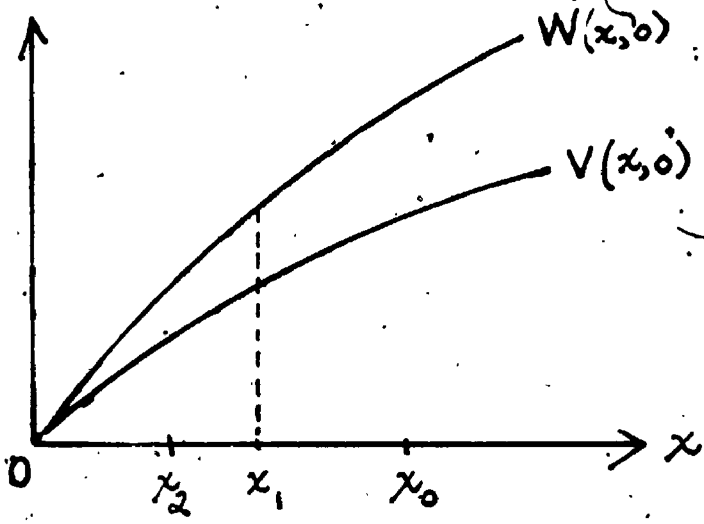
We shall now prove that $W_x(x,t) > V_x(x,t)$ for all $x > 0$ and all $t \geqq 0$. Without any loss of generality, we can set $t = 0$.

(a) Let $x_1 > 0$. Then clearly $W(x_1,0) > V(x_1,0)$. This last strict inequality together with the fact that $W(0,0) = V(0,0) = 0$ mean that we can find x_2, $0 < x_2 < x_1$, such that

$$W_x(x_2,0) > V_x(x_2,0)$$

We claim that

$$W_x(x,0) > V_x(x,0) \quad \text{for all } x \geqq x_2 \tag{A3.1}$$



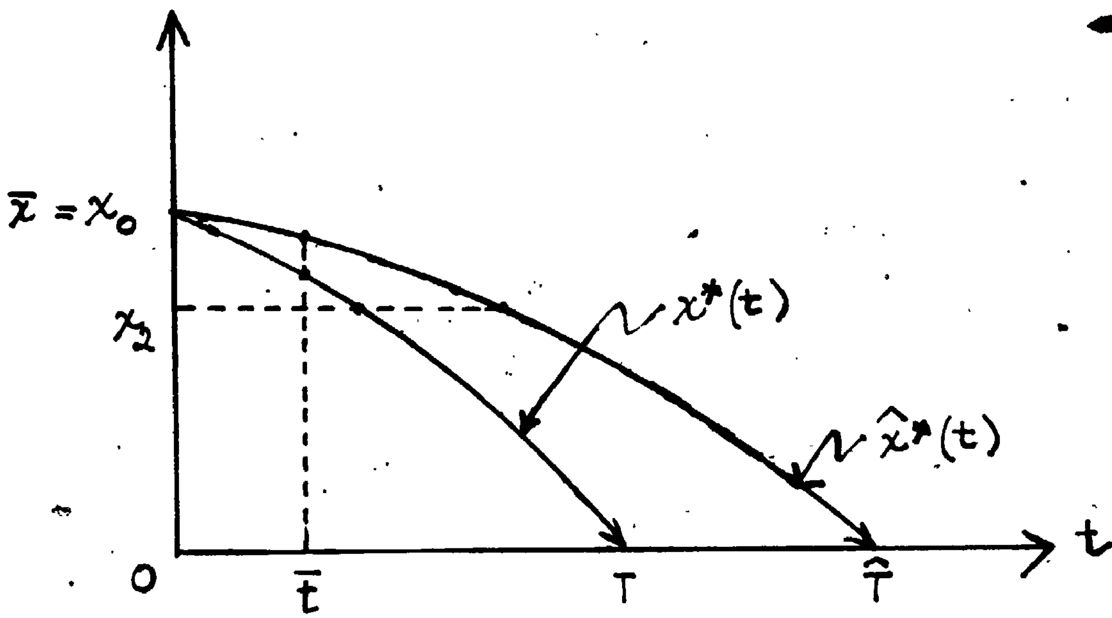
Once (A3.1) is established, we can let $x_1 \to 0$ to force $x_2 \to 0$ and obtain a proof of the lemma. Thus all the following effort is devoted to proving (A3.1).

(b) Indeed, if (A3.1) is not true, then we can find $x_0 > x_2$ such that

$$\text{(A3.2)} \qquad W_x(x_0,0) = V_x(x_0,0) \quad \text{and}$$

$$W_x(x,0) > V_x(x,0) \quad \text{for all } x \text{ in } [x_2,x_0)$$

Now let q^*: $t \to q^*(t)$ be the solution to the problem of monopoly resource extraction under stochastic entry when the monopolist's initial stock is $\bar{x} = x_0$. Here, we recall that q^* is defined by (7) of section 3. Furthermore, we shall let $x^*(t)$ denote the monopolist's remaining stock at time t under q^*.



Next, let $\hat{q}^*$: $t \to \hat{q}^*(t)$ be the monopolist's optimal extraction strategy under the assumption that entry occurs at time zero and the monopolist's initial stock is $\bar{x} = x_0$. We shall let $\hat{x}^*(t)$ denote the monopolist's remaining stock at time t under $\hat{q}^*$.

It is well known that $\hat{\pi}'(\hat{q}^*(0)) = V_x(x_0,0)$. Furthermore, by (4) of section 3, we must have $\pi'(q^*(0)) \leq W_x(x_0,0) = V_x(x_0,0)$. Hence, by assumption ($\hat{f}$3), we must have

(A3.3) $\quad q^*(0) > \hat{q}^*(0)$.

This last strict inequality together with the fact that $x_0 > x_2$ then mean that we can find $\bar{t} > 0$, sufficiently small, so that

(A3.4) $\hat{x}^*(t) > x^*(t) > x_2$ for all t in $(0,\bar{t}]$

If the monopolist begins at time zero with the initial stock $\bar{x} = x_0$, then his optimal expected discounted profit is given by

$$\text{(A3.5)} \quad W(x_0,0) = \int_0^\infty \left(\int_0^t e^{-rs}\pi(q^*(s))ds + V(x^*)t),t)\right)dF(t,q^*)$$

$$= \int_0^{\bar{t}} \left(\int_0^t e^{-rs}\pi(q^*(s))ds + V(x^*(t),t)\right)dF(t,q^*)$$

$$+ \int_{\bar{t}}^\infty \left(\int_0^t e^{-rs}\pi(q^*(s))ds + V(x^*(t),t)\right)dF(t,q^*)$$

$$= I_1 + I_2$$

Because $q^*(t)$, $t \geq \bar{t}$, must also be optimal from $(x^*(\bar{t}),\bar{t})$ on, we must have

$$\text{(A3.6)} \quad W(x^*(\bar{t}),\bar{t}) = \int_{\bar{t}}^\infty \left(\int_{\bar{t}}^t e^{-rs}\pi(q^*(s))ds + V(x^*(t),t)\right)\frac{dF(t,q^*)}{G(\bar{t},q^*)}$$

Here we recall that $F(t,q^*)$ is defined by (1) of section 2.3 and $G(t,q^*) = 1-F(t,q^*)$.

Now using (A3.6) to evaluate I_2, we can rewrite (A3.5) as

(A3.7) $$W(x_0,0) = \int_0^{\bar{t}} \left(\int_0^t e^{-rs}\pi(q^*(s))ds+V(x^*(t),t)\right)dF(t,q^*)$$
$$+ G(\bar{t},q^*)\left(\int_0^{\bar{t}} e^{-rs}\pi(q^*(s))ds+W(x^*(\bar{t}),\bar{t})\right)$$

Let $\epsilon > 0$ be given. If the monopolist begins at time zero with the initial fixed stock $\bar{x} = x_0 + \epsilon$, then his optimal expected discounted profit must satisfy the following inequality

(A3.8) $$W(x_0+\epsilon,0) \geq \int_0^{\bar{t}} \left(\int_0^t e^{-rs}\pi(q^*(s))ds+V(x^*(t)+\epsilon,t)\right)dF(t,q^*)$$
$$+ G(\bar{t},q^*)\left(\int_0^{\bar{t}} e^{-rs}\pi(q^*(s))ds+W(x^*(\bar{t})+\epsilon,\bar{t})\right)$$

Here, we note that the right side of (A3.8) is obtained in exactly the same way we use to derive the right side of (A3.7). More specifically, the right side of (A3.8) is nothing other than the expected discounted profit enjoyed by the monopolist if he starts at time zero with the initial stock $x_0+\epsilon$, follows q^* inside $[0,\bar{t})$, then continues optimally afterward from the new initial condition $(x^*(\bar{t})+\epsilon,\bar{t})$. We also recall here that q^* is the optimal extraction strategy from the initial condition $(x_0,0)$ and $x^*(t)$ is the remaining stock under q^*.

Subtracting (A3.7) from (A3.8), dividing the resulting expression by ϵ, then letting $\epsilon \to 0$, we obtain

(A3.9) $$W_x(x_0,0) \geqq \int_0^{\bar{t}} V_x(x^*(t),t)\,dF(t,q^*) + G(\bar{t},q^*)W_x(x^*(\bar{t}),\bar{t}) = J_1 + J_2$$

By (A3.4), $V_x(x^*(t),t) > V_x(\hat{x}^*(t),t)$ for all t in $(0,\bar{t}]$. Furthermore, because $x_2 < x^*(\bar{t}) < x_0$, we must have $W_x(x^*(\bar{t}),t) > V_x(x^*(\bar{t}),\bar{t})$ by (A3.2). Furthermore, also by (A3.4), we must have $V_x(x^*(\bar{t}),\bar{t}) > V_x(\hat{x}^*(\bar{t}),\bar{t})$. Hence we must have

(A3.10) $$I_1+I_2 > \int_0^{\bar{t}} V_x(\hat{x}^*(t),t)\,dF(t,q^*) + G(\bar{t},q^*)V_x(\hat{x}^*(\bar{t}),\bar{t}) \doteq V_x(x_0,0) = W_x(x_0,0)$$

because $V_x(x_0,0) = V_x(\hat{x}^*(t),t)$ for all $t \leqq \hat{T}$ and (A3.2). Clearly (A3.9) and (A3.10) are mutually contradictory.

q.e.d.

CHAPTER THREE

RESOURCE EXTRACTION AND EXPLORATION
UNDER MONOPOLY

1. INTRODUCTION

Until recently, most studies on the economics of nonrenewable resources have neglected the exploration process as a means to augment proven reserves. Any extractive firm which does not actively engage in any exploration program will soon deplete its proven reserves and cease to exist. It is well known that exploration is an extremely risky venture. The risky nature of exploration arises from two fundamental factors: (1) the uncertain location of a mineral deposit and (2) the uncertain size of a mineral deposit to be searched for.

Pindyck [1978] studied exploration, but not in the context of uncertainty. Gilbert [1979] and Pindyck [1980] touched on the subject of exploration; their models, however, are not grounded on the chance distribution of the resource on the surface of the earth.

Arrow and Chang [1979] provided the first serious treatment of the problem of simultaneously extracting and exploring for a nonrenewable resource. These two authors considered an exploration region A of finite area and assumed that the number of mines in any subregion B of A has a Poisson distribution with parameter proportional to the area of B. Exploration is carried out by sweeping the region A; the rate of exploratory effort at any time t is the instantaneous area swept. An exploration strategy thus becomes a function of time, specifying the instantaneous area swept at each time t. Once an exploration strategy is chosen, the cumulative area swept up to any time t is completely determined and, given the Poisson distribution of mines in each given area, this cumulative area uniquely determines the probability distribution of the number of mines discovered. In this fashion, these two authors managed to transform the problem of simultaneous extraction and exploration into a standard problem of impulsive stochastic control with the arrival of the mines following a nonuniform Poisson process on the time axis; the distribution of this nonuniform Poisson process, of course, depends on the chosen exploration strategy. The optimal solution requires positive extraction at all times; exploratory activities, however, are only carried out intermittently and impulsively. Furthermore, over a period of time the price has no trend, i.e., the classic Hotelling rule that price rises exponentially at the rate of discount ceases to be valid.

In this chapter, we build a model of resource extraction and exploration under monopoly with the exploration process being modelled in the spirit of the theory of optimal search (for an exposition, see Saaty [1973] and Stone [1975]). While the model of Arrow and Chang [1979] treats every spot in the exploration region equally, our model is spatially more discriminatory in the sense that the more promising a location is, the more exploratory effort it receives.

Our main contributions in this chapter include the following results:

(i) If the initial proven reserve $\bar{x}$ is large, the monopolist will not explore at the beginning. On the other hand, if $\bar{x}$ is small, exploration will be undertaken immediately and will be carried out without any interruption until either a discovery is made or at least until the proven reserve $\bar{x}$ is exhausted. This result is given by Theorem 2.

(ii) In the case where only the location of the potential deposit is uncertain while its size is known with absolute certainty, the discounted marginal revenue, associated with the exploitation of the proven reserve $\bar{x}$, increases through time as long as a discovery has not been made. At the exact time of discovery, the monopolist's optimal extraction rate will take an upward jump, which is brought about by the increase in proven reserve engendered by the discovery. This result is the essence of Theorem 4.

(iii) An increase in the uncertainty about the size of the potential deposit, in the sense of a mean-preserving spread, reduces exploration. This result is represented by Theorem 5.

(iv) Intuitively speaking, we expect that the uncertainty concerning the size of the potential deposit will call for a more conservative strategy in exploiting the proven reserve $\bar{x}$. Theorem 7 confirms this result when $\bar{x}$ is large.

(v) Compared with the optimal extraction strategy under the assumption that the monopolist is not allowed to explore, the optimal extraction strategy under simultaneous extraction and exploration requires extracting more of the proven reserve $\bar{x}$ at the beginning, and, in case of no discovery, exhausts $\bar{x}$ sooner. That is, having the likelihood of being able to augment his proven reserve by exploration, the monopolist accelerates the exploitation of the initial proven reserve $\bar{x}$. This result is embodied in Theorem 3.

(vi) Compared with the social optimum, the monopolist underexplores. This result is given by Theorem 6.

The plan of this chapter is as follows. In section 2, we present our model. Section 3 transforms the formal model of resource extraction and exploration under monopoly into a standard problem of optimal control. In section 4, we give the main results. Section 5 contains some concluding remarks.

2. THE MODEL

2.1 Stochastic Specification of the Exploration Process

To give some structure to the exploration process, we visualize an exploration region A of the earth surface. From the geological knowledge of A, we suspect that there is a chance α, $0 < \alpha \leq 1$, that a single mineral deposit lies hidden somewhere in this region. The likely location of this deposit is summarized by a nonnegative measurable function

$$m: a \to m(a)$$

defined on A such that for any measurable subset B of A,

$$\int_B m(a)da$$

gives the probability that the deposit is located inside B, given that it really exists in A.

Concerning the uncertain size of the potential deposit, we shall represent this uncertain size by a random variable S. The distribution of S is assumed to be known. Furthermore, we shall also assume that the uncertainty about S is completely resolved at the time of discovery.

In addition to the stochastic specification of the exploration region A, we also need to describe the search technology.

For each location a in the exploration region A, let $b(.,a)$: $y \to b(y,a)$ be a function defined for all non-negative y. Here we interpret $b(y,a)$ as the probability of discovering S at location a, given that it lies hidden at a and an amount of effort y is spent searching for it at a. We shall assume that

$$b(0,a) = 0 \ , \ b(\infty,a) = 1$$

$$0 < b_1(y,a) < +\infty, \ b_{11}(y,a) < 0$$

Here b_1 and b_{11} are the first and second partial derivatives of b with respect to y.

By an exploration strategy u, we mean a nonnegative function

$$u: \quad (t,a) \to u(t,a)$$

with u(t,a) representing the instantaneous effort applied to location a at time t. For any $t < +\infty$, the probability, under u, of discovering the potential deposit inside the time interval [0,t] is given by

$$P(t,u) = \alpha \int_A m(a)b(y(t,a),a)da$$

Here $y(t,a) = \int_0^t u(s,a)ds$ = the cumulative effort spent up to time t, under u.

The probability, under u, of failing to discover the potential deposit is then given by

$$1 - \lim_{t \to +\infty} P(t,u)$$

As for exploration costs, we shall let c(v) denote the cost at any time t, as a function of the total instantaneous effort v spent at time t over the entire exploration region. We shall assume that $c: \; v \to c(v)$ is convex and strictly increasing, and $c(0) = 0$.

2.2 The Market Demand Curve

We shall let $f: \; Q \to f(Q)$ denote the market demand curve, as a function of output Q. The following assumptions are imposed upon f:

(f1) $0 < f(0) < +\infty$

$f(Q) > 0$, $f'(Q) < 0$ for all $Q \geq 0$

(f2) the total revenue curve π: $Q \to \pi(Q)$ is strictly concave and achieves a maximum at $Q = Q_{max} > 0$.

2.3 A Formal Statement of the Problem of Resource Extraction and Exploration Under Monopoly

At this juncture, we have assembled enough machinery to give a formal statement of the problem of resource extraction and exploration under monopoly as follows.

Suppose that the monopolist's initial proven reserve is $\bar{x} \geq 0$, his extraction cost is zero, and his objective is to maximize expected discounted profit. Find an extraction strategy q: $t \to q(t)$ and an exploration strategy u: $(t,a) \to u(t,a)$ to maximize

$$\int_0^\infty \left[\int_0^t e^{-rs}[\pi(q(s))-c(v(s))]ds + EV(x(t)+S,t)\right]dP(t,u)$$

$$+ [1 - \lim_{t\to+\infty} P(t,u)] \int_0^\infty e^{-rs}[\pi(q(s))-c(v(s))]ds$$

$$= I_1 + I_2$$

subject to

$$dx/dt = -q(t) \quad ; \quad x(0) = \bar{x}$$

$$q(t) \geqq 0 \ , \ x(t) \geqq 0 \quad \text{for all } t$$

$$v(t) = \int_A u(t,a)\,da$$

$V(x(t)+S,t)$ = the optimal discounted profit associated with optimally depleting the fixed stock $x(t)+S$, starting from the time of discovery t.

$EV(x(t)+S,t)$ = the expected value of $V(x(t)+S,t)$, taken with respect to the distribution of the random variable S, the uncertain size of the potential deposit

$P(t,u)$ = the probability of discovering the potential deposit inside the time interval $[0,t]$, $t<+\infty$, as defined at the end of section 2.1.

r = the market rate of interest

We note that the objective function is composed of two terms: the first term, I_1, takes into account the event of discovering the potential deposit; the second term, I_2, takes into account the event of failing to discover the potential deposit.

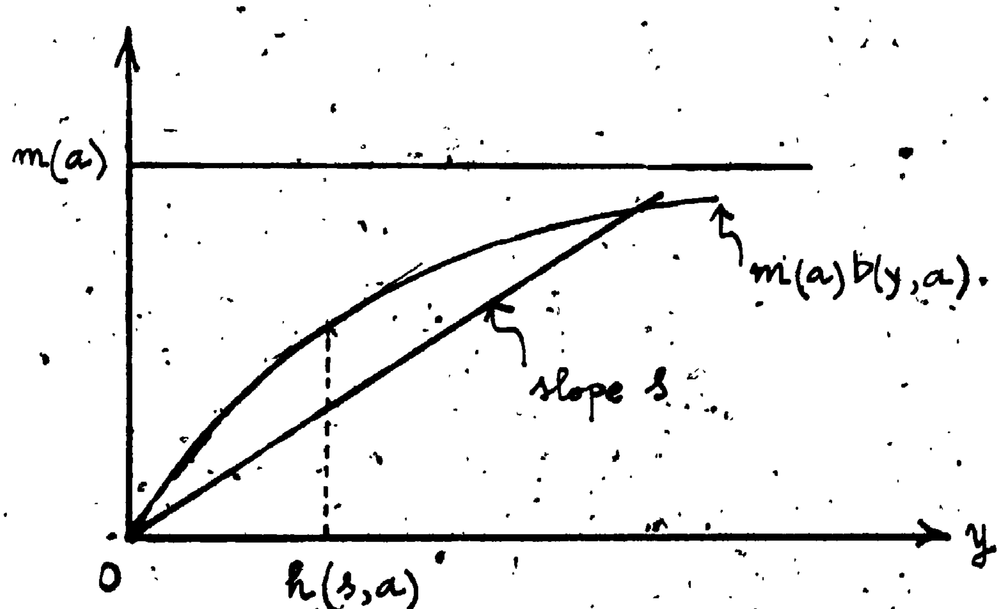
3. REDUCTION OF THE PROBLEM OF RESOURCE EXTRACTION AND EXPLORATION UNDER MONOPOLY TO A STANDARD PROBLEM OF OPTIMAL CONTROL

The problem of resource extraction and exploration under monopoly, as formally stated in section 2.3 seems difficult to be tackled in its original form. However, using a result due to Stone [1975], we manage to transform the original problem into a standard problem of optimal control. The argument (Stone [1975], Theorem 2.2.4) gives an efficient search strategy for the static one-period search problem and can be summarized as follows.

Let s be a given positive number. For each location a in A, the assumed concavity of b(y,a) means that the maximization problem

$$\max_{y \geq 0} m(a)b(y,a) - sy$$

has a unique solution h(s,a). This solution is clearly continuous and monotone decreasing in s, for each given a.



Furthermore, $h(s,a) = 0$ if $s \geqq m(a)b_1(0,a)$ and $\lim_{s \to 0} h(s,a) = +\infty$.

The function

$$U: \quad s \to U(s) = \int_A h(s,a)\,da$$

is clearly defined for all $s > 0$. Using the properties of $h(s,a)$, just mentioned, we see immediately that U is continuous and monotone decreasing. Furthermore,

$$\lim_{s \to 0} U(s) = +\infty$$

$$U(s) = 0 \quad \text{if } s \geqq \sup_a m(a)b_1(0,a)$$

We shall assume that $\sup_a m(a)b_1(0,a) < +\infty$. This assumption means that we can find $s_{max} < +\infty$ such that U is strictly monotone decreasing in $(0,s_{max})$ and $U(s_{max}) = 0$.

By a search strategy for the one-period problem, we mean a function of two variables a and $z \geqq 0$

$$J: \quad (a,z) \to J(a,z)$$

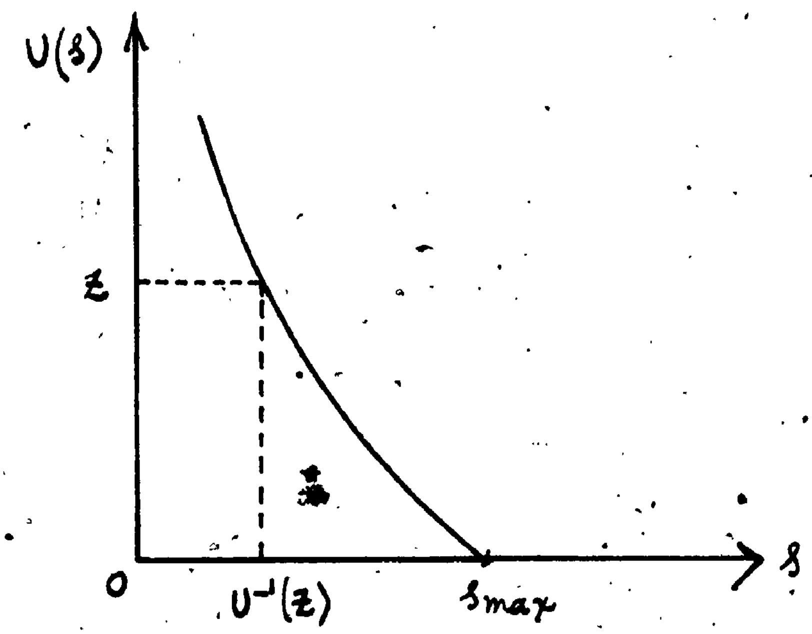
which allocates to location a the amount of effort $J(a,z)$ such that

$$\int_A J(a,z)\,da \leqq z$$

Here, we interpret z as the total amount of effort to be spent over the entire exploration region A.

Let J* be the search strategy for the one-period problem defined as follows

$$J^*: \quad (a,z) \to J^*(a,z) = h(U^{-1}(z),a)$$



By definition, $\int_A h(U^{-1}(z),a)\,da = U(U^{-1}(z)) = z.$

Hence, $\int_A J^*(a,z)\,da = z.$ Furthermore, $J^*(a,z)$ is monotone increasing in z, for each given a.

Now, for any given $z > 0$, we claim that the search strategy J* maximizes the probability of discovering the potential deposit for the one-period problem, given that the total effort allowed for searching over the entire exploration region A is z. Indeed, let J be any search strategy, then using the definition of h(s,a), $s = U^{-1}(z)$, we must have

$$m(a)b(J^*(a,z),a) - sJ^*(a,z) \geqq$$

$$m(a)b(J(a,z),a) - sJ(a,z)$$

i.e.,

$$\int_A m(a)b(J^*(a,z),a)da \geqq \int_A m(a)b(J(a,z),a)da + s\int_A [J^*(a,z) - J(a,z)]da$$

In the preceding inequality, the left side is the probability of discovery under J*, the first term on the right side is the probability of discovery under J; both probabilities are conditioned on the event that the potential deposit really exists in the exploration region A. The second term in the right side of the preceding inequality is clearly nonnegative due to the total effort constraint.

Now using J*, we define a function of $z \geqq 0$ as follows

$$F: \quad z \rightarrow F(z) = \int_A m(a)b(J^*(a,z),a)da$$

The function F, as just defined, gives the maximum probability of discovery for the one-period problem as a function of z, the total effort available for exploration, given that the potential deposit really exists in the exploration region A.

Using the search strategy J^* defined above, we shall now reduce the original statement of the problem of resource extraction and exploration under monopoly, as presented in section 2.3, to a more manageable form as follows.

Let $u: (t,a) \to u(t,a)$ be any exploration strategy. Here $u(t,a)$ is the instantaneous effort that u allocates to location a at time t. If we let

$$v(t) = \int_A u(t,a)\,da \quad z(t) = \int_0^t v(s)\,ds \quad ,$$

then $v(t)$ gives the instantaneous effort over the entire exploration region at time t and $z(t)$ gives the cumulative effort over the entire exploration region up to time t, if u is adopted.

Next, let $\tilde{u}: (t,a) \to \tilde{u}(t,a)$ be the exploration strategy defined by

$$\tilde{y}(t,a) = \int_0^t \tilde{u}(s,a)\,ds = J^*(a,z(t)) .$$

Here $\tilde{u}(t,a)$ is the instantaneous effort that $\tilde{u}$ allocates to location a at time t.

By its definition, $\tilde{u}$ allocates the same cumulative effort up to any time t as u does to any location a. Furthermore,

$$\tilde{v}(t) = \int_A \tilde{u}(t,a)da = v(t)$$

i.e., $\tilde{u}$ and u spend the same instantaneous effort over the entire exploration region at each time t.

Therefore $\tilde{u}$ costs the same as u. Moreover, by its definition, $\tilde{u}$ gives the maximum probability of discovery in [0,t] for any given function v: $t \to v(t)$ which specifies how much effort to spend at each instant over the entire exploration region. Hence, the original formulation of the problem of resource extraction and exploration under monopoly is equivalent to the following optimal control problem.

Find an extraction strategy q: $t \to q(t)$ and an exploration expenditure strategy v: $t \to v(t)$, with q(t) representing the extraction rate at time t and v(t) representing the instantaneous effort spent over the entire exploration region at time t, to maximize the following expected discounted profit

$$\int_0^\infty [\int_0^t e^{-rs}[\pi(q(s))-c(v(s))]ds + EV(x(t)+S,t)]dP(t,v)$$

$$+ [1 - \lim_{t\to+\infty} P(t,v)] \int_0^\infty e^{-rs}[\pi(q(s))-c(v(s))]ds$$

subject to

$$q(t) \geqq 0 , \; v(t) \geqq 0 , \; x(t) \geqq 0 \text{ for all } t$$

$$dx/dt = -q(t) \ , \ x(0) = \bar{x}$$

$P(t,v)$ = the probability of discovering the potential deposit inside the time interval $[0,t]$, $t < +\infty$

$= \alpha F(z(t))$

$$dz/dt = v(t) \ , \ z(0) = 0$$

$F(z)$ = the maximum probability of discovery for the one-period search problem if z is the total effort available for exploration, given that the potential deposit really exists in the exploration region A

$EV(x(t)+S,t)$ is the same as in section 2.3

r = the market rate of interest

4. THE OPTIMAL SOLUTION

Instead of starting at time zero with the proven reserve $\bar{x}$ and zero cumulative effort, we shall take as the initial condition (x,z,t) with $x \geq 0$, $z \geq 0$, $t \geq 0$. Here we shall interpret x as the proven reserve at time t and z as the cumulative effort already spent without success up to time t. We shall let

$$W: \ (x,z,t) \rightarrow W(x,z,t)$$

denote the optimal value function, i.e., the expected discounted profit yielded by the optimal extraction and exploration strategies for the initial condition (x,z,t) is given by W(x,z,t).

Assuming that W is sufficiently smooth, we can use a standard argument of Dynamic Programming to derive the following Hamilton-Jacobi-Bellman equation for the problem of resource extraction and exploration under monopoly (the derivation is given in the appendix of this chapter)

$$(1) \qquad \max_{\xi \geqq 0,\, \theta \geqq 0} \begin{bmatrix} \pi(\xi) e^{-rt} - \xi W_x(x,z,t) + W_t(x,z,t) \\ + \theta [W_z(x,z,t) + \alpha g(z)(Y(x,t) - W(x,z,t))] \\ - c(\theta) e^{-rt} \end{bmatrix} = 0$$

where

$$(2) \qquad g(z) = F'(z)/[1-F(z)]$$

$$(3) \qquad Y(x,t) = EV(x+S,t)$$

The function g(z), as defined by (2), gives the conditional instantaneous probability of discovery at cumulative effort level z, given that the amount z has already been spent without success. Also, Y(x,t) is the expected value of V(x+S,t); the expectation is taken with respect to the distribution of the random variable S. For our own use later, we shall let

$$(4) \qquad \bar{Y} = Y(0,0) = EV(S,0)$$

Carrying out the maximization of the left side of equation (1), we obtain the following first order necessary conditions

$$(5) \qquad \pi'(\xi)e^{-rt} = W_x(x,z,t)$$

$$(6) \qquad c'(\theta)e^{-rt} = L(x,z,t) \quad \text{if } L(x,z,t) > c'(0)e^{-rt}$$

$$= 0, \text{ otherwise}$$

where $L(x,z,t) = W_z(x,z,t) + \alpha g(z)(Y(x,t) - W(x,z,t))$.

We shall let $\xi^*(x,z,t)$ and $\theta^*(x,z,t)$ denote the unique solutions of (5) and (6), respectively.

Equation (5) embodies the well known result that along the optimal trajectory the shadow price of the proven reserve W_x must be equal to the discounted marginal revenue. We shall show later that when no exploration is undertaken during any time interval, the shadow price remains constant, i.e., marginal revenue rises exponentially at the market rate of interest.

Equation (6) is more novel. It asserts that whenever exploration is carried out, exploratory effort is increased until the marginal cost $c'(\theta)e^{-rt}$ is equal to the marginal gain $L(x,z,t)$. This marginal gain is composed of two terms: (1) the shadow price of cumulative effort $W_z(x,z,t)$ which represents the gain in the optimal value function yielded when cumulative effort is increased one unit from the initial level z and (2) the instantaneous gain $\alpha g(z)[Y(t,x)-W(x,z,t)]$ which results if a discovery is made. On the other hand, if $c'(0)e^{-rt} \geq L(x,z,t)$, i.e., if the gain is less than the cost, then no exploratory activities should be undertaken.

The preceding results come directly from the Hamilton-Jacobi-Bellman equation (1). To learn more about the optimal solution we shall now investigate the movement of the shadow prices of the proven reserve and the cumulative effort along the optimal trajectory. To this end, let us consider the following system of differential equations

$$(7) \qquad \frac{dx^*}{dt} = -\xi^*(x^*(t),z^*(t),t) = -q^*(t)$$

$$\frac{dz^*}{dt} = \theta^*(x^*(t),z^*(t),t) = v^*(t)$$

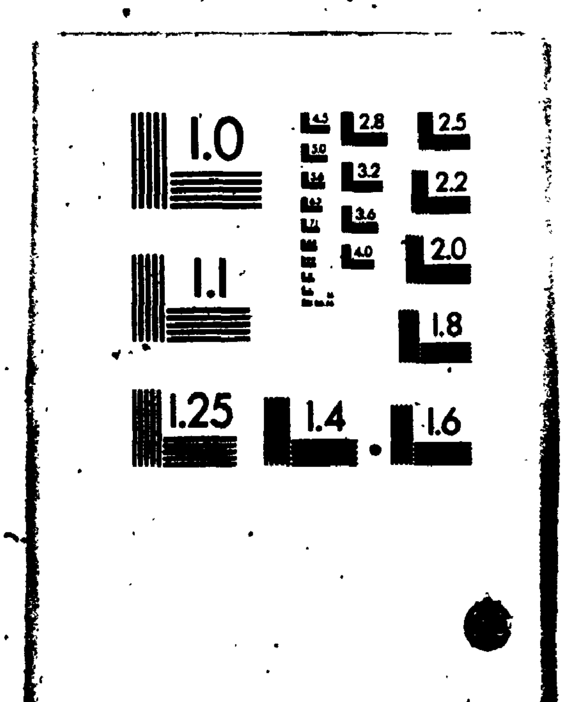
$$x^*(0) = \bar{x}\ , \ y^*(0) = 0$$

That is, given that no discovery has been made, along the optimal trajectory

$x^*(t)$ denotes the remaining stock of the proven reserve at time t, and $q^*(t)$ denotes the extraction rate at time t

$z^*(t)$ denotes the cumulative effort up to time t, and $v^*(t)$ denotes the instantaneous exploratory effort at time t.

2 OF / DE 2



If we let $\lambda(t) = W_x(x^*(t),z^*(t),t)$ and $n(t) = W_z(x^*(t),z^*(t),t)$, then by a standard argument of Dynamic Programming, we can show that these shadow prices obey the following differential equations (the derivatives are given in the appendix of this chapter)

(8) $d\lambda/dt = -\alpha v^*(t)g(z^*(t))[Y_x(x^*(t),t) - \lambda(t)]$

(9) $dn/dt = -\alpha v^*(t) \begin{bmatrix} g'(z^*(t))[Y(x^*(t),t)-W(x^*(t),z^*(t),t)] \\ -g(z^*(t)n(t) \end{bmatrix}$

Before presenting our major results, we need to prove the following lemma.

Lemma

If $g'(\hat{z}) < 0$ for some $z = \hat{z}$, then $W_z(x,\hat{z},t) \leqq 0$ for any x,t.

Proof

For any $y \geqq 0$, $z \geqq 0$, let $F(y|z)$ denote the conditional probability of discovery, given that an amount of cumulative effort z has already been spent without success, and an extra amount of effort y will now be spent, searching for the potential deposit again. That is,

$$F(y|z) = \frac{F(y+z)-F(z)}{1-F(z)}.$$

The partial derivative

$$F_z(y|z) = [g(y+z)-g(z)]/[\frac{1-F(y+z)}{1-F(z)}$$

is clearly negative at $z = \hat{z}$ if $g'(\hat{z}) < 0$. Here, we recall that $g(z)$ is defined by (2). Thus,

$$F(y|\hat{z}) > F(y|\hat{z}+\Delta z)$$

for all $y > 0$ and $\Delta z > 0$ sufficiently small. That is, the conditional probability of discovery from the cumulative effort level $\hat{z}$ is higher than the conditional probability of discovery from the cumulative effort level $\hat{z}+\Delta z$. This last result obviously implies $W(x,\hat{z},t) > W(x,\hat{z}+\Delta z,t)$, i.e., $W_z(x,\hat{z},t) \leqq 0$.

q.e.d.

We are now ready to prove a theorem for the special case of pure exploration, i.e., when $\bar{x} = 0$.

<u>Theorem 1</u>

Suppose that the monopolist's initial proven reserve is $\bar{x} = 0$. Next, let $\bar{z}$ denote the optimal terminal cumulative effort. That is, as long as no discovery has been made, exploration is carried out until cumulative effort z reaches $\bar{z}$ at which time all exploratory activities are terminated.

(a) If $\alpha g(z)\overline{Y} > c'(0)$ for all $z \geqq 0$, then $\overline{z} = +\infty$.

(b) If, for some $\hat{z}$, the two inequalities $g'(\hat{z}) < 0$ and $\alpha g(\hat{z})\overline{Y} \leqq c'(0)$ are true, then $\overline{z} < +\infty$. Furthermore, if $\overline{z}$ is also positive, then

$$\alpha g(\overline{z})\overline{Y} = c'(0)$$

That is, at the terminal cumulative effort $\overline{z}$, the marginal gain is equal to the marginal cost.

<u>Proof</u>

(a) For any $z \geqq 0$, if $\alpha g(z)\overline{Y} > c'(0)$, then for $\varepsilon > 0$ sufficiently small, it is still true that

$$\alpha g(z)\overline{Y} > c(\varepsilon)/\varepsilon$$

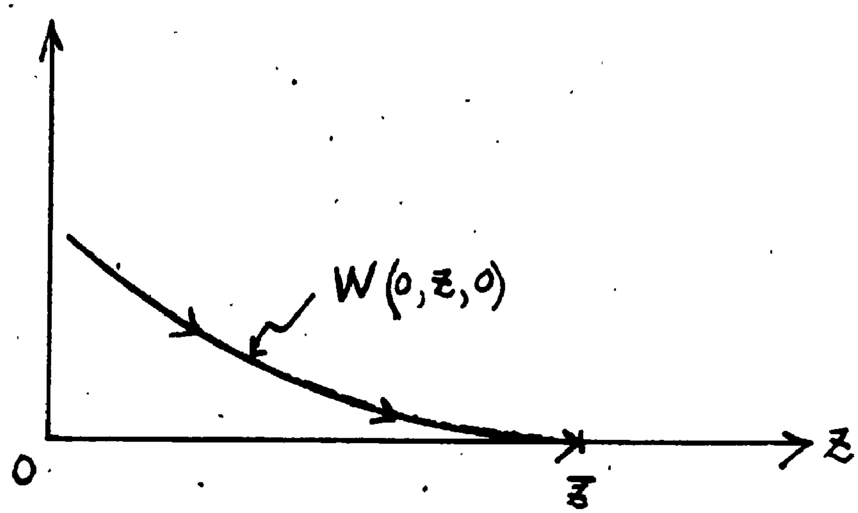
Next, let $\Delta t > 0$ be given and let $v(t)$ be the exploration strategy obtained by choosing $v(t) = \varepsilon$ inside the infinitesimal time interval $[0,\Delta t]$, then continuing optimally afterward. The expected discounted profit yielded by v is

$$\alpha g(z)\overline{Y}\,\varepsilon\Delta t - c(\varepsilon)\Delta t + (1-\alpha g(z)\varepsilon\Delta t)W(0,z+\varepsilon\Delta t,\Delta t)$$
$$+ 0(\Delta t) \geqq [(\alpha g(z)\overline{Y} - \frac{c(\varepsilon)}{\varepsilon})\varepsilon + \frac{0(\Delta t)}{\Delta t}]\Delta t$$

For $\Delta t > 0$ sufficiently small, the right side of the preceding inequality is positive, i.e., $W(0,z,0) > 0$. Therefore, if $\alpha g(z)\overline{Y} > c'(0)$ for all $z \geqq 0$, we must have $W(0,z,0) > 0$ for all $z \geqq 0$, proving that as long as the potential deposit has not been found, it is not optimal to terminate exploration.

(b) If $g'(\hat{z}) < 0$, then $W_z(0,\hat{z},t) \leqq 0$ by the Lemma. This last result together with the inequality $\alpha g(\hat{z})\overline{Y} \leqq c'(0)$ then imply that the unique value of θ that maximizes the left side of the Hamilton-Jacobi-Bellman equation is $\theta = 0$. Therefore, $W(0,\hat{z},t) = 0$, i.e., if cumulative effort reaches $\hat{z}$, all exploratory activities will be terminated. This proves $\overline{z} \leqq \hat{z}$.

Furthermore, if $\overline{z} > 0$, then by the way $\overline{z}$ is defined,



$$W(0,z,0) > 0 \text{ , if } z < \overline{z}$$

$$W(0,\overline{z},0) = 0$$

That is, optimal expected discounted profit is always positive as long as cumulative effort is still below $\overline{z}$, the optimal terminal cumulative effort, and at $z = \overline{z}$, the optimal expected discounted profit must be equal to zero. Therefore, $W_z(0,\overline{z},0) = 0$.

Now, we prove that $\alpha g(\overline{z})\overline{Y} = c'(0)$. Indeed, if $\alpha g(\overline{z})\overline{Y} < c'(0)$, then for z in a left neighborhood of $\overline{z}$ this strict inequality still holds. This result together with the result $W_z(0,\overline{z},0) = 0$, proved in the preceding paragraph, then imply that

$$\alpha g(z)\overline{Y} - c'(0) + W_z(0,z,0) < 0$$

for z in a left neighborhood of $\overline{z}$. The preceding inequality means that the unique value of θ that maximizes the left side of the Hamilton-Jacobi-Bellman equation is $\theta = 0$. Hence, $W_t(0,z,0) = 0$; i.e., $W(0,z,0) = 0$, contradicting the definition of z. Therefore, the strict inequality $\alpha g(\overline{z})\overline{Y} < c'(0)$ cannot hold. On the other hand, if $\alpha g(\overline{z})\overline{Y} > c'(0)$, then the argument presented in (a) means $W(0,\overline{z},0) > 0$, which also contradicts the definition of $\overline{z}$. Thus we must have $\alpha g(\overline{z})\overline{Y} = c'(0)$, proving (b).

q.e.d.

The next theorem deals with the case $\overline{x} > 0$, i.e., simultaneous extraction and exploration.

Theorem 2

Suppose that $\bar{x} > 0$, $0 \leq \bar{z} < +\infty$ and $g(z)$ is strictly decreasing in z.

(a) If $\bar{x}$ is small, then extraction and exploration are simultaneously started at time zero and, given that no discovery is made, exploration will be continuously carried out at least until $\bar{x}$ is exhausted.

(b) If $\bar{x}$ is large, only extraction begins at time zero. Exploration is only started when the proven reserve is reduced to a level low enough. Furthermore, in the beginning, when only extraction is carried out, the marginal revenue rises exponentially at the market rate of interest.

Proof

First, we reproduce the Hamilton-Jacobi-Bellman equation (1) as follows

$$\max_{\xi\geq 0,\theta\geq 0}\begin{bmatrix}\pi(\xi)e^{-rt} - \xi W_x(x,z,t) + W_t(x,z,t) \\ \\ + \theta L(x,z,0)e^{-rt} - c(\theta)e^{-rt}\end{bmatrix} = 0$$

where

$$\begin{aligned} L(x,z,t) &= W_z(x,z,t) + \alpha g(z)[Y(x,t) - W(x,z,t)] \\ &= L(x,z,0)e^{-rt} \end{aligned}$$

Next, we let $H(x,z) = L(x,z,0) - c'(0)$. Then the optimal exploration rate $\theta^*(x,z,t)$ is positive if $H(x,z) > 0$. Otherwise, $\theta^*(x,z,t) = 0$. Therefore, it is necessary to investigate the sign of $H(x,z)$.

Let z, $0 \leq z < \bar{z}$, be given. Then it is clear that $H(0,z) > 0$. Furthermore, it is also true that $H(x,z) < 0$ for x large enough. Indeed, as $x \to +\infty$,

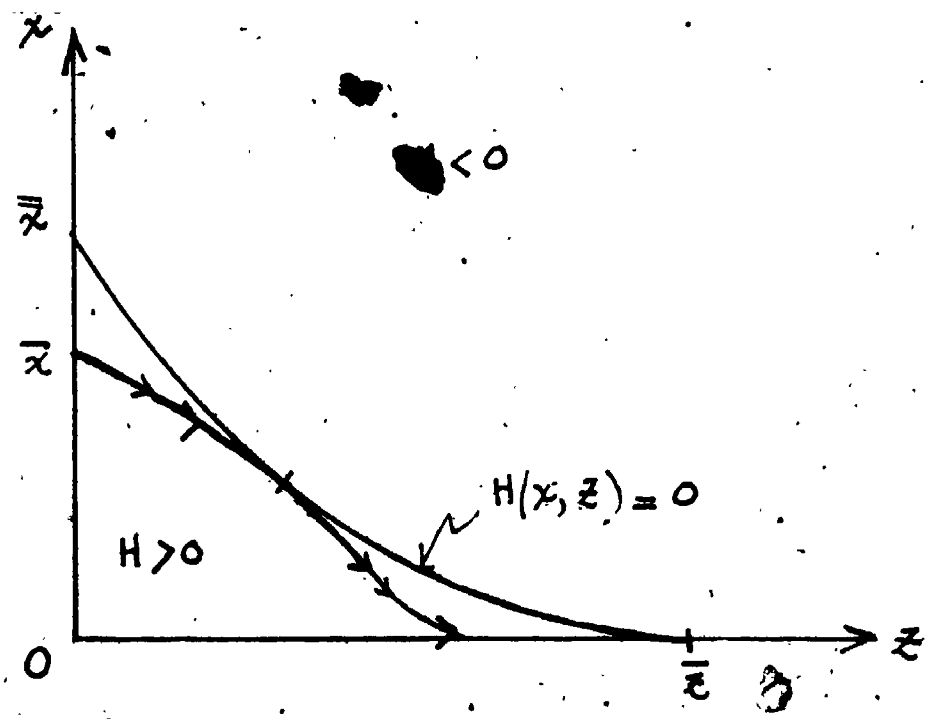
$$L(x,z,t) \to W_z(x,z,t) \leq 0$$

because both $Y(x,t)$ and $W(x,z,t) \to \frac{\pi}{r} (Q_{max}) e^{-rt}$. The inequality is due to the assumption that $g(z)$ is strictly decreasing, as asserted by the lemma. Thus

$$\lim_{x\to\infty} H(x,z) = W_z(x,z,0) - c'(0) < 0$$

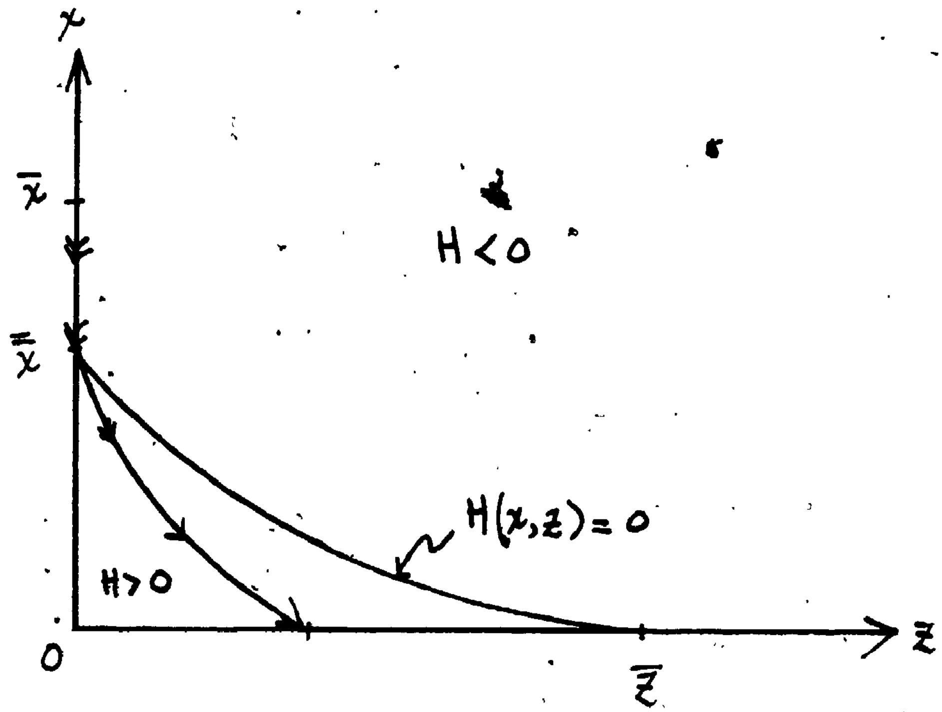
The preceding inequality and $H(0,z) > 0$, taken together, imply that for any given z, $0 \leq z < \bar{z}$, there exists a finite value of x such that $H(x,z) = 0$. If we assume that such an x is unique then the functional relationship $H(x,z) = 0$ defines a curve in the region $0 \leq x < +\infty$, $0 \leq z < \bar{z}$, such that for any (x,z) below the curve $H(x,z) = 0$ exploration always takes place, otherwise, no exploratory activities will be undertaken. We shall let $\bar{\bar{x}}$ denote the assumed unique value of x such that $H(\bar{\bar{x}},0) = 0$. Clearly, $\bar{\bar{x}} > 0$.

(a) For $\bar{x} < \bar{\bar{x}}$, $H(\bar{x},0) > 0$, i.e., both extraction and exploration begin at the same time. Furthermore, the optimal trajectory in the (x,z)-plane always stays below the curve H=0. Indeed, the moment the optimal trajectory enters the region H < 0, all exploratory activities will stop and extraction will move this trajectory immediately



vertically downward back into the region H > 0. This proves that if $\bar{x} \leq \bar{\bar{x}}$, exploration begins at time zero and will never be terminated before the proven reserve $\bar{x}$ is exhausted.

(b) For $\bar{x} > \bar{\bar{x}}$, $H(\bar{x},0) < 0$. That is exploration will not begin until the initial stock has been reduced to $\bar{\bar{x}}$.



During the initial stage, i.e., $x^*(t) > \bar{\bar{x}}$, $w^*(t) = 0$ and, by (8), $\lambda(t)$ is constant. This means that marginal revenue rises exponentially at the market rate of interest for all t such that $x^*(t) > \bar{\bar{x}}$.

q.e.d.

Now if the monopolist engages in an exploration program, he might be able to increase his working deposits beyond the initial proven reserve $\bar{x}$. Hence, we can expect that he will accelerate his depletion of the original stock $\bar{x}$. This is indeed the case as confirmed by the following theorem.

Theorem 3

Suppose that $\overline{x} > 0$, $\overline{z} > 0$. Next, let $\hat{q}(t)$ be the optimal extraction strategy for the monopolist under the assumption he is not allowed to explore. Then

(a) $q^*(0) > \hat{q}(0)$

(b) $T^* < \hat{T}$

Here $\hat{T}$ is the exact time $\hat{q}$ exhausts $\overline{x}$ and T^* is the exact time q^* exhausts $\overline{x}$ if no discovery is made. We recall that q^* is the optimal extraction strategy for the monopolist under simultaneous extraction and exploration, as already defined by (7).

Proof

(a) To prove $q^*(0) > \hat{q}(0)$, we suppose the contrary, i.e., $q^*(0) \leqq \hat{q}(0)$.

Now it is well known that $\hat{q}$ satisfies the following condition for all time t

$$(10) \qquad \pi'(\hat{q}(t))e^{-rt} = V_x(\hat{x}(t),t) = V_x(\overline{x},0)$$

where $\hat{x}(t)$ is the remaining stock at time t, under $\hat{q}$. Now using the definition of $Y(x,t)$, as given by (3), and the strict concavity of $V(x,t)$ as a function of x, we must have

(11) $Y_x(x,t) = EV_x(x+S,t) < V_x(x,t)$

for all $x \geqq 0$ all $t \geqq 0$.

Furthermore, using (5), (7), (8), we must have

(12) $\pi'(q^*(t))e^{-rt} = \lambda(t)$

Now if $q^*(0) \leqq \hat{q}(0)$, using the strict concavity of π, (10), (11), and (12), we must have

(13) $\lambda(0) \geqq V_x(\bar{x},0) > Y_x(\bar{x},0)$

This last strict inequality together with (8) imply that $\lambda(t)$ is strictly increasing in a neighborhood of time $t = 0$. That is, $q^*(t) < \hat{q}(t)$ for t sufficiently small. We shall show that $q^*(t) \leqq \hat{q}(t)$ for all $t \leqq \hat{T}$. Indeed, let $t_1 > 0$ be the first time such that $q^*(t_1) = \hat{q}(t_1)$. Then by (10) and (12) we must have

(14) $\lambda(t_1) = V_x(\hat{x}(t_1),t_1)$

Using the definition of t_1, we must have $\hat{x}(t_1) < x^*(t_1)$. Hence the strict concavity of $V(x,t)$ implies

(15) $V_x(\hat{x}(t_1),t_1) > V_x(x^*(t_1),t_1)$

If we combine (14) with (15), taking into consideration (11), we obtain

(16) $\lambda(t_1) > V_x(x^*(t_1),t_1) > Y_x(x^*(t_1),t_1)$

This last strict inequality together with (8) then imply that $\lambda(t)$ is strictly increasing in a right neighborhood of t_1. Therefore, $q^*(t) \leqq \hat{q}(t)$ for all $t \leqq \hat{T}$ with strict inequality holding during at least some positive time interval, contradicting the stock constraint $\bar{x}$.

(b) To show $T^* < \hat{T}$, we proceed as follows. First, from part (a) we have already shown $q^*(0) > \hat{q}(0)$, i.e., $x^*(t)$ is below $\hat{x}(t)$ in a right neighborhood of $t = 0$. Next, we want to show $x^*(t) \leqq \hat{x}(t)$ for all $t \leqq T$. Indeed, let $t_1 > 0$ be any time such that $x^*(t_1) = \hat{x}(t_1)$, then applying the result of (a) to the problem with initial proven reserve $x^*(t_1)$, we can assert that $q^*(t_1) > \hat{q}(t_1)$, i.e., $x^*(t)$ dips below $\hat{x}(t)$ in a right neighborhood of t_1. This final result means that $x^*(t) \leqq \hat{x}(t)$ for all $t \leqq \hat{T}$. Hence $T^* \leqq \hat{T}$. If $T^* = \hat{T}$, then we can also find a sequence $(t_n) \to \hat{T}$, $t_n < \hat{T}$, such that

(17) $$0 < q^*(t_n) < \hat{q}(t_n).$$

The strict inequality (17) together with (10) and (12) imply

(18) $$\lambda(t_n) > V_x(\hat{x}(t_n), t_n) = V_x(0, \hat{T}) = f(0)e^{-r\hat{T}}$$

Hence in the limit we must have

(19) $$\lambda(\hat{T})e^{r\hat{T}} > f(0)$$

a contradiction because the undiscounted shadow price of the proven reserve cannot exceed the choke price f(0).

q.e.d.

If the monopolist's problem is to deplete a known fixed stock, then the Hotelling Rule dictates that marginal revenue must rise exponentially at the market rate of interest. This result ceases to be valid when the model also incorporates exploration. However, when there is only uncertainty about the location but no uncertainty about the size of the potential deposit, then the discounted marginal revenue associated with depleting the initial proven reserve will increase through time before a discovery is made, as shown by the following theorem.

Theorem 4

If the probability distribution of S is degenerate, say $S = \bar{S} > 0$ with probability one, then

(a) discounted marginal revenue increases through time before a discovery is made

(b) at the exact time of discovery τ, the optimal post-discovery extraction strategy dictates an upward jump from the original optimal extraction strategy q^* which is followed before discovery.

Proof

(a) To prove this part of Theorem 4, we only need to show $\frac{d\lambda}{dt} \geqq 0$ by virtue of (8), and this is equivalent to showing $Y_x(x^*(t),t) < \lambda(t)$.

Using the Hamilton-Jacobi-Bellman equation (1), we know that $q^*(t)$ must satisfy the following inequality

$$\pi(q^*(t))e^{-rt} - q^*(t)\pi'(q^*(t))e^{-rt} \leqq - W_t(x,z,t) = rW(x,z,t) \tag{20}$$

If we let $\hat{q}(t)$ denote the monopolist's optimal strategy for depleting the known fixed stock $x+\bar{S}$, then a similar version of the Hamilton-Jacobi-Bellman equation (1) can also be obtained for $\hat{q}$ without the terms involving exploration. Then exactly in the same way (20) is obtained, we have the following equation

$$\pi(\hat{q}(t))e^{-rt} - \hat{q}(t)\pi'(\hat{q}(t))e^{-rt} = - V_t(x+\bar{S},t) = rV(x+\bar{S},t) \tag{21}$$

In (20) and (21), x is the remaining proven reserve at time t and z is the cumulative effort at t. Clearly, $V(x+\bar{S},t) > W(x,z,t)$. Hence

$$\pi(q^*(t))-q^*(t)\pi'(q^*(t)) < \pi(\hat{q}(t))-\hat{q}(t)\pi'(\hat{q}(t)) \tag{22}$$

By the strict concavity of π, the function $\pi(Q)-Q\pi'(Q)$ is strictly increasing. Hence (22) implies

(23) $\quad q^*(t) < \hat{q}(t)$

Now using the facts that $\pi'(q^*(t))e^{-rt} = \lambda(t)$ as given by (5) and $\pi'(\hat{q}(t))e^{-rt} = V_x(x+\bar{S},t) = Y_x(x,t)$ as given by (10) and (11) we must have $\lambda(t) > Y_x(x^*(t),t)$ as desired.

(b) Let τ denote the uncertain time a discovery occurs. Before τ, the extraction rate $q^*(t)$ satisfies (5), i.e., $\pi'(q^*(t))e^{-rt} = \lambda(t)$. After discovery, the extraction rate $\hat{q}(t)$ satisfies $\pi'(\hat{q}(t))e^{-rt} = Y_x(\hat{x}(t),t) = V_x(\hat{x}(t)+\bar{S},t)$ where $\hat{x}(\tau) = x^*(\tau)$. Hence the argument in part (a) means $\hat{q}(\tau) > q^*(\tau)$.

q.e.d.

The next result deals with the effect of the uncertainty about the size of the potential deposit on exploration.

<u>Theorem 5</u>

An increase in the uncertainty about the size S of the potential deposit, in the sense of a mean-preserving spread, reduces exploration.

Proof

Let $\overline{z} > 0$ be the terminal cumulative effort as defined in Theorem 1. Then by (b) of Theorem 1,

$$\alpha g(\overline{z})\overline{Y} = c'(0)$$

where $\overline{Y} = EV(S,0)$, as defined by (4). Because V is strictly concave in its first argument, a mean-preserving spread in the distribution of S will reduce $\overline{Y}$, hence $\overline{z}$.

q.e.d.

It is well known that the monopolist tends to over-conserve in depleting a known fixed stock; the comparison being made with respect to the social optimum. In the case of simultaneous extraction and exploration we prove that the monopolist underexplores as follows.

Theorem 6

Let $\overline{x} \geq 0$ be the initial proven reserve. Then, under simultaneous extraction and exploration, the monopolist underexplores in the sense that, conditioned on the assumption of no discovery, the terminal cumulative effort under monopoly $\overline{z}$ is lower than the terminal cumulative effort $\overline{z}_s$ under the social optimum. Here, social welfare is measured by the sum of consumer and producer surplus.

Proof

Let $\overline{Y}_s$ be the social welfare version of $\overline{Y}$ as defined by (4). Then obviously $\overline{Y}_S > \overline{Y}$, and the same argument used in proving Theorem 5 can be used to show $\overline{z}_S > \overline{z}$.

q.e.d.

Gilbert [1979] investigated the problem of optimally depleting an uncertain stock which is currently available for exploitation. This author (Gilbert [1979], Theorem 1) proved that the optimal extraction rate at time zero is lower under stock uncertainty than under stock certainty. In the following theorem, we deal with the effect that the uncertainty about the size of the potential deposit exerts on the exploitation pattern of the proven reserve. The uncertainty we consider here is thus the size uncertainty of a prize which might or might not be discovered at a cost in the future, not the size uncertainty of a stock currently available for exploitation that was investigated by Gilbert [1979]. Our result might be considered as a generalization of Theorem 1 of Gilbert [1979].

Theorem 7

Let $\overline{S}$ be the expected value of S, the random variable representing the uncertain size of the potential deposit. Next, let $\overline{q}$: $t \to \overline{q}(t)$ and $\overline{v}$: $t \to \overline{v}(t)$ be the monopolist's optimal extraction strategy and optimal exploration strategy under stock certainty, i.e., under the assumption that the size of the potential deposit is known with absolute certainty to be equal to $\bar{S}$.

If the optimal terminal cumulative effort $\bar{z}$ is positive, and if the initial proven reserve $\bar{x}$ is sufficiently large, then

$$\bar{q}(0) > q^*(0)$$

Here, we recall from (7) that $q^*: t \to q^*(t)$ and $v^*: t \to v^*(t)$ are the monopolist's optimal extraction strategy and optimal exploration strategy under stock uncertainty, i.e., under the assumption that the size of the potential deposit is uncertain and represented by the random variable $\hat{S}$.

Proof

(a) Let $\bar{W}: (x,z,t) \to \bar{W}(x,z,t)$ be the monopolist's optimal value function under stock certainty, (i.e., $\bar{W}(x,z,t)$ is his optimal expected discounted profit if he begins at time t with proven reserve x and cumulative effort z already spent, without success, searching for $\bar{S}$. To prove Theorem 7, we claim that it is sufficient to establish the following inequality

(24) $\bar{W}(\bar{x},0,0) > W(\bar{x},0,0)$.

(b) Indeed, if the initial proven reserve $\bar{x}$ is sufficiently large, then part (b) of Theorem 2 implies that exploration will not begin at time zero whether the potential deposit has size $\bar{S}$ or uncertain size S, i.e., $\bar{v}(0) = v^*(0) = 0$.

For q*, the Hamilton-Jacobi-Bellman equation reduces to

(25) $\pi(q^*(0)) - q^*(0)W_x(\bar{x},0,0) + W_t(\bar{x},0,0) = 0$

at time t = 0.

Now, using (5) and the fact that $W_t(x,z,t) = -rW(x,z,t)$, we can rewrite (25) as

(26) $r\bar{W}(\bar{x},0,0) = \pi(q^*(0)) - q^*(0)\pi'(q^*(0))$

A similar version of (26) also holds for $\bar{q}$. That is,

(27) $r\bar{W}(\bar{x},0,0) = \pi(\bar{q}(0)) - \bar{q}(0)\pi'(\bar{q}(0))$

Therefore, if (24) is true, the following inequality must hold

(28) $\pi(\bar{q}(0)) - \bar{q}(0)\pi'(\bar{q}(0)) > \pi(q^*(0)) - q^*(0)\pi'(q^*(0))$

Now, using the assumed concavity of π, we can readily verify that $\pi(Q) - Q\pi'(Q)$ is increasing in Q. This last result together with (28) then imply $\bar{q}(0) > q^*(0)$, as desired.

(c) Thus, we shall now attempt to establish (24). To this end, let $(q,v): t \to (q(t),v(t))$ be any pair of feasible extraction and exploration strategies. We shall let $E(q,v,\bar{S})$ and $E(q,v,S)$ denote the monopolist's expected discounted profit under (q,v) when the potential deposit has size $\bar{S}$ and uncertain size S, respectively. We can evaluate $E(q,v,\bar{S})$ and $E(q,v,S)$ by using the monopolist's objective function, as formulated in section 3, and obtain the following result

$$E(q,v,\bar{S}) - E(q,v,S) =$$
$$\int_0^\infty [V(x(t)+\bar{S},t) - EV(x(t)+S,t)]dP(t,v)$$
$$> 0$$

The strict inequality is due to the fact that $V(x,t)$ is strictly concave in x. This strict inequality clearly implies (24), as desired.

q.e.d.

5. CONCLUSION

The model of the exploration process we build in this paper represents an analytical framework to deal directly with location and size uncertainty. Every other thing being equal, our model requires more effort to be spent where the potential deposit is likely located. The model can be generalized to deal with multiple potential deposits in the exploration region A or multiple exploration regions. We have used our model to study the effects that the potential deposit has on the extraction pattern of the proven reserve. We also have shown that the exploration pattern under monopoly is not socially optimum.

APPENDIX

A1. THE HAMILTON-JACOBI-BELLMAN EQUATION

Let $W: (x,z,t) \to W(x,z,t)$ denote the monopolist's optimal value function. That is, $W(x,z,t)$ is his optimal expected discounted profit, given that he begins at time t with proven reserve x and cumulative effort z already spent unsuccessfully searching for the potential deposit.

Next, let $q: s \to q(s)$, $s \geq t$, be any feasible extraction strategy, and $v: s \to v(s)$ be any exploration strategy. We shall assume that both q and v are piecewise and right continuous. Hence we can find $\Delta t \to 0$ sufficiently small so that both q and v are continuous inside the time interval $[t,t+\Delta t]$. Under v, the probability of discovering the potential deposit inside the time interval $[t,t+\Delta t]$ is given by

$$\text{(A1.1)} \qquad \alpha\left[\frac{F'(z)}{1-F(z)}\,\Delta z + 0(\Delta z)\right]$$

where $F(z)$, already defined in section 3, is the probability of discovering the potential deposit as a function of cumulative effort z and $z = \int_t^{t+\Delta t} v(s)ds$.

If we define $g(z) = F'(z)/(1-F(z))$ and use the fact that $\Delta z = v(t)\Delta t + 0(\Delta t)$, we can rewrite (A1.1) as

(A1.2) $\alpha g(z)v(t)\Delta t + 0(\Delta t)$

Now if the monopolist adopts q and v inside the infinitesimal time interval $[t,t+\Delta t]$ but continues optimally from the new initial condition $(x+\Delta x, z+\Delta z, t+\Delta t)$, $\Delta x = -\int_t^{t+\Delta t} q(s)ds$, then he will obtain the following expected discounted profit

$$\text{(A1.3)} \quad e^{-rt}(\pi(q(t))-c(v(t)))\Delta t$$

$$+ \alpha g(z)v(t)\Delta t(EV(x+\Delta x+S,t+\Delta t))$$

$$+ (1-\alpha g(z)v(t)\Delta t)W(x+\Delta x,z+\Delta z,t+\Delta t)+0(\Delta t)$$

$$\leq W(x,z,t)$$

In the second term in the left side of (A1.3), $V(x+\Delta x+S, t+\Delta t)$, already defined in section 2.3, is the monopolist's optimal discounted profit associated with depleting the known fixed stock $x+\Delta x+S$, starting from time $t+\Delta t$, i.e., $V(x+\Delta x+S,t+\Delta t)$ gives the post-discovery optimal discounted profit, given that the potential deposit is discovered and its size is resolved to be S. The expectation in the second term in the left side of (A1.3) is taken with respect to the distribution of the random variable S. We shall define $Y(x,t) = EV(x+S,t)$, and use this definition to rewrite (A1.3) as

$$\text{(A1.4)} \quad e^{-rt}(\pi(q(t))-c(v(t)))\Delta t + \alpha g(z)v(t)\Delta t\, Y(x+\Delta x,t+\Delta t)$$
$$+ (1-\alpha g(z)v(t)\Delta t)W(x+\Delta x,z+\Delta z,t+\Delta t)+0(\Delta t)$$
$$\leqq W(x,z,t)$$

with equality holding if q and v are optimal inside [t,t+Δt].

A rearrangement of (A1.4) gives us

$$\text{(A1.5)} \quad e^{-rt}\pi(q(t))\Delta t + W(x+\Delta x,z+\Delta z,t+\Delta t)-W(x,z,t)$$
$$+\Delta t\begin{bmatrix}\alpha g(z)v(t)(Y(x+\Delta x,t+\Delta t)-W(x+\Delta x,z+\Delta z,t+\Delta t)) \\ - c(v(t))e^{-rt}\end{bmatrix}$$
$$+ 0(\Delta t)$$
$$\leqq 0$$

with equality holding if q and v are optimal inside [t,t+Δt].

If W is sufficiently smooth, we can divide (A1.5) by Δt, then let Δt → 0 to obtain

$$\text{(A1.6)} \quad e^{-rt}\pi(q(t))-q(t)W_x(x,z,t)+v(t)W_z(x,z,t)$$
$$+ W_t(x,z,t) + \alpha g(z)v(t)(Y(x,t)-W(x,z,t))$$
$$- c(v(t))e^{-rt} \leqq 0$$

with equality holding if q and v are optimal inside [t,t+Δt]. These results can be expressed more compactly as

$$\text{(A1.7)} \qquad \max_{\xi \geq 0, \theta \geq 0} \begin{bmatrix} \pi(\xi)e^{-rt} - \xi W_x(x,z,t) + W_t(x,z,t) \\ +\theta[W_z(x,z,t) + \alpha g(z)(Y(x,t) - W(x,z,t))] \\ - c(\theta)e^{-rt} \end{bmatrix} = 0$$

Equation (A1.7) is exactly the Hamilton-Jacobi-Bellman equation (1) of section 4.

A2. THE ADJOINT EQUATIONS

Let $\xi^*(x,z,t)$ and $\theta^*(x,z,t)$ be the values of ξ and θ, respectively, that maximize the left side of (A1.7). Next, pick an arbitrary initial condition $I_o = (x_o, z_o, t_o)$. For the moment, let I_o be fixed and consider a function of three variables $(x,z,t) \geq 0$, defined as follows

$$M: \quad (x,z,t) \rightarrow M(x,z,t)$$

where

$$\text{(A2.1)} \qquad \begin{aligned} M(x,z,t) = {} & \pi(\xi^*(I_o))e^{-rt} - \xi^*(I_o)W_x(x,z,t) \\ & + W_t(x,z,t) \\ & + \theta^*(I_o)[W_z(x,z,t) + \alpha g(z)(Y(x,t) - W(x,z,t))] \\ & - c(\theta^*(I_o))e^{-rt} \end{aligned}$$

We note that the right side of (A2.1) is nothing other than the left side of (A1.7) with ξ and θ replaced by $\xi^*(I_o)$

and $\theta^*(I_o)$, respectively.

By its definition, $M(x,z,t) \leqq 0$ for all (x,z,t), and achieves a maximum at (x_o,z_o,t_o). Therefore, its partial derivatives with respect to x,z,t must vanish at x_o,z_o,t_o, i.e.,

(A2.2)
$$
\begin{aligned}
M_x(x_o,z_o,t_o) &= -\xi^*(x_o,z_o,t_o)W_{xx}(x_o,z_o,t_o) \\
&+ W_{tx}(x_o,z_o,t_o) \\
&+ \theta^*(x_o,z_o,t_o)W_{zx}(x_o,z_o,t_o) \\
&+ \alpha g(z_o)\theta^*(x_o,z_o,t_o)[Y_x(x_o,t) \\
&- W'_x(x_o,z_o,t_o)] \\
&= 0
\end{aligned}
$$

(A2.3)
$$
\begin{aligned}
M_z(x_o,z_o,t_o) &= -\xi^*(x_o,z_o,t_o)W_{xz}(x_o,z_o,t_o) \\
&+ W_{tz}(x_o,z_o,t_o) \\
&+ \theta^*(x_o,z_o,t_o)W_{zz}(x_o,z_o,t_o) \\
&+ \theta^*(x_o,z_o,t_o)\begin{bmatrix} \alpha g'(z_o Y(x_o,t_o) \\ -\alpha g'(z_o)W(x_o,z_o,t_o) \\ -\alpha g(z_o)W_z(x_o,z_o,t_o) \end{bmatrix} \\
&= 0
\end{aligned}
$$

Equations (A2.2) and (A2.3) hold for any arbitrary initial condition (x_o, z_o, t_o), in particular, for any point $(x^*(t), z^*(t), t)$ on the optimal trajectory. That is, along the optimal trajectory, (A2.2) and (A2.3) take the following forms

(A2.4) $$-q^*(t)W_{xx}(x^*(t),z^*(t),t) + W_{tx}(x^*(t),z^*(t),t) + v^*(t)W_{zx}(x^*(t),z^*(t),t) = -\alpha g(z^*(t))v^*(t)[Y_x(x^*(t),t)-W_x(x^*(t),z^*(t),t)]$$

(A2.5) $$-q^*(t)W_{xz}(x^*(t),z^*(t),t) + W_{tz}(x^*(t),z^*(t),t) + v^*(t)W_{zz}(x^*(t),z^*(t),t) = -\alpha v^*(t)\begin{bmatrix} g'(z^*(t))[Y(x^*(t),t)-W(x^*(t),z^*(t),t)] \\ -g(z^*(t))W_z(x^*(t),z^*(t),t) \end{bmatrix}$$

Here, we recall that $x^*(t)$, $z^*(t)$, $q^*(t)$, $v^*(t)$ are as defined by (7) of section 4.

Finally, let $\lambda(t) = W_x(x^*(t),z^*(t),t)$ and $\eta(t) = W_z(x^*(t),z^*(t),t)$, then $d\lambda/dt$ and $d\eta/dt$ are given by the left sides of (A2.4) and (A2.5), respectively, i.e.,

(A2.6) $$d\lambda/dt = -\alpha g(z^*(t))v^*(t)[Y_x(x^*(t),t)-\lambda(t)]$$

(A2.7) $$d\eta/dt = -\alpha v^*(t)\begin{bmatrix} g'(z^*(t))[Y(x^*(t),t)-W(x^*(t),z^*(t),t)] \\ -g(z^*(t))\eta(t) \end{bmatrix}$$

Equations (A2.6) and (A2.7) are exactly equations (8) and (9), respectively, of section 4.

CHAPTER FOUR

INFORMATION SPILLOVER IN THE EXPLORATION FOR A NONRENEWABLE RESOURCE

1. INTRODUCTION

In this chapter we investigate the problem of information spillover in the exploration for a nonrenewable resource. As is well known, the discovery of oil and gas on a piece of land provides information about the geology of adjacent lots, and sometimes the spillover can spread over hundreds of square miles. Even a dry hole provides useful information about the surrounding tracts. This problem of information spillover represents a classical example of externalities.

Because it is almost impossible for a prospector to conceal information about his exploratory activities and also because markets for information can never be expected to exist, there is a strong incentive for each leaseholder just to sit back and wait for the outcomes of the exploratory efforts of the other leaseholders to obtain free information about his own tract. The model we build in this chapter represents an attempt to study this kind of behavior.

Our approach is that of a two-period non-cooperative game played by two leaseholders. Our model can readily be generalized to the case when there are more than two leaseholders. An infinite time horizon version of the game is formulated, but not solved, in the appendix of this chapter. The model we build contains an explicit formulation of the exploration process and a precise specification of the information spillover phenomenon. Specifically, we want to investigate the following three questions: (1) which player will explore first and, in the process, provide the other player with the free information spillover?; (2) compared with the social optimum, does the market solution involve underexploration or overexploration?; (3) what are the private and social values of the information spillover?

Our main contributions in this chapter include the following results:

(i) The solution of the game that we would like to propose suggests that the player who has a lower risk limit will explore first and, in the process, provide the other player with the free information spillover. On the other hand, if both players have the same risk limits, then our proposed solution suggests that both of them would use the same mixed strategy. These results are contained in the theorem given in section 4.2.

(ii) Compared with the social optimum, the market might generate too much or too little exploration. In general, the solution of the game predicts underexploration. However, in some unusual situations, the non-cooperative solution leads to an inefficient use of the information externalities and results in overexploration. These results are discussed in section 5.

(iii) We show how the private and social values of the information spillover are evaluated; their expressions are given in section 5.

The plan of this chapter is as follows. In section 2, the technical aspects of the geological model are presented. In section 3, we evaluate the economic value of the information spillover for each player. In section 4, we formulate and solve the two-period waiting game of information spillover; the main theorem of this chapter is presented in section 4.2. In section 5, we discuss the private and social values of the information spillover, and address the question of whether the market generates too much or too little exploration. Section 6 contains some concluding remarks. In the appendix to this chapter, we give a sketch of the formulation of an infinite time horizon version of the game of information spillover. Finally, the calculations performed to evaluate some posterior probabilities are also delegated to the appendix.

2. THE TECHNICAL ASPECTS OF THE INFORMATION SPILLOVER

2.1 Stochastic Specification of Each Individual Lot

In our model there are two players. Player i, $i = 1,2$, owns a piece of land A_i. We shall assume that A_1 and A_2 are adjacent to each other, and there is a single geological structure lying beneath both A_1 and A_2.

In A_i, $i = 1,2$, it is suspected that there might be a single mineral deposit, say D_i, lying hidden somewhere. Because of their contiguity, we expect that D_1 and D_2 are not independent, i.e., their characteristics must be stochastically correlated in some manner. We shall now describe, in more precise terms, the stochastic relationship that exists between D_1 and D_2. The description involves specifying a joint probability distribution over the characteristics of D_1 and D_2. For each deposit D_i, we assume that a complete characterization of this deposit requires specifying (1) its probable existence and (2), given its existence, its exact location as well as its size and extraction cost.

(a) First, we consider the probable existence of D_1 and D_2. There are four exhaustive possibilities: (1) both D_1 and D_2 exist, (2) D_1 exists but D_2 does not, (3) D_2

exists but D_1 does not, (3) neither D_1 nor D_2 exists. We shall assume that the probabilities of these four events are given by the following matrix

	D_2 exists	D_2 does not exist
D_1 exists	α_{11}	α_{12}
D_1 does not exist	α_{21}	α_{22}

We require that $\alpha_{ij} \geqq 0$ and $\sum \alpha_{ij} = 1$, $i,j = 1,2$. If D_i exists, we expect that there is a high probability that D_j also exists. Hence we shall assume that both α_{12} and α_{21} are small. This assumption means that the conditional probability that D_i exists, given that D_j exists, is almost equal to unity. We shall let

$$\alpha_1 = \alpha_{11} + \alpha_{12} \quad \text{and} \quad \alpha_2 = \alpha_{11} + \alpha_{21}$$

denote the prior probabilities that D_1 and D_2 exist, respectively.

(b) Second, we consider the probable location of D_1 and D_2. To this end, let

(2) $\quad m: \ a \to m(a) \ , \ a = (a_1,a_2) \in A = A_1 \times A_2 .$

be a nonnegative measurable function defined on A such that for any measurable subset B of A,

$$\int_B m(a)da$$

gives the probability that (D_1, D_2) is located in B, given that they really exist.

From the function m, just mentioned, we can derive two density functions m_1 and m_2 such that m_i, i = 1,2, describes the probable location of D_i only in terms of A_i. That is

(3) $$m_i:\ a_i \to m_i(a_i)\ ,\ a_i \in A_i\ ,\ i = 1,2$$

is a nonnegative measurable function defined on A_i such that for any measurable subset B_i of A_i,

$$\int_{B_i} m_i(a_i)da_i$$

gives the probability that D_i is located in B_i, given that this deposit really exists.

(c) Third, we consider the uncertain sizes and uncertain extraction costs of D_1 and D_2. For each i = 1,2, let S_i and c_i be two nonnegative random variables representing the uncertain size and uncertain extraction cost of D_i. Furthermore, let

(4) $$P' = P(S_1, c_1, S_2, c_2)$$

be the joint probability distribution of S_1, c_1, S_2, c_2. The marginal probability distribution of (S_i, c_i), $i = 1,2$, can be obtained from (4) and will be denoted by

(5) $$P_i = P_i(S_i, c_i) \;,\; i = 1,2$$

2.2 The Exploration Process

The discussion and the results of this subsection are all conditioned on the assumption that D_i really exists.

First, we describe the search technology. For each location a_i in A_i, let

$$b_i(\cdot, a_i): \quad y_i \to b_i(y_i, a_i)$$

be a strictly increasing and concave function of $y_i \geq 0$ such that $b_i(0, a_i) = 0$ and $b_i(\infty, a_i) = 1$. Here we shall interpret $b_i(y_i, a_i)$ as the probability of discovering D_i, given that it is located at a_i and an amount of effort y_i is spent at a_i searching for D_i.

By an exploration program for player i, we mean a non-negative function of $a_i \in A_i$, $z_i \geq 0$

$$J_i: \quad (a_i, z_i) \to J_i(a_i, z_i)$$

which allocates the amount of effort $J_i(a_i, z_i)$ to location a_i such that

(6) $$\int_{A_i} J_i(a_i,z_i)da_i \leqq z_i$$

Here, we interpret z_i as the total amount of effort player i plans to spend for exploring A_i. The preceding inequality expresses the total effort constraint under J_i.

For each given z_i, the probability of discovering D_i, under J_i, is given by

(7) $$\text{Prob}[J_i(.,z_i)] = \int_{A_i} b_i(J_i(a_i,z_i),a_i)m_i(a_i)da_i$$

Now, the argument of section 3, Chapter 3, can also be used here to obtain a unique exploration program $J_i^*: (a_i,z_i) \to J_i^*(a_i,z_i)$, which satisfies the total effort constraint (6) and maximizes the probability of discovering D_i, for any given z_i. That is

(8) $$\text{Prob}[J_i^*(.,z_i)] = F_i(z_i) \geqq \text{Prob}[J_i(.,z_i)]$$

for any J_i and any z_i.

The function F_i: $z_i \to F_i(z_i)$, defined by (7)', gives the maximum probability of discovering D_i in terms of the total effort z_i available for exploration. Using the assumed concavity of $b_i(.,a_i)$, we can easily verify that F is also strictly concave.

Before moving on, we would like to emphasize that (7) and (8) are evaluated in terms of the prior density function

m_i, i.e., if player i ignores the possible information spillover that might be generated by player j, $i \neq j$, $i,j = 1,2$.

2.3 Specification of the Information Spillover

Because we are not interested in the time path of the exploration process, we shall assume that exploration is a one-shot trial the result of which will not be known until the next period. That is, if player i decides to explore his lot A_i in period t, then the outcome of his exploratory activities will be known in period t+1, and if no discovery is made he will never try again. We shall represent the outcome of player i's exploration program by a binary random variable s_i

$$s_i = \begin{cases} 1 & \text{if a discovery is made} \\ 0 & \text{if the outcome is a failure} \end{cases}$$

In the case $s_i = 1$, we shall assume that the location, the size S_i and the extraction cost c_i of the deposit in A_i will become public knowledge.

If player i is the first one to explore, then the probability distribution of s_i, as a function of the total effort z_i he plans to spend for exploration, is given by

$$\text{(9)} \qquad \text{Prob}\{s_i=1\} = \alpha_i F_i(z_i)$$

$$\text{Prob}\{s_i=0\} = 1 - \alpha_i F_i(z_i)$$

Here we recall that α_i is the prior probability that the potential deposit D_i really exists and F_i is as defined by (8).

We shall now describe the effects, on A_j, of the information spillover generated by the exploratory activities undertaken in A_i.

(a) First, let $\hat{\alpha}_j$ denote the posterior probability that the potential deposit D_j really exists after the outcome of the exploratory activities in A_i is revealed. Clearly, $\hat{\alpha}_j$ depends on (s_i,z_i), i.e., $\hat{\alpha}_j = \hat{\alpha}_j(s_i,z_i)$. In the appendix of this chapter, we show that

$$\text{(10)} \qquad \hat{\alpha}_j(s_i,z_i) = \alpha_{11}/\alpha_i \quad \text{if } s_i = 1$$

$$= \frac{\alpha_{11}G_i(z_i) + \alpha_{ji}}{\alpha_i G_i(z_i)+\alpha_{ji}+\alpha_{22}} \quad \text{if } s_i = 0$$

where $G_i(z_i) = 1 - F_i(z_i)$, $i \neq j$, $i,j = 1,2$.

We can readily verify that $\hat{\alpha}_j(0,0) = \alpha_j$. Using the assumption that α_{12} and α_{21} are sufficiently small, we can also show that $\hat{\alpha}_j(1,z_i) \cong 1$ and $\hat{\alpha}_j(0,z_i)$ is strictly decreasing in z_i. In other words, a discovery in A_i almost guarantees the existence of D_j, and the more effort that player i spends unsuccessfully for exploration, the lower is the chance that D_j really exists.

(b) Second, let $\hat{m}_j$, $\hat{P}_j$ denote the posterior density and posterior distribution summarizing the probable location, the uncertain size and extraction cost of the potential deposit D_j, all evaluated with the help of the information spillover generated in A_i.

If the exploratory activities in A_i are a failure, then we assume that player j does not gain any further information concerning the location, size, and extraction cost of the potential deposit in A_j. In this case, $\hat{m}_j$ and $\hat{P}_j$ are the same as their respective priors, i.e.,

(11) $$\hat{m}_j = m_j \ , \ \hat{P}_j = P_j$$

where m_j is given by (3) and P_j is given by (5).

On the other hand, if the exploratory activities in A_i result in a discovery, then the location a_i, the size S_i, and the extraction cost c_i of the deposit D_i will be completely revealed and become public knowledge. In this case, $\hat{m}_j$ and $\hat{P}_j$ are the conditional density and conditional probability of m_j and P_j, given a_i and (S_i, c_i), i.e.,

(12) $$\hat{m}_j(.|a_i): \quad a_j \to \hat{m}_j(a_j|a_i) = m(a)/m_i(a_i)$$

$$\hat{P}_j(S_j, c_j|S_i, c_i) = P(S_1, c_1, S_2, c_2)/P_i(S_i, c_i)$$

for $i,j = 1,2$, $i \neq j$. Here $a = (a_1, a_2)$, and m, m_i, P, P_i are given by (2), (3), (4), (5), respectively.

Now, we recall from (8) that, conditioned on the existence of D_j, $j = 1,2$, $F_j(z_j)$ gives the maximum probability of discovering the deposit D_j as a function of the total effort z_j when the probable location of D_j is represented by the prior density m_j. Therefore, when the probable location of D_j is represented by $\hat{m}_j(.|a_i)$, the posterior density of m_j, we shall use the symbol

$$F_j(z_j|\hat{m}_j(.|a_i)) \ , \ i,j = 1,2 \ , \ i \neq j \tag{13}$$

to denote the corresponding maximum conditional probability of discovering D_j as a function of z_j, given that D_j really exists in A_j.

3. THE ECONOMIC VALUE OF THE INFORMATION SPILLOVER

Let p denote the constant market price of the nonrenewable resource in each period $t = 0,1,2$. The market rate of interest and the cost per unit exploratory effort are denoted by r and e, respectively. Because we are not interested in risk aversion, we shall assume that the objective of each player is to maximize expected discounted profit.

Now, if player i decides to explore in period t = 0, then he will choose z_i, the total amount of effort he plans to spend for exploration, to maximize the following expected discounted profit

$$(14) \qquad \pi_i(z_i) = \frac{\alpha_i F_i(z_i)}{1+r} \int S_i(P-c_i)dP_i(S_i,c_i) - ez_i$$

We shall let z_i^* denote the unique value of z_i that maximizes (14). Then z_i^* satisfies the following first order condition

$$(15) \qquad F_i'(z_i^*) \geqq \frac{e(1+r)}{\alpha_i \int S_i(P-c_i)dP_i(S_i,c_i)}, \quad i = 1,2$$

with equality holding if $z_i^* > 0$.

For our own use later, we shall let

$$(16) \qquad \pi_i^* = \pi_i(z_i^*) \ , \ i = 1,2.$$

Next, we shall evaluate player i's optimal expected discounted profit, given that he enjoys the information spillover generated by player j when the latter explores A_j in period t = 0. Now we note that if player j decides to explore in period t = 0, then his optimal total exploratory effort will be z_j^*, as given by the first order condition (15). However, for our own use later, we shall evaluate player i's optimal expected discounted profit for any arbitrary total effort z_j that player j might spend in exploring A_j, conditioned on the assumption that player j explores while player i does not explore in period t = 0. There are two possibilities to consider: $s_j=0$ and $s_j=1$.

First, if $s_j=0$, i.e., if no discovery is made in A_j, then the posterior probability that D_i exists is $\hat{\alpha}_i(0,z_j)$, as given by (10), while the probable location, the uncertain size and uncertain extraction cost of D_i remain to be represented by the priors m_i and P_i, respectively, as assumed by (11). In this case, the problem that player i has to solve is to find $z_i \geqq 0$ to maximize the following expected discounted profit

$$(17) \qquad \hat{\alpha}_i(0,z_j)F_i(z_i) \int \frac{S_i(P-c_i)}{(1+r)^2} \, dP_i(S_i,c_i) - \frac{ez_i}{1+r}$$

In evaluating (17), we note that the information spillover is only available in period t=1, and the outcome of player i's exploratory activities, undertaken in period t=1, will be revealed in period t=2. These facts account for the discounting factors in (17). We shall let $h_i(0,z_j)$, $i \neq j$, $i,j = 1,2$, denote the maximum of (17). That is, $h_i(0,z_j)$ gives the optimal expected discounted profit for player i, given that he receives the information spillover generated by player j, who explores his lot A_j first and spends the amount of total effort z_j without making any discovery. Because $\hat{\alpha}_i(0,z_j)$ is strictly decreasing in z_j, it is clear that $h_i(0,z_j)$ is strictly decreasing in z_j.

Second, if $s_j=1$, i.e., if D_j is discovered, then its location a_j, its size and extraction cost (S_j,c_j) are revealed. In this case, player i has to find $z_i \geqq 0$ to maximize the following expected discounted profit

$$(18) \qquad \hat{\alpha}_i(1,z_j)F_i(z_i|\hat{m}_i(.|a_j))\int \frac{S_i(P-c_i)}{(1+r)^2}\,d\hat{P}_i(S_i,c_i|S_j,c_j) - \frac{ez_i}{1+r}$$

In (18), $\hat{\alpha}_i(1,z_j)$, $\hat{m}_i(.|a_j)$, $\hat{P}_i(S_i,c_i|S_j,c_j)$ are the revised estimates of the chance that D_i really exists, the probable location of D_i, the uncertain size and uncertain extraction cost of D_i, all evaluated in terms of the outcome $(s_j=1, a_j,S_j,c_j)$. Also, $F_i(z_i|\hat{m}_i(.|a_j))$ is as defined by (13).

Now by (10) we have $\hat{\alpha}_i(1,z_j) = \alpha_{11}/\alpha_j$, independent of z_j, i.e., (18) does not depend on z_j. Hence the maximum of (18), denoted by $h_i(1,z_j|a_j,S_j,c_j)$, depends only on (a_j,S_j,c_j). If we let

$$h_i(1,z_j) = \int \left[\int h_i(1,z_j|a_j,S_j,c_j)\,dP_j(S_j,c_j)\right]m_j(a_j)\,da_j$$

then $h_i(1,z_j)$ gives the optimal expected discounted profit for player i, given that he waits for the information spillover generated by player j, who spends the amount of total effort z_j and discovers the deposit D_j. Furthermore,

$h_i(1,z_j)$ does not depend on z_j. However, to avoid potential notational confusion, we shall retain the variable z_j in $h_i(1,z_j)$.

Intuitively speaking, we expect that $h_i(1,z_j) > h_i(0,z_j)$, i.e., a discovery in A_j is more valuable for player i than a failure in A_j. Therefore, we shall assume that $h_i(1,z_j) > h_i(0,z_j)$. This assumption can be justified on the following two grounds. First, we recall from the discussion in part (a) of section 2.3 that $\hat{\alpha}_i(1,z_j) \cong 1$ while $\hat{\alpha}_i(0,z_j) < \alpha_i$, i.e., a discovery in A_j increases substantially the chance that D_i really exists while a failure in A_j reduces the chance that D_i really exists. Second, a discovery in A_j yields further information about the location, size, extraction cost of the potential deposit D_i while a failure in A_j reveals nothing beyond the priors m_i, $P_i(S_i,c_i)$. In the extreme case when a discovery in A_j yields perfect information concerning the location, size, extraction cost of the deposit D_i, it is obvious that $h_i(1,z_j) > h_i(0,z_j)$.

We are finally ready to give an expression for the optimal expected discounted profit enjoyed by player i, given that he waits for the information spillover generated by player j when the latter decides to spend the amount of total effort z_j exploring his lot A_j first, as follows

(19) $$h_i(z_j) = \alpha_j F_j(z_j) h_i(1,z_j) + [1-\alpha_j F_j(z_j)] h_i(0,z_j)$$

$i,j = 1,2, \; i \neq j$.

The derivative of $h_i(z_j)$ with respect to z_j is given by,

(20) $$h_i'(z_j) = \alpha_j F_j'(z_j)[h_i(1,z_j) - h_i(0,z_j)] + [1-\alpha_j F_j(z_j)] h_i'(0,z_j)$$

In deriving (20), we recall that $h_i(1,z_j)$ does not depend on z_j. The first term in the right side of (20) is positive; the second term in the right side of (20) is negative. Therefore, as it stands, the sign of $h_i'(z_j)$ is ambiguous. However, we expect that the more total effort player j plans to spend in exploring his lot A_j, the higher will be the benefit, for player i, of the information spillover that player j generates. Therefore, we shall assume that

(21) $$h_i'(z_j) > 0 \;, \; i,j = 1,2, \; i \neq j.$$

Now if player j decides to explore first, then his optimal total exploratory effort is z_j^*, as given by (15). In this case, the optimal expected discounted profit for player i, given that he waits in period t=0 to obtain the information spillover generated by player j, is given by

(22) $\hat{\pi}_i = h_i(z_j^*)$, $i \neq j$, $i,j = 1,2$

and the economic value, for player i, of the information spillover generated by player j is then given by

(23) $I_i = \hat{\pi}_i - \pi_i^*$ $i = 1,2$

4. THE WAITING GAME OF INFORMATION SPILLOVER FOR THE TWO-PERIOD TIME HORIZON

4.1 Formulation of the Game

In any period t, t = 0,1,2, player i can either start exploring his lot A_i or just sit back, hoping that he might obtain the free information spillover generated by player j, $j \neq i$, in case the latter decides to explore his lot A_j in period n. We shall represent the choice available to player i in any period t by a binary decision variable $d_i(t)$, where

$$d_i(t) = \begin{cases} 1 & : \text{explore} \\ 0 & : \text{wait} \end{cases}$$

Now, in section 2.3, we have assumed that exploration is a one-shot trial, i.e., if player i decides to explore his lot A_i in period t, then he will not try again if no discovery is made. This assumption means that $d_i(t) = 1$ for at most one t. Furthermore, we shall allow player i to make only one choice in any period. That is, he cannot first choose $d_i(t) = 0$, then change his mind to $d_i(t) = 1$ in the same period, and vice versa.

We shall now determine the payoff matrix for the two-period waiting game of information spillover.

First, if both players decide to explore in period $t=0$, then their payoffs will be given by π_1^* and π_2^*. Second, if player i explores in period $t=0$ while player j waits in this period, then the payoff for player i is π_i^* and the payoff for player j is $\hat{\pi}_j$. Here, we recall that π_i^*, $i = 1,2$, is given by (16), and $\hat{\pi}_i$, $i = 1,2$, is given by (22). Finally, if both players do not explore in period $t=0$, then the status quo will be carried into period $t=1$. If $\pi_i^* = 0$, then player i will not explore in period $t=1$. On the other hand, if $\pi_i^* > 0$, then player i will certainly explore in period $t=1$, if he has not done so before. The reason is simple: exploration lasts one period and the waiting game ends after period $t=2$. Therefore, for the two-period game, the payoffs for the two players are given by $\pi_1^*/(1+r)$ and $\pi_2^*/(1+r)$ if they both decide not to explore in period $t=0$.

We can summarize the results of the preceding paragraph in the following payoff matrix

		player 2	
		$d_2(0) = 1$	$d_2(0) = 0$
player 1	$d_1(0) = 1$	π_1^* π_2^*	π_1^* $\hat{\pi}_2$
	$d_1(0) = 0$	$\hat{\pi}_1$ π_2^*	$\frac{\pi_1^*}{1+r}$ $\frac{\pi_2^*}{1+r}$

4.2 Solution of the Game

Having determined the payoff matrix, we shall now proceed to solve the waiting game of information spillover. There are several possibilities to consider.

(a) If $\pi_i^* = 0$, then it is clear that player i will never be the first one to explore. Hence, if $\pi_1^* = \pi_2^* = 0$, neither player will explore first, i.e., exploration will never be undertaken in A_1 and A_2.

(b) If $\pi_i^* > 0$ and $\pi_j^* = 0$, $i \neq j$, $i,j = 1,2$, then player i will explore first and provide player j with the information spillover.

(c) When $\pi_1^* > 0$, $\pi_2^* > 0$, the behavior of the two players cannot be determined from these quantities alone; we need to take into consideration the value of the information spillover I_i, $i = 1,2$, for each player. First, if $I_i = \hat{\pi}_i - \pi_i^* \leqq 0$ for both $i = 1,2$, then both players will explore in period t=0. Second, if $I_i \leqq 0$ and $I_j > 0$, $i,j = 1,2$, $i \neq j$, then player i will explore first and provide player j with the free information spillover. Finally, if $I_1 > 0$, $I_2 > 0$, then both players have strong incentive to wait in period t=0. All our effort in the rest of section 4.2 is devoted to solving the game for this last case, i.e., when $0 < \pi_i^* < \hat{\pi}_i$ for both i = 1,2.

The waiting game of information spillover has two pure-strategy equilibrium points and one mixed-strategy equilibrium point in the sense defined by Nash [1951]. The two pure-strategy equilibrium points correspond to the cases when player i explores while player j waits in period t=0, $i \neq j$, i,j = 1,2. We shall exhibit the mixed-strategy equilibrium point later as we proceed.

We are thus confronted with a multiplicity problem: our game has three different equilibrium points. To resolve this multiplicity problem, Harsanyi [1982] has proposed a mathematical procedure, called the tracing procedure, to pick out one equilibrium point as the solution of the game. However, we shall not use Harsanyi's tracing procedure to find the solution of our game because of the complicated computations required by the tracing procedure. Instead, we shall suggest how to pick one equilibrium point as a plausible solution for the game of information spillover by adapting a decision rule, proposed by Zeuthen [1930] and reinterpreted by Harsanyi [1977], in the following manner.

Now in our game player i, i = 1,2, has two pure strategies: $d_i(0) = 1$, i.e., explore in period t=0, and $d_i(0) = 0$, i.e., wait in period t=0. If player i knows which pure strategy player j will use, his problem will be very simple: Indeed, if $d_j(0) = 1$, then player i should choose $d_i(0) = 0$, and if $d_j(0) = 0$, then player i should choose $d_i(0) = 1$. However, in the absence of this knowledge, player i, as a Bayesian expected-utility maximizer, must start with assigning subjective probabilities to the two possible pure strategies that player j can make. Therefore, we shall let

$$p_j = \text{prob}\{d_j(0) = 0\}$$

be the probability, as subjectively perceived by player i, that player j will not explore in period t=0.

If player i chooses to explore in period t=0, he is guaranteed the payoff π_i^*, regardless of what player j will do. On the other hand, if player i chooses not to explore in period t=0, then he will obtain the payoff $\pi^*/(1+r)$ with probability p_j in the case player j chooses $d_j(0) = 0$, and the payoff $\hat{\pi}_i$ with probability $1-p_j$ in the case player j chooses $d_j(0) = 1$. Thus player i will decide to wait in period t=0, i.e., to choose the pure strategy $d_i(0) = 0$ with probability one, only if

$$p_j \, \pi_i^*/(1+r) + (1-p_j)\hat{\pi}_i > \pi_i^*$$

i.e., if

$$(24) \qquad p_j < r_i = \frac{\hat{\pi}_i - \pi_i^*}{\hat{\pi}_i - \pi_i^*/(1+r)}, \quad i,j = 1,2, \; i \neq j$$

The quantity r_i, as defined in (24), represents player i's risk limit, since it is the highest risk that player i is willing to take, by waiting in period t=0, in order to obtain the information spillover that player j might generate. As just shown by (24), if $p_j < r_i$, player i will wait in period t=0. On the other hand, if $p_j > r_i$, i.e., if the probability, as subjectively perceived by player i, that player j will not explore in period t=0 exceeds his risk limit r_i, then player i will explore in period t=0 with probability one. In the special case when $p_j = r_i$, player i is indifferent between the two pure strategies $d_i(0) = 1$ and $d_i(0) = 0$, i.e., any strategy, pure or mixed, will give him the same payoff π_i^*.

The argument in the preceding paragraph also implies that if player i chooses the pure strategy $d_i(0) = 0$ with probability r_j, $i \neq j$, $i,j = 1,2$, and the pure strategy $d_i(0) = 1$ with probability $1-r_j$, then we have a mixed-strategy equilibrium point for our game. Furthermore, there is no mixed-strategy equilibrium point other than the one just mentioned.

To pick out a plausible solution for the game among the three equilibrium points, we propose that player i will explore first and, in the process, provide player j with the free information spillover if $r_i < r_j$. Intuitively, we can visualize a situation in which each player waits for the other one to explore first. As time passes, the situation grows more tense, and the chance that neither player will explore first increases, until player i (whose risk limit has been reached) gives up and explores first.

On the other hand, if $r_i = r_j$, then it means that there is some symmetry in our game in the sense that both players have the same risk limit. Furthermore, because the behavior of the two players now only depends on the risk limits r_i and r_j, the equality $r_i = r_j$ means that it is reasonable to assume that their optimal strategies must be symmetric. Therefore, the two asymmetric pure-strategy equilibrium points will be ruled out. That is, when $r_i = r_j$, we shall choose the mixed-strategy equilibrium point as the solution of the game, i.e., player i will not explore in period t=0 with probability $r_j = r_i$ and will explore in period t=0 with probability $1-r_j = 1-r_i$, $i,j = 1,2$, $i \neq j$.

We now summarize the results obtained in section 4.2 in the following theorem.

Theorem

For the two-period waiting game of information spillover,

(a) If $\pi_1^* = \pi_2^* = 0$, exploration will never be undertaken in A_1 and A_2.

(b) If $\pi_i^* > 0$, $\pi_j^* = 0$, $i \neq j$, $i,j = 1,2$, then player i will explore first and provide player j with the information spillover.

(c) If $0 < \pi_i^*$ for both i = 1 and i = 2, there are several possibilities to consider.

(c1) First, if $I_i = \hat{\pi}_i - \hat{\pi}_i^* \leq 0$ for both i = 1 and i = 2, then each player will ignore the information spillover that the other player might generate and explore in period t=0.

(c2) Second, if $I_i \leq 0$ and $I_j > 0$, $i,j = 1,2$, $i \neq j$, then player i will explore first and provide player j with the information spillover.

(c3) Finally, if $I_i > 0$, $I_j > 0$, $i,j = 1,2$, $i \neq j$, then player i will explore first and provide player j with the information spillover when $r_i < r_j$. When $r_i = r_j$, each player will use the same mixed strategy: wait in period t=0 with probability $r_i = r_j$ and explore in period t=0 with probability $1-r_i = 1-r_j$. Here, we recall that r_i, i = 1,2, is player i's risk limit, as defined by (24).

5. THE PRIVATE AND SOCIAL VALUES OF THE INFORMATION SPILLOVER

5.1 The Private Value of the Information Spillover

The solution of the waiting game of information spillover, as given in section 4.2, determines the private value of the information spillover for each player. If player i explores first, he will generate the information spillover and obtain no benefit from this information spillover. On the other hand, if player j explores first while player i waits, then the private value of the information spillover for player i is equal to I_i, as defined by (23).

5.2 The Social Value of the Information Spillover

If player i is the one that generates the information spillover for player j, then their joint expected discounted profit, as a function of the total exploratory effort z_i that player i plans to spend for exploration, is given by

(25) $$H_i(z_i) = \pi_i(z_i) + h_j(z_i) \; , \; i \neq j, \; i,j = 1,2$$

where $\pi_i(z_i)$ and $h_j(z_i)$ are given by (14) and (19), respectively.

Now if the solution of the waiting game of information spillover requires player i to explore first, then his optimal total exploratory effort is z_i^*. In the case $z_i^* > 0$, we have

$$H_i'(z_i^*) = h_j'(z_i^*) > 0$$

by (21). That is because player i cannot capture all the benefits derived from his exploratory activities, he will not go far enough to maximize the joint expected discounted profit of both of them. Therefore, if we let $\overline{z}_i$ be the value of z_i that maximizes $H_i(z_i)$, then $\overline{z}_i > z_i^*$ if $z_i^* > 0$, $i = 1,2$.

If we accept the market price p as a measure of the social welfare that society attaches to the nonrenewable resource under consideration, then the welfare associated with the social optimum is

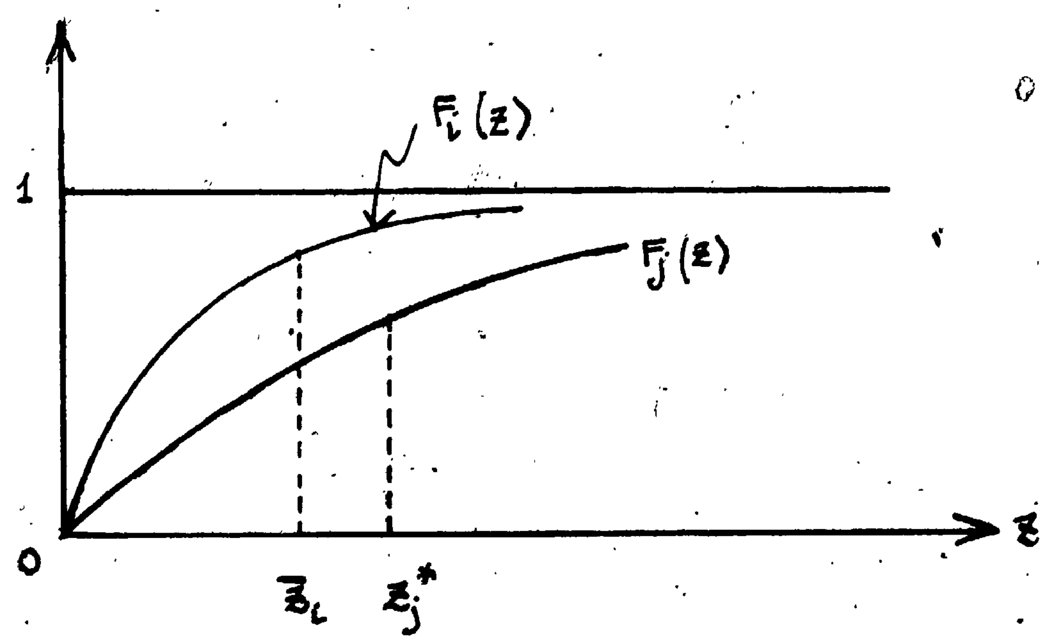
$$\overline{H} = \max \{H_1(\overline{z}_1), H_2(\overline{z}_2)\}$$

It might happen that $\overline{H} > 0$ but $\pi_1^* = \pi_2^* = 0$. In this case, the solution of our game predicts that no exploration will be undertaken in A_1 and A_2 while the social optimum requires some exploration to be carried out. This means that the market generates too little exploration.

On the other hand, the market might also generate too much exploration. To illustrate this situation, we visualize two lots A_i and A_j such that (i) $\pi_i^* = 0$, $\pi_j^* > 0$,

(ii) $\alpha_i = \alpha_j$, (iii) $F_i(z) > F_j(z)$ for all $z > 0$, (iv) $F_i'(z_j^*)$ is sufficiently small, and (v) a discovery in A_i will guarantee the existence of D_j and give perfect information concerning the location, size, and extraction cost of D_j.

To justify (i) we think of the situation when S_i is small and S_j is large. If the prior density m_j, which summarizes the probable location of D_j, is more diffuse than the prior density m_i, which summarizes the probable location of D_i, then it is much more difficult to find D_j than to find D_i, i.e., $F_i(z) > F_j(z)$ for all $z > 0$. This scenario justifies (iii). To justify (iv), pick any distribution function F_i which increases rapidly at first, but levels off quickly.



Now if (i) is true, then the solution of our game predicts that player j will explore first and provide player i with the information spillover.

On the other hand, using (ii), (iii) and (v), we see immediately that to maximize social welfare, it is better to explore A_i first. This is because for any given $z > 0$, if z is spent in A_i then the probability of discovering both D_i and D_j is $\alpha_i F_i(z)$, which is greater than $\alpha_j F_j(z)$, the probability of discovering both D_i and D_j if z is spent in A_j. Furthermore, if (iv) is true, then $\bar{z}_i$, the value of z_i which maximizes (25), will satisfy the strict inequality $\bar{z}_i < z_j^*$. This last strict inequality means that the market solution generates too much exploration.

Finally, we note that the social value of the information spillover is given by the difference between the social welfare evaluated when the externality is internalized and the social welfare evaluated when the externality is ignored, i.e.,

$$SVIS = \bar{H} - \pi_1^* - \pi_2^*$$

6. CONCLUSION

In this chapter, we have presented a game-theoretical model to study the problem of information spillover generated in the exploration for a nonrenewable resource. We also discussed the private and social values of the information spillover and addressed the question of whether the market underexplores or overexplores. We also suggested how a plausible solution for the game might be chosen among the set of Nash equilibrium points. More specifically, we proposed that the player with the lower risk limit would explore first and provide the other player with the free information spillover.

APPENDIX

In the first half of this appendix, we show how the posterior probabilities $\hat{\alpha}_i$ and $\hat{\alpha}_j$, as given by (10) of section 2.3, are obtained. In the second half of this appendix, we present an infinite time horizon version of the game of information spillover.

A1. THE POSTERIOR PROBABILITIES $\hat{\alpha}_i$ AND $\hat{\alpha}_j$

In section 2.3, we have defined $\hat{\alpha}_i = \hat{\alpha}_i(s_j, z_j)$ as the posterior probability that the potential deposit D_i really exists immediately after the outcome of player j's exploratory activities is revealed, given that player j is the first one to explore. However, to avoid any inconsistency that might arise, we should express the posterior probability that the potential deposit D_i really exists as a function of both (s_j, z_j) and (s_i, z_j), i.e.,

$$\text{(A1.1)} \qquad \hat{\alpha}_i = \hat{\alpha}_i(s_1, s_2, z_1, z_2)$$

$$= \text{Prob}\{D_i \text{ exists} | s_1, s_2, z_1, z_2\}$$

To obtain the posterior probability $\hat{\alpha}_i$ given by (10) of section 2.3, just set $s_i = 0$, $z_i = 0$ in (A1.1). We shall now perform the necessary computations to evaluate (A1.1).

For each $i = 1,2$, let e_i be the binary random variable such that $e_i = 1$ represents the event that D_i exists and $e_i = 0$ represents the event that D_i does not exist. We recall that the prior distribution of (e_1,e_2) is given by the matrix in part (a) of section 2.1. The random vector (e_1,e_2,s_1,s_2) might assume one of the following nine exhaustive values:

$$(e_1 = 1,\ e_2 = 1,\ s_1 = 1,\ s_2 = 1)$$
$$(e_1 = 1,\ e_2 = 1,\ s_1 = 1,\ s_2 = 0)$$
$$(e_1 = 1,\ e_2 = 1,\ s_1 = 0,\ s_2 = 1)$$
$$(e_1 = 1,\ e_2 = 1,\ s_1 = 0,\ s_2 = 0)$$
$$(e_1 = 1,\ e_2 = 0,\ s_1 = 1,\ s_2 = 0)$$
$$(e_1 = 1,\ e_2 = 0,\ s_1 = 0,\ s_2 = 0)$$
$$(e_1 = 0,\ e_2 = 1,\ s_1 = 0,\ s_2 = 1)$$
$$(e_1 = 0,\ e_2 = 1,\ s_1 = 0,\ s_2 = 0)$$
$$(e_1 = 0,\ e_2 = 0,\ s_1 = 0,\ s_2 = 0)$$

with probabilities $\alpha_{11}F_1(z_1)F_2(z_2)$, $\alpha_{11}F_1(z_1)[1-F_2(z_2)]$, $\alpha_{11}[1-F_1(z_1)]F_2(z_2)$, $\alpha_{11}[1-F_1(z_1)][1-F_2(z_2)]$, $\alpha_{12}F_1(z_1)$, $\alpha_{12}[1-F_1(z_1)]$, $\alpha_{21}F_2(z_2)$, $\alpha_{21}[1-F_2(z_2)]$, α_{22}, respectively.

First, we evaluate $\hat{\alpha}_1(s_1,s_2,z_1,z_2)$. If $s_1 = 1$, then obviously $\hat{\alpha}_1 = 1$. If $s_1 = 0$, there are two possibilities to consider: $s_2 = 1$ and $s_2 = 0$.

(a) If $s_2 = 1$, then

$$\hat{\alpha}_1(0,1,z_1,z_2) = \text{Prob}\{e_1=1 | s_1=0, s_2=1\}$$

$$= \frac{\text{Prob}\{e_1=1, s_1=0, s_2=1\}}{\text{Prob}\{s_1=0, s_2=1\}}$$

where $\text{Prob}\{e_1=1, s_1=0, s_2=1\} = \alpha_{11}[1-F_1(z_1)]F_2(z_2)$, and

$$\text{Prob}\{s_1=0, s_2=1\} = \alpha_{11}[1-F_1(z_1)]F_2(z_2) + \alpha_{21}F_2(z_2)$$

i.e.,

$$\text{(A1.2)} \qquad \hat{\alpha}_1(0,1,z_1,z_2) = \frac{\alpha_{11}[1-F_1(z_1)]}{\alpha_{11}[1-F_1(z_1)]+\alpha_{21}}$$

$$= \frac{\alpha_{11}}{\alpha_2} \text{ when } z_1 = 0$$

(b) If $s_2 = 0$, then

$$\hat{\alpha}_1(0,0,z_1,z_2) = \frac{\text{Prob}\{e_1=1 | s_1=s_2=0\}}{\text{Prob}\{s_1=s_2=0\}}$$

where $\text{Prob}\{e_1=1, s_1=s_2=0\} = \alpha_{11}[1-F_1(z_1)][1-F_2(z_2)]$, and

$$\text{Prob}\{s_1=s_2=0\} = \alpha_{11}[1-F_1(z_1)][1-F_2(z_2)] + \alpha_{12}[1-F_1(z_1)] + \alpha_{21}[1-F_2(z_2)] + \alpha_{22}$$

i.e.,

$$\text{(A1.3)} \qquad \hat{\alpha}_1(0,0,z_1,z_2) = \frac{\alpha_{11}G_2(z_2)}{\alpha_2 G_2(z_2)+\alpha_{12}+\alpha_{22}} \text{ when } z_1 = 0$$

Here $\alpha_2 = \alpha_{11} + \alpha_{21}$, $G_2(z_2) = 1 - F_2(z_2)$.

Second, to evaluate $\hat{\alpha}_2$, we can proceed in the same manner used to evaluate $\hat{\alpha}_1$ and obtain

$$\text{(A1.4)} \qquad \hat{\alpha}_2(1,0,z_1,z_2) = \frac{\alpha_{11}}{\alpha_1} \text{ when } z_2 = 0$$

$$\text{(A1.5)} \qquad \hat{\alpha}_2(0,0,z_1,z_2) = \frac{\alpha_{11}G_1(z_1)}{\alpha_1 G_1(z_1)+\alpha_{21}+\alpha_{22}} \text{ when } z_2 = 0$$

Here $\alpha_1 = \alpha_{11} + \alpha_{12}$, $G_1(z_1) = 1-F_1(z_1)$.

A2. THE FINITE TIME HORIZON VERSION OF THE GAME OF INFORMATION SPILLOVER

For each $i = 1,2$, let γ_i: $t \to \gamma_i(t)$ be a function of t such that $0 \leqq \gamma_i(t) \leqq 1$, $t = 0,1,2,\ldots$. Here we interpret $\gamma_i(t)$ as the probability that player i will decide not to explore in period t, given that neither player has explored in any period before t. As just defined, γ_i is a behavior strategy for player i in the game of information spillover when the time horizon is infinite.

Once γ_1 and γ_2 are chosen, the payoff of player i, $i = 1,2$, is completely determined and is given by

$$\text{(A2.1)} \qquad W_i(\gamma_1,\gamma_2) = \sum_{t=0}^{\infty} \frac{\pi_i^* \xi_i(t)}{(1+r)^t} + \sum_{t=0}^{\infty} \frac{\hat{\pi}_i \xi_j(t)}{(1+r)^t} .$$

Here $i,j = 1,2$ and $i \neq j$. Furthermore, π_i^* and $\hat{\pi}_i$ are as defined by (16) and (22) of section 3, respectively, and $\xi_i(t)$ is the probability that player i will be the first one to explore in period t, i.e.,

(A2.2) $$\xi_i(t) = [1-\gamma_i(t)] \prod_{s=0}^{t-1} \gamma_1(s)\gamma_2(s) \quad , \; i = 1,2$$

A pair of behavior strategies (γ_1^*, γ_2^*) is a Nash equilibrium if γ_i^* is a best reply for player i against γ_j^*, $i,j = 1,2$, $i \neq j$. To solve the infinite time horizon version of the game of information spillover, we must locate such a pair.

CHAPTER FIVE

RESOURCE EXTRACTION UNDER OLIGOPOLY: THE NONCOOPERATIVE SOLUTION

1. INTRODUCTION

The pattern of resource extraction has been investigated extensively for two extreme market structures: monopoly and perfect competition. The intermediate case in which the supply side consists of only a few firms has not received much attention. Recently, several authors such as Salant [1976, 1982], Lewis and Schmalensee [1979, 1980] have modified the conventional theory of oligopoly to study the problem of resource extraction when the number of extractive firms is small enough. In this chapter, we present some further results on the problem of resource extraction under oligopoly, adopting the same approach of the authors just mentioned.

In our model, the supply side consists of N firms, each characterized by an ordered pair $(\bar{c}_i, \bar{x}_i)$, $i = 1, \ldots, N$, where $\bar{c}_i$ and $\bar{x}_i$ represent the constant unit extraction cost and the initial fixed stock of the ith firm, respectively. Exploration is not considered in our model. The model we build in this chapter follows the differential game approach. To conform with the terminology of game theory we shall address the ith firm as player i.

We suppose that each player announces, at the beginning, an extraction strategy

$$q_i: \; t \to q_i(t) \; , \; i = 1,\ldots,N$$

satisfying his own initial fixed stock constraint. An N-tuple $q = (q_1,\ldots,q_N)$ of extraction strategies representing the joint action of the N players might not always be stable because a departure from this joint action alone by some player i might increase his profit. A Cournot-Nash equilibrium $(q_1^*,\ldots,q_N^*) = q^*$ occurs when a departure from this joint action alone by any single player will not increase his profit. Such an equilibrium, if it exists, will be taken as a noncooperative solution to our differential game of resource extraction under oligopoly.

Our main contributions in this chapter include the following results:

(i) A new and more rigorous proof of the existence of a noncooperative solution under less restrictive assumptions (Theorem 5).

(ii) A proof of the uniqueness of the noncooperative solution and an algorithm to find this solution, when suitably imposing an extra condition on the market demand curve (Theorem 6). Unlike the algorithm, used by Salant [1982] in a computerized Cournot-Nash model of the World Oil Market to locate a noncooperative solution, which does not guarantee convergence even for linear market demand curves, our algorithm, as described in Theorem 6, does guarantee monotone convergence. As a by-product of this algorithm, we also obtain an alternative proof of the existence of a noncooperative solution.

The plan of this chapter is as follows. In section 2, the differential game is presented. In section 3, we investigate an auxiliary static game which plays a fundamental role in solving the problem of resource extraction under oligopoly. Our main results are presented in section 4.

2. THE MODEL

2.1 The Market Demand Curve

Let $f: Q \to f(Q)$ denote the market demand curve as a function of industry output Q. This demand curve is assumed to satisfy the following conditions

(f1) $f(Q) > 0$, $f'(Q) < 0$ for all $Q \geq 0$

$0 < \bar{c}_i < f(0) < +\infty$, $i = 1,\ldots,N$. Here $\bar{c}_i$ is the unit extraction cost for player i .

(f2) $\pi(Q) = Qf(Q)$ is strictly concave

(f3) f^{-1}, the inverse of f, is defined for all positive prices.

The assumptions (f1), (f2), (f3) are sufficient to guarantee the existence of a noncooperative solution. We shall always assume that these assumptions are satisfied by the market demand curve. To obtain stronger results such as uniqueness or an algorithm to find the unique noncooperative solution, we need another extra assumption

(f4) $-f'(Q)/f(Q)$ is monotone increasing in Q

Many demand curves satisfy (f4), e.g., linear market demand curves.

2.2 The Noncooperative Solution

For each player i, i = 1,...,N, let $q_i: \ t \to q_i(t)$ be a feasible extraction strategy, i.e.,

q_i is a nonnegative function of time t

$$\int_0^\infty q_i(t)\,dt \leqq \bar{x}_i$$

The discounted profit enjoyed by player i under the joint action $q = (q_1,\ldots,q_N)$ is given by

$$H_i(q) = \int_0^\infty e^{-rt}[f(Q(t))-\bar{c}_i]q_i(t)\,dt \ , \ i = 1,\ldots,N$$

where r is the market rate of interest and $Q(t) = q_1(t)+\ldots+q_N(t)$.

We say that an N-tuple of feasible extraction strategies $q^* = (q_1^*,\ldots,q_N^*)$ is a noncooperative solution for the problem of resource extraction under oligopoly if for each player $i = 1,\ldots,N$,

$$H_i(q^*) \geqq H_i(q_1^*,\ldots,q_i,\ldots,q_N^*)$$

for any feasible extraction strategy q_i. In other words, q_i^* is an optimal strategy for player i when player $j \neq i$ choose q_j^*, $j = 1,\ldots,N$.

The following theorem gives a general characterization of such a noncooperative solution.

<u>Theorem 1</u>

For an N-tuple of extraction strategies $(q_1^*,\ldots,q_N^*)$ to be a noncooperative solution, it is necessary and sufficient that

(a) There exists N constants $\lambda_i > 0$, $i = 1,\ldots,N$; such that for any player i at any time t,

$$e^{-rt}[f(Q^*(t))-\bar{c}_i]q_i^*(t) - \lambda_i q_i^*(t) \geqq$$

$$e^{-rt}[f(v_i + \sum_{j\neq i} q_j^*(t))-\bar{c}_i]v_i - \lambda_i v_i$$

for any $v_i \geqq 0$. Here $Q^*(t) = q_1^*(t)+\ldots+q_N^*(t)$

(b) $\int_0^\infty q_i^*(t)\,dt = \bar{x}_i$

Furthermore, if q^* is a noncooperative solution, then the multipliers $\lambda_1,\ldots,\lambda_N$ are unique.

Proof

The proof of the necessary part can be readily obtained by invoking the Pontryagin Maximum Principle. To prove the sufficiency part, take any feasible extraction strategy q_i for player i, then by (a) we must have

$$e^{-rt}[f(Q^*(t))-\bar{c}_i]q_i^*(t) - \lambda_i q_i^*(t) \geqq$$

$$e^{-rt}[f(q_i(t) + \sum_{j\neq i} q_j^*(t))-\bar{c}_i]q_i(t) - \lambda_i(t)$$

Integrating this inequality with respect to t, and taking into account (b) as well as the stock constraint on q_i, we obtain

$$\int_0^\infty e^{-rt}[f(Q^*(t))-\bar{c}_i]q_i^*(t)\,dt \geqq$$

$$\int_0^\infty e^{-rt}[f(q_i(t) + \sum_{j\neq i} q_j^*(t))-c_i]q_i(t)\,dt$$

proving that q_i^* is optimal against q_j^*, $j\neq i$, $j = 1,\ldots,N$.

q.e.d.

3. AN AUXILIARY STATIC GAME

3.1 Statement of the Auxiliary Static Game

Let $c_1,\ldots,c_N$ be N positive numbers. We say that an N-vector $v^* = (v_1^*,\ldots,v_N^*)$ is a Cournot-Nash equilibrium for the auxiliary static game associated with costs $c_1,\ldots,c_N$, if for each player $i = 1,\ldots,N$, v_i^* is a solution to the following maximization problem

$$i \max_{v_i \geq 0} \; v_i f(v_i + \sum_{j\neq i} v_j^*) - c_i v_i$$

Theorem 2

For each given positive cost vector $c = (c_1,\ldots,c_N)$, there exists a unique Cournot-Nash equilibrium $v^* = (v_1^*,\ldots,v_N^*)$, say $v^* = v^*(c)$, for the auxiliary static game. This equilibrium is a continuous function of c and its components are given by

$$(1) \qquad \begin{aligned} v_i^* &= 0 \text{ if } f(||v^*||) \leq c_i \\ &= -[f(||v^*||)-c_i]/f'(||v^*||) \text{ if } f(||v^*||) > c_i \end{aligned}$$

Here $||v^*|| = v_1^* + \ldots + v_N^*$.

Proof

(a) Let $v = (v_1,\ldots,v_N)$, $v_i \geq 0$, $i = 1,\ldots,N$. By Lemma 1 of Chapter One, the function of $Q_i \geq 0$

$$h(\cdot, \sum_{j\neq i} v_j): \; Q_i \to h(Q_i, \sum_{j\neq i} v_j) = Q_i f(Q_i + \sum_j v_j)$$

is strictly concave. Hence the maximization problem

$$i \max_{Q_i \geq 0} Q_i f(Q_i + \sum_{j\neq i} v_j) - c_i Q_i$$

has a unique solution $Q_i = a_i(\sum_{j\neq i} v_j, c_i)$. Clearly, $0 \leq a_i(\sum_{j\neq i} v_j, c_i) \leq f^{-1}(c_i)$. Furthermore, the uniqueness of $a_i(\sum_{j\neq i} v_j, c_i)$ together with a standard continuity argument imply that a_i is continuous in its arguments.

Now let $c = (c_1, \ldots, c_N)$ be any positive cost vector. The function of v

$$A(\cdot, c): \; v \to A(v,c) = (A_1(v,c), \ldots, A_N(v,c))$$

defined by

$$A_i(v,c) = a_i(\sum_{j\neq i} v_j, c_i) \;, \; i = 1, \ldots, N$$

is clearly continuous and maps the compact, convex set $[0, f^{-1}(c_1)] \times \ldots \times [0, f^{-1}(c_N)]$ into itself. Therefore, by the Brouwer fixed point theorem, it must have a fixed point v^*. That is,

$$v^* = A(v^*, c) = (A_1(v^*, c), \ldots, A_N(v^*, c))$$

By its definition, $A_i(v,c)$ is player i's best reply to v_j, $j\neq i$, $j = 1, \ldots, N$. Hence $v^* = (v_1^*, \ldots, v_N^*)$ is a Cournot-Nash equilibrium for the auxiliary static game associated with the cost vector c.

(b) If $f(||v^*||) > c_i$, then obviously $v_i^* > 0$, and because v_i^* is optimal against v_j^*, $j \neq i$, $j = 1,\ldots,N$, the first order necessary condition gives

$$(2) \qquad v_i^* = -[f(||v^*||)-c_i]/f'(||v^*||)$$

On the other hand, if $f(||v^*||) \leqq c_i$, then clearly $v_i^* = 0$.

(c) We shall now show that v^* is unique. If $c_i \geqq f(0)$ for all $i = 1,\ldots,N$, then the only Cournot-Nash equilibrium is $v^* = 0$. So we only need to consider the case $c_i < f(0)$, for at least one player i. In this case, it is clear that any equilibrium v^* must be nonzero. Without any loss of generality, we can order the players according to their costs, i.e., $0 < c_i \leqq \ldots \leqq c_N$. By this ordering, it can be readily verified that $v_1^* \geqq \ldots \geqq v_N^*$. Hence, we can find a unique integer k such that

$$v_i^* > 0 \quad \text{if } i \leqq k$$
$$\quad = 0 \quad \text{otherwise}$$

Now, summing v_i^* over $i = 1,\ldots,k$, and using (2) as well as the fact $v_1^*+\ldots+v_k^* = v_1^*+\ldots+v_N^* = ||v^*||$, we obtain the following expression

$$(3) \qquad kf(||v^*||) + ||v^*||f'(||v^*||) = c_1 + \ldots + c_k$$

Next, let $u^* = (u_1^*, \ldots, u_N^*)$ be another Cournot-Nash equilibrium for the auxiliary static game. Then $u_1^* \geqq \ldots \geqq u_N^*$, and we can find a unique integer ℓ such that

$$u_i^* > 0 \quad \text{if } i \leqq \ell$$
$$= 0 \quad \text{otherwise} .$$

Furthermore, a version of (3) also holds for u^*, i.e.,

$$\ell f(||u^*||) + ||u^*|| \; f'(||u^*||) = c_1 + \ldots + c_\ell \tag{4}$$

There are two possibilities to consider: $k = \ell$ and $k \neq \ell$.

(i) If $k = \ell$, then (3) and (4) imply

$$kf(||v^*||)+||v^*||f'(||v^*||) = kf(||u^*||)+||u^*||f'(||u^*||) \tag{5}$$

Now, we note that the following function of $Q \geqq 0$

$$\begin{aligned} Q \to kf(Q) + Qf'(Q) &= (k-1)f(Q) + f(Q) + Qf'(Q) \\ &= (k-1)f(Q) + \pi'(Q) \end{aligned} \tag{6}$$

is strictly decreasing due to the assumed strict concavity of π. This fact together with (5) then imply

$$||v^*|| = ||u^*|| \tag{7}$$

The same argument used in (b) to evaluate v_i^* can be used to evaluate u_i^*. Therefore, we must have $u_i^* = v_i^*$, $i = 1, \ldots, N$, by virtue of (7).

(ii) If $k \neq \ell$, say $k < \ell$, then using the definitions of k and ℓ, we must have $v_\ell^* = 0$ and $u_\ell^* > 0$, i.e., $f(||v^*||) \leqq c_\ell < f(||u^*||)$, which then implies

(8) $$||u^*|| < ||v^*||$$

We have shown that the function defined by (6) is strictly decreasing. This result together with (8) then imply

(9) $$kf(||u^*||)+||u^*||f'(||u^*||)>kf(||v^*||)+||v^*||f'(||v^*||)$$
$$= c_1 + \ldots + c_k$$

The equality in (9) is due to (3).

Because $u_i^* > 0$ for $i \leqq \ell$, we must have $f(||u^*||) > c_i$, $i \leqq \ell$. Hence, if we add $(\ell-k)f(||u^*||)$ to the left side of (9) and $c_{k+1},\ldots,c_\ell$ to the right side of (9) we have the following strict inequality

(10) $$\ell f(||u^*||) + ||u^*||f'(||u^*||) > c_1 + \ldots + c_\ell$$

Clearly (10) contradicts (4). Therefore, the possibility $k \neq \ell$ cannot happen. This proves that v^* is unique for any given cost vector c.

(d) We have just shown that for any given cost vector c, there exists a unique Cournot-Nash equilibrium $v^* = v^*(c)$ for the auxiliary static game. Using the uniqueness of

$v^*(c)$ and a standard continuity argument, it can easily be shown that v^*: $c \to v^*(c)$ is continuous.

q.e.d.

<u>Theorem 3</u>

For each $c = (c_1,...,c_N) > 0$, let $v^*(c)$ be the corresponding unique Cournot-Nash equilibrium for the auxiliary static game. Then $v^*(c)$ gives the output vector of the whole industry as a function of the cost vector c and the behavior of $v^*(c)$ can be described as follows:

(a) An increase in the cost c_i of player i, ceteris paribus, decreases total industry output.

(b) A decrease in the cost c_i of player i, ceteris paribus, increases total industry output.

Mathematically, we can represent (a) and (b) by the following inequality

$$(11) \quad \sum_{j=1}^{N} \Delta v_j^*/\Delta c_i \leqq 0$$

where $\Delta v_j^* = v_j^*(c+\Delta c_i e_i) - v_j^*(c)$, $j = 1,...,N$

e_i = the N-vector with 1 at the ith position and zero elsewhere

Δc_i = the change in cost of player i.

Furthermore, (11) is strict in case (a) if $v_i^*(c) > 0$. In case (b), (11) is strict if $f(||v^*(c)||) \geqq c_i$.

Proof

(a) Let $\Delta c_i > 0$ be the increase in the cost of player i. The new cost vector is then $c+\Delta c_i e_i$, and the change in output of each player is given by

$$\Delta v_j^* = v_j^*(c+\Delta c_i e_i) - v_j^*(c) \quad , \; j = 1,\ldots,N$$

If $v_i^*(c) = 0$, then obviously the new equilibrium is the same as the old equilibrium, i.e., $\Delta v_j^* = 0$, $j = 1,\ldots,N$, and (a) is clearly true.

If $v_i^*(c) > 0$, we shall now show that $\sum_{j=1}^{N} \Delta v_j^* < 0$. The proof is by reductio ad absurdum, i.e., suppose

$$(12) \qquad \sum_{j=1}^{N} \Delta v_j^* \geqq 0$$

Now let J be the subset of players such that j is in J iff $v_j^*(c) + \Delta v_j^* > 0$. We claim that if j is in J, then it is also true that $v_j^*(c) > 0$. First, if player i is in J, then it is trivially true that $v_i^*(c) > 0$, by assumption. Next, if $j \neq i$, j belongs to J, then, using the definition of J, we must have $v_j^*(c) + \Delta v_j^* > 0$, i.e.,

$$(13) \qquad c_j < f(||v^* + \Delta v^*||) \leqq f(||v^*||)$$

In (13), $||v^*|| = \sum_{j=1}^{N} v_j^*(c)$ and $||v^*+\Delta v^*|| = \sum_{j=1}^{N} [v_j^*(c)+\Delta v_j^*]$, and the second inequality is due to assumption (12).

Clearly (13) implies that $v_j^*(c) > 0$, proving our claim.

Now, using (1) of Theorem 2 to evaluate $v_j^*(c)$ for each j in J, we have

(14) $$v_j^*(c) = -[f(||v^*||) - c_j]/f'(||v^*||)$$

Summing (14) over J, rearranging the result, we obtain the following expression

(15) $$|J|f(||v^*||) + (\sum_{j \in J} v_j^*(c))f'(||v^*||) = \sum_{j \in J} c_j$$

Here $|J|$ denotes the number of players in J.

By the definition of J, $v_j^*(c) + \Delta v_j^* > 0$ for all j in J. Hence using the same calculation to obtain (15), we have

(16) $$|J|f(||v^* + \Delta v^*||) + ||v^*+\Delta v^*||f'(||v^*+\Delta v^*||)$$

$$= \sum_{j \in J} (c_j + \delta_{ij}\Delta c_i)$$

Here $\delta_{ij} = 1$ if $i = j$

$= 0$ if $i \neq j$

If player i belongs to J, then the right side of (15) is strictly less than the right side of (16). This result together with the fact that $||v^*|| \geq \sum_{j \in J} v_j^*(c)$ imply the following strict inequality

$$(17) \qquad |J|f(||v^*||) + ||v^*||f'(||v^*||) < |J|f(||v^*+\Delta v^*||) + ||v^*+\Delta v^*||f'(||v^*+\Delta v^*||).$$

Now we recall that the function defined by (6) is strictly decreasing. Applying this result to (17), we obtain $||v^*|| > ||v^*+\Delta v^*||$: a clear contradiction to assumption (12).

On the other hand, if player i does not belong to J, then (15) and (16) are equal, and, again, using the fact that the function defined by (6) is strictly decreasing, we must have $||v^*+\Delta v^*|| = \sum_{j \in J} v_j^*(c)$. Moreover, because $v_i^*(c) > 0$ and player i does not belong to J, we must have $\sum_{j \in J} v_j^*(c) < ||v^*||$. That is $||v^*+\Delta v^*|| < ||v^*||$: contradicting (12) again. This completes the proof of part (a) of Theorem 3.

(b) To prove part (b) of Theorem 3, let $\Delta c_i < 0$, and let $\Delta v_j^* = v_j^*(c+\Delta c_i e_i)-v_j^*(c)$. Now, using the same argument as in part (a) of the proof, but in the opposite direction, i.e., from $c+\Delta c_i e_i$ to c, we obtain immediately

$$(18) \qquad \sum_{j=1}^{N} \Delta v_j^*/\Delta c_i \leq 0$$

Finally, we show that (18) is a strict inequality if $f(||v^*||) \geqq c_i$. To prove this, suppose the contrary, i.e., suppose that (18) is an equality. This means that $\sum_{j=1}^{N} \Delta v_j^* = 0$, i.e.,

$$(19) \qquad ||v^*|| = ||v^*+\Delta v^*||$$

Now, using (1) of Theorem 2 to evaluate $v_j^*(c)$ and $v_j^*(c)+\Delta v_j^*$, for all $j \neq i$, and taking into account (19), we see immediately that $\Delta v_j^* = 0$ for all $j \neq i$. This last result together with (19) imply

$$(20) \qquad \Delta v_i^* = 0$$

If $f(||v^*||) \geqq c_i$, then we also have $f(||v^*+\Delta v^*||) \geqq c_i > c_i+\Delta c_i$, by (19). Hence, using (1) of Theorem 2, we have

$$(21) \qquad v_i^*(c)+\Delta v_i^* = -\frac{[f(||v^*+\Delta v^*||)-(c_i+\Delta c_i)]}{f'(||v^*+\Delta v^*||)}$$

$$- [f(||v^*||)-c_i]/f'(||v^*||)$$

The strict inequality in (21) is obtained by noting that $\Delta c_i < 0$ and by using (19). Finally, because $f(||v^*||) \geqq c_i$, we can use (1) of Theorem 2 again to evaluate $v_i^*(c)$ and obtain

$$(22) \qquad v_i^*(c) = - [f(||v^*||)-c_i]/f'(||v^*||)$$

Taken together, (21) and (22) certainly contradict (20). The proof of part (b) of Theorem 3 is now complete.

q.e.d.

Theorem 3 asserts that a change in the cost of a single player alone has an unambiguous effect on total industry output. However, the effects on the outputs of the individual players are not clearcut. Intuitively, we expect that an increase in the cost of player i, ceteris paribus, decreases his output and increases the outputs of the others. A decrease in the cost of player i is expected to have the opposite result. Indeed, this intuition is confirmed by the following theorem if the market demand curve also satisfies the extra assumption (f4).

Theorem 4

Suppose that the market demand curve also satisfies (f4). Then an increase in the cost c_i of player i, ceteris paribus, reduces his output and increases the outputs of all the other players. A decrease in the cost c_i of player i, ceteris paribus, has the opposite effect. Mathematically,

(a) $\Delta v_i^* / \Delta c_i \leqq 0$

(b) $\Delta v_j^* / \Delta c_i \geqq 0 \ , \ j = 1,\ldots,N \ , \ j \neq i$

Furthermore, (a) is strict if initially the output $v_i^*(c)$ of player i was positive and (b) is strict if both $v_i^*(c)$ and $v_j^*(c)$ were positive.

Here $\Delta v_j^* = v_j^*(c+\Delta c_i e_i) - v_j^*(c)$, $j = 1,\ldots,N$.

<u>Proof</u>

First, we consider the case $\Delta c_i > 0$. If $v_i^*(c) = 0$, then the new equilibrium is identical with the original equilibrium and the theorem is obviously true. If $v_i^*(c) > 0$, then by (11) of Theorem 3 we must have $\sum_{j=1}^{N} \Delta v_j^* < 0$.

For any $j \neq i$, if $v_j^*(c) = 0$, then (b) is certainly true for this player. On the other hand, if $v_j^*(c) > 0$, then by (1) it is given by

$$(23) \qquad v_j^*(c) = -\frac{f(||v^*||)-c_j}{f'(||v^*||)}$$

and because $\sum_{j=1}^{N} \Delta v_j^* < 0$ we must have

$$(24) \qquad v_j^* + \Delta v_j^* = -\frac{f(||v^* + \Delta v^*||)-c_j}{f'(||v^* + \Delta v^*||)} > 0.$$

where $\Delta v^* = (\Delta v_1^*,\ldots,\Delta v_N^*)$.

Now if (f4) is also satisfied, then the function of Q

$$-\frac{f(Q)-c_j}{f'(Q)} = -\frac{1-c_j/f(Q)}{f'(Q)/f(Q)}$$

is clearly decreasing in Q. Hence, a comparison of (23) and (24) yields $\Delta v_j^* > 0$, as desired. Thus we have proved (b). To prove (a), we note that $\sum_{j=1}^{N} \Delta v_j^* < 0$ and $\Delta v_j^* \geq 0$, $j \neq i$. Hence $\Delta v_i^* < 0$.

To prove the theorem for the case $\Delta c_i < 0$, we proceed as above but starting from the initial cost vector $c + \Delta c_i e_i$.

q.e.d.

4. THE NONCOOPERATIVE SOLUTION: EXISTENCE, UNIQUENESS, AND COMPUTATION

For each N-vector $\lambda = (\lambda_1,\ldots,\lambda_N) > 0$, let

$$q_i^*(\lambda,.): \quad t \to q_i^*(\lambda,t) \quad , \; i = 1,\ldots,N$$

be the extraction strategy for player i, defined by

$$q_i^*(\lambda,t) = v_i^*(\bar{c} + \lambda e^{rt})$$

Here we recall that $\bar{c} = (\bar{c}_1,\ldots,\bar{c}_N)$ with $\bar{c}_i$, $i = 1,\ldots,N$, as the unit extraction cost of player i mentioned in section 1 and v_i^* has already been defined by Theorem 2.

By the sufficiency part of Theorem 1, the N-tuple $q^*(\lambda,.) = (q_1^*(\lambda,.),\ldots,q_N^*(\lambda,.))$ is a noncooperative solution for the problem of resource extraction under oligopoly if the initial fixed stock of each player i is equal to

$$I_i(\lambda) = \int_0^\infty q_i^*(\lambda,t)dt \quad , \; i = 1,\ldots,N$$

To show the existence and uniqueness of a noncooperative solution for the problem of resource extraction under oligopoly when player i owns the initial fixed stock $\bar{x}_i$ with unit extraction cost $\bar{c}_i$ we need some preliminary results.

Lemma 1

Let $\lambda > 0$ be any N-vector. For some player i, let $\Delta\lambda_i \neq 0$ be such that $\lambda_i + \Delta\lambda_i > 0$. Then

(a) $\sum_{j=1}^{N} \frac{\Delta I_j}{\Delta\lambda_i} \leqq 0$ with strict inequality holding if $I_i(\lambda) > 0$

(b) If the market demand curve also satisfies (f4), then

$\frac{\Delta I_j}{\Delta\lambda_i} \geqq 0$ for all $j \neq i$, with strict inequality holding if both $q_i^*(\lambda,t)$ and $q_j^*(\lambda,t)$ are positive for some time t,

$\frac{\Delta I_i}{\Delta\lambda_i} \leqq 0$, with strict inequality holding if $I_i(\lambda) > 0$.

Here $\Delta I_j = I_j(\lambda+\Delta\lambda_i e_i) - I_j(\lambda)$, $j = 1,\ldots,N$.

Proof

Let $\Delta q_j^*(t) = q_j^*(\lambda+\Delta\lambda_i e_i,t) - q_j^*(\lambda,t)$, $j = 1,\ldots,N$

then

$$\Delta I_j = \int_0^\infty \Delta q_j^*(t)dt \, .$$

The quantity $\Delta q_j^*(t)$, at each time t, behaves according to Theorems 3 and 4 as $\Delta\lambda_i$ changes. Applying these two theorems to $\Delta q_j^*(t)$, ($j = 1,\ldots,N$, at each instant t, then performing the integration to evaluate ΔI_j, we obtain (a) and (b) immediately.

q.e.d.

Lemma 2

Let $x_i > 0$, $i = 1,\ldots,N$. Then there exists at most one N-vector $\lambda = (\lambda_1,\ldots,\lambda_N) > 0$ such that

$$I_i(\lambda) = x_i \, , \quad i = 1,\ldots,N.$$

Proof

To prove Lemma 2, suppose the contrary, i.e.,

$$I(\lambda^1) = I(\lambda^2) = x = (x_1,\ldots,x_N)$$

for two N-vectors λ^1, λ^2 which are distinct from each other. We shall let $\Delta\lambda = \lambda^2 - \lambda^1 = (\Delta\lambda_1,\ldots,\Delta\lambda_N)$. Then $\Delta\lambda_i \neq 0$ for at least one i. There are two possibilities to consider

(i) $\Delta\lambda_i \geqq 0$ for all $i = 1,\ldots,N$ with strict inequality holding for at least one i. In this case, if we apply part (a) of Lemma 1 N times we obtain the following chain of inequalities with at least one strict inequality

$$||I(\lambda^1)|| \geqq ||I(\lambda^1+\Delta\lambda_1 e_1)|| \geqq ||I(\lambda^1+\Delta\lambda_1 e_1+\Delta\lambda_2 e_2)||$$

$$\geqq \ldots \geqq ||I(\lambda^1+\Delta\lambda_1 e_1 + \ldots + \Delta\lambda_N e_N)||$$

$$= ||I(\lambda^2)||$$

contradicting the assumption $I(\lambda^1) = I(\lambda^2)$.

(ii) $\Delta\lambda_i > 0$ for $i = 1,\ldots,k$ and $\Delta\lambda_i \leqq 0$ for $i = k+1,\ldots,N$. In this case, if we transform λ^1 into $\lambda^1+\Delta\lambda_1 e_1 + \ldots + \Delta\lambda_k e_k$, then the argument in (i) just presented implies

(25) $$||I(\lambda^1)|| > ||I(\lambda^1+\Delta\lambda_1 e_1 + \ldots + \Delta\lambda_k e_k)||$$

Furthermore, by (b) of Lemma 1, we also have

(26) $$I_i(\lambda^1) \leqq I_i(\lambda^1+\Delta\lambda_1 e_1 + \ldots + \Delta\lambda_k e_k)$$

for $i = k+1,\ldots,N$. Hence (25) and (26) taken together imply

(27) $$\sum_{i=1}^{k} I_i(\lambda^1+\Delta\lambda_1 e_1 + \ldots + \Delta\lambda_k e_k) < \sum_{i=1}^{k} I_i(\lambda^1).$$

Next, we transform $\lambda^1+\Delta\lambda_1 e_1 + \ldots + \Delta\lambda_k e_k$ into λ^2. Taking into account the fact that $\Delta\lambda_i \leqq 0$, $i = k+1,\ldots,N$, and using (b) of Lemma 1, we must have

(28) $$I_i(\lambda^1+\Delta\lambda_1 e_1 + \ldots + \Delta\lambda_k e_k) \geqq I_i(\lambda^2)$$

for $i = 1,\ldots,k$.

Taken together, (27) and (28) imply

$$(29) \qquad \sum_{i=1}^{k} I_i(\lambda^1) > \sum_{i=1}^{k} I_i(\lambda^2)$$

contradicting the assumption $I(\lambda^1) = I(\lambda^2)$.

q.e.d.

Finally, we are ready to give a proof of the existence and uniqueness of a noncooperative solution to the problem of resource extraction under oligopoly.

<u>Theorem 5</u>

If the market demand curve satisfies (f1), (f2), (f3), then there exists an N-vector $\bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_N) > 0$ such that

$$I(\bar{\lambda}) = \bar{x}$$

Furthermore, if the market demand curve also satisfies (f4), then by Lemma 2, $\bar{\lambda}$ is unique.

<u>Proof</u>

First, let

$$A = \{\lambda \geqq 0 \mid I_i(\lambda) \leqq \bar{x}_i \text{ for each } i = 1,\dots,N\}.$$

We claim that there exists a positive number α such that if $\lambda = (\lambda_1,\dots,\lambda_N)$ belongs to A, then $\lambda_i > \alpha$ for each

$i = 1, \ldots, N$. Indeed, if this is not the case, then we can find a sequence $(\lambda^n) = ((\lambda_1^n, \ldots, \lambda_N^n))$ such that for some index i, $\lim_{n\to\infty} \lambda_i^n = 0$. This last statement clearly implies that

$$\lim_{n\to\infty} \sum_{j=1}^{N} I_j(\lambda^n) = +\infty,$$

contradicting the definition of A.

Next, let $\lambda > 0$ and define a function of λ as follows

$$F: \lambda \to F(\lambda) = (F_1(\lambda), \ldots, F_N(\lambda))$$

where $F_i(\lambda)$ is the shadow price of the following optimal control problem

$$\max \int^{\infty} e^{-rt}[f(q_i(t) + \sum_{j\neq i} q_j^*(\lambda,t)) - \bar{c}_i]q_i(t)dt$$

subject to $q_i(t) \geqq 0$, $\int_0^{\infty} q_i(t)dt = \bar{x}_i$.

Clearly F is continuous.

Finally, let $G: \lambda \to G(\lambda) = (G_1(\lambda), \ldots, G_N(\lambda))$ be the function defined as follows

$$G_i(\lambda) = \max\{F_i(\lambda), \alpha\}, \ i = 1, \ldots, N.$$

The function G, thus defined, is continuous and maps the convex compact set $[\alpha, f(0)]^N$ into itself. By the Brouwer fixed point theorem, there exists a $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_N)$ such that $\bar{\lambda} = G(\bar{\lambda})$. We claim that $\bar{\lambda}_i > \alpha$ for each $i = 1,\ldots,N$. Indeed, if $\bar{\lambda}_i = \alpha$ for some i, then using the definition of A, we must have $I_j(\bar{\lambda}) > \bar{x}_j$ for some player j. This last strict inequality clearly implies that $F_j(\bar{\lambda}) > \bar{\lambda}_j \geq \alpha$, i.e., $G_j(\bar{\lambda}) = F_j(\bar{\lambda}) > \bar{\lambda}_j$, contradicting the result that $\bar{\lambda}$ is a fixed point of G. Therefore, we have proved that $\bar{\lambda}_i > \alpha$ for each $i = 1,\ldots,N$, which means that $\bar{\lambda}$ is also a fixed point of F. That is, $\bar{\lambda} = F(\bar{\lambda})$. Hence, $I(\bar{\lambda}) = \bar{x}$, proving the existence of a noncooperative equilibrium.

q.e.d.

Theorem 6

Suppose that the market demand curve also satisfies (f4). Next, let $\lambda^o = (\lambda_1^o,\ldots,\lambda_N^o)$ be any positive N-vector such that $I(\lambda^o) \leqq \bar{x}$. A good candidate for λ^o is readily available if we choose $\lambda_1^o = \ldots = \lambda_N^o = f(0)$.

Now, let (λ^n), $n = 0,1,\ldots$, be the sequence of N-vectors defined inductively as follows. Let $\lambda^n = (\lambda_1^n,\ldots,\lambda_N^n)$ be given. Then for each $i = 1,\ldots,N$, the ith component of λ^{n+1} is the unique positive number λ_i^{n+1} such that

$$I_i(\lambda_1^n,\ldots,\lambda_{i-1}^n, \lambda_i^{n+1}, \lambda_{i+1}^n,\ldots,\lambda_N^n) = \bar{x}_i .$$

The uniqueness of λ_i^{n+1} is guaranteed by the second strict inequality of part (b) of Lemma 1.

The sequence (λ^n), thus defined, is monotone decreasing and its limit $\bar{\lambda}$ satisfies $I(\bar{\lambda}) = \bar{x}$.

Proof

Because $I_i(\lambda^o) \leqq \bar{x}_i$, we must have $\lambda_i^1 \leqq \lambda_i^o$ by part (b) of Lemma 1. Hence, $\lambda^1 \leqq \lambda^o$. Furthermore, because $I_i(\lambda_1^o, \ldots, \lambda_{i-1}^o, \lambda_i^1, \lambda_{i+1}^o, \ldots, \lambda_N^o) = \bar{x}_i$ and $\lambda_j^1 \leqq \lambda_j^o$ for all $j \neq i$, we must have $I_i(\lambda^1) \leqq \bar{x}_i$ for each $i = 1, \ldots, N$ by part (b) of Lemma 1. Therefore, we have proved $\lambda^1 \leqq \lambda^o$ and $I(\lambda^1) \leqq I(\lambda^o) \leqq \bar{x}$. An induction argument then establishes that (λ^n) is monotone decreasing and $I(\lambda^n) \leqq \bar{x}$ for each $n = 0, 1, \ldots$. If we let $\bar{\lambda}$ be the limit of (λ^n), then $I(\bar{\lambda}) \leqq \bar{x}$. Using the fact that $I: \lambda \rightarrow I(\lambda)$ is continuous, we can readily verify that $I(\bar{\lambda}) = \bar{x}$.

q.e.d.

REFERENCES

Arrow, K.J., and Chang, S.S.L., "Optimal Pricing, Use, and Exploration of Uncertain Natural Resource Stocks," in P.T. Liu, ed., Dynamic Optimization and Applications to Economics (New York: Plenum Press, 1979).

Dasgupta, P., and Stiglitz, J.E., "Market Structure and Resource Extraction Under Uncertainty," The Scandinavian Journal of Economics, 1981, Vol. 83, No. 2, 318-33.

Gilbert, R.J., "Dominant Firm Pricing in a Market for an Exhaustible Resource," The Bell Journal of Economics, (Autumn 1978), 385-96.

__________, "Optimal Depletion of an Uncertain Stock," Review of Economic Studies, 1979, Vol. 45, 47-58.

__________, and Goldman, S., "Potential Competition and the Monopoly Price of an Exhaustible Resource," Journal of Economic Theory, 1978, Vol. 17, 319-31.

Harsanyi, J.C., "Rational Behavior and Bargaining Equilibrium in Games and Social Situations," (Cambridge University Press, 1977).

__________, "Papers in Game Theory," (D. Reidel Publishing Company, 1982).

Hoel, M., "Resource Extraction, Substitute Production, and Monopoly," Journal of Economic Theory, 1978, Vol. 19, 28-37.

Lewis, T.R., and Schmalensee, R., "Cartel Oligopoly Pricing of Nonrenewable Natural Resources," in P.T. Liu, ed., Dynamic Optimization and Applications to Economics (New York: Plenum Press, 1979).

__________, and __________, "On Oligopolistic Markets for Nonrenewable Natural Resources," Quarterly Journal of Economics, November 1980, 475-91.

Long, N.V., "Resource Extraction Under Uncertainty About Possible Nationalization," Journal of Economic Theory, 1975, Vol. 10, 42-53.

Nash, J.F., "Noncooperative Games," Annals of Mathematics, 54 (1981), 286-95.

Newbery, D.M.G., "Stochastic Limit Pricing," The Bell Journal of Economics, (Spring 1978), 260-69.

Pindyck, R.S., "Optimal Exploration and Production of a Non-renewable Resource," Journal of Political Economy, 1978, Vol. 86, 841-62.

__________, "Uncertainty and Exhaustible Resource Markets," Journal of Political Economy, 1980, Vol. 88, No. 6, 1203-25.

Saaty, T.L., "Topics in Behavioral Mathematics," Lectures given at the 1973 MAA Summer Seminar, Williams College, Williamstown, Massachusetts, Mathematical Association of America, 1973.

Salant, S.W., "Exhaustible Resources and Industrial Structure: A Nash-Cournot Approach to the World Oil Market," Journal of Political Economy, 1976, Vol. 84, No. 5, 1079-93.

__________, "Imperfect Competition in the World Oil Market: A Computerized Nash-Cournot Model," (Lexington Books, 1982).

Solow, R.M., "The Economics of Natural Resources and the Resources of Economics," American Economic Review, May 1974, 1-14.

Stone, L.D., "The Theory of Optimal Search," (Academic Press, 1975).

Weinstein, M.C., and Zeckhauser, R.J., "The Optimal Consumption of Depletable Natural Resources," Quarterly Journal of Economics, August 1975, 371-92.

Zeuthen, J.F., "Problems of Monopoly and Economic Welfare," (Routledge & Kegan Paul, 1930).

END

27 03 85

FIN