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THE CATEGORY OF NODE-AND-CHOICE FORMS FOR EXTENSIVE-FORM GAMES

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Abstract. It would be useful to have a category of extensive-form games whose isomorphisms specify equivalences between games. Toward this goal, Streufert (2016) introduced the category of node-and-choice preforms, where a "preform" is a rooted tree together with choices and information sets. This paper takes another step: It introduces the category of node-and-choice forms, where a "form" is a preform augmented with players. In addition, it derives some consequences of the category's morphisms and provides a characterization of the category's isomorphisms.

1. Introduction

Category theory has been used to systematize many other subject areas. There is, for example, the category of graphs whose morphisms allow one to systematically compare graphs. Similarly, it would be useful to have a category of extensive-form games whose morphisms would allow one to systematically compare extensive-form games.[1] As yet, little has been done. Lapitsky (1999) and Jiménez (2014) define categories of normal-form games. Machover and Terrington (2014) defines a category for simple voting games. Finally, Vannucci (2007) defines categories of various kinds of games, but in its category of extensive-form games, every morphism merely maps a game to itself.

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[1] Extensive-form games are not readily comparable with the games defined in the theoretical computer-science literature. Categories of such games are developed in McCusker (2000), Abramsky, Jagadeesan, and Malacaria (2000), and Hyland and Ong (2000).

Building a category of extensive-form games with nontrivial morphisms is a large project because each extensive-form game has so many components: each is a rooted tree with choices, information sets, players, chance probabilities, and preferences. Accordingly, Streufert (2016, henceforth `S16`) took a first step by introducing a category of preforms, where a "preform" is a rooted tree with choices and information sets. The present paper takes a second step by introducing a category of forms, where a "form" is a preform augmented with players.

In particular, Section 2 introduces (node-and-choice)[2] forms. Each form incorporates an `S16` preform with its node set T and choice set C. In addition, each form specifies an allocation $(C_i)_i$ of those choices to the players i in the player set I. Proposition 2.1 shows that this allocation of choices across players entails [a] an allocation of decision nodes across players and [b] an allocation of information sets across players.

Section 3 introduces form morphisms. Each morphism takes an "old" form to "new" form. Each morphism incorporates an `S16` preform morphism which transforms the old form's preform into the new form's preform. In addition, it specifies a function which transforms old players to new players. Proposition 3.1 shows that such a form morphism entails [a] a certain relationship between the old allocation of decision nodes and the new allocation of decision nodes, and [b] a certain relationship between the old allocation of information sets and the new allocation of information sets.

Section 4 introduces the category of (node-and-choice)[2] forms. It incorporates the forms of Section 2 and the form morphisms of Section 3. Theorem 1 shows that the category is well-defined. Further, Theorem 2 characterizes the category's isomorphisms. In particular, it shows that a morphism is an isomorphism iff its node, choice, and player transformation functions are bijections. Both directions of this characterization are useful. The reverse direction helps one to establish that a morphism is an isomorphism. Meanwhile, the forward direction helps one to derive further properties of isomorphisms. For example,

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[2] I use the adjective "node-and-choice" to distinguish [a] the preforms and forms in `S16` and here from [b] the less abstract preforms and forms in Streufert (2015b, 2015c). Here in this paper, "preform" always means "node-and-choice preform", and "form" always means "node-and-choice form".

Proposition 4.2 uses the forward direction to show how a form isomorphism preserves [a] the allocation of choices, [b] the allocation of decision nodes, and [c] the allocation of information sets.

Work is currently underway to prove that this paper's node-and-choice forms are general enough to encompass the formulations used in the extensive-form games of von Neumann and Morgenstern (1944), Kuhn (1953), Osborne and Rubinstein (1994), Alós-Ferrer and Ritzberger (2013), and Streufert (2015b). When that work is completed, fundamental equivalences across these various formulations can be established as isomorphisms within the category of node-and-choice forms.

Further, the obvious sequel to this paper is to develop a category of extensive-form games. That category of games will incorporate this paper's category of forms just as this paper's category of forms incorporates `S16`'s category of preforms. In addition, the category of games will specify chance probabilities and player preferences. Work on this is also underway.

2. Forms

2.1. Definition

This paragraph recalls the definition of a node-and-choice preform from `S16` Section 2.1. Let T be a set and call $t \in T$ a *node*. Let C be a set and call $c \in C$ a *choice*. A *(node-and-choice)*[2] *preform* $\mathit{\Pi}$ is a triple $(T, C, \otimes)$ such that

$$(\exists F \subseteq T \times C)(\exists t^o \in T) \quad \otimes \text{ is a bijection from } F \text{ onto } T \smallsetminus \{t^o\}\ , \tag{1a}$$

$$(T, p) \text{ is a tree oriented toward } t^o \quad \text{where } p := \{(t^\sharp, t) | (\exists c)(t, c, t^\sharp) \in \otimes\}\ , \text{ and} \tag{1b}$$

$$\mathcal{H} \text{ partitions } F^{-1}(C) \quad \text{where } \mathcal{H} := \{F^{-1}(c) | c\}\ . \tag{1c}$$

`S16` Section 2 discusses this definition in detail. There, $\otimes$ is called the *node-and-choice operator*, F is called the *feasibility* correspondence, t^o is called the *root* node, p is called the *immediate-predecessor* function, $F^{-1}(C)$ is called the set of *decision nodes*, and $\mathcal{H}$ is called the collection of *information sets*.

This paragraph introduces the definition of a node-and-choice form. Let I be a set and call $i \in I$ a *player*. For each i, let C_i be a set and call $c_i \in C_i$ a choice of player i. A *(node-and-choice)*[2] *form* Φ is a quadruple $(T, I, (C_i)_i, \otimes)$ such that

(2a) $$\Pi := (T, C, \otimes) \text{ is a preform where } C := \cup_i C_i \ ,$$

(2b) $$(C_i)_i \text{ is an indexed prepartition of } C \ , \text{ and}$$

(2c) $$(\forall i)(\forall t) \ F(t) \subseteq C_i \text{ or } F(t) \cap C_i = \varnothing \ ,$$

where an *indexed prepartition* $(A_i)_i$ of a set A is a function from I to the power set of A such that $(\forall i \neq j) \ A_i \cap A_j = \varnothing$ and $\cup_i A_i = A$. Note that (2b) is equivalent to $(\forall i \neq j) \ C_i \cap C_j = \varnothing$ because (2a) defines C to be $\cup_i C_i$.

Essentially, a form $(T, I, (C_i)_i, \otimes)$ allocates the choices in the preform $(T, C, \otimes)$ to the players i by means of their choice sets C_i. Accordingly, a preform can be understood as a one-player form [to be precise, $(T, C, \otimes)$ is a preform iff $(T, \{1\}, (C), \otimes)$ is a form, where $(C_i)_i = (C)$ is taken to mean $C_1 = C$].

The following proposition shows that the allocation $(C_i)_i$ of choices to players entails [a] an allocation of decision nodes to players and [b] an allocation of informations sets to players. The proposition's proof relies heavily on (2c).

To define terms, consider any player i. $F^{-1}(C_i)$ is player i's set of decision nodes. Further, define

(3) $$\mathcal{H}_i = \{ \ F^{-1}(c) \mid c \in C_i \ \} \ .$$

$\mathcal{H}_i$ is player i's collection of information sets. By inspecting (3), $\cup \mathcal{H}_i = F^{-1}(C_i)$. Further, (3) and (1c) imply that $\mathcal{H}_i$ consists of nonempty disjoint sets. Thus the last two sentences imply that

(4) $$\mathcal{H}_i \text{ partitions } F^{-1}(C_i) \ .$$

This is like (1c), but for player i. It states that player i's collection of information sets partitions her set of decision nodes.

Proposition 2.1. *Suppose* $(T, I, (C_i)_i, \otimes)$ *is a form (2) with its* C *(2a),* F *(1a),* $\mathcal{H}$ *(1c), and* $(\mathcal{H}_i)_i$ *(3). Then*

(a) $(F^{-1}(C_i))_i$ *is an indexed prepartition of* $F^{-1}(C)$*.*

(b) $(\mathcal{H}_i)_i$ *is an indexed prepartition of* $\mathcal{H}$*. (Proof A.1.)*

Incidentally, equation (2b) and the above definition of an indexed prepartition allow a player to have an empty C_i. This can be useful.

For example, it admits the possibility of a vacuous chance player. In particular, one could [1] require that the player set I always contains a chance player i^o, and then [2] set $C_{i^o} = \varnothing$ to model the special case of no randomness. In the special case [2], one would have [a] $F^{-1}(C_{i^o}) = \varnothing$ (that the chance player has no decision nodes), and [b] $\mathcal{H}_{i^o} = \varnothing$ (that the chance player has no information sets). The possibility of a vacuous chance player can simplify notation (e.g. Streufert (2015a, page 38, last paragraph)).

Finally, notice that a form Φ determines many entities. In particular, Φ is itself a quadruple $(T, I, (C_i)_i, \otimes)$. Further, it determines C (2a), Π (2a), F (1a), t^o (1a), p (1b), $\mathcal{H}$ (1c), and $(\mathcal{H}_i)_i$ (3). In addition, the preform Π determines k, $\prec$, $\preccurlyeq$, $\mathcal{Z}$, $\mathcal{Z}_{\mathsf{ft}}$, $\mathcal{Z}_{\mathsf{inft}}$, E, and q by S16 (3)–(7) and (9) (some of these additional entities make occasional appearances in this paper's proofs).

2.2. Examples

Here are three example forms. Define the "two-player centipede" form Φ^{two} by

(5)
$$\begin{aligned} T^{\mathsf{two}} &= \{1,2,3\} \cup \{\bar{1},\bar{2}\} \ , \\ I^{\mathsf{two}} &= \{1,2\} \ , \\ (\forall i)\ C_i^{\mathsf{two}} &= \{i\mathsf{go}, i\mathsf{stop}\} \ , \text{ and} \\ \otimes^{\mathsf{two}} &= \{\ (1,1\mathsf{stop},\bar{1}),\ (1,1\mathsf{go},2),\ (2,2\mathsf{stop},\bar{2}),\ (2,2\mathsf{go},3)\ \} \ . \end{aligned}$$

Define the "many-player centipede" form Φ^{many} by

(6)
$$\begin{aligned} T^{\mathsf{many}} &= \{1,2,3,...\} \cup \{\bar{1},\bar{2},\bar{3},...\} \ , \\ I^{\mathsf{many}} &= \{1,2,3,...\} \ , \\ (\forall i)\ C_i^{\mathsf{many}} &= \{i\mathsf{stop}, i\mathsf{go}\} \ , \text{ and} \\ \otimes^{\mathsf{many}} &= \{\ (1,1\mathsf{stop},\bar{1}),\ (1,1\mathsf{go},2),\ (2,2\mathsf{stop},\bar{2}),\ (2,2\mathsf{go},3),\ ...\ \} \ . \end{aligned}$$

Finally, define the "planner centipede" form Φ^{planner} by

(7)
$$\begin{aligned} T^{\mathsf{planner}} &= \{1,2,3,...\} \cup \{\bar{1},\bar{2},\bar{3},...\} \ , \\ I^{\mathsf{planner}} &= \{\mathsf{planner}\} \ , \\ C_{\mathsf{planner}}^{\mathsf{planner}} &= \{1\mathsf{stop}, 1\mathsf{go}, 2\mathsf{stop}, 2\mathsf{go},\ ...\ \} \ , \text{ and} \\ \otimes^{\mathsf{planner}} &= \{\ (1,1\mathsf{stop},\bar{1}),\ (1,1\mathsf{go},2),\ (2,2\mathsf{stop},\bar{2}),\ (2,2\mathsf{go},3),\ ...\ \} \ . \end{aligned}$$

These examples will appear again.

3. Morphisms

3.1. Definition

As in S16 Section 3.1, a *preform morphism* α is a quadruple $[\mathit{\Pi},\mathit{\Pi}',\tau,\delta]$ such that $\mathit{\Pi} = (T,C,\otimes)$ and $\mathit{\Pi}' = (T',C',\otimes')$ are preforms (1),

(8a) $$\tau{:}T{\to}T' \ ,$$

(8b) $$\delta{:}C{\to}C' \ , \text{ and}$$

(8c) $$\{\ (\tau(t),\delta(c),\tau(t^\sharp)) \mid (t,c,t^\sharp){\in}{\otimes}\ \} \subseteq \otimes' \ .$$

A *(form) morphism* β is a quintuple $[\mathit{\Phi},\mathit{\Phi}',\tau,\iota,\delta]$ such that $\mathit{\Phi} = (T,I,(C_i)_i,\otimes)$ and $\mathit{\Phi}' = (T',I',(C'_{i'})_{i'},\otimes')$ are forms (2),

(9a) $$[\mathit{\Pi},\mathit{\Pi}',\tau,\delta] \text{ is a preform morphism where}$$
$$\mathit{\Phi} \text{ determines } \mathit{\Pi} \text{ (2a) and } \mathit{\Phi}' \text{ determines } \mathit{\Pi}' \text{ (2a) ,}$$ [3]

(9b) $$\iota{:}I{\to}I' \ , \text{ and}$$

(9c) $$(\forall i)\ \delta(C_i) \subseteq C'_{\iota(i)} \ .$$

S16 Propositions 3.1 and 3.2 show that a preform morphism has many properties. Thus (9a) implies that a form morphism inherits all these properties.

In addition, a form specifies an allocation $(C_i)_i$ of choices to players. Proposition 2.1 showed that this implies [a] an allocation $(F^{-1}(C_i))_i$ of decision nodes to players and [b] an allocation $(\mathcal{H}_i)_i$ of information sets to players. Accordingly, a morphism between two forms implies relationships between the two forms' allocations of [a] decision nodes and [b] information sets. These relationships are the subject of the following proposition.

Proposition 3.1. *Suppose* $[\mathit{\Phi},\mathit{\Phi}',\tau,\iota,\delta]$ *is a morphism (9), where* $\mathit{\Phi} = (T,I,(C_i)_i,\otimes)$ *determines* F *(1a) and* $(\mathcal{H}_i)_i$ *(3), and where* $\mathit{\Phi}' = (T',I',(C'_{i'})_{i'},\otimes')$ *determines* F' *(1a) and* $(\mathcal{H}'_{i'})_{i'}$ *(3). Then the following hold.*

(a) $(\forall i)\ \tau(F^{-1}(C_i)) \subseteq (F')^{-1}(C'_{\iota(i)})$.

(b) $(\forall i)(\forall H{\in}\mathcal{H}_i)(\exists H'{\in}\mathcal{H}'_{\iota(i)})\ \tau(H) \subseteq H'$. *(Proof B.2.)*

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[3] This phrase is equivalent to defining $\mathit{\Pi}{=}(T,\cup_i C_i,\otimes)$ and $\mathit{\Pi}'{=}(T',\cup_{i'} C'_{i'},\otimes')$. Hence a quintuple $[\mathit{\Phi},\mathit{\Phi}',\tau,\iota,\delta]$ is a form morphism iff [a] $\mathit{\Phi} = (T,I,(C_i)_i,\otimes)$ and $\mathit{\Phi}' = (T',I',(C'_{i'})_{i'},\otimes')$ are forms and [b] (8a)–(8c) and (9b)–(9c) hold when $C = \cup_i C_i$ and $C' = \cup_{i'} C'_{i'}$.

Incidentally, Proposition 3.1 extends similar results for preform morphisms. In particular, part (a) about decision nodes extends `S16` Proposition 3.2(b), and part (b) about information sets extends `S16` Proposition 3.2(j).

3.2. Examples

The following four examples illustrate the definition of a morphism $[\Phi,\Phi',\tau,\iota,\delta]$. They also show that there is no logical connection between [a] the injectivity of ι and [b] the injectivity of τ and δ.

Every morphism from Φ^{two} to Φ^{many} has an injective ι. This follows from Remark B.3, which shows that every morphism from Φ^{two} to Φ^{many} satisfies $\iota(1){+}1=\iota(2)$. A morphism in which τ and δ are injective is

$$\begin{gathered}\tau(\mathsf{1}){=}\mathsf{6},\ \tau(\bar{\mathsf{1}}){=}\bar{\mathsf{6}},\ \tau(\mathsf{2}){=}\mathsf{7},\ \tau(\bar{\mathsf{2}}){=}\bar{\mathsf{7}},\ \tau(\mathsf{3}){=}\mathsf{8},\\ \iota(1){=}6,\ \iota(2){=}7,\ \text{and}\\ \delta(\mathsf{1stop}){=}\mathsf{6stop},\ \delta(\mathsf{1go}){=}\mathsf{6go},\ \delta(\mathsf{2stop}){=}\mathsf{7stop},\ \delta(\mathsf{2go}){=}\mathsf{7go}.\end{gathered}\tag{10}$$

As required by (8c),

$$\begin{gathered}\{\ (\tau(t),\delta(c),\tau(t^\sharp))\mid(t,c,t^\sharp){\in}\otimes^{\mathsf{two}}\ \}\\ =\{\ (\mathsf{6},\mathsf{6stop},\bar{\mathsf{6}}),\ (\mathsf{6},\mathsf{6go},\mathsf{7}),\ (\mathsf{7},\mathsf{7stop},\bar{\mathsf{7}}),\ (\mathsf{7},\mathsf{7go},\mathsf{8})\ \}\\ \subseteq\otimes^{\mathsf{many}}\ .\end{gathered}$$

Further, as required by (9c), $\delta(C_1^{\mathsf{two}})=\{\mathsf{6stop},\mathsf{6go}\}$ is a subset of $C_{\iota(1)}^{\mathsf{many}}=C_6^{\mathsf{many}}$, and $\delta(C_2^{\mathsf{two}})=\{\mathsf{7stop},\mathsf{7go}\}$ is a subset of $C_{\iota(2)}^{\mathsf{many}}=C_7^{\mathsf{many}}$. A morphism in which τ and δ are non-injective is

$$\begin{gathered}\tau(\mathsf{1}){=}\mathsf{6},\ \tau(\bar{\mathsf{1}}){=}\tau(\mathsf{2}){=}\mathsf{7},\ \tau(\bar{\mathsf{2}}){=}\tau(\mathsf{3}){=}\mathsf{8},\\ \iota(1){=}6,\ \iota(2){=}7,\quad\text{and}\\ \delta(\mathsf{1stop}){=}\delta(\mathsf{1go}){=}\mathsf{6go},\ \delta(\mathsf{2stop}){=}\delta(\mathsf{2go}){=}\mathsf{7go}.\end{gathered}$$

As required by (8c),

$$\begin{gathered}\{\ (\tau(t),\delta(c),\tau(t^\sharp))\mid(t,c,t^\sharp){\in}\otimes^{\mathsf{two}}\ \}\\ =\{\ (\mathsf{6},\mathsf{6go},\mathsf{7}),\ (\mathsf{7},\mathsf{7go},\mathsf{8})\ \}\\ \subseteq\otimes^{\mathsf{many}}\ .\end{gathered}$$

Further, as required by (9c), $\delta(C_1^{\mathsf{two}})=\{\mathsf{6go}\}$ is a subset of $C_{\iota(1)}^{\mathsf{many}}=C_6^{\mathsf{many}}$, and $\delta(C_2^{\mathsf{two}})=\{\mathsf{7go}\}$ is a subset of $C_{\iota(2)}^{\mathsf{many}}=C_7^{\mathsf{many}}$.

Every morphism from Φ^{two} to Φ^{planner} is non-injective simply because Φ^{two} has two players and Φ^{planner} has one. A morphism in which τ and

δ are injective is

$$\tau(\mathsf{1})=\mathsf{6},\ \tau(\bar{\mathsf{1}})=\bar{\mathsf{6}},\ \tau(\mathsf{2})=\mathsf{7},\ \tau(\bar{\mathsf{2}})=\bar{\mathsf{7}},\ \tau(\mathsf{3})=\mathsf{8},$$
$$\iota(\mathsf{1})=\iota(\mathsf{2})=\mathsf{planner},\ \text{and} \tag{11}$$
$$\delta(\mathsf{1stop})=\mathsf{6stop},\ \delta(\mathsf{1go})=\mathsf{6go},\ \delta(\mathsf{2stop})=\mathsf{7stop},\ \delta(\mathsf{2go})=\mathsf{7go}.$$

A morphism where τ and δ are non-injective is

$$\tau(\mathsf{1})=\mathsf{6},\ \tau(\bar{\mathsf{1}})=\tau(\mathsf{2})=\mathsf{7},\ \tau(\bar{\mathsf{2}})=\tau(\mathsf{3})=\mathsf{8},$$
$$\iota(\mathsf{1})=\iota(\mathsf{2})=\mathsf{planner},\quad \text{and}$$
$$\delta(\mathsf{1stop})=\delta(\mathsf{1go})=\mathsf{6go},\ \delta(\mathsf{2stop})=\delta(\mathsf{2go})=\mathsf{7go}.$$

4. The category **ncForm**

4.1. Definition

This paragraph and the following theorem define the category **ncForm**, which is called the *category of node-and-choice forms*. Let an object be a (node-and-choice)[2] form $\varPhi = (T, I, (C_i)_i, \otimes)$. Let an arrow be a form morphism $\beta = [\varPhi, \varPhi', \tau, \iota, \delta]$. Let source, target, identity, and composition be

$$\beta^{\mathsf{src}} = [\varPhi, \varPhi', \tau, \iota, \delta]^{\mathsf{src}} = \varPhi\ ,$$
$$\beta^{\mathsf{trg}} = [\varPhi, \varPhi', \tau, \iota, \delta]^{\mathsf{trg}} = \varPhi'\ ,$$
$$\mathsf{id}_{\varPhi} = \mathsf{id}_{(T,I,(C_i)_i,\otimes)} = [\varPhi, \varPhi, \mathsf{id}^{\mathsf{Set}}_T, \mathsf{id}^{\mathsf{Set}}_I, \mathsf{id}^{\mathsf{Set}}_{\cup_i C_i}]\ ,$$
$$\text{and } \beta'\circ\beta = [\varPhi', \varPhi'', \tau', \iota', \delta']\circ[\varPhi, \varPhi', \tau, \iota, \delta]$$
$$= [\varPhi, \varPhi'', \tau'\circ\tau, \iota'\circ\iota, \delta'\circ\delta]\ ,$$

where $\mathsf{id}^{\mathsf{Set}}$ is the identity in **Set**.

Theorem 1. ncForm *is a category. (Proof C.1.)*

Recall from (2a) that a form incorporates a preform by definition. Also recall from (9a) that a form morphism incorporates a preform morphism by definition. These two observations can be used to define a functor from **ncForm** to **ncPreform** (`S16` Section 3.2). Details are provided in Remark C.2. This functor is "forgetful" in the sense of Simmons (2011, page 76).

4.2. Isomorphisms

The following theorem characterizes the isomorphisms in **ncForm**. The forward half of (a) and all of (b) are proved with relatively abstract arguments. In contrast, the reverse half of (a) is proved by means of the special structure of node-and-choice forms. There the obstacle is proving that $[\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}]$ is a morphism.

Both directions of part (a)'s characterization are useful. The forward direction helps one to derive further properties of isomorphisms. It is used, for example, in the proof of Proposition 4.2 below. Meanwhile, the reverse direction can help one to establish that a morphism is an isomorphism.

Theorem 2. *Suppose $\beta = [\Phi, \Phi', \tau, \iota, \delta]$ is a morphism. Then the following hold.*

(a) β is an isomorphism iff τ, ι, and δ are bijections.

(b) If β is an isomorphism, $\beta^{-1} = [\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}]$. (Proof C.4.)

Theorem 2(a) resembles `S16` Theorem 2 (second sentence), which showed that a morphism in **ncPreform** is an isomorphism iff τ and δ are bijections. These two theorems easily lead to Corollary 4.1.

Corollary 4.1. *Suppose $[\Phi, \Phi', \tau, \iota, \delta]$ is a morphism, where Φ determines Π and Φ' determines Π'. Then the following hold.*

(a) $[\Pi, \Pi', \tau, \delta]$ is a preform isomorphism iff τ and δ are bijections.

(b) $[\Phi, \Phi', \tau, \iota, \delta]$ is an isomorphism iff [1] $[\Pi, \Pi', \tau, \delta]$ is a preform isomorphism and [2] ι is a bijection. (Proof C.5.)

`S16` Proposition 3.3 showed that a preform isomorphism has many properties. Condition [1] of Corollary 4.1(b)'s forward direction[4] shows that a form isomorphism inherits all those properties. In addition, the following proposition shows how a form isomorphism preserves each player's choices, decision nodes, and information sets. (If C_i is empty, then $\delta|_{C_i}$, $C'_{\iota(i)}$, $\tau|_{F^{-1}(C_i)}$, $(F')^{-1}(C'_{\iota(i)})$, $\tau|_{\mathcal{H}_i}$, and $\mathcal{H}'_{\iota(i)}$ are all empty as well.)

[4]The result, that a form isomorphism implies an underlying preform isomorphism, could also be shown abstractly via the forgetful functor of Remark C.2.

Proposition 4.2. *Suppose $[\Phi, \Phi', \tau, \iota, \delta]$ is an isomorphism, where $\Phi = (T, I, (C_i)_i, \otimes)$ determines F and $(\mathcal{H}_i)_i$, and $\Phi' = (T', I', (C'_{i'})_{i'}, \otimes')$ determines F' and $(\mathcal{H}'_{i'})_{i'}$. Then the following hold.*

(a) $(\forall i)$ $\delta|_{C_i}$ *is a bijection from* C_i *onto* $C'_{\iota(i)}$.

(b) $(\forall i)$ $\tau|_{F^{-1}(C_i)}$ *is a bijection from* $F^{-1}(C_i)$ *onto* $(F')^{-1}(C'_{\iota(i)})$.

(c) $(\forall i)$ $\tau|_{\mathcal{H}_i}$ *is a bijection from* $\mathcal{H}_i$ *onto* $\mathcal{H}'_{\iota(i)}$.[5] *(Proof C.7.)*

Incidentally, Proposition 4.2 extends similar results for preforms. In particular, part (b) concerning decision nodes extends `S16` Proposition 3.3(c), and part (c) concerning information sets extends `S16` Proposition 3.3(h).

4.3. Example

Consider the morphism from Φ^{many} (6) to Φ^{planner} (7) defined by

$$\begin{aligned} &(\forall t)\ \tau(t) = t\ , \\ &(\forall i)\ \iota(i) = \mathsf{planner}\ ,\ \text{and} \\ &(\forall c)\ \delta(c) = c\ . \end{aligned} \tag{12}$$

Here τ and δ are bijections (in fact they are identity functions). Thus by Corollary 4.1(a), the two preforms underlying Φ^{many} and Φ^{planner} are isomorphic (in fact the two are equal). Meanwhile, ι is not injective. Accordingly, the morphism changes nothing except to merge all the players of Φ^{many} into the single player of Φ^{planner}.

Morphism (11) from Φ^{two} to Φ^{planner} is the composition of morphism (10) from Φ^{two} to Φ^{many} followed by morphism (12) from Φ^{many} to Φ^{planner}. The first is an injective morphism (i.e. an embedding), and the second merges the players together.

Appendix A. For Forms

Proof A.1 (for Proposition 2.1). *(a).* By rearrangements and the definition of C,

$$\begin{aligned} \cup_i F^{-1}(C_i) &= \cup_i \{\ t \mid (\exists c{\in}C_i)(t,c){\in}F\ \} \\ &= \{\ t \mid (\exists c{\in}\cup_i C_i)(t,c){\in}F\ \} \\ &= \{\ t \mid (\exists c{\in}C)(t,c){\in}F\ \} \\ &= F^{-1}(C)\ . \end{aligned}$$

[5] Here the symbol τ is overloaded: for any $H \in \mathcal{H}_i$, $\tau(H) := \{\tau(t) | t{\in}H\}$.

Thus it remains to show that $(\forall i^1{\neq}i^2)\ F^{-1}(C_{i^1}){\cap}F^{-1}(C_{i^2}) = \varnothing$. Accordingly, suppose there existed $i^1 \neq i^2$, $c^1 \in C_{i^1}$, $c^2 \in C_{i^2}$, and t, such that $t \in F^{-1}(c^1){\cap}F^{-1}(c^2)$. Then

$$c^1 \in F(t) \text{ and} \tag{13a}$$

$$c^2 \in F(t)\ . \tag{13b}$$

(13a), $c^1 \in C_{i^1}$, and (2c) together imply $F(t) \subseteq C_{i^1}$. Similarly, (13b), $c^2 \in C_{i^2}$, and (2c) together imply $F(t) \subseteq C_{i^2}$. Since $F(t) \neq \varnothing$ by (13a), the last two sentences imply $C_{i^1}{\cap}C_{i^2} \neq \varnothing$. This and $i^1 \neq i^2$ together violate (2b).

(b). By rearrangements and the definitions of $(\mathcal{H}_i)_i$, C, and $\mathcal{H}$,

$$\begin{aligned}
\cup_i\mathcal{H}_i &= \cup_i\{\ F^{-1}(c)\mid c{\in}C_i\ \} \\
&= \{\ F^{-1}(c)\mid c{\in}\cup_iC_i\ \} \\
&= \{\ F^{-1}(c)\mid c{\in}C\ \} \\
&= \mathcal{H}\ .
\end{aligned}$$

Thus it remains to show that $(\forall i^1{\neq}i^2)\ \mathcal{H}_{i^1}{\cap}\mathcal{H}_{i^2} = \varnothing$. Accordingly, suppose there existed $i^1 \neq i^2$ and H such that $H \in \mathcal{H}_{i^1}{\cap}\mathcal{H}_{i^2}$. Then

$$H \subseteq \cup\mathcal{H}_{i^1} = \cup\{F^{-1}(c)|c{\in}C_{i^1}\} = F^{-1}(C_{i^1})\ , \tag{14a}$$

where the set inclusion follows from $H \in \mathcal{H}_{i^1}$, the first equality follows from the definition of $\mathcal{H}_{i^1}$, and the last equality is a rearrangement. Similarly,

$$H \subseteq \cup\mathcal{H}_{i^2} = \cup\{F^{-1}(c)|c{\in}C_{i^2}\} = F^{-1}(C_{i^2})\ . \tag{14b}$$

Since $i^1 \neq i^2$, part (a) implies that $F^{-1}(C_{i^1})$ and $F^{-1}(C_{i^2})$ are disjoint. Thus (14a) and (14b) imply $H = \varnothing$. This contradicts (4) because $H \in \mathcal{H}_{i^1}$ and because every member of a partition is nonempty. □

Appendix B. For Morphisms

Lemma B.1. [6] *Suppose $\alpha = [\Pi, \Pi', \tau, \delta]$ is a preform morphism, where $\Pi = (T, C, \otimes)$ determines F and where $\Pi' = (T', C', \otimes')$ determines F'. Then the following hold.*

(a) $\tau(F^{-1}(c)) \subseteq (F')^{-1}(\delta(c))$.

(b) Suppose α is an isomorphism. Then $\tau(F^{-1}(c)) = (F')^{-1}(\delta(c))$.

[6] This lemma excerpts parts of proofs from `S16`. In particular, the proof of part (a) rearranges part of Proof B.4's argument for `S16` Proposition 3.2(j), and the proof of the part (b) rearranges part of the argument for `S16` Lemma B.8(a).

Proof. (a). I argue

$$\begin{aligned}\tau(F^{-1}(c)) &= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } t{\in}F^{-1}(c)\ \}\\ &= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (t,c){\in}F\ \}\\ &\subseteq \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (\tau(t),\delta(c)){\in}F'\ \}\\ &= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (t',\delta(c)){\in}F'\ \}\\ &\subseteq \{\ t' \mid (t',\delta(c)){\in}F'\ \}\\ &= (F')^{-1}(\delta(c))\ .\end{aligned}$$

The first inclusion follows from (11a) of `S16` Proposition 3.1(a). The second inclusion holds because $\tau(T) \subseteq T'$ by (8a). The equalities are rearrangements.

(b). I argue

$$\begin{aligned}\tau(F^{-1}(c)) &= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } t{\in}F^{-1}(c)\ \}\\ &= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (t,c){\in}F\ \}\\ &= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (\tau(t),\delta(c)){\in}F'\ \}\\ &= \{\ t' \mid (\exists t)\ t'{=}\tau(t) \text{ and } (t',\delta(c)){\in}F'\ \}\\ &= \{\ t' \mid (t',\delta(c)){\in}F'\ \}\\ &= (F')^{-1}(\delta(c))\ .\end{aligned}$$

The third equality holds by `S16` Proposition 3.3(b). The fifth holds because τ is a bijection by `S16` Theorem 2 (second sentence). The remaining equalities are rearrangements. □

Proof B.2 (for Proposition 3.1)**.** *(a).* Take any i. I argue

$$\begin{aligned}\tau(F^{-1}(C_i)) &= \cup\{\ \tau(F^{-1}(c)) \mid c{\in}C_i\ \}\\ &\subseteq \cup\{\ (F')^{-1}(\delta(c)) \mid c{\in}C_i\ \}\\ &\subseteq \cup\{\ (F')^{-1}(c') \mid c'{\in}C'_{\iota(i)}\ \}\\ &= (F')^{-1}(C'_{\iota(i)})\ .\end{aligned}$$

The first equation is a rearrangement, the first set inclusion follows from Lemma B.1(a), the second set inclusion follows from (9c), and the second equality is a rearrangement.

(b). Take any i and any $H \in \mathcal{H}_i$. By the definition of $\mathcal{H}_i$, there exists $c \in C_i$ such that $H = F^{-1}(c)$. Let $H' = (F')^{-1}(\delta(c))$. By (9c), $\delta(c) \in C'_{\iota(i)}$. Thus by the definition of $\mathcal{H}'_{\iota(i)}$, $H' \in \mathcal{H}'_{\iota(i)}$. Thus it suffices

to argue

$$\tau(H) = \tau(F^{-1}(c)) \subseteq (F')^{-1}(\delta(c)) = H' \ .$$

The first equality follows from the definition of c, the set inclusion follows from Lemma B.1(a), and the second equality is the definition of H'. □

Remark B.3. *Consider the examples Φ^{two} (5) and Φ^{many} (6). Further suppose $[\Phi^{\mathsf{two}}, \Phi^{\mathsf{many}}, \tau, \iota, \delta]$ is a form morphism. Then $\iota(1){+}1 = \iota(2)$.*

Proof. As in (5) and (6), let $\Phi^{\mathsf{two}} = (T^{\mathsf{two}}, I^{\mathsf{two}}, (C_i^{\mathsf{two}})_i, \otimes^{\mathsf{two}})$ and $\Phi^{\mathsf{many}} = (T^{\mathsf{many}}, I^{\mathsf{many}}, (C_i^{\mathsf{many}})_i, \otimes^{\mathsf{many}})$.

This paragraph shows that

$$(\forall i{\in}\{1,2\})\ \tau(i) = \iota(i) \ . \tag{15}$$

Take $i \in \{1,2\}$. Note

$$\delta(i\mathsf{stop}) \in \delta(C_i^{\mathsf{two}}) \subseteq C_{\iota(i)}^{\mathsf{many}} = \{\iota(i)\mathsf{stop}, \iota(i)\mathsf{go}\} \ , \tag{16}$$

where the set membership follows from the definition of C_i^{two} in (5), the set inclusion follows from (9c), and the equality follows from the definition of $C_{\iota(i)}^{\mathsf{many}}$ in (6). Further, since $(i, i\mathsf{stop}, \bar{i}) \in \otimes^{\mathsf{two}}$ by the definition of $\otimes^{\mathsf{two}}$ in (5), (8c) implies

$$(\tau(i), \delta(i\mathsf{stop}), \tau(\bar{i})) \in \otimes^{\mathsf{many}} \ . \tag{17}$$

(16) and (17) imply that

$$(\tau(i), \iota(i)\mathsf{stop}, \tau(\bar{i})) \in \otimes^{\mathsf{many}} \ \text{ or } \\ (\tau(i), \iota(i)\mathsf{go}, \tau(\bar{i})) \in \otimes^{\mathsf{many}} \ .$$

By the definition of $\otimes^{\mathsf{many}}$ in (6), either eventuality implies that $\tau(i) = \iota(i)$.

By (9a), $[\Pi^{\mathsf{two}}, \Pi^{\mathsf{many}}, \tau, \delta]$ is a preform morphism for $\Pi^{\mathsf{two}} = (T^{\mathsf{two}}, \cup_i C_i^{\mathsf{two}}, \otimes^{\mathsf{two}})$ and $\Pi^{\mathsf{many}} = (T^{\mathsf{many}}, \cup_i C_i^{\mathsf{many}}, \otimes^{\mathsf{many}})$. Use (1b) to derive p^{two} from Π^{two} and p^{many} from Π^{many}. By the definition of $\otimes^{\mathsf{two}}$ in (5), $1 = p^{\mathsf{two}}(2)$. Thus by `S16` Proposition 3.2(c) at $m{=}1$, $\tau(1) = p^{\mathsf{many}}(\tau(2))$. Thus by the definition of $\otimes^{\mathsf{many}}$ in (6), $\tau(1){+}1 = \tau(2)$. Thus by (15), $\iota(1){+}1 = \iota(2)$. □

Appendix C. For ncForm

Proof C.1 (for Theorem 1)**.** The next two paragraphs draw upon S16 Theorem 1, which showed that **ncPreform** is a well-defined category.

This paragraph shows that, for every form Φ, id_Φ is a form morphism (9). Accordingly, take any form $\Phi = [T, I, (C_i)_i, \otimes]$. By definition,

$$\mathsf{id}_\Phi = [\Phi, \Phi, \mathsf{id}_T^{\mathsf{Set}}, \mathsf{id}_I^{\mathsf{Set}}, \mathsf{id}_C^{\mathsf{Set}}] \ ,$$

where $C = \cup_i C_i$. (9a) requires that the quadruple $[\Pi, \Pi, \mathsf{id}_T^{\mathsf{Set}}, \mathsf{id}_C^{\mathsf{Set}}]$ is a morphism in **ncPreform**, where Φ determines Π (2a). This holds a fortiori because [1] the quadruple equals id_Π in **ncPreform** since [2] Π is a preform by (2a). (9b) requires that $\iota{:}I{\rightarrow}I$, which holds because $\iota = \mathsf{id}_I^{\mathsf{Set}}$. (9c) requires that $(\forall i)\ \delta(C_i) \subseteq C_{\iota(i)}$, which holds with equality because $\iota = \mathsf{id}_I^{\mathsf{Set}}$ and $\delta = \mathsf{id}_C^{\mathsf{Set}}$.

This paragraph shows that, for any two morphisms β and β', $\beta'{\circ}\beta$ is a morphism (9). Accordingly, take any two morphisms $\beta = [\Phi, \Phi', \tau, \iota, \delta]$ and $\beta' = [\Phi', \Phi'', \tau', \iota', \delta']$, where $\Phi = (T, I, (C_i)_i, \otimes)$, $\Phi' = (T', I', (C'_{i'})_{i'}, \otimes')$, and $\Phi'' = (T'', I'', (C''_{i''})_{i''}, \otimes'')$. By definition,

$$\beta'{\circ}\beta = [\Phi, \Phi'', \tau'{\circ}\tau, \iota'{\circ}\iota, \delta'{\circ}\delta] \ .$$

(9a) requires that the quadruple $[\Pi, \Pi'', \tau'{\circ}\tau, \delta'{\circ}\delta]$ is a morphism in **ncPreform**, where Φ determines Π and Φ'' determines Π''. This holds a fortiori because [1] the quadruple equals $[\Pi', \Pi'', \tau', \delta']{\circ}[\Pi, \Pi', \tau, \delta]$ in **ncPreform**, where Φ' determines Π', since [2] $[\Pi, \Pi', \tau, \delta]$ and $[\Pi', \Pi'', \tau', \delta']$ are morphisms in **ncPreform** by (9a) for β and β'. To see (9b), note that $\iota{:}I{\rightarrow}I'$ by (9b) for β, and that $\iota'{:}I'{\rightarrow}I''$ by (9b) for β'. Hence $\iota'{\circ}\iota{:}I{\rightarrow}I''$, which is (9b) for $\beta'{\circ}\beta$. To show that (9c) holds for $\beta'{\circ}\beta$, take any i. I argue

$$\delta'(\delta(C_i)) \subseteq \delta'(C'_{\iota(i)}) \subseteq C''_{\iota'{\circ}\iota(i)} \ .$$

The first inclusion holds because $\delta(C_i) \subseteq C'_{\iota(i)}$ by (9c) for β, applied at i. The second inclusion holds by (9c) for β', applied at $i' = \iota(i)$.

The previous two paragraphs have established the well-definition of identity and composition. The unit and associative laws are immediate. Thus **ncForm** is a category (e.g. Awodey (2010, page 4, Definition 1.1)). □

Remark C.2. *Define* F *from* **ncForm** *to* **ncPreform** *by*

$$\mathsf{F}_0 : (T, I, (C_i)_i, \otimes) \mapsto (T, \cup_i C_i, \otimes) \textit{ and}$$
$$\mathsf{F}_1 : [\varPhi, \varPhi', \tau, \iota, \delta] \mapsto [\mathsf{F}_0(\varPhi), \mathsf{F}_0(\varPhi'), \tau, \delta] \ .$$

Then F *is a well-defined functor.*

Proof. By (2a), F_0 maps any form into a preform. By (9a), F_1 maps any form morphism into a preform morphism. Thus it suffices to show that F preserves source, target, identity, and composition (Awodey (2010, page 8, Definition 1.2)). This is done in the following four paragraphs.

[1] Take any $\beta = [\varPhi, \varPhi', \tau, \iota, \delta]$. Then

$$\begin{aligned} \mathsf{F}_1(\beta)^{\mathsf{src}} &= \mathsf{F}_1([\varPhi, \varPhi', \tau, \iota, \delta])^{\mathsf{src}} \\ &= [\mathsf{F}_0(\varPhi), \mathsf{F}_0(\varPhi'), \tau, \delta]^{\mathsf{src}} \\ &= \mathsf{F}_0(\varPhi) \\ &= \mathsf{F}_0([\varPhi, \varPhi', \tau, \iota, \delta]^{\mathsf{src}}) \\ &= \mathsf{F}_0(\beta^{\mathsf{src}}) \ , \end{aligned}$$

where the first equation holds by the definition of β, the second by the definition of F_1, the third by the definition of src in **ncPreform**, the fourth by the definition of src in **ncForm**, and the fifth by the definition of β.

[2] Take any $\beta = [\varPhi, \varPhi', \tau, \iota, \delta]$. Then $\mathsf{F}_1(\beta)^{\mathsf{trg}} = \mathsf{F}_0(\beta^{\mathsf{trg}})$ can be shown by replacing src with trg in the preceding paragraph.

[3] Take any $\varPhi = (T, I, (C_i)_i, \otimes)$ and let $C = \cup_i C_i$. Note $\mathsf{F}_0(\varPhi) = (T, C, \otimes)$ by the definitions of $\varPhi$, F_0, and C. Then

$$\begin{aligned} & \mathsf{F}_1(\mathsf{id}_{\varPhi}) \\ =\ & \mathsf{F}_1([\varPhi, \varPhi, \mathsf{id}_T^{\mathsf{Set}}, \mathsf{id}_I^{\mathsf{Set}}, \mathsf{id}_C^{\mathsf{Set}}]) \\ =\ & [\mathsf{F}_0(\varPhi), \mathsf{F}_0(\varPhi), \mathsf{id}_T^{\mathsf{Set}}, \mathsf{id}_C^{\mathsf{Set}}] \\ =\ & [(T, C, \otimes), (T, C, \otimes), \mathsf{id}_T^{\mathsf{Set}}, \mathsf{id}_C^{\mathsf{Set}}] \\ =\ & \mathsf{id}_{(T,C,\otimes)} \\ =\ & \mathsf{id}_{\mathsf{F}_0(\varPhi)} \ , \end{aligned}$$

where the first equality holds by the definition of id in **ncForm**, the second by the definition of F_1, the third by the previous sentence, the fourth by the definition of id in **ncPreform**, and the last by the previous sentence.

[4] Take any $\beta = [\Phi, \Phi', \tau, \iota, \delta]$ and $\beta' = [\Phi', \Phi'', \tau', \iota', \delta']$. Note that since F_1 is well-defined by the first paragraph, $\mathsf{F}_1([\Phi, \Phi', \tau, \iota, \delta]) = [\mathsf{F}_0(\Phi), \mathsf{F}_0(\Phi'), \tau, \delta]$ and $\mathsf{F}_1([\Phi', \Phi'', \tau', \iota', \delta']) = [\mathsf{F}_0(\Phi'), \mathsf{F}_0(\Phi''), \tau', \delta']$ are preform morphisms. Then

$$\begin{aligned}
&\mathsf{F}_1(\beta' \circ \beta) \\
&= \mathsf{F}_1([\Phi', \Phi'', \tau', \iota', \delta'] \circ [\Phi, \Phi', \tau, \iota, \delta]) \\
&= \mathsf{F}_1([\Phi, \Phi'', \tau' \circ \tau, \iota' \circ \iota, \delta' \circ \delta]) \\
&= [\mathsf{F}_0(\Phi), \mathsf{F}_0(\Phi''), \tau' \circ \tau, \delta' \circ \delta] \\
&= [\mathsf{F}_0(\Phi'), \mathsf{F}_0(\Phi''), \tau', \delta'] \circ [\mathsf{F}_0(\Phi), \mathsf{F}_0(\Phi'), \tau, \delta] \\
&= \mathsf{F}_1([\Phi', \Phi'', \tau', \iota', \delta'] \circ \mathsf{F}_1[\Phi, \Phi', \tau, \iota, \delta] \\
&= \mathsf{F}_1(\beta') \circ \mathsf{F}_1(\beta) \ ,
\end{aligned}$$

where the first equality holds by the definitions of β and β', the second by the definition of $\circ$ in **ncForm**, the third by the definition of F_1, the fourth by the previous sentence and by the definition of $\circ$ in **ncPreform**, the fifth by the definition of F_1, and the sixth by the definitions of β and β'. □

Lemma C.3. *Suppose that $[\Phi, \Phi', \tau, \iota, \delta]$ is a morphism, where $\Phi = (T, I, (C_i)_i, \otimes)$ and $\Phi' = (T', I', (C'_{i'})_{i'}, \otimes')$. Further suppose that ι and δ are bijections. Then the following hold.*

(a) $(\forall i)$ $\delta|_{C_i}$ is a bijection from C_i onto $C'_{\iota(i)}$.

(b) $(\forall i')$ $\delta^{-1}|_{C'_{i'}}$ is a bijection from $C'_{i'}$ onto $C_{\iota^{-1}(i')}$.

Proof. (a). Let $C = \cup_i C_i$ and $C' = \cup_{i'} C'_{i'}$. Take any i. Then $\delta|_{C_i}$ is a function from C_i because δ is a function from C by (8b). It is injective because δ is a bijection by assumption. It is into $C'_{\iota(i)}$ by (9c). Thus it remains to show that $C'_{\iota(i)} \setminus \delta(C_i) = \varnothing$.

Accordingly, suppose $c' \in C'_{\iota(i)} \setminus \delta(C_i)$. Since $c' \in C'_{\iota(i)} \subseteq C'$ and since δ is a bijection by assumption, $\delta^{-1}(c')$ exists. Further, since $c' \notin \delta(C_i)$, there is $j \neq i$ such that $\delta^{-1}(c') \in C_j$. Thus by (9c), $\delta(\delta^{-1}(c')) \in C'_{\iota(j)}$. Hence $c' \in C'_{\iota(j)}$. This and the definition of c' imply $c' \in C'_{\iota(i)} \cap C'_{\iota(j)}$. This contradicts (2b) for Φ' because $i \neq j$ and because ι is a bijection by assumption.

(b). This paragraph shows

$$(18) \qquad (\forall i)\ \delta^{-1}|_{C'_{\iota(i)}} \text{ is a bijection from } C'_{\iota(i)} \text{ onto } C_i \ .$$

Accordingly, take any i. By part (a),

$$\delta|_{C_i} \text{ is a bijection from } C_i \text{ onto } C'_{\iota(i)} \ .$$

This has two implications. First,

$$(\delta|_{C_i})^{-1} \text{ is a bijection from } C'_{\iota(i)} \text{ onto } C_i \ .$$

Second, because δ is a bijection, $(\delta|_{C_i})^{-1} = \delta^{-1}|_{C'_{\iota(i)}}$. The previous two sentences imply (18) at i.

Finally, the result follows from (18) because ι is a bijection. □

Proof C.4 (for Theorem 2)**.** Throughout this proof let the components of Φ be $(T, I, (C_i)_i, \otimes)$, define $C = \cup_i C_i$, let the components of Φ' be $(T', I', (C'_{i'})_{i'}, \otimes')$, and define $C' = \cup_{i'} C'_{i'}$.

The forward half of (a) and all of (b). Suppose that β is an isomorphism (Awodey (2010, page 12, Definition 1.3)). Recall that $\beta = [\Phi, \Phi', \tau, \iota, \delta]$ and let $\beta^{-1} = [\Phi^*, \Phi^{**}, \tau^*, \iota^*, \delta^*]$. Then

$$\text{(19a)} \quad [\Phi^*, \Phi^{**}, \tau^*, \iota^*, \delta^*]\circ[\Phi, \Phi', \tau, \iota, \delta] = \mathsf{id}_\Phi = [\Phi, \Phi, \mathsf{id}^{\mathsf{Set}}_T, \mathsf{id}^{\mathsf{Set}}_I, \mathsf{id}^{\mathsf{Set}}_C]$$

and

$$\text{(19b)} \quad [\Phi, \Phi', \tau, \iota, \delta]\circ[\Phi^*, \Phi^{**}, \tau^*, \iota^*, \delta^*] = \mathsf{id}_{\Phi'} = [\Phi', \Phi', \mathsf{id}^{\mathsf{Set}}_{T'}, \mathsf{id}^{\mathsf{Set}}_{I'}, \mathsf{id}^{\mathsf{Set}}_{C'}],$$

where the first equality in both lines holds by the definition of β^{-1}, and the second equality in both lines holds by the definition of id.

The well definition of $\circ$ in (19a) implies

$$\text{(20a)} \qquad \Phi^* = \Phi' \ .$$

The well definition of $\circ$ in (19b) implies

$$\text{(20b)} \qquad \Phi^{**} = \Phi \ .$$

The third component of (19a) implies that $\tau^*\circ\tau = \mathsf{id}^{\mathsf{Set}}_T$. The third component of (19b) implies that $\tau\circ\tau^* = \mathsf{id}^{\mathsf{Set}}_{T'}$. The last two sentences imply that τ is a bijection from T onto T' and that

$$\text{(20c)} \qquad \tau^* = \tau^{-1} \ .$$

Similarly, the fourth components of (19a) and (19b) imply that ι is a bijection from I onto I' and that

$$\text{(20d)} \qquad \iota^* = \iota^{-1} \ .$$

Similarly again, the fifth components of (19a) and (19b) imply that δ is a bijection from C onto C' and that

$$\delta^* = \delta^{-1} \; . \tag{20e}$$

The previous three sentences have shown that τ, ι, and δ are bijections. Further,

$$\beta^{-1} = [\Phi^*, \Phi^{**}, \tau^*, \iota^*, \delta^*] = [\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}] \; ,$$

where the first equality follows from the definition of β^{-1} near the start of the previous paragraph, and where the second equality follows from (20a)–(20e).

The reverse half of (a). Suppose that τ, ι, and δ are bijections. Define $\beta^* = [\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}]$.

This paragraph shows that β^* is a morphism. Derive Π from Φ and Π' from Φ'. By (9) for β^*, it must be shown that

$$[\Pi', \Pi, \tau^{-1}, \delta^{-1}] \text{ is a preform morphism} \; , \tag{21a}$$

$$\iota^{-1}{:}I' \to I \; , \text{ and} \tag{21b}$$

$$(\forall i')\; \delta^{-1}(C'_{i'}) \subseteq C_{\iota^{-1}(i')} \; . \tag{21c}$$

$[\Pi, \Pi', \tau, \delta]$ is a preform morphism by (9a) for β. Thus (21a) holds by the lemma in the `S16` addendum. (21b) is immediate. (21c) holds with equality by Lemma C.3(b).

Finally,

$$\beta^* \circ \beta = [\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}] \circ [\Phi, \Phi', \tau, \iota, \delta] = \mathsf{id}_\Phi \;\; \text{and}$$
$$\beta \circ \beta^* = [\Phi, \Phi', \tau, \iota, \delta] \circ [\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}] = \mathsf{id}_{\Phi'} \; .$$

Thus β is an isomorphism. □

Proof C.5 (for Corollary 4.1)**.** *(a)* $[\Pi, \Pi', \tau, \delta]$ is a preform morphism by (9a). Thus part (a) follows from the second sentence of `S16` Theorem 2.

(b) This follows immediately from part (a) and Theorem 2(a). □

Lemma C.6. *Suppose* $[\Phi, \Phi', \tau, \iota, \delta]$ *is a morphism, where* $\Phi = (T, I, (C_i)_i, \otimes)$ *determines* $(\mathcal{H}_i)_i$*, and where* $\Phi' = (T', I', (C'_{i'})_{i'}, \otimes')$ *determines* $(\mathcal{H}'_{i'})_{i'}$*. Further suppose* τ *and* δ *are bijections. Then* $(\forall i)(\forall H \in \mathcal{H}_i)\; \tau(H) \in \mathcal{H}'_{\iota(i)}$.

Proof. Take any i and any $H \in \mathcal{H}_i$. Then there exists $c \in C_i$ such that $H = F^{-1}(c)$. I argue

$$\tau(H) = \tau(F^{-1}(c)) = (F')^{-1}(\delta(c)) \in \mathcal{H}'_{\iota(i)} \ .$$

The first equality holds by the definition of c. To see the second, let Φ determine Π and let Φ' determine Π'. Then $[\Pi, \Pi', \tau, \delta]$ is a preform isomorphism by Corollary 4.1(a). Thus the second equality holds by Lemma B.1(b). Finally, the set membership holds because $\delta(c) \in C'_{\iota(i)}$ by (9c). □

Proof C.7 (for Proposition 4.2)**.** Theorem 2(a) implies that τ, ι, and δ are bijections.

(a). This follows from Lemma C.3(a).

(b). Take any i. I argue

$$\begin{aligned} \tau(F^{-1}(C_i)) &= \cup\{\ \tau(F^{-1}(c)) \mid c{\in}C_i\ \} \\ &= \cup\{\ (F')^{-1}(\delta(c)) \mid c{\in}C_i\ \} \\ &= \cup\{\ (F')^{-1}(c') \mid c'{\in}C'_{\iota(i)}\ \} \\ &= (F')^{-1}(C'_{\iota(i)}) \ . \end{aligned}$$

The first equality is a rearrangement. To see the second, derive Π from Φ and Π' from Φ'. By Corollary 4.1(b), $[\Pi, \Pi', \tau, \delta]$ is an isomorphism. Thus the second equality follows from Lemma B.1(b). The third equality holds by part (a). The fourth equality is a rearrangement.

(c). Take any i. Lemma C.6 implies that $\tau|_{\mathcal{H}_i}$ is a well-defined function from $\mathcal{H}_i$ into $\mathcal{H}'_{\iota(i)}$. It is injective because τ is injective. To show that it is surjective, take any $H' \in \mathcal{H}_{\iota(i)}$. Since $[\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}]$ is an isomorphism by Theorem 2(b), Lemma C.6 can be applied to $[\Phi', \Phi, \tau^{-1}, \iota^{-1}, \delta^{-1}]$. Therefore $H' \in \mathcal{H}_{\iota(i)}$ implies $\tau^{-1}(H') \in \mathcal{H}_{\iota^{-1}\circ\iota(i)}$. Hence $\tau^{-1}(H') \in \mathcal{H}_i$. This implies that $\tau(\tau^{-1}(H')) = H'$ is in the range of $\tau|_{\mathcal{H}_i}$. □

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