1974

Generalization Of The Production Side Of The Trade Model

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GENERALIZATION OF THE PRODUCTION SIDE OF THE TRADE MODEL

by

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Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
London, Ontario

June, 1974

ABSTRACT

Most of the literature in international trade theory has been developed in terms of a simple model with two goods, two factors, no transportation costs, etc. Without claiming complete generalization in all directions, the purpose of our thesis is to provide some extensions to the model.

The thesis' three chapters provide three additions to the basic model. The first chapter develops the general case of n goods, m factors around the production possibility set and its shape. Necessary and sufficient conditions for convexity of the set are derived. We also review each of the main theorems in the Heckscher-Ohlin literature in terms of the production set. Among the various theorems we prove, two are worth mentioning: first we establish that duality between the Stolper-Samuelson and Samuelson-Rybczynski effect is true for any n and m, provided the number of goods is

not duality between the Stolper-Samuelson and Samuelson-Rybczynski effect is true for any n and m, provided the number of goods is not superior to the number of factors. The second noteworthy theorem we prove considers the effect of a change in commodity prices on production. We demonstrate that as soon as the number of goods is superior to two, and, even with a strictly convex production set, it is possible to have gross complementarity along the production set; that is, when the price of a good increases, the production of another good might also increases.

In the second chapter, we relax the assumption that factors are perfectly mobile between industries. Instead we assume that capital is completely immobile. We are thus studying a particular case of the model with three goods and two factors. We review each of the main theorems in the Heckscher-Ohlin model and provide a discussion of technological progress. We arrive at a weaker version of the Heckscher-Ohlin theorem. We prove that if a country has absolute advantage in one specific factor, relative to a given amount of labor, that country will export the good most intensive in that factor. We also show that the Stolper-Samuelson and Samuelson-Rybczynski theorems are valid for the specific factors of production.

The third chapter introduces a third good and a third factor in the model. Furthermore it is assumed that the other good is transportation services, which are required to import and/or export. We show how the offer curve will move depending on whether a country transports its exports, its imports or part of both. By introducing a trading partner, we are able to consider the equilibrium before and after the introduction of transportation costs. An interesting result we get relates to a reduction in the amount of transportation services required to transport each good; the question we ask is if all the other conditions to get factor price equalization are present, will we have factor prices moving unambiguously towards equality in the two countries? The answer is no; in fact we show that is is even not possible to state that commodity prices will tend to equality.

ACKNOWLEDGEMENT

I would like to extend special thanks to Professor J.R. Melvin. He provided financial support while I was a graduate student at the University of Western Ontario. He also was very helpful and encouraging in my writing of the thesis. He provided helpful and detailed comments and suggestions for each of the various drafts of this thesis. The final version owes much to his labor. I also thank my other advisors, V.S. Rao and D.Burgess, for helpful comments and suggestions. My thanks extend to the participants of the international trade workshop at the University of Western Ontario for listening to the presentation of the various papers which constituted the draft to this thesis. I appreciated the help of J.Bisciglia to improve the english presentation. Finally, and not the least, my wife was very good in giving encouragement and typing the various and final drafts of the thesis.

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INTRODUCTION

It has now become a cliché to say that most of the pure theory of international trade has been built around the two-by-two model. This was mentioned so often that it is now beginning not to be true anymore. By consulting the bibliography the reader will have a small idea of the vast amount of research that has been conducted in that general area.

Nevertheless, it still remains true that there are many areas in which extensions to the theory have yet to be made. In fact, these are so many that it would be beyond the goal of this thesis to attempt to make all possible extensions. It is also not clear, furthermore, that one could even list all the questions, for theory is such that when one question is resolved, another is raised and consequently new frontiers of research are opened.

From a general point of view, most of the research on international trade theory has been conducted by an analysis of its possible determinants. Following Melvin's approach[1], these determinants can be found by listing the set of conditions under which there would be no trade. The minimum list contains five conditions:

(1) Tastes are the same for every country.

(2) Production functions differ among goods but are the same between countries.

(3) Every production function is homogeneously linear.

(4) Endowments are the same for every country.

(5) Perfect competition prevails in all markets.

If any of these conditions are relaxed, then there is a basis for trade. As soon as such a basis is established, it is clear that other assumptions which will prevent other barriers to trade from arising, must be made. For example, an assumption about the cost of transportation must be considered; if it is high enough it can discourage trade, even though there is a basis for profitable exchange between the two countries.

Supposing that only the first condition is relaxed, then the following case can be considered: two countries, A and B, possess the same production possibility curve for the two goods X and Y, and, A's set of indifference curves is closer to the X axis than B's. In autarky, the price of X in terms of Y will be higher in A. By allowing trade, the people in A see a cheaper price for X in B. Also, the price of Y in terms of X will be higher in B. By introducing trade, the pattern of exchange is clear; A will export Y and import X. It can thus be stated that if trade is caused by a difference in tastes, a country will import the good towards which its tastes are biased.

A lot of research has been conducted on the effects of relaxation of anyone of the other conditions. The fourth one, which postulates that countries differ in endowments, has been traditionnally called the determinant of trade and it has received ample consideration in international trade theory. However, only recently has it been placed in its proper perspective, i.e., it is simply one of the possible determinants of trade. From a theoretical point of view, all of the other four conditions offer as valid an explanation of trade as difference in endowments.

It will be assumed nonetheless, that trade is caused by a difference in endowments. This stand is taken because most of the literature on international trade theory is built around this assumption. This approach, known as the Heckscher-Ohlin model permits us to make generalizations in the same manner that was developed and used decades ago.

It is important to emphasize however that this model relaxes only one of the five possible conditions in order to have no trade. This will help show the tremendous amount of research that can still be done in international trade theory.

Given that all other conditions hold, it is possible to dichotomize the model into a production side on the one hand and a demand side on the other. Every student of economics is familiar with the diagram showing a tangency solution between the production possibility curve and an indifference curve.

This thesis focuses its attention on the production side of the model, thus following the generally accepted approach found in international trade theory. This does not mean however that the demand side is unimportant: in fact, it is equally important because it it through the interaction between the two which makes a solution possible.

Even though the demand side is as important as the production side in determining equilibrium, it was relatively ignored in previous extension of the Hecksher-Ohlin model. This can be explained as follows: in order to study the properties of equilibrium in the light of various changes in parameters, only those parameters affecting the production side were studied for the following two reasons: firstly, the parameters of the production side were more directly observable than those of the demand side (tastes); secondly, when the theory was given rigourous treatment, the very existence of the community indifference curve was questioned whilst the concept of the production possibility curve was readily accepted and made its way into the introductory textbook.

This thesis is structured into three chapters, each being almost independent of the other two. The content of each chapter offers both different direction and possible extensions to the theory of international trade. Chapters II and III analyse two different models in detail while the role of chapter one shall be explained later on in this section.

A special case of the two-good-three factor model is considered in chapter two. There, it will be assumed that capital is immobile between industries; from this it is deduced that each sector possesses a specific type of capital. Jones (1971) studied this model because it presented a very special case where certain mathematical properties could not hold at the general level. In chapter two, Jones' work will be extended by studying the effect of relaxing the assumption where factors are internally mobile; this shall be done by assuming that one of the factors is completely immobile. The reader is referred to that particular chapter in order to follow the process utilised, the additions to Jones' contribution, and also the theoretical results obtained.

Chapter three will relax the assumption that transportation costs are non-existent. Most of the studies[2] in this area made the following simplification: transportation costs were defined as that portion of good X or Y that would be used in landing the rest of X or Y in the foreign country. This evaporation model was used in order to avoid considering a model with three goods which would have made the geometric derivation difficult. Chapter three will relax the evaporation hypothesis and will assume a third factor in order to obtain a strictly concave production set. It will then be possible to see the various possible configurations of equilibrium and to relate them to previous research in this area.

Chapter one provides the unifying elements for the entire thesis. That chapter was originally intended to be much shorter, the plan having been simply to present a summary of the research conducted on the general model by relating the extension to the concept of the production possibility curve. While constructing chapters two and three, two points were raised, points which justified the considerable extension of the first chapter.

The first point raised forced a link between chapter one and chapter two: if one considers the usual presentation of the general model with n goods and m factors, the application of two of the main theorems, Stolper-Samuelson and the Samuelson-Rybczynski, must be done according to matrix A, the matrix of per unit requirements. These two theorems were extended in previous studies to the case of many goods and many factors, provided that the goods and the factors were in equal numbers. By applying the derivation to the case where they were unequal, the first theorem would require m to be greater than n and the second one would require the reverse, thus infering that both theorems could apply only in the special case where the number of goods is equal to the number of factors.

Chapter II will present, however, a model with two goods and three factors, and with the aid of differential techniques, it will be shown that not only are both theorems valid, but also that the duality found in the n X n case is still valid. The explanation of the apparent contradiction will be found in the fourth section of the first chapter. There, the importance of Kuga's work (1972) will be shown, for it is through his study that the conditions for matrix A to be independent of the endowment vector, shall be shown.

The second point raised, forced the addition of a section to unite chapter one with chapter three. In a case where two goods are involved and the price of the first rises, it then follows that the production of that good increases while the production of the other decreases. Will the same result occur if the number of goods is increased by only one? If this is so, then there are a number of possibilities which cannot be ignored in a model with three goods. In the sixth section of chapter I, it shall be shown that it is possible to have complementarity along the production possibility surface as soon as more than two goods are involved.

The final section of chapter one will present a summary of our work conducted on the production surface. Since this thesis is structured in such a way that each chapter is accompanied with an introduction and a conclusion, the reader is referred the each individual chapter for further information on their content.

Footnotes to the introduction

1 Professor Melvin has described this approach in various places. For example, see: Melvin, J.R. "Increasing returns to scale as a determinant of trade." Canadian Journal of Economics, Vol. II, No. 3, Aug. 1969, pp. 389-402.

2 Both Mundell and Samuelson have presented studies on the evaporation model. See the bibliography for exact reference.

THE PRODUCTION SET AND ITS IMPLICATIONS IN

INTERNATIONAL TRADE THEORY

The concept of the production possibility curve is basic to international trade theory. Together with the community indifference curve, these form the basis for the analysis of what has been labelled the Heckscher-Ohlin model of international trade.

Other approaches to the question of the determinants of trade use this concept intensively. Chipman (1965), in the second part of his survey, studied the generalization of a community indifference curve and his discussion of the production possibility curve (Section 2.4) was done only through the two-by-two case. Quirk and Saposnik (1966) made extensions beyond the two-by-two case and were able to prove the convexity of the production set for the case of two goods with n factors. Lancaster[1] discussed some aspects of the n goods, m factors case, but some of his conclusions are false, as will be demonstrated later on. Samuelson (1967) offers a very brief and indirect study of the general (n by m) case. Hong (1970) implicitly presented the case where n exceeds m while Melvin (1968) considered explicitly the case of three goods and two factors.

It is our intention to show that the principle theorems of international trade theory can be integrated into a discussion on the production possibility set. The general case, n goods and m factors, and, the shape of the transformation, are the principle points of interest.

This discussion will permit the attainment of the following three goals: it will permit, firstly, a tighter integration of the various theorems; it will offer, secondly, more insight into the more general case and will illustrate the importance of the relationship between the number of goods and the number of factors; and these results will be useful thirdly, in encouraging further research.

This chapter is divided into eight sections. The first introduces the notation. By extending the Melvin (1971) technique, the second section introduces the geometrical derivation of the production possibility at the two-by-two level and a general derivation at the n X m level. The conditions under which the production sets are strictly convex, both locally and globally, is discussed in the third section[2]. The fourth section discusses two of the basic theorems of international trade theory, the Stolper-Samuelson and the Samuelson-Rybczynski theorems and it shall be shown that the duality found at the n X n level can be extended to the case where the number of goods is not equal to the number of factors, as long as the former is smaller than the latter. The factor price equalization theorem is briefly discussed in the fifth section. The sixth section presents a study on the relationship between the changes in commodity prices and output, while the seventh considers possible pattern of trade, the Heckscher-Ohlin theorem. The final section presents the conclusion to the first chapter.

Throughout the chapter, it is assumed that the production functions are linear homogeneous, continuous and strictly quasi-concave and that full employment of all factors and perfect competition prevail in all markets.

Section I Notation

The model can be described as follows: there are n goods, $(x_1,\ldots\ldots,x_n)$ with their respective prices $(p_1,\ldots\ldots,p_n)$. There are m factors and we denote the quantity of factor i used in the production of one unit of j as a_{ij}. Each factor has its own price $(W_1,\ldots\ldots,W_m)$ and is in fixed total supply $(v_1,\ldots\ldots,v_n)$. In vector column notation we define respectively the above as X, P, A_j, W, and V.

Let us introduce the following:

$A = \{A_1, A_2 \ldots\ldots A_n\}$ (size n X m)

f^i: The production function of the ith good

f^i_j: $\dfrac{\partial f^i}{\partial a_{ij}}$

F^i: The matrix of second order partial derivative for the production function of the ith good.

F_i: An n X n matrix with f^i_j in the i, jth element and zeros everywhere else.

I_n: An identity matrix of size n X n.

I^i_k: An identity matrix of size k with the ith row deleted.

$^{j}I_k$: An identity matrix of size k with the jth column deleted.

More specific notation may be used in each section and it will be defined as we proceed.

Section II

In this section we would like to show the relationship between the general transformation surface and the geometrical derivation of the curve. Three methods have been presented. The first one by Savosnick (1958) involves the box diagram and the contract curve. It does not readily lend itself to generalization, neither does the second method, by Travis (1964). The third by Melvin (1971), can be transformed more easily to higher dimensions and ties in closely with other generalizations of the Heckscher-Ohlin model.[3]

Let M be a diagonal matrix of size N, with the ith element along the diagonal defined as the maximum of the ith good that can be produced with input V. We define

$$m_{ii} = f_i (V).$$

Thus M corresponds to the value of the isoquants at the endowment point. The next step consists in choosing a vector of factor prices, say W_1. Let A_i^* be the cost minimizing point at the tangency of the isocost hyperplane, defined by W_1, and the m_{ii} isoquants.[4]

Such a point can be derived for every good $(i = 1, \ldots\ldots, N)$. Putting together these vectors for the n goods, we get a matrix A* of size m X n. The parallelogram method as used by Melvin consists of finding the solution to the svstem

$$A^* \qquad \alpha = V \tag{1}$$

$$0 < \alpha < 1 \tag{2}$$

where equation (2) is a vector equation. If this system possesses a solution, it is possible to find a point on the transformation surface as

$$X = M\alpha$$

By varying factor prices we can vary matrix A*, thus α and generate the whole set.

Before discussing the circumstances under which equation (1) will have a solution, we introduce the following theorem

Theorem 1. If the production functions are linear homogeneous and quasi-concave, then for any given set of factor prices there will be a unique set of relative commodity prices, provided all goods are produced.

Proof. Let us choose arbitrarily good 1 as our numéraire. In perfect competition we have in equilibrium

Price = Marginal costs = Minimum average cost

$$P_1 = \frac{A_1^* W_1'}{m_{11}} \qquad P_i = \frac{A_i^* W_1'}{m_{ii}} \quad (i = 2, \ldots, N)$$

Thus

$$\frac{P_i}{P_1} = \frac{A_i^* \; W_1'}{A_1^* \; W_1'} \; \frac{m_{11}}{m_{ii}}$$

m_{11} and m_{ii} are given, so is W_1; now if A_i^* and A_1^* are unique for W_1 then $\frac{P_i}{P_1}$ is also unique. The proof that A_i^* is unique for a given W ($W \neq 0$) for linear homogeneous, quasi-concave production functions can be found in Chipman (1964, P.11).

Let us now examine under which condition will (1) have a solution, subject to (2). A is an m X n matrix. For the case where n> m (more goods than factors) the solution will never be unique. The case where n< m, is analyzed below. We now ~~proceed to~~ the special case where n = m, since most of the extension of the Heckscher-Ohlin model has been conducted along the lines of this special case.

For the case where n = m, we shall refer to all dimensionality as n. The solution to (1) will be

$$\alpha = A^{*-1}V \qquad (3)$$

i.e., if A* is invertible and (2) is satisfied, then at a given W (thus at a given and unique set of relative prices, from Theorem 1) there will be a unique level of output.

Let us first seek the condition for the non-singularity of A*. The column of A* must be independent if A^{*-1} is to exist. The case where one column is proportional to another is known as a factor intensity reversal ray. This is also a possibility at higher dimensions. Furthermore, there is the possibility of one column being equal to a weighted sum of others. For A^{*-1} to exist, the following condition must hold. The cone of diversification (5) must form a basis for R^N.(6) This is just another way of stating that the columns of A have to be independent.

The following step is to seek the condition under which (2) will be satisfied. This condition takes the familiar form that V (the endowment ray) must lie within the cone of diversification, formed by the columns of A*. If the endowment ray lies outside the cone of diversification then the particular set of factor prices which generated the matrix A* will not be feasible because that set requires negative output for some products.

If A* is singular then we know that (1) will not have a unique solution and the production set will have linear segments. This will be the same as in the case where $n > m$.

For the case where $n < m$, (1) can have one, an infinity, or no solution. In order to obtain a unique solution, it is necessary that the columns of A* are independent. As in the previous case the endowment ray must lie within the cone of diversification in order that (2) be satisfied. If both conditions are met then we will have

$$\alpha = (A^{*'}A^{*})^{-1} A^{*'} V \qquad (4)$$

This is true because, if the columns of A^* are independent, then A^* is of rank n, so is $A^{*'}$, then $(A^{*'}A^*)$ is non-singular, positive definite.[7]

The independence of the columns of matrix A^* is a necessary condition to have a unique solution. However, it is not sufficient. Consider the following counter example

$$A^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad V = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix}$$

Therefore $\alpha = \begin{pmatrix} 14/3 \\ 11/3 \end{pmatrix}$

It is obvious that α does not satisfy (2); also (1) is not satisfied

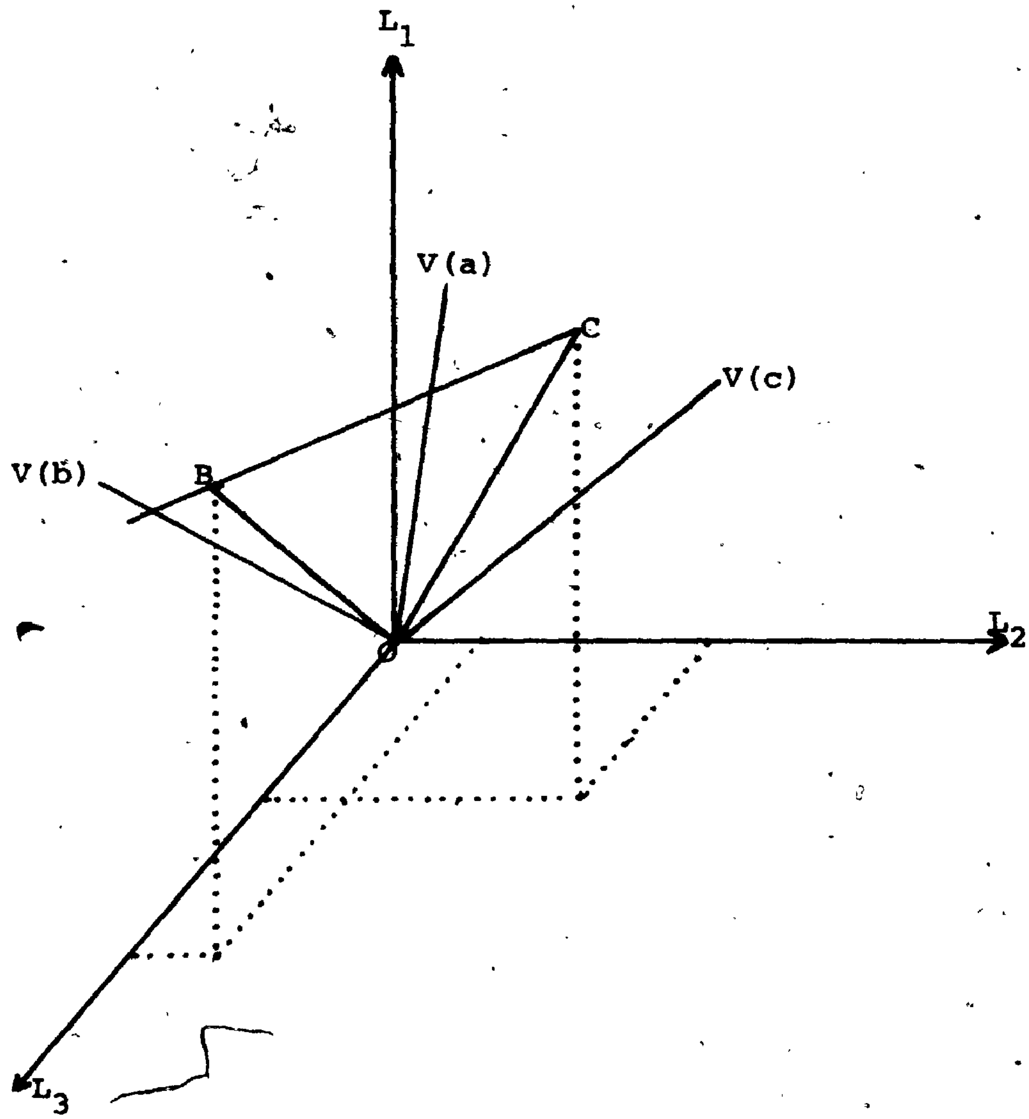
$$A^*\alpha = \begin{pmatrix} 14/3 \\ 11/3 \\ 25/3 \end{pmatrix} \neq V$$

The reason for this result is that the three equations are not consistent: the addition of the first and the second will give

$$\alpha_1 + \alpha_2 = 9$$

a result that is not consistent with the third equation.

FIGURE I



The Figure I offers a graphical representation of the case where there are two goods and three factors. The three factors are L_1, L_2, L_3, and the two points B and C represent the columns of matrix A. B and C are the tangency points between the two isoquants and their respective isocost plane, for a given set of factor prices. We then have to consider the endowment ray OV. The following three cases are therefore possible:

a) OV cuts the line BC between points B and C, in which case (1) will have a solution satisfying (2);

b) OV cuts the line BC, extended outside the segment BC, in which case (1) will have a solution but (2) will not be satisfied;

c) OV does not cut either BC or its extension, in which case (1) will not have a solution.

As can readily be seen (a) implies that V lies within the cone of diversification. This is also generalized to higher levels of dimensionality. It is therefore false to conclude that every set of factor prices will lead to a point on the transformation surface. As there are only N goods, out of the R^M positive orthant of factor prices space, we will generate a production set located in R^N.

The case where n>m (more goods than factors), has been discussed by Hong (1970). He points out that there will be a basic indeterminacy in production. This is also true according to the present approach because the transformation surface has linear subspace. An example can be found in Melvin's treatment of the three goods two factors model (1968). Linear segments of the transformation surface will however occur not only when n>m, but also as long as the columns of A* are linearly dependent, provided that (1) and (2) are mutually satisfied.

Suppose that only K (K<N) of the columns of A* are linearly independent at a given set of factor prices. Without loss of generality we can assume those to be the first K columns. Defining

$$A_K^{*}\alpha_K = \sum_{j=K}^{N} \alpha_j A_j^{*} \qquad (5)$$

$$A^{*} = \{A^{*}_1, \ldots, A^{*}_{K-1}, A^{*}_K\}$$

$$\alpha' = (\alpha_1, \ldots, \alpha_{K-1}, \alpha_K)$$

If $A^*\alpha = V$ has a solution satisfying (2), we can solve for α'_K. We then substitute in (5) and see that it is defining a hyperplane for the $a_j(j = K, \ldots, N)$. Also note that the choice of A^*_K is quite arbitrary. The principle is still valid. We can choose any of the non-linearly dependent columns of A to aggregate with the other.

We can now see how this geometrical extension relates to the usual mathematical derivations. Our method comes from the solution to the following problem: minimize $\sum_{i=1}^{M} V_i W_i$, i.e., minimize total factors payment (subject to the full employment constraint). Another method, adopted by Lancaster (1968, p.118), consists of maximizing $\sum_{j=1}^{N} p_j x_j$, subject to the full employment constraint. Before proceeding to the following section, we note the following theorem, from Dorfman, Samuelson, and Solow (1958, p.370).

<u>Theorem II</u>

"Whenever we solve a pair of duals problems, the maximum value of the linear form being maximized (here $\sum_{1}^{N} p_j x_j$) equals the minimum value of the form being minimized (here $\sum_{1}^{M} r_i v_i$)."

Section III

In this section, we will analyze the conditions under which the production set will be convex and strictly convex. We will consider both local and global convexity.

Definitions

1. Convex set - A set S is convex if

$X_1 \in S$ and $X_2 \in S \Rightarrow \theta X_1 + (1 - \theta) X_2 \in S$ for any $0 \leq \theta \leq 1$.

2. Production set.

$$(S) - S = \{(X_1 \ldots X_n) \;/\; X_1 \leq f_1(A_1), X_2 \leq f_2(A_2), \ldots, X_n \leq f_N(A_N), \; \Sigma A_i = \bar{v}\}$$

3. Concave function - A function $f: \bar{R}_N \rightarrow \bar{R}$ is said to be a concave function if $\bar{R}_N$ is a convex set and

$$Z_1, Z_2 \in \bar{R}_N \Rightarrow f(\alpha Z_1 + (1-\alpha) Z_2) \leq \alpha f(Z_1) + (1-\alpha) f(Z_2)$$

We can first consider convexity. Afterwards, we will concentrate on strict convexity. We present the following theorem, extended from Quirk and Saposnik (1966).

<u>Theorem III.</u> (Global convexity) If f_i $(i = 1 \ldots N)$ are concave, then S is a convex set.

<u>Proof</u>

Assume $X \in S$, $Y \in S$

define $\bar{X} = tX + (1-t)Y$ $\qquad 0 \leq t \leq 1$

we need to show that $\bar{X} \in S$

we have

$$X_1 \leq f_1(t_{11}v_1, t_{12}v_2 \ldots t_{1m}v_m)$$
$$X_2 \leq f_2(t_{21}v_1, t_{22}v_2 \ldots t_{2m}v_m) \qquad (6)$$
$$X_n \leq f_n(t_{n1}v_1, t_{n2}v_2 \ldots t_{nm}v_m)$$

where

$$0 \leq t_{ij} \leq 1 \qquad i = 1 \ldots N$$
$$j = 1 \ldots m \qquad (7)$$

also $\sum_i t_{ij} = 1 \qquad j = 1 \ldots m$

We have X_i equals the ith element in X and v_i the ith element in V.

(6) means that for each X_i there will be a higher level of production of that good which will be feasible, and (7) states that these levels will be all together feasible.

Similarly we have

$$Y_1 \le f_1(\theta_{11}v_1, \ldots \theta_{1m}v_m)$$

$$\vdots$$

$$Y_n \le f_n(\theta_{n1}v_1, \ldots \theta_{nm}v_m)$$

$$\sum_i \theta_{ij} = 1 \qquad j = 1 \ldots M$$

$$0 \le \theta_{ij} \le N \qquad i = 1 \ldots N$$
$$j = 1 \ldots M$$

now

$$\bar{X}_1 = tX_1 + (1 - t)Y_1$$

$$\le t f_1 (t_{11}v_1, \ldots, t_{1m}v_n) + (1-\theta) f_1(\theta_{11}v_1, \ldots, \theta_{1m}v_m)$$

$$\le f_1 \quad (t t_{11} + (1-t) \theta_{11})v_1, \ldots\ldots, (t t_{1m} + (1-t)\theta_{1m})v_m$$

The last result follows from the definition of convexity.

Generally we will have

$$\bar{X}_i = t X_i + (1-t) Y_i$$

$$\le t f_i(t_{i1}v_1, \ldots, t_{im}v_m) + (1-t) f_i({}_{i1}v_1, \ldots, {}_{im}v_n)$$

$$\le f_i \quad (t t_{i1} + (1-t) \theta_{i1})v_1 \ldots\ldots (t t_{im} + (1-t)\theta_{im})W_m$$

Define $a_{ij} = t t_{ij} + (1-t)\theta_{ij}$

we have

$$\sum_i a_{ij} = t \sum_i t_{ij} + (1-t) \sum_i \theta_{ij} = 1 \qquad (8)$$

also $0 \leq a_{ij} \leq 1$ because even if θ_{ij} and t_{ij} are at their maximum (1), a_{ij} is equal to 1; if they are at their minimum (0), a_{ij} is equal to 0.

Thus

$$X_1 \leq f_1 (a_{11}v_1, \ldots, a_{1m}v_m) = Z_1$$

$$X_2 \leq f_2 (a_{21}v_1, \ldots, a_{2m}v_m) = Z_2$$

$$X_n \leq f_n (a_{n1}v_1, \ldots, a_{nm}v_m) = Z_n$$

or $\overline{X} \leq Z$

but on account of (8), Z is feasible: thus $Z \in S$ which implies $\overline{X} \in S$.

Q.D.E.

Theorem III states that for any number of goods, as long as production functions are concave, the production set will be convex. Homogeneous production functions of degree less or equal to 1 being concave, we can apply the following corollary to theorem III.

Corollary 1

Any set of production functions which are homogeneous of degree $K \leq 1$ will generate a globally convex production set.

We now turn to local results. We are interested in the case where the set will be strictly convex. Let us ~~extend Lancaster's work~~ (1968) and then depart from it in order to show that strict convexity requires that (1) has a unique solution. Kelly (1969) arrived at the same conclusion as in this thesis. For a more detailed explanation, the reader should consult Kelly's results.

Consider the problem of maximizing $\sum_{i=1}^{N} p_i x_i$ subject to $\sum_{j=1}^{M} A_j^{**} = V$, where A_j^{**} is the vector of input required to produce x_j.

We can form the Lagrangian of the problem

$$L(A_1^{**},\ldots,A_n^{**}) = \sum_{i=1}^{N} p_i f^i - \sum_{r=1}^{M} \mu_r (A_r^{**} - V) \qquad (9)$$

The first order conditions can be derived

$$\frac{\delta L}{\delta A^{**}_{ij}} = p_i f^i_j \quad - \quad \mu_j \quad = \quad 0 \qquad i = 1 \ldots N \qquad (10)$$

$$\frac{\delta L}{\delta \mu_r} \quad = \quad \sum_{r=1}^{N} \quad (A^{**}_r \quad - V) \quad = \quad 0 \qquad (11)$$

Note that (11) is a vector equation of size m X 1. Also, it is well-known that the μ's will be proportional to the ω's. By an appropriate choice of weight, we can make our lagrangian multipliers equal to the factor prices so that (10) can be replaced by

$$p_i f^i_j \quad - \quad \omega_j \quad = \quad 0 \qquad \begin{array}{l} i = 1 \ldots n \\ j = 1 \ldots m \end{array} \qquad (10')$$

This result is useful for other sections.

A study on strict convexity demands special interest in the second order conditions rather than in the first; it is necessary that "the quadratic form based on the matrix of second order partial derivatives of L (with respect to the A^{**}_{ij} only) be always negative". (8)

Lancaster derived from this, the following result: "It is sufficient for a regular (strictly convex) transformation surface in an economy consisting entirely of industries with constant returns to scale that: a) every production function be concave-contoured; and b) no two production functions have the same relative factor intensities anywhere" (1968, p.134).

The result according to him applies to the $n \times m$ case without any reference to their relative size. From the previous section however, we know that when $n > m$, there is a basic indeterminacy in production. A counter-example to Lancaster's claim can be found in Melvin (1968).

We can derive

$$\frac{\partial^2 L}{\partial a_r^j \partial a_s^K} = 0 \quad j \neq K \qquad j, K = 1 \ldots N \quad s, r = 1 \ldots m$$

$$\frac{\partial^2 L}{\partial a_r^j \partial a_s^j} = P_j f_{rs}^j \qquad j = 1 \ldots n \quad s, r, = 1 \ldots m$$

where the superscript j and K stand for the jth and Kth good respectively.

The double stars superscript was dropped for purposes of clarity. Before proceeding, we would like to present the following definition and Lemma 2, from Quirk and Saposnik (1966).

A function f: $\overline{R}_N \to \overline{R}$ is said to be a <u>locally quasi-concave</u> function at Z^o if $\overline{R}_N$ is a convex set and $\sum_i \sum_j f_{ij} h_i h_j \leq 0$ for any vector h satisfying $\sum_j f_j h_j = 0$.

<u>Lemma 2</u>

"Assume that f possesses continuous second partial derivatives at point Z^o. Furthermore, assume that f is homogeneous of degree K −1, then f is locally quasi-concave at Z^o."

From lemma 2 it can readily be seen that for homogeneous linear locally quasi-concave production functions the unbordered Hessian will be a negative semi-definite matrix.

Let us define Y as a vector with m X n elements. The first m elements are dA_1^{**}; the elements from m+1 to the 2m are dA_2^{**} and so on. Y is thus the vector of differentials of matrix A^{**}.

Let

$$B = \begin{bmatrix} P_1F^1 & 0 & \cdots & 0 \\ 0 & P_2F^2 & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & P_NF_N \end{bmatrix}.$$

Thus Y is a vector of mn elements and B is a square matrix of dimension mn X mn. The quadratic form in which we are interested, is

$$Y'BY = \sum_j P_j (Y^j)' F^j Y^j \qquad (12)$$

subject to

$$\sum_j Y^j = 0 \qquad (13)$$

Where (13) is the full-employment condition.

we can write (12) explicitly as

$$Y'BY = \sum_{k=1}^{N} \sum_{i=1}^{M} \sum_{j=1}^{M} f^k \quad Y^k \quad Y^k \qquad (12')$$

The correction of Lancaster's error will be done along the same lines as Kelly's. However,he discusses the presentation in terms of (12') while our discussion is kept at the level of vectors. We next present this theorem, an adaptation from Lancaster's (1968, p.127). We use ∇f as the vector of partial derivative of f.

<u>Theorem IV</u> For a homogeneous production function of degree ρ, with continuous first and second order derivatives, the following relation holds:

for $f = f(A^*)$

$(\rho - 1)\ \nabla f = A^*F$

<u>Proof</u> From Euler's Theorem

$\rho f(A^*) = A^{*'}\nabla f$

differentiating with respect to A^*: $\rho\nabla f = \nabla f + A^{*'}F$

or $(\rho - 1)\ \nabla f = A^{*'}F$

Q.D.E.

Thus, when $\rho = 1$, we have $A^{**'}F^i = 0$

Let U represent a value for dA^{**} such that it follows a contour of f, which lies along an isoquant. Since f is concave contoured, we have

$$U'FU < 0 \text{ unless } U = A^{**} \qquad (14)$$

Let us define $Y^i = U^i + \lambda A_i^{**}$

Then,

$$Y^{i'}F^{i}Y^{i} = (U^{i}+\lambda A_{i}^{**})' F^{i} (U^{i}+ \lambda A_{i}^{**})$$

$$= U^{i'} F^{i}U^{i}+2\lambda A_{i}^{**'} F^{i}U^{i} + \lambda^{2} A_{i}^{**'}F^{i}A_{i}^{**} \tag{15}$$

$$= U^{i'} F^{i}U^{i}$$

by Theorem IV.

Returning to the quadratic form in (12) it is noted that some of the vectors Y^{i} can be set to equaling to zero. At most, n-2 can be neutralized in this fashion without all of Y becoming a zero vector. Suppose we leave only Y^{j} and Y^{k} non zero; we thus have

$$Y' BY = Y^{j'} P_{j} F^{j}Y^{j} + Y^{k'}P_{k} F^{k} Y^{k}$$

$$= U^{j'} P_{j} F^{j}U^{j} + U^{k'}P_{k} F^{k} U^{k}$$

also from (13) we can derive $Y^{j}= -Y^{k}$

$$\text{Thus } Y'BY= U' (P_{j}F^{j}+ P_{k} F^{k})U \tag{16}$$

in order that (16) be zero, we need $U'F^{j}U= U'F^{k}U= 0$; i.e., we need both. Thus, both $U = \gamma_{i} A_{i}^{**}$, $i = j,k$, where the γ_{i} are constant, are needed.

$$A_{i}^{**} \neq \gamma A_{k}^{**} \tag{17}$$

The pre-cited condition will guarantee that we do not have $U = \gamma_i P_i^{**}$ $i = j, k$.

If (17) is satisfied, then it is impossible to obtain a value of U which will make both terms of (16) zero, because if, for example, $U = P_i^{**}$, then $U \neq \gamma_K P_K^{**}$ which implies that $U' F^k U < 0$.

At this point Lancaster concluded and stated that he had proven his theorem which was mentioned above. However, consider the statement after equation (15). If Y^i equals zero for n-2 industries, at the very most, it can apply to any number equal to or smaller than n-2.

Consider a particular sector, the j^{th}. Also consider a new Y defined as

$$Y^{\ell} = \theta_1 Y^1 + \theta_2 Y^2 + \dots \theta_n Y^n \qquad (18)$$

where the θ's are weights with at least two non zero and $\theta_j = 0$. We can also define

$$P^{\ell} F^{\ell} = \theta_1 P_1 F^1 + \theta_2 P_2 F^2 + \dots \theta_N P_N F^N \qquad (19)$$

Upon reviewing Lancaster's exercises, the following conditions are required in order to arrive at a strictly convex production set:

$$A_j^{**} \neq \gamma A_\ell^{**}$$

where $$A_\ell^{**} = \theta_1 A_1^{**} + \theta_2 A_2^{**} + \ldots + \theta_N A_N^{**} \qquad (20)$$

Thus A_j^{**} must not be proportional to any weighted sum of other A_i^{**}. By including the proportionality constant into the weights, we arrive at the result that the columns of matrix A^{**} must be independent.

However if matrix A^{**} is non-singular, then matrix A must possess the same quality.

The following theorem is thus proven:

Theorem V. In order that the production set be strictly convex in an economy consisting entirely of industries with constant returns to scale that (a) every production function be concave-contoured; and (b) the column of A be independent.

In the special case where $n = m$; theorem V implies that A be invertible. Geometrical interpretation clarifies the difference in results between Lancaster's and the present. He found the condition for strict concavity when there is parallel movement to all axes except two, while the present thesis demonstrated the condition for movement in any direction.[9]

Let us now turn to the proof of another theorem, which deals more specifically with a globally strictly convex transformation surface.

Theorem VI. The transformation surface will be strictly convex if and only if at every point on the surface the columns of matrix A are independent.

Proof.

Necessity.

Suppose one column of A is linearly dependent on the other, it must then be shown that the production transformation surface will not be strictly convex. We have

$$A_j = \sum_{i=1}^{N} \gamma_i \; A_i \, , \quad \gamma_j = 0 \tag{21}$$

let α^1 be a solution to (1) satisfying (α) or,

$$\alpha_1^1 \; A_1 + \alpha_2^1 \; A_2 \ldots + \alpha_j^1 \, A_j + \ldots \alpha_N^1 \, A_N = V \tag{22}$$

From (21) we define

$$\theta \, \alpha_j^1 = \sum_{i=1}^{N} \theta \quad \alpha_j^1 \quad \gamma_i \, A_i \tag{23}$$

it is then possible to substitute into (22) and obtain

$$(\theta \, \alpha_j^1 \quad \gamma_1 + \alpha_1^1) \; A_1 + (\theta\alpha_j^1 \quad \gamma_2 + \alpha_2^2) \; A_2 \ldots + (1-\theta)\alpha_j^1 A_j \ldots$$

$$+ (\theta \quad \alpha_j^1 \quad \gamma_N + \alpha_N) \quad A_N = V$$

or

$$\alpha_1^2 \quad A_1 + \ldots + \alpha_N^2 \quad A_N = V \tag{24}$$

where

$$\alpha_i^2 = (\theta \, \alpha_j^1 \, \gamma_i + \alpha_i^1)$$

thus $\quad A \, \alpha^2 = V$

By an appropriate choice of θ we can get α^2 to satisfy (2) so that α^1 and α^2 are solutions, which are on the same hyperplane, since W has not changed. Thus, the set is not strictly convex.

Sufficiency

The duality result obtained in the work done by Dorfman, Samuelson and Solow brings sufficiency to the theorem. To arrive at any form of valid conclusion, it is necessary to show; if the transformation surface has linear subspace, then the columns of at least one A will not be independent.

It is well-known that under our assumptions the slope of the transformation surface going parallel to all axes except two will be equal to the reciprocal of the price ratio of these two goods.(10) The hyperplane tangent can therefore be called the price hyperplane. Thus, if the transformation surface has linear segments, then, at the given set of prices, the solution to the problem

$$\max \sum_{i=1}^{N} P_i X_i$$

will have more than one solution.

this means however that the dual problem

$$\min \sum_{i=1}^{M} w_i v_i$$

will have more than one solution, which in turn implies that $A\alpha = V$ will have more than one solution and from this it follows that A has linearly dependent columns.

Q.D.E.

The exact curvature of the transformation surface will depend on more than one factor. Melvin (1971) mentions two: "The first is the degree of difference in the intensities of the two goods and the second one is the elasticity of production for the two goods, i.e., the curvature of the isoquants."(11) Both he and Hsiao (1971) reached different conclusions regarding the effect of the elasticity of substitution. They were both corrected by Scarth and Warne (1971).

We would like to show that there is a definite relationship between the elasticity of substitution and the curvature of the production possibility set. In the two factors case, the total elasticity of substitution is a meaningful concept. For the general case, however, there is no such relationship. It is therefore necessary to use the concept of partial elasticities of substitution.

Consider (12) which shows the curvature only if the a_{ij} are changed. Thus with P and W constant, we can state that

$$\nabla f_i' \, dA_i = 0 \tag{25}$$

If the stars upperscript are dropped, it is then possible to rewrite (12) as

$$Y' BY = \sum_i dA_i' P_i F^i dA_i \tag{26}$$

Let us consider only one element in (26); F^i is the unbordered Hessian matrix. The following transformations can be performed : define $dA_i^* = \begin{pmatrix} 0 \\ dA_i \end{pmatrix}$, a vector of order (m + 1) X (1). Let H^i be (12)

$$H^i = \begin{bmatrix} 0 & \nabla f_i' \\ \nabla f_i & F^i \end{bmatrix}$$

Then $dA_i' P_i F^i dA_i = dA_i^* P_i H_i dA_i^*$

because of (25).

Thus

$$dA_i' P_i F^i dA_i = dA_i^{*\prime} P_i ((H^i)^{-1})^{-1} dA_i^* \tag{27}$$

Now

$$(H_i)^{-1} = \frac{(\text{Adj } H^i)'}{H^i} = \frac{1}{H^i} \begin{bmatrix} * & * & \cdots & * \\ * & H_{11} & \cdots & H_{m1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ * & H_{1m} & \cdots & H_{mm} \end{bmatrix}$$

where H_{rs} is the cofactor of the element in the r^{th} row and s^{th} column of H. Stars replace the elements which are not needed.

This transformation permits the use of the definition of partial elasticities of substitution as proposed by Allen.(13)

$$\sigma_{rs} = \frac{A_i' \nabla f_i}{a_r \quad a_s} \frac{H_{rs}}{H} .$$

Using Euler's theorem $A' \nabla f_i = Y_{io}$

then,

$$\frac{H_{rs}}{H} = \frac{\sigma_{rs} \; a_r \; a_s}{Y_{io}}$$

And combining this to (26) we thus have, when dropping the i upperscript:

$$H^{-1} = \frac{1}{Y_{io}} \begin{bmatrix} * & * & \cdot & \cdot & \cdot & * \\ * & \sigma_{11}a_1^2 & \cdot & \cdot & \cdot & \sigma_{n1}a_m a_1 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ * & \sigma_{1m}a_1 a_m & \cdot & \cdot & \cdot & \sigma_{mm}a_m^2 \end{bmatrix}$$

Let the matrix of partial elasticities of substitution be denoted by Σ. It is a symmetric matrix. Define $(i\ A)$ as an $m \times m$ diagonal matrix with an element of A_i along its diagonal. Thus,

$$H^{-1} = \frac{1}{Y_{io}} \begin{bmatrix} * & * \\ * & (i\ A)\ \Sigma\ (i\ A) \end{bmatrix} \tag{29}$$

Using (29) in (27) we thus have

$$dA_i' \ P_i F^i dA_i = dA_i^{*\prime} \left(P_i Y_{io} \begin{bmatrix} * & * \\ * & (i\ A)\ \Sigma^i (i\ A) \end{bmatrix}^{-1} \right) dA_i^* \tag{30}$$

By substituting (30) into (26) a relationship between the elasticity of substitution and the curvature of the production possibility surface can be obtained. The analysis of (30) in greater detail is beyond the scope of this thesis. The result was introduced only in order to show that a definite relationship exists between the elasticity of substitution along the isoquant and the production possibility surface.

Section IV The Stolper-Samuelson and Samuelson-Rybczynski Theorems.

In this section, we consider the Stolper-Samuelson and the Samuelson-Rybczynski Theorems. The latter studies the relationship between a change in outputs and a change in the endowment of one factor. The Stolper-Samuelson theorem considers the effect of a change in the price of a commodity on factor prices. More precisely, it states that if the price of a good increases, the return to the factor used most intensively in the production of that good will increase more than proportionally with respect to the original price increase.

Following Chipman (1969), both theorems can be classified into a strong and weak form: let

$\ln w_i = \omega_i$, $\ln p_i = \pi_i$

then

a) Weak Stolper-Samuelson criterion

$$\frac{\delta \omega i}{\delta \pi i} \to 0 \quad i = 1 \ldots n$$

that is: "this expresses the fact that the ith factor price will increase more than the ith commodity price, which is equivalent to the condition that it rises relative to all commodity prices;this is the criterion which makes it possible to avoid the index number problem."[14]

b) Strong Stolper-Samuelson criterion

$$\frac{\delta \omega i}{\delta \pi j} \begin{matrix} > 0 & i = j \\ < 0 & i \neq j \end{matrix} \qquad i, j = 1 \ldots n$$

c) Weak Samuelson-Rybczynski criterion

$$\frac{\delta Xi}{\delta Vi} > 0 \qquad i = 1 \ldots n$$

This means that "there will exist an association between commodities and factors such that an increase in the endowment of the jth factor will increase the production of the jth commodity".[15]

d) Strong Samuelson-Rybczynski criterion

$$\frac{\delta Xi}{\delta Vj} \begin{matrix} > 0 & i = j \\ < 0 & i \neq j \end{matrix} \qquad i, j = 1 \ldots n$$

The discussions on the theorem have always been considered in the case where the number of goods and factors are equal. Kemp, Uekawa and Wegge (1973) have extended the theorems to the case where intermediate products are possible, retaining, however, the restriction on the equality between the number of goods and primary factors.

With all goods produced, the system of equation can be presented as follows

$$A \quad x = V \tag{31}$$

$$A' \quad w \leqq P \tag{32}$$

(31) states that $\sum_{j=1}^{N} a_{ij} X_j = V_i \quad i = 1 \ldots m$

(32) states that $\sum_{i=1}^{N} a_{ij} w_i \leqq p_j \quad j = 1 \ldots n$

In the more general case, if one good is not produced, then the inequality sign will be valid in (26). With all goods produced, the equality sign holds.

Suppose m = n: Assuming that A is independent of V, it is possible to derive from (31)

$$\frac{dX}{dV} = A^{-1} \tag{33}$$

By transposing both sides of (32), the following is obtained after differentiation

$$(dw')A + w' \, dA = dP' \tag{34}$$

The second term of (34) vanishes, for it is an application of the Wong-Viner "envelope theorem" given by Samuelson (1947). In his 1953 paper, he sketches the proof as follows: the second term will be equal to $\sum_k w_k\ (\delta a_j / \delta w_k)$; it is " seen to vanish by virtue of Euler's theorem applied to the homogeneous function of order zero representing aj and where the reciprocity relations have been utilized".[16] We are thus left with

$$\frac{dw'}{dP} = A^{-1} \tag{35}$$

From (33) and (35) the fundamental duality relation is derived, i.e.,

$$\frac{\delta X_i}{\delta V_j} = \frac{\delta w_j}{\delta P_i} \tag{36}$$

The papers by Chipman (1969), Kemp and Wegge (1969) and Uekawa (1971) in the bibliography, present the conditions whether necessary and/or sufficient, which verify either the strong or the weak version of both theorem.

If m and n are unequal, is it possible to derive from (31) and (32) the following relations?

$$\frac{dX}{dV'} = (A'A)^{-1} A' \tag{37}$$

$$\frac{dw}{dP} = (AA')^{-1} A \tag{38}$$

These two relations cannot both hold at the same time unless A is square, i.e., m equals n. In fact, it is possible for both to be invalid. With m>n, (38) cannot hold because matrix (AA') is of size m X m and cannot be of rank n; even if the rank condition to have (37) is satisfied, this equation will not hold for the case where m is greater than n. This is due to the fact that in order to derive (37) from (31) we need to assume that A is independent of V. Kuga (1972) showed that for the case where m >n, A is dependent on V.[17]

For m <n, then (37) is not valid because $(A'A)$ is of size n X n. However, (38) is still valid. Note that in the case where commodities outnumber factors, we cannot use (38) to get some insight into the value of the matrix (dX/dV'). As will be shown later, this comes from the fact that the duality between the two effects does not hold anymore.

The case of A depending on V is illustrated by Jones' paper (1971), where he considers a model with three factors and two goods. Furthermore, he assumes that two of these factors enter into the production of only one good. Using the full employment conditions, he is able to derive a system of three equations with three unknowns. According to our notation, these equations are:

$$u_{11} w_1 + u_{31} w_3 = P_1$$

$$u_{22} w_2 + u_{32} w_3 = P_2$$

$$\frac{u_{31} v_1}{u_{11}} + \frac{u_{32} v_2}{u_{22}} = v$$

In that case, factor prices depend on endowments.

Samuelson (1953) states that the duality between the two theorems hold for the general case. However, his proof is too sketchy and it is possible to show how a proof of the duality between the two effects can be integrated into the discussion on the transformation surface. Given our original assumptions, it can be proven:

Theorem VII

If the transformation surface is strictly concave, (36) holds for the case where $m > n$.

Let ∇f^i be the vector of first partial derivative of f^i; 10' can be transformed into matrix notation; the full employment conditions and the set of production functions can be used. All of this is needed to obtain

$$x_i = f^i(a_i) \qquad i = 1 \ldots N \tag{39}$$

$$\sum_{i=1}^{N} a_{ij} = v_j \qquad j = 1 \ldots M \tag{40}$$

$$W = P_i \nabla f^i \qquad i = 1 \ldots N \tag{41}$$

(39), (40) and (41) can be viewed as both the set of solutions to the maximization process in (9) and also as the implied dual relations.(18) In both cases the lagrangian multipliers are replaced by the goods or commodity prices.(19)

System (39) represents, in other words, the set of production functions, (40) the full employment conditions, and (41), the equilibrium conditions in the production side of the economy. We have thus $N + M + M \times N$ equations in the same number of unknowns (X, W and the a_{ij}). The parameters are V and P.

In order to carry out total differentiation, the systems (39) to (41) must have a unique solution. This is due to the fact that the partial derivatives must be evaluated at the equilibrium point. It is for this reason that it is assumed that the production surface is strictly convex. Only in that case will the system have a unique solution. If, on the other hand, the surface is not strictly convex, then it will possess linear segments and the system will offer an infinite number of solutions. If (40) is written as

$$\sum_{i=1}^{N} \frac{a_{ij}}{x_i} x_i = v_j \qquad j = 1 \ldots M \qquad (41')$$

the result is (31). (31) must therefore possess a unique solution, thus requiring that $M \geq N$. For this reason only this case is considered.

Let us differentiate (39) to (41) totally.

$$dx_i = \sum_{j=1}^{M} \nabla f_j^i \; da_{ij} \qquad i = 1 \ldots \qquad (42)$$

$$dv_j = \sum_{i=1}^{N} da_{ij} \qquad j = 1 \ldots M \qquad (43)$$

$$dw = P_i F^i \; da_i + \nabla f^i dp_i \qquad i = 1 \ldots N \qquad (44)$$

(42) to (44) are next written into matrix form: define

$$B = \begin{bmatrix} I_{(N)} & O_{(N \times M)} & -F_1 & \cdots & -F_N \\ O_{(M \times N)} & O_{(M \times M)} & I_M & \cdots & I_M \\ \vdots & I_M & -P_1 F^1 & \cdots & O \\ \vdots & \vdots & \vdots & & \vdots \\ O_{(M \times N)} & I_M & O & \cdots & -P_N F^N \end{bmatrix}$$

$$Y_{(N+M+N \times M) \times 1} = \begin{bmatrix} dX \\ dW \\ da_1 \\ \vdots \\ da_N \end{bmatrix} \qquad Z_{(N+M+M \times N) \times 1} = \begin{bmatrix} O_{(N \times 1)} \\ dV_{(M \times 1)} \\ O \\ \vdots \\ O \end{bmatrix} + \begin{bmatrix} O_{(M+N) \times 1} \\ \nabla f_1 dp_1 \\ \vdots \\ \nabla f_N dp_N \end{bmatrix}$$

(16) and (18) can be rewritten as

$$B \, y = Z \qquad (45)$$

Let us first consider the effect of a change in endowment on output. In other words $\frac{dx_i}{dv_j}$ is under study. In that case the following is obtained:

$$dv_K = 0 \quad dp_i = 0 \quad K = 1 \ldots M \quad i = 1 \ldots N$$
$$K \neq j$$

By using Cramer's rule, ~~and defining~~ a matrix C as matrix B with the ith column replaced by a column of all zero, except for the N + j element which is equal to unity, the following is obtained:

$$\frac{\partial x_i}{\partial v_j} = \frac{|C|}{|B|} \qquad (46)$$

The horizontal lines define a determinant. We first take the expansion of C along the first N columns except the jth. Since these columns all show unity along the diagonal, and zero everywhere else, all these are simply eliminated, together with their respective row. The i column and the ith row leave out the first N columns and the first N rows. The ith row will be as follows:

$$0_{1X(1+iM)} \quad -\nabla f^{i}{}_{1XM} \quad 0_{1X\,(N-i)M} \; .$$

the ith column will have 1 in the (j+1) element. If we expand the rows in the first column (the original ith) and we multiply the first row by (-1), we get

$$|C| = (-1)^j \begin{vmatrix} 0_{1X(1+iM-1)} & f^i_{1XM} & & 0_{1X(N-i)XN} \\ 0_{(M-1)XM} & I^j_N & \cdots\cdots\cdots\cdots & I^j_N \\ I_M & -P_1F^1 & \cdots\cdots\cdots\cdots & 0 \\ & & & \\ I_M & 0 & \cdots\cdots\cdots\cdots & P_NF^N \end{vmatrix} \qquad (47)$$

Let us now turn to $\frac{\partial w_j}{\partial p_i}$. We define matrix D as B N + jth column replaced by all zeros except the vector ∇f_i from the N + M + Mi + 1 element to the N + M + M (1 + i), we have

$$\frac{\partial w_j}{\partial p_i} = \frac{|D|}{|B|}$$

We can make an expansion along the first N columns which all have a unity on the main diagonal. This does not change the value of the determinant. It simply eliminates the first N rows and columns. Next, the jth column of the new matrix can be placed in the first column thus pushing the original column of the first j-1 column into the second column and the second into the third one and so on.

$$|D| = \begin{vmatrix} 0_{(1+iM-1) \times 1} & 0_{M \times (M-1)} & I_M & \cdots & I_M \\ \nabla f^i_{(M \times 1)} & {}^jI_M & -P_1F^1 & \cdots & 0 \\ 0 & & & & \\ 0 & {}^jI_M & 0 & & -P_NF^N \end{vmatrix}$$

Since the Hessian matrices F^i (i=1 ...N) are symmetric, it is clear that in their reduced form both (47) and (48) are the same because

$$|D| = |C'|$$

Our factors can always be labelled in such a way that j is even and thus the following is obtained

$$|D| = |C'| = |C|$$

which gives the required result

$$\frac{\partial x_i}{\partial v_j} = \frac{|C|}{|B|} = \frac{|D|}{|B|} = \frac{\partial w_j}{\partial p_i} \qquad (49)$$

Section V: The factor price equalization theorem

The theorem which stipulates that free trade will equalize not only commodity prices but also factor prices, has, for a long time, been at the heart of the debate among the economists who are interested in the pure theory of international trade. The discussion has even lead to some important discoveries in the field of mathematics (for example, Gale and Nikaido [1965]). The literature contains such an abundance of summaries and survey in that area, that it was deemed unnecessary to repeat the work here.

There is however one point to note here. Most of the studies have been conducted according to the method of global inversion of costs functions. Kuga (1972) has taken quite a different approach; he is interested in the condition for dw/dV to vanish, i.e., factor prices are independent of endowments. Since A depends solely on W, then his condition implies that A is independent of V. This condition is therefore necessary in order that the Stolper-Samuelson and the Samuelson-Rybczynski theorems may hold. He also shows that " The non-joint case when factors outnumber commodities never meets this condition" (p.723). He states : " The condition for [dw/dV] to vanish is

$$(10) \qquad J_{*} = J_{**}(J_{***})^{-1}(J_{**})'$$

where $J_{*} = [T_{ij}]$, $T_{ij} = \frac{\delta^2 T}{\delta V_i \delta V_j}$

$J_{**} = [T_{i,m-i+j}]$, $T_{i,m-i+j} = \frac{\delta^2 T}{\delta V_i \delta Q_j}$

and $J_{***} = [T_{m-1+j,m-1+i}]$

$$T_{m-1+j,\, m-1+i} = \frac{\delta^2 T}{\delta Q_j \delta Q_i}\text{."}$$

He presents the following economic interpretation: the right-hand side, which refers to the direct effect, represents "such changes in factor-prices without adjustment in Qj's". The left-hand side, called the adjustment effect represents "the possible amount of adjustment in the Wi's through Qj's corresponding to the discrepancies $(J_{**})'$.".(21) It is thus necessary and sufficient that the two effects cancel each other.

By using this novel approach, he is able to consider the case where N > m.

He states :

"Theorem 4 : If commodities outnumber factors in the non-joint production case, the full rank conditions of all the m n submatrices of the input-coefficient matrix is enough to ensure the factor price equalization. " Kuga, (1972, p.732).

The factor price equalization theorem was very briefly considered in the fourth section. The results obtained by Kuga which were discussed in length in the previous section, were placed in proper perspective.

Section VI

Because most of the pure international trade theory has been developed by means of the two-by-two model, there is a basic relationship which seems to have been overlooked. It concerns the effect of a change in the price of a commodity on the output. With only two goods, strict concavity of the transformation curve will guarantee that if the price of one good relative to the other is increased, the production of the first will increase while the production of the second will decrease. It is not clear, though, that this will occur when the number of goods is increased.

In order to clarify that issue, the following question must be asked : Does it necessarily follow that the quantity of produced good decreases when the price of another increases, or is it possible to increase the quantity of that good when the price of the other increases? A study on the possibility of gross complementarity along the production surface can clarify the issue. The sign pattern of the following matrix is therefore needed:

$$\frac{dX}{dP'}_{n \times n} = \left[\frac{dx_i}{dp_j}\right]_{n \times n} \qquad (50)$$

The relation between this question and gross complementarity is apparent. It is well-known that if the number of goods is greater than two, strictly quasi-convexity of the isoquants will not guarantee gross substitutability.

Yet, the relationship which must be derived has not been fully explored. There are two reasons for this. First, the transformation surface has been used mainly in a two goods model. In that case, the matrix in (50) possesses a positive element on the main diagonal and a negative element elsewhere, this being an obvious conclusion from the concavity of the production possibility curve. Secondly, the analogy to demand theory has long been recognized in production theory. It has been used, however, to derive demand of factors of production, and not aggregate supply relations.

In order to derive the relationship, it is necessary to use (39) to (41) of Section IV, i.e., from the differential of the solution to the constrained maximization problem. $dv_j = 0$ $(j=1....m)$ is fixed and the sign of $\delta Xi/\delta Pj$ is sought by doing the appropriate simplification, it is possible to reduce the expression to the following determinant :

$$\frac{\delta Xi}{\delta Pj} = \left[\begin{array}{cccc} 0\ldots\ldots0 & & \nabla f_i' & 0\ldots\ldots0 \\ 0 & & & \\ \vdots & 0 & I_M \ldots\ldots\ldots\ldots\ldots & I_N \\ 0 & I_M & -P_1F^1 \ldots\ldots\ldots & 0 \\ \nabla f_j & \vdots & \vdots & \\ 0 & \vdots & \vdots & \\ 0 & I_M & 0 \ldots\ldots\ldots\ldots & -P_NF^N \end{array}\right] \qquad (51)$$

The first row of (51) is composed of $(i\text{x}m)+1$ zeros while the first column is made of $(i\text{x}n)+1$ zeros, before $\nabla f_i'$ and ∇f_j respectively. Instead of evaluating (51) directly, this procedure can be used and it follows more closely the techniques traditionally employed to derive the other theorems in international trade theory. The full-employment conditions is stated as

$$Ax = V$$

Differentiating totally and using the fact that endowments do not vary[22], we get

$$AdX + (dA)\,X = 0 \qquad (52)$$

We can partition the matrix (dA) into the differential of every row: dAi; i.e., $dA = [\,dA_1 \ldots\ldots dA_N\,]$.

With all goods produced, equation (53) holds

$$W = p_i \nabla f^i \qquad (53)$$

we can derive from (53)

$$\frac{\delta Ai}{\delta w} = p_i F^{-1}(i) \qquad (54)$$

Where $F(i)^{-1}$ is the inverse of matrix F for the ith good. Note that (54) gives a matrix of partial derivatives. Using the chain rule we can derive

$$dAi = \frac{\delta Ai}{\delta w}\;\frac{dw}{dP}\quad dP \qquad (55)$$

in other words

$$da_{ij} = \sum_{k=1}^{M} \frac{\delta a_{ij}}{\delta \omega k} dw_k$$

$$\text{Thus } da_{ij} = \sum_{k=1}^{M} \sum_{r=1}^{M} \frac{\delta a_{ij}}{\delta w_k} \frac{\delta w_k}{\delta P_r} dP_r \quad i=1....M$$

By rewriting the above equation in matrix form, the relation (55) can be obtained

$$\text{Let } S = \left[\frac{\delta w}{\delta p'}\right] \tag{56}$$

Using (54) and (56) in (55) we get

$$dA_i = p_i F^{-1} \quad (i) \quad S\,dP = C(i)dP$$

Define $I_{m \times mn}$ as a matrix of size (m X mn) composed of n identities matrix of size m, one beside the other. B is a matrix of size mn X nn such that

$$B = \begin{bmatrix} C\ (i) & 0 \ .....0 \\ 0 & C(2).....0 \\ . & . \quad\quad . \\ . & . \quad\quad . \\ . & . \quad\quad . \\ . & . \quad\quad . \\ 0 & 0.....C(n) \end{bmatrix}$$

We can also define dP* as a vector of size nn X 1 such that

$$dP^* = \begin{pmatrix} x_1 & dP \\ x_2 & dP \\ \cdot & \\ x_n & dP \end{pmatrix}$$

Thus $(dA)\ X = I_{(m\ X\ mn)} B\ dP^*$

from (53) the following is obtained

$$dX = (A'A)^{-1} A'\ I_{(m\ X\ mn)} B\ dP^*$$

Let D be a matrix of size nn X n such that

$$D = \begin{bmatrix} x_1 & I_n \\ \cdot & \\ \cdot & \\ \cdot & \\ \cdot & \\ \cdot & \\ x_n & I_n \end{bmatrix}$$

Then

$$\frac{dX'}{dP} = -(A'A)^{-1}\ A'\ I_{m\ X\ mn} BD$$

S can be multiplied by $I_{(m \times mn)}$. The resulting matrix can be multiplied by B and separated again such that

$$\frac{dX_{\prime}}{dP} = I_{(n \times mn)} M\, I_{(nn \times n)}$$

where

$$M = \begin{bmatrix} M(1) & 0 & 0 \\ 0 & & \\ 0 & M(n) & \\ \multicolumn{3}{l}{M(i) = x_i\, (A'A)^{-1} A'\, F^{-1}(i)\, S} \end{bmatrix}$$

define M (ij) as the jth row of M (i).

then

$$\frac{dX_{\prime}}{dP} = - \sum_{j=1}^{N} M\,(1j) \text{------} \sum_{j=1}^{N} M\,(nj) \qquad (57)$$

Consider the two-by-two case.[23] Then,

$$M(i) = p_i x_i\ A^{-1}\, F(i)^{-1}\ (A^{-1})$$

whether A^{-1} has positive elements along the main diagonal and negative elsewhere, or vice-versa, the sign pattern of (57) will still be the same:

$$\frac{dX_{\prime}}{dP} = \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

When more than two factors are considered, thepossibility of gross complementarity between factor and the fact that the strong Samuelson-Rybczynski result might not hold, leads to the possibility of having positive elements that are not on the main diagonal of (dX/dP').

Section VII The Heckscher-Ohlin Theorem

Briefly, the Heckscher-Ohlin Theorem states that a country will export the good which uses more intensively its abundant factor. As soon as factor intensity is considered, ratios enter the discussion; the usual two factors being capital and labor, the theorem states that the country with the higher capital labor ratio will export the capital intensive good. In order that the theorem be valid, the capital intensive good must be the same in every possible set of factor prices. It is well-known that the theorem might not be valid if factor intensities are reversed when a specified set of factor prices are given.

Also, something about demand must be known; it is not only necessary that all countries face the same set of community indifference curves, but it is also necessary that these curves be homothetic. See Robinson (1956) on this point.

Warne (1971) extended the theorem to the three-by-three case. His first theorem states:

" If a good is the only good most (least) intensive in a factor, any country abundant (scarce) in that factor will export (import) that good".

The proceedure is as follows: since every country faces the same set of homothetic indifference curves, all countries will consume goods in the same proportions, provided they face the same price vector. Thus, the application of the theorem consists of comparing production ratios, i.e., if country A is producing relatively more of good X_i, it will be exporting that good. Let V^{ω} be the world endowment of factors. That is

$$V^{\omega} = \sum_{i=1}^{K} V^{i}$$

At a given set of commodity prices, the solution to $AX = V^{\omega}$ will determine the consumption ray for each country because every country consumes in the same proportion, and trade balances for every commodity.

Consider equation (58)

$$\tilde{X}^{\omega} = A^{-1}V^{\omega} \tag{58}$$

Note that when V is multiplied by a constant, and when assuming that the conditions which assure that A is independent of V hold, it is possible to multiply X by the same constant. In other words, the production ray is homogeneous of degree zero in relative endowments.

Equation (58) shows a relationship between endowment and output; with A independent of V. A Taylor expansion can be applied to (58) and the result is

$$X^1 = X^\omega + \frac{\delta X}{\delta V} (V^1 - V^\omega) \qquad , (59)$$

Equation (59) is true because of the assumption that A is independent of A; all higher order terms of derivative vanish. For the result to be meaningful, it is imperative that A^{-1} exists; from Section 1 this implies a strictly convex production set. It is necessary to make sure that the technique is applicable, and this is only possible in the case where the first partial derivative exists everywhere.

In that case, and only in that case, can the points by the differential path be connected. If the partial derivative is to exist everywhere, A^{-1} must exist everywhere, which is sufficient to guarantee factor price equalization.

Through the information derived in (59) and also because of the fact that the production ray is homogeneous of degree zero in endowments, it is possible to multiply V^{ω} and/or V^{1} by a constant without changing the ratios between any two elements of X^{ω} and/or X^{1}. This possibility can be used as follows: by defining $V^{\omega *}$ as the result of V^{ω} multiplied by V_i^1 / V_i^{ω}, so that in the ith position along $V_i^{\omega *}$, V_i^1 is obtained. Constructing a Taylor expansion as in (59), $V_i^1 - V_i^{\omega *}$ is eliminated. Thus, instead of having n non-zero elements in $V^1 - V^{\omega *}$ n-1 is the result.

Thus, for the two-by-two case, we can derive a relation similar to (59) for country A and country B. By substructing the two equations the result is

$$X^A - X^B = \left.\frac{\delta X}{\delta V}\right|_{V^{\omega}} (V^A - V^B) \qquad (60)$$

By multiplying V^B, one factor can be eliminated and using K and L as our two factors, (60) will read

$$(X^A - X^B) = \frac{\delta X}{\delta V'} \Bigg|_{L^\omega} \begin{pmatrix} L^A - \dfrac{L_B K^A}{K^B} \\ K^A - K^A \end{pmatrix}$$

if the strong Samuelson result holds, the sign pattern of $\delta X / \delta V'$ will be $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$; if $L^A - L^B K^A / K^B > 0$ (61)

then $X_1^A > X_1^B$ and $X_2^A < X_2^B$; this implies that $X_1^A / X_2^A > X_1^B / X_2^B$ which in turn implies that A is exporting X_1 and importing X_2, X_1 being the L intensive good in that case. Rearranging (61) we get

$$\frac{L^A}{K^A} > \frac{L^B}{K^B}$$

the well-known factor intensity hypothesis.

It is always possible to eliminate one factor from (59). However only one can be eliminated. As soon as the number of factors is increased beyond 2, further restriction must be placed on A^{-1} in order to obtain a meaningfull result: this is how Warne worked with the 3 X 3 case.

CONCLUSION

The reader will have recognized some of the similarities between our discussion and the one presented by Samuelson (1953). Since that publication however, many new ideas and contributions were made and, where possible, these were integrated into the chapter.

Rather than summarizing each section, a few general conclusions can be drawn from the content of the chapter. First, there is the fact that the theory here, followed directly the path drawn out by the classical economists. Even though the demand side, which was presented in the discussion on community indifference surface is as important as the production side, that side having been summarized in the discussion on the transformation surface, the primary target of studies has been the production side: all the theorems discussed here were related to the production side.

Secondly it is important to emphasize the result obtained in Section VI which asserts that it is possible to have gross complementarity along the Transformation surface, with more than two goods. This permits the tracing of an offer curve in a three good model. This model shows that case where two goods are kept at the same price whilst the price of the third is increased does not necessarily imply a decrease in the production of both. This result can be useful for the study of transportation costs, if it is assumed that there is now, in the system, a third good called transportation services.

Also note that some theorems are no longer valid if applied at a higher dimension than at the two-by-two model. The reasons for this are as follows: some of these theorems rest on the special properties of two-by-two matrices; in that context, the definition of factor intensity becomes very relevant. However, when another dimension is considered, the special property of the matrix is lost and the result no longer applies.

All this information seems to point to the idea that other approaches to international trade theory must be investigated, i.e., other determinants of trades must be found.

Also, a change in the question underlying all the research done in international trade theory seems to be worth considering. Rather than asking what will the pattern of international trade be?, the question should perhaps ask[66] how much benefit is obtained through trades?

FOOTNOTES TO CHAPTER I

1. Lancaster, Mathematical Economics, MacMillan, (1968), pp.118-134.

2. We will include a discussion on the relation between the elasticity of substitution and the curvature of the production set.

3. Note with Melvin "all three methods are really variants of one another" 1970, p.287, ff.2.

4. See Chipman, (1964), p.11, for a proof of the uniqueness of A_i^* for a given W.

5. The cone of diversification is defined as the cone formed by the vector joining the cost minimization points and the origin.

6. R^n is used here to represent the n dimensional Euclidean space.

7. These results are taken from Goldberger (1964), p.36.

8. Lancaster, (1968), p.124.

9. I would like to thank Professor Melvin for pointing out this geometrical explanation.

10. That is $dx_j/dx_k = - p_k/p_j$ for $dx_i = 0$

    $i,j = 1 \ldots n \quad k \neq j\ ,\ i \neq k$

11. Melvin (1971), p.289.

12. Note that H is not the bordered Hessian matrix which is equal to :

$$\begin{matrix} F^i & \nabla f^i \\ \nabla f_i & 0 \end{matrix}$$

13. Allen, (1967), p.504.

14. Chipman, (1969), p.401.

15. Chipman, (1969), p.402.

16. Samuelson, (1953), p.8

17. See Kuga, (1972), p.732, Theorem 3.

18. See Lancaster, (1968), p.122-124.

19. We are assuming that all goods are produced.

20. Kuga, (1972), p.726-727.

21. Kuga, (1972), p.727.

22. As mentioned in Section V, it is quite possible for A to depend on endowments.

23. In that case $S = A^{-1}$.

CHAPTER II

SPECIFIC FACTORS OF PRODUCTION AND INTERNATIONAL TRADE THEORY*

Trade theorists usually treat capital as a homogeneous factor of production which can be substituted freely and perfectly between industries. The purpose of this paper is to investigate whether or not the traditional theorems of international trade can still be proven, when this assumption is relaxed. The assumption can be relaxed in different ways; Kenen (1965) treats capital as a factor of production that must be applied to land and labour, before these factors can be used in production.. It is then possible to explain the "Leontief paradox" arguing that the United States imports in reality land intensive goods instead of capital intensive goods.

The assumption that capital is a homogeneous factor of production shall be relaxed by assuming that each good is produced with a specific sort of capital and that there is no possibility of substitution between the two.

This model has recently been treated by Jones (1971). He uses it to show that factor prices will not be equalized through trade. Jones also uses the model to discuss Kenen's argument and Peter Temin's (1966) "remarks concerning American and British technology in the mid-nineteenth century".(1) Jones also discusses both the effect of a change in endowments on output and also the effect of a change in commodity prices on factor rewards. His work shall be extended by presenting demonstration of these results, and we will also investigate the effect of endowment and commodity price changes on the other variables of the model. In the Appendix, we will present alternative proof of his results.

The Chapter is divided as follows. Section I, presents the model. Section II discusses the pattern of trade in a world with two countries. Section III treats the Factor Price Equalization theorem. Sections

IV to VI discuss three comparative static statements; Section IV presents a study on the effects of a change in endowments, and the effects of a change in prices is studied in Section V, while the effects of the introduction of technological progress is analysed in Section VI. Section VII presents our conclusions. Sections in the Appendix are numbered according to the section of the paper to which they are related.

Section I: The Model

Let us first consider a closed economy producing two goods X and Y. Each good uses a common factor L and a specific factor in production: K will be the specific factor associated with X, while T will be the specific factor associated with Y. It is possible to use K and T as capital units which have been firstly embodied into their industry and which cannot be transferred secondly from one to the other in this world where there is no depreciation. The technologies which are opened to producers of each good, are summarized by the following production functions which are homogeneous of degree one, continuous, and quasi-concave:

$$X = f(L_X, K_X) \qquad (1)$$

$$Y = g(L_Y, T_Y) \qquad (2)$$

the following properties of the partial derivatives of these functions are assumed

$$f_i > 0 \quad f_{ii} < 0 \quad f_{ij} > 0 \quad i = L_X, K_X \quad (1a)$$
$$j = L_X, K_X \ (i \neq j)$$

$$g_i > 0 \quad g_{ii} < 0, \quad g_{ij} > 0 \quad i = L_Y, \ K_Y \qquad (2a)$$

$$j = L_Y, \ T_Y \ (i \neq j)$$

where the subscripts refer to the partial derivatives of the production functions f and g.

Next it is assumed that the country is endowed with a given fixed amount of each factor ($\bar{L}$, $\bar{K}$ and $\bar{T}$). There is, furthermore, no friction in the model, so that full-employment prevails at all times. Thus,

$$L_X + L_Y = \bar{L} \qquad (3)$$

$$K_X = \bar{K} \qquad (4)$$

$$T_Y = \bar{T} \qquad (5)$$

It is assumed that perfect competition prevails in all markets. Factors of production will thus be paid according to the value of their marginal products. Letting w, r and s be the nominal per unit returns to L, K and T respectively, the following can be written and this will close the production side of our model:

$$w = P_X f_L \quad (6)$$

$$w = P_Y g_L \quad (7)$$

$$r = P_X f_K \quad (8)$$

$$s = P_Y g_T \quad (9)$$

To close the model we need the demand side. We assume that tastes of the community can be summarized in a well-behaving community utility function such that the indifference curves are homothetic.[(2),(3)]

$$U = U(X, Y) \quad (10)$$

$$\frac{U_X}{U_Y} = -\frac{P_X}{P_Y} \quad (11)$$

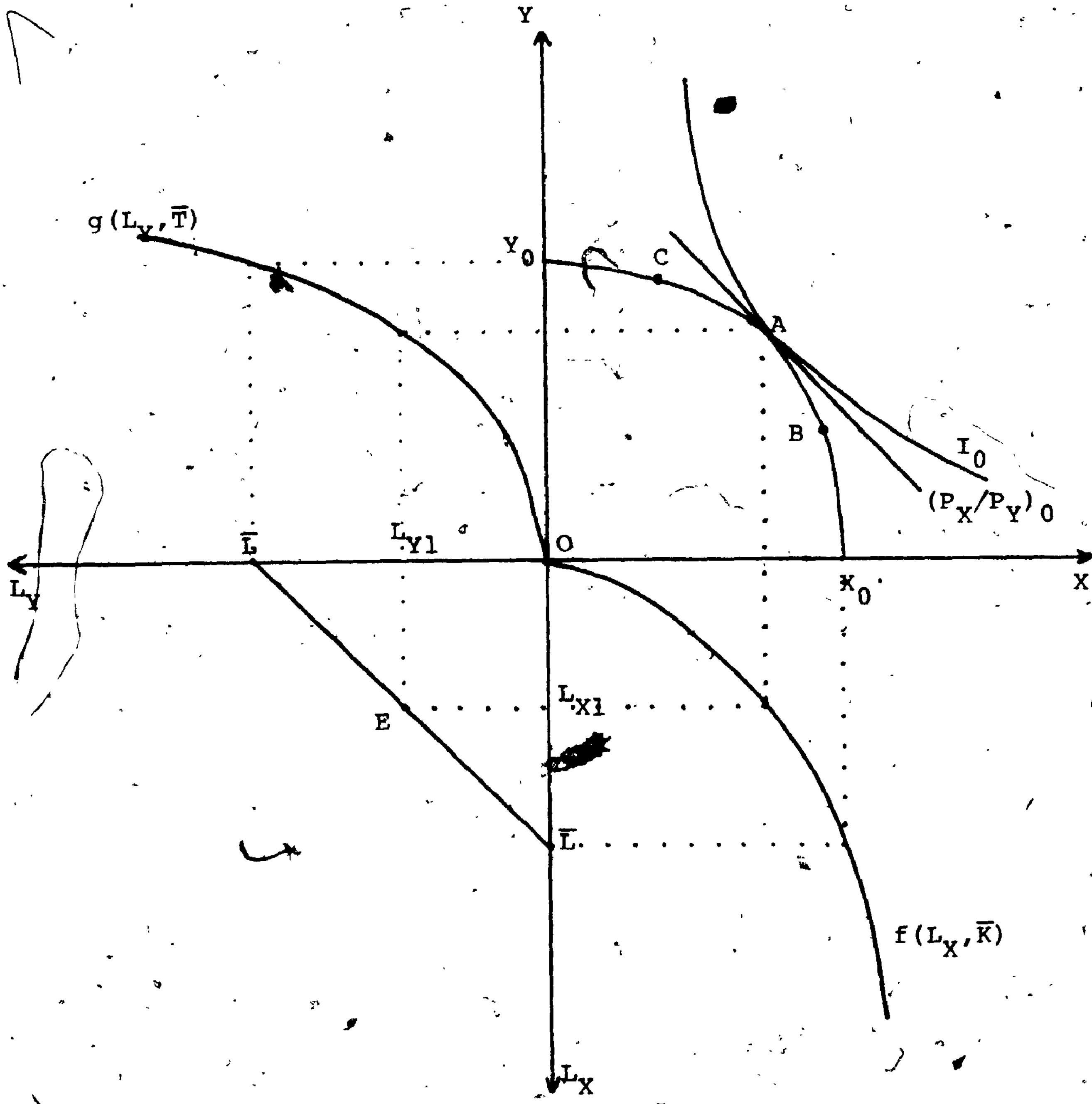
Equation (10) represents the utility function of the community; (11) shows the equilibrium condition of the demand side of the economy. In perfect competition, a maximization of society satisfaction will be achieved at a point of tangency between an indifference curve and the relevant price line.

A systen of eleven equations for the same number of unknowns is thus obtained: X, Y, L_X, L_Y, K_X, T_Y, w, s, r, U, P_X/ P_Y. Since our model is a special case of the general one used by Katzner (1972), because all the conditions for such a model are met, it therefore follows that a competitive equilibrium will exist and be unique in our model.

The following geometrical representation of the system can be used. Since only L can vary between industries, this reduces the task considerably, because it makes work at the two dimensional levels possible. With the help of Figure I we can derive the production possibility curve Y_OAX_O. It is shown in the Appendix that the slope at any point along this curve will be equal to the negative of the price ratio. It is furthermore demonstrated that the function is locally strictly convex. As can be seen from Figure I, global convexity of the production possibility curve can be established from the concavity of the two production functions.(4)

In the third quadrant of Figure I, $\overline{LL}$ is plotted thus showing the various combinations of L_X and L_Y that will keep L fully employed. In the second and fourth

FIGURE I



quadrants the two production functions are plotted under the assumption that K and T are fixed.

By moving from one end of LL to the other and by reading off from the production functions the quantities of X and Y produced, it is thus possible to generate the production possibility curve Y_OAX_O. For example, if labour is divided between industries in the manner shown by point E (L_Y and L_X), production will be placed at point A. Next determine the autarky equilibrium by introducing the set of community indifference curves in quadrant I. Point A represents the equilibrium, where I_O is tangent to the transformation curve. The common slope is equal to the negative of the price ratio (P_X/P_Y).

The relation between the price ratio and the factor ratio can now be studied for each industry. Consider Figure I. It can be observed that the price ratio increases as points move away from A and close in towards B. At the same time the production of X will increase while Y will be produced in lesser quantities. Less labour will be used in the Y industry while more will be used in the X

2 OF/DE 3

industry. Since T and K are fixed, this will imply that the (L_X/K) ratio will increase while the (L_Y/T) will decrease. In Figure II, the relationship between (P_X/P_Y) and (L_X/K) is represented as ϕ, while ψ represents the relationship between (P_X/P_Y) and (L_Y/T). In the Appendix it is proven that ψ will be negatively sloped while ϕ will be positively sloped and that the two are parallel as shown in that Figure.(5)

Similarly the relationship between w/r and L_X/K and between w/s and L_Y/T can be derived. Consider Figure III. The isoquant X_O is arbitrarily selected. The selection is not important because of our assumption of homogeneity of the production function. Consider point B. The slope of the isoquant is $(w/r)_O$. The slope of ray OB is K/L_X. As points move from B to C, w/r and K/L_X decrease. Thus w/r and L_X/K will move in opposite directions. This relationship is plotted as χ in Figure IV. Note that it is independent of endowments. A similar relationship (π) can be derived between w/s and L_Y/T.

FIGURE II

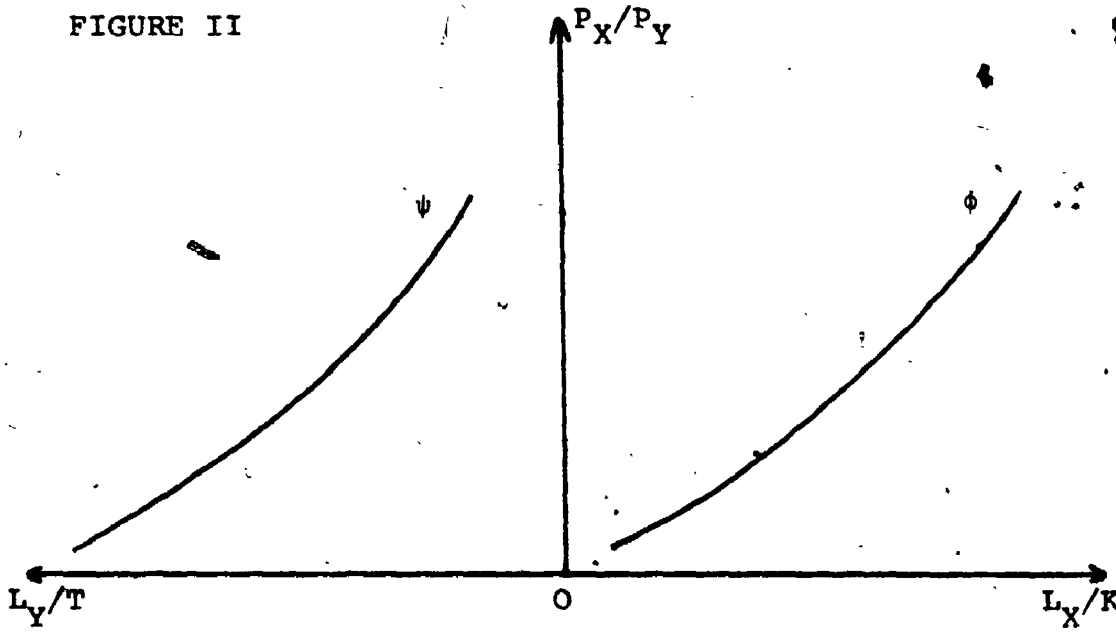


FIGURE III

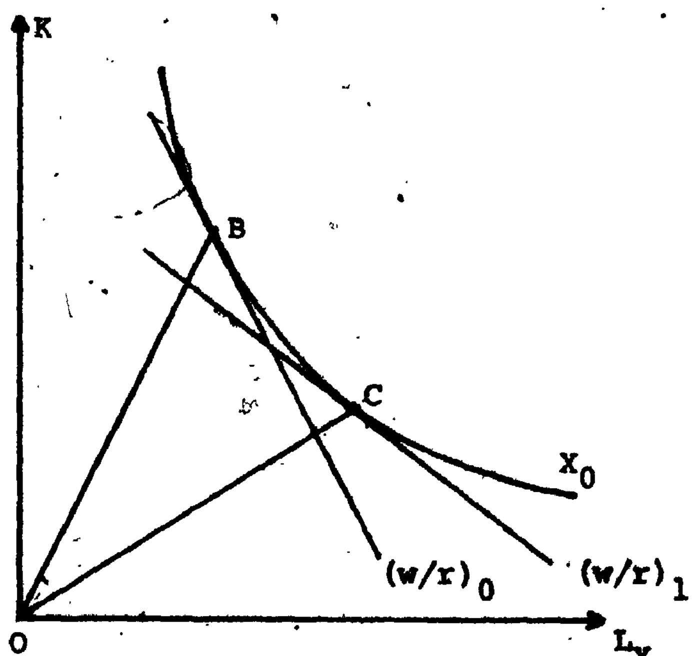
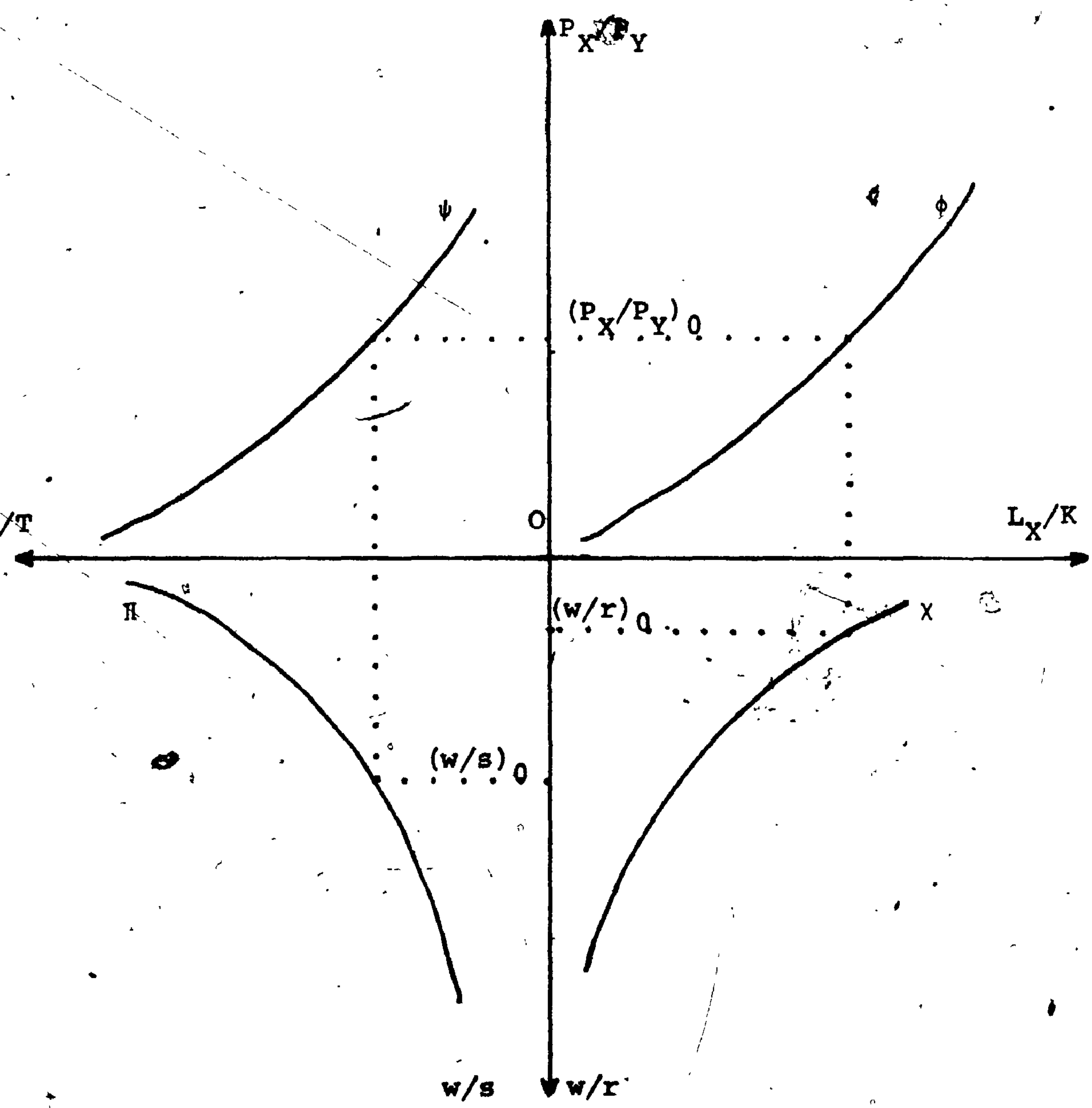


FIGURE IV



Also included in Figure IV are the functions ψ and ϕ which were already defined above. Through this, the distribution of factor and their relative returns can be determined for a given price ratio.

Let the price ratio at point A in Figure I be $(P_X/P_Y)_O$. The distribution of labour between X and Y will be L_{K1} and L_{Y2}. In Figure IV we see that associated to $(P_X/P_Y)_O$ will be the factor return ratios $(w/s)_O$ and $(w/r)_O$. Labour earning the same return in both industries we thus have the result that $r_O > s_O$ i.e., the nominal return to capital is higher in the X than in the Y industry.

It can also be shown that the relations ϕ, ψ, χ and π are monotonic. As long as (1a) and (2a) hold it is clear that π and χ will be monotonic. Similarly ϕ and ψ are monotonic as long as the production possibility curve is strictly concave. Here we are only considering the cases where both goods are produced. The monotonicity of these four relations will mean that for a given set of commodity prices, factor prices will be uniquely determined.

This does not necessarily mean that factor price will be equalized. As we will show, ψ and ϕ do depend on endowments.

This model, which has just been defined can be used to test the traditional conclusions of trade theory. More specifically, the Samuelson-Rybczynski, the Stolper-Samuelson, the Heckscher-Ohlin, and the factor price equalization theorems will be examined through this model. The role of technological change will also be considered.

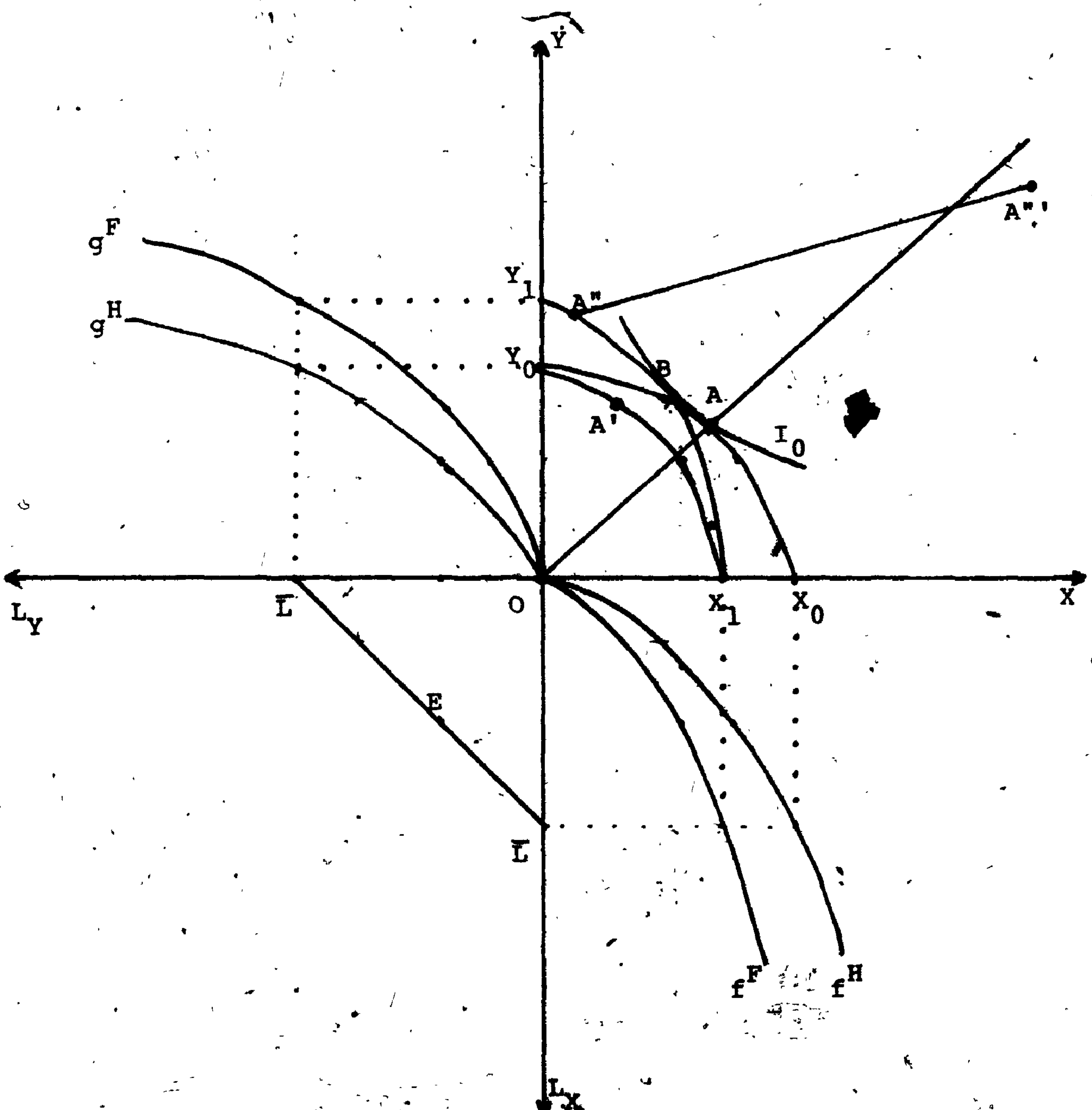
Section II - The Pattern of Trade.

Let us now introduce a second country and allow trade to take place.[6] By definition X is K intensive, while Y is intensive. A special case will be first examined. Let it be assumed that both countries have exactly the same amount of labour. It is furthermore assumed that the foreign country has absolutely more of factor T, while the domestic economy has absolutely more of K . We shall refer to the domestic economy as H and the foreign economy as F.

In Figure V, LL found in the third quadrant can be used to represent the full-employment of labour in both countries. Since F has more T, g (L_Y, T) will be higher for F than H. Because H has more K, f(L_X,K) will be higher for H.

The superscript to the production function indicates the country; for example, g^H is the production function of good Y for country H. Let point A

FIGURE V



be the autarky equilibrium for country H. It is now necessary to show the autarky equilibrium point for country H. This will allow us to establish a basis for trade and this basis is such that the traditional result holds.

The first one consists in considering a decrease in K, i.e., a shift from f^H to f^F. The production possibility curve will shift from Y_oAX_o to $Y_oA^1X_1$. As will be shown later, a decrease in K will decrease X and increase Y for the same price ratio. Point A^1 corresponds to the point on the new curve which has the same slope as it had at A. Secondly, the movement from g^H to g^F is considered. The production possibility curve for F is thus $Y_1A^{11}X_1$. At point A^{11} the slope of the transformation surface is the same as at point A^1 and A. The following relationship can be shown; if an increase in T is to shift from g^H to g^F, at constant commodity prices, the production of Y increases and the production of X decreases. Since the indifference curves are homothetic, consumption will still occur within ray OA. In autarky, there will thus be an excess demand of X and an excess supply of Y. The price of X will increase while the price of Y will decrease. The autarky equilibrium for F is thus at a point like B where X is relatively more expensive than Y. Once we allow trade, country F will export Y, the good which is intensive in its abundant factor. Country H will export X, the good using its abundant factor. This result is consistant with the usual result of the Heckscher-Ohlin model.

It is clear that the assumption of the same endowment of labour makes our case a very special one. To generalize our result, the property which states, that the production possibility curve is homogeneous in all endowments, must be used. Figure V shows that if the endowment vector of country F is multiplied by a constant, production will still occur along OB in autarky. Thus the same conclusion as in the previous case can be deduced.

It is therefore possible to state the following more general result: if a country has an absolute advantage in one factor and an absolute disadvantage in another, relatively to the same endowment of labour, it will export the good using the factor in which it has an absolute advantage. We will say H will export X if

$$\frac{K^H}{L_i^H} > \frac{K^F}{L_i^F}$$

and $$\frac{T^H}{L_i^H} < \frac{T^F}{L_i^F}$$

where L_i^H is a given level of L for H and L_i^F is a given level of L for F.

Can this result be generalized further? Is it possible to argue that country F exports X if,

$$\frac{(K)^F}{(T)} > \frac{(K)^H}{(T)} \qquad (12)$$

as in the two-by-two model where only relative endowments determine the pattern of trade? The answer to this question is obviously no. The result cannot be generalized; however it cannot be proven wrong.

Further generalization cannot be made because the production possibility curve is not homogeneous in L alone but in all endowments. Consider Figure V. Suppose that the endowment of labour is increased for F. In general it cannot be expected that production will still occur along ray OA^{11}. Suppose it is moving instead along segment $A^{11}A^{111}$. If the endowment of labour in F is such that A^{111} is the production point, the conclusion will be reversed and the country which has a greater amount of T in absolute size will export the K intensive good. On the other hand, as long as no portion of the line $A^{11}A^{111}$ intersects ray OA or its extension, the traditional result will hold.

There are three factors which will determine the direction the line $A^{11}A^{111}$ may take. It can be shown that

$$\frac{d\left(\frac{Y}{X}\right)}{d\,(L)} = \frac{w}{Y^2(P_X f_{LL} + P_Y g_{LL})}\left(Y \frac{P_Y}{P_Y} g_{LL} - X \frac{P_X}{P_Y} f_{LL}\right) \qquad (13).$$

The three factors can be seen as the slope of the transformation surface (P_X/P_Y), where we are on the surface (X/Y) and the curvature of both production functions (f_{LL} and g_{LL}). This greatly restricts the possibility of generating further results. Note that the relative endowments of T and K will affect the result since they will affect g_{LL} and f_{LL} respectively.

In summary, it is possible to state that the traditional result can be shown to hold, i.e., that the T abundant country exports the T intensive good.(7) The condition can be worked out as,

$$\frac{K_H}{L_H} > \frac{K_F}{L_F} \quad \text{and} \quad \frac{T_H}{L_H} < \frac{T_F}{L_F} \qquad (14)$$

In (14) the relative endowment of labour which is used to compare both K and T is chosen to be unity.

Section III - Factor Price Equalization.

In this type of world, factor prices will not be equalized. This is one of Jones' main points. Factor prices do not depend on commodity prices alone. They also depend on endowments. In terms of our model consider Figure IV. As endowments are changed, ϕ and ψ will shift while π and $_\sigma X$ are independent of endowments. It is even possible to conceive a case where factor prices are equal before trade and unequal after.

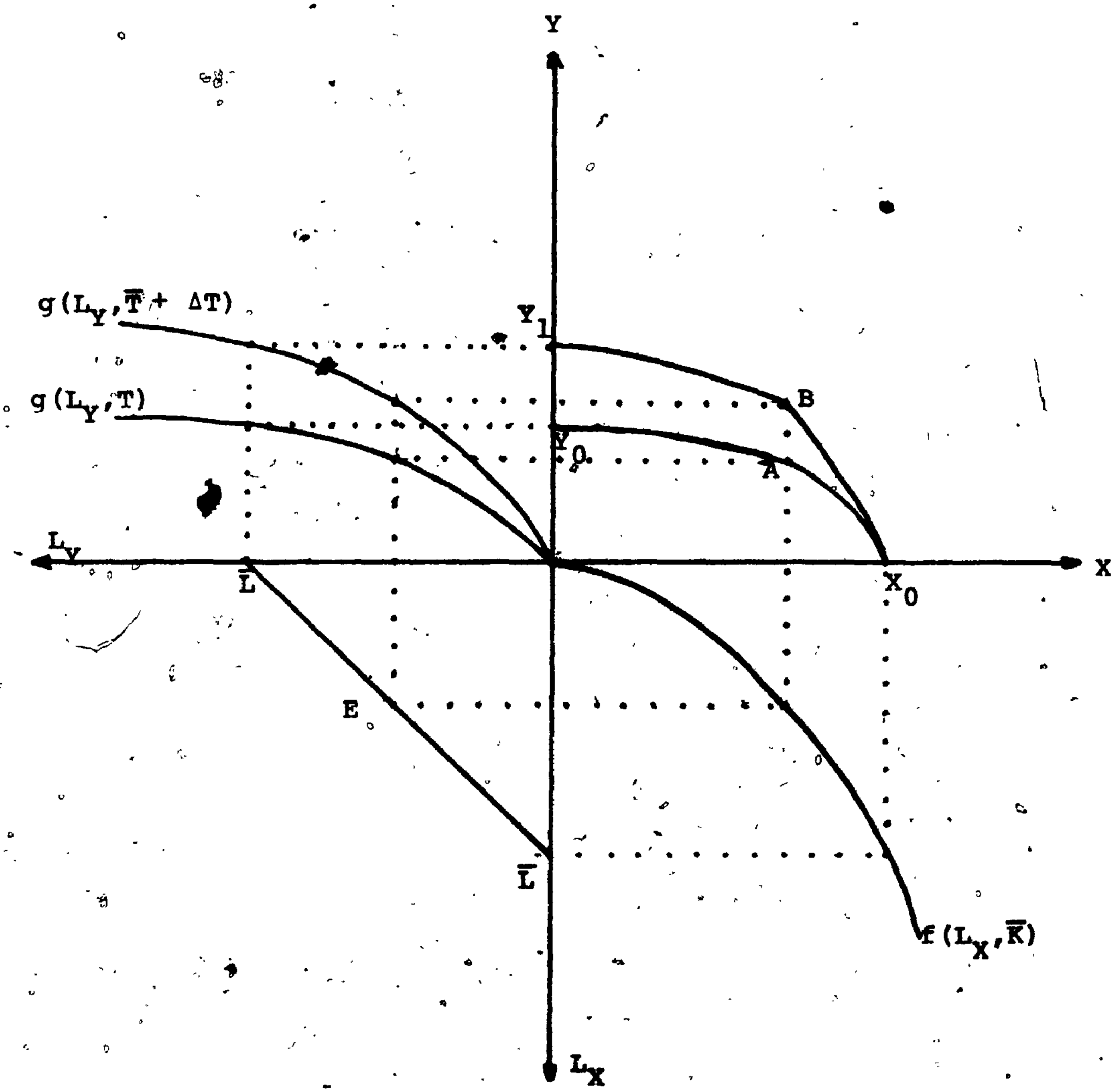
Section IV - The Samuelson-Rybczynski Theorem

The first comparative static problem considered, studies the effect of a change in the endowment of one factor at constant prices. Changes in $\bar{T}$, $\bar{L}$ and K will be considered respectively.

In Figure VI it is assumed that an increase in $\bar{T}$ has shifted the production function of Y from $g(L_Y, \bar{T})$ to $g(L_Y, \bar{T} + \Delta\bar{T})$. This means an upward shift of the production possibility curve from X_oAY_o to X_oBY_1, since the maximum amount of X which can be produced cannot vary.

Kemp (1969, p.14) states the Samuelson-Rybczynski theorem for the two-by-two case as follows: " An increase in the endowment of any factor results in an expansion of whichever industry is relatively intensive in its use of that factor and in a decline in the output of the other industry ", at fixed terms of trade. In the case of an increase in $\bar{T}$, it is clear that Y is the T intensive good, thus resulting in an increase in Y and in a decrease in X, if the theorem is still

FIGURE VI



to be valid.

Consider point B in Figure VI. If the increase in T would not change the production of X, then the new situation would be B. But from (2a) movement from A to B will tend to increase the marginal physical product of labour in Y (q_L) while remaining constant in X. At constant prices this means a higher wage in the Y industry as taken from (6) and (7). Thus labour will move from X to Y, decreasing X (increasing MPL_X) and increasing Y (decreasing MPL_Y) until the wage rate returns to equality. A mathematical proof is given in the Appendix.[8]

Next consider an increase in $\bar{K}$. It is clear that the argument presented above about $\bar{T}$ will apply exactly in the same manner. Thus, an increase in $\bar{K}$ will increase X and decrease Y, at constant prices. The Appendix gives a rigorous proof.[9]

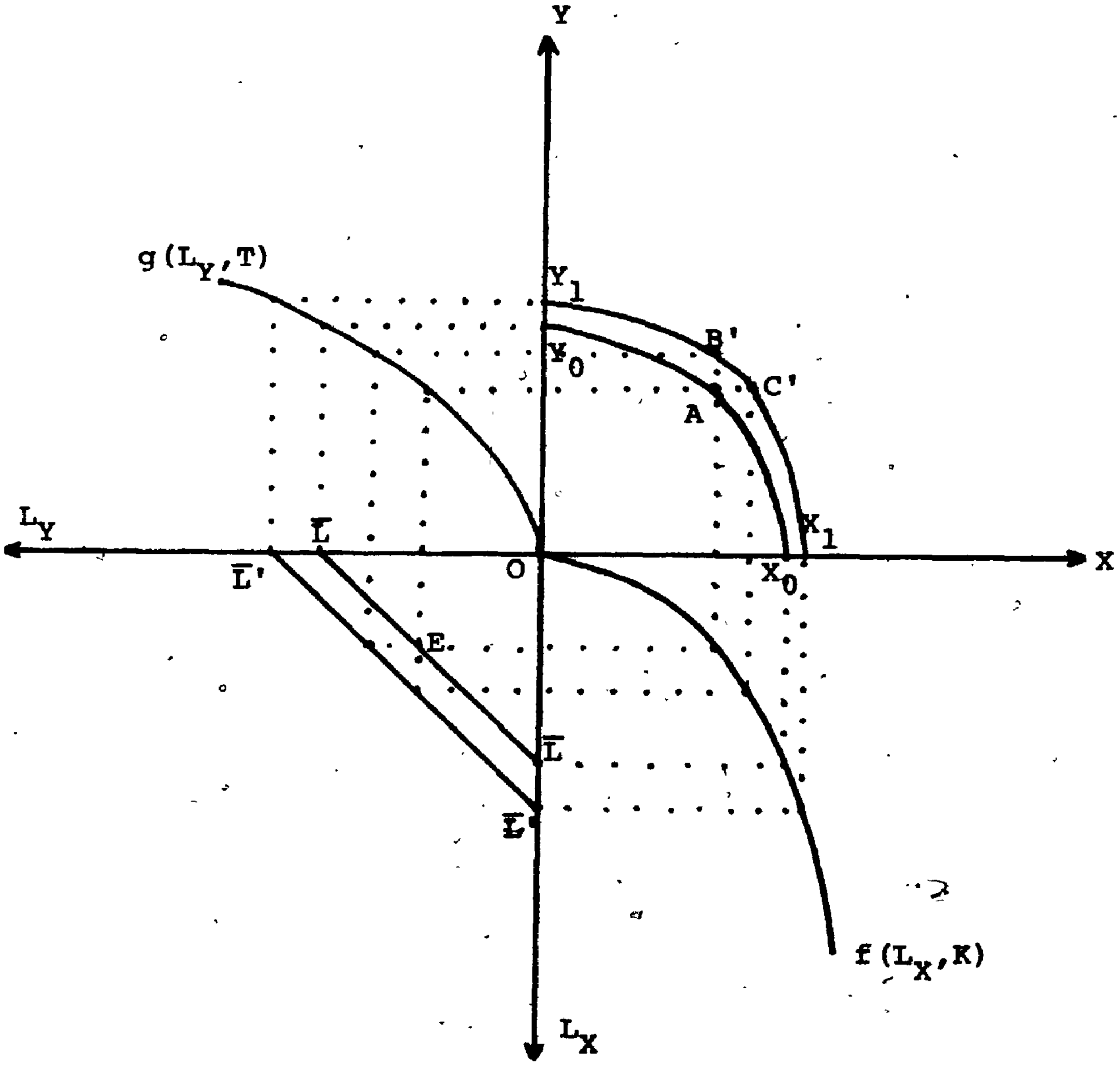
Thus, a change in the endowment of a specific factor will lead to a change in the same direction for the good using this factor and in the opposite direction for the other good. Thus, the Samuelson-Rybczynski theorem is valid in the case of an increase in the endowment of a specific factor.

Finally, a change in the endowment of labour can be considered. Figure VII considers an increase in labour from $\overline{LL}$ to $\overline{L'L'}$. As neither $\overline{K}$ nor $\overline{T}$ vary, g $(L,\overline{T})$ and f $(L, \overline{K})$ will remain in their original position. The production possibility curve will thus shift from Y_0AX_0 to Y_1 B'C'X. As noted in Section II, the new curve is not homogeneous to the first one.

It is possible to show that an increase in $\overline{L}$ will lead to an increase in the production of both goods. Starting from the equilibrium point A, it can be noted that if industry Y receives all the increase in labour, there would be a shift from A to B'; but at this point, MPL_Y has declined while MPL_X is constant. Based on (6) and (7), this means that at constant prices the wage rate in Y would be lower and thus there will be a movement of labour from Y to X. Thus, X increases so that the new equilibrium be to the right of B'.

Similarly, if the X industry were receiving all the increase in labour, there would be a shift from A to C'; but at this point MPL_X has declined while MPL_Y is constant. Based again on (6) and (7), the

FIGURE VII



wage rate is lower in X; if prices are constant, we would thus have a movement of labour from X to Y, and an increase in Y, and, therefore, the new equilibrium must occur at the left of C'.

From this it follows that both X and Y will increase. This is confirmed by results (A.24) and (A.25) in the Appendix. It must be noted that the new equilibrium is not necessarily along ray OA extended. In the appendix (A.28), the condition necessary for this result to occur is derived.

We can summarize our results in the following theorem:

Theorem I

(a) An increase (decrease) in the endowment of one of the specific factors of production will lead to an increase (decrease) in the production of the good using this specific factor and to a decrease (increase) in the production of the other good. These results verify the validity of the Samuelson-Rybczynski theorem in the case of the specific factors.

(b) An increase (decrease) in the common factor (L) will lead to an increase (decrease) in the production of both goods. This is possible because it is impossible to define a labour intensive good in the model.

Section V - The Stolper-Samuelson Theorem.

Kemp (1969, p.17) states the Stolper-Samuelson theorem as" an increase in the price of any commodity gives rise to an increase in the real reward of whichever factor is used relatively intensively in the production of that commodity, and to a decline in the real reward of the other factor." If the theorem is to ve valid, it must be shown that an increase in the price of good X will lead to an increase in the real return to factor K while leading to a decrease in the real return to factor T, since X is by definition K intensive, while Y is T intensive. Since it is impossible to define a labour intensive commodity, it is difficult to use the theorem to discuss the return to labour.

Two different types of results must obtain order to clarify the issue: the effect of a change in prices on both nominal and on real return to factors must be obtained. Equations (A.31) to (A.47) in the Appendix contain the mathematical calculations and results which were used as a basis for our conclusions.

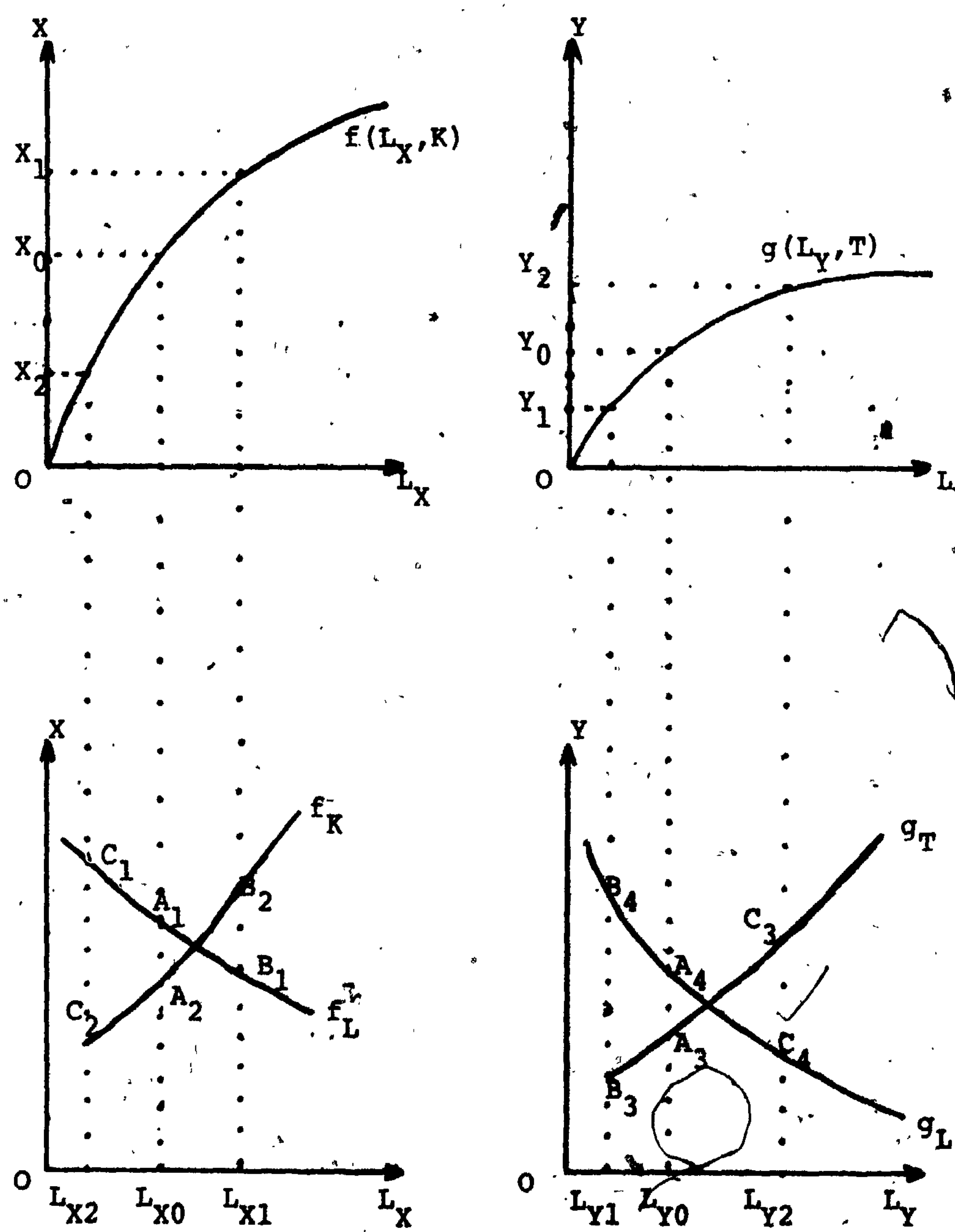
Suppose the price of X increases, then, the production of X will increase while the production of Y will decrease. Consider Figure VIII where f, f_L, f_K, g, g_L, g_T are represented. The increase in X, an increase from X_o to X_1, will mean that f_K increases from A_2 to B_2 while f_L decreases from A_1 to B_1.

The decrease in Y, a decrease from Y_o to Y_1, will imply that g_L increases from A_4 to B_4 while g_T decreases from A_3 to B_3. It is possible to rewrite (6) to (9) as,

$$\frac{w}{P_X} = f_L \qquad \frac{r}{P_X} = f_K \qquad (15)$$

Thus if the real return to K increases, so must its nominal return. Also, if the real return to T decreases and, since P_Y is constant, so must its nominal return. The real return to labour increases in the Y industry while it decreases in the X industry. Since P_Y is constant, this must imply that w increases.

FIGURE VIII


X
X1
X0
X2
O
f(LX,K)
LX
Y
Y2
Y0
Y1
O
g(LY,T)
LY
X
fK
C1
A1
B2
B1
C2
A2
fL
O
LX2
LX0
LX1
LX
Y
gT
B4
C3
A4
A3
C4
B3
gL
O
LY1
LY0
LY2
LY

A similar argument can be applied to show the effect of an increase in the price of Y. The results can be summarized in the following theorem:

Theorem V

(a) An increase (decrease) in the price of one good will lead to an increase (decrease) in the nominal return of labour and the specific factor used in the production of that good, and to a decrease (increase) in the return of the specific factor of the other industry.

(b) An increase (decrease) in the price of one good will lead to a decrease (increase) in the real wage of that industry, while it will increase (decrease) the real wage of the other. Similarly, it will lead to an increase (decrease) in the real return to the specialized factor used in the production of that good and to a decrease (increase) in the real return of the specialized factor used in the production of the other good.

(c) Let us define a . over a variable as a proportionate change. For example, $\hat{w} = dw/w$. Jones (1971) showed the following relation to be true in this model: if we assume that P_X increases proportionally more than P_Y we have

$$\dot{r} > \dot{P}_X > \dot{w} > \dot{P}_Y > \dot{w}$$

It can be shown finally that the basic duality relationship [11] in the system is still present, i.e., the effect of an increase in the amount of factor i on the price of good j, will be equal to the effect of an increase in the price of good j on the return to factor i (i = L, K, T j = X, Y).

Section VI - Technological progress

Technological improvement can take many different forms; better machineries and more skilled labour are but two examples.

To stay within our two goods economy, it is assumed that the change involves simply a new way of combining the existing factors of production of one or both of the two goods.

We want to establish the effect of an improvement on output and factor returns. Prices are assumed to remain fixed. To recognize the possibility of technical improvement, (1) and (2) are rewritten and (17) is introduced

$$X = f\ (\ aL_X\ ,\ cK_X\ ) \qquad (1")$$

$$Y = g\ (\ vL_Y\ ,\ tT_Y\ ) \qquad (2")$$

$$a = c = v = t = 1 \qquad (17)$$

Shifts parameters have been introduced in our production functions. The next step consists of finding the effect of changing a, c, v and t. (A.51) to (A.62) derive the mathematical results for changes in these parameters in the Appendix.

The following is a summary of the results obtained in the Appendix. It presents an analysis of the different pure case.

TABLE I

Implication of an improvement in the first industry (X).

	L_X	L_Y	w	r	s	X	Y
Pure labour saving $da > 0$	↓	↑	↓	↑	↑	↑	↑
Pure K saving $dc > 0$	↑	↓	↑	?	↓	↑	↓
Neutral $da = dc > 0$	?	?	?	?	?	↑	?

TABLE 2

Implication of an improvement in the second industry (Y).

	L_X	L_Y	w	r	s	X	Y
Pure labour saving $dv > 0$	↑	↓	↓	↑	↑	↑	↑
Pure T saving $dt > 0$	↓	↑	↑	↓	?	↓	↑
Neutral $dv = dt > 0$	?	?	?	?	?	?	↑

TABLE 3

Implication of an improvement in both industries.

	L_X	L_Y	w	r	s	X	Y
Pure labour saving $da = dv = du > 0$	?	?	↓	↑	↑	↑	↑

A pure labour saving improvement is the first point to be analysed in Table I. There is an increase in a. Since the total amount of labour to be allocated is fixed, any increase or decrease in L_X will be matched by a change in L_Y in the opposite direction of the same magnitude. Consider Figure IX. X_1 was the equilibrium output of X before the improvement. The X industry was using factors at A. The pure labour saving change will shift the isoquants to the left such that the slope at B on the new isoquant (not drawn) is equal to the slope at A. There is thus some amount of labour (AB) that can be used to increase output.

Let us suppose that X was to receive all the newly released labour such that the industry uses factor at A again. It is clear that the slope of the new isoquant at A will be lower than at B. This will mean that the relative return of labour in X (w/r) has decreased while w/r is constant. Thus, labour will move from the X to the Y industry. Since there has been no change in Y, the right-hand side portion of Figure VIII is still valid. As L_Y increases, the real wage decreases

FIGURE IX

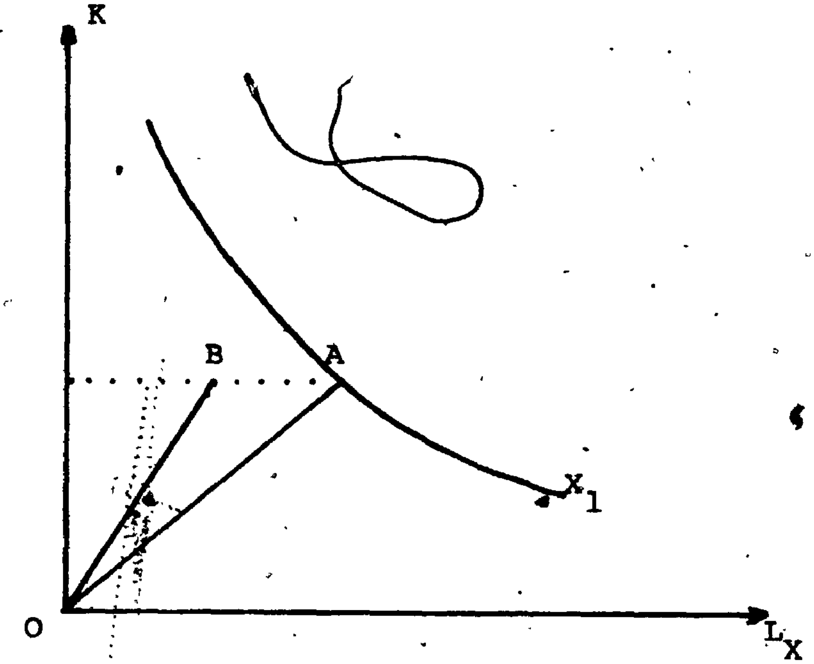
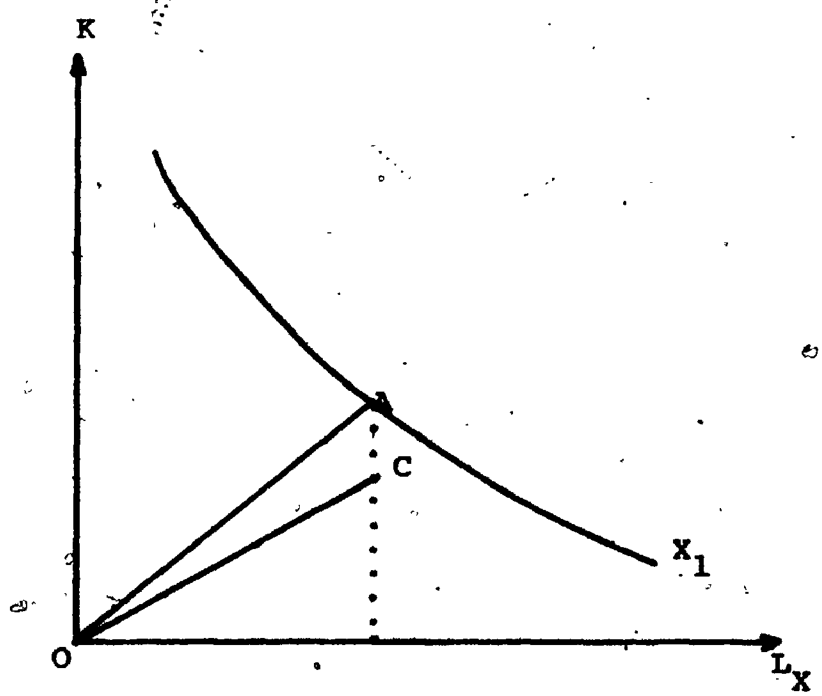


FIGURE X



and the real return to T increases. Since P_Y is fixed, this implies that w decreases and s increases. It is not possible however to show diagrammatically what happens to r. Result (A.56) in the Appendix shows that it increases.

The second point to be analysed in Table I is a pure capital saving improvement in the second industry. Consider Figure X. The change will shift the isoquant downward such that the same level of output can be achieved at C after the change, as opposed to A before the change, while remaining, in both situations, at the same factor w/r ratio. There is thus, AC of K released. The X industry will obviously take the released K, being the sole user of that factor.

It is also possible to suppose that the production of Y does not change. The X industry would still be producing at A. But the slope of the new isoquant at A would be higher than at C. Thus, there would be an increase in the relative return to labour in the X industry while the relative return to labour in the Y industry would not change. Labour would move from Y to X. So, as L_X increases, L_Y decreases. Using the right-hand side of Figure VIII, w and w/P_Y increase while s and s/P_Y decrease.

r can either increase or decrease. This is possible for the following reason: the increase in L_X will tend to increase r on the one hand, and on the other hand, X industry will ask to pay a lower price for the factor in order to take the AC of K released.

It is also possible to analyze in Table I, a neutral technological change by simply renumbering the isoquant. Two different effects can be observed; they consist of a pure labour saving plus a pure K saving where both involve changes of the same magnitude. As shown in the Appendix the net effect is equal to the sum of the two. Because the pure labour saving and the pure K saving have opposite effects on all variables except one, in this case X, the effect of neutral change will be uncertain for all except one. What is certain however, is that Y, L_Y and s will move in the same direction. For example, Y will decrease if the negative effect of a K improvement is greater than the positive effect of a labour saving improvement.

The explanation of Table 2 is symmetric to the one presented for Table 1.[12] Thus, for example, an improvement which is K saving will have the same effect on Y, as an improvement which is T saving, will have on X.

Table 3 presents the results of (A.62). It illustrates the case where labour saving improvements occur in both industries. As shown in the Appendix, the effect is equal to the sum of the two separate labour saving improvements.[13] Thus, Table 3 is simply the "sum" of the first row of Table 1 and the first row of Table 2.

Since labour saving improvement increase both X and Y, it is clear that the composite effect will also increase both. On the other hand, a pure labour saving improvement in the X industry will decrease L_X. Similarly, an improvement in the Y industry decreases L_Y. Thus, in one case L_i goes up, and in the other, it goes down (i = X, Y). Depending on which of the opposite effects dominate, the composite effect on either L_X or L_Y will be positive or negative. We can examine, for example, the condition where the composite effect is positive for L_X, i.e., a net increase.

$$\frac{dL_X}{du} = \frac{P_Y \; g_{LL} \; L_Y - P_X \; f_{LL} \; L_X}{P_Y \; g_{LL} + P_X \; f_{LL}} > 0 \qquad (18)$$

This requires

$$\frac{P_X}{P_Y} \quad \frac{f_{LL}}{g_{LL}} \quad \frac{L_X}{L_Y} > 1 \qquad (19)$$

Equation (19) states that there will be three factors governing whether L_X will increase or decrease. The first concerns the relative price of the two goods; the higher the price of X, the better the chance that L_X will increase. The second concerns the relative curvature of the two production functions, while the third deals with the relative endowment of labour between industries.

The rest of Table 3 is easily explained. Since there is more of each good, and since K and T are fixed, it is clear that both will benefit. Thus, r and s will increase.

Finally, the results of the section on the technological improvement and the results of the section on change in endowments can be brought together in order to permit new extensions. Consider first a K saving improvement. Having more efficient units of K is equivalent to having more units of K, helping the same level of efficiency. Result (A.49) indeed shows that the two effects are equal up to a multiplicative constant. The demonstration is also valid for a T improvement and an increase in T.

It is also shown that a pure labour saving improvement in both industries (Table 3) is equivalent, up to a constant, to an increase in $\bar{L}$, except for the effect on L_X and L_Y. Thus, let dc be a pure K saving change in X, let dt be a pure T saving change in Y, let dn be a pure L saving change in X and Y. We have shown that

$$\frac{di}{dc} = \bar{K} \frac{di}{d\bar{K}} \qquad i = X, Y, L_X, L_Y, w, r, s \tag{20}$$

$$\frac{dj}{dt} = \bar{T} \frac{dj}{d\bar{T}} \qquad j = X, Y, L_X, L_Y, w, r, s \tag{21}$$

$$\frac{dk}{dn} = \bar{L} \frac{dk}{d\bar{L}} \qquad k = X, Y, w, r, s \tag{22}$$

Using these results and Tables 1,2 and 3, plus (A.26)

for $\frac{dL_X}{d\bar{L}}$ and $\frac{dL_Y}{d\bar{L}}$, we obtain

TABLE 4

Summary of the effect of a change in endowments

	L_X	L_Y	w	r	s	X	Y
Increase in $\bar{K}$	↑	↓	↑	?	↓	↑	↓
Increase in $\bar{T}$	↓	↑	↑	↓	?	↓	↑
Increase in $\bar{L}$	↑	↑	↓	↑	↑	↑	↑

Theorem IV

(a) A pure technological improvement in the specific factor used by an industry, at constant price, will increase the output of that industry and will, as a consequence, force decrease on the output of the other industry. Labour will move from the non-improving industry to the improving. The wage rate will increase, and thus, the real wage will also increase. The return to the specific factor in the other industry will decrease.

(b) A pure labour saving improvement in one industry will increase the production of both goods. The return to both specific factors will increase, while the return to labour will decrease.

(c) A neutral technological improvement in one industry will be equal to the sum of the labour saving and the specific factor saving improvement in that industry. It will increase the output of that industry.

(d) A pure labour saving improvement in both industries will be equal to the sum of the separate improvements in each industry. The production of both goods increases, together with the return to both specific factors. The wage rate will go down.

(e) The effect of an improvement in one specific factor will be equal to the effect of an increase in that factor multiplied by a positive constant ($\overline{T}$ for an improvement in the second industry, $\overline{K}$ for the X industry). This also holds for the effect on X, Y, L_X, L_Y, w, r, s. The effect of a labour saving improvement in both industries will be equal to $\overline{L}$ times the effect of an increase in $\overline{L}$ on X, Y, w, r, s.

Section VII Conclusion.

A model in which capital is fixed in each industry and where there is no substitution possible has been analysed. Labour however, is perfectly mobile between industries. At the end of Sections III, IV and V, theorems summarizing the conclusions have been presented. Let us briefly recall our results.

With respect to a change in endowments, the Samuelson-Rybczynski theorem was valid when there was an increase in either T or K. Also, since it is not possible at the present to define a labour intensive good, an increase in the endowment of labour increases the production of both goods.

The Stolper-Samuelson theorem is valid in this type of model. An increase in the price of good X will increase the return to K, since X is K intensive, and will decrease the return to T, since Y is T intensive. Similarly, an increase in the price of Y will increase the return to T and decrease the return to K.

The effects of introducing technological progress are summarized in Tables 1, 2 and 3. It was also found that a technological improvement of K or T has the same effect as K or T times respectively the effect of increasing K or T.

It was demonstrated that the Heckscher-Ohlin theorem is not valid in general. The validity of the theorem requires that each country have an absolute advantage in either K or T, relative to the same amount of labour. If this condition is not met, it does not mean that the theorem is not valid; it simply implies that a counter example can be constructed.

It was also shown that the introduction of free trade in this type of world will leave factor prices unequalized.

An attempt to solve the "Leontief Paradox", can be made at this point if there really is a Paradox. Assume there are two goods; manufacturing goods, using capital and labour and agricultural goods, using land

and labour. Vis-a-vis the rest of the world, the U.S. might be capital intensive, but the effect of possessing a varied labour force can be such that the U.S. is simply improving the capital intensive good.

Appendix.

Section A.I

Let us first restate the model.

$$X = f(L_K, \bar{K}) \qquad (1)$$

$$Y = g(L_Y, T) \qquad (2)$$

$$L_X + L_Y = \bar{L} \qquad (3)$$

$$K_X = \bar{K} \qquad (4)$$

$$T_Y = \bar{T} \qquad (5)$$

$$w = P_X f_L \qquad (6)$$

$$w = P_Y g_L \qquad (7)$$

$$r = P_X f_K \qquad (8)$$

$$s = P_Y g_T \qquad (9)$$

$$U = U(X, Y) \qquad (10)$$

$$\frac{U_X}{U_Y} = \frac{-P_X}{P_Y} \qquad (11)$$

also, $f_i > 0 \quad f_{ii} < 0 \quad f_{ij} > 0 \quad i \neq j$ (1a)

$g_i > 0 \quad g_{ii} < 0 \quad g_{ij} > 0 \quad i \neq j$ (2a)

We first show that the slope of the transformation curve will be equal to the negative of the price ratio: first differentiating (1) through (5) we get

$$dX = f_L \, dL_X + f_X \, dK \tag{A.1}$$

$$dY = g_L \, dL_Y + g_T \, dT \tag{A.2}$$

$$dL_X + dL_Y = 0 \tag{A.3}$$

$$dT_Y = dK_X = 0 \tag{A.4}$$

Using (A.3) and (A.4) in (A.1) and (A.2) and dividing we get

$$\frac{(dY)}{(dX)} = \frac{-g_L}{f_L} \tag{A.5}$$

using (6) and (7)

$$\frac{(dY)}{(dX)} = \frac{-\left(\frac{w}{P_Y}\right)}{\left(\frac{w}{P_X}\right)} = -\frac{P_X}{P_Y} \tag{A.6}$$

Next it is possible to show that the transformation curve will be concave. Immediate use of (A.4) can be made

$$\frac{d^2Y}{dX^2} = \frac{d\left(\frac{dY}{dX}\right)}{dX} = - \frac{d \frac{(g_L)}{(f_L)}}{d_X} = -$$

$$\left[\frac{1}{f_L \; dL_X} \quad \frac{g_{LL} dL_Y}{f_L} - \frac{g_L}{(f_L)^2} f_{LL} dL_X \right] =$$

$$+ \left[\frac{g_{LL}}{(f_L)^2} + \frac{g_L \quad f_{LL}}{(f_L)^3} \right] \qquad (A.7)$$

Using (1a) and (2a)

$$\frac{d^2Y}{dX^2} < 0 . \qquad (A.8)$$

The relationship between P_X/P_Y and L_X/K and L_T/T can next be shown because Φ is positively sloped, and Ψ is negatively sloped in the text.

Dividing (6) and (7) and rearranging, the following is obtained:

$$\frac{P_X}{P_Y} = \frac{g_L}{f_L} .$$

Thus,

$$\frac{d\Psi}{d\left(\frac{L_Y}{T}\right)} = \frac{d \frac{(P_X)}{(P_Y)}}{d \frac{(L_Y)}{T}} .$$

$$= \frac{d\left(\frac{g_L}{f_L}\right)}{dL_Y} \quad \text{using (A.4)}$$

$$= \frac{1}{dL_Y}\left[\frac{g_{LL}\, dL_Y}{f_L} - \frac{g_L}{(f_L)^2}\, f_{LL}\, dL_X\right]$$

Using (A.3), (1a) and (2a)

$$\frac{d\Psi}{d\left(\frac{L_Y}{T}\right)} = \left[\frac{g_{LL}}{f_L} + \frac{g_L}{(f_L)^2}\, f_{LL}\right] < 0 \qquad (A.9)$$

For, Φ, the following is obtained:

$$\frac{d\,(\Phi)}{d\,\frac{(L_X)}{K}} = \frac{d\,\frac{(P_X)}{(P_Y)}}{d\,\frac{(L_X)}{K}}$$

$$= \frac{d\,\frac{(g_L)}{f_L}}{dL_X} \quad \text{using (A.4)}$$

$$= \frac{1}{dL_X}\left[\frac{g_{LL}}{f_L}\, dL_Y - \frac{-g_L}{(f_L)^2}\, f_{LL}\, dL_X\right]$$

Using (A.3), (1a) and (2a)

$$\frac{d\Phi}{d\frac{(L_Y)}{T}} = -\left[\frac{+g_{LL}}{f_L} + \frac{g_L}{(f_L)^2}\, f_{LL}\right] > 0 \quad (A.10)$$

Note from result (A.9) and (A.10) that

$$\frac{d\Psi}{d\frac{(L_Y)}{T}} = - \frac{d\Phi}{d\frac{(L_X)}{K}} \qquad \text{(A.11)}$$

So the two curves, as shown in the text will be parallel, i.e., have the same slope.

Next, relationships between w/r and L_X/K and between w/s and L_Y/T are considered. This is done to show that Ψ and χ in the text (Figure III) are negatively sloped.

$$\frac{d\Psi}{d\frac{(L_Y)}{T}} = \frac{d\frac{w}{s}}{dL_Y} \quad \text{using (A.4)}$$

$$= \frac{d\frac{g_L}{g_K}}{dL_Y} \quad \text{using (14) in the text}$$

$$= \frac{1}{dL_Y}\left[\frac{g_{LL}dL_Y}{g_T} - \frac{g_L}{(g_T)^2}\, g_{TL}\, dL_Y\right]$$

Using (1a) and (2a)

$$= \left[\frac{g_{LL}}{g_T} - \frac{g_L}{(g_T)^2}\, g_{TL}\right] < 0 \qquad \text{(A.12)}$$

Also

$$\frac{d X}{d\left(\frac{L_X}{K}\right)} = \frac{d\left(\frac{w}{r}\right)}{dL_X} \quad \text{using (A.4)}$$

$$= \frac{d\left(\frac{f_L}{f_K}\right)}{dL_X} \quad \text{using (13) in the text}$$

$$= \frac{1}{dL_X}\left[\frac{f_{LL}}{f_K} dL_X - \frac{f_L}{(f_K)^2} f_{KL}\, dL_X\right]$$

Using (1a) and (2a)

$$\frac{dX}{d\left(\frac{L_X}{K}\right)} = \left[\frac{f_{LL}}{f_K} - \frac{f_L}{(f_K)^2} f_{KL}\right] < 0 \quad \text{(A.13)}$$

Section A.IV - Change in endowments.

To investigate the effect of a change in endowment, equation (1) to (9) in the text is differentiated. Since prices are considered as constants, it is possible to use this information directly; also, by doing this, the demand side of the system does not need to be differentiated.

Thus

$$dK_X = \overline{dK} = dK$$

$$dT_Y = \overline{dT} = dT$$

And the following is obtained:

$$dK - f_L \, dL_X = f_K \, dK_X \qquad (1')$$

$$dY - g_L \, dL_Y = g_T \, dT_Y \qquad (2')$$

$$dL_X + dL_Y = \overline{dL} \qquad (3')$$

$$dw - P_X \, f_{LL} \, dL_X = P_X \, f_{LK} \, dK_X \qquad (6')$$

$$dw - P_Y \, g_{LL} \, dL_Y = P_Y \, g_{LT} \, dT_Y \qquad (7')$$

$$ds - P_Y \, g_{TL} \, dL_Y = P_Y \, g_{TT} \, dT_Y \qquad (8')$$

$$dr - P_X \, f_{KL} \, dL_X = P_X \, f_{KK} \, dK_X \qquad (9')$$

The system can be arranged in a matrix by defining

$$\begin{bmatrix} 1 & 0 & -f_L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -g_L & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -P_X f_{LL} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -P_Y g_{LL} & 1 & 0 & 0 \\ 0 & 0 & -P_X f_{KL} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -P_Y g_{TL} & 0 & 0 & 1 \end{bmatrix} = A$$

All the vectors used will become column vectors. In order to save space these will be presented in rows by putting the transpose sign.

$$z' = (dX \quad dY \quad dL_X \quad dL_Y \quad dw \quad dr \quad ds)$$

$$b_1' = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$$

$$b_2' = (f_K \quad 0 \quad 0 \quad P_X f_{LK} \quad 0 \quad P_K f_{KK} \quad 0)$$

$$b_3' = (0 \quad g_T \quad 0 \quad 0 \quad P_Y g_{LT} \quad 0 \quad P_Y g_{TT})$$

Thus,

$$Az = b_1\, d\bar{L} + b_2\, dK + b_3\, dT \qquad \text{(A.14)}$$

a) Let $d\bar{L} = dT = 0,\ dK > 0$

Then:

$$z_1 = A^{-1} b_2 = \frac{z}{dK} \qquad \text{(A.15)}$$

The following can be found:

$$A^{-1} = \frac{1}{\det.A} \begin{bmatrix} \det.A & 0 & P_Y f_L g_{LL} & -f_L & f_L & 0 & 0 \\ 0 & \det.A & P_X g_L f_{LL} & g_L & -g_L & 0 & 0 \\ 0 & 0 & P_X g_{LL} & -1 & 1 & 0 & 0 \\ 0 & 0 & P_X f_{LL} & 1 & -1 & 0 & 0 \\ 0 & 0 & (P_X P_Y f_{LL} g_{LL}) & (P_Y g_{LL}) & (P_X f_{LL}) & & \\ 0 & 0 & (P_X P_Y f_{KL} g_{LL}) & (-P_X f_{KL}) & (P_X f_{KL}) & \det.A & 0 \\ 0 & 0 & (P_X P_Y f_{LL} g_{LT}) & (P_Y g_{LT}) & (-P_Y g_{LT}) & 0 & \det.A \end{bmatrix}$$

$$\det.A = P_X f_{LL} + P_Y g_{LL} < 0$$

Thus,

$$\frac{dX}{dK} = \frac{f_K(\det.A) - f_L P_X f_{LK}}{\det A} > 0 \qquad \text{(A,16)}$$

$$\frac{dY}{dK} = \frac{P_X\, g_L\, f_{LK}}{\det.A} < 0 \qquad \text{(A.17)}$$

Also,

$$\frac{dL_X}{dK} > 0, \quad \frac{dL_Y}{dK} < 0 \quad \frac{dw}{dK} > 0, \quad \frac{dr}{dK} \gtrless 0, \quad \frac{ds}{dK} < 0 \qquad \text{(A.18)}$$

b) Let $d\bar{L} = dK = 0 \quad dT > 0$

Then :

$$z_2 = A^{-1} b_3 = \frac{z}{dT} \qquad (A.19)$$

Thus ,

$$\frac{dX}{dT} = \frac{P_Y f_L g_{LT}}{det.A} < 0 \qquad (A.20)$$

$$\frac{dY}{dT} = \frac{g_T \ (det.A) - g_L \ P_Y \ g_{LT}}{det.A} > 0 \qquad (A.21)$$

Also ,

$$\frac{dL_X}{dT} < 0 \ , \frac{dL_Y}{dT} > 0 \ , \frac{dw}{dT} > 0 \ , \frac{dr}{dT} < 0 \ , \frac{ds}{dT} \gtrless 0 \qquad (A.22)$$

c) Let $dK = dT = d\bar{L} > 0$

Then :

$$z_3 = A^{-1} b_1 = \frac{z}{d\bar{L}} \qquad (A.23)$$

Thus ,

$$\frac{dX}{d\bar{L}} = \frac{f_L \ \ P_Y \ \ g_{LL}}{det.A} > 0 \qquad (A.24)$$

$$\frac{dY}{d\bar{L}} = \frac{g_L \ \ P_X \ \ f_{LL}}{det.A} > 0 \qquad (A.25)$$

Also

$$\frac{dL_X}{d\bar{L}} > 0 \quad , \frac{dL_Y}{d\bar{L}} > 0 \ , \frac{dw}{d\bar{E}} < 0 \quad , \frac{dr}{d\bar{L}} > 0 \ , \frac{ds}{d\bar{L}} > 0 \qquad \text{(A.26)}$$

Under what condition will the transformation curve be parallel? Thus, in Figure V, we want to know the conditions such that a point on the new surface on ray OA extended has the same slope. The following is necessary:

$$\frac{d \left(\frac{X}{Y} \right)}{dL} = 0$$

or,

$$\frac{1}{Y} \frac{dX}{d\bar{L}} \quad \frac{X}{Y^2} \cdot \frac{dY}{dL} = 0$$

Using (A.18) and (A.19) the common factor

$$\frac{1}{Y \left(P_Y g_{LL} + P_X f_{LL} \right)}$$

can be eliminated and the following is obtained:

$$f_L P_Y g_{LL} = \frac{X}{Y} g_L P_X f_{LL} \qquad \text{(A.27)}$$

since,

$$\frac{P_X}{P_Y} = \frac{g_L}{f_L} \ ,$$ we can write

$$\left(\frac{P_Y}{P_X}\right)^2 g_{LL} = \frac{X}{Y} f_{LL} \qquad (A.28)$$

Section A.V - Change in the price of goods.

In this section the effect of an increase in the price of one good is analysed.

Since endowments are fixed, then,

$$dK = dT = d\bar{L} = 0 ,$$

differentiating totally (1), (2), (3), (6), (7), (8), (9) and organizing that information in matrix form, the following is obtained:

$$A_z = b_4\, dP_X + b_5\, dP_Y \qquad (A.29)$$

$$b_4' = (0 \quad 0 \quad 0 \quad f_L \quad 0 \quad f_K \quad 0 \quad)$$

$$b_5' = (0 \quad 0 \quad 0 \quad 0 \quad g_L \quad 0 \quad g_T \quad)$$

a) $dP_Y = 0 \qquad dP_X > 0$

$$z_4 = A^{-1} b_4 = \frac{z}{dP_X} \qquad (A.30)$$

The following can be found:

$$\frac{dw}{dP_X} = \frac{f_L\, P_Y\, g_{LL}}{\det.A} > 0 \qquad (A.31)$$

$$\frac{dr}{dP_X} = \frac{f_K\,(\text{det.A}) - P_X f_L\, f_{KL}}{\text{det.A}} > 0 \qquad \text{(A.32)}$$

$$\frac{ds}{dP_X} = \frac{P_Y\, f_K\, q_{TL}}{\text{det.A}} < 0 \qquad \text{(A.33)}$$

also

$$\frac{dX}{dP_X} > 0, \frac{dY}{dP_X} < 0, \frac{dL_X}{dP_X} > 0, \frac{dL_Y}{dP_X} < 0 \qquad \text{(A.34)}$$

b) $dP_X = dP_Y > 0$

$$z_5 = A^{-1}\, b_5 = \frac{z}{dP_Y} \qquad \text{(A.35)}$$

Thus

$$\frac{dw}{dP_Y} = \frac{g_L\, P_X\, f_{LL}}{\text{det.A}} > 0 \qquad \text{(A.36)}$$

$$\frac{dr}{dP_Y} = \frac{P_X\, g_L\, f_{KL}}{\text{det.A}} < 0 \qquad \text{(A.37)}$$

$$\frac{ds}{dP_Y} = \frac{g_T(\text{det.A}) - P_Y\, q_L\, g_{LT}}{\text{det.A}} > 0 \qquad \text{(A.38)}$$

also

$$\frac{dX}{dP_Y} < 0, \frac{dY}{dP_Y} > 0, \frac{dL_X}{dP_Y} < 0, \frac{dL_Y}{dP_Y} > 0 \qquad \text{(A.39)}$$

These results are just preliminary results since both the effect on nominal return, and also the effect in the real reward to factors, are studied.

c) Let P_Y be constant and P_X increases, then,

$$\frac{d\left(\frac{w}{P_X}\right)}{dP_X} = \frac{1}{P_X}\left[\frac{dw}{dP_X} - \frac{w}{P_X}\right]$$

$$= \frac{f_L}{P_X}\left[\frac{P_X\, f_{LL}}{\det.A}\right] < 0 \qquad (A.40)$$

Using (A.31) and (6)

$$\frac{d\left(\frac{w}{P_Y}\right)}{dP_X} = \frac{1}{P_Y}\quad\frac{dw}{dP_X} > 0 \qquad (A.41)$$

Using (A.31)

$$\frac{d\left(\frac{r}{P_X}\right)}{dP_X} = \frac{1}{P_X}\left[\frac{dr}{dP_X} - \frac{r}{P_X}\right]$$

$$= -\frac{f_L\ f_{KL}}{\det.A} > 0 \qquad (A.42)$$

Using (A.32) and (9)

$$\frac{d\left(\frac{s}{P_Y}\right)}{dP_X} = \frac{1}{P_Y}\quad\frac{ds}{dP_X} < 0 \quad \text{from (A.33)} \qquad (A.43)$$

d) In a similar fashion the case where P_Y increases while P_X is constant is considered. Thus,

$$\frac{d\left(\frac{w}{P_X}\right)}{dP_Y} = \frac{1}{P_X} \frac{dw}{dP_Y} > 0 \qquad (A.44)$$

$$\frac{d\left(\frac{w}{P_Y}\right)}{dP_Y} = \frac{g_L}{P_Y}\left[\frac{-P_Y g_{LL}}{det.A}\right] < 0 \qquad (A.45)$$

$$\frac{d\left(\frac{s}{P_Y}\right)}{dP_Y} = \frac{-g_L g_{LT}}{det.A} > 0 \qquad (A.46)$$

$$\frac{d\left(\frac{r}{P_X}\right)}{dP_Y} = \frac{1}{P_X} \frac{dr}{dP_Y} < 0 \qquad (A.47)$$

The duality relationship between the Stolper-Samuelson and the Samuelson-Rybczynski theorems can be observed by pairing the following relationship:

(A.16) and (A.32) give $\frac{dX}{dK} = \frac{dr}{dP_X}$

(A.21) and (A.38) give $\frac{dY}{dT} = \frac{ds}{dP_Y}$

(A.17) and (A.37) give $\frac{dY}{dK} = \frac{dr}{dP_Y}$

(A.20) and (A.33) give $\frac{dX}{dT} = \frac{ds}{dP_X}$

(A.25) and (A.36) give $\frac{dY}{dL} = \frac{dw}{dP_Y}$

(A.24) and (A.31) give $\frac{dX}{dL} = \frac{dw}{dP_X}$

i.e., the effect of an increase in the amount of factor i on the output of good j will be equal to the effect of an increase in the price of good j on the return to factor i. (i = L,K,T   j = X,Y)

Section A.VI - Technological Change

To introduce technological progress, (1) and (2) are rewritten as

$$X = f(aL_X, cK_X) \quad (1")$$

$$Y = g(tL_Y, vT_Y) \quad (2")$$

$$a = c = t = v = 1 \quad (15)$$

The next step consists of differentiating (1"), (2") and (3) to (9) totally. The use of (4) and (5) which state that $dK_X = dT_Y = 0$, permits the elimination of two unknowns and two equations. Also, (15) and $dP_X = dP_Y = 0$ is used.

The resulting system can be arranged in matrix form:

$$AZ = b_6\,da + b_7\,dc + b_8\,dt + b_9\,dv \quad , \qquad (A.48)$$

where

$$b_6' = (\; f_L L_X \quad 0 \quad 0 \quad P_X f_{LL} L_X \quad 0 \quad P_X f_{KL} L_X \quad 0 \;)$$

$$b_7' = (\; f_K K \quad 0 \quad 0 \quad P_X F_{LK} K \quad 0 \quad P_X f_{KK} K \quad 0 \;)$$

$$b_8' = (\; 0 \quad g_L L_Y \quad 0 \quad 0 \quad P_Y g_{LL} L_Y \quad 0 \quad P_Y g_{LT} L_Y \;)$$

$$b_9' = (\; 0 \quad g_T T \quad 0 \quad 0 \quad P_Y g_{LT} T \quad 0 \quad P_Y g_{TT} T \;)$$

Note that

$$b_7 = Kb_2 \qquad (A.49)$$

$$b_9 = Tb_3 \qquad (A.50)$$

A pure labour-saving improvement in the first industry is defined as $da > 0$, $dc = 0$. A pure capital-saving improvement will likewise be $dc > 0$, $da = 0$. A neutral improvement will be defined as $da = dc > 0$. It can be generally defined that a labour-saving improvement is one where,

$$da > dc > 0$$

also, a capital-saving improvement is one where

$$dc > da > 0$$

Case I - Pure labour-saving improvement in the first industry.

We have,

$$da \quad 0 \quad , \quad dc = dt = dv = 0$$

From (A.48)

$$z_6 = A^{-1} b_6 = \frac{z}{da}$$

Thus;

$$\frac{dX}{da} = \frac{f_L \; L_X \; P_Y \; g_{LL}}{\det.A} > 0 \qquad (A.51)$$

$$\frac{dY}{da} = \frac{g_L \; P_X \; f_{LL} \; L_X}{\det.A} > 0 \qquad (A.52)$$

$$\frac{dL_X}{da} = \frac{-P_X \; f_{LL} \; L_X}{\det.A} < 0 \qquad (A.53)$$

$$\frac{dL_Y}{da} = \frac{dL_X}{da} > 0 \qquad (A.54)$$

$$\frac{dw}{da} = \frac{P_X \; P_Y \; L_X \; f_{LL} \; g_{LL}}{\det.A} < 0 \qquad (A.55)$$

$$\frac{dr}{da} = \frac{P_Y \; g_{LL} \; P_X \; f_{KL} \; L_X}{\det.A} > 0 \qquad (A.56)$$

$$\frac{ds}{da} = \frac{P_X \; P_Y \; L_X \; f_{LL} \; q_{TL}}{\det.A} > 0 \qquad (A.57)$$

<u>Case II</u> - Pure capital-saving improvement, in the first industry.

We have .

$$dc \quad 0 \quad , \; da = dv = dt = 0$$

From (A.48)

$$z_7 = A^{-1} b_7$$

but from (A.49)

$$z_7 = A^{-1} b_2 K = K z_1 = \frac{z}{dc}$$

From (A.16). (A.17) and (A.18) we can find the sign of z_7 as

$$\text{sign } z_7' = (\; + \; - \; + \; - \; + \; ? \; - \;) \qquad (A.58)$$

<u>Case III</u> - Neutral technological change in the first industry.

We have

$$dn = da = dc > 0 \; , \; dv = dt = 0$$

From (A.48)

$$- z_8 = A^{-1} (b_6 + b_7) = \frac{z}{dn}$$

Thus

$$z_8 = z_6 + z_7 \tag{A.59}$$

Using (A.51) to (A.58) we have

$$\text{sign } z_8' = (+ \ ? \ ? \ ? \ ? \ ? \ ? \ ) \tag{A.60}$$

dX/dn is the only sign known with certainty. If (A.59) is explicitly written, it could be shown that all the others except dr/dn depend on the sign of the expression ($f_{LL}L_X + f_{LK}K$). If this expression is positive, then

$$\frac{dY}{dn} < 0 \quad \frac{dL_X}{dn} > 0 \quad \frac{dL_Y}{dn} < 0 \quad \frac{dw}{dn} > 0 \quad \frac{ds}{dn} < 0$$

It is possible to show that whatever is the sign of the expression, dY/dn, ds/dn and dL_Y/dn will be of the same sign.

Let us now turn to the case where there is technological improvement in the second industry. Instead of going through the same set of calculations, we can proceed as follows. In the system (1"), (2") and (6) to (9) interchange (1") and (a"), (6) and (7), (8) and (9). In matrix A, the dr and ds columns are interchanged; the same thing is done to the dL_X and dL_Y columns. This column and row manipulation will not change the solution to the system. Our terms define the relation $\leftrightarrow$ as meaning: "we now write... instead of...and vice versa".

$$\begin{array}{lll} q \leftrightarrow f & K \leftrightarrow T & K \leftrightarrow Y \\ a \leftrightarrow t & c \leftrightarrow v & r \leftrightarrow s \end{array} \qquad (A.61)$$

With these changes, the system is the same as in the case of improvements in the first industry. The solutions will thus be the same. Using (A.61) the results can be tabulated in the following manner:

TABLE I

	Effect on	X	Y	L_X	L_Y	w	r	s
CASE IV	Pure labour savings $dv > 0 \quad dt = 0$	↑	↑	↑	↓	↓	↑	↑
CASE V	Pure capital savings $dt > 0 \quad dv = 0$	↓	↑	↓	↑	↑	↓	?
CASE VI	Neutral $dt = dv > 0$	?	↑	?	?	?	?	?

<u>Case VII</u> - Equal pure labour savings improvement in both industries.

We have

$$da = dv = du > 0 \quad dc = dt = 0$$

From (A.48)

$$z_{12} = A^{-1}(b_6 + b_8) = A^{-1}b_6 + A^{-1}b_8$$

$$= z_6 + z_9 \qquad \text{(A.61a)}$$

where z_9 is the first row of Table I.

We can use Table I and (A.51) to (A.57) to obtain

$$\text{sign } z_{12} = (+ \; + \; ? \; ? \; - \; + \; +) \qquad \text{(A.62)}$$

Footnotes

* I would like to thank Professor J.R.Melvin [illegible] obtaining a Killam Senior Research Fellowship which helped finance the research for this chapter. My thanks are also extended to the participants of the International Trade Workshop at the University of Western Ontario for comments on an earlier draft.

1. Jones (1971), page 4.

2. i.e., along any ray from the origin in the output space, the slope of the indifference curves will be constant. Note that we assume that the indifference curves are strictly convex everywhere.

3. In the introduction to the thesis, we briefly discuss the conditions necessary for such curves to exist.

4. A function f (X) is said to be concave if we let $\bar{X} = X_1 + t\,(1 - \alpha)\,X_2 \quad 0 < \alpha < 1$ then $f(\bar{X}) < \alpha f(X_1) + (1 - \alpha) F(X_2)$.

5. By parallel we mean that the absolute value of the slopes are equal, at every price ratio, provided both goods are produced.

6. We assume that trade is caused by difference in endowments. The two countries are assumed to have the same indifference curves.

7. By the traditional result we refer to the conclusions derived from the Heckscher-Ohlin model.

8. See (A.24) and (A.25).

9. See (A.16) and (A.17).

10. Note that B is the production point not the consumption point.

Footnotes (continued)

11. See Jones (1965).

12. See (A.61) and the following text for proof.

13. See (A.61a).

CHAPTER III

TRANSPORTATION COSTS IN INTERNATIONAL TRADE THEORY:

A GEOMETRICAL ANALYSIS

Most of the pure theory of international trade has been carried out under the assumption that there are no transportation costs. The reasons for this are numerous. First, it is one of the sets of simplifying assumptions which is made in order to derive the basic features of the economic system. Second, the incorporation of a third good, namely transportation services, requires a system with at least three goods. This makes the geometric treatment difficult.

The problem has received some consideration. Samuelson's paper (1952) discusses the effect of transportation costs in terms of the transfer problem. Mundell (1957) treated the problem geometrically. Both used a "drastic but very useful assumption regarding the nature of transportation costs: transport costs are met by wastage of a proportion of the goods traded."(1) Our aim here is to relax this assumption.

For this purpose it is assumed that transportation services (T) are provided by an industry which uses factors of production. Furthermore, there are two other goods X and Y and all production is carried out under constant returns to scale. Three factors of production are considered; it is assumed firstly, that perfect competition exists in every market; technology is secondly diffused freely; it is also diffused, thirdly, instantaneously (i.e., all countries face the same production function for any good).

Tastes are assumed to be represented by a set of well-behaved[2] indifference curves and the level of well-being for the community is assumed to depend only on X and Y and not on T[3]. The assumption that the indifference curves are homothetic is carried over from previous chapters.

Transportation costs are ignored within a country. To keep our results in line with the Heckscher-Ohlin literature, it is assumed that countries differ in factor endowments. We will confine our attention to the two countries case except for Section V where we assume more than two countries.

In order to avoid unnecessary confusion, we will briefly explain the technique used. We will derive the offer curve by looking at all possible situations of equilibria. This does not mean that it will be an equilibria. We will ask the following question: for a given set of commodity prices, what quantities would the producers be willing to supply? For the same set of commodity prices and given the quantity produced totally, we can find the equilibrium on the demand side. The difference between the two sides will show the amount of excess demand or excess supply for every good.

As utility does not depend on T, there will be an excess supply of T equal to the full amount of production of that good. Only if the amount needed to transport x or y is equal to the amount of T produced, can an equilibrium result. This can only be found when the other country is introduced. So, when considering the offer curve for one country, it is necessary to consider all points on the transformation surface as a possible candidate for equilibrium, even if the domestic demand for T is always zero.

The order of this Chapter is as follows: Section I presents the model and discusses the existence of equilibrium. Section II derives geometrically the equilibrium for the three basic cases: the first case occurs when each country is engaging in the transportation of exports only; the second one occurs when each country is transporting its import only; finally, the third, a combination of one and two occurs when each country is doing both. Many intermediary cases are ignored.(4) In Section III we examine the effect of technological improvement in the T industry in terms of the three cases stated above. Section IV briefly discusses the effect of our analysis on the Heckscher-Ohlin literature. Section V covers the gains from trade theorem, and the last section presents our conclusions.

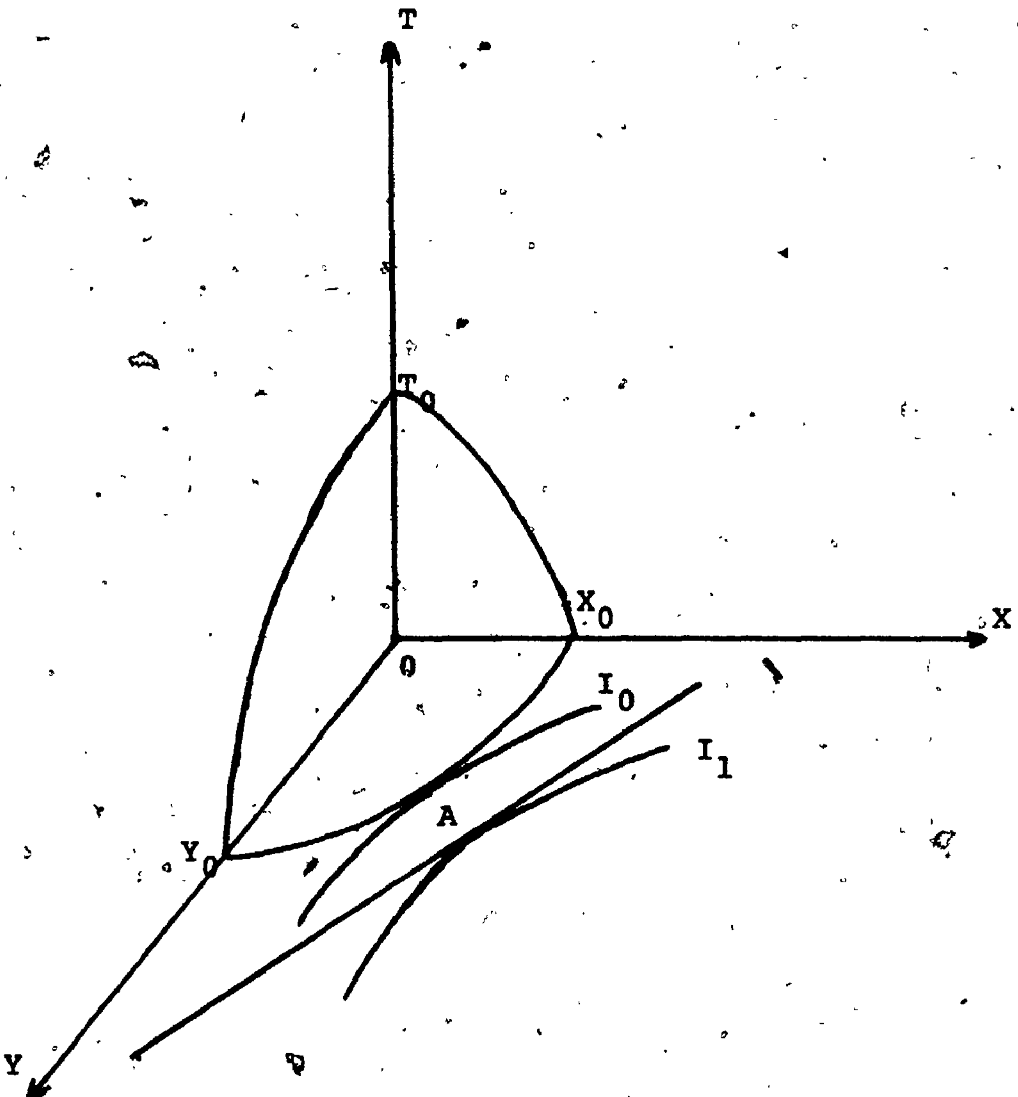
Section I. The Model

Using the three production functions, the full-employment assumption, and, the fact that factors are paid the value of their marginal products in perfect competition, it is possible to derive a production possibility surface which is assumed to be bowed out everywhere, i.e., strictly concave.(5) In such a model, the price plane will be tangent to the surface, and the tangency will be unique. The demand side is represented by a set of indifference surfaces where each of them is perpendicular to the X, Y plane. The production possibility surface and the set of indifference surfaces are graphicaly represented in Figure I. The autarky equilibrium is at point A where no T is being produced.

When trade is allowed, some T will be required. Depending on commodity prices, production will be placed at the point of tangency of the production set and the price plane. Consumption will be placed at a point of tangency with an indifference surface on the X, Y plane, i.e., the domestic consumption of T will be zero. In a two country case, equilibrium can occur at this point only if the amount of T produced is equal to the amount required.

The model must also include technological requirements which state how many units of T are required to transport given quantities of X or Y a given distance (d). We have

FIGURE I



$$T_X = T_X(X,d) \tag{1}$$

$$T_Y = T_Y(Y,d) \tag{2}$$

These functions are assumed to have the following properties:

$$\frac{\partial T_X}{\partial i} > 0 \quad i = X,d \tag{1a}$$

$$\frac{\partial T_Y}{\partial j} > 0 \quad j = Y,d \tag{2a}$$

Let T_X^{ij} and T_Y^{ij} be the cost of transportation one unit of the good from country i to j. We must have in equilibrium

$$P_X^j = P_X^i + T_X^{ij} \tag{3}$$

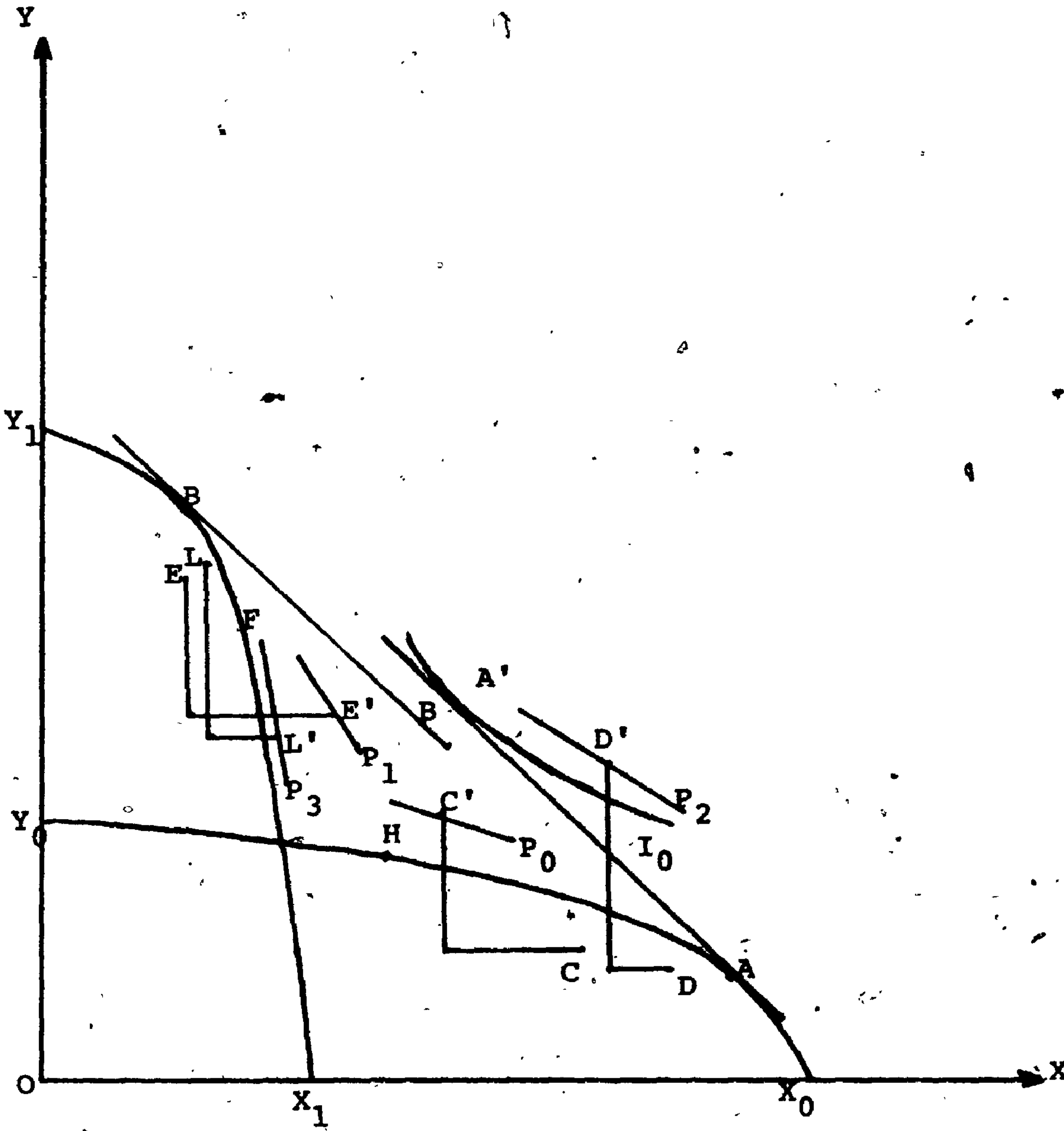
$$P_Y^j = P_Y^i + T_Y^{ij} \tag{4}$$

In a general equilibrium analysis T_X^{ij} and T_Y^{ij} are determined within the model. In the usual partial equilibrium analysis they can be taken as constants.

Let us look at the possible equilibrium situation. The three dimensional production surface is assumed to be strictly convex everywhere; this implies that to any point in the (X,Y) plane there will be a unique level of T associated. In Figure II we draw YoAXo as the production possibility curve for country H, associated with a level of T equal to zero. For country F the set of production points associated with zero T are Y_1BX_1. These will represent the production possibility curves for the two countries when transportation costs are absent. If transportation costs are inexistent the trading equilibrium will be as follows: country H produces at A and consumes at A', while country F produces at B and consumes at B'. When transportation costs are introduced such a trading pattern is impossible: there will obviously be no T to transport X and Y between H and F. Let the two autarky equilibrium be H for country H and F for country F.

Starting from the two autarky equilibriums, it is clear that there will be a basis for trade. The price ratio is higher in H than Y. In order for trade

FIGURE II



to actually take place, some T will have to be produced, We picture two possible equilibriums.

A first possibility is to have country H producing at C and consuming at C', while country F is producing at E and consuming at E'. Both E and C are associated with positive level of T and equilibrium will be attained if the level of T produced is the exact amount required to transpost the amount CK of Y and KC' of X.

A second possibility is to have H producing at D and consuming at D', while F is producing at L and consuming at L'. In that case, the result shows that country H is better off in a world with free trade and transportation costs. Note that country F is better off at L than at the autarky point. This can be shown as follows: the line P_3 is really the base of a three dimensional price plane tangent to the production possibility surface. Because the transformation surface is strictly convex, the price plane will not intersect the surface anywhere (except at the point of tangency).

Since L is on the price plane, Y_1FBX_1 is all interior to the line P_3 extended. In that case F is interior to P_3 thus L is superior to F.

Note also that both countries cannot be made better off with transportation costs when compared to the case where such costs were absent. This is because some resources are released to produce T. The available resources for X and Y are thus reduced. In order for both countries to be better off with transportation costs, we would need the total X and Y to be greater than without transportation costs.

To construct the offer curve, two basic sets of relationships must be derived. Let us assume that the country under consideration (H) will be exporting X and importing Y. In Figure I, let the price of T be fixed at zero. By varying the price ratio of X to Y and reporting the excess demand of Y and excess supply of X, we can construct O_H in Figure III. This schedule can be identified with the offer curve for country H when transportation costs are absent. The first set of

FIGURE III

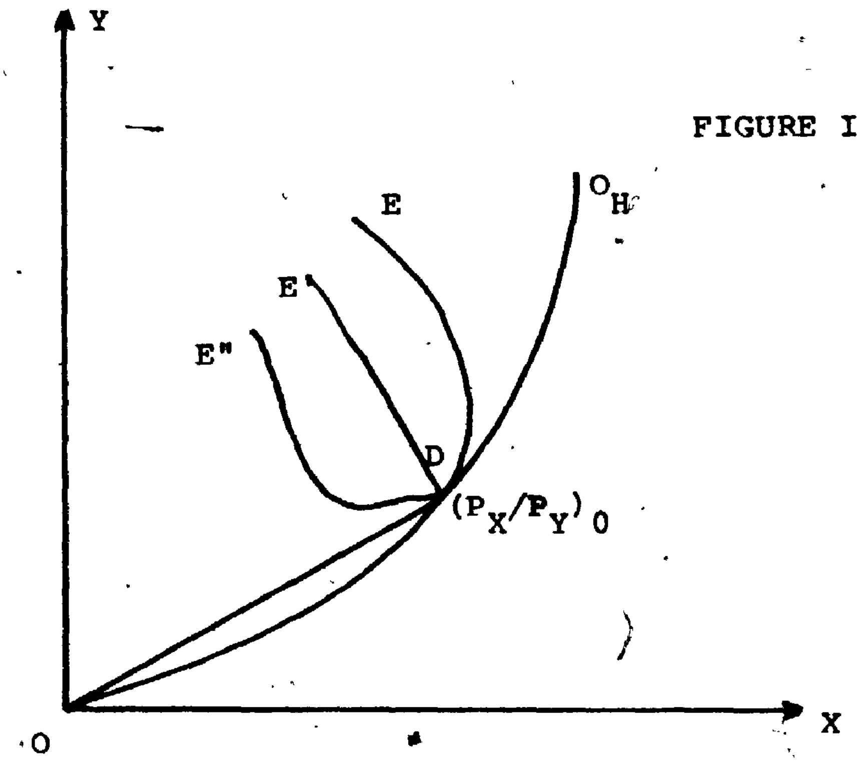
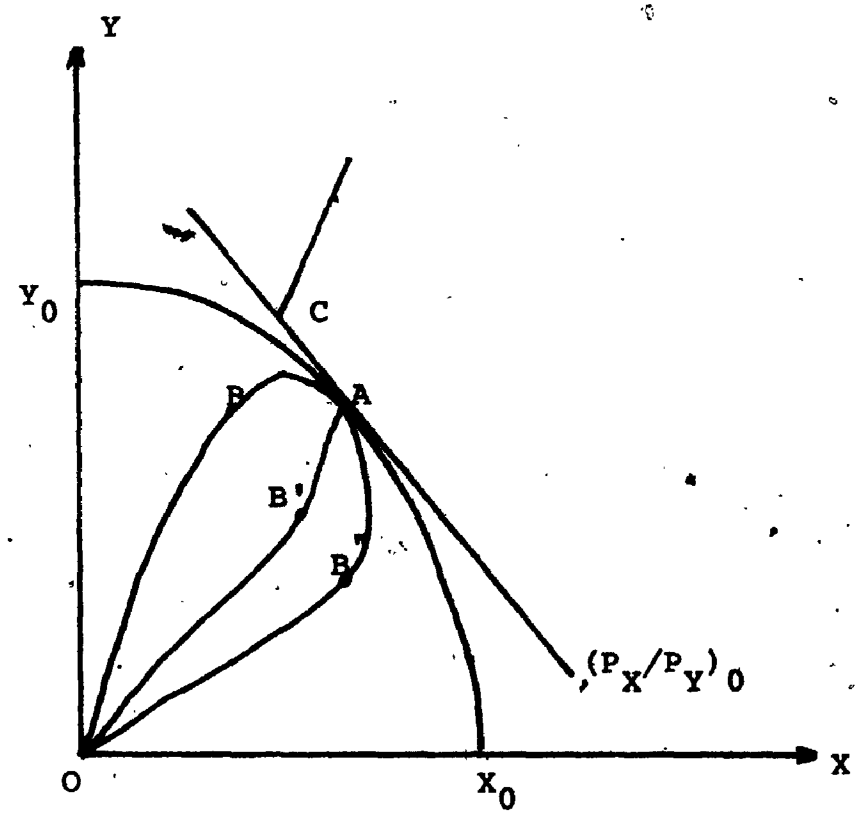


FIGURE IV



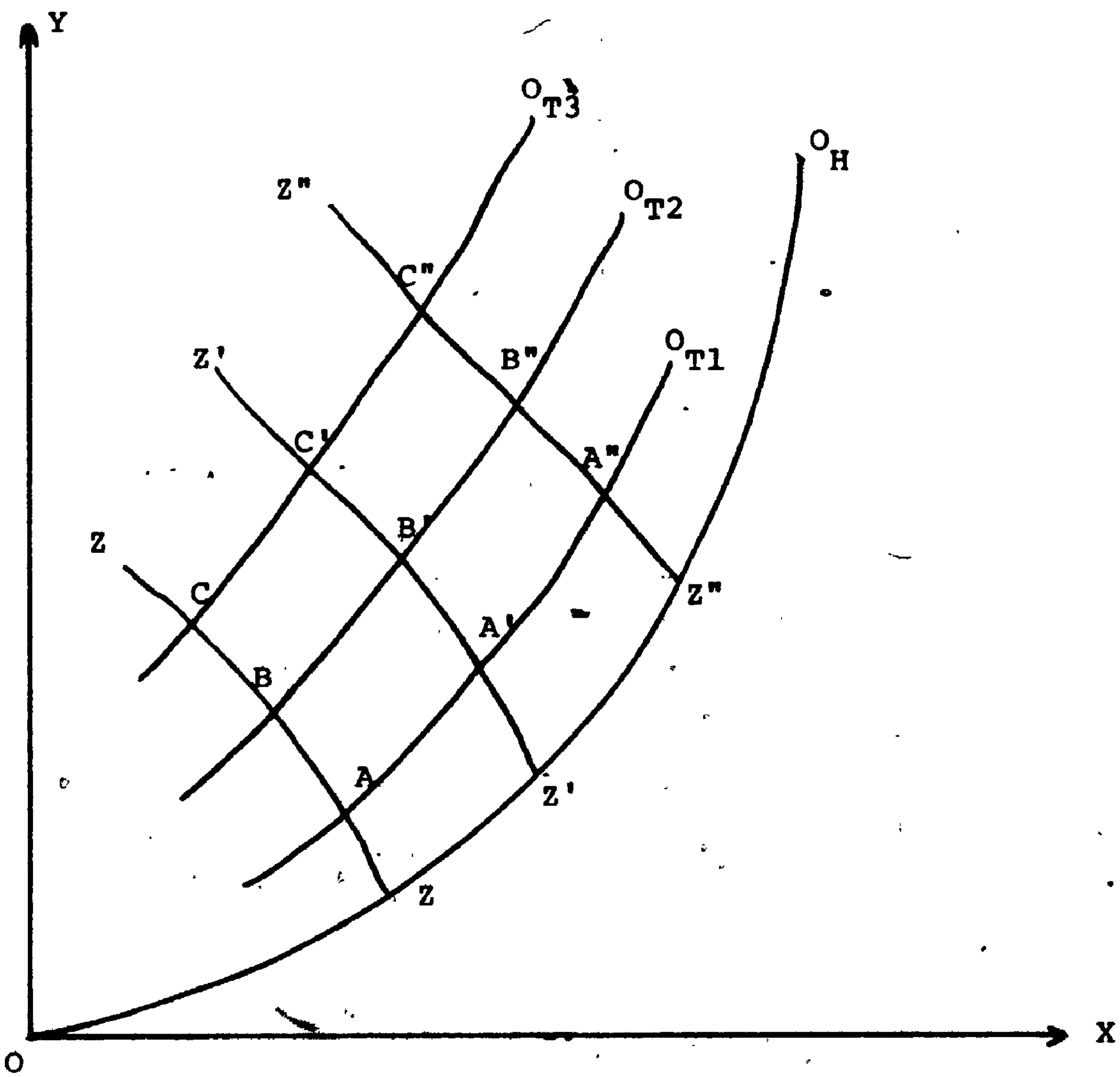
relations are derived in order to discover what happens to the excess demand of Y and excess sypply of X as the price of T varies while P_X and P_Y are kept constant. For example, price ratio $(P_X / P_Y)_o$ can be chosen.

In Figure IV, YoAXo represents the production possibility curve when T is zero. It is thus the base of the production possibility surface in Figure I. With $(P_X / P_Y)_o$, production is placed at A and consumption at C. This information is considered in order to discover what happens to the production of X and Y as P_T moves from zero to infinity. Since (P_X/P_Y) is not changing, consumption will occur along ray OC, because our assumption shows that indifference curves are homothetic. It is clear that, as the price of T becomes large enough, only T will be produced. Thus, the production of X and Y will move from point A to the origin. Because we have more than two goods, it is possible that there will be gross complementarity between goods on the production set.(6) The gross complementarity can occur between T and Y in which case ABO results. If the complementarity lies between T and X, we will have AB"O.

The case where no complementarity exists is AB'O. Thus, as P_T increases, the production of X and Y will move according to one of these three possibilities: the consumption point will be along OC extended; if no gross complementarity exists, as the prices of T increases there will be a decrease in the excess supply of X and an increase in the excess demand of Y; in Figure III there will be a north-west movement from D along say DE'. When there is gross complementarity on the production side, it is possible (but not necessary) that such curves be positively sloped in some range like DE (corresponding to ABO) and DE" (corresponding to AB"O).

We can now derive the second set of relationships needed. The schedule associating the excess demand of Y to the excess supply of X, for a given amount of T is sought. For example, in Figure III, O_H corresponds to the case where T = O. These schedules can be derived with the one derived above. Consider Figure V. ZZ, Z'Z', Z"Z" ..., are schedules, each one of which is associated with a particular price ratio for X and Y.

FIGURE V



On each of these curves we find the points which are associated with T = 1, T = 2, T = 3 etc... By joining these points we are able to derive O_{T1}, O_{T2}, O_{T3} which correspond to the points of excess demand of Y and excess supply of X when T is equal to 1,2 and 3 respectively.(8) These curves will not intersect. If they were to intersect this would mean that a point like A" is associated with a higher level of T than B". This situation is not possible in the model.(9)

We are now able to derive the offer curve for the three different cases stated in the introduction. This will be our task in the next section. Before doing this we would like to briefly consider the question of the existence of equilibrium.

Hadley and Kemp (1966) proved the existence of an equilibrium for a competitive economy in the case where a transportation good is produced. Woodland (1968) extended their results to the case where different routings are possible. Since our model is a special case of the one used by Hadley and Kemp, their results are applicable to our model. We can thus state that a competitive equilibrium exists in our model.

Section II - The offer curves and equilibrium for three different cases.

In this section we will consider a world where country H exports X and country F exports Y. The three cases stated in the introduction will be analysed in turn. At the end of this section we will compare our results with the "evaporation" model.

Case I - Each country is using its T for export only.

To analyze this case, we can use the O_{Ti} curves derived in Figure V. We reproduce three such curves O_{T_1}, O_{T_2}, O_{T_3}, in Figure VI. Consider X_o. From equation (1) suppose that T_1 is required to transport X_o from H to F. Then, A will be on the offer curve, being on O_{T_1}. Similarly, to transport X_1, suppose T_2 is necessary. Thus B will also be on the offer curve, being on O_{T_2}. We can derive the offer curve repeating the operation for each point. The inclusion of transportation costs will thus shift the offer curve to the left in that case. In Figure VII we represent such a

FIGURE VI

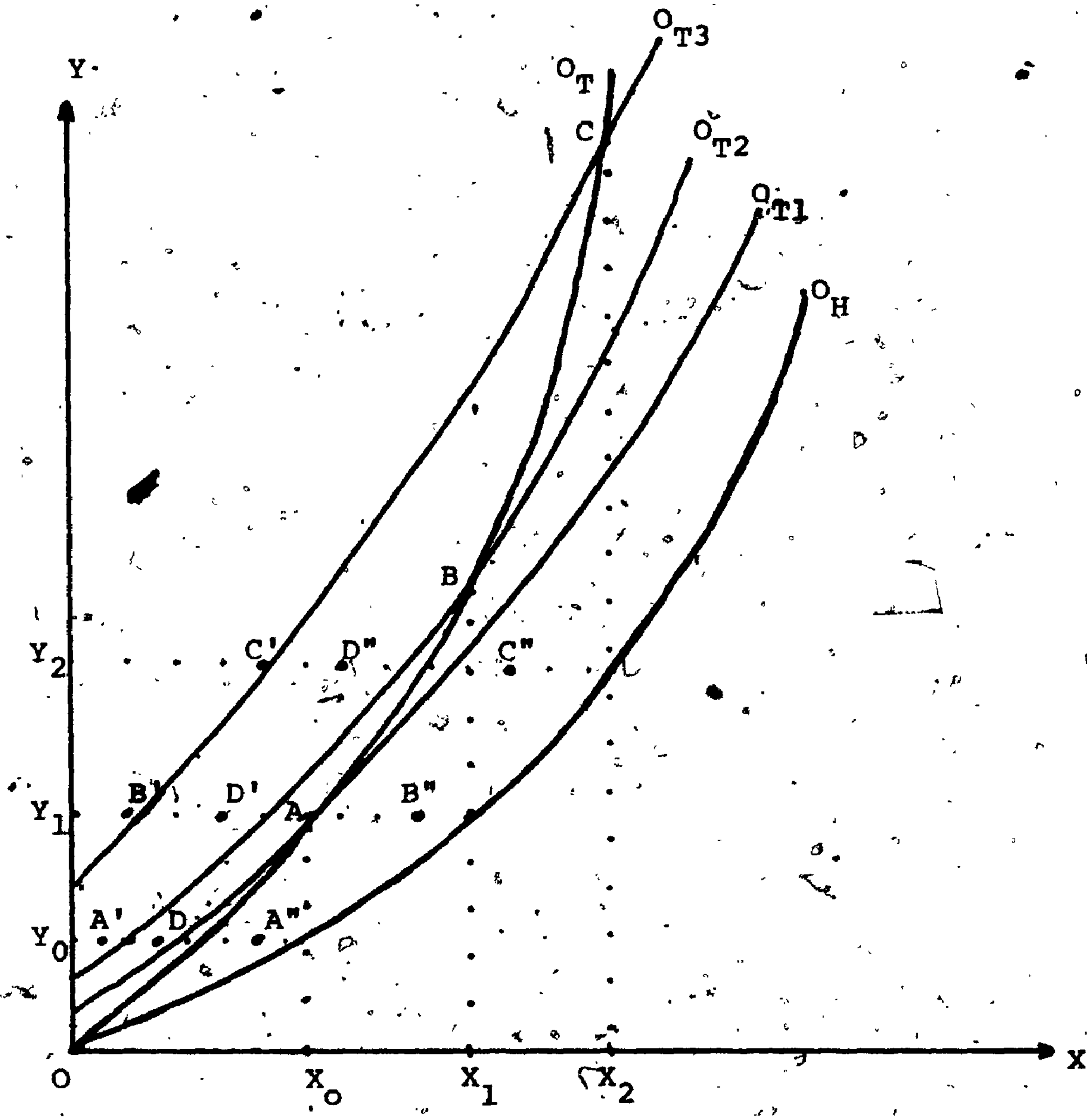
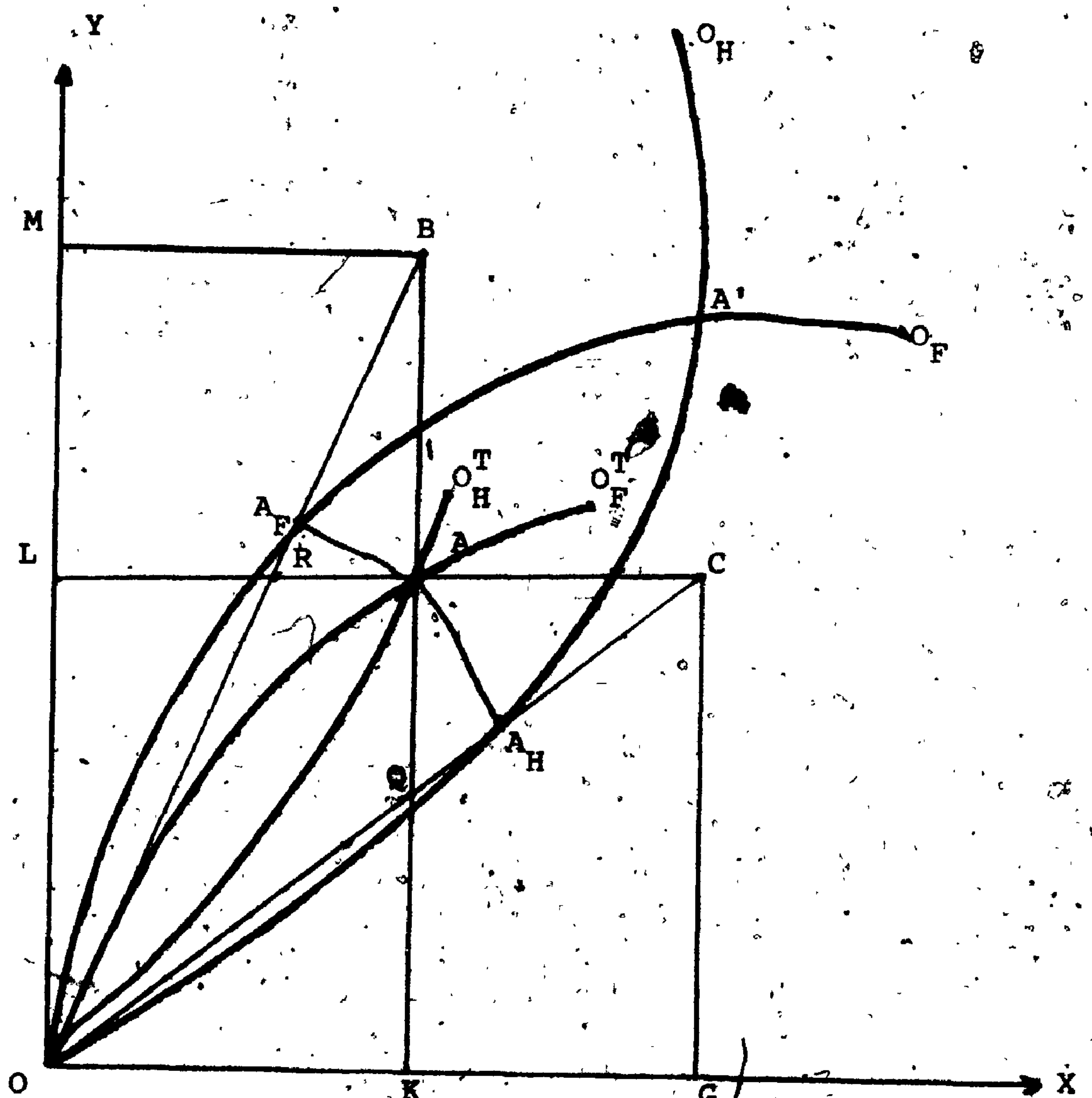


FIGURE VII



shift for country H by O_H^T. The offer curves for country F can be added to the same diagram. O_F will be the offer curve for F when transportation costs are absent. O_F^T will be the offer curve when F is transporting only its exports of Y to H. The new equilibrium will be at A when country H will be exporting OK of X plus the amount of T required to transport it to F in exchange of OL of Y plus the amount of T required to transport it from F to H. Let OA_H and OA_F be the two domestic price ratio[10] associated with point A. The curves AA_H and AA_F are curves like ZZ' in Figure V.

Draw LA parallel to OX and KA parallel to OY. Let point B be at the intersection of OA_F and KA extended. Point G is at the intersection of LA and OA_H extended. Point R is where OB intersects LA and point Q where KA and AC meet. BM and CG are perpendicular to their axis. We thus have,

$$P^F = \frac{(P_X)}{(P_Y)}^F = \frac{BK}{OK} = \frac{OM}{OK} \qquad (3)$$

$$P^H = \frac{(P_X)}{(P_Y)} = \frac{CG}{OG} = \frac{AK}{OG} = \frac{QK}{OK} \qquad (4)$$

$$\frac{P_X^F}{P_Y^H} = \frac{AK}{OK} \qquad (5)$$

Let us now find the value of T in terms of X and Y for both countries. First, consider country H. The country is exporting OK of X plus T in return of AK of Y. At domestic prices the OK of X would get only QK of Y in return.(11) Thus in equilibrium this means that the T required to transport OK from H to F is receiving AQ in terms of Y. To get AK of Y (= CG) people in H would have to give away OG of X. Now, only OK of X is given. Thus, in equilibrium KG of X (= AC) has to be given away in order to get T. Thus, AC is the value of T for H in terms of Y.

Similarly for country F. AR will be the value of T in terms of X and AB in terms of Y.

Let us derive some of the equilibrium price relations. We can write

$$P^F = \frac{BK}{OK} = \frac{BA}{OK} + \frac{AK}{OK} \quad (6)$$

but $P^F = \frac{BA}{AR}$.

thus, $BA = P^F AR$

Substituting in (6), we get, after rearranging,

$$P^F = \frac{AK}{OK}\left(\frac{1}{1 - \frac{AR}{OK}}\right)$$

from (5)

$$\frac{P_X^F}{P_Y^F} = \frac{P_X^F}{P_Y^H}\left(\frac{1}{1 - \frac{AR}{OK}}\right)$$

thus[12],

$$P_Y^H = P_Y^F\left(\frac{1}{1 - \frac{AR}{OK}}\right) = \frac{P_Y^F}{k_Y} \quad (7)$$

Now AR is the value of T_Y^F in terms of X (the import), OK is total import. Thus, AR/OK will be the proportion of total import which goes to T. So k_Y will be the proportion of total imports which goes to Y.

Similarly, we can derive,

$$P^H = \frac{CG}{OG} = \frac{CG}{OK} \quad \frac{OK}{OG}$$

but CG = AK

$$P^H = \frac{AK}{OK} \quad \frac{OK}{OG}$$

from (5) we get

$$\frac{P_X^H}{P_Y^H} = \frac{P_X^F}{P_Y^H} \quad \frac{OK}{OG}$$

So,

$$P_X^F = \frac{P_X^H}{k_X} \qquad (8)$$

where $k_X = \frac{OK}{OG}$

The value of OL (the import) when considering X at domestic prices, is OG. In equilibrium, exporter will only receive a value of OK. Thus k_X can be interpreted as the proportion of total import for H which goes to X.

Combining (7) and (8) we get

$$P^F = \frac{1}{k_X \; k_Y} P^H \qquad (9)$$

Result (9) requires further considration. Mundell,[13] finds the same result when considering the "evaporation" model. He defined k_X and k_Y as the proportion of X and Y that are landed in the importing country. He studies the case where "transport costs are incured only in the good of the exporting country, which means that some X is used up in shipping X and some Y is used up in shipping Y."[14] This can be associated with our case where each country transports only its exports. The difference between his results and ours can be explained in the "evaporation" model k_X and k_Y which are given constants or can be made variables by assuming a more general technology like (1) and (2). They will nevertheless not be, as in our model, part of the solution, depending on prices and endowments. Furthermore, only X is sacrificed to transport X, according to a stated technology in the "evaporation" model. In our model both X and Y will be sacrificed to release factors to produce T.[15]

Case II - Each country is using its T for import only.

In that case, country H will be producing enough T to transport its import of Y, according to (2). Similarly F will produce T to transport its import of X, according to (1). We can derive the two country offer curve in the same way as in the previous case. Suppose that in Figure VI we assume that T_1, T_2 and T_3 are required to transport Y_o, Y_1, Y_2 as well as X_o, X_1, X_2, then the offer curve for country H will be OA'B'C'. It can be derived in the same way as O_T. However, less than T_1 can be required to transport Y_2 (thus Y_1 and Y_o). In that case we can have a situation like A"B"C".

On the other hand, a simple comparaison of the amount of T required to transport either quantities at a point on O_T will not be sufficient to decide whether the curve for the second case is more to the right or to the left of O_T. For example, Y_o, Y_1 and Y_2 can require less T than X_o, X_1 and X_2 respectively. Nevertheless we can still have a situation like DD'D".(16) Whether the offer curve in the second case is to the right or to the left of O_T, will depend not only on (1) and (2) but also on the full model.

We can use Figure VII to represent the shifts in the offer curve for this second case. If we were to draw both cases in the same diagram, the equilibria would not be the same generally. All the relationship which was derived for the first case will carry through in the second one. In other words, (7), (8), and (9) are still valid. The equilibrium prices will be different but not the relationship among them. The comments which were made for the first case, apply equally well here.

Case III - Each country is using its T for some of its import and some of its export.

In that case, the total T required is given by

$$T^i = T^i_X + T^i_Y, \quad i = F, H \tag{10}$$

One way to find how much T will be required, is to assume that a country is transporting a fixed proportion of its imports and exports. The "evaporation" model can be extended easily to take this case into account. Or, we can simply assume that total T produced (T^H+T^F) has to be equal to the total T required according to (10).

The amount of its import and/or exports which a country will transport, depends on the specific parameters of the model, to the same degree as all the other endogeneous variables. A geometrical representation of this last general case is almost impossible.

On the other hand, it is easy to show that whenever transportation costs are included, there will be a leftward shift in the offer curve for H. Consider point A in Figure VI. We can state that T_1 is the amount of T necessary to transport that portion of X_0 and X_1 which country H will move. Thus, even in this general case, we can use Figure VII to represent the new equilibrium. This implies that (7), (8), and (9) will carry through. It is important to note that this third case will give a solution which is different from the previous two. It is not possible to compare the three cases exactly, unless we know the specific form of the functions in the model.

Section III. - Technological improvement in the T industry.

There are two kinds of technological change which could be considered, the first one can be described as the same amount of T which can be produced with less of each input. This is the case generally treated in the literature on technological change. The second type can be described as less T, being required to transport a given amount of X or Y. That is, there is a change in the functional forms (1) and/or (2). We will treat this second case because it focuses on the peculiarity of good T (being a transportation service).

In this discussion we would like to see if it is possible to state that factor prices will tend towards equality, as the model returns to the no transportation costs case.

Assume there is an improvement in the transportation of X only. This means that the same amount of X can be transported with less T for the same distance.

From Figure VI, it is clear that if a country transports X, its offer curve which includes transport costs will shift towards its offer curve without transport costs. We will refer to case 1, 2, and 3 in the same manner as was done in the previous Section.

Consider Figure VII. In the first case, only H transports X. Then O_H^T will shift to the right while O_F^T will be the offer curve for F. The new situation is pictured in VIIIa. In the second case, only F transports X. In that case O_F^T will shift towards O_F. O_H^T will not change. The new equilibrium is represented in VIIIb. Finally, in the third case both offer curves shift and the new situation is represented in VIIIc. Since the conclusions reached in each case will be the same, we will only cover the first one.

After the improvement, equilibrium will move from A to B. As long as O_F^T is positively sloped, there will be an increase in the trading volume of each good. If the curves like ZZ in Figure V are negatively sloped for both countries (17), we can state unambiguously that commodity price ratios will tend towards equality. For example, in Figure VIIIa the domestic price ratio of X to Y will go from OC to OD in F and from OE to OM in H. However, if curves like ZZ are positively sloped in certain regions (like DE and DE" in Figure II) it is possible for these price ratios to become more unequal. In Figure IX we represent such a case. O_H, O_F, A and B

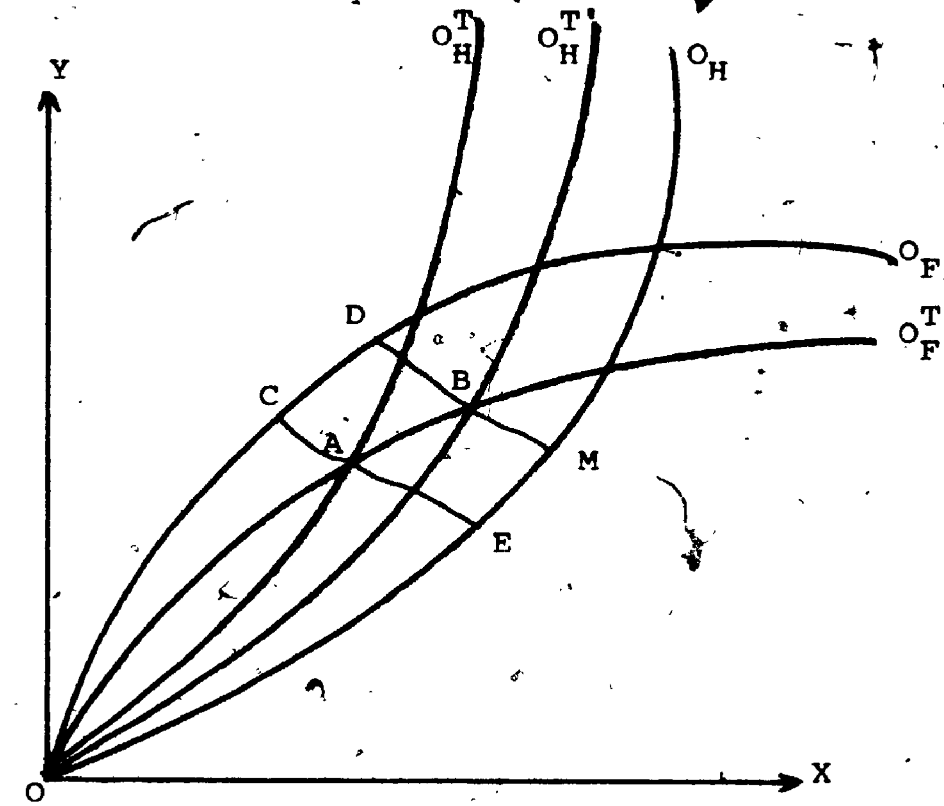


FIGURE VIIIa

FIGURE VIIIb

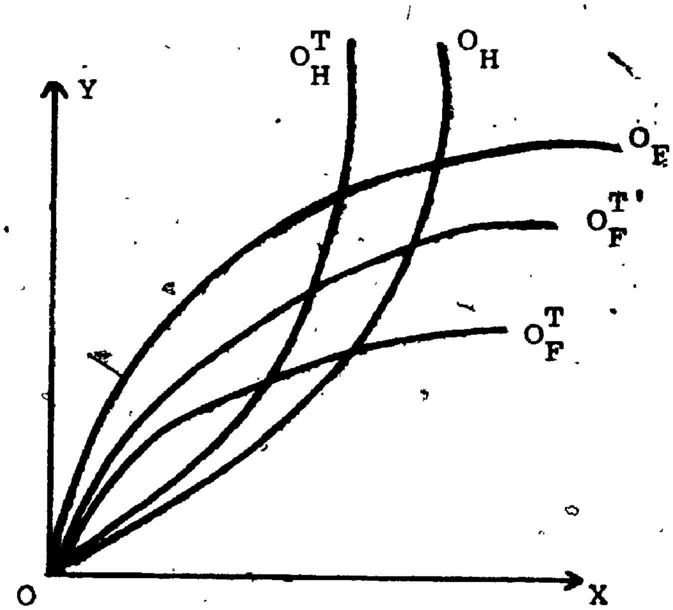


FIGURE VIIIc

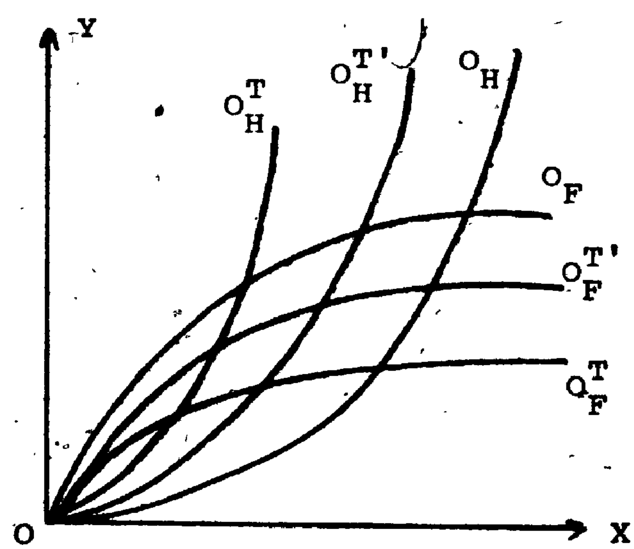
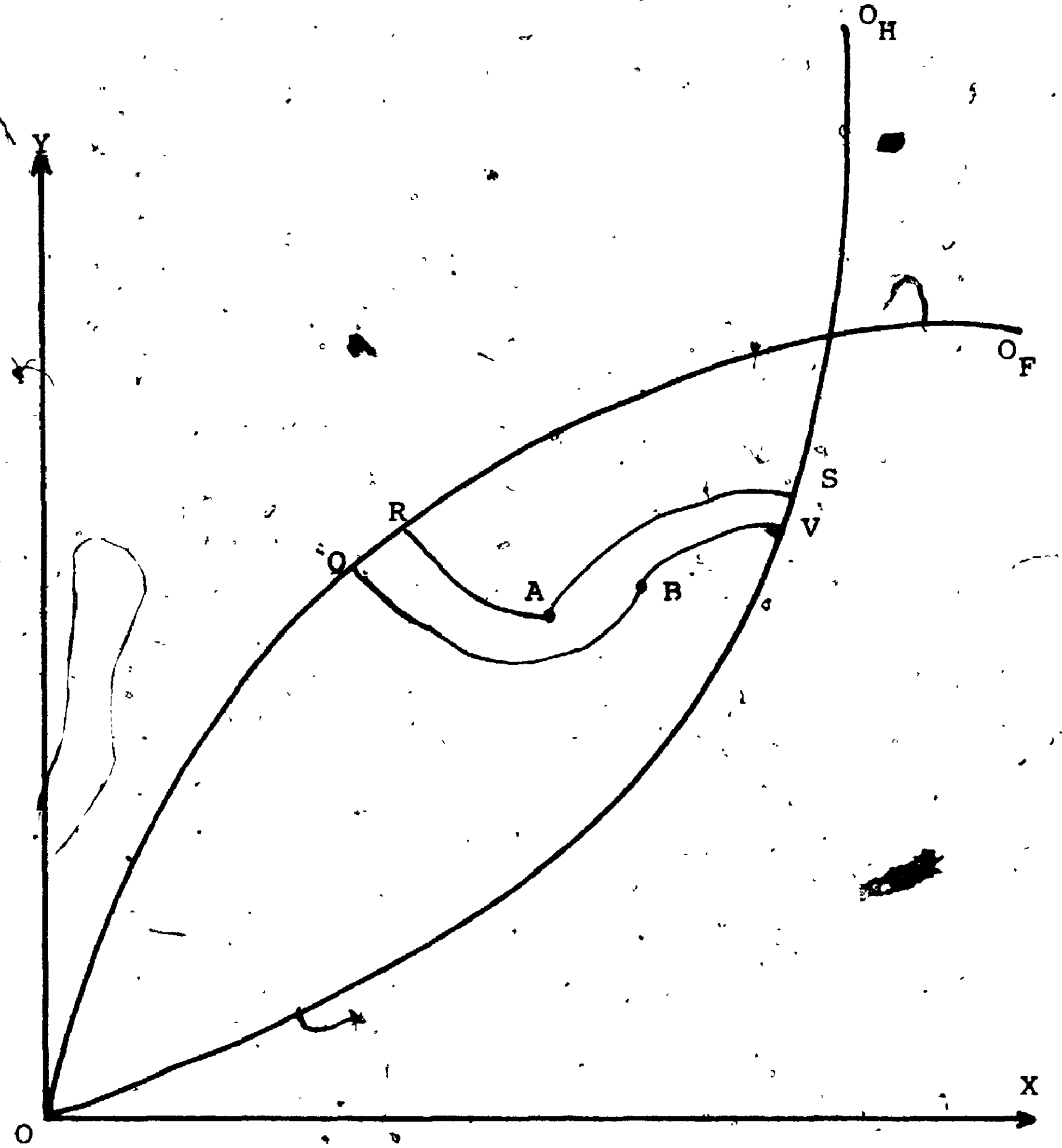


FIGURE IX



are reported from VIIIa. We omit O_H^T, O_H^T, and O_F^T. In this case the price ratio of X to Y will increase in F (from OR to OQ) and decrease in H (from OS to OV). Thus the inequality between prices will be accentuated.

The question on commodity prices unambiguously tending to equality must offer a positive answer in order to find out whether factor prices will tend to equality as commodity prices tend to equality. The second part of the question becomes irrelevant if the first one cannot be answered positively.

Section IV - Some related theorems.

In this section we will briefly discuss the effects of including transportation costs on four of the theorems derived in the two-by-two model. We will consider the Heckscher-Ohlin, Stolper-Samuelson, Samuelson-Rybczynski and Factor Price Equalization theorems.

If the mapping of commodity prices into factor prices is univalent, factor prices will not be equalized because commodity prices are not. On the other hand, if the mapping is not univalent, it is possible to find

3 OF/DE 3

a case where the same set of factor prices will be associated to two different sets of commodity prices. Conditions might be such that these commodity prices are equilibrium values. In such a case factor price equalization will simply be of an accidental nature and, therefore, not in accordance with the spirit of the theorem.

The validity of the Stolper-Samuelson and the Samuelson-Rybczynski theorems does not depend on whether T is a transportation good or simply another good. So, if conditions are such that these theorems are valid in the three-by-three case, the fact that T is a transportation service will not change this conclusion. In other words, the fact that T is transport, affects the model through (1), (2) and the demand side, not the production possibility surface on which both theorems depend.(18)

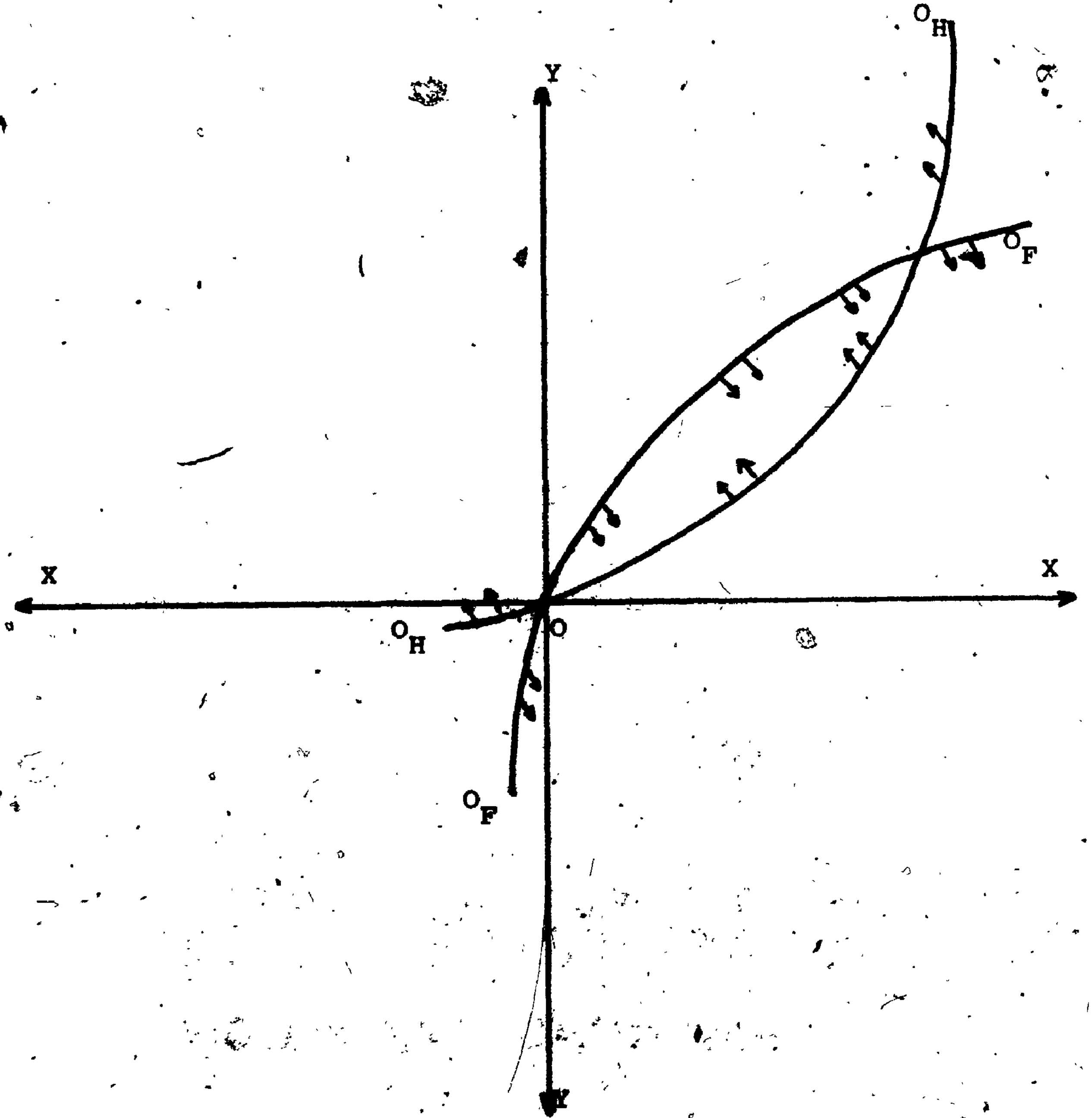
Finally, we can note that if the Heckscher-Ohlin theorem is valid before T is introduced, it is still valid after.(19) Consider the case where (1) and (2) are such that no T is required to transport either X or Y. It is clear that in Figure VII, O_H and O_F will be the relevant offer curves. We reproduce these schedules in Figure X. We also include the portion of the offer curves which is usually left out. T will reverse the trade pattern if an equilibrium can be found in quadrant III.

In Figure X, curves like ZZ of Figure V will always be to the left of O for H and to the right of O_F for F. Thus, once T is included, the offer curves will shift in the same direction. This makes it impossible for the two modified offer curves to have any common point in quadrant III. This proves that even high transportation requirements will not reverse the pattern of trade which would exist, if no such assumption would be required. At most, it can discourage trade to take place.

Section V - The gains from trade.

Can it be possible that, on account of the presence of transportation costs, a country actually looses, if it engages in trading relations? The answer to this question is no, because there are no transport costs domestically, and, given the other assumptions of the model, the price plane will be tangent to the production

FIGURE X



set. Return to Figure I; the production possibility surface is strictly bowed out. The price plane, which is also the consumption plane, will thus always include points which are superior to A, the autarky equilibrium. A country will therefore always gain from trade.

Up to now, we have been considering the case where only two countries are involved in trade. Let us now assume that there are many countries. In that case, it is clearly possible that a country exports only T in exchange for X and Y. Is it possible for a country that imports T to be worse off? Here, we want to investigate whether the situation pictured in Figure XI is a possibility. The country is producing at A, exchanging AB of X and BC of Y to get T, which is necessary to carry the exchange of CE of Y for ED of X. Suppose Y_oAX_o is the case when no T is produced; then the situation is not one of equilibrium. There is a positive price on T, internationally (as reflected by AB and BC being non zero). The domestic price is zero. There will thus be an increase in the domestic price of T up to the point where equilibrium is reached. This occurs when the consumption set is tangent to the production possibility surface. Consumption will be somewhere along GG, which includes completely the autarky consumption set Y_oAX_o.[20]

Section VI - Conclusions.

We have considered geometrically the effect of including a produced transportation good in the model of international trade. We have not started from the usual two-by-two model because, we would have a model with two factors and three goods giving us a production set with linear segments.[21]

We found that the introduction of transportation costs will shift the offer curve towards the axis of the imported good. This result is similar to the one derived in an "evaporation" model of the kind proposed by Samuelson and Mundell. One thing that we have not done, was to compare the two shifts. This is due to the fact that a simple comparaison is not possible. The "evaporation" model will give a technical constant which represents the rate at which X and/or Y can be transformed into transportation services. In our model, the rate at which X and/or Y can be transformed into Y will not be given, but will instead be dependent on the full model.

We described the trading equilibrium according to three basic cases. We were able to show that results (7), (8), and (9) were valid for all three cases. An interesting point was that (9) is derived by Mundell in terms of the "evaporation" model. In fact (7) and (8) can be related directly to (4) and (3). Consider (7) for example; it states that

$$\underline{P^H_Y} = \frac{P^F_Y}{k_Y}$$

or

$$P^F_Y = k_Y\ P^H_Y$$

k_Y is the proportion of import that goes to Y. Suppose that k_Y is 60% and P^H_Y is $1.00, then the transportation industry is receiving 40¢ to transport one unit of Y. P^F_Y being 60¢, we get (4) directly. The same thing occurs in (8) and (3). Comparaison of (7), (8), and (4), (3) will give us, in relation to Figure VII:

$$t^{HF}_X = P^F_X \quad \frac{AC}{OG} \qquad (10)$$

$$t^{FH}_Y = P^H_Y \quad \frac{AR}{OK} \qquad (11)$$

AC and AR are the values of T in terms of X for H and F respectively. IK is H export and OG is the domestic value of its import in terms of X.

We also briefly discussed some of the theorems derived in the usual two-by-two model. We saw that the validity of the Stolper-Samuelson and the Samuelson-Rybczynski theorems does not depend on T being a produced transported good. Factor prices will not in general be equalized. We showed that the introduction of T will not reverse the trading pattern. In the last section we saw that there will still be gains from international trade discarding a possible counter-example as a non equilibrium situation.

Footnotes

* I would like to thank Professor J.R.Melvin for obtaining a Senior Killam Research Scholarship which helped support the research for this paper.

1. Mundell (1957), page 65.

2. That is, strictly quasi-concave indifference curves.

3. We ignore the possibility of T being a consumed good, for example, travel or tourism.

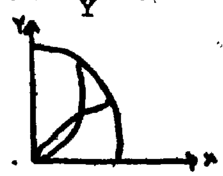
4. For example, when only one good requires transportation or when a country is doing both, while the other is either only importing or only exporting.

5. See Chapter I for a discussion of this topic.

6. See the third Section of Chapter I.

8. Such curves cannot be called "offer curves" because they are not equilibrium points. I would like to thank Professor Melvin for this point.

9. To have such a possibility would require that the schedule associated with a given P_X and P_Y and varying amount of T be of the following form:



this is impossible given our assumptions.

10. These are referred to as the f.o.b. (Free on board) prices compared to the c.i.f. (Cost insurance freight) prices.

11. This is because OA_H is the domestic price ratio for country H.

12. Note that as transportation costs tend towards zero, commodity prices tend to equality.

Footnotes (continued)

13. Mundell (1957), page 70.

14. Mundell (1957), pages 68-69.

15. Unless segment $A_H A$ in Figure VII is positively sloped.

16. Suppose that the second partial derivative of (1) with respect to X goes from positive to negative while the second partial of (2) with respect to Y has the reverse pattern. It is possible to devise a case where the two offer curves will intersect.

17. i.e., there is no gross complementarity on the production side of the model. Or, if it exists, it is not strong enough to generate a positively sloped ZZ.

18. For a further discussion of the conditions under which these theorems will be valid in the case where there are three factors and three goods, see Chipman (1969).

19. Remember there are three factors in the model.

20. Hartwick (1972) has found results of the opposite kind.

21. See Melvin (1971) for a treatment of this case.

CONCLUSION

This thesis concentrated on the production side of the model. The introduction offered some reasons for this choice. We could not however extend the production side in all directions and, this imposed a more modest contribution. We did contribute however three chapters which are quite different by nature. The reader will remember that in the introduction we showed how the three chapters were linked; recapitulating, Chapters I and II formed the first link while Chapters I and III formed the other.

Chapter I was intended mainly, to summarize many of the contributions made on the generalization of the model with two goods and two factors. One of the main themes of this chapter dealt with the principal theorems in the Pure Theory of International Trade which could be integrated into a discussion of the shape of the production possibility surface. One point can be brought out more clearly about the two-by-two model: if we assume that any endowment ray generates a strictly convex production set, all the theorems hold, i.e., the Samuelson-Rybczynski, Stolper-Samuelson, Heckscher-Ohlin, Factor price-equalization theorems. This occurs on account of a very special property of the two-by-two definite matrices; this property concerns a positive (negative) definite matrix of size two-by-two, where the elements along the diagonal of the inverse matrix will be positive (negative) while the diagonal elements will be negative (positive). When we leave the two-by-two world, and move to higher dimensionalities, positive or negative definiteness does not guarantee this property of the inverse matrix. Other restrictions have to be imposed.

Chipman (1969) worked out the conditions under which the property holds in a three-by-three matrix. Kemp (1969) extended the discussion to the four-by-four case. Generalization on the three cases leads to this observation; as dimensionality is increased, the conditions which would guarantee the above property to the inverse matrix would have to be more restrictive: what was sufficient for the n by n case no longer applies to the (n + 1) by (n + 1).

We can add to this, Kugas's finding (1972) which stipulates that when factors outnumber goods, the per unit requirements do depend on endowments; it is also to be remembered that when goods outnumber factors there is a basic indeterminacy in production. It is necessary to have recourse to arbitrary assumptions; for example Hong (1969) assumes that the solution will be such, so as to minimize the value of transaction between countries. All these mentioned above, present a rather unhappy future for international trade theory. Are all the theorems likely to hold only for the two-by-two case with some possible extension to the case where m equals n? One look at chapter II will reassure us. We consider there, a model with two goods and three factors and all the theorems remained mostly valid. The matrix approach, however, used in the discussion of the n x n case, will generally not be valid anymore at higher levels.

In chapter II, we studied the effect of relaxing the assumption that capital was perfectly mobile between industries. We assumed that each industry had a specific factor of production. Jones used the same model not to study the effect of perfect immobility of a factor, but for it is the simpler model where factor prices depend on endowments. To his early results we added a geometrical treatment, and a fairly intensive study of the effect of technological change. For a summary of the findings the reader should consult the conclusion to chapter II.

In chapter III, we analyzed a model where transportation costs were not assumed to be zero. Unlike previous studies, we relaxed the wastage hypothesis, assuming instead, that transportation services had to be produced like any other commodity. This implies that a model with three goods had to be considered. We assumed three factors in order to have a strictly convex production set. The conclusion to the chapter summarizes our findings; we will not repeat it here.

We would like to stress however a particular point which was derived in chapter III. This model is a primary example of an over-determined system. In order to arrive at a unique solution we had to add further restrictions or conditions. We had to assume what each country was transporting. In the case of a more general over-determined system, Hong assumed that trade will take place at the point which minimizes total international transactions. These are two cases where the usual system will not generate a unique solution. There is a difference between the two cases: in the case of more goods than factors, the indeterminacy will basically follow from the fact that one good can be "produced" by using linear combination of other goods. For transportation costs, considering the cases where we stated explicitly which good each country transports is a way to avoid to firstly make statement about the price of transportation services between different countries and secondly to draw three dimensional pictures.

There are many areas opened for research and we can point to a few emerging from this thesis.

First the specific factor of production model could be extended to the case where we have many goods and many common factors. One interesting model to analyze would be the case where each good has a specific factor and there are n common factors to the n goods. It could well be that this model is not that much different from the n by n case.

In chapter III we stated that the absence of transportation costs was one of the simplifying assumptions made in order to arrive at the basic feature of the economic system. It should be worthwhile to relax other simplifying assumptions one by one or in groups. Much research has been done in that area, but there are still many unexplored issues leaving many possibilities for further research.

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