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Strong Cesaro Summability Factors

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STRONG CESARO
SUMMABILITY FACTORS

by

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ABSTRACT

This thesis is concerned with the Strong Cesaro Method of order p, $[C,k]_p$.

Necessary and sufficient conditions on a sequence $\{\varepsilon_n\}$ are given so that $\sum_{n=o}^{\infty} a_n\varepsilon_n$ is summable $[C,k]_1$ whenever $\sum_{n=o}^{\infty} a_n$ is summable $[C,k]_1$; necessary conditions on a sequence $\{\varepsilon_n\}$ are given so that $\sum_{n=o}^{\infty} a_n\varepsilon_n$ is summable $[C,k]_p$ whenever $\sum_{n=o}^{\infty} a_n$ is summable (C,k) and hence are necessary for $\sum_{n=o}^{\infty} a_n\varepsilon_n$ to be summable $[C,k]_p$. It is shown that these conditions are sufficient for $\sum_{n=o}^{\infty} a_n\varepsilon_n$ to be summable (C,k) for $k \geq 1$ and sufficient for $\sum_{n=o}^{\infty} a_n\varepsilon_n$ to be summable $[C,k]_p$ for $k = 1,2,3,\ldots$ whenever $\sum_{n=o}^{\infty} a_n$ is summable $[C,k]_p$.

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While this thesis was in preparation it was brought to my attention that Theorem 2.1 was included in a paper entitled Matrix Transformations of Strongly Summable Series by B. Kuttner and B. Thorpe which was submitted for publication in the Summer of 1974. My proof of this theorem was completely independent of theirs. Indeed, I had no idea that anyone else was working on this problem.

In this introductory chapter I will sketch their proof. Their main result, from which the proof of Theorem 2.1 is deduced, is the following Theorem.

In order that a matrix A map the convergence field of $[C,1]_p$ into that of $[C,1]_1$ it is necessary and sufficient that the following conditions hold.

(i) For each $\nu \geq 0$ there exists α_ν such that

$$2^{-n} \sum_n |a_{m-1,\nu} - \alpha_\nu| \to 0 \text{ as } n \to \infty$$

(ii) there exists a constant α such that

$$2^{-n} \sum_n \left| \sum_{\nu=0}^{\infty} a_{m-1,\nu} - \alpha \right| \to 0 \text{ as } n \to \infty$$

and (iii) (a) if $0 < p \leq 1$

$$\sum_{\mu=0}^{\infty} 2^{\frac{\mu}{p}-n} \max_\mu \left| \sum_{m \in \sigma_n} a_{m-1,\nu-1} \right| = O(1)$$

uniformly in σ_n

or (b) if $1 < p < \infty$.

$$\sum_{\mu=o}^{\infty} 2^{\frac{\mu-n}{p}} \{\sum_\mu | \sum_{m\varepsilon\sigma_n} a_{m-1,\nu-1}|^q\}^{\frac{1}{q}} = O(1)$$

uniformly in σ_n where $\frac{1}{p} + \frac{1}{q} = 1$

and σ_n is any subset of integers contained in $[2^n, 2^{n+1})$.

They use this theorem to prove the following theorem;

Let $\alpha > 0$, $\beta > 0$ and E be a matrix. Suppose that for all $x\varepsilon[C,1]_p$

$$(E\, C_{\alpha-1}^{-1})x = E\, (C_{\alpha-1}^{-1}\, x)$$

(this includes the assertion that both sides exist). Then in order that $E:[C,\alpha]_p \rightarrow [C,\beta]_1$ it is necessary and sufficient that

$$A = C_{\beta-1}\, E\, C_{\alpha-1}^{-1} \text{ maps } [C,1]_p \text{ to } [C,1]_1$$

That is (i), (ii) and (iii) (a) or (b) must hold.

They then use this last theorem to obtain the result in Theorem 2.1 by showing that the conditions on $\{\varepsilon_n\}$ imply (i), (ii) and (iii) (a) hold for $0 < k \leq 1$ and

$$a_{n\nu} = \frac{A_\nu^{k-1}}{A_n^{k-1}} \sum_{\mu=\nu}^{n} \varepsilon_\mu\, A_{\mu-\nu}^{-k-1}\, A_{n-\mu}^{k-1}.$$

CHAPTER 1

1.1 *INTRODUCTION*

Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers and $\{\varepsilon_n\}$ be a sequence of real numbers. If A and B denote two summability methods then we say $\{\varepsilon_n\}$ is a summability factor and write $\{\varepsilon_n\} \in [A:B]$ if whenever $\sum_{n=0}^{\infty} a_n$ is summable by the method A then $\sum_{n=0}^{\infty} a_n \varepsilon_n$ is summable by the method B.

The problem of finding necessary and sufficient conditions so that $\{\varepsilon_n\} \in [A:B]$ is well known. For example Bosanquet [1] gives the necessary and sufficient conditions

(i) $\varepsilon_n = O(n^{\rho-k})$

and (ii) $\sum_{n=\rho}^{\infty} n^k \,|\Delta^{k+1} \varepsilon_n| < \infty$

so that for integral values of ρ and k with $\rho \geq k$

$\{\varepsilon_n\} \in [(C,k):(C,\rho)]$.

In the same paper he gives references to other papers dealing with summability factors for ordinary Cesàro summability.

In this thesis we shall be dealing with Cesàro and Strong Cesàro methods of summability which are defined in the following way. For n an integer and k a real number set

$$A_n^k = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \dfrac{(k+1)\,(k+2)\cdots(k+n)}{n!} & n > 0, \end{cases}$$

and for k > − 1 set

$$\sigma_n^k = \frac{1}{A_n^k} \sum_{\nu=0}^{n} A_{n-\nu}^k \, a_\nu .$$

Then we say that $\sum_{n=0}^{\infty} a_n$ is summable to s by the Cesàro Method, (C,k), if

$$\lim_{n\to\infty} \sigma_n^k = s.$$

Following Hyslop [2] we say $\sum_{n=0}^{\infty} a_n$ is summable to s by the Strong Cesàro Method with index p, $[C,k]_p$ $(k > 0,\ 1 \leq p < \infty)$ if

$$\sum_{\nu=0}^{n} | \sigma_\nu^{k-1} - s |^p = o(n).$$

The symbols M and $M_i : i = 1,2,\cdots$ are used throughout the thesis to denote positive constants, independent of the variables under consideration, but not necessarily having the same value at each occurence.

We use the standard convention:

$$f = O(g) \quad \text{if } |f| < Mg$$

$$f = o(g) \quad \text{if } f/g \to 0.$$

By the symbol $\sum_r$ we mean the sum is to be taken over the interval $[2^r, 2^{r+1})$ and similarly $\max_r$ means the maximum is to be taken over $[2^r, 2^{r+1})$. If we have $\max_r X_{n,m}$ we mean the dependent variable to be the second subscript.

The theorems and lemmas in the thesis are numbered chapterwise and independently. That is, Theorem 2.1 is the first theorem of Chapter Two and similarly Lemma 2.1 is the first lemma of Chapter Two. Relations are numbered according to the chapter and section in which they occur. For example (3.2.1) is the first relation in the second section of Chapter Three.

1.2 *THE RELATIONSHIP BETWEEN THE CESARO AND STRONG CESARO METHODS*

In this section we prove the following theorem.

Theorem 1.1

A series $\sum_{n=o}^{\infty} a_n$ is summable $[C,k]_p$ $(1 \leq p < \infty)$ if, and only if, it is summable (C,k) and

$$\sum_{\nu=o}^{n} \nu^p \mid \sigma_\nu^k - \sigma_{\nu-1}^k \mid^p = o(n).$$

This theorem was first proved by Hyslop [2] and we follow his proof. In order to prove the theorem we require two lemmas the first due to Winn [3] and the second to Hyslop [2].

Lemma 1.1

If $\sum_{n=o}^{\infty} a_n$ is summable to s $[C,k]_1$ then it is summable to s (C,k).

Proof

$$\left|\sum_{\mu=o}^{n} A_{n-\mu}^{k} a_{\mu} - s A_{n}^{k}\right|$$

$$= \left|\sum_{\nu=o}^{n} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} a_{\mu} - s A_{n}^{k}\right|$$

$$= \left|\sum_{\nu=o}^{n} A_{\nu}^{k-1} \sigma_{\nu}^{k-1} - s \sum_{\nu=o}^{n} A_{\nu}^{k-1}\right|$$

$$\leq \sum_{\nu=o}^{n} A_{\nu}^{k-1} \left|\sigma_{\nu}^{k-1} - s\right|$$

$$= A_{n}^{k-1} \sum_{\nu=o}^{n} \left|\sigma_{\nu}^{k-1} - s\right| - \sum_{\nu=1}^{n} A_{\nu}^{k-2} \sum_{\mu=o}^{\nu-1} \left|\sigma_{\mu}^{k-1} - s\right|$$

$$= 0\ (n+1^{k-1})\ o\ (n) - \sum_{\nu=1}^{n} 0\ (\nu+1^{k-2})\ o\ (\nu)$$

$$= o\ (n^{k}).$$

Therefore $\left|\frac{1}{A_{n}^{k}} \sum_{\mu=o}^{n} A_{n-\mu}^{k} a_{\mu} - s\right| = o\ (1).$

That is $\sigma_{n}^{k} \to s$ as $n \to \infty$.

Proof

The case when $\varepsilon = 1$ is trivial. Otherwise

$$\frac{1}{(\nu+1)^{\varepsilon}} \sum_{\mu=o}^{\nu} (\mu+1)^{\varepsilon-1} |a_\mu|$$

$$= \frac{1}{(\nu+1)^{\varepsilon}} \{ \sum_{\mu=o}^{\nu} [(\mu+1)^{\varepsilon-1} - (\mu+2)^{\varepsilon-1}] \sum_{r=o}^{\mu} |a_r| + (\nu+2)^{\varepsilon-1} \sum_{\mu=o}^{\nu} |a_\mu| \}$$

$$\leq \frac{M_1}{(\nu+1)^{\varepsilon}} \sum_{\mu=o}^{\nu} (\mu+1)^{\varepsilon-2} \sum_{r=o}^{\mu} |a_r| + \frac{M_2}{\nu+1} \sum_{\mu=o}^{\nu} |a_\mu|$$

$$\leq \frac{M_1 m}{(\nu+1)} \sum_{\mu=o}^{\nu} (\mu+1)^{\varepsilon-1} + M_2 m$$

$$\leq M_3 m.$$

Lemma 2.2

Let $p \geq 1$. If $\sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu|^p = o(n)$

then $\sum_{\nu=o}^{n} |\frac{1}{A_\nu^{k+1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k} x_\mu|^p = o(n)$.

Proof

This is Hyslop's [2] Theorem 4.

and hence for p > 1 by Minkowski's inequality

$$\{\sum_{\nu=0}^{n} \nu^p \mid \sigma_\nu^k - \sigma_\nu^{k-1}|^p\}^{\frac{1}{p}} \leq \{k^p \sum_{\nu=0}^{n} \mid \sigma_\nu^k \mid^p\}^{\frac{1}{p}}$$

$$+ \{k^p \sum_{\nu=0}^{n} \mid \sigma_\nu^{k-1}|^p\}^{\frac{1}{p}}$$

By hypothesis the second term is $o(n^{\frac{1}{p}})$ and since $\sigma_\nu^k = o(1)$ the first term is also $o(n^{\frac{1}{p}})$.

Hence when n p > 1

$$\sum_{\nu=0}^{n} \nu^p | \sigma_\nu^k - \sigma_\nu^{k-1} |^p = o(n).$$

If p = 1,

$$\sum_{\nu=0}^{n} \nu \, |\sigma_\nu^k - \sigma_{\nu-1}^k| \leq k \sum_{\nu=0}^{n} |\sigma_\nu^k| + k \sum_{\nu=0}^{n} |\sigma_\nu^{k-1}|$$

$$= o(n).$$

Hence the conditions are necessary.

Now, suppose the two conditions hold with $\sum_{n=0}^{\infty} a_n$ summable to 0 (C,k). Writing (1.1.1) in the form

$$k \, \sigma_\nu^{k-1} = k \, \sigma_\nu^k + \nu \, \{\sigma_\nu^k - \sigma_{\nu-1}^k\},$$

when p > 1, Minkowski's inequality gives

$$\left[\sum_{\nu=0}^{n} | k \sigma_{\nu}^{k-1} |^{p}\right]^{\frac{1}{p}} \leq \left[\sum_{\nu=0}^{n} | k \sigma_{\nu}^{k} |^{p}\right]^{\frac{1}{p}} + \left[\sum_{\nu=0}^{n} \nu^{p} | \sigma_{\nu}^{k} - \sigma_{\nu-1}^{k} |^{p}\right]^{\frac{1}{p}}$$

The second term is $o(n^{\frac{1}{p}})$ and so also is the first since $\sigma_{\nu}^{k} = o(1)$.

When $p = 1$ the proof, as in the case of necessity, is obvious.

§1.3 *A SUMMABILITY FACTOR THEOREM*

This section deals with a summability factor theorem proved by Kuttner and Maddox [4] Theorem 2(b). We follow their proof. We begin by setting

$$\Delta^{k}\varepsilon_{n} = \sum_{\rho=n}^{\infty} A_{\rho-n}^{-k-1} \varepsilon_{\rho} \tag{1.3.1}$$

when the series converges and let

$$M_{r}^{k} = \max_{r} | \Delta^{k}\varepsilon_{n} |.$$

We prove the following theorem:

Theorem 1.2

$\{\varepsilon_n\} \in [[C,k]_1;(C,k)]$ if and only if

(i) $\varepsilon_n = O(1)$

and

(ii) $\sum_{r=0}^{\infty} 2^{rk} M_r^k < \infty$.

In order to prove this theorem we require the following lemmas. Lemmas 1.3, 1.4 and 1.5 are Lemmas 4, 5 and 6 in Kuttner and Maddox [4]. For $p = 1$ Lemma 1.6 is stated but not explicitly proven in their Theorem 2(b).

Lemma 1.3

Let $k > 0$. If conditions (i), (ii) of Theorem 1.2 are satisfied, then

$$\varepsilon_n = \Delta^{-k}(\Delta^k \varepsilon_n) + d, \qquad (1.3.2)$$

where d is a constant.

Proof

Supposing that $k > 0$, it has been proved by Andersen [see [5], Theorem 1] that if the series defining $\Delta^{-k}C_n$ converges for some n then it converges for all n, and that we then have

$$\Delta^{-k}C_n = \begin{cases} o(n^{1-k}) & (0 < k < 1) \\ o(1) & (k \geq 1). \end{cases} \qquad (1.3.3)$$

Further, the general solution of the equation

$$\Delta^k \varepsilon_n = C_n$$

is then $\varepsilon_n = \Delta^{-k} C_n + P_s(n)$

where $P_s(n)$ is an arbitrary polynomial of degree not exceeding s, s being the greatest integer less than k. We apply these results with $C_n = \Delta^k \varepsilon_n$, noting that condition (ii) implies that for fixed n (and hence by 1.3.3 for all n)

$$\sum_{\rho=n}^{\infty} |A_{\rho-n}^{k-1} \Delta^k \varepsilon_\rho| \leq \sum_{r=o}^{\infty} \sum_r |A_{\rho-n}^{k-1}| \Delta^k \varepsilon_\rho|$$

$$\leq M\{ \sum_{r=o}^{\infty} 2^{r(k-1)} M_r^k \sum_r 1\}$$

$$< \infty.$$

Thus

$$\varepsilon_n = \Delta^{-k}(\Delta^k \varepsilon_n) + P_s(n),$$

and it follows from (1.3.3) and (i) that $P_s(n)$ is a constant.

Lemma 1.4

Let $0 < k < 1$. Let

$$T_{n,\rho}^k = \sum_{\mu=o}^{\min(n,\rho)} A_{n-\mu}^{k-1} A_{\rho-\mu}^{k-1} A_\mu^{-k-1}$$

Then, for $n \geq 1$, $\rho \geq 1$ we have

$$T_{n,\rho}^k < 0.$$

Proof

By symmetry, it is enough to consider the case in which $\rho \geq n$. ~~Since $n \geq 1$, we have~~

$$\sum_{\mu=0}^{n} A_{n-\mu}^{k-1} \quad A_{\mu}^{-k-1} = 0. \qquad (1.3.4)$$

The term $\mu = 0$ in the sum (1.3.4) is positive, and the remaining terms are all negative. Since

$$A_{\rho-\mu}^{k-1}$$

is a positive increasing function of μ for $0 \leq \mu \leq \rho$ (and hence, *a fortiori*, for $0 \leq \mu \leq n$) the conclusion is now evident.

Lemma 1.6

Let $k > 0$. Let

$$S_{n,\rho}^{k} = \sum_{\mu=0}^{\min(n,\rho)} A_{n-\mu}^{k} \quad A_{\rho-\mu}^{k-1} \quad A_{\mu}^{-k-1}.$$

Then, for fixed k we have, uniformly for $n \geq 0$, $\rho \geq 1$,

$$S_{n,\rho}^{k} = \begin{cases} O((n+1)^{k-1}) & (n \geq \rho); \\ O((\rho+1)^{k-1}) & (\rho > n). \end{cases}$$

Proof

We remark that

$$S^k_{n,o} = A^k_n, \qquad (1.3.5)$$

so that the exclusion of the case $p = 0$ is essential.

Let us write $f^k(x,y)$ for the generating function

$$f^k(x,y) = \sum_{n=o}^{\infty} \sum_{\rho=o}^{\infty} S^k_{n,\rho}\, x^n y^\rho; \qquad (1.3.6)$$

the series (1.3.6) clearly converges absolutely for $|x| < 1$, $|y| < 1$. A straightforward calculation shows that

$$f^k(x,y) = (1 - x)^{-k-1}(1 - y)^{-k}(1 - xy)^k.$$

Hence

$$f^{k+1}(x,y) = \left[\frac{(1 - xy)}{(1 - x)(1 - y)}\right] f^k(x,y) = \left[\frac{1}{1 - x} + \frac{1}{1 - y} - 1\right] f^k(x,y).$$

Comparing coefficients we find that

$$S^{k+1}_{n,\rho} = \sum_{\nu=o}^{n} S^k_{\nu,\rho} + \sum_{\mu=o}^{\rho-1} S^k_{n,\mu} \qquad (1.3.7)$$

Using (1.3.5) to deal with the term $\mu = 0$ in the second sum, we deduce at once that if the lemma is true for a given $k > 0$ then it is true for $k + 1$. Thus it is enough to prove the lemma for the case $0 < k \leq 1$. But, for $n \geq 0$, $\rho \geq 1$,

$$S^1_{n,\rho} = 1,$$

so that the result is trivial when $k = 1$. We may therefore suppose, throughout the rest of the proof of this lemma, that $0 < k < 1$.

Now it is clear that, with the notation of Lemma 1:4,

$$\left[S^k_{n,\rho} - S^k_{n-1,\rho} \right] = T^k_{n,\rho}$$

It therefore follows from that lemma that, for fixed ρ, $S^k_{n,\rho}$ is a decreasing function of n for $n \geq 0$. Since

$$S^k_{0,\rho} = A^{k-1}_\rho$$

the truth of the lemma in the case $n \leq 2\rho$ is now evident. Let us therefore suppose that $n > 2\rho$. We write

$$S^k_{n,\rho} = A^k_n \sum_{\mu=0}^{\rho} A^{k-1}_{\rho-\mu} A^{-k-1}_\mu$$

$$+ \sum_{\mu=0}^{\rho} \{A^k_{n-\mu} - A^k_n\} A^{k-1}_{\rho-\mu} A^{-k-1}_\mu \qquad (1.3.8)$$

The first sum on the right of (1.3.8) vanishes (since $\rho \geq 1$).

Since, uniformly for $0 \leq \mu \leq \rho$, $n > 2\rho$,

$$\left[A^k_{n-\mu} - A^k_n\right] = O((n+1)^{k-1}\mu),$$

it follows easily that the second sum on the right of (1.3.8) is

$$O((n+1)^{k-1}).$$

The proof of the lemma is thus completed.

Lemma 1.6

Let $k > 0$.

(a) In order that $\{\epsilon_n\} \in [[C,k]_\rho ; (C,k)]$, $1<p<\infty$ it is sufficient that uniformly for r, $2^r \leq n < 2^{r+1}$

(i) $\sum_{s=0}^{r} 2^{s/p} (\sum_s |\alpha_{n\nu}|^q)^{\frac{1}{q}} = O(1)$

(ii) $\alpha_{n\nu} \to \alpha_{\cdot\nu}$ as $n \to \infty$

and (b) in order that $\{\epsilon_n\} \in [[C,k]_1 ;(C,k)]$ it is sufficient that

(i) for r, uniformly in $2^r \leq n < 2^{r+1}$

$$\sum_{s=0}^{r} 2^s \max_s |\alpha_{n\nu}| = O(1)$$

and that

(ii) for fixed ν, $\alpha_{n\nu}$ tends to a limit as $n \to \infty$

where

$$\alpha_{n\nu} = \begin{cases} \dfrac{A_\nu^{k-1}}{A_n^k} \displaystyle\sum_{\mu=\nu}^{n} A_{n-\mu}^k A_{\mu-\nu}^{-k-1} \varepsilon_\mu & \nu \leq n \\ 0 & \nu > n \end{cases}$$

Proof

(a) Let $1 < p < \infty$. There is clearly no loss in generality in assuming $\sum_{n=0}^{\infty} a_n = 0\ [C,k]_p$. Let

$$v_n = \frac{1}{A_n^k} \sum_{\mu=0}^{n} A_{n-\mu}^k a_\mu \varepsilon_\mu$$

$$= \frac{1}{A_n^k} \sum_{\mu=0}^{n} A_{n-\mu}^k \varepsilon_\mu \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{-k-1} A_\nu^{k-1} \sigma_\nu^{k-1}$$

$$= \frac{1}{A_n^k} \sum_{\nu=0}^{n} A_\nu^k \sum_{\mu=\nu}^{n} A_{n-\mu}^k A_{\mu-\nu}^{-k-1} \varepsilon_\mu \sigma_\nu^{k-1}$$

$$= \sum_{\nu=0}^{n} \alpha_{n\nu} \sigma_\nu^{k-1}.$$

Given $\varepsilon > 0$ since $\sum_{n=0}^{\infty} a_n = 0\ [C,k]_p$ there is an r_o such that for all $s \geq r_o$, $(\frac{1}{2^s} \sum_s |\sigma_\nu^{k-1}|^p)^{\frac{1}{p}} < \varepsilon$ and since $\alpha_{n\nu} \to \alpha_\nu$ there exists n_o such that for all $m \geq n \geq n_o$

$$\sum_{s=0}^{r_o} 2^{\frac{s}{p}} (\sum_s |\alpha_{m\nu} - \alpha_{n\nu}|^q)^{\frac{1}{q}} (\frac{1}{2^s} \sum_s |\sigma_\nu^{k-1}|^p)^{\frac{1}{p}} < \varepsilon$$

Let $2^r \leq m < 2^{r+1}$. Then

$$|\sum_{\nu=0}^{m} \alpha_{m\nu} \sigma_\nu^{k-1} - \sum_{\nu=0}^{n} \alpha_{n\nu} \sigma_\nu^{k-1}|$$

$$= |\sum_{\nu=0}^{m} (\alpha_{m\nu} - \alpha_{n\nu}) \sigma_\nu^{k-1}|$$

$$\leq \sum_{s=0}^{r} (\sum_s |\alpha_{m\nu} - \alpha_{n\nu}|^q)^{\frac{1}{q}} . (\sum_s |\sigma_\nu^{k-1}|^p)^{\frac{1}{p}}$$

$$= \sum_{s=0}^{r} 2^{\frac{s}{p}} (\sum_s |\alpha_{m\nu} - \alpha_{n\nu}|^q)^{\frac{1}{q}} (\frac{1}{2^s} \sum_s |\sigma_\nu^{k-1}|^p)^{\frac{1}{p}}$$

$$= \sum_{s=0}^{r_o} 2^{\frac{s}{p}} (\sum_s |\alpha_{m\nu} - \alpha_{n\nu}|^q)^{\frac{1}{q}} (\frac{1}{2^s} \sum_s |\sigma_\nu^{k-1}|^p)^{\frac{1}{p}}$$

$$+ \sum_{s=r_o+1}^{r} 2^{\frac{s}{p}} (\sum_s |\alpha_{m\nu} - \alpha_{n\nu}|^q)^{\frac{1}{q}} (\frac{1}{2^s} \sum_s |\sigma_\nu^{k-1}|^p)^{\frac{1}{p}}$$

$$\leq \varepsilon\, 2^{r_o} \sum_{s=o}^{r_o} (\textstyle\sum_s |\sigma_\nu^{k-1}|^p)^{\frac{1}{p}} + \varepsilon \displaystyle\sum_{s=o}^{r} 2^{s/p} (\textstyle\sum_s |\alpha_{m\nu}|^q)^{\frac{1}{q}}$$

$$+ \varepsilon \sum_{s=o}^{r} 2^{s/p} \ (\textstyle\sum_s |\alpha_{n\nu}|^q)^{\frac{1}{q}}$$

$$< M.\varepsilon.$$

(b) The proof of (b) is similar to that of (a).

Definition:

A sequence $\{n_s\}$ of positive integers is said to be *rapidly increasing* if, for some constant $C > 1$,

$$\frac{n_{s+1}}{n_s} \geq C .$$

We now prove Theorem 1.2.

Suppose, that (i) is false. Then we can find a rapidly increasing sequence $\{n_s\}$ such that

$$|\varepsilon(n_s)| \to \infty \tag{1.3.9}$$

as $s \to \infty$. Define

$$\sigma_n^{k-1} = \begin{cases} \pm \dfrac{n_s}{\varepsilon(n_s)} & (n = n_s) \\ 0 & (\text{otherwise}). \end{cases}$$

It follows from (1.3.9) that, however the signs are chosen, $\sum_{n=0}^{\infty} a_n$ is summable $[C,k]_1$ to 0. Now $a_{n_s} = \sum_{\mu=0}^{n_s} A_{n_s-\mu}^{-k-1} A_\mu^{k-1} \sigma_\mu^{k-1}$.

Suppose we have chosen the signs of σ_μ^{k-1} for $\mu < n_s$. The possible values of a_{n_s} will differ by $2A_{n_s}^{k-1} \sigma_{n_s}^{k-1}$ so we can choose the sign of $\sigma_{n_s}^{k-1}$, so that

$$|a_{n_s}| \geq A_{n_s}^{k-1} |\sigma_{n_s}^{k-1}|.$$

Hence

$$|a_{n_s} \varepsilon_{n_s}| \geq A_{n_s}^{k-1} \left| \frac{n_s}{\varepsilon_{n_s}} \cdot \varepsilon_{n_s} \right|$$

$$\geq M n_s^k.$$

It is then clear that

$$a_n\varepsilon_n \neq o(n^k),$$

so that $\sum_{n=0}^{\infty} a_n\varepsilon_n$ cannot be summable (C,k).

It remains only to prove that (ii) is necessary. We prove the equivalent result that (ii)', for any rapidly increasing sequence $\{n_s\}$

$$\sum_{s=0}^{\infty} n_s^k |\Delta^k \varepsilon(n_s)| < \infty ,$$

is necessary. Since (i) has been proved necessary, we suppose this satisfied. Let v_n denote the (C,k) mean of $\sum_{n=0}^{\infty} a_n\varepsilon_n$. Then

$$v_n = \frac{1}{A_n^k} \sum_{\mu=0}^{n} A_{n-\mu}^k \varepsilon_\mu a_\mu$$

$$= \frac{1}{A_n^k} \sum_{\mu=0}^{n} A_{n-\mu}^k \varepsilon_\mu \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{-k-1} A_\nu^{k-1} \sigma_\nu^{k-1}$$

$$= \sum_{\nu=0}^{n} \alpha_{n,\nu} \sigma_\nu^{k-1} , \tag{1.3.10}$$

where

$$\alpha_{n,\nu} = \frac{A_\nu^{k-1}}{A_n^k} \sum_{\mu=\nu}^{n} A_{n-\mu}^k A_{\mu-\nu}^{-k-1} \varepsilon_\mu . \tag{1.3.11}$$

Now let $\{n_s\}$ be any fixed rapidly increasing sequence. We consider the special case in which

$$\sigma_\nu^{k-1} = \begin{cases} n_s\beta_s & (\nu = n_s) \\ 0 & (\text{otherwise}), \end{cases}$$

so that the transformation (1.3.10) becomes

$$v_n = \sum_{s=0}^{\infty} \alpha(n,n_s)n_s\beta_s. \qquad (1.3.12)$$

~~It is clear that, whenever $\beta_s \to 0$, $\sum_{n=0}^{\infty} a_n$ is summable $[C,k]_1$;~~ thus a necessary condition for $\{\varepsilon_n\} \varepsilon [[C,k]_1;(C,k)]$ is that the transformation (1.3.12) should be such that $\{v_n\}$ converges whenever $\beta_s \to 0$. For this, by Theorem II.1, Statement 4° on page 12 of Peyerimhoff [6], it is necessary that, for each fixed s,

$$\alpha(n,n_s) \to \alpha_s$$

say, as $n \to \infty$, and that

$$\sum_{s=0}^{\infty} n_s|\alpha_s| < \infty. \qquad (1.3.13)$$

But it clearly follows from (1.3.11) and (i) that, for each fixed ν,

$$\alpha_{n,\nu} \to A_{\nu}^{k-1} \Delta^{k} \varepsilon_{\nu}$$

as $n \to \infty$; and hence (1.3.13) is equivalent to (ii)'.

Now suppose, then, that the conditions of that theorem are satisfied. It then follows from Lemma 1.4 that (1.3.2) holds. But by Lemma 1.1 a series summable $[C,k]_1$ is necessarily summable (C,k); hence any constant belongs to $[[C,k]_1;(C,k)]$. Thus it is enough to consider the case in which the constant d of (1.3.2) is equal to 0.

There is clearly no loss of generality in supposing that the $[C,k]_1$ sum of $\sum_{n=0}^{\infty} a_n$ is 0. Thus we are given that

$$\sum_r |\sigma_n^{k-1}| = o(2^r) \tag{1.3.14}$$

and we have to prove that the sequence $\{t_n\}$ converges, where

$$t_n = \frac{1}{A_n^k} \sum_{\mu=0}^{n} A_{n-\mu}^{k} a_\mu \varepsilon_\mu.$$

Now we have

$$t_n = \frac{1}{A_n^k} \sum_{\mu=0}^{n} A_{n-\mu}^{k} \varepsilon_\mu \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{-k-1} A_\nu^{k-1} \sigma_\nu^{k-1}$$

$$= \sum_{\nu=0}^{n} \alpha_{n,\nu} \sigma_\nu^{k-1}, \tag{1.3.15}$$

where

$$\alpha_{n,\nu} = \frac{A_\nu^{k-1}}{A_n^k} \sum_{\mu=\nu}^{n} A_{n-\mu}^{k} A_{\mu-\nu}^{-k-1} \varepsilon_\mu$$

$$= \frac{A_\nu^{k-1}}{A_n^k} \beta_{n,\nu}, \tag{1.3.16}$$

say. In view of Lemma 1.6 it is clear that the conclusion will follow if we show that

(a) for large r, uniformly in $2^r \leq n < 2^{r+1}$,

$$\sum_{s=0}^{r} 2^s \max_s |\alpha_{n,\nu}| = O(1) \tag{1.3.17}$$

and that

(b) for fixed $\nu, \alpha_{n,\nu}$ tends to a limit as $n \to \infty$.

We note that, expressed in terms of $\beta_{n,\nu}$, (1.3.17) and (b) are equivalent respectively to

$$\sum_{s=0}^{r} 2^{sk} \max_s |\beta_{n,\nu}| = O(2^{rk}); \qquad (1.3.18)$$

$$\beta_{n,\nu} / A_n^k \to \beta_\nu \qquad (1.3.19)$$

(say) as $n \to \infty$.

Now it follows from (1.3.2) that

$$\beta_{n,\nu} = \sum_{\mu=\nu}^{n} A_{n-\mu}^{k} A_{\mu-\nu}^{-k-1} \sum_{\rho=\mu}^{\infty} A_{\rho-\mu}^{k-1} \Delta^k \varepsilon_\rho$$

$$= \sum_{\rho=\nu}^{\infty} S_{n-\nu,\rho-\nu}^{k} \Delta^k \varepsilon_\rho, \qquad (1.3.20)$$

with the notation of Lemma 1.5. Now let $\beta_{n,\nu}^{i}$ $(i = 1,2,3)$ denote respectively the contributions to the sum (1.3.20) of the term $\rho = \nu$, the terms with $\nu + 1 \leq \rho \leq n$, and the terms with $\rho > n$. It will clearly be enough to show that (1.3.18) and (1.3.19) are satisfied with $\beta_{n,\nu}$ replaced by each of $\beta_{n,\nu}^{i}$ $(i = 1,2,3)$.

First consider the case $i = 1$. It follows from (1.3.6) that

$$\beta_{n,\nu}^{1} = A_{n-\nu}^{k} \Delta^k \varepsilon_\nu .$$

It is clear that, uniformly in $0 \le \nu \le n$, $2^r \le n < 2^{r+1}$,

$$\beta^1_{n,\nu} = O(2^{rk}\ |\Delta^k \varepsilon_\nu|)$$

and (1.3.18) therefore follows at once from (ii). Further, (1.3.19) is evident. Next, it follows from Lemma 1.5 that

$$\beta^2_{n,\nu} = O\ \{(n+1-\nu)^{k-1} \sum_{\rho=\nu+1}^{\mu} |\Delta^k \varepsilon_\rho|\}\ . \qquad (1.3.21)$$

Suppose that $2^r \le n < 2^{r+1}$; $2^s \le \nu < 2^{s-1}$ (so that $s \le r$).

If $s \le r - 2$, then $n > 2\nu$, so that

$$(n+1-\nu)^{k-1} = O((n+1)^{k-1}) = O(2^{r(k-1)}): \qquad (1.3.22)$$

hence

$$\beta^2_{n,\nu} = O(2^{r(k-1)} \sum_{t=s}^{r} 2^t M^k_t). \qquad (1.3.23)$$

If $s = r,\ r-1$, then (1.3.22) is valid only when $k \ge 1$. However, it clearly follows from (1.3.21) that

$$\beta^2_{n,\nu} = O\{(n+1-\nu)^k \max_{\nu+1 \le \rho \le n} |\Delta^k \varepsilon_\rho|\}.$$

and we deduce that (1.3.23) still holds in these cases. Hence

$$\sum_{s=0}^{r} 2^{sk} \max_s |\beta^2_{n,\nu}| = O\{2^{r(k-1)} \sum_{s=0}^{r} 2^{sk} \sum_{t=s}^{r} 2^t M^k_t\}.$$

$$= O\{2^{r(k-1)} \sum_{t=o}^{r} 2^t M_t^k \sum_{s=o}^{t} 2^{sk}\}$$

$$= O\{2^{rk} \sum_{t=o}^{r} 2^{tk} M_t^k = O(2^{rk})\},$$

by (ii). Also it follows from (1.3.23) that, for fixed ν,

$$\beta_{n,\nu}^2 = O(n^{k-1}) + O(1),$$

which gives (1.3.19) (with $\beta_\nu = 0$).

~~Finally, again by Lemma 1.5, we have, for $2^r \leq n < 2^{r+1}$,~~

$$\beta_{n,\nu}^3 = O\{ \sum_{t=r}^{\infty} M_t^k \sum (p+1-\nu)^{k-1} \}$$

where the inner sum is taken over the range

$$2^t \leq p < 2^{t+1}; \quad p \geq n + 1.$$

It thus does not exceed the same sum taken over the range

$$\nu < p < 2^{t+1}+\nu,$$

which is $O(2^{tk})$. It follows from (ii) that $\beta_{n,\nu}^3$ is bounded for all n,ν, and hence certainly satisfies (1.3.18) and (1.3.19).

CHAPTER 2

2.1 *INTRODUCTION*

In this chapter we will prove that the conditions on $\{\varepsilon_n\}$ of Theorem 1.2 are also sufficient to show that whenever $\sum_{n=o}^{\infty} a_n$ is summable $[C,k]_1$,

$$\sum_{\nu=o}^{n} \left| \frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \varepsilon_\mu \right| = o(n).$$

Combining this result with the results of Theorems 1.1 and 1.2 gives the following:

$$\{\varepsilon_n\} \in [[C,k]_1;(C,k)] \text{ if and only if}$$

$$\{\varepsilon_n\} \in [[C,k]_1;[C,k]_1].$$

2.2 *SOME LEMMAS*

Lemma 2.1

$$\text{If } \quad m = \sup_{\nu \geq o} \frac{1}{\nu+1} \sum_{\mu=o}^{\nu} |a_\mu| < \infty \text{ and } \varepsilon > o$$

then for $\nu \geq o$

$$\frac{1}{(\nu+1)^\varepsilon} \sum_{\mu=o}^{\nu} (\mu+1)^{\varepsilon-1} |a_\mu| \leq Mm$$

Proof

The case when $\varepsilon = 1$ is trivial. Otherwise

$$\frac{1}{(\nu+1)^{\varepsilon}} \sum_{\mu=0}^{\nu} (\mu+1)^{\varepsilon-1} |a_\mu|$$

$$= \frac{1}{(\nu+1)^{\varepsilon}} \{ \sum_{\mu=0}^{\nu} [(\mu+1)^{\varepsilon-1} - (\mu+2)^{\varepsilon-1}] \sum_{r=0}^{\mu} |a_r| + (\nu+2)^{\varepsilon-1} \sum_{\mu=0}^{\nu} |a_\mu| \}$$

$$\leq \frac{M_1}{(\nu+1)^{\varepsilon}} \sum_{\mu=0}^{\nu} (\mu+1)^{\varepsilon-2} \sum_{r=0}^{\mu} |a_r| + \frac{M_2}{\nu+1} \sum_{\mu=0}^{\nu} |a_\mu|$$

$$\leq \frac{M_1 m}{(\nu+1)} \sum_{\mu=0}^{\nu} (\mu+1)^{\varepsilon-1} + M_2 m$$

$$\leq M_3 m.$$

Lemma 2.2

Let $p \geq 1$. If $\sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} x_\mu|^p = o(n)$

then $\sum_{\nu=0}^{n} |\frac{1}{A_\nu^{k+1}} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k} x_\mu|^p = o(n)$.

Proof

This is Hyslop's [2] Theorem 4.

Lemma 2.3

Let $p \geq 1$, $x_n = \frac{1}{n+1} \sum_{\nu=o}^{n} y_\nu$ then

$$\sum_{\nu=o}^{n} \left| \frac{1}{A_\nu^{k+1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k} y_\mu \right|^p = o(n) \text{ if and only if}$$

$$\sum_{\nu=o}^{n} \left| \frac{1}{A_\nu^{k}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu \right|^p = o(n).$$

Proof

Since the (C,k) matrices are Hausdorff and the product method (C,k)(C,1) is equivalent to the (C,k+1) method, Peyerimhoff [6] Theorem II;22, we can apply Corollary 2 of Theorem 5 in Borwein [8] using the (C,1) matrix as P, the product matrix (C,k)(C,1) as Y and the (C,k+1) matrix as Z to obtain the if statement and then interchange Y and Z to obtain the only if part.

The following Lemma is Lemma 5 in Borwein [9].

Lemma 2.4

For $y > w > v > 0$; $0 < k < 1$

$$\int_v^w (y-t)^{k-2}\,dt < (1-k)^{-\frac{1}{2}} (w-v)(y-w)^{\frac{k}{2}-1} (y-v)^{\frac{k}{2}-1}$$

Proof

The result is obtained on taking the root of the product of the following two inequalities:

$$\int_v^w (y-t)^{k-2}\,dt < (y-v)^k \int_v^w (y-t)^{-2}\,dt$$

$$= (w-v)\,(y-w)^{-1}\,(y-v)^{k-1}$$

$$\int_v^w (y-t)^{k-2} = (1-k)^{-1} \{y-v-(y-w)\,(\tfrac{y-v}{y-w})^k\}\,(y-w)^{k-1}\,(y-v)^{-1}$$

$$< (1-k)^{-1}\,(w-v)(y-w)^{k-1}\,(y-v)^{-1}.$$

Lemma 2.5

For $0 \le \nu \le \rho$ and $0 < k < 1$

$$\Big|\sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} A_{\rho-\mu}^{k-1} \mu a_\mu\Big| \le M \{A_{\rho-\nu}^{k-1} \Big|\sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu\Big| + A_{\rho-\nu}^{\frac{k}{2}-1} \sum_{\mu=0}^{\nu} A_{\rho-\mu}^{\frac{k}{2}-1} \Big|\sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r\Big|\}.$$

Proof

By Lemma 2.4 for $y \geq w \geq v \geq 0$ and $0 < k < 1$

$$\int_v^w (y+1-t)^{k-2}dt < (1-k)^{-\frac{1}{2}}(w+1-v)(y+1-w)^{\frac{k}{2}-1}\,(y+1-v)^{\frac{k}{2}-1}$$

and hence for $n \geq m \geq v \geq 0$ we have

$$\sum_{\mu=\nu}^{m} |A_{n-\mu}^{k-2}| \leq M\,\{A_{m-\nu}^{1}\,A_{n-m}^{\frac{k}{2}-1}\,A_{n-\nu}^{\frac{k}{2}-1}\}.$$

Now for $m > \nu \geq 0$

$$\sum_{\xi=0}^{\nu} \sum_{\mu=0}^{\xi} A_{\xi-\mu}^{k-1}\mu a_\mu \sum_{\gamma=\nu+1}^{m} A_{m-\gamma}^{k-1}\,A_{\gamma-\xi}^{-k-1}$$

$$= \sum_{\mu=0}^{\nu} \mu a_\mu \sum_{\xi=\mu}^{\nu} A_{\xi-\mu}^{k-1}\left(\sum_{\gamma=\xi}^{m} A_{m-\gamma}^{k-1}\,A_{\gamma-\xi}^{-k-1} - \sum_{\gamma=\xi}^{\nu} A_{m-\gamma}^{k-1}\,A_{\gamma-\xi}^{-k-1}\right)$$

$$= \sum_{\mu=0}^{\nu} \mu a_\mu \sum_{\xi=\mu}^{\nu} A_{\xi-\mu}^{k-1}\left(A_{m-\xi}^{-1} - \sum_{\gamma=\xi}^{\nu} A_{m-\gamma}^{k-1}\,A_{\gamma-\xi}^{-k-1}\right)$$

$$= -\sum_{\mu=0}^{\nu} \mu a_\mu \sum_{\xi=\mu}^{\nu} A_{\xi-\mu}^{k-1} \sum_{\gamma=\xi}^{\nu} A_{m-\gamma}^{k-1}\,A_{\gamma-\xi}^{-k-1}$$

$$= -\sum_{\mu=0}^{\nu} \mu a_\mu \sum_{\gamma=\mu}^{\nu} A_{m-\gamma}^{k-1} \sum_{\xi=\mu}^{\gamma} A_{\xi-\mu}^{k-1}\,A_{\gamma-\xi}^{-k-1}$$

$$= - \sum_{\mu=0}^{\nu} \mu a_\mu \sum_{\gamma=\mu}^{\nu} A_{m-\gamma}^{k-1} A_{\gamma-\mu}^{-1}$$

$$= - \sum_{\mu=0}^{\nu} \mu a_\mu A_{m-\mu}^{k-1}$$

and so for $n > m \geq 0$

$$\left| \sum_{\nu=0}^{m-1} A_{n-\nu}^{k-2} \sum_{\mu=0}^{\nu} A_{m-\mu}^{k-1} \mu a_\mu \right|$$

$$\leq \sum_{\nu=0}^{m-1} |A_{n-\nu}^{k-2}| \sum_{\xi=0}^{\nu} \left| \sum_{\mu=0}^{\xi} A_{\xi-\mu}^{k-1} \mu a_\mu \right| \sum_{\gamma=\nu+1}^{m} A_{m-\gamma}^{k-1} |A_{\gamma-\xi}^{-k-1}|$$

$$\leq \sum_{\xi=0}^{m-1} \left| \sum_{\mu=0}^{\xi} A_{\xi-\mu}^{k-1} \mu a_\mu \right| \sum_{\gamma=\xi+1}^{m} A_{m-\gamma}^{k-1} |A_{\gamma-\xi}^{-k-1}| \sum_{\nu=\xi}^{\gamma-1} |A_{n-\nu}^{k-2}|$$

$$\leq M \sum_{\xi=0}^{m-1} \left| \sum_{\mu=0}^{\xi} A_{\xi-\mu}^{k-1} \mu a_\mu \right| \sum_{\gamma=\xi+1}^{m} A_{m-\gamma}^{k-1} |A_{\gamma-\xi}^{-k-1}| \, A_{\gamma-\xi}^{1} \, A_{n-\gamma}^{\frac{k}{2}-1} A_{n-\xi}^{\frac{k}{2}-1}$$

$$\leq M \sum_{\xi=0}^{m-1} \left| \sum_{\mu=0}^{\xi} A_{\xi-\mu}^{k-1} \mu a_\mu \right| A_{n-\xi}^{\frac{k}{2}-1} \sum_{\gamma=\xi}^{m} A_{m-\gamma}^{k-1} A_{\gamma-\xi}^{-k} A_{n-\gamma}^{\frac{k}{2}-1}$$

$$\leq M A_{n-m}^{\frac{k}{2}-1} \sum_{\xi=0}^{m-1} \left| \sum_{\mu=0}^{\xi} A_{\xi-\mu}^{k-1} \mu a_\mu \right| A_{n-\xi}^{\frac{k}{2}-1} A_{m-\xi}^{0} .$$

$$= M\ A_{n-m}^{\frac{k}{2}-1} \sum_{\xi=o}^{m} \left| \sum_{\mu=o}^{\xi} A_{\xi-\mu}^{k-1} \mu a_\mu \right| A_{n-\xi}^{\frac{k}{2}-1} .$$

So for $\rho \geq \nu \geq 0$

$$\left| \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} A_{\rho-\mu}^{k-1} \mu a_\mu \right| = \left| A_{\rho-\nu}^{k-1} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu + \sum_{\mu=o}^{\nu-1} (A_{\rho-\mu}^{k-1} - A_{\rho-\mu-1}^{k-1}) \sum_{r=o}^{\mu} A_{\nu-r}^{k-1} r a_r \right|$$

$$\leq A_{\rho-\nu}^{k-1} \left| \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \right| + \sum_{\mu=o}^{\nu-1} |A_{\rho-\mu}^{k-2}| \left| \sum_{r=o}^{\mu} A_{\nu-r}^{k-1} r a_r \right|$$

$$\leq M \left[A_{\rho-\nu}^{k-1} \left| \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \right| + A_{\rho-\nu}^{\frac{k}{2}-1} \sum_{\mu=o}^{\nu+1} A_{\rho-\mu}^{\frac{k}{2}-1} \left| \sum_{r=o}^{\mu} A_{\mu-r}^{k-1} r a_r \right| \right].$$

Theorem 2.1

Let $k > 0$ then

$\{\varepsilon_n\} \in [[C,k]_1;[C,k]_1]$ if and only if

(i) $\sum_{r=o}^{\infty} 2^{rk} M_r^k < \infty$

and (ii) $\varepsilon_n = O(1)$

where

$$M_r^k = \max_r |\Delta^k \varepsilon_n|.$$

Proof

Suppose $\{\varepsilon_n\} \in [[C,k]_1;[C,k]_1]$ then by Lemma 1.1, $\{\varepsilon_n\} \in [[C,k]_1;(C,k)]$.

Hence by Theorem 1.2 the conditions are necessary. By Theorem 1.2 the conditions are sufficient for $\{\varepsilon_n\} \in [[C,k];(C,k)]$, and hence by Theorem 1.1, since $|\nu\{\sigma_\nu^k - \sigma_{\nu-1}^k\}| = \frac{1}{A_\nu^k}|\sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1}\mu a_\mu|$ we need only show that (i) and (ii) together with

$$\sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1}\mu a_\mu| = o(n)$$

imply

$$\sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1}\mu a_\mu \varepsilon_\mu| = o(n).$$

By Lemma 1.3 we may suppose

$$\varepsilon_n = \Delta^{-k} (\Delta^k \varepsilon_n)$$

$$= \sum_{\rho=n}^{\infty} A_{\rho-n}^{k-1} \Delta^k \varepsilon_\rho.$$

We first note that condition (ii) implies that

$$\sum_{\mu=0}^{\infty}(\mu+1)^{k-1}|\Delta^k\varepsilon_\mu| = \sum_{r=0}^{\infty} \sum_r(\mu+1)^{k-1}|\Delta^k\varepsilon_\mu|$$

$$\leq \sum_{r=0}^{\infty} M_r^k \sum_r(\mu+1)^{k-1}$$

$$\leq 2\sum_{r=0}^{\infty} 2^{rk} M_r^k < \infty$$

and so $\mu^k|\Delta^k\varepsilon_\mu| = o(1)$.

Case 1

Let k = 1.

$$\sum_{\nu=0}^{n}\left|\frac{1}{\nu+1}\sum_{\mu=0}^{\nu}\mu a_\mu\varepsilon_\mu\right| = \sum_{\nu=0}^{n}\left|\frac{1}{\nu+1}\sum_{\mu=0}^{\nu}\mu a_\mu\sum_{\rho=\mu}^{\infty}\Delta^k\varepsilon_\rho\right|$$

$$\leq \sum_{\nu=0}^{n}\left|\frac{1}{\nu+1}\sum_{\mu=0}^{\nu}\mu a_\mu\sum_{\rho=\mu}^{\nu}\Delta^k\varepsilon_\rho\right| + \sum_{\nu=0}^{n}\left|\frac{1}{\nu+1}\sum_{\mu=0}^{\nu}\mu a_\mu\sum_{\rho=\nu+1}^{\infty}\Delta^k\varepsilon_\rho\right|$$

$$\leq \sum_{\nu=0}^{n}\left|\frac{1}{\nu+1}\sum_{\rho=0}^{\nu}\Delta^k\varepsilon_\rho\sum_{\mu=0}^{\rho}\mu a_\mu\right| + \sum_{\nu=0}^{n}\frac{1}{\nu+1}\left|\sum_{\rho=\nu+1}^{\infty}\Delta^k\varepsilon_\rho\right|\left|\sum_{\mu=0}^{\nu}\mu a_\mu\right|$$

$$\leq \sum_{\nu=0}^{n} \frac{1}{\nu+1} \sum_{\rho=0}^{\nu} |\Delta^k \varepsilon_\rho| |\sum_{\mu=0}^{\rho} \mu a_\mu| + \sum_{\nu=0}^{n} \frac{1}{\nu+1} \sum_{\rho=\nu+1}^{\infty} |\Delta^k \varepsilon_\rho| |\sum_{\mu=0}^{\nu} \mu a_\mu|$$

$$\leq M \left[\sum_{\nu=0}^{n} \frac{1}{\nu+1} \sum_{\rho=0}^{\nu} \frac{1}{\rho+1} |\sum_{\mu=0}^{\rho} \mu a_\mu| + \sum_{\nu=0}^{n} \frac{1}{\nu+1} |\sum_{\mu=0}^{\nu} \mu a_\mu| \right].$$

$$= o(n).$$

Case 2

Let $0 < k < 1$.

$$\sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \varepsilon_\mu| = \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \sum_{\rho=\mu}^{\infty} A_{\rho-\mu}^{k-1} \Delta^k \varepsilon_\rho|$$

$$\leq \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\rho=0}^{\nu} \Delta^k \varepsilon_\rho \sum_{\mu=0}^{\rho} A_{\rho-\mu}^{k-1} A_{\nu-\mu}^{k-1} \mu a_\mu|$$

$$+ \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\rho=\nu}^{\infty} \Delta^k \varepsilon_\rho \sum_{\mu=0}^{\nu} A_{\rho-\mu}^{k-1} A_{\nu-\mu}^{k-1} \mu a_\mu|$$

$$\leq M \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \sum_{\rho=0}^{\nu} (\rho+1)^{-k} |\sum_{\mu=0}^{\rho} A_{\rho-\mu}^{k-1} A_{\nu-\mu}^{k-1} \mu a_\mu|$$

$$+ \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} |\sum_{\rho=\nu}^{\infty} \Delta^k \varepsilon_\rho \sum_{\mu=0}^{\nu} A_{\rho-\mu}^{k-1} A_{\nu-\mu}^{k-1} \mu a_\mu|$$

$$= M \sum\nolimits_1 + \sum\nolimits_2.$$

Now by Lemma 2:5

$$\sum_1 \leq M\left[\sum_{\nu=o}^{n} \frac{1}{A_\nu^k} \sum_{\rho=o}^{\nu} (\rho+1)^{-k} A_{\nu-\rho}^{k-1} \left|\sum_{\mu=o}^{\rho} A_{\rho-\mu}^{k-1} \mu a_\mu\right|\right.$$

$$\left. + \sum_{\nu=o}^{n} \frac{1}{A_\nu^k} \sum_{\rho=o}^{\nu} (\rho+1)^{-k} A_{\nu-\rho}^{\frac{k}{2}-1} \sum_{\mu=o}^{\rho} A_{\nu-\mu}^{\frac{k}{2}-1} \left|\sum_{r=o}^{\mu} A_{\mu-r}^{k-1} \mu a_\mu\right|\right]$$

$$= M\left[\sum_1^1 + \sum_1^2\right]$$

Now

$$\sum_1^1 = \sum_{\nu=o}^{n} \frac{1}{A_\nu^k} \sum_{\rho=o}^{\nu} (\rho+1)^{-k} A_{\nu-\rho}^{k-1} \left|\sum_{\mu=o}^{\rho} A_{\rho-\mu}^{k-1} \mu a_\mu\right|$$

$$= \sum_{\rho=o}^{n} (\rho+1)^{-k} \left|\sum_{\mu=o}^{\rho} A_{\rho-\mu}^{k-1} \mu a_\mu\right| \sum_{\nu=\rho}^{n} \frac{A_{\nu-\rho}^{k-1}}{A_\nu^k}$$

$$\leq (n+1)^\varepsilon \sum_{\rho=o}^{n} (\rho+1)^{-k} \left|\sum_{\mu=o}^{\rho} A_{\rho-\mu}^{k-1} \mu a_\mu\right| \sum_{\nu=\rho}^{\infty} \frac{A_{\nu-\rho}^{k-1}}{(\nu+1)^\varepsilon A_\nu^k} \qquad 0 < \varepsilon < 1$$

$$\leq M(n+1)^\varepsilon \sum_{\rho=o}^{n} (\rho+1)^{-k} \left|\sum_{\mu=\rho}^{\rho} A_{\rho-\mu}^{k-1} \mu a_\mu\right| \int_\rho^\infty (t+1-\rho)^{k-1} (t+1)^{-k-\varepsilon} dt$$

$$= M(n+1)^\varepsilon \sum_{\rho=o}^{n} (\rho+1)^{-k} \left|\sum_{\mu=o}^{\rho} A_{\rho-\mu}^{k-1} \mu a_\mu\right| (\rho+1)^{-\varepsilon}$$

$$= M(n+1)^{\varepsilon} \sum_{\rho=0}^{n} (\rho+1)^{-k-\varepsilon} \left| \sum_{\mu=0}^{\rho} A_{\rho-\mu}^{k-1} \mu a_{\mu} \right|$$

$$= o(n) \quad \text{since by Lemma 2:1}$$

$$\overline{\lim_{n\to\infty}} (n+1)^{\varepsilon-1} \sum_{\rho=0}^{n} (\rho+1)^{-k-\varepsilon} \left| \sum_{\mu=0}^{\rho} A_{\rho-\mu}^{k-1} \mu a_{\mu} \right|$$

$$= \overline{\lim_{n\to\infty}} \frac{1}{(n+1)^{1-\varepsilon}} \sum_{\rho=0}^{n} (\rho+1)^{(1-\varepsilon-1)} (\rho+1)^{-k} \left| \sum_{\mu=0}^{\rho} A_{\rho-\mu}^{k-1} \mu a_{\mu} \right|$$

$$= \overline{\lim_{n\to\infty}} \frac{1}{n+1} \sum_{\rho=0}^{n} (\rho+1)^{-k} \left| \sum_{\mu=0}^{\rho} A_{\rho-\mu}^{k-1} \mu a_{\mu} \right| \quad \text{since } 1 - \varepsilon > 0$$

$$= 0.$$

$$\Sigma_1^2 = \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \sum_{\rho=0}^{\nu} (\rho+1)^{-k} A_{\nu-\rho}^{\frac{k}{2}-1} \sum_{\mu=0}^{\rho} A_{\nu-\mu}^{\frac{k}{2}-1} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right|$$

$$= \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{\frac{k}{2}-1} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right| \sum_{\rho=\mu}^{\nu} (\rho+1)^{-k} A_{\nu-\rho}^{\frac{k}{2}-1}$$

$$= \sum_{\mu=0}^{n} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right| \sum_{\nu=\mu}^{n} \frac{A_{\nu-\mu}^{\frac{k}{2}-1}}{A_\nu^k} \sum_{\rho=\mu}^{\nu} (\rho+1)^{-k} A_{\nu-\rho}^{\frac{k}{2}-1}$$

$\sum_1^2 = O(n)$ will follow if we prove that

$$\sum_{\nu=\mu}^{\infty} \frac{A_{\nu-\mu}^{\frac{1}{2}k-1}}{A_{\nu}^{k}} \sum_{\rho=\mu}^{\nu} (\rho+1)^{-k} A_{\nu-\rho}^{\frac{1}{2}k-1} = O\{(\mu+1)^{-k}\}. \qquad (2.2.1)$$

For $\nu \leq 2\mu$, the inner sum is

$$O\{(\mu+1)^{-k} \sum_{\rho=\mu}^{\nu} A_{\nu-\rho}^{\frac{1}{2}k-1}\} = O\{(\mu+1)^{-k} A_{\nu-\mu}^{\frac{1}{2}k}\}$$

For $\nu > 2\mu$, the inner sum is

$$O\{(\nu+1)^{\frac{1}{2}k-1} \sum_{\mu<\rho\leq\frac{1}{2}\nu} (\rho+1)^{-k}\} + O\{(\nu+1)^{-k} \sum_{\frac{1}{2}\nu<\rho<\nu} A_{\nu-\rho}^{\frac{1}{2}k-1}\}$$

$$= O\{(\nu+1)^{-\frac{1}{2}k}\}$$

The result (2.2.1) now follows

Hence $\sum_1 = o(n)$.

Also by Lemma 2.5

$$\Sigma_2 = \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \left| \sum_{\rho=\nu}^{\infty} \Delta^k \varepsilon_\rho \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} A_{\rho-\mu}^{k-1} \mu a_\mu \right|$$

$$= O\left[\sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \left| \sum_{\rho=\nu}^{\infty} \Delta^k \varepsilon_\rho A_{\rho-\nu}^{k-1} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \right| \right.$$

$$\left. + \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \sum_{\rho=\nu}^{\infty} |\Delta^k \varepsilon_\rho| A_{\rho-\nu}^{\frac{k}{2}-1} \sum_{\mu=0}^{\nu} A_{\rho-\mu}^{\frac{k}{2}-1} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right| \right]$$

$$= O\left[\Sigma_2^1 + \Sigma_2^2 \right].$$

Now

$$\Sigma_2^1 = \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \left| \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \sum_{\rho=\nu}^{\infty} A_{\rho-\nu}^{k-1} \Delta^k \varepsilon_\rho \right|$$

$$= O\left[\sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \left| \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \right| \right]$$ since the inner sum is

$\Delta^{-k}(\Delta^k \varepsilon_\nu) = O(1)$ by 1.3.2 and ii

$= o(n)$ by assumption.

Finally

$$\sum\nolimits_2^2 = \sum_{\nu=0}^{n} \frac{1}{A_\nu^k} \sum_{\rho=\nu}^{\infty} |\Delta^k \varepsilon_\rho| A_{\rho-\nu}^{\frac{k}{2}-1} \sum_{\mu=0}^{\nu} A_{\rho-\mu}^{\frac{k}{2}-1} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right|$$

$$\leq \sum_{\mu=0}^{n} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right| \sum_{\nu=\mu}^{\infty} \frac{1}{A_\nu^k} \sum_{\rho=\nu}^{\infty} A_{\rho-\nu}^{\frac{k}{2}-1} A_{\rho-\mu}^{\frac{k}{2}-1} (\rho+1)^k$$

$$= 0\left[\sum_{\mu=0}^{n} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right| \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{\frac{1}{2}k-1} (\nu+1)^{\frac{1}{2}k} \right]$$

$$= 0\left[\sum_{\mu=0}^{n} \frac{1}{(\mu+1)^k} \left| \sum_{r=0}^{\mu} A_{\mu-r}^{k-1} r a_r \right| \right]$$

$= o(n)$ by assumption.

Therefore

$\sum_2 = o(n)$ and hence

$$\sum_{\nu=0}^{\mu} \left| \frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \varepsilon_\mu \right| = o(n).$$

Case 3

$k > 1$

We note that in Cases 1 and 2 we only used the conditions that $\varepsilon_n = \sum_{\rho=n}^{\infty} A_{\rho-n}^{k-1} h_\rho$ where $\{h_\rho\}$ is a sequence

satisfying the conditions $(\rho+1)^{k}|h_\rho| = O(1)$

and $\sum_{\rho=o}^{\infty} (\rho+1)^{k-1}|h_\rho| = O(1)$.

Suppose the Theorem to be true with k replaced by k-1

i.e. we assume that if $\{k_n\}$ is a sequence such that

$$k_n = \sum_{\rho=n}^{\infty} A_{\rho-n}^{k-2} h_\rho \text{ where } \sum_{\rho=o}^{\infty} (\rho+1)^{k-2}|h_\rho| = O(1)$$

and $$(\rho+1)^{k-1}|h_\rho| = O(1)$$

and if $$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} \mu a_\mu\right| = o(n) \quad \text{then we have}$$

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} \mu a_\mu k_\mu\right| = o(n).$$

Let $P_n = \sum_{\rho=n}^{\infty} A_{\rho-n}^{k-2}(\rho+k)h_\rho$.

Then $$\sum_{\mu=o}^{\infty} (\mu+1)^{k-2}|(\mu+k)h_\mu| = O\left[\sum_{\mu=o}^{\infty} (\mu+1)^{k-1}|h_\mu|\right]$$

$$< \infty$$

and

$$(\mu+1)^{k-1}|(\mu+k)h_\mu| = 0\left[(\mu+1)^k|h_\mu|\right]$$

$$= 0(1).$$

Hence $\{P_n\}$ satisfies the conditions with k replaced by k-1.

We also note that

$$\varepsilon_{n+1} = \sum_{\rho=n+1}^{\infty} A^{k-1}_{\rho-n-1} h_\rho$$

$$= \sum_{\rho=n+1}^{\infty} h_\rho \sum_{\mu=n+1}^{\rho} A^{k-2}_{\mu-n-1}$$

$$= \sum_{\mu=n+1}^{\infty} A^{k-2}_{\mu-n-1} \sum_{\rho=\mu}^{\infty} h_\rho \, ,$$

the interchange being justified by the convergence of

$$\sum_{n=0}^{\infty} (\mu+1)^{k-1}|h_\mu|.$$

$$\sum_{\mu=0}^{\infty} (\mu+1)^{k-2}\left|\sum_{\rho=\mu}^{\infty} h_\rho\right|$$

$$\leq \sum_{\rho=0}^{\infty} |h_\rho| \sum_{\mu=0}^{\rho} (\mu+1)^{k-2}$$

$$\leq \sum_{\rho=0}^{\infty} (\rho+1)^{k-1}|h_\rho| < \infty$$

and $$(\mu+1)^{k-1}\left|\sum_{\rho=\mu}^{\infty} h_\rho\right|$$

$$\leq \sum_{\rho=o}^{\infty} (\rho+1)^{k-1}|h_\rho|$$

$$< \infty$$

Therefore $\{\varepsilon_{n+1}\}$ also satisfies the conditions with k replaced by $k-1$.

Further

$$\sum_{\mu=o}^{n} \mu a_\mu \varepsilon_\mu = \varepsilon_{n+1} \sum_{r=o}^{n} ra_r + \sum_{\mu=o}^{n} (\varepsilon_\mu - \varepsilon_{\mu+1}) \sum_{r=o}^{\mu} ra_r$$

$$= \varepsilon_{n+1} \sum_{r=o}^{n} ra_r + \sum_{\mu=o}^{n} \left(\sum_{r=o}^{\mu} ra_r\right) \sum_{\rho=\mu}^{\infty} A_{\rho-\mu}^{k-2} h_\rho$$

$$= \varepsilon_{n+1} \sum_{r=o}^{n} ra_r + \sum_{\mu=o}^{n} \left(\sum_{r=o}^{\mu} ra_r\right) \sum_{\rho=\mu}^{\infty} \frac{1}{\mu+1}\left[A_{\rho-\mu}^{k-1}(\rho+k) + (1-k)\, A_{\rho+\mu}^{k-1}\right] h_\rho$$

$$= (n+1)x_n\varepsilon_{n+1} + \sum_{\mu=o}^{n} P_\mu x_\mu + (1-k)\sum_{\mu=o}^{n} \varepsilon_\mu x_\mu$$

where $x_n = \frac{1}{n+1}\sum_{r=o}^{n} ra_r$.

Therefore

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k}\sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1}\mu a_\mu \varepsilon_\mu\right|$$

$$= \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k}\sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2}\sum_{\rho=o}^{\mu} \rho a_\rho \varepsilon_\rho\right|$$

$$= \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k}\sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2}\left\{(\mu+1)x_\mu\varepsilon_{\mu+1} + \sum_{r=o}^{\mu} x_r P_r + (1-k)\sum_{r=o}^{\mu} x_r\varepsilon_r\right\}\right|$$

$$\leq \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k}\sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2}(\mu+1)x_\mu\varepsilon_{\mu+1}\right|$$

$$+ \sum_{\nu=o}^{\mu} \left|\frac{1}{A_\nu^k}\sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r P_r\right|$$

$$+ (k-1)\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k}\sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r \varepsilon_r\right|$$

Given that

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} r a_r\right| = o(n)$$

we have by Lemma (2.3) that

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{r=o}^{\nu} A_{\nu-r}^{k-2} x_r\right| = o(n)$$

and hence by our induction hypothesis that

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{r=o}^{\nu} A_{\nu-r}^{k-2} x_r P_r\right| = o(n).$$

Hence by Lemma (2.2)

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r P_r\right| = o(n).$$

Similarly

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{r=o}^{\nu} A_{\nu-r}^{k-2} x_r \varepsilon_r\right| = o(n).$$

and

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r \varepsilon_r\right| = o(n).$$

Further

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2}(\mu+1)\, x_\mu \varepsilon_{\mu+1}\right|$$

$$= \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} \{(k+\nu)\, A_{\nu-\mu}^{k-2} - (k-1)\, A_{\nu-\mu}^{k-1}\}\, x_\mu \varepsilon_{\mu+1}\right|$$

$$\leq \sum_{\nu=o}^{n} \left|\frac{k+\nu}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} x_\mu \varepsilon_{\mu+1}\right|$$

$$+ \; (k-1) \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu \varepsilon_{\mu+1}\right|$$

$$= \; k \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} x_\mu \varepsilon_{\mu+1}\right|$$

$$+ \; (k-1) \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu \varepsilon_{\mu+1}\right|.$$

And so

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2}(\mu+1)\, x_\mu \varepsilon_{\mu+1}\right| = o(n).$$

Therefore

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \varepsilon_\mu\right| = o(n)$$

and the Theorem is proved.

CHAPTER 3

3.1 *INTRODUCTION*

In this chapter we deal with the case where $p > 1$. We develop necessary conditions for $\{\varepsilon_n\} \in [[C,k]_p;(C,k)]$ for $k > 0$ and show that these conditions are sufficient for $\{\varepsilon_n\} \in [[C,k]_p;(C,k)]$ for $k \geq 1$ and for $k = 1,2,3,\cdots$ they are sufficient for $\{\varepsilon_n\} \in [[C,k]_p;[C,k]_p]$. Hence if $k = 1,2,3\cdots$ we have

$$\{\varepsilon_n\} \in [[C,k]_p;(C,k)] \text{ if and only if } \{\varepsilon_n\} \in [[C,k]_p;[C,k]_p].$$

3.2 *THE NECESSITY*

Lemma 3.1 is Theorem 66, page 122 in Hardy [7] while Definitions 3.1 and 3.2 together with Lemma 3.2 are found in Borwein [10].

Lemma 3.1

$\sum_{n=0}^{\infty} x_n$ is summable (C,k) if and only if

$$\sum_{n=1}^{\infty} (nA_n^k)^{-1} \sum_{\nu=0}^{n} A_{n-\nu}^{k-1} \nu x_\nu \text{ is convergent.}$$

Definition 3.1

w_p is the space of real sequences $x = \{x_n\}$ for which there is a number $\ell = \ell_x$ such that

$$\sum_{n=0}^{N} |x_n - \ell|^p = o(N),$$

with norm

$$||x|| = \sup_{N\geq 0} \{\frac{1}{N+1} \sum_{n=0}^{N} |x_n|^p\}^{\frac{1}{p}}.$$

Definition 3.2

Given any real sequence $a = \{a_n\}$ we define a sequence $\{m_n(a,p)\}$ by:

$$m_n(a,p) = \begin{cases} \sup\limits_{2^n \leq \nu < 2^{n+1}} |\nu a_\nu| & \text{if } p = 1, \\ \left(\dfrac{1}{2^n} \sum\limits_{\nu=2^n}^{2^{n+1}-1} |\nu a_\nu|^q \right)^{\frac{1}{q}} & \text{if } p > 1. \end{cases}$$

Definition 3.3

χ_k is the normed linear space of all sequences $x = \{x_n\}$ such that $\sum_{n=0}^{\infty} x_n$ is summable $[C,k]_p$ with norm

$$||x|| = \sup_{n \geq 0} \left(\frac{1}{n+1} \sum_{\nu=0}^{\mu} |\sigma_\nu^{k-1}|^p \right)^{\frac{1}{p}}.$$

χ_k^a is the subspace of χ_k consisting of those sequences $x = \{x_n\}$ such that x_n is eventually zero

(i) $x_n = 0$ for $n < a$

and

(ii) $\sum_{n=0}^{\infty} n|x_n| < \infty$.

Lemma 3.2

(i) If f is a linear functional on w_p, then there is a real number α and a real sequence $a = \{a_n\}$ such that

$$f(x) = \alpha l_x + \sum_{n=1}^{\infty} a_n x_n \qquad (3.2.1)$$

for every $x = \{x_n\} \varepsilon w_p$ and

$$\sum_{n=o}^{\infty} m_n(a,p) < \infty. \qquad (3.2.2)$$

(ii) If α is a real number and $a = \{a_n\}$ is a real sequence satisfying (3.2.2), then (3.2.1) defines a linear functional f on w_p with

$$||f|| \leq |\alpha| + 2^{\frac{1}{p}} \sum_{n=o}^{\infty} m_n(a,p),$$

and the series in (3.2.1) is absolutely convergent for every $x = \{x_n\} \varepsilon w_p$.

Lemma 3.3

If $\sum_{n=o}^{\infty} x_n \varepsilon_n$ is summable (C,k) for some $k \geq 1$ whenever $x \varepsilon \chi_k$ then $f(x) = \sum_{n=o}^{\infty} x_n \varepsilon_n$ defines a continuous linear functional on χ_k^C for some $C \geq 1$.

Proof

Suppose not. We define a sequence $\{n_s\}$ of integers and a sequence $\{x^s\}$ of sequences as follows.

Let $n_o = 1$ and suppose $n_1, n_2, \cdots, n_{s-1}, x^1, \cdots, x^{s-1}$ have been chosen so that $x^r \varepsilon \chi_k^{n_{r-1}}$ for $r = 1, 2, \cdots, s-1$.

Since f is not continuous in $\chi_k^{n_{s-1}}$ there is an $x^s \varepsilon \chi_k^{n_{s-1}}$ such that $||x^s|| < 2^{-s}$ and $f(x^s) \geq 1$.

Define $n_s = 2n_{s-1} + \left[\sum_{\mu=1}^{s} \sum_{\nu=1}^{\infty} \nu |x_\nu^\mu \varepsilon_\nu| \right]$ where [g] denotes the integer part of g.

Now define a sequence $x = \{x_n\}$

by $x_m = x_m^1 + \cdots + x_m^s$ when $1 \leq m \leq n_s$

and $s = 1, 2, \cdots$.

Then for any $k \geq 1$ and $s = 1,2,\cdots$

$$\left|\sum_{n=1}^{n_s} (nA_n^k)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{k-1} \nu x_\nu \varepsilon_\nu\right|$$

$$= \left|\sum_{\mu=1}^{s} \sum_{n=1}^{n_s} (nA_n^k)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{k-1} \nu x_\nu^\mu \varepsilon_\nu\right|$$

$$\geq \sum_{\mu=1}^{s} \sum_{n=1}^{\infty} (nA_n^k)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{k-1} \nu x_\nu \varepsilon_\nu - \sum_{\mu=1}^{s} \sum_{n=n_s}^{\infty} (nA_n^k)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{k-1} \nu \, |x_\nu \varepsilon_\nu|$$

$$\geq \sum_{\mu=1}^{s} \sum_{\nu=1}^{\infty} \nu x_\nu \varepsilon_\nu \sum_{n=\nu}^{\infty} (nA_n^k)^{-1} A_{n-\nu}^{k-1} - \sum_{\mu=1}^{s} \sum_{\nu=1}^{\infty} \nu |x_\nu \varepsilon_\nu| \sum_{n=n_s}^{\infty} (nA_n^k)^{-1} A_{n-\nu}^{k-1}$$

$$\geq M_1 \left(\sum_{\mu=1}^{s} \sum_{\nu=1}^{\infty} x_\nu \varepsilon_\nu \right) - M_1 \left(\sum_{\mu=1}^{s} \sum_{\nu=1}^{\infty} \nu |x_\nu \varepsilon_\nu| \sum_{n=n_s}^{\infty} n^{-2} \right)$$

$$\geq M_2 \sum_{\mu=1}^{s} f(x^\mu) - M_3 \sum_{\mu=1}^{s} \frac{1}{n_s} \sum_{\nu=1}^{\infty} \nu |x_\nu \varepsilon_\nu|$$

$$\geq M_2 s - M_3.$$

$\sum_{n=0}^{\infty} x_n \varepsilon_n$ is not bounded (C,k) for any $k \geq 1$ and hence is not summable (C,k) for $k \geq 1$. However

$$\overline{\lim_{n\to\infty}}\left(\frac{1}{n}\sum_{\nu=0}^{n}\left|\frac{1}{A_\nu^k}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}\mu x_\mu\right|^p\right)^{\frac{1}{p}}$$

$$\leq \overline{\lim_{n\to\infty}}\sum_{r=1}^{s}\left(\frac{1}{n}\sum_{\nu=0}^{n}\left|\frac{1}{A_\nu^k}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}\mu x_\mu^r\right|^p\right)^{\frac{1}{p}}$$

$$+ \overline{\lim_{n\to\infty}}\sum_{r=s+1}^{\infty}\left(\frac{1}{n}\sum_{\nu=0}^{n}\left|\frac{1}{A_\nu^k}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}\mu x_\mu^r\right|^p\right)^{\frac{1}{p}}$$

$$= \overline{\lim_{n\to\infty}}\sum_{r=s+1}^{\infty}\left(\frac{1}{n}\sum_{\nu=0}^{n}\left|\frac{1}{A_\nu^k}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}\mu x_\mu^r\right|^p\right)^{\frac{1}{p}} \quad *$$

$$= \overline{\lim_{n\to\infty}}\sum_{r=s+1}^{\infty}\left(\frac{1}{n}\sum_{\nu=0}^{n}\left|\frac{1}{A_\nu^k}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}(\mu-\nu-k+\nu+k)x_\mu^r\right|^p\right)^{\frac{1}{p}}$$

$$\leq \overline{\lim_{n\to\infty}}\sum_{r=s+1}^{\infty}\left(\frac{1}{n}\sum_{\nu=0}^{n}\left|\frac{\nu+k}{A_\nu^k}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}x_\mu^r\right|^p\right)^{\frac{1}{p}}$$

$$+ \overline{\lim_{n\to\infty}}\sum_{r=s+1}^{\infty}\left(\frac{1}{n}\sum_{\nu=0}^{n}\left|\frac{1}{A_\nu^k}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}(\mu-\nu-k)x_\mu^r\right|^p\right)^{\frac{1}{p}}$$

$$\leq M_3\,\overline{\lim_{n\to\infty}}\sum_{r=s+1}^{\infty}\left(\frac{1}{n}\sum_{\nu=0}\left|\frac{1}{A_\nu^{k-1}}\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{k-1}x_\mu^r\right|^p\right)^{\frac{1}{p}}$$

* The first s terms are zero since they are (C,1) means of null sequences.

$$+ \overline{\lim_{n\to\infty}} \sum_{r=s+1}^{\infty} \{\frac{k}{n} \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k}x_\mu^r|^p\}^{\frac{1}{p}}$$

$$\leq M_3 \sum_{r=s+1}^{\infty} ||x^r|| + \overline{\lim_{n\to\infty}} \sum_{r=s+1}^{\infty} \{\frac{k}{n} \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k}x_\mu^r|^p\}^{\frac{1}{p}}$$

$$\leq M_3 \sum_{r=s+1}^{\infty} 2^{-r} + \overline{\lim_{n\to\infty}} \sum_{r=s+1}^{\infty} \{\frac{k}{n} \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} x_r \sum_{\gamma=\mu}^{\nu} A_{\gamma-\mu}^{k-1}|^p\}^{\frac{1}{p}}$$

$$= M_3 2^{-s} + \overline{\lim_{n\to\infty}} \sum_{r=s+1}^{\infty} \{\frac{k}{n} \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\gamma=0}^{\nu} \sum_{\mu=0}^{\gamma} A_{\gamma-\mu}^{k-1}x_\mu^r|^p\}^{\frac{1}{p}}$$

$$= M_3 2^{-s} + \overline{\lim_{n\to\infty}} \sum_{r=s+1}^{\infty} \{\frac{k}{n} \sum_{\nu=0}^{n} |\frac{1}{A_\nu^k} \sum_{\gamma=0}^{\nu} A_\gamma^{k-1} \frac{1}{A_\gamma^{k-1}} \sum_{\mu=0}^{\gamma} A_{\gamma-\mu}^{k-1}x_\mu^r|^p\}^{\frac{1}{p}}$$

$$= M_3 2^{-s} + \overline{\lim_{n\to\infty}} \sum_{r=s+1}^{\infty} \{\frac{k}{n} \sum_{\nu=0}^{n} \frac{1}{\nu} \sum_{\gamma=0}^{\nu} |\frac{1}{A_\gamma^{k-1}} \sum_{\mu=0}^{\gamma} A_{\gamma-\mu}^{k-1}x_\mu^r|^p\}^{\frac{1}{p}}$$

$$\leq M_3 2^{-s} + \overline{\lim_{n\to\infty}} \sum_{r=s+1}^{\infty} \{\frac{k}{n} \sum_{\nu=0}^{n} M_4^p ||x^r||^p\}^{\frac{1}{p}} ,$$

since $[C,k]_p \Rightarrow [C,k]_1$. That is we have

$$\{\frac{1}{n} \sum_{\nu=0}^{n} |\frac{1}{A_\nu^{k-1}} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1}x_\mu|\} \leq \frac{1}{n^1} \{\sum_{\nu=0}^{n} |\frac{1}{A_\nu^{k-1}} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1}x_\mu|^p\}^{\frac{1}{p}}\{\sum_{\nu=0}^{n} 1\}^{\frac{1}{q}}$$

$$= \{\frac{1}{n} \sum_{\nu=0}^{n} |\frac{1}{A_\nu^{k-1}} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1}x_\mu|^p\}^{\frac{1}{p}}.$$

$$\leq (M_3 + M_4)\, 2^{-s}.$$

Therefore

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu x_\mu\right|^p = o(n).$$

Now let $m > n > o$. Then

$$\overline{\lim_{\substack{m\to\infty \\ n\to\infty}}} \left|\frac{1}{m+1} \sum_{\nu=o}^{m} \frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu - \frac{1}{n+1} \sum_{\nu=o}^{n} \frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu\right|$$

$$= \overline{\lim_{\substack{m\to\infty \\ n\to\infty}}} \left| \sum_{r=s+1}^{\infty} \left[\frac{1}{m+1} \sum_{\nu=o}^{m} \frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu^r - \frac{1}{n+1} \sum_{\nu=o}^{n} \frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu^r\right]\right|.$$

$$\leq 2 \sum_{r=s+1}^{\infty} ||x^r|| = 2^{1-s}.$$

Therefore $\sum_{n=o}^{\infty} x_n$ is summable (C,k). Hence $\sum_{n=o}^{\infty} x_n$ is summable $[C,k]_p$ and this contradicts the hypothesis of the lemma.

Lemma 3.4

For $a \geq 1$, the general continuous linear functional on X_k^a is given by an equation of the form

$$F(x) = \sum_{\nu=1}^{\infty} x_\nu \sum_{A=\nu}^{\infty} A_{A-\nu}^{k-1} h_A + \alpha \ell_x$$

where α is a constant and

$$\sum_{n=o}^{\infty} m_n (h,p) < \infty \text{ where}$$

$$m_n (h,p) = \begin{cases} \sup_n |A_\nu^k h_\nu| & p = 1 \\ \left(\frac{1}{2^n} \sum_n |A_\nu^k h_\nu|^q\right)^{\frac{1}{q}} & p > 1 \end{cases}$$

and ℓ_x is the (C,k) sum of $\sum_{n=o}^{\infty} x_n$.

Proof

The equation

$$y_n = \frac{1}{A_n^{k-1}} \sum_{\nu=o}^{n} A_{n-\nu}^{k-1} x_\nu$$

defines a linear isometric transformation from X_k^a into a vector subspace of w_p. By the Hahn-Banach Theorem and Lemma 3,2 the ~~general continuous linear functional in X_k^a has the form~~

$$F(x) = \alpha \ell_x + \sum_{n=0}^{\infty} \alpha_n y_n$$

$$= \alpha \ell_x + \sum_{n=0}^{\infty} \alpha_n \frac{1}{A_n^{k-1}} \sum_{\nu=0}^{n} A_{n-\nu}^{k-1} x_\nu$$

where $\sum_{n=0}^{\infty} m_n(\alpha,p) < \infty$

$$m_n(\alpha,p) = \begin{cases} \sup_n |\nu\alpha_\nu| & p = 1 \\ \left(\frac{1}{2^n} \sum_n |\nu\alpha_\nu|^q\right)^{\frac{1}{q}} & p > 1 \end{cases}$$

Hence

$$F(x) = \alpha \ell_x + \sum_{\nu=0}^{\infty} x_\nu \sum_{n=\nu}^{\infty} A_{n-\nu}^{k-1} \frac{\alpha_n}{A_n^{k-1}}$$

the inversion being justified since

$$\sum_{n=\nu}^{\infty} A_{n-\nu}^{k-1} \left|\frac{\alpha_n}{A_n^{k-1}}\right| \leq M_1 \sum_{n=\nu}^{2\nu} A_{n-\nu}^{k-1} \frac{1}{A_n^k} + M_1 \sum_{n=2\nu}^{\infty} |\alpha_n|$$

$$\leq M_2 \nu^{-k} \sum_{n=\nu}^{2\nu} A_{n-\nu}^{k-1} + M_1 \sum_{n=0}^{\infty} |\alpha_n|$$

$$\leq M_3 + M_1 \sum_{n=0}^{\infty} |\alpha_n|$$

$$< \infty$$

and $\sum_{n=o}^{\infty} |\alpha_n| < \infty$.

The result follows by defining

$$h_n = \frac{\alpha_n}{A_n^{k-1}}.$$

Theorem 3.1

Let $k \geq 1$. If $\sum_{n=o}^{\infty} x_n \varepsilon_n$ is summable $[C,k]_p$ whenever $\sum_{n=o}^{\infty} x_n$ is summable $[C,k]_p$ then there is an integer $a \geq 1$ such that

$$\varepsilon_\nu = \alpha + \sum_{n=\nu}^{\infty} A_{n-\nu}^{k-1} h_\nu \qquad \nu = a + 1, a + 2, \cdots$$

with

$$\sum_{n=o}^{\infty} m_n (h,p) < \infty,$$

where

$$m_n(h,p) = \begin{cases} \sup_n |A_\nu^k h_\nu| & p = 1 \\ \left(\frac{1}{2^n} \sum_n |A_\nu^k h_\nu|^q\right)^{\frac{1}{q}} & p > 1. \end{cases}$$

Proof

By Lemmas 3.3 and 3.4 there is an integer $a > 1$ such that that for all $x \varepsilon\, X_k^a$

$$\sum_{n=o}^{\infty} x_n \varepsilon_n = \sum_{\nu=o}^{\infty} x_\nu \sum_{n=\nu}^{\infty} A_{n-\nu}^{k-1} h_n + \alpha \ell_x$$

where $h = [h_n]$ satisfies the conditions of the Theorem. The result follows since for $i = 1,2,\cdots$ the sequence $x = [x_n]$ defined by

$$x_n = \begin{cases} 1 & n = a + i \\ 0 & \text{otherwise} \end{cases}$$

is in χ_k^a and $\ell_x = 1$.

3.3 *THE SUFFICIENCY*

Theorem 3.2

Let $k = 1,2,3\cdots\cdots$.

If $$\sum_{\nu=o}^{n} \left(\frac{1}{A_\nu^k} \left| \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \right| \right)^p = o(n)$$

and $\varepsilon_\mu = \sum_{\rho=\mu}^{\infty} A_{\rho-\mu}^{k-1} h_\rho$ where

(i) $$\frac{1}{2^r} \sum_r \left(A_\rho^k |h_\rho| \right)^q = O(1)$$

and (ii) $\sum_{\rho=0}^{\infty} A_{\rho}^{k-1}|h_{\rho}| < \infty$

then $$\sum_{\nu=0}^{n}\left(\frac{1}{A_{\nu}^{k}}\left|\sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1}\mu a_{\mu}\varepsilon_{\mu}\right|\right)^{p} = o(n).$$

Proof.

By induction.

Case 1.

$k = 1.$

$$\left(\frac{1}{2^r}\sum_r\{\frac{1}{\nu+1}|\sum_{\mu=0}^{\nu}\mu a_{\mu}\varepsilon_{\mu}|\}^p\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{2^r}\sum_r\{\frac{1}{\nu+1}|\sum_{\mu=0}^{\nu}\mu a_{\mu}\sum_{\rho=\mu}^{\infty}h_{\rho}|\}^p\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{2^r}\sum_r\{\frac{1}{\nu+1}|\sum_{\mu=0}^{\nu}\mu a_{\mu}\{\sum_{\rho=\mu}^{\nu}h_{\rho} + \sum_{\rho=\nu+1}^{\infty}h_{\rho}|\}\}^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{2^r}\sum_r\{\frac{1}{\nu+1}|\sum_{\mu=0}^{\nu}\mu a_{\mu}\sum_{\rho=\mu}^{\nu}h_{\rho}|\}^p\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{2^r}\sum_r\{\frac{1}{\nu+1}|\sum_{\mu=0}^{\nu}\mu a_{\mu}\sum_{\rho=\nu+1}^{\infty}h_{\rho}|\}^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{2^r} \sum_r \{\frac{1}{\nu+1} | \sum_{\rho=o}^{\nu} h_\rho \sum_{\mu=o}^{\rho} \mu a_\mu |\}^p\right)^{\frac{1}{p}}$$

$$+ \left(\frac{1}{2^r} \sum_r \{\frac{1}{\nu+1} | \sum_{\mu=o}^{\nu} \mu a_\mu | \sum_{\rho=\nu+1}^{\infty} |h_\rho|\}^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{2^{r(1+p)}} \sum_r \{\sum_{s=o}^{r} \sum_s |h_\rho| \, |\sum_{\mu=\rho}^{\rho} \mu a_\mu|\}^p\right)^{\frac{1}{p}}$$

$$+ \sum_{\rho=o}^{\infty} |h_\rho| \left(\frac{1}{2^r} \sum_r \{\frac{1}{\nu+1} | \sum_{\mu=o}^{\nu} \mu a_\mu |\}^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{2^{r(1+p)}} \{\sum_{s=o}^{r} \left(\sum_s \{(\rho+1)|h_\rho|\}^q\right)^{\frac{1}{q}} \left(\sum_s \{\frac{1}{\rho+1} | \sum_{\mu=o}^{\rho} \mu a_\mu |\}^p\right)^{\frac{1}{p}}\}. \ \sum_r 1\right)^{\frac{1}{p}}$$

$$+ o(1)$$

$$= \left(\frac{1}{2^{r(1+p)}} \{\sum_{s=o}^{r} O(2^{s/q}). \ O(2^{s/p}). \ o(1)\}^p . \ 2^r\right)^{\frac{1}{p}} + o(1)$$

$$= \left(\frac{o(1)}{2^{rp}} \{\sum_{s=o}^{r} O(2^{s/p + s/q})\}^p\right)^{\frac{1}{p}} + o(1)$$

$$= \left(\frac{o(1)}{2^{rp}} . \ O(2^{rp})\right)^{\frac{1}{p}} + o(1)$$

$$= o(1).$$

Case 2.

k > 1.

Assume the Theorem with k replaced by k-1.

Let $P_n = \sum_{\rho=n}^{\infty} A_{\rho-n}^{k-2} (\rho+k)\, h_\rho$.

Then $\sum_{\rho=0}^{\infty} A_\rho^{k-2} (\rho+k)|h_\rho|$

$$= O\left(\sum_{\rho=0}^{\infty} A_\rho^{k-1}|h_\rho|\right) < \infty$$

and $\frac{1}{2^r} \sum_r \left(A_\rho^{k-1} (\rho+k)|h_\rho|\right)^q$

$$= O\left(\frac{1}{2^r} \sum_r \{A_\rho^k |h_\rho|\}^q\right)$$

$$= O(1).$$

Therefore P_n satisfies the conditions of the Theorem with k replaced by k-1.

Also

$$\varepsilon_{n+1} = \sum_{\rho=n+1}^{\infty} A_{\rho-n-1}^{k-1} h_\rho$$

$$= \sum_{\rho=n+1}^{\infty} h_\rho \sum_{\mu=n+1}^{\rho} A_{\mu-n-1}^{k-2}$$

$$= \sum_{\mu=n+1}^{\infty} A_{\mu-n-1}^{k-2} \sum_{\rho=\mu}^{\infty} h_\rho,$$

the interchange being justified by absolute convergence with

$$\sum_{\rho=o}^{\infty} A_\rho^{k-2} \left| \sum_{\mu=\rho}^{\infty} h_\mu \right|$$

$$\leq \sum_{\mu=o}^{\infty} |h_\mu| \sum_{\rho=o}^{\mu} A_\mu^{k-2}$$

$$= \sum_{\mu=o}^{\infty} A_\mu^{k-1} |h_\mu| < \infty$$

and

$$\frac{1}{2^s} \sum_s \left(A_\rho^{k-1} \left| \sum_{\mu=\rho}^{\infty} h_\mu \right| \right)^q$$

$$\leq \frac{1}{2^s} \sum_s \left(\sum_{\mu=\rho}^{\infty} A_\mu^{k-1} |h_\mu| \right)^q$$

$$= 0\left(\frac{1}{2^s} \sum_s 1\right)$$

$$= 0(1).$$

Therefore $\{\varepsilon_{n+1}\}$ satisfies the conditions of the Theorem with k replaced by k-1.

Now

$$\sum_{\mu=o}^{n} \mu a_\mu \varepsilon_\mu$$

$$= (n+1)\, x_n \varepsilon_{n+1} + \sum_{\mu=o}^{n} x_\mu P_\mu + (1-k) \sum_{\mu=o}^{n} x_\mu \varepsilon_\mu$$

where

$$x_n = \frac{1}{\mu+1} \sum_{r=o}^{n} r a_r.$$

Therefore

$$\{\sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \varepsilon_\mu|^p\}^{\frac{1}{p}}$$

$$= \{\sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} \sum_{\rho=o}^{\mu} \rho a_\rho \varepsilon_\rho|^p\}^{\frac{1}{p}}$$

$$= \{\sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} \{(\mu+1) x_\mu \varepsilon_{\mu+1} + \sum_{r=o}^{\mu} x_r P_r + (1-k) \sum_{r=o}^{\mu} x_r \varepsilon_r\}|^p\}^{\frac{1}{p}}$$

$$\leq \{ \sum_{\nu=o} |\frac{1}{A_\nu^k} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} (\mu+1) x_\mu \varepsilon_{\mu+1}|^p \}^{\frac{1}{p}}$$

$$+ \{ \sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r P_r|^p \}^{\frac{1}{p}}$$

$$+ \{ (k-1)^p \sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r \varepsilon_r|^p \}^{\frac{1}{p}}.$$

Given that

$$\sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} r a_r|^p = o(n)$$

we have by Lemma (2.3) that

$$\sum_{\nu=o}^{n} |\frac{1}{A_\nu^{k-1}} \sum_{r=o}^{\nu} A_{\nu-r}^{k-2} x_r|^p = o(n)$$

and hence by our induction hypothesis that

$$\sum_{\nu=o}^{n} |\frac{1}{A_\nu^{k-1}} \sum_{r=o}^{\nu} A_{\nu-r}^{k-2} x_r P_r|^p = o(n).$$

Hence by Lemma (2.2)

$$\sum_{\nu=o}^{n} |\frac{1}{A_\nu^k} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r P_r|^p = o(n).$$

Similarly

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{r=o}^{\nu} A_{\nu-r}^{k-2} x_r \varepsilon_r\right|^p = o(n).$$

and

$$\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k}} \sum_{r=o}^{\nu} A_{\nu-r}^{k-1} x_r \varepsilon_r\right|^p = o(n).$$

Further

$$\{\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2}(\mu+1)\, x_\mu \varepsilon_{\mu+1}\right|^p\}^{\frac{1}{p}}$$

$$= \{\sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k}} \sum_{\mu=o}^{\nu} (k+\nu)\, A_{\nu-\mu}^{k-2} - \frac{(k-1)}{k} A_{\nu-\mu}^{k-1} x_\mu \varepsilon_{\mu+1}\right|^p\}^{\frac{1}{p}}$$

$$\leq \{\sum_{\nu=o}^{n} \left|\frac{k+\nu}{A_\nu^{k}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} x_\mu \varepsilon_{\mu+1}\right|^p\}^{\frac{1}{p}}$$

$$+ \{\frac{(k-1)^p}{k} \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu \varepsilon_{\mu+1}\right|^p\}^{\frac{1}{p}}$$

$$= \{k^p \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k-1}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-2} x_\mu \varepsilon_{\mu+1}\right|^p\}^{\frac{1}{p}}$$

$$+ \{\frac{(k-1)^p}{k} \sum_{\nu=o}^{n} \left|\frac{1}{A_\nu^{k}} \sum_{\mu=o}^{\nu} A_{\nu-\mu}^{k-1} x_\mu \varepsilon_{\mu+1}\right|^p\}^{\frac{1}{p}}.$$

And so

$$\sum_{\nu=0}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-2}(\mu+1)\, x_\mu \varepsilon_{\mu+1}\right|^p = o(n).$$

Therefore

$$\sum_{\nu=0}^{n} \left|\frac{1}{A_\nu^k} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{k-1} \mu a_\mu \varepsilon_\mu\right|^p = o(n)$$

and the Theorem is proved.

Theorem 3.3

Let $k \geq 1$.

If

$$\varepsilon_n = \sum_{\rho=\mu}^{\infty} A_{\rho-\mu}^{k-1}\, h_\rho$$

where

$$\sum_{s=0}^{\infty} 2^{s(k-\frac{1}{q})} \left(\textstyle\sum_s |h_\rho|^q\right)^{\frac{1}{q}} < \infty$$

then

$$\{\varepsilon\} \in [[C,k]_p;(C,k)].$$

Proof

By Lemma 1.6 we must prove (i) $\alpha_{n\nu} \to \alpha_\nu$ as $n \to \infty$

and (ii) Uniformly in r, for $n \in [2^r, 2^{r+1})$

$$\sum_{s=0}^{r} 2^{\frac{s}{p}} \cdot (\sum_s |\alpha_{n\nu}|^q)^{\frac{1}{q}} = O(1)$$

where $$\alpha_{n\nu} = \begin{cases} \dfrac{A_\nu^{k-1}}{A_n^k} \displaystyle\sum_{\mu=\nu}^{n} A_{n-\mu}^{k} A_{\mu-\nu}^{-k-1} \varepsilon_\mu \\ 0 & \nu > n \end{cases}$$

or equivalently,

$$\text{(i)* } \frac{1}{A_n^k} \beta_{n\nu} \to \beta_\nu \quad \text{as} \quad n \to \infty$$

and (ii)* Uniformly in r, for $n \varepsilon [2^r, 2^{r+1})$

$$\sum_{s=0}^{r} 2^{s(k-\frac{1}{q})} (\sum_s |\beta_{n\nu}|^q)^{\frac{1}{q}} = O(2^{rk})$$

where $$\beta_{n\nu} = \begin{cases} \dfrac{A_n^k}{A_\nu^{k-1}} \alpha_{n\nu} & \nu \leq n \\ 0 & \nu > n \end{cases}$$

Using the notation of Lemma 1.5 we have

$$\beta_{n\nu} = \sum_{\rho=\nu}^{\infty} S_{n-\nu,\rho-\nu}^{k} h_\rho \tag{3.3.1}$$

and β_n^i ($i = 1,2,3$) denoting respectively the contribution to the sum (3.3.1) of the term $\rho = \nu$, the terms with $\nu + 1 \leq \rho \leq n$, and the terms with $\rho > n$. By Minkowski's Inequality it will suffice to show that (i)* and (ii)* hold for each of the $\beta_{n\nu}^i$.

Case 1

$i = 1$.

Since $\beta_{n\nu}^1 = A_{n-\nu}^k h_\nu$ we have (i)* holding with $\beta_\nu^1 = h_\nu$.

It is clear that, uniformly in $0 \leq \nu \leq n$, $2^r \leq n < 2^{r+1}$,

$$\beta_{n\nu}^1 = O\{2^{rk} h_\nu\}$$

and hence

$$\sum_{s=0}^{r} 2^{s(k-\frac{1}{q})} (\textstyle\sum_s |\beta_{n\nu}|^q)^{\frac{1}{q}}$$

$$= O\{\sum_{s=0}^{r} 2^{s(k-\frac{1}{q})} (\textstyle\sum_s |h_\nu|^q)^{\frac{1}{q}}$$

$$= O(1).$$

Therefore (ii)* holds for $\beta_{n\nu}^1$.

Case 2

$$i = 2$$

By Lemma 1.6

$$\beta^2_{n\nu} = 0\ \{(n+1-\nu)^{k-1} \sum_{\rho=\nu+1}^{n} |h_\rho|\}.$$

Then uniformly in $0 \le \nu \le n$, $2^r \le n < 2^{r+1}$

$$(n+1-\nu)^{k-1} = 0(n+1)^{k-1} = 0(2^{r(k-1)})$$

whenever $k \ge 1$.

Therefore

$$\beta^2_{n\nu} = 0\ \{2^{r(k-1)} \sum_{\rho=\nu+1}^{n} |h_\rho|\}$$

$$= 0\ \{2^{r(k-1)} \sum_{t=s}^{r} \sum_t |h_\rho|\} \quad \text{where } \nu\varepsilon\ [2^s, 2^{s+1})$$

with $s \le r$.

Hence

$$\sum_{s=o}^{r} 2^{s(k-\frac{1}{q})} (\sum_s |\beta^2_{n\nu}|^q)^{\frac{1}{q}}$$

$$= 0\ \{2^{r(k-1)} \sum_{s=o}^{r} 2^{s(k-\frac{1}{q})}.\ 2^{\frac{s}{q}}.\ \sum_{t=s}^{r} \sum_t |h_\rho|\}$$

$$= 0\ \{2^{r(k-1)} \sum_{t=o}^{r} \textstyle\sum_t |h_\rho| \sum_{s=o}^{t} 2^{sk}\}$$

$$= 0\ \{2^{r(k-1)} \sum_{t=o}^{r} 2^{tk} (\textstyle\sum_t |h_\rho|)\}$$

$$= 0\ \{2^{r(k-1)} \sum_{t=o}^{r} 2^{tk} (\textstyle\sum_t |h_\rho|^q)^{\frac{1}{q}} (\textstyle\sum_t 1)^{\frac{1}{p}}\}$$

$$= 0\ \{2^{r(k-1)} \sum_{t=o}^{r} 2^{tk}.\ 2^{\frac{t}{p}} (\textstyle\sum_t |h_\rho|^q)^{\frac{1}{q}}\}$$

$$= 0\ \{2^{rk} \sum_{t=o}^{r} 2^{t(k-\frac{1}{q})} (\textstyle\sum_t |h_\rho|^q)^{\frac{1}{q}}\}$$

$$= 0\ \{2^{rk}\}\ .$$

So that (ii)* holds for $\beta^2_{n\nu}$.

Also for fixed ν

$$\beta^2_{n\nu} = 0\ \{2^{r(k-1)} \sum_{t=s}^{r} (\textstyle\sum_t |h_\rho|^q)^{\frac{1}{q}} (\textstyle\sum_t 1)^{\frac{1}{p}}\}$$

$$= 0\ \{2^{r(k-1)} \sum_{t=s}^{r} 2^{\frac{t}{p}} (\textstyle\sum_t |h_\rho|^q)^{\frac{1}{q}}\}$$

$$= 0\ \{2^{r(k-1)} \sum_{t=o}^{r} 2^{t(k-\frac{1}{q})} (\sum_t |h_\rho|^q)^{\frac{1}{q}}\}$$

$$= 0\ \{2^{r(k-1)}\}.$$

Therefore (i)* holds for $\beta^2_{n\nu}$ with $\beta^2_\nu = 0$.

Case 3

$i = 3$. We have for $n \varepsilon [2^r, 2^{r+1})$

$$\beta^3_{n\nu} = \sum_{\rho=n+1}^{\infty} S^k_{n-\nu,\rho-\nu}\ h_\rho$$

$$= 0\ \{ \sum_{\rho=n+1}^{\infty} (\rho+1-\nu)^{k-1} |h_\rho|$$

$$= 0\ \{ \sum_{t=r}^{\infty} (\sum_t |h_\rho|^q)^{\frac{1}{q}} (\sum_t (\rho+1-\nu)^{(k-1)p})^{\frac{1}{p}}\}$$

$$= 0\ \{ \sum_{t=r}^{\infty} 2^{t(k-1)} . 2^{\frac{t}{p}} (\sum_t |h_\rho|^q)^{\frac{1}{q}}\}$$

$$= 0\ \{ \sum_{t=r}^{\infty} 2^{t(k-\frac{1}{q})} (\sum_t |h_\rho|^q)^{\frac{1}{q}}\}$$

$$= 0(1).$$

Hence, clearly, (i)* and (ii)* hold for $\beta^3_{n\nu}$.

Theorem 3.4

Let $k = 1,2,3,\cdots$.

If $\varepsilon_n = \sum_{\rho=n}^{\infty} A_{\rho-n}^{k-1} h_\rho$

where $\sum_{r=\rho}^{\infty} 2^{r(k-\frac{1}{q})} (\sum_r |h_\rho|^q)^{\frac{1}{q}} < \infty$

then $\{\varepsilon_n\} \varepsilon [[C,k]_p;[C,k]_p]$.

Proof

This follows from Theorems 3.2 and 3.3 via Theorem 1.2.

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