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Dynamics Of The Liquid Outer Core Of The Earth

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DYNAMICS OF THE LIQUID OUTER CORE OF THE EARTH

by

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ABSTRACT

A general mathematical theory is developed for the dynamics of the earth. The facts that the outer core is liquid and the earth is rotating are taken into account so that the hydrodynamic equations for the outer core are correct to first order in the ellipticities of the surfaces of equal density. To separate the variables in the equations of motion, the method of spherical harmonic expansion is used. The resulting ordinary linear differential equations show coupling among spheroidal and toroidal fields. However, the nature of the coupling is such that for spheroidal deformation of a specific degree, it is sufficient to consider only the coupling effects from toroidal fields of neighboring degrees.

Using earth models with uniform polytropic cores, we applied the theory to free spheroidal oscillations of the earth of degree two as well as the earth tides. Two types of free core oscillations are found to exist for all three earth models used. The first type we call the 'core modes' has spheroidal fields as the dominant components in the outer core and frequency spectra characteristic of the density stratification in the outer core. The second type we call the 'toroidal modes' has large toroidal fields in

the outer core and consequently sizable spheroidal fields in the mantle due to the ellipticity of the core-mantle boundary. The frequency of a toroidal mode is found to be very insensitive to the density distribution in the outer core.

Due to the existence of free oscillations, the tidal response of the earth exhibits resonance patterns. Two important cases are found: 1. Resonance of diurnal tides at a toroidal mode of period 23.88337 hours. This effect is observed astronomically through the nutations associated with diurnal tides. Good agreements between the observations and the present theoretical results demonstrate the dynamic effects of the liquid core. 2. Resonance of semi-diurnal tides at a core mode of period about 12 hours. The period of this core mode depends strongly on the density stratification in the outer core. Therefore, this resonance is important for the study of the core structure.

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NOMENCLATURE

Notation	Description
A	equatorial moment of inertia
C	polar moment of inertia
$\underline{F}$	$= (F_r, F_\theta, F_\psi)$ external force
${}_rF_n$, ${}_\theta F_n$, ${}_\psi F_n$	radial coefficients of $\underline{F}$ under spherical harmonic expansion
F	defined by equations (3.14) and (3.15)
F_n	radial coefficient of F under spherical harmonic expansion
G	gravitational constant
H_n	radial coefficient of W_a under spherical harmonic expansion
I_{xx}, I_{xy}, I_{xz}, I_{yy}, I_{yz}, I_{zz}	principal moments and products of inertia
I_1, I_2	integrals
K	defined by equation (4.36)
K_1	defined by equation (4.38)
$\underline{L}$	$= (L_x, L_y, L_z)$ external torque

Notation	Description
$\underline{M}$	$= (M_x, M_y, M_z)$ angular momentum
$\Delta M_x, \Delta M_y, \Delta M_z$	relative angular momentum of the outer core
P	hydrostatic pressure
P_n^m	$= P_n^m (\cos\theta)$ associated Legendre polynomial of degree n, azimuthal number m
Q_n	radial coefficient of the change in normal gravitational flux density under spherical harmonic expansion
R	defined by equations (3.10) and (3.11)
R_n	radial coefficient of R under spherical harmonic expansion
S	defined by equations (3.12) and (3.13)
S_n	radial coefficient of S under spherical harmonic expansion
T_n	radial coefficient of toroidal displacement of degree n
T_{ij}	total stress tensor
U_n	radial coefficient of radial displacement of degree n

Notation	Description
V_n	radial coefficient of tangential displacement of degree n for spheroidal field
W	total gravitational potential
W_a	additional potential due to deformation
W_c	centrifugal potential
W_m	gravitational potential of the earth due to the undeformed mass
W_r	component of W_o' with spherical symmetry
W_T	the tesseral potential defined by equation (2.7)
W_t	defined by (2.12)
W_o	$= W_m + W_c$ gravitational potential of the earth under hydrostatic equilibrium
X_n	defined by equation (3.17)
Y_n	radial coefficient of tangential stress for spheroidal field
Y_n^m	$Y_n^m = Y_n^m(\theta,\psi)$ spherical harmonic of degree n, azimuthal number m
Z_n	radial coefficient of radial stress
b	defined by equation (2.28)

Notation	Description
d	radius of the equivalent spherical earth
e	ellipticity of the surface of equal density within the earth
f	defined by equation (3.54)
g	gravity of the equivalent spherical earth
h, k, l	love numbers
$\bar{n}$	external normal to the equipotential surface of W_o
p q	components of the rotation vector defined by equation (2.3)
r	radial distance from center of earth
$\underline{r}$	$= (x_1, x_2, x_3) = (x, y, z) = (r, \theta, \psi)$ coordinate system
t	time
$\underline{u}$	$= (u_1, u_2, u_3) = (u_x, u_y, u_z) = (u_r, u_\theta, u_\psi)$ displacement vector
$\underline{\Omega}$	$= (\Omega_1, \Omega_2, \Omega_3) = (\Omega_x, \Omega_y, \Omega_z) = (p, q, \omega)$ $= (\Omega_r, \Omega_\theta, \Omega_\psi)$ rotation vector of the earth
α	defined by equation (2.35)
β	defined by equation (4.35)

Notation	Description
δ_{ij}, δ^j_i	Kronecker delta
ε	constant related to nutation or free wobble; defined by equation (2.4)
ε_o	value of ε for rigid earth
η	displacement normal to the equipotential surface
η_n	radial coefficient of η under spherical harmonic expansion
λ	Lamé's constant
λ_s	Lamé's constant for the equivalent spherical earth
μ	rigidity
μ_s	rigidity for the equivalent spherical earth
ν	defined by equation (4.45)
ζ	defined by equation (3.53)
ρ	density at any time
ρ_o	density of the earth under hydrostatic equilibrium
ρ_s	density of the equivalent spherical earth
σ	angular frequency of oscillation

Notation	Description
τ_{ij}	additional stress relative to the state of hydrostatic equilibrium
ω	component of the rotation vector; angular frequency of diurnal rotation
Δ	dilatation
Δ_n	radial coefficient of dilatation under spherical harmonic expansion
∇	vector operator; gradient
∇^2	Laplacian
$(')$	a prime over W_o, ρ_o, λ, μ, ρ_s, λ_s, μ_s means derivative along the external normal of equipotential surface of W_o
$(\dot{})$	a dot over any quantity means derivative along radius of the earth

CHAPTER 1

INTRODUCTION

1.1 Dynamics of the Earth

The study of the dynamic behaviour of an elastic earth extends back more than one hundred years. The interest in such studies arises mainly from the influence they have upon the better understanding of the internal constitution of the earth. Early works by Lord Kelvin, H. Lamb, G.H. Darwin, A.E.H. Love, Lord Rayleigh, and others, as summarized by R. Stoneley (1961), were mainly concerned with the free and forced vibrations of a simple, uniform hypothetical earth. The forced oscillations, such as diurnal earth tides and the associated nutations were the most discussed due to the availability of astronomical data. The free oscillation of the earth was of limited interest at that time because the phenomenon was not observed until 1954 when H. Benioff identified a 57 minute period as the fundamental period of free oscillation from the record of Kamchatka earthquake of 1952. However, during the past 15 years, with the advent of computers and observational instruments, the problem of the free oscillations of the earth has been solved with increasing details. The state of progress is summed up in the work of P.C. Luh (1974) where a perturbation technique has been employed to tackle the effects of rotation and asphericities of the earth.

1.2 Dynamics of the Liquid Core

In 1910, H. Poincaré treated the dynamics of an incompressible fluid enclosed in an ellipsoidal rigid shell. The results, when applied to the earth, showed that the presence of the liquid would shorten the period of free (Chandler) wobble and reduce the amplitude of the 18.66 year principal nutation. However, in subsequent studies of the elastic deformations of the earth, the dynamic effect of the liquid core was neglected (e.g., Lamb 1932). The hydrodynamic equations for the liquid were obtained from the elastic theory for a spherically symmetric, non-rotating solid by simply setting the rigidity to zero. The very recent theories of free oscillation of the earth (e.g., Luh 1974) do incorporate the ellipticity and rotation of the earth by means of perturbation schemes. However, in the perturbation theory, the unperturbed eigenfunctions are derived from a spherically symmetric, non-rotating earth.

The eigenfunctions in the liquid core will therefore be purely spheroidal (Smylie and Mansinha 1971), and despite the perturbation scheme, toroidal fields will remain non-existent. It will be shown in this thesis that for the real earth, toroidal fields do exist in the core due to the ellipticity and rotation of the earth. In fact, in some cases, toroidal fields dominate the displacement in the liquid core, even though the displacement in the mantle is largely spheroidal. The applicability of a perturbation theory based on a spherical, non-rotating earth is therefore in doubt.

The observed Chandler period of 14 months can be conveniently explained by assigning a rigidity to the "equivalent" earth (Munk and MacDonald 1960, page 28). But the discrepancy between the observed ($9''.2030 \pm 0''.0023$) and theoretical ($9''.2232 \pm 0''.0012$ for a rigid earth) amplitudes for the principal nutation cannot be explained in this way. In fact, consideration of elasticity leaves the theoretical value practically unchanged (Jeffreys 1949). Therefore the entire problem of diurnal earth tides and nutations should be reconsidered, using the proper hydrodynamical theory for the liquid core. Physically, the reason for the revision is simple. Due to the absence of rigidity, the liquid core is capable of rotation relative to the mantle, which, together with the ellipticity of the core-mantle boundary, leads to the application of additional stresses at the base of the mantle. Consequently, the solution in the mantle will be changed. The same argument applies to other type of oscillation of the earth.

In Chapter 4, the dynamic effects of the liquid core are shown through the theoretical spectrum of free core oscillations. Diurnal earth tides have periods close to a free core oscillation of period 23.88337 hours. The resulting resonance is shown to agree with the observations as well as Molodensky's results.

1.3 Previous Work on the Dynamics of the Liquid Core

The theory of elastic deformation for a self-gravitating earth was first formulated by A.E.H. Love (1911). However, numerical solution for a reasonably realistic earth model was absent until 1950 when H. Takeuchi published the paper entitled "On the Earth Tide of the Compressible Earth of Variable Density and Elasticity". The solution, although only the static limit to an actually dynamic problem, gave a good approximation for the solid mantle and enabled H. Jeffreys and R.O. Vicente to attack the problem of diurnal earth tides and nutations. With Takeuchi's (1950) solution for the mantle, H. Jeffreys and R.O. Vicente (1957a, 1957b), applied Poincaré's theory to the liquid core, assuming the medium is incompressible. The results change the discrepancies between the observed and theoretical amplitudes of nutations in the right direction but with unsatisfactory amplitudes. Moreover, the variational method they employed lacked logical clarity in the sense that some of the mathematical steps are not obvious, and the degree of approximation cannot easily be visualized.

M.S. Molodensky proposed an analogous theory in 1961 which we will rederive from our general theory as a particular case. The theory, taking into account the compressibility of the liquid core, and correct to first order in the ellipticity of the earth, exhibits mathematical simplicity and agrees well with the observations. However, there are certain drawbacks. Firstly, the theory is particularly constructed for the diurnal earth tides only and therefore cannot readily be generalized to other problems. Secondly, although results from the present work show otherwise, the theory is apparently valid only for earth models with Adams and Williamson (1923) cores (see section 4.6.1).

The dynamic behaviour for earth models with polytropic cores (see section 4.6.1) was discussed by C.L. Pekeris and Y. Accad (1972). With the earth being spherically symmetric and non-rotating, Love numbers and spectra of free core oscillations were derived for uniformly stable, neutral, and unstable core models. One important result is that the uniformly unstable and neutral core models exhibit no free core oscillations while the uniformly stable core model has an unlimited number of free core oscillations. However, it will be shown in section 4.6 that for the real earth, the situation is completely different.

1.4 The Present Work

The aim of the present work is to construct a general theory which deals with the dynamic effects of the liquid core upon free and forced oscillations of the earth. The block diagram (Figure 1) illustrates the logical organization of the theory.

In the liquid outer core, the ellipticity and rotation of the earth are taken into account so that the resulting hydrodynamic equations are correct to first order in both displacement and ellipticity. These equations are general in that they impose no limitations on the structure of the liquid outer core, and are applicable to any type of spherical harmonic oscillations.

In the mantle and inner core, due to the existence of large rigidity, the effects of ellipticity and rotation are relatively small and are neglected in the present work.

It is known that any vector may be expanded in spheroidal and toroidal fields (Copson 1935). We therefore employ the method of spherical harmonic

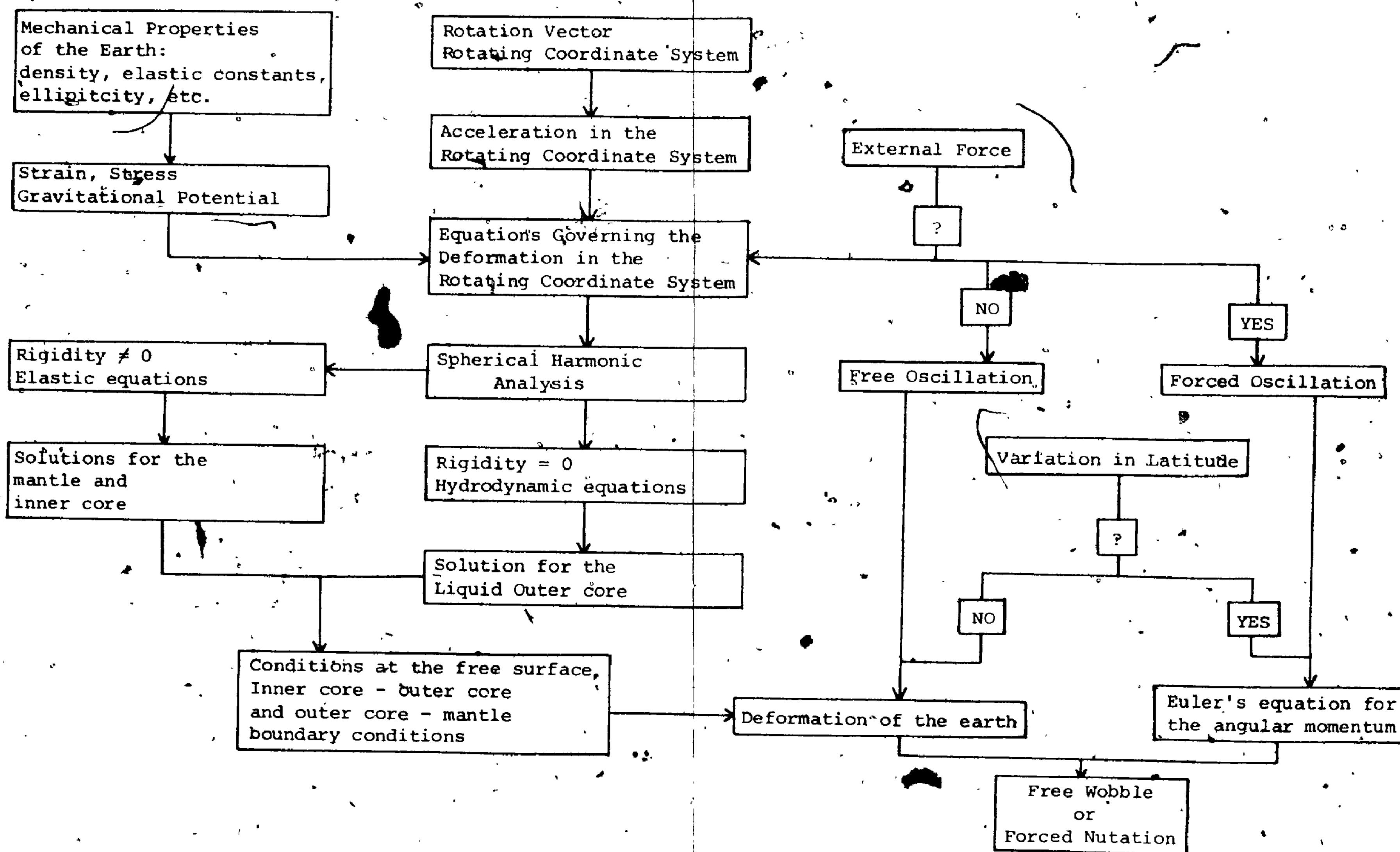

Figure 1 Logical Organization of the Theory

expansion to solve the general equations of deformation. In the outer core, coupling occurs between the spheroidal and toroidal fields. Therefore, to render the numerical solution possible, an approximation is necessary. It is found that the coupling constant is the ellipticity of the earth. Since the equations of deformation are correct to first order in ellipticity, displacement fields of the order of ellipticity and smaller are neglected.

In the mantle and inner core, due to the neglect of effects of ellipticity and rotation, spheroidal and toroidal displacement fields are separable. In the present work, we shall consider only deformations of the earth in which the displacement field in the mantle and inner core is spheroidal, and we shall call them spheroidal deformations of the earth. It is worth noting that the purely spheroidal fields in the mantle and inner core are accompanied by both toroidal and spheroidal fields in the outer core.

1.5 Numerical Calculations

As a particular case of the general theory, we study the second degree spheroidal deformations of the earth. Due to the different character of the approximations, oscillations of sectorial, tesseral, and zonal harmonics are treated separately. Tidal Love numbers and spectra of free core oscillations are derived for various earth models with uniform polytropic cores.

Two types of free core oscillations are found. The "core modes", have the dominant spheroidal displacement confined mainly to the liquid core. These are the "core oscillations" first discussed by C.L. Pekeris, Z. Alterman, and H. Jarosch (1963). The "toroidal modes", on the other hand, exhibit the same characteristics as the elastic normal modes except that there exist large toroidal fields in the liquid outer core. An important toroidal mode is the tesseral free oscillation with a period of 23.88337 hours. This mode, in addition to giving rise to a nearly diurnal free wobble of the earth, leads to the resonance effect for diurnal earth tides and nutations.

The core modes by themselves are barely observable on the surface of the earth. However, their strong dependence on the density stratification of the core and the possibility of observing them through resonance make them interesting. A sectorial core mode occurs at a period of about 12 hours for uniformly stable cores. A close examination of the semi-diurnal (sectorial) tidal responses is therefore recommended.

CHAPTER 2

GENERAL THEORY

In this chapter, the general theory of deformation of the earth is described. The basio equations of motion are given in section 2.1. The frame of reference in which these equations are given is described in section 2.2. In section 2.3, the gravitational potential and stress distributions are described assuming small displacements. The conditions in the undisturbed earth are discussed in section 2.4. We then write the equations of motion in section 2.5 in terms of displacements.

To complete the solution, we also need the equation of motion of the rotating frame of reference in space when the deformation leads to variation of latitude. This is Euler's equation for the angular momentum and is discussed in section 2.6.

2.1 Equations of Deformation

Let $\underline{r} = (x_1, x_2, x_3)$ be a spatial coordinate system rotating in space at an angular velocity $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

To describe the dynamical behaviour in this coordinate system, let the displacement vector of a particle which is initially at $\underline{r}$ be given by $\underline{u} = (u_1, u_2, u_3)$, where $u_i = u_i(t, \underline{r})$, and $u_i(0, \underline{r}) = 0$. Then the acceleration of the particle in vector notation is

$$\frac{d^2\underline{u}}{dt^2} = \frac{\partial^2\underline{u}}{\partial t^2} + 2\underline{\Omega} \times \frac{\partial \underline{u}}{\partial t} + \frac{\partial \underline{\Omega}}{\partial t} \times \underline{r} + (\underline{\Omega} \cdot \underline{r})\, \underline{\Omega} - (\underline{\Omega} \cdot \underline{\Omega})\, \underline{r} + \left(\frac{\partial \underline{u}}{\partial t} \cdot \nabla\right) \frac{\partial \underline{u}}{\partial t}, \qquad (2.1)$$

where the term $2\underline{\Omega} \times \frac{\partial \underline{u}}{\partial t}$ is the acceleration of Coriolis and the rest comprises what is generally called the acceleration of transport. $\partial/\partial t$ is $(d/dt)_{\text{rotating frame}}$.

The last term in the right hand side of (2.1) is non-linear in $\underline{u}$. However, we shall consider only small displacements so that this term may be neglected.

The equations of motion are given in tensor notation by

$$\rho \frac{d^2 u_i}{dt^2} = \rho F_i + \rho \frac{\partial W}{\partial x_i} + \frac{\partial}{\partial x_j} T_{ji}, \qquad (2.2)$$

where ρ is the density, $\underline{F} = (F_1, F_2, F_3)$ the (external) force density, W the potential of self-gravitation, and T_{ij} the stress tensor. Notice that Einstein's summation convention is used.

2.2 The Frame of Reference

The choice of the rotating coordinate system is arbitrary. But it must be attached to the earth in some way. If the earth were rigid, we could choose the coordinate system which rotates with the earth. Unfortunately, the earth is deformable; winds, ocean currents, fluid core, tidal distortions, and in the geologic time scale, convection in the mantle complicate the problem. Therefore in studying the dynamical behaviour of the deforming earth, we must choose a set of rigid axes which are kinematically defined. Three possible choices are described by W.H. Munk and G.J.F. MacDonald (1960, page 10).

i) Tisserand's mean axes of body

These axes are defined so that the relative angular momentum within the frame vanishes. If the earth were rigid, this coordinate system would have been the one that "rotates with the earth".

ii) The geographic axes

These axes are attached in a prescribed way to the observatories. The possibility of relative motion of the observatories can be avoided by choosing suitable locations for the observatories. If the relative motion cannot be avoided, the relation between the rigid axes and the observatories may be prescribed. Geophysical observations, astronomical observations, etc. are referred to this coordinate system.

iii) The principal axes, or axes of figure

These axes are defined so that the "mean" products of inertia vanish.

The principal axes and the mean axes of body lead to mathematical simplicity. However, possible relative motion between these axes and the observatories must be corrected for (Munk and Macdonald 1960, p. 11).

We shall discuss our choice of the reference frame in section 2.6. For the time being, we assume the angular velocity of this rotating frame in space is given by

$$\underline{\Omega} = (p, q, \omega) \tag{2.3}$$

in right handed cartesian coordinates (x, y, z). Here ω is the angular frequency of diurnal rotation, and p and q are due to deviation from diurnal rotation when the oscillation of the earth leads to variation of latitude. If the frequency of oscillation is σ, we can write

$$\begin{aligned} p &= \omega\varepsilon \cos \sigma t, \\ q &= \omega\varepsilon \sin \sigma t, \end{aligned} \tag{2.4}$$

where ε is a small constant, the angle between the angular velocity and the axis of diurnal rotation.

The equations of motion (2.2) are conveniently given in spherical coordinates (r, θ, ψ). We shall take the z-axis as the polar axis. Then θ is the colatitude and ψ

the east longitude.

The angular velocity $\underline{\Omega}$ given in (2.3) may be transformed into spherical coordinates using the covariant law,

$$\Omega_i' = \frac{\partial x_j}{\partial x_i'} \Omega_j$$

We find

$$\begin{aligned} \Omega_r &= p \sin\theta \cos\psi + q \sin\theta \sin\psi + \omega \cos\theta, \\ \Omega_\theta &= p \cos\theta \cos\psi + q \cos\theta \sin\psi - \omega \sin\theta, \\ \Omega_\psi &= - p \sin\psi + q \cos\psi. \end{aligned} \tag{2.5}$$

Then, to first order in ε and $\underline{u}$,

$$\begin{aligned} \underline{\Omega} \times \frac{\partial \underline{u}}{\partial t} &= \left(- \omega \sin\theta \frac{\partial u_\psi}{\partial t}, - \omega \cos\theta \frac{\partial u_\psi}{\partial t}, \omega \cos\theta \frac{\partial u_\theta}{\partial t} + \right. \\ &\qquad \left. \omega \sin\theta \frac{\partial u_r}{\partial t}\right), \\ \frac{\partial \underline{\Omega}}{\partial t} \times \underline{r} &= \left(0, r \frac{\partial \Omega_\psi}{\partial t}, - r \frac{\partial \Omega_\theta}{\partial t}\right); \\ (\underline{\Omega} \cdot \underline{r})\, \underline{\Omega} &= r\, \Omega_r\, \underline{\Omega}, \\ (\underline{\Omega} \cdot \underline{\Omega})\, \underline{r} &= \Omega^2\, \underline{r}. \end{aligned} \tag{2.6}$$

Using (2.6) in (2.1), we get

$$\frac{d^2u_r}{dt^2} = \frac{\partial^2 u_r}{\partial t^2} - 2\omega \sin\theta \frac{\partial u_\psi}{\partial t} - \frac{2}{3}\omega\sigma\varepsilon r\, P_2^1 (\cos\theta) \cos(\sigma t - \psi) -$$

$$\frac{\partial}{\partial r}\left(W_c + \frac{\sigma+\omega}{\omega} W_T\right), \tag{2.7}$$

$$\frac{d^2u_\theta}{dt^2} = \frac{\partial^2 u_\theta}{\partial t^2} - 2\omega \cos\theta \frac{\partial u_\psi}{\partial t} + \frac{8}{5}\omega\sigma\varepsilon r \frac{1}{\sin\theta} P_1^1 (\cos\theta)$$

$$\cos(\sigma t - \psi) - \frac{4}{15}\omega\sigma\varepsilon r \frac{1}{\sin\theta} P_3^1 (\cos\theta) \cos(\sigma t - \psi) -$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(W_c + \frac{\sigma+\omega}{\omega} W_T\right), \tag{2.8}$$

$$\frac{d^2u_\psi}{dt^2} = \frac{\partial^2 u_\psi}{\partial t^2} + 2\omega \sin\theta \frac{\partial u_r}{\partial t} + 2\omega\cos\theta \frac{\partial u_\theta}{\partial t} -$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\psi}\left(W_c + \frac{\sigma+\omega}{\omega} W_T\right), \tag{2.9}$$

where $P_2^1 (\cos\theta)$ is associated Legendre polynomial of degree 2 and azimuthal number 1,

$$W_c = \frac{1}{2}\omega^2 r^2 \sin^2\theta \tag{2.10}$$

is the centrifugal potential, and

$$W_T = -\frac{1}{3}\varepsilon\omega^2 r^2 P_2^1 (\cos\theta)\cos(\sigma t - \psi) \tag{2.11}$$

is the potential in the form of the tesseral harmonic arising from a variation of latitude (Melchior 1966, p.368).

We shall write

$$W_t = -\frac{1}{3}\varepsilon\omega^2 r^2. \tag{2.12}$$

2.3 The Potential W and the Stress T_{ij}

2.3.I Assumptions

For the sake of simplicity, we shall make the following assumptions about the undisturbed state of the earth:

i) In the undisturbed state, the earth is in hydrostatic equilibrium.

The earth is subjected to the potential W_o,

$$W_o = W_m + W_c \tag{2.13}$$

where W_m is the undisturbed gravitational potential which satisfies

$$\nabla^2 W_m = -4\pi G\,\rho_o \tag{2.14}$$

with ρ_o the undisturbed density.

The fundamental equation of hydrostatics then states

$$\nabla P = \rho_o\,\nabla W_o, \tag{2.15}$$

where P is the hydrostatic pressure.

ii) In the undisturbed state, the equipotential surfaces coincide with the surfaces of equal density, equal incompressibility, and equal rigidity.

Let a prime over a quantity indicate its derivative along the external normal of the equipotential surface, then the assumption states

$$\frac{\partial \rho_o}{\partial x_i} = \frac{\rho_o'}{W_o'} \frac{\partial W_o}{\partial x_i} \qquad (2.16)$$

and similar expressions for other elastic properties of the earth.

iii) The dynamic stress-strain relation is perfectly elastic and isotropic.

This implies that the additional stress τ_{ij} due to small deformation can be given by

$$\tau_{ij} = \lambda\Delta \, \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (2.17)$$

where λ and μ are Lamé's constants,

$$\Delta = \text{div } (\underline{u}) \qquad (2.18)$$

the dilatation, and δ_{ij} the Kronecker delta,

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

2.3.2 Expressions for W and T_{ij}

The gravitational potential W and the stress T_{ij} for a spherically symmetric, non-rotating earth under small oscillation were first given by A.E.H. Love (1911). For the real earth, W and T_{ij} can be deduced in a similar way.

Due to the deformation, there is a variation in volume density. The equation of continuity states that

$$\rho - \rho_o = - \operatorname{div} (\rho \underline{u}). \tag{2.19}$$

Let η be the work done by the deformation,

$$\eta = \underline{u} \cdot \operatorname{grad} W_o$$

$$= u_r \frac{\partial W_o}{\partial r} + u_\theta \frac{1}{r} \frac{\partial W_o}{\partial \theta} + u_\psi \frac{1}{r \sin \theta} \frac{\partial W_o}{\partial \psi}, \tag{2.20}$$

then (2.19) can be written as

$$\rho - \rho_o = - \rho_o \Delta - \frac{\rho_o'}{W_o'} \eta, \tag{2.21}$$

where equation (2.16) has been used.

The variation in volume density leads to a change in gravitational potential, W_a, which satisfies the Poisson equation

$$\nabla^2 W_a = -4\pi G (\rho' - \rho_o') = 4\pi G \left(\rho_o \Delta + \frac{\rho_o'}{W_o'} \eta \right). \qquad (2.22)$$

The total potential of self-gravitation in the disturbed state is then given by

$$W = W_m + W_a = W_o + W_a - W_c. \qquad (2.23)$$

The stress T_{ij} consists of the initial hydrostatic stress and the additional stress τ_{ij} given by (2.17).

The initial hydrostatic pressure at a material point (x_1, x_2, x_3) in the deformed state is given by the hydrostatic pressure at the point $(x_1 - u_1, x_2 - u_2, x_3 - u_3)$ in the initial state.

Now, to first order in $\underline{u}$,

$$P(x_1 - u_1, x_2 - u_2, x_3 - u_3) = P(x_1, x_2, x_3) - \underline{u} \cdot \text{grad } P.$$

Using (2.15) and (2.20),

$$P(x_1 - u_1, x_2 - u_2, x_3 - u_3) = P - \rho_o \eta.$$

Thus, we have

$$T_{ij} = -(P - \rho_o \eta)\ \delta_{ij} + \lambda\Delta\ \delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad (2.24)$$

2.4 The Potential W_o in the Undisturbed Earth

Due to the centrifugal potential W_c, the equipotential surfaces within the earth are not spherical. In fact, correct to first order in the ellipticities of surfaces of equal density, W_o can be written as

$$W_o = W_r(r) + e(r)\ g(r)\ r \sin^2 \theta, \qquad (2.25)$$

where

$$g = -\frac{d}{dr} W_r(r) \doteq - W_o' \qquad (2.26)$$

is the gravity, and $e(r)$ the ellipticity (see Appendix A, equations A.17 and A.18).

We shall write for convenience,

$$W_o = W_r + b(r) \sin^2 \theta, \qquad (2.27)$$

with

$$b(r) = eg\ r. \qquad (2.28)$$

The ellipticity $e(r)$ can be determined in two ways. One is to solve the classical Clairaut's equation with the surface value determined from observation (Jeffreys 1959). This, however, is in conflict with our assumption of initial hydrostatic equilibrium, as the figure of the earth deviates slightly from that of hydrostatic equilibrium.

The second method is to consider the centrifugal force as a disturbing force to an initially spherically symmetric earth and calculate the resulting deformation. The equations that govern the deformation are equivalent to Clairaut's equation. But the advantage is that hydrostatic theory can be strictly followed.

The detailed theory for the second method is given in Appendix A.

Since the earth deviates from hydrostatic equilibrium by only about 0.5% (Jeffreys, 1963), the ellipticities calculated from the two methods differ at best by the same amount. With our theory correct to first order in ellipticity, such a small difference can well be neglected.

As the equipotential surfaces are assumed to coincide with surfaces of equal density, incompressibility, and rigidity, the following expressions are correct to first order in ellipticity.

$$\rho_o = \rho_s(r) + \frac{\rho_s'}{W_o'} b(r) \sin^2\theta,$$

$$\lambda = \lambda_s(r) + \frac{\lambda_s'}{W_o'} b(r) \sin^2\theta, \qquad (2.29)$$

$$\mu = \mu_s(r) + \frac{\mu_s'}{W_o'} b(r) \sin^2\theta,$$

where the subscript s indicates the value of a quantity in the spherically symmetric earth.

2.5 Equations of Deformation in Terms of Displacement

Using (2.21) and (2.23), we can write

$$\rho \frac{\partial W}{\partial x_i} = \rho\left(-\frac{\partial W_c}{\partial x_i} + \frac{\partial W_o}{\partial x_i} + \frac{\partial W_a}{\partial x_i}\right)$$

$$= -\rho\frac{\partial W_c}{\partial x_i} + \left(\rho_o - \rho_o\Delta - \frac{\rho_o'}{W_o'}\eta\right)\frac{\partial W_o}{\partial x_i} + \rho_o\frac{\partial W_a}{\partial x_i}, \qquad (2.30)$$

neglecting second order terms.

Using (2.15) and (2.24), with the help of (2.16), we get

$$\frac{\partial T_{ji}}{\partial x_j} = - \rho_o \frac{\partial W_o}{\partial x_i} + \rho_o \frac{\partial \eta}{\partial x_i} + \frac{\rho_o}{W_o} \eta \frac{\partial W_o}{\partial x_i} + \frac{\partial}{\partial x_i} \left[\frac{\lambda \Delta}{\rho_o}\right] +$$

$$\frac{\lambda \rho_o'}{\rho_o^2 W_o'} \Delta \frac{\partial W_o}{\partial x_i} + \mu' u_i' + \frac{\mu'}{W_o'} \left[\frac{\partial \underline{u}}{\partial x_i} \cdot \text{grad } W_o\right] +$$

$$\mu \left[\nabla^2 u_i + \frac{\partial \Delta}{\partial x_i}\right]. \qquad (2.31)$$

Substituting (2.7), (2.8), (2.9), (2.30), and (2.31) in (2.2), we get

$$\frac{\partial^2 u_r}{\partial t^2} - 2\omega \sin \theta \frac{\partial u_\psi}{\partial t} - \frac{2}{3} \omega\sigma\varepsilon r \, P_2^1 (\cos \theta) \cos (\sigma t - \psi) =$$

$$F_r + \frac{\partial}{\partial r} \left[W_a + \frac{\sigma+\omega}{\omega} W_T + \eta + \frac{\lambda \Delta}{\rho_o}\right] + \alpha(r) \Delta \frac{\partial W_o}{\partial r} +$$

$$\frac{1}{\rho_o} \left\{\mu' u_r' + \frac{\mu'}{W_o'} \left[\frac{\partial \underline{u}}{\partial r} \cdot \nabla W_o\right] + \mu \left[\nabla^2 u_r + \frac{\partial \Delta}{\partial r}\right]\right\}, \qquad (2.32)$$

$$\frac{\partial^2 u_\theta}{\partial t^2} - 2\omega \cos \theta \frac{\partial u_\psi}{\partial t} + \frac{8}{5} \omega\sigma\varepsilon r \frac{1}{\sin \theta} P_1^1 (\cos \theta) \cos (\sigma t - \psi) -$$

$$\frac{4}{15} \omega\sigma\varepsilon r \frac{1}{\sin \theta} P_3^1 (\cos \theta) \cos (\sigma t - \psi) =$$

$$F_\theta + \frac{1}{r} \frac{\partial}{\partial \theta} \left[W_a + \frac{\sigma+\omega}{\omega} W_T + \eta + \frac{\lambda \Delta}{\rho_o}\right] + \alpha(r) \frac{1}{r} \Delta \frac{\partial W_o}{\partial \theta} +$$

$$\frac{1}{\rho_o}\left\{\mu' u_\theta' + \frac{\mu'}{W_o'}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\cdot \nabla W_o\right) + \mu\left(\nabla^2 u_\theta + \frac{1}{r}\frac{\partial \Delta}{\partial \theta}\right)\right\}, \quad (2.33)$$

$$\frac{\partial^2 u_\psi}{\partial t^2} + 2\omega \sin\theta \frac{\partial u_r}{\partial t} + 2\omega \cos\theta \frac{\partial u_\theta}{\partial t} =$$

$$F_\psi + \frac{1}{r \sin\theta}\frac{\partial}{\partial \psi}\left[W_a + \frac{\sigma+\omega}{\omega} W_T + \eta + \frac{\lambda\Delta}{\rho_o}\right] +$$

$$\alpha(r)\frac{1}{r\sin\theta}\Delta\frac{\partial W_o}{\partial \psi} + \frac{1}{\rho_o}\left\{\mu' u_\psi' + \frac{\mu'}{W_o'}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \psi}\cdot \nabla W_o\right) + \mu\left(\nabla^2 u_\psi + \frac{1}{r\sin\theta}\frac{\partial \Delta}{\partial \psi}\right)\right\}, \quad (2.34)$$

where

$$\alpha(r) = \frac{\lambda \rho_o'}{\rho_o^2 W_o'} - 1. \quad (2.35)$$

Equations (2.32), (2.33), and (2.34) are the general equations of deformation in the rotating coordinate system. The same set of equations was derived by M.S. Molodensky (1961) in cartesian coordinates.

In the outer core where the rigidity μ is assumed to vanish, the last terms in (2.32), (2.33), and (2.34) are identically zero.

2.6 Euler's Equation for the Angular Momentum

The Euler's equation of motion is

$$\left.\frac{d\underline{M}}{dt}\right|_{\text{rotating frame}} + \underline{\Omega} \times \underline{M} = \underline{L}, \tag{2.36}$$

where $\underline{M}$ is the angular momentum, $\underline{\Omega}$ the angular velocity of the rotating frame in space, and $\underline{L}$ the external torque (Munk and MacDonald 1960, page 9).

In cartesian coordinates, with $\underline{\Omega}$ given by (2.3), the components of angular momentum to first order in ε are

$$\begin{aligned} M_x &= \omega(I_{xx}\,\varepsilon \cos \sigma t - I_{xz}) + \Delta M_x, \\ M_y &= \omega(I_{yy}\,\varepsilon \sin \sigma t - I_{yz}) + \Delta M_y, \\ M_z &= \omega\, I_{zz} + \Delta M_z, \end{aligned} \tag{2.37}$$

where I_{ij} are the moments of inertia, ΔM_x, ΔM_y, and ΔM_z are components of relative angular momentum due to motions in the rotating frame of reference.

The possible choices of rotating frame of reference for the earth have been discussed in section 2.2. To simplify the mathematical expressions for the products of inertia and the relative angular momentum, we must choose between the mean axes of body and the principal axes. By choosing the mean axes of body, we can make the relative angular momentum identically zero. However, astronomical observations are referred to the geographic axes which are attached to the mantle and consequently characterize the rotation of the mantle only. We therefore choose the principal axes of the earth as our rotating frame of reference.

To evaluate ΔM_x, ΔM_y, and ΔM_z, we observe that in the mantle and inner core, the effects of ellipticity and rotation of the earth have been neglected (section 1.4). We can therefore consider oscillations of the earth with purely spheroidal deformation in the mantle and inner core. For these oscillations, the contributions from mantle and inner core to the relative angular momentum vanish (section 3.6). The relative angular momentum is therefore due only to motions in the outer core relative to the rotating frame of reference (the principal axes).

In vector notation,

$$\Delta \underline{M} = \left[\Delta M_x,\ \Delta M_y,\ \Delta M_z\right] = \int_{\text{outer core}} \rho_o\, \underline{r} \times \frac{\partial \underline{u}}{\partial t}\, d\tau . \tag{2.38}$$

The products of inertia are due to the redistributions of volume density $(\rho - \rho_o) = -\rho_o \Delta - \frac{\rho_o'}{W_o}\eta$, and the surface density $\frac{\rho\eta}{W_o}$, at each surface of discontinuity;

$$I_{xz} = \int_{\tau} (\rho - \rho_o) \; xz \, d\tau + \sum_s \int_s \frac{\rho n}{W_o} \, xz \, ds, \qquad (2.39)$$

and similar expression for I_{yz}.

Using (2.37), (2.38), and (2.39) in (2.36), a relation between ε and the displacement $\underline{u}$ is obtained.

Equations (2.22), (2.32), (2.33), (2.34) and (2.36) form the complete set of equations for the dynamical problem in hand.

If we assume the earth is rigid,

$$\Delta M_x = \Delta M_y = \Delta M_z = I_{xy} = I_{yz} = I_{xz} = 0.$$

Let

$$I_{xx} = I_{yy} = A, \qquad I_{zz} = C, \qquad (2.40)$$

$$\underline{L} = (L_x, L_y, o), \qquad L_x = - L \sin \sigma t, \qquad L_y = L \cos \sigma t, \qquad (2.41)$$

then (2.36) becomes

$$\varepsilon_o - \frac{\sigma+\omega}{\omega} \frac{A}{C} \varepsilon_o = \frac{-1}{\omega^2 C} L, \qquad (2.42)$$

where ε_o is the constant ε for the rigid earth.

The interpretation of the angular momentum $\underline{M}$ given above is slightly incorrect in that possible rotations of the inner core relative to the principal axes have not been taken into account. Due to its ellipticity, the inner core is capable of free wobble or nutation relative to the mantle. Let the amplitude of nutation or free wobble of the inner core relative to the principal axes be ε_I, and let the principal moment of inertia of the inner core be I_I, then a term of the form

$$\Delta M_I = (\omega\varepsilon_I\, I_I \cos \sigma t,\ \omega\varepsilon_I\, I_I \sin \sigma t,\ o)$$

must be added to the angular momentum $\underline{M}$. However, ε_I is of the order of ε or smaller,

$$\varepsilon_I \leq \varepsilon.$$

Also

$$I_I \sim 10^{-3} A.$$

Therefore ΔM_I can be neglected.

CHAPTER 3

EXPANSION IN SPHERICAL HARMONICS

To facilitate the solution of the equations of motion described in chapter 2, the displacements are expanded in spherical harmonics $Y_n^m(\theta,\psi)$.

Due to the ellipticity and the Coriolis force there are couplings among the displacement fields with different 'n'. However, equations with different 'm' can be separated. Therefore, it is only necessary to expand the displacements in spherical harmoncs Y_n^m with m fixed.

Some relevant properties of spherical harmonics are given in section 1. A sufficient form of expansion is given in section 2. The inclusion of toroidal fields in addition to the spheroidal fields is necessary in the liquid core, but not in the mantle and inner core, especially when we are mostly interested in the dynamical behaviour of the liquid core. The treatments of the expansion for the liquid outer core and mantle and inner core are discussed separately in section 3 and 4. Due to the discontinuity in mechanical properties at the outer core boundaries, it is necessary to connect the displacement fields, stress distributions and potentials on both sides of the boundaries. This is discussed in section 5. Finally, the angular momentum is given in terms of displacement fields in section 6.

3.1 Spherical Harmonics $Y_n^m(\theta,\psi)$ of Degree n and Azimuthal Number m

The spherical harmonic function Y_n^m is related to the associated Legendre polynomial $P_n^m(\cos\theta)$ by

$$Y_n^m(\theta,\psi) = (-)^m \left(\frac{2n+1}{4\pi}\,\frac{(n-m)!}{(n+m)!}\right)^{1/2} P_n^m(\cos\theta)\, e^{im\psi}, \qquad (3.1)$$

for $n > m > 0$, and

$$Y_n^{-m}(\theta,\psi) = (-)^m\, Y_n^{m\,*}(\theta,\psi), \quad (n > m > 0), \qquad (3.2)$$

where $Y_n^{m\,*}$ is the complex conjugate of Y_n^m.

The associated Legendre polynomials are defined by

$$P_n^m(u) = \frac{(1-u^2)^{1/2\,m}}{2^n n!}\,\frac{d^{n+m}}{du^{n+m}}(u^2-1)^n, \quad \left.\begin{matrix} n = 0 \\ m = 0 \\ -1<u<1 \end{matrix}\right\} n \qquad (3.3)$$

The orthogonal relations are

$$\int_{-1}^{1} P_k^m(u)\, P_n^m(u)\, du = \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}\,\delta_{kn}. \qquad (3.4)$$

The recursion relations are

$$(2n + 1)\, u\, P_n^m(u) = (n + 1 - m)\, P_{n+1}^m(u) + (n + m)\, P_{n-1}^m(u), \tag{3.5}$$

$$(1 - u^2)\frac{d}{du} P_n^m(u) = -\frac{n(n+1-m)}{2n+1} P_{n+1}^m(u) + \frac{(n+1)(n+m)}{2n+1} P_{n-1}^m(u), \tag{3.6}$$

which are also valid when n = 0, when the convention $P_{-1}^o = 0$ is realized.

Some particular values of $P_n^m(u)$ are given as follows:

$$P_n^o(1) = 1, \quad P_n^o(-1) = (-1)^n,$$
$$P_n^m(1) = P_n^m(-1) = 0, \text{ if } m > 0. \tag{3.7}$$

The spherical harmonics are given here in complex version. However, it is clear we can avoid complex variables by replacing $e^{im\psi}$ with $\cos m\psi$ and $\sin m\psi$.

3.2 The General Form of Expansion

For a particular m > 0, the displacement is expanded in the following way

$$u_r = \sum_n U_n(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\psi),$$

$$u_\theta = \sum_n V_n(r)\,\frac{\partial}{\partial\theta} P_n^m(\cos\theta)\cos(\sigma t - m\psi) -$$

$$m\sum_n T_n(r)\,\frac{1}{\sin\theta}\, P_n^m(\cos\theta)\cos(\sigma t - m\psi), \quad (3.8)$$

$$u_\psi = m\sum_n V_n(r)\,\frac{1}{\sin\theta}\, P_n^m(\cos\theta)\sin(\sigma t - m\psi)$$

$$\sum_n T_n(r)\,\frac{\partial}{\partial\theta} P_n^m(\cos\theta)\sin(\sigma t - m\psi).$$

In the right hand side of (3.8), the first term represents the spheroidal field, and the second term the toroidal field. The summation is over $m < n < \infty$.

The general representation of time and ψ dependence is $e^{i(\sigma t - m\psi)}$. Here we take the physically significant real part. Also σ is taken as negative so that a positive m is related to a retrograde motion.

With the displacement expanded as in (3.8), the change in potential W_a, the work done by deformation η, the dilatation Δ, and the body force $\underline{F}$ are expanded as follows

$$W_a = \sum_n H_n(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\psi),$$

$$\eta = \sum_n \eta_n(r)\, P_n^m(\cos\theta)\cos(\sigma t - m\psi),$$

$$\Delta = \sum_n \Delta_n (r) P_n^m (\cos\theta) \cos(\sigma t - m\psi),$$

$$F_r = \sum_n {}_rF_n (r) P_n^m (\cos\theta) \cos(\sigma t - m\psi),$$

$$F_\theta = \sum_n {}_\theta F_n (r) \frac{\partial}{\partial\theta} P_n^m (\cos\theta) \cos(\sigma t - m\psi), \qquad (3.9)$$

$$F_\psi = m \sum_n {}_\psi F_n (r) \frac{1}{\sin\theta} P_n^m (\cos\theta) \sin(\sigma t - m\psi).$$

Substituting (3.8) and (3.9) into (2.22), (2.32), (2.33), and (2.34), we can obtain an infinite set of linear ordinary differential equations by using the recursion and orthogonality relations of associated Legendre polynomials. Theoretically this set of differential equations can then be solved for U_n, V_n, T_n, H_n, $n = m, \infty$, (in terms of ε, if $\varepsilon \neq 0$.) However, in practice, depending on the form of the body force, simplifications must be made.

In the next two sections we shall derive the differential equations for the outer core, and the mantle and inner core.

We note here that when free wobble or forced nutation is involved, Euler's equation for the angular momentum of the earth must be added for a complete solution.

3.3 Hydrodynamic Equations

In the outer core, $\mu = 0$, and the equations of motion (2.32), (2.33), and (2.34) become

$$R = \frac{\partial^2 u_r}{\partial t^2} - 2\omega \sin\theta \frac{\partial u_\psi}{\partial t} - \frac{2}{3} \omega\sigma\varepsilon r P_2^1 (\cos\theta) \cos(\sigma t - \psi), \tag{3.10}$$

$$R = F_r + \frac{\partial}{\partial r}\left(W_a + \frac{\omega+\sigma}{\omega} W_T + \eta + \frac{\lambda\Delta}{\rho_o}\right) + \alpha(r)\,\Delta \frac{\partial W_o}{\partial r}, \tag{3.11}$$

$$S = \frac{\partial^2 u_\theta}{\partial t^2} - 2\omega \cos\theta \frac{\partial u_\psi}{\partial t} + \frac{8}{5} \omega\sigma\varepsilon r \frac{P_1^1(\cos\theta)}{\sin\theta} \cos(\sigma t - \psi) -$$

$$\frac{4}{15} \omega\sigma\varepsilon r \frac{P_3^1(\cos\theta)}{\sin\theta} \cos(\sigma t - \psi), \tag{3.12}$$

$$S = F_\theta + \frac{1}{r}\frac{\partial}{\partial\theta}\left(W_a + \frac{\omega+\sigma}{\omega} W_T + \eta + \frac{\lambda\Delta}{\rho_o}\right) + \frac{\alpha(r)}{r}\,\Delta \frac{\partial W_o}{\partial\theta} \tag{3.13}$$

$$F = \frac{\partial^2 u_\psi}{\partial t^2} + 2\omega \sin\theta \frac{\partial u_r}{\partial t} + 2\omega \cos\theta \frac{\partial u_\theta}{\partial t}, \tag{3.14}$$

$$F = F_\psi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\psi}\left(W_a + \frac{\omega+\sigma}{\omega} W_T + \eta + \frac{\lambda\Delta}{\rho_o}\right) +$$

$$\frac{\alpha(r)}{r\sin\theta}\,\Delta \frac{\partial W_o}{\partial\psi}. \tag{3.15}$$

W_a satisfies the Poisson equation (2.22),

$$\nabla^2 W_a = 4\pi G \left(\rho_o \Delta + \frac{\rho_o'}{W_o'} \eta\right). \tag{3.16}$$

Let us expand R, S, F and $\frac{\lambda\Delta}{\rho_o}$ as follows

$$R = \Sigma\, R_n\,(r)\, P_n^m\,(\cos\theta)\,\cos\,(\sigma t - m\psi),$$

$$S = \frac{1}{\sin\theta}\,\Sigma\, S_n\,(r)\, P_n^m\,(\cos\theta)\,\cos\,(\sigma t - m\psi), \tag{3.17}$$

$$F = \frac{1}{\sin\theta}\,\Sigma\, F_n\,(r)\, P_n^m\,(\cos\theta)\,\sin\,(\sigma t - m\psi),$$

$$\frac{\lambda\Delta}{\rho_o} = \Sigma\, X_n\,(r)\, P_n^m\,(\cos\theta)\,\cos\,(\sigma t - m\psi).$$

Using (2.29) and (3.9), we get

$$X_n(r) = \frac{\lambda_s}{\rho_s}\,\Delta_n + \frac{\lambda_s}{\rho_s}\left(\frac{\dot{\lambda}_s}{\lambda_s} - \frac{\dot{\rho}_s}{\rho_s}\right)\frac{b(r)}{W_o'}\left(-\frac{(n-1-m)(n-m)}{(2n-3)(2n-1)}\,\Delta_{n-2} + \frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)}\,\Delta_n - \frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\,\Delta_{n+2}\right). \tag{3.18}$$

Now from (3.10),

$$R_n\,(r) = -\sigma^2\,U_n + 2\omega\sigma\left(\frac{(n-1)(n-m)}{2n-1}\,T_{n-1} - m\,V_n - \frac{(n+2)(n+1+m)}{2n+3}\,T_{n+1}\right) - \frac{2}{3}\,\omega\sigma\varepsilon r\,\delta_n^2\,\delta_m^1. \tag{3.19}$$

From (3.11),

$$R_n(r) = \frac{d}{dr}\left[H_n + \frac{\omega+\sigma}{\omega} W_t \; \delta_n^2 \; \delta_m^1 + \eta_n + X_n\right] - \alpha g \, \Delta_n + {}_r F_n +$$

$$\alpha \frac{db}{dr}\left[- \frac{(n-1-m)(n-m)}{(2n-3)(2n-1)} \Delta_{n-2} + \frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)} \Delta_n -\right.$$

$$\left.\frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)} \Delta_{n+2}\right]. \qquad (3.20)$$

From (3.12),

$$S_{n-1}(r) = \frac{(n-3)(n-2-m)(n-1-m)}{(2n-5)(2n-3)} \; 2\omega\sigma \; T_{n-3} +$$

$$\left[- \frac{n-1-m}{2n-3}\right] \left[2m\omega\sigma + (n-2)\;\sigma^2\right] V_{n-2} +$$

$$\left[- \frac{n(n-1)-3m^2}{(2n-3)(2n+1)} \; 2\omega\sigma + m\sigma^2\right] T_{n-1} +$$

$$\left[- \frac{n+m}{2n+1}\right] \left[2m\omega\sigma - (n+1)\;\sigma^2\right] V_n +$$

$$\left[- \frac{(n+2)(n+m)(n+1+m)}{(2n+1)(2n+3)}\right] 2\omega\sigma \; T_{n+1} +$$

$$\frac{8}{5}\,\omega\sigma\varepsilon r \;\delta_n^2\;\delta_m^1 - \frac{4}{15}\,\omega\sigma\varepsilon r\;\delta_n^4\;\delta_m^1. \qquad (3.21)$$

From (3.13),

$$S_{n-1}(r) = \left[\frac{(n-2)(n-1-m)}{2n-3}\right] \frac{1}{r} \left[H_{n-2} + \frac{\omega+\sigma}{\omega} W_t \;\delta_n^4\;\delta_m^1 + \right.$$

$$\left.\eta_{n-2} + X_{n-2} + r\;{}_\theta F_{n-2}\right] +$$

$$\left[-\frac{(n+1)(n+m)}{2n+1}\right] \frac{1}{r} \left[H_n + \frac{\omega+\sigma}{\omega} W_t \delta_n^2 \delta_m^1 + \eta_n + X_n + r\ {}_\theta F_n\right] + \left[-\frac{(n-3-m)(n-2-m)(n-1-m)}{(2n-7)(2n-5)(2n-3)}\right] \frac{2\alpha b}{r} \Delta_{n-4} +$$

$$\left[+\frac{n+m}{(2n-1)(2n+1)}\right]\left[\frac{2(n^2+n-1+m^2)}{2n+3} - \frac{(n-1-m)(n-1+m)}{2n-3}\right]$$

$$\frac{2\alpha b}{r} \Delta_n + \left[-\frac{n-1-m}{(2n-3)(2n-1)}\right]\left[\frac{(n-m)(n+m)}{2n+1} - \frac{2(n^2-3n+1+m^2)}{2n-5}\right] \frac{2\alpha b}{r} \Delta_{n-2} +$$

$$\left[-\frac{(n+m)(n+1+m)(n+2+m)}{(2n+1)(2n+3)(2n+5)}\right] \frac{2\alpha b}{r} \Delta_{n+2}. \qquad (3.22)$$

From (3.14),

$$F_n(r) = \left[-\frac{(n-2)(n-1-m)(n-m)}{(2n-3)(2n-1)}\right] 2\omega\sigma\ V_{n-2} +$$

$$\left[\frac{n-m}{2n-1}\right]\left[m\ 2\omega\sigma + (n-1)\ \sigma^2\right] T_{n-1} +$$

$$\left[\frac{n(n+1)-3m^2}{(2n-1)(2n+3)}\ 2\omega\sigma - m\ \sigma^2\right] V_n +$$

$$\left[\frac{n+1+m}{2n+3}\right]\left[m\ 2\omega\sigma - (n+2)\ \sigma^2\right] T_{n+1} +$$

$$\left[\frac{(n+3)(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\right] 2\omega\sigma\ V_{n+2} +$$

$$\left[\frac{(n-1-m)(n-m)}{(2n-3)(2n-1)}\right] 2\omega\sigma\ U_{n-2} +$$

$$\left(-\frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)}\right).2\omega\sigma\ U_n +$$

$$\left(\frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\right)2\omega\sigma\ U_{n+2}. \tag{3.23}$$

From (3.15),

$$F_n\ (r) = \frac{m}{r}\left[H_n + \frac{\omega+\sigma}{\omega}\ W_t'\ \delta_n^2\ \delta_m^1 + \eta_n + X_n + r\ {}_\psi F_n\right]. \tag{3.24}$$

Expanding (3.16), we get

$$\ddot{H}_n + \frac{2}{r}\dot{H}_n - \frac{n(n+1)}{r^2}\ H_n = 4\pi G\rho_o\ \alpha\ \Delta_n + \frac{4\pi G\rho_o'}{W_o'}\ (\eta_n + X_n). \tag{3.25}$$

Also, Δ_n is given by

$$\Delta_n^{(r)} = \dot{U}_n + \frac{2}{r}\ U_n - \frac{n(n+1)}{r}\ V_n, \tag{3.26}$$

and since

$$\eta = u_r\ \frac{\partial W_o}{\partial r} + u_\theta\ \frac{1}{r}\ \frac{\partial W_o}{\partial \theta} + u_\psi\ \frac{1}{r\ \sin\ \theta}\ \frac{\partial W_o}{\partial \psi},$$

$$\eta_n\ (r) = \left[-\frac{(n-1-m)(n-m)}{(2n-3)(2n-1)}\right]\left[\dot{b}\ (r)\ U_{n-2} - (n-2)\ \frac{2b}{r}\ V_{n-2}\right] +$$

$$m\left(-\frac{n-m}{2n-1}\right)\ \frac{2b}{r}\ T_{n-1} +$$

$$\left(-\frac{n(n+1)-3m^2}{(2n-1)(2n+3)}\right)\ \frac{2b}{r}\ V_n +$$

$$\left(\frac{2(n^2+n-1+m^2)}{(2n-1)(2n+3)}\ \dot{b} - g\right)\ U_n +$$

$$m\left(-\frac{n+1+m}{2n+3}\right)\frac{2b}{r}T_{n+1}+$$

$$\left(-\frac{(n+1+m)(n+2+m)}{(2n+3)(2n+5)}\right)\left(\dot{b}\,U_{n+2}+(n+3)\frac{2b}{r}V_{n+2}\right). \qquad (3.27)$$

Equations (3.18) to (3.27) with $n = m, + \infty$ form the complete set of differential equations for the deformation in the outer core. Discussions on the formalism are in Appendix C.

We notice that the coupling among the displacement fields are such that spheroidal and toroidal fields do not appear with the same degree n.

3.4 Equations for the Mantle and Inner Core.

In the mantle and inner core, due to the rather large rigidity, the effects of ellipticity and rotation are small. In fact, for low frequency oscillations of the earth the problem can be treated with static equilibrium theory without introducing significant error (Jeffreys 1959, page 211). In the present work, we neglect terms due to ellipticity and rotation in the equations of motion. Then the deformation is governed by a set of 6th-order linear, ordinary differential equations (Smylie and Mansinha 1971).

$$\dot{U}_n = -\frac{2\lambda}{\lambda+2\mu}\frac{1}{r}U_n+\frac{1}{\lambda+2\mu}Z_n+\frac{n(n+1)\lambda}{\lambda+2\mu}\frac{1}{r}V_n,$$

$$\dot{Z}_n = \left(-\rho_o\sigma^2-\frac{4}{r}\rho_o g+4\mu\frac{3\lambda+2\mu}{\lambda+2\mu}\frac{1}{r^2}\right)U_n$$

$$- \frac{4\mu}{\lambda+2\mu} \frac{1}{r} Z_n - \rho_o \left[Q_n + \frac{\sigma+\omega}{\omega} \frac{dW_t}{dr} \delta_n^2 \delta_m^1 \right] - \rho_o \, {}_rF_n +$$

$$\left[\frac{n(n+1)}{r} \rho_o g - \frac{2n(n+1)\mu}{\lambda+2\mu} \frac{(3\lambda+2\mu)}{r^2} \right] V_n + \frac{n(n+1)}{r} Y_n,$$

$$\dot{V}_n = - \frac{1}{r} U_n + \frac{1}{r} V_n + \frac{1}{\mu} Y_n, \qquad (3.28)$$

$$\dot{Y}_n = \left[\frac{1}{r} \rho_o g - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu) r^2} \right] U_n - \frac{\lambda}{\lambda+2\mu} \frac{1}{r} Z_n -$$

$$\frac{\rho_o}{r} \left[H_n + \frac{\omega+\sigma}{\omega} W_t \delta_n^2 \delta_m^1 \right] - \rho_o \, {}_\theta F_n + \left\{ - \rho_o \sigma^2 + \right.$$

$$\left. \frac{2\mu}{\lambda+2\mu} \left[(2n^2+2n-1) \lambda + 2(n^2+n-1)\mu \right] \frac{1}{r^2} \right\} V_n - \frac{3}{r} Y_n,$$

$$\dot{H}_n = 4\pi G \rho_o U_n + Q_n,$$

$$\dot{Q}_n = - 4\pi G \rho_o \frac{n(n+1)}{r} V_n + \frac{n(n+1)}{r^2} H_n - \frac{2}{r} Q_n,$$

where U_n is the radial displacement, Z_n the change in normal stress, V_n the transverse displacement, Y_n the change in transverse stress, H_n the change in gravitational potential, and Q_n the change in gravitational flux density.

3.5 Boundary Conditions

Due to the ellipticities of surfaces of equal density, the boundary conditions for a spherically symmetric earth must be modified. The condition for the radial component of a quantity is to be replaced by the condition for its component normal to the equipotential surface, and the condition for the transverse component by the condition for the component tangential to the equipotential surface. However, the effects of ellipticity have been neglected in the solutions for inner core and mantle. As a consequence, errors of the order of ellipticity cannot be avoided in the boundary conditions.

A) At the inner core-outer core and outer core-mantle boundaries, where mechanical properties are discontinuous, the conditions on the deformation are as follows.

i) The normal displacement is continuous.

$$\eta_n \ (C) = W_o' \ U_n \ (M). \tag{3.29}$$

ii) The change in normal stress is continuous.

$$\lambda \ \Delta_n \ (C) = Z_n \ (M) \tag{3.30}$$

iii) The transverse displacement can be discontinuous.

iv) The change in transverse stress is continuous.

Since in the outer core the transverse stress vanishes,

$$Y_n \ (C) = 0. \tag{3.31}$$

v) The change in gravitational potential is continuous.

$$H_n(C) = H_n(M). \tag{3.32}$$

vi) The change in gravitational flux density is continuous.

$$H_n'(C) - 4\pi G\rho_o \frac{\eta_n}{W_o'}(C) = Q_n(M). \tag{3.33}$$

In the above, (C) stands for quantities evaluated in the outer core, and (M) those in the mantle or inner core. Notice that these conditions involve errors of the order of the ellipticity.

B) At the origin, the solutions of (3.28) must be regular. Since (3.28) is a set of 6th-order differential equations, there are only three independent solutions in the inner core with ε as a parameter.

C) At the free surface, the changes in stress must vanish, and the change in gravitational potential becomes harmonic. These are

$$Z_n(d) = Y_n(d) = 0, \tag{3.34}$$

$$H_n(d) + \frac{d}{n+1} Q_n(d) = 0, \tag{3.35}$$

where d is the radius of the earth.

3.6 Euler's Equation of Motion

In section 2.6, we have given the Eulerian equation of motion in its general form. Here we shall derive the expression for the angular momentum in terms of the displacement fields.

Equation (2.38) gives

$$\Delta M_x = \int_{\text{OUTER CORE}} \rho_o \left[y \frac{\partial u_z}{\partial t} - z \frac{\partial u_y}{\partial t} \right] d\tau. \qquad (3.36)$$

Now

$$u_z = \cos\theta\, u_r - \sin\theta\, u_\theta,$$
$$u_y = \sin\theta \sin\psi\, u_r + \cos\theta \sin\psi\, u_\theta + \cos\psi\, u_\psi. \qquad (3.37)$$

Using (3.8) in (3.37), we find

$$\frac{\partial u_z}{\partial t} = -\sigma \sin(\sigma t - m\psi) \left[\sum_n U_n \cos\theta\, P_n^m - \sum_n V_n \sin\theta \frac{\partial P_n^m}{\partial\theta} + m \sum_n T_n P_n^m \right],$$

$$\frac{\partial u_y}{\partial t} = -\sigma \sin(\sigma t - m\psi) \sin\psi \left[\sum_n U_n \sin\theta\, P_n^m + \sum_n V_n \cos\theta \frac{\partial P_n^m}{\partial\theta} - m \sum_n T_n \frac{\cos\theta}{\sin\theta} P_n^m \right] +$$

$$\cos(\sigma t - m\psi)\cos\psi\left\{m\sum_n V_n \frac{1}{\sin\theta} P_n^m - \sum_n T_n \frac{\partial P_n^m}{\partial\theta}\right\}. \tag{3.38}$$

Substituting (3.38) into (3.36), we get

$$\Delta M_x = \sigma\int_{\text{OUTER CORE}} \rho_o r^3 \sum_n V_n \left\{\sin\theta \frac{\partial P_n^m}{\partial\theta} \sin\psi \sin(\sigma t - m\psi) - m\cos\theta P_n^m \cos\psi\cos(\sigma t - m\psi)\right\} dr\, d\theta\, d\psi +$$

$$\sigma\int_{\text{OUTER CORE}} \rho_o r^3 \sum_n T_n \left\{- m P_n^m \sin\psi \sin(\sigma t - m\psi) + \sin\theta\cos\theta \frac{\partial P_n^m}{\partial\theta} \cos\psi\cos(\sigma t - m\psi)\right\} dr\, d\theta\, d\psi. \tag{3.39}$$

Expanding $\sin(\sigma t - m\psi)$ and $\cos(\sigma t - m\psi)$, and using the orthogonality relations of $\sin m\psi$ and $\cos m\psi$, we find upon integration over ψ,

$$\Delta M_x = 0, \quad \text{if } m \neq 1. \tag{3.40}$$

For $m = 1$, (3.39) becomes

$$\Delta M_x = -\pi\sigma\cos\sigma t\left\{\sum_n \int_c^b \rho\, r^3 V_n\, dr \int_0^\pi \left(\sin\theta \frac{\partial P_n^1}{\partial\theta} + \cos\theta P_n^1\right) d\theta\right\} + \pi\sigma\cos\sigma t\left\{\sum_n \int_c^b \rho r^3\, T_n\, dr \int_0^\pi\right.$$

$$\left(P_n^1 + \sin\theta\cos\theta\,\frac{\partial P_n^1}{\partial\theta}\right) d\theta \Bigg\}. \qquad (3.41)$$

Consider the integral I_1,

$$I_1 = \int_0^\pi \left(\sin\theta\,\frac{\partial P_n^1}{\partial\theta} + \cos\theta\, P_n^1\right) d\theta.$$

Integrating by parts, we get

$$I_1 = \sin\theta\, P_n^1 \Big|_0^\pi - \int_0^\pi \cos\theta\, P_n^1\, d\theta + \int_0^\pi \cos\theta\, P_n^1\, d\theta$$

$$= 0.$$

Consider the integral I_2

$$I_2 = \int_0^\pi \left(P_n^1 + \sin\theta\cos\theta\,\frac{\partial P_n^1}{\partial\theta}\right) d\theta.$$

Again, integration by parts gives

$$I_2 = \int_0^\pi \left(P_n^1 - \cos 2\theta\, P_n^1\right) d\theta + \sin\theta\cos\theta\, P_n^1 \Big|_0^\pi$$

$$= 2\int_0^\pi \sin^2\theta\, P_n^1\, d\theta$$

$$= 2\int_{-1}^1 P_1^1(u)\, P_n^1(u)\, du.$$

Using the orthogonality relation of $P_n^m(u)$, we get

$$I_2 = \begin{cases} \frac{8}{3}, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1. \end{cases}$$

Therefore, for $m = 1$,

$$\Delta M_x = \frac{8\pi}{3} \sigma \cos \sigma t \int_c^b \rho r^3 T_1 \, dr, \tag{3.42}$$

and for $m \neq 1$,

$$\Delta M_x = 0.$$

Similarly

$$\Delta M_y = \begin{cases} \frac{8\pi}{3} \sigma \sin \sigma t \int_c^b \rho r^3 T_1 \, dr, & \text{if } m = 1 \\ 0, & \text{if } m \neq 1 \end{cases} \tag{3.43}$$

Equation (2.39) gives

$$I_{xz} = \int_\tau (\rho - \rho_o) \, xz d\tau + \sum_s \int_s \frac{\rho_o \eta}{W_o'} xz ds \tag{3.44}$$

Using the equation of continuity (2.19) and the Poisson equation (2.22), we can write (3.44) as

$$I_{xz} = - \frac{1}{4\pi G} \int_\tau \nabla^2 W_a \, xz d\tau + \sum_s \int_s \frac{\rho_o \eta}{W_o'} xz ds. \tag{3.45}$$

The function xz is harmonic,

$$\nabla^2 xz = 0. \tag{3.46}$$

Applying Green's theorem to the first integral, we get

$$\int_\tau \nabla^2 W_a \, xz d\tau = \sum_s \int_s \left(\frac{\partial W_a}{\partial \bar{n}} xz - W_a \frac{\partial (xz)}{\partial \bar{n}} \right) ds,$$

where $\bar{n}$ is the outward normal to the surface S.

Neglecting ellipticities of the surfaces, we have

$$\frac{\partial W_a}{\partial \bar{n}} \doteq \frac{\partial W_a}{\partial r},$$

$$\frac{\partial (xz)}{\partial \bar{n}} = \frac{2}{r} xz.$$

Thus I_{xz} becomes

$$I_{xz} = \frac{1}{4\pi G} \sum_s \int_s \left\{ \frac{2}{r} W_a - \left(\frac{\partial W_a}{\partial r} - 4\pi G \frac{\rho_o \eta}{W_o} \right) \right\} xzds.$$

Now since W_a and $\frac{\partial W_a}{\partial r} - 4\pi G \frac{\rho_o \eta}{W_o}$ are continuous across each surface of discontinuity,

$$I_{xz} = \frac{1}{4\pi G} \int_{\text{FREE SURFACE}} \left\{ \frac{2}{r} W_a - \left(\frac{\partial W_a}{\partial r} - 4\pi G \frac{\rho_o \eta}{W_o} \right) \right\} xzds. \tag{3.47}$$

Since

$$W_a = \sum_n H_n P_n^m (\cos \theta) \cos (\sigma t - m\psi),$$

and

$$\frac{\partial W_a}{\partial r} - 4\pi G \frac{\rho_o \eta}{W_o} = \sum_n Q_n P_n^m (\cos\theta) \cos(\sigma t - m\psi),$$

(3.47) becomes

$$I_{xz} = \frac{1}{4\pi G} \sum_n \left[r^3 (2H_n - Q_n r) J_n \right], \tag{3.48}$$

where

$$J_n = \frac{1}{3} \int_0^\pi \int_0^{2\pi} P_n^m (\cos\theta)\, P_2^1 (\cos\theta) \cos(\sigma t - m\psi)$$

$$\cos\psi \sin\theta\, d\theta\, d\psi. \tag{3.49}$$

Carrying out the integration, we find

$$J_n = \begin{cases} 0, & \text{if } n \neq 2, \text{ or } m \neq 1, \\ \frac{4}{5}\pi \cos\sigma t, & \text{if } n = 2 \text{ and } m = 1. \end{cases}$$

Therefore,

$$I_{xz} = \begin{cases} 0, & \text{if } m \neq 1, \\ \frac{\cos\sigma t}{5G} d^3 \left[2\, H_2(d) - d\, Q_2(d) \right], & \text{if } m = 1 \text{ and } n = 2, \end{cases} \tag{3.50}$$

where d is the radius of the earth.

$$I_{yz} = \begin{cases} 0, & \text{if } m \neq 1, \\ \frac{1}{5G} \sin \sigma t \; d^3 \left[2H_2(d) - d\, Q_2(d) \right], & \text{if } m = 1 \text{ and } n = 2. \end{cases} \tag{3.51}$$

Using (3.42), (3.43), (3.50), and (3.51) in (2.37), we can write

$$M_x = M \cos \sigma t, \qquad M_y = M \sin \sigma t, \tag{3.52}$$

where

$$\zeta = \left(\frac{8\pi}{3} \int_c^b \rho r^3 \; T_1 \; dr \right) \delta_m^1 , \tag{3.53}$$

$$f = \left\{ \frac{1}{5G} d^3 \left[2H_2(d) - d\, Q_2(d) \right] \right\} \delta_m^1 , \tag{3.54}$$

and

$$M = \omega (A\varepsilon - f) + \sigma\zeta. \tag{3.55}$$

Using (2.41) and (3.52) in (2.36), the Eulerian equation of motion becomes

$$\varepsilon - \frac{\sigma+\omega}{\omega^2} \frac{M}{C} = \frac{-1}{\omega^2 C} L. \tag{3.56}$$

Equation (3.56) should be compared with equation (2.42).

CHAPTER 4

FREE SPHEROIDAL OSCILLATIONS OF DEGREE 2 AND EARTH TIDES

For numerical computation, the general theory developed in chapter 2 and 3 must be simplified. The method of simplification is described in section 4.1. The simplified equations, together with the boundary conditions are given in section 4.2, 4.3, and 4.4. As a check on our derivation, Molodensky's theory for diurnal earth tides is derived from our theory in section 4.5. The numerical results are discussed in section 4.6 and 4.7.

4.1 Simplified Mathematical Theory

We have seen that the hydrodynamic equations for the liquid core differ from the classical treatment in that there exist interactions among spheroidal and toroidal displacement fields of various degrees. As a result, the differential equations governing the motion form a set of infinite order ordinary differential equations. Therefore, to facilitate numerical computations, we must truncate the infinite set. We observe first that there exists no coupling among displacement fields with different azimuthal numbers (Dahlen 1969). Second, the direct coupling constant among spheroidal displacement fields with different degrees (but same azimuthal number) is the ellipticity. Thus, if we are interested in spheroidal deformation of a specific degree (in this case, degree 2), we can well neglect the contributions from spheroidal displacement fields of any other degrees.

On the other hand, numerical calculations show that the coupling of toroidal fields to the spheroidal fields depends strongly on the frequency of oscillation. Fortunately, we have found that when we are interested in spheroidal deformations of degree 2, it is sufficient to consider only the contributions from toroidal fields of degrees 1 and 3. With these considerations in mind, we shall write down the differential equations which are sufficiently accurate and at the same time numerically convenient. Further discussions of the truncation are given in Appendix D.

4.2 Hydrodynamic Equations

4.2.1 Free Spheroidal Oscillations of Degree 2, Azimuthal number 1.

Putting $n = 2$ and 4 respectively and $m = 1$ in (3.18) - (3.27), we obtain the following relations,

$$\Delta_2 = \dot{U}_2 + \frac{2}{r} U_2 - \frac{6}{r} V_2, \tag{4.1}$$

$$\dot{X}_2 = \frac{\lambda_s}{\rho_s} \Delta_2 + \frac{4}{7} \frac{\lambda_s}{\rho_s} \left(\frac{\dot{\lambda}_s}{\lambda_s} - \frac{\dot{\rho}_s}{\rho_s}\right) \frac{b(r)}{W_o} \Delta_2, \tag{4.2}$$

$$\eta_2 = -\frac{2b}{r} \left(\frac{1}{3} T_1 + \frac{1}{7} V_2 + \frac{4}{7} T_3\right) + \left(\frac{4}{7} \dot{b} - g\right) U_2, \tag{4.3}$$

$$F_2 = \left(\frac{2}{3} \omega\sigma + \frac{1}{3} \sigma^2\right) T_1 + \left(\frac{2}{7} \omega\sigma - \sigma^2\right) V_2 +$$

$$\left(\frac{8}{7} \omega\sigma - \frac{16}{7} \sigma^2\right) T_3 - \frac{8}{7} \omega\sigma U_2, \tag{4.4}$$

$$F_4 = -\frac{24}{35} \omega\sigma V_2 + \left(\frac{6}{7} \omega\sigma + \frac{9}{7} \sigma^2\right) T_3 + \frac{12}{35} \omega\sigma U_2, \tag{4.5}$$

$$S_1 = \left(\frac{2}{5} \omega\sigma + \sigma^2\right) T_1 + \left(-\frac{6}{5} \omega\sigma - \frac{9}{5} \sigma^2\right) V_2 - \frac{96}{35} \omega\sigma T_3 +$$

$$\frac{8}{5} \omega\sigma\varepsilon r, \tag{4.6}$$

$$S_3 = \frac{4}{15} \omega\sigma T_1 + \left(-\frac{4}{5} \omega\sigma - \frac{4}{5} \sigma^2\right) V_2 + \left(-\frac{2}{5} \omega\sigma + \sigma^2\right) T_3 -$$

$$\frac{4}{15} \omega\sigma\varepsilon r, \tag{4.7}$$

$$R_2 = -\sigma^2 U_2 + \frac{2}{3} \omega\sigma T_1 - 2\omega\sigma V_2 - \frac{32}{7} \omega\sigma T_3 -$$

$$\frac{2}{3} \omega\sigma\varepsilon r, \tag{4.8}$$

$$rF_2 = H_2 + \eta_2 + X_2, \tag{4.9}$$

$$rF_4 = H_4 + \eta_4 + X_4, \tag{4.10}$$

$$rS_1 = -\frac{9}{5}(H_2 + \eta_2 + X_2) + \frac{24}{35}\alpha\, b\, \Delta_2, \tag{4.11}$$

$$rS_3 = \frac{4}{5}(H_2 + \eta_2 + X_2) - \frac{25}{9}(H_4 + \eta_4 + X_4) + \frac{4}{15}\alpha\, b\, \Delta_2, \tag{4.12}$$

$$R_2 = \frac{d}{dr}(H_2 + \eta_2 + X_2) - \alpha g\, \Delta_2 + \frac{4}{7}\alpha\, b\, \Delta_2. \tag{4.13}$$

Eliminating F_2, S_1, H_2, η_2 and X_2 from (4.4), (4.6), (4.11) and (4.9), we obtain

$$(\omega\sigma + \sigma^2)\, T_1 - \frac{3}{7}\omega\sigma\, V_2 - \frac{3}{7}(\omega\sigma + 6\sigma^2)\, T_3 = \frac{9}{7}\omega\sigma\, U_2 + \frac{3}{7}\frac{\alpha b}{r}\Delta_2 - \omega\sigma\varepsilon r. \tag{4.14}$$

Eliminating F_2, F_4, S_3, H_2, η_2, X_2, H_4, η_4, and X_4 from (4.4), (4.5), (4.7), (4.9), (4.10), and (4.12), we get

$$-(\omega\sigma + \sigma^2)\, T_1 - 11\,\omega\sigma\, V_2 + 4\,(\omega\sigma + 6\sigma^2)\, T_3 = -7\,\omega\sigma\, U_2 - \frac{\alpha b}{r}\Delta_2 + \omega\sigma\varepsilon r. \tag{4.15}$$

Substituting (4.2) and (4.3) into (4.9), we get

$$- \frac{2b}{r}\left(\frac{1}{3}T_1 + \frac{1}{7}V_2 + \frac{4}{7}T_3\right) - rF_2 =$$

$$\left(-\frac{4}{7}b + g\right)U_2 - \frac{\lambda_s}{\rho_s}\left[1 + \frac{4}{7}\left(\frac{\dot{\lambda}_s}{\lambda_s} - \frac{\dot{\rho}_s}{\rho_s}\right)\frac{b}{W_o'}\right]\Delta_2 - H_2. \quad (4.16)$$

Eliminating R_2, H_2, η_2 and X_2 from (4.9), (4.13) and (4.8), we obtain

$$\frac{d}{dr}(rF_2) =$$

$$-\sigma^2 U_2 - \alpha\left(g + \frac{4}{7}\dot{b}\right)\Delta_2 + \omega\sigma\left(\frac{2}{3}T_1 - 2V_2 - \frac{32}{7}T_3\right) + \frac{2}{3}\omega\sigma\varepsilon r. \quad (4.17)$$

Putting n = 2 in (3.25), and using (4.9), we obtain

$$\ddot{H}_2 + \frac{2}{r}\dot{H}_2 + \left(\frac{4\pi G\rho_s'}{W_o'} - \frac{6}{r^2}\right)H_2 = -4\pi G\rho_s\alpha\Delta_2 + \frac{4\pi G\rho_s'}{W_o'}rF_2. \quad (4.18)$$

(4.14) - (4.18) and (4.4), and (4.1) can be solved for U_2, V_2, H_2, T_1 and T_3 in terms of ε and 4 other free constants.

4.2.2 Free Spheroidal Oscillations of Degree 2, Azimuthal Number 2

Similar treatments as in 4.2.1 lead to the following equations.

$$\Delta_2 = \dot{U}_2 + \frac{2}{r} U_2 - \frac{6}{r} V_2, \tag{4.19}$$

$$\frac{1}{2} F_2 = - \frac{6}{7} \omega\sigma\ U_2 + \left(- \frac{2}{7} \omega\sigma - \sigma^2\right) V_2 + \left(\frac{10}{7} \omega\sigma - \frac{10}{7} \sigma^2\right) T_3, \tag{4.20}$$

$$4\omega\sigma\ V_2 + (-5\omega\sigma - 15\sigma^2)\ T_3 = 2\omega\sigma\ U_2 - \frac{\alpha b}{r} \Delta_2, \tag{4.21}$$

$$\left(4 \frac{b}{r^2} + 2\omega\sigma + 7\sigma^2\right) V_2 + \left(- 20 \frac{b}{r^2} - 10\omega\sigma + 10\sigma^2\right) T_3 =$$

$$\left(- 6 \frac{\dot{b}}{r} - 6\omega\sigma + 7 \frac{g}{r}\right) U_2 - 7 \frac{\lambda_s}{r\rho_s} \Delta_2 - \frac{1}{r} H_2, \tag{4.22}$$

$$\frac{1}{2} \frac{d}{dr} (rF_2) = - \sigma^2 U_2 - 4\omega\sigma\ V_2 - \frac{40}{7} \omega\sigma\ T_3 - \alpha \left(\frac{6}{7} \dot{b} - g\right) \Delta_2, \tag{4.23}$$

$$\ddot{H}_2 + \frac{2}{r} \dot{H}_2 + \left(\frac{4\pi G\rho_o'}{W_o'} - \frac{6}{r^2}\right) H_2 = - 4\pi G\rho_o\ \alpha\ \Delta_2 +$$

$$\frac{4\pi G\rho_o'}{W_o'}\ r \frac{1}{2} F_2 \tag{4.24}$$

(4.19) to (4.24) can be solved for U_2, V_2, H_2, T_3 in terms of 4 free constants. We also have

$$\eta_2 = \left(\frac{6}{7} \dot{b} - g\right) U_2 + \frac{4}{7} \frac{b}{r} V_2 - \frac{20}{7} \frac{b}{r} T_3. \tag{4.25}$$

4.2.3 Free spheroidal Oscillations of Degree 2, Azimuthal Number 0

Following the similar treatments as in 4.2.1, we obtain

$$\Delta_2 = \dot{U}_2 + \frac{2}{r} U_2 - \frac{6}{r} V_2, \tag{4.26}$$

$$7\sigma^2 T_1 + 6\omega\sigma V_2 - 18\sigma^2 T_3 = 10\omega\sigma U_2, \tag{4.27}$$

$$4\omega\sigma V_2 - 5\sigma^2 T_3 = 2\omega\sigma U_2, \tag{4.28}$$

$$- 14\omega\sigma T_1 + \left(- 12 \frac{b}{r^2} + 21\sigma^2\right) V_2 - 24\omega\sigma T_3 =$$

$$\left(- 10 \frac{\dot{b}}{r} + 21 \frac{g}{r}\right) U_2 + \left(- 21 \frac{\lambda_s}{r\rho_s} + 2 \frac{\alpha b}{r}\right) \Delta_2 -$$

$$21 \frac{1}{r} H_2, \tag{4.29}$$

$$14\omega\sigma \dot{T}_1 - 21\sigma^2 \dot{V}_2 + 24\omega\sigma \dot{T}_3 + 2 \frac{\alpha b}{r} \dot{\Delta}_2 =$$

$$\frac{1}{r} (14\omega\sigma T_1 + 21\sigma^2 V_2 - 96\omega\sigma T_3 - 21\sigma^2 U_2) +$$

$$\left\{21 \frac{\alpha g}{r} - 10 \frac{\dot{\alpha b}}{r} - 2 \frac{\alpha b}{r^2} - 2 \frac{d}{dr} \left(\frac{\alpha b}{r}\right)\right\} \Delta_2, \tag{4.30}$$

$$\ddot{H}_2 + \frac{2}{r} \dot{H}_2 + \left(\frac{4\pi G\rho_o'}{W_o'} - \frac{6}{r^2}\right) H_2 = \frac{4\pi G\rho_o'}{W_o'} r \left(\frac{2}{3}\omega\sigma T_1 -\right.$$

$$\sigma^2 V_2 + \frac{8}{7}\omega\sigma T_3\Big) + \left(\frac{4\pi G\rho_o'}{W_o'} \cdot \frac{2}{21} b\alpha - 4\pi G\rho_o \alpha\right) \Delta_2, \quad (4.31)$$

$$\eta_2 = -\frac{4}{7}\frac{b}{r} V_2 + \left(\frac{10}{21} b - g\right) U_2. \quad (4.32)$$

(4.26) - (4.31) can be solved for U_2, V_2, H_2, T_1 and T_3 in terms of 4 free constants.

4.2.4 Remarks on the Equations given in Sections 4.2.1, 4.2.2, and 4.2.3.

A. The forms of equations for free spheroidal oscillations of degree 2 and azimuthal number -1 and -2 are identical with those given in 4.2.1, and 4.2.2 respectively provided we replace the frequency σ by $-\sigma$.

B. The quantity H_2 in sections 4.2.2 and 4.2.3 represents the radial coefficient of the additional potential arising from redistribution of mass. But H_2 in section 4.2.1 includes also the quantity $\frac{\omega+\sigma}{\omega} W_t$ (r) (see equation (2.7), (2.9), (2.10), (2.11)).

C. Although the equations in 4.2.1, 4.2.2, and 4.2.3 are written for free oscillations, they are also valid for earth tides except in these cases, the quantity H_2 includes also the radial coefficient of the external disturbing potential. Such a treatment is possible because for earth tides, the disturbing potentials satisfy the Laplace equation. (For treatise on earth tides, see Melchior, 1966).

D. Spheroidal oscillations of degree 2, azimuthal number 1 are accompanied by variations in latitude. In this case, equation (3.56) is needed for a complete solution.

4.3 Solutions for the Mantle and Inner Core

Because the ellipticity and rotation are neglected, the set of equations (3.28) with $n = 2$ is applicable for all azimuthal numbers.

The external force terms ${}_rF_2$, ${}_\theta F_2$, and ${}_\psi F_2$ are absent for free oscillations. However, for earth tides, the body forces exist. Since the body forces are derived from potentials which satisfy the Laplace equation, the radial coefficient of the external potential and its derivative may be included in H_2 and Q_2 respectively.

4.4 Boundary Conditions

The boundary conditions given in section 3.5 determine the solution completely except in the cases $m = 1$ and -1 where the remaining free constant ε is to be determined by the Euler's equation (3.56).

But it must be remembered that in the condition (3.35), the H_2 and Q_2 represent the change in gravitational potential and change in gravitational flux density due only to the redistribution of mass. They bear different meaning as compared to the H_2 and Q_2 in the differential equations given in sections 4.2, and 4.3.

4.5 Molodensky's Theory for Diurnal Earth Tides and Nutations

While our theory is applicable to any density stratification of the liquid core, Molodensky's theory is valid only for an Adams-Williamson core (Adams and Williamson, 1923). Therefore let us assume

$$\alpha(r) = \frac{\lambda\rho_o'}{\rho_o^2 W_o'} - 1 = 0 \qquad (4.33)$$

in the liquid core.

Next, we observe that for most of the diurnal earth tides $\frac{\omega+\sigma}{\omega} << 1$, hence in view of the forms of the equations (4.14) and (4.15), we have

$$V_2 \sim \frac{\omega+\sigma}{\omega} T_1,$$

$$U_2 \sim \frac{\omega+\sigma}{\omega} T_1,$$

$$T_3 \sim \frac{\omega+\sigma}{\omega} T_1$$

in order of magnitude.

Therefore (4.4) may be approximated by

$$F_2 \doteq \left(\frac{2}{3}\,\omega\sigma + \frac{1}{3}\,\sigma^2\right) T_1. \tag{4.34}$$

Using (4.34) in (4.17), we get

$$\left(\frac{2}{3}\,\omega\sigma + \frac{1}{3}\,\sigma^2\right) \frac{d}{dr}(rT_1) \doteq \frac{2}{3}\,\omega\sigma\, T_1 + \frac{2}{3}\,\omega\sigma\varepsilon r.$$

The last equation means

$$T_1 = -\beta r, \tag{4.35}$$

where β is the resonant parameter in Molodensky's theory.

Let us write

$$K(r) = H_2 - rF_2, \tag{4.36}$$

then (4.18) becomes

$$\ddot{K} + \frac{2}{r}\dot{K} + \left(\frac{4\pi G\rho_s{}'}{W_o{}'} - \frac{6}{r^2}\right) K = 0. \tag{4.37}$$

This is equation (30) in Molodensky's paper.

We notice from Appendix A, the function $b(r) = \varepsilon g r$ also satisfies (4.37), we can therefore write

$$2b(r) = -K_1(r). \tag{4.38}$$

Now, (4.14) and (4.15) give

$$9V_2 = -\ 3U_2 + 5\left(\frac{\omega+\sigma}{\omega} T_1 + \varepsilon r\right) =$$

$$-\ 3U_2 + 5\left(\varepsilon - \frac{\omega+\sigma}{\omega}\beta\right) r. \qquad (4.39)$$

Substituting (4.39) into (4.1), we get

$$\Delta_2 = \frac{1}{r^4}\frac{d}{dr}\left(r^4\ U_2\right) - \frac{10}{3}\left(\varepsilon - \frac{\omega+\sigma}{\omega}\beta\right). \qquad (4.40)$$

To relate U_2 to η_2, we neglect terms of the order $\frac{\omega+\sigma}{\omega}$ in (4.3), and obtain

$$\eta_2 \doteq \frac{2}{3}\beta\ b(r) - g\ U_2 .$$

Upon using (4.38), the last equation becomes

$$U_2 = \frac{1}{W_o'}\left(\eta_2 + \frac{1}{3}\beta\ K_1\right). \qquad (4.41)$$

Now, using (4.36) and (4.33) in (4.9), we can write

$$\Delta_2 = -\frac{\rho_s'}{\rho_s W_o'}\left(\frac{1}{3} K + \eta_2\right). \qquad (4.42)$$

Combining (4.40), (4.41), and (4.42), we get

$$\frac{d}{dr}\left(\frac{\rho_s r^4 \eta_2}{W_o'} + \frac{1}{3}\beta\frac{\rho_s r^4 K_1}{W_o'}\right) + \frac{1}{3}\frac{\rho_s' r^4}{W_o'}(K - \beta K_1) =$$

$$\frac{10}{3} r^4 \rho_s\left(\varepsilon - \frac{\omega+\sigma}{\omega}\beta\right). \qquad (4.43)$$

But from (4.37),

$$r^4 \frac{\rho_s'}{W_o'} (K - \beta K_1) = - \frac{1}{4\pi G} \frac{d}{dr} \left\{ r^6 \frac{d}{dr} \left(\frac{K - \beta K_1}{r^2} \right) \right\}.$$

Therefore (4.43) becomes.

$$\left\{ \frac{3}{W_o'} \rho_s r^4 \eta_2 + \frac{\beta}{W_o'} \rho_s r^4 K_1 - \frac{r^6}{4\pi G} \frac{d}{dr} \left(\frac{K - \beta K_1}{r^2} \right) \right\}_c^b =$$

$$5\nu \int_c^b \rho_s r^4 \, dr. \qquad (4.44)$$

This is the equation (39) in Molodensky's paper, except that here the constant ν is given by

$$\nu = 2\left(- \frac{\omega+\sigma}{\omega} \beta + \varepsilon \right), \qquad (4.45)$$

while in Molodensky's theory

$$\nu = 2\left(\frac{\omega+\sigma}{\sigma} \beta - \frac{\omega}{\sigma} \varepsilon \right). \qquad (4.46)$$

However, since $\sigma \sim \omega$, the resulting error is small.

4.6 Numerical Calculations

Numerical computation was done using the CDC CYBER 73 Computer at the University of Western Ontario. Spline interpolation and fourth order Runge-Kutta methods were used for integrations of the differential equations. The step size was varied in such a way as to keep the ratio between the step size and the initial radius (the independent variable) at each step constant. A value of 0.004 or smaller for the ratio is found to give stable integrations. The numerical procedures are given in Appendix B.

4.6:1 Earth Models

The three earth models with uniform polytropic cores ($\alpha = +0.2, 0.0, -0.2$) listed in table 1, and plotted in figure 2, 3a, and 3b are originally given by Pekeris and Accad (1972). Here, a slight modification is made to allow for a solid inner core. Also, for the model with uniformly unstable core ($\alpha = -0.2$), the density in the core is slightly increased as the original density used by Pekeris and Accad leads to a deficiency in the total mass of the earth.

The interest in using these earth models stems from the fact that the function $\alpha(r) = \frac{\lambda\rho_o'}{\rho_o^2 W_o'} - 1$ determines the stability of the outer core (Smylie 1974). The restoring force, when a particle is suddenly displaced radially, is proportional to $\alpha(r)$. Thus when $\alpha(r) = 0$ (Adams and Williamson 1923), the core is in neutral equilibrium; when $\alpha(r) > 0$, the core is stable; and when $\alpha(r) < 0$, the core is unstable. An immediate implication is that an unstable, or neutrally stable core is incapable of free oscillations. This, and some other

TABLE 1. EARTH MODEL M_3 WITH UNIFORM POLYTROPIC CORES FOR α = +0.2, 0.0, −0.2

(See equation (2.30) for definition of α*) (Pekeris and Accad, 1972) Modified

To Allow for a Solid Inner Core

r km	C_p km sec^{-1}	C_s km sec^{-1}	M_3 ρ_0 gm.cm^{-3}	α = +0.2 ρ_0 gm cm^{-3}	α = 0.0 ρ_0 gm cm^{-3}	α = −0.2+ ρ_0 gm cm^{-3}
6371	6.30	3.56	2.840			
6338	6.30	3.55	2.840			
6338	8.16	4.65	3.386			
6311	8.15	4.60	3.474			
6271	8.00	4.40	3.488			
6221	7.85	4.35	3.462			
6171	8.05	4.40	3.413			
6071	8.50	4.60	3.374			
5958	9.06	5.00	3.569			
5871	9.60	5.30	3.812			
5771	10.10	5.60	4.047			
5671	10.50	5.90	4.215			
5571	10.90	6.15	4.373			
5471	11.30	6.30	4.502			
5371	11.40	6.35	4.619			
5171	11.80	6.50	4.852			
4971	12.05	6.60	4.955			
4771	12.30	6.75	5.040			
4571	12.55	6.85	5.066			
4371	12.80	6.95	5.072			
4171	13.00	7.00	5.085			
3971	13.20	7.10	5.090			
3771	13.45	7.20	5.092			
3571	13.70	7.25	5.086			
3491	13.70	7.20	5.239			
3473	13.65	7.20	5.279			
3473	8.04		10.087	9.795	10.020	10.246
3123	8.44		10.637	10.449	10.573	10.693
2776	8.90		11.082	11.023	11.051	11.073
2429	9.31		11.478	11.517	11.457	11.392
2082	9.63		11.809	11.939	11.799	11.657
1738	9.88		12.079	12.293	12.084	11.876
1388	10.08		12.290	12.581	12.314	12.052
1318.6	10.11		12.321	12.630	12.354	12.082
1297.8	10.11		12.330	12.645	12.365	12.091
1283.9	10.17		12.337	12.654	12.373	12.097
1249.2	10.48		12.352	12.677	12.390	12.110
1249.2	10.48	3.16	12.352	12.677	12.390	12.110
1214.5	10.76	3.16	12.368	12.697	12.407	12.123
1179.8	10.93	3.16	12.382	12.717	12.422	12.135
1145.1	11.04	3.16	12.400	12.735	12.437	12.146
1110.4	11.09	3.16	12.412	12.753	12.451	12.156
[illegible]	11.12	3.16	12.429	12.770	12.464	12.166
1041.0	11.13	3.16	12.443	12.786	12.477	12.176
[illegible]	11.15	3.16	12.501	12.860	12.536	12.221
[illegible]	11.17	3.16	12.551	12.921	12.584	12.257
520.5	11.17	3.16	12.590	12.968	12.621	12.285
347.6	11.16	3.16	12.614	13.003	12.648	12.306
173.5	11.15	3.16	12.629	13.023	12.665	12.318
0.0	11.15	3.16	12.635	13.030	12.670	12.322

* α is equal to α* given in Pekeris and Accad (1972)

+ For the model with α = −0.2, the density in the core is slightly modified to conserve mass.

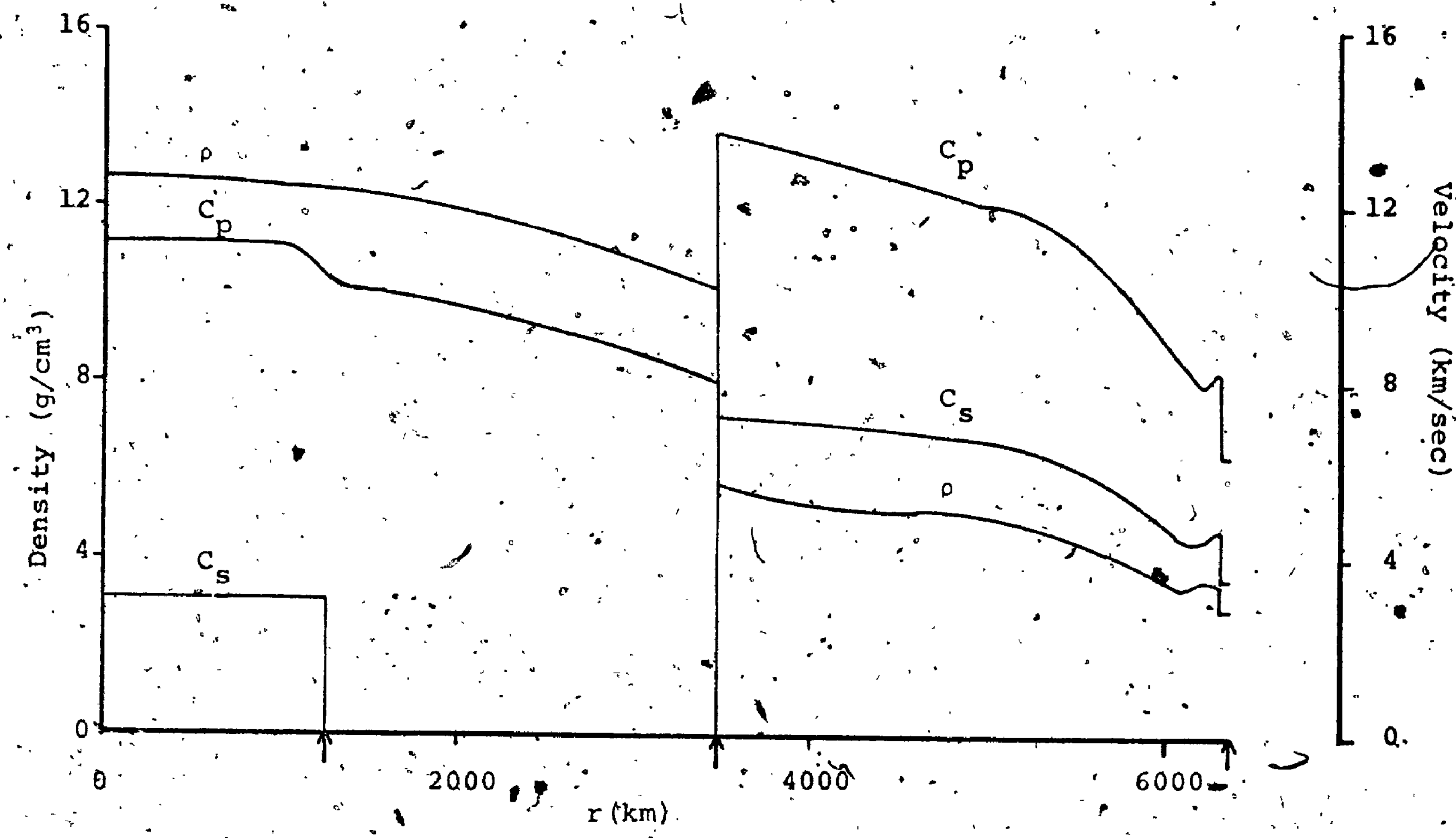

Figure 2. Mechanical properties of the earth model M_3 (Pekeris 1966)

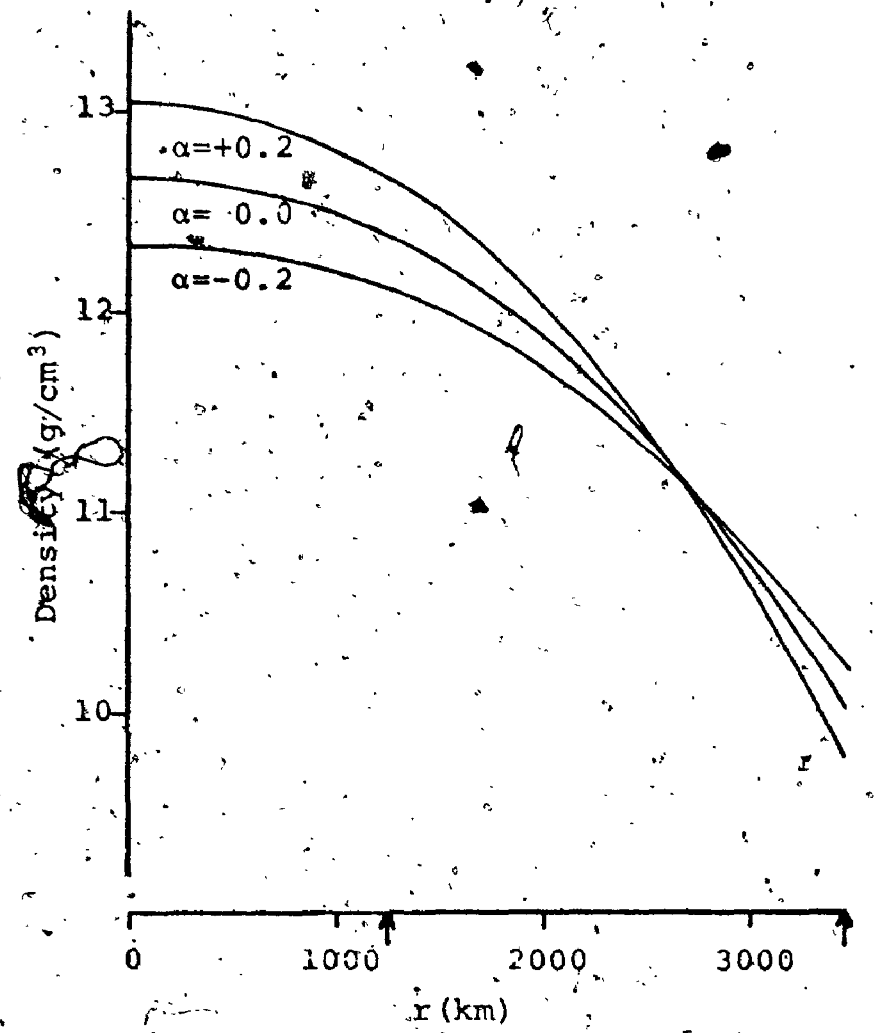

Figure 3a. Density of the uniform polytropic cores given in Table 1.

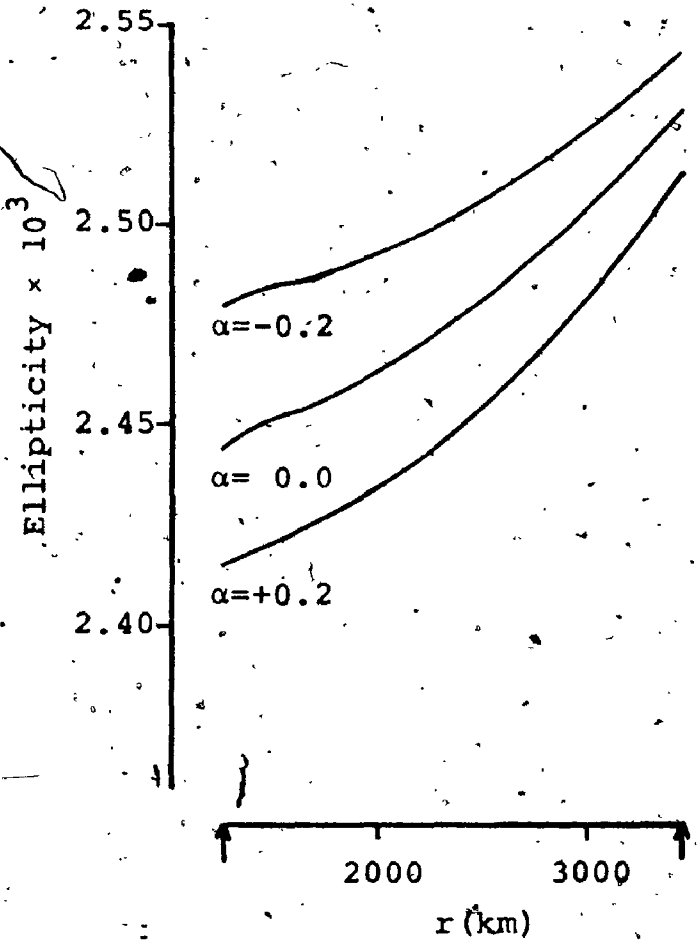

Figure 3b Ellipticity in the outer core.

characteristic dynamic responses of the polytropic cores, have been demonstrated by Pekeris and Accad.

However, the parameter α determines stability for the liquid outer core only when the earth is non-rotating. It is clear from equation (2.1) that in a rotating earth, the acceleration of a particle assumes a complicated form. The stability of the outer core no longer relates to α in a simple way. In fact, as we shall show in the following subsections, all three types of uniform polytropic cores are capable of free oscillations.

4.6.2 Free Core Oscillations

The free core oscillations have periods longer than the fundamental period of free elastic oscillation of the earth (Pekeris, Alterman, and Jarosch 1963), and have displacements and stresses mainly confined to the liquid outer core. Table 2 shows the periods of free spheroidal core oscillations of degree 2. The dependence of the period on the core model, as well as on the azimuthal number m is manifest. The strong dependence on the azimuthal number indicates that the effects of ellipticity and rotation are large and cannot be neglected. In the table, the free periods are calculated up to 28 hours. It is possible to extend the calculation to infinite period. But at long periods, the dynamic theory can be

TABLE 2 PERIODS IN HOURS OF FREE CORE OSCILLATIONS FOR THE POLYTROPIC EARTH MODELS

$\alpha = +0.2$					$\alpha = 0.0$					$\alpha = -0.2$				
M = 2	M = 1	M = 0	M = -1	M = -2	M = 2	M = 1	M = 0	M = -1	M = -2	M = 2	M = 1	M = 0	M = -1	M = -2
0.8857	0.8894	0.8982	0.9068	0.9128	0.8869	0.8906	0.8981	0.9081	0.9131	0.8869	0.8906	0.8993	0.9081	0.9131
6.432	6.585	6.883	7.258	7.743	10.086	11.789	10.227	10.445	12.289	14.727	9.493	11.820	11.898	17.148
8.865	8.992	9.400	10.055	11.508	18.977	14.508	15.227	13.945		15.915	10.664	12.618	12.618	23.977
12.198	11.182	12.383	14.820	19.055		17.023	17.977	15.388		19.008	12.883	15.477	13.992	
14.848	12.477	14.637	15.192			23.883		16.352		21.320	13.664	15.993	14.570	
16.134	14.492	15.642	15.994					18.743		21.976	15.086	16.008	15.555	
17.008	17.417	17.523	17.524					22.805		23.165	15.727	17.320	15.962	
22.537	18.382	19.142	21.008					26.149		25.415	17.522	18.008	19.368	
23.242	19.462	20.241	22.962								20.211	19.586	21.836	
24.705	20.132	24.686	24.274								22.400	22.852	23.462	
27.038	23.883	26.571									23.883		24.789	
	25.332										27.931		26.680	
	27.802												27.336	

1. These are free spheroidal oscillations of degree N = 2, and azimuthal number $-N \leq M \leq N$, with periods greater than the fundamental elastic mode.

2. The fundamental elastic mode is included to show the splitting effect due to the liquid core.

3. All the periods are calculated to within 0.008 hour except the fundamental one which is calculated to within .0007 hour.

4. Periods greater than 28 hours are not included.

approximated by the static theory and is therefore unnecessary. Periods of the fundamental elastic modes are shown on the first row. They are included to show the slight perturbing effects of ellipticity and rotation upon the dynamical response of the liquid core.

Consider free spheroidal oscillations of the earth of degree 2 and azimuthal number 1 for the earth model with a uniformly stable core of $\alpha = +0.2$. We notice from section 4.2.1 that this type of free oscillation is associated with a free wobble of amplitude ε. Figure 4.5.6 and 7 show the toroidal displacement T_1 (equivalent to simple rotation), normal spheroidal displacement η_2, normal stress Z_2, and change in gravitational potential H_2 respectively. The normalization factor is indicated by the value of ε.

From these 4 figures, we may classify the 5 free oscillations into 3 groups:

Group A. Elastic Modes

This group is represented by the free oscillation with period 0.88928 hours. The liquid outer core responds as a solid. Although the core rotates (curve 1 in Figure 4) slightly with respect to the mantle, the motion in the liquid core is predominately spheroidal and determined by the elastic properties.

Curve	T (hour)	$\varepsilon \times 10^{11}$
1	0.88928	1000.
2	6.58385	1.
3	8.99421	0.333
4	11.18246	0.5
5	23.88337	0.002

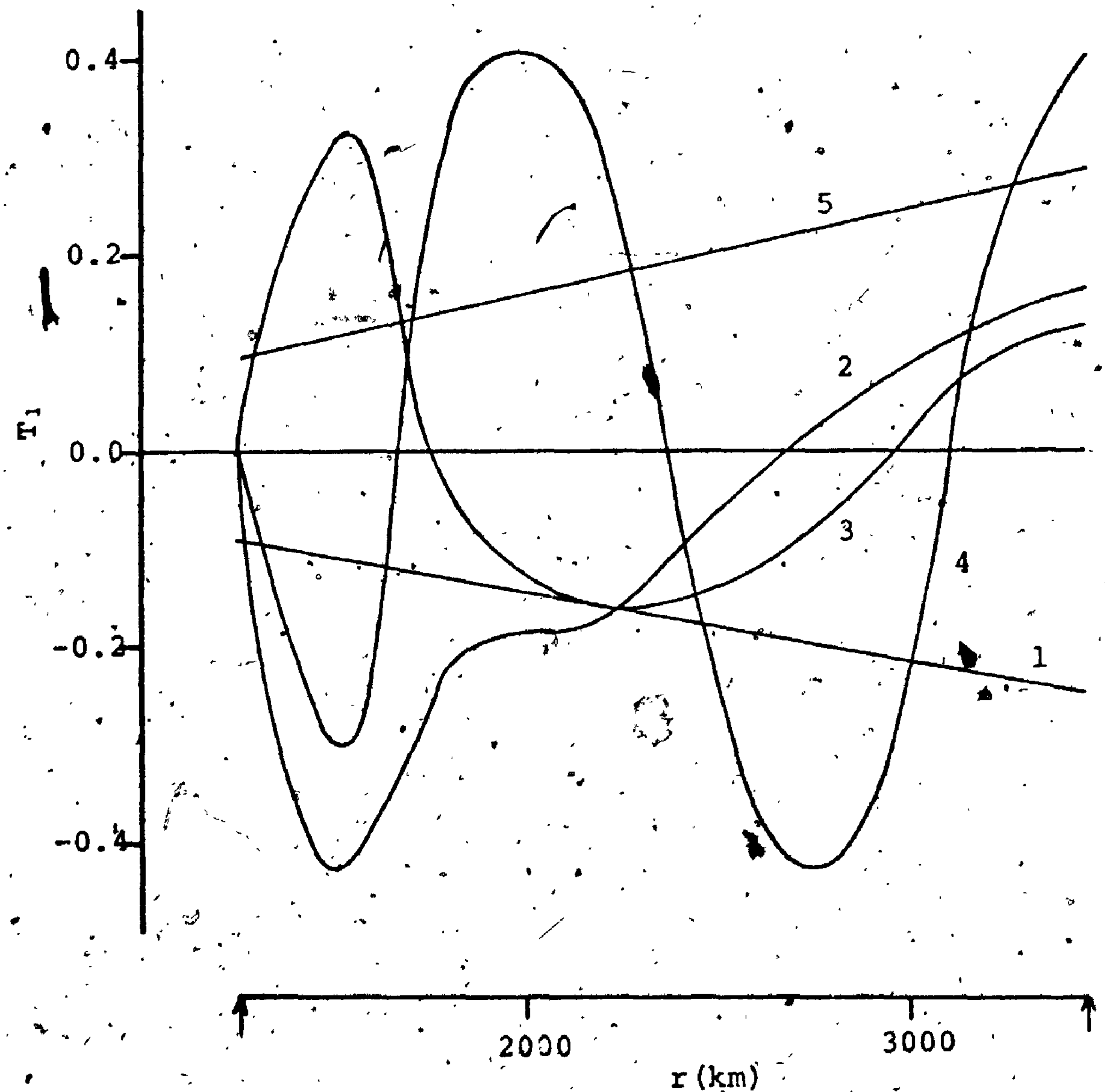

Figure 4. The toroidal displacement T_1 in the outer core for the free spheroidal oscillations for N=2,M=1 for the earth model with $\alpha=+0.2$. Normalization is indicated by the amplitude of free wobble ε.

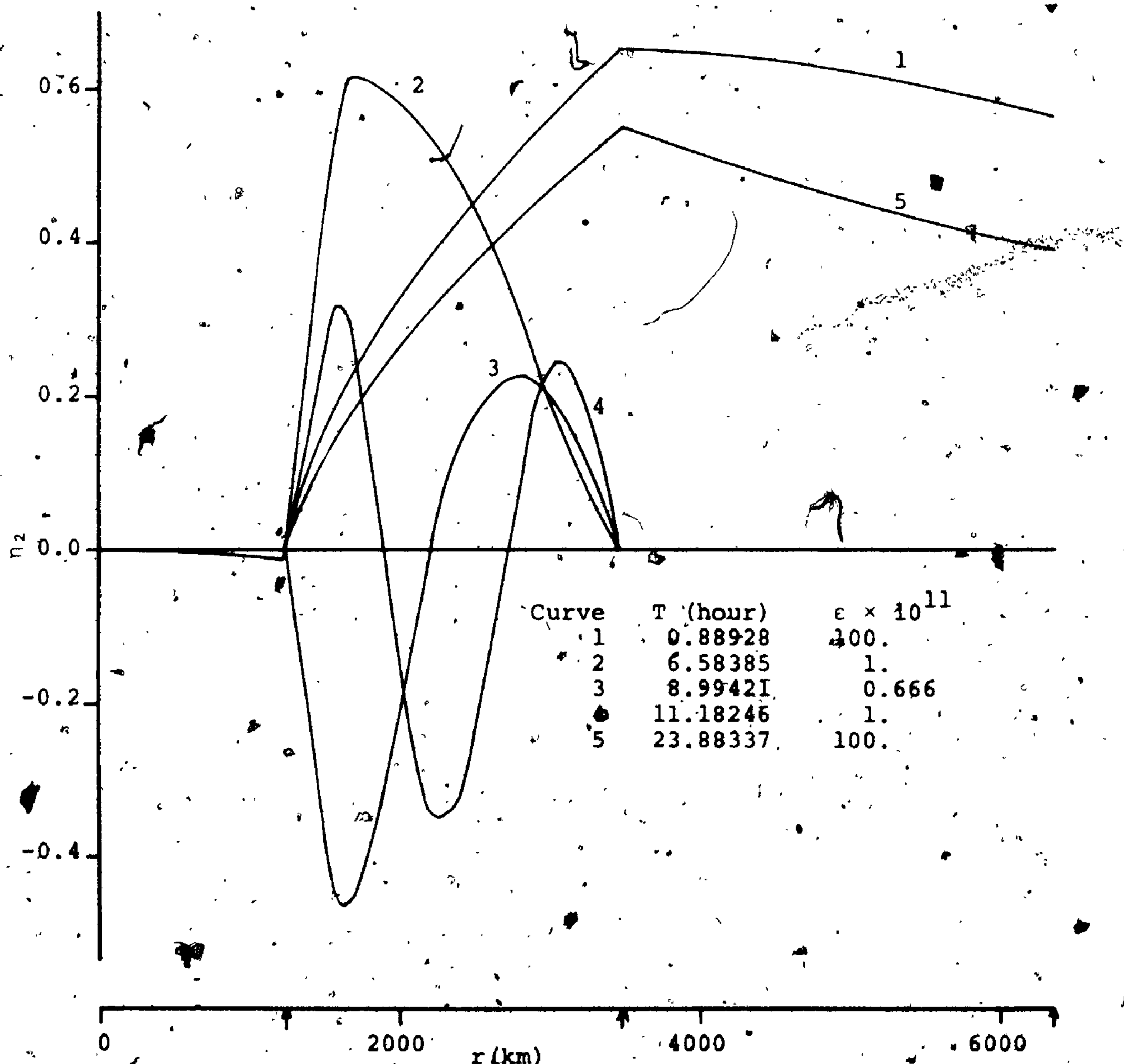

Figure 5. The normal displacement η_2 for the free spheroidal oscillations for N=2,M=1 for the earth model with α=+0.2. Normalization is indicated by the amplitude of free wobble ε.

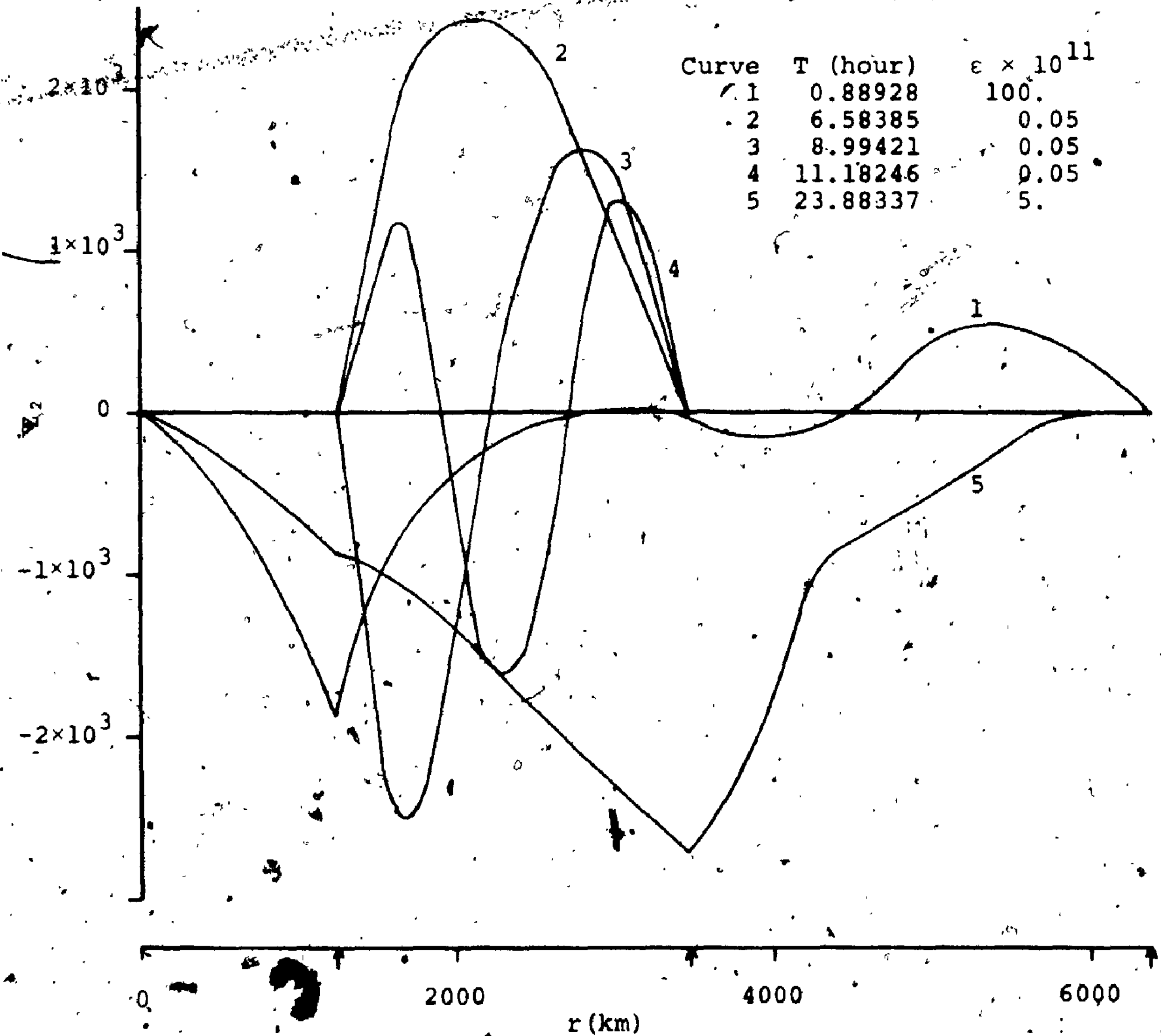

Figure 6. The normal stress Z_2 for the free spheroidal oscillations for N=2,M=1 for the earth model with α=+0.2. Normalization is indicated by the amplitude of free wobble ε.

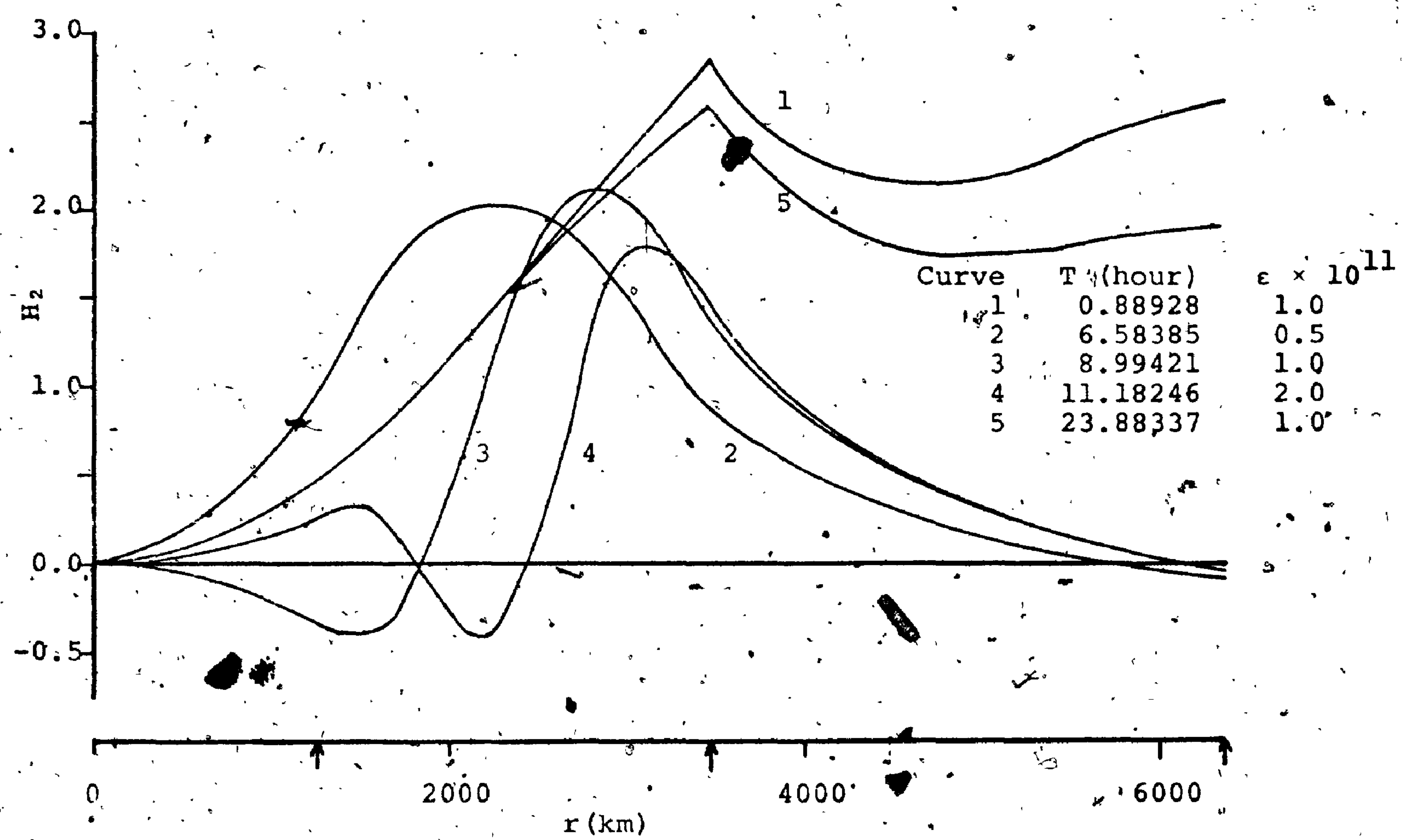

Figure 7. The change in gravitational potential H_2 for the free spheroidal oscillations for N=2, M=1 for the earth model with α=+0.2. Normalization is indicated by the amplitude of free wobble ε.

Group B. Core Modes

This group is represented by the free oscillations with periods 6.58385, 8.99421, and 11.18246 hours. The free oscillations in this group are the type first discovered by Alterman et al. (1959). The normal displacement in the mantle is induced by the toroidal displacement in the liquid core. Since the toroidal displacement is of the same magnitude as the normal displacement in the outer core, normal displacement in the mantle is of the order of ellipticity compared to that in the outer core. In view of this, observation of these modes may be difficult unless resonance occurs.

Group C. Toroidal Modes

An example of this type is the free oscillation with period 23.88337 hours. The spheroidal part of the displacement resembles those of the elastic modes. But in the liquid outer core, there exists a large toroidal field of degree 2 and azimuthal number 1. This T_1 toroidal field assumes the form of a rigid rotation of the entire liquid outer core as can be seen from Fig. 4. Moreover, if we compare curve 5 on Fig. 4 with that on Fig. 5, we see that the T_1 field is 2.5 x 10^4 times larger than the normal displacement η_2. We call this free core oscillation a T_1-mode.

The importance of this T_1-mode lies in the fact that the period is close to those of the diurnal earth tides and hence leads to a resonance effect. This we shall discuss in the next subsection.

In view of the form of the equations (4.14) and (4.15), there is the possibility of a T_3-mode with period about 144 hours. The dominating field in the outer core will then be toroidal field of degree 3 and azimuthal number 1. But there is no tesseral earth tide with such a long period. Therefore computation was not attempted.

A similar classification can be made for free spheroidal oscillations of degree $N = 2$ but different azimuthal numbers. However, for the period range under investigation, T_1 or T_3-modes exist only for the azimuthal number +1. Inspection of Table 2 shows that the elastic modes varies slightly with the core model, the core modes depend strongly on the core model, while the T_1-mode is practically independent of the core model. Thus for the study of the structure of the core, core modes can give us the best information. Unfortunately, due to their characteristics, they are hardly observable. The only exception is the core mode $N = 2$, $M = 2$, period $T = 12.1980$ hours for the uniformly stable core model, where resonance may be possible due to the semi-diurnal tides.

4.6.3 Diurnal (Tesseral) Earth Tides and Nutations

The principal components of the diurnal tides are given in Table 3 (see Melchior 1966, 1971) together with the associated nutations.

The importance of these tides lies in the facts that resonance occurs near the T_1-mode, and the nutations can be observed astronomically.

In Table 3, both the observed and theoretical (for a rigid earth) amplitudes for nutations are given. The discrepancy between the two values can only be explained by the dynamical effects of the liquid core as we have mentioned in section 1.2.

Figure 8, 9, and 10 illustrate the response of the earth models (α = +0.2, 0.0, -0.2) to the diurnal tides ψ_1 and S_1 which lie on either sides of the T_1-mode. The variations of the normal displacement η_2 and gravitational potential H_2 are clearly observable.

Table 4 shows the Love numbers of the principal diurnal tides, and Figure 11 shows the Love numbers as functions of the period (hypothetical, since tidal potentials do not exist throughout the frequency range). Notice that the asymptotic behaviours of the Love numbers near the periods of free oscillations change form at the T_1-mode.

TABLE 3 PRINCIPAL DIURNAL EARTH TIDES AND ASTRONOMICAL NUTATIONS

	DIURNAL TIDES				NUTATIONS AMPLITUDE OBSERVED (O) THEORETICAL (T)*	
SYMBOL	DOODSON'S ARGUMENT	FREQUENCY IN DEGREES PER HOUR	NORTH SOUTH COMPONENT OF ACCELERATION AT THE EQUATOR IN GALS	PERIOD IN SIDEREAL DAYS	IN LONGITUDE	IN OBLIQUITY
Q_1	135.655	13.3986609	+ 5.9428	9.157938		
	195.455	16.6834763	- 0.2561			
C_1	145.555	13.9430356	+31.0391	13.698192	0"093 (O)	0"097 (O)
OO_1	185.555	16.1391017	- 1.3366	Fortnightly	0"0876 (T)	0"0944 (T)
M_1	155.655	14.4966940	- 2.4410	27.629992		
J_1	175.455	15.5854433	- 2.4410			
π_1	162.556	14.9178647	+ 0.8474	122.082681		
	168.554	15.1642724	- 0.0362			
P_1	163.555	14.9589314	+14.4815	183.121117	0"529 (O)	0"575 (O)
	167.555	15.1232059	- 0.6226	semi-annual	0"5104 (T)	0"5558 (T)
S_1	164.556	15.0000020	- 0.3484	366.259758		
ψ_1	166.554	15.0821353	- 0.3484			
	165.575	15.0454814	- 0.1268	3408.493577		
	165.545	15.0388622	+ 0.8647	6816.987155	6"848 (O)	9"203 (O)
	165.565	15.0432751	+ 5.9148	principal	6"8672 (T)	9"2232 (T)
K_1	165.555	15.0410686	+43.6898			

Reference: Melchior (1971)

*The theoretical amplitudes of nutations for a rigid earth are those given by Molodensky (1961)

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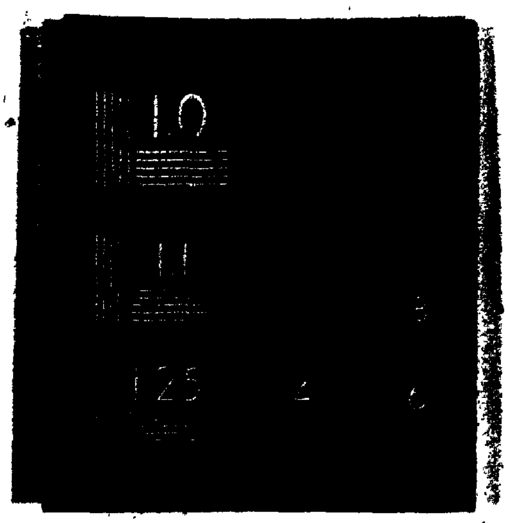

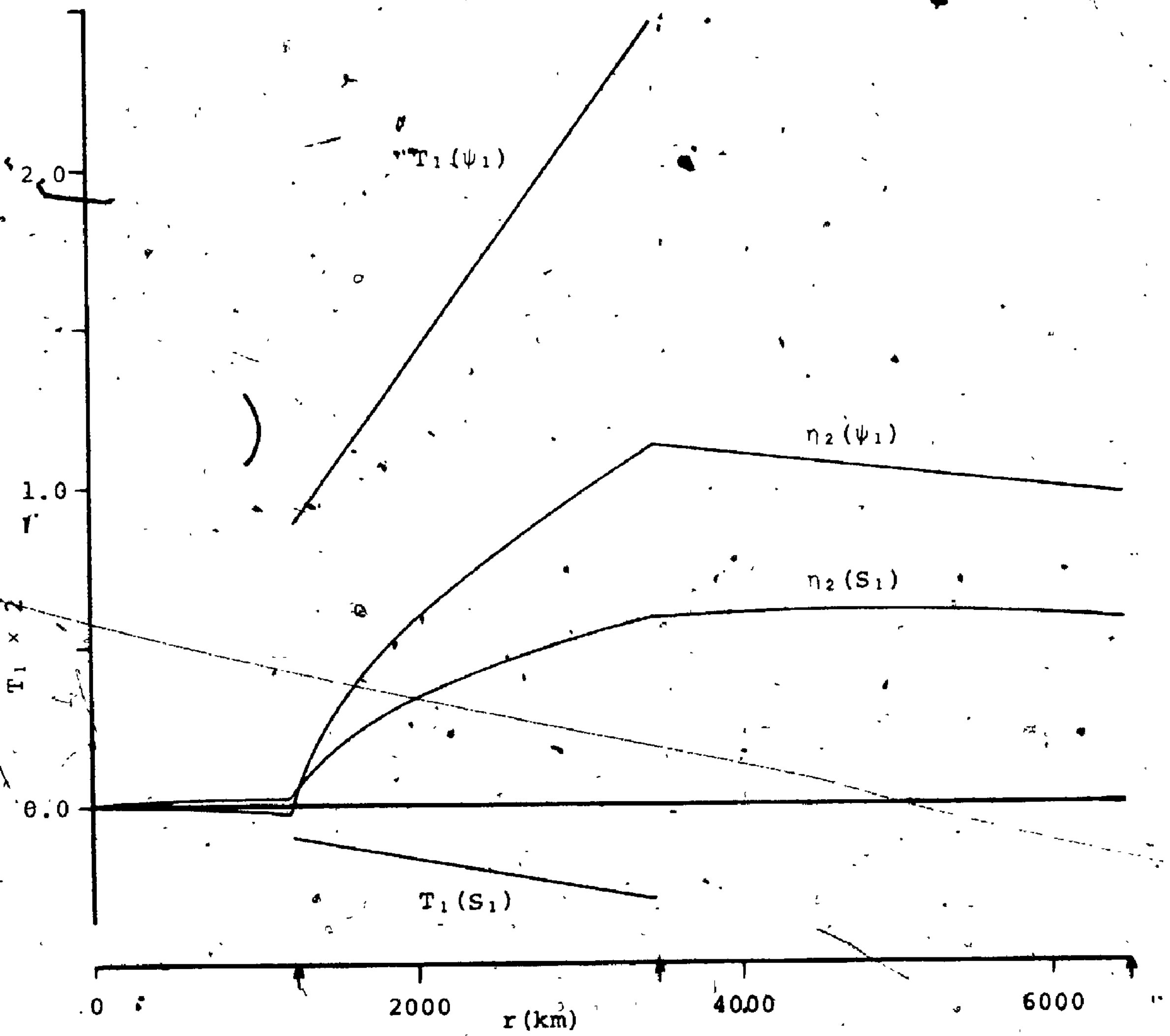

Figure 8. The normal displacement η_2 and toroidal displacement T_1 for the diurnal tides ψ_1 and S_1 for earth models with α=+0.2, 0.0, -0.2. Amplitude of the tidal potential is set to unity at the free surface.

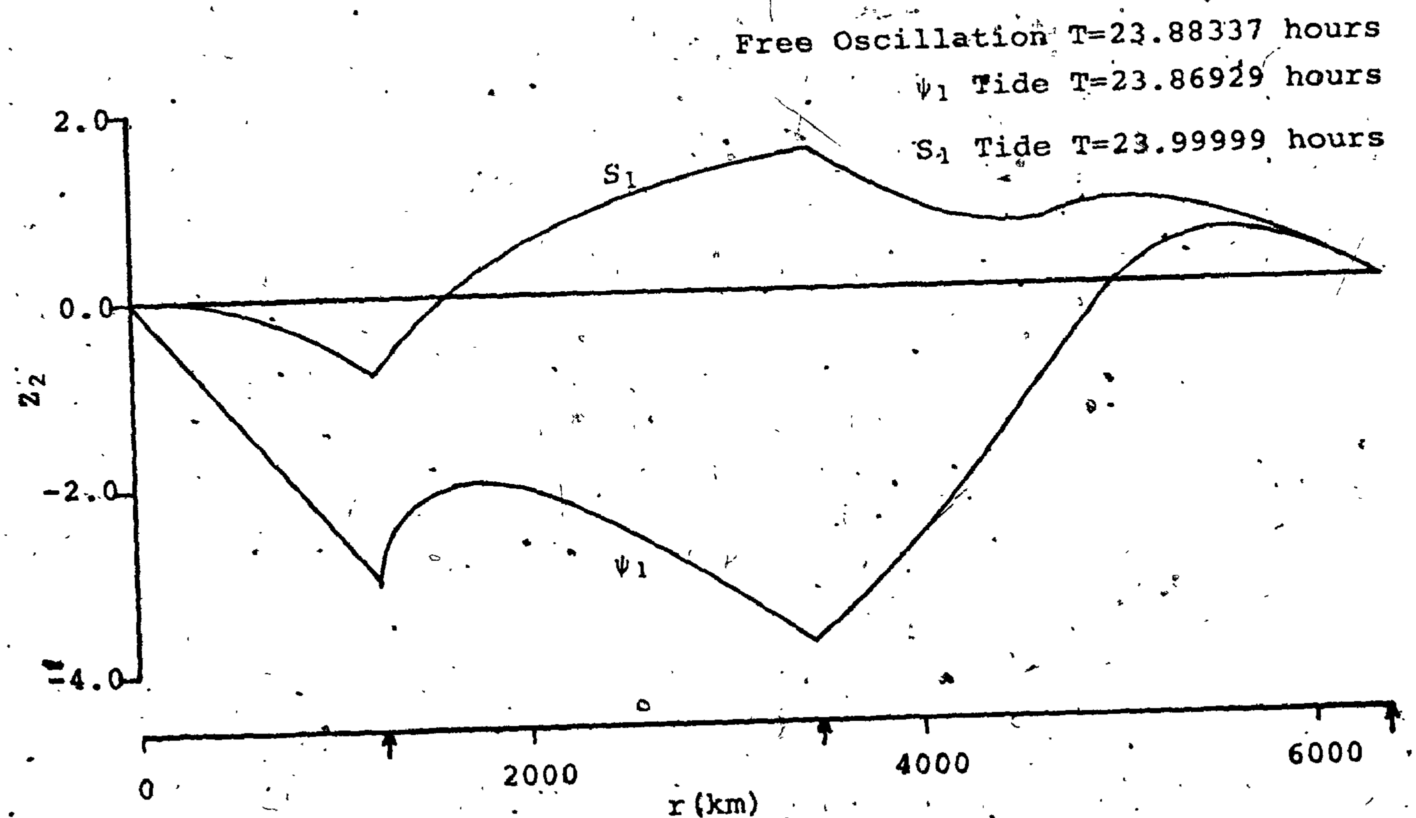

Figure 9. The normal stress Z_2 for the diurnal tides ψ_1 and S_1 for the earth models with $\alpha=+0.2$, 0.0, -0.2. Amplitude of the tidal potential is set to unity at the free surface.

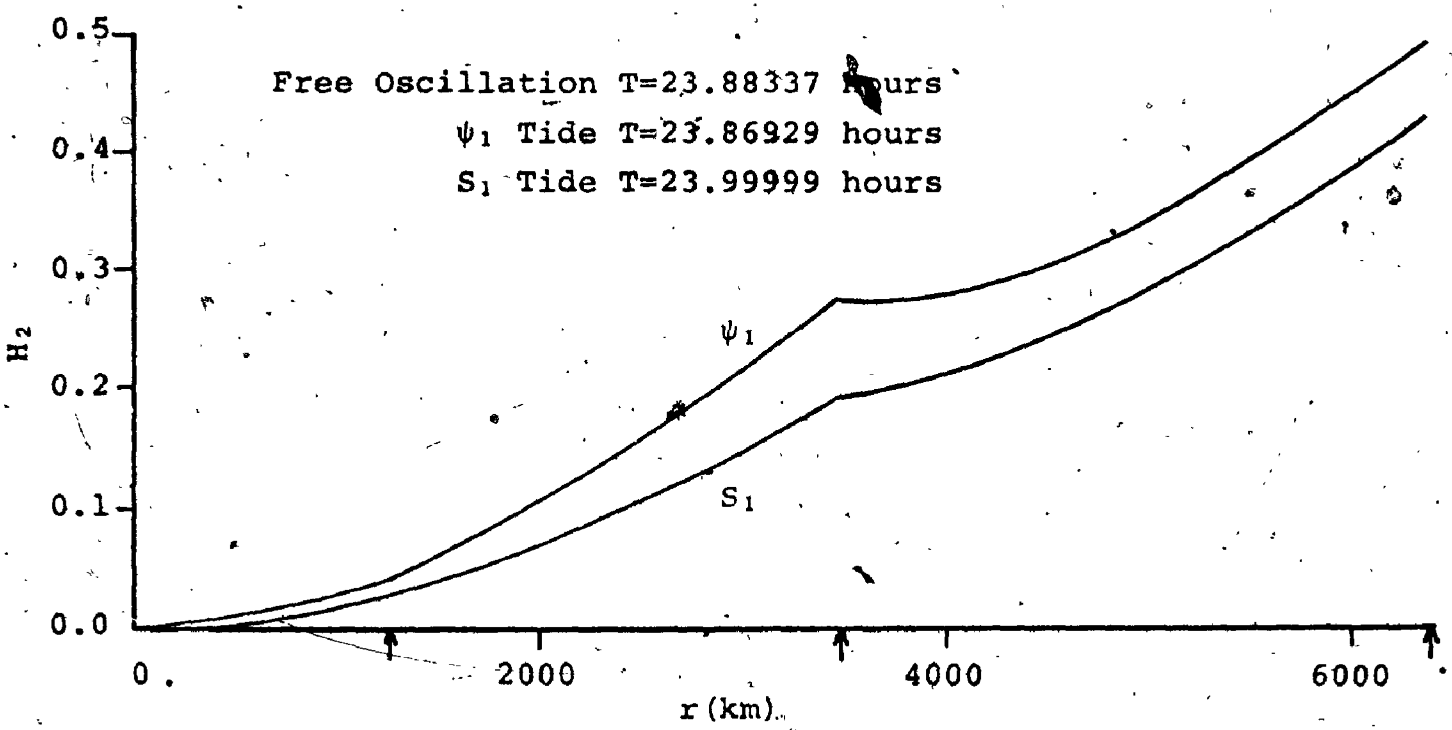

Figure 10. The change in gravitational potential H_2 for the diurnal tides ψ_1 and S_1 for the earth models with α=+0.2, 0.0, -0.2. Amplitude of the tidal potential is set to unity at the free surface.

TABLE 4 DIURNAL TIDAL LOVE NUMBERS

SYMBOL	DOODSON'S ARGUMENT	$\alpha = +0.2$ h	$\alpha = +0.2$ k	$\alpha = +0.2$ l	$\alpha = 0.0$ PRESENT THEORY h	$\alpha = 0.0$ PRESENT THEORY k	$\alpha = 0.0$ PRESENT THEORY l	$\alpha = 0.0$ MOLODENSKY'S THEORY h	$\alpha = 0.0$ MOLODENSKY'S THEORY k	$\alpha = 0.0$ MOLODENSKY'S THEORY l	$\alpha = -0.2$ h	$\alpha = -0.2$ k	$\alpha = -0.2$ l
Q_1	135.655	0.6096	0.2994	0.08422	0.6101	0.3002	0.08431	0.6096	0.2995	0.09427	0.6103	0.3009	0.09441
O_1	145.555	0.6092	0.2991	0.08421	0.6093	0.2998	0.09434	0.6090	0.2993	0.09431	0.6096	0.3005	0.08444
M_1	155.655	0.6062	0.2978	0.08436	0.6069	0.2986	0.09442	0.6067	0.2983	0.09440	0.6072	0.2992	0.09452
π_1	162.556	0.5928	0.2911	0.09479	0.5932	0.2917	0.09487	0.5936	0.2918	0.08484	0.5935	0.2924	0.08496
P_1	163.555	0.5861	0.2877	0.08501	0.5865	0.2884	0.08509	0.5871	0.2886	0.08505	0.5868	0.2890	0.08517
S_1	164.556	0.5719	0.2806	0.08547	0.5723	0.2813	0.08555	0.5734	0.2817	0.0855	0.5726	0.2819	0.08563
	165.545	0.5272	0.2583	0.08693	0.5278	0.2590	0.28700	0.5302	0.2601	0.08690	0.5284	0.2597	0.08706
K_1	165.555	0.5214	0.2554	0.08712	0.5221	0.2561	0.08718	0.5246	0.2573	0.08709	0.5227	0.2569	0.08724
	165.565	0.5148	0.2521	0.08734	0.5155	0.2528	0.08739	0.5182	0.2541	0.08729	0.5162	0.2536	0.08745
	165.575	0.5071	0.2482	0.08759	0.5079	0.2490	0.08764	0.5108	0.2604	0.08754	0.5087	0.2498	0.08769
ψ_1*	166.554	0.9379	0.4633	0.07350	0.9486	0.4696	0.07333	0.9296	0.4600	0.07393	0.9598	0.4761	0.07314
φ_1	167.555	0.6696	0.3294	0.08228	0.6706	0.3304	0.08236	0.6680	0.3291	0.08243	0.6715	0.3315	0.08244
	168.554	0.6434	0.3163	0.08313	0.6441	0.3172	0.08322	0.6426	0.3164	0.08326	0.6447	0.3180	0.08330
J_1	175.455	0.6173	0.3032	0.08399	0.6177	0.3040	0.08408	0.6173	0.3038	0.08409	0.6183	0.3047	0.08415
OO_1	185.555	0.6144	0.3018	0.08408	0.6148	0.3025	0.08418	0.6146	0.3026	0.08420	0.6131	0.3030	0.08449
	195.455	0.6136	0.3014	0.08412	0.6139	0.3021	0.08421	0.6138	0.3023	0.08424	0.6141	0.3028	0.08432

* Free oscillation of the earth occurs at a frequency between tides with Doodson's arguments 165.575 and 166.554. This is true for all three polytropic models.

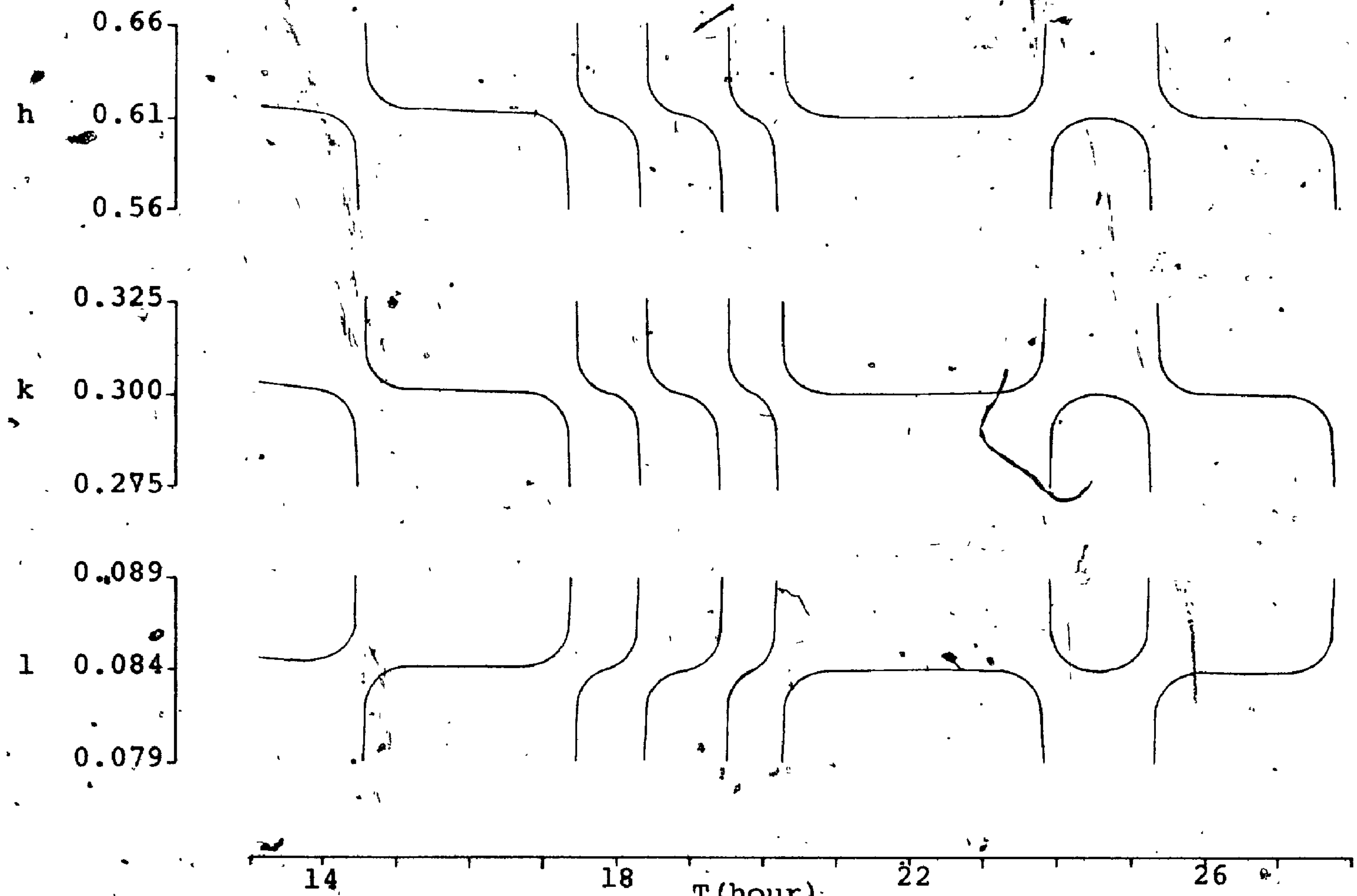

Figure 11. The tesseral (N=2,M=1) tidal Love numbers as functions of period for the earth model with α=+0.2.

Table 5 gives the amplitudes of nutations calculated from the present theory and Molodensky's theory. The results from both theories agree well and they also agree with the observed values (see Table 3).

4.6.4 Semi-diurnal (Sectorial) Tides

The principal semi-diurnal earth tides are given in Table 6 (see Melchior 1966).

Figure 12-17 show the responses of the earth models to the semi-diurnal tides λ_2 and L_2. For the earth model with uniformly stable core (α = +0.2), resonance occurs. This is illustrated by the behaviours of h_2 and Z_2 in Figures 12 and 13. On the other hand, for the earth models with unstable and neutrally stable cores, there exists no resonance. These characteristic behaviours are reflected in Love numbers given in Table 7. For earth models with α = 0.0 and α = -0.2, the Love numbers are practically independent of frequency, while for the earth model with α = +0.2, the Love numbers vary with frequency. Clearly, careful analysis of the observed semi-diurnal tidal response may offer us a clue to the structure of the liquid core.

To illustrate the response of the earth to forced sectorial oscillations, the Love numbers are plotted against period in Figure 18. Again, we observe the asymptotic

TABLE 5 THEORETICAL AMPLITUDES OF NUTATIONS

EARTH MODEL	DOODSON'S ARGUMENT	DIURNAL TIDES $\frac{\varepsilon - \varepsilon_o}{\varepsilon_o} \times 10^3$ PRESENT THEORY	MOLODENSKY THEORY	NUTATIONS PERIOD IN SIDEREAL DAYS	AMPLITUDE PRESENT THEORY (P) MOLODENSKY (M) LONGITUDE	OBLIQUITY
	145.555 185.555	+28.4 +73.0		13.698192	0"0899 (P)	0"0973 (P)
α = -0.2	163.555 167.555	+36.0 +89.1		183.121117	0"5274 (P)	0"5768 (P)
	165.545 165.565	+ 3.29 - 3.77		6816.987155	6"8330 (P)	9"1968 (P)
	145.555 185.555	+28.4 +76.9	+28.2 +72.2	13.698192	0"0899 (P) 0"0899 (M)	0"0973 (P) 0"0974 (M)
α = 0.0	163.555 167.555	+35.9 +88.3	+34.5 +84.1	183.121117	0"5274 (P) 0"5273 (M)	0"5768 (P) 0"5758 (M)
	165.545 165.565	+ 3.29 - 3.78	+ 3.17 - 3.64	6816.987155	6"8328 (P) 6"8342 (M)	9"1966 (P) 9"1977 (M)
	145.555 185.555	+28.6 +76.2		13.698192	0"0899 (P)	0"0973 (P)
α = +0.2	163.555 167.555	+35.8 +87.4		183.121117	0"5274 (P)	0"5768 (P)
	165.545 165.565	+ 3.30 - 3.79		6816.987155	6"8327 (P)	9"1965 (P)

TABLE 6 PRINCIPAL SEMI-DIURNAL AND LONG PERIOD TIDES

SYMBOL	DOODSON'S ARGUMENT	FREQUENCY IN DEGREES PER HOUR	AMPLITUDE * OF THE TIDAL POTENTIAL
	SEMI-DIURNAL COMPONENTS		
$2N_2$	235.755	27.895355	+0.02301
μ_2	237.555	27.968208	+0.02777
N_2	245.655	28.439730	+0.17307
ν_2	247.455	28.512583	+0.03303
M_2	255.555	28.984104	+0.90812
λ_2	263.655	29.455625	-0.00670
L_2	265.455	29.528479	-0.02567
T_2	272.557	29.958933	+0.02479
S_2	273.555	30.000000	+0.42286
R_2	274.554	30.041067	-0.00354
$^mK_2$	275.555	30.082137	-0.00354
$^sK_2$	275.555	30.082137	+0.03648
	LONG PERIOD COMPONENTS		
M_o	055.555	0.000000	+0.50458
S_o	055.555	0.000000	+0.23411
S_a	056.554	0.041067	+0.01176
S_{sa}	057.555	0.082137	+0.07287
M_m	065.455	0.544375	+0.08254
M_f	075.555	1.098033	+0.15642

* The amplitude is given in the form of a dimensionless coefficient which is to be multiplied by $G_D = 26206\ cm^2/sec^2$

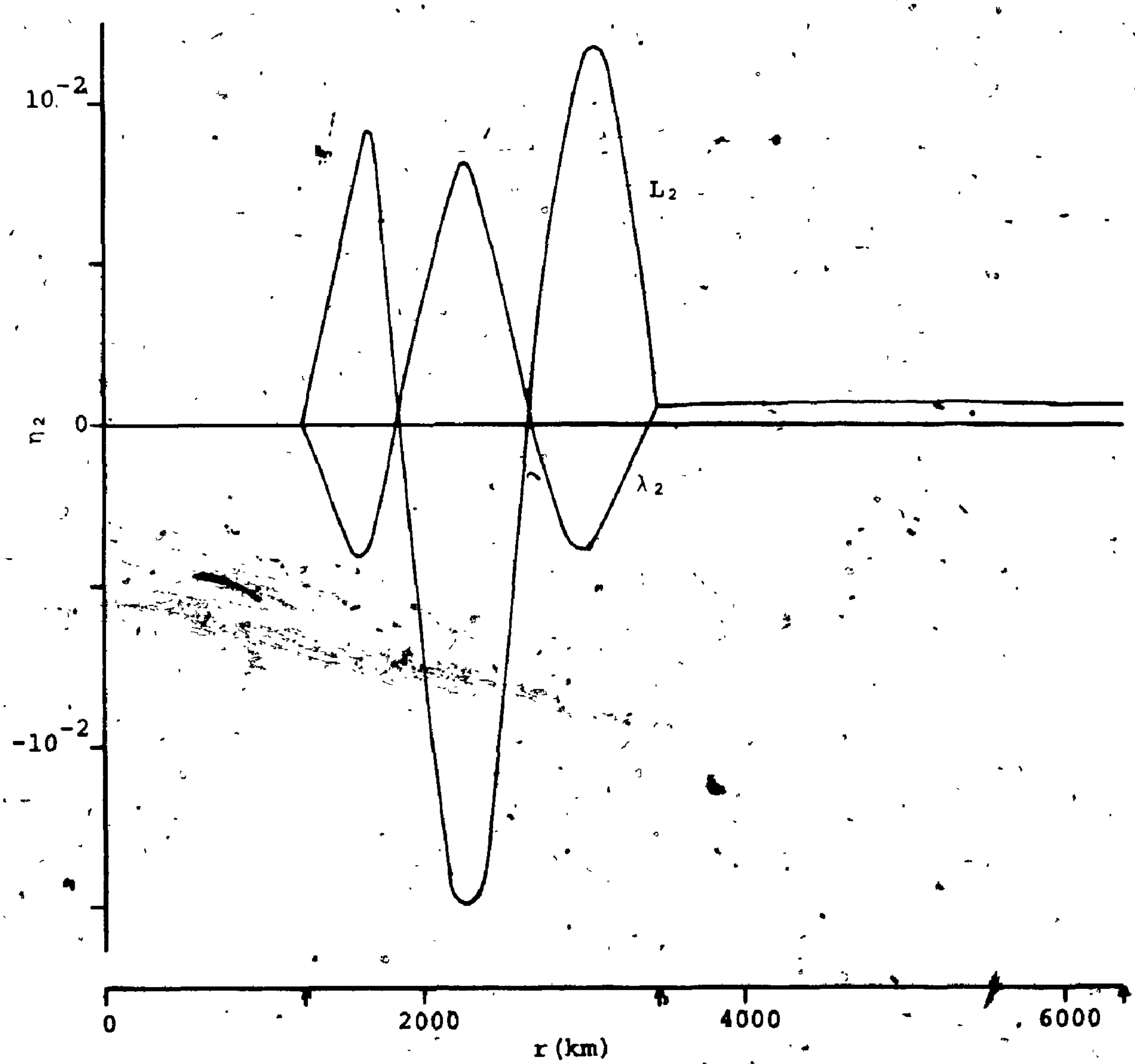

Figure 12. The normal displacement η_2 for the semi-diurnal tides λ_2 and L_2 for the earth model with $\alpha=+0.2$. Amplitude of the tidal potential is set to unity at the free surface.

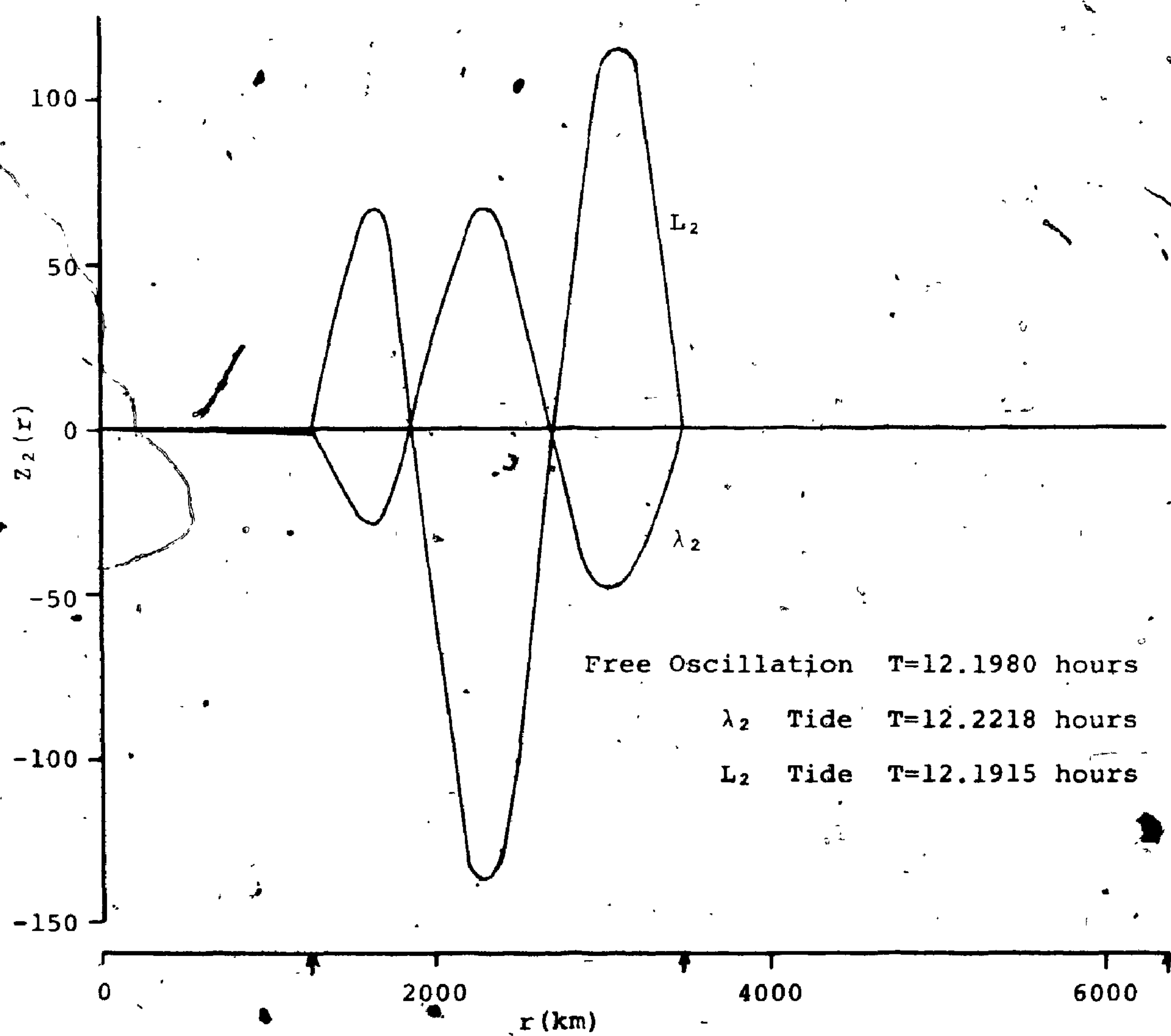

Figure 13. The normal stress Z_2 for the semi-diurnal tides λ_2 and L_2 for the earth model with $\alpha=+0.2$. Amplitude of the tidal potential is set to unity at the free surface.

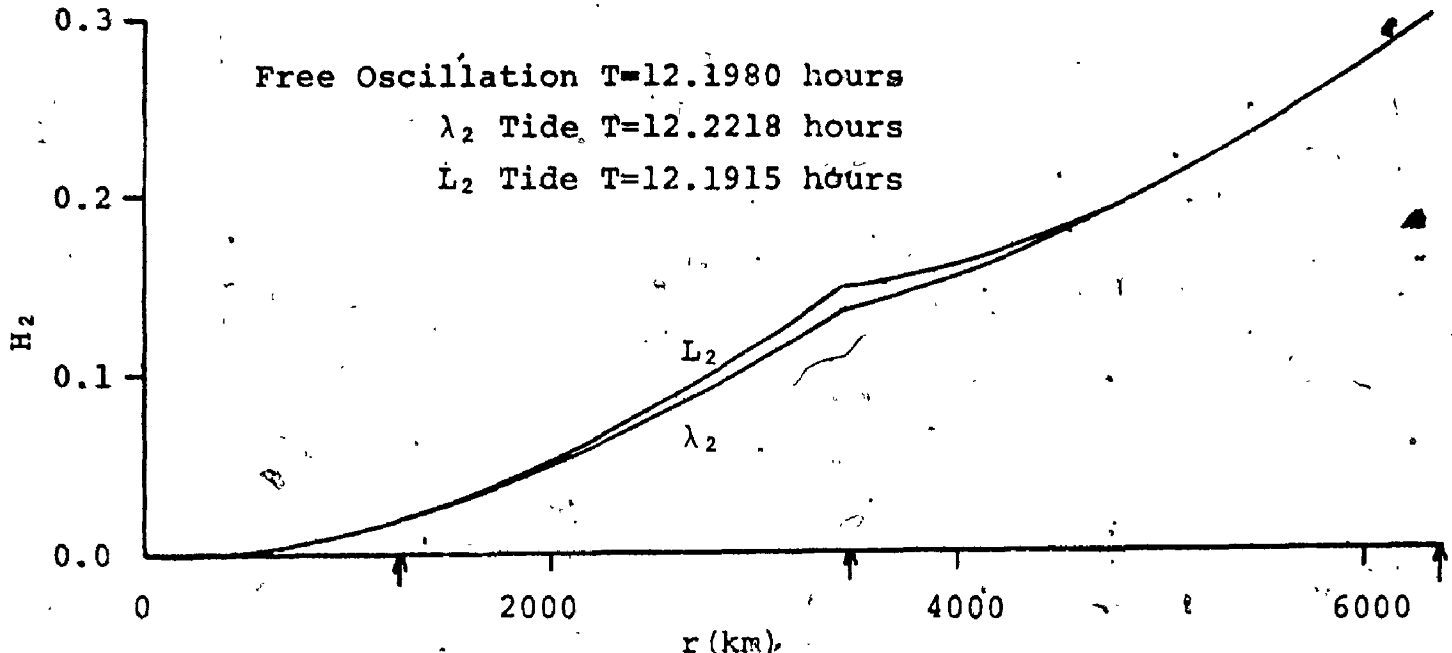

Figure 14. The change in gravitational potential H_2 for the semi-diurnal tides λ_2 and L_2 for the earth model with $\alpha=+0.2$. Amplitude of the tidal potential is set to unity at the free surface.

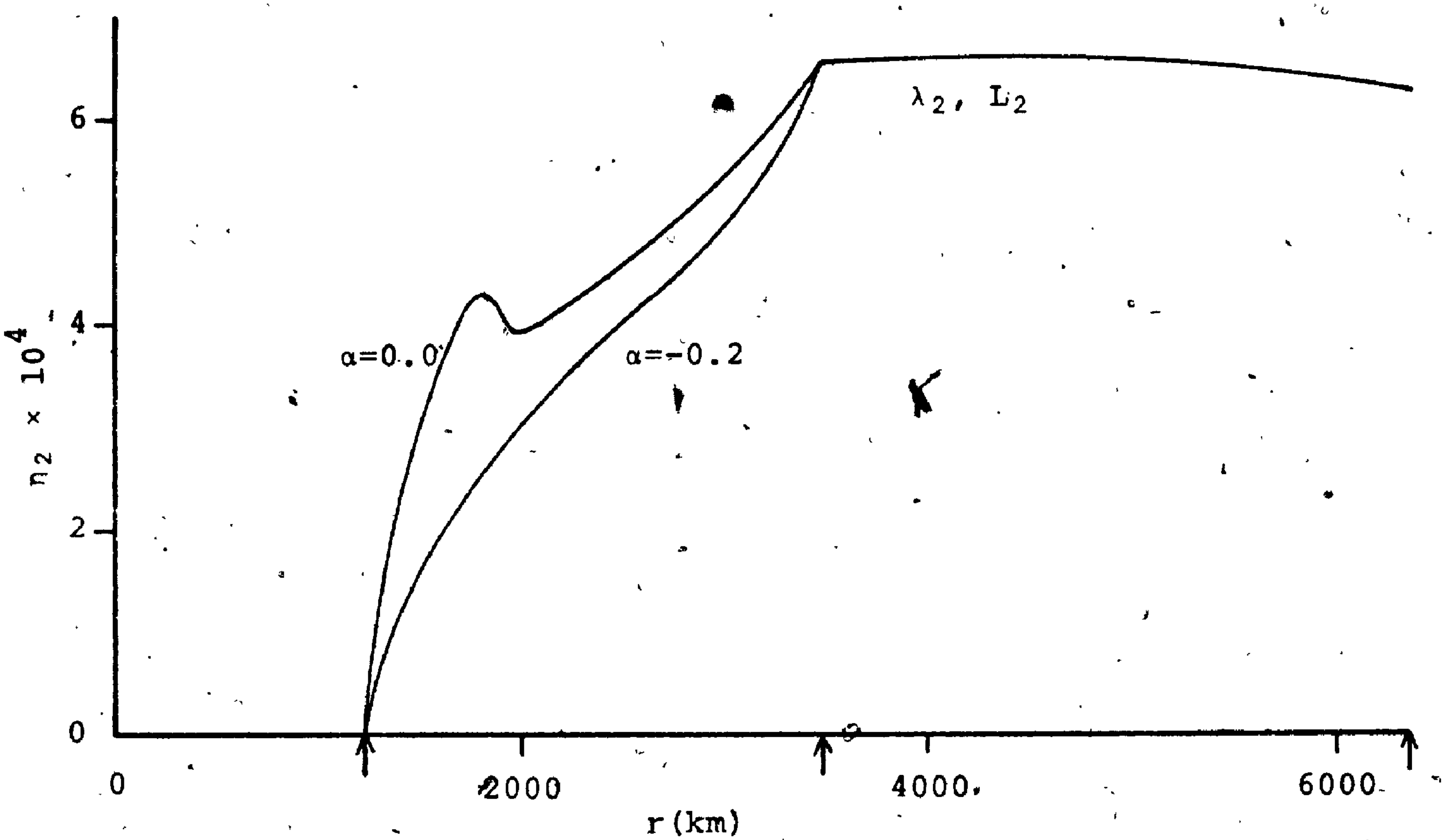

Figure 15. The normal displacement η_2 for the semi-diurnal tides λ_2 and L_2 for the earth models with $\alpha= -0.2,\ 0.0$. Amplitude of the tidal potential is set to unity at the free surface.

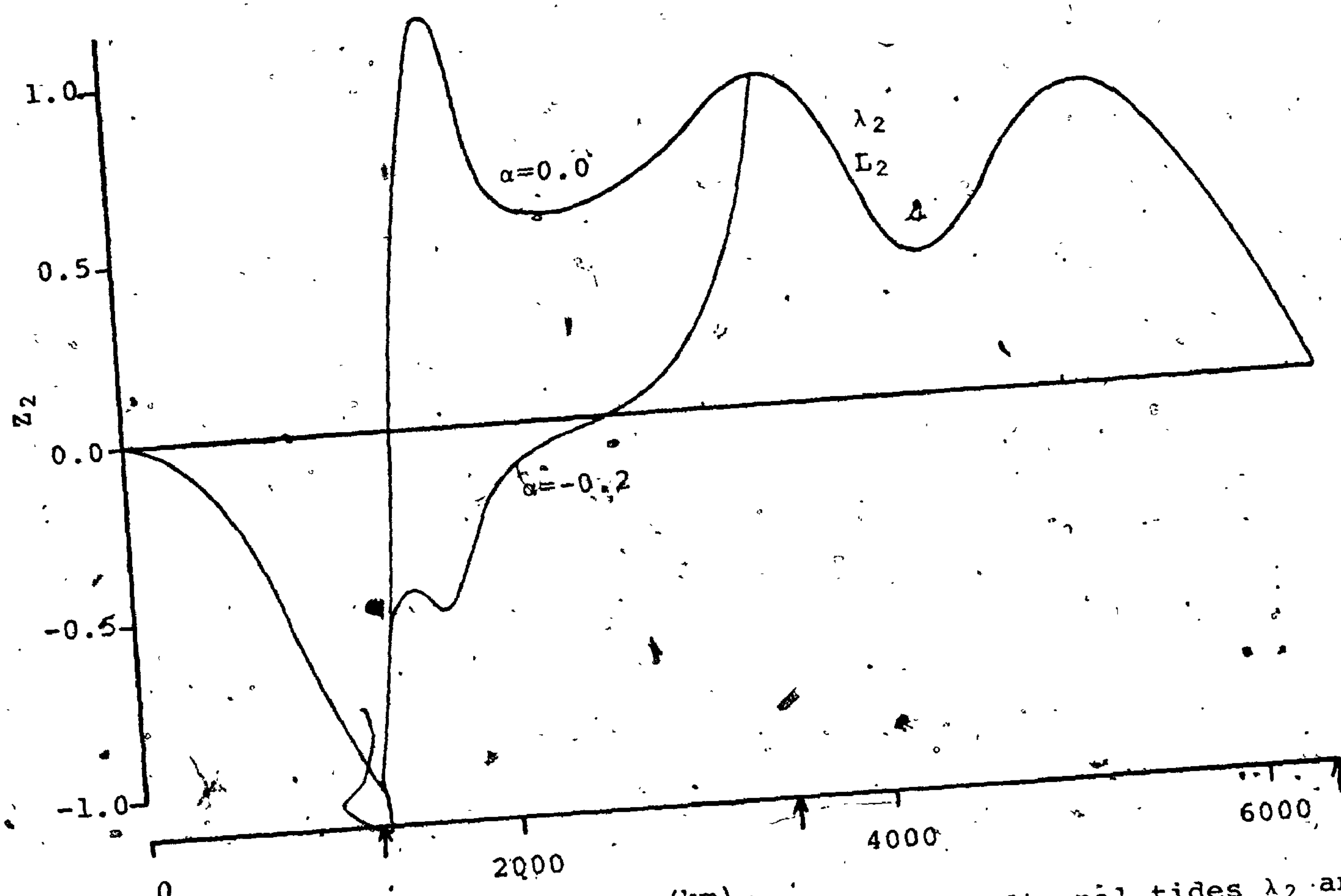

Figure 16. The normal stress Z_2 for the semi-diurnal tides λ_2 and L_2 for the earth models with $\alpha=-0.2$, 0.0. Amplitude of the tidal potential is set to unity at the free surface.

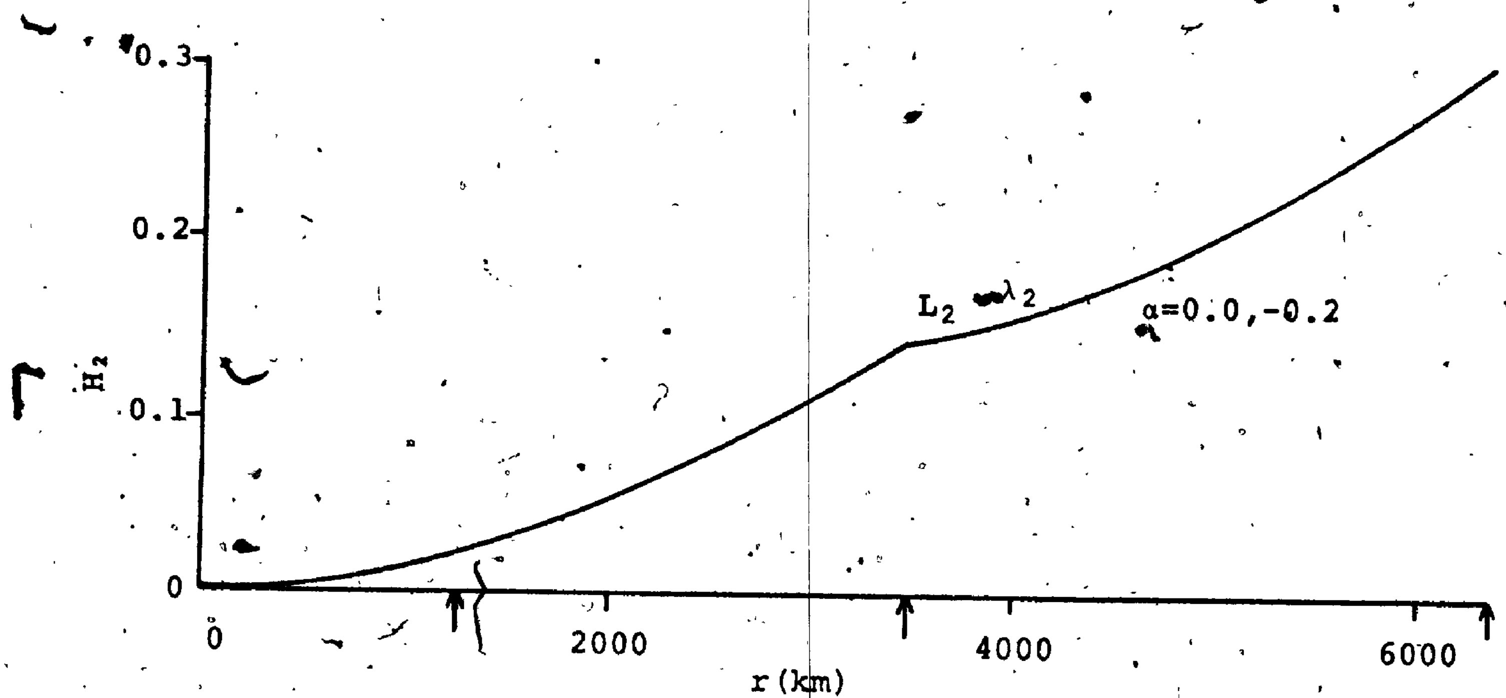

Figure 17. The change in gravitational potential H_2 for the semi-diurnal tides λ_2 and L_2 for the earth models with $\alpha=-0.2,\ 0.0$. Amplitude of the tidal potential is set to unity at the free surface.

TABLE 7 SEMI-DUIRNAL TIDAL LOVE NUMBERS

SYMBOL	DOODSON'S ARGUMENT	$\alpha = 0.2$ h	$\alpha = 0.2$ k	$\alpha = 0.2$ l	$\alpha = 0.0$ h	$\alpha = 0.0$ k	$\alpha = 0.0$ l	$\alpha = -0.2$ h	$\alpha = -0.2$ k	$\alpha = -0.2$ l
$2N_2$	235.755	0.6140	0.3015	0.08424	0.6139	0.3022	0.08437	0.6142	0.3029	0.08447
μ_2	237.555	0.6140	0.3015	0.08423	0.6140	0.3022	0.08437	0.6142	0.3029	0.08448
N_2	245.655	0.6143	0.3016	0.08422	0.6140	0.3022	0.08438	0.6143	0.3029	0.08448
ν_2	247.455	0.6143	0.3016	0.08422	0.6141	0.3022	0.08438	0.6143	0.3029	0.08448
M_2	255.55	0.6148	0.3017	0.08418	0.6141	0.3023	0.08439	0.6144	0.3030	0.08449
λ_2	263.655	0.6232	0.3027	0.08330	0.6142	0.3023	0.08439	0.6145	0.3030	0.08449
L_2 *	265.455	0.5936	0.2991	0.08645	0.6142	0.3023	0.08439	0.6145	0.3030	0.08449
T_2	272.556	0.6131	0.3015	0.08439	0.6143	0.3023	0.08440	0.6146	0.3030	0.08450
S_2	273.555	0.6132	0.3015	0.08438	0.6143	0.3023	0.08440	0.6146	0.3030	0.08450
R_2	274.554	0.6133	0.3015	0.08437	0.6143	0.3023	0.08440	0.6146	0.3031	0.08450
K_2	275.555	0.6134	0.3016	0.08437	0.6143	0.3023	0.08440	0.6146	0.3031	0.08450

* For the earth model with $\alpha = +0.2$, free oscillation occurs at a frequency between λ_2 and L_2 tides. This is reflected in the behaviours of Love numbers.

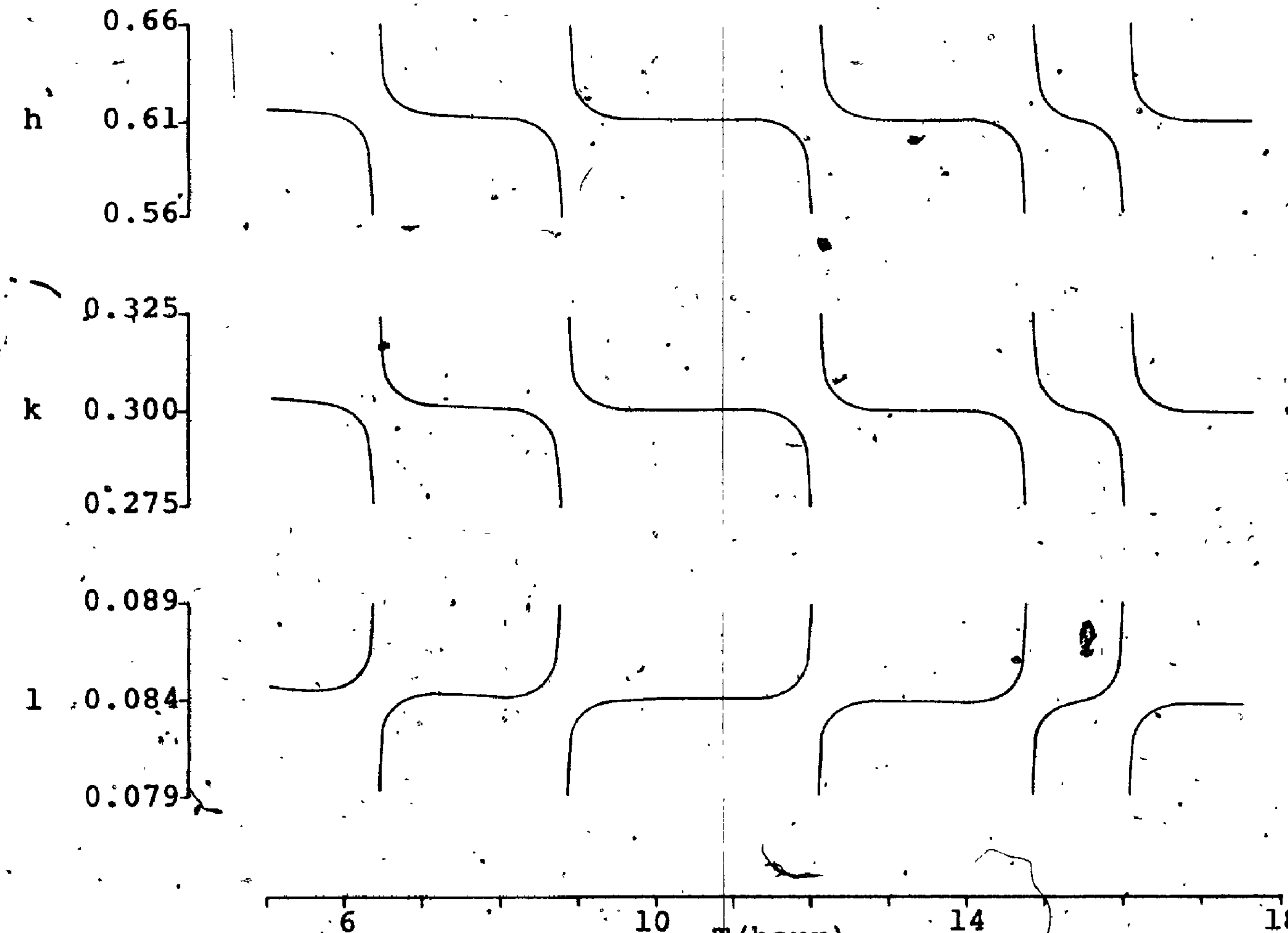

Figure 18. The sectorial (N=2,M=2) tidal Love numbers as functions of period for the earth model with α=+0.2.

behaviours.

4.6. Long Period (Zonal) Tides

The principal zonal tides are given in Table 6.

Due to their long periods, static theory is valid. The curves on Figure 19, 20, and 21 are plotted for the tide M_f. But in fact, curves for other long period tides are indistinguishable from them. This is reflected by the Love numbers given in Table 8, which are almost identical to the static limit as exhibited by the Love numbers for M_o and S_o.

However, if the external potential existed at higher frequencies, the zonal tidal Love numbers would behave the way as shown on Figure 22.

4.7 Summary of the Numerical Results

The dynamical response of the liquid outer core depends on its density stratification. But all three types of stratifications (stable, neutrally stable, unstable) are capable of free oscillations.

The free spheroidal oscillations of the earth can be classified into three groups, the elastic modes, the core modes, and the toroidal modes. The elastic modes are governed by the elastic properties of the earth and have

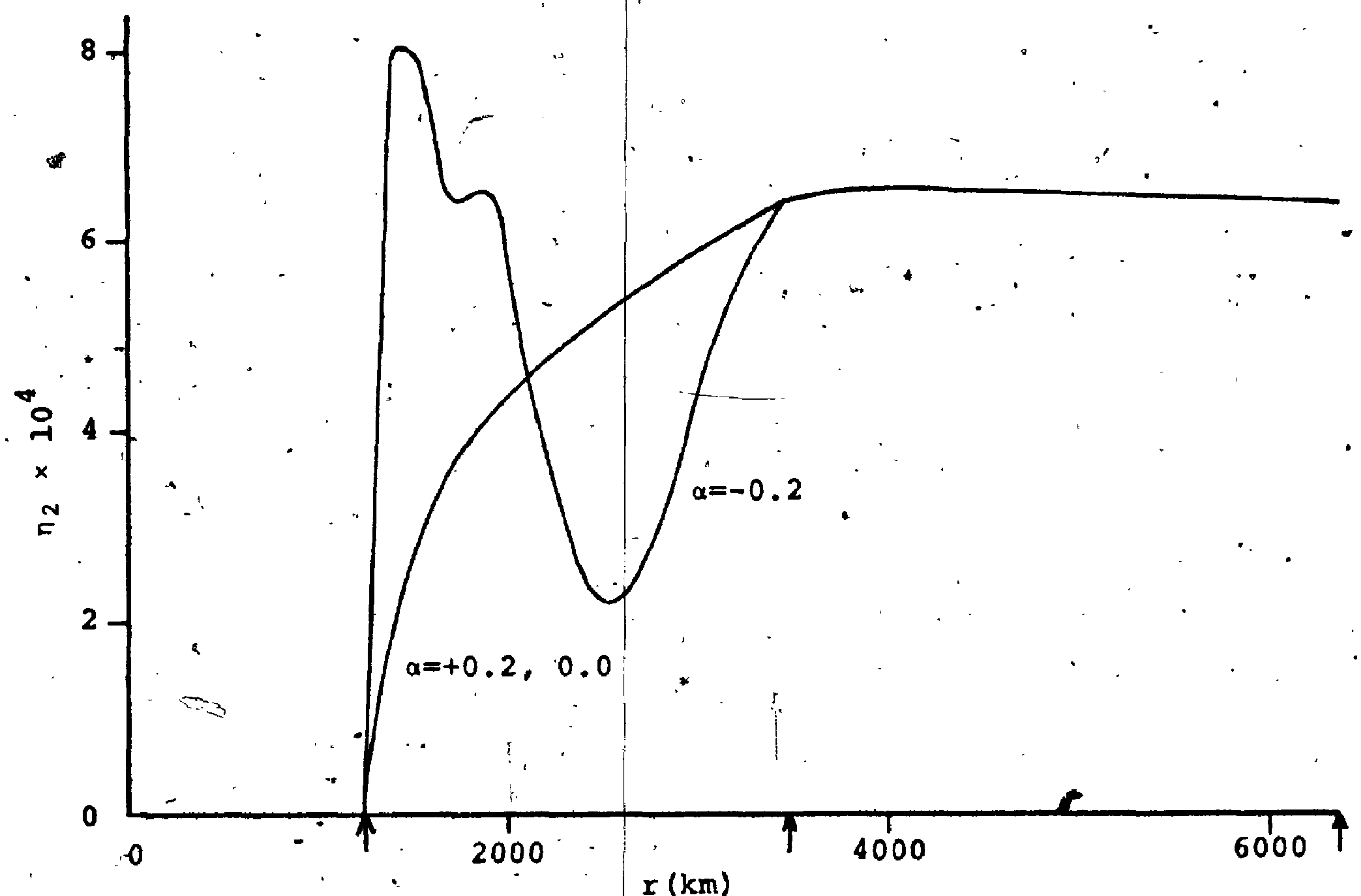

Figure 19. The normal displacement η_2 for the long period tide M_f for the earth models with α=+0.2, 0.0, -0.2. Amplitude of the tidal potential is set to unity at the free surface.

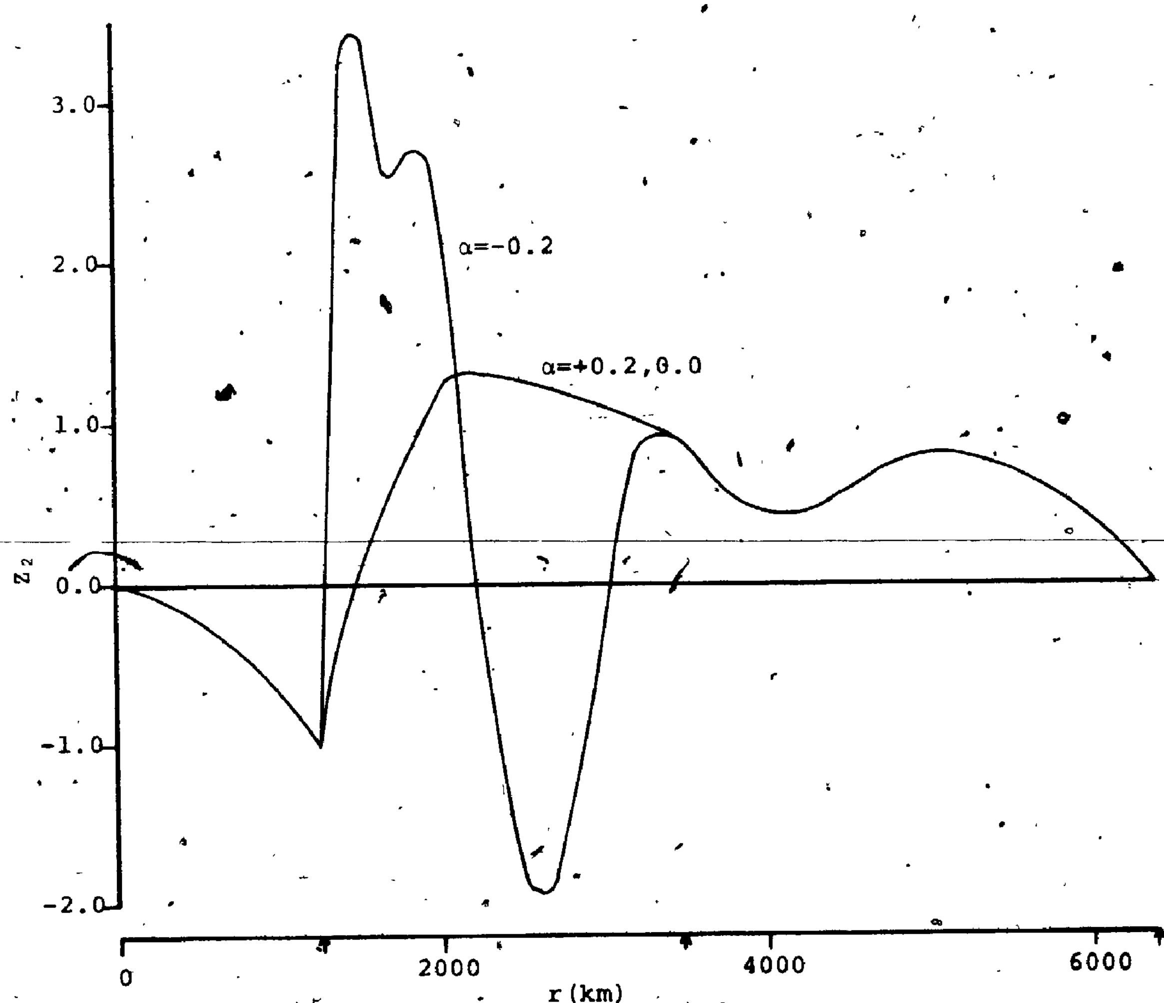

Figure 20. The normal stress Z_2 for the long period tide M_f for earth models with α=+0.2,0.0,-0.2. Amplitude of the tidal potential is set to unity at the free surface.

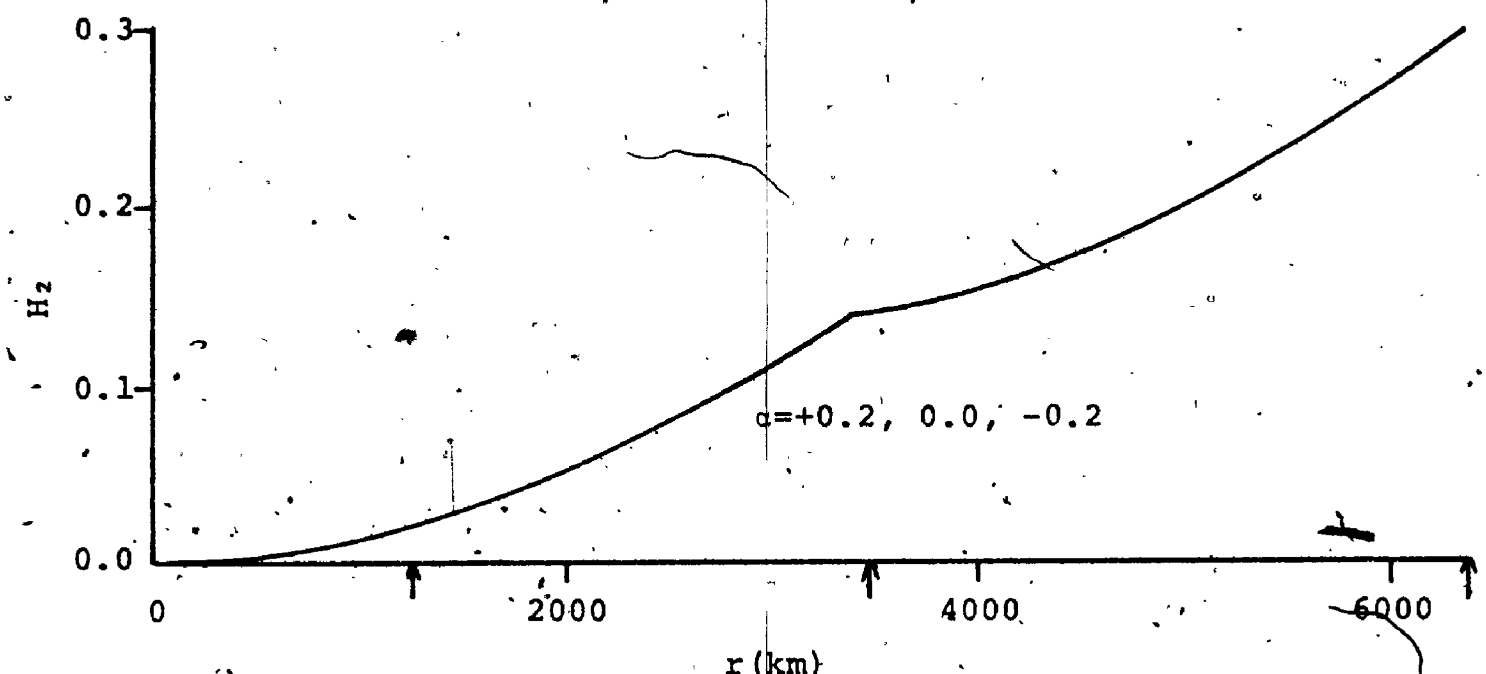

Figure 21. The change in gravitational potential H_2 for the long period tide M_f for the earth models with α=+0.2, 0.0, -0.2. Amplitude of the tidal potential is set to unity at the free surface.

TABLE 8 LONG PERIOD TIDAL LOVE NUMBERS

SYMBOL	DODSON'S ARGUMENT	$\alpha = +0.2$ h	k	l	$\alpha = 0.0$ h	k	l	$\alpha = -0.2$ h	k	l
M_o +	055.555	0.6110	0.3000	0.08410	0.6112	0.3007	0.08420	0.6114	0.3014	0.08431
S_o +	065.555	0.6110	0.3000	0.08410	0.6112	0.3007	0.08420	0.6114	0.3014	0.08431
S_a	056.554	0.6111	0.3001	0.08411	0.6113	0.3008	0.08421	0.6115	0.3015	0.08432
S_{sa}	057.555	0.6111	0.3001	0.08411	0.6113	0.3008	0.08421	0.6115	0.3015	0.08432
M_m	065.455	0.6111	0.3001	0.8411	0.6113	0.3008	0.08421	0.6115	0.3015	0.08432
M_f	075.555	0.6111	0.3001	0.8411	0.6113	0.3008	0.08421	0.6115	0.3015	0.08432

+ Mo and So are permanent deformation (frequency = 0).

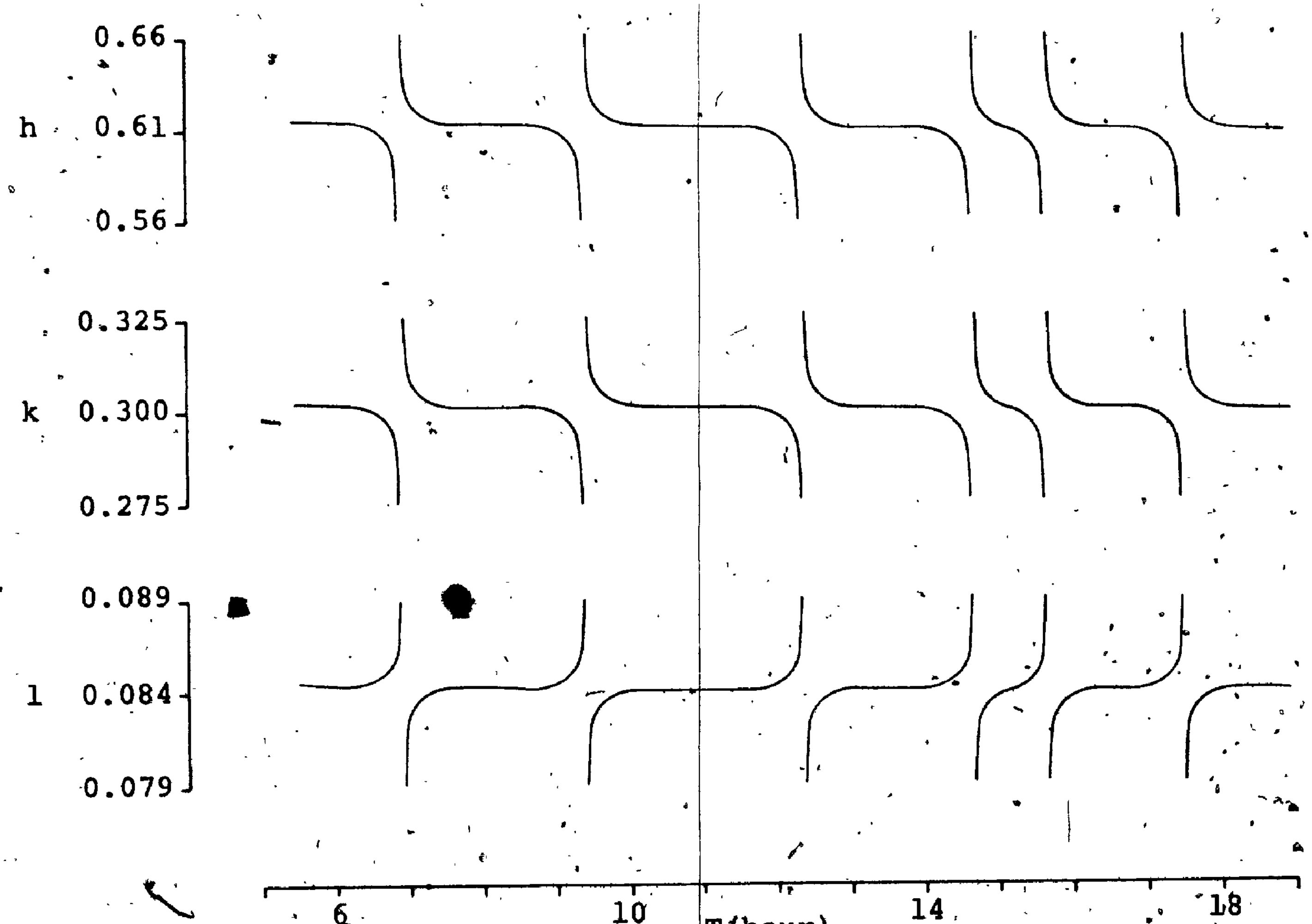

Figure 22. The zonal (N=2,M=0) tidal Love numbers as functions of period for the earth model with $\alpha=+0.2$.

periods shorter than or equal to roughly 53.7 minutes.

The core modes are mainly governed by the gravitational force in the liquid outer core and have periods extending from roughly 7 hours to infinity. The core motion is primarily spheroidal. These modes are the only ones which depend strongly on the structure of the outer core.

The toroidal modes are due to the rotation of the outer core relative to the solid mantle. In the case of T_1 mode, for example, the outer core rotates as a solid as exhibited by curve 5 in figure 4.

Under the applications of external forces, resonances occur near the frequencies of free oscillations. The validity of the present theory is demonstrated through diurnal earth tides which have frequencies near a T_1-mode with period equal to 23.88337 hours, the resonance effect being observed astronomically, through the associated nutations.

Since core modes depend strongly on the nature of density stratification in the liquid core, they should be investigated more thoroughly for a better understanding of the core. A core mode with azimuthal number $m = 2$ and period 12.1980 hours exists for the stable core with $\alpha = +0.2$ and not for unstable and neutral stable cores. Therefore semi-diurnal tides deserve a detailed study.

APPENDIX A

ROTATIONAL DEFORMATION

The problem we have here is to find the ellipticities of surfaces of equal density within the earth, assuming the earth is in hydrostatic equilibrium.

The classical method consists of solving Clairaut's differential equation

$$\frac{d^2e}{dr^2} - \frac{n(n+1)}{r^2} e + \frac{8\pi G\rho_o}{g} \left(\frac{de}{dr} + \frac{e}{r}\right) = 0, \qquad (A.1)$$

with the arbitrary constant determined by the observed surface ellipticity.

In (A.1), e is the ellipticity, ρ_o the density, g the gravity, and n = 2.

Detailed theory can be seen in The Earth (Jeffreys, 1959). One inconsistency in this theory is that equation (A.1) is derived from hydrostatic theory, but the observed surface ellipticity is not the hydrostatic value.

In the following, a self-consistent method is described.

The problem of ellipticity is equivalent to the problem of deformation when an initially spherically symmetric earth is subjected to rotation.

Therefore, the deformation theory developed in Chapter 2 and 3 can be applied.

In this case, the external potential is the centrifugal potential.

$$W_e = \frac{1}{2} \omega^2 r^2 \sin^2 \theta, \quad (A.2)$$

which can be written as

$$W_e = W_R + W_2, \quad (A.3)$$

with

$$W_R = \frac{1}{3} \omega^2 r^2 P_o^o (\cos \theta), \quad (A.4)$$

$$W_2 = K (r) P_2^o (\cos \theta), \quad (A.5)$$

$$K = - \frac{1}{3} \omega^2 r^2. \quad (A.6)$$

The purely radial displacement resulting from W_R is small due to the large incompressibility of the earth. Therefore, as far as ellipticity is concerned, the effect of W_R can be neglected.

The potential W_2 is a solid spherical harmonic of $n = 2$, $m = 0$. Due to its static character ($\sigma = 0$), there is no coupling among the displacement fields.

$$U_r = U_2(r)\, P_2^o\ (\cos\theta),$$

$$U_\theta = V_2(r)\, \frac{\partial}{\partial\theta} P_2^o\ (\cos\theta), \qquad (A.7)$$

$$U_\psi = 0.$$

$$W_a = \dot{H}_2(r)\, P_2^o\ (\cos\theta). \qquad (A.8)$$

$$F_r = \frac{\partial W_2}{\partial r} = \frac{d}{dr} K(r)\, P_2^o\ (\cos\theta)$$

$$F_\theta = \frac{1}{r}\frac{\partial W_2}{\partial\theta} = \frac{1}{r} K(r)\, \frac{\partial}{\partial\theta} P_2^o\ (\cos\theta).$$

Due to the assumption of hydrostatic equilibrium, we must set $\mu = 0$. Then equations (2.32), (2.33) and (2.34) become

$$D_2 = H_2 + K, \qquad (A.9)$$

$$D_2 + \frac{\lambda}{\rho_o}\Delta_2 - g\, U_2 = 0, \qquad (A.10)$$

$$\frac{d}{dr}\left[D_2 + \frac{\lambda}{\rho_o}\Delta_2 - g\, U_2\right] + \alpha\, g\, \Delta_2 = 0. \qquad (A.11)$$

And (2.21) becomes

$$\ddot{D}_2 + \frac{2}{r}\dot{D}_2 + \left(\frac{4\pi G\rho_o{}'}{W_o{}'} - \frac{6}{r^2}\right) D_2 = -4\pi G\rho_o\, \alpha\, \Delta_2. \qquad (A.12)$$

The boundary condition (3.34) and (3.35) are

$$
\left.
\begin{aligned}
&Z_z = \lambda\Delta_2 = 0, \\
&H_2 + \frac{d}{3}(\dot{H}_2 - 4\pi G\rho_o U_2) = 0,
\end{aligned}
\right\} \tag{A.13}
$$

at the free surface.

Notice that Δ_2 is related to U_2, and V_2 by (3.26),

$$\Delta_2 = \dot{U}_2 + \frac{2}{r} U_2 - \frac{6}{r} V_2. \tag{A.14}$$

Equations (A.9) - (A.14) should completely determine the solutions in principle. However, a careful study of the equations shows that some precautions must be taken. Equations (A.10) and (A.11) lead to the condition

$$\alpha\Delta_2 = 0, \tag{A.15}$$

where α is given by (2.35),

$$\alpha(r) = \frac{\rho_o'}{\rho_o^2}\frac{\lambda}{W_o'} - 1. \tag{A.16}$$

If $\alpha \neq 0$ throughout the entire earth, then $\Delta_2 \equiv 0$, and the deformation is completely governed by the gravitational force. However, if $\alpha = 0$, an arbitrary constant can appear in the mathematical solutions. Therefore, the physical argument that Δ_2 is continuous when α approaches zero is necessary. Then for any $\alpha(r)$, we can take $\Delta_2 = 0$. Thus U_2 is related to D_2 by

$$U_2 = \frac{1}{g} D_2, \quad (A.17)$$

and the ellipticity is given by

$$e(r) = \frac{1}{r} \left\{ U_r'\left(\theta = \frac{\pi}{2}\right) - U_r(\theta = 0) \right\},$$

or

$$e(r) = -\frac{3}{2} \frac{U_2}{r} = -\frac{3}{2} \frac{D_2}{rg} . \quad (A.18)$$

We notice that the equivalence of (A.1) and (A.12) can be shown by substituting (A.18) into (A.1).

APPENDIX B

NUMERICAL INTEGRATION

1. Initial Values

The center of the earth is a singular point for the set of differential equations (3.28). To avoid this singular point, we must start the numerical integrations at a finite distance r_i from the center. The approximate solutions of (3.28) at a sufficiently small r_i can be obtained in several ways. If we assume that the earth is homogeneous within r_i, exact solutions of (3.28) for $r < r_i$ can be found (Love 1911). The three independent solutions are BB_1, BB_2, and BB_3 given below.

(i) BB_1

$$_1U_n = n\ _1V_n + a\ r^{n+1}\ \psi_{n+1}(\alpha r),$$

$$_1Z_n = n\ _1Y_n - r^n \left[\mu\ a \left(2(n+2)\ \psi_{n+1}(\alpha r) + \psi_n(\alpha r) \right) + (\lambda + \mu)\ \frac{\alpha^2}{4\pi G\rho_0}\ \psi_n(\alpha r) \right],$$

$$_1V_n = r^{n-1} \left[\frac{a}{\alpha^2}\ \psi_n(\alpha r) + \left(\frac{a}{n\alpha^2} - \frac{1}{4\pi G\rho_0 n} \right) \psi_{n-1}(\alpha r) \right],$$

$$ {}_1Y_n = \mu \, r^{n-2} \left\{ \frac{a}{\alpha^2} \left[\left(\frac{\alpha^2 r^2}{n} - 2(n-2) \right) \right. \right. \quad \text{(B.1)} $$

$$ \left. \psi_n(\alpha r) - \frac{2}{n} \psi_{n-1}(\alpha r) \right] - \frac{1}{4\pi G \rho_0} $$

$$ \left. \left[\frac{\alpha^2 r^2}{n} \psi_n(\alpha r) + \frac{2(n-1)}{n} \psi_{n-1}(\alpha r) \right] \right\}, $$

$$ {}_1H_n = r^n \psi_n(\alpha r), $$

$$ {}_1Q_n = r^{n-1} \left[\alpha^2 r^2 \psi_{n+1}(\alpha r) + n \psi_n(\alpha r) \right] - $$

$$ 4\pi G \rho_0 \, {}_1U_n, $$

where $-\alpha^2$ is the negative root of the equation

$$ \mu(\lambda + \mu) x^2 + \left[\frac{16}{3} \pi G \rho_0^2 \mu + (\lambda + 3\mu) \sigma^2 \rho_0 \right] x - $$

$$ \left[n(n+1) \left(\frac{4}{3} \pi G \rho_0^2 \right)^2 - \frac{16}{3} \pi G \rho_0^2 \sigma^2 \rho_0 - \sigma^4 \rho_0^2 \right] = 0, \quad \text{(B.2)} $$

$$a = \frac{\alpha^2}{4\pi G\rho_0}\left[1 + \frac{4\pi G\rho_0^2 n}{\mu\ \alpha^2}\right],$$

and

$$\psi_n(x) = \left\{\frac{1}{x}\ \frac{d}{dx}\right\}^n \frac{\sin x}{x}.$$

(ii) BB_2

$$_2U_n = n\ _2V_n + b\ r^{n+1}\ \chi_{n+1}(\beta r),$$

$$_2Z_n = n\ _2Y_n - r^n\left[\mu\ b\left[2(n+2)\ \chi_{n+1}(\beta r) - \chi_n(\beta r)\right] - (\lambda + \mu)\frac{\beta^2}{4\pi G\rho_0}\ \chi_n(\beta r)\right],$$

$$_2V_n = r^{n-1}\left[\frac{b}{\beta^2}\ \chi_n(\beta r) - \left(\frac{b}{n\beta^2} - \frac{1}{4\pi G\rho_0 n}\right)\ \chi_{n-1}(\beta r)'\right],$$

(B.3)

$$_2Y_n = \mu\ r^{n-2}\left\{-\frac{b}{\beta^2}\left[\left(\frac{\beta^2 r^2}{n} + 2(n+2)\right)\ \chi_n(\beta r) - \right.\right.$$

$$\frac{2}{n}\,\chi_{n-1}(\beta r)\Bigg] + \frac{1}{4\pi G\rho_0}\left[\frac{\beta^2\,r^2}{n}\,\chi_n(\beta r) + \right.$$

$$\left.\frac{2(n-1)}{n}\,\chi_{n-1}(\beta r)\right]\Bigg\},$$

$${}_2H_n = r^n\,\chi_n(\beta r),$$

$${}_2\Omega_n = r^{n-1}\left[\beta^2\,r^2\,\chi_{n+1}(\beta r) + n\,\chi_n(\beta r)\right] - 4\pi G\rho_0\,{}_2U_n,$$

where β^2 is the positive root of the equation (B.2),

$$b = \frac{\beta^2}{4\pi G\rho_0}\left[1 - \frac{4\pi G\rho_0^2 n}{3\mu\,\beta^2}\right],$$

and

$$\chi_n(x) = \left[\frac{1}{x}\,\frac{d}{dx}\right]^n \frac{\sinh x}{x}.$$

(iii) BB_3

$${}_3U_n = n\,{}_3V_n,$$

$$_3Z_n = n \; _3Y_n,$$

$$_3V_n = c \; r^{n-1},$$

$$_3Y_n = 2(n-1) \; \mu \; c,$$

$$_3H_n = r^n, \qquad \text{(B.4)}$$

$$_3Q_n = n \; r^{n-1} - 4\pi G\rho_0 \; _3U_n,$$

where

$$c = \frac{3}{4\pi G\rho_0 \; n} .$$

We must point out here that the three independent solutions BB_1, BB_2, and BB_3 given above apply only when the equation (B.2) possesses a positive and a negative root. When σ is sufficiently large, both of the roots of (B.2) will be negative. In this case the solution BB_2 will also assume the form of BB_1. Finally when σ is such that

$$n(n+1) \; \left(\frac{4}{3} \pi G \; \rho_0^2 \right)^2 - \frac{16}{3} \pi G \rho_0^2 \; \sigma^2 \; \rho_0 = 0,$$

the equation (B.2) possesses a zero root. In this case the solution BB_2 must be replaced by BB_4 which is given by

$$
\begin{aligned}
{}_4U_n &= n\ {}_4V_n + d\ r^{n+1}, \\
{}_4Z_n &= n\ {}_4Y_n + r^n \left[(\lambda + \mu) \frac{2n+3}{2\pi G \rho_0} - \mu\, d \right], \\
{}_4V_n &= \left(\frac{1}{4\pi G\rho_0} + \frac{n+3}{3\sigma^2} \right) r^{n+1} + e\ r^{n-1}, \\
{}_4Y_n &= \mu\ r^{n-2} \left\{ \left[\frac{2n+3}{2\pi G\rho_0} - (n+2)\ d \right] r^2 + 2(n-1)\ e \right\}, \\
{}_4H_n &= r^{n+2}, \\
{}_4Q_n &= (n+2)\ r^{n+1} - 4\pi G\rho_0\ {}_4U_n,
\end{aligned}
\tag{B.5}
$$

where

$$d = \frac{1}{2\pi G\rho_0} - \frac{2}{3}\frac{n}{\sigma^2},$$

and

$$e = -\frac{2n+3}{2\pi G\rho_0^2 \left(\sigma^2 - \frac{4}{3}\pi G\rho_0 n \right)} \left[\lambda + 2\mu + \frac{\mu}{n} \left(\frac{3\sigma^2}{4\pi G\rho_0} + 4 \right) \right].$$

Another way to obtain the approximate solutions of (3.28) at small r_i is to expand U_n, Z_n, etc. in power serieses. Retaining terms to 4th power in r_i, we find in the force free cases,

$$U_n = n\, a_{31}\, r_i + a_{13}\, r_i^3,$$

$$Z_n = n \left(-\,(n-2)\, \lambda + 2\, \mu \right) a_{31} + a_{22}\, r_i^2,$$

$$V_n = a_{31}\, r_i + a_{33}\, r_i^3,$$

$$Y_n = n\, \mu\, a_{31} + a_{42}\, r_i^2, \qquad \text{(B.6)}$$

$$H_n = a_{52}\, r_i^2 + a_{54}\, r_i^4,$$

$$Q_n = 2\, (-\,2\, n\, \pi\, G\, \rho_0\, a_{31} + a_{52})\, r_i + a_{63}\, r_i^3,$$

where

$$\det = (n^2 + n)\, \lambda + (n^2 + n - 6)\, \mu,$$

$$k = \rho_0 \left\{ a_{52} + \rho_0 \left(\sigma^2 - \frac{2}{3} n^2\, \pi\, G \right) a_{31} \right\},$$

$$4 \det a_{22} = -\,4\, (n^2 + n)\, \lambda\, k + \mu \left\{ \left((n^2 + n)^2 + \right.\right.$$

$$10\,(n^2 + n) - 120\Big]\,\lambda + 24\,(n^2 + n - 6)\,\mu\Big\},$$

$$4 \det a_{33} = 4\,k + \Big[\,20\,\lambda - (n^2 + n - 34)\,\mu\,\Big]\,a_{13},$$

$$2 \det a_{42} = 4\,\mu\,k + \mu\,\Big[\,2\,(n^2 + n + 10)\,\lambda + (n^2 + n +$$

$$22)\,\mu\,\Big]\,a_{13},$$

$$14 \det a_{54} = \pi\,G\,\rho_0\,\Big[-4\,n\,(n+1)\,k + (n^2 + n + 6)$$

$$(n^2 + n - 20)\,\mu\,a_{13}\Big],$$

$$7 \det a_{63} = 2\,\pi\,G\,\rho_0\,\Big\{-4\,n\,(n+1)\,k + \Big[-14\,(n^2 +$$

$$n)\,\lambda + \Big(\,(n^2 + n)^2 - 15\,(n^2 + n) - 114\,\Big)\Big]\,a_{13}\Big\}.$$

The three independent constants in (B.6) are a_{31}, a_{52}, and a_{13}.

We note here that (B.6) can be obtained from (B.1) to (B.5) by power series expansion. In the present work, (B.1) to (B.5) have been used.

The approximate solutions for (3.28) at r_i for forced oscillations are harder to obtain because particular solutions must be found. However, in the present work, this problem can be avoided. We deal only with external forces which are derivable from potentials satisfying the Laplace equation. Such external forces can be made implicit in (3.28) by including the corresponding external potentials in H_n and Q_n. The problem is then equivalent to a force free case.

2. Integration by Runge-Kutta Method

The propagator matrix formalism (Gilbert and Backus 1966) was used in the numerical integration. The three independent constants are propagated from r_i to the top of the inner core where one of the free constants is determined by the requirement that the transverse stress vanishes. The remaining two free constants are propagated through the outer core. At the bottom of the mantle a free constant is introduced to account for the transverse displacement. The new set of three free constants are then propagated through the mantle and determined at the free surface by the conditions (3.34) and (3.35).

At this point, we must bring attention to the equations given in section 4.2.1. Due to the non-vanishing constant ε, particular integrals must be evaluated in the outer core. Also, due to the continuity conditions at the outer core-mantle boundary, the constant ε is propagated through the mantle as a free constant. This additional constant is determined by the condition (2.42).

APPENDIX C

THE NOTATIONS AND FORMALISM OF THE DIFFERENTIAL EQUATIONS FOR THE OUTER CORE

In the studies of the free oscillations of the earth, the differential equations governing the deformations are generally presented in a standard form (e.g., Smylie and Mansinha 1971) using the y notations introduced by Alterman, Jarosch, and Pekeris (1959). However, in the present work, it is immediately obvious from an inspection of the equations given in section 3.3 that such a formalism is no longer convenient. The necessity for changing the formalism arises mainly from the effects of introducing the ellipticity. For example, due to the ellipticity, the functions $\eta_n(r)$ and $X_n(r)$ do not have analogies in the y notations. Expanding these functions using (3.27) and (3.18) respectively would unnecessarily lengthen the differential equations. Moreover, since η_n and X_n, and their first derivatives both appear in the differential equations, a reduction of the equations to the standard form would involve tedious algebraic manipulations. These are the obvious reasons that we put the equations in the present forms.

In fact, even if it is possible to put the present equations in the standard form, there is not much meaning in doing so. The free core oscillations, which is the principal subject of the present study, are so strongly influenced by the ellipticity that comparisons of the results with those obtained from a spherical earth are not only impossible but also meaningless. We shall discuss this point in details in Appendix D.

The following identifications may be helpful for readers who are familiar with the y notations.

$$U_n = y_1^n,$$

$$\lambda_s \Delta_n = y_2^n,$$

$$V_n = y_3^n,$$

$$H_n = y_5^n,$$

$$\dot{H}_n - 4 \pi G \rho_s U_n = y_6^n.$$

APPENDIX D

THE TRUNCATION OF THE HYDRODYNAMIC EQUATIONS

I. In the present work, the truncation of the differential equations for the outer core is governed by the approximation made in the mantle and inner core where we neglect the effects of the rotation and ellipticity of the earth. Let us consider a specific example. Suppose we are interested in a spheroidal oscillation of the earth of degree 2 and azimuthal number m, then in the mantle and inner core, the coupling effects from the spheroidal fields S_ℓ^m ($\ell \neq n$) and toroidal fields T_ℓ^m (ℓ any) are neglected. The continuity conditions at the outer core boundaries then force us to neglect the S_ℓ^m ($\ell \neq n$) in the outer core. Now since the toroidal fields T_ℓ^m ($\ell \leq n-3$, or $\ell \geq n+3$) are coupled to S_n^m through S_ℓ^m ($\ell \neq n$), we must also neglect their effects. This leaves us with the coupling effects from T_{n-1}^m and T_{n+1}^m. At this point we must note that the coupling among the displacements T_{n-1}^m, S_n^m, and T_{n+1}^m is of zero order in ellipticity (and / or rotation) and not of first order. Smylie (1974) neglected the coupling effects from the toroidal fields. This leads to zero order errors in the solutions for the outer core and hence zero order errors in the periods of the gravitational undertones.

II. The solutions for the mantle and inner core involve errors of the order of ellipticity due to the neglect of the effects of rotation and ellipticity. As a consequence, in treating the inner core-outer core and outer core-mantle boundary conditions, the same amount of error can be allowed. This is why we can consider the bottom of the mantle as spherical. However, we cannot do the same at the top of the outer core because in the outer core, the effects of ellipticity is of zero order. For example, in the case of the nearly diurnal free spheroidal oscillation with a period of 23.883 hours (p.74), the toroidal field T_1^1 in the outer core is roughly 2.5×10^4 times larger than the spheroidal field S_2^1 (see Figures 4 and 5). Since the ellipticity is about 2.5×10^{-3} at the top of the outer core, T_1^1 contributes to the displacement normal to the core-mantle boundary 62.5 times more than S_2^1 does. In this case, obviously, the ellipticity of the outer core boundary cannot be neglected.

III. At periods comparable to a day, the ellipticity and rotation of the earth play about equal roles in determining the deformation of the outer core. This is obvious from the equation (4.16). The quantity b/r^2 is of the order of 3 ω. Therefore, as the period of oscillation increases the ellipticity gradually becomes the dominating factor in determining the deformation of the outer core. Crossley (1974) did not take the effects of ellipticity into account. It is therefore not surprising that he obtains drastically different results as compared to the present ones. Smylie (1974) and Crossley (1974) both suggested the existence of an upper limit for the period of gravitational undertones, while the present work does not. The difference obviously arises from the consideration of the effects of the ellipticity.

IV. The toroidal modes of free oscillation of the earth discussed in section 4.6.2 are inertial oscillations of the outer core as considered by Greenspan (1965), Aldridge and Toomre (1969), and Aldridge (1972). However, for the real earth, the inertial oscillations are strongly affected by the compressibility and ellipticity of the outer core, as well as the elasticity of the mantle. This is clearly demonstrated in figures 4, 5, 6, and 7 (curve 5). Apart from the rotation of the outer core relative to the mantle and inner core, the deformation of the earth for the T_1 toroidal mode resembles those of elastic modes. Obviously, it is insufficient to consider the outer core as imcompressible and the mantle as rigid.

The existence of the 23.883 hour inertial oscillation can be predicted from the forms of the equations (4.14) and (4.15). The factor $\omega\sigma + \sigma^2$ appears in front of T_1 in both equations. T_1 is therefore expected to become very large when σ approaches $-\omega$. The readers are reminded that we have taken σ negative throughout the present work.

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