1975

On Integral Hausdorff And Abel-type Methods Of Summability

Elizabeth Christine Heagy

ON INTEGRAL HAUSDORFF AND ABEL-TYPE METHODS OF SUMMABILITY

by

Elizabeth Christine Heagy

Department of Mathematics

Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
London, Ontario

June, 1975

ABSTRACT

Integral Hausdorff methods of summability were investigated by W. W. Rogosinski in papers published in 1942. Some of his results are examined and extended. Integral Abel-type summability is defined and a scale of inclusions obtained. The behaviour of the product of integral Hausdorff and Abel-type methods is studied and it is shown that the product is commutative. Some integral logarithmic methods of summability are defined and are related to the integral Abel-type scale previously obtained. Finally, strong summability for integral methods is studied, with particular emphasis on strong integral Abel-type summability.

ACKNOWLEDGEMENTS

I should like to express my gratitude to Dr. B. L. R. Shawyer for his guidance, encouragement and friendship. I am also indebted to other members of the summability group at the University of Western Ontario for their assistance.

My thanks go to Janet Williams who typed this thesis.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Integral Hausdorff methods of summability were investigated by W. W. Rogosinski in papers published in 1942. We examine and extend some of his results. Integral Abel-type summability is defined and a scale of inclusions obtained. The behaviour of the product of integral Hausdorff and Abel-type methods is studied and it is shown that the product is commutative. Some integral logarithmic methods are defined and we relate them to the integral Abel-type scale previously obtained. Finally, strong summability for integral methods is studied, with particular emphasis on strong integral Abel-type summability.

1.1 CONVENTIONS AND NOTATION.

Theorems appearing with letters (for example, Theorem D) are results due to other authors and are not proved in this thesis. Theorems which we prove are numbered within each chapter and section. For example, Theorem 4.2.3 is the third theorem of the second section of Chapter 4. Lemmata are numbered in a similar manner.

The symbols σ and τ are used to denote arbitrary real numbers. We emphasize that σ and τ are finite.

Throughout the thesis, the symbol f denotes a real-valued function with domain $[0,\infty)$. We require that such an f be bounded on any finite interval of the form $[0,x]$. Occasionally g is used to denote another such function.

1.2 INTEGRAL METHODS OF SUMMABILITY.

Much of traditional summability has been concerned with methods applied to infinite series or sequences (see, for example, [13]). In this thesis we are concerned with a general class of summability methods applied to functions. An *integral method of summability* $\hat{T}$[1] is of the following form (see, for example, [13, Ch. III]). Let

$$\begin{aligned} T(y) &= T_f(y) \\ &= \int_{\Omega} f(x)\,d\xi(x,y) \end{aligned}$$

where Ω is some subset of $[0,\infty)$ and $d\xi(x,y)$ indicates some type of integration with respect to some function of x and y. The integral may be a Cauchy-Lebesgue integral or a Lebesgue-Stieltjes integral, depending on the nature of f and ξ. If $T(y)$ exists (in some sense) for all $y > 0$

---

[1] Throughout this thesis, a method denoted in the form $\hat{T}$ is always an integral method. A method denoted in the form T is a method applied to series or sequences.

and if $T(y) \to \sigma$ as $y \to \infty$, then we say that f is $\hat{T}$-*summable* to σ and we write,

$$f(x) \to \sigma \ (\hat{T}).$$

For convenience of notation, $T(y)$ is sometimes written in the form (see, for example, [13, Ch. III])

$$T(y) = \int_0^\infty c(y,x) f(x) dx.$$

Here $c(y,x)$ is known as the *kernel* of the transformation.

When considering a method applied to a sequences $\{s_n\}$ we are always interested in the behaviour of the transform as $n \to \infty$. To avoid possible ambiguity with an integral method, the symbol y is reserved for a parameter tending to infinity; that is, we are concerned with the behaviour of $T(y)$ as $y \to \infty$. Occasionally we shall write $T(\frac{1}{t})$ to indicate that we are using the parameter $t \to 0^+$.

For two integral methods $\hat{T}_1$ and $\hat{T}_2$, the symbol

$$\hat{T}_1 \supseteq \hat{T}_2$$

indicates that whenever $f(x) \to \sigma \ (\hat{T}_2)$ we also have $f(x) \to \sigma \ (\hat{T}_1)$. The symbol

$$\hat{T}_1 \supsetneq \hat{T}_2$$

indicates that $\hat{T}_1 \supseteq \hat{T}_2$ and that there is at least one function which is $\hat{T}_1$-summable but not $\hat{T}_2$-summable. We write

$$\hat{T}_1 = \hat{T}_2$$

when we have for every f $T_{1_f}(y) = T_{2_f}(y)$ for all $y > 0$. The symbol

$$\hat{T}_1 \simeq \hat{T}_2$$

indicates that $f(x) \to \sigma\ (\hat{T}_1)$ if and only if $f(x) \to \sigma\ (\hat{T}_2)$, but does not imply any relationship between $T_1(y)$ and $T_2(y)$. We use similar notation to describe sequence and series methods.

All products of integral methods in this thesis are are iteration products. For example, if

$$T_1(y) = \frac{1}{y}\int_0^y f(t)\,dt$$

and $$T_2(y) = \int_0^\infty e^{-u/y} f(u)\,du,$$

then the method $\hat{T} = \hat{T}_1\hat{T}_2$ has transform

$$\begin{aligned} T(y) &= T_1T_2(y) \\ &= \frac{1}{y}\int_0^y T_2(t)\,dt \\ &= \frac{1}{y}\int_0^y \int_0^\infty e^{-u/t} f(u)\,du\,dt. \end{aligned}$$

We briefly mention one problem associated with integral methods which is not encountered with series or sequence methods: the lack of a limitation theorem for integrals. If a series $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$. Similarly, if a sequence $\{s_n\}$ converges, then we know s_n tends to some finite limit as $n \to \infty$. However, if $\int_0^\infty F(x)\,dx$ exists, even as a Lebesgue integral, we know nothing about the magnitude of $F(x)$ as $x \to \infty$.

1.3. REGULARITY.

A series (or sequence) method T is said to be *regular* (see, for example, [13, p. 43]) if every convergent series (or sequence) is T-summable to its limit. Necessary and sufficient conditions for the regularity of such methods are well known (see, for example, [13, Ch. III]).

An integral method $\hat{T}$ is *regular* (see, for example, [13, p. 50]) if every function converging to a finite limit at infinity is $\hat{T}$-summable to that limit. The following theorem states sufficient conditions for regularity of integral methods.

THEOREM A. [13, Th. 6] *Let* $\hat{T}$ *be an integral method with transform*

$$T(y) = \int_0^\infty c(y,x)f(x)\,dx.$$

If (i) *for some constant* K

$$\int_0^\infty |c(y,x)|\,dx < K$$

for all $y > 0$,

(ii) *for any finite constant* $Y > 0$

$$\int_0^Y |c(y,x)|\,dx \to 0 \textit{ as } y \to \infty,$$

and (iii)

$$\int_0^\infty c(y,x)\,dx \to 1 \textit{ as } y \to \infty,$$

then $f(x) \to \sigma$ $(\hat{T})$ *whenever* $f(x) \to \sigma$ *as* $x \to \infty$; *that is,* $\hat{T}$ *is regular.*

We remark that Hardy requires f to be bounded for this theorem. By prior assumption we have f bounded on any finite interval. Since we suppose that $f(x) \to \sigma$ as $x \to \infty$, f is thus bounded on all of $[0,\infty)$.

Necessary conditions for regularity are given by:

THEOREM B. [13, p. 61] *Let* $\hat{T}$ *be an integral method with transform*

$$T(y) = \int_0^\infty c(y,x) f(x)\,dx.$$

Suppose $f(x) \to \sigma\ (\hat{T})$ *whenever* $f(x) \to \sigma$ *as* $x \to \infty$; *that is,* $\hat{T}$ *is regular. Then*

(i) *there exists a constant* K *such that*

$$\int_0^\infty |c(y,x)|\,dx < K$$

for y *sufficiently large,*

(ii) *for any finite constant* $Y > 0$,

$$\int_0^Y c(y,x)\,dx \to 0 \text{ as } y \to \infty,$$

and (iii)
$$\int_0^\infty c(y,x)\,dx \to 1 \text{ as } y \to \infty.$$

Hardy's requirement of f bounded is again satisfied.

Hardy [13, p. 62] states necessary and sufficient conditions for regularity of integral methods. For our purposes, however, Theorems A and B are more convenient than the most general theorem.

CHAPTER 2

INTEGRAL HAUSDORFF METHODS

2.1. INTRODUCTION.

Following Rogosinski [23], we define the *integral Hausdorff method of summability* as follows. Let $\chi \in BV\,[0,1]$; that is, let χ be a function of bounded variation on the closed interval $[0,1]$. We extend χ to the entire real line by requiring

$$\chi(t) = \chi(0) \text{ for } t \in (-\infty,0)$$

and $$\chi(t) = \chi(1) \text{ for } t \in (1,\infty).$$

For $y > 0$, we define

$$H(y) = H_\chi(y) = H_{f;\chi}(y) = \int_0^1 f(yt)\,d\chi(t),$$

where the integral is the Lebesgue-Stieltjes integral[1] over the closed interval $[0,1]$. If $H(y)$ exists for all $y > 0$ and if $H(y) \to \sigma$ as $y \to \infty$, we say that f is $\hat{H}$-*summable*, or, $\hat{H}_\chi$-*summable*, to σ and we write

$$f(x) \to \sigma \ (\hat{H}) \text{ or } f(x) \to \sigma \ (\hat{H}_\chi),$$

whichever is convenient.

By prior assumption, we have f bounded on all

---

[1] For a discussion of the Lebesgue-Stieltjes integral, see Appendix 1.

finite intervals of the form $[0,x]$. We now require that f be Borel measurable (see Appendix 1). These conditions are sufficient to guarantee the existence of $H_{f;\chi}(y)$ for any $\chi \in BV[0,1]$ and any $y > 0$.

Rogosinski restricts his integral Hausdorff methods to regular methods. We relax the definition to include some non-regular methods. Regularity will be investigated in the next section.

The sequence-to-sequence Hausdorff method is defined as follows ([15,16]; see also [13, Ch. XI]). For a sequence $\{a_n\}$, we define

$$\Delta^0 a_n = a_n$$

$$\Delta^1 a_n = \Delta a_n = a_n - a_{n+1},$$

$$\Delta^k a_n = \Delta(\Delta^{k-1} a_n), \quad \text{for } k = 1,2,\ldots .$$

Given a sequence $\{d_n\}$, we define a triangular matrix $M = [a_{n\nu}]$ where $a_{n\nu} = \binom{n}{\nu} \Delta^{n-\nu} d_\nu$. A sequence $s = \{s_n\}$ is said to be H-summable to σ if Ms has limit σ.

By a well-known theorem (see, for example, [13, Th. 208]), the method H is regular if and only if $\{d_n\}$ is a regular moment sequence; i.e.,

$$d_n = \int_0^1 t^n d\phi(t)$$

where $\phi \in BV[0,1]$, $\phi(1) - \phi(0) = 1$ and $\phi(0^+) = \phi(0)$. We use H_ϕ to denote this method, when convenient.

For any function $\phi \in BV[0,1]$, we can define the methods H_ϕ and $\hat{H}_\phi$. We will refer to these as the related sequence-to-sequence Hausdorff and integral Hausdorff methods determined by ϕ.

In this chapter, we will examine properties of integral Hausdorff methods. We will find that the behaviour of the sequence-to-sequence Hausdorff methods often establishes the behaviour of the related integral Hausdorff methods.

2.2. PROPERTIES OF INTEGRAL HAUSDORFF METHODS

We now investigate necessary and sufficient conditions for regularity of integral Hausdorff methods.

LEMMA 2.2.1. *Let* $\chi \in BV[0,1]$. *The following are equivalent.*

(i) $\chi(0^+) = \chi(0)$; *that is,* χ *is continuous at zero.*

(ii) $\lim_{\varepsilon\to 0^+} \int_0^\varepsilon d\chi(t) = 0.$

(iii) $\lim_{\varepsilon\to 0^+} \int_0^\varepsilon |d\chi(t)| = 0.$

PROOF. That (i) and (ii) are equivalent follows from the definition of $\int_0^\varepsilon d\chi(t)$. Since

$$\left|\int_0^\varepsilon d\chi(t)\right| \le \int_0^\varepsilon |d\chi(t)|,$$

(iii) implies (ii). It remains to show that (i) implies (iii).

Let $v(x)$ denote the variation of χ on $[0,x]$. Since $\chi \in BV[0,1]$, there exist monotone non-decreasing functions p and q such that

$$\chi(x) - \chi(0) = p(x) - q(x)$$

and

$$v(x) = p(x) + q(x).$$

Hence

$$2p(x) = v(x) + \chi(x) - \chi(0)$$

and

$$2q(x) = v(x) - \chi(x) + \chi(0).$$

Since χ is continuous at zero if and only if v is continuous at zero (see, for example, [24, Ch. 6]), it follows that p and q are also continuous at zero. Hence

$$\int_0^\varepsilon |d\chi(t)| = \int_0^\varepsilon dp(t) + \int_0^\varepsilon dq(t) \quad \text{(p,q monotone non-decreasing)}$$

$$= p(\varepsilon^+) - p(0^-) + q(\varepsilon^+) - q(0^-)$$

$$= v(\varepsilon^+) - v(0^-)$$

$$\to 0 \text{ as } \varepsilon \to 0.$$

This completes the proof of the lemma.

THEOREM 2.2.1. *The method* $\hat{H}_\chi$ *is regular if and only if*

$$\chi(0^+) = \chi(0)$$

and

$$\chi(1) - \chi(0) = 1.$$

PROOF. We use Theorems A and B. For $y > 0$, we have

$$H(y) = H_\chi(y) = H_{f;\chi}(y) = \int_0^1 f(yt)\,d\chi(t)$$

$$= \int_0^y f(x)\,d\chi\left(\frac{x}{y}\right).$$

Hence

$$\int_0^y \left|d\chi\left(\frac{x}{y}\right)\right| = \int_0^1 |d\chi(t)|,$$

the variation of χ on $[0,1]$, is a constant independent of y. Thus the first condition of both theorems is satisfied.

By Lemma 2.2.1, the second conditions are equivalent to

$$\chi(0^+) = \chi(0).$$

Since

$$\int_0^y d\chi\left(\frac{x}{y}\right) = \int_0^1 d\chi(t) = \chi(1) - \chi(0),$$

the third condition of both theorems is satisfied if and only if

$$\chi(1) - \chi(0) = 1$$

This completes the proof of the theorem.

We remark that these conditions for regularity of integral Hausdorff methods are identical to those for sequence-to-sequence Hausdorff methods.

If

$$\chi(t) = \frac{1}{2}\{\chi(t^-) + \chi(t^+)\} \text{ for } t \in (0,1),$$

then χ is uniquely determined up to an additive constant. We will assume henceforth that χ satisfies this condition and that $\chi(0) = 0$. Hence an integral Hausdorff method $\hat{H}$ will correspond to a unique χ. We will refer to such a χ as being *normalized*. We then have the following result.

THEOREM 2.2.2. *The method* $\hat{H}_\chi$ *is regular if and only if*

$$\chi(0^+) = \chi(0) = 0$$

and

$$\chi(1) = 1.$$

Hardy [13, p. 276] has proved this result. He requires, however, that f be continuous on finite intervals of the form [0,x]. He evidently restricts himself to the Riemann-Stieltjes integral.

Let $\hat{H}_\chi$ be an integral Hausdorff method. We define a related method $\tilde{H}_\chi$ as follows. For $y > 0$, let

$$\tilde{H}(y) = \tilde{H}_{f,\chi}(y) = \int_0^1 f(yt)\,|d\chi(t)|.$$

If $\tilde{H}(y) \to \sigma$ as $y \to \infty$, we say that f is $\tilde{H}$-*summable* or $\tilde{H}_\chi$-*summable* to σ, and we write

$$f(x) \to \sigma \ (\tilde{H}) \text{ or } f(x) \to \sigma \ (\tilde{H}_\chi),$$

as is convenient. We observe that $\tilde{H}_\chi$ is itself an integral Hausdorff method $\hat{H}_\phi$, where

$$\phi(t) = \int_0^t |d\chi(u)|,$$

the variation of χ on $[0,t]$, is a function of bounded variation. Even if $\hat{H}_\chi$ is regular, $\tilde{H}_\chi$ need not be regular. The following theorem shows that for certain χ, $\tilde{H}_\chi$ does preserve convergence to zero.

THEOREM 2.2.3. *Let* $\hat{H}_\chi$ *be an integral Hausdorff method with* $\chi(0^+) = \chi(0)$. *If* $f(x) \to 0$ *as* $x \to \infty$, *then* $f(x) \to 0 \ (\tilde{H}_\chi)$.

PROOF. Suppose $f(x) \to 0$ as $x \to \infty$. Let

$$V = \int_0^1 |d\chi(t)|,$$

the total variation. Given $\varepsilon > 0$, there is a constant K such that

$$|f(x)| < \frac{\varepsilon}{3V}$$

for $x \geq K$. For $y > K$, we have

$$\tilde{H}_{f;\chi}(y) = \int_0^1 f(yt)\,|d\chi(t)|$$

$$= \int_0^{K/y} f(yt)\,|d\chi(t)| + \int_{K/y}^1 f(yt)\,|d\chi(t)|$$

$$- f(K)\,\Delta\chi(K/y),$$

where

$$\Delta\chi(x) = \chi(x^+) - \chi(x^-).$$

Now, $|f(K)||\Delta\chi(K/y)| < \frac{\varepsilon}{3V} \cdot V = \frac{\varepsilon}{3}$.

Since for $t \in [K/y,1]$ we have $yt \geq \frac{yK}{y} = K$, it follows that

$$\left|\int_{K/y}^{1} f(yt)\,|d\chi(t)|\right| \leq \frac{\varepsilon}{3V} \cdot V = \frac{\varepsilon}{3}.$$

Finally,

$$\left|\int_{0}^{K/y} f(yt)\,|d\chi(t)|\right| \leq \int_{0}^{K/y} |f(yt)|\,|d\chi(t)|$$

$$\leq \max_{x\in[0,K]} |f(x)| \int_{0}^{K/y} |d\chi(t)|$$

$$< \frac{\varepsilon}{3} \text{ for } y \text{ sufficiently large.}$$

Hence $\hat{H}_{f;\chi}(y) \to 0$ as $y \to \infty$; that is, $f(x) \to 0$ $(\hat{H}_\chi)$.

COROLLARY. *Let* $\hat{H}_\chi$ *be a regular integral Hausdorff method. If* $f(x) \to 0$ *as* $x \to \infty$, *then* $f(x) \to 0$ $(\hat{H}_\chi)$.

PROOF. By Lemma 2.2.1, the regularity of $\hat{H}_\chi$ implies $\chi(0^+) = \chi(0)$, and we can apply the theorem.

We now examine the behaviour of the product of two integral Hausdorff methods.

THEOREM 2.2.4. *Let* $\hat{H}_\chi$, $\hat{H}_\psi$ *be integral Hausdorff methods. Then for* $y > 0$, $H_\chi H_\psi(y) = H_\psi H_\chi(y)$.

PROOF. We have

$$\begin{aligned}
H_\chi H_\psi(y) &= \int_0^1 H_\psi(yx)\,d\chi(x) \\
&= \int_0^1 \int_0^1 f(yxt)\,d\psi(t)\,d\chi(x) \\
&= \int_0^1 \int_0^1 f(yxt)\,d\chi(x)\,d\psi(t) \\
&= \int_0^1 H_\chi(yt)\,d\psi(t) \\
&= H_\psi H_\chi(y).
\end{aligned}$$

Since the integrals are absolutely convergent, the interchange of integrals is valid. This completes the proof of the theorem.

As an immediate corollary, we have:

THEOREM 2.2.5. *Integral Hausdorff methods commute; that is, for any integral Hausdorff methods* $\hat{H}_\chi$ *and* $\hat{H}_\psi$,

$$\hat{H}_\chi \hat{H}_\psi = \hat{H}_\psi \hat{H}_\chi .$$

~~We emphasize that $\hat{H}_\chi$ and $\hat{H}_\psi$ need not be regular and that the product methods are equal, not merely equivalent.~~

Rogosinski [23] has proved the following results. Although he defines his integral Hausdorff methods so as to be regular, examination of the proofs shows this to be unnecessary (see also Appendix 2).

THEOREM C. *Let* $\hat{H}_\chi$ *and* $\hat{H}_\psi$ *be integral Hausdorff methods.*

(i) *Integral Hausdorff methods commute, with*

$$\hat{H}_\psi \hat{H}_\chi = \hat{H}_\chi \hat{H}_\psi .$$

(ii) *The product of two integral Hausdorff methods is itself an integral Hausdorff method.*

(iii) *If* $H_\phi = H_\psi H_\chi$, *then* $\hat{H}_\phi = \hat{H}_\psi \hat{H}_\chi$.

We note that (i) is an alternate to Theorem 2.2.4, but its proof involves proving the stronger result (ii). That the product of two sequence-to-sequence Hausdorff methods is again a sequence-to-sequence Hausdorff method is familiar (see, for example, [13, Ch. XI]).

The final theorem of this section is again due to Rogosinski and deals with the relative strengths of Hausdorff methods. First, we need a definition. For k = 1,2,3,... , the *moment of order* k of the Hausdorff methods H_χ and $\hat{H}_\chi$ is defined to be (see, for example, [13, Ch. XI])

$$\mu_k = \int_0^1 t^k d\chi(t),$$

THEOREM D. [22,23] *Let* H_1 *and* H_2 *be sequence-to-sequence Hausdorff methods. If* $H_1 \subseteq H_2$ *and at most a finite number of the moments of* H_1 *vanish, then there exists a unique regular sequence-to-sequence Hausdorff method* H_3 *such that*

$$H_2 = H_3 H_1.$$

Further,

$$\hat{H}_2 = \hat{H}_3 \hat{H}_1.$$

Hence,

$$\hat{H}_1 \subseteq \hat{H}_2.$$

Again, regularity is not necessary for the proof.

Rogosinski [23] shows that the converse to the final result of Theorem D is not true. A counterexample is provided by the Euler method. For $\alpha \geq 1$, let

$$\chi_\alpha(t) = \begin{cases} 0 & \text{for } 0 \leq t < \frac{1}{\alpha} \\ 1 & \text{for } \frac{1}{\alpha} \leq t \leq 1 \end{cases}.$$

Now the integral Euler method $(\hat{E},\alpha)$ is given by

$$E_\alpha(y) = \int_0^1 f(yt)\,d\chi_\alpha(t)$$

$$= f\left(\frac{y}{\alpha}\right), \text{ for } y > 0.$$

Hence for $\alpha \geq 1$, the methods $(\hat{E},\alpha)$ are identical and are merely convergence, in that there are no functions which do not converge which are $(\hat{E},\alpha)$-summable for any $\alpha \geq 1$. For $\alpha > 1$, the sequence-to-sequence Euler methods (E,α) are not equivalent to convergence and for $\alpha' > \alpha$ satisfy

$$(E,\alpha) \subsetneq (E,\alpha')$$

(see, for example, [13, pp. 179-180]). Since, for $k = 1,2,3,\ldots$

$$\mu_k = \int_0^1 t^k\,d\chi_\alpha(t) = \alpha^{-k},$$

which never vanishes, the moment condition of Theorem D is satisfied. Thus we have, for $\alpha' > \alpha \geq 1$,

$$(\hat{E},\alpha') \subseteq (\hat{E},\alpha),$$

but $(E,\alpha') \not\subseteq (E,\alpha)$, even though the moments of (E,α') never vanish.

2.3 INTEGRAL CESARO-TYPE METHODS.

Throughout this section, suppose that $\alpha > 0$, $\beta > -1$, $\gamma > 0$, and $\delta > -1$.

The *integral Cesàro-type method* is defined as follows (see, for example, [3]). For $y > 0$, let

$$C_{\alpha,\beta}(y) = C_{f;\alpha,\beta}(y)$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \int_0^1 (1-t)^{\alpha-1} t^{\beta} f(yt)\,dt.$$

If $C_{\alpha,\beta}(y)$ exists for all $y > 0$ and if $C_{\alpha,\beta}(y) \to \sigma$ as $y \to \infty$, then we say that f is $(\hat{C},\alpha,\beta)$*-summable* to σ and we write

$$f(x) \to \sigma \quad (\hat{C},\alpha,\beta).$$

We also define the integral Cesàro method $(\hat{C},\alpha)$ to be $(\hat{C},\alpha,0)$, with

$$C_{\alpha}(y) = C_{f;\alpha}(y)$$

$$= \alpha \int_0^1 (1-t)^{\alpha-1} f(yt)\,dt,$$

(see, for example, [13, p. 110; 23]). We could require that f be only Lebesgue measurable (see, for example [20]).

By a simple change of variable, we obtain

$$C_{\alpha,\beta}(y) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)y^{\alpha+\beta}} \int_0^y (y-t)^{\alpha-1} t^{\beta} f(t)\,dt.$$

Borwein [7] has defined the sequence-to-sequence Cesàro-type method (C,ρ,β) for $\beta > -1$, $\rho+\beta > -1$, as follows. Given a sequence $\{s_n\}$, let

$$s_n^{\rho,\beta} = \frac{1}{\binom{n+\rho+\beta}{n}} \sum_{\nu=0}^{n} \binom{n-\nu+\rho-1}{n-\nu} \binom{\nu+\beta}{\nu} s_{\nu}.$$

If $s_n^{\rho,\beta} \to \sigma$ as $n \to \infty$, we say that $\{s_n\}$ is (C,ρ,β)-summable to σ, and we write

$$s_n \to \sigma \quad (C,\rho,\beta).$$

The case $\beta = 0$ gives the familiar (C,ρ) method (see, for example, [13, Ch. V]). The regularity of (C,ρ,β) and (C,ρ) are familiar.

Letting

$$\phi(t) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \int_0^t (1-x)^{\alpha-1} x^{\beta} dx,$$

we see that $(\hat{C},\alpha,\beta)$ is an integral Hausdorff method, $\hat{H}_\phi$. Since $\phi(0^+) = \phi(0) = 0$ (by definition) and $\phi(1) = 1$, by Theorem 2.2.2, $(\hat{C},\alpha,\beta)$ is a regular integral Hausdorff method.

We now show that the behaviour of integral Cesàro-type methods parallels that of sequence-to-sequence Cesàro-type methods. Borwein [7] has proved

THEOREM E. $(C,\alpha,\beta) \simeq (C,\alpha)$.

Using this, we show

THEOREM 2.3.1. $(\hat{C},\alpha,\beta) \simeq (\hat{C},\alpha)$.

PROOF. The result follows immediately from Theorems E and D, provided at most a finite number of the moments of (C,α) and (C,α,β) vanish. We show, in fact, that none of these moments vanish. For $k = 1,2,3,\ldots$ the moment of order k of (C,α,δ) is

$$\mu_k = \frac{\Gamma(\alpha+\delta+1)}{\Gamma(\alpha)\Gamma(\delta+1)} \int_0^1 (1-t)^{\alpha-1} t^{k+\delta} dt$$

$$= \frac{\Gamma(\alpha+\delta+1)\Gamma(\alpha)\Gamma(k+\delta+1)}{\Gamma(\alpha)\Gamma(\delta+1)\Gamma(\alpha+\delta+k+1)}$$

$$\neq 0.$$

Letting $\delta = 0$ and β, we see that (C,α) and (C,α,β) have no moments which vanish. This completes the proof of the theorem.

The following theorem is again the integral analogue of a sequence-to-sequence theorem by Borwein [7]. He has proved, for $\beta > -1$ and $\rho+\beta > -1$

$$(C,\rho,\beta)(C,\beta) = (C,\rho+\beta).$$

THEOREM 2.3.2. $\quad (\hat{C},\alpha,\gamma)(\hat{C},\gamma) = (\hat{C},\alpha+\gamma).$

Hence $\quad (\hat{C},\alpha)(\hat{C},\gamma) \simeq (\hat{C},\alpha+\gamma).$

PROOF. We have

$$C_{\alpha,\gamma}C_{\gamma}(y) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha)\Gamma(\gamma+1)y^{\alpha+\gamma}} \int_0^y (y-t)^{\alpha-1} t^{\gamma} \frac{\gamma}{t^{\gamma}} \int_0^t (t-x)^{\gamma-1} f(x)\,dx\,dt$$

$$= \frac{(\alpha+\gamma)\Gamma(\alpha+\gamma)\ \gamma}{\Gamma(\alpha)\gamma\Gamma(\gamma)y^{\alpha+\gamma}} \int_0^y f(x) \int_x^y (y-t)^{\alpha-1}(t-x)^{\gamma-1}\,dt\,dx,$$

the absolute convergence of the integrals allowing us to change the order of integration. Now

$$\int_x^y (y-t)^{\alpha-1}(t-x)^{\gamma-1}dt = \int_0^{y-x} [(y-x)-u]^{\alpha-1}u^{\gamma-1}du$$

$$= \int_0^1 [(y-x) - (y-x)t]^{\alpha-1}[(y-t)t]^{\gamma-1}(y-x)\,dt$$

$$= (y-x)^{\alpha+\gamma-1} \int_0^1 (1-t)^{\alpha-1}t^{\gamma-1}dt$$

$$= (y-x)^{\alpha+\gamma-1} \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}$$

Hence

$$C_{\alpha,\gamma}C_{\gamma}(y) = \frac{\alpha+\gamma}{y^{\alpha+\gamma}}\int_0^y (y-x)^{\alpha+\gamma-1}f(x)\,dx$$

$$= C_{\alpha+\gamma}(y).$$

Thus $$(\hat{C},\alpha,\gamma)(\hat{C},\gamma) = (\hat{C},\alpha+\gamma).$$

By Theorem 2.3.1, we also have

$$(\hat{C},\alpha)(\hat{C},\gamma) \simeq (\hat{C},\alpha+\gamma).$$

An immediate corollary of Theorems 2.3.1 and 2.3.2, is

THEOREM 2.3.3. *For* $\alpha > \gamma > 0$ *and* $\beta,\delta > -1$,

$$(\hat{C},\gamma,\delta) \subseteq (\hat{C},\alpha,\beta).$$

Theorem D could also be used to prove this result. The similar theorem for sequence-to-sequence Cesàro methods is also true (see, for example, [7 ;13, Th. 43]).

CHAPTER 3

INTEGRAL ABEL-TYPE METHODS

3.1 INTRODUCTION.

Throughout this chapter, we suppose that $\lambda > -1$.

We define the *integral Abel-type method of order* λ as follows (see Jakimovski [34]). For $y > 0$, let

$$A_\lambda(y) = A_{f;\lambda}(y)$$

$$= \frac{1}{\Gamma(\lambda+1)} \int_0^\infty e^{-x} x^\lambda f(xy)\,dx.$$

If $A_\lambda(y)$ exists as a Cauchy-Lebesgue integral[1] and if $A_\lambda(y) \to \sigma$ as $y \to \infty$, then we say that f is $\hat{A}_\lambda$-*summable* to σ and we write

$$f(x) \to \sigma \quad (\hat{A}_\lambda).$$

The method $\hat{A}_0$ is the Laplace transform (see, for example, [10, 31]). Shawyer and Yang [27] utilize the $\hat{A}_\lambda$ transform but do not explicitly define the method.

The sequence-to-function Abel-type method A_μ was defined by Jakimovski [34]. For a sequence $\{s_n\}$, if

$$(1-x)^{\mu+1} \sum_{n=0}^\infty \binom{n+\mu}{n} s_n x^n$$

---

[1]By this we mean $\int_0^\infty = \lim_{R\to\infty} \int_0^R$, where $\int_0^R$ is a Lebesgue integral.

is convergent for all $x \in (0,1)$ and tends to σ as $x \to 1^-$ in $(0,1)$, then we say that the sequence is A_μ-summable to σ and we write

$$s_n \to \sigma \ (A_\mu).$$

It is clear that $\hat{A}_\lambda$ is not an integral Hausdorff method and that A_μ is not a sequence-to-sequence Hausdorff method.

By a simple change of variable we obtain

$$A_\lambda(y) = \frac{1}{\Gamma(\lambda+1)y^{\lambda+1}} \int_0^\infty e^{-u/y}u^\lambda f(u)\,du.$$

Another convenient form is

$$A_\lambda(\tfrac{1}{t}) = \frac{t^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-ut}u^\lambda f(u)\,du.$$

Here we are interested in the behaviour of $A_\lambda(\frac{1}{t})$ as $\frac{1}{t} \to \infty$; that is, as $t \to 0^+$.

3.2 PROPERTIES OF INTEGRAL ABEL-TYPE METHODS.

Borwein [3] has shown that for $\mu > -1$, the sequence-to-function Abel-type method A_μ is regular. We prove a similar result for integral Abel-type methods.

THEOREM 3.2.1. *The method* $\hat{A}_\lambda$ *is regular.*

PROOF. Since

$$A_\lambda(y) = \frac{1}{\Gamma(\lambda+1)y^{\lambda+1}} \int_0^\infty e^{-u/y}u^\lambda f(u)\,du,$$

we have

$$\frac{1}{\Gamma(\lambda+1)y^{\lambda+1}}\int_0^\infty e^{-u/y}u^\lambda du = \frac{1}{\Gamma(\lambda+1)}\int_0^\infty e^{-x}x^\lambda dx$$
$$= 1.$$

Let $Y > 0$. Then

$$\frac{1}{\Gamma(\lambda+1)y^{\lambda+1}}\int_0^Y e^{-u/y}u^\lambda du = \frac{1}{\Gamma(\lambda+1)}\int_0^{Y/y} e^{-x}x^\lambda dx$$
$$\to 0 \text{ as } y \to \infty.$$

Hence, by Theorem A, $\hat{A}_\lambda$ is a regular method of summability.

In defining the integral Abel-type method, we have employed the Cauchy-Lebesgue integral. Unlike the Lebesgue integral, the Cauchy-Lebesgue integral is not absolutely convergent, nor does it possess a simple general theorem enabling us to change orders of integration where needed.[1] To avoid these problems, we will henceforth require that $A_\lambda(y)$ exist as a *Lebesgue integral* (see, also [10, p. 11]).

If $A_\lambda(y)$ exists for all $y > 0$, we say that $\hat{A}_\lambda$ is *applicable*, or, *applicable to* f, as is convenient. If $A_\lambda(y)$ exists and is bounded for all $y > 0$, we say that f is $\hat{A}_\lambda$*-bounded* and we write

$$f(x) = O(1)(\hat{A}_\lambda).$$

We will show that integral Abel-type methods have a natural scale of inclusions. We first prove two lemmata.

LEMMA 3.2.1. *Let* $\lambda > \mu > -1$ and $y > 0$. *If* $A_\lambda(y)$ *exists, then* $A_\mu(y)$ *exists.*

[1]See Appendix 3.

PROOF. It is sufficient to assume $f(x) \geq 0$ for all $x \in [0,\infty)$. We have

$$A_\mu(y) = \frac{1}{\Gamma(\mu+1)y^{\mu+1}} \int_0^\infty e^{-x/y} x^\mu f(x)\,dx$$

$$= \frac{1}{\Gamma(\mu+1)y^{\mu+1}} \int_0^1 e^{-x/y} x^\mu f(x)\,dx$$

$$+ \frac{1}{\Gamma(\mu+1)y^{\mu+1}} \int_1^\infty e^{-x/y} x^\lambda x^{\mu-\lambda} f(x)\,dx$$

$$\leq \frac{1}{\Gamma(\mu+1)y^{\mu+1}} \int_0^1 e^{-x/y} x^\mu f(x)\,dx$$

$$+ \frac{1}{\Gamma(\mu+1)y^{\mu+1}} \int_1^\infty e^{-x/y} x^\lambda f(x)\,dx$$

$$< \infty .$$

This completes the proof of the lemma.

LEMMA 3.2.2. (cf. [3]) *Let* $\lambda > \mu > -1$ and $y \geq 0$. *If* $A_\lambda(y)$ *exists, then*

$$A_\mu(y) = C_{\lambda-\mu,\mu} A_\lambda(y).$$

PROOF. By Lemma 3.2.1, $A_\mu(y)$ exists. Now

$$C_{\lambda-\mu,\mu} A_\lambda(y)$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\Gamma(\mu+1)y^\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^\mu \frac{1}{\Gamma(\lambda+1)t^{\lambda+1}} \int_0^\infty e^{-x/t} x^\lambda f(x)\,dx\,dt$$

$$= \frac{1}{\Gamma(\lambda-\mu)\Gamma(\mu+1)y^\lambda} \int_0^y \left(\frac{y}{t}-1\right)^{\lambda-\mu-1} \frac{1}{t^2} \int_0^\infty e^{-x/t} x^\lambda f(x)\,dx\,dt$$

$$= \frac{1}{\Gamma(\lambda-\mu)\Gamma(\mu+1)y^\lambda} \int_0^\infty x^\lambda f(x) \int_0^y \left(\frac{y}{t}-1\right)^{\lambda-\mu-1} \frac{e^{-x/t}}{t^2}\,dt\,dx.$$

Since the integrals are absolutely convergent, we are able to change the order of integration.

Letting $z = \frac{x}{t} - \frac{x}{y}$ and hence $dz = -\frac{x}{t^2}dt$, we obtain

$$\int_0^y \left(\frac{y}{t}-1\right)^{\lambda-\mu-1} \frac{e^{-x/t}}{t^2}dt = \int_0^\infty \left(\frac{y}{x} z\right)^{\lambda-\mu-1} \frac{e}{x}^{-z-x/y} dz$$

$$= \frac{y^{\lambda-\mu-1}}{x^{\lambda-\mu}} e^{-x/y} \int_0^\infty z^{\lambda-\mu-1}e^{-z}dz$$

$$= \frac{y^{\lambda-\mu-1}e^{-x/y}}{x^{\lambda-\mu}} \Gamma(\lambda-\mu).$$

Substituting this simplification, we obtain the desired result

$$C_{\lambda-\mu,\mu}A_\lambda(y) = A_\mu(y).$$

If $\hat{A}_\lambda$ is applicable (to f) for all $\lambda > -1$, we say that $\hat{A}$ *is applicable (to* f). The following theorem is a direct consequence.

THEOREM 3.2.2. (i) *If* $\hat{A}$ *is applicable, then* $\hat{A}$ *is applicable.*

(ii) *Let* $\lambda > \mu > -1$. *If* $\hat{A}$ *is applicable, then*

$$\hat{A}_\mu = (\hat{C},\lambda-\mu,\mu)\hat{A}_\lambda.$$

The more general version of this theorem with Cauchy-Lebesgue integrals, has been proved by Jakimovski [34].

Since $(\hat{C},\lambda-\nu,\nu)$ is a regular method, we have as a corollary:

If $\lambda > \mu > -1$, *then* $\hat{A}_\mu \supseteq \hat{A}_\lambda$.

We will now show that this is a strict inclusion; that is, there exists a function which is $\hat{A}_\mu$-summable but not $\hat{A}_\lambda$-summable. We need the following familiar theorem (see, for example, [19, Theorem 249]).

THEOREM F. (Riemann-Lebesgue) *Suppose that* F *is Lebesgue integrable over* $(-\infty,\infty)$ *and that* G *is bounded and Lebesgue measurable in* $(-\infty,\infty)$. *Suppose also that there is a constant* c *such that for all* x, $G(x+c) = -G(x)$. *Then*

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} F(x)G(kx)\,dx = 0.$$

Using this result, we are able to show:

THEOREM 3.2.3. (cf. [3]) *Let* $\lambda > -1$. *Then there exists a function which is* $\hat{A}_\mu$*-summable to zero for all* μ *with* $\lambda > \mu > -1$ *but which is not* $\hat{A}_\lambda$*-summable.*

PROOF. For all real x we define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where
$$a_n = \begin{cases} 0 & \text{for n even} \\ \dfrac{(-1)^{[n/2]}\,\Gamma(\lambda+1)}{n!\;\Gamma(\lambda+n+1)} & \text{for n odd.} \end{cases}$$

Since
$$|f(x)| \le \sum_{n=0}^{\infty} |a_n|\,|x|^n$$
$$\le \sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$
$$= e^{|x|}$$
$$< \infty,$$

f(x) exists for all real x. Hence

$$A_{f;\lambda}(y) = \frac{1}{\Gamma(\lambda+1)} \int_0^\infty e^{-u} u^\lambda f(uy)\,du$$

$$= \frac{1}{\Gamma(\lambda+1)} \int_0^\infty e^{-u} u^\lambda \sum_{n=0}^\infty a_n u^n y^n du$$

$$= \sum_{n=0}^\infty \frac{a_n}{\Gamma(\lambda+1)} y^n \int_0^\infty e^{-u} u^{\lambda+n} du$$

$$= \sum_{n=0}^\infty \frac{a_n}{\Gamma(\lambda+1)} \Gamma(\lambda+n+1) y^n$$

$$= \sum_{m=0}^\infty \frac{(-1)^m y^{2m+1}}{(2m+1)!}$$

$$= \sin y.$$

We observe that

$$\sum_{m=0}^\infty \frac{|y|^{2m+1}}{(2m+1)!} \le \sum_{n=0}^\infty \frac{|y|^n}{n!}$$

$$= e^{|y|} < \infty.$$

Hence the sum resulting from the interchange of order of integration and summation is absolutely convergent, and the interchange is thus valid.

We now have $A_{f;\lambda}(y) = \sin y$. Since $\sin y$ does not tend to a limit as $y \to \infty$, f is not $\hat{A}_\lambda$-summable. However, by Theorem 3.2.2,

$$A_{f;\mu}(y) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\Gamma(\mu+1)y^\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^\mu \sin t\,dt$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\Gamma(\mu+1)} \int_0^1 (1-u)^{\lambda-\mu-1} u^\mu \sin(uy)\,du$$

$$\to 0 \text{ as } y \to \infty, \text{ by Theorem F.}$$

Hence $f(x) \to 0$ $(\hat{A}_\mu)$. This completes the proof of the theorem.

Summarizing, we have the following theorem (cf. [3]).

THEOREM 3.2.4. (i) *Let* $\lambda > \mu > -1$. *Then*

$$\hat{A}_\mu \supsetneqq \hat{A}_\lambda .$$

(ii) *Let* $\alpha > \gamma > 0$. *Then*

$$(\hat{C},\alpha) \supsetneqq (\hat{C},\gamma).$$

PROOF. ~~The first result follows immediately from~~ Theorem 3.2.3.

We now prove the second result. Let $\delta > \alpha - 1$. By Theorem 3.2.2 and 2.3.1, if $\hat{A}$ is applicable,

$$\hat{A}_{\delta-\gamma} = (\hat{C},\gamma,\delta-\gamma)\hat{A}_\delta \simeq (\hat{C},\gamma)\hat{A}_\delta$$

and

$$\hat{A}_{\delta-\alpha} = (\hat{C},\alpha,\delta-\alpha)\hat{A}_\delta \simeq (\hat{C},\alpha)\hat{A}_\delta .$$

By Theorem 3.2.3, there is a function f such that $f(x) \to 0$ $(\hat{A}_{\delta-\alpha})$ and f is not $\hat{A}_{\delta-\gamma}$-summable. Letting $g(y) = A_{f;\delta}(y)$, we have $g(x) \to 0$ $(\hat{C},\alpha)$ and g is not $(\hat{C},\gamma)$-summable.

CHAPTER 4

SOME RELATIONS BETWEEN INTEGRAL ABEL-TYPE AND HAUSDORFF METHODS

4.1 INTRODUCTION.

Throughout this chapter, we assume that $\lambda > -1$.

In this chapter we investigate the product of integral Abel-type and Hausdorff methods. Among other results, we find that the product is commutative.

4.2 RESULTS.

THEOREM 4.2.1. *Let* $\hat{H}$ *be an integral Hausdorff method. If* $A_\lambda(y)$ *exists for some* $y > 0$, *then*

$$HA_\lambda(y) = A_\lambda H(y).$$

Moreover, if $\hat{A}$ *is applicable,*

$$\hat{H}\hat{A}_\lambda = \hat{A}_\lambda \hat{H};$$

that is, integral Abel-type and Hausdorff methods commute.

PROOF. Let $\hat{H} = \hat{H}_\chi$. Then

$$HA_\lambda(y) = \int_0^1 \frac{1}{\Gamma(\lambda+1)} \int_0^\infty e^{-u} u^\lambda f(yut)\, du\, d\chi(t)$$

$$= \frac{1}{\Gamma(\lambda+1)} \int_0^\infty e^{-u} u^\lambda \int_0^1 f(yut)\, d\chi(t)\, du$$

$$= A_\lambda H(y).$$

To justify the change of order of integration, we prove that $HA_\lambda(y)$ is absolutely convergent. To show this, it is sufficient to prove that

$$\int_0^\infty e^{-u}u^\lambda |f(yut)|du$$

is bounded with respect to t, for y fixed (cf. [31, p. 181]). Letting $g(ut) = f(yut)$, since y is fixed, we will examine

$$\frac{1}{t^{\lambda+1}} \int_0^\infty e^{-u/t}u^\lambda |g(u)|du.$$

Let $\varepsilon > 0$ be given. For $t \in [\varepsilon,\infty)$, we have

$$\frac{1}{t^{\lambda+1}} \int_0^\infty e^{-u/t}u^\lambda |g(u)|du$$

$$\leq \frac{1}{\varepsilon^{\lambda+1}} \int_0^\infty e^{-u/\varepsilon}u^\lambda |g(u)|du$$

which is bounded independently of t.

Suppose now that $t < \varepsilon$. We first observe that

$$\int_0^x u^\lambda |g(u)|du = \int_0^x e^{u/\varepsilon}e^{-u/\varepsilon}u^\lambda |g(u)|du$$

$$\leq e^{x/\varepsilon} \int_0^x e^{-u/\varepsilon}u^\lambda |g(u)|du$$

$$\leq e^{x/\varepsilon} \int_0^\infty e^{-u/\varepsilon}u^\lambda |g(u)|du$$

$$= e^{x/\varepsilon} K,$$

where K is a constant independent of x but dependent on ε. Let R be given. Let $S > R$. Integrating by parts, we obtain

$$\int_R^S e^{-u/t}u^\lambda|g(u)|du$$

$$= e^{-u/t}\int_0^u x^\lambda|g(x)|dx\Big|_{u=R}^{u=S} + \int_R^S \frac{e^{-u/t}}{t}\int_0^u x^\lambda|g(x)|dx\,du$$

$$\le e^{-S/t}\int_0^S x^\lambda|g(x)|dx + \int_R^S \frac{e^{-u/t}}{t}\int_0^u x^\lambda|g(x)|dx\,du$$

$$\le e^{-S/t}e^{S/\varepsilon}K + \frac{1}{t}\int_R^S e^{-u/t}e^{u/\varepsilon}K\,du$$

$$= K e^{-S(1/t-1/\varepsilon)} + \frac{K}{t}\int_R^S e^{-u(1/t-1/\varepsilon)}du$$

$$\le K e^{-S(1/t-1/\varepsilon)} + \frac{K}{t}\frac{e^{-R(1/t-1/\varepsilon)}}{(1/t-1/\varepsilon)}$$

$$\to \frac{K}{t}\frac{e^{-R(1/t-1/\varepsilon)}}{(1/t-1/\varepsilon)} \text{ as } S \to \infty.$$

We therefore have

$$\frac{1}{t^{\lambda+1}}\int_R^\infty e^{-u/t}u^\lambda|g(u)|du \le \frac{K\,e^{-R(1/t-1/\varepsilon)}}{t^{\lambda+1}(1/t-1/\varepsilon)}$$

$$\to 0 \text{ as } t \to 0^+.$$

Our final task is then to show that

$$\frac{1}{t^{\lambda+1}}\int_0^R e^{-u/t}u^\lambda|g(u)|du$$

is bounded as $t \to 0^+$. Integrating by parts, we have

$$\int_0^R e^{-u/t}u^\lambda|g(u)|du$$

$$= e^{-u/t}\int_0^u x^\lambda|g(x)|dx\Big|_{u=0}^{u=R} + \frac{1}{t}\int_0^R e^{-u/t}\int_0^u x^\lambda|g(x)|dx\,d\mu$$

$$\le e^{-R/t}\int_0^R x^\lambda|g(x)|dx + \frac{1}{t}\int_0^R e^{-u/t}u^{\lambda+1}\frac{1}{u^{\lambda+1}}\int_0^u x^\lambda|g(x)|dx\,du.$$

Examining the first integral, we have

$$e^{-R/t}\int_0^R x^\lambda |g(x)|dx \le e^{-R/t}e^{R/\varepsilon}K$$

$$= K\, e^{-R(1/t-1/\varepsilon)}.$$

The second integral is less than or equal to

$$\frac{1}{t}\int_0^R e^{-u/t}u^{\lambda+1}du \quad \max_{u\in[0,R]} \frac{1}{u^{\lambda+1}}\int_0^u x^\lambda |g(x)|dx$$

$$= t^{\lambda+1}\Gamma(\lambda+2) \quad \max_{u\in[0,R]} \frac{1}{u^{\lambda+1}}\int_0^u x^\lambda |g(x)|dx$$

Now we have

$$\max_{u\in[0,R]} \frac{1}{u^{\lambda+1}}\int_0^u x^\lambda |g(x)|dx$$

$$\le \max_{u\in[0,R]} |g(u)| \cdot \max_{u\in[0,R]} \frac{1}{u^{\lambda+1}}\int_0^u x^\lambda dx$$

$= K_1$, where K_1 is a constant dependent on R alone.

Thus we have

$$\frac{1}{t^{\lambda+1}}\int_0^R e^{-u/t}u^\lambda |g(u)|du \le \frac{K\, e^{-R(1/t-1/\varepsilon)}}{t^{\lambda+1}} + \frac{K_1 t^{\lambda+1}\Gamma(\lambda+2)}{t^{\lambda+1}}$$

$$\to K_1\Gamma(\lambda+2) \quad \text{as } t \to 0^+.$$

This completes the proof of the theorem.

The sequence-to-function analogue was proved by Borwein [3]. The next theorem parallels a sequence-to-function result of Jakimovski [18]. It has been proved with Cauchy-Lebesgue integrals by Jakimovski [34].

THEOREM 4.2.2. *Let* $\hat{H}$ *be a regular integral Hausdorff method. Then* $\hat{A}_\lambda \hat{H} \supseteq \hat{A}_\lambda$.

PROOF. This result is immediate from Theorem 4.2.1 and the regularity of $\hat{H}$.

The following theorem gives a stronger inclusion.

THEOREM 4.2.3. (cf. [3]) *Let* $\hat{H}_\chi$ *be a regular integral Hausdorff method with* $\chi(t)$ *absolutely continuous in* $[0,1]$. *Then*

$$\hat{A}_\lambda \hat{H}_\chi \supsetneqq \hat{A}_\lambda .$$

PROOF. By Theorem 4.2.2, we have

$$\hat{A}_\lambda \hat{H}_\chi \supseteq \hat{A}_\lambda .$$

We show by example that

$$\hat{A}_\lambda \hat{H}_\chi \neq \hat{A}_\lambda .$$

In Theorem 3.2.3, we defined a function f such that $A_{f;\lambda}(y) = \sin y$. Now

$$\begin{aligned} A_\lambda H_{f;\chi}(y) &= H_\chi A_{f;\lambda}(y), \text{ by Theorem 4.2.1} \\ &= \int_0^1 \sin(yt)\, d\chi(t) \\ &= \int_0^1 \sin(yt)\chi'(t)\,dt, \text{ by the absolute continuity of } \chi. \\ &\to 0 \text{ as } y \to \infty, \text{ by the Riemann-Lebesgue Theorem (Theorem F).} \end{aligned}$$

Hence $f(x) \to 0$ $(\hat{A}_\lambda \hat{H})$ but $f(x)$ is not $\hat{A}_\lambda$-summable, since $\sin y$ does not tend to a limit as $y \to \infty$. This completes the proof of the theorem.

THEOREM 4.2.4. (cf. [7]) *Let* $\hat{H}_1$ *and* $\hat{H}_2$ *be integral Hausdorff methods with* $\hat{H}_1$ *regular. If* $\hat{A}$ *is applicable, then*

$$\hat{A}_\lambda \hat{H}_2 \hat{H}_1 \supseteq \hat{A}_\lambda \hat{H}_2.$$

PROOF. Using Theorems 4.2.2, 4.2.1 and 2.2.5, we have

$$\hat{A}_\lambda \hat{H}_2 \subseteq \hat{H}_1 \hat{A}_\lambda \hat{H}_2 = \hat{A}_\lambda \hat{H}_1 \hat{H}_2 = \hat{A}_\lambda \hat{H}_2 \hat{H}_1.$$

THEOREM 4.2.5. (cf. [7]) *Let* $\hat{H}_1$ *and* $\hat{H}_2$ *be integral Hausdorff methods, with* $\hat{H}_1 \supseteq \hat{H}_2$. *If* $\hat{A}$ *is applicable, then*

$$\hat{A}_\lambda \hat{H}_1 \supseteq \hat{A}_\lambda \hat{H}_2.$$

PROOF. Using Theorem 4.2.1 twice, we have

$$\hat{A}_\lambda \hat{H}_2 = \hat{H}_2 \hat{A}_\lambda \subseteq \hat{H}_1 \hat{A}_\lambda = \hat{A}_\lambda \hat{H}_1.$$

Our final theorem of this chapter relates integral Abel-type and Cesàro-type methods.

THEOREM 4.2.6. (cf. [7]) *Let* $\alpha > 0$ *and* $\beta > -1$. *If* $\hat{A}$ *is applicable, then* $\hat{A}_\lambda \supsetneqq (\hat{C},\alpha,\beta)$.

PROOF. By Theorem 3.2.2,

$$\begin{aligned} \hat{A}_\lambda &\doteq (\hat{C},\alpha,\lambda)\hat{A}_{\lambda+\alpha} \\ &= \hat{A}_{\lambda+\alpha}(\hat{C},\alpha,\lambda), \text{ by Theorem 4.2.1} \\ &\supseteq (\hat{C},\alpha,\lambda) \\ &\underset{\sim}{\sim} (\hat{C},\alpha,\beta), \text{ by Theorem 2.3.1.} \end{aligned}$$

In View of Theorem 3.2.4, we have

$$\hat{A}_\lambda \supsetneqq (\hat{C},\alpha,\beta).$$

CHAPTER 5

THE METHOD $\hat{A}_{-1}$

5.1 INTRODUCTION.

We define the *integral Abel-type method* $\hat{A}_{-1}$ as follows. For $y > 0$, let

$$A_{-1}(y) = A_{f;-1}(y) = \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(x+1)/y}}{x+1} f(x)\,dx.$$

If $A_{-1}(y)$ exists (as a Cauchy-Lebesgue integral) for all $y > 0$, and if $A_{-1}(y) \to \sigma$ as $y \to \infty$, then we say that f is $\hat{A}_{-1}$-*summable* to σ and we write

$$f(x) \to \sigma \ (\hat{A}_{-1}).$$

This method is the integral analogue of the sequence-to-function method L defined by Hardy [13, p. 81] (see also Borwein [2]) as follows. Let $\{s_n\}$ be a sequence of complex numbers. If

$$\frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n x^{n+1}}{n+1} \to \sigma \text{ as } x \to 1^- \text{ in } (0,1),$$

then we say that $\{s_n\}$ is L-convergent to σ and we write

$$s_n \to \sigma \ (L).$$

Henceforth we will write A_{-1} in place of L.

Another convenient form of the $\hat{A}_{-1}$-transform is

$$A_{-1}(\tfrac{1}{t}) = \frac{1}{\log(1+\frac{1}{t})} \int_0^\infty \frac{e^{-t(u+1)}}{u+1} f(u)\,du.$$

Here we are interested in the behaviour as $\frac{1}{t} \to \infty$; that is, as $t \to 0^+$.

5.2 BASIC PROPERTIES.

As is the case for the sequence-to-function method A_{-1} [2], the integral Abel-type method $\hat{A}_{-1}$ is regular.

THEOREM 5.2.1. *The method* $\hat{A}_{-1}$ *is a regular method of summability.*

PROOF. We use Theorem A. We have

$$\int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\,dx = \int_{1/y}^\infty \frac{e^{-z}}{z}\,dz$$

$$= \int_t^\infty \frac{e^{-z}}{z}\,dz, \text{ where } t = \frac{1}{y}.$$

Integrating by parts, we obtain

$$\int_t^\infty \frac{e^{-z}}{z}\,dz = e^{-z}\log z\,\Big|_t^\infty + \int_t^\infty e^{-z}\log z\,dz$$

$$= -e^{-t}\log t + O(1), \text{ since } \int_0^\infty e^{-z}\log z\,dz \text{ exists.}$$

Hence $\displaystyle\int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\,dx = e^{-1/y}\log y + O(1)$.

Therefore, we have

$$\frac{1}{\log(1+y)}\int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\,dx = e^{-1/y}\{1 + o(1)\}$$

$$\to 1 \text{ as } y \to \infty.$$

To show that for some constant K

$$\frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\, dx < K \quad \text{for all } y > 0,$$

we show now that

$$\lim_{y\to 0^+} \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\, dx = 0.$$

We use L'Hospital's rule. We have

$$\int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\, dx = -\int_0^z \frac{du}{\log u}$$

where $z = e^{-1/y}$ and $u = e^{-(x+1)/y}$. Hence

$$\frac{d}{dy}\int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\, dx = \frac{1}{y^2} e^{-1/y} \frac{d}{dz} - \int_0^z \frac{du}{\log u}$$

$$= \frac{e^{-1/y}}{y^2} \frac{-1}{\log z}$$

$$= \frac{e^{-1/y}}{y^2}\, y$$

$$= \frac{e^{-1/y}}{y}.$$

We also have

$$\frac{d}{dy} \log(1+y) = \frac{1}{1+y}.$$

We thus obtain

$$\frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(x+1)/y}}{x+1}\, dx \sim (1+y)\, \frac{e^{-1/y}}{y} \quad \text{as } y \to 0^+$$

$$\to 0 \quad \text{as } y \to 0^+.$$

Also, for $Y \geq 0$,

$$\frac{1}{\log(1+y)}\int_0^Y \frac{e^{-(x+1)/y}}{x+1}\,dx \leq \frac{1}{\log(1+y)}\int_0^Y dx$$

$$\to 0 \text{ as } y \to \infty.$$

Hence, by Theorem A, $\hat{A}_{-1}$ is a regular integral method of summability.

Because of the problems involved in using the Cauchy-Lebesgue integral, we again restrict ourselves to the Lebesgue integral. If $A_{-1}(y)$ exists (as a Lebesgue integral) we say that $\hat{A}_{-1}$ is *applicable*, or, $\hat{A}_{-1}$ is *applicable to* f, as is convenient.

Our next theorem shows that any finite interval of the range of integration may be ignored.

THEOREM 5.2.2. *Let* M *be a fixed positive constant. Then*

(i) *there exists a constant* K *such that*

$$\int_0^M \frac{e^{-(x+1)/y}}{x+1} f(x)\,dx < K \quad \textit{for all } y > 0, \textit{ and}$$

(ii) $$\frac{1}{\log(1+y)}\int_0^M \frac{e^{-(x+1)/y}}{x+1} f(x)\,dx \to 0 \textit{ as } y \to \infty.$$

PROOF. We have

$$\left|\int_0^M \frac{e^{-(x+1)/y}}{x+1} f(x)\,dx\right| \leq \int_0^M \frac{e^{-(x+1)/y}}{x+1} |f(x)|\,dx$$

$$\leq \int_0^M |f(x)|\,dx$$

$$< K, \text{ independent of } y.$$

The second result follows immediately, since $\log(1+y) \to \infty$ as $y \to \infty$. This completes the proof of the theorem.

LEMMA 5.2.1. *Let* δ *be real and* $y > 0$. *Let*

$$g(x) = \begin{cases} \dfrac{f(x)}{x+\delta} & \textit{for } x \geq |\delta| + 1 \\ 0 & \textit{otherwise.} \end{cases}$$

If $A_{f;-1}(y)$ *exists, then* $A_{g;-1}(y)$ *exists.*

PROOF. In view of Theorem 5.2.2, we need examine only $\int_M^\infty$, where $M \geq |\delta| + 1$. We have

$$\left|\int_M^\infty \frac{e^{-(x+1)/y}}{x+1} g(x)\,dx\right| \leq \int_M^\infty \frac{e^{-(x+1)/y}|f(x)|}{(x+1)(x+\delta)}\,dx$$

$$\leq \frac{1}{M+\delta} \int_M^\infty \frac{e^{-(x+1)/y}|f(x)|}{x+1}\,dx$$

$$< \infty \text{ by hypothesis.}$$

This completes the proof of the lemma.

Our next theorem relates the $\hat{A}_{-1}$-summability of $f(x)$ and $\frac{f(x)}{x+\delta}$.

THEOREM 5.2.3. (cf. [2]) *Let* δ *be real. Let*

$$g(x) = \begin{cases} \dfrac{f(x)}{x+\delta} & \textit{for } x \geq |\delta| + 1 \\ 0 & \textit{otherwise.} \end{cases}$$

If $f(x) \to \sigma\ (\hat{A}_{-1})$, *then* $g(x) \to 0\ (\hat{A}_{-1})$.

PROOF. By Lemma 5.2.1, $\hat{A}_{-1}$ is applicable to g. Let

$$\phi(t) = \int_M^\infty \frac{e^{-t(x+\delta)}}{x+1} f(x)\,dx,$$

where M is a constant such that $M > |\delta| + 1$. Then

$$\frac{1}{\log(1+\frac{1}{t})}\,\phi(t) = \frac{1}{\log(1+\frac{1}{t})}\int_M^\infty \frac{e^{-t(x+\delta)}}{x+1}\, f(x)\,dx\,.$$

$$= \frac{e^{-t(\delta-1)}}{\log(1+\frac{1}{t})}\int_M^\infty \frac{e^{-t(x+1)}}{x+1}\, f(x)\,dx$$

$$\to 0 \text{ as } t \to 0^+,$$

using Theorem 5.2.2 and the fact that $f(x) \to \sigma\,(\hat{A}_{-1})$. Hence there exists a constant K such that for $t \in (0,1)$, we have $$|\phi(t)| \le K\,|\log(1+\tfrac{1}{t})|.$$

We have

$$A_{g;-1}(\tfrac{1}{t}) \sim \frac{1}{\log(1+\frac{1}{t})}\int_M^\infty \frac{e^{-t(x+1)}}{x+1}\, g(x)\,dx, \quad \text{as } t \to 0^+$$

$$= \frac{e^{-t(1-\delta)}}{\log(1+\frac{1}{t})}\int_M^\infty e^{-t(x+\delta)}\frac{f(x)}{(x+\delta)(x+1)}\,dx$$

$$= \frac{e^{-t(1-\delta)}}{\log(1+\frac{1}{t})}\int_M^\infty \frac{f(x)}{x+1}\int_t^\infty e^{-z(x+\delta)}\,dz\,dx$$

$$= \frac{e^{-t(1-\delta)}}{\log(1+\frac{1}{t})}\int_t^\infty \int_M^\infty \frac{e^{-z(x+\delta)}}{x+1}\, f(x)\,dx\,dz,$$

the absolute convergence of the integrals enabling us to change the order of integration. Hence

$$A_{g;-1}(\tfrac{1}{t}) \sim \frac{e^{-t(1-\delta)}}{\log(1+\frac{1}{t})}\int_t^\infty \phi(z)\,dz$$

$$\sim \frac{1}{\log(1+\frac{1}{t})}\int_t^\infty \phi(z)\,dz, \quad \text{as } t \to 0^+.$$

~~Let $\varepsilon \in (0,1)$ be given. Then for $0 < t < \varepsilon$,~~

$$A_{g;-1}\left(\frac{1}{t}\right) \sim \frac{1}{\log(1+\frac{1}{t})}\int_t^\varepsilon \phi(t)\,dz + \frac{1}{\log(1+\frac{1}{t})}\int_\varepsilon^\infty \phi(z)\,dz.$$

Since $A_{g;-1}(y)$ exists for all $y > 0$, $A_{g;-1}(\frac{1}{\varepsilon})$ exists. Hence $\int_\varepsilon^\infty \phi(z)\,dz$ exists and is independent of t. Therefore

$$\frac{1}{\log(1+\frac{1}{t})}\int_\varepsilon^\infty \phi(z)\,dz \to 0 \quad \text{as } t \to 0^+.$$

We also have

$$\left|\frac{1}{\log(1+\frac{1}{t})}\int_t^\varepsilon \phi(z)\,dz\right| \le \frac{1}{\log(1+\frac{1}{t})}\int_t^\varepsilon |\phi(z)|\,dz$$

$$\le \frac{K}{\log(1+\frac{1}{t})}\int_t^\varepsilon \log(1+\frac{1}{t})\,dz$$

$$\le \frac{K}{\log(1+\frac{1}{t})}\log(1+\frac{1}{t}) \cdot \varepsilon$$

$$= K\varepsilon.$$

Thus, given $\varepsilon > 0$, we have

$$\limsup_{t\to 0^+} \left|A_{g;-1}\left(\frac{1}{t}\right)\right| \le K\varepsilon,$$

where K is independent of ε. Therefore

$$\lim_{t\to 0^+} A_{g;-1}\left(\frac{1}{t}\right) = 0;$$

that is, $g(x) \to 0$ $(\hat{A}_{-1})$. This completes the proof of the theorem.

We now introduct the notion of translativity. Let $\{s_n\}$ be a sequence. Let $t_0 = 0$ and let $t_n = s_{n-1}$ for $n = 1,2,\ldots$. Let M be a method of summability operating on sequences. Then M is *right translative* if $\{t_n\}$ is M-summable whenever $\{s_n\}$ is M-summable. Also, M is *left translative* if $\{s_n\}$ is M-summable whenever $\{t_n\}$ is M-summable. If M is both left and right translative, then it is said to be *translative* (see, for example [33, pp. 69-70; 13, conditions γ and δ, p. 95]).

An integral method $\hat{M}$ is called *translative* if for any real number δ, $f(x) \to \sigma\ (\hat{M})$ if and only if $f(x+\delta) \to \sigma\ (\hat{M})$.

THEOREM 5.2.4. (cf. [2]) *The method $\hat{A}_{-1}$ is translative; that is, for any real number δ*

$$f(x) \to \sigma\ (\hat{A}_{-1}) \textit{ if and only if } f(x+\delta) \to \sigma\ (\hat{A}_{-1}).$$

PROOF. For convenience, we define $f(x) = 0$ for $x < 0$. Let δ be a real number. Let $g(x) = f(x+\delta)$. Suppose $f(x) \to \sigma\ (\hat{A}_{-1})$. Then for $y > 0$, we have

$$\frac{f(u+\delta)}{u+1} e^{-(u+1)/y} = \frac{f(x)}{x+1-\delta} e^{-(x+1-\delta)/y}, \text{ where } u = x+\delta,$$

$$= e^{\delta/y} \frac{f(x) e^{-(x+1)/y}}{x+1} \{1 + \frac{\delta}{x+1-\delta}\} .$$

Hence we obtain

$$A_{g;-1}(y) = \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+1)/y}}{u+1} g(u)du$$

$$\sim \frac{e^{\delta/y}}{\log(1+y)} \int_M^\infty \frac{e^{-(x+1)/y}}{x+1} \{1 + \frac{\delta}{x+1-\delta}\} f(x)dx,$$

by Theorem 5.2.2, where M is a constant such that $M > |\delta| + 2$. We now have

$$A_{g;-1}(y) \sim \frac{1}{\log(1+y)} \int_M^\infty \frac{e^{-(x+y)/y}}{x+1} f(x)\,dx$$

$$+ \frac{\delta}{\log(1+y)} \int_M^\infty \frac{e^{-(x+1)/y} f(x)}{(x+1)\ (x+1-\delta)}\,dx$$

$\to \sigma + 0$, by hypothesis and Theorem 5.2.3.

Therefore $f(x+\delta) \to \sigma\ (\hat{A}_{-1})$. Since δ was arbitrary, the result follows.

The preceding theorem shows that we may define the transform for the $\hat{A}_{-1}$ method by

$$\frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+\delta)/y}}{u+\delta} f(u)\,du$$

for any positive δ. We shall continue, however, to use $\delta = 1$.

THEOREM 5.2.5. *Let* $\lambda > -1$. *If* $\hat{A}_\lambda$ *is applicable, then* $\hat{A}_{-1}$ *is applicable.*

PROOF. Let $y > 0$. Since we are dealing with Lebesgue integrals we may assume that f is non-negative. Now, using the translativity of $\hat{A}_{-1}$, we have

$$\int_1^\infty \frac{e^{-x/y}}{x} f(x)\,dx = \int_1^\infty e^{-x/y} x^\lambda f(x) \frac{1}{x^{\lambda+1}}\,dx$$

$$\le \int_1^\infty e^{-x/y} x^\lambda f(x)\,dx$$

$< \infty$ by hypothesis.

We now extend the definition of $\hat{A}$ being applicable to mean $\hat{A}_\lambda$ is applicable for all $\lambda \ge -1$.

5.3 RELATIONS WITH INTEGRAL HAUSDORFF METHODS.

In this section, we examine the product of the integral Abel-type method $\hat{A}_{-1}$ and integral Hausdorff methods.

We need a result due to Borwein [2].

THEOREM G. *Let* $\hat{H}_\chi$ *be a regular integral Hausdorff method. For* $x \geq 0$, *let* $g(x)$ *be a continuous function. If* $g(x) \to \sigma$ *as* $x \to \infty$, *then*

$$\frac{1}{\log(1+y)} \int_0^1 \log(1+yt)\, g(yt)\, d\chi(t) \to \sigma \text{ as } y \to \infty.$$

We can now prove

THEOREM 5.3.1. (cf. [2]) *Let* $\hat{H}_\chi$ *be a regular integral Hausdorff method. Then*

$$\hat{A}_{-1}\hat{H}_\chi \supseteq \hat{A}_{-1}.$$

PROOF. Suppose $f(x) \to \sigma$ $(\hat{A}_{-1})$. By Theorem 5.2.2, we may assume $f(x) = 0$ for $x \leq 1$. Now

$$A_{-1}H(y) = \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+1)/y}}{u+1} H(u)du$$

$$\sim \frac{1}{\log(1+y)} \int_1^\infty \frac{e^{-u/y}}{u} H(u)du, \text{ by Theorem 5.2.4,}$$

$$= \frac{1}{\log(1+y)} \int_1^\infty \frac{e^{-u/y}}{u} \int_0^1 f(ut)d\chi(t)du$$

$$= \frac{1}{\log(1+y)} \int_0^1 \int_1^\infty \frac{e^{-u/y}}{u} f(ut)du\, d\chi(t),$$

using the absolute convergence of the integrals,

$$= \frac{1}{\log(1+y)} \int_0^1 \int_t^\infty \frac{e^{-x/(yt)}}{x} f(x)\, dx\, d\chi(t)$$

$$= \frac{1}{\log(1+y)} \int_0^1 \int_1^\infty \frac{e^{-x/(yt)}}{x} f(x)dx\, d\chi(t),$$

since $f(x) = 0$ for $x \le 1$,

$$= \frac{1}{\log(1+y)} \int_0^1 \log(1+yt)\left\{\frac{1}{\log(1+yt)} \int_1^\infty \frac{e^{-x/(yt)}}{x} f(x)dx\right\}d\chi(t)$$

$$= \frac{1}{\log(1+y)} \int_0^1 \log(1+yt)g(yt)d\chi(t)$$

where g is the function given by

$$g(x) = \frac{1}{\log(1+x)} \int_1^\infty \frac{e^{-u/x}}{u} f(u)du.$$

We now show that g is continuous. It is clear that for any given $\varepsilon > 0$, $g(x)$ is continuous on $[\varepsilon,\infty)$. It remains to prove that $g(x)$ is continuous as $x \to 0^+$. We show, in fact, that $g(x) \to 0$ as $x \to 0^+$ (cf. [31, p. 181]). For $t > 1$,

$$\int_1^t \frac{|f(u)|}{u}\,du = \int_1^t e^{u/\varepsilon} e^{-u/\varepsilon} \frac{|f(u)|}{u}\,du$$

$$\le e^{t/\varepsilon} \int_1^t e^{-u/\varepsilon} \frac{|f(u)|}{u}\,du$$

$$\le e^{t/\varepsilon} \int_1^\infty e^{-u/\varepsilon} \frac{|f(u)|}{u}\,du$$

$$= e^{t/\varepsilon} \cdot K,$$

where K is independent of t but dependent on ε. We also observe that $x \sim \log(1+x)$ as $x \to 0^+$. Let $y > \varepsilon$ and let $R > 1$. Integrating by parts we obtain

$$\int_1^R \frac{e^{-u/y}}{u} |f(u)|\,du$$

$$= e^{-u/y} \int_1^u \frac{|f(t)|}{t}\,dt \Big|_{u=1}^{u=R} + \int_1^R \frac{e^{-u/y}}{y} \int_1^u \frac{|f(t)|}{t}\,dt\,du$$

$$= e^{-R/y} \int_1^R \frac{|f(t)|}{t}\,dt + \frac{1}{y} \int_1^R e^{-u/y} \int_1^u \frac{|f(t)|}{t}\,dt\,du$$

$$\le e^{-R/y} e^{R/\varepsilon} K + \frac{1}{y} \int_1^R e^{-u/y} e^{u/\varepsilon} K\,du$$

$$= K e^{-R(1/y-1/\varepsilon)} + \frac{K}{y} \int_1^R e^{-u(1/y-1/\varepsilon)}\,du$$

$$= K e^{-R(1/y-1/\varepsilon)} + \frac{K}{y} \frac{-e^{-u(1/y-1/\varepsilon)}}{1/y - 1/\varepsilon} \Big|_{u=1}^{u=R}$$

$$= K e^{-R(1/y-1/\varepsilon)} + \frac{K}{y} \left\{ \frac{e^{-(1/y-1/\varepsilon)} - e^{-R(1/y-1/\varepsilon)}}{1/y - 1/\varepsilon} \right\}.$$

Hence we have

$$\frac{1}{y}\int_1^R \frac{e^{-u/y}}{u}\,|f(u)|\,du \le \frac{K\,e^{-R(1/y-1/\varepsilon)}}{y}$$

$$+\frac{K}{y^2}\left\{\frac{e^{-(1/y-1/\varepsilon)} - e^{-R(1/y-1/\varepsilon)}}{1/y - 1/\varepsilon}\right\}$$

Therefore, letting $R \to \infty$, we have

$$\frac{1}{y}\int_1^\infty \frac{e^{-u/y}}{u}\,|f(u)|\,du \le \frac{K}{y^2}\,\frac{e^{-(1/y-1/\varepsilon)}}{1/y - 1/\varepsilon}$$

$$\to 0 \text{ as } y \to 0^+.$$

We have thus show g continuous.

But $\quad g(x) \sim A_{f;-1}(x) \to \sigma$ as $x \to \infty$.

Hence, by Theorem G, $A_{-1}H(g) \to \sigma$ as $y \to \infty$.

CHAPTER 6

MOMENT FUNCTIONS

6.1 INTRODUCTION.

Let $\phi \in BV[0,1]$. Then for $x \geq 0$, the m-*function* or *moment function* μ is defined ([16]; see also [22]) by

$$\mu(x) := \int_0^1 t^x \, d\phi(t).$$

If in addition

$$\mu(x) \geq \delta \int_0^1 t^x |d\phi^*(t)| \quad \text{for } x \geq X_0 \geq 0,$$

where δ is a constant with $1 \geq \delta > 0$, X_0 is constant and

$$\phi^*(t) = \begin{cases} 0 & \text{for } t = 0 \\ \frac{1}{2}\{\phi(t^-) + \phi(t^+)\} - \phi(0) & \text{for } t \in (0,1) \\ \phi(1) - \phi(0) & \text{for } t = 1, \end{cases}$$

then μ is an $\overline{m}$-*function* (cf. [5]).

We observe that replacing ϕ by ϕ^* does not affect the value of the moment function; that is,

$$\int_0^1 t^x d\phi(t) = \int_0^1 t^x d\phi^*(t).$$

We also remark that if ϕ is the function associated with an integral Hausdorff method $\hat{H}_\phi$ (not necessarily regular), then by previous assumptions, $\phi = \phi^*$.

In this chapter we briefly examine some properties of $\overline{m}$-functions needed in the following chapter to prove that for $\lambda > -1$

$$\hat{A}_{-1} \supsetneqq \hat{A}_{\lambda} .$$

6.2 RESULTS.

LEMMA 6.2.1. (cf. [5]) *Any* m*-function converging to a positive limit is an* $\overline{m}$*-function.*

PROOF. Suppose that

$$\mu(x) = \int_0^1 t^x d\phi(t)$$

$$= \int_0^1 t^x d\phi^*(t)$$

$$\to \tau \text{ as } x \to \infty, \text{ where } \tau > 0.$$

Then there exists an $x_0 \geq 0$ such that $\mu(x) \geq \frac{\tau}{2}$ whenever $x \geq x_0$. Let δ be a constant such that $\delta \in (0,1]$ and

$$\frac{\tau}{2\delta} \geq \int_0^1 |d\phi^*(t)| .$$

Then for $x \geq x_0$ we have

$$\delta \int_0^1 t^x |d\phi^*(t)| \leq \delta \int_0^1 |d\phi^*(t)|$$

$$\leq \frac{\tau}{2}$$

$$\leq \mu(x) .$$

Hence, μ is an $\overline{m}$-function.

LEMMA 6.2.2. (cf. [5])

(i) *The sum of* m*-functions is an* m*-function.*

(ii) *The function* 1 *is an* m*-function and an* $\overline{m}$*-function.*

PROOF. (i) Suppose

$$\mu(x) = \int_0^1 t^x d\phi(t)$$

and

$$\nu(x) = \int_0^1 t^x d\psi(t).$$

Then

$$\mu(x) + \nu(x) = \int_0^1 t^x d\{\phi(t) + \psi(t)\}$$

$$= \int_0^1 t^x d\chi(t)$$

where

$$\chi(t) = \phi(t) + \psi(t) \in BV[0,1].$$

Hence $\mu + \nu$ is an m-function.

(ii) Let

$$\chi(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & t = 1. \end{cases}$$

Then

$$1 = \int_0^1 t^x d\chi(t).$$

Hence 1 is an m-function. By Lemma 6.2.1, we have 1 is an $\overline{m}$-function.

THEOREM H. ([22], see also [4]) *Let* c_0 *be a constant. If* F(s) *is an analytic function of* $s = \rho + ri$ *in the region* $\rho > c_0$, *and if for all* $c > c_0$ *there is a constant* K *such that*

$$\int_{-\infty}^{\infty} |F(c + it)|^2 dt < K,$$

then for $\rho > c_0$,

$$F(s) = \int_0^1 t^s \phi(t)\,dt,$$

where $t^c\phi(t)$ *is Lebesgue integrable on* [0,1] *for all* $c > c_0$.

EXAMPLE. (cf. [4]) *For* $\lambda \geq 1$,

$$\left(\frac{x}{x+1}\right)^\lambda - 1 = \int_0^1 t^x\, \Phi\,(t)\,dt,$$

where $t^c\Phi(t)$ *is Lebesgue integrable on* [0,1] *for all* $c > 0$.

PROOF. Let $s = \rho + ir$. Let

$$F(s) = \left(\frac{s}{s+1}\right)^\lambda - 1.$$

Hence for $\rho > 0$, $F(s)$ is analytic. Let $c > 0$. Then we have

$$|F(c+it)| = \left|\frac{(c+it)^\lambda}{(c+1+it)^\lambda} - 1\right|$$

$$= \left|\frac{(c+it)^\lambda - (c+1+it)^\lambda}{(c+1+it)^\lambda}\right|$$

$$= \frac{\lambda \left|\int_c^{c+1} (x+it)^{\lambda-1}\,dx\right|}{|c+1+it|^\lambda}$$

$$\leq \frac{\lambda \int_c^{c+1} |x+it|^{\lambda-1}\,dx}{|c+1+it|^\lambda}$$

$$\leq \frac{\lambda \max\limits_{x\in[c,c+1]} |x+it|^{\lambda-1}}{|c+1+it|^\lambda}$$

$$= \frac{\lambda \max\limits_{x\in[c,c+1]} (x^2+t^2)^{(\lambda-1)/2}}{((c+1)^2+t^2)^{\lambda/2}}$$

$$= \frac{\lambda \left((c+1)^2 + t^2\right)^{(\lambda-1)/2}}{\left((c+1)^2 + t^2\right)^{\lambda/2}}$$

$$= \frac{\lambda}{\left((c+1)^2 + t^2\right)^{1/2}} .$$

Thus we obtain

$$\int_{-\infty}^{\infty} |F(c+it)|^2 dt \le \lambda^2 \int_{-\infty}^{\infty} \frac{dt}{(c+1)^2 + t^2}$$

$$\le \lambda^2 \int_{-\infty}^{\infty} \frac{dt}{1+t^2} ,$$

a constant independent of c. Hence by Theorem H, $\left(\frac{x}{x+1}\right)^{\lambda} - 1$ has representation

$$\left(\frac{x}{x+1}\right)^{\lambda} - 1 = \int_0^1 t^x \Phi(t)\,dt$$

where $t^c\Phi(t)$ is Lebesgue integrable on [0,1] for all $c > 0$. This completes the example.

We remark that since we have been unable to prove $\Phi(t)$ is Lebesgue integrable on [0,1], we cannot claim that $\left(\frac{x}{x+1}\right)^{\lambda} - 1$ is an m-function.

CHAPTER 7

A SCALE OF ABEL-TYPE METHODS

7.1. INTRODUCTION.

In this chapter, we prove the full scale of Abel-type methods, showing that for $\lambda > \mu \geq -1$,

$$\hat{A}_\mu \supsetneqq \hat{A}_\lambda.$$

7.2 WATSON'S METHOD.

Adapting a method defined by Watson [30, p. 41], for $\lambda > -1$, we define the method $\hat{J}_\lambda$ as follows. For $y > 0$, let

$$J_\lambda(y) = \frac{1}{y^\lambda \log(1+y)} \int_0^y (y-x)^\lambda \frac{e^{-1/x}}{x} f(x)\,dx.$$

If $J_\lambda(y)$ exists as a Lebesgue integral for all positive y and if $J_\lambda(y) \to \sigma$ as $y \to \infty$, then we say that f is $\hat{J}_\lambda$-*summable* to σ and we write

$$f(x) \to \sigma \quad (\hat{J}_\lambda).$$

THEOREM 7.2.1. (cf. [30, §4.6]) *Suppose* $\lambda > -1$. *For* $x > 0$, *let*

$$g(x) = \left(\frac{x}{x+1}\right)^\lambda f(x).$$

If $\hat{A}$ *is applicable to* f, *then*

$$\hat{J}_\lambda \hat{A}_{f;\lambda} = \hat{A}_{g;-1}.$$

PROOF. Let $y > 0$. We have

$$J_\lambda A_{f;\lambda}(y)$$

$$= \frac{1}{y^\lambda \log(1+y)} \int_0^y (y-x)^\lambda \frac{e^{-1/x}}{x} \cdot \frac{1}{\Gamma(\lambda+1)x^{\lambda+1}} \int_0^\infty e^{-u/x} u^\lambda f(u)\,du\,dx$$

$$= \frac{1}{\Gamma(\lambda+1)y^\lambda \log(1+y)} \int_0^\infty u^\lambda f(u) \int_0^y (y-x)^\lambda \frac{e^{-(u+1)/x}}{x^{\lambda+2}}\,dx\,du,$$

assuming the change of order of integration is valid. Examining the inner integral, we obtain

$$\int_0^y (y-x)^\lambda \frac{e^{-(u+1)/x}}{x^{\lambda+2}}\,dx = y^\lambda \int_0^y \left(\frac{1}{x} - \frac{1}{y}\right)^\lambda \frac{e^{-(u+1)/x}}{x^2}\,dx$$

$$= y^\lambda \int_0^\infty z^\lambda e^{-(u+1)z-(u+1)/y}\,dz$$

$$= y^\lambda e^{-(u+1)/y} \int_0^\infty \frac{x^\lambda e^{-x}}{(u+1)^{\lambda+1}}\,dx$$

$$= \frac{y^\lambda}{(u+1)^{\lambda+1}} e^{-(u+1)/y}\Gamma(\lambda+1).$$

Hence

$$J_\lambda A_{f;\lambda}(y) = \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+1)/y}}{u+1} \left[\left(\frac{u}{u+1}\right)^\lambda f(u)\right] du$$

$$= A_{g;-1}(y)$$

where $$g(x) = \left(\frac{x}{x+1}\right)^\lambda f(x).$$

To justify the change of order of integration, we show that $A_{g;-1}(y)$ is absolutely convergent. For $\lambda \geq 0$, we have

$$|A_{g;-1}(y)| \le \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^\lambda |f(u)|\ du$$

$$\le \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+1)/y}}{u+1} |f(u)|\ du$$

$$< \infty \text{ since } \hat{A}_{-1} \text{ is applicable to } f.$$

We now consider the case $-1 < \lambda < 0$. Let $\varepsilon > 0$ be given. Then we have

$$\frac{1}{\log(1+y)} \int_\varepsilon^\infty \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^\lambda |f(u)|\ du$$

$$\le \left(\frac{\varepsilon}{\varepsilon+1}\right)^\lambda \frac{1}{\log(1+y)} \int_\varepsilon^\infty \frac{e^{-(u+1)/y}}{u+1} |f(u)|\ du$$

$$< \infty .$$

We also have

$$\int_0^\varepsilon \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^\lambda |f(u)|\ du \le \frac{e^{-1/y}}{(\varepsilon+1)^\lambda} K \int_0^\varepsilon u^\lambda\ du,$$

where $|f(u)| \le K$ for $u \in [0,\varepsilon]$

$$= \frac{e^{-1/y}}{(\varepsilon+1)^\lambda} K \frac{\varepsilon^{\lambda+1}}{\lambda+1}$$

$$< \infty .$$

We therefore have $A_{g;-1}(y)$ is absolutely convergent, and the interchange is valid.

This completes the proof of the theorem.

Using the preceding theorem, we are now able to prove:

THEOREM 7.2.2. (cf. [30, §4.6]) *For $\lambda > -1$, $\hat{J}_\lambda$ is a regular integral method.*

PROOF. We use Theorem A. Letting $f(x) = 1$ for all x, $\hat{A}$ is clearly applicable to f. We have

$$A_{f;\lambda}(x) = \frac{1}{\Gamma(\lambda+1)x^{\lambda+1}} \int_0^\infty e^{-u/x} u^\lambda du$$
$$= 1.$$

Using Theorem 7.2.1, with $g(x) = \left(\frac{x}{x+1}\right)^\lambda$, we obtain

$$J_{f;\lambda}(y) = J_\lambda A_{f;\lambda}(y)$$
$$= A_{g;-1}(y).$$

But $g(x) \to 1$ as $x \to \infty$ and $\hat{A}_{-1}$ is regular (Theorem 5.2.1). Therefore $J_{f;\lambda}(y) \to 1$ as $y \to \infty$ for $f = 1$.

We must show $J_{f;\lambda}(y)$ is bounded as $y \to 0^+$ for $f = 1$. For $\lambda \geq 0$,

$$\frac{1}{\log(1+y)} \int_0 \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^\lambda du$$
$$\leq \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+1)/y}}{u+1} du$$
$$= A_{f;-1}(y)$$

which remains bounded as $y \to 0^+$. We now consider the case $-1 < \lambda < 0$. Let $\varepsilon > 0$ be given. Then we have

$$\frac{1}{\log(1+y)} \int_\varepsilon^\infty \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^\lambda du$$
$$\leq \left(\frac{\varepsilon}{\varepsilon+1}\right)^\lambda \frac{1}{\log(1+y)} \int_\varepsilon^\infty \frac{e^{-(u+1)/y}}{u+1} du$$
$$\leq \left(\frac{\varepsilon}{\varepsilon+1}\right)^\lambda A_{f;-1}(y),$$

which remains bounded as $y \to 0^+$. Now we have

$$\int_0^{\varepsilon} \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^{\lambda} du \le \frac{e^{-1/y}}{(\varepsilon+1)^{\lambda}} \int_0^{\varepsilon} u^{\lambda}\, du$$

$$= \frac{e^{-1/y}}{(\varepsilon+1)^{\lambda}} \frac{\varepsilon^{\lambda+1}}{\lambda+1} .$$

Hence we obtain

$$\frac{1}{\log(1+y)} \int_0^{\varepsilon} \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^{\lambda+1} du \le \frac{e^{-1/y}}{\log(1+y)} \frac{\varepsilon^{\lambda+1}}{(\lambda+1)(\varepsilon+1)^{\lambda}}$$

$$\to 0 \quad \text{as } y \to 0^+ .$$

We have therefore shown that for $f = 1$, $J_{f;\lambda}(y)$ remains bounded as $y \to 0^+$.

It remains to show that the second condition of Theorem A is satisfied. Let Y be a fixed positive constant. For $\lambda \ge 0$, we have $(y-x)^{\lambda} \le y^{\lambda}$ for $x \in [0,Y]$ and $y > Y$. Thus, for $\lambda \ge 0$

$$\frac{1}{y^{\lambda}\log(1+y)} \int_0^{Y} (y-x)^{\lambda} \frac{e^{-1/x}}{x}\, dx \le \frac{1}{\log(1+y)} \int_0^{Y} \frac{e^{-1/x}}{x}\, dx$$

$$\to 0 \text{ as } y \to \infty.$$

For $-1 < \lambda < 0$, we have $\left(\frac{y-x}{y}\right)^{\lambda} \le 2^{-\lambda}$ for $x \in [0,Y]$ and $y > 2Y$. Hence, for $-1 < \lambda < 0$,

$$\frac{1}{y^{\lambda}\log(1+y)} \int_0^{Y} (y-x)^{\lambda} \frac{e^{-1/x}}{x}\, dx \le \frac{2^{-\lambda}}{\log(1+y)} \int_0^{Y} \frac{e^{-1/x}}{x}\, dx$$

$$\to 0 \text{ as } y \to \infty.$$

Therefore, for $\lambda > -1$, $\hat{J}_{\lambda}$ is a regular method of summability.

An immediate corollary of Theorems 7.2.1 and 7.2.2 is

THEOREM 7.2.3. *Suppose* $\lambda > -1$. *For* $x > 0$, *let* $g(x) = \left(\frac{x}{x+1}\right)^{\lambda} f(x)$. *If* $f(x) \to \sigma\ (\hat{A}_{\lambda})$, *then* $g(x) \to \sigma\ (\hat{A}_{-1})$.

7.3 THE MAIN RESULT

THEOREM 7.3.1. (cf. [4]) *Let* $\lambda \geq 1$. *If* $f(x) \to \sigma\ (\hat{A}_{-1})$, *then* $\left(\frac{x}{x+1}\right)^{\lambda} f(x) \to \sigma\ (\hat{A}_{-1})$.

PROOF. Let $g(x) = \left(\frac{x}{x+1}\right)^{\lambda} f(x)$. In view of Theorem 5.2.2, we may assume that $f(x) = 0$ for $x < 1$. Suppose first that $f(x) \to 0\ (\hat{A}_{-1})$. We must show $g(x) \to 0\ (\hat{A}_{-1})$.

By the example of Chapter 6, we have

$$\left(\frac{x}{x+1}\right)^{\lambda} - 1 = \int_0^1 t^x \Phi(t)\, dt$$

where $t^c\Phi(t)$ is Lebesgue integrable on $[0,1]$ for all $c > 0$. We observe that $\Phi(t)$ is thus Lebesgue integrable on $[\varepsilon,1]$ for any $\varepsilon > 0$. Changing variables we obtain

$$\left(\frac{x}{x+1}\right)^{\lambda} - 1 = \int_0^{\infty} e^{-x/z}\, \Psi(z)\, dz,$$

where $\Psi(z) = \frac{\Phi(e^{-1/z})}{z^2}$. We note that $\Psi(z)$ is Lebesgue integrable on $[\varepsilon,\infty)$ for any $\varepsilon > 0$. In particular, we have

$$\int_1^{\infty} |\Psi(z)|\, dz < \infty.$$

By Theorem 5.2.4 and the fact that $f(x) = 0$ for $x < 1$, we obtain

$$A_{g;-1}(y) \sim \frac{1}{\log(1+y)} \int_1^\infty \frac{e^{-u/y}}{u} \left(\frac{u}{u+1}\right)^\lambda f(u)\, du$$

$$= \frac{1}{\log(1+y)} \int_1^\infty \frac{e^{-u/y}}{u} f(u) \int_0^\infty e^{-u/z}\Psi(z)\, dz\, du$$

$$+ \frac{1}{\log(1+y)} \int_1^\infty \frac{e^{-u/y}}{u} f(u)\, du.$$

The second integral is merely $A_{f;-1}(y)$, which by assumption tends to zero as $y \to \infty$.

Now we have

$$\left|\int_0^\infty \Psi(z) \int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\, du\, dz\right|$$

$$\leq \int_0^\infty |\Psi(z)|\, e^{-1/z} \int_1^\infty \frac{e^{-u/y}}{u} |f(u)|\, du\, dz$$

$$= \int_0^\infty |\Psi(z)|\, e^{-1/z}\, dz \cdot \int_1^\infty \frac{e^{-u/y}}{u} |f(u)|\, du$$

$$< \infty .$$

This shows that the function obtained by interchanging the order of integration in the first integral is absolutely convergent. We may therefore perform this interchange. We thus obtain

$$A_{g;-1}(y) \sim \frac{1}{\log(1+y)} \int_0^\infty \Psi(z) \int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\, du\, dz.$$

Let Y be a constant greater than 1 which will be specified later. Then for $y > Y$ we have

$$A_{g;-1}(y) \sim \frac{1}{\log(1+y)} \int_0^Y \Psi(z) \int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\, du\, dz$$

$$+ \frac{1}{\log(1+y)} \int_Y^\infty \Psi(z) \int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\, du\, dz.$$

Examining the first integral, we have

$$\left|\frac{1}{\log(1+y)}\int_0^Y \Psi(z)\int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\,du\,dz\right|$$

$$\le \frac{1}{\log(1+y)}\int_0^Y |\Psi(z)|\, e^{-1/z}\int_1^\infty \frac{e^{-u/Y}}{u}|f(u)|\,du\,dz$$

$$= \frac{1}{\log(1+y)}\int_0^Y |\Psi(z)|\, e^{-1/z}dz \cdot \int_1^\infty \frac{e^{-u/Y}}{u}|f(u)|\,du$$

$= \frac{1}{\log(1+y)} K$, where K is a constant depending on Y, but independent of y,

$\to 0$ as $y \to \infty$.

The second integral satisfies

$$\left|\frac{1}{\log(1+y)}\int_Y^\infty \Psi(z)\int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\,du\,dz\right|$$

$$\le \int_Y^\infty \frac{|\Psi(z)|}{\log(1+y)}\left|\int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\,du\right|dz$$

$$\le \int_Y^\infty \frac{|\Psi(z)|}{\log(1+\frac{yz}{y+z})}\left|\int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\,du\right|dz,$$

since $y > \frac{yz}{y+z}$. We also have $\frac{1}{y} + \frac{1}{z} \le \frac{2}{Y}$, and hence $\frac{yz}{y+z} \ge \frac{Y}{2}$.

Substituting this result in our inequality, we obtain

$$\left|\frac{1}{\log(1+y)}\int_Y^\infty \Psi(z)\int_1^\infty \frac{e^{-u(1/y+1/z)}}{u} f(u)\,du\,dz\right|$$

$$\le \int_Y^\infty |\Psi(z)| \sup_{x\ge\frac{Y}{2}}\left|\frac{1}{\log(1+x)}\int_1^\infty \frac{e^{-u/x}}{u} f(u)\,du\right|dz$$

$$\le \int_1^\infty |\Psi(z)|\,dz \cdot \sup_{x\ge\frac{Y}{2}}\left|\frac{1}{\log(1+y)}\int_1^\infty \frac{e^{-u/x}}{u} f(u)\,du\right|$$

$$= K_1 \sup_{x\ge\frac{Y}{2}} A_{f;-1}(x),$$

where K_1 is a constant independent of y and Y. Since $f(x) \to 0$ $(\hat{A}_{-1})$, this second integral may be made as small as desired by choosing Y sufficiently large.

Combining these results, we have shown that $g(x) \to 0$ $(\hat{A}_{-1})$.

Suppose now that $f(x) \to \sigma$ $(\hat{A}_{-1})$. Then we have $f(x) - \sigma \to 0$ $(\hat{A}_{-1})$. Using the preceding section of the proof, we have

$$\left(\frac{x}{x+1}\right)^{\lambda}(f(x) - \sigma) \to 0 \quad (\hat{A}_{-1});$$

that is, $$\left\{\left(\frac{x}{x+1}\right)^{\lambda} f(x) - \left(\frac{x}{x+1}\right)^{\lambda}\sigma\right\} \to 0 \quad (\hat{A}_{-1}).$$

But $\left(\frac{x}{x+1}\right)^{\lambda}\sigma \to \sigma$ as $x \to \infty$ and $\hat{A}_{-1}$ is regular. Therefore $\left(\frac{x}{x+1}\right)^{\lambda} f(x) \to \sigma$ $(\hat{A}_{-1})$. This completes the proof of the theorem.

THEOREM 7.3.2. *Suppose* $\lambda > -1$. *Then* $f(x) \to \sigma$ $(\hat{A}_{-1})$ *if and only if* $\left(\frac{x}{x+1}\right)^{\lambda} f(x) \to \sigma$ $(\hat{A}_{-1})$.

PROOF. We first prove the necessity. Suppose $f(x) \to \sigma$ $(\hat{A}_{-1})$. For $\lambda \geq 1$, we have $\left(\frac{x}{x+1}\right)^{\lambda} f(x) \to \sigma$ $(\hat{A}_{-1})$, by Theorem 7.3.1. For $0 \leq \lambda < 1$, we observe that

$$\left(\frac{x}{x+1}\right)^{\lambda} f(x) = \left(\frac{x+1}{x}\right)\left(\frac{x}{x+1}\right)^{\lambda+1} f(x)$$

$$= \left(\frac{x}{x+1}\right)^{\lambda+1} f(x) + \left(\frac{x}{x+1}\right)^{\lambda+1} \frac{f(x)}{x}.$$

Now $\lambda + 1 \geq 1$. Hence $\left(\frac{x}{x+1}\right)^{\lambda+1} f(x) \to \sigma$ $(\hat{A}_{-1})$. By Theorem 5.2.3, we also have

$$\left(\frac{x}{x+1}\right)^{\lambda+1} \frac{f(x)}{x} \to 0 \quad (\hat{A}_{-1}).$$

Hence we obtain $\left(\frac{x}{x+1}\right)^{\lambda} f(x) \to \sigma \ (\hat{A}_{-1})$. For $-1 < \lambda < 0$, we again use the relation

$$\left(\frac{x}{x+1}\right)^{\lambda} f(x) = \left(\frac{x+1}{x}\right)\left(\frac{x}{x+1}\right)^{\lambda+1}$$

to obtain the desired result.

We now prove the sufficiency. Suppose $\left(\frac{x}{x+1}\right)^{\lambda} f(x) \to \sigma \ (\hat{A}_{-1})$. For $-1 < \lambda < 1$, we note that

$$f(x) = \left(\frac{x}{x+1}\right)^{-\lambda} \cdot \left\{\left(\frac{x}{x+1}\right)^{\lambda} f(x)\right\},$$

where $-\lambda > -1$. Hence, by necessity of this theorem, we have $f(x) \to \sigma \ (\hat{A}_{-1})$ for $-1 < \lambda < 1$. Suppose now $\lambda \geq 1$. Then $\lambda = \lambda_0 + \lambda_1$, where λ_0 is a positive integer and $-1 < \lambda_1 \geq 0$. We have now

$$\left(\frac{x}{x+1}\right)^{\lambda} f(x) = \left(\frac{x}{x+1}\right)^{\lambda_0} \left(\frac{x}{x+1}\right)^{\lambda_1} f(x).$$

But $$\left(\frac{x}{x+1}\right)^{\lambda_1} f(x) = \left(\frac{x+1}{x}\right)^{\lambda_0} \left(\frac{x}{x+1}\right)^{\lambda} f(x)$$

$$= [1 + \frac{\lambda_0}{x} + \cdots + \frac{1}{x^{\lambda_0}}]\left[\left(\frac{x}{x+1}\right)^{\lambda} f(x)\right],$$

$$\to \sigma \ (\hat{A}_{-1})$$

by repeated application of Theorem 5.2.3. Since $-1 < \lambda_1 \leq 0$, by using the first part of the proof of the sufficiency, we have $f(x) \to \sigma \ (\hat{A}_{-1})$.

By Theorems 7.2.3, 7.3.2 and 3.2.4, we have established a full scale of inclusions for Abel-type methods.

THEOREM 7.3.3. (cf. [2]) *For* $\lambda > \mu \geq -1$,

$$\hat{A}_{\mu} \supsetneqq \hat{A}_{\lambda}.$$

CHAPTER 8

A LOGARITHMIC METHOD OF SUMMABILITY

8.1 DEFINITION AND PROPERTIES.

Borwein [7] has defined a logarithmic method of summability (L,α) for $\alpha > 0$ as follows. Given a series $\sum a_n$, let

$$a_n^{\alpha} = \frac{1}{\binom{n+\alpha}{n}} \sum_{\nu=0}^{n} \binom{n-\nu+\alpha-1}{n-\nu} a_{\nu}.$$

If
$$\frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} a_n^{\alpha} x^{n+1} \to \alpha\sigma \quad \text{as } x \to 1^-,$$

then we say that $\sum a_n$ is (L,α)-*summable* to σ or $s_n \to \sigma\ (L,\alpha)$, and we write

$$\sum a_n = \sigma \ (L,\alpha), \text{ or } s_n \to \sigma \ (L,\alpha),$$

where $s_n = \sum_{\nu=0}^{n} a_{\nu}$. He has proved that

$$(L,\alpha) \simeq A_{-1}(C,\alpha-1).$$

We use this relationship to define the integral logarithmic method.

For $\alpha \geq 1$, we define the *integral logarithmic method* $(\hat{L},\alpha)$ by

$$(\hat{L},\alpha) = \begin{cases} \hat{A}_{-1} & \text{for } \alpha = 1 \\ \hat{A}_{-1}(\hat{C},\alpha-1) & \text{for } \alpha > 1. \end{cases}$$

It is clear that $(\hat{L},\alpha)$ is regular.

THEOREM 8.1.1. *Let* $\hat{H}$ *be a regular integral Hausdorff method. Then for* $\alpha \geq 1$,

$$(\hat{L},\alpha)\,\hat{H} \supseteq (\hat{L},\alpha)$$

PROOF. Theorem 5.3.1 gives the result for $\alpha = 1$. For $\alpha > 1$, we have

$$\begin{aligned}(\hat{L},\alpha)\,\hat{H} &= \hat{A}_{-1}\,(\hat{C},\alpha-1)\,\hat{H} \\ &= \hat{A}_{-1}\,\hat{H}\,(\hat{C},\alpha-1), && \text{by Theorem 2.2.5} \\ &\supseteq \hat{A}_{-1}\,(\hat{C},\alpha-1), && \text{by Theorem 5.3.1} \\ &= (\hat{L},\alpha).\end{aligned}$$

The next theorem gives a scale of integral logarithmic methods and relates it to the integral Abel-type methods.

THEOREM 8.1.2. (cf. [7]) *Suppose* $\beta > \alpha \geq 1$ *and* $\lambda > -1$. *Then*

$$(\hat{L},\beta) \supseteq (\hat{L},\alpha) \supseteq \hat{A}_{-1} \supsetneq \hat{A}_{\lambda}.$$

PROOF. By Theorem 5.3.1, $(\hat{L},\alpha) \supseteq \hat{A}_{-1}$ for $\alpha \geq 1$. It remains to show that $(\hat{L},\beta) \supseteq (\hat{L},\alpha)$ for $\beta > \alpha \geq 1$. For $\alpha = 1$, we have

$$\begin{aligned}(L,\beta) &= \hat{A}_{-1}\,(\hat{C},\beta-1) \\ &\supseteq \hat{A}_{-1}, && \text{by Theorem 5.3.1} \\ &= (\hat{L},1)\end{aligned}$$

For $\alpha > 1$, we observe that

$$\begin{aligned}(\hat{L},\beta) &= \hat{A}_{-1}\,(\hat{C},\beta-1) \\ &= \hat{A}_{-1}\,(\hat{C},\beta-\alpha,\alpha-1)\,(\hat{C},\alpha-1), && \text{by Theorem 2.3.2} \\ &\supseteq \hat{A}_{-1}\,(\hat{C},\alpha-1), && \text{by Theorem 5.3.1} \\ &= (\hat{L},\alpha).\end{aligned}$$

CHAPTER 9

STRONG INTEGRAL SUMMABILITY

9.1. INTRODUCTION.

We now study strong integral methods of summability based on summability methods encountered in previous chapters.

The first strong sequence-to-sequence method of summability was strong Cesàro summability of order 1, introduced in 1916 by Fekete [11]. Strong Cesàro summability of any positive order was defined by Winn [32] in 1933. Various authors have since extended the notion of strong summability to other methods. Borwein [6] defines strong summability for matrix methods of a general type.

The strong sequence-to-sequence method $[P,Q]_\theta$ is defined for $\theta > 0$ as follows [6]. Let $\{s_n\}$ be a sequence and let

$$Q = [q_{n,\nu}] \text{ and } P = [p_{n,\nu}]$$

be matrices, with $p_{n,\nu} \geq 0$ for $n,\nu = 0,1,2,\ldots$.

Let
$$\tau_n = \sum_{\nu=0}^{\infty} q_{n,\nu} s_\nu .$$

If
$$\sum_{\nu=0}^{\infty} p_{n,\nu} |\tau_\nu - \sigma|^\theta$$

is defined for each n and tends to zero as $n \to \infty$, then we say that $\{s_n\}$ is $[P,Q]_\theta$-*summable* to σ and we write

$$s_n \to \sigma \ [P,Q]_\theta .$$

Various types of strong summability for sequence-to-function methods have been defined. For example, Srivastava [28,29] gives the following definition. Let $\{s_n\}$ be a sequence and let $\{\phi_n(x)\}$ be a sequence of functions. Let

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n(x) s_n .$$

If $\phi(x)$ exists for all $x > 0$ and if $\phi(x) \to \sigma$ as $x \to \infty$, then we say that $\{s_n\}$ is ϕ-*summable* to σ. Given a sequence-to-function method ϕ, strong ϕ-summability with index θ is defined. If for some $M > 0$

$$\frac{1}{y} \int_M^y \left| x \frac{d\phi(x)}{dx} \right|^\theta dx \to 0 \quad \text{as } y \to \infty$$

and if $\{s_n\}$ is ϕ-summable, then we say that $\{s_n\}$ is strongly ϕ-summable with index θ.

We now consider function-to-function methods of strong summability. We first note that a sequence-to-function method may be regarded as a function-to-function method. Given a sequence $\{s_n\}$, we define a function by

$$f(n) = s_n, \quad n = 0,1,2,\ldots .$$

Using this convention, we will henceforth regard sequence-to-function methods as function-to-function methods.

Shawyer [26] has defined *strong summability for a general class of function-to-function methods*. Let $\hat{P}$ be an integral method of summability given by the transform

$$P(y) = P_f(y) = \int_0^\infty p(y,t) f(t)\,dt$$

where $p(y,t) \geq 0$ for all $y > 0$ and all $t > 0$. Let $\hat{Q}$ be a function-to-function method. Let $\theta > 0$. For $y > 0$, we define

$$P_{|Q-\sigma|^\theta}(y) = \int_0^\infty p(y,t)\,|Q_f(t) - \sigma|^\theta dt.$$

If $P_{|Q-\sigma|^\theta}(y)$ exists for all $y > 0$ and if $P_{|Q-\sigma|^\theta}(y) \to 0$ as $y \to \infty$, we say that f is *strongly summable* $[\hat{P},\hat{Q}]_\theta$ to σ and we write

$$f(x) \to \sigma \; [\hat{P},\hat{Q}]_\theta.$$

We emphasize that the left-hand member appearing in the symbol $[\hat{P},\hat{Q}]_\theta$ must satisfy $p(y,t) \geq 0$ for all $y > 0$ and all $t > 0$.

In the case of an integral Hausdorff method $\hat{H}_\chi$,

$$\int_0^\infty p(y,t)f(t)dt$$

will be interpreted as

$$\int_0^y f(x)d\chi(\frac{x}{y}),$$

that is, as

$$\int_0^1 f(yt)d\chi(t).$$

The condition

$$p(y,t) \geq 0$$

will be written as $\quad d\chi(t) \geq 0,$

meaning $\chi(t)$ is non-decreasing;

that is, $\quad \hat{H}_\chi = \tilde{H}_\chi.$

We will use $\hat{Q}$ or $\hat{R}$ to indicate any function-to-function method throughout this chapter.

9.2. BASIC PROPERTIES.

THEOREM 9.2.1. (cf. [6]) *Let* $\theta > 0$.

(i) *If* $\hat{P}$ *and* $\hat{Q}$ *are regular, then* $[\hat{P},\hat{Q}]_\theta$ *is regular.*

(ii) *If* $\hat{P}_1 \geq \hat{P}_2$, *then*

$$[\hat{P}_1,\hat{Q}]_\theta \geq [\hat{P}_2,\hat{Q}]_\theta.$$

(iii) *Strong summability is linear; that is,*

(a) *if* $f(x) \to \sigma\ [\hat{P},\hat{Q}]_\theta$, *then for any real number* α, $\alpha f(x) \to \alpha\sigma\ [\hat{P},\hat{Q}]_\theta$, *and*

(b) *if* $f(x) \to \sigma\ [\hat{P},\hat{Q}]_\theta$ *and* $g(x) \to \tau\ [\hat{P},\hat{Q}]_\theta$, *then* $f(x) + g(x) \to \sigma + \tau\ [\hat{P},\hat{Q}]_\theta$.

PROOF. Results (i), (ii) and (iii) (a) follow directly from the definition of strong summability. We now examine the final result. We have

$$P_{|Q_{f+g} - \sigma - \tau|^\theta}(y)$$

$$= \int_0^\infty p(y,t)|Q_{f+g}(t) - \sigma - \tau|^\theta dt$$

$$= \int_0^\infty p(y,t)|Q_f(t) - \sigma + Q_g(t) - \tau|^\theta dt$$

$$\leq 2^\theta \int_0^\infty p(y,t)\{|Q_f(t) - \sigma|^\theta + |Q_g(t) - \tau|^\theta\}dt^{1}$$

---

[1]We use a convenient form of the triangle inequality: for $c > 0$, $|a+b|^c \leq 2^c\{|a|^c + |b|^c\}$.

$$= 2^{\theta}\{P_{|Q_f-\sigma|^{\theta}}(y) + P_{|Q_g-\tau|^{\theta}}(y)\}$$

$$\to 0 \text{ as } y \to \infty$$

We now quote two theorems due to Shawyer.

THEOREM I. [26, cf. 6] *Let* $\theta > \eta > 0$.

(i) *Suppose that for some constant* M,

$$\int_0^{\infty} p(y,t)\,dt < M$$

independent of y. *Then*

$$[\hat{P},\hat{Q}]_{\eta} \supseteq [\hat{P},\hat{Q}]_{\theta}.$$

(ii) *Suppose* $\hat{P}$ *is regular. Then*

$$[\hat{P},\hat{Q}]_{\eta} \supseteq [\hat{P},\hat{Q}]_{\theta}.$$

THEOREM J. [26, cf. 6] *Suppose that* $\hat{P}$ *is regular. Then*

(i) *for* $\theta > 0$,

$$[\hat{P},\hat{Q}]_{\theta} \supseteq \hat{Q}, \qquad \textit{and}$$

(ii) *for* $\theta \geq 1$,

$$\hat{P}\hat{Q} \supseteq [\hat{P},\hat{Q}]_{\theta}.$$

The next theorem gives a result for product methods.

THEOREM 9.2.2. *Let* $\hat{P}$ *and* $\hat{Q}$ *be regular integral methods with both* $p(y,t) \geq 0$ *and* $q(y,t) \geq 0$ *for all* $y > 0$ *and all* $t > 0$. *Let* $\theta \geq 1$. *Then*

$$[\hat{P},\hat{Q}\hat{R}]_{\theta} \supseteq [\hat{P}\hat{Q},\hat{R}]_{\theta}.$$

PROOF. Suppose $f(x) \to \sigma\ [\hat{P}\hat{Q},\hat{R}]_{\theta}$. We have

$$P_{|QR-\sigma|^\theta}(y) = \int_0^\infty p(y,t)|QR(t) - \sigma|^\theta dt$$

$$= \int_0^\infty p(y,t)\left|\int_0^\infty q(t,x)R_f(x)dx - \sigma\right|^\theta dt$$

$$= \int_0^\infty p(y,t)\left|\int_0^\infty q(t,x)\{R_f(x) - \sigma\}dx + h(t)\right|^\theta dt,$$

where $h(t) \to 0$ as $t \to \infty$, since $\hat{Q}$ is regular. Using the triangle inequality, we obtain

$$P_{|QR-\sigma|^\theta}(y) \le 2^\theta \int_0^\infty p(y,t)\left|\int_0^\infty q(t,x)\{R_f(x) - \sigma\}dx\right|^\theta dt$$

$$+ 2^\theta \int_0^\infty p(y,t)|h(t)|^\theta dt.$$

The second integral on the right side tends to zero as $y \to \infty$ since $h(t) \to 0$ as $t \to \infty$ and $\hat{P}$ is regular.

By Hölder's inequality, we have

$$\left|\int_0^\infty q(t,x)\{R_f(x) - \sigma\}dx\right|^\theta$$

$$\le \left\{\left|\int_0^\infty q(t,x)|R_f(x) - \sigma|^\theta dx\right|^{1/\theta} \left|\int_0^\infty q(t,x)dx\right|^{1-\frac{1}{\theta}}\right\}^{\theta}[1]$$

$$\le M \int_0^\infty q(t,x)|R_f(x) - \sigma|^\theta dx,$$

$$= M\, Q_{|R-\sigma|^\theta}(t),$$

where M is a constant independent of t, since $\hat{Q}$ is regular.

Hence

[1] When $\theta = 1$, the second integral does not appear.

$$\int_0^\infty p(y,t)\left|\int_0^\infty q(t,x)\{R_f(x) - \sigma\}dx\right|^\theta dt$$

$$\le M \int_0^\infty p(y,t) \int_0^\infty q(t,x)\,|R_f(x) - \sigma|^\theta dx$$

$$= M\, PQ_{|R-\sigma|^\theta}(y)$$

$$\to 0 \text{ as } y \to \infty.$$

This completes the proof of the theorem.

Our next result investigates necessary and sufficient conditions for strong summability.

THEOREM 9.2.3. (cf. [6]) *Let* $\hat{P}$ *be a regular method. Let* $\theta \geq 1$. *Then the following are equivalent.*

(i) $f(x) \to \sigma\ [\hat{P},\hat{Q}]_\theta$.

(ii) $f(x) \to \sigma\ (\hat{P}\hat{Q})$ *and* $g(x) \to 0\ (\hat{P})$, *where*

$$g(x) = |Q_f(x) - PQ_f(x)|^\theta.$$

PROOF. We first show that (i) implies (ii). Suppose $f(x) \to \sigma\ [\hat{P},\hat{Q}]_\theta$. By Theorem J, we have $f(x) \to \sigma\ (\hat{P}\hat{Q})$; that is, $PQ_f(x) \to \sigma$ as $x \to \infty$. Hence

$$|PQ_f(x) - \sigma|^\theta \to 0 \text{ as } x \to \infty.$$

But $\hat{P}$ is regular. Therefore $|PQ_f(x) - \sigma|^\theta \to 0\ (\hat{P})$. Now for $y > 0$,

$$P_g(y) = \int_0^\infty p(y,t)\,|Q_f(t) - PQ_f(t)|^\theta dt$$

$$= \int_0^\infty p(y,t)\,|Q_f(t) - \sigma - PQ_f(t) + \sigma|^\theta dt$$

$$\le 2^\theta \int_0^\infty p(y,t)\{|Q_f(t) - \sigma|^\theta + |PQ_f(t) - \sigma|^\theta\}dt$$

$$= 2^\theta\{P_{|Q_f-\sigma|^\theta}(y) + P_{|PQ_f-\sigma|^\theta}(y)\}$$

$$\to 0 \text{ as } y \to \infty;$$

that is, $g(x) \to 0\ (\hat{P})$.

We now prove that (ii) implies (i). Suppose $f(x) \to \sigma\ (\hat{P}\hat{Q})$ and $g(x) \to 0\ (\hat{P})$. Since $\hat{P}$ is regular, we have $|PQ_f(x) - \sigma|^\theta \to 0\ (\hat{P})$.

Now for $y > 0$,

$$P_{|Q_f-\sigma|^\theta}(y) = \int_0^\infty p(y,t)\,|Q_f(t) - \sigma|^\theta dt$$

$$= \int_0^\infty p(y,t)\,|Q_f(t) - PQ_f(t) + PQ_f(t) - \sigma|^\theta dt$$

$$\le 2^\theta \int_0^\infty p(y,t)\{|Q_f(t) - PQ_f(t)|^\theta + |PQ_f(t) - \sigma|^\theta\}dt$$

$$= 2^\theta\{P_g(y) + P_{|PQ_f-\sigma|^\theta}(y)\}$$

$$\to 0 \text{ as } y \to \infty.$$

9.3. STRONG SUMMABILITY WITH INTEGRAL HAUSDORFF METHODS.

In this section we investigate strong summability involving integral Hausdorff methods.

LEMMA 9.3.1. (cf. [6]) *Let* $\hat{H}_\chi$ *be an integral Hausdorff method and let* $\theta \geq 1$. *Then for* $y > 0$

$$|H_f(y)|^\theta \leq M^{\theta-1} \hat{H}_{|f|^\theta}(y)$$

where
$$M = \int_0^1 |d\chi(t)|$$

is the variation of χ *on* $[0,1]$.

PROOF. For $y > 0$, we have

$$|H_f(y)|^\theta = \left|\int_0^1 f(yt)\,d\chi(t)\right|^\theta$$

$$\leq \left\{\int_0^1 |f(yt)|\,|d\chi(t)|\right\}^\theta$$

$$\leq \left\{\left[\int_0^1 |f(yt)|^\theta |d\chi(t)|\right]^{1/\theta} \left[\int_0^1 |d\chi(t)|\right]^{1-\frac{1}{\theta}}\right\}^{\theta}\ {}^{1}$$

$$= M^{\theta-1}\int_0^1 |f(yt)|^\theta |d\chi(t)|$$

$$= M^{\theta-1} \hat{H}_{|f|^\theta}(y).$$

We use this lemma to prove:

THEOREM 9.3.1. (cf. [6]) *Let* $\hat{H}_\psi$ *and* $\hat{H}_\chi$ *be regular integral Hausdorff methods with* $d\psi(t) \geq 0$. *Let* $\theta \geq 1$. *Then*

$$[\hat{H}_\psi, \hat{H}_\chi \hat{Q}]_\theta \supseteq [\hat{H}_\psi, \hat{Q}]_\theta.$$

PROOF. Suppose $f(x) \to \sigma\ [\hat{H}_\psi, \hat{Q}]_\theta$. Letting $g(x) = f(x) - \sigma$, we have $g(x) \to 0\ [\hat{H}_\psi, \hat{Q}]_\theta$. Now for $y > 0$,

---

[1] When $\theta = 1$, the second integral does not appear.

$$H_{|H_\chi Q_g - 0|^\theta;\psi}(y) = \int_0^1 |H_\chi Q_g(yt)|^\theta d\psi(t)$$

$$= \int_0^1 |H_{Q_g;\chi}(yt)|^\theta d\psi(t)$$

$$\le \int_0^1 M^{\theta-1} \tilde{H}_{|Q_g|^\theta;\chi}(yt)\, d\psi(t),$$

by Lemma 9.3.1, where M is constant,

$$= M^{\theta-1} \int_0^1 \int_0^1 |Q_g(xyt)|^\theta |d\chi(x)|\, d\psi(t)$$

$$= M^{\theta-1} \int_0^1 \int_0^1 |Q_g(xyt)|^\theta d\psi(t)\, |d\chi(x)|,$$

by Theorem 2.2.4,

$$= M^{\theta-1} \tilde{H}_{H_{|Q_g|^\theta;\psi};\chi}(y)$$

$\to 0$ as $y \to \infty$, by Theorem 2.2.3.

This completes the proof of the theorem.

Let $\hat{H}$ be an integral Hausdorff method. For convenience, we shall say that $\hat{H}$ (or H) satisfies the *finite moment condition* if at most finitely many of the moments of $\hat{H}$ (or H) vanish.

THEOREM 9.3.2. (cf. [6]) *Let* $\hat{H}_\chi$ *be a regular integral Hausdorff method with* $d\chi(t) \ge 0$. *Let* $\hat{H}_1$ *and* $\hat{H}_2$ *be integral Hausdorff methods such that* $H_1 \supseteq H_2$ *and* H_2 *satisfies the finite moment condition. Let* $\theta \ge 1$. *Then*

$$[\hat{H}_\chi, H_1]_\theta \supseteq [\hat{H}_\chi, \hat{H}_2]_\theta.$$

PROOF. By Theorem D, there is a regular integral Hausdorff method $\hat{H}_3$ such that $\hat{H}_1 = \hat{H}_3\hat{H}_2$. Hence we obtain by Theorem 9.3.1,

$$[\hat{H}_\chi, \hat{H}_1]_\theta = [\hat{H}_\chi, \hat{H}_3\hat{H}_2]_\theta$$
$$\supseteq [\hat{H}_\chi, \hat{H}_2]_\theta .$$

As a corollary to Theorem 9.3.2 we have:

THEOREM 9.3.3. *Let* $\hat{H}_\chi$ *be a regular integral Hausdorff method with* $d\chi(t) \geq 0$. *Let* $\hat{H}_1$ *and* $\hat{H}_2$ *be integral Hausdorff methods both satisfying the finite moment property and with* $H_1 \simeq H_2$. *Let* $\theta \geq 1$. *Then*

$$[\hat{H}_\chi, \hat{H}_1]_\theta \simeq [\hat{H}_\chi, \hat{H}_2]_\theta .$$

Under the hypotheses of Theorems 9.3.2 and 9.3.3, we obtain the relations

$$[\hat{H}_\chi, \hat{H}_1\hat{Q}]_\theta \supseteq [\hat{H}_\chi, \hat{H}_2\hat{Q}]_\theta$$

and

$$[\hat{H}_\chi, \hat{H}_1\hat{Q}]_\theta \simeq [\hat{H}_\chi, \hat{H}_2\hat{Q}]_\theta ,$$

respectively.

9.4. STRONG INTEGRAL CESARO SUMMABILITY.

We examine some strong summability methods involving Cesaro summability, and then consider strong Cesàro summability.

THEOREM 9.4.1. (cf. [6]) *Let* $\alpha, \beta > 0$, $\theta > \eta > 0$ *and* $\beta\theta > \alpha\eta > 0$. *Then*

$$[(\hat{C},\beta),\hat{Q}]_\eta \supseteq [(\hat{C},\alpha),\hat{Q}]_\theta .$$

PROOF. Suppose $f(x) \to \sigma\ [(\hat{C},\alpha),\hat{Q}]_\theta$. Let

$$g(x) = Q_f(x) - \sigma .$$

Then $|g(x)|^\theta \to 0\ (\hat{C},\alpha)$. Using Hölder's inequality, we have

$$C_{|Q_f-\sigma|^\eta;\beta}(y) = C_{|g|^\eta;\beta}(y)$$

$$= \frac{\beta}{y^\beta}\int_0^y (y-t)^{(\alpha-1)\eta/\theta}|g(t)|^\eta (y-t)^{\beta-1-(\alpha-1)\eta/\theta}dt$$

$$\leq \frac{\beta}{y^\beta}\left\{\int_0^y [(y-t)^{(\alpha-1)\eta/\theta}|g(t)|^\eta]^{\theta/\eta}dt\right\}^{\eta/\theta}$$

$$\cdot\left\{\int_0^y [(y-t)^{\beta-1-(\alpha-1)\eta/\theta}]^{\frac{\theta}{\theta-\eta}}dt\right\}^{\frac{\theta-\eta}{\theta}}$$

$$= O(1)\left\{\frac{1}{y^\alpha}\int_0^y (y-t)^{\alpha-1}|g(t)|^\theta dt\right\}^{\eta/\theta}$$

$$\cdot\left\{\frac{1}{y^{\frac{\beta\theta-\alpha\eta}{\theta-\eta}}}\int_0^y (y-t)^{\frac{\beta\theta-\alpha\eta}{\theta-\eta}-1}dt\right\}^{\frac{\theta-\eta}{\theta}}$$

$$= O(1)\ \{C_{|g|^\theta;\alpha}(y)\}^{\eta/\theta}$$

$$\to 0 \text{ as } y \to \infty .$$

This completes the proof of the theorem.

THEOREM 9.4.2. (cf. [6]) *Let* $\hat{H}_\chi$ *be an integral Hausdorff method. Let* $\theta \geq 1$. *Suppose* f *is absolutely continuous. Then the following are equivalent:*

(i) $f(x) \to \sigma\ [(\hat{C},1),\hat{H}_\chi]_\theta$.

(ii) $f(x) \to \sigma\ ((\hat{C},1)\hat{H}_\chi)$ *and*

$$xf'(x) \to 0\ [(\hat{C},1),(\hat{C},1)\hat{H}_\chi]_\theta.$$

PROOF. In view of Theorem 9.2.3, it is sufficient to prove $xf'(x) \to 0\ [(\hat{C},1),(\hat{C},1)\hat{H}'_\chi]_\theta$

is equivalent to

$$g(x) \to 0\ (\hat{C},1),$$

where $g(x) = |H_{f;\chi}(x) - C_{H_{f;\chi};1}(x)|^\theta$.

Now, integrating by parts, we obtain

$$\frac{1}{t}\int_0^t xuf'(xu)dx = \frac{1}{tu}\int_0^{tu} zf'(z)dz$$

$$= \frac{1}{tu}\{[zf(z)]_0^{tu} - \int_0^{tu} f(z)dz\}$$

$$= f(tu) - \frac{1}{t}\int_0^t f(xu)dx.$$

Hence for $y > 0$, we have

$$\frac{1}{y}\int_0^y \left|\frac{1}{t}\int_0^t \int_0^1 xuf'(xu)d\chi(u)dx\right|^\theta dt$$

$$= \frac{1}{y}\int_0^y \left|\int_0^1 \frac{1}{t}\int_0^t xuf'(xu)dxd\chi(u)\right|^\theta dt,$$

by Theorem 2.2.4,

$$= \frac{1}{y}\int_0^y \left|\int_0^1 f(tu)d\chi(u) - \frac{1}{t}\int_0^1 f(xu)dxd\chi(u)\right|^\theta dt$$

$$= \frac{1}{y}\int_0^y |g(t)|^\theta dt$$

$$= C_{|g|^\theta;1}(y).$$

This completes the proof of the theorem.

We remark that Theorem 9.4.2 parallels Srivastava's definition of strong sequence-to-function summability.

For $\alpha, \theta > 0$, *the strong integral Cesàro method of order* $\alpha+1$ *with index* θ is defined by

$$[\hat{C}, \alpha+1]_\theta = [(\hat{C}, 1), (\hat{C}, \alpha)]_\theta.$$

This is analogous to the definition of strong sequence-to-sequence Cesàro summability given by Fekete [11], Winn [32] and Hyslop [12] (see also [6]), as

$$[C, \rho+1]_\theta = [(C, 1), (C, \rho)]_\theta, \quad \text{for } \rho > -1.$$

In view of Theorem 9.3.3, we restrict ourselves to Cesàro methods, with one index, rather than Cesàro-type methods, with two indices.

THEOREM 9.4.3. (cf. [6]) *Let* $\alpha > 0$ *and* $\theta \geq 1$. *Suppose* f *is absolutely continuous. Then the following are equivalent.*

(i) $f(x) \to \sigma \ [\hat{C}, \alpha+1]_\theta$.

(ii) $f(x) \to \sigma \ (\hat{C}, \alpha+1)$ *and*

$$xf'(x) \to 0 \ [\hat{C}, \alpha+2]_\theta.$$

PROOF. The result follows immediately from Theorems 9.4.2, 2.3.2 and 9.3.3.

For $\alpha, \theta > 0$, the strong integral Hölder method of summability is defined by [cf. 6]

$$[\hat{H}, \alpha+1]_\theta = [(\hat{H}, 1), (\hat{H}, \alpha)]_\theta.$$

We have the following result, from Theorem 9.3.3.

THEOREM 9.4.4. (cf. [6]) *For* $\alpha > 0$ *and* $\theta \geq 1$,

$$[\hat{C},\alpha+1]_\theta \simeq [\hat{H},\alpha+1]_\theta;$$

that is, for $\theta \geq 1$, *strong integral Cesàro and strong integral Hölder summability are equivalent.*

CHAPTER 10

STRONG INTEGRAL ABEL-TYPE SUMMABILITY

10.1. INTRODUCTION.

Harington and Hyslop [14] and Flett [12] have defined forms of strong sequence-to-function Abel-type summability. Rizvi [21, pp. 29 and 45] (see also [8]) has defined strong Abel-type summability as follows. For $\theta > 0$,

$$[A_\lambda]_\theta = [(C,1), A_{\lambda+1}]_\theta.$$

We define *strong integral Abel-type summability* in a similar manner. For $\theta > 0$, $\lambda > -2$ and $\alpha > 0$,

$$[\hat{A}_\lambda]^\alpha_\theta = [(\hat{C},\alpha), \hat{A}_{\lambda+1}]_\theta.$$

Strong Borel-type summability is defined in a somewhat similar manner [26].

10.2. RESULTS.

We first gather some simple results.

THEOREM 10.2.1. (cf. [26])

(i) *For* $\beta > \alpha > 0$, $\lambda > -2$ *and* $\theta > 0$,

$$[\hat{A}_\lambda]^\beta_\theta \supseteq [\hat{A}_\lambda]^\alpha_\theta.$$

(ii) *For* $\alpha, \beta > 0$, $\lambda > -2$, $\theta > \eta > 0$ *and* $\beta\theta > \alpha\eta > 0$;

$$[\hat{A}_\lambda]^\beta_\eta \supseteq [\hat{A}_\lambda]^\alpha_\theta.$$

(iii) *For* $\alpha > 0$, $\lambda > \mu > -2$ *and* $\theta \geq 1$,

$$[\hat{A}_\mu]^\alpha_\theta \supseteq [\hat{A}_\lambda]^\alpha_\theta.$$

(iv) *For* $\alpha, \beta > 0$, $\lambda > \mu > -2$, $\theta \geq 1$, $\theta > \eta > 0$ *and* $\beta\theta > \alpha\eta > 0$,

$$[\hat{A}_\mu]^\beta_\eta \supseteq [\hat{A}_\lambda]^\alpha_\theta.$$

PROOF. The first result is trivial. The second follows from Theorem 9.4.1. We now prove the third result. By Theorems 3.2.2 and 9.3.1, we have

$$\begin{aligned}[\hat{A}_\mu]^\alpha_\theta &= [(\hat{C},\alpha),\hat{A}_{\mu+1}]_\theta \\ &= [(\hat{C},\alpha),(\hat{C},\lambda-\mu,\mu+1)\hat{A}_{\lambda+1}]_\theta \\ &\supseteq [(\hat{C},\alpha),\hat{A}_{\lambda+1}]_\theta \\ &= [\hat{A}_\lambda]^\alpha_\theta.\end{aligned}$$

The final result follows from parts (ii) and (iii). We have

$$\begin{aligned}[\hat{A}_\mu]^\beta_\eta &\supseteq [\hat{A}_\mu]^\alpha_\theta \\ &\supseteq [\hat{A}_\lambda]^\alpha_\theta.\end{aligned}$$

This completes the proof of the theorem.

THEOREM 10.2.2. (cf. [26]) *Let* $\alpha, \beta > 0$, $\eta > \theta > 1$, $\alpha\eta > \beta\theta > 0$, $\gamma > \frac{\alpha}{\theta} - \frac{\beta}{\eta} > 0$, $\gamma > \alpha\left(\frac{1}{\theta} - \frac{1}{\eta}\right) > 0$ *and* $\lambda - \gamma > -2$. *Then*

$$[\hat{A}_{\lambda-\gamma}]^\beta_\eta \supseteq [\hat{A}_\lambda]^\alpha_\theta.$$

PROOF. We are required to prove

$$[(\hat{C},\beta),\hat{A}_{\lambda-\gamma+1}]_\eta \supseteq [(\hat{C},\alpha),\hat{A}_{\lambda+1}]_\theta.$$

Suppose $f(x) \to \sigma\ [(\hat{C},\alpha),\hat{A}_{\lambda+1}]_\theta$. Since all the methods involved are regular, we may assume $\sigma = 0$. We also have

$$[(\hat{C},\beta),\hat{A}_{\lambda-\gamma+1}]_\eta = [(\hat{C},\beta),(\hat{C},\gamma,\lambda+1-\gamma)\hat{A}_{\lambda+1}]_\eta$$

$$\sim [(\hat{C},\beta),(\hat{C},\gamma)\hat{A}_{\lambda+1}]_\eta,$$

using Theorems 3.2.2 and 9.3.3. It is therefore sufficient to prove

$$f(x) \to 0\ [(\hat{C},\beta),(\hat{C},\gamma)\hat{A}_{\lambda+1}]_\eta .$$

Let
$$a(x) = A_{f;\lambda+1}(x) = \frac{1}{\Gamma(\lambda+2)}\int_0^\infty e^{-u}u^{\lambda+1}f(ux)\,du.$$

We must show

$$\frac{\beta}{y^\beta}\int_0^y (y-t)^{\beta-1}\left|\frac{\gamma}{t^\gamma}\int_0^t (t-x)^{\gamma-1}\ a(x)\,dx\right|^\eta dt \to 0 \quad \text{as } y \to \infty.$$

Using Hölder's inequality with indices η, $\frac{\theta\eta}{\eta-\theta}$ and $\frac{\theta}{\theta-1}$, we obtain

$$\left|\int_0^t (t-x)^{\gamma-1}\ a(x)\ dx\right|$$

$$\leq \left\{\int_0^t [|a(x)|^{\theta/\eta}(t-x)^{\alpha^{**}}]^\eta dx\right\}^{1/\eta}$$

$$\cdot \left\{\int_0^t \left[|a(x)|^{\frac{\theta(\eta-\theta)}{\theta\eta}}(t-x)^{(\alpha-1)\frac{\eta-\theta}{\theta\eta}}\right]^{\frac{\theta\eta}{\eta-\theta}} dx\right\}^{\frac{1}{\theta}-\frac{1}{\eta}}$$

$$\cdot \left\{\int_0^t \left[(t-x)^{(\alpha^*-1)\left(\frac{\theta-1}{\theta}\right)}\right]^{\frac{\theta}{\theta-1}} dx\right\}^{1-\frac{1}{\theta}}$$

where α^* and α^{**} are given by

$$0 < \alpha^* \left(1 - \frac{1}{\theta}\right) < \gamma - \frac{\alpha}{\theta} + \frac{\beta}{\eta}$$

$$0 < \alpha^* \left(1 - \frac{1}{\theta}\right) < \gamma - \alpha \left(\frac{1}{\theta} - \frac{1}{\eta}\right)$$

and
$$\alpha^{**} = \gamma-1 - (\alpha-1)\left(\frac{\eta-\theta}{\theta\eta}\right) - (\alpha^*-1)\left(\frac{\theta-1}{\theta}\right)$$

$$= \gamma-1 - (\alpha-1)\left(\frac{1}{\theta} - \frac{1}{\eta}\right) - (\alpha^*-1)\left(1 - \frac{1}{\theta}\right).$$

Hence

$$\left|\frac{1}{t^\gamma}\int_0^t (t-x)^{\gamma-1} a(x)\,dx\right|^\eta \le \left\{\frac{1}{t^{\alpha^{**}\eta+1}}\int_0^t |a(x)|^\theta (t-x)^{\alpha^{**}\eta} dx\right\}$$

$$\cdot \left\{\frac{1}{t^\alpha}\int_0^t |a(x)|^\theta (t-x)^{\alpha-1} dx\right\}^{\eta/\theta - 1}$$

$$\cdot \left\{\frac{1}{t^{\alpha^*}}\int_0^t (t-x)^{\alpha^*-1} dx\right\}^{\eta - \eta/\theta}.$$

Now
$$\frac{1}{t^{\alpha^*}}\int_0^t (t-x)^{\alpha^*-1} dx = O(1),$$

independent of t, for $\alpha^* > 0$. Further

$$\frac{\alpha}{t^\alpha}\int_0^t |a(x)|^\theta (t-x)^{\alpha-1} dx$$

$$= C_{|A_{\lambda+1}-0|^\theta;\alpha}(t)$$

$$\to 0 \text{ as } t \to \infty,$$

since $f(x) \to 0\ [(\hat{C},\alpha)\hat{A}_{\lambda+1}]_\theta$.

Therefore
$$\frac{1}{t^\alpha}\int_0^t |a(x)|^\theta (t-x)^{\alpha-1} dx = O(1),$$

independent of t.

We now show that $\alpha^{**}\eta+1 > 0$. We have

$$\alpha^{**}\eta + 1 = \eta[\gamma-1 - (\alpha-1)(\tfrac{1}{\theta} - \tfrac{1}{\eta}) - (\alpha^{*}-1)(1-\tfrac{1}{\theta})] + 1$$

$$> \eta\ [\gamma-1 - (\alpha-1)(\tfrac{1}{\theta} - \tfrac{1}{\eta}) - \gamma + \alpha\ (\tfrac{1}{\theta} - \tfrac{1}{\eta}) + 1 - \tfrac{1}{\theta}] + 1$$

$$= 0.$$

Therefore

$$\frac{1}{t^{\alpha^{**}\eta+1}} \int_0^t |a(x)|^{\theta}(t-x)^{\alpha^{**}\eta}dx$$

is defined. We also have

$$\alpha^{**}\eta+1 > \eta\ [\gamma-1 - (\alpha-1)(\tfrac{1}{\theta} - \tfrac{1}{\eta}) - \gamma + \tfrac{\alpha}{\theta} - \tfrac{\beta}{\eta} + 1 - \tfrac{1}{\theta}] + 1$$

$$= \alpha-\beta.$$

We now have

$$\frac{O(1)}{y^{\beta}} \int_0^y (y-t)^{\beta-1} \frac{1}{t^{\alpha^{**}\eta+1}} \int_0^t |a(x)|^{\theta}(t-x)^{\alpha^{**}\eta}\ dxdt$$

$$= O(1)\ C_{\beta}C_{|A_{\lambda+1}|^{\theta};\alpha^{**}\eta+1}(y)$$

$$= O(1)\ C_{|A_{\lambda+1}|^{\theta};\beta+\alpha^{**}\eta+1}(y), \text{ by Theorem 2.3.2}$$

$$\to 0 \text{ as } y \to \infty, \text{ since } \beta + \alpha^{**}\eta + 1 > \alpha$$

and $f(x) \to 0\ [(\hat{C},\alpha),\hat{A}_{\lambda+1}]_{\theta}$. This completes the proof of the theorem.

THEOREM 10.2.3. (cf. [9]) *Let* $\lambda > -2$, $\theta > 1$ *and* $1 > \gamma > 1/\theta$. *Then*

$$(\hat{C},\tilde{\gamma})\hat{A}_{\lambda+1} \supseteq [\hat{A}_{\lambda}]_{\theta}^{1}$$

If further $\lambda + 1 - \gamma > -1$, *then*

$$\hat{A}_{\lambda+1-\gamma} \supseteq [\hat{A}_{\lambda}]_{\theta}^{1}.$$

PROOF. Suppose $f(x) \to \sigma\ [\hat{A}_\lambda]^1_\theta$. We may assume that $\sigma = 0$. Let $a(x) = A_{\lambda+1}(x)$. Let $\frac{1}{\eta} + \frac{1}{\theta} = 1$ and let $0 < \varepsilon < \gamma - \frac{1}{\theta}$. Then for $y > 0$, using Hölder's inequality we have

$$|C_\gamma A_\lambda(y)| = \left|\frac{\gamma}{y^\gamma}\int_0^y (y-t)^{\gamma-1}a(t)\,dt\right|$$

$$\le \gamma\left\{\frac{1}{y^{\varepsilon\theta+1}}\int_0^y (y-t)^{\varepsilon\theta}|a(t)|^\theta dt\right\}^{1/\theta}$$

$$\cdot\left\{\frac{1}{y^{\eta(\gamma-1-\varepsilon)+1}}\int_0^y (y-t)^{\eta(\gamma-1-\varepsilon)}dt\right\}^{1/\eta}$$

$$= O(1)\, C_{|A_{\lambda+1}|^\theta;\varepsilon\theta+1}(y), \text{ since } \eta(\gamma-1-\varepsilon) > -1$$

$$\to 0 \text{ as } y \to \infty \text{ since } \varepsilon\theta+1 > 1.$$

This completes the proof of the theorem.

Our final theorem relates strong integral Abel-type and Hausdorff summability. We just need some notation. For an integral Hausdorff method $\hat{H}_\chi$, for $\lambda > -2$ and $\theta, \alpha > 0$, we define (cf. [8])

$$[\hat{A}_\lambda \hat{H}_\chi]^\alpha_\theta = [(C,\alpha), \hat{A}_{\lambda+1}\hat{H}_\chi]_\theta.$$

THEOREM 10.2.4. (cf. [8]) *Let $\alpha > 0$, $\lambda > -2$, and $\theta \ge 1$. Suppose $\hat{H}_\chi$ is an integral Hausdorff method. Then*

$$[\hat{A}_\lambda \hat{H}_\chi]^\alpha_\theta \supseteq [\hat{A}_\lambda]^\alpha_\theta.$$

PROOF. We have, by Theorems 2.2.5 and 9.3.1,

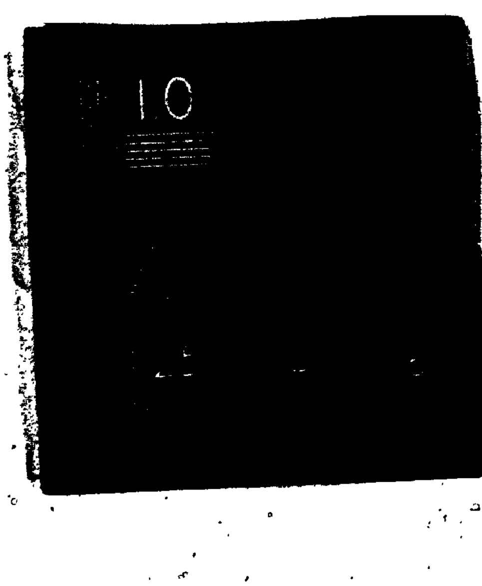
$$[\hat{A}_\lambda \hat{H}_\chi]^\alpha_\theta = [(\hat{C},\alpha),\hat{A}_{\lambda+1}\hat{H}_\chi]_\theta$$

$$= [(\hat{C},\alpha),\hat{H}_\chi\hat{A}_{\lambda+1}]_\theta$$

$$\supseteq [(\hat{C},\alpha),\hat{A}_{\lambda+1}]_\theta$$

$$= [\hat{A}_\lambda]_\theta .$$

2 OF/DE 2



APPENDIX 1

LEBESGUE-STIELTJES INTEGRATION

We wish to clarify what Rogosinski [23] means when he writes

$$\int_0^1 f(xt)\, d\chi(t)$$

where $\chi \in BV[0,1]$ and f is Borel measurable and bounded on every interval of the form $[0,x]$. He describes it as a Lebesgue-Stieltjes integral. We shall examine the Lebesgue-Stieltjes integral as discussed in Ash [1] and Rudin [25] and obtain the properties needed for integral Hausdorff methods.

We use $\mathbb{R}$ and $\mathbb{R}^*$ to denote the real numbers and the extended real numbers, respectively.

A collection S of subsets of a space X is called a *σ-algebra* in X if S satisfies the following three conditions.

(i) $X \in S$.

(ii) If $A \in S$, then $A^c \in S$, where A^c denotes the complement of A with respect to X.

(iii) If $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in S$ for $n = 1,2,3,\ldots$, then $A \in S$.

Given a topological space X, there is a smallest σ-algebra containing all the open sets of X. The members of this σ-algebra are called the *Borel sets* of X. We use $\mathcal{B}$ to denote the Borel sets of $\mathbb{R}$.

A *positive measure* on a σ-algebra S is a map $\mu : S \to [0,\infty]$ for which $\mu(A) < \infty$ for some $A \in S$ and which is countable additive; that is, when $A_1, A_2, A_3, \ldots$ is a countable collection of disjoint sets in S, then

$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n).$$

In particular [1, p. 23], a *Lebesgue-Stieltjes measure* on $\mathbb{R}$ is a measure μ on $\mathcal{B}$ such that $\mu(I) < \infty$ for all bounded intervals $I \subseteq \mathbb{R}$.

Let $F:\mathbb{R} \to \mathbb{R}$ be a monotone non-decreasing function. Ash [1, p. 23] calls such a map a *distribution function* if it is also right continuous; that is, $F(x) = F(x^+)$ for all $x \in \mathbb{R}$. He then proves the following result [1, p. 24].

Let F *be a distribution function on* $\mathbb{R}$ *and let*

$$\mu(a,b] = F(b) - F(a) \text{ for } a < b.$$

Then there is a unique extension of μ *to a Lebesgue-Stieltjes measure on* $\mathbb{R}$.

Hence, starting with a monotone non-decreasing function F, Ash defines a right continuous function F_1 by $F_1(x) = F(x^+)$. He then obtains a unique Lebesgue-Stieltjes measure μ_1 satisfying for $a < b$

$$\begin{aligned}\mu_1(a,b] &= F_1(b) - F_1(a)\\ &= F(b^+) - F(a^+).\end{aligned}$$

Rudin defines a left-continuous function F_2, by $F_2(x) = F(x^-)$ and obtains [25, Th. 8.14] a unique Lebesgue-Stieltjes measure μ_3 satisfying for $a < b$

$$\mu_3[a,b) = F_2(b) - F_2(a)$$
$$= F(b^-) - F(a^-).$$

It is easy to show that μ_1 and μ_2 agree on intervals and hence must agree on all Borel sets. Using μ to denote this common measure, we have

$$\mu(a,b] = F(b^+) - F(a^+)$$
$$\mu(a,b) = F(b^-) - F(a^+)$$
$$\mu[a,b) = F(b^-) - F(a^-)$$

and

$$\mu[a,b] = F(b^+) - F(a^-).$$

Note that

$$\mu(a) = F(a^+) - F(a^-)$$
$$= \Delta F(a).$$

Hence μ does not depend on the actual value of F at any point of discontinuity of F. Therefore Rogosinski [23] may normalize in the fashion

$$F_3(x) = \begin{cases} F(0) & \text{for } x \le 0 \\ \frac{1}{2}\{F(x^-) + F(x^+)\} & \text{for } x \in (0,1) \\ F(1) & \text{for } x \ge 1. \end{cases}$$

The Lebesgue-Stieltjes integral is now defined as in Ash or Rudin. It is applied to Borel measurable functions; that is, to functions $f: \mathbb{R} \to \mathbb{R}$ for which $f^{-1}(V) \in \mathcal{B}$ for all open $V \subseteq \mathbb{R}$. One result which we use is the following.

FUBINI THEOREM. [1, p. 103; 25, Th. 7.8]. *Let* μ_1, μ_2 *be Lebesgue-Stieltjes measures; let* $A_1, A_2 \in \mathcal{B}$ *and let* f *be*

Borel measurable. If

$$\int_{A_1}\int_{A_2} |f(x,y)|\,d\mu_2(y)\,d\mu_1(x) < \infty,$$

then

$$\int_{A_1}\int_{A_2} f(x,y)\,d\mu_2(y)\,d\mu_1(x)$$

$$= \int_{A_2}\int_{A_1} f(x,y)\,d\mu_1(x)\,d\mu_2(y).$$

The symmetric result is also true.

We observe at this time that the symbol

$$\int_a^b f\,d\mu$$

is ambiguous, since it can be regarded as

$$\int_A f\,d\mu$$

where A is any one of (a,b), [a,b), [a,b] or (a,b]. When we write

$$\int_a^b f\,d\mu$$

we will mean

$$\int_{[a,b]} f\,d\mu$$

Hence for $a < b < c$,

$$\int_a^c f\,d\mu = \int_a^b f\,d\mu + \int_b^c f\,d\mu - f(b)\mu(b).$$

To interpret Rogosinski's integral, we observe that for a given $\chi \in BV[0,1]$, there are monotone non-decreasing functions p and q such that

$$\chi(t) = p(t) - q(t).$$

From these, we can obtain Lebesgue-Stieltjes measures μ^+ and μ^- and define for $A \in \mathcal{B}$

$$\int_A f(t)\,d\chi(t) = \int_A f\,d\mu^+ - \int_A f\,d\mu^- .$$

If, as is the case for an integral Hausdorff method, A is a finite interval and f is bounded on finite intervals, this integral is always defined, since we never have a "$\infty - \infty$" situation on the right side.

We note that the familiar Riemann-Stieltjes integral

$$\int_0^1 t^n d\chi(t)$$

agrees with our Lebesgue-Stieltjes integral

$$\int_{[0,1)} t^n d\chi(t)$$

since we have imposed the condition

$$\chi(t) = \begin{cases} \chi(0) & \text{for } t < 0 \\ \chi(1) & \text{for } t > 1. \end{cases}$$

APPENDIX 2

SOME COMMENTS ON THEOREM C

We examine some aspects of the proof of Theorem C in Rogosinski's paper [23].

Let $\phi_1, \phi_2 \in BV[0,1]$ and be normalized so that for $i = 1,2$,

$$\phi_i(t) = \begin{cases} \phi_i(0) & \text{for } t \le 0 \\ \frac{1}{2}\{\phi_i(t^-) + \phi_i(t^+)\} & \text{for } t \in (0,1) \\ \phi_i(1) & \text{for } t \ge 1. \end{cases}$$

He states the following formula, for f Borel measurable and bounded on finite intervals of the form [0,x]. For

$$y > 0, \qquad H_{\phi_2} H_{\phi_1}(y) = \int_0^1 \int_0^1 f(yxz)\, d\phi_1(x)\, d\phi_2(z)$$

$$= \int_0^1 f(yz)\, d\phi_{21}(z)$$

where

$$\phi_{21}(z) = \int_0^1 \phi_1\left(\frac{z}{x}\right) d\phi_2(x)$$

$$= \int_z^1 \phi_1\left(\frac{z}{x}\right) d\phi_2(x)$$

$$+ \phi_1(1)\{\phi_2(z) - \phi_2(0)\}.$$

Using this formula, he proves that $\phi_{21}(z) = \phi_{12}(z)$ at $z = 0$, $z = 1$ and at points of continuity of $\phi_{21}(z)$ in $(0,1)$.

He then has shown that $H_{\phi_2}H_{\phi_1}(y) = H_{\phi_1}H_{\phi_2}(y)$. We can prove this relation directly, using Fubini's Theorem (see Appendix 1). We do need his formula, however, to show that the product of two integral Hausdorff methods is itself an integral Hausdorff method. We now prove the formula.

We first show that $\phi_{21}(z) \in BV[0,1]$. This follows directly from a lemma due to Tatchell[1] which gives us

$$\int_0^1 |d\phi_{21}(z)| \leq \sup_{0<z<1} \int_0^1 |d_x\phi_1(\tfrac{z}{x})| \cdot \int_0^1 |d\phi_2(t)|$$

$$= \sup_{0<z<1} \int_z^1 |d_u\phi_1(u)| \cdot \int_0^1 |d\phi_2(t)|$$

$$\leq V_1V_2,$$

where V_1 and V_2 are the total variations on $[0,1]$ of ϕ_1 and ϕ_2 respectively. Hence $\phi_{21}(z) \in BV[0,1]$. The second expression for $\phi_{21}(z)$ is derived from the first by integration.

We now prove the formula for step functions. Let $p \in (0,1]$ be a fixed constant. Define

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,p) \\ 1 & \text{for } x \in [p,1]. \end{cases}$$

Now, supressing the parameter y, which is constant within the formula, we have,

---

[1]J. B. Tatchell, "A theorem on absolute Riesz summability," J. London Math. Soc., 29(1954), 49-59.

$$\int_0^1 f(t)\, d\phi_{21}(t)$$

$$= \int_{[p,1]} d\phi_{21}(t)$$

$$= \phi_{21}(1) - \phi_{21}(p^-)$$

$$= \phi_1(1)\{\phi_2(1) - \phi_2(0)\} - \int_{[p,1]} \phi_1(\tfrac{p^-}{t})\, d\phi_2(t)$$

$$- \phi_1(1)\{\phi_2(p^-) - \phi_2(0)\}$$

$$= \phi_1(1)\{\phi_2(1) - \phi_2(p^-)\} - \int_{[p,1]} \phi_1(\tfrac{p^-}{t})\, d\phi_2(t) .$$

We also obtain

$$\int_0^1 \int_0^1 f(xz)\, d\phi_1(x)\, d\phi_2(z) = \int_{[p,1]} \int_{[p/z,1]} d\phi_1(x)\, d\phi_2(z)$$

$$= \int_{[p,1]} \{\phi_1(1) - \phi_1(\tfrac{p^-}{z})\}\, d\phi_2(z)$$

$$= \phi_1(1)\{\phi_2(1) - \phi_2(p^-)\}$$

$$- \int_{[p+1]} \phi_1(\tfrac{p^-}{z})\, d\phi_2(z)$$

$$= \int_0^1 f(t)\, d\phi_{21}(t) .$$

The result for general functions f which are Borel measurable follows by a limiting process, using a sequence of step functions approximating f.

APPENDIX 3

CAUCHY-LEBESGUE INTEGRATION

The Cauchy-Lebesgue integral is defined as the limit of Lebesgue integrals, just as the Cauchy-Riemann integral is the limit of Riemann integrals. For example, we might have

$$\int_0^\infty F(t)\,dt = \lim_{R\to\infty} \int_0^R F(t)\,dt,$$

where $\int_0^R F(t)\,dt$ is a Lebesgue integral. We note that if $\int_0^\infty F(t)\,dt$ is absolutely convergent, then it is a Lebesgue integral. Since absolute convergence of the iterated integral is needed to apply a Fubini Theorem, we have this tool for interchanging order of integration only when the Cauchy-Lebesgue integrals reduce to Lebesgue integrals. Hence we encounter the problem of finding an *ad hoc* method for each specific case to justify interchanging order of integration for Cauchy-Lebesgue integrals. It is for this reason that we restrict ourselves to the Lebesgue integral.

REFERENCES

1. R. A. Ash, *Real Analysis and Probability*, Academic Press, 1972.

2. D. Borwein, "A logarithmic method of summability," *J. London Math. Soc.*, 33(1958), 212-220.

3. D. Borwein, "On a scale of Abel-type summability methods," *Proc. Cambridge Philos. Soc.*, 53(1957), 318-322.

4. D. Borwein, "On methods of summability based on integral functions. II," *Proc. Cambridge Philos. Soc.*, 56(1960), 125-131.

5. D. Borwein, "On methods of summability based on power series," *Proc. Royal Soc. Edinburgh*, 64(1957), 342-349.

6. D. Borwein, "On strong and absolute summability," *Proc. Glasgow Math. Assoc.*, 4(1960), 122-139.

7. D. Borwein, "Theorems on some methods of summability," *Quart. J. Math. Oxford Ser.*, 9(1958), 310-316.

8. D. Borwein and J. H. Rizvi, "On strong summability," *J. Reine Angew. Math.*, 260(1973), 119-126.

9. D. Borwein and B.L.R. Shawyer, "On Borel-type methods, II," *Tôhoku Math. J.*, 19(1967) 232-237.

10. G. Doetsch, *Introduction to the Theory and Application of the Laplace Transform*, Springer-Verlag, 1974.

11. M. Fekete, "Vizsagálatok a Fourier-sorokról," *Math. és termész. ért.*, 34(1916), 759-786.

12. T. M. Flett, "Some remarks on strong summability," *Quart. J. Math. Oxford Ser.*, 10(1959), 115-139.

13. G. H. Hardy, *Divergent Series*, Oxford, 1949.

14. C. F. Harington and J. M. Hyslop, "An analogue of strong summability of Abel's summability method," *Proc. Edinburgh Math. Soc.*, 5(1953), 28-34.

15. F. Hausdorff, "Summationsmethoden und Momentfolgen. I," *Math. Z.*, 9(1921), 74-109.

16. F. Hausdorff, "Summationsmethoden und Momentfolgen. II," *Math. Z.*, 9(1921), 280-299.

17. J. M. Hyslop, "Note on the strong summability of series," *Proc. Glasgow Math. Assoc.*, 1(1951-3), 16-20.

18. A. Jakimovski (A. Amir), "Some relations between the methods of summability of Abel, Borel, Cesàro, Hölder and Hausdorff," *J. Analyse Math.*, 3(1954), 346-381.

19. H. Kestelman, *Modern Theories of Integration*, 2nd Revised Edition, Dover, 1960.

20. B. Kuttner, "On translated 'quasi-Cesàro' summability," *Proc. Cambridge Philos. Soc.*, 62(1966), 705-712.

21. S. J. H. Rizvi, "Abel-type summability," *Ph.D. Thesis, University of Western Ontario*, 1969.

22. W. W. Rogosinski, "On Hausdorff's methods of summability," *Proc. Cambridge Philos. Soc.*, 38(1942), 166-192.

23. W. W. Rogosinski, "On Hausdorff's methods of summability. II," *Proc. Cambridge Philos. Soc.*, 38(1942), 344-363.

24. W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 2nd Edition, 1964.

25. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 2nd Edition, 1974.

26. B. L. R. Shawyer, "On strong functional summability," (personal manuscript communication).

27. B. L. R. Shawyer and G. S. Yang, "On the relation between the Abel-type and Borel-type methods of summability," *Proc. Amer. Math. Soc.*, 26(1970), 323-328.

28. P. Srivastava, "On strong Rieszian summability of infinite series," *Proc. Nat. Inst. Sci. India*, 23(1957), 58-71.

29. P. Srivastava, "On the concept of strong summability," *Proc. Nat. Inst. Sci. India*, 26(1960), 545-552.

30. B. B. Watson, "Tauberian theorems on a scale of Abel-type summability methods," *Ph.D. Thesis, University of Western Ontario*, 1974.

31. D. V. Widder, *The Laplace Transform*, Princeton University Press, 2nd Printing, 1946.

32. C. E. Winn, "On strong summability for any positive order," *Math. Z.*, 37(1933), 481-492.

33. K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren*, Springer-Verlag, 1970.

34. A. Jakimovski (A. Amir), "Some remarks on Tauberian theorems," *Quart. J. Math. Oxford Ser.*, 9 (1958), 114-131.