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A Study Of Partially-ordered Rings

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A STUDY OF PARTIALLY-ORDERED RINGS

by

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ABSTRACT

The first part of this thesis is devoted to the study of a class of ℓ-rings (lattice-ordered rings) which properly contains the class of f-rings. The second part is a study of a structure theory of a commutative partially-ordered ring which contains no non-zero positive nilpotent elements.

In Chapter II, a new class of ℓ-rings is introduced, and it is shown that an ℓ-ring having zero ℓ-radical is an f-ring if and only if it is in this class. Some classes of ℓ-rings having the ℓ-radical equal to the set of all nilpotent elements of the rings are investigated. In these classes of ℓ-rings, it is proved that an ℓ-ideal is a prime ℓ-ideal if and only if it is a ring prime ideal.

In Chapter III, some properties of square archimedean ℓ-rings are studied. For example, it is shown that (1) in a square archimedean ℓ-ring, the ℓ-radical is equal to the ℓ-prime radical. (2) in a square archimedean ℓ-ring A, the set $\bar{\bar{N}}(A) = \{x \in A \mid (x \vee -x)^n = 0$ for some $n\}$ is an ℓ-ideal if and only if $\bar{\bar{N}}(A)$ is equal to the ℓ-radical, and the set of all nilpotent elements $\bar{N}(A)$ is an ℓ-ideal if and only if $\bar{N}(A)$ equal to the ℓ-radical. (3) in a square archimedean pseudo f-ring, the set of all nilpotent elements is always an ℓ-ideal, hence is equal to the ℓ-radical.

In Chapter IV, Johnson radical of ℓ-rings is studied. The concept of a faithful, irreducible ℓ-module is introduced and, under certain conditions, the rings having a faithful, irreducible ℓ-module are shown to be the ℓ-primitive rings. The relation between $\mathcal{R}(A)$, the Johnson radical of A and $\mathcal{R}(A_{nxn})$, where A_{nxn} is the ring of nxn matrices with entries from A, (the ordering on A_{nxn} take to be the canonical one), is investigated.

In Chapter V, the concept of an m-filet is defined in a commutative partially-ordered ring without positive nilpotent elements. This concept is used to give necessary and sufficient conditions for a commutative partially-ordered ring to be o-isomorphic to a direct sum of strict rings.

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INTRODUCTION

The first part of this thesis is devoted to the study of a class of lattice-ordered rings which properly contains the class of f-rings. It is a continuation of the study initiated in the papers by Birkhoff and Pierce [5], D.G. Johnson [13], J. Diem [7] and Steinberg [18]. The second part is a study of a structure theory of commutative partially-ordered rings which contain no non-zero positive nilpotent elements.

In Chapter I we present the necessary background material for our study.

In [7], Diem has shown that the following conditions are equivalent for an ℓ-ring (lattice-ordered ring) with zero ℓ-radical: i) A is an f-ring; ii) A is a subdirect union of totally-ordered rings without non-zero divisors of zero; iii) $(x \vee 0)a(-x \vee 0) = 0$ for all x, a in A; iv) if a, b, c are in A and $a \geq 0$, then $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$ and v) $(x \vee 0)(-x \vee 0) = 0$ for all x in A. In Chapter II, we prove that, for an ℓ-ring with zero ℓ-radical, these conditions are equivalent to vi) $a(x \vee 0)a(-x \vee 0)a = 0$ for all x, a in A with $a \geq 0$. It is easily seen that in any f-ring the conditions iii), iv), v) and vi) hold. In [5] Birkhoff and Pierce have shown that in any f-ring the ℓ-radical is equal to the set of nilpotent elements and in [13] D.G. Johnson has shown that an ℓ-ideal P in an f-ring is a prime ℓ-ideal if and only if P is a ring prime ideal. In Chapter II we also generalize these two results: that is, in any ℓ-ring which satisfies one of the conditions iii), iv), v) or vi) the ℓ-radical is equal to the set of all nilpotent elements and an ℓ-ideal P is a prime ℓ-ideal if and only if it is a ring prime ideal.

Chapter III is a study of square archimedean ℓ-rings. In a square archimedean ℓ-ring A, the ℓ-radical coincides with the prime radical of A, and hence the square archimedean pseudo f-rings form another class of ℓ-rings having the property that the ℓ-radical is equal to the set of all nilpotent elements. We will show that for a square archimedean ℓ-ring the ℓ-radical is the set of all nilpotent elements if and only if the latter set is an ℓ-ideal.

In [13], Johnson introduced an analogue of the Jacobson radical for f-rings, and in [18], Steinberg defined three different generalizations of the Jacobson radical for the classes of all ℓ-rings. Denoting these various "radicals" by P_{m_o}, J, and $\mathcal{R}$, Steinberg [18] showed that $P_{m_o}(A) \subseteq J(A) \subseteq \mathcal{R}(A)$ for any ℓ-ring A, and that these three ideals are equal under certain conditions. In chapter IV we shall show that $J(A) = \mathcal{R}(A)$ under more general conditions, and that $P(A) \subseteq P_{m_o}(A)$. Also, we will define the concept of a faithful, irreducible ℓ-module, and, for the class of pseudo f-rings, we will relate this concept to Steinbergs work. In chapter IV we will also investigate the relation between $\mathcal{R}(A)$ and $\mathcal{R}(A_{nxn})$, where A_{nxn} is the ring of nxn matrices with entries from A, the ordering on A_{nxn} taken to be the canonical one.

In chapter V we define an equivalence relation on the positive cone of a commutative partially-ordered ring without positive non-zero nilpotent elements. The equivalence classes will be called m-filets. The m-filets so defined give a disjunctive and distributive lattice. We say a ring A has Jaffard's property if the set M of all minimal m-filets of A is non-empty and satisfies the following condition; for every $f \in A^+$ (the positive cone of A) and for every $\bar{a} \in M$ there exists $f_{\bar{a}} \in A^+$ such that i) $f_{\bar{a}} \leq f$, ii) $\overline{f_{\bar{a}}} \leq \bar{a}$, iii) $\overline{f-f_{\bar{a}}} \wedge \bar{a} = \bar{o}$, (where $\bar{a}$ represents the m-filet containing a). The main theorem is the following:

> Let A be a commutative partially-ordered ring which satisfy the condition that $x^2 = xy = y^2$ implies $x = y$ for all x,y in A^+. For A^+ to be o-isomorphic to the direct sum of a family of strict cones (of A) it is necessary and sufficient that (1) the lattice of m-filets be lattice isomorphic to the lattice of a finite subset of a set; (2) A^+ have Jaffard's property. Moreover, if A is directed then A is o-isomorphic to the direct sum of a family of strict rings.

Throughout the thesis, definitions, theorems, propositions, corollaries, examples and remarks are numbered by two integers. The first integer represents the number of the chapter. For example, Proposition 3.20 is found in chapter 3, immediately following Example 3.19.

CHAPTER I

PRELIMINARIES

In this chapter we present those definitions and results in the theory of lattice-ordered groups, partially-ordered rings and latticed-ordered rings which we will need in the sequel. None of these results will be proved; the reader is referred to [4], [5], [7], [8] and [13] for proofs. Our notation is the same as [7] and [13]. The term ring will always mean associative ring and not necessarily possessing an identity.

<u>DEFINITION 1.1.</u> A *partially-ordered group* is a group G which is partially ordered and in which $a \leq b$ implies $x + a + y \leq x + b + y$ for all x, y in G. If G is a lattice under this partial order, then G is called a *lattice-ordered group*. If G is totally ordered, then G is called a *totally-ordered group*.

Let G be a partially-ordered group. An element b of G is said to be *positive* if $b \geq 0$. The set of all positive elements of G is denoted by G^+. If G is a lattice-ordered group and $a \varepsilon G$, then the *absolute value* of a is $|a| = a \vee (-a)$, the *positive part* of a is $a^+ = a \vee 0$, and the *negative part* of a is $a^- = (-a) \vee 0$.

PROPOSITION 1.2. Let G be an abelian lattice-ordered group and let a, b and c in G. Then

i) $a + b = (a \vee b) + (a \wedge b)$;

ii) $a + (b \vee c) = (a + b) \vee (a + c)$ and $a + (b \wedge c) = (a + b) \wedge (a + c)$;

iii) $(-a) \wedge (-b) = -(a \vee b)$ and $(-a) \vee (-b) = -(a \wedge b)$;

iv) $|a + b| \leq |a| + |b|$ and $|a - b| \geq ||a| - |b||$;

v) $a = a^+ - a^-$;

vi) $a^+ \wedge a^- = 0$;

vii) $|a| = a^+ + a^-$;

viii) if a is not zero, and n is a non-zero integer, then na is not zero;

ix) if $a + b = 0$ and $a, b \geq 0$, then $a = b = 0$; and

x) if $a, b, c \geq 0$ and $a \leq b + c$, then there are elements $b_1, c_1 \geq 0$ such that $b_1 \leq b$, $c_1 \leq c$, and $a = b_1 + c_1$.

PROPOSITION 1.3. Let G and G' be lattice-ordered groups and let $f : G \to G'$ be a group homomorphism. Then the following are equivalent:

i) $f(a \vee b) = f(a) \vee f(b)$ for any a, b in G;

ii) $f(a \wedge b) = f(a) \wedge f(b)$ for any a, b in G;

iii) $f(a^+) = (f(a))^+$ for any a in G:

iv) $f(|a|) = |f(a)|$; for any a in G;

v) if a, b in G and $a \wedge b = 0$, then $f(a) \wedge f(b) = 0$.

A group homomorphism between lattice-ordered groups that satisfies any one, and hence all of i), ii), iii), iv) and v) of Proposition 1.3 will be called an *ℓ-homomorphism*. The kernel of an ℓ-homomorphism is an ℓ-subgroup of G in the sense of :

DEFINITION 1.4. An *ℓ-subgroup* of a lattice-ordered group G is a normal subgroup H of G that satisfies;

$$a \in H \text{ and } |b| \leq |a| \text{ imply } b \in H .$$

If H is any ℓ-subgroup of G, then the difference group G/H can be made into a lattice-ordered group by defining $a + H \in (G/H)^+$ if and only if $a^- \in H$.

DEFINITION 1.5. A partially-ordered group G is said to be *archimedean* if for every pair a, b of elements of G, with $a \neq 0$, there is an integer n such that $na \not\leq b$.

<u>DEFINITION 1.6.</u> A *partially-ordered ring* A is a ring A in which a partial order has been defined so that;

i) $a \geq b$ implies $a + c \geq b + c$ for all c in A; and

ii) $a \geq 0$ and $b \geq 0$ imply $ab \geq 0$.

A partially-ordered ring A is a *lattice-ordered ring* (ℓ-ring) if A as partially ordered set is a lattice, and A is a *totally-ordered ring* if A as a partially-ordered set is totally ordered.

<u>PROPOSITION 1.7.</u> Let A be a partially-ordered ring and let A^+ be the set of all positive elements in A. Then

i) $A^+ \cap (-A^+) = \{0\}$;

ii) $A^+ + A^+ \subseteq A^+$;

iii) $A^+ \cdot A^+ \subseteq A^+$ and

iv) if $a, b \in A$, then $a \geq b$ if and only if $a - b \in A^+$.

Conversely, if P is any subset of a ring A that satisfies i), ii) and iii), then the relation on A defined by iv) makes A into a partially-ordered ring with $A^+ = P$. Also, if $x \vee 0$ exists for each $x \in A$, then A is an ℓ-ring ([4]).

Let A and A' be partially-ordered rings. An order preserving homomorphism of A into A' is called an o-*homomorphism*. It is clear that a homomorphism $\phi: A \to A'$ is an o-homomorphism if and only if $\phi(A^+) \subseteq (A')^+$. If in addition, $\phi(A) = A'$ and $\phi(A^+) = (A')^+$, we call ϕ an o-*epimorphism* of A onto A'. If ϕ and its inverse ϕ^{-1} are both o-epimorphism, ϕ is an o-isomorphism, and the partially-ordered rings A and A' are o-*isomorphic*.

<u>DEFINITION 1.8.</u> A ring ideal I of an ℓ-ring A is an ℓ-*ideal* if $a \in A$, $b \in I$ and $|a| \leq |b|$ imply $a \in I$.

Every ℓ-ideal in a lattice-ordered ring is the kernel of an ℓ-homomorphism. If I is an ℓ-ideal of an ℓ-ring A, then quotient ring A/I is an ℓ-ring with $(A/I)^+ = \{x + I | x^- \in I\}$. Moreover, the natural map from A onto A/I is an ℓ-homomorphism. Also, if f is an ℓ-homomorphism from A onto A', and if I is the kernel of f, then A/I is ℓ-isomorphic to A'.

<u>PROPOSITION 1.9.</u> In any ℓ-ring A we have:

i) $|ab| \leq |a||b|$ for all $a, b \in A$;

ii) for all $a, b, c \in A$ and $a \geq 0$ $a(b \vee c) \geq ab \vee ac$,

$(b \vee c)a \geq ba \vee ca$, $a(b \wedge c) \leq ab \wedge ac$, and

$(b \wedge c)a \leq ba \wedge ca$.

If S is a subset of an ℓ-ring A, then the ℓ-ideal generated by S will be denoted by $<S>$. If I, J are ℓ-ideals, then $<I + J> = I + J$ ([13]). The product of two ℓ-ideals I and J is $IJ = \{ \sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J \}$.

A (right, left, two-sided) ℓ-ideal I of an ℓ-ring A is said to be *proper* if $I \neq A$. If I is such that it is contained in no other proper (right, left, two-sided) ℓ-ideal, then I is said to be a *maximal* (right, left, two-sided) ℓ-ideal.

DEFINITION 1.10. A proper ℓ-ideal P of an ℓ-ring A is *prime* if for ℓ-ideals I and J of A, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

A non-zero ℓ-ring A is called *prime* if $\{0\}$ is a prime ℓ-ideal.

DEFINITION 1.11. A lattice-ordered ring A is said to be ℓ-*simple* if $A^2 \neq 0$ and if A contains no non-zero proper ℓ-ideals.

PROPOSITION 1.12. ([7], 2.4) If P is an ℓ-ideal of an ℓ-ring A for which $A^+ \setminus P$ is closed under multiplication, then P is a prime ℓ-ideal.

<u>DEFINITION 1.13.</u> A non-empty subset M of an ℓ-ring A is an m-*system* if for all a, b in M, there exists an $x \varepsilon A^+$ such that $axb \varepsilon M$.

<u>DEFINITION 1.14.</u> ([5], p45). Let A be an ℓ-ring. Then the ℓ-radical of A is the subset $N(A) = \{a \varepsilon A \mid$ there is a positive integer n such that $x_0|a|\cdot x_1 \cdot |a| \cdot \ldots \cdot x_{n-1}|a| \; x_n = 0$ for all $x_0, x_1, \ldots x_n \varepsilon A\}$ of A.

The ℓ-radical of an ℓ-ring A, denoted by $N(A)$, is a nil ℓ-ideal which is the join of all nilpotent ℓ-ideals of A ([5]).

<u>PROPOSITION 1.15.</u> ([7], 2.16, p76). If A is an ℓ-ring, then $N(A) = \{a \varepsilon A \mid$ there exists a positive integer n such that $(x|a|)^n x = 0$ for all $x \varepsilon A^+\}$.

<u>DEFINITION 1.16.</u> The P-*radical*, denoted by $P(A)$, of an ℓ-ring A is the intersection of all the prime ℓ-ideals of A. Set $P(A) = A$ if A has no prime ℓ-ideals.

<u>PROPOSITION 1.17.</u> ([7], 2.9). Let A be an ℓ-ring. Then $P(A)$ is a nil ℓ-ideal containing $N(A)$.

<u>THEOREM 1.18.</u> ([7], 2.15, p76). Let A be an ℓ-ring. Then $P(A) = \{a \varepsilon A \mid$ any m-system containing $|a|$ contains $0\}$.

THEOREM 1.19. ([7], 2.13, p75). Let A be an ℓ-ring. Then $N(A/N(A))$ is zero if and only if $N(A) = P(A)$. Hence $N(A)$ is zero if and only if $P(A)$ is zero.

DEFINITION 1.20. (1) An ℓ-ring A is called a *distributive ℓ-ring* (d-ring) if $a(b\wedge c) = ab\wedge ac$, $(b\wedge c)a = ba\wedge ca$ for all $a, b, c \in A$ and $a \geq 0$. (2) An ℓ-ring A is called an *f-ring* if and only if it satisfies the condition $z^+x^+\wedge x^- = x^+z^+\wedge x^- = 0$ for all $z, x \in A$.

We note that a d-ring A also satisfies $a(b\vee c) = ab\vee ac$ and $(b\vee c)a = ba\vee ca$ for all a, b, c A and $a \geq 0$, since $-(x\vee y) = -x\wedge -y$ for all $x, y \in A$. Moreover, an ℓ-ring A is an f-ring if and only if $a\wedge b = 0$ and $c \geq 0$ imply $ca\wedge b = ac\wedge b = 0$ for all a, b, c in A. (see [5], p56).

PROPOSITION 1.21. ([7], 4.1, 4.2, 4.3). Let A be a d-ring, or a ring which satisfies either one of the identities $x^+a\,x^- = 0$ or $x^+x^- = 0$. Then an ℓ-ideal P of A is prime if and only if A/P is totally ordered and has no non-zero divisors of zero.

PROPOSITION 1.22. ([7], 2.17, 2.18). If I is an ℓ-ideal of an ℓ-ring A, then $N(I) = N(A) \cap I$ and $P(I) = P(A) \cap I$.

PROPOSITION 1.23. ([7], Corollary 4.6). Let A be an ℓ-ring which is a d-ring or satisfies either one of the identities $x^+ a\, x^- = 0$ or $x^+ x^- = 0$. Then $P(A) = \{x \in A \mid x \text{ is nilpotent}\}$.

PROPOSITION 1.24. ([7], 2.5). An ℓ-ideal P of an ℓ-ring A is prime if and only if $a, b \in A^+$ and $aA^+b \subseteq P$ imply $a \in P$ or $b \in P$.

Let $\{A_\alpha : \alpha \in \Lambda\}$ be a non-empty family of ℓ-rings, and let $A = \prod_{\alpha \in \Lambda} A_\alpha$ the cartesian product of the A_α. Define addition and multiplication in A componentwise. Then A is a ring. We partially order A by decreeing that $(a_\alpha) \geq (b_\alpha)$ if $a_\alpha \geq b_\alpha$ for all $a \in A$.

Then A is an ℓ-ring in which

$$(a_\alpha) \vee (b_\alpha) = (a_\alpha \vee b_\alpha) \quad \text{and} \quad (a_\alpha) \wedge (b_\alpha) = (a_\alpha \wedge b_\alpha).$$

The ℓ-ring A is called the *complete direct union* of the family $\{A_\alpha : \alpha \in \Lambda\}$. The map P_α from A onto A_α defined by $P_\alpha((a_\alpha)) = a_\alpha$ is called the α-th projection and is an ℓ-homomorphism. A sub ℓ-ring A' of the complete direct union A is called a *subdirect union* of the family $\{A_\alpha : \alpha \in \Lambda\}$ if P_α restricted to A' is onto A_α for each $\alpha \in \Lambda$.

<u>THEOREM 1.25.</u> An ℓ-ring A is isomorphic to a subdirect union of the family $\{A_\alpha : \alpha \in \Lambda\}$ of ℓ-rings if and only if there is a family $\{I_\alpha : \alpha \in \Lambda\}$ of ℓ-ideals in A such that $\bigcap\{I_\alpha : \alpha \in \Lambda\}$ is zero, and A/I_α is isomorphic to A_α for each $\alpha \in \Lambda$.

<u>THEOREM 1.26.</u> ([5]). A lattice-ordered ring A is an f-ring if and only if A is isomorphic to a subdirect union of totally ordered rings.

<u>PROPOSITION 1.27.</u> ([5]). Let A be an f-ring and let $a, b, c \in A$. Then

i) $a^2 \geq 0$;

ii) if $a \geq 0$, then $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$; $a(b \wedge c) = ab \wedge ac$ and $(b \wedge c)a = ba \wedge ca$;

iii) $|ab| = |a|\,|b|$;

iv) $a^+a^- = 0$;

v) $a^+b\,a^- = 0$.

<u>DEFINITION 1.28.</u> An ℓ-ring A is called *subdirectly irreducible* if the intersection of all of the non-zero ℓ-ideals of A is not $\{0\}$.

PROPOSITION 1.29. ([13], p189). If A is a subdirectly irreducible totally ordered ring with zero ℓ-radical, then A is ℓ-simple.

THEOREM 1.30. ([13], Theorem 4.4, p174). An f-ring A is prime if and only if $A \neq \{0\}$ and A is a totally-ordered ring without non-zero divisors of zero.

PROPOSITION 1.31. ([7], 2.4, p75). An ℓ-ring A has zero ℓ-radical if and only if it is a subdirect union of prime ℓ-rings.

THEOREM 1.32. ([7], Theorem 4.4, p81). Let A be an ℓ-ring with zero ℓ-radical. Then the following are equivalent:

i) A is an f-ring;

ii) A is a subdirect union of totally-ordered rings with no non-zero divisors of zero;

iii) $x^+ a x^- = 0$ for all x, $a \in A$;

iv) if $a, b, c \in A$ with $a \geq 0$, then $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$; and

v) $x^+ x^- = 0$ for all $x \in A$.

CHAPTER II

SOME CLASSES OF LATTICE-ORDERED RINGS AND THEIR ℓ-RADICAL PROPERTIES

The main purpose of this chapter is to prove that for an ℓ-ring A with one of the following conditions:

(α) $x^+ a x^- = 0$ for all x, $a \in A$;

(β) if a, b, c in A and $a \geq 0$, then $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$;

(γ) $x^+ x^- = 0$ for all $x \in A$;

(δ) $ax^+ax^-a = 0$ for all $a \in A^+$, $x \in A$,

the ℓ-radical is equal to the set of all nilpotent elements. If an ℓ-ring A satisfies any one of (α), (β), (γ) or (δ) then an ℓ-ideal P of A is a prime ℓ-ideal if and only if it is a ring prime ideal. For an ℓ-ring with zero ℓ-radical the conditions (α), (β), (γ) and (δ) are equivalent.

An ℓ-ring which satisfies the condition (β) has been called a distributive ℓ-ring or d-ring ([18], [20]). We will sometimes call an ℓ-ring a D-ring (after Diem) if it satisfies the condition (α), orthogonal ℓ-ring if it satisfies the condition (γ) and h-ring (helping) if it satisfies the condition (δ).

PROPOSITION 2.1. Let A be an ℓ-ring. Then the following conditions are equivalent:

(δ) $ax^+ax^-a = o$ for all $a\in A^+$, $x\in A$;

(δ)' $ax^+bx^-c = o$ for all $a, b, c\in A^+$, $x\in A$;

(δ)" $ax^+bx^-c = o$ for all $a, b, c, x\in A$.

PROOF. (δ) implies (δ)'. For any $a, b, c, \in A^+$ and $x\in A$, $0 \leq ax^+bx^-c \leq (a\vee b\vee c)x^+(a\vee b\vee c)x^-(a\vee b\vee c) = 0$ by (δ). Hence $ax^+bx^-c = 0$ for all $a, b, c\in A^+$ and $x\in A$.

(δ)' implies (δ)". For any a, b, c, x in A, $ax^+bx^-c = (a^+ - a^-)x^+(b^+ - b^-)x^-(c^+ - c^-) = a^+x^+b^+x^-c^+ - a^-x^+b^+x^-c^+ - a^+x^+b^-x^-c^+ + a^-x^+b^-x^-c^+ - a^+x^+b^+x^-c^- + a^-x^+b^+x^-c^- + a^+x^+b^-x^-c^- - a^-x^+b^-x^-c^- = 0$ by (δ)'. Hence $ax^+bx^-c = 0$ for all $a, b, c, x\in A$.

(δ)" implies (δ). Trivial.

PROPOSITION 2.2. Let A and B be two ℓ-rings and $\phi: A \to B$ an ℓ-homomorphism of A onto B. Then B has the property (α), (β), (γ) or (δ) if A has the property (α), (β), (γ) or (δ) respectively.

PROOF. i) Assume A has the property (α) : $a^+ A\, a^- = 0$ for all $a \in A$. Let $x, y \in B$, then there exist a and $b \in A$ such that $\phi(a) = x$, $\phi(b) = y$. Then $x^+ y\, x^- = (x \vee 0) y (-x \vee 0) =$ $(\phi(a) \vee \phi(0)) \phi(b) (\phi(-a) \vee \phi(0)) = \phi((a \vee 0) b (-a \vee 0)) = \phi(a^+ b\, a^-) = \phi(0) = 0.$

ii) Assume A has the property (β) : $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$ for a, b, c in A and $a \geq 0$. If a', b', c' in B with $a' \geq 0$, then there exists $a, b, c \in A$ with $a \geq 0$ such that $\phi(a) = a'$, $\phi(b) = b'$, and $\phi(c) = c'$. We have $a'(b' \vee c') =$ $\phi(a)(\phi(b) \vee \phi(c)) = \phi(a)\phi(b \vee c) = \phi(a(b \vee c)) = \phi(ab \vee ac) =$ $\phi(ab) \vee \phi(ac) = \phi(a)\phi(b) \vee \phi(a)\phi(c) = a'b' \vee a'c'$. A similar proof gives $\phi(b' \vee c')a' = b'a' \vee c'a'$.

iii) Assume A has the property (γ) : $a^+ a^- = 0$ for all $a \in A$. Let $x \in B$, then there exists $a \in A$ such that $\phi(a) = x$. Now $x^+ x^- = (x \vee 0)(-x \vee 0) = (\phi(a) \vee \phi(0))(\phi(-a) \vee \phi(0)) = [\phi(a \vee 0)][\phi(-a \vee 0)] =$ $[\phi(a^+)][\phi(a^-)] = \phi(a^+ a^-) = \phi(0) = 0.$

iv) Assume A has the property (δ) : $x a^+ x a^- x = 0$ for all $x \in A^+$, $a \in A$. Let $y, b \in B$ then there exists x, a in A such that

$\phi(x) = y$, $\phi(a) = b$. We have $yb^+yb^-y = y(b\vee 0)y(-b\vee 0)y =$ $\phi(x)(\phi(a)\vee\phi(0))\phi(x)(-\phi(a)\vee\phi(0))\phi(x) = \phi(x)\phi(a\vee 0)\phi(x)\phi(-a\vee 0)\phi(x)$ $= \phi(x)\phi(a^+)\phi(x)\phi(a^-)\phi(x) = \phi(xa^+xa^-x) = 0$. This completes the proof of the lemma.

PROPOSITION 2.3. Let $\{A_\lambda : \lambda\varepsilon\Lambda\}$ be a non-empty family of ℓ-rings, and let $A = \prod_{\lambda\varepsilon\Lambda} A_\lambda$ be the ℓ-ring endowed with the direct union order (see Chapter I). Then A is an ℓ-ring satisfying the conditions (α), (β), (γ) or (δ) if and only if A_λ satisfies the same conditions for every $\lambda\varepsilon\Lambda$.

PROOF. Assume A satisfies the condition (α) (respectively (β), (γ) or (δ)). Since the λ-th projection P_λ is an ℓ-homomorphism onto A_λ for every $\lambda\varepsilon\Lambda$, by Lemma 2.2 A_λ satisfies the condition (α) (respectively (β), (γ) or (δ)).

Conversely, let A_λ satisfy the condition (α) for every $\lambda\varepsilon\Lambda$. Let $x = (x_\lambda)$, $a = (a_\lambda)\varepsilon A$, where x_λ, $a_\lambda \varepsilon A_\lambda$, then $x^+ = (x\vee 0) =$ $(x_\lambda)\vee(0) = (x_\lambda\vee 0) = (x_\lambda^+)$ and $x^- = -x\vee 0 = (-x_\lambda)\vee(0) = (-x_\lambda\vee 0)$ $= (x_\lambda^-)$. We have $x^+ax^- = (x_\lambda^+)(a_\lambda)(x_\lambda^-) = (x_\lambda^+ a_\lambda x_\lambda^-) = (0)$. The same argument works for the other cases.

DEFINITION 2.4. An ℓ-ideal I in an ℓ-ring A is called an *f-ideal* if $a\wedge b\varepsilon I$ and $r\varepsilon A^+$ imply $ra\wedge b$, $ar\wedge b\varepsilon I$.

We have the following properties. The intersection of any family of f-ideals is an f-ideal. In any f-ring every ℓ-ideal is an f-ideal. If an ℓ-ring A is not an f-ring, then A contains a smallest non-zero f-ideal S and it is generated by the set of elements of the form $|r|a^+ \wedge a^-$, $a^+|r| \wedge a^-$ where r, a run through all the elements in A. Every ℓ-ideal containing an f-ideal is an f-ideal. Moreover, an ℓ-ideal I of an ℓ-ring is an f-ideal if and only if the quotient ordered ring A/I is an f-ring (see the note after the Definition 1.20).

PROPOSITION 2.5. An ℓ-ideal I in an ℓ-ring A is an f-ideal if and only if $a \wedge b = 0$ and $r \in A^+$ imply $ra \wedge b$, $ar \wedge b \in I$.

PROOF. Necessity is obvious. To prove the sufficiency, suppose the condition is satisfied and $a \wedge b \in I$, then $\bar{a} \wedge \bar{b} = \bar{0}$ in the quotient ℓ-ring A/I. Let $\bar{x} = (\bar{a} - \bar{b})$, then $\bar{x}^+ = (\bar{a} - \bar{b})^+ = \bar{a}$ and $\bar{x}^- = (\bar{a} - \bar{b})^- = \bar{b}$ (by Proposition 1.2ii). But $x^+ \wedge x^- = 0$ and hence for $r \in A^+$, $rx^+ \wedge x^- \in I$ and so $\bar{0} = \overline{rx^+ \wedge x^-} = \overline{rx^+} \wedge \overline{x^-} = \bar{r}\,\overline{x^+} \wedge \overline{x^-}$. Since $(\bar{x})^+ = (\overline{x^+})$ and $(\bar{x})^- = (\overline{x^-})$ for every $x \in A$, we have $\bar{0} = \bar{r}\,\overline{x^+} \wedge \overline{x^-} = \bar{r}(\bar{x})^+ \wedge (\bar{x})^- = \bar{r}(\bar{a}) \wedge (\bar{b}) = \overline{ra \wedge b}$ and so $ra \wedge b \in I$. Similarly, $ar \wedge b \in I$.

LEMMA 2.6. Let A, B be two ℓ-rings and $f : A \to B$ be an ℓ-homomorphism from A into B. Then $f^{-1}(G) = \{x \in A \mid f(x) \in G\}$, where G is an f-ideal of B, is an f-ideal of A.

PROOF. It is clear that $f^{-1}(G)$ is an ℓ-ideal of A. Now let $a, b \in A$ such that $a \wedge b = 0$, then for every $r \in A^+$, we have $f(ra \wedge b) = f(ra) \wedge f(b) = f(r)\ f(a) \wedge f(b) \in G$, $f(ar \wedge b) = f(ar) \wedge f(b) = f(a)\ f(r) \wedge f(b) \in G$. Hence $f^{-1}(G)$ is an f-ideal of A.

Let I be an f-ideal of an ℓ-ring A and for a positive integer n, let $I_n = \{x \in A | x^n \in I\}$. The set of all nilpotent elements of A will be denoted by $\overline{N}(A)$.

PROPOSITION 2.7. For every positive integer n, I_n is an f-ideal. Also, $I_n^n \subseteq I$.

PROOF. Recall that $Z_n = \{\bar{x} \in A/I | \bar{x}^n = \bar{0}\}$ is an f-ideal of the f-ring A/I and that $Z_n^n = \{\bar{0}\}$. ([5], p63, Theorem 16). The ideal I_n is the pre-image of Z_n under the natural map θ: $A \to A/I$. Hence I_n is an f-ideal. As $\theta(I_n^n) \subseteq (\theta(I_n))^n = \{\bar{0}\}$, we have $I_n^n \subseteq I$.

THEOREM 2.8. Let A be an ℓ-ring containing a nilpotent f-ideal I. Then $N(A) = P(A) = \overline{N}(A)$.

PROOF. Since $I^m = \{0\}$ for some $m > 0$, it follows that $(I_n^n)^m \subseteq I^m = \{0\}$. So I_n is a nilpotent ideal. The set of nilpotent elements of A is contained in $\bigcup_{n>0} I_n \subseteq N(A)$ and so the set of all nilpotent elements is contained in N(A). Since N(A) is contained in the set of all nilpotent elements for any ℓ-ring, hence $N(A)$ is equal to the set of all nilpotent elements in A.

PROPOSITION 2.9. Let A be an ℓ-ring which satisfies the condition (β), (i.e., A is a d-ring). Then the set $P = \{x \in A | AxA = \{0\}\}$ is an f-ideal which is nilpotent.

PROOF. It is clear that $P = \{x \in A | A^+xA^+ = \{0\}\}$ and P is an ideal. Also $x \in P$ if and only if $A|x|A = \{0\}$; for let $a, b \in A^+$ and $0 = a|x|b = a(x^+ + x^-)b = a\, x^+ b + a\, x^- b$ we have $a\, x^+ b = 0 = a\, x^- b$. Thus $axb = ax^+b - ax^-b = 0$. Conversely, let $x \in P$, then $d\, x\, d = 0$ for all $d \in A^+$. Now $d|x|d' \leq (d \vee d')|x|(d \vee d') = |(d \vee d')x(d \vee d')| = 0$; thus $A^+|x|A^+ = \{0\}$. This shows that P is an ℓ-ideal. Now let $x \wedge y = 0$, $r \geq 0$ then $d(rx \wedge y)d' = drxd' \wedge dyd' \leq (dr \vee d)xd' \wedge (dr \vee d)yd'$ $= (dr \vee d)\cdot(x \wedge y)d' = 0$. Hence P is an f-ideal. Clearly $P^3 = \{0\}$ and so P is nilpotent.

THEOREM 2.10. Let A be an ℓ-ring which satisfies the condition (β). Then $N(A) = P(A) = \overline{N}(A)$.

PROOF. This is a consequence of Theorem 2.8 and Proposition 2.9.

LEMMA 2.11. Let A be an ℓ-ring which satisfies the condition $a^+A\, a^- = \{0\}$ for all $a \in A$. Then for any a, b in A we have:

i) $a^4 = (a^+)^4 + (a^-)^4 \geq 0$;

ii) $a^4 = 0$ if and only if $(a^+)^4 = (a^-)^4 = 0$;

iii) $(ab)^2 \geq 0$; and

iv) $a\, A^+\, a \geq 0$.

PROOF. i) $a^4 = (a^+ - a^-)^4 = [(a^+ - a^-)^2]^2 = ((a^+)^2 + (a^-)^2 - a^+a^- - a^-a^+)^2$ $= (a^+)^4 + (a^-)^4 \geq 0$.

ii) Follows from the equation $a^4 = (a^+)^4 + (a^-)^4$.

iii) $(ab)^2 = [(a^+ - a^-)(b^+ - b^-)]^2 = (a^+b^+ - a^+b^- - a^-b^+ + a^-b^-)^2$

$= a^+b^+a^+b^+ + a^+b^-a^+b^- + a^-b^+a^-b^+ + a^-b^-a^-b^- \geq 0$.

iv) Let $b \in A^+$, then $aba = (a^+ - a^-)\, b\, (a^+ - a^-) =$ $(a^+b - a^-b)(a^+ - a^-) = a^+ba^+ - a^+b\, a^- - a^-b\, a^+ + a^-b\, a^- =$ $a^+b\, a^+ + a^-b\, a^- \geq 0$.

LEMMA 2.12. ([7], 3.5, p78). Let A be an ℓ-ring in which the square of every element is positive, and let $a, b \in A^+$ with $a^2 = b^2 = 0$. Then $ab = ba = 0$.

PROPOSITION 2.13. Let A be an ℓ-ring satisfying the condition $a^+a^- = 0$, for all $a \in A$. Then $a^2 = b^2 = 0$ implies $ab = ba = 0$,

$(a+b)^2 = (a-b)^2 = 0$ and $|a|^2 = 0 = |b|^2$.

PROOF. Let A be an ℓ-ring with the condition $a^+a^- = 0$ for all $a \in A$. Then $a^2 = (a^+ - a^-)^2 = (a^+)^2 + (a^-)^2 - a^+a^- - a^-a^+ = (a^+)^2 + (a^-)^2 \geq 0$. If $a^2 = 0$, then $(a^+)^2 = (a^-)^2 = 0$.

Now suppose $a^2 = b^2 = 0$ then $(a^+)^2 = (a^-)^2 = (b^+)^2 = (b^-)^2 = 0$ and by Lemma 2.12 $ab = ba = 0$. Moreover, $(a \pm b)^2 = a^2 \pm ab \pm ba + b^2 = 0$; $|a|^2 = (a^+ + a^-)^2 = (a^+)^2 + (a^-)^2 + a^+a^- + a^-a^+ = 0$. Similarly, $|b|^2 = 0$.

LEMMA 2.14. Let A be an ℓ-ring satisfying the condition $a^+A\,a^- = \{0\}$ for all a in A or $a^+a^- = 0$ for all a in A. Then $(rm^+ \wedge m^-)^2 = (m^+r \wedge m^-)^2 = 0$ for all $m \in A$ and $r \in A^+$.

PROOF. Let $x = rm^+ \wedge m^-$, $y = m^+r \wedge m^-$, then $0 \leq x^2 = (rm^+ \wedge m^-)(rm^+ \wedge m^-) \leq (rm^+ \wedge m^-)rm^+ \wedge (rm^+ \wedge m^-)m^- \leq rm^+ \cdot rm^+ \wedge m^-rm^+ \wedge rm^+m^- \wedge m^-m^- = 0$, since we have $m^-rm^+ = 0$ or $rm^+m^- = 0$. Similarly, $y^2 = 0$.

LEMMA 2.15. Let A be an ℓ-ring satisfying the condition $a^+A\,a^- = 0$ for all a in A. Then $(rm^+ \wedge m^-)A(rm^+ \wedge m^-) = \{0\}$, $(m^+r \wedge m^-)A(m^+r \wedge m^-) = \{0\}$, for all $r \in A^+$.

PROOF. Let $x = rm^+ \wedge m^-$, $y = m^+ r \wedge m^-$ with $r \varepsilon A^+$. Then for $p \varepsilon A^+$ we have $0 \leq xpx = (rm^+ \wedge m^-)p(rm^+ \wedge m^-) \leq (rm^+ p \wedge m^- p)(rm^+ \wedge m^-) \leq rm^+ p(rm^+ \wedge m^-) \wedge m^- p(rm^+ \wedge m^-) \leq rm^+ prm^+ \wedge rm^+ pm^- \wedge m^- prm^+ \wedge m^- pm^- = 0$. Now if $a \varepsilon A$, then $xax = x(a^+ - a^-)x = xa^+ x - xa^- x = 0$. Thus $xAx = 0$. Similarly, $yAy = 0$.

PROPOSITION 2.16. Let A be ℓ-ring satisfying the condition $a^+ A a^- = 0$ for all a in A. Then the set $I = \{a \varepsilon A | AaAaA = \{0\}\} = \{a \varepsilon A | A^+ a A^+ a A^+ = \{0\}\}$ is a nilpotent ℓ-ideal.

PROOF. It is easy to see that $\{a \varepsilon A | AaAaA = \{0\}\} = \{a \varepsilon A | A^+ a A^+ a A^+ = \{0\}\}$. Since $A^+ a A^+ a A^+ = A^+(a^+ - a^-)A^+(a^+ - a^-)A^+ = A^+ a^+ A^+ a^+ A^+ + A^+ a^- A^+ a^- A^+ - A^+ a^+ A^+ a^- A^+ - A^+ a^- A^+ a^+ A^+ = A^+ a^+ A^+ a^+ A^+ + A^+ a^- A^+ a^- A^+$, we have $a \varepsilon I$ if and only if $a^+, a^- \varepsilon I$, and so $a \varepsilon I$ if and only if $|a| \varepsilon I$. Whence if $|y| \leq |x|$ with $x \varepsilon I$, then $|x| \varepsilon I$ and from $0 \leq A^+ |y| A^+ |y| A^+ \leq A^+ |x| A^+ |x| A^+ = 0$ we have $y \varepsilon I$.

Now let $a, b \varepsilon I$ and $a \geq 0$, $b \geq 0$, then by Lemma 2.11 iv), $0 \leq A^+(a - b)A^+(a - b)A^+ = (A^+ a A^+ - A^+ b A^+)(a A^+ - b A^+) = A^+ a A^+ a A^+ - A^+ a A^+ b A^+ - A^+ b A^+ a A^+ + A^+ b A^+ b A^+ = -A^+ a A^+ b A^+ - A^+ b A^+ a A^+ \leq 0$. Hence $A^+ a A^+ b A^+ = A^+ b A^+ a A^+ = 0$. For $a, b \varepsilon I$, we have $a^+, a^-, b^+, b^- \varepsilon I$ and hence $A^+ a^+ A^+ b^+ A^+ = A^+ a^+ A^+ b^- A^+ = A^+ a^- A^+ b^+ A^+ = A^+ a^- A^+ b^- A^+ = 0$, thus $A^+(a - b)A^+(a - b)A^+ = A^+(a^+ - a^- - b^+ + b^-)A^+(a^+ - a^- - b^+ + b^-)A^+ = 0$, and so $a - b \in I$. It is clear that for any $a \in I$ and $r \in A$ we have $ar, ra \in I$. This shows that I is an ℓ-ideal. The ℓ-ideal I is a nilpotent ideal, for $I^5 = \{0\}$.

<u>THEOREM 2.17.</u> Let A be an ℓ-ring satisfying the condition $a^+ A a^- = 0$ for all a in A. Then $N(A) = P(A) = \overline{N}(A)$.

<u>PROOF.</u> The ℓ-ideal $I = \{a \in A | AaAaA = \{0\}\}$ defined in Proposition 2.16 is a nilpotent ℓ-ideal. By Lemma 2.15, I contains the elements of the form $rm^+ \wedge m^-$, $m^+ r \wedge m^-$ for $m \in A$, $r \in A^+$, hence I is an f-ideal. By Theorem 2.8 $N(A)$ is the set of all nilpotents, hence $N(A) = P(A) = \overline{N}(A)$.

<u>PROPOSITION 2.18.</u> Let A be an ℓ-ring. Then the following are equivalent:

i) A satisfies the identity $a^+ a^- = 0$ for all a in A;

ii) $a \wedge b = 0$ implies $ab = 0$ for all a, b in A;

iii) $(a \wedge b)^2 + ab = a(a \wedge b) + (a \wedge b)b$ for all a, b in A;

iv) $(a \vee b)^2 + ab = a(a \vee b) + (a \vee b)b$ for all a, b in A.

PROOF. iii) implies ii). Clear.

ii) implies i). Since $a^+ \wedge a^- = 0$ for all $a \varepsilon A$, by ii) $a^+a^- = 0$.

i) implies iii). If $a \wedge b = 0$, then $x^+ = (a-b)^+ = (a-b) \vee 0 = (a - b) \vee (a - a) = a + (-b \vee -a) = a + [-(b \wedge a)] = a$ and $x^- = (a - b)^- = -(a - b) \vee 0 = (b - a) \vee (b-b) = b + (-a \vee -b) = b + [-(a \wedge b)] = b$. By i) $ab = x^+x^- = 0$ and hence iii) holds. Now assume $a \wedge b = r \neq 0$. Then $a \wedge b - r = 0$ and so $(a - r) \wedge (b - r) = 0$. Since $y^+ = [(a - r) - (b - r)]^+ = a - r$ and $y^- = [(a - r) - (b - r)]^- = b - r$, we have by i) $y^+y^- = (a - r)(b - r) = ab - ar - rb + r^2 = 0$. Thus $ab + r^2 = ar + rb$, i.e., $(a \wedge b)^2 + ab = a(a \wedge b) + (a \wedge b)b$. This completes the proof of i) implies iii).

iii) equivalent iv). This follows from the fact that $(-a \vee -b) = -(a \wedge b)$ and $(-a \wedge -b) = -(a \vee b)$ for all $a, b \varepsilon A$.

LEMMA 2.19. Let A be an ℓ-ring satisfying the condition $x^+x^- = 0$ for all $x \varepsilon A$. Then the set $J = \{a \varepsilon A | a^2 = 0\}$ has the following properties:

i) $a \varepsilon J$ if and only if $-a \varepsilon J$;

ii) $a \varepsilon J$ if and only if $a^+, a^- \varepsilon J$;

iii) $a \varepsilon J$ if and only if $|a| \varepsilon J$;

iv) $a, b \in J$ implies $a - b \in J$;

v) $a, b \in J$ implies $ab = 0$, hence $J^2 = 0$;

vi) J contains the set of the elements of the forms
$|r|a^+ \wedge a^-$, $a^+|r| \wedge a^-$, where a, r run through A;

vii) if $|b| \leq |a|$, $a \in J$, then $b \in J$.

PROOF. i). Clear. Also ii) and iii) are obvious, for $a^2 = |a|^2 = (a^+)^2 + (a^-)^2$, and iv), v) and vi) follow from Proposition 2.13, and Lemma 2.14. Finally if $|b| \leq |a|$, where $a \in J$, then $0 \leq |b^2| \leq |b|^2 \leq |a|^2 = 0$ by iii) and vii) holds.

LEMMA 2.20. Let A be an ℓ-ring satisfying the condition $x^+x^- = 0$ for all $x \in A$. If every element of A is nilpotent, (i.e., if $A = \overline{N}(A)$), then $N(A) = P(A) = \overline{N}(A) = A$.

PROOF. The set $J = \{a \in A | a^2 = 0\}$ is the set $\{a \in A | |a|^2 = 0\}$ by Lemma 2.19, iii). For any $a, b \in J$, $a - b \in J$ by Lemma 2.19, iv). Now let $a \in J$ and $r \in A^+$. From $(ra - a)^2 \geq 0$, we get $(ra)^2 - ara \geq 0$. Hence $ara \leq (ra)^2 = rara$. We have

$$\begin{aligned} ara &\leq rara \\ &\leq r^2ara \\ &\leq r^3ara \text{ etc} \\ &\leq r^n ara \text{ for all } n \geq 1 \text{ and } r \in A^+. \end{aligned}$$

Since $\overline{N}(A) = A$, $r^n = 0$ for some positive integer n, it follows that $(ra)^2 = 0$. Now for any $r \in A$, $|ra|^2 \leq (|r||a|)^2 = 0$. Hence $ra \in J$ for any $r \in A$ and $a \in J$. Similarly, $ar \in J$. Thus J is a nilpotent f-ideal by Lemma 2.19 v) and vi). Now the lemma follows from Theorem 2.8.

<u>THEOREM 2.21.</u> Let A be an ℓ-ring satisfying the condition $x^+x^- = 0$ for all $x \in A$. Then $N(A) = P(A) = \overline{N}(A)$.

<u>PROOF.</u> Let $P(A) = P$. From Proposition 1.23 $P = \overline{N}(A)$. Also P is an ℓ-ideal of A and P may be considered as an ℓ-subring of A. By Proposition 1.22 $N(P) = N(A) \cap P = N(A)$ since $N(A) \subseteq P$. But every element of P is nilpotent, hence by Lemma 2.20 $N(P) = P$. Thus $N(A) = P = \overline{N}(A)$.

<u>LEMMA 2.22.</u> Let A be an ℓ-ring with the following property: if $x \in A^+$ and $a \in A$ then $(xa)^2x \in A^+$. Then $I_2 = \{a \in A | (x\,|a|)^2x = 0$ for all $x \in A^+\}$ is a nilpotent ℓ-ideal of A.

<u>PROOF.</u> Let $a, b \in I_2$. Hence for every $x \in A^+$, $(x|a|)^2x = 0$ and $(x|b|)^2x = 0$. From $(x|a| - x|b|)^2x \in A^+$, it follows that $x|a|x|b|x = 0$ and $x|b|x|a|x = 0$. We have $0 \leq (x|a - b|)^2x \leq (x|a| + x|b|)^2x = 0$ for all $x \in A^+$. Hence $a - b \in I_2$. Also if $a \in I_2$ and $r \in A$ then for all

$x \in A^+$, $0 \leq (x|ra|)^2x \leq (x|r||a|)^2x \leq [(x|r| \vee x)|a|]^2(x|r|\vee x) = 0$. Hence $ra \in I_2$. Similarly, $ar \in I_2$. Thus I_2 is an ideal. Now for any $y \in A$, and $a \in I_2$, the condition $|y| \leq |a|$ implies $(x|y|)^2x \leq (x|a|)^2x$ for any $x \in A^+$, thus I_2 is an ℓ-ideal. For any a, b, c, d, and e in I_2, we have $|a|$, $|b|$, $|c|$, $|d|$, and $|e|$ in I_2 and so $u = |a|\vee|b|\vee|c|\vee|d|\vee|e| \in I_2$. From $0 \leq |abcde| \leq |a||b||c||d||e| \leq u^5 = (|u|)^5 = 0$ we have $I_2^5 = \{0\}$. This shows that I_2 is a nilpotent ℓ-ideal of A.

COROLLARY 2.23. Let A be a positive square ring, i.e., $a^2 \geq 0$ for all $a \in A$. Then $I_2 = \{a \in A|(x|a|)^2x = 0$ for all $x \in A^+\}$ is a nilpotent ℓ-ideal.

COROLLARY 2.24. Let A be an ℓ-ring satisfying the condition $x^+a\,x^- = 0$ for all x, a in A. Then I_2 is a nilpotent ℓ-ideal.

PROOF. In an ℓ-ring satisfying the condition $x^+a\,x^- = 0$ for all x, a in A, we have $(ab)^2 \geq 0$ for all a, b in A by Lemma 2.11. Hence for any $x \in A^+$ and $a \in A$, we have $(xa)^2x \in A^+$ and hence the Corollary follows.

LEMMA 2.25. Let A be an ℓ-ring satisfying the condition (δ): $a\,x^+a\,x^-a = 0$ for all $a \in A^+$ and $x \in A$. Then I_2 is a nilpotent f-ideal.

PROOF. We first show that if $x\varepsilon A$ and $a\varepsilon A^+$ then $(ax)^2 a\varepsilon A^+$. Indeed $(ax)^2 a = (a x^+ - a x^-)^2 a = (ax^+)^2 a - ax^+ax^-a - ax^-ax^+a + (ax^-)^2 a = (ax^+)^2 a + (a x^-)^2 \varepsilon A^+$. using the condition (δ). Thus by Lemma 2.22 I_2 is a nilpotent ℓ-ideal. We now show that I_2 is an f-ideal. Let a, $r\varepsilon A^+$ and $x\varepsilon A$, we have $[a(rx^+ \wedge x^-)]^2 a = [a(rx^+ \wedge x^-)] \cdot [a(rx^+ \wedge x^-)]a \leq (arx^+)(ax^-)a = arx^+ax^-a = 0$ by Lemma 2.1. Hence $rx^+ \wedge x^- \varepsilon I_2$. Similarly, $x^+r \wedge x^- \varepsilon I_2$ and so I_2 is an f-ideal.

THEOREM 2.26. Let A be an ℓ-ring satisfying the condition $a x^+ a x^- a = 0$ for all $x\varepsilon A$ and $a\varepsilon A^+$. Then $N(A) = P(A) = \overline{N}(A)$.

PROOF. The Theorem follows from Lemma 2.25 and Theorem 2.8.

LEMMA 2.27. Let A be a prime ℓ-ring (i.e., $\{0\}$ is a prime ℓ-ideal). Then the condition $x^+a x^- = 0$ for all $a\varepsilon A^+$, $x\varepsilon A$ is equivalent to the condition (δ): $a x^+ax^-a = 0$ for all $a\varepsilon A^+$ and $x\varepsilon A$.

PROOF. That the condition $x^+ax^- = 0$ for all $a\varepsilon A^+$ and $a\varepsilon A$ implies (δ) is trivial.

Now assume A satisfies (δ). For any a, b in A^+ and x in A, we have $0 \leq (x^+ax^-)b(x^+ax^-) \leq x^+(a\vee b)x^-(a\vee b)x^+(a\vee b)x^- = 0$ by (δ). Since b is any element in A^+ and hence $(x^+ax^-)A^+(x^+ax^-) = \{0\}$. By Proposition 1.24, $x^+ax^- = 0$ for all $a\varepsilon A^+$, $x\varepsilon A$ since A is a prime ℓ-ring. This completes the proof of the lemma.

PROPOSITION 2.28. Let A be a prime ℓ-ring. Then A is a totally-ordered ring without non-zero divisors of zero if and only if $a x^+ a x^- a = 0$ for all $a \in A^+$ and $x \in A$.

PROOF. This is a consequence of Proposition 1.21 and Lemma 2.27.

PROPOSITION 2.29. Let A be an ℓ-ring satisfying the condition (δ): $a x^+ a x^- a = 0$ for all $a \in A^+$ and $x \in A$. Then an ℓ-ideal P is a prime ℓ-ideal if and only if A/P is a totally-ordered ring without non-zero divisors of zero.

PROOF. Assume A satisfies (δ). By Lemma 2.2 A/P satisfies the condition (δ). Since A/P is a prime ℓ-ring, by Proposition 2.28, A/P is a totally-ordered ring without non-zero divisors of zero.

Conversely, if A/P has no non-zero divisors of zero, then by Proposition 1.12 P is a prime ℓ-ideal.

THEOREM 2.30. Let A be an ℓ-ring with zero ℓ-radical. Then A satisfies the condition (δ): $a x^+ a x^- a = 0$ for all $a \in A^+$ and $x \in A$, if and only if A is a subdirect union of totally-ordered rings without non-zero divisors of zero.

PROOF. Since A is an ℓ-ring with zero ℓ-radical, by Proposition 1.31 A is a subdirect union of a family $\{A_\alpha\}_{\alpha\varepsilon\Gamma}$ of prime ℓ-rings. By Proposition 2.2 A_α satisfies the condition (δ) for every $\alpha\varepsilon\Gamma$. A_α is a totally-ordered ring without non-zero divisors of zero for every $\alpha\varepsilon\Gamma$ by Proposition 2.28.

The converse is trivial by Proposition 2.3.

COROLLARY 2.31. Let A be an ℓ-ring with zero ℓ-radical. Then the following are equivalent:

i) A is an f-ring;

ii) A is a subdirect union of totally-ordered rings without any non-zero divisors of zero;

iii) $x^+ a\, x^- = 0$ for all $x, a\varepsilon A$;

iv) if a, b, c, A, $a \geq 0$, then $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$;

v) $x^+ x^- = 0$ for all $x\varepsilon A$;

vi) $a\, x^+ a\, x^- a = 0$ for all $a\varepsilon A^+$, $x\varepsilon A$.

PROOF. This is a consequence of Theorem 2.30 and Theorem 1.32.

<u>COROLLARY 2.32.</u> Let A be a prime ℓ-ring (i.e., $\{0\}$ is a prime ℓ-ideal). Then the following are equivalent:

i) A is an f-ring;

ii) A is a totally-ordered ring without non-zero divisors of zero;

iii) $x^{+}a\,x^{-} = 0$ for all $a, x \in A$;

iv) if $a, b, c \in A$ and $a \geq 0$, then $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$;

v) $x^{+}x^{-} = 0$ for all $x \in A$; and

vi) $ax^{+}ax^{-}a = 0$ for all $a \in A^{+}$ and $x \in A$.

<u>PROOF.</u> Since $\{0\}$ is a prime ℓ-ideal $N(A) = \{0\}$. By Corollary 2.31, i), iii), iv), v) and vi) are equivalent. Also, since an f-ring is prime if and only if it is a totally-ordered ring without non-zero divisors of zero by Theorem 1.30. Hence the corollary follows.

We note that in the Corollary 2.31 and Corollary 2.32, iii) is the condition (α), iv) is the condition (β), v) is the condition (γ) and vi) is the condition (δ).

PROPOSITION 2.33. Let A be an ℓ-ring which satisfies one of the conditions (α), (β), (γ) or (δ). Then every ℓ-ideal P which is a prime ℓ-ideal is a ring prime ideal.

PROOF. Suppose P is an ℓ-ideal which is a prime ℓ-ideal, then A/P is totally-ordered ring without non-zero divisors of zero by Proposition 1.21 and 2.28. Let $a, b \in A$ such that $aAb \subseteq P$, and $a \notin P$, then $(a + P)(ab + P) = a(ab) + P = 0 + P$ and $a + P \neq 0 + P$. Since A/P contains no non-zero divisors of zero, $(ab + P) = 0 + P$ and again $b + P = 0 + P$. Thus $b \in P$. This shows that P is a ring prime ideal.

PROPOSITION 2.34. Let A be an ℓ-ring which satisfies one of the conditions (α), (β), (γ) or (δ). Then an ℓ-ideal P is a prime ℓ-ideal if and only if $ab \in P$ implies $a \in P$ or $b \in P$.

PROOF. Necessity is clear from the proof of Proposition 2.33. Now we prove the sufficiency. If P is an ℓ-ideal with the property, that is $ab \in P$ implies $a \in P$ or $b \in P$, then A/P contains no non-zero divisors of zero. Since $(A/P)^+ \setminus \overline{0}$ is closed under multiplication, hence $\{\overline{0}\}$ is a prime ℓ-ideal of A/P by Proposition 1.12. Thus P is a prime ℓ-ideal of A.

<u>PROPOSITION 2.35.</u> Let I be an ℓ-ideal of an ℓ-ring A. If A/I satisfies one of the conditions (α), (β), (γ) or (δ), then $\sqrt{I} = \{a \in A \mid a^n \in I$ for some positive integer $n\}$ is an ℓ-ideal.

<u>PROOF.</u> Assume A/I satisfies one of the conditions (α), (β), (γ) or (δ). Consider the natural ℓ-homomorphism ϕ: $A \to A/I = \bar{A}$. Since the ℓ-radical $N(\bar{A})$ is the set of all nilpotent elements of $\bar{A}$ by Theorems 2.10, 2.17, 2.21 and 2.26, we have $\phi^{-1}(N(\bar{A})) = \{a \in A \mid a^n \in I$ for some positive integer $n\} = \sqrt{I}$ is an ℓ-ideal of A by Lemma 2.6.

<u>COROLLARY 2.36.</u> Let A be an ℓ-ring which satisfies one of the conditions (α), (β), (γ) or (δ). Then $\sqrt{I}$ is an ℓ-ideal for every ℓ-ideal I.

<u>PROOF.</u> A consequence of Lemma 2.2 and Proposition 2.35.

<u>REMARK 2.37.</u> In a general ℓ-ring A, $\sqrt{I}$ may not be an ℓ-ideal. For example; in Example 3.3, $\{(0, 0)\}$ is an ℓ-ideal, but $\sqrt{\{(0, 0)\}} = \{(a, -a) \in A \mid a \in R\}$ is not an ℓ-ideal of A.

<u>PROPOSITION 2.38.</u> Let A be an ℓ-ring which satisfies one of the conditions (α), (β), (γ) or (δ). If P is an ℓ-ideal then the following are equivalent:

i) P is an ℓ-prime ideal;

ii) $\sqrt{P} = P$ and $a \wedge b = 0$ implies $a \in P$ or $b \in P$.

PROOF. i) implies ii). $\sqrt{P} \supseteq P$ is clear. Since P is a prime ℓ-ideal A/P satisfies one of the conditions (α), (β), (γ) or (δ) by Lemma 2.2, we have A/P a totally-ordered ring without non-zero divisors of zero by Propositions 1.27 and 2.29. If $x \in \sqrt{P}$ then $x^n \in P$ for some n (and $(\bar{x})^n = \bar{0}$). Since A/P contains no non-zero divisors of zero we have $\bar{x} = \bar{0}$, i.e., $x \in P$. Hence $\sqrt{P} = P$. Assume $a \wedge b = 0$, then $\bar{a} \wedge \bar{b} = (a + P) \wedge (b + P) = a \wedge b + P = \bar{0}$. We have $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$, i.e., $a \in P$ or $b \in P$.

ii) implies i). We note that the condition "$a \wedge b = 0$ implies $a \in P$ or $b \in P$" is equivalent to the condition "$a \wedge b \in P$ implies $a \in P$ or $b \in P$". Now assume $\sqrt{P} = P$ and $a \wedge b \in P$ implies $a \in P$ or $b \in P$. Let $I \nsubseteq P$, $J \nsubseteq P$ where I, J two ℓ-ideals of A, then there exists $a \in I^+ \setminus P$ and $b \in J^+ \setminus P$ (since $a \notin P$ if and only if $|a| \notin P$). If $IJ \subseteq P$ then $0 \leq (a \wedge b) \leq a$, b and hence $0 \leq (a \wedge b)^2 \leq ab \in IJ \subseteq P = \sqrt{P}$. Thus $a \wedge b \in P$ a contradiction. Hence $IJ \nsubseteq P$. This shows that P is a prime ℓ-ideal of A.

THEOREM 2.39. Let A be an ℓ-ring which satisfies one of the conditions (α), (β), (γ) or (δ). If P is an ℓ-ideal of A, then the following are equivalent:

i) if $a, b \in A$ and $ab \in P$, then $a \in P$ or $b \in P$;

ii) if I, J are two ideals (ring ideals) of A such that $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$;

iii) if I, J are ℓ-ideals of A and $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$;

iv) A/P is a totally-ordered ring without non-zero divisors of zero;

v) $\sqrt{P} = P$ and $a \wedge b = 0$ implies $a \in P$ or $b \in P$.

PROOF. A consequence of Proposition 1.21, 2.29, 2.33, 2.34 and 2.38.

In some classes of ℓ-rings, the ring has no non-zero nilpotent elements if the ring has no non-zero positive nilpotent elements. Examples of such classes are the classes of ℓ-rings satisfying the conditions, (α), (β) or (γ). This is true since for any positive integer n, $x^{4n} = (x^+)^{4n} + (x^-)^{4n}$ holds for the ℓ-rings satisfying the condition (α), $|x^n| = |x|^n$ holds for the ℓ-rings satisfying the condition (β) and $x^{2n} = (x^+)^{2n} + (x^-)^{2n}$ holds for the ℓ-ring satisfying the condition (γ). Hence an ℓ-ring which contains no non-zero positive nilpotent elements, and satisfies any one of the conditions (α), (β) or (γ) is an f-ring.

In [6], Diem has given some examples which satisfy the conditions (α), (β) or (γ). We now present two examples to show that an ℓ-ring satisfying (α), (β), (γ) and (δ) may not be an f-ring; an ℓ-ring satisfies the condition (δ) but does not satisfy any of the conditions (α), (β) or (γ).

EXAMPLE 2.40. An ℓ-ring satisfying (α), (β), (γ) and (δ) that is not an f-ring.

Let R be the set of all even integers with usual order. Let A be the cartesian product of two copies of R with the operations and order in A as follows: for a_1, a_2, b_1 and b_2 in R,

i) $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$;

ii) $(a_1, a_2)(b_1, b_2) = (a_2b_2, 0)$; and

iii) $(a_1, a_2) \geq (0, 0)$ if $a_1 \geq 0$ and $a_2 \geq 0$.

Then it is easy to see that A is an ℓ-ring. It is clear that $|(a_1, a_2)| = (|a_1|, |a_2|)$ for all $(a_1, a_2)\varepsilon A$ and A is an archimedean commutative ℓ-ring. Since the product of any three elements in A is zero, the ring A satisfies the conditions (α) and (δ). Now for any $(a_1, a_2)\varepsilon A$, $(a_1, a_2)^+(a_1, a_2)^- =$ $[(a_1, a_2)\vee(0, 0)][-(a_1, a_2)\vee(0, 0)] = [(a_1, a_2)\vee(0, 0)][(-a_1-a_2)\vee(0,0)]$ $= (a_1\vee 0, a_2\vee 0)(-a_1\vee 0, -a_2\vee 0) = ((a_2\vee 0)(-a_2\vee 0),0) = (0, 0)$, since

a_2 is an even integer. Thus the ring A satisfies the condition (γ). For any $x = (a_1, a_2)$ and $y = (b_1, b_2)$ in A, $|xy| = |(a_1, a_2)(b_1, b_2)| = |(a_2b_2, 0)| = (|a_2||b_2|, 0) = (|a_1|, |a_2|)(|b_1|, |b_2|) = |(a_1, a_2)||(b_1, b_2)| = |x||y|$. By ([5], Lemma 1, p58) the ring A satisfies the condition (β). The ℓ-ring A is not an f-ring. For $(0, 2)\wedge(2,0) = (0, 0)$ but $(0, 2)(0, 2)\wedge(2,0) = (4,0)\wedge(2,0) = (2,0) > (0,0)$.

<u>EXAMPLE 2.41.</u> An ℓ-ring which satisfies the condition (δ), but which satisfies none of the conditions (α), (β) nor (γ).

Let $A = \{aX + bY + cY^2 + dY^3 + eY^4 \mid a,b,c,d,e \in Z\}$ where $Y^5 = 0$, $X^2 = XY = XY^2 = XY^3 = XY^4 = Y^2X = Y^3X = Y^4X = 0$, $YX = Y^4$ and Z is the ring of integers with the usual order. Then the semi-group ring A is an ℓ-ring if we define $A^+ = \{aX + bY + cY^2 + dY^3 + eY^4 \mid a \geq 0, b \geq 0, c \geq 0, d \geq 0 \text{ and } e \geq 0\}$. Since $A^5 = \{0\}$, the ring A satisfies the condition (δ). This ring satisfies none of the conditions (α), (β) or (γ). To prove this, let $m = -Y + Y^2$, then $m^+ = Y^2$, $m^- = Y$ and $m^+(Y)m^- = Y^2(Y)(Y) = Y^4 \neq 0$; $m^+m^- = Y^2Y = Y^3 \neq 0$. Hence A satisfies neither (α) nor (γ). Since $Y[(X-Y^3)\vee Y] = Y(X + Y) = Y^2 + Y^4$ and $Y(X - Y^3)\vee Y(Y) = (-Y^4 + Y^4)\vee Y^2 = Y^2$, it does not satisfy the condition (β).

CHAPTER III

SQUARE ARCHIMEDEAN ℓ-RINGS

The main aim of this chapter is to prove that, for a square archimedean ℓ-ring A, (and, in particular, for a positive square ℓ-ring), (i) $N(A) = P(A)$, (ii) the set $\overline{N}(A)$ of all nilpotent elements is an ℓ-ideal if and only if $N(A) = P(A) = \overline{N}(A)$, and (iii) $\overline{\overline{N}}(A) = \{x \in A \mid |x|^n = 0$ for some $n\}$ is an ℓ-ideal if and only if $N(A) = P(A) = \overline{\overline{N}}(A)$.

<u>DEFINITION 3.1.</u> An ℓ-ring A is said to be a *square archimedean ℓ-ring* if it satisfies one of the following obviously equivalent conditions:

i) if $x, y \in A^+$ then there exists a positive integer $n = n(x,y)$ such that $xy + yx \leq n(x^2 + y^2)$;

ii) if $x, y \in A$, then there exists a positive integer $n = n(x,y)$ such that $xy + yx \leq n(|x|^2 + |y|^2)$; and

iii) if $x, y \in A^+$, then there exists a non-negative integer $n = n(x,y)$ such that $n(x^2 + y^2) + (x-y)^2 \geq 0$.

We have the following properties:

i) any ℓ-subring of a square archimedean ℓ-ring is a square archimedean ℓ-ring;

ii) if A is a square archimedean ℓ-ring, B an ℓ-ring, and if f is a homomorphism, such that $f(A^+) = B^+$, $f(A) = B$, then B is a square archimedean ℓ-ring;

iii) let $\{A_\lambda : \lambda \varepsilon \Lambda\}$ be a non-empty family of square archimedean ℓ-rings and let $A = \sum_{\lambda \varepsilon \Lambda} A_\lambda$ the direct sum of the family with the order defined as follows: $(a_\lambda) \geq 0$ if and only if $a_\lambda \geq 0$ for each $\lambda \varepsilon \Lambda$, then A is a square archimedean ℓ-ring; and

iv) every positive square ℓ-ring $(x^2 \geq 0,\ x \varepsilon A)$ is a square archimedean ℓ-ring.

__LEMMA 3.2.__ Let A be an ℓ-ring. If $a, b \in A$, then
(i) $|a \wedge b| \leq |a| \vee |b|$; (ii) $|a \vee b| \leq |a| \vee |b|$.

__PROOF.__ (i) Let $a, b \varepsilon A$, then $a \wedge b \leq a \leq |a| \leq |a| \vee |b|$ and $-(a \wedge b) = (-a) \vee (-b) \leq |a| \vee |b|$. Hence $|a \wedge b| \leq |a| \vee |b|$.

ii) Since $a \vee b \leq |a| \vee |b|$ and $-(a \vee b) = (-a) \wedge (-b) \leq -a \leq |a| \leq |a| \vee |b|$ we have $|a \vee b| \leq |a| \vee |b|$.

We will often use the following notations in this chapter: for an ℓ-ring A,

$$\overline{N} = \overline{N}(A) = \{x\varepsilon A \mid x^n = 0 \text{ for some positive integer } n\},$$

$$\overline{\overline{N}} = \overline{\overline{N}}(A) = \{x\varepsilon A \mid |x|^n = 0 \text{ for some positive integer } n\},$$

$$\overline{\overline{N}}_m = \overline{\overline{N}}_m(A) = \{x\varepsilon A \mid |x|^m = 0\}, \text{ where } m \text{ is a positive integer.}$$

It is clear that in any ℓ-ring A, $\overline{\overline{N}}(A) \subseteq \overline{N}(A)$ and that, for certain classes of ℓ-rings, (for example the ℓ-rings satisfying one of the conditions (α), (β), (γ) or (δ)), $\overline{\overline{N}}(A) = \overline{N}(A)$. The following example will show that in general $\overline{\overline{N}}(A) \neq \overline{N}(A)$.

<u>EXAMPLE 3.3.</u> Let Z be the ring of integers with the usual order. Let $A = Z \times Z$ with the operations and order as follows: for a, b, c and $d\varepsilon Z$,

i) $(a, b) + (c, d) = (a+c, b+d)$;

ii) $(a, b) \cdot (c, d) = ((a+b)(c+d), 0)$;

iii) $(a, b) \geq (0, 0)$ if and only if $a \geq 0$, $b \geq 0$,

It is easy to see that A is an ℓ-ring with $\overline{\overline{N}}(A) = \{(0, 0)\}$ and $\overline{N}(A) = \{(a, -a) \mid a\varepsilon Z\}$.

LEMMA 3.4. Let A be a square archimedean ℓ-ring and $x, y \varepsilon A^+$. Then for all $m \geq 1$, $(x + y)^{(2^m)} \leq \lambda_m (x^{(2^m)} + y^{(2^m)})$ and $(xy)^{(2^{m-1})} \leq \mu_m (x^{(2^m)} + y^{(2^m)})$ for some positive integers λ_m and μ_m.

PROOF. We proceed by induction on m. Let $x, y \varepsilon A^+$. If $m = 1$, then $(x+y)^2 = x^2 + y^2 + xy + yx \leq (x^2+y^2) + n(x^2+y^2) = (n+1)(x^2+y^2)$ for some n and $xy \leq xy + yx \leq u(x^2+y^2)$ for some u (by the square archimedean condition). Assume the assertion holds for $m - 1$, that is, assume

$$(x+y)^{2^{m-1}} \leq \lambda_{m-1}(x^{2^{m-1}} + y^{2^{m-1}}) \text{ say ,}$$

then

$$\begin{aligned} (x+y)^{2^m} &= ((x+y)^{2^{m-1}})^2 \leq \lambda_{m-1}^2(x^{2^{m-1}}+y^{2^{m-1}})^2 \\ &= \lambda_{m-1}^2(x^{2^m} + y^{2^m} + x^{2^{m-1}}y^{2^{m-1}} + y^{2^{m-1}}x^{2^{m-1}}) \\ &\leq \lambda_{m-1}^2(x^{2^m} + y^{2^m} + n'(x^{2^m} + y^{2^m})) \end{aligned}$$

for some positive integer n', (by the square archimedean property), and hence

$$(x+y)^{2^m} \leq \lambda_{m-1}^2(n'+1)(x^{2^m} + y^{2^m}) .$$

Similarly, if

$$(xy)^{2^{m-2}} \leq u(x^{2^{m-1}} + y^{2^{m-1}}), \text{ then}$$

$$\begin{aligned}(xy)^{2^{m-1}} &\leq u^2(x^{2^{m-1}} + y^{2^{m-1}})^2 \\ &= u^2(x^{2^m} + y^{2^m} + x^{2^{m-1}}y^{2^{m-1}} + y^{2^{m-1}}x^{2^{m-1}}) \\ &\leq u^2(x^{2^m} + y^{2^m} + \mu(x^{2^m} + y^{2^m}))\end{aligned}$$

for some positive integer μ (by square archimedean property), and hence

$$(xy)^{2^{m-1}} \leq u^2(+1)(x^{2^m} + y^{2^m}) .$$

This completes the proof.

LEMMA 3.5. Let A be a square archimedean ℓ-ring. Then $\bar{\bar{N}}(A)$ is a sublattice subring of A and $\bar{\bar{N}}(A)$ is also a square archimedean ℓ-ring.

PROOF. If $x, y \in (\bar{\bar{N}}(A))^+$, then $x^{2^m} = 0$ and $y^{2^m} = 0$ for some $m \geq 1$. Hence $(x+y)^{2^m} \leq \lambda_m(x^{2^m} + y^{2^m}) = 0$ and $(xy)^{2^{m-1}} \leq \mu_m(x^{2^m} + y^{2^m}) = 0$

for some positive integers λ_m and μ_m by Lemma 3.4. That $\bar{\bar{N}}(A)$ is a subring now follows from $|x + y| \leq |x| + |y|$ and $|xy| \leq |x||y|$. Clearly $\bar{\bar{N}}(A)$ is a convex subring of A. Now if $a, b \in (\bar{\bar{N}}(A))^+$ then $0 \leq a \wedge b \leq a$ and so $a \wedge b \in \bar{\bar{N}}(A)$; $a \vee b = (a+b) - (a \wedge b)$, hence $a \vee b \in \bar{\bar{N}}(A)$. If $a, b \in \bar{\bar{N}}(A)$, then since $|a \wedge b| \leq |a| \vee |b|$, $|a \vee b| \leq |a| \vee |b|$ and $\bar{\bar{N}}(A)$ is convex, we have $a \wedge b$, $a \vee b \in \bar{\bar{N}}(A)$. It is clear that $\bar{\bar{N}}(A)$ is a square archimedean ℓ-ring.

<u>PROPOSITION 3.6.</u> Let A be a square archimedean ℓ-ring in which every element of A is nilpotent. Then $\bar{\bar{N}}_2(A)$ is a nilpotent ℓ-ideal of A.

<u>PROOF.</u> Let $\bar{\bar{N}}_2 = \bar{\bar{N}}_2(A)$. From Lemma 3.4, $a, b \in \bar{\bar{N}}_2$ implies $a - b \in \bar{\bar{N}}_2$. Let $a \in (\bar{\bar{N}}_2)^+$ and $r \in A^+$. Then $(ar)a + a(ar) \leq n\,((ar)^2 + a^2)$ for some $n \geq 1$ and since $a^2 = 0$,

$$
\begin{aligned}
ara &\leq n(arar) \\
&\leq n^2(arar^2) \\
&\leq n^3(arar^3) \quad \text{etc.} \\
&\leq n^m(arar^m) \quad \text{for all } m \geq 1.
\end{aligned}
$$

Since r is nilpotent by assumption, we have

$$ara = 0$$

Thus ar and $ra \varepsilon \bar{\bar{N}}_2$.

Now if $a \varepsilon \bar{\bar{N}}_2$, $r \varepsilon A$ and let $a = a^+ - a^-$, $r = r^+ - r^-$ then $a^+, a^- \leq |a|$, whence $a^+, a^- \varepsilon (\bar{\bar{N}}_2)^+$. We have ra and $ar \varepsilon \bar{\bar{N}}_2$. Thus $\bar{\bar{N}}_2$ is an ideal. It is clear that $\bar{\bar{N}}_2$ is an ℓ-ideal. Now we show that $(\bar{\bar{N}}_2)^2 = \{0\}$. Suppose $a, b \varepsilon (\bar{\bar{N}}_2)^+$, then $0 \leq ab + ba \leq n\ (a^2 + b^2) = 0$. Hence $ab = ba = 0$. If $a, b \varepsilon \bar{\bar{N}}_2$ then $a^+, a^-, b^+, b^- \varepsilon (\bar{\bar{N}})^+$ and $a^+b^- = 0 = a^+b^- = a^-b^+ = a^-b^-$. Thus $ab = 0$. This completes the proof of the proposition.

<u>PROPOSITION 3.7.</u> Let A be a square archimedean ℓ-ring in which every element is a nilpotent. Then for every $m \geq 1$, $\bar{\bar{N}}_{2m}$ is a nilpotent ℓ-ideal of A and $(\bar{\bar{N}}_{2m})^{2^m} = \{0\}$, where $\bar{\bar{N}}_{2m} = \bar{\bar{N}}_{2m}(A)$.

<u>PROOF.</u> We proceed by induction on m.

For $m = 1$, the result is true by Proposition 3.6. Assume the assertion holds for $m - 1$. Let $A^* = A/\bar{\bar{N}}_2$, then A^* is also a square archimedean ℓ-ring and every element of A^* is nilpotent. By the induction hypothesis $\bar{\bar{N}}_{2m-1}(A^*)$ is an ℓ-ideal of A^* such that $\bar{\bar{N}}^{2^{m-1}}_{2^{m-1}} = \{0\}$ in A^*. Now $\bar{\bar{N}}_{2^m}(A) = \theta^{-1}(\bar{\bar{N}}_{2^{m-1}}(A^*))$, where

$$\theta : A \to A/\bar{\bar{N}}_2$$

and so $\bar{\bar{N}}_{2^m}(A)$ is an ℓ-ideal of A. Also

$$\begin{aligned} \bar{\bar{N}}^{2^{m-1}}_{2^m}(A) &= \theta^{-1}(\bar{\bar{N}}^{2^{m-1}}_{2^{m-1}}(A^*)) \\ &= \theta^{-1}(\bar{0}) \\ &= \bar{\bar{N}}_2 \end{aligned}$$

Hence $\bar{\bar{N}}^{2^m}_{2^m} = \bar{\bar{N}}^2_2 = \{0\}$.

PROPOSITION 3.8. Let A be a square archimedean ℓ-ring in which every element of A is a nilpotent. Then $N(A) = P(A) = A$.

PROOF.

$$\bigcup_{n=1}^{\infty} \bar{\bar{N}}_{2^n}(A) \subseteq N(A) \subseteq P(A) \subseteq A \subseteq \bigcup_{n=1}^{\infty} \bar{\bar{N}}_{2^n}(A) .$$

THEOREM 3.9. Let A be any square archimedean ℓ-ring. Then $N(A) = P(A)$.

PROOF. Let $B = P(A)$ then B is an ℓ-ring which is square archimedean and every element of B is a nilpotent. By Proposition 3.8 $N(B) = P(B) = B$. But $N(B) = B \cap N(A)$ and so $B \subseteq N(A)$. Thus $P(A) = B = N(A)$ and hence $P(A) = N(A)$.

The following corollary is another proof of Theorem 2.21.

COROLLARY 3.10. Let A be an ℓ-ring satisfying condition (γ) (i.e., $x^+x^- = 0$ for all $x \in A$). Then $N(A) = P(A) = \bar{\bar{N}}(A) = \overline{N}(A)$.

PROOF. Since A satisfies the condition (γ), A is a positive square ℓ-ring and hence a square archimedean ℓ-ring. By Theorem 3.9 $N(A) = P(A)$. Thus by Proposition 1.23, $N(A) = P(A) = \bar{\bar{N}}(A) = \overline{N}(A)$.

THEOREM 3.11. Let A be a square archimedean ℓ-ring. Then a necessary and sufficient condition for $N(A) = P(A) = \bar{\bar{N}}(A)$ is that $\bar{\bar{N}}(A)$ be an ℓ-ideal of A.

PROOF. Necessity is clear. For the sufficiency, suppose $\bar{\bar{N}}(A) = \bar{\bar{N}}$ is an ℓ-ideal of A. Since every element of $\bar{\bar{N}}$ is a nilpotent, we have $N(\bar{\bar{N}}) = P(\bar{\bar{N}}) = \bar{\bar{N}}$ by Proposition 3.8. But $N(\bar{\bar{N}}) = \bar{\bar{N}} \cap N(A)$, hence $\bar{\bar{N}} \subseteq N(A)$. Since $N(A) \subseteq P(A) \subseteq \bar{\bar{N}}$ for any ℓ-ring, we have

$$\bar{\bar{N}} = N(A) = P(A) .$$

THEOREM 3.12. Let A be a square archimedean ℓ-ring. Then a necessary and sufficient condition for $N(A) = P(A) = \bar{\bar{N}}(A) = \bar{N}(A)$ is that $\bar{N}(A)$ be an ℓ-ideal of A.

PROOF. Necessity is clear. For the sufficiency, suppose $\bar{N}(A)$ is an ℓ-ideal of A. If $x \in \bar{N}(A)$, then $|x| \in \bar{N}(A)$ and so $|x|^n = 0$ for some positive integer n. Thus $x \in \bar{\bar{N}}(A)$. This shows that $\bar{N}(A) = \bar{\bar{N}}(A)$. By Theorem 3.11 $N(A) = P(A) = \bar{\bar{N}}(A) = \bar{N}(A)$.

In [5], p60, Birkhoff and Pierce have shown that a d-ring A is positive square if and only if it is an orthogonal ℓ-ring, (i.e., $x^+x^- = 0$ for all $x \in A$). We have the following generalization.

PROPOSITION 3.13. Let A be a d-ring. Then A is a square archimedean ℓ-ring if and only if A is an orthogonal ℓ-ring.

PROOF. Assume A is square archimedean and suppose $x, y \in A$ and $x \wedge y = 0$, then $x \geq 0$, $y \geq 0$. By the square archimedean property, $0 \leq xy + yx \leq \lambda\ (x^2 + y^2)$ for some positive integer λ. It follows that

$$0 \leq xy \leq \lambda(x^2 + y^2) \quad \text{and so}$$

$$\begin{aligned} 0 \leq xy &= xy \wedge (\lambda(x^2 + y^2) \leq \lambda[xy \wedge (x^2 + y^2)] \\ &\leq \lambda[xy \wedge x^2 + xy \wedge y^2] \\ &= \lambda[x(y \wedge x) + (x \wedge y)y] \\ &= 0\ . \end{aligned}$$

Thus, by Proposition 2.18 A is an orthogonal ℓ-ring.

The converse is trivial, for an orthogonal ℓ-ring is a positive square ℓ-ring and a positive square ℓ-ring is a square archimedean ℓ-ring.

COROLLARY 3.14. Let A be a d-ring. Then A is a square archimedean ℓ-ring if and only if A is a positive square ℓ-ring.

PROPOSITION 3.15. Let A be a square archimedean ℓ-ring. If I is a right ℓ-ideal of A and $x \in A^+$ is such that $x^2 \in I$, then $xI \subseteq I$.

PROOF. For $y \in I^+$, then $0 \le xy + yx \le \lambda(x^2 + y^2)$ for some positive integer λ (by the square archimedean property). Then $0 \le xy \le \lambda(x^2 + y^2)$. Since $x^2 + y^2 \in I$ and I is convex, we have $xy \in I$. Now for any $y \in I$, $y = y^+ - y^-$ and $y^+, y^- \in I$. Hence $xy \in I$.

THEOREM 3.16. Let A be a square archimedean ℓ-ring with an identity $1 > 0$. Then $|x| \le m \cdot 1$ for some positive integer m if $|x|^k = 0$ for some positive integer k.

PROOF. We proceed by induction on k. For $k = 2$, i.e., $|x|^2 = 0$, then by the square archimedean property $|x| \le 2|x| = |x| + |x| \le \lambda(|x|^2 + 1) = \lambda \cdot 1$ for some positive integer λ. Now assume the assertion holds for $j < k$ $(k > 2)$, that is for $j < k$, $|x|^j = 0$

implies $|x| \leq m\cdot 1$ for some $m \geq 1$. Assume $|x|^k = 0$. Let $k' = k$ if k is even and $k' = k+1$ if k is odd. Since $(|x|^2)^{k'/2} = 0$ and $k'/2 < k$, by hypothesis $|x|^2 \leq n\cdot 1$ for some $n \geq 1$. Now by the square archimedean property $|x| \leq 2|x| \leq \lambda'(|x|^2 + 1) \leq \lambda'(n\cdot 1 + 1)$ $= \lambda' n\cdot 1 + \lambda'\cdot 1 = (\lambda' n + \lambda')\cdot 1$.

We now generalize some results of Diem [7], and give an example to show that one of the theorems found in [7] can not be generalized.

LEMMA 3.17. Let A be a square archimedean ℓ-ring. Then for $x, y \in A^+$ with $x^2 = y^2 = 0$ we have $xy = yx = 0$.

PROOF. Since $x, y \in A^+$, there exists a positive integer λ such that $0 \leq xy + yx \leq \lambda(x^2 + y^2)$ by square archimedean property. We have $xy = yx = 0$, for $x^2 = y^2 = 0$ by assumption.

We recall that a non-zero ℓ-ring A is called an ℓ-domain if $A^+ \setminus \{0\}$ is closed under multiplication.

LEMMA 3.18. Let A be a prime ℓ-ring which is square archimedean. Then A is an ℓ-domain if and only if $a, b \in A$, $a \wedge b = 0$, and $ab = 0$ imply $ba = 0$.

PROOF. See [7], p79, Lemma 3.6.

PROPOSITION 3.19. Let A be an ℓ-ring with $N(A) = \{0\}$ which is a square archimedean ℓ-ring and in which disjoint elements commute. Then A is a subdirect union of square archimedean ℓ-domains in which disjoint elements commute.

PROOF. Since $N(A) = \{0\}$, A is a subdirect union of a family $\{A_\alpha, \alpha\varepsilon\Gamma\}$ of prime ℓ-rings by [7], 2.14. Since the properties of being square archimedean and having disjoint elements commute are preserved under ℓ-homomorphisms, each A_α is square archimedean and disjoint elements commute. Thus by Lemma 3.18 each A_α is an ℓ-domain.

PROPOSITION 3.20. Let A be a square archimedean ℓ-ring in which disjoint elements commute. The $P(A) = \bar{\bar{N}}(A)$.

PROOF. Since $P(A/P(A)) = \{0\}$, $A/P(A)$ is a subdirect union of ℓ-domains by Proposition 3.19. It is clear that $A/P(A)$ has no non-zero positive nilpotent elements. Hence $a\varepsilon A^+$ and $a^n = 0$ for some positive integer n imply that $a\varepsilon P(A)$. Now if $x\varepsilon\bar{\bar{N}}(A)$, then $|x|^n = 0$ for some n and so $|x|\varepsilon P(A)$. Thus $x\varepsilon P(A)$. It follows that $P(A) = \bar{\bar{N}}(A)$.

THEOREM 3.21. Let A be a square archimedean ℓ-ring in which disjoint elements commute. Then $N(A) = P(A) = \bar{\bar{N}}(A)$.

<u>PROOF</u>. The theorem follows from Proposition 3.20 and Theorem 3.9.

In [7], p79, Diem posed the following question: Is it true that, in every prime ℓ-ring A in which the square of every element is positive, $a, b \in A$, $a \wedge b = 0$, and $ab = 0$ always imply $ba = 0$.

In view of Theorem 3.11, if we knew that for an arbitrary prime ℓ-ring A which is positive square, $\bar{\bar{N}}(A)$ was an ℓ-ideal of A then, for $a, b \in A$, $a \quad b = 0$ and $ba = 0$, we would have $(ba)^2 = (ba)(ba) = b(ab)a = 0$. But $N(A) = P(A) = \bar{\bar{N}}(A) = \{0\}$ and so $ba \in \bar{\bar{N}}(A) = \{0\}$. Hence the following question is interesting: Does there exist an ℓ-prime positive square ring for which $\bar{\bar{N}}$ is not an ℓ-ideal.

The following example will show that in general a square archimedean ℓ-ring may not be a positive square ring.

<u>EXAMPLE 3.22</u>. (Diem, [6]) Let A be the group direct sum of two copies of the integers Z with usual order and define the multiplication and order in A as follows: for a_1, a_2, b_1 and $b_2 \in Z$,

i) $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_1b_2 + b_1a_2)$; and

ii) $(a_1, a_2) \geq (0, 0)$ if $a_2 \geq a_1 \geq 0$.

Then A is a commutative ℓ-ring with $P(A) = \{(0, a_2) | a_2 \varepsilon Z\}$. Moreover, $A/P(A)$ is an f-ring. It is easy to check that A is square archimedean and also archimedean. The element $(1, 0)$ is an idempotent and so A is not a positive square ring. The element $(0, 1) > 0$ has the properties; $(0,1)^2 = (0, 1)(0, 1) = (0, 0)$ and $(0,1)(1,1) = (0,1) \neq (0,0)$. This shows that $(0,1)^2 = (0,0)$ and $(0,1)A \neq \{(0,0)\}$. Thus the following Diem's Theorem cannot be generalized to the square archimedean ℓ-rings which are also archimedean ℓ-rings.

<u>THEOREM 3.23.</u> (Diem, [7]) If A is an archimedean ℓ-ring which is positive square then, (i) $x \varepsilon A^+$ and $x^2 = 0$ imply $xA = Ax = \{0\}$, (ii) $P(A)\, A^2 = A^2 P(A) = P(A)^3 = \{0\}$.

<u>THEOREM 3.24.</u> Let $P(A)$ be the ℓ-prime radical of an ℓ-ring A, Then the following are equivalent:

i) $A/P(A)$ is an f-ring;

ii) $A/P(A)$ satisfies the condition (α);

iii) $A/P(A)$ satisfies the condition (β);

iv) $A/P(A)$ satisfies the condition (γ); and

v) $A/P(A)$ satisfies the condition (δ).

<u>PROOF.</u> Since $N(A/P(A)) \subseteq P(A/P(A)) = \{0\}$ by ([7], 2.9), hence $A/P(A)$ has zero ℓ-radical. By Theorem 2.31, the above conditions are equivalent.

DEFINITION 3.25. An ℓ-ring A is called a *pseudo f-ring* if it satisfies any one of the conditions in Theorem 3.24.

PROPOSITION 3.26. ([6], p47, 2.1) Let A be an ℓ-ring. Then the following are equivalent:

i) if P is a prime ℓ-ideal of A, then A/P is an f-ring;

ii) if P is a prime ℓ-ideal of A, then A/P is totally-ordered without non-zero divisors of zero;

iii) A is a pseudo f-ring; and

iv) A/P(A) is a subdirect union of totally-ordered rings without non-zero divisors of zero.

COROLLARY 3.27. ([6], p53, 2.8) Let A be a pseudo f-ring, then $\overline{N}(A) \subseteq P(A)$ (whence $\overline{N}(A) = P(A)$).

We note that a pseudo f-ring may not be a square archimedean ℓ-ring (c.f. [6], Example 2.6, p51). Also a pseudo f-ring which is a square archimedean ℓ-ring need not satisfy any of the conditions (α), (β), (γ) or (δ) (see [6], Example 2.7, p52), Nevertheless we have the following:

<u>THEOREM 3.28.</u> Let A be a pseudo f-ring which is a square archimedean ℓ-ring. Then $N(A) = P(A) = \bar{\bar{N}}(A) = \overline{N}(A)$.

<u>PROOF.</u> Since A is a pseudo f-ring, by Corollary 3.27 $\overline{N}(A) = P(A)$ and since A is a square archimedean ℓ-ring, $N(A) = P(A)$ by Theorem 3.9. Hence $N(A) = P(A) = \bar{\bar{N}}(A) = \overline{N}(A)$.

<u>PROPOSITION 3.29.</u> ([5], p56) Let A be an ℓ-ring with $1 > 0$. Then $B(A) = \{x \in A \mid |x| \leq n.1$ for some positive integer $n\}$ is a sub ℓ-ring (a subring which is sublattice) of A which is an f-ring.

<u>PROOF.</u> Let x, $y \in B(A)$, then $|x| \leq n.1$, $|y| \leq m.1$ for some positive integers m,n. $|x-y| \leq |x| + |y| \leq n.1 + m.1 = (m+n).1$, $|xy| \leq |x||y| \leq (n.1)(m.1) = mn.1$. Hence $B(A)$ is a subring of A. Since $|x \vee y| \leq |x| \vee |y| \leq n.1 \quad m.1 \leq (m+n).1$ and $|x \wedge y| \leq |x| \vee |y| \leq (m+n).1$, $B(A)$ is a sub ℓ-ring. It is an f-ring, for, if $z \geq 0$ and $x \wedge y = 0$, then $z \leq u.1$, where u is a positive integer and $zx \leq (u.1)x = ux$, $xz \leq x(u.1) = ux$. Hence $xz \wedge y \leq ux \wedge y = 0$, $zx \wedge y \leq ux \wedge y = 0$. Thus $B(A)$ is an f-ring.

From Theorem 3.16, we note that for any ℓ-ring with an identity $1 > 0$, which is a square archimedean ℓ-ring, $P(A) \subseteq B(A)$ holds.

LEMMA 3.30. Let A be an ℓ-ring and S a non-empty subset of A. Then the polar set of S, $S^{\perp} = \{x \in A \mid |x| \wedge |s| = 0 \text{ for all } s \in S\}$ is an ℓ-subgroup of A. (A normal subgroup D of an ℓ-group A is called an ℓ-subgroup if $|x| \leq |y|$, where $y \in D$, implies $x \in D$).

PROOF. Suppose $x,y \in S^{\perp}$, then $|x| \wedge |s| = 0$, $|y| \wedge |s| = 0$ for all $s \in S$. Since $|x-y| \leq |x| + |y|$ for any $x,y \in A$, we have $(x-y) \in S^{\perp}$.

PROPOSITION 3.31. (Heinriksen & Isbell, [10]). Let A be an ℓ-ring. Then A is an f-ring if and only if $S^{\perp}$ is an ℓ-ideal for each non-empty subset S of A.

PROOF. If A is an f-ring, then clearly $S^{\perp}$ is an ℓ-ideal for any non-empty subset of A. Conversely, suppose A is not an f-ring, then A is not a totally ordered ring. Hence there exists $x, y \in A$ such that x and y are not comparable. Thus $(x-y)^{+} \wedge (x-y)^{-} = 0$ and $(x-y)^{+} \neq 0$, $(x-y)^{-} \neq 0$. Let $M = \{(a,b) \in A \times A \mid a \wedge b = 0 \text{ and } a \neq 0,\ b \neq 0\}$. The set M in this case is not empty. Since A is not an f-ring by assumption. There exists $(a,b) \in M$ and $r \in A^{+}$ such that $ar\ b \neq 0$ or $ra \wedge b \neq 0$. Take $S = \{b\}$, then we have $S^{\perp}$ is not a ring ideal of A. This completes the proof of the theorem.

PROPOSITION 3.32. Let A be a square archimedean ℓ-ring with $1 > 0$. Then $\bar{\bar{N}}(A) \cap \{1\}^{\perp} = \{0\}$ and $B(A) \cap \{1\}^{\perp} = \{0\}$, where $B(A) = \{x \in A \mid x \leq m.1 \text{ for some } m \in Z^{+}\}$.

PROOF. We prove $B(A) \cap \{1\}^{\perp} = \{0\}$ first. Let $a \in B(A) \cap \{1\}^{\perp}$, then $|a| \wedge 1 = 0$ and $|a| \leq m.1$ for some $m \in Z^{+}$. We have $|a| = |a| \wedge m.1 = 0$. Thus $a = 0$ and so $B(A) \cap \{1\}^{\perp} = \{0\}$. For the first case, since $\bar{\bar{N}}(A) \subseteq B(A)$ by Theorem 3.16, hence $\bar{\bar{N}}(A) \cap \{1\}^{\perp} = \{0\}$.

COROLLARY 3.33. Let A be a square archimedean ℓ-ring with $1 > 0$. If $A \setminus \{m.1 \mid m \in Z\} \subseteq \bar{\bar{N}}(A)$, then $1 \wedge x = 0$ implies $x = 0$ (i.e., 1 is a weak order unit).

For any ℓ-ideal I of a ℓ-ring A, we define the following set,

$$\sqrt{\bar{\bar{I}}} = \{x \in A \mid |x|^{n} \in I \text{ for some } n\} \quad \text{and recall that}$$

$$\sqrt{I} = \{x \in A \mid x^{n} \in I \text{ for some } n\}.$$

As we will see these two sets have some connection with the sets $\bar{\bar{N}}(A)$ and $\bar{N}(A)$. If A is a square archimedean ℓ-ring, then it follows from Lemma 3.4 that $\sqrt{\bar{\bar{I}}}$ is a subring of A for any ℓ-ideal I of A. It is clear that $\sqrt{\bar{\bar{I}}}$ is always a convex set.

Now consider an arbitrary ℓ-ring A. If I is an ℓ-ideal and ϕ: $A \to A/I$ the natural ℓ-homomorphism, then

$$\sqrt{\overline{I}} = \phi^{-1}(\bar{\bar{N}}(A/I)) \quad \text{and}$$

$$\sqrt{I} = \phi^{-1}(\bar{N}(A/I)).$$

Since the preimage of an ℓ-ideal of an ℓ-homomorphism is an ℓ-ideal, we have the following Proposition.

<u>PROPOSITION 3.34.</u> Let A be an ℓ-ring which satisfies one of the conditions (α), (β), (γ), (δ) or is a square archimedean pseudo f-ring. Then $\sqrt{\overline{I}} = \sqrt{I}$ is always an ℓ-ideal of A for any ℓ-ideal I of A.

<u>PROOF.</u> Since an ℓ-ring A satisfies one of the conditions (α), (β), (γ), (δ) or is a square archimedean pseudo f-ring, $\bar{\bar{N}}(A/I) = \bar{N}(A/I) = N(A/I)$ is an ℓ-ideal of A/I. Hence $\sqrt{\overline{I}} = \sqrt{I}$ is an ℓ-ideal of A.

In the relation $\sqrt{\overline{I}} = \phi^{-1}(\bar{\bar{N}}(A/I))$, if for any ℓ-ideal I in a square archimedean ℓ-ring A, $\sqrt{\overline{I}}$ is an ℓ-ideal, then $\bar{\bar{N}}(A)$ is an ℓ-ideal. Now if A has zero ℓ-radical, then $a \wedge b = 0$ and $ab = 0$ imply $ba = 0$. We have the following theorem.

<u>THEOREM 3.35.</u> Let A be a square archimedean ℓ-ring with zero ℓ-radical. If $\sqrt{I}$ is always an ℓ-ideal for any ℓ-ideal I of A, then A is a subdirect sum of ℓ-domains.

<u>PROOF.</u> Since A has zero ℓ-radical, by Proposition 1.31 A is a subdirect union of ℓ-prime rings $\{A_\alpha\}$, $\alpha\varepsilon\Lambda$. Each A_α is a square archimedean and $a\wedge b = 0$, $ab = 0$ imply $ba = 0$. Thus by Lemma 3.18 A_α is an ℓ-domain for each $\alpha\varepsilon\Lambda$.

CHAPTER IV

JOHNSON RADICAL FOR A CLASS OF LATTICE-ORDERED RINGS

In [13], Johnson introduced an analogue of the Jacobson radical for f-rings, and in [18], Steinberg defined three different generalizations of the Jacobson radical for the class of all ℓ-rings. Denoting these various "radicals" by P_{m_o}, J, and $\mathcal{R}$, Steinberg [18] showed that $P_{m_o}(A) \subseteq J(A) \subseteq \mathcal{R}(A)$ for any ℓ-ring A, and that these three ideals are equal under certain conditions. We shall show that $J(A) = \mathcal{R}(A)$ under more general conditions and that $P(A) \subseteq P_{m_o}(A)$. Also, we will define the concept of a faithful, irreducible ℓ-module, and, for the class of pseudo f-rings, we will relate this concept to Steinberg's work. In this chapter we will also investigate the relation between $\mathcal{R}(A)$ and $\mathcal{R}(A_{nxn})$, where A_{nxn} is the ring of nxn matrices with entries from A, the ordering on A_{nxn} taken to be the canonical one.

<u>DEFINITION 4.1.</u> (i) A right ℓ-ideal I of an ℓ-ring A is said to be *modular* if there exists a left identity modulo I, $e \in A$ such that $x-ex \in I$ for all $x \in A$.

ii) A right ℓ-ideal I of a ℓ-ring A is said to be *regular* if there exists a left identity modulo I, $e \in A^+$ such that $x-ex \in I$ for all $x \in A$.

We note that if e is a left identity modulo I, then e^n is a left identity modulo I for any positive integer n. Since any orthogonal ℓ-ring $(x^+x^- = 0)$ is a positive square ring and in an ℓ-ring which satisfies the condition (α), every element x satisfies $x^4 \geq 0$, for these classes of ℓ-rings the notions of modular and regular are the same. Steinberg has shown this for the class of d-rings (the ℓ-rings satisfy the condition (β)).

Let $<(1-a)A>_r$ denotes the right ℓ-ideal generated by the set $(1-a)A$.

<u>DEFINITION 4.2.</u> An element $a \geq 0$ of an ℓ-ring A is said to be *right ℓ-quasi regular* (right ℓ-QR) if $<(1-a)A>_r = A$.

Let $\Omega(A) = \{a \in A | <(1-|a|)A> = A\}$, then $a \in A^+ \cap \Omega(A)$ if and only if $|a| \leq \sum_{i=1}^{n} |x_i - ax_i| + \sum_{j=1}^{m} |z_j - az_j||r_j|$ for some x_i, z_i, r_j in A $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$. Also $a \in \Omega(A)$ if and only if $|a| \in \Omega(A)$ and $|a| \in \Omega(A)$ if $|a|^n = 0$ for some positive integer n.

In general $\Omega(A)$ may not be a ring ideal and it may contain idempotent elements. Also $\bar{\bar{N}}(A) \subseteq \Omega(A)$ and if $\bar{N}(A) = A$, then $\Omega(A) = A$.

<u>DEFINITION 4.3.</u> A right ℓ-ideal I of an ℓ-ring A is a *right ℓ-QR ℓ-ideal* if $I^+ \subseteq \Omega(A)$. An ℓ-ring A is said to be *ℓ-QR ℓ-ring* if $A = \Omega(A)$.

In general a right ℓ-QR right ideal may not be a right ℓ-QR ℓ-ring. (see Example 7, [18], p159).

<u>LEMMA 4.4.</u> ([18]). Let A be an ℓ-ring and I a right ℓ-ideal in A. Then (i) $(I:A) = \{a \in A \mid |x||a| \in I \text{ for all } x \in A\}$ is an ℓ-ideal in A. (ii) if I is modular, then (I:A) is the largest ℓ-ideal of A contained in I.

<u>DEFINITION 4.5.</u> An ℓ-ring A is said to be *ℓ-primitive* if there exists a **regular** maximal right ℓ-ideal M such that $(M:A) = \{0\}$. An ℓ-ideal P is an *ℓ-primitive ℓ-ideal* if A/P is an ℓ-primitive ℓ-ring.

We now want to relate some results concerning Steinberg's work [18]. In his work, for any ℓ-ring A, in fact $P_{m_o}(A)$ is the largest ℓ-QR ℓ-subring of A, (it is an ℓ-ideal of A), $J(A)$ is the largest right ℓ-QR ℓ-ideal of A and $R(A)$ is the largest right ℓ-QR right ℓ-ideal of A. Since $P(A) \subseteq \bar{\bar{N}}(A) \subseteq \Omega(A)$ and $P(A)$ is an ℓ-QR ℓ-ring, hence $P(A) \subseteq P_{m_o}(A)$.

<u>THEOREM 4.6.</u> ([18], 4.3.18). Let A be an ℓ-ring such that $A/P_{m_o}(A)$ satisfies one of the conditions (α), (β) or (γ). Then $P_{m_o}(A) = J(A) = R(A)$.

Since $P(A/P_{m_o}(A)) = \{0\}$ implies $N(A/P_{m_o}(A)) = \{0\}$ by Theorem 1.19, by Corollary 2.31, $A/P_{m_o}(A)$ satisfies any of the conditions (α), (β) or (γ) if and only if $A/P_{m_o}(A)$ satisfies the condition (δ). Thus Theorem 4.6 holds true for the class of ℓ-rings satisfying the condition (δ).

If an ℓ-ring A satisfies (α) (respectively (β), (γ), (δ)) then $A/P(A)$ satisfies (α) (respectively (β), (γ), (δ)). Hence an ℓ-ring A which satisfies one of the conditions (α), (β), (γ) or (δ) is a pseudo f-ring. Since $P(A) \subseteq P_{m_o}(A)$ for any ℓ-ring A, the natural map from $A/P(A)$ to $A/P_{m_o}(A)$ is an ℓ-homomorphism. Hence if A is a pseudo f-ring then $A/P_{m_o}(A)$ satisfies one of the conditions (α), (β), (γ) or (δ). Thus we have the following.

<u>THEOREM 4.7.</u> If A is a pseudo f-ring, then $P_{m_o}(A) = J(A) = \mathcal{R}(A)$.

<u>DEFINITION 4.8.</u> If A is an ℓ-ring and M an abelian ℓ-group, then M is said to be an A-*ℓ-module* if a law of composition is defined on $M \times A$ into M which satisfies for $g_1, g_2 \in M$ and $a, b \in A$;

i) $(g_1 + g_2)a = g_1a + g_2a$;

ii) $g_1(a + b) = g_1a + g_1b$;

iii) $g_1(ab) = (g_1a)b$; and

iv) $M^+A^+ \subseteq M^+$.

DEFINITION 4.9. Let M_A be an A-ℓ-module. An ℓ-subgroup H of M is said to be an A-ℓ-*submodule* if $HA \subseteq H$.

DEFINITION 4.10. An A-ℓ-module M_A is said to be an *irreducible module* if M_A is simple, (i.e., $\{0\}$ and M are the only A-ℓ-submodules) and there exists $m \in M^+$ such that $mA = M$.

DEFINITION 4.11. An A-ℓ-module M_A is said to be *faithful* if $|m|\,|r| = 0$ for all $m \in M$ implies $r = 0$.

PROPOSITION 4.12. If A is an ℓ-primitive ring, then A has a faithful, irreducible A-ℓ-module G.

PROOF. Suppose A is an ℓ-primitive ring, then by definition there is a maximal regular right ℓ-ideal M such that $(M:A) = \{0\}$. Then $G = A/M = \{a+M \mid a \in A\}$ is an abelian lattice-ordered group. Define $(g+M)a$ by $ga+M$ for all $g \in G$ and $a \in A$. We have for $g, g_1, g_2, a, b \in A$;

i) $[(g_1 + M) + (g_2 + M)]a = (g_1 + M)a + (g_2 + M)a$,

ii) $(g + M)(a + b) = (g + M)a + (g + M)b$,

iii) $(g + M)ab = [(g + M)a]b$,

iv) $(g + M) \in G^+$, $a \in A^+$ then $(g + M)a \in G^+$.

Thus G is an A-ℓ-module. Now the A-ℓ-submodules of G are of the form M'/M for some $M \subseteq M' \subseteq A$. Since M is a maximal regular right ℓ-ideal we have $M' = M$ or $M' = A$. Now let e be a left identity modulo M. For any $x \in A$ we have $(e+M)x = ex + M = x+M$. Hence $(e+M)A = G$, and $e+M$ is a positive element in G. This shows that G is an irreducible A-ℓ-module. It remains to show that G is a faithful module. Let $r \in A$ and $|a+M||r| = \overline{0}$ for all $a \in A$. Then $|a+M||r| = (|a|+M)|r| = (|a||r|)+M = \overline{0}$. This means that $|a||r| \in M$ for all $a \in A$. Since $(M:A) = \{0\}$, we have $r = 0$. Thus G is a faithful A-ℓ-module.

<u>LEMMA 4.13.</u> If G is an A-ℓ-module, then $|ga| \leq |g||a|$ for all $g \in G$ and $a \in A$.

<u>PROOF.</u> $|ga| = |(g^+ - g^-)(a^+ - a^-)| \leq |g^+a^+| + |g^+a^-| + |g^-a^+| + |g^-a^-|$ $= g^+a^+ + g^+a^- + g^-a^+ + g^-a^- = (g^+ + g^-)(a^+ + a^-) = |g||a|$.

<u>THEOREM 4.14.</u> If an ℓ-ring A has a faithful, irreducible A-ℓ-module, then A is a prime ℓ-ring.

<u>PROOF.</u> Let G be a faithful, irreducible, A-ℓ-module, and I_1, I_2 be non-zero ℓ-ideals in A. Then, by the assumption that G is faithful, there are elements $u_1, u_2 \in G^+$ such that $u_1|i_1| = |u_1||i_1| \neq 0$,

for some $i_1 \in I_1$ and $u_2|i_2| = |u_2||i_2| \neq 0$ for some $i_2 \in I_2$. The set $<u_1I_1> = \{u \in G|\ |u| \leq |u_1a|$ for some $a \in I_1$ is an A-ℓ-submodule of G. Indeed if $h, k \in <u_1I_1>$ then $|h| \leq |u_1a|$, $|k| \leq |u_1b|$ for some $a, b \in I_1$, we have $|h-k| \leq |h| + |k| \leq |u_1a| + |u_1b| \leq |u_1||a| + |u_1||b| = |u_1|(|a|+|b|) = ||u_1|(|a| + |b|)| = |u_1(|a|+|b|)|$, where $|a| + |b| \in I_1$; and for every $r \in A$, $|hr| \leq |h||r| \leq |u_1a||r| \leq u_1(|a||r|) = |u_1(|a||r|)|$, $|a||r| \in I_1$. Since G is irreducible and $u_1I_1 \neq \{0\}$, we have $<u_1I_1> = G$. Similarly $<u_2I_2> = G$.

Choose $b \in I_2$ such that $u_2b \neq 0$. Since $u_2 \in G = <u_1I>$, there is an element $a \in I_1$ such that $|u_2| \leq |u_1a|$. Then $0 \neq |u_2b| \leq |u_2||b| \leq |u_1a||b| \leq u_1(|a||b|)$ and so $|a||b| \neq 0$ and $|a||b| \in I_1I_2$. Thus A is a prime ℓ-ring.

<u>COROLLARY 4.15.</u> ([18], 4.3.10) Let A be an ℓ-ring, then every ℓ-primitive ideal P is a prime ℓ-ideal.

<u>COROLLARY 4.16.</u> Every ℓ-primitive ring is a prime ℓ-ring.

<u>PROPOSITION 4.17.</u> Let A be a pseudo f-ring. If A has a faithful, irreducible A-ℓ-module, then A is totally-ordered ring without non-zero divisors of zero.

PROOF. Since A has a faithful, irreducible A-ℓ-module, by Theorem 4.14, A is a prime ℓ-ring. By Proposition 3.26 A is a totally-ordered ring without non-zero divisors of zero.

PROPOSITION 4.18. Let A be an ℓ-ring with a faithful, irreducible A-ℓ-module G and let $e \in G^{+}$ satisfy $eA = G$. Suppose A is a pseudo f-ring. Then the map $\alpha : A \to G$ defined by $\alpha(a) = ea$ is an ℓ-homomorphism of the totally-ordered additive group of A onto G. The kernel of this homomorphism, $I_e = \{a \in A \mid ea = 0\}$, is a maximal regular right ℓ-ideal of A.

PROOF. By Proposition 4.17 A is a totally-ordered ring without non-zero divisors of zero. If $a \in A^{+}$, then $ea \in G^{+}$. Since A is a totally-ordered ring, $a \wedge b = 0$ implies either $a = 0$ or $b = 0$. Hence either $ea = 0$ or $eb = 0$. Thus $ea \wedge eb = 0$. It is clear that α is an ℓ-group homomorphism of A onto G. The kernel I_e is an ℓ-subgroup of the ordered additive group of A, and I_e is a right ideal of A. Since G is irreducible we have that I_e a maximal right ℓ-ideal of A. The right ℓ-ideal I_e is a regular ideal, for there is an element $u \in A$ such that $eu = e$. Now from the Proposition 4.19 (below), $|e||u| = |e|$ and hence $e|u| = e$. For any $a \in A$ we have $e(a - |u|a) = ea - e|u|a = 0$ and so $a - |u|a \in I_e$. Thus $|u|$ is a left identity modulo I_e.

PROPOSITION 4.19. Let A be a pseudo f-ring. If G is a faithful irreducible A-ℓ-module, then G is a totally-ordered group and $|ga| = |g||a|$ for every $g \in G$ and every $a \in A$.

PROOF. G is a totally-ordered group, since it is a homomorphic image of the totally-ordered additive group A. The second statement is obvious.

LEMMA 4.20. Let A be a pseudo f-ring. Let G be a faithful, irreducible A-ℓ-module and suppose $e \in G^+$ satisfies $eA = G$. If $0 \neq g_1 \in G^+$, then $Ig_1 = \{a \in A | g_1 a = 0\} = I_e$.

PROOF. The set Ig_1 is a right ideal. It is also a right ℓ-ideal, for if $|b| \leq |a|$ with $a \in Ig_1$, then $0 \leq |g_1 b| \leq |g_1||b| \leq |g_1||a| = |g_1 a| = 0$ so $b \in Ig_1$. Since the set of all right ℓ-ideals forms a chain and I_e is the maximal right ℓ-ideal, we have $Ig_1 \subseteq I_e$.

Now if $Ig_1 \neq I_e$, then $g_1 I_e \neq \{0\}$. The set $<g_1 I_e> = \{a \in G | |g| \leq |g_1 a|$, for some $a \in I_e\}$ clearly forms an A-ℓ-submodule of G. Since $g_1 I_e \neq \{0\}$, we have $<g_1 I_e> = G$. Let $u \in A^+$ be a left identity modulo I_e. Then $u \notin I_e$ and $u \geq a$ for every $a \in I_e$, (since A is a totally-ordered ring, $0 < u < a$ would imply $u \in I_e$, a contradiction). Thus $u \notin Ig_1$, so $0 \neq g_1 \in G$ and there exists $a \in I_e$

such that $g_1 u \leq g_1 a$. Thus $g_1(a-u) \geq 0$ and so $a-u \geq 0$, i.e., $a \geq u$. But then $a = u$ so $u \in I_e$, (since $a \in I_e$), which is a contradiction.

<u>PROPOSITION 4.21.</u> Let A be a pseudo f-ring. If A has a faithful, irreducible, A-ℓ-module G, then A contains no non-zero proper right ℓ-ideal.

<u>PROOF.</u> By Proposition 4.18 A is a totally-ordered ring and I_e is the unique maximal right ℓ-ideal which is regular. Now if $a \in I_e$ then $g_1 a = 0$ for all $g_1 \in G^+$, since by Lemma 4.20 $Ig_1 = I_e$. We have then $ga = (g^+ - g^-)a = g^+a - g^-a - g^-a = 0$ for all $g \in G$ and so $|ga| = |g||a| = 0$. Since G is a faithful A-ℓ-module, $a = 0$. Thus $I_e = \{0\}$ and $\{0\}$ is a maximal right ℓ-ideal in A.

<u>COROLLARY 4.22.</u> Let A be a pseudo f-ring. If A has a faithful, irreducible A-ℓ-module, then A is a totally-ordered ring with identity and $\mathcal{R}(A) = \{0\}$.

<u>PROOF.</u> By Proposition 4.21 $\mathcal{R}(A) = \{0\}$ and by Proposition 4.17 A is a totally-ordered ring. Let e be a left identity modulo $\{0\}$, then e is a left identity of A. Since A contains no non-zero divisors of zero, e is a two-sided identity of A.

THEOREM 4.23. Let A be a pseudo f-ring. Then every maximal regular right ℓ-ideal is a two-sided ℓ-ideal.

PROOF. Let M be a maximal regular right ℓ-ideal of A. Then $(M:A) = P$ is the largest ℓ-ideal contained in M and A/P is an ℓ-primitive ring. Thus A/P has a faithful, irreducible A-ℓ-module. Since any homomorphic image of a pseudo f-ring is a pseudo f-ring ([6], 2.9, p54) we have A/P a pseudo f-ring. Hence by Proposition 4.21 A/P contains no non-zero proper right ℓ-ideals. Since M/P is a proper right ℓ-ideal in A/P, we have $M/P = \{0\}$, i.e., $M = P$.

From Theorem 4.23, we have that, in a pseudo f-ring A, $\mathfrak{R}(A)$ is a two-sided ℓ-ideal. Moreover, if M is a maximal regular right ℓ-ideal, then M is two-sided and $(M:A) \subseteq M$ implies that $(M:A) = M$. Hence M is an ℓ-primitive ℓ-ideal. Thus $\mathfrak{R}(A)$ is an intersection of ℓ-primitive ℓ-ideals.

THEOREM 4.24. Let A be a pseudo f-ring. Then the following are equivalent:

i) A is ℓ-primitive;

ii) A is ℓ-simple, totally-ordered and has an identity;

iii) A has a faithful, irreducible A-ℓ-module.

PROOF. By Proposition 4.12 (i) implies (iii) and by Proposition 4.21 and Corollary 4.22 (iii) implies (ii). Now prove (ii) implies (i).

Suppose A is ℓ-simple totally-ordered ring and has an identity, since $1 > 0$ every right ℓ-ideal is a regular right ℓ-ideal and $A^2 \neq \{0\}$. The set

$$\Lambda = \{I \subseteq A \mid \{0\} \subseteq I \subsetneq A, \ \ I \text{ a right } \ell\text{-ideal}\}$$

is not empty, since $\{0\} \in \Lambda$. If $\{I_\alpha \mid \alpha\in\Gamma\}$ is any chain of members of Λ, then $\bigcup_{\alpha\in\Gamma} I_\alpha \in \Lambda$, for if $\bigcup_{\alpha\in\Gamma} I_\alpha = A$, then $1 \in I_{\alpha_o}$ for some α_o, which is a contradiction. By Zorn's Lemma A has a maximal right ℓ-ideal. Let M be any maximal right ℓ-ideal, then $(M:A) = \{0\}$, for $(M:A)$ is a two-sided ℓ-ideal such that $(M:A) \subseteq M \neq A$. Hence $(M:A) = \{0\}$.

DEFINITION 4.25. A right ℓ-ideal I of an ℓ-ring A is called a *Von Neumann right ℓ-ideal* if for any $a \in A$ there exists $b \in A$ such that $aba \equiv a(I)$. An ℓ-ring A is called a *Von Neumann ℓ-ring* if $\{0\}$ is a Von Neumann ideal.

PROPOSITION 4.26. Every totally-ordered commutative Von Neumann ℓ-simple ring A with identity is a field.

PROOF. Suppose $0 \neq a \in A$ a non-unit. Then a is a zero divisor, for since A is Von Neumann there exists $u \in A$ such that $a(ua - 1) = 0$. Let $U = \{x \in A | ax = 0\}$, then $U \neq \{0\}$ and $U \neq A$. It is clear that U is an ℓ-ideal, a contradiction.

THEOREM 4.27. Let A be a commutative pseudo f-ring. Then A is Von Neumann and ℓ-primitive if and only if it is a field.

PROOF. The sufficiency is clear. Now prove the necessity. By Theorem 4.24 A is a totally-ordered ℓ-simple ring with identity. Again by Proposition 4.26 A is a field.

PROPOSITION 4.28. Every Von Neumann f-ring A is J-semisimple (i.e., $J(A) = 0$).

PROOF. Let A be an f-ring, then by Theorem 4.7 $J(A) = \mathcal{R}(A)$. If $a \in J(A)$ then $\langle a \rangle \subseteq J(A)$, since $J(A)$ is a two-sided ℓ-ideal, where $\langle a \rangle$ is the smallest ℓ-ideal contains a. Since A is a von Neumann, there exists $x \in A$ such that $axa = a$ and $(ax)(ax) = ax$. We have $e^2 = e = ax$ and $e \in \langle a \rangle \subseteq J(A)$, a contradiction ([13], p180, Corollary 28).

We now give an example to show that the condition "Von Neumann" is necessary in the Theorem 4.27 and the converse of Proposition 4.28 is not true.

EXAMPLE 4.29. Let Z be a ring of integers with usual ordered. Then Z is a commutative ℓ-simple totally-ordered ring with identity. It is not a field. Moreover, Z is J-semisimple but not von Neumann. The same example shows that a J-semisimple ring with descending chain condition for ℓ-ideals may not be a von Neumann ring.

It is well known in ring theory that every right maximal modular ideal of an arbitrary ring is a right von Neumann ideal [19]. But this is not the case for arbitrary ℓ-rings. For example the lexicographic ordering of $Z[\lambda]$, where Z is the ring of integers with usual order, has the unique maximal ℓ-ideal $\langle\lambda\rangle = \{a_o+a_1\lambda+\ldots+a_n\lambda^n \in Z[\lambda] \mid a_o = 0\}$. The maximal ℓ-ideal $\langle\lambda\rangle$ is not von Neumann ℓ-ideal. On the other hand the ring $Q[\lambda]$, with lexicographic order, is not a von Neumann totally-ordered ring but the unique maximal ℓ-ideal $\langle\lambda\rangle = \{a_o + a_1\lambda + \ldots + a_n\lambda^n \in Q[\lambda] \mid a_o = 0\}$ is a von Neumann ℓ-ideal.

As we have defined that for arbitrary ℓ-ring A, $\mathcal{R}(A)$ is the intersection of all maximal regular right ℓ-ideals of A and for the class of pseudo f-rings $\mathcal{R}(A)$ coincides with $J(A)$, the intersection of all ℓ-primitive ℓ-ideals of A. Now we will show that $\mathcal{R}(A)$, where A is the nxn matrix ring over an ℓ-ring R, is the collection of all matrices with entries from $\mathcal{R}(R)$.

Let R be an arbitrary ℓ-ring and A be the nxn matrix ring over R. If we define the positive cone of A to be $A^+ = \{(a_{ij}) \in A \mid a_{ij} \geq 0\}$, then A is a lattice-ordered ring. We have the following facts:

i) $(a_{ij}) \vee (b_{ij}) = (a_{ij} \vee b_{ij})$, $(a_{ij}) \wedge (b_{ij}) = (a_{ij} \wedge b_{ij})$;

ii) $|(a_{ij})| = (|a_{ij}|)$.

Throughout the rest of this chapter the ordering of the $n \times n$ matrix ring A over an ℓ-ring R, always means this ordering.

PROPOSITION 4.30. Let A be the $n \times n$ matrix ring $(n \geq 2)$ over a lattice-ordered ring R. Then A can never satisfy (β) or (γ) if $R^2 \neq \{0\}$, (α) if $R^3 \neq \{0\}$, nor can it satisfy (δ) if $R^5 \neq \{0\}$.

PROOF. i). We note that the condition (β) is equivalent to $|ab| = |a||b|$ for all a, b, in A. (see [5], Lemma 1, p58). Assume $(R)^2 \neq \{0\}$ and let a, b be in R so that $a \neq 0$, $b \neq 0$ and $ab \neq 0$. Take $x = (a_{ij})$, $y = (b_{ij})$, where $a_{11} = a$, $a_{12} = a$, $a_{ij} = 0$ for all other entries, $b_{11} = b$, $b_{21} = -b$, $b_{ij} = 0$ for all other entries. Then $|xy| = |(a_{ij})(b_{ij})| = |(c_{ij})|$ with $c_{11} = 0$; $|x||y| = (|a_{ij}|)(|b_{ij}|) = (d_{ij})$ with $d_{11} = ab + ab \neq 0$. Hence $|xy| \neq |x||y|$. Thus A does not satisfy the condition (β).

ii). Assume $(R)^2 \neq \{0\}$ and let $a \neq 0$, $b \neq 0$ such that $ab \neq 0$. Take $x = (a_{ij})$ with $a_{11} = a$, $a_{1n} = -b$, $a_{ij} = 0$ for all other entries. Then $x^+ = (a_{ij})$ with $a_{11} = a$ and all other entries zero and $x^- = (a_{ij})$ with $a_{1n} = b$ and all other entries over zero. We have $x^+x^- = (c_{ij})$ with $c_{1n} = ab \neq 0$.

iii) Assume $(R)^3 \neq \{0\}$ and let $a \neq 0$, $b \neq 0$, $c \neq 0$ with $abc \neq 0$. Let $x = (a_{ij})$ with $a_{11} = a$, $a_{1n} = -c$ and zero elsewhere, $y = (b_{ij})$ with $b_{11} = b$ and zero elsewhere, then $x^+ = (a_{ij})$ with $a_{11} = a$ and zero elsewhere, $x^- = (a_{ij})$ with $a_{1n} = b$ and zero elsewhere. We thus have $x^+yx^- = (d_{ij})$ with $d_{1n} = abc \neq 0$. Hence $x^+ A x^- \neq \{0\}$.

iv) Assume $(R)^5 \neq 0$ and let $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$ and $e \neq 0$ with $abcde \neq 0$. Let $x = (x_{ij})$ with $x_{11} = b$, $x_{1n} = -d$ and zero elsewhere; $y = (y_{ij})$ with $y_{11} = a$ and zero elsewhere; $z = (z_{ij})$ with $z_{11} = c$ and zero elsewhere; $w = (w_{ij})$ with $w_{nn} = e$ and zero elsewhere. Then $y\, x^+z\, x^-w = (v_{ij})$ with $v_{1n} = abcde \neq 0$. Hence A does not satisfy the condition (δ) by Lemma 2.1.

<u>PROPOSITION 4.31.</u> Let A be the nxn matrix ℓ-ring over an ℓ-ring R. Let I_i, $i = 1, 2, \ldots, n$, be right ℓ-ideals of R. Then $\bar{I} = \{(a_{ij}) \in A \mid a_{ij} \in I_i,\ i = 1, 2, \ldots, n,\ a_{ij} \in R\}$ is a right ℓ-ideal of A. Moreover, if each I_i is a regular ideal, then $\bar{I}$ is a regular ideal.

<u>PROOF.</u> It is clear that $\bar{I}$ is a ring right ideal of A. If $|(b_{ij})| \leq |(a_{ij})|$ where $(a_{ij}) \in \bar{I}$, then $(|b_{ij}|) \leq (|a_{ij}|)$ and so $|b_{ij}| \leq |a_{ij}|$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. Since I_i is ℓ-ideal for $i = 1, 2, \ldots, n$, we have $b_{ij} \in I_i$ and so $(b_{ij}) \in \bar{I}$. This shows that $\bar{I}$ is an ℓ-ideal. Now assume I_i is a regular ℓ-ideal with left identity $e_i > 0$, $i = 1, 2, \ldots, n$.

Let $\overline{e} = (e_{ij})$ where $e_{ii} = e_i$ and $e_{ij} = 0$ for $i \neq j$ (it is clear that $\overline{e} > \overline{0}$), then for every $(x_{ij}) \varepsilon A$ we have $(x_{ij}) - \overline{e}(x_{ij}) = (x_{ij} - e_i x_{ij}) \varepsilon \overline{I}$. thus $\overline{I}$ is a regular ℓ-ideal.

PROPOSITION 4.32. Let A be the nxn matrix ℓ-ring over an ℓ-ring R. If $\overline{I}$ is a right ℓ-ideal of A, then for fixed λ and μ, $(\overline{I})_{\lambda\mu} = \{b \varepsilon R | b = a_{\lambda\mu}$ is an entry in the (λ,μ)th position of a matrix (a_{ij}) in $\overline{I}\}$ is a right ℓ-ideal of R. Moreover, if $\overline{I}$ is a regular right ℓ-ideal of A, then $(\overline{I})_{\lambda\mu}$ is a regular right ℓ-ideal of R, $\lambda = 1, 2, \ldots, n$; $\mu = 1, 2, \ldots, n$.

PROOF. Let $a', b' \varepsilon (\overline{I})_{\lambda\mu}$. Then there exist two matrices $(a_{ij}), (b_{ij}) \varepsilon \overline{I}$ such that $a_{\lambda\mu} = a'$, $b_{\lambda\mu} = b'$. Since $(a_{ij} - b_{ij}) = (a_{ij}) - (b_{ij}) \varepsilon \overline{I}$ with $a' - b'$ in the $(\lambda-\mu)$th position of $(a_{ij} - b_{ij})$, we have $a' - b' \varepsilon (\overline{I})_{\lambda\mu}$. Now let $r \varepsilon R$ and $a' \varepsilon (\overline{I})_{\lambda\mu}$, let $(c_{ij}) \varepsilon A$ with $c_{\mu\lambda} = r$ and zero elsewhere and $(a_{ij}) \varepsilon \overline{I}$ with $a_{\lambda\mu} = a'$. Then $(d_{ij}) = (a_{ij})(c_{ij}) \varepsilon \overline{I}$ with $d_{\lambda\mu} = a'r$. Hence $a'r \varepsilon (\overline{I})_{\lambda\mu}$. This shows that $(\overline{I})_{\lambda\mu}$ is a right ring ideal of R for any λ and μ. To show that $(\overline{I})_{\lambda\mu}$ is a right ℓ-ideal, let $|b'| \leq |a'|$ with $a' \varepsilon (\overline{I})_{\lambda\mu}$, then there exists a matrix $(u_{ij}) \varepsilon \overline{I}$ with $u_{\lambda\mu} = a'$. Take (b_{ij}) to be the matrix such that $b_{ij} = a_{ij}$ for all $i \neq \lambda$, $j \neq \mu$ and $b_{\lambda\mu} = b'$. Then $|(b_{ij})| \leq |(a_{ij})|$ and since $\overline{I}$ is a right ℓ-ideal, we have $(b_{ij}) \varepsilon \overline{I}$. This implies that $b' \varepsilon (\overline{I})_{\lambda\mu}$ and hence $(\overline{I})_{\lambda\mu}$ is a right ℓ-ideal of R.

To show the second part, let us assume that $\overline{I}$ is a regular right ℓ-ideal of A with a left identity $\overline{e} = (e_{ij})(\overline{e} > \overline{0})$ modulo $\overline{I}$. For fixed λ and μ, we now show that $e_{\lambda\lambda} > 0$ is a left identity modulo $(\overline{I})_{\lambda\mu}$. Since $\overline{I}$ is regular for any (x_{ij}) in A we have $(x_{ij}) - \overline{e}(x_{ij}) = (x_{ij}) - (e_{ij})(x_{ij}) = (x_{ij} - \sum_{k=1}^{n} e_{ik}x_{kj}) \in \overline{I}$.

For any element x in R, let $\overline{x} = (x_{ij})$, where $x_{\lambda\mu} = x$ and x_{rs} is zero elsewhere, then $x_{\lambda\mu} - \sum_{k=1}^{n} e_{\lambda k}x_{k\mu} = x_{\lambda\mu} - e_{\lambda\lambda}x_{\lambda\mu}$ and this is in $(\overline{I})_{\lambda\mu}$. Hence $e_{\lambda\lambda}$ is a left identity modulo $(\overline{I})_{\lambda\mu}$. Thus $(\overline{I})_{\lambda\mu}$ is a regular right ℓ-ideal of R for all λ and μ.

<u>PROPOSITION 4.33.</u> Let A be the nxn matrix ℓ-ring over an ℓ-ring R. If M is a maximal regular right ℓ-ideal, then for a fixed λ, $\overline{M}_\lambda = \{(a_{ij}) | a_{\lambda j} \in M,\ a_{ij} \in R\}$ is a maximal regular right ℓ-ideal of A.

<u>PROOF.</u> By Proposition 4.31, $\overline{M}_\lambda$ is a regular right ℓ-ideal of A. Let $\overline{M}_\lambda \subsetneq \overline{M}'$, then there exists j_0 such that $M = (\overline{M}_\lambda)_{\lambda j_0} \subsetneq (\overline{M}')_{\lambda j_0}$. Since $(\overline{M}')_{\lambda j_0}$ is a regular right ℓ-ideal of R by Proposition 4.32, which properly contains M and so $(\overline{M}')_{\lambda j_0} = R$.

Now assume $e > 0$ is a left identity modulo M. Take $\overline{e} = (e_{ij}) \in \overline{M}'$ with $e_{\lambda j_0} = e$, zero elsewhere and for any $k \neq j_0$, let $\overline{h} = (d_{ij}) \in A$ with $d_{j_0 k} = e$, zero elsewhere. Then we have $\overline{e}\overline{h} = (e_{ij})(d_{ij}) = (g_{ij}) \in \overline{M}'$ with $g_{\lambda k} = e^2 \in (\overline{M})'_{\lambda k}$. Since $e^2 \notin M$ (for if so then $e \in M$, a contradiction) we get $(\overline{M})'_{\lambda k} = R$, for any $k \neq j_0$. Thus $\overline{M}' = A$ and so $\overline{M}_\lambda$ is a maximal regular right ℓ-ideal of A.

If $\bar{M}$ is a maximal regular right ℓ-ideal of A, then it is clear that for some λ, $(\bar{M})_{\nu\mu} = R$ for all $\nu \neq \lambda$, and for $\mu = 1,2,\ldots,n$.

THEOREM 4.34. Let $\bar{M}$ be a maximal regular right ℓ-ideal of A, and (for a fixed λ), let $(\bar{M})_{\nu\mu} = R$ for all $\nu \neq \lambda$, $\mu = 1, 2, \ldots, n$. Then for some maximal regular right ℓ-ideal M of R, $M = (\bar{M})_{\lambda\mu}$, $\mu = 1, 2, \ldots, n$.

PROOF. For any $r \in R$ and $m \in (\bar{M})_{\lambda j}$, for any i, let $\bar{m} = (m_{ij}) \in \bar{M}$, with $m = m_{\lambda j}$, and $\bar{r} = (r_{ij})$ with $r = r_{ji}$, zero elsewhere. Then $\bar{m}\bar{r} = (d_{ij}) \in \bar{M}$, with $mr = d_{\lambda i}$ and hence $(\bar{M})_{\lambda j}R \subseteq (\bar{M})_{\lambda i}$ for any i and j. We now show $\sum_{i=1}^{n} (\bar{M})_{\lambda i} \neq R$. For if $\sum_{i=1}^{n} (\bar{M})_{\lambda i} = R$, then $RR = (\sum_{i=1}^{n} (\bar{M})_{\lambda i})R \subseteq (\bar{M})_{\lambda 1}$ for all i. But then $e_{\lambda\lambda}^2 \in \bar{M}_{\lambda 1}$ where $e_{\lambda\lambda}$ is a left identity modulo $\bar{M}_{\lambda 1}$ by Proposition 4.32, which is impossible. Thus $\sum_{i=1}^{n} (\bar{M})_{\lambda i} \neq R$. Since every proper regular right ℓ-ideal is contained in a maximal regular right ℓ-ideal say N, we have $\sum_{i=1}^{n} (\bar{M})_{\lambda i} \subseteq N$. Hence $\bar{M} \subseteq \bar{N}$, where $(\bar{N})_{\lambda i} = N$, $i = 1, 2, \ldots, n$ and $(\bar{N})_{\nu i} = R$ for $\nu \neq \lambda$. Since $\bar{M}$ is a maximal regular right ℓ-ideal by assumption we have $\bar{M} = \bar{N}$, and $(\bar{M})_{\lambda\mu} = N$ for each μ.

THEOREM 4.35. Let A be the nxn matrix ℓ-ring over an ℓ-ring R. Let the $\mathcal{R}$-radical of R be $\mathcal{R}(R)$, then the $\mathcal{R}$-radical of A, $\mathcal{R}(A)$

is the set $\{(a_{ij}) \in A \mid a_{ij} \in \mathcal{R}(R)\} = [\mathcal{R}(R)]_{nxn}$.

<u>PROOF.</u> Since by Proposition 4.33, for any maximal regular right ℓ-ideal M of R, $\overline{M}_\lambda = \{(a_{ij}) \mid a_{\lambda j} \in M,\ a_{ij} \in R\}$ is a maximal regular right ℓ-ideal of A, we have $[\mathcal{R}(R)]_{nxn} = \bigcap_{\substack{\lambda=1,2,\ldots,n \\ M \in \Lambda}} \overline{M}_\lambda \supseteq \mathcal{R}(A)$, where Λ is the set of all maximal modular right ℓ-ideals of R. By Theorem 4.34 for any maximal regular right ℓ-ideal $\overline{M}$ of A, there exists a maximal regular right ℓ-ideal M of R and a λ such that $(\overline{M})_{\lambda\mu} = M$, and $(\overline{M})_{\nu\mu} = R$ for all $\nu \neq \lambda$, $\mu = 1, 2, \ldots, n$, we have $\bigcap_{\substack{\lambda=1,2,\ldots,n \\ M \in \Lambda}} \overline{M}_\lambda \subseteq \mathcal{R}(A)$. Hence $\mathcal{R}(A) = [\mathcal{R}(R)]_{nxn} = \bigcap_{\substack{\lambda=1,2,\ldots,n \\ M \in \Lambda}} \overline{M}_\lambda$.

CHAPTER V

DECOMPOSITION OF COMMUTATIVE PARTIALLY-ORDERED RINGS INTO A DIRECT SUM OF STRICT RINGS

In this chapter we will define an equivalence relation on the positive cone of a commutative partially-ordered ring without positive non-zero nilpotent elements. The equivalence classes will be called m-filets. These differ, in general, from Jaffard's filets. The m-filets so defined form a disjunctive and distributive lattice. The main theorem of this chapter is the following: Let A be a commutative partially-ordered ring which satisfies the condition that $x^2 = xy = y^2$ implies $x = y$ for all x, y in A^+. Then for A^+ to be o-isomorphic to a direct sum of a family of strict cones (of A) it is necessary and sufficient that (1) the lattice of m-filets be lattice isomorphic to the lattice of finite subsets of a set; (2) A^+ have Jaffard's property (Definition 5.16). Moreover, if A is directed then A is o-isomorphic to a direct sum of a family of strict rings.

In this chapter all rings are commutative partially-ordered rings without non-zero positive nilpotent elements, i.e., $x \in A^+$ $x^n = 0$ implies $x = 0$. The positive cone of A is $A^+ = \{x \in A | x \geq 0\}$; for each $a \in A^+$, we define the set $E(a) = \{y \in A^+ | ya = 0\}$, i.e., $E(a)$

is the annihilator set of a in A^+. We define a binary relation "$\sim$" on A^+ as follows: For $a, b \in A^+$, $a \sim b$ if and only if $E(a) = E(b)$. It is easy to see that "$\sim$" is an equivalence relation. Let F_A be the set of equivalence classes defined by this equivalence relation. As usual $\bar{a}$ will denote the class of a and will be called the m-filet or m-carrier of a. The prefix m is used to distinguish the role of multiplication in the definition. We introduce an order relation on the set F_A by defining $\bar{a} \leq \bar{b}$ if and only if $E(a) \supseteq E(b)$. clearly this relation is independent of the representatives a and b and F_A is an ordered set. It is clear that $\bar{0} \leq \bar{a}$ for all a in A^+ and $a \leq b$ implies $\bar{a} \leq \bar{b}$ for all a, b in A^+.

<u>PROPOSITION 5.1.</u> The partially-ordered set $(F_A, \leq)$ is a lattice where (1) $\bar{a} \wedge \bar{b} = \overline{ab}$ and (2) $\bar{a} \vee \bar{b} = \overline{a+b}$ for all a, b in A^+.

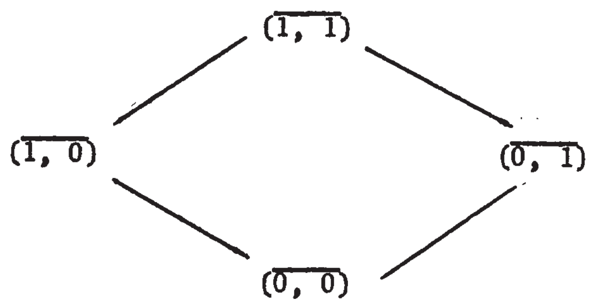
<u>PROOF.</u> (1) Since $E(ab) \supseteq E(a)$, $E(b)$ and so $\overline{ab} \leq \bar{a}$, $\bar{b}$. Now assume $\bar{x} \leq \bar{a}$, $\bar{b}$. Let $y \in E(ab)$ then $yab = o$ and so $ya \in E(b) \subseteq E(x)$. We have $yax = 0$ whence $yx \in E(a) \subseteq E(x)$. By definition $yx^2 = 0$ and so $(yx)^2 = y^2x^2 = 0$. Since A contains no non-zero positive nilpotent elements, we have $yx = 0$. Therefore $y \in E(x)$, and hence $E(ab) \subseteq E(x)$, i.e., $\bar{x} \leq \overline{ab}$. Thus $\bar{a} \wedge \bar{b} = \overline{ab}$.

(2) $\overline{a+b} \geq \overline{a}, \overline{b}$ is obvious, for $0 = x(a+b) = xa + xb$ implies $xa = xb = 0$, since the elements involved are non-negative. Now let $\overline{c} \geq \overline{a}$, $\overline{c} \geq \overline{b}$ and $x \in E(c)$. Since $E(c) \subseteq E(a) \cap E(b)$, it follows that $xa = xb = 0$ and so $x(a+b) = 0$, i.e., $x \in E(a+b)$. Thus $\overline{c} \geq \overline{a+b}$ and $\overline{a+b} = \overline{a} \vee \overline{b}$.

<u>EXAMPLE.5.2.</u> Let S be the semi-group consisting of two elements X and Y with the multiplication $XY = YX = X^2 = Y^2 = X$. The semigroup ring $A = Z(S)$ with positive cone $A^+ = \{aX + bY | a \geq 0, b \geq 0\}$ is a commutative ℓ-ring (lattice-ordered ring, see example 3.3), where Z is the ring of integers with the usual order. In this ring A^+ has no nilpotent elements but A itself has nilpotents. (For example, $X - Y$ is nilpotent).

<u>EXAMPLE 5.3.</u> Consider the group ring $A = Q(G)$, where Q is the ordered rational field and G any finite abelian group. The set $A^+ = \{ \sum_{g \in G} r_g g | r_g \geq 0\}$ is a cone in A, hence A is a partially-ordered ring without non-zero nilpotent elements, for the ring A is semi-simple. Let $0 \neq r = \sum r_g g \in A^+$ and $y = \sum s_g g \in A^+$ such that $ry = 0 = (\sum r_g g)(\sum s_g g)$. It is clear that all s_g must be zero. Hence $E(r) = \{0\}$ and so $\overline{r} = \overline{1}$ for all $r \neq 0$ in A^+. Thus the ring Q(G) has only two m-filets $\{\overline{1}, \overline{0}\}$ with $\overline{0} \leq \overline{1}$.

<u>EXAMPLE 5.4.</u> Let Z be the ring of integers with the usual order. The ordered direct product $A = Z \times Z$ is a partially-ordered ring with the positive cone $A^+ = \{(a_1, a_2) | a_1 \geq 0, \ a_2 \geq 0\}$. It is clear that A contains no non-zero nilpotent elements. This ring has four m-filets and F_A has the following structure.



F_A is ℓ-isomorphic to the lattice of subsets of a 2 element-set.

The above examples are characteristic of a general situation and are suggestive of the following definition.

<u>DEFINITION 5.5.</u> A partially-ordered ring A is a *strict ring* if the positive cone A^+ of A admits only the trivial zero divisor. A^+ itself will be called a *strict cone*.

<u>REMARKS 5.6.</u> (1) Examples 5.2 and 5.3 indicate that A may have non-zero zero divisors even though A^+ has none. (2) $\overline{x} = \overline{0}$ if and only if $x = 0$, for $x \notin E(x)$.

DEFINITION 5.7. A lattice-ordered set L with least element 0 is said to be *disjunctive lattice* if for any $x, y \in L$, $x \not\leq y$ then there exists $z \neq 0$ such that $z \leq x$ and $z \wedge y = 0$.

PROPOSITION 5.8. The lattice of m-filets $(F_A, \vee, \wedge)$ is (1) distributive and (2) disjunctive lattice.

PROOF. (1) $\bar{a} \wedge (\bar{b} \vee \bar{c}) = \bar{a} \wedge (\overline{b + c}) = \overline{a(b+c)} = \overline{ab + ac} = \overline{ab} \vee \overline{ac} = (\bar{a} \wedge \bar{b}) \vee (\bar{a} \wedge \bar{c})$.

(2) If $\bar{a}, \bar{b} \in F_A$ and $\bar{a} \not\leq \bar{b}$ then $E(a) \not\supseteq E(b)$; so there exists $d \in E(b)$ such that $d \notin E(a)$. Let $c = ad \neq 0$, then $\bar{c} \neq \bar{0}$. We have $\bar{c} \wedge \bar{b} = \overline{cb} = \bar{0}$, for $cb = adb = 0$. It is clear that $\bar{c} \leq \bar{a}$, since $x \in A^+$. $xa = 0$ implies $xc = xad = 0$.

DEFINITION 5.9. An m-filet $\bar{a} \in F_A$ is called a *minimal m-filet* or *atom* if it is not zero and $\bar{0} \leq \bar{x} < \bar{a}$ implies $\bar{x} = \bar{0}$.

The set of all minimal m-filets will be denoted by M. For $\bar{a} \in F_A$, let $M(\bar{a}) = \{\bar{b} \in M \mid \bar{b} \leq \bar{a}\}$.

PROPOSITION 5.10. (1) $M(\bar{a} \vee \bar{b}) = M(\bar{a}) \cup M(\bar{b})$. (2) $M(\bar{a} \wedge \bar{b}) = M(\bar{a}) \cap M(\bar{b})$.

PROOF. (1) That $M(\bar{a} \vee \bar{b}) \supseteq M(\bar{a}) \cup M(\bar{b})$ is trivial. Let $\bar{m} \in M(\bar{a} \vee \bar{b})$ then $\bar{m} = \bar{m} \wedge (\bar{a} \vee \bar{b}) = (\bar{m} \wedge \bar{a}) \vee (\bar{m} \wedge \bar{b})$. If $\bar{m} \notin M(\bar{a}) \cup M(\bar{b})$ then $\bar{m} \wedge \bar{a} = \bar{0}$, $\bar{m} \wedge \bar{b} = \bar{0}$, since $\bar{m}$ is a minimal m-filet, which is a contradiction. Hence $\bar{m} \in M(\bar{a}) \cup M(\bar{b})$. Thus $M(\bar{a} \vee \bar{b}) \subseteq M(\bar{a}) \cup M(\bar{b})$, and hence (1) follows.

(2) That $M(\bar{a} \wedge \bar{b}) \subseteq M(\bar{a}) \cap M(\bar{b})$ is trivial. Let $\bar{m} \in M(\bar{a}) \cap M(\bar{b})$ then $\bar{m} \wedge (\bar{a} \wedge \bar{b}) = (\bar{m} \wedge \bar{a}) \wedge \bar{b} = \bar{m} \wedge \bar{b} = \bar{m}$. Thus $\bar{m} \in M(\bar{a} \wedge \bar{b})$. Hence $M(\bar{a} \wedge \bar{b}) = M(\bar{a}) \cap M(\bar{b})$.

PROPOSITION 5.11. Let F_A be the lattice of m-filets of a ring A, then the following properties are equivalent:

(1) $\bar{a} \neq \bar{0}$ implies $M(\bar{a}) \neq \phi$.

(2) $M(\bar{a}) \subseteq M(\bar{b})$ implies $\bar{a} \leq \bar{b}$.

(3) $M(\bar{a}) = M(\bar{b})$ implies $\bar{a} = \bar{b}$.

(4) $M(\bar{a}) \cap M(\bar{b}) = \phi$ implies $\bar{a} \wedge \bar{b} = \bar{0}$.

(5) $\bar{a}$ is the l.u.b. of $M(\bar{a})$.

(6) The map $\theta : F_A \to 2^{M}$ defined by $\theta(\bar{a}) = M(\bar{a})$

is a one to one lattice homomorphism from F_A to the family of subsets of M.

PROOF. (1) implies (2). Suppose $\bar{a} \not\leq \bar{b}$, let $\bar{c} \in F_A$ such that $\bar{c} \neq \bar{0}$, $\bar{c} \leq \bar{a}$, $\bar{c} \wedge \bar{b} = \bar{0}$; then $M(\bar{c}) \neq \phi$ by (1). If $\bar{m} \in M(\bar{c})$ then $\bar{m} \leq \bar{c} \leq \bar{a}$ where $\bar{m} \in M(\bar{a})$ and $\bar{0} \leq \bar{m} \wedge \bar{b} \leq \bar{c} \wedge \bar{b} = \bar{0}$ shows $\bar{m} \notin M(\bar{b})$.

(2) implies (3). Trivial (3) implies (4). If $\bar{a} \wedge \bar{b} \neq \bar{0}$ then we have $M(\bar{a}) \cap M(\bar{b}) = M(\bar{a} \wedge \bar{b}) \neq M(\bar{0}) = \phi$.

(4) implies (5). Suppose $\bar{c} \geq \bar{m}$ for all $\bar{m} \in M(\bar{a})$ and $\bar{a} \not\leq \bar{c}$. Then by disjunctive property there exists $\bar{d} \neq \bar{0}$, $\bar{d} \leq \bar{a}$, $\bar{d} \wedge \bar{c} = \bar{0}$. Thus we have $\bar{d} \wedge \bar{a} = \bar{d} \neq \bar{0}$ and by (4) $M(\bar{d}) \cap M(\bar{a}) \neq \phi$. Let $\bar{m} \in M(\bar{d}) \cap M(\bar{a})$ since $\bar{m} \leq \bar{c}$ so $\bar{m} \leq \bar{c} \wedge \bar{d} = \bar{0}$ which is a contradiction. Hence $\bar{a} \leq \bar{c}$ and this shows that $\bar{a}$ is the l.u.b. of $M(\bar{a})$.

(5) implies (6). The map θ so defined is a lattice homomorphism, for $\theta(\bar{a} \wedge \bar{b}) = M(\bar{a} \wedge \bar{b}) = M(\bar{a}) \cap M(\bar{b})$, $\theta(\bar{a} \vee \bar{b}) = M(\bar{a} \vee \bar{b}) = M(\bar{a}) \cup M(\bar{b})$. Moreover, if $\theta(\bar{a}) = \theta(\bar{b})$ we have $\bar{a} = \text{Sup}\{\bar{m} | \bar{m} \in M(\bar{a})\} = \text{Sup}\{\bar{m} | \bar{m} \in M(b)\} = \bar{b}$.

(6) implies (1). This is trivial for the map θ is a one-to-one map.

PROPOSITION 5.12. Let F_A be the lattice of m-filets of a ring A. Consider the conditions:

(1) for any $\bar{b}, \bar{a}_i \in F_A$, $i = 1, 2, \ldots$, then $\bar{a}_1 \leq \bar{a}_2 \leq \cdots \leq \bar{b}$ implies $\bar{a}_n = \bar{a}_{n+1} = \ldots$, for some n;

(2) i) for any $\bar{a}_i \in F_A$, $i = 1, 2, \ldots,$ and $\bar{a}_1 \geq \bar{a}_2 \geq \ldots \geq \bar{0}$ implies $\bar{a}_n = \bar{a}_{n+1} = \ldots,$ for some n, and

ii) if $\bar{a}, \bar{b}, \bar{d} \in F_A$, where $\bar{a} \geq \bar{b} \geq \bar{d}$, then there exists $\bar{c} \in F_A$ such that $\bar{c} \vee \bar{b} = \bar{a}$, $\bar{c} \wedge \bar{b} = \bar{d}$;

(3) if $\bar{a} \neq \bar{0}$ then $M(\bar{a})$ is a non-empty finite set;

(4) F_A is isomorphic to the set of all finite subsets of a set.

Then (1) implies (2) and (1), (3) and (4) are equivalent.

PROOF. (1) implies (2). We prove the second part ii) first. It is sufficient to prove for the case $\bar{d} = \bar{0}$, for if $\bar{c}' \in F_A$ such that $\bar{c}' \wedge \bar{b} = \bar{0}$, $\bar{c}' \vee \bar{b} = \bar{a}$ then take $\bar{c} = \bar{c}' \vee \bar{d}$ we have $\bar{c} \wedge \bar{b} = (\bar{c}' \vee \bar{d}) \wedge \bar{b} = (\bar{c}' \wedge \bar{b}) \vee (\bar{d} \wedge \bar{b}) = \bar{d}$, $\bar{c} \vee \bar{b} = \bar{c}' \vee \bar{d} \vee \bar{b} = (\bar{c}' \vee \bar{b}) \vee \bar{d} = \bar{a} \vee \bar{d} = \bar{a}$. Now let $\bar{a} \geq \bar{b} \geq \bar{0}$. Since $\bar{a} \not\leq \bar{b}$ (if $\bar{a} = \bar{b}$, take $\bar{c} = \bar{0}$) by disjunctive property there exists $\bar{c}_1 \neq 0$, $\bar{c}_1 \leq \bar{a}$, $\bar{c}_1 \wedge \bar{b} = \bar{0}$. Let $\bar{b}_1 = \bar{c}_1 \vee \bar{b}$ and if $\bar{b}_1 \neq \bar{a}$ then $\bar{a} \not\leq \bar{b}_1$ ($\because$ $\bar{a} \geq \bar{b}_1$). Again by disjunctive property there exists $\bar{c}_2 \neq \bar{0}$, $\bar{c}_2 \leq \bar{a}$, $\bar{c}_2 \wedge \bar{b}_1 = \bar{0}$. Let $\bar{b}_2 = \bar{c}_2 \vee \bar{b}_1 = (\bar{c}_2 \vee \bar{c}_1) \vee \bar{b} \leq \bar{a}$. By continuing this process we will have $\bar{b}_1 \leq \bar{b}_2 \leq \bar{b}_3 \leq \ldots \leq \bar{a}$. The ascending sequence is strict, for if $\bar{b}_n = \bar{b}_{n+1}$ then $\bar{c}_{n+1} \leq \bar{b}_n$ ($\because$ $\bar{c}_{n+1} \wedge \bar{b}_n = \bar{0}$ implies $\bar{c}_{n+1} = \bar{0}$) a contradiction, and so $\bar{b}_n = \bar{a}$ for some n. By (1) there exists $\bar{c} \in F_A$ such that $\bar{c} \vee \bar{b} = \bar{a}$ and $\bar{c} \wedge \bar{b} = \bar{0}$. There is a similar argument to prove (2) i). Let $\bar{a}_1 \geq \bar{a}_2 \geq \ldots \geq \bar{0}$. For any $\bar{a}_n$, $\bar{a}_1 \geq \bar{a}_n \geq 0$, there exists $\bar{b}_n$ such that $\bar{b}_n \wedge \bar{a}_n = \bar{0}$, $\bar{b}_n \vee \bar{a}_n = \bar{a}_1$. Since

$\bar{b}_n = \bar{b}_n \wedge \bar{a}_1 = \bar{b}_n \wedge (\bar{b}_{n+1} \vee \bar{a}_{n+1}) = (\bar{b}_n \wedge \bar{b}_{n+1}) \vee (\bar{b}_n \wedge \bar{a}_{n+1}) \leq (\bar{b}_n \wedge \bar{b}_{n+1}) \vee (\bar{b}_n \wedge \bar{a}_n) = (\bar{b}_n \wedge \bar{b}_{n+1}) \leq \bar{b}_{n+1}$, so $\bar{b}_n = \bar{b}_n \wedge \bar{b}_{n+1} \leq \bar{b}_{n+1}$. We have $\bar{b}_2 \leq \bar{b}_3 \leq \dots \leq \bar{a}_1$, and by (1) $\bar{b}_n = \bar{b}_{n+1} = \dots$, for some n. This implies that $\bar{a}_n = \bar{a}_{n+1} = \dots$, for the complement is unique.

(1) implies (3). The condition (2) i) implies $M(\bar{a}) \neq \phi$ for all $\bar{a} \neq \bar{0}$. Now if $M(\bar{a})$ is infinite say $\{\bar{m}_1, \bar{m}_2, \dots\}$, then $\bar{m}_1 < \bar{m}_1 \vee \bar{m}_2 < \bar{m}_1 \vee \bar{m}_2 \vee \bar{m}_3 < \dots \leq \bar{a}$ contradict to (1), hence the set $M(\bar{a})$ is finite.

(3) implies (4). Since (3) implies the condition (1) in Proposition 5.11, and so the map $\theta: F_A \to 2^M$ define by $\theta(\bar{a}) = M(\bar{a})$ in (6) of Proposition 5.11, is one-to-one. The image is just the family of the finite subsets of the set M. Thus F_A is isomorphic to the family of finite subsets of a set.

(4) implies (1). Trivial.

Let A be a partially-ordered ring with psoitive cone A^+. We are interested in commutative rings satisfying the following condition:

(*) $x^2 = xy = y^2$ implies $x = y$ for all $x, y \in A^+$.

<u>PROPOSITION 5.13.</u> (1) Let A be a commutative partially-ordered ring which has no non-zero nilpotents, then A satisfies condition (*). (2) A commutative partially-ordered ring with (*) condition has no positive nilpotent elements.

PROOF. (1) If $x^2 = xy = y^2$ then $(x - y)^2 = x^2 - 2xy + y^2 = 0$. Since the ring A has no non-zero nilpotent elements, we have $x - y = 0$. Thus $x = y$.

(2) i). If $a^2 = 0$, $a \geq 0$ then $a^2 = a \cdot 0 = 0^2$ and by (*) condition we have $a = 0$. ii). If $a^3 = 0$, then $a^4 = 0$. Now if $a^4 = 0$, then $(a^2)^2 = a^2 0 = 0^2$, and by (*) condition we have $a^2 = 0$. Thus $a = 0$ by i). iii). Now prove by induction on the exponent of a. Assume $a^k = 0$ implies $a = 0$ for all $n \geq k (n \geq 3)$. Assume $a^{n+1} = 0$. Let $i = n + 1$ if $n + 1$ is even, $i = n + 2$ if $n + 1$ is not even, then $(a^{i/2})^2 = (a^{i/2}) \cdot 0 = 0^2$. By induction hypothesis $a^{i/2} = 0$. Hence $a^{n+1} = 0$ implies $a = 0$ for all n.

EXAMPLE 5.14. Let $A = \{a + bX | a, b \in Z\}$ where $X^2 = 0$. If we define the addition componentwise and multiplication as usual, where Z is the ring of integers with the usual order, then A is a commutative ring. The set $A^+ = \{a + bX | a > 0\} \cup \{0\}$ is a positive cone for A and A is a partially-ordered ring. The ring A has no nilpotent elements in A^+ but A itself has infinitely many nilpotent elements and A satisfies the (*) condition. The ring so defined is a direct ring (see the following definition).

DEFINITION 5.15. A partially-ordered ring is said to be *direct* if for any a, b in A, there exists an element c in A such that $c \geq a$, $c \geq b$.

We note that a partially-ordered ring A is directed if and only if $A = A^+ - A^+$, Where A^+ is the positive cone of A. (See [15], p18).

<u>DEFINITION 5.16.</u> We say a commutative partially-ordered ring A has *Jaffard's property* if the set M of minimal m-filets of A is non-empty and satisfies the following condition: to every $f \in A^+$ and $\bar{a} \in M$ there exists $f_{\bar{a}} \in A^+$ such that i) $f_{\bar{a}} \leq f$, ii) $\bar{f}_{\bar{a}} \leq \bar{a}$, iii) $\overline{f-f_{\bar{a}}} \wedge \bar{a} = \bar{0}$. $f_{\bar{a}}$ will be called the *Jaffard projection* of f with respect to $\bar{a}$.

<u>PROPOSITION 5.17.</u> Let A be a partially-ordered (commutative) ring with positive cone A^+. If A has Jaffard's property and satisfies condition (*), then the Jaffard projection of f with respect to the minimal m-filet $\bar{a}$ is uniquely determined.

<u>PROOF.</u> Suppose $f_{\bar{a}}$ and g satisfy i), ii), and iii). Then $f = g + h$ with $g \leq f$, $\bar{g} \leq \bar{a}$ and $\bar{h} \wedge \bar{a} = \bar{0}$. Since $\bar{f}_{\bar{a}} \leq \bar{a}$ and from $\overline{f-f_{\bar{a}}} \wedge \bar{a} = \bar{0}$, we have $\overline{f-f_{\bar{a}}} \wedge \bar{f}_{\bar{a}} = \bar{0}$, whence $(f-f_{\bar{a}})f_{\bar{a}} = 0$. Similarly $gh = 0$. From $\bar{h} \wedge \bar{a} = \bar{0}$ and $\bar{f}_{\bar{a}} \leq \bar{a}$ we have $\bar{f}_{\bar{a}} \wedge \bar{h} = \overline{f_{\bar{a}}h} = \bar{0}$, thus $f_{\bar{a}} \cdot h = 0$. Similarly $(f-f_{\bar{a}})g = 0$. If we multiply $f = g+h = f_{\bar{a}} + (f-f_{\bar{a}})$ by g we obtain $g^2 = f_{\bar{a}}g$. Similarly we obtain $f_{\bar{a}}g = f_{\bar{a}}^2$. By (*) condition we have $g = f_{\bar{a}}$.

LEMMA 5.18. Let A be a commutative partially-ordered ring with positive cone A^+. For $\bar{a} \in M$ the set $P_{\bar{a}} = \{x \in A^+ | \bar{x} \leq \bar{a}\}$ is a strict cone of A.

PROOF. (i) $0 \in P_{\bar{a}}$. (ii) Since $P_{\bar{a}} \subseteq A^+$, $P_{\bar{a}} \cap (-P_{\bar{a}}) = \{0\}$. (iii) If $x, y \in P_{\bar{a}}$, then $\overline{x + y} = \bar{x} \vee \bar{y} \leq \bar{a}$. Hence $x + y \in P_{\bar{a}}$. Thus $P_{\bar{a}} + P_{\bar{a}} \subseteq P_{\bar{a}}$. (iv) If $x, y \in P_{\bar{a}}$, then $\overline{xy} = \bar{x} \wedge \bar{y} \leq \bar{a}$. Hence $xy \in P_{\bar{a}}$. Thus $P_{\bar{a}} \cdot P_{\bar{a}} \subseteq P_{\bar{a}}$. (v) If $x, y \in P_{\bar{a}}$ such that $xy = 0$ then $\bar{x} \wedge \bar{y} = \overline{xy} = \bar{0}$. Now if both $\bar{x}, \bar{y} = \bar{a}$ then $a = 0$ this is impossible, since $\bar{a} \in M$, and hence $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$, i.e., $x = 0$ or $y = 0$.

Note that $P_{\bar{a}}$ is also a module cone. For if $r \in A^+$, $x \in P_{\bar{a}}$, then $\overline{rx} = \bar{r} \wedge \bar{x} \leq \bar{r} \wedge \bar{a} \leq \bar{a}$. Hence $rx \in P_{\bar{a}}$.

Let $\{P_i\}$, $i \in I$ be a family of cones in a partially-ordered ring A. A non-empty subset $R \subseteq A^+$ is the direct sum of cones $\{P_i\}$, $i \in I$, denoted by $R = \sum_{i \in I} P_i$, if for each $r \in R$, there exist $p_i \in P_i$ $i \in I$, $p_i \neq 0$ only for a finite number of $i \in I$ such that $r = \sum p_i$ and the expression is unique.

THEOREM 5.19. Let A be a commutative partially-ordered ring satisfying the condition (*). For A^+ to be o-isomorphic to the direct sum of a family of strict cones (of A) it is necessary and sufficient that (1) the lattice of m-filets be lattice isomorphic to the lattice of finite subsets of a set S, and (2) A^+ have Jaffard's property. Moreover, if A is directed then A is o-isomorphic to the direct sum of a family of strict rings.

PROOF. We prove the sufficiency first. By assumption $M \neq \phi$ and from Lemma 5.18, $P_{\bar{a}}$ is a strict cone for every $\bar{a} \in M$. We now show that the sum $\sum_{\bar{a} \in M} P_{\bar{a}}$ is direct. Suppose $x \in P_{\bar{a}} \cap P_{\bar{b}}$ where $\bar{a} \neq \bar{b}$ then $\bar{x} \leq \bar{a}$, $\bar{x} \leq \bar{b}$, whence $\bar{x} \leq \bar{a} \wedge \bar{b} = \bar{0}$, this implies that $x = 0$. Now if $x = x_{\bar{a}} + x_{\bar{b}} + \ldots + x_{\bar{d}} = y_{\bar{a}} + y_{\bar{b}} + \ldots + y_{\bar{d}}$, then $x_{\bar{a}} \in P_{\bar{a}}$ and so $\overline{x_{\bar{a}}} \leq \bar{a}$. It is clear that $x_{\bar{a}} \leq x$. Also $\overline{x - x_{\bar{a}}} \wedge \bar{a} = \overline{x_{\bar{b}} + \ldots + x_{\bar{d}}} \wedge \bar{a} = (\overline{x_{\bar{b}}} \vee \ldots \vee \overline{x_{\bar{d}}})\ \bar{a} = (\overline{x_{\bar{b}}} \wedge \bar{a}) \vee \ldots$

$\vee (\overline{x_{\bar{d}}} \wedge a)$, since F_A is a distributive lattice by Proposition 5.8. As $\bar{a}, \bar{b}, \ldots \bar{d} \in M$, we have $\overline{x_{\bar{b}}} \wedge \bar{a} \leq \bar{b} \wedge \bar{a} = \bar{0}$, ..., $\overline{x_{\bar{d}}} \wedge \bar{a} \leq \bar{d} \wedge \bar{a} = \bar{0}$. Hence $\overline{x - x_{\bar{a}}} \wedge \bar{a} = \bar{0}$. This shows that $x_{\bar{a}}$ is a Jaffard Projection of x with respect to $\bar{a}$. Similarly $y_{\bar{a}}$ is a Jaffard Projection of x with respect to $\bar{a}$. Since the projection is unique by Proposition 5.17, we have $x_{\bar{a}} = y_{\bar{a}}$ and so the sum $\sum_{\bar{a} \in M} P_{\bar{a}}$ is direct. It is clear that $\sum_{\bar{a} \in M} P_{\bar{a}} \subseteq A^+$. Now we want to show $A^+ \subseteq \sum_{\bar{a} \in M} P_{\bar{a}}$. Let $0 \neq y \in A^+$, then $\bar{y} \in F_A$. By (1) there exists only a finite number of minimal m-filets $\{\bar{a}_i\}_{i=1}^{n}$ such that $\bar{a}_i \leq \bar{y}$. By Jaffard's property to each $\bar{a}_i$ corresponds $y_i \in A^+$ such that $y_i \leq y$, $\bar{y}_i \leq \bar{a}_i$ and $\overline{y - y_i} \wedge \bar{a}_i = \bar{0}$. Now proceed by induction on n. If $n = 1$ then $\bar{y} \in M$ and so $y \in P_{\bar{y}}$. Now assume the proposition holds for $n - 1$ and y has n minimal filets $\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n\}$ such that $\bar{y} \geq \bar{a}_i$, $i = 1, 2, \ldots, n$. For $\bar{a}_n \in M$ and $y \in A^+$, by Jaffard's property there exists $y \in A^+$ such that $y_n \leq y$, $\bar{y}_n \leq \bar{a}_n$, $\overline{y - y_n} \wedge \bar{a}_n = \{0\}$. Since $0 \leq y - y_n \leq y$ we have $\overline{y - y_n} \leq \bar{y}$. We have $M(\overline{y - y_n}) \subsetneq M(\bar{y})$ and by induction hypothesis $y - y_n \in \sum P_{\bar{a}}$. Hence $y \in \sum P_{\bar{a}}$, since $y_n \in P_{\bar{a}_n}$.

Consequently $y \in \sum_{\bar{a} \in M} P_{\bar{a}}$ for all $y \in A^+$.

If $y, z \in \sum P_{\bar{a}}$, then $(y + z)_{\bar{a}} = y_{\bar{a}} + z_{\bar{a}}$, $\bar{a} \in M$, since $\sum P_{\bar{a}}$ is a direct sum of cones. To show that $y, z \in \sum P_{\bar{a}}$ implies $(yz)_{\bar{a}} = y_{\bar{a}} z_{\bar{a}}$ for all $\bar{a} \in M$, let $(yz)_{\bar{a}}$, $y_{\bar{a}}$, $z_{\bar{a}}$ be the unique projections of yz, y, z respectively, $y = y_{\bar{a}} + (y - y_{\bar{a}})$, $z = z_{\bar{a}} + (z - z_{\bar{a}})$, $yz = y_{\bar{a}} z_{\bar{a}} + y_{\bar{a}}(z - z_{\bar{a}}) + (y - y_{\bar{a}}) z_{\bar{a}} + (y - y_{\bar{a}})(z - z_{\bar{a}})$.

Since $\overline{y_{\bar{a}} z_{\bar{a}}} = \bar{y}_{\bar{a}} \wedge \bar{z}_{\bar{a}} \leq \bar{a}$ and so $y_{\bar{a}} z_{\bar{a}} \in P_{\bar{a}}$. $\overline{y_{\bar{a}}(z - z_{\bar{a}})} = \bar{y}_{\bar{a}} \wedge (\overline{z - z_{\bar{a}}})$

and this is equal to $\bar{a} \wedge (\overline{z - z_{\bar{a}}})$ or $\bar{0} \wedge (\overline{z - z_{\bar{a}}})$. We know in either case it is zero. Hence $y_{\bar{a}}(z - z_{\bar{a}}) = 0$. Similarly $z_{\bar{a}}(y - y_{\bar{a}}) = 0$. Now $\overline{(y - y_{\bar{a}})(z - z_{\bar{a}})} = (\overline{y - y_{\bar{a}}}) \wedge (\overline{z - z_{\bar{a}}}) \neq \bar{a}$, for otherwise $\bar{0} = (\overline{y - y_{\bar{a}}}) \wedge (\overline{z - z_{\bar{a}}}) \wedge \bar{a} = \bar{a} \wedge \bar{a} = \bar{a}$, a contradiction. Hence $(y - y_{\bar{a}})(z - z_{\bar{a}}) \notin P_{\bar{a}}$. We claim that $u = (y - y_{\bar{a}})(z - z_{\bar{a}})$ has no contribution to $P_{\bar{a}}$, for if this is not the case let us call it $u_{\bar{a}}$; then $u_{\bar{a}} \leq u$, $\bar{u}_{\bar{a}} \leq \bar{a}$, since $\bar{u}_{\bar{a}} \leq \bar{u}$ we have $\bar{0} \leq \bar{u}_{\bar{a}} \leq \bar{u} \wedge \bar{a} =$

$\overline{(y - y_{\bar{a}})(z - z_{\bar{a}})} \wedge \bar{a} = \overline{(y - y_{\bar{a}})} \wedge \overline{(z - z_{\bar{a}})} \wedge \bar{a} = \bar{0}$. Thus $u_{\bar{a}} = 0$. We have $(yz)_{\bar{a}} = y_{\bar{a}} z_{\bar{a}}$. Hence the sum is a ring direct sum of cones.

Now assume that A is directed; $A = A^+ - A^+$. The set $D_{\bar{a}} = P_{\bar{a}} - P_{\bar{a}}$ is a ring for all $\bar{a} \in M$. Indeed $x = x_1 - x_2$, $y = x_3 - x_4$, $x_1, x_2, x_3, x_4 \in P_{\bar{a}}$, then $x + y = (x_1 + x_3) - (x_2 + x_4)$ and $xy = (x_1x_3 + x_2x_4) - (x_2x_3 + x_1x_4)$ belongs to $D_{\bar{a}}$. In fact $D_{\bar{a}}$ is an ideal of A. It is clear $A = \sum_{\bar{a} \in M} D_{\bar{a}}$. To see that the sum $\sum_{\bar{a} \in M} D_{\bar{a}}$ is direct, let $\lambda \in D_{\bar{a}} \cap \sum_{\substack{\bar{b} \in M \\ \bar{b} \neq \bar{a}}} D_{\bar{b}}$, then $\lambda = x_{\bar{a}} - y_{\bar{a}} = \sum_{\bar{b} \neq \bar{a}} (x_{\bar{b}} - y_{\bar{b}})$

where $x_{\bar{a}}, y_{\bar{a}} \in P_{\bar{a}}$ and $x_{\bar{b}}, y_{\bar{b}} \in P_{\bar{b}}$. We have $x_{\bar{a}} + y_{\bar{b}} + \dots + y_{\bar{c}} = y_{\bar{a}} + x_{\bar{b}} + \dots + x_{\bar{c}}$. It follows that $x_{\bar{a}} = y_{\bar{a}}, \dots, x_{\bar{c}} = y_{\bar{c}}$, for the expression in unique. Thus $\lambda = 0$ and so $D_{\bar{a}} \cap \sum_{\substack{\bar{b} \in M \\ \bar{b} \neq \bar{a}}} D_{\bar{b}} = 0$. Let

$u = u_1 - u_2$, $v = v_1 - v_2$, $u_1, u_2, v_1, v_2 \in A^+$, then $uv = u_1v_1 + u_2v_2 - (v_1u_2) - (v_2u_1)$. For any $x \in A$ if we let $x_{\bar{a}}$ be the unique component of x in $D_{\bar{a}}$, $\bar{a} \in M$, then $(u \pm v)_{\bar{a}} = u_{\bar{a}} \pm v_{\bar{a}}$ for any $u, v \in A$. Hence $(uv)_{\bar{a}} = (u_1v_1)_{\bar{a}} + (u_2v_2)_{\bar{a}} - (v_1u_2)_{\bar{a}} - (u_1v_2)_{\bar{a}} = (u_1)_{\bar{a}}(v_1)_{\bar{a}} + (u_2)_{\bar{a}}(v_2)_{\bar{a}} - (v_1)_{\bar{a}} - (u_2)_{\bar{a}} - (u_1)_{\bar{a}}(v_2)_{\bar{a}} = u_{\bar{a}}v_{\bar{a}}$. Since $P_{\bar{a}}$ is a cone for $D_{\bar{a}}$, $\bar{a} \in M$, hence $D_{\bar{a}}$ is a partially-ordered ring for every $\bar{a} \in M$. Thus $A = \sum_{\bar{a} \in M} D_{\bar{a}}$ a direct sum of partially-ordered rings with direct product order. It is clear that the positive elements correspond to the positive elements.

Now prove the necessity. (1) Let A_i, $i \in I$ be strict rings and let $A = \sum_{i \in I} A_i$. The positive cone of A is $A^+ = \{(r_i)_{i \in I} \mid r_i \in A_i, r_i \geq 0\}$.

It is clear that A is a partially-ordered ring without non-zero positive nilpotnet elements. Let $r = (r_i)_{i\varepsilon I} \varepsilon A$, and define $p_i : A \to A_i$ by $p_i(r) = r_i \varepsilon A_i$. We also define supp(r) to be $\{i \varepsilon I | p_i(r) \neq 0\}$. Let $r = (r_i)_{i\varepsilon I}$, $s = (s_i)_{i\varepsilon I}$ be in A^+. Since each A_i is strict, $E(r) \subseteq E(s)$ if and only if $p_i(s) \neq 0$ implies $p_i(r) \neq 0$ for all $i \varepsilon I$. We have $\overline{r} \geq \overline{s}$ if and only if $p_i(s) \neq 0$ implies $p_i(r) \neq 0$ for all $i \varepsilon I$, or equivalently $\overline{r} \geq \overline{s}$ if and only if $\text{Supp}(r) \supseteq \text{Supp}(s)$. Hence the lattice of m-filets of A is lattice isomorphic to the lattice of the family of finite subsets of I. (2) We observe that $\overline{r}$ is a minimal m-filet if and only if there is $i \varepsilon I$ such that $p_i(r) \neq 0$ and $p_j(r) = 0$ for all $i \neq j$. Let us denote $M = \{\delta_i\}_{i\varepsilon I}$. Now let $f \varepsilon A^+$, $\overline{\delta}_j \varepsilon M$. Case (I) if $\overline{\delta}_j \not\leq \overline{f}$ take $f_{\overline{\delta}_j} = 0$. It is clear that $0 \leq f$, $\overline{0} \leq \overline{\delta}_j$ and $\overline{f} \wedge \overline{\delta}_j = \overline{0}$. Case (II) if $\overline{\delta}_j \leq \overline{f}$, we may assume $p_i(\overline{\delta}_j) = r_i = p_i(f)$ for those $i \varepsilon I$ such that $r_i \neq 0$. In this case we take $f_{\overline{\delta}_j} = \delta_j$. Thus $\delta_j \leq f$, $\overline{\delta}_j = \delta_j$ and $\overline{f - \delta_j} \wedge \overline{\delta}_j = \overline{0}$. This completes the proof of the theorem (The same proof works for the case $A^+ = \sum_{i\varepsilon I} (A^+)_i$).

<u>LEMMA 5.20</u>. Let A be an f-ring without positive nilpotent elements. Then $a \wedge b = 0$ if and only if $ab = 0$ for all a, b in A^+.

PROOF. Since A is an f-ring, we have $a \wedge b = 0$ implies $ab = 0$. Conversely, if $ab = 0$ then $0 \leq a \wedge b \leq a, b$ whence $0 \leq (a \wedge b)^2 \leq ab = 0$. Thus $a \wedge b = 0$.

In a commutative f-ring without positive nilpotent elements, the definition of m-filet is equivalent to the definition of filet for lattice-ordered abelian group, ([15], p32). Thus $P_{\bar{a}} = T = \{b \in A | \bar{b} \leq \bar{a}, b \geq 0\}$ (See [15], p37), and $G(a) = T \cup (-T) = P_{\bar{a}} - P_{\bar{a}} = D_{\bar{a}}$ a totally-ordered ring for every minimal filet $\bar{a}$.

We have the following Corollary.

COROLLARY 5.21. Let A be a commutative f-ring without positive nilpotent elements. Then A has Jaffard's property and the lattice of m-filets is lattice isomorphic to the lattice of finite subsets of a set S if and only if A is o-isomorphic to a direct sum of totally-ordered rings without non-zero divisors of zero.

REFERENCES

1. Anderson, F.W., *On f-rings with the ascending chain condition*, Proc. Amer. Math. Soc., 13 (1962), 715-721.

2. Anderson, F.W., *Lattice-ordered rings of quotients*, Can. J. Math., 8 (1956), 335-361.

3. Birkhoff, G., *On the structure of abstract algebras*, Proc. Camb. Phil. Soc., 31 (1935), 432-454.

4. Birkhoff, G., *Lattice Theory*, Rev. ed., Colloquium Publication no. 25, Amer. Math. Soc., New York, 1948.

5. Birkhoff, G. and Pierce, R.S., *Lattice-ordered rings*, An. Acad. Brasil. Ci., 28 (1956), 41-69.

6. Diem, J.E., *A radical for lattice-ordered rings*, Thesis, Purdue University, (1965).

7. Diem, J.E., *A radical for lattice-ordered rings*, Pacific J. Math. 25 (1968) 71-82.

8. Fuchs, L., *Partially-ordered Algebraic Systems*, Addison-Wesley Publishing Company, Reading, Mass., 1963.

9. Heinriksen, M. and Johnson, D.G., *On the structure of a class of archimedean lattice-ordered algebras,* Fund. Math., 50 (1961), 73-94.

10. Heinriksen, M. and Isbell, J.R., *Lattice-ordered rings and functions rings,* Pacific J. Math. 12 (1962), 533-565.

11. Jacobson, N., *Structure of Rings,* Colloquium Publication no. 37, Amer. Math. Soc., Providence, 1956.

12. Jaffard, P., *Contribution à L'Étude des Groupes Ordonnês,* Journal des Math. Pures et Appl., 32 (1953), 203-280.

13. Johnson, D.G., *A structure theory for a class of lattice-ordered rings,* Acta Math., 104 (1960), 163-215.

14. Pierce, R.S., *Radicals in function rings,* Duke J. Math., 71 (1949), 823-833.

15. Ribenboim, P., *Theorie des groupes ordonnes,* Universided Nacional del sur, Bhaia Blanca, 1963.

16. Ribenboim, P., *On ordered modules,* J.f.d. reine u. angew. Math. 225 (1967), 120-146.

17. Ribenboim, P., *Rings and Modules,* Interscience Publishers, a division of John Wiley & Sons, 1967.

18. Steinberg, S.L., *Lattice-ordered rings and modules,* Thesis, University of Illinois (1970).

19. Thierrin, G., *Quelques caractérisations du radical d'un anneau*, Can. Math. Bull. vol. 10, no. 5, (1967), 643-647.

20. Viswanathan, T.M., *Order modules of fractions*, J. f. d. reine u. angew. Math. 235 (1969), 78-107.

21. Weinberg, E.C., *On the scarcity of lattice-ordered matrix rings*, Pacific J. Math., vol. 19, no. 3, (1966), 561-571.

22. Weinberg, E.C., *Lecture notes on ordered algebraic systems*, University of Illinois, 1968.