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# Contributions To Structural Inference And Behrens-fisher Problem

Kiong Doong Ling

# CONTRIBUTIONS TO STRUCTURAL INFERENCE AND BEHRENS-FISHER PROBLEM

by

Kiong-Doong Ling
Department of Mathematics

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# ABSTRACT

The well-known Behrens-Fisher problem is concerned with statistical inference about the difference between the means of two independently distributed normal populations. This problem has been studied by many people using different methods of inference. The present thesis considers the Behrens-Fisher problems in the light of the structural method of inference for the following different cases:

Case (i): two independent normal populations with no assumptions on the standard deviations;

Case (ii): two independent normal populations under the condition that the ratio of standard deviations is known;

Case (iii): bivariate normal population with no assumption on the covariance matrix;

Case (iv): bivariate normal population under the condition that both the correlation coefficient and the ratio of the standard deviations are known;

Case (v): a generalization to $k(\geq 3)$ independent normal populations with known ratios of standard deviations;

Case (vi): a multivariate generalization to two independent multivariate normal populations (having the same number of components) with no assumptions on the covariance matrices; and

Case (vii): a related Behrens-Fisher problem of obtaining the structural distribution for the difference of two location parameters of two independently distributed negative exponential populations.

In addition, the present thesis also deals with the distributions of

(i) the maximum likelihood estimators of the correlation coefficients when samples arise from bivariate normal distributions having (a) equal variances; and (b) equal means and equal variances; and

(ii) the maximum likelihood estimators of the correlation matrices when samples arise from multivariate normal distributions with zero means when (a) variances are equal; and (b) a fixed number of variances have a same unknown value while the remaining ones are equal to a different unknown value.

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# CHAPTER 1

# INTRODUCTION

1.1. SUMMARY: In this chapter, a brief description of the Bayesian method of inference, the Fiducial method of inference, and the Structural method of inference is given. Since the present thesis is concerned with the Structural method of inference, this method is discussed in greater details. A comparison of these three methods of inference, and a brief introduction to the well-known Behrens-Fisher problem are also given. Finally the problems dealt with in this thesis are stated briefly in the last section of this chapter.

1.2. BAYESIAN METHOD OF INFERENCE: Let $x_1, x_2, \ldots, x_n$ be a set of observations from a continuous random variable X having a distribution depending on an unknown parameter $\theta$. In the literature several different methods have been employed to make inference about the unknown parameter $\theta$. They are the usual standard method of inference -- which includes point estimation, confidence interval and testing of hypothesis, the Bayesian method of inference, the Fiducial method of inference, and the Structural method of inference. The last three methods of inference provide distribution for the unknown parameter $\theta$, and this distribution is used as the basis of inference about the unknown parameter $\theta$.

The Bayesian method of inference, using Bayes' Theorem, was developed by Jeffreys (1948). It assumes an *a priori* distribution of the unknown parameter, and views statistical inference as a method of combining the *a priori* distribution with the sample information to arrive at an *a posteriori* distribution for the unknown parameter. Therefore the problem of statistical inference, using Bayesian method, is effectively by stating the *a posteriori* distribution and so the choice of an *a priori* distribution for the unknown parameter becomes the main object of the Bayesian method of inference.

Suppose now X is a random variable whose probability density function (pdf in short) $f(x|\theta)$ depends on an unknown parameter $\theta$. If $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ is a set of observations from X, and $p(\theta)$ is an *a priori* density for $\theta$, then the *a posteriori* distribution $g(\theta|\underset{\sim}{x})$ of $\theta$ is proportional to

$$p(\theta)f(\theta|\underset{\sim}{x}) = p(\theta) \prod_{i=1}^{n} f(x_i|\theta)$$

where $f(\theta|\underset{\sim}{x})$ is the likelihood function for $\theta$. The constant of proportionality can be easily obtained by integrating out $\theta$ over the parameter space.

1.3. FIDUCIAL METHOD OF INFERENCE: The Fiducial method of inference was introduced by Fisher in 1930. The main object of the method is to derive the Fiducial distribution of the unknown parameter without assuming any *a priori* distribution for the parameter. But the application of Fiducial method of inference is restricted to variables whose distributions must belong to Koopman-Darmoris class of distributions. The Fiducial distribution for the unknown parameter is usually obtained by means of a pivotal quantity. A pivotal quantity is a function of a sufficient statistic and the parameter that has a fixed distribution independent of the true value of the parameter. The three-step procedure of arriving at a Fiducial distribution is summarized as follows:

(i) choose a pivotal quantity with a fixed distribution associated with it;

(ii) substitute the observed value of the sufficient statistic into the pivotal quantity; and

(iii) transfer this distribution of the pivotal quantity to that of the parameter.

As an example, suppose $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ is a set of observations from normal distribution $N(\mu, 1)$. Then $t = \bar{x}-\mu$, $\bar{x}$ the sample mean, is a pivotal quantity which is distributed according to normal distribution $N(0, 1/n)$. The Fiducial distribution for the parameter $\mu$ is thus distributed according to normal distribution $N(\bar{x}, 1/n)$.

The Fiducial distribution for the parameter is the basis of inference about the parameter in the Fiducial method of inference.

1.4. STRUCTURAL METHOD OF INFERENCE: The Structural method of inference was introduced by Fraser in a series of papers (1961a, b, 1964a,b, 1966, 1967a,b) and in a book (1968). The basis of inference in this method is the error variable, introduced to describe the relationship between the response variable and the unknown parameter. The point of view taken is, that a response variable x, obtained from a process operating under stable conditions, is derived from an unknown transformation $\theta$, operating on realized but unknown value of an error variable e. The error variable e, describing the unidentified sources of variation of the process -- the internal error of the system -- is assumed to have a known distribution on the space $\mathcal{X}$. The transformation $\theta$ belongs to a unitrary group of transformations on $\mathcal{X}$. (A group G of transformations on $\mathcal{X}$ is called unitrary if $g_1x = g_2x$ implies $g_1 = g_2$ for any $g_1, g_2 \varepsilon G$, and $x \varepsilon \mathcal{X}$.) The response variable x and the realized error e are related by the following equation

$$x = \theta e. \tag{1.4.1}$$

The above description can be summarized by the structural model

$$\begin{cases} x = \theta e \\ f(e)de \end{cases} \tag{1.4.2}$$

The structural model has two parts: (i) an error variable having a known distribution on the space $\mathcal{X}$; and (ii) the structural equation (1.4.1) describing the relationship of a realized value e from the error variable, the known response x, and the unknown quantity $\theta$ in the unitrary group G of transformations on $\mathcal{X}$.

The following definitions and assumption are essential for the analysis of the structural model (1.4.2).

<u>Definition</u>: The orbit Gx of a response value x is the set of all the pre-images of x under all the transformations of G:

$$\begin{aligned} Gx &= \{g^{-1}x : g\varepsilon G\} \\ &= \{gx : g\varepsilon G\} . \end{aligned}$$

The orbit of x gives the information that the values of e which could have given rise to the response x.

<u>Definition</u>: A transformation [x] from the space $\mathcal{X}$ to the group is called a transformation variable if

$$[gx] = g[x]$$

for all $g\varepsilon G$ and $x\varepsilon\mathcal{X}$.

The [x]'s can be considered as a new coordinate of the points x on the orbit Gx. Furthermore, a transformation variable [x] defines a reference point

$$D(x) = [x]^{-1}x$$

on the orbit Gx. Note that

$$\begin{aligned} D(x) &= [x]^{-1}x \\ &= [x]^{-1}g^{-1}gx \\ &= (g[x])^{-1}gx \\ &= [gx]^{-1}gx \\ &= D(gx). \end{aligned}$$

Thus reference point D(x) on each of the orbit Gx is uniquely determined by the transformation variable [x], and so the set of all reference points indexes the class of all orbits.

The relation

$$x = [x]D(x)$$

shows that every point x' on the orbit Gx can be obtained from the reference point by a transformation in G; and it also indicates that a transformation variable [x] can alternately be defined by first choosing a reference point D(x) on the orbit, and then letting [x] be the unique transformation in G that transforms the point D(x) to x.

Assumption: $\mathcal{X}$ is an open subset in Euclidean space $R^n$; G is an open subset in Euclidean space $R^L$, $L \leq n$; and the transformations

$$\tilde{g} = gh, \quad \tilde{x} = ghx$$

are continuously differentiable with respect to g, h and x.

The assumption implies that G is a locally compact topological group, endowed with the usual topology inherited from $R^L$.

Invariant Differentials: The use of invariant differentials in analysing the structural model is very helpful. The existence of invariant measures is guaranteed by the above assumption. A measure $\mu(\cdot)$ on the group G is said (Halmos (1950)) to be a left invariant measure (left Haar measure) if

$$\mu(A) = \mu(gA)$$

for all elements $g$ in $G$ and all Borel sets $A$ contained in $G$; and where $gA$ is defined as follows:

$$gA = \{g\theta : g \varepsilon G\} .$$

The uniqueness of left invariant measure, unique in the sense that any two left invariant measures differ by a constant, was established in measure theory. For a left invariant measure $\mu(\cdot)$, a unique right invariant measure (right Haar measure) $\nu(\cdot)$ can be defined:

$$\nu(A) = \mu(A^{-1})$$

where

$$A^{-1} = \{g^{-1} : g \varepsilon A\}$$

for any Borel set $A$ in $G$. For a given transformation $g$, let $\mu_g(\cdot)$ be a new measure defined by

$$\mu_g(A) = \mu(Ag)$$

for every Borel set $A$ in $G$ and where $Ag$ is the Borel set

$$\{\theta g : \theta \varepsilon A\} .$$

This measure, constructed from the left invariant measure $\mu(\cdot)$ and the transformation $g$, can be easily shown to satisfy the property of being a left invariant measure. Therefore by the uniqueness property we know that the measures $\mu_g(\cdot)$ and $\mu(\cdot)$ differ only by a constant, which of course depends on $g$ as follows:

$$\Delta(g) = \mu_g(A)/\mu(A)$$
$$= \mu(Ag)/\mu(A).$$

This positive real-valued function $\Delta(\cdot)$ defined on G is called the modular function of the group G. The following properties of the modular function and the its relationship between invariant measures can be easily established:

$$\Delta(i) = 1,\ \Delta(gh) = \Delta(g)\Delta(h),\quad \Delta(g^{-1}) = \Delta(g)^{-1};$$
$$\nu(gA) = \Delta(g)^{-1}\nu(A), \text{for all Borel set A in G,}$$

where i is the identity element of the group G. The relationship between the left and the right invariant measures can also be expressed by means of differentials:

$$d\mu(\cdot) = \Delta(\cdot)d\nu(\cdot),\ \text{and}\ d\nu(\cdot) = \Delta(\cdot)^{-1}d\mu(\cdot)\ .$$

Let $m(\cdot)$ be an invariant measure defined on the space $\mathcal{X}$ such that

$$m(gB) = m(B)$$

for all Borel sets B in $\mathcal{X}$ and g in the group G. The terminology "invariant differentials" is used for the invariant measures constructed from the volume elements dx, dg and the Jacobian of the transformations in G. A detail discussion on the construction of invariant differentials has been given by James (1954) and Fraser (1968).

Suppose now the error variable e has a density f with respect to the invariant measure $m(\cdot)$ on the space $\mathcal{X}$:

$$f(e)dm(e) .$$

The conditional distribution of [e] given the orbit Gx (=Ge), labelled by its reference point D(x), can be derived from invariant properties as:

$$K(D(x))f([e]D(x))d\mu[e] . \tag{1.4.3}$$

Note that there exists an one-to-one correspondence between points on the orbit Gx and elements in the group G of transformations. The structural distribution for $\theta$ given the orbit Gx, or simply the response x, can now be obtained by transferring the density (1.4.3) for [e] on Gx to the corresponding element on G:

$$K(D(x))f(\theta^{-1}x)\Delta(\theta^{-1}[x])d\mu(\theta), \tag{1.4.4}$$

since

$$\begin{aligned} d\mu[e] &= d\mu(\theta^{-1}[x]) \\ &= \Delta([x])d\mu(\theta^{-1}) \\ &= \Delta([x])d\nu(\theta) \\ &= \Delta(\theta^{-1}[x])d\mu(\theta) . \end{aligned}$$

If the density f of the error variable e is given, with respect to the Lebesgue measure, as f(e)de, then the expression (1.4.4) becomes

$$K(D(x))f(\theta^{-1}x)J_N(\theta^{-1}x)\Delta(\theta^{-1}[x])d\mu(\theta) \tag{1.4.5}$$

where

$$J_N(x) = \left|\frac{\partial[x]x'}{\partial x'}\right|_{x'=D(x)}$$

is used a compensating factor to produce the invariant differential $dm(\cdot)$ on $\mathcal{X}$.

The uniqueness of structural distribution on the group space for a given structural model has been pointed out by Fraser; but a detailed proof of this property is not given anywhere. A proof of the uniqueness property will be given in Chapter 2.

1.5. A COMPARISON OF THE THREE METHODS OF INFERENCE: It is well-known that for a given *a periori* distribution $p(\theta)$ for the parameter $\theta$, the Bayesian method of inference leads to a unique *a posteriori* distribution $g(\theta|x)$ which is used as the basis of statistical inference about $\theta$. Therefore the difficulty of the Bayesian method of inference lies in the choice of a particular *a priori* distribution to represent the prior knowledge about $\theta$. Different *a priori* distributions could lead to different *a posteriori* distributions. The Fiducial method of inference could also lead to a multiplicity of Fiducial distributions based on the same set of data depending on various choices of the pivotal quantity. Mauldon (1955) has provided an example where infinitely many different pivotal quantities could be chosen to give infinitely many different Fiducial distributions. Examples in which the Fiducial distribution is not unique is also given by Tukey (1957).

The relationship between the Fiducial and the Bayesian methods of inference was obtained by Grundy (1956) and Lindley (1958). Grundy provides a class of one-parameter distributions, for which the sample sum is a sufficient statistic for the parameter in sample of any size. He proves that the resulting Fiducial distribution for the parameter does not coincide with the *a posteriori* distribution, derived by the Bayesian method, for any given *a priori* distribution. Lindley shows that a Fiducial distribution is equivalent to *a posteriori* distribution if, and only if the random variable X with parameter $\theta$ can be transformed to Y and $\mu$ respectively so that $\mu$ is the location parameter of Y. In this case the Fiducial distribution is the same as the Bayesian *a posteriori* distribution obtained by using uniform *a priori* distribution for $\mu$.

The following criteria has been proposed by Lindley (1958) and Sprott (1960) for the investigation of the consistency of Fiducial distributions:

Criterion I (Lindley): A Bayesian analysis for a first sample, using the Fiducial distribution obtained from a second sample as *a priori* distribution, should yield a result coincide with the Fiducial distribution obtained directly from the combined sample;

Criterion II (Sprott): A Bayesian analysis for a sample, using the Fiducial distribution obtained from another sample as *a priori* distribution, should yield a result independent of the order of the combination; and

Criterian III (Sprott): Consider two independent distributions both involving the same parameter. A Bayesian analysis for a sample from one distribution, using the Fiducial distribution obtained from a sample from the other distribution as *a priori* distribution, should yield a result independent of the order of combination.

An example is provided by Lindley showing the Criterion I, and hence Criterion II or III, is not fulfilled.

Lindley and Sprott considered random variable X whose distribution belongs to Koopman-Darmois class of distributions. They showed that the Fiducial distribution is consistent, under the above three criteria, if and only if, the random variable X can be transformed to a random variable which has either a normal distribution with location parameter or a gamma distribution with scale parameter.

It will be shown in Chapter 2 that structural distributions of these parameters evidently satisfy all the three criteria mentioned above.

Like the Fiducial method of inference, the Structural method of inference has its limitation too. For some problems, it is not possible to find a proper structural model, so that they cannot be solved by this method. Generalization of the structural method of inference has been studied by Fraser (1962, 1966, 1970). Applications of the structural distributions have appeared in papers by Haq (1968), Fraser and Haq (1969), Maxwell (1969) and Whitney (1970).

Confidence sets for the parameter based on Fiducial method, Bayesian method and Structural method are in general different from the confidence intervals of Neyman (1937). This has been mentioned by Kendall and Stuart (1961).

We would like to conclude this section with a remark by Lehmann (1959) -- "Statistical inference is concerned with methods of using this observational material to obtain information concerning the distribution of X or the parameter $\theta$ with which it is labelled. ... The need for statistical analysis stems from the fact that the distribution of X, and hence some aspect of the situation underlying the mathematical model, is not known".

1.6. BEHRENS-FISHER PROBLEM: Let $\underset{\sim}{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})$, $i = 1,2$, be samples of size $n_i (\geq 2)$ drawn from independent normal populations $X_i$ with respective means $\mu_i$ and standard deviations $\sigma_i (> 0)$. The problem of making statistical inference about the difference between the means was first posed by W. Behrens in 1929. In the literature this problem is well-known as the Behrens-Fisher problem.

Fisher (1935) proposed the so-called "Behrens-Fisher Distribution" as a solution to the problem of estimating and testing the difference between the means of two independent normal populations with different standard deviations. A Behrens-Fisher distribution is a distribution of a random variable which is a linear combination of two independently distributed random variables both having Student's t-distributions with not necessarily equal numbers of degrees of freedom. His argument was essentially based on the Fiducial method of inference. For the case in which the standard deviations of the populations are equal, the Fiducial distribution for the difference of the means has been shown to be a modified student's t-distribution.

Jeffreys (1948) used the Bayesian method of inference to arrive at the same results. A review on the Behrens-Fisher problem and its Bayesian solution was given by Patil (1964). Brown (1967) uses a method which he calls as "secondarily Bayes' method" to obtain a solution to the Behrens-Fisher problem. He assumes *a priori* distributions for nusiance parameters and obtain estimates from the *a posteriori* distributions of the desired statistics which are induced by the *a priori* distributions.

The problem of testing the hypothesis of equality of two means of two independent normal random variables has received a great deal of attention. Wald (1955) proposed four criteria for determination of non-randomized critical regions, or tests, and showed that for equal sample size, the critical regions must satisfy

$$\left| \frac{\sqrt{n}(\bar{x}_1 - \bar{x}_2)}{(s_1^2+s_2^2)^{1/2}} \right| > \phi(s_1^2/s_2^2)$$

where $\bar{x}_i = \sum_{j=1}^{n} x_{ij}/n$, $s_i^2 = \sum_{j=1}^{n} (x_{ij}-x_i)^2$, and $\phi(\cdot)$ is a function to be determined. For large sample sizes he had proven that the constant function $\phi(s_1^2/s_2^2) = c$ yields the asymptotically most powerful unbiased region. Test of this form has also been examined by Fisher (1939) and Welch (1949). Extension of Wald's result to the case of unequal sample sizes is considered by Romanovskaja (1965). McCullough, Gurland and Rossenberg (1960), and Gurland and McCullough (1962) have employed a preliminary test of equality of variance before proceeding to test of equality of means. Analytic theory of tests for the Behrens-Fisher problem have been examined by Linnik (1963a,b, 1965, 1966), Linnik and Salaevskii (1963), Linnik, Romanovskii and Sudakov (1964), and many others. The multivariate analogue to Sceffe's (1943) solution for the Behrens-Fisher problem is given by Anderson (1957). Structurally, the Behrens-Fisher problem has been studied by Fraser (1961a,b).

The present thesis is mainly concerned with the solution of the Behrens-Fisher problem by the Structural method under different situations. These problems are stated in the next section.

1.7. STATEMENT OF PROBLEMS: The present thesis deals with the following variations of the Behrens-Fisher problem.

(A) Behrens-Fisher Problem-Independent Populations: Let $\underset{\sim}{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n_1})$ and $\underset{\sim}{x}_2 = (x_{21}, x_{22}, \ldots, x_{2n_2})$ be samples from independent normal distributions $X_1$ and $X_2$ with respective means $\mu_1$ and $\mu_2$, and standard deviations $\sigma_1$ and $\sigma_2$. Corresponding to these samples we can associate the following structural models (measurement models):

$$\begin{cases} x_{ij} = \mu_i + \sigma_i e_{ij}, \quad j = 1, 2, \ldots, n_i, \\ \prod_{j=1}^{n_i} [(2\pi)^{-1/2} \exp\{-e_{ij}^2/2\} de_{ij}] \end{cases}$$

for each i = 1, 2. This problem deals with the structural distribution for the difference of the means, $\mu = \mu_1 - \mu_2$, say, based on the complete sets of observations $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, for the following two cases:

(i) with no condition on the standard deviations; and

(ii) under the condition that the ratio $\sigma = \sigma_2/\sigma_1$ of the standard deviations is known.

A related Behrens-Fisher problem for two independent negative exponential distributions is also examined. Let the distribution $X_1$ and $X_2$ be negative exponential distributions with respective location parameters $\mu_1$ and $\mu_2$, and scale parameters $\sigma_1$ and $\sigma_2$. The structural distribution for the difference of location parameters $\mu = \mu_1 - \mu_2$, based on Type II censored observations, are considered for the following three cases:

(i) when both scale parameters $\sigma_1$ and $\sigma_2$ are known;

(ii) when the ratio $\sigma_2/\sigma_1$ of the scale parameters is known; and

(iii) when both scale parameters $\sigma_1$ and $\sigma_2$ are unknown.

(B) A Generalization of the Behrens-Fisher Problem - Independent Populations: This problem is a generalization to the problem (A). For each i = 1, 2, ..., k, let $\underset{\sim}{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})$ be a sample of size $n_i (\geq 2)$ from normal distribution $X_i$ with mean $\mu_i$ and standard deviation $\sigma_i$. The distributions $x_i$'s are assumed to be mutually independent. Or equivalently we consider the following structural models:

$$\begin{cases} x_{ij} = \mu_i + \sigma_i e_{ij}, \; j = 1, 2, \ldots, n_i \\ \prod_{j=1}^{n_i} f_i(e_{ij}) de_{ij} \end{cases}$$

where

$$f_i(t) = (2\pi)^{-1/2} \exp\{- t^2/2\}$$

for each i = 1, 2, ..., k. The main object of this problem is to obtain the joint structural distribution for the (k-1) differences of two means:

$$\mu_i^* = \mu_1 - \mu_i, \; i = 2, 3, \ldots, k,$$

under the condition that the (k-1) ratios

$$\sigma_i^* = \sigma_i / \sigma_1, \; i = 2, 3, \ldots, k,$$

of the corresponding standard deviations are known. The case where k = 3 is discussed in greater details.

(C) Multivariate Behrens-Fisher Problem: Let $\underset{\sim}{x}^{(i)} = (x_{1\alpha}^{(i)}, x_{2\alpha}^{(i)}, \ldots, x_{p\alpha}^{(i)})$, $\alpha = 1, 2, \ldots, n_i$, $i = 1, 2$, be samples from two independent p-variate normal distributions with means vector $\underset{\sim}{\mu}_i$ and covariance matrix $\sum_i$. These samples can be considered structurally as:

$$\begin{cases} X_i = \theta E_i \\ f(E_i)dE_i = (2\pi)^{-n_i p/2} \exp\{-\sum_{j,\alpha}(e_{j\alpha}^{(i)})^2/2)\} \prod_{j,\alpha} de_{j\alpha}^{(i)} \end{cases}$$

where

$$X_i = \begin{pmatrix} 1 & \cdots & 1 \\ x_{11}^{(i)} & \cdots & x_{1n_i}^{(i)} \\ \vdots & & \vdots \\ x_{p1}^{(i)} & \cdots & x_{pn_i}^{(i)} \end{pmatrix}, \quad E_i = \begin{pmatrix} 1 & \cdots & 1 \\ e_{11}^{(i)} & \cdots & e_{1n_i}^{(i)} \\ \vdots & & \vdots \\ e_{p1}^{(i)} & \cdots & e_{pn_i}^{(i)} \end{pmatrix}$$

and

$$\theta_i = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mu_{i1} & c_{11}^{(i)} & \cdots & c_{1p}^{(i)} \\ \vdots & \vdots & & \vdots \\ \mu_{ip} & c_{p1}^{(i)} & \cdots & c_{pp}^{(i)} \end{pmatrix} = \begin{pmatrix} 1 & \underset{\sim}{0} \\ \underset{\sim}{\mu}_i & c_i \end{pmatrix}, \quad i = 1, 2,$$

where $|c_i| > 0$, is an element of the positive affine group of transformations on $R^p$. The problem is to derive the structural distribution for the difference of the two mean vectors, $\underset{\sim}{\mu} = \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2$, based on the complete sets of observations $\underset{\sim}{x}_{\alpha}^{(i)}$, $\alpha = 1, 2, \ldots, n_i$, $i = 1, 2$.

(D) Behrens-Fisher Problem - Dependent Populations: Let $(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})$ be sample of size $n(\geq 2)$ from a bivariate normal distribution $(X_1, X_2)$ with mean vector $\underset{\sim}{\mu}' = (\mu_1, \mu_2)$ and covariance matrix

$$\begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix} .$$

The associated structural model corresponding to the set of observations is

$$\left\{ \begin{array}{l} \begin{pmatrix} 1 \\ x_{1j} \\ x_{2j} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma_1 & \gamma \\ \mu_2 & \alpha & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_{1j} \\ e_{2j} \end{pmatrix} , \quad j=1,2,\ldots,n, \\ (2\pi)^{-n}\exp\{-\frac{1}{2}\sum_{j=1}^{n}(e_{1j}^2+e_{2j}^2)\}\prod_{j=1}^{n} de_{1j}de_{2j} \end{array} \right.$$

where the submatrix

$$\begin{pmatrix} \sigma_1 & \gamma \\ \alpha & \sigma_2 \end{pmatrix}$$

with $\sigma_1, \sigma_2 > 0$, has determinant $>0$ (i.e. $\sigma_1\sigma_2-\alpha\gamma>0$). This problem is concerned with the structural distribution for the difference of the means $\mu = \mu_1 - \mu_2$, based on the complete set of observations, for the following two cases:

(i) with no assumption on the covariance matrix of $(X_1, X_2)$; and

(ii) under the condition that both the correlation coefficient $\rho$ and the ratio $\sigma_2/\sigma_1$ of the standard deviations are known.

Note that for case (ii), we have $\alpha = 0$, and the pdf of the error variables $(e_{11}, e_{21}), (e_{12}, e_{22}), \ldots, (e_{1n}, e_{2n})$ is replaced by

$$\prod_{j=1}^{n} \{[2\pi(1-\rho^2)^{1/2}]^{-1}\exp(-\frac{1}{2(1-\rho^2)}[e_{1j}^2-2\rho e_{1j}e_{2j}+e_{2j}^2])de_{1j}de_{2j}\}.$$

In addition, this thesis also deals with the distributions of correlation coefficients and correlation matrices as indicated below.

(E) <u>Distributions of Some Correlation Coefficients</u>: Let $(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})$ be a sample of size n from a bivariate normal distribution $(X_1, X_2)$ with means $\mu_1$ and $\mu_2$, and covariance matrix

$$\begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}$$

where $|\rho| < 1$. The maximum likelihood estimator (MLE for short) for $\rho$ is

$$\hat{\rho} = 2\sum(x_{1j}-\bar{x}_1)(x_{2j}-\bar{x}_2)/\sum[(x_{1j}-\bar{x}_1)^2+(x_{2j}-\bar{x}_2)^2]$$

where

$$n\bar{x}_i = \sum x_{ij}, \quad i = 1, 2$$

and $\sum = \sum_{j=1}^{n}$ .

If we know that the means of the two marginal distribution are equal, i.e., $\mu_1 = \mu_2$, then the MLE for $\rho$ becomes

$$\rho^* = 2\sum(x_{1j} - \bar{x})(x_{2j} - \bar{x})/\sum[(x_{1j} - \bar{x})^2 + (x_{2j} - \bar{x})^2]$$

where $2n\bar{x} = \sum(x_{1j} + x_{2j})$.

This problem is to derive the distributions for $\rho$ and $\rho^*$.

(F) Distributions of Some Correlation Matrices: Let $\underset{\sim}{x}_i = (x_{1i}, x_{2i}, \ldots, x_{pi})$, $i = 1, 2, \ldots, n$, be a sample of size n from a p-variate normal distribution $X = (X_1, X_2, \ldots, X_p)$ with mean vector $\underset{\sim}{0}$, and covariance matrix $\sum$. If the variances of all the marginal distributions $X_i$'s are equal, then $\sum = \sigma^2 P$ for some positive real number $\sigma$. Note that P is the correlation matrix of X. The MLE for P is the following random symmetric matrix:

$$\hat{P} = \begin{pmatrix} 1 & & & * \\ r_{12} & 1 & & \\ \vdots & & \ddots & \\ r_{1p} & \cdots & r_{(p-1)p} & 1 \end{pmatrix}$$

where

$$r_{ij} = p\sum(x_{ik}x_{jk})/[\sum_{i=1}^{p}(\sum x_{ik}^2)], \quad 1 \le i < j \le p ,$$

and $\sum$ stands for summation over k from 1 to n.

Now, suppose that $\alpha$, $1 \leq \alpha < p$, of the variances of the marginal distributions $X_i$'s are equal, say $\sigma_1^2$, and all the rest of them equal to another value $\sigma_2^2 (\neq \sigma_1^2)$. By rearranging the order of $X_i$'s in X, we can, without loss of generality, assume that

$$\sum = DPD'$$

where D is a p×p diagonal matrix of the form

$$D = \text{diag}\,(\overbrace{\sigma_1, \ldots, \sigma_1}^{\alpha}, \overbrace{\sigma_2, \ldots, \sigma_2}^{p-\alpha}),\ \sigma_1, \sigma_2 > 0$$

and P is the correlation matrix of the random variable X. The MLE for P is the following random symmetric matrix

$$P^* = \left(\begin{array}{cccc|cccc} 1 & & & & & & * & \\ r_{12} & 1 & & & & & & \\ \vdots & & \ddots & & & & & \\ r_{1\alpha} & \cdots & r_{(\alpha-1)\alpha} & 1 & & & & \\ \hline r_{1(\alpha+1)} & \cdots & & r_{\alpha(\alpha+1)} & 1 & & & \\ \vdots & & & \vdots & \vdots & \ddots & & \\ r_{1p} & \cdots & & r_{\alpha p} & r_{(\alpha+1)p} & \cdots & r_{(p-1)p} & 1 \end{array}\right)$$

where

$$r_{ij} = \begin{cases} \alpha S_{ij}/(\sum_{i=1}^{\alpha} S_{ii}), & 1 \leq i < j \leq \alpha \\ \{\alpha(p-\alpha)\}^{1/2} S_{ij}/\{(\sum_{i=1}^{\alpha} S_{ii})(\sum_{i=\alpha+1}^{p} S_{ii})\}^{1/2}, & 1 \leq i \leq \alpha < j \leq p \\ (p-\alpha) S_{ij}/(\sum_{i=\alpha+1}^{p} S_{ii}), & \alpha < i < j \leq p \end{cases}$$

where

$$S_{ij} = \sum_{k=1}^{n} x_{ik} x_{jk} .$$

The problem is to derive the distributions for $\hat{P}$ and $P^*$.

# CHAPTER 2

# SOME RESULTS IN STRUCTURAL DISTRIBUTION

2.1. INTRODUCTION: In this chapter, some elementary results in structural distributions are given. These results include the following:

(i) Uniqueness of the structural distribution;

(ii) Consistency of the structural distribution in the light of criteria proposed by Lindley (1958) and Sprott (1960);

(iii) Structural distributions for independent structural models;

(iv) Structural distributions over subgroup spaces of general composite measurement models;

(v) Structural distributions based on Type II censored responses; and

(vi) Structural distributions for some transformed structural models.

2.2. ON THE UNIQUENESS OF STRUCTURAL DISTRIBUTION: In this section, we give a proof of the uniqueness property for structural distribution over the group space for a general structural model. In other words, we prove that the structural distribution, based on the complete set of responses, does not depend on the choice of a transformation variable.

Consider a general structural model

$$\begin{cases} \underset{\sim}{x} = \theta \underset{\sim}{e} \\ f(\underset{\sim}{e})d\underset{\sim}{e} = \prod_{i=1}^{n} f(e_i)de_i \end{cases}$$

where $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, $\underset{\sim}{e} = (e_1, e_2, \ldots, e_n)$ and $\theta\underset{\sim}{e} = (\theta e_1, \theta e_2, \ldots, \theta e_n)$. Then the structural distribution for $\theta$, based on $\underset{\sim}{x}$, is given by (See (1.4.5) of Chapter 1)

$$g(\theta:\underset{\sim}{x})d\theta = K(D(\underset{\sim}{x}))f(\theta^{-1}\underset{\sim}{x})J_N(\theta^{-1}\underset{\sim}{x})\Delta(\theta^{-1}[\underset{\sim}{x}])d\mu(\theta).$$

This pdf for $\theta$ can be rewritten as follows:

(2.2.1) $$g(\theta:\underset{\sim}{x})d\theta = K(\underset{\sim}{x})\bar{f}(\theta:\underset{\sim}{x})d\mu(\theta)$$

where

$$\bar{f}(\theta:\underset{\sim}{x}) = f(\theta^{-1}\underset{\sim}{x})J_N(\theta^{-1}:\underset{\sim}{x})\Delta(\theta^{-1})$$

$$K(\underset{\sim}{x}) = K(D(\underset{\sim}{x}))J_N([x]:D(\underset{\sim}{x}))\Delta([\underset{\sim}{x}]),$$

and

$$J_N(\theta:\underset{\sim}{x}) = \left|\frac{\partial\theta\underset{\sim}{x}}{\partial\underset{\sim}{x}}\right| .$$

Note that $K(\underset{\sim}{x})$ serves as the normalizing constant factor for $g(\theta:\underset{\sim}{x})d\mu(\theta)$ to a probability density so that we have

$$K(\underset{\sim}{x})^{-1} = \int_G \bar{f}(\theta:\underset{\sim}{x})d\mu(\theta) .$$

It is clear that the factor $\bar{f}(\theta:\underset{\sim}{x})d\mu(\theta)$ does not depend on the choice of a transformation variable. This implies that the normalizing constant factor depends only on the responses $\underset{\sim}{x}$.

Therefore it follows that structural distribution for $\theta$, based on the responses $\underset{\sim}{x}$, is unique.

Furthermore, from (2.2.1) we conclude the following:

"When a structural model is given, the structural distribution for $\theta$, based on the responses $\underset{\sim}{x}$, can be obtained directly without introducing any transformation variable. We need only to calculate the left invariant differential $d\mu(\theta)$, the modular function $\Delta(\theta)$ and the jacobian $J_N(\theta^{-1}:\underset{\sim}{x})$."

The following example is given here to illustrate the usefulness of the above conclusion. Also the structural distribution obtained will be used for future study in obtaining structural distributions for the differences of two means in Chapter 3.

Example: Consider the measurement model:

$$\begin{cases} x_i = \mu + \sigma e_i, \; i = 1, 2, \ldots, n \\ (2\pi)^{-n/2} \exp\left\{- \sum_{i=1}^{n} e_i^2/2\right\} \prod_{i=1}^{n} de_i \end{cases}$$

where $(\mu,\sigma)$ belongs to the positive affine group G on $R^n$. The distribution for $\mu$ and $\sigma$, based on $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, has been obtained by Fraser (1961b). This is given here in a slightly different form:

$$\begin{aligned} g(\mu,\sigma:\underset{\sim}{x})d\mu d\sigma &= K(\underset{\sim}{x})\exp\left\{-\frac{1}{2} \sum_{i=1}^{n} (x_i-\mu)/\sigma\right\}\sigma^{-(n+1)}d\mu d\sigma \\ &= K(\underset{\sim}{x})\exp\left\{-n[(\bar{x}-\mu)^2+s^2]/(2\sigma^2)\right\}\sigma^{-(n+1)}d\mu d\sigma \end{aligned}$$

where $n\bar{x} = \sum_{i=1}^{n} x_i$, $ns^2 = \sum_{i=1}^{n} (x_i-\bar{x})^2$. The normalizing constant factor $K(\underset{\sim}{x})$ is given by

$$K(\underset{\sim}{x})^{-1} = \pi^{1/2}(n/2)^{-n/2}\Gamma(n-1)/2)s^{-(n-1)/2} .$$

2.3. CONSISTENCY OF THE STRUCTURAL DISTRIBUTION: The main object of this section is to prove the following proposition:

"The structural distribution is consistent in the light of Criteria I to III of Chapter 1 proposed by Lindley and Sprott".

It is clear that Criterion III is weaker than Criterion II and Criterion II is weaker than Criterion I. Therefore it is sufficient to prove that the structural distribution is consistent in the light of Criterion III only. Before proceeding to the proof, let us first recall Criterion III.

Criterion III deals with two independent distributions both involving the same parameter. The Fiducial distribution is obtained from a sample of one distribution. Then using this Fiducial distribution as *a priori* distribution for a Bayesian analysis for a sample from the other distribution should yield a result independent of the order of combination.

Now we proceed to give the proof. Let us consider the following two general structural models

$$\begin{cases} x_{ij} = \theta e_{ij}, \ j = 1, 2, \ldots, n \\ \prod_{j=1}^{n_i} f_i(e_{ij})de_{ij} \end{cases}$$

$i = 1, 2$. Then the structural distribution for $\theta$, based on a first set of responses, say $\underset{\sim}{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n_1})$, is

$$g_1(\theta:\underset{\sim}{x}_1)d\theta = K(\underset{\sim}{x}_1)\prod_{j=1}^{n_1} f_1(\theta^{-1}x_{1j})J_{N_1}(\theta^{-1}:\underset{\sim}{x}_1)\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)} .$$

The likelihood function for $\theta$, based on the other set of responses $\underset{\sim}{x}_2 = (x_{21}, x_{22}, \ldots, x_{2n_2})$, is

$$\prod_{i=1}^{n_2} f_2(\theta^{-1}x_{2i})\cdot J_{N_2}(\theta^{-1}:\underset{\sim}{x}_2) .$$

Hence a Bayesian analysis for the responses $\underset{\sim}{x}_2$, using the structural distribution $g_1(\theta:\underset{\sim}{x}_1)d\theta$ as *a priori* distribution, yield the *a posteriori* distribution

$$g(\theta:\underset{\sim}{x}_1, \underset{\sim}{x}_2)d\theta = K(\underset{\sim}{x}_1, \underset{\sim}{x}_2)\prod_{i=1}^{n_2} f_2(\theta^{-1}\underset{\sim}{x}_{2i})\cdot J_{N_2}(\theta^{-1}:\underset{\sim}{x}_2)\cdot \prod_{j=1}^{n_1} f_1(\theta^{-1}\underset{\sim}{x}_{1j})\cdot J_{N_1}(\theta^{-1}:\underset{\sim}{x}_1)\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)} . \tag{2.3.1}$$

The proof will be complete if we can prove (2.3.1) is the structural distribution for $\theta$, based on the combined sample $\underset{\sim}{x} = (\underset{\sim}{x}_1, \underset{\sim}{x}_2)$, derived from the following combined model:

$$\begin{cases} x'_j = \theta e'_j \\ \prod_{j=1}^{n_1} f_1(e'_j)\cdot \prod_{j=n_1+1}^{n_1+n_2} f_2(e'_j)\cdot \prod_{j=1}^{n_1+n_2} de'_j \end{cases}$$

where

$$x_j' = x_{1j},\ e_j' = e_{1j}' \quad \text{for } j = 1, 2, \ldots, n_1,$$

and

$$x_{n_1+j}' = x_{2j},\ e_{n_1+j}' = e_{2j} \text{ for } j = 1, 2, \ldots, n_2.$$

The structural distribution for θ, based on the combined responses $\underset{\sim}{x}' = (x_1', x_2', \ldots, x_n')$, derived from the above combined structural model is

$$(2.3.2) \qquad g'(\theta:\underset{\sim}{x}')d\theta = K(\underset{\sim}{x}') \prod_{j=1}^{n_1} f_1(\theta^{-1}x_j') \cdot \prod_{j=n_1+1}^{n_1+n_2} f_2(\theta^{-1}x_j') \cdot$$

$$J_N(\theta^{-1}:\underset{\sim}{x}')\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)},$$

where $N = n_1 + n_2$. Note that

$$J_N(\theta^{-1}:\underset{\sim}{x}') = J_{N_1}(\theta^{-1}:\underset{\sim}{x}_1) \cdot J_{N_2}(\theta^{-1}:\underset{\sim}{x}_2),$$

$$\prod_{j=1}^{n_1} f_1(\theta^{-1}x_j') = \prod_{j=1}^{n_1} f_1(\theta^{-1}x_{1j}),$$

and

$$\prod_{j=n_1+1}^{n_1+n_2} f_2(\theta^{-1}x_j') = \prod_{i=1}^{n_2} f_2(\theta^{-1}x_{2i}).$$

Hence (2.3.2) becomes

$$g'(\theta:\underset{\sim}{x}')d\theta = K(\underset{\sim}{x}') \prod_{i=1}^{n_2} f_2(\theta^{-1}x_{2i}) \cdot J_{N_2}(\theta^{-1}:\underset{\sim}{x}_2) \cdot$$

$$\prod_{j=1}^{n_1} f_1(\theta^{-1}x_{1j}) \cdot J_{N_1}(\theta^{-1}:\underset{\sim}{x}_1)\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)}$$

which is identical to (2.3.1). Thus the proof is complete.

## 2.4. A RESULT IN STRUCTURAL DISTRIBUTIONS FOR TWO OR MORE INDEPENDENT STRUCTURAL MODELS:

First of all, we define the independence of two or more structural models.

Definition: Two or more structural models are said to be mutually independent if the corresponding error variables are mutually independent.

The main result of this section is:

"For two or more mutually independent structural models, with not necessarily equal numbers of responses, the joint structural pdf over the direct product of the group spaces is the product of the structural pdf over the corresponding group space."

It is sufficient to prove the above result for the case having only two independent structural models. Let us consider the following two independent structural models:

$$(2.4.1) \quad \begin{cases} x_{ij} = \theta_i e_{ij}, \quad j = 1, 2, \ldots, n_i, \ \theta_i \varepsilon G_i , \\ f_i(\underset{\sim}{e}) d\underset{\sim}{e}_i = \prod_{j=1}^{n_i} f_i(e_{ij}) de_{ij} \end{cases}$$

$i = 1, 2$. These independent structural models can be rewritten as a single model as follows:

$$\begin{cases} X = \theta E \\ f(E)dE = \prod_{j=1}^{n_1} f_1(e_{1j}) de_{1j} \cdot \prod_{j=1}^{n_2} f_2(e_{2j}) de_{2j} \end{cases}$$

where

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n_1} & 0 & \cdots & 0 \\ 0 & & 0 & x_{21} & \cdots & x_{2n_2} \end{pmatrix}, \quad E = \begin{pmatrix} e_{11} & \cdots & e_{1n_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & e_{21} & \cdots & e_{2n_2} \end{pmatrix}$$

and

$$\theta = \begin{pmatrix} \theta_1 & 0 \\ & \\ 0 & \theta_2 \end{pmatrix}$$

is an element of the direct product $G = G_1 \times G_2$ of the two unitrary groups of transformations $G_1$ and $G_2$. The unitrary property of G follows directly from the unitrary property of $G_1$ and $G_2$. The structural distribution for $\theta$, based on X, is

$$(2.4.2) \quad g(\theta_1, \theta_2 : X) d\theta_1 d\theta_2 = K(X) f(\theta^{-1}X) J_N(\theta^{-1}:X) \Delta(\theta^{-1}) \frac{d\theta}{J_L(\theta)},$$

where $N = n_1 + n_2$ and $L = L_1 + L_2$. We note that

$$(2.4.3) \quad \begin{cases} J_N(\theta^{-1}:X) = J_{N_1}(\theta_1^{-1}:\underset{\sim}{x}_1) \cdot J_{N_2}(\theta_2^{-1}:\underset{\sim}{x}_2) \\ \Delta(\theta^{-1}) = \Delta_1(\theta_1^{-1}) \cdot \Delta_2(\theta_2^{-1}) \\ J_L(\theta) = J_{L_1}(\theta_1) \cdot J_{L_2}(\theta_2) \end{cases}$$

where $\underset{\sim}{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n_1})$, $\underset{\sim}{x}_2 = (x_{21}, x_{22}, \ldots, x_{2n_2})$, and the subscripts i, i = 1, 2, of $\Delta_i(\cdot)$, $J_{L_i}(\cdot)$ refer to the corresponding functions the models (2.4.1). The identities (2.4.3) follow directly from the fact that elements of G operate component-wisely on X, and also the product of any two transformation of G operate component-wisely as well. Now

$$f(\theta^{-1}X) = \prod_{j=1}^{n_1} f_1(\theta_1^{-1}x_{1j})\cdot \prod_{j=1}^{n_2} f_2(\theta_2^{-1}x_{2j}).$$

Therefore (2.4.2) can be rewritten as

$$g(\theta_1,\theta_2:X)d\theta_1 d\theta_2 = K(X)\cdot \prod_{j=1}^{n_1} f_1(\theta_1^{-1}x_{1j})\cdot J_{N_1}(\theta_1^{-1}:\underset{\sim}{x}_1)\Delta_1(\theta_1^{-1})\frac{d\theta_1}{J_{L_1}(\theta_1)}\cdot$$

$$\prod_{j=1}^{n_2} f_2(\theta_2^{-1}x_{2j})\cdot J_{N_2}(\theta_2^{-1}:\underset{\sim}{x}_2)\Delta_2(\theta_2^{-1})\frac{d\theta_2}{J_{L_2}(\theta_2)}$$

$$= g_1(\theta_1:\underset{\sim}{x}_1)d\theta_1\cdot g_2(\theta_2:\underset{\sim}{x}_2)d\theta_2$$

where $g_i(\theta_i:\underset{\sim}{x}_i)d\theta_i$, $i = 1, 2$, are the structural pdf for the corresponding structural models (2.4.1). Thus we complete the proof.

## 2.5. STRUCTURAL DISTRIBUTIONS OVER CERTAIN SUBGROUP SPACES OF A GENERAL COMPOSITE MEASUREMENT MODEL:

Consider a general composite measurement model

$$(2.5.1)\qquad \begin{cases} X = \theta E \\ \\ f(E)dE = f(e_{11},\dots,e_{1n},e_{21},\dots,e_{2n})\cdot \prod_{i=1}^{n} de_{1i}de_{2i} \end{cases}$$

where

$$X = \begin{pmatrix} 1 & \cdots & 1 \\ x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \end{pmatrix}, \quad E = \begin{pmatrix} 1 & \cdots & 1 \\ e_{11} & \cdots & e_{21} \\ e_{21} & \cdots & e_{2n} \end{pmatrix}$$

and $\theta$ belongs to the group $G_1$ of transformations on $R^{2n}$

$$G_1 = \left\{ \theta = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma_1 & 0 \\ \mu_2 & 0 & \sigma_2 \end{pmatrix} : -\infty < \mu_1, \mu_2 < \infty; \sigma_1, \sigma_2 > 0 \right\} .$$

The structural model (2.5.1) can also be written in the alternative form: the general transformation $\theta$ being expressed differently as

$$\theta' = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma_1 & 0 \\ \mu_2 & 0 & \eta\sigma_1 \end{pmatrix}, -\infty < \mu_1, \mu_2 < \infty, \sigma_1, \eta > 0.$$

We call this "new" model as structural model (2.5.1'), and denote by G' the set of all $\theta'$. This "new" model is the same as the original model (2.5.1) with relabelling of the transformation $\theta$ of the group $G_1$. Hence the structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\eta$, based on X, derived from

(a) the structural model (2.5.1'); and

(b) the structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma_2$

by applying the following substitutions

(2.5.2)
$$\begin{cases} \mu_i = \mu_i, \ i = 1, 2 \\ \sigma_1 = \sigma_1 \\ \eta = \sigma_2/\sigma_1 \end{cases}$$

are identical.

The main object of this section is to prove the following: "The structural distributions for $\theta$ over some subgroup spaces of $G_1$ obtained from

(a) the subgroup model directly; and

(b) the structural distribution for $\theta$ derived from the full-group model (2.5.1) by imposing the appropriate condition; are identical."

The structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$, and $\sigma_2$, based on X, derived from model (2.5.1) is

(2.5.3) $$g(\mu_1,\mu_2,\sigma_1,\sigma_2:X)\prod_{i=1}^{2} d\mu_i d\sigma_i$$

$$= K(X)f\left(\frac{x_{11}-\mu_1}{\sigma_1}, \ldots, \frac{x_{1n}-\mu_1}{\sigma_1}, \frac{x_{21}-\mu_2}{\sigma_2}, \ldots, \frac{x_{2n}-\mu_2}{\sigma_2}\right) \cdot$$

$$\cdot(\sigma_1\sigma_2)^{-(n+1)} \cdot \prod_{i=1}^{2} d\mu_i d\sigma_i \ .$$

Applying the substitution (2.5.2), we obtain the structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\eta$:

(2.5.4) $$g(\mu_1,\mu_2,\sigma_1,\eta:X)d\mu_1 d\mu_2 d\sigma_1 d\eta$$

$$= K(X)f\left(\frac{x_{11}-\mu_1}{\sigma_1}, \ldots, \frac{x_{1n}-\mu_1}{\sigma_1}, \frac{x_{21}-\mu_2}{\eta\sigma_1}, \ldots, \frac{x_{2n}-\mu_2}{\eta\sigma_1}\right) \cdot$$

$$\cdot \ \sigma_1^{-(2n+1)}\eta^{-(n+1)} d\mu_1 d\mu_2 d\sigma_1 d\eta$$

since the jacobian of the substitution (2.5.2) is $\sigma_1$. By conditioning $\eta = 1$ to the structural distrinution (2.5.4), we obtain

$$g(\mu_1,\mu_2,\sigma_1:X,\sigma_1=\sigma_2)d\mu_1 d\mu_2 d\sigma_1$$

$$= K'(X)f\left(\frac{x_{11}-\mu_1}{\sigma_1}, \ldots, \frac{x_{1n}-\mu_1}{\sigma_1}, \frac{x_{21}-\mu_2}{\sigma_1}, \ldots, \frac{x_{2n}-\mu_2}{\sigma_1}\right) \cdot$$

$$\cdot\sigma_1^{-(2n+1)} d\mu_1 d\mu_2 d\sigma_1 \ .$$

For the subgroup model of (2.5.1), where $\theta$ belongs to the subgroup

$$G_2 = \{\theta \varepsilon G_1 : \sigma_2 = \sigma_1\} .$$

we have

$$J_N(\theta^{-1}:X) = \sigma_1^{-2n}, \ \Delta(\theta^{-1}) = \sigma^2 \text{ and } J_L(\theta) = \sigma_1^3 .$$

Hence the structural distribution for $\mu_1$, $\mu_2$ and $\sigma_1$, based on X and derived from this subgroup model, coincides with the structural distribution given above. This proves the above proposition for the subgroup $G_2$ of $G_1$.

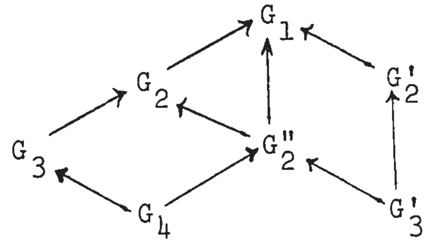
Let

$$G_3 = \{\theta \varepsilon G_2 : \sigma_1 = 1\};$$
$$G_4 = \{\theta \varepsilon G_3 : \mu_1 = \mu_2\};$$
$$G_2' = \{\theta \varepsilon G_1 : \mu_1 = \mu_2 = 0\} ;$$
$$G_3' = \{\theta \varepsilon G_2' : \sigma_1 = \sigma_2\};$$

and

$$G_2'' = \{\theta \varepsilon G_1 : \mu_1 = \mu_2, \ \sigma_1 = \sigma_2\}.$$

If the arrow "→" is interpreted as "is a subgroup of", then we can easily verify the following relationship between subgroups defined above given by the diagram below:



The proposition on structural distributions over subgroups is valid if the full group space and the subgroup space are any two groups appearing in the above diagram. These proofs are similar to the one given above and so are omitted. Structural inference, based on a given structural model, when outside information is available has been considered by Fraser (1968)- (See Sections 6 and 7 of Chapter two).

The above proposition is not valid in general for any subgroup of group of transformations as can be seen from the following counter example.

<u>Example</u>: Consider a general location-progression model

$$\left\{\begin{array}{l} \begin{pmatrix} 1 & \cdots & 1 \\ x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma_1 & 0 \\ \mu_2 & k & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ e_{11} & \cdots & e_{1n} \\ e_{21} & \cdots & e_{2n} \end{pmatrix} \\ \\ f(\underset{\sim}{e})d\underset{\sim}{e} = f(e_{11}, \ldots, e_{1n}, e_{21}, \ldots, e_{2n}) \prod_{i=1}^{n} de_{1i}de_{2i} \end{array}\right.$$

where the transformation $\theta$ belongs to the location-progression group

$$G = \left\{ \theta = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma_1 & 0 \\ \mu_2 & k & \sigma_2 \end{pmatrix} : -\infty < \mu_1, \mu_2, k < \infty; \sigma_1, \sigma_2 > 0 \right\} .$$

For this structural model, we have

$$J_N(\theta^{-1}:\underset{\sim}{x}) = (\sigma_1\sigma_2)^{-n}, \Delta(\theta^{-1}) = \sigma_2^2 \text{ and } J_L(\theta) = \sigma_1^2\sigma_2^3 .$$

Hence the structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$, $\sigma_2$ and k, based on the complete set of responses $\underset{\sim}{x}$, is given by

$$g(\mu_1,\mu_2,\sigma_1,\sigma_2,k:\underset{\sim}{x})d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 dk$$

$$= k(\underset{\sim}{x})f(\theta^{-1}\underset{\sim}{e})J_N(\theta^{-1}:\underset{\sim}{x})\Delta(\theta^{-1})d\mu(\theta)$$

$$= k(\underset{\sim}{x})f\left(\frac{x_{11}-\mu_1}{\sigma_1},\ldots,\frac{x_{1n}-\mu_1}{\sigma_2},\frac{\sigma_1 x_{21}-kx_{11}+k\mu_1-\sigma_1\mu_2}{\sigma_1\sigma_2},\ldots,\frac{\sigma_1 x_{2n}-kx_{1n}+k\mu_1-\sigma_1\mu_2}{\sigma_1\ \sigma_2}\right).$$

$$\cdot(\sigma_1\sigma_2)^{-n}\sigma_2^2\ \frac{d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 dk}{\sigma_1^2\sigma_2^3} .$$

By conditioning k = 0 to the last pdf we obtain

$$(2.5.5)\qquad g(\mu_1,\mu_2,\sigma_1,\sigma_2:\underset{\sim}{x})d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

$$= k(x)f\left(\frac{x_{11}-\mu_1}{\sigma_1},\ldots,\frac{x_{1n}-\mu_1}{\sigma_1},\frac{x_{21}-\mu_2}{\sigma_1},\ldots,\frac{x_{2n}-\mu_2}{\sigma_2}\right)\sigma_1^{-(n+2)}\sigma_2^{-(n+1)}d\mu_1 d\mu_2 d\sigma_1 d\sigma_2.$$

On the other hand, if $\theta = G' = \{\theta\varepsilon G:k = 0\}$ a subgroup of G, the location-progression model becomes the composite measurement model (2.5.1). For this model, the structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma_2$, based on $\underset{\sim}{x}$, is given by (2.5.3), which is clearly not the same as (2.5.5).

## 2.6. STRUCTURAL DISTRIBUTIONS BASED ON TYPE II CENSORED RESPONSES:

In many practical situations it happens that only censored samples can be obtained for one reason or another. Analysis based on censored samples has received a great deal of attention in the literature. Contributed articles in this area up to 1961 can be found in a book edited by Sarhan and Greeberg (1962). Censoring is classified into two types: namely Type I and Type II. Type I censored samples are referred to samples such that no observations above or below a fixed value can be obtained. Type II censored samples refer to samples such that a proportion of the original full samples are cenosred. In this section we combine the theory of Structure inference and the basic distribution theory of order statistic to derive structural distributions based on Type II censored responses. To do this, the following definition is necessary.

Definition: A structural model

$$(2.6.1) \qquad \begin{cases} x_i = \theta e_i, \quad i = 1, 2, \ldots, n, \quad \theta \varepsilon G, \\ \\ \prod_{i=1}^{n} f(e_i) de_i \end{cases}$$

is called an order-preserving structural model if

$$x \leq x' \text{ implies } \theta x \leq \theta x'$$

for any $x, x' \varepsilon \mathcal{X}$ and any $\theta \varepsilon G$.

For simple measurement models, $\theta x = \theta + x$. Therefore simple measurement models are order-preserving since it is always true that $x \leq x'$ implies $\theta + x \leq \theta + x'$. Some other examples of order-preserving structural models are multiplicative measurement models and measurement models.

The object of this section is

(i) to prove the following proposition:

"For any order-preserving structural model, the structural distribution based on ordered responses is the same as the structural distribution based on unordered responses".

(ii) to derive structural distribution based on Type II censored responses; and

(iii) to provide some examples which are needed for future study in Chapter 3.

Let us first take up (ii). Let the responses $x_i$'s be arranged according to magnitude as $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$. The unknown realized values of $e_i$'s can be arranged in the same manner. A general Type II censored responses is

$$(2.6.2) \qquad \underset{\sim}{x} = (x_{(k_1)},\ldots,x_{(\ell_1)},x_{(k_2)},\ldots,x_{(\ell_2)},\ldots,x_{(k_r)}\ldots x_{(\ell_r)})$$

where $k_i$ and $\ell_i$, $i = 1, 2, \ldots, r$, are integers such that

$$1 \leq k_1 < \ell_1 < k_2 < \ell_2 < \ldots < k_r < \ell_r \leq n.$$

The ordered responses $\underset{\sim}{x}(\cdot)$ corresponds to the case $r = 1$, $k_1 = 1$ and $\ell_1 = n$. The structural distribution for $\theta$, based on Type II responses $\underset{\sim}{x}$, is derived from the following structural model induced from the structural model (2.6.1):

$$(2.6.3)\quad \begin{cases} \underset{\sim}{x} = \theta \underset{\sim}{e} \\ \bar{f}(\underset{\sim}{e})d\underset{\sim}{e} = CF(e_{(k_1)})^{k_1-1} \cdot \prod_{i=1}^{r-1} \{F(e_{(k_{i+1})}) - F(e_{(\ell_i)})\}^{k_{i+1}-\ell_i-1} \cdot \\ \{1 - F(e_{(\ell_r)})\}^{n-\ell_r} \cdot \prod_{i=1}^{r} \left\{ \prod_{j=k_i}^{i} f(e_{(j)})de_{(j)} \right\} \end{cases}$$

where

$$C = n!\{(k_1-1)![\prod_{i=1}^{r-1}(k_{i+1}-\ell_i-1)!](n-\ell_r)!\}^{-1} ,$$

and

$$F(t) = \int_{-\infty}^{t} f(y)dy.$$

The structural distribution for $\theta$, based on $\underset{\sim}{x}$, derived from the induced structural model (2.6.3) is

$$(2.6.4)\qquad g(\theta:\underset{\sim}{x})d\theta = K(\underset{\sim}{x})\bar{f}(\theta^{-1}\underset{\sim}{x})J_{N*}(\theta^{-1}:\underset{\sim}{x})\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)}$$

where $N^* = \sum_{i=1}^{r}(\ell_i - k_i + 1)$. The normalizing constant factor $K(\underset{\sim}{x})$ may be obtained by integration

$$K(\underset{\sim}{x})^{-1} = \int_G \bar{f}(\theta^{-1}\underset{\sim}{x})J_{N*}(\theta^{-1}:\underset{\sim}{x})\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)} .$$

Now we proceed to the proof of the proposition (i). We find that the structural distribution for $\theta$, based on the ordered responses $\underset{\sim}{x}(\cdot)$, is given by (2.6.4) with $r = 1$, $k_1 = 1$ and $\ell_1 = n$. So the desired pdf for $\theta$ is

$$(2.6.5)\quad g(\theta:\underset{\sim}{x}(\cdot))d\theta = K(\underset{\sim}{x}(\cdot))\bar{f}(\theta^{-1}\underset{\sim}{x}(\cdot))J_N(\theta^{-1}:\underset{\sim}{x}(\cdot))\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)}$$

$$= K(\underset{\sim}{x}(\cdot))\cdot\prod_{i=1}^{n} f(\theta^{-1}\underset{\sim}{x}_{(i)})\cdot J_N(\theta^{-1}:\underset{\sim}{x}(\cdot))\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)}\ .$$

The structural distribution for $\theta$, based on the unordered responses $\underset{\sim}{x}' = (x_1, x_2, \ldots, x_n)$, is

$$(2.6.6)\qquad g(\theta:\underset{\sim}{x}')d\theta = K(\underset{\sim}{x}')\prod_{i=1}^{n} f(\theta^{-1}x_i)\cdot J_N(\theta^{-1}:\underset{\sim}{x}')\Delta(\theta^{-1})\frac{d\theta}{J_L(\theta)}\ .$$

Now since

$$\prod_{i=1}^{n} f(\theta^{-1}x_i) = \prod_{i=1}^{n} f(\theta^{-1}x_{(i)}),\ \text{and}\ J_N(\theta^{-1}:\underset{\sim}{x}') = J_N(\theta^{-1}:\underset{\sim}{x}(\cdot))$$

it follows that (2.6.5) is identical with (2.6.6). Hence the proof of the proposition is complete.

We conclude this section by giving two examples.

<u>Example 2.6.7</u>: Simple measurement model

$$\begin{cases} x_i = \mu + e_i, \quad e_i > 0,\ i = 1, 2, \ldots, n \\ f(e_1, \ldots, e_n)\prod_{i=1}^{n} de_i = \exp\left\{ -\sum_{i=1}^{n} e_i \right\}\cdot\prod_{i=1}^{n} de_i\ . \end{cases}$$

The structural distribution for $\mu$, based on censored responses $\underset{\sim}{x}$ given by (2.6.2), is easily obtained from (2.6.4):

$$g(\mu:\underset{\sim}{x})d\mu\,\alpha\,(1-\exp\{-(x_{(k_1)}-\mu)\})^{k_1-1}\cdot(\exp\{-(x_{(\ell_r)}-\mu)\})^{n-\ell_r}\cdot$$

$$\prod_{i=1}^{r-1}[\exp\{-(x_{(\ell_i)}-\mu)\}-\exp\{-(x_{(k_{i+1})}-\mu)\}]^{k_{i+1}-\ell_i-1}\cdot$$

$$\prod_{i=1}^{r}[\exp\{-\sum_{j=k_i}^{\ell_i}(x_{(j)}-\mu)\}]d\mu$$

$$\alpha\,(1-\exp\{-(x_{(k_1)}-\mu)\})^{k_1-1}\cdot\exp\{a\mu\}d\mu,\ \mu < x_{(k_1)},$$

where

$$a = (n-\ell_r) + \sum_{i=1}^{r-1}(k_{i+1}-\ell_i-1) + \sum_{i=1}^{r}(\ell_i-k_i+1)$$

$$= n - k_1 + 1.$$

The normalizing constant factor $K(\underset{\sim}{x})$ is

$$K(\underset{\sim}{x})^{-1} = \int_{-\infty}^{x_{(k_1)}}(1-\exp\ \mu\cdot\exp\{-x_{(k_1)}\})^{k_1-1}\cdot\exp\{(n-k_1+1)\mu\}d\mu$$

$$= \int_{0}^{1}(1-t)^{k_1-1}(t\exp x_{(k_1)})^{n-k_1}\cdot\exp x_{(k_1)}d\mu$$

$$= \exp\{(n-k_1+1)x_{(k_1)}\}\cdot\beta(k_1,\ n-k_1+1).$$

The substitution $t = \exp\{\mu - x_{(k_1)}\}$ is used in the preceeding simplification. Thus we have

$$(2.6.8)\quad g(\mu:\underset{\sim}{x})d\mu=\frac{(1-\exp(\mu-x_{(k_1)})^{k_1-1}\cdot\exp\{(n-k_1+1)\mu\}\cdot d\mu}{\exp\{(n-k_1-1)x_{(k_1)}\}\cdot\beta(k_1,n-k_1+1)},\ \mu < x_{(k_1)},$$

as the structural distribution for $\mu$ based on $\underset{\sim}{x}$.

Remark: It is of interest to note that (2.6.8) does not involve $r$, $\ell_1$, and $k_i$, $\ell_i$, $i = 2, 3, \ldots, r$. Therefore we conclude:

(i) "The structural distributions for $\mu$ based on the following two different Type II censored responses

$$\underset{\sim}{x} = (x_{(k_1)},\ldots,x_{(\ell_1)},\ldots,x_{(k_r)},\ldots,x_{(\ell_r)}),\ 1 \leq k_1 < \ell_1 < \ldots < k_r < \ell_r \leq n,$$

and

$$\underset{\sim}{x}' = (x_{(k_1')},\ldots,x_{(\ell_1')},\ldots,x_{(k_s')},\ldots,x_{(\ell_s')}),\ 1 \leq k_1' < \ell_1' < \ldots < k_s' < \ell_s' \leq n,$$

are identical if, and only if $k_1 = k_1'$."

(ii) The structural distribution $\mu$, based on the complete set of responses (ordered or unordered), is

$$g(\mu : \underset{\sim}{x})d\mu = n \exp \{n(\mu - x_{(1)})\}d\mu, \quad \mu < x_{(1)} \,.$$

Example 2.6.9: Measurement model

$$\begin{cases} x_i = \mu + \sigma e_i, \ i = 1, 2, \ldots, n \\ f(e_1, \ldots, e_n) \prod_{i=1}^{n} de_i = \exp\left\{ - \sum_{i=1}^{n} e_i \right\} \cdot \prod_{i=1}^{n} de_i, \quad e_i > 0. \end{cases}$$

In this example we wish to derive the structural distribution for $\mu$ and $\sigma$ based on the following Type II doubly censored responses.

$$\underset{\sim}{x} = (x_{(k)}, x_{(k+1)}, \ldots, x_{(\ell)})$$

where $k$ and $\ell$ are integers such that $1 \leq k < \ell \leq n$. It follows from (2.6.4) that the structural distribution for $\mu$ and $\sigma$ based on $\underset{\sim}{x}$ is

$$g(\mu,\sigma:\underset{\sim}{x})d\mu d\sigma \alpha \left(1-\exp\left\{-\frac{x_{(k)}-\mu}{\sigma}\right\}\right)^{k-1}\left(\exp -\left\{\frac{x_{(\ell)}-\mu}{\sigma}\right\}\right)^{n-\ell}\cdot$$

$$\exp\left\{-\sum_{i=k}^{\ell}\frac{x_{(i)}-\mu}{\sigma}\right\}\cdot\sigma^{-(\ell-k+1)-1}d\mu d\sigma$$

$$\alpha\left(1-\exp\left\{-\frac{x_{(k)}-\mu}{\sigma}\right\}\right)^{k-1}\exp\left\{-[A(\underset{\sim}{x}) - (n-k+1)\mu]/\sigma\right\}\sigma^{-(\ell-k+2)}d\mu d\sigma$$

for $-\infty < \mu < x_{(k)}$, $\sigma > 0$; and where

$$A(\underset{\sim}{x}) = (n-\ell)x_{(\ell)} + \sum_{i=k}^{\ell} x_{(i)} .$$

The normalizing constant factor $K(\underset{\sim}{x})$ is given by

$$K(\underset{\sim}{x})^{-1}=\int_0^\infty\int_{-\infty}^{x_{(k)}}\left(1-\exp\left\{-\frac{x_{(k)}-\mu}{\sigma}\right\}\right)^{k-1}\exp\left\{-[A(\underset{\sim}{x})-(n-k+1)]/\sigma\right\}\sigma^{-(\ell-k+2)}d\mu d\sigma.$$

To carry out the above integration, we make the following substitution:

$$\begin{cases} t = \mu/\sigma \\ z = 1/\sigma \end{cases}$$

which has jacobian $z^{-3}$. The integration reduces to

$$K(\underset{\sim}{x})=\int_0^\infty\int_{-\infty}^{x_{(k)}z}(1-\exp\{-x_{(k)}z+t\})^{k-1}\exp\{-A(\underset{\sim}{x})z+(n-k+1)t\}z^{\ell-k+1}dtdz$$

$$=\int_0^\infty \exp\{-A(\underset{\sim}{x})z\}z^{\ell-k-1}\int_{-\infty}^{x_{(k)}z}(1-\exp\{-x_{(k)}z+t\})^{k-1}\exp\{(n-k+1)t\}dtdz$$

$$=\int_0^\infty \exp\{-A(\underset{\sim}{x})z\}z^{\ell-k-1}\int_0^1 (1-y)^{k-1}(y\ \exp\{x_{(k)}z\})^{n-k}\exp\{x_{(k)}z\}dydz$$

$$=\int_0^\infty \exp\{-[A(x)-(n-k+1)x_{(k)}]z\}z^{\ell-k-1}dz\cdot\int_0^1 (1-y)^{k-1}y^{n-k}dy$$

$$=\frac{\Gamma(\ell-k)\beta(k,\ n-k+1)}{[A(x)-(n-k+1)x_{(k)}]^{\ell-k}}$$

since $A(\underset{\sim}{x}) - (n-k+1)x_{(k)} = (n-\ell)x_{(\ell)}+\sum_{i=k}^{\ell} x_{(i)} - (n-k+1)x_{(k)}$

$$= (n-\ell)(x_{(\ell)}-x_{(k)}) + \sum_{i=k}^{\ell} (x_{(i)}-x_{(k)})$$

$$> 0 .$$

The preceeding simplification involves a substitution

$y = \exp\{ -x_{(k)}z + t\}$.

Hence we have

(2.6.10) $\quad g(\mu,\sigma:x)d\mu d\sigma$

$$= \frac{[A(\underset{\sim}{x})-(n-k+1)x_{(k)}]^{\ell-1}}{\Gamma(\ell-k)\beta(k,\ n-k+1)}(1 - \exp\{-(x_{(k)}-\mu)/\sigma\}).$$

$$\exp\ \{-[A(\underset{\sim}{x}) - (n-k+1)\mu]/\sigma\}\sigma^{-(\ell-k+2)}d\mu d\sigma$$

for $-\infty < \mu < x_{(k)}$, $\sigma > 0$.

## 2.7. STRUCTURAL DISTRIBUTIONS FOR TRANSFORMED STRUCTURAL MODELS:

A general multiplicative measurement model

$$(2.7.1)\quad \begin{cases} x_i = \sigma e_i, \quad e_i > 0, \ i = 1, 2, \ldots, n \\ f(e_1, e_2, \ldots, e_n) \prod_{i=1}^{n} de_i \end{cases}$$

with $\sigma > 0$, is transformed into the following simple measurement model

$$(2.7.2)\quad \begin{cases} x_i' = \mu + e_i', \quad i = 1, 2, \ldots, n \ , \quad -\infty < \mu < \infty, \\ f(\exp e_1', \ldots \exp e_n') \prod_{i=1}^{n} \exp e_i' de_i' \end{cases}$$

by the transformation $t \to \ell n\ t$. Note that $x_i'=\ell n x_i$, $e_i'=\ell n e_i$, for $i=1,2,\ldots,n$, and $\mu=\ell n\sigma$. Similarly a general simple measurement model

$$(2.7.3)\quad \begin{cases} y_i = \mu + e_i, \ i = 1, 2, \ldots, n, \quad -\infty < \mu < \infty, \\ f(e_1, \ldots, e_n) \prod_{i=1}^{n} de_i \end{cases}$$

is transformed into the following multiplicative measurement model

$$(2.7.4)\quad \begin{cases} y' = \sigma e_i' \\ f(\ell n\ e_i', \ldots, \ell n\ e_n') \prod_{i=1}^{n} e_i'^{-1} de_i' \end{cases}$$

by the transformation $t \to \exp\{t\}$. Note that $y_i' = \exp y_i$, $e_i' = \exp e_i$, $i = 1, 2, \ldots, n$, and $\sigma = \exp \mu$.

In this section we wish to prove the following propositions.

(i) "For the structural model (2.7.1), the structural distribution for $\mu(= \ell n\ \sigma)$, based on $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, or equivently $\underset{\sim}{x}' = (x_1', x_2', \ldots, x_n')$, derived from

(a) the structural distribution for $\sigma$, based on $\underset{\sim}{x}$, by applying the transformation $\sigma = \exp \mu$; and

(b) the transformed structural model (2.7.2) directly; are identical."

(ii) "For the structural model (2.7.3), the structural distribution for $\sigma( = \exp \mu)$, based on $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n)$, or equivently $\underset{\sim}{y}' = (y_1', y_2', \ldots, y_n')$, derived from

(a) the structural distribution for $\mu$, based on $\underset{\sim}{y}$, by applying the transformation $\mu = \ln \sigma$; and

(b) the transformed structural model (2.7.4) directly; are identical."

First, let us prove proposition (i). The structural distribution for $\sigma$, based on $\underset{\sim}{x}$ and derived from the structural model (2.7.1), is

$$g(\sigma:\underset{\sim}{x})d\sigma = K(\underset{\sim}{x})f(x_1/\sigma, \ldots, x_n/\sigma)\sigma^{-(n+1)}d\sigma .$$

Therefore, the structural for $\mu$, derived by using (a), is

$$(2.7.5) \quad g(\mu:\underset{\sim}{x})d\mu = K(\underset{\sim}{x})f(x_1 \exp\{-\mu\}, \ldots, x_n\exp\{-\mu\})\exp\{-n\mu\}d\mu.$$

On the other hand, the structural distribution for $\mu$, based on $\underset{\sim}{x}'$ and derived from the structural model (2.7.2), is

$$g(\mu:\underset{\sim}{x}')d\mu = K(\underset{\sim}{x}')f(\exp\{x_1'-\mu\},\ldots,\exp\{x_n'-\mu\})\cdot \prod_{i=1}^{n} \exp\{x_i-\mu\}d\mu$$

$$= K'(\underset{\sim}{x}')f(\exp x_1'\cdot\exp\{-\mu\},\ldots,\exp x_n'\cdot\exp\{-\mu\})\exp\{-n\mu\}d\mu.$$

Since $x_i = \exp\{x_i'\}$, it follows that the last expression is the same as (2.7.5). Hence we complete the proof for proposition (i).

<u>Remark</u>: Proposition (i) can be extended in the following two directions. First, if a Type II censored response is used instead of the complete set of responses, then the transformation $t \to \ell n\ t$ will produce a corresponding Type II censored transformed response for the transformed structural model. The proposition (i) is still valid if censored response is used. Also, if we have a general compositive multiplicative measurement model whose error variables take only positive values, then the transformation $t \to \ell n\ t$ transforms the model into a transformed composite simple measurement model. The structural distribution for μ's derived by (a) and (b) are again identical. Proofs for these two extensions are simple generalizations of the proof given above and so they are omitted.

Next, we proceed to prove proposition (ii). The structural distribution for μ, based on $\underset{\sim}{y}$ and derived from the structural model (2.7.3), is

$$g(\mu:\underset{\sim}{y})d\mu = K(\underset{\sim}{y})f(y_1-\mu, \ldots, y_n-\mu)d\mu.$$

Therefore the structural distribution for σ, based on $\underset{\sim}{y}$ and derived by using (a) is

$$g(\sigma:\underset{\sim}{y})d\sigma = K(\underset{\sim}{y})f(y_i - \ell n\ \sigma, \ldots, y_n - \ell n\ \sigma)\frac{d\sigma}{\sigma}\ . \tag{2.7.6}$$

On the other hand, the structural distribution for σ, based on $\underset{\sim}{y}'$ and derived directly from the structural model (2.7.4), is

$$g(\sigma:\underset{\sim}{y}')d\sigma = K(\underset{\sim}{y}')f(\ell n\ y_1'/\sigma, \ldots, \ell n\ y_n'/\sigma)\prod_{i=1}^{n}(y_i'/\sigma)^{-1}\cdot\sigma^{-(n+1)}d\sigma$$

$$= K'(\underset{\sim}{y}')f(\ell n\ y_i' - \ell n\ \sigma, \ldots, \ell n\ y_n' - \ell n\ \sigma)\sigma^{-1}\ d\sigma.$$

Since $y_i = \ell n\ y_i'$, i = 1, 2, ..., n, it follows that the last expression is identical with (2.7.6). The proof of proposition (ii) is thus completed.

Remark: Similar extension of the proposition (ii) to the case when censored responses, and the case for a general composite simple measurement model can be easily proved.

We conclude this chapter by an example.

Example: Simple measurement model

$$\begin{cases} x_i = \mu + e_i, \ i = 1, 2, \ldots, n, \quad -\infty < \mu < \infty , \\ \prod_{i=1}^{n} [\exp(e_i - \exp\{e_i\}) de_i] . \end{cases}$$

The purpose of this example is to obtain the structural distribution for $\sigma$(=exp $\mu$) based on a singly Type II response $\underset{\sim}{x} = (x_{(1)},\ldots,x_{(k)})$, $1 < k \leq n$, at the right. Instead of looking at the above model, we consider the transformed mutliplicative model:

$$\begin{cases} x_i' = \sigma e_i', \quad i = 1, 2, \ldots, n \\ \prod_{i=1}^{n} [\exp(\ell n\ e_i' - \exp\{\ell n\ e_i'\}) \cdot de_i'/e_i'] = \exp\{-\prod_{i=1}^{n} e_i'\} \prod_{i=1}^{n} de_i' \end{cases}$$

where $x_i' = \exp\{x_i\}$, $e_i' = \exp\{e_i\}$, and $\sigma = \exp\{\mu\}$. The structural distribution for $\sigma$, based on $\underset{\sim}{y}$ (or equivently $\underset{\sim}{y}' = (y_1', \ldots, y_k')$, is obtained by using (2.6.4) as follows:

$$g(\sigma:\underset{\sim}{y}')d\sigma = K(\underset{\sim}{y}')\ [1 - F(y'_{(k)})]^{n-k} \cdot \prod_{i=1}^{k} f(y'_{(i)}/\sigma) \cdot \sigma^{-(k+1)} d\sigma$$

$$= K(\underset{\sim}{y}')\exp\{-(n-k)y'_{(k)}/\sigma\}\exp\{-\sum_{i=1}^{k} y'_{(i)}/\sigma\}\sigma^{-(k+1)} d\sigma$$

$$= K(\underset{\sim}{y}')\exp\{-A(\underset{\sim}{y}')/\sigma\}\sigma^{-(k+1)}d\sigma$$

where $A(\underset{\sim}{y}') = (n-k)y'_{(k)} + \sum_{i=1}^{k} y'_{(i)}$. The normalizing constant factor $K(\underset{\sim}{y}')$ is

$$K(\underset{\sim}{y}') = \int_0^\infty \exp\ \{-A(y')/\sigma\}\sigma^{-(k+1)}d\sigma$$

$$= \int_0^\infty \exp\ \{-A(\underset{\sim}{y}')t\}t^{k+1}\ \frac{dt}{t^2}$$

$$= \Gamma(k)/A(\underset{\sim}{y}')^k\ .$$

Hence we have

$$g(\sigma:\underset{\sim}{y}')d\sigma = \frac{\left\{(n-k)y'_{(k)}+\sum_{i=1}^{k}y'_{(i)}\right\}^k}{\Gamma(k)}\ \exp\left\{-[(n-k)y'_{(k)}+\sum_{i=1}^{k}y'_{(i)}]/\sigma\right\}.$$

$$\sigma^{-(k+1)}d\sigma.$$

# CHAPTER 3

# BEHRENS-FISHER PROBLEMS

3.1. <u>INTRODUCTION</u>: The present chapter employs the structural method of inference to investigate Behrens-Fisher problems under several different assumptions. Sets of observations are viewed as responses of the associated structural models (See Section 1.7 of Chapter 1).

3.2. <u>BEHRENS-FISHER PROBLEM I -- INDEPENDENT POPULATIONS WITH NO ASSUMPTION ON STANDARD DEVIATIONS</u>: Consider the following two independent measurement models

$$(3.2.1)\qquad \begin{cases} x_{ij} = \mu_i + \sigma_i e_{ij}, \quad j = 1, 2, \ldots, n_i, \\ f_i(\underset{\sim}{e}_i)d\underset{\sim}{e}_i = (2\pi)^{-n_i/2}\exp\left\{-\sum_{j=1}^{n_i} e_{ij}^2/2\right\}\cdot \prod_{j=1}^{n_i} de_{ij} \end{cases}$$

for each $i = 1, 2$. We wish to derive the structural distribution for the difference of two means, $\mu = \mu_1 - \mu_2$, say, based on the complete sets of responses $\underset{\sim}{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})$, $i = 1, 2$. The structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is (See Section 2.4 of Chapter 2)

$$g(\mu_1,\mu_2,\sigma_1,\sigma_2:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

$$\alpha\ \exp\left\{-\sum_{i=1}^{2} \frac{n_i}{2\sigma_i^2}\left[(\mu_i-\bar{x}_i)+s_i^2\right]\right\}\sigma_1^{-(n_1+1)}\sigma_2^{-(n_2+1)}d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

where $n_i\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}$, $n_i s_i^2 = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2$, $i = 1, 2$. Since no assumption is made on the standard deviations $\sigma_1$ and $\sigma_2$, it follows that the structural distribution for $\mu_1$ and $\mu_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, can be obtained by integrating out $\sigma_1$ and $\sigma_2$ over the range $(0, \infty)$. Note that

$$\int_0^\infty \exp\left\{- \frac{n_i}{2\sigma_i^2} [(\mu_i - \bar{x}_i)^2 + s_i^2]\right\}\sigma_i^{-(n_i+1)} d\sigma_i$$

$$= \frac{1}{2}\int_0^\infty \exp\left\{- \frac{n_i}{2} [(\mu_i - \bar{x}_i)^2 + s_i^2]t\right\}t^{n_i/2-1}dt$$

$$= \frac{1}{2}\Gamma(n_i/2)\left\{\frac{n_i}{2}[(\mu_i - \bar{x}_i)^2 + s_i^2]\right\}^{-n_i/2}$$

$$= \frac{1}{2}\Gamma(n_i/2)(\frac{n_i}{2})^{-n_i/2}s_i^{-n_i}\left\{1+(\mu_i - \bar{x}_i)^2/s_i^2\right\}^{-n_i/2} .$$

Therefore the structural distribution for $\mu_1$ and $\mu_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

(3.2.2) $$g(\mu_1,\mu_2:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1 d\mu_2 \propto \left\{1 + (\mu_1-\bar{x}_1)^2/s_1^2\right\}^{-n_1/2} \cdot \left\{1+(\mu_2-\bar{x}_2)^2/s_2^2\right\}^{-n_2/2}d\mu_1 d\mu_2 .$$

This implies that the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

$$g(\mu:\underset{\sim}{x})d\mu \propto \int_{-\infty}^{\infty}\left\{1 + (\mu_1-\bar{x}_1)^2/s_1^2\right\}^{-n_1/2} \cdot \left\{1+(\mu_1-\mu-\bar{x}_2)^2/s_2\right\}^{-n_2/2}d\mu_1 \cdot d\mu.$$

From (3.2.2) we note that, if we introduced new variables

$$t_i = (n_i-1)^{1/2}(\mu_i - \bar{x}_i)/s_i,\ i = 1, 2,$$

then $t_1$ and $t_2$ are variables having Student's t-distributions with $(n_1-1)$ and $(n_2-1)$ degrees of freedom respectively. Now define

$$r^2 = s_1^2/(n_1-1) + s_2^2/(n_2-1)$$

and

$$\tan\theta = [s_2/(n_2-1)^{1/2}]/[s_1^2/(n_1-1)^{1/2}].$$

Then we have (See Figure 3.1)

$$r\cos\theta = r\cdot[s_1/(n_1-1)^{1/2}]/r$$
$$= s_1/(n_1-1)^{1/2}$$

and similarly $$r\sin\theta = s_2/(n_2-1)^{1/2}$$

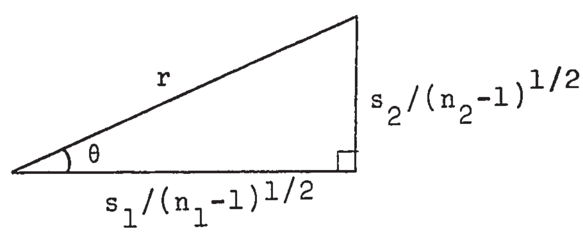



FIGURE 3.1.

Hence the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, can be represented in the form

$$\mu = \mu_1 - \mu_2$$
$$= (\bar{x}_1-\bar{x}_2) + r(t_1\cos\theta - t_2\sin\theta)$$

where the distribution for the variable

$$z = t_1\cos\theta - t_2\sin\theta$$

is usually known as Behrens-Fisher distribution. Tables for percentage points and cumulative probabilities for Behrens-Fisher distributions have been published by Fisher and Yates (1957) and Weir (1966). Welch (1947) gave an approximation to Behrens-Fisher distribution based on a single Student's t-distribution with f degrees of freedom where f is given by

$$f = \left\{\left(\sum_{i=1}^{2} s_i^2/n_i\right)^2 - 2\left(\sum_{i=1}^{2} s_i^4/[n_i(n_i+1)]\right)\right\} / \left(\sum_{i=1}^{2} s_i^4/[n_i^2(n_i+1)]\right) .$$

Patil (1965) gives a similar approximation to Behrens-Fisher distribution z based on a single Student's t-distribution. She chooses a constant h and number f so that hz have the same second and fourth culmulants as that of the student's t-distribution with f degrees of freedom. She also proves that this t-approximation for z is sufficiently accurate for most purposes unless degrees of freedom of t's in z are very small.

3.3. BEHRENS-FISHER PROBLEM II -- INDEPENDENT POPULATIONS UNDER THE CONDITION THAT THE RATIO OF STANDARD DEVIATIONS IS KNOWN: The structural model (3.2.1) is again used in this section. Here we imposed the condition that the ratio $\sigma_2/\sigma_1$ of the standard deviations is known, say equals $c > 0$. The structural distribution for $\mu = \mu_1 - \mu_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is derived here under the above condition. Some properties of a multivariate t-distribution are quoted from Cornish (1954) paper for the present and future investigation.

The structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

(3.3.1) $$g(\mu_1,\mu_2,\sigma_1,\sigma_2:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

$$\alpha \exp\left\{-\sum_{i=1}^{2}\frac{n_i}{2\sigma_i}\left[(\mu_i-\bar{x}_i)^2+s_i^2\right]\right\}\sigma_1^{-(n_1+1)}\sigma_2^{-(n_2+1)}d\mu_1 d\mu_2 d\sigma_1 d\sigma_2 \;.$$

Applying the substitution

$$\begin{cases}\mu_i = \mu_i,\; i = 1,\,2\\ \sigma_1 = \sigma_1\\ \sigma \;= \sigma_2/\sigma_1\end{cases}$$

having jacobian $\sigma_1$, to (3.3.1), we obtain the structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma$ based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$:

$$g(\mu_1,\mu_2,\sigma_1,\sigma:\underset{\sim}{x}_1,\,\underset{\sim}{x}_2)d\mu_1 d\mu_2 d\sigma_1 d\sigma$$

$$\alpha\exp\left\{-\frac{n_1}{2\sigma_1^2}\left[(\mu_1-\bar{x}_1)^2+s_1^2\right]-\frac{n_2}{2\sigma_1^2\sigma^2}\left[(\mu_2-\bar{x}_2)+s_2^2\right]\right\}\sigma_1^{-(n_1+n_2+1)}\sigma^{-(n_2+1)}\cdot$$

$$\cdot d\mu_1 d\mu_2 d\sigma_1 d\sigma \;.$$

Therefore by conditioning $\sigma = \sigma_2/\sigma_1 = c$ in the last probability density and then integrate out $\sigma_1$ we obtain the structural distribution for $\mu_1$ and $\mu_2$ based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$:

$$g(\mu_1,\mu_2:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)d\mu_1 d\mu_2 \alpha\int_0^\infty \exp\left\{-A(\mu_1,\mu_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)/\sigma_1^2\right\}\sigma_1^{-(n_1+n_2+1)}d\sigma_1\cdot$$

$$\cdot d\mu_1 d\mu_2$$

where

$$A(\mu_1,\mu_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2,c) = \frac{n_1}{2}[(\mu_1-\bar{x}_1)^2+s_1^2]+\frac{n_2}{2c^2}[(\mu_2-\bar{x}_2)^2+s_2^2] > 0 .$$

Note that

$$g(\mu_1,\mu_2:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)d\mu_1 d\mu_2$$

$$\alpha\int_0^\infty \exp\left\{-A(\mu_1,\mu_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)/\sigma_1^2\right\}\sigma_1^{-(n_1+n_2+1)}d\sigma_1\cdot d\mu_1 d\mu_2$$

$$\alpha[A(\mu_1,\mu_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2,,c)]^{-(n_1+n_2)/2}d\mu_1 d\mu_2$$

$$\alpha\frac{d\mu_1 d\mu_2}{\left\{s^2+n_1(\mu_1-\bar{x}_1)^2+n_2(\mu_2-\bar{x}_2)^2/c^2\right\}^{(n_1+n_2)/2}}$$

$$\alpha\frac{d\mu_1 d\mu_2}{\left\{1+\frac{n_1(\mu_1-\bar{x}_1)^2}{s^2}+\frac{n_2(\mu_2-\bar{x}_2)^2}{c^2s^2}\right\}^{(n_1+n_2)/2}}$$

where $s^2 = n_1s_1^2 + n_2s_2^2/c^2$. Putting

$$t_1^* = (n_1 + n_2 - 2)^{1/2}(\mu_1 - \bar{x}_1)/s,\ t_2^* = (n_1 + n_2 - 2)^{1/2}(\mu_2-\bar{x}_2)/s$$

and

$$R^{*-1} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2/c^2 \end{pmatrix}$$

we find that

$g(\mu_1, \mu_2 : \underset{\sim}{x}, \underset{\sim}{x}_2, c) d\mu_1 d\mu_2$

$$\alpha \frac{d\mu_1 d\mu_2}{\left\{1+(t_1^*, t_2^*)R^{*-1}(t_1^*, t_2^*)'/(n_1+n_2-2)\right\}^{(n_1+n_2)/2}} .$$

Hence the variables $t_1^*$ and $t_2^*$ has a bivariate t-distribution characterized by the matrix $R^{*-1}$ (see Cornish (1954)).

The following results of multivariate t-distributions are quoted from Cornish (1954) paper. Let $\underset{\sim}{t}' = (t_1, t_2, \ldots, t_n)$ be a multivariate t-distribution of order n and characterized by the matrix $R^{-1}$. Then the pdf of $\underset{\sim}{t}$ is given by

$$\frac{\Gamma(\frac{v+n}{2})|R|^{-1/2}}{(\pi v)^{n/2}\Gamma(v/2)} \left(1 + \underset{\sim}{t}'\underset{\sim}{R}^{-1}t/v\right)^{-(n+v)/2} d\underset{\sim}{t} .$$

Cornish has proven the following:

(i) The limiting for the distribution $\underset{\sim}{t}$, as $v \to \infty$, is a multivariate normal distribution with mean vector $\underset{\sim}{0}$ and covariance matrix R.

(ii) Suppose $\underset{\sim}{x} = H\underset{\sim}{t}$ are any $p(< n)$ linearly independent linear functions of the $t_i$'s. Then $\underset{\sim}{x}$ has a multivariate t-distribution of order p characterized by the matrix $(HRH')^{-1}$. In particular, the marginal distribution of $\underset{\sim}{t}_1 = (t_1, t_2, \ldots, t_r)$, $r < n$, is a multivariate t-distribution of order r characterized by the matrix $R_1^{-1}$ , where $R_1$ is the leading $r \times r$ submatrix in R.

(iii) The pdf of the conditional distribution of $t_1$, given $\underset{\sim}{t}_2' = (t_{r+1}, \ldots, t_n) = \underset{\sim}{a}'$, is

$$\frac{\Gamma(\frac{v+n}{2})|R_1|^{1/2}}{(\pi v)^{r/2}\Gamma(\frac{v+n-r}{2})}\left\{1 + \underset{\sim}{a}'(R_2 - R_3'R_1^{-1}R_3)\underset{\sim}{a}'/v\right\}^{(v+n-r)/2}.$$

$$\left\{1+ \frac{(\underset{\sim}{t}_1+R_1^{-1}R_3\underset{\sim}{a})'R_1(t_1+R_3'R_1^{-1}\underset{\sim}{a}) + \underset{\sim}{a}'(R_2-R_3'R_1^{-1}R_3)\underset{\sim}{a}}{v}\right\}^{-(v+n)/2} d\underset{\sim}{t}_1$$

where $R_1$, $R_2$ and $R_3$ are submatrices of $R^{-1}$ given by

$$R^{-1} = \begin{pmatrix} R_1 & R_3 \\ R_3' & R_2 \end{pmatrix} .$$

Now we return to the derivation of the structural distribution for $\mu$ based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$. Define two new variables $t_1$ and $t_2$ as follows

$$(t_1,t_2)' = H(t_1^*,t_2^*)'$$

where

$$H = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} .$$

Then the variables $t_1$ and $t_2$ has a bivariate t-distribution characterized by the matrix $R^{-1}$ where R is given by

$$R = HR^*H'$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/n_1 & 0 \\ 0 & c^2/n_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/n_1+c^2/n_2 & -c^2/n_2 \\ -c^2/n_2 & c^2/n_2 \end{pmatrix} .$$

Therefore the marginal distribution for $t_1$ has a student's t-distribution with $(n_1+n_2-2)$ degrees of freedom and characterized by $(1/n_1 + c^2/n_2)^{-1}$. That is the pdf of $t_1$ is given by

$$g(t_1:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)dt \propto \frac{dt_1}{\left\{1+t_1^2/[(n_1+n_2-2)(1/n_1+c^2/n_2)]\right\}^{(n_1+n_2-1)/2}} .$$

But $t_1 = (n_1+n_2-2)^{1/2} [\mu-(\bar{x}_1-\bar{x}_2)]/s$, so that the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

$$(3.3.2) \quad g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)d\mu \propto \frac{d\mu}{\left\{1+(\mu-\bar{x}_1+\bar{x}_2)^2/[s^2(1/n_1+c^2/n_2)]\right\}^{(n_1+n_2-1)/2}}$$

$$\propto \frac{d\mu}{\left\{1+\dfrac{n_1n_2(\mu-\bar{x}_1+\bar{x}_2)^2}{c^2s^2(n_1+n_2/c^2)}\right\}^{(n_1+n_2-1)/2}} .$$

The constant of proportionality $K(n_1, n_2, c)$ for (3.3.2) is

$$K(n_1,n_2,c) = \Gamma\left(\frac{n_1+n_2-1}{2}\right)\left\{\pi^{1/2}\Gamma\left(\frac{n_1+n_2-2}{2}\right)c^2s^2(n_1+n_2/c^2)/(n_1n_2)\right\}^{-1} .$$

Now if we define

$$t(n_1,n_2,c) = (n_1+n_2-2)^{1/2}(\mu-\bar{x}_1+\bar{x}_2)/[c^2s^2(n_1+n_2/c^2)/(n_1n_2)]^{1/2}$$

then it follows from (3.3.2) that $t(n_1, n_2, c)$ has a student's t-distribution with $(n_1 + n_2 -2)$ degrees of freedom. Consequently, the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, can be rewritten in the form

$$\mu = (\bar{x}_1-\bar{x}_2) - \left\{\frac{c^2s^2(n_1+n_2/c^2)}{n_1n_2(n_1+n_2-2)}\right\}^{1/2} t(n_1,n_2,c) \quad .$$

The following three special cases for the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are given here for later comparison.

(i) When $n_1 = n_2 = n$, the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, can be rewritten in the form

$$\mu = (\bar{x}_1-\bar{x}_2) - \left\{\frac{c^2(s_1^2+s_2^2/c^2)(1+1/c^2)}{2(n-1)}\right\}^{1/2} t(n,n,c)$$

and whose pdf is given by

$$K(n,n,c) \frac{d\mu}{\left\{1+\dfrac{(\mu-\bar{x}_1+\bar{x}_2)^2}{c^2(1+1/c^2)(s_1^2+s_2^2/c^2)}\right\}^{(2n-1)/2}} \quad .$$

(ii) When $n_1 = n_2 = n$ and $c = 1$, the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, can be rewritten in the form

$$\mu = (\bar{x}_1-\bar{x}_2) - \left\{\frac{s_1^2+s_2^2}{(n-1)}\right\}^{1/2} t(n,n,1)$$

and whose pdf is given by

$$K(n,n,1) \frac{d\mu}{\left\{1+\frac{1}{2}(\mu-\bar{x}_1+\bar{x}_2)/(s_1^2+s_2^2)\right\}^{(2n-1)/2}} \quad .$$

(iii) When $c = 1$, the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, can be rewritten in the form

$$\mu = (\bar{x}_1-\bar{x}_2) - \left\{\frac{(n_1+n_2)(n_1s_1^2+n_2s_2^2)}{n_1n_2(n_1+n_2-2)}\right\}^{1/2} t(n_1,\ n_2,\ 1)$$

and whose pdf is given by

$$K(n_1,n_2,1)\frac{d\mu}{\left\{1+\frac{n_1n_2(\mu-\bar{x}_1+\bar{x}_2)^2}{(n_1+n_2)(n_1s_1^2+n_2s_2^2)}\right\}^{(n_1+n_2-1)/2}} .$$

The available t-tables can be used for constructing structural interval for $\mu$. Furthermore, we note that the particular case (iii) with $c = 1$, gives the result the same as that obtained by the Fiducial method. This result was also obtained by Pitman (1939).

3.4. <u>BEHRENS-FISHER PROBLEM III -- DEPENDENT POPULATIONS WITH NO ASSUMPTION ON THE COVARIANCE MATRIX</u>: The last two sections are concerned with the structural distributions for the differences of two means based on the complete sets of observations taken from independent normal distributions. In this section and the next we consider the case in which the two distributions are correlated. This section deals with the case with no assumption on the covariance matrix. That is we wish to obtain the structural distribution for $\mu$, based on $\underset{\sim}{x}_1 = (x_{11}, \ldots, x_{1n})$ and $\underset{\sim}{x}_2 = (x_{21}, \ldots, x_{2n})$, from the following structural model

$$\begin{cases} \begin{pmatrix} 1 \\ x_{1j} \\ x_{2j} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma_1 & \gamma \\ \mu_2 & \alpha & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_{1j} \\ e_{2j} \end{pmatrix} , \ j = 1, 2, \ldots, n, \\ (2\pi)^{-n} \exp\left\{- \sum_{j=1}^{n} (e_{1j}^2 + e_{2j}^2)\right\} \prod_{j=1}^{n} de_{1j} de_{2j} , \end{cases}$$

where $\sigma_1$, $\sigma_2 > 0$ and the matrix

$$\begin{pmatrix} \sigma_1 & \gamma \\ \alpha & \sigma_2 \end{pmatrix}$$

has determinant $> 0$. Fraser (1968) (For details see page 241) has shown that the structural distribution for $\mu_1$ and $\mu_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

$$g(\mu_1, \mu_2 : \underset{\sim}{x}_1, \underset{\sim}{x}_2) d\mu_1 d\mu_2 \alpha \{1 + n(\underset{\sim}{m}(\underset{\sim}{x}) - \underset{\sim}{\mu})' \underset{\sim}{s}(\underset{\sim}{x})^{-1} (\underset{\sim}{m}(\underset{\sim}{x}) - \mu)]^{-n/2} d\mu_1 d\mu_2$$

where $\underset{\sim}{\mu}' = (\mu_1, \mu_2)$, $\underset{\sim}{m}(\underset{\sim}{x})' = (\bar{x}_1, \bar{x}_2)$ and

$$\underset{\sim}{S}(\underset{\sim}{x}) = \begin{pmatrix} x_{11} - \bar{x}_1 & \cdots & x_{1n} - \bar{x}_1 \\ x_{21} - \bar{x}_2 & \cdots & x_{2n} - \bar{x}_2 \end{pmatrix} \begin{pmatrix} x_{11} - \bar{x}_1 & \cdots & x_{1n} - \bar{x}_1 \\ x_{21} - \bar{x}_2 & \cdots & x_{2n} - \bar{x}_2 \end{pmatrix}'$$

$$= \begin{pmatrix} nS_1^2 & nS_{12} \\ nS_{12} & nS_2^2 \end{pmatrix} .$$

Let $t_1^* = (n-2)^{1/2}(\mu_1-\bar{x}_1)$, $t_2^* = (n-2)^{1/2}(\mu_2-\bar{x}_2)$, and

$$R^{*-1} = n\underset{\sim}{S}(\underset{\sim}{x})^{-1}$$
$$= a^{-1}\begin{pmatrix} s_2^2 & -s_{12} \\ -s_{12} & s_1^2 \end{pmatrix}, \quad a = s_1^2 s_2^2 - s_{12}^2 .$$

Then the variables $t_1^*$ and $t_2^*$ has a bivariate t-distribution characterized by the matrix $R^{*-1}$. If $t_1$ and $t_2$ are variables defined by

$$(t_1,t_2)' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} (t_1^*, t_2^*),$$

then the variables $t_1$ and $t_2$ has a bivariate t-distribution characterized by the matrix $R^{-1}$ where R is given by

$$R = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} R^* \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} s_1^2-2s_{12}+s_2^2 & s_{12}-s_2^2 \\ s_{12}-s_2^2 & s_1^2 \end{pmatrix} .$$

Therefore the marginal distribution for $t_1$ has a Student's t-distribution with (n-2) degrees of freedom and is characterized by $(s_1^2-2s_{12}+s_2^2)^{-1}$.

But $t_1 = t_1^* - t_2^* = (n-2)^{1/2}(\mu - \bar{x}_1 + \bar{x}_2)$, so that the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

$$g(\mu : \underset{\sim}{x}_1, \underset{\sim}{x}_2)\,d\mu \;\alpha\; \frac{d\mu}{\left\{1 + \dfrac{(\mu - \bar{x}_1 + \bar{x}_2)^2}{s_1^2 - 2s_{12} - s_2^2}\right\}^{(n-1)/2}} \quad .$$

The constant of proportionality is

$$\Gamma\left(\frac{n-1}{2}\right)\left\{\pi^{1/2}\,\Gamma\left(\frac{n-2}{2}\right)(s_1^2 - 2s_{12} + s_2^2)^{1/2}\right\}^{-1} \quad .$$

Furthermore, we note that the variable

$$t = (n-2)^{1/2}(\mu - \bar{x}_1 + \bar{x}_2)/(s_1^2 - 2s_{12} + s_2^2)^{1/2}$$

has a Student's t-distribution with (n-2) degrees of freedom. Hence the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, can be rewritten in the form

$$\mu = (\bar{x}_1 - \bar{x}_2) - \{(s_1^2 - 2s_{12} + s_2^2)/(n-2)\}^{1/2}\,t \ .$$

This structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is slightly different from the usual paired t-test distribution

$$T = (n-1)^{1/2}(\bar{x}_1 - \bar{x}_2 - \mu)/(s_1^2 - 2s_{12} + s_2^2)^{1/2}$$

which has a Student's t-distribution with (n-1) degrees of freedom.

3.5. BEHRENS-FISHER PROBLEM IV -- DEPENDENT POPULATIONS WITH THE CONDITION THAT BOTH THE CORRELATION COEFFICIENT AND THE RATIO OF STANDARD DEVIATIONS ARE KNOWN: In this section we wish to derive the structural distribution for the difference of two means $\mu = \mu_1 - \mu_2$ based on samples $\underset{\sim}{x}$ from a bivariate normal distribution in which

(i) the correlation coefficient $\rho$; and

(ii) the ratio $\sigma = \sigma_2/\sigma_1$ of the standard deviations, are

known. More precisely, the derivation for the desired structural distribution is based on the following structural model

$$\begin{cases} \begin{pmatrix} 1 \\ x_{1j} \\ x_{2j} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma_1 & 0 \\ \mu_2 & 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_{1j} \\ e_{2j} \end{pmatrix}, \quad j = 1,2,\ldots,n, \\ \left\{2\pi(1-\rho^2)^{1/2}\right\}^{-n} \exp\left\{-\frac{1}{2(1-\rho^2)} \sum_{j=1}^{n} (e_{1j}^2 - 2\rho e_{1j}e_{2j} + e_{2j}^2)\right\} \\ \cdot \prod_{j=1}^{n} de_{1j}de_{2j} \end{cases}$$

where $-\infty < \mu_1, \mu_2 < \infty$, and $\sigma_1, \sigma_2 > 0$. The structural distribution for $\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma_2$, based on $\underset{\sim}{x} = (\underset{\sim}{x}_1, \underset{\sim}{x}_2)$, $\underset{\sim}{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$, is

$$g(\mu_1,\mu_2,\sigma_1,\sigma_2 : \underset{\sim}{x})d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

$$\alpha \exp\left\{ -a^2 \sum_{j=1}^{n} [(\frac{x_{ij}-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_{1j}-\mu_1}{\sigma_1})(\frac{x_{2j}-\mu_2}{\sigma_2}) + (\frac{x_{2j}-\mu_1}{\sigma_2})^2 ]\right\} \frac{d\mu_1 d\mu_2 d\sigma_1 d\sigma_2}{(\sigma_1 \sigma_2)^{n+1}}$$

where $a^2 = \{2(1-\rho^2)\}^{-1}$. Making use of the substitution

$$\begin{cases} \mu_i = \mu_i, \quad i = 1,2 \\ \sigma_1 = \sigma_1 \\ \sigma = \sigma_2/\sigma_1 \end{cases}$$

and then conditioning on $\sigma = c$, we obtain the structural distribution for $\mu_1$, $\mu_2$ and $\sigma_1$ based on $\underset{\sim}{x}$ and given $\sigma = c$:

$$g(\mu_1,\mu_2\sigma:\underset{\sim}{x},c)d\mu_1 d\mu_2 d\sigma_1 \alpha \exp\left\{-a^2A^2(\underset{\sim}{x},\mu_1,\mu_2,c)/\sigma_1^2\right\} \frac{d\mu_1 d\mu_2 d\sigma_1}{\sigma_1^{2n+1}}$$

where

$$A^2(\underset{\sim}{x},\mu_1,\mu_2,c) = \sum_{j=1}^{n}[(x_{1j}-\mu_1)^2-2\rho(x_{1j}-\mu)(x_{2j}-\mu_2)/c + (x_{2j}-\mu_2)^2/c^2]$$

$$= n\left\{s_1^2+(\mu_1-\bar{x}_1)^2-2\rho[s_{12}+(\mu_1-\bar{x}_1)(\mu_2-\bar{x}_2)]+s_2^2+(\mu_2-\bar{x}_2)^2/c^2\right\}$$

Therefore the structural distribution for $\mu_1$ and $\mu_2$, based on $\underset{\sim}{x}$, is

$$g(\mu_1,\mu_2:\underset{\sim}{x},c)d\mu_1 d\mu_2 \alpha \int_o^\infty \exp\left\{-a^2A^2(\underset{\sim}{x},\mu_1,\mu_2,c)/\sigma_1^2\right\} \frac{d\sigma_1 \cdot d\mu_1 d\mu_2}{\sigma_1^{2n+1}}$$

$$\alpha \quad \frac{d\mu_1 d\mu_2}{\{a^2A^2(\underset{\sim}{x},\mu_1,\mu_2,c)\}^n}$$

$$\alpha \frac{d\mu}{\left\{s^2+(\mu_1-\bar{x}_1)^2-2\rho(\mu_1-\bar{x}_1)(\mu_2-\bar{x}_2)/c+(\mu_2-\bar{x}_2)^2/c^2\right\}^n}$$

$$\alpha \frac{d\mu_1 d\mu_2}{\left\{1+(t_1^*,t_2^*)R^{*-1}(t_1^*,t_2^*)'/(2n-2)\right\}^{(2n-2+2)/2}}$$

where $s^2 = s_1^2 - 2\rho s_{12}/c + s_2^2/c^2$, $t_1^* = (2n-2)^{1/2}(\mu_1-\bar{x}_1)/s$, $t_2^* = (2n-2)^{1/2}(\mu_2-\bar{x}_2)/s$, and $R^{*-1}$ is the matrix

$$\begin{pmatrix} 1 & -\rho/c \\ -\rho/c & 1/c^2 \end{pmatrix} .$$

Therefore the variables $t_1^*$ and $t_2^*$ have a bivariate t-distribution characterized by the matrix $R^{*-1}$. Note that

$$R^* = \begin{pmatrix} 1 & -\rho/c \\ -\rho/c & 1/c^2 \end{pmatrix}^{-1}$$

$$= \begin{bmatrix} 1/(1-\rho^2) & \rho c/(1-\rho^2) \\ \rho c/(1-\rho^2) & c^2/(1-\rho^2) \end{bmatrix}.$$

Let $t_1 = t_1^* - t_2^*$, and $t_2 = t_2^*$. Then $t_1$ and $t_2$ has a bivariate t-distribution characterized by $R^{-1}$ where $R^{-1}$ is

$$R = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} R^* \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1-2\rho c+c^2)/(1-\rho^2) & (\rho c-c^2)/(1-\rho^2) \\ (\rho c-c^2)/(1-\rho^2) & c^2/(1-\rho^2) \end{pmatrix} .$$

Therefore the marginal distribution for $t_1$ has a Student's t-distribution with (2n-2) degrees of freedom characterized by $[(1-\rho^2)/(1-2\rho c+c^2)]$. Since $t_1 = t_1^*-t_2^* = (2n-2)^{1/2}(\mu-\bar{x}_1+\bar{x}_2)/s$, it follows that the structural distribution for $\mu$, based on $\underset{\sim}{x}$, is

$$g(\mu:\underset{\sim}{x},c)d\mu \;\alpha\; \frac{d\mu}{\{1+(\mu-\bar{x}_1+\bar{x}_2)^2(1-\rho^2)/s^2c^2(1-2\rho/c+1/c^2)\}^{(2n-1)/2}} .$$

The constant of proportionality $K(\rho,n,c)$ is

$$K(\rho,n,c) = \Gamma(\frac{2n-1}{2})\left\{\pi^{1/2}\Gamma(\frac{2n-2}{2})[\frac{1-\rho^2}{s^2c^2(1-2\rho/c+1/c^2)}]^{1/2}\right\}^{-1} .$$

Furthermore, we note that the variable

$$t(\rho,n,c) = (2n-2)^{1/2}(\mu-\bar{x}_1+\bar{x}_2)(1-\rho^2)^{1/2}/\{sc(1-2\rho/c+1/c^2)^{1/2}\}$$

has a Student's t-distribution with (2n-2) degrees of freedom. Therefore the structural distribution for $\mu$, based on $\underset{\sim}{x}$, can be rewritten in the form

$$\mu = (\bar{x}_1-\bar{x}_2) - \left\{\frac{sc(1-2\rho/c+1/c^2)^{1/2}}{[(2n-2)(1-\rho^2)]^{1/2}}\right\} t(\rho,n,c).$$

The following two particular cases are of interest:

(i) When $\rho = 0$, we have

$$\mu = x_1-\bar{x}_2 - \left\{\frac{c^2(1+1/c^2)}{(2n-2)} (s_1^2+s_2^2/c^2)\right\}^{1/2}t(0,n,c);$$

and

(ii) When $\rho = 0$ and $c = 1$, we have

$$\mu = \bar{x}_1-\bar{x}_2 - \{(s_1^2+s_2^2)/(n-1)\}^{1/2}t(0,n,1)$$

which agree with the results of the particular cases (i) and (ii) of Section 3.3 respectively.

## 3.6. A GENERALIZATION OF BEHRENS-FISHER PROBLEM FOR INDEPENDENT POPULATIONS WITH KNOWN RATIOS OF STANDARD DEVIATIONS:

In this section we consider a generalization of the Behrens-Fisher problem considered in Section 3.3. Let k be a fixed integer greater than two. Suppose we have the following k independent measurement models:

$$\begin{cases} x_{ij} = \mu_i + \sigma_i e_{ij}, \; j = 1, 2, \ldots, n_i, \\ (2\pi)^{-n_i/2} \exp\left\{- \sum_{j=1}^{n_i} e_{ij}^2/2\right\} \prod_{j=1}^{n_i} de_{ij} \end{cases}$$

for each $i = 1,2,\ldots,k$. Based on the above structural models, our purpose is to derive the structural distribution for (k-1) differences of two means $\mu_i^* = \mu_1 - \mu_i$, $i = 2, 3, \ldots, k$, under the condition that the (k-1) ratios $\sigma_i^* = \sigma_i/\sigma_1$, $i = 2,3,\ldots,k$, of the corresponding standard deviations are known. The case for k = 3, is considered with greater details.

The structural distribution for $\mu_1, \ldots, \mu_k, \sigma_1, \ldots, \sigma_k$, based on $\underset{\sim}{x}_i = (x_{i1}, \ldots, x_{in_i})$, $i = 1,2,\ldots,k$, is

$$g(\mu_1,\ldots,\mu_k,\sigma_1,\ldots,\sigma_k : \underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k) \prod_{i=1}^{k} d\mu_i d\sigma_i$$

$$\alpha \exp\left\{- \sum_{i=1}^{k} \frac{n_i}{2\sigma_i^2} [(\mu_i - \bar{x}_i)^2 + s_i^2]\right\} \cdot \prod_{i=1}^{k} \left\{\sigma_i^{-(n_i+1)} d\mu_i d\sigma_i\right\} .$$

Applying the substitutions

$$\begin{cases} \mu_i = \mu_i, \ i = 1, 2, \ldots, k \\ \sigma_1 = \sigma_1 \\ \sigma_i^* = \sigma_i/\sigma_1, \ i = 2, 3, \ldots, k \end{cases}$$

having jacobian $\sigma_1^{(k-1)}$, we obtain the structural distribution for $\mu_i$, $i = 1, 2, \ldots, k$, $\sigma_1$ and $\sigma_i^*$, $i = 2, 3, \ldots, k$, based on $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k$, as

$$g(\mu_1, \ldots, \mu_k, \sigma_1, \sigma_2^*, \ldots, \sigma_k^* : \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k) \prod_{i=1}^{k} d\mu_i \cdot d\sigma_1 \cdot \prod_{i=2}^{k} d\sigma_i^*$$

$$\alpha \exp\left\{ - \sum_{i=1}^{k} \frac{n_i}{2(\sigma_1\sigma_i^*)^2} [(\mu_i - \bar{x}_i)^2 + s_i^2] \right\} \sigma_1^{k-1} \prod_{i=1}^{k} (\sigma_1\sigma_i^*)^{-(n_i+1)} d\mu_i \cdot d\sigma_1 \cdot \prod_{i=2}^{k} d\sigma_i^*$$

where $\sigma_1^* = 1$. By conditioning on $\sigma_i^* = c_i$, $i = 2, 3, \ldots, k$, $c_1 = 1$, we obtain the structural distribution for $\mu_i$, $i = 1,2,\ldots,k$, and $\sigma_1$, based on $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k$, as

$$g(\mu_1, \ldots, \mu_k, \sigma_1 : \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k, \underset{\sim}{c}) d\sigma_1 \cdot \prod_{i=1}^{k} d\mu_i$$

$$\alpha \exp\left\{ - \sum_{i=1}^{k} \frac{n_i}{2\sigma_1^2 c_i^2} [(\mu_i - \bar{x})^2 + s_i^2] \right\} \sigma_1^{-(N+1)} d\sigma_1 \cdot \prod_{i=1}^{k} d\mu_i$$

where $\underset{\sim}{c} = (c_2, c_3, \ldots, c_k)$ and $N = \sum_{i=1}^{k} n_i$. By eliminating $\sigma_1$ from the last expression, we obtain

$$g(\mu_1, \ldots, \mu_k : \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k, \underset{\sim}{c}) \prod_{i=1}^{k} d\mu_i$$

$$= \int_0^\infty g(\mu_1, \ldots, \mu_k, \sigma_1 : \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k, \underset{\sim}{c}) d\sigma_1 \cdot \prod_{i=1}^{k} d\mu_i$$

$$\propto \frac{\prod_{i=1}^{k} d\mu_i}{\left\{\sum_{i=1}^{k} \frac{n_i}{c_i^2}\left[(\mu_i-\bar{x}_i)^2+s_i^2\right]\right\}^{N/2}}$$

$$\propto \frac{\prod_{i=1}^{k} d\mu_i}{\left\{1+\sum_{i=1}^{k} n_i(\mu_i-\bar{x}_i)^2/(c_i^2 s^2)\right\}^{N/2}}$$

$$\propto \frac{\prod_{i=1}^{k} d\mu_i}{\left\{1+t^{*\prime}R^{*-1}t^{*}/(N-k)\right\}^{(N-k+k)/2}}$$

where $s^2 = \sum_{i=1}^{k} n_i s_i^2/c_i^2$, $t^{*\prime} = (t_1^*, \ldots, t_k^*)$, $t_i^* = (N-k)^{1/2}(\mu_i-\bar{x}_i)/s^2$, $i = 1, 2, \ldots, k$ and $R^{*-1}$ is a $k\times k$ diagonal matrix

$$R^{*-1} = \mathrm{diag}(n_1, n_2/c_2^2, \ldots, n_k/c_k).$$

Therefore the variables $t_i^*$, $i = 1, 2, \ldots, k$ has a multivariate t-distribution of order k and characterized by the matrix $R^{*-1}$. Define

$$(t_1, t_2, \ldots, t_k)' = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & & 0 \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 \ldots & 0 & 1 \end{pmatrix} (t_1^*, t_2^*, \ldots, t_k^*)'$$

$$= A(t_1^*, t_2^*, \ldots, t_k^*)', \text{ say.}$$

Then the variables $t_1$, $t_2$, ..., $t_k$ has a multivariate t-distribution of order k and characterized by the matrix $R^{-1}$, where

$$R = AR^*A'$$

$$= \begin{pmatrix} 1/n_1+c_2^2/n_2 & & & & * \\ 1/n_1 & 1/n_1+c_3^2/n_3 & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \\ 1/n_1 & \ldots & & 1/n_1+c_k^2/n_k & \\ 1/n_1 & \ldots & & 1/n_1 & c_k^2/n_k \end{pmatrix}$$

$$= \left(\begin{array}{c|c} R_1 & * \\ \hline 1/n_1 \ldots 1/n_1 & c_k^2/n_k \end{array}\right)$$

It follows that the marginal distribution for $t_1, t_2, \ldots, t_{k-1}$ is a multivariate t-distribution of order (k-1) and characterized by the matrix $R_1^{-1}$. But

$$t_i = t_1^* - t_{i+1}^*$$
$$= (N-k)^{1/2}(\mu_{i+1}^* - \bar{x}_1 + \bar{x}_{i+1}), \mu_{i+1}^* = \mu_1 - \mu_{i+1}, \quad i=1,2,\ldots,(k-1),$$

so that the structural distribution for $\mu_2^*, \ldots, \mu_k^*$, based on $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k$, is given by

$$g(\mu_2^*, \ldots, \mu_k^* : \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k, \underset{\sim}{c}) \prod_{i=2}^{k} d\mu_i^*$$

$$\alpha \frac{\prod_{i=2}^{k} d\mu_i^*}{\left\{1 + \underset{\sim}{t}' R_1^{-1} \underset{\sim}{t}/(N-k)\right\}^{(N-k+k-1)/2}}$$

where $\underset{\sim}{t} = (t_1, t_2, \ldots, t_{k-1})$. The constant of proportionality is

$$\frac{\Gamma(\frac{n-1}{2}) |R_1|^{-1/2} (N-k)^{(k-1)/2}}{\left\{\pi(N-k)\right\}^{(k-1)/2} \Gamma(\frac{N-k}{2}) s^{k-1}}$$

$$= \frac{\Gamma(\frac{n-1}{2}) |R_1|^{-1/2}}{(\pi s^2)^{(k-1)/2} \Gamma(\frac{N-k}{2})} .$$

Dunnett and Sobel (1954) have also considered multivariate generalization of t-distribution. In that paper, they obtained expressions for probability integral for the bivariate case:

$$P_n(h,k;\rho) = \int_{-\infty}^{k} \int_{-\infty}^{h} g_n(t_1, t_2 : \rho) dt_1 dt_2$$

where

$$g_n(t_1,t_2:\rho) = \frac{1}{2\pi(1-\rho^2)}\left\{1+\frac{t_1^2-2\rho t_1 t_2+t_2^2}{n(1-\rho^2)}\right\}^{-(n+2)/2} .$$

The expressions are:

$$P_n(h,k;\rho) = \frac{1}{2\pi}\text{ arc tan }\frac{\sqrt{(1-\rho^2)}}{-\rho}$$

$$+\frac{k}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}n}\frac{\Gamma(j-\frac{1}{2})}{\Gamma(j)}\frac{1}{(1+k^2/n)^{j-1/2}}[1+\text{sgn}(h-\rho k)I_{x(n,h,k)}(\tfrac{1}{2},j-\tfrac{1}{2})]$$

$$+\frac{h}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}n}\frac{\Gamma(j-\frac{1}{2})}{\Gamma(j)}\frac{1}{(1+h^2/n)^{j-1/2}}[1+\text{sgn}(k-\rho h)I_{x(n,k,h)}(\tfrac{1}{2},j-\tfrac{1}{2})],$$

for even n; and

$$P_n(h,k;\rho)$$

$$=\frac{1}{2\pi}\text{ arc tan}\left\{\sqrt{n}\left[\frac{-(h+k)(hk+\rho n)-(hk-n)\sqrt{(h^2-2\rho hk+k^2+n[1-\rho^2])}}{(hk-n)(hk+\rho n)-n(h+k)\sqrt{(h^2-2\rho hk+k^2+n[1-\rho^2])}}\right]\right\}$$

$$+\frac{k}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}(n-1)}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\frac{1}{(1+k^2/n)^j}[1+\text{sgn}(h-\rho k)I_{x(n,h,k)}(\tfrac{1}{2},j)]$$

$$+\frac{h}{4\sqrt{(n\pi)}}\sum_{j=1}^{\frac{1}{2}(n-1)}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\frac{1}{(1+h^2/n)^j}[1+\text{sgn}(k-\rho h)I_{x(n,k,h)}(\tfrac{1}{2},j)],$$

for odd n, where

$$\text{sgn}(x)=\begin{cases}+1, & \text{if } x \geq 0,\\ -1, & \text{otherwise;}\end{cases}$$

$$x(m,h,k)=\frac{(h-\rho k)^2}{(h-\rho k)^2+(1-\rho^2)(m+k^2)},$$

$$I_x(\tfrac{1}{2},j-\tfrac{1}{2}) = \frac{2}{\pi}\text{ arc tan }\sqrt{(\frac{x}{1-x})}+\frac{2}{\pi}\sqrt{\left\{x(1-x)\right\}}\sum_{i=0}^{j-2}\frac{4^i(i!)^2}{(2i+1)!}(1-x)^i,$$

and

$$I_x(\tfrac{1}{2},j) = \sqrt{x} \sum_{i=o}^{j-1} \frac{(2i)!}{4^i (i!)^2} (1-x)^i \ .$$

They also provided numerical values for $P_n(h,k;\rho)$ with h=k=t. These values are tablulated in two tables for t = 0.00(0.25)2.50(0.50)10.0 and n=1(1)30(3)45(15)120, 150, 300, 600, $\infty$ , for $\rho$ = 0.50 and -0.50.

We note from the symmetry of $g_n(t_1, t_2;\rho)$ that $P_n(h,k;\rho) = P_n(k,h;\rho)$ for any h and k. We have computed numerical values for $P_n(h,k;\rho)$ for selected values of h,k=0.00(0.25)3.50, n = 1(1)30, and $\rho$ = -0.75(0.25)0.75. Only those numerical values for h = k = t are tabulated in tables given at the end of this section. All computations were carried out by the IBM 7040 computer with all inputs and outputs to eight decimal places. They are rounded to five decimal places in the tables given below. In comparing some of these numerical values with those tabluated in Tables 1 and 2 of Dunnett and Sobel (1954), we find that our results agree with theirs except for a few cases which differ mostly by one or two units in the fifth decimal places. On the differences is as close as one unit in the second decimal place. For example

| N | $\rho$ | t | $P_n(t,t;\rho)$ | |
|---|---|---|---|---|
| 1 | -0.50 | 1.75 | 0.69338093 | 0.69336 |
| 1 | -0.50 | 2.00 | 0.72582472 | 0.71332 |
| 14 | -0.50 | 0.75 | 0.55031971 | 0.55033 |
| 2 | 0.50 | 0.75 | 0.60699224 | 0.60697 |
| 6 | 0.50 | 3.00 | 0.97892956 | 0.97894 |

where the values in the last column are taken from Dunnett and Sobel's paper, while the fourth column are our computed values given with eight decimal places before rounding.

The relationship between h, P, $\rho$ and n, where P is the quantities given in the bodies of the following tables, are given by

$$P = \int_{-\infty}^{h}\int_{-\infty}^{h}\left\{2\pi(1-\rho^2)\right\}^{-1}\cdot\left\{1+\frac{u^2-2\rho uv+v^2}{n(1-\rho^2)}\right\}^{-(n+2)/2}dudv.$$

Probability Integral of Bivariate t-distribution with $\rho=-0.75$

| n/h | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 |
|---|---|---|---|---|---|---|---|
| 1 | 0.21598 | 0.33209 | 0.43569 | 0.51995 | 0.58657 | 0.63918 | 0.68113 |
| 2 | 22598 | 35838 | 48231 | 58585 | 66797 | 73167 | 78081 |
| 3 | 22992 | 36896 | 50170 | 61387 | 70285 | 77113 | 82270 |
| 4 | 23202 | 37465 | 51232 | 62936 | 72217 | 79287 | 84549 |
| 5 | 23331 | 37819 | 51902 | 63918 | 73443 | 80660 | 85974 |
| 6 | 23419 | 38062 | 52363 | 64597 | 74290 | 81604 | 86946 |
| 7 | 23482 | 38237 | 52699 | 65093 | 74908 | 82292 | 87650 |
| 8 | 23530 | 38371 | 52956 | 65472 | 75380 | 82815 | 88183 |
| 9 | 23568 | 38476 | 53158 | 65771 | 75752 | 83226 | 88600 |
| 10 | 23598 | 38560 | 53321 | 66012 | 76053 | 83558 | 88935 |
| 11 | 23623 | 38629 | 53455 | 66212 | 76300 | 83831 | 89210 |
| 12 | 23643 | 38688 | 53568 | 66379 | 76508 | 84060 | 89440 |
| 13 | 23661 | 38737 | 53664 | 66521 | 76685 | 84254 | 89634 |
| 14 | 23676 | 38779 | 53747 | 66644 | 76837 | 84421 | 89802 |
| 15 | 23689 | 38816 | 53819 | 66750 | 76970 | 84567 | 89947 |
| 16 | 23701 | 38849 | 53882 | 66844 | 77086 | 84694 | 90074 |
| 17 | 23711 | 38877 | 53938 | 66927 | 77189 | 84807 | 90186 |
| 18 | 23720 | 38903 | 53988 | 67001 | 77280 | 84907 | 90286 |
| 19 | 23728 | 38926 | 54033 | 67068 | 77363 | 84997 | 90375 |
| 20 | 23735 | 38946 | 54073 | 67127 | 77437 | 85078 | 90456 |
| 21 | 23742 | 38965 | 54109 | 67182 | 77504 | 85152 | 90529 |
| 22 | 23748 | 38982 | 54143 | 67231 | 77565 | 85218 | 90595 |
| 23 | 23753 | 38997 | 54173 | 67276 | 77621 | 85280 | 90655 |
| 24 | 23758 | 39011 | 54201 | 67318 | 77672 | 85336 | 90711 |
| 25 | 23763 | 39025 | 54227 | 67356 | 77720 | 85387 | 90762 |
| 26 | 23767 | 39037 | 54251 | 67391 | 77763 | 85435 | 90809 |
| 27 | 23771 | 39048 | 54273 | 67424 | 77804 | 85479 | 90852 |
| 28 | 23775 | 39058 | 54293 | 67455 | 77842 | 85520 | 90893 |
| 29 | 23778 | 39068 | 54312 | 67483 | 77877 | 85558 | 90931 |
| 30 | 23781 | 39077 | 54330 | 67510 | 77910 | 85594 | 90966 |

Probability Integral of Bivariate t-distribution with $\rho$=-0.75

| n/h | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 |
|---|---|---|---|---|---|---|---|
| 1 | 0.71503 | 0.74283 | 0.76594 | 0.78541 | 0.80200 | 0.81628 | 0.82869 |
| 2 | 81885 | 84852 | 87191 | 89056 | 90560 | 91786 | 92796 |
| 3 | 86144 | 89061 | 91270 | 92957 | 94257 | 95270 | 96067 |
| 4 | 88419 | 91256 | 93337 | 94872 | 96013 | 96867 | 97514 |
| 5 | 89821 | 92581 | 94556 | 95972 | 96992 | 97732 | 98273 |
| 6 | 90765 | 93460 | 95349 | 96672 | 97600 | 98254 | 98718 |
| 7 | 91442 | 94082 | 95902 | 97150 | 98006 | 98594 | 99001 |
| 8 | 91951 | 94544 | 96306 | 97494 | 98293 | 98830 | 99192 |
| 9 | 92346 | 94900 | 96614 | 97752 | 98504 | 99000 | 99328 |
| 10 | 92662 | 95182 | 96855 | 97952 | 98666 | 99128 | 99427 |
| 11 | 92920 | 95411 | 97049 | 98111 | 98792 | 99226 | 99503 |
| 12 | 93135 | 95600 | 97209 | 98240 | 98893 | 99304 | 99562 |
| 13 | 93316 | 95759 | 97341 | 98346 | 98976 | 99367 | 99609 |
| 14 | 93471 | 95895 | 97453 | 98436 | 99045 | 99419 | 99646 |
| 15 | 93606 | 96011 | 97549 | 98511 | 99103 | 99462 | 99678 |
| 16 | 93723 | 96113 | 97633 | 98577 | 99152 | 99498 | 99704 |
| 17 | 93826 | 96202 | 97705 | 98633 | 99195 | 99529 | 99726 |
| 18 | 93918 | 96280 | 97769 | 98683 | 99231 | 99555 | 99744 |
| 19 | 94000 | 96351 | 97826 | 98727 | 99264 | 99579 | 99760 |
| 20 | 94074 | 96414 | 97877 | 98765 | 99292 | 99599 | 99774 |
| 21 | 94140 | 96470 | 97922 | 98800 | 99318 | 99617 | 99787 |
| 22 | 94200 | 96522 | 97963 | 98831 | 99341 | 99633 | 99798 |
| 23 | 94256 | 96568 | 98001 | 98860 | 99361 | 99647 | 99807 |
| 24 | 94306 | 96611 | 98035 | 98885 | 99379 | 99660 | 99816 |
| 25 | 94352 | 96650 | 98066 | 98909 | 99396 | 99671 | 99823 |
| 26 | 94395 | 96686 | 98094 | 98930 | 99411 | 99682 | 99830 |
| 27 | 94435 | 96720 | 98121 | 98950 | 99425 | 99691 | 99837 |
| 28 | 94471 | 96751 | 98145 | 98968 | 99438 | 99700 | 99842 |
| 29 | 94506 | 96779 | 98167 | 98984 | 99450 | 99708 | 99848 |
| 30 | 94538 | 96806 | 98188 | 99000 | 99461 | 99715 | 99852 |

Probability Integral of Bivariate t-distribution with $\rho$=-0.50

| n/h | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 |
|---|---|---|---|---|---|---|---|
| 1 | 0.26044 | 0.36554 | 0.46120 | 0.54022 | 0.60327 | 0.65333 | 0.69338 |
| 2 | 26993 | 38889 | 50259 | 59968 | 67778 | 73892 | 78635 |
| 3 | 27368 | 39817 | 51957 | 62478 | 70974 | 77566 | 82580 |
| 4 | 27568 | 40313 | 52878 | 63863 | 72750 | 79605 | 84747 |
| 5 | 27691 | 40620 | 53455 | 64740 | 73881 | 80900 | 86111 |
| 6 | 27774 | 40829 | 53851 | 65345 | 74663 | 81794 | 87046 |
| 7 | 27835 | 40980 | 54139 | 65788 | 75237 | 82449 | 87727 |
| 8 | 27880 | 41095 | 54359 | 66126 | 75675 | 82948 | 88245 |
| 9 | 27916 | 41185 | 54531 | 66393 | 76021 | 83342 | 88651 |
| 10 | 27945 | 41257 | 54670 | 66608 | 76302 | 83661 | 88978 |
| 11 | 27968 | 41316 | 54784 | 66786 | 76533 | 83923 | 89247 |
| 12 | 27988 | 41366 | 54880 | 66936 | 76727 | 84144 | 89472 |
| 13 | 28005 | 41408 | 54962 | 67063 | 76893 | 84331 | 89663 |
| 14 | 28019 | 41444 | 55032 | 67172 | 77036 | 84493 | 89827 |
| 15 | 28031 | 41476 | 55093 | 67268 | 77160 | 84633 | 89970 |
| 16 | 28042 | 41503 | 55147 | 67352 | 77269 | 84757 | 90095 |
| 17 | 28052 | 41528 | 55194 | 67426 | 77366 | 84866 | 90205 |
| 18 | 28061 | 41549 | 55236 | 67492 | 77452 | 84963 | 90304 |
| 19 | 28068 | 41569 | 55274 | 67551 | 77529 | 85051 | 90392 |
| 20 | 28075 | 41586 | 55308 | 67605 | 77599 | 85129 | 90471 |
| 21 | 28081 | 41602 | 55339 | 67653 | 77663 | 85201 | 90543 |
| 22 | 28087 | 41617 | 55367 | 67698 | 77720 | 85266 | 90608 |
| 23 | 28092 | 41630 | 55393 | 67738 | 77773 | 85325 | 90668 |
| 24 | 28097 | 41642 | 55417 | 67775 | 77822 | 85380 | 90723 |
| 25 | 28102 | 41653 | 55438 | 67810 | 77866 | 85430 | 90774 |
| 26 | 28106 | 41663 | 55458 | 67841 | 77908 | 85477 | 90820 |
| 27 | 28109 | 41673 | 55477 | 67871 | 77946 | 85520 | 90863 |
| 28 | 28113 | 41682 | 55494 | 67898 | 77982 | 85560 | 90903 |
| 29 | 28116 | 41690 | 55510 | 67923 | 78015 | 85597 | 90941 |
| 30 | 28119 | 41698 | 55526 | 67947 | 78046 | 85632 | 90976 |

Probability Integral of Bivariate t-distribution with $\rho$=-0.50

| n/h | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 |
|---|---|---|---|---|---|---|---|
| 1 | 0.72582 | 0.75247 | 0.77465 | 0.79334 | 0.80928 | 0.82301 | 0.83495 |
| 2 | 82320 | 85202 | 87479 | 89296 | 90763 | 91960 | 92947 |
| 3 | 86365 | 89222 | 91391 | 93050 | 94330 | 95328 | 96114 |
| 4 | 88547 | 91341 | 93396 | 94914 | 96043 | 96890 | 97531 |
| 5 | 89902 | 92631 | 94588 | 95993 | 97007 | 97742 | 98280 |
| 6 | 90820 | 93492 | 95368 | 96684 | 97607 | 98259 | 98721 |
| 7 | 91482 | 94103 | 95914 | 97157 | 98010 | 98597 | 99002 |
| 8 | 91981 | 94559 | 96314 | 97499 | 98295 | 98831 | 99193 |
| 9 | 92369 | 94911 | 96619 | 97755 | 98506 | 99001 | 99328 |
| 10 | 92681 | 95191 | 96859 | 97954 | 98667 | 99128 | 99428 |
| 11 | 92935 | 95418 | 97052 | 98112 | 98793 | 99227 | 99503 |
| 12 | 93148 | 95605 | 97211 | 98241 | 98894 | 99305 | 99562 |
| 13 | 93327 | 95763 | 97343 | 98347 | 98976 | 99367 | 99609 |
| 14 | 93481 | 95898 | 97455 | 98436 | 99045 | 99419 | 99647 |
| 15 | 93614 | 96014 | 97550 | 98512 | 99103 | 99462 | 99678 |
| 16 | 93730 | 96115 | 97633 | 98577 | 99152 | 99498 | 99704 |
| 17 | 93832 | 96204 | 97706 | 98633 | 99195 | 99529 | 99726 |
| 18 | 93924 | 96282 | 97770 | 98683 | 99232 | 99555 | 99744 |
| 19 | 94005 | 96352 | 97826 | 98727 | 99264 | 99579 | 99760 |
| 20 | 94078 | 96415 | 97877 | 98765 | 99292 | 99599 | 99774 |
| 21 | 94144 | 96472 | 97923 | 98800 | 99318 | 99617 | 99787 |
| 22 | 94204 | 96523 | 97964 | 98831 | 99341 | 99633 | 99798 |
| 23 | 94259 | 96569 | 98001 | 98860 | 99361 | 99647 | 99807 |
| 24 | 94309 | 96612 | 98035 | 98885 | 99379 | 99660 | 99816 |
| 25 | 94356 | 96651 | 98066 | 98909 | 99396 | 99671 | 99823 |
| 26 | 94398 | 96687 | 98094 | 98930 | 99411 | 99682 | 99830 |
| 27 | 94438 | 96720 | 98121 | 98950 | 99425 | 99691 | 99837 |
| 28 | 94474 | 96751 | 98145 | 98968 | 99438 | 99700 | 99842 |
| 29 | 94508 | 96780 | 98168 | 98985 | 99450 | 99708 | 99848 |
| 30 | 94540 | 96807 | 98189 | 99000 | 99461 | 99715 | 99852 |

Probability Integral of Bivariate t-distribution with ρ=-0.25

| n/h | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 |
|---|---|---|---|---|---|---|---|
| 1 | 0.29975 | 0.39758 | 0.48688 | 0.56118 | 0.62081 | 0.66833 | 0.70645 |
| 2 | 30907 | 41957 | 52517 | 61617 | 69004 | 74825 | 79363 |
| 3 | 31276 | 42829 | 54074 | 63920 | 71958 | 78250 | 83067 |
| 4 | 31472 | 43293 | 54915 | 65184 | 73596 | 80152 | 85107 |
| 5 | 31594 | 43581 | 55440 | 65982 | 74638 | 81361 | 86396 |
| 6 | 31676 | 43776 | 55799 | 66531 | 75358 | 82198 | 87282 |
| 7 | 31735 | 43918 | 56060 | 66933 | 75886 | 82810 | 87929 |
| 8 | 31780 | 44025 | 56258 | 67239 | 76290 | 83279 | 88421 |
| 9 | 31815 | 44109 | 56413 | 67480 | 76608 | 83648 | 88808 |
| 10 | 31844 | 44176 | 56539 | 67675 | 76866 | 83947 | 89120 |
| 11 | 31867 | 44231 | 56642 | 67835 | 77079 | 84194 | 89378 |
| 12 | 31886 | 44278 | 56728 | 67970 | 77258 | 84401 | 89593 |
| 13 | 31903 | 44317 | 56802 | 68085 | 77410 | 84577 | 89776 |
| 14 | 31917 | 44351 | 56865 | 68184 | 77541 | 84730 | 89934 |
| 15 | 31929 | 44380 | 56920 | 68270 | 77656 | 84862 | 90071 |
| 16 | 31940 | 44406 | 56968 | 68345 | 77756 | 84978 | 90191 |
| 17 | 31949 | 44428 | 57010 | 68412 | 77845 | 85081 | 90297 |
| 18 | 31958 | 44449 | 57048 | 68471 | 77925 | 85173 | 90392 |
| 19 | 31965 | 44467 | 57082 | 68525 | 77996 | 85255 | 90477 |
| 20 | 31972 | 44483 | 57113 | 68573 | 78060 | 85330 | 90553 |
| 21 | 31978 | 44498 | 57141 | 68617 | 78119 | 85397 | 90622 |
| 22 | 31984 | 44511 | 57166 | 68657 | 78172 | 85459 | 90685 |
| 23 | 31989 | 44523 | 57189 | 68693 | 78220 | 85515 | 90743 |
| 24 | 31994 | 44535 | 57210 | 68726 | 78265 | 85566 | 90796 |
| 25 | 31998 | 44545 | 57230 | 68757 | 78306 | 85614 | 90845 |
| 26 | 32002 | 44555 | 57248 | 68786 | 78344 | 85658 | 90890 |
| 27 | 32006 | 44564 | 57264 | 68812 | 78379 | 85699 | 90931 |
| 28 | 32009 | 44572 | 57280 | 68837 | 78412 | 85737 | 90970 |
| 29 | 32012 | 44579 | 57294 | 68859 | 78443 | 85772 | 91006 |
| 30 | 32015 | 44587 | 57308 | 68881 | 78471 | 85805 | 91040 |

Probability Integral of Bivariate t-distribution with $\rho$=-0.25

| n/h | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 |
|---|---|---|---|---|---|---|---|
| 1 | 0.73739 | 0.76283 | 0.78402 | 0.80190 | 0.81715 | 0.83030 | 0.84173 |
| 2 | 82900 | 85674 | 87869 | 89624 | 91042 | 92200 | 93155 |
| 3 | 86720 | 89488 | 91594 | 93207 | 94455 | 95428 | 96196 |
| 4 | 88791 | 91510 | 93516 | 95001 | 96107 | 96938 | 97568 |
| 5 | 90082 | 92748 | 94666 | 96046 | 97043 | 97768 | 98298 |
| 6 | 90961 | 93577 | 95422 | 96718 | 97630 | 98274 | 98731 |
| 7 | 91596 | 94169 | 95952 | 97180 | 98025 | 98606 | 99008 |
| 8 | 92076 | 94612 | 96343 | 97515 | 98305 | 98837 | 99197 |
| 9 | 92451 | 94954 | 96642 | 97768 | 98513 | 99005 | 99330 |
| 10 | 92752 | 95226 | 96878 | 97964 | 98672 | 99131 | 99429 |
| 11 | 92998 | 95448 | 97067 | 98120 | 98796 | 99229 | 99504 |
| 12 | 93204 | 95632 | 97223 | 98247 | 98897 | 99306 | 99563 |
| 13 | 93378 | 95787 | 97353 | 98352 | 98979 | 99368 | 99609 |
| 14 | 93528 | 95919 | 97464 | 98440 | 99047 | 99420 | 99647 |
| 15 | 93657 | 96033 | 97558 | 98515 | 99104 | 99462 | 99678 |
| 16 | 93770 | 96132 | 97640 | 98580 | 99153 | 99498 | 99704 |
| 17 | 93870 | 96219 | 97712 | 98636 | 99196 | 99529 | 99726 |
| 18 | 93959 | 96296 | 97775 | 98685 | 99232 | 99556 | 99744 |
| 19 | 94039 | 96365 | 97832 | 98729 | 99265 | 99579 | 99761 |
| 20 | 94110 | 96427 | 97882 | 98767 | 99293 | 99599 | 99775 |
| 21 | 94175 | 96483 | 97927 | 98802 | 99318 | 99617 | 99787 |
| 22 | 94234 | 96534 | 97967 | 98833 | 99341 | 99633 | 99798 |
| 23 | 94287 | 96580 | 98004 | 98861 | 99361 | 99647 | 99807 |
| 24 | 94336 | 96622 | 98038 | 98886 | 99380 | 99660 | 99816 |
| 25 | 94382 | 96660 | 98069 | 98910 | 99397 | 99671 | 99823 |
| 26 | 94423 | 96696 | 98097 | 98931 | 99412 | 99682 | 99830 |
| 27 | 94462 | 96729 | 98124 | 98951 | 99426 | 99691 | 99837 |
| 28 | 94498 | 96759 | 98148 | 98969 | 99439 | 99700 | 99842 |
| 29 | 94531 | 96788 | 98170 | 98985 | 99450 | 99708 | 99848 |
| 30 | 94562 | 96814 | 98191 | 99001 | 99461 | 99715 | 99852 |

Probability Integral of t-distribution with ρ=0.00

| n/h | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 |
|---|---|---|---|---|---|---|---|
| 1 | 0.33735 | 0.42963 | 0.51344 | 0.58333 | 0.63959 | 0.68454 | 0.72065 |
| 2 | 34659 | 45081 | 54960 | 63490 | 70444 | 75949 | 80255 |
| 3 | 35025 | 45920 | 56424 | 65635 | 73196 | 79148 | 83727 |
| 4 | 35219 | 46367 | 57212 | 66808 | 74716 | 80920 | 85638 |
| 5 | 35340 | 46644 | 57703 | 67546 | 75680 | 82046 | 86846 |
| 6 | 35421 | 46832 | 58039 | 68054 | 76346 | 82824 | 87678 |
| 7 | 35480 | 46968 | 58282 | 68424 | 76833 | 83394 | 88284 |
| 8 | 35525 | 47071 | 58467 | 68706 | 77206 | 83830 | 88747 |
| 9 | 35560 | 47152 | 58612 | 68927 | 77499 | 84173 | 89110 |
| 10 | 35588 | 47217 | 58729 | 69106 | 77737 | 84451 | 89404 |
| 11 | 35611 | 47270 | 58825 | 69254 | 77933 | 84681 | 89646 |
| 12 | 35630 | 47314 | 58905 | 69378 | 78098 | 84874 | 89849 |
| 13 | 35647 | 47352 | 58973 | 69483 | 78238 | 85038 | 90021 |
| 14 | 35661 | 47385 | 59032 | 69574 | 78359 | 85179 | 90170 |
| 15 | 35673 | 47413 | 59083 | 69653 | 78464 | 85302 | 90299 |
| 16 | 35683 | 47437 | 59128 | 69722 | 78556 | 85411 | 90412 |
| 17 | 35693 | 47459 | 59167 | 69783 | 78638 | 85507 | 90513 |
| 18 | 35701 | 47479 | 59203 | 69838 | 78711 | 85592 | 90602 |
| 19 | 35709 | 47496 | 59234 | 69887 | 78777 | 85669 | 90682 |
| 20 | 35715 | 47512 | 59263 | 69931 | 78836 | 85738 | 90754 |
| 21 | 35722 | 47526 | 59288 | 69971 | 78889 | 85801 | 90820 |
| 22 | 35727 | 47539 | 59312 | 70007 | 78938 | 85858 | 90879 |
| 23 | 35732 | 47551 | 59333 | 70040 | 78983 | 85910 | 90934 |
| 24 | 35737 | 47561 | 59353 | 70071 | 79024 | 85958 | 90984 |
| 25 | 35741 | 47571 | 59371 | 70099 | 79062 | 86002 | 91030 |
| 26 | 35745 | 47581 | 59388 | 70125 | 79096 | 86043 | 91072 |
| 27 | 35749 | 47589 | 59403 | 70149 | 79129 | 86081 | 91112 |
| 28 | 35752 | 47597 | 59418 | 70172 | 79159 | 86116 | 91148 |
| 29 | 35755 | 47604 | 59431 | 70193 | 79187 | 86149 | 91182 |
| 30 | 35758 | 47611 | 59444 | 70212 | 79213 | 86180 | 91214 |

Probability Integral of t-distribution With $\rho$=0.00

| n/h | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 |
|---|---|---|---|---|---|---|---|
| 1 | 0.75000 | 0.77416 | 0.79430 | 0.81129 | 0.82580 | 0.83831 | 0.84918 |
| 2 | 83621 | 86266 | 88362 | 90041 | 91398 | 92507 | 93423 |
| 3 | 87213 | 89863 | 91884 | 93436 | 94637 | 95576 | 96317 |
| 4 | 89163 | 91775 | 93708 | 95143 | 96214 | 97020 | 97631 |
| 5 | 90381 | 92949 | 94804 | 96142 | 97111 | 97817 | 98335 |
| 6 | 91211 | 93738 | 95526 | 96787 | 97676 | 98306 | 98753 |
| 7 | 91812 | 94302 | 96035 | 97233 | 98058 | 98628 | 99023 |
| 8 | 92267 | 94724 | 96411 | 97556 | 98330 | 98853 | 99207 |
| 9 | 92623 | 95052 | 96699 | 97800 | 98532 | 99017 | 99338 |
| 10 | 92909 | 95313 | 96926 | 97991 | 98687 | 99140 | 99434 |
| 11 | 93144 | 95526 | 97109 | 98143 | 98809 | 99236 | 99508 |
| 12 | 93339 | 95703 | 97260 | 98266 | 98907 | 99312 | 99566 |
| 13 | 93505 | 95852 | 97387 | 98369 | 98987 | 99373 | 99612 |
| 14 | 93648 | 95979 | 97494 | 98455 | 99054 | 99423 | 99649 |
| 15 | 93772 | 96089 | 97586 | 98528 | 99111 | 99466 | 99680 |
| 16 | 93880 | 96185 | 97665 | 98592 | 99159 | 99501 | 99705 |
| 17 | 93975 | 96269 | 97735 | 98647 | 99201 | 99532 | 99727 |
| 18 | 94060 | 96344 | 97797 | 98695 | 99237 | 99558 | 99745 |
| 19 | 94136 | 96410 | 97852 | 98738 | 99269 | 99581 | 99761 |
| 20 | 94205 | 96470 | 97901 | 98776 | 99297 | 99601 | 99775 |
| 21 | 94266 | 96524 | 97945 | 98809 | 99322 | 99618 | 99787 |
| 22 | 94323 | 96573 | 97984 | 98840 | 99344 | 99634 | 99798 |
| 23 | 94374 | 96618 | 98021 | 98868 | 99364 | 99648 | 99808 |
| 24 | 94421 | 96658 | 98054 | 98893 | 99382 | 99661 | 99816 |
| 25 | 94465 | 96696 | 98084 | 98916 | 99399 | 99672 | 99824 |
| 26 | 94505 | 96730 | 98112 | 98937 | 99414 | 99683 | 99831 |
| 27 | 94542 | 96762 | 98137 | 98956 | 99428 | 99692 | 99837 |
| 28 | 94576 | 96792 | 98161 | 98974 | 99440 | 99701 | 99843 |
| 29 | 94608 | 96819 | 98183 | 98990 | 99452 | 99709 | 99848 |
| 30 | 94638 | 96845 | 98203 | 99005 | 99463 | 99716 | 99853 |

Probability Integral of Bivariate t-distribution With ρ=0.25

| n/h | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 |
|---|---|---|---|---|---|---|---|
| 1 | 0.37549 | 0.46308 | 0.54187 | 0.60745 | 0.66027 | 0.70251 | 0.73649 |
| 2 | 38468 | 48369 | 57639 | 65617 | 72127 | 77288 | 81335 |
| 3 | 38832 | 49187 | 59034 | 67635 | 74699 | 80276 | 84578 |
| 4 | 39026 | 49623 | 59783 | 68734 | 76115 | 81925 | 86360 |
| 5 | 39146 | 49892 | 60250 | 69424 | 77011 | 82971 | 87484 |
| 6 | 39227 | 50076 | 60569 | 69898 | 77629 | 83693 | 88258 |
| 7 | 39286 | 50208 | 60800 | 70243 | 78081 | 84221 | 88822 |
| 8 | 39330 | 50309 | 60975 | 70506 | 78425 | 84624 | 89251 |
| 9 | 39365 | 50387 | 61112 | 70712 | 78697 | 84942 | 89590 |
| 10 | 39393 | 50450 | 61223 | 70879 | 78916 | 85199 | 89862 |
| 11 | 39416 | 50502 | 61314 | 71016 | 79098 | 85411 | 90087 |
| 12 | 39435 | 50545 | 61390 | 71131 | 79250 | 85590 | 90276 |
| 13 | 39451 | 50582 | 61455 | 71229 | 79379 | 85741 | 90436 |
| 14 | 39465 | 50614 | 61510 | 71313 | 79490 | 85872 | 90574 |
| 15 | 39477 | 50641 | 61559 | 71386 | 79587 | 85986 | 90694 |
| 16 | 39488 | 50665 | 61601 | 71451 | 79673 | 86086 | 90799 |
| 17 | 39497 | 50686 | 61639 | 71507 | 79748 | 86174 | 90893 |
| 18 | 39505 | 50705 | 61672 | 71558 | 79815 | 86253 | 90976 |
| 19 | 39513 | 50722 | 61702 | 71604 | 79876 | 86324 | 91050 |
| 20 | 39520 | 50738 | 61729 | 71644 | 79930 | 86388 | 91117 |
| 21 | 39526 | 50751 | 61753 | 71682 | 79979 | 86446 | 91178 |
| 22 | 39531 | 50764 | 61775 | 71715 | 80024 | 86498 | 91233 |
| 23 | 39536 | 50775 | 61796 | 71746 | 80066 | 86547 | 91284 |
| 24 | 39541 | 50786 | 61814 | 71775 | 80103 | 86591 | 91331 |
| 25 | 39545 | 50796 | 61831 | 71801 | 80138 | 86632 | 91373 |
| 26 | 39549 | 50805 | 61847 | 71825 | 80170 | 86669 | 91413 |
| 27 | 39553 | 50813 | 61862 | 71847 | 80200 | 86704 | 91450 |
| 28 | 39556 | 50821 | 61875 | 71868 | 80228 | 86737 | 91484 |
| 29 | 39559 | 50828 | 61888 | 71888 | 80254 | 86767 | 91515 |
| 30 | 39562 | 50834 | 61900 | 71906 | 80278 | 86796 | 91545 |

Probability Integral of Bivariate t-distribution With $\rho$=0.25

| n/h | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 |
|---|---|---|---|---|---|---|---|
| 1 | 0.76412 | 0.78687 | 0.80585 | 0.82187 | 0.83555 | 0.84735 | 0.85761 |
| 2 | 84503 | 86997 | 88976 | 90561 | 91845 | 92895 | 93762 |
| 3 | 87863 | 90366 | 92279 | 93749 | 94890 | 95782 | 96486 |
| 4 | 89685 | 92156 | 93990 | 95355 | 96375 | 97144 | 97728 |
| 5 | 90822 | 93256 | 95020 | 96297 | 97223 | 97899 | 98395 |
| 6 | 91597 | 93996 | 95700 | 96906 | 97758 | 98363 | 98794 |
| 7 | 92159 | 94525 | 96180 | 97328 | 98121 | 98670 | 99052 |
| 8 | 92584 | 94923 | 96535 | 97635 | 98380 | 98885 | 99228 |
| 9 | 92917 | 95231 | 96807 | 97867 | 98573 | 99042 | 99354 |
| 10 | 93184 | 95477 | 97022 | 98048 | 98721 | 99161 | 99447 |
| 11 | 93404 | 95677 | 97196 | 98193 | 98838 | 99253 | 99518 |
| 12 | 93587 | 95844 | 97340 | 98311 | 98932 | 99326 | 99574 |
| 13 | 93742 | 95984 | 97460 | 98409 | 99010 | 99385 | 99618 |
| 14 | 93876 | 96104 | 97562 | 98492 | 99074 | 99434 | 99655 |
| 15 | 93992 | 96208 | 97649 | 98562 | 99129 | 99475 | 99685 |
| 16 | 94093 | 96299 | 97725 | 98623 | 99175 | 99509 | 99710 |
| 17 | 94183 | 96379 | 97792 | 98676 | 99215 | 99539 | 99731 |
| 18 | 94262 | 96449 | 97851 | 98722 | 99250 | 99565 | 99749 |
| 19 | 94334 | 96512 | 97903 | 98763 | 99281 | 99587 | 99764 |
| 20 | 94398 | 96569 | 97950 | 98800 | 99308 | 99606 | 99778 |
| 21 | 94456 | 96620 | 97992 | 98832 | 99333 | 99624 | 99790 |
| 22 | 94509 | 96667 | 98030 | 98862 | 99354 | 99639 | 99800 |
| 23 | 94557 | 96709 | 98065 | 98888 | 99374 | 99653 | 99810 |
| 24 | 94601 | 96748 | 98096 | 98913 | 99391 | 99665 | 99818 |
| 25 | 94642 | 96783 | 98125 | 98935 | 99408 | 99676 | 99826 |
| 26 | 94679 | 96816 | 98152 | 98955 | 99422 | 99686 | 99832 |
| 27 | 94714 | 96846 | 98176 | 98974 | 99436 | 99696 | 99838 |
| 28 | 94746 | 96874 | 98199 | 98991 | 99448 | 99704 | 99844 |
| 29 | 94777 | 96900 | 98220 | 99007 | 99459 | 99712 | 99849 |
| 30 | 94805 | 96925 | 98240 | 99021 | 99470 | 99719 | 99854 |

Probability Integral of Bivariate t-distribution With ρ=0.50

| n/h | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 |
|---|---|---|---|---|---|---|---|
| 1 | 0.41678 | 0.50000 | 0.57384 | 0.63497 | 0.68411 | 0.72338 | 0.75495 |
| 2 | 42592 | 52017 | 60699 | 68114 | 74145 | 78925 | 82671 |
| 3 | 42955 | 52818 | 62036 | 70018 | 76550 | 81702 | 85680 |
| 4 | 43148 | 53245 | 62755 | 71052 | 77869 | 83231 | 87326 |
| 5 | 43267 | 53510 | 63202 | 71701 | 78702 | 84196 | 88362 |
| 6 | 43348 | 53689 | 63507 | 72146 | 79275 | 84862 | 89074 |
| 7 | 43407 | 53819 | 63728 | 72470 | 79693 | 85348 | 89592 |
| 8 | 43451 | 53917 | 63895 | 72716 | 80012 | 85719 | 89987 |
| 9 | 43486 | 53994 | 64027 | 72909 | 80263 | 86011 | 90297 |
| 10 | 43513 | 54056 | 64133 | 73065 | 80466 | 86247 | 90547 |
| 11 | 43536 | 54107 | 64220 | 73194 | 80633 | 86442 | 90753 |
| 12 | 43555 | 54150 | 64292 | 73301 | 80773 | 96605 | 90926 |
| 13 | 43572 | 54186 | 64354 | 73393 | 80893 | 86744 | 91073 |
| 14 | 43585 | 54217 | 64407 | 73472 | 80995 | 86864 | 91199 |
| 15 | 43598 | 54243 | 64453 | 73540 | 81085 | 86968 | 91309 |
| 16 | 43608 | 54267 | 64494 | 73600 | 81163 | 87060 | 91406 |
| 17 | 43617 | 54288 | 64530 | 73653 | 81233 | 87141 | 91491 |
| 18 | 43626 | 54306 | 64561 | 73701 | 81295 | 87213 | 91567 |
| 19 | 43633 | 54323 | 64590 | 73743 | 81350 | 87278 | 91635 |
| 20 | 43640 | 54338 | 64616 | 73781 | 81401 | 87336 | 91696 |
| 21 | 43646 | 54351 | 64639 | 73816 | 81446 | 87389 | 91752 |
| 22 | 43652 | 54364 | 64660 | 73847 | 81487 | 87438 | 91802 |
| 23 | 43657 | 54375 | 64680 | 73876 | 81525 | 87482 | 91849 |
| 24 | 43661 | 54385 | 64697 | 73903 | 81560 | 87522 | 91891 |
| 25 | 43665 | 54395 | 64714 | 73927 | 81592 | 87560 | 91930 |
| 26 | 43669 | 54404 | 64729 | 73950 | 81621 | 87594 | 91967 |
| 27 | 43673 | 54412 | 64743 | 73971 | 81649 | 87626 | 92000 |
| 28 | 43676 | 54419 | 64756 | 73990 | 81674 | 87656 | 92031 |
| 29 | 43680 | 54426 | 64768 | 74008 | 81698 | 87683 | 92060 |
| 30 | 43683 | 54433 | 64779 | 74025 | 81720 | 87709 | 92087 |

Probability Integral of Bivariate t-distribution With $\rho$=0.50

| n/h | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 |
|---|---|---|---|---|---|---|---|
| 1 | 0.78063 | 0.80178 | 0.81943 | 0.83432 | 0.84704 | 0.85801 | 0.86755 |
| 2 | 85607 | 87919 | 89754 | 91225 | 92417 | 93392 | 94197 |
| 3 | 88720 | 91038 | 92812 | 94178 | 95238 | 96067 | 96723 |
| 4 | 90402 | 92692 | 94394 | 95662 | 96612 | 97329 | 97874 |
| 5 | 91449 | 93706 | 95344 | 96533 | 97397 | 98028 | 98493 |
| 6 | 92163 | 94387 | 95972 | 97096 | 97893 | 98459 | 98864 |
| 7 | 92679 | 94875 | 96415 | 97487 | 98230 | 98745 | 99104 |
| 8 | 93070 | 95241 | 96743 | 97771 | 98470 | 98945 | 99268 |
| 9 | 93375 | 95525 | 96995 | 97986 | 98650 | 99092 | 99386 |
| 10 | 93621 | 95751 | 97194 | 98155 | 98788 | 99202 | 99473 |
| 11 | 93823 | 95936 | 97355 | 98289 | 98897 | 99289 | 99540 |
| 12 | 93991 | 96089 | 97487 | 98399 | 98985 | 99357 | 99593 |
| 13 | 94134 | 96219 | 97598 | 98490 | 99057 | 99413 | 99635 |
| 14 | 94256 | 96330 | 97693 | 98567 | 99117 | 99459 | 99669 |
| 15 | 94362 | 96425 | 97774 | 98633 | 99168 | 99497 | 99697 |
| 16 | 94455 | 96509 | 97845 | 98689 | 99212 | 99530 | 99721 |
| 17 | 94538 | 96582 | 97906 | 98739 | 99250 | 99558 | 99741 |
| 18 | 94611 | 96648 | 97961 | 98782 | 99282 | 99582 | 99758 |
| 19 | 94676 | 96706 | 98009 | 98820 | 99311 | 99603 | 99773 |
| 20 | 94735 | 96758 | 98053 | 98854 | 99337 | 99621 | 99786 |
| 21 | 94788 | 96805 | 98092 | 98885 | 99360 | 99637 | 99797 |
| 22 | 94837 | 96848 | 98127 | 98912 | 99380 | 99652 | 99807 |
| 23 | 94881 | 96887 | 98159 | 98937 | 99399 | 99665 | 99816 |
| 24 | 94922 | 96923 | 98189 | 98960 | 99415 | 99677 | 99824 |
| 25 | 94959 | 96956 | 98216 | 98981 | 99430 | 99687 | 99831 |
| 26 | 94993 | 96986 | 98240 | 99000 | 99444 | 99697 | 99837 |
| 27 | 95025 | 97014 | 98263 | 99017 | 99457 | 99706 | 99843 |
| 28 | 95055 | 97040 | 98284 | 99033 | 99469 | 99714 | 99849 |
| 29 | 95083 | 97064 | 98304 | 99048 | 99479 | 99721 | 99853 |
| 30 | 95108 | 97087 | 98322 | 99062 | 99489 | 99728 | 99858 |

Probability Integral of Bivariate t-distrihution With $\rho = 0.75$

| n/h | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 |
|---|---|---|---|---|---|---|---|
| 1 | 0.46654 | 0.54517 | 0.61356 | 0.66957 | 0.71434 | 0.75000 | 0.77863 |
| 2 | 47565 | 56494 | 64537 | 71313 | 76781 | 81093 | 84465 |
| 3 | 47927 | 57280 | 65820 | 73104 | 79010 | 83643 | 87207 |
| 4 | 48119 | 57699 | 66509 | 74075 | 80228 | 85038 | 88699 |
| 5 | 48238 | 57959 | 66938 | 74683 | 80995 | 85916 | 89634 |
| 6 | 48319 | 58135 | 67230 | 75100 | 81521 | 86520 | 90274 |
| 7 | 48377 | 58262 | 67442 | 75402 | 81905 | 86961 | 90740 |
| 8 | 48421 | 58359 | 67603 | 75632 | 82197 | 87296 | 91094 |
| 9 | 48456 | 58434 | 67729 | 75813 | 82427 | 87560 | 91371 |
| 10 | 48484 | 58495 | 67830 | 75959 | 82613 | 87773 | 91595 |
| 11 | 48506 | 58545 | 67913 | 76079 | 82766 | 87948 | 91779 |
| 12 | 48525 | 58587 | 67983 | 76179 | 82894 | 88096 | 91933 |
| 13 | 48542 | 58622 | 68042 | 76265 | 83003 | 88221 | 92064 |
| 14 | 48555 | 58653 | 68093 | 76338 | 83097 | 88329 | 92177 |
| 15 | 48567 | 58679 | 68137 | 76402 | 83179 | 88422 | 92275 |
| 16 | 48578 | 58702 | 68176 | 76458 | 83250 | 88505 | 92361 |
| 17 | 48587 | 58723 | 68210 | 76507 | 83314 | 88577 | 92437 |
| 18 | 48596 | 58741 | 68240 | 76551 | 83370 | 88642 | 92504 |
| 19 | 48603 | 58757 | 68268 | 76591 | 83421 | 88700 | 92565 |
| 20 | 48610 | 58772 | 68292 | 76627 | 83467 | 88753 | 92619 |
| 21 | 48616 | 58785 | 68315 | 76659 | 83508 | 88800 | 92669 |
| 22 | 48621 | 58797 | 68335 | 76688 | 83546 | 88844 | 92714 |
| 23 | 48626 | 58808 | 68353 | 76715 | 83580 | 88883 | 92755 |
| 24 | 48631 | 58818 | 68370 | 76740 | 83612 | 88920 | 92793 |
| 25 | 48635 | 58828 | 68386 | 76763 | 83641 | 88953 | 92828 |
| 26 | 48639 | 58836 | 68401 | 76784 | 83668 | 88984 | 92860 |
| 27 | 48643 | 58844 | 68414 | 76803 | 83693 | 89013 | 92889 |
| 28 | 48646 | 58852 | 68426 | 76821 | 83716 | 89040 | 92917 |
| 29 | 48649 | 58859 | 68438 | 76838 | 83738 | 89064 | 92943 |
| 30 | 48652 | 58865 | 68449 | 76854 | 83758 | 89088 | 92967 |

Probability Integral of Bivariate t-distribution With ρ= 0.75

| n/h | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 |
|---|---|---|---|---|---|---|---|
| 1 | 0.80188 | 0.82101 | 0.83696 | 0.85043 | 0.86192 | 0.87183 | 0.88045 |
| 2 | 87102 | 89176 | 90822 | 92141 | 93209 | 94082 | 94804 |
| 3 | 89926 | 91998 | 93582 | 94802 | 95748 | 96488 | 97074 |
| 4 | 91443 | 93484 | 95001 | 96131 | 96978 | 97616 | 98102 |
| 5 | 92384 | 94392 | 95850 | 96908 | 97677 | 98240 | 98654 |
| 6 | 93022 | 95000 | 96409 | 97409 | 98119 | 98623 | 98984 |
| 7 | 93484 | 95435 | 96803 | 97756 | 98418 | 98877 | 99197 |
| 8 | 93832 | 95760 | 97095 | 98009 | 98632 | 99055 | 99343 |
| 9 | 94104 | 96012 | 97318 | 98200 | 98791 | 99185 | 99448 |
| 10 | 94323 | 96213 | 97494 | 98349 | 98913 | 99284 | 99526 |
| 11 | 94502 | 96377 | 97637 | 98468 | 99010 | 99360 | 99586 |
| 12 | 94651 | 96513 | 97754 | 98566 | 99088 | 99421 | 99632 |
| 13 | 94778 | 96627 | 97853 | 98646 | 99152 | 99471 | 99670 |
| 14 | 94887 | 96725 | 97936 | 98715 | 99206 | 99512 | 99700 |
| 15 | 94981 | 96810 | 98008 | 98773 | 99251 | 99546 | 99726 |
| 16 | 95063 | 96884 | 98070 | 98823 | 99290 | 99575 | 99747 |
| 17 | 95136 | 96949 | 98125 | 98867 | 99323 | 99600 | 99765 |
| 18 | 95201 | 97007 | 98173 | 98905 | 99353 | 99621 | 99780 |
| 19 | 95259 | 97058 | 98216 | 98939 | 99378 | 99640 | 99793 |
| 20 | 95311 | 97104 | 98255 | 98969 | 99401 | 99656 | 99805 |
| 21 | 95358 | 97146 | 98289 | 98996 | 99421 | 99671 | 99815 |
| 22 | 95401 | 97184 | 98320 | 99021 | 99439 | 99684 | 99824 |
| 23 | 95440 | 97218 | 98349 | 99043 | 99456 | 99696 | 99832 |
| 24 | 95476 | 97250 | 98375 | 99063 | 99471 | 99706 | 99839 |
| 25 | 95509 | 97279 | 98399 | 99081 | 99484 | 99716 | 99845 |
| 26 | 95539 | 97306 | 98420 | 99098 | 99496 | 99724 | 99851 |
| 27 | 95568 | 97330 | 98441 | 99114 | 99508 | 99732 | 99857 |
| 28 | 95594 | 97353 | 98459 | 99128 | 99518 | 99739 | 99861 |
| 29 | 95618 | 97374 | 98477 | 99141 | 99528 | 99746 | 99866 |
| 30 | 95641 | 97394 | 98493 | 99153 | 99537 | 99752 | 99870 |

3.7. MULTIVARIATE BEHRENS-FISHER PROBLEM: This section deals with a multivariate generalization to the Behrens-Fisher Problem considered in Section 3.2. Let $(X_1^{(i)}, X_2^{(i)}, \ldots, X_p^{(i)})$, i = 1, 2, be two independent p-variate normal distributions with mean vectors $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$, and covariance matrices $\Sigma_1$ and $\Sigma_2$ respectively. This problem is concerned with the derivation of the structural distribution for the difference of two means vectors, $\underset{\sim}{\mu} = \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2$, based on the complete sets of observations. More precisely, we consider the following two independent multivariate models

$$\begin{cases} X_i = \theta_i E_i \\ f(E_i)dE_i = (2\pi)^{-n_i p/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{n_i}(e_{jk}^{(i)})^2\right\} \prod_{j=1}^{p}\prod_{k=1}^{n_i} de_{jk}^{(i)} \end{cases},$$

where

$$X_i = \begin{pmatrix} 1 & \cdots & 1 \\ x_{11}^{(i)} & \cdots & x_{1n_i}^{(i)} \\ \vdots & & \\ x_{p1}^{(i)} & \cdots & x_{pn_i}^{(i)} \end{pmatrix}, \quad E_i = \begin{pmatrix} 1 & \cdots & 1 \\ e_{11}^{(i)} & \cdots & e_{1n_i}^{(i)} \\ \vdots & & \\ e_{p1}^{(i)} & \cdots & e_{pn_i}^{(i)} \end{pmatrix}$$

and

$$\theta_i = \left(\begin{array}{c|ccc} 1 & 0 & & 0 \\ \hline \mu_{i1} & c_{11}^{(i)} & \cdots & c_{1p}^{(i)} \\ \vdots & \vdots & & \vdots \\ \mu_{ip} & c_{p1}^{(i)} & \cdots & c_{pp}^{(i)} \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline \underset{\sim}{\mu}_i & c_i \end{array}\right)$$

with $|c_i| > 0$, is an element of the positive affine group on $R^p$ for each i = 1, 2. The structural distribution for $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$, based on $X_1$ and $X_2$, is (See Fraser (1968) page 241)

$$g(\underset{\sim}{\mu}_1,\underset{\sim}{\mu}_2:X_1,X_2)d\underset{\sim}{\mu}_1 d\underset{\sim}{\mu}_2=K_1K_2\{1+n_1(\underset{\sim}{m}_1(x_1)-\underset{\sim}{\mu}_1)'S_1(x_1)^{-1}(\underset{\sim}{m}_1(x_1)-\underset{\sim}{\mu}_1)\}^{-n_1/2}\cdot$$

$$\{1+n_2(\underset{\sim}{m}_2(X_2)-\underset{\sim}{\mu}_2)'S_2(X_2)^{-1}(\underset{\sim}{m}_2(X_2)-\underset{\sim}{\mu}_2)\}^{-n_2/2}d\mu_1 d\mu_2$$

where

$$\underset{\sim}{m}_i(X_i)' = (\bar{x}_1^{(i)},\ldots,\bar{x}_p^{(i)}),\ \bar{x}_\alpha^{(i)} = \sum_{k=1}^{n_i} x_{\alpha k}^{(i)}/n_i,\ \alpha = 1, 2, \ldots, p,$$

$$S_i(X_i) = \begin{pmatrix} x_{11}^{(i)}-\bar{x}_1^{(i)} & \cdots & x_{1n_i}^{(i)}-\bar{x}_1^{(i)} \\ \vdots & & \vdots \\ x_{p1}^{(i)}-\bar{x}_p^{(i)} & \cdots & x_{pn_i}^{(i)}-\bar{x}_p^{(i)} \end{pmatrix}\begin{pmatrix} x_{11}^{(i)}-\bar{x}_1^{(i)} & \cdots & x_{1n_i}^{(i)}-\bar{x}_1^{(i)} \\ \vdots & & \vdots \\ x_{p1}^{(i)}-\bar{x}_p^{(i)} & \cdots & x_{pn_i}^{(i)}-\bar{x}_p^{(i)} \end{pmatrix}'$$

$$= (n_i s_{jk}^{(i)}),$$

and

$$K_i = \frac{A_{n_i-p}\cdot n_i^{p/2}}{A_{n_i}|S_i(X_i)|^{1/2}},\quad A_f = (2\pi)^{f/2}\Gamma(f/2).$$

The structural distribution for $\mu$, based on $X_1$ and $X_2$, is

$$g(\underset{\sim}{\mu}:X_1,X_2)d\underset{\sim}{\mu} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(\underset{\sim}{\mu}_1,\underset{\sim}{\mu}_2:X_1,X_2)d\mu_1\cdot d\mu$$

where $\mu_2$ of the integrand is replaced by $\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}$. Further, we note that for each i = 1, 2, the structural distribution for $\underset{\sim}{\mu}_i$, based on $X_i$, is distributed according to a relocated and rescaled multivariate t-distribution. Therefore from the result obtained in Section (3.1) and the fact that the marginal distribution of multivariate t-distribution is again a multivariate t-distribution, we obtain

$$\underset{\sim}{\mu} = \underset{\sim}{m}_1(X_1)-\underset{\sim}{m}_2(X_2) + \begin{pmatrix} r_1t_1^{(1)}\cos\theta_1 - r_1t_2^{(2)}\sin\theta_1 \\ \vdots \\ r_pt_p^{(1)}\cos\theta_p - r_pt_p^{(2)}\sin\theta_p \end{pmatrix},$$

where

$$t_\alpha^{(i)} = (n_i-1)^{1/2}(\mu_{i\alpha} - \bar{x}_\alpha^{(i)})/s_{\alpha\alpha}^{(i)} ,$$

$$r_\alpha^2 = (s_{\alpha\alpha}^{(1)})^2/(n_1-1)+(s_{\alpha\alpha}^{(2)})^2/(n_2-1),$$

and

$$\tan\theta_\alpha = [s_{\alpha\alpha}^{(2)}/(n_2-1)^{1/2}]/[s_{\alpha\alpha}^{(1)}/(n_1-1)^{1/2}]$$

$i = 1, 2$, $\alpha = 1, 2, \ldots, p$. Each of the $t_\alpha^{(i)}$'s is a Student's t-distribution with $(n_i-1)$ degrees of freedom. It is of interest to notice that for $p = 1$, the result obtained here agrees with that derived in Section 3.2.

3.8. A RELATED BEHRENS-FISHER PROBLEM: This problem deals with the structural distribution for the difference of two location parameters from negative exponential distributions.

Let $X_1$ and $X_2$ be two independent negative exponential distributions:

$$f(x_i) = (1/\sigma_i) \exp \{-(x_i-\mu_i)/\sigma_i\}, \quad x_i > \mu_i,$$

where $\mu_i$ and $\sigma_i$ are respectively the location and scale parameters of $X_i$, i = 1, 2. Let $\underset{\sim}{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})$, $n_i \geq 2$, be a sample of size $n_i$ drawn from the distribution $X_i$, i = 1, 2. Our object is to obtain the structural distribution for the difference of two location parameters, $\mu = \mu_1 - \mu_2$, say, under the following three different situations:

(i) when both scale parameters $\sigma_1$ and $\sigma_2$ are known;

(ii) when the ratio $\sigma_2/\sigma_1$ of scale parameters is known, and

(iii) when both scale parameters $\sigma_1$ and $\sigma_2$ are unknown.

These three cases will be taken up in order.

(i) <u>When both scale parameters $\sigma_1$ and $\sigma_2$ are known</u>: In this case, we can assume, without loss of generality, that $\sigma_1 = \sigma_2 = 1$. The structural distribution for $\mu$ is based on the following independent structural models

$$\begin{cases} x_{ij} = \mu_i + e_{ij}, \quad e_{ij} > 0,\ j = 1, 2, \ldots, n_i, \\ \exp\left\{-\sum_{j=1}^{n_i} e_{ij}\right\} \prod_{j=1}^{n_i} de_{ij} \end{cases}$$

i = 1, 2. Let

$$\underset{\sim}{x}_1 = (x_{1(k_1)}, \ldots, x_{1(\ell_1)}, \ldots, x_{1(k_r)}, \ldots x_{1(\ell_r)}), 1 \leq k_1 < \ell_1 < \ldots < k_r < \ell_r \leq n_1,$$

and

$$\underset{\sim}{x}_2 = (x_{2(k_1')} \ldots, x_{2(\ell_1')} \ldots x_{2(k'_s)} \ldots x_{2(\ell_s')}), \ 1 \leq k_1' < \ell_1' < \ldots < k_s' < \ell_s' \leq n_2,$$

be respectively multiply Type II censored responses for $X_i$, i = 1, 2. From the independence of the above two structural models and the result of Example (2.6.7), we conclude that the structural distribution for $\mu_1$ and $\mu_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

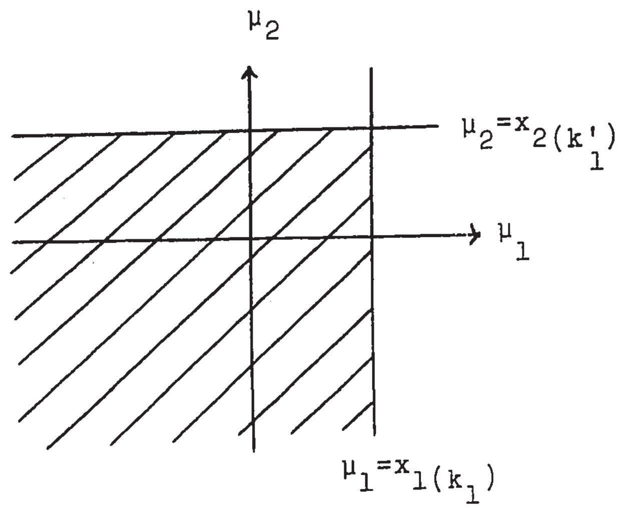
$$g(\mu_1,\mu_2:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1 d\mu_2 = K(1-\exp\{-x_{1(k_1)}+\mu_1\})^{k_1-1}(1-\exp\{-x_{2(k_1')}+\mu_2\})^{k_1'-1}.$$

$$\exp\{(n_1-k_1+1)\mu_1+(n_2-k_1'+1)\mu_2\}d\mu_1 d\mu_2,$$

for $-\infty < \mu_1 < x_{1(k_1)}$ and $-\infty < \mu_2 < x_{2(k_1')}$, and where

$$K = \frac{\exp\{-(n_1-k_1+1)x_{1(k_1')}-(n_2-k_1'+1)x_{2(k_1')}\}}{\beta(k_1,n_1-k_1+1)\cdot\beta(k_1',n_2-k_1'+1)} .$$

The following diagram gives the region for which points $(\mu_1,\mu_2)$ having




positive densities. The substitution $\mu = \mu_1 - \mu_2$ gives the structural distribution for $\mu_1$ and $\mu$ based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$:

$$g(\mu_1,\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1 d\mu=K(1-\exp\{-x_{1(k_1)}+\mu_1\})^{k_1-1}\cdot(1-\exp\{-x_{2(k_1')}+\mu_1-\mu\})^{k_1'-1}$$

$$\exp\{(n_1-k_1+1)\mu_1+(n_2-k_1'+1)(\mu_1-\mu)\}d\mu_1 d\mu$$

for $-\infty < \mu_1 < x_{1(k_1)}$, $-\infty < \mu < \infty$. The desired structural distribution is then obtained by integrating out $\mu_1$. For $\mu \geq \mu_o = x_{1(k_1)} - x_{2(k_2')}$, we have

$$g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu = \int_{-\infty}^{x_{1(k_1)}} g(\mu_1,\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1 \cdot d\mu \; .$$

Making use of the substitution

$$t = \exp\{-x_{1(k_1)} + \mu_1\}$$

the last integral reduces to

$$g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu = K\exp\{Nx_{1(k_1)}-(n_2-k_1'+1)\mu\}\cdot$$

$$\int_o^1 (1-t)^{k_1-1}(1-t\ \exp\{x_{1(k_1)}-x_{2(k_1')}-\mu\})^{k_1'-1} t^{N-1} dt\cdot d\mu$$

$$= K\ \exp\{Nx_{1(k_1)}-(n_2-k_1'+1)\mu\}\cdot$$

$$\int_o^1 (1-t)^{k_1-1}\Big[\sum_{i=o}^{k_1'-1} (-1)^i \binom{k_1'-1}{i} \exp\{i(x_{1(k_1)}-x_{2(k_1')}-\mu)\} t^{N+i-1} dt d\mu$$

$$= K\ \exp\{Nx_{1(k_1)}-(n_2-k_1'+1)\mu\}\sum_{i=o}^{k_1'-1} (-1)^i \binom{k_1'-1}{i} \exp\{i(x_{1(k_1)}-x_{2(k_1')}-\mu)\}\cdot$$

$$\cdot\beta(k_1,N+i)$$

where $N = n_1+n_2-(k_1+k_1')+2$. The preceeding simplification involves the expanding of the term $(1-t\exp\{x_{1(k_1)}-x_{2(k_1')}-\mu\})$ according to the Binomial Theorem, and then the integration is carried out term-by-term. Similarly for $\mu \leq \mu_o$, we have

$$g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu = \int_{-\infty}^{\mu+x_{2(k_1')}} g(\mu_1,\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1\cdot d\mu.$$

Applying the substitution

$$t = \exp\{-x_{2(k_1')}+\mu_1-\mu\}$$

and carrying out the integration in the same manner as before, we obtain

$$g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu=K\exp\{Nx_{2(k_1')}+(n_1-k_1+1)\mu\}\sum_{i=o}^{k_1-1}(-1)^i\binom{k_1-1}{i}\cdot$$

$$\cdot\exp\{i(x_{2(k_1')}-x_{1(k_1)}+\mu\}\beta(k_1,N+i).$$

Combining together we have the structural distribution for $\mu$ based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$:

$$g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu$$

$$=\begin{cases} K\exp\{Nx_{2(k_1')}+(n_1-k_1+1)\mu\}\sum\limits_{i=o}^{k_1-1}(-1)^i\binom{k_1-1}{i}\exp\{i(x_{2(k_1')}-x_{1(k_1)}+\mu)\}\cdot \\ \quad\cdot\beta(k_1,N+i) \qquad\qquad \text{for } \mu \leq \mu_o, \\ K\exp\{Nx_{1(k_1)}+(n_2-k_1'+1)\mu\}\sum\limits_{i=o}^{k_1'-1}(-1)^i\binom{k_1'-1}{i}\exp\{i(x_{1(k_1)}-x_{2(k_1')}-\mu)\}\cdot \\ \quad\cdot\beta(k_1',N+i) \qquad\qquad \text{for } \mu \geq \mu_o . \end{cases}$$

In particular, when $k_1 = k_1' = 1$, we obtain the structural distribution for $\mu$ based on the complete sets of responses (ordered or unordered):

$$g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu=\begin{cases}\dfrac{n_1 n_2}{n_1+n_2}\exp\{n_1(x_{2(1)}-x_{1(1)}+\mu)\}, & \mu \le \mu_o\\[2ex] \dfrac{n_1 n_2}{n_1+n_2}\exp\{n_2(x_{1(1)}-x_{2(1)}-\mu)\}, & \mu \ge \mu_o\ ,\end{cases}$$

where $\mu_o = x_{1(1)}-x_{2(1)}$.

(ii) <u>When the ratio $\sigma_2/\sigma_1$ of scale parameters is known</u>: In this case, the desired structural distribution is derived from the following independent structural models:

$$(3.8.1)\qquad \begin{cases}x_{ij} = \mu_i + \sigma_i e_{ij},\ e_{ij} > 0,\ j = 1, 2, \ldots, n_i,\\ \exp\{-\sum_{j=1}^{n_i} e_{ij}\}\prod_{i=1}^{n_i} de_{ij},\end{cases}$$

$i = 1,2$. Let $\underset{\sim}{x}_1 = (x_{1(1)}, x_{1(2)},\ldots,x_{1(k)})$, $2 \le k \le n_1$, $\underset{\sim}{x}_2 = (x_{2(1)}, x_{2(2)}, \ldots, x_{2(\ell)})$, $2 \le \ell \le n_2$, be Type II singlely censored responses at the right. We wish to derive the structural distribution for $\mu = \mu_1-\mu_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, under the condition that $\sigma_2/\sigma_1 = c$ for some positive known real number c. From the results of Example (2.6.9), we have the structural distribution for $\mu_1,\mu_2,\sigma_1$ and $\sigma_2$ based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$:

$$g(\mu_1,\mu_2,\sigma_1,\sigma_2:\underset{\sim}{x}_1,\underset{\sim}{x}_2)d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

$$\alpha \exp\left\{-(n_1-k)\frac{x_{1(k)}-\mu}{\sigma_1} - \sum_{i=1}^{k} \frac{x_{1(i)}-\mu_1}{\sigma_1} - (n_2-\ell)\frac{x_{2(\ell)}-\mu_2}{\sigma_2} - \sum_{i=1}^{\ell} \frac{x_{2(i)}-\mu_2}{\sigma_2}\right\}.$$

$$\sigma_1^{-(k+1)}\sigma_2^{-(\ell+1)} d\mu_1 d\mu_2 d\sigma_1 d\sigma_2$$

for $-\infty < \mu_1 < x_{1(1)}$, $-\infty < \mu_2 < x_{2(1)}$, $\sigma_1 > 0$ and $\sigma_2 > 0$.

Then by applying the substitution

$$\begin{cases} \mu_1 = \mu_1 \\ \mu = \mu_1-\mu_2 \\ \sigma_1 = \sigma_1 \\ \sigma = \sigma_2/\sigma_1 \end{cases},$$

and then conditioning on $\sigma = c$, we obtain the structural distribution for $\mu_1,\mu,\sigma_1$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$:

$$g(\mu_1,\mu,\sigma_1:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)d\mu_1 d\sigma_1 d\mu$$

$$\alpha\exp\left\{-[(n_1-k)(x_{1(k)}-\mu_1)+\sum_{i=1}^{k}(x_{1(i)}-\mu_1)]/\sigma_1 -[(n_2-\ell)(x_{2(\ell)}-\mu_1+\mu)/c-\sum_{i=1}^{\ell}(x_{2(i)}-\mu_1+\mu)/c]/\sigma\right\}\cdot\sigma_1^{-(k+\ell+1)}d\mu_1 d\sigma_1 d\mu,$$

for $-\infty < \mu_1 < x_{1(1)}$, $-\infty < \mu < \infty$ and $\sigma_1 > 0$. Putting

$$n_1\bar{x}_1^* = (n_1-k)x_{1(k)} + \sum_{i=1}^{k} x_{1(i)},$$

and

$$n_2\bar{x}_2^* = (n_2-\ell)x_{2(\ell)}+ \sum_{i=1}^{\ell} x_{2(i)},$$

then

$$(n_1-k)(x_{1(k)}-\mu_1) + \sum_{i=1}^{k}(x_{1(i)}-\mu) = n_1(\bar{x}_1^*-\mu_1),$$

and

$$(n_2-\ell)(x_{2(\ell)}-\mu_1+\mu) + \sum_{i=1}^{\ell}(x_{2(i)}-\mu_1+\mu) = n_2(\bar{x}_2^*-\mu_1+\mu).$$

Thus we have

$$g(\mu_1,\mu,\sigma:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)\,d\mu_1 d\sigma_1 d\mu \propto \exp\{-A(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\mu_1,\mu,c)/\sigma\}\cdot$$

$$\cdot\sigma_1^{-(k+\ell+1)}d\sigma_1 d\mu_1 d\mu,$$

where $A(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\mu_1,\mu,c)=n_1(\bar{x}_1^*,\mu_1)+n_2(\bar{x}_2^*-\mu_1+\mu)/c$. Hence by integrating out $\sigma_1$ over the range $(0,\infty)$, we obtain

$$g(\mu_1,\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)\,d\mu_1 d\mu \propto \int_0^\infty \exp\{-A(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\mu_1,\mu,c)/\sigma\}\sigma_1^{(k+\ell+1)}d\sigma_1\cdot d\mu_1 d\mu$$

$$\propto \{A(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\mu_1,\mu,c)\}^{-(k+\ell)}.$$

Finally, by eliminating $\mu_1$, we obtain the desired structural distribution:

$$g(\mu:\underset{\sim}{x}_1,\underset{\sim}{x}_2,c)\,d\mu \propto \begin{cases} \displaystyle\int_{-\infty}^{\mu+x_{2(1)}}\{A(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\mu_1,\mu,c)\}^{-(k+\ell)}d\mu_1\cdot d\mu, & \mu \le \mu_o \\ \displaystyle\int_{-\infty}^{x_{1(1)}}\{A(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\mu_1,\mu,c)\}^{-(k+\ell)}d\mu_1\cdot d\mu, & \mu \ge \mu_o \end{cases}$$

where $\mu_o = x_{1(1)} - x_{2(1)}$. Now

$$\int_{-\infty}^{x_{1(1)}} \{A(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \mu_1, \mu, c)\}^{-(k+\ell)} d\mu_1$$

$$= \int_{-\infty}^{x_{1(1)}} \frac{d\mu_1}{\{n_1(\bar{x}_1^* - \mu_1) + n_2(\bar{x}_2^* - \mu_1 + \mu)/c\}^{k+\ell}}$$

$$= \frac{(n_1 + n_2/c)^{-1}}{(k + \ell - 1)} \{n_1(\bar{x}_1^* - x_{1(1)}) + n_2(\bar{x}_2^* - x_{1(1)} + \mu)/c\}^{-(k+\ell-1)};$$

and similarly

$$\int_{-\infty}^{\mu + x_{2(1)}} \{A(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \mu_1, \mu, c\}^{-(k+\ell)} d\mu_1$$

$$= \frac{(n_1 + n_2/c)^{-1}}{(k + \ell - 1)} \{n_1[\bar{x}_1^* - (\mu + x_{2(1)})] + n_2[\bar{x}_2^* - (\mu + x_{2(1)}] + \mu]/c\}^{-(k+\ell-1)}.$$

Therefore the structural distribution for $\mu$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, is

$$g(\mu : \underset{\sim}{x}_1, \underset{\sim}{x}_2, c)d\mu = \begin{cases} \dfrac{K d\mu}{\{n_1(\bar{x}_1^* - \mu - x_{2(1)}) + n(\bar{x}_2^* - x_{2(1)})/c\}^{k+\ell-1}}, & \mu \leq \mu_o, \\[2ex] \dfrac{K d\mu}{\{n_1(\bar{x}_1^* - x_{1(1)}) + n_2(\bar{x}_2^* - x_{1(1)} + \mu)/c\}^{k+\ell-1}}, & \mu \geq \mu_o \end{cases}$$

where the normalizing constant factor K is given by

$$K^{-1} = \int_{-\infty}^{\mu_o} \frac{d\mu}{\{n_1(\bar{x}_1^* - \mu - x_{2(1)}) + n_2(\bar{x}_2^* - x_{2(1)})/c\}^{k+\ell-1}}$$

$$+ \int_{\mu_o}^{\infty} \frac{d\mu}{\{n_1(\bar{x}_1^* - x_{1(1)}) + n_2(\bar{x}_2^* - x_{1(1)} + \mu)/c\}^{k+\ell-1}}$$

$$=[(k+\ell-2)n_1]^{-1}\{n_1(\bar{x}_1^*-x_{1(1)}+x_{2(1)}-x_{2(1)})+n_2(\bar{x}_2^*-x_{2(1)})/c\}^{-(k+\ell-2)}$$

$$+c[(k+\ell-2)n_2]^{-1}\{n_1(\bar{x}_1^*-x_{1(1)})+n_2(\bar{x}_2^*-x_{1(1)}+x_{1(1)}-x_{2(1)})/c\}^{-(k+\ell-2)}$$

$$=\frac{n_1c+n_2}{(k+\ell-2)n_1n_2}\{n_1(\bar{x}_1^*-x_{1(1)})+n_2(\bar{x}_2^*-x_{2(1)})/c\}^{-(k+\ell-2)}.$$

The particular case $k=n_1$, $\ell = n_2$ and $c = 1$ of the above result agrees with the result obtained by Pitman (1939).

(iii) <u>When both scale parameters $\sigma_1$ and $\sigma_2$ are unknown</u>: The structural model (3.8.1) is used here again. Here, our object is to find the structural distribution for $\mu$ based on the complete sets of responses. Let $\underset{\sim}{x}_i=(x_{i1},x_{i2},\ldots,x_{in_i})$, and $x_{i(1)} = \min \underset{\sim}{x}_i$. The structural distribution for $\mu_i$, $\sigma_i$ based on $\underset{\sim}{x}_i$ are

$$g(\mu_i,\sigma_i:\underset{\sim}{x}_i)d\mu_i d\sigma_i=K_i\exp\{-n_i(\bar{x}_i-\mu_i)/\sigma_i\}\sigma_i^{-(n_i+1)}d\mu_i d\sigma_i,$$

for $\mu_i < x_{i(1)}$, $\sigma_i > 0$, where $K_i=n_i(n_i-1)[n_i(\bar{x}_i-x_{i(1)})]^{n_i-1}/\Gamma(n_i)$, $n_i\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}$, $i = 1, 2$. Note that

$$\int_0^\infty \exp\{-n_i(\bar{x}_i-\mu_i)/\sigma_i\}\sigma_i^{-(n_i+1)}d\sigma_i$$

$$=\int_0^\infty \exp\{-n_i(\bar{x}_i-\mu_i)t\}t^{n_i-1}dt$$

$$=\{n_i(\bar{x}_i-\mu_i)\}^{-n_i}\Gamma(n_i).$$

Therefore the structural distribution for $\mu_1, \mu_2$, based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ is

$$g(\mu_1, \mu_2 : \underset{\sim}{x}_1, \underset{\sim}{x}_2) d\mu_1 d\mu_2 = K\{(\bar{x}_1-\mu_1)^{n_1}(\bar{x}_2-\mu_2)^{n_2}\}^{-1} d\mu_1 d\mu_2$$

for $-\infty < \mu_1 < x_{1(1)}$, $-\infty < \mu_2 < x_{2(1)}$, and

$$K = K_1 \cdot K_2 n_1^{-n_1} n_2^{-n_2} \Gamma(n_1)\Gamma(n_2)$$

$$= (n_1-1)(n_2-1)(\bar{x}_1-x_{1(1)})^{n_1-1}(\bar{x}_2-x_{2(1)})^{n_2-1}.$$

Hence, by letting $\mu = \mu_1-\mu_2$, $\mu_1 = \mu_1$, we obtain the desired structural distribution for $\mu$ based on $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$:

$$g(\mu : \underset{\sim}{x}_1, \underset{\sim}{x}_2) d = \begin{cases} K\int_{-\infty}^{\mu+x_{2(1)}} \{(\bar{x}_1-\mu_1)^{n_1}(\bar{x}_2-\mu_1+\mu)^{n_2}\}^{-1} d\mu_1 \cdot d\mu, & \mu \le \mu_o \\ K\int_{-\infty}^{x_{1(1)}} \{(\bar{x}_1-\mu_1)^{n_1}(\bar{x}_2-\mu_1+\mu)^{n_2}\}^{-1} d\mu_1 \cdot d\mu, & \mu \ge \mu_o \end{cases}$$

where $\mu_o = x_{1(1)} - x_{2(1)}$ .

# CHAPTER 4

# DISTRIBUTIONS OF SOME SAMPLE CORRELATION COEFFICIENTS AND SAMPLE CORRELATION MATRICES

4.1. INTRODUCTION: In this chapter, we are mainly concerned with the distribution of

(i) two sample correlation coefficients; and

(ii) two sample correlation matrices.

The method employed here is the so-called method of "likelihood modulation". To do this, a conditional structural model is introduced in each case in order to provide a marginal likelihood function for the parameter concerned. Therefore, in the following section, we give an introduction to conditional structural model and marginal likelihood.

## 4.2. AN INTRODUCTION TO CONDITIONAL STRUCTURAL MODEL AND MARGINAL LIKELIHOOD:

A conditional structural model

$$\begin{cases} x = \theta e \\ f(e:\lambda)de \end{cases}$$

is a model that is partly structural and partly classical. The error variable e has a distribution depends on an additional quantity $\lambda$ which is unknown. If the additional quantity $\lambda$ is known, then the conditional structural model becomes an ordinary structural model. In this section, we obtain the marginal likelihood function for $\lambda$ based on the orbit.

Let $[\underset{\sim}{x}]$ be a transformation variable for the conditional structural model. The conditional distribution for [e], given the orbit $D(e) = D(\underset{\sim}{x}) = D$, is

$$K_\lambda(D)f([e]D:\lambda)\frac{J_N(\underset{\sim}{e})}{J_L(\underset{\sim}{e})}\, d[\underset{\sim}{e}] \tag{4.2.1}$$

which is usually depending on the unknown quantity $\lambda$ (for details see Fraser (1968)). Thus the marginal pdf for the orbit D can be obtained by dividing the full pdf $f(\underset{\sim}{e}:\lambda)d\underset{\sim}{e}$ by the conditional pdf (4.2.1):

$$\frac{1}{K_\lambda(D)}\ \frac{J_L(\underset{\sim}{e})}{J_N(\underset{\sim}{e})}\ \frac{d\underset{\sim}{e}}{d[\underset{\sim}{e}]}\ .$$

Therefore the marginal pdf based on the differentials at the point $\underset{\sim}{x}$ rather than at $\underset{\sim}{e}$ on the orbit D is

$$\frac{1}{K_\lambda(D)}\ \frac{J_L(\underset{\sim}{x})}{J_N(\underset{\sim}{x})}\ \frac{d\underset{\sim}{x}}{d[\underset{\sim}{x}]}\ .$$

Hence the marginal likelihood function for $\lambda$ based on D is

$$L(D:\lambda) = R^+(D)/K_\lambda(D)$$

where $R^+(D)$ is the mapping that carries any orbit D to the set $(0,\infty)$.

4.3. <u>DISTRIBUTION OF A SAMPLE CORRELATION COEFFICIENT I</u>: Let $(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})$ be a sample of size n drawn from a bivariate normal distribution $(X_1, X_2)$ with mean $\mu_1$ and $\mu_2$ and covariance matrix

$$\sum = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} .$$

The pdf for $(X_1, X_2)$ is

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{- \frac{1}{2(1-\rho^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{x_1-\mu_1}{\sigma_1}\cdot\frac{x_2-\mu_2}{\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

It is well-known that the MLE $\hat{\rho}$ for $\rho$ is

$$\hat{\rho} = S_{12}(\underset{\sim}{x})/[S_1(\underset{\sim}{x})S_2(\underset{\sim}{x})]$$

where $nS_{12}(\underset{\sim}{x}) = \sum(x_{1j}-\bar{x}_1)(x_{2j}-\bar{x}_2)$

$nS_i^2(\underset{\sim}{x}) = \sum(x_{ij}-\bar{x}_i)^2, \; i = 1,2$

and $n\bar{x}_i = \sum x_{ij}$, $i = 1, 2$, and where $\sum = \sum_{j=1}^{n}$. The general distribution for $\hat{\rho}$ has been derived by Fisher (1915) by means of geometrical approach. It has been obtained also by Fraser (1968) by the method of likelihood modulation of the special distribution (i.e., distribution for $\hat{\rho}$ when $\rho = 0$).

Now, if the variances of the two marginal distributions are equal, that is $\sigma_1^2 = \sigma_2^2$, then MLE for $\rho$ becomes

$$r = \rho^* = 2S_{12}(\underset{\sim}{x})/S^2(\underset{\sim}{x})$$

where $S^2(\underset{\sim}{x}) = S_1^2(\underset{\sim}{x}) + S_2^2(\underset{\sim}{x})$. The general distribution for r has been given by DeLury (1938) under the assumption that $\sigma_1^2 = \sigma_2^2$. Mehta and Gurland (1969) also obtain the general distribution for r where the assumption that $\sigma_1^2 = \sigma_2^2$ is released. In this section we show how the general distribution for r, under the condition $\sigma_1^2 = \sigma_2^2$, can be obtained by the method of likelihood modulation.

We associate the above sample with the following conditional structural model

$$\left\{\begin{array}{l} \begin{pmatrix} 1 \\ x_{1i} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \sigma & 0 \\ \mu_2 & 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 \\ e_{1i} \\ e_{2i} \end{pmatrix}, \quad i = 1, 2, \ldots, n, \\ \prod_{i=1}^{n} \left[ \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ - \frac{e_{1i}^2 - 2\rho e_{1i} e_{2i} + e_{2i}^2}{2(1-\rho^2)} \right\} de_{1i} de_i \right] \end{array}\right.$$

with additional quantity $\rho$. Take

$$[\underset{\sim}{x}] = \begin{pmatrix} 1 & 0 & 0 \\ \bar{x}_1 & S(x) & 0 \\ \bar{x}_2 & 0 & S(x) \end{pmatrix}$$

be our transformation variable. The reference point $D = (\underset{\sim}{d}_1, \underset{\sim}{d}_2)$ is

$$\underset{\sim}{d}_i = \left(\frac{x_{i1}-\bar{x}_i}{S(\underset{\sim}{x})}, \frac{x_{i2}-\bar{x}_i}{S(\underset{\sim}{x})}, \ldots, \frac{x_{in}-\bar{x}_i}{S(\underset{\sim}{x})}\right)$$

$$= \left(\frac{e_{i1}-\bar{e}_i}{S(\underset{\sim}{e})}, \frac{e_{i2}-\bar{e}_i}{S(\underset{\sim}{e})}, \ldots, \frac{e_{in}-\bar{e}_i}{S(\underset{\sim}{e})}\right),$$

$i = 1, 2$. The conditional distribution for $[\underset{\sim}{e}]$, given the orbit D, is

$$\bar{f}(\bar{e}_1, \bar{e}_2, S(e) : \underset{\sim}{d}_1, \underset{\sim}{d}_2) d\bar{e}_1 d\bar{e}_2 dS(\underset{\sim}{e})$$

$$= K_\rho(\underset{\sim}{d}_1, \underset{\sim}{d}_2) \prod_{j=1}^{n} f(\bar{e}_1 + S(\underset{\sim}{e}) d_{1j}, \bar{e}_2 + S(\underset{\sim}{e}) d_{2j}) S(\underset{\sim}{e})^{2n-3} d\bar{e}_1 d\bar{e}_2 dS(\underset{\sim}{e})$$

$$= K_\rho(\underset{\sim}{d}_1, \underset{\sim}{d}_2) [2\pi(1-\rho^2)^{1/2}]^{-n} \exp\{-y^2\} S(\underset{\sim}{e})^{2n-3} d\bar{e}_1 d\bar{e}_2 dS(\underset{\sim}{e})$$

where

$$y^2 = \{2(1-\rho^2)\}^{-1} \sum \{(\bar{e}_1 + S(\underset{\sim}{e}) d_{1j})^2 - 2\rho(\bar{e}_1 + S(\underset{\sim}{e}) d_{1j})(\bar{e}_2 + S(\underset{\sim}{e}) de_{2j})$$

$$+ (\bar{e}_2 + S(e) d_{2j})^2\} .$$

Note that we have

$$\text{(i)} \quad \sum(\bar{e}_i + S(\underset{\sim}{e}) d_{ij})^2 = \sum(\bar{e}_i^2 + S(\underset{\sim}{e})^2 d_{ij}^2 + 2\bar{e}_i S(\underset{\sim}{e}) d_{ij})$$

$$= n\bar{e}_i^2 + S(\underset{\sim}{e})^2 \sum d_{ij}^2 ,$$

since $\sum d_{ij} = 0$;

$$\text{(ii)}\ \sum(d_{1j}^2+d_{2j}^2) = \sum[(e_{1j}-\bar{e}_1)^2+(e_{2j}-\bar{e}_2)^2]/S(e)^2$$

$$= (S_1(\underset{\sim}{e})^2+S_2(\underset{\sim}{e})^2)/S(\underset{\sim}{e})^2$$

$$= n;$$

and

$$\text{(iii)}\sum(\bar{e}_1+S(\underset{\sim}{e})d_{1j})(\bar{e}_2+S(\underset{\sim}{e})d_{2j})=\sum(\bar{e}_1\bar{e}_2+S(\underset{\sim}{e})^2 d_{1j}d_{2j}+\bar{e}_2 d_{1j}+\bar{e}_1 d_{2j})$$

$$= n(\bar{e}_1\bar{e}_2+rS(\underset{\sim}{e})^2/2)$$

since

$$\sum d_{1j}d_{2j} = n\{\sum(e_{1j}-\bar{e}_1)(e_{2j}-\bar{e}_2)/n\}/S(\underset{\sim}{e})^2$$

$$= nr/2 .$$

Therefore

$$y^2 = \{2(1-\rho^2)\}^{-1}\cdot\{n\bar{e}_1^2+n\bar{e}_2+nS(\underset{\sim}{e})^2-2\rho n(\bar{e}_1\bar{e}_2+rS(\underset{\sim}{e})^2/2)\}$$

$$= \{2(1-\rho^2)\}^{-1}\cdot\{n(\bar{e}_1-2\rho\bar{e}_1\bar{e}_2+\bar{e}_2)^2 + nS(\underset{\sim}{e})^2(1-\rho r)\}.$$

The normalizing constant factor $K_\rho(\underset{\sim}{d}_1, \underset{\sim}{d}_2)$ can be obtained by integration:

$$K_\rho(\underset{\sim}{d}_1,\underset{\sim}{d}_2)^{-1} = \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\bar{f}(\bar{e}_1,\bar{e}_2,S(\underset{\sim}{e}):\underset{\sim}{d}_1,\underset{\sim}{d}_2)d\bar{e}_1 d\bar{e}_2 dS(\underset{\sim}{e})}{K_\rho(\underset{\sim}{d}_1, \underset{\sim}{d}_2)}$$

$$= (2\pi(1-\rho^2)^{1/2})^{-n}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \exp\left\{-\frac{n(\bar{e}_1^2-2\rho\bar{e}_1\bar{e}_2+\bar{e}_2^2)+nS(\underset{\sim}{e})^2(1-\rho_r^2)}{2(1-\rho^2)}\right\}$$

$$\cdot S(\underset{\sim}{e})^{2n-3}d\bar{e}_1 d\bar{e}_2 dS(\underset{\sim}{e})$$

$$=[(2\pi)(1-\rho^2)^{1/2}]^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{n(\bar{e}_1^2-2\rho\bar{e}_1\bar{e}_2+\bar{e}_2^2)}{2(1-\rho)^2}\right\}d\sqrt{n}\bar{e}_1 d\sqrt{n}\bar{e}_2 \cdot$$

$$n^{-1}(2\pi)^{-(n-1)}(1-\rho^2)^{-(n-1)/2}\int_0^{\infty}\exp\left\{-\frac{n(1-\rho r)S(\underset{\sim}{e})^2}{2(1-\rho^2)}\right\}S(\underset{\sim}{e})^{2n-3}dS(\underset{\sim}{e})$$

$$=\frac{1}{n(2\pi)^{n-1}(1-\rho^2)^{(n-1)/2}}\left[\frac{2(1-\rho^2)}{n}\right]^{(2n-3)/2}\int_0^{\infty}\exp\left\{-(1-\rho r)z\right\}z^{(2n-3)/2}.$$

$$\frac{2(1-\rho^2)\cdot\sqrt{n}\cdot z^{-1/2}}{2n[2(1-\rho^2)]^{1/2}}\,dz$$

$$=\frac{(1-\rho^2)^{(n-1)/2}}{(2\pi)^{n-1}\cdot n^{n+1}}\int_0^{\infty}\exp\left\{-(1-\rho r)z\right\}z^{(n-1)-1}dz$$

$$=\frac{(1-\rho^2)^{(n-1)/2}\Gamma(n-1)}{(2\pi)^{n-1}\cdot n^{n+1}(1-\rho r)^{n-1}}\ .$$

The preceeding simplification involves two steps. First, express the integral into the product of two integrals: the first of which is unity. In calculating the second integral the substitution

$$z = nS(\underset{\sim}{e})^2/[2(1-\rho^2)]$$

is made. Therefore the marginal likelihood function for $\rho$ based on the orbit $\underset{\sim}{D}$ is

$$L(\underset{\sim}{d}_1,\underset{\sim}{d}_2:\rho) = R^+(\underset{\sim}{D})/K_\rho(\underset{\sim}{d}_1,\underset{\sim}{d}_2).$$

This marginal likelihood function for $\rho$ can also be expressed as a ratio relative to that of $\rho = 0$:

$$L^*(\underset{\sim}{d}_1, \underset{\sim}{d}_2 : \rho) = K_o(\underset{\sim}{d}_1, \underset{\sim}{d}_2)/K_\rho(\underset{\sim}{d}_1, \underset{\sim}{d}_2)$$

$$= (1-\rho^2)^{(n-1)/2}(1-\rho r)^{-(n-1)} .$$

Hence, if $h(r:0)dr$ is the marginal pdf for $r$ with $\rho = 0$, then

$$h(r:\rho)dr = L^*(\underset{\sim}{d}_1, \underset{\sim}{d}_2 : \rho)h(r:0)dr, \qquad |r| < 1$$

is the marginal pdf for $r$ for the general correlation coefficient $\rho$. For $\rho = 0$, it has been found (See DeLury (1938) or Mehta and Gurland (1969)) that

$$h(r:0)dr = \pi^{-1/2}\Gamma(n/2)\Gamma(\tfrac{n-1}{2})^{-1}(1-r^2)^{(n-3)/2}dr, \; |r| < 1.$$

Hence the general distribution for $r$, with correlation coefficient $\rho$, is

$$h(r:\rho)dr = \pi^{-1/2}\Gamma(n/2)\Gamma(\tfrac{n-1}{2})^{-1}(1-r)^{(n-3)/2}(1-\rho^2)^{(n-1)/2}(1-\rho r)^{-(n-1)},$$

$$|r| < 1.$$

## 4.4. DISTRIBUTION FOR A SAMPLE CORRELATION COEFFICIENT II:

Suppose we have the same sample as given in the last section. In addition, we assume that the means of the two marginal distributions are equal, i.e. $\mu_1 = \mu_2$. Then the MLE for $\rho$ is

$$r = \rho^* = 2\sum(x_{1j}-\bar{x})(x_{2j}-\bar{x})/[nS(\underset{\sim}{x})^2]$$

where $\bar{x} = \frac{1}{2}(\bar{x}_1+\bar{x}_2)$ and $\bar{x}_1, \bar{x}_2$ and $S(x)^2$ are the same as those given the last section and $\sum = \sum_{j=1}^{n}$. In this section we wish to derive

(i) the distribution for r when $\rho = 0$ by the usual method;

and (ii) the distribution for r, for general correlation coefficient $\rho$, by means of the likelihood modulation.

Pitman (1939b) has obtained the following relation, when $\rho = 0$,

$$W = (1 + r)/(1 - r)$$
$$= \sum(u_j - \bar{u})^2/\sum(v_j)^2$$

where $u_j = x_{1j} + x_{2j}$ and $v_j = x_{1j} - x_{2j}$ are independent normal variables both having variances equal to $2\sigma^2$. Hence W has the same distribution as $\chi^2_{n-1}/\chi'^2_n$ where $\chi^2_{n-1}$ and $\chi'^2_n$ are independent variables each distributed like chi-square distribution with (n-1) and n degrees of freedom respectively. Therefore the pdf for W is (Cramér page 241)

$$f(W)dW = \Gamma(\tfrac{2n-1}{2})\left\{\Gamma(\tfrac{n-1}{2})\Gamma(\tfrac{n}{2})\right\}^{-1} W^{\frac{n-1}{2}-1}(1+W)^{-\frac{2n-1}{2}}dW .$$

Since the jacobian of the substitution $W = (1+r)/(1-r)$ is $2/(1-r)^2$, it follows that the pdf for r with $\rho = 0$ is

$$h(r:0)dr = K \frac{[(1+r)/(1-r)]^{(n-1)/2-1}}{[1+(1+r)/(1-r)]^{(2n-1)/2}} (1-r)^{-2}dr, \ |r| < 1$$

where $K = 2\Gamma(\frac{2n-1}{2}) \{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})\}^{-1}$. After a little simplification we obtain

$$h(r:0)dr = \frac{\Gamma(\frac{2n-1}{2})2^{-(2n-1)/2+1}}{\Gamma(\frac{n-1}{2}) \quad \Gamma(\frac{n}{2})} (1+r)^{(n-1)/2-1} \cdot (1-r)^{n/2-1} dr,$$

$$|r| < 1.$$

This completes (i). In order to derive the general distribution for r, we consider the following conditional structural distribution

$$\left\{\begin{array}{l} \begin{pmatrix} 1 \\ x_{1j} \\ x_{2j} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu & \sigma & 0 \\ \mu & 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 \\ e_{1j} \\ e_{2j} \end{pmatrix}, \ j = 1, 2, \ldots, n, \\ \prod_{j=1}^{n} [(2\pi(1-\rho^2)^{1/2})^{-1} \exp\left\{- \dfrac{e_{1j}^2 - 2\rho e_{1j} e_{2j} + e_{2j}^2}{2(1-\rho^2)}\right\} de_{1j} de_{2j}] \end{array}\right.$$

For a transformation variable, we choose

$$[\underset{\sim}{x}] = \begin{pmatrix} 1 & 0 & 0 \\ \bar{x} & S(x) & 0 \\ \bar{x} & 0 & S(x) \end{pmatrix}$$

which lead to the reference $D = (\underset{\sim}{d}_1, \underset{\sim}{d}_2)$ of the orbit:

$$\underset{\sim}{d}_i = \left(\frac{x_{i1}-\bar{x}}{S(\underset{\sim}{x})}, \frac{x_{i2}-\bar{x}}{S(\underset{\sim}{x})}, \ldots, \frac{x_{in}-\bar{x}}{S(\underset{\sim}{x})}\right)$$

$$= \left(\frac{e_{i1}-\bar{e}}{S(\underset{\sim}{e})}, \frac{e_{i2}-\bar{e}}{S(\underset{\sim}{e})}, \ldots, \frac{e_{in}-\bar{e}}{S(\underset{\sim}{e})}\right) .$$

The conditional distribution for $[\underset{\sim}{e}]$, given the orbit $\underset{\sim}{D}$, is

$$\bar{f}(\bar{e}_1, S(\underset{\sim}{e}) : \underset{\sim}{d}_1, \underset{\sim}{d}_2) d\bar{e} dS(\underset{\sim}{e})$$

$$= K_\rho(\underset{\sim}{d}_1, \underset{\sim}{d}_2) \prod_{j=1}^{n} f(\bar{e}+S(\underset{\sim}{e})d_{1j}, \bar{e}+S(\underset{\sim}{e})d_{2j}) S(\underset{\sim}{e})^{2n-2} d\bar{e} dS(\underset{\sim}{e})$$

$$= K_\rho(\underset{\sim}{d}_1, \underset{\sim}{d}_2)\{2\pi(1-\rho^2)^{1/2}\}^{-1} \exp\{-y^2\} S(\underset{\sim}{e})^{2n-2} d\bar{e} dS(\underset{\sim}{e})$$

where

$$y^2 = [2(1-\rho^2)]^{-1} \sum\{[\bar{e}+S(\underset{\sim}{e})d_{1j}]^2 - 2\rho(\bar{e}+S(\underset{\sim}{e})d_{1j})(\bar{e}+S(\underset{\sim}{e})d_{2j}) + (\bar{e}+S(\underset{\sim}{e})d_{2j})^2\}.$$

Note that

(i) $\sum[(\bar{e}+S(\underset{\sim}{e})d_{1j})^2 + (\bar{e}+S(\underset{\sim}{e})d_{2j})^2]$

$$= 2n\bar{e} + S(e)^2 \sum(d_{1j}^2 + d_{2j}^2)$$

since $\sum(d_{1j}+d_{2j}) = 0$;

(ii) $\sum(d_{1j}^2+d_{2j}^2) = \sum\{(e_{1j}-\bar{e})^2 + (e_{2j}-\bar{e})^2\}/(S(\underset{\sim}{e}))^2$

$$= n;$$

and

(iii) $\sum(\bar{e}+S(\underset{\sim}{e})d_{1j})(\bar{e}+S(e)d_{2j})=n\bar{e}^2+S(\underset{\sim}{e})^2\sum d_{1j}d_{2j}+S(\underset{\sim}{e})\sum(d_{1j}+d_{2j})$

$$= n\bar{e}^2+nS(\underset{\sim}{e})^2 r/2$$

since

$$\sum d_{1j}d_{2j} = \sum \frac{n}{S(\underset{\sim}{e})^2} \frac{(e_{1j}-e)(e_{2j}-e)}{n}$$

$$= nr/2.$$

Hence we have

$$y^2 = \frac{1}{2(1-\rho)^2}\{2n(1-\rho)\bar{e}^2+nS(\underset{\sim}{e})(1-\rho r)\} .$$

The normalizing constant factor $K_\rho(\underset{\sim}{d}_1,\underset{\sim}{d}_2)$ can be obtained by integration:

$$K_\rho(\underset{\sim}{d}_1,\underset{\sim}{d}_2)^{-1}=\int_0^\infty\int_{-\infty}^\infty \frac{\bar{f}(\bar{e},S(\underset{\sim}{e}):\underset{\sim}{d}_1\underset{\sim}{d}_2)d\bar{e}dS(\underset{\sim}{e})}{K_\rho(\underset{\sim}{d}_1,\underset{\sim}{d}_2)}$$

$$=\{2\pi(1-\rho^2)^{1/2}\}^{-n}\int_0^\infty\int_{-\infty}^\infty \exp\left\{-\frac{2n(1-\rho)\bar{e}^2+nS(\underset{\sim}{e})^2(1-\rho r)}{2(1-\rho^2)}\right\} \cdot$$

$$\cdot S(\underset{\sim}{e})^{2n-2}d\bar{e}dS(\underset{\sim}{e})$$

$$=\int_0^\infty \frac{1}{(2\pi)^{1/2}}\exp\left\{-\frac{1}{2}\frac{2n\bar{e}^2}{1+\rho}\right\} d\sqrt{\frac{2n}{1+\rho}}\,\bar{e} .$$

$$\frac{1}{(2\pi)^{n-1/2}(1-\rho^2)^{n/2}}\left(\frac{1+\rho}{2n}\right)^{1/2}\int_0^\infty \exp\left\{-\frac{n(1-\rho r)S(\underset{\sim}{e})^2}{2(1-\rho^2)}\right\}$$

$$\cdot S(\underset{\sim}{e})^{2n-2}dS(\underset{\sim}{e})$$

$$= \frac{1}{(2\pi)^{n-1/2}(1-\rho^2)^{n/2}}\left(\frac{1+\rho}{2n}\right)^{1/2}\frac{1}{2}\int_0^\infty \exp\left\{-\frac{n(1-\rho r)}{2(1-\rho^2)}t\right\}t^{\frac{2n-1}{2}-1}dt$$

$$= \frac{(1+\rho)^{1/2}\Gamma(\frac{2n-1}{2})[2(1-\rho^2)]^{(2n-1)/2}}{2(2\pi)^{(n-1/2)}(1-\rho^2)^{n/2}[n(1-\rho r)]^{(2n-1)/2}} .$$

Hence the marginal likelihood function for $\rho$, based on the orbit $\underset{\sim}{D}$, is

$$L(\underset{\sim}{d}_1,\underset{\sim}{d}_2:\rho) = R^+(\underset{\sim}{D})/K_\rho(\underset{\sim}{d}_1,\underset{\sim}{d}_2),$$

or when expressed as a ratio relative to $\rho = 0$, is

$$L^*(\underset{\sim}{d}_1,\underset{\sim}{d}_2:\rho) = K_o(\underset{\sim}{d}_1,\underset{\sim}{d}_2)/K_\rho(\underset{\sim}{d}_1,\underset{\sim}{d}_2)$$

$$= (1-\rho^2)^{(n-1)/2}(1+\rho)^{1/2}(1-\rho r)^{-(n-1/2)}.$$

Therefore the general distribution for r, with correlation coefficient $\rho$, is

$$h(r:\rho)dr = L^*(\underset{\sim}{d}_1,\underset{\sim}{d}_2:\rho)h(r:0)dr$$

$$= \frac{\Gamma(\frac{2n-1}{2})(1-\rho^2)^{(n-1)/2}(1+\rho)^{1/2}}{2^{(2n-3)/2}\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})}(1+r)^{(n-1)/2-1}\cdot(1-r)^{n/2-1}(1-\rho r)^{-(n-1/2)}dr$$

for $|r| < 1$.

4.5. <u>DISTRIBUTION FOR A SAMPLE CORRELATION MATRIX I</u>: Let $\underset{\sim}{x}_i = (x_{1i}, x_{2i}, \ldots, x_{pi})$, i = 1, 2, ..., n be a sample of size n, $n \geq p$, drawn from a p-component vector variable $X = (X_1, X_2, \ldots, X_p)$, $p > 2$, distributed according to multivariate normal distribution with means $\underset{\sim}{\mu}$ and covariance matrix $\sum$. It is well-known that the MLE for $\rho_{ij}$, $i < j$, are given by (See Anderson (1966)):

$$r_{ij} = S_{ij}/(S_{ii}S_{jj})^{1/2}$$

where

$$S_{ij} = \sum(x_{ik} - \bar{x}_i)(\bar{x}_{jk} - \bar{x}_k),$$

$$n\bar{x}_i = \sum x_{ik}$$

and where $\sum$ stands for summation over k from 1 to n throughout the rest of this chapter. Fisher (1962) has obtained the general pdf for $r_{ij}$, $i < j$, which is expressed in an integral form. Ali, Fraser and Lee (1970) show how this sample can be associated with a conditional model, and show also that the marginal likelihood analysis of this conditional model can give the distribution of $r_{ij}$, $i < j$, for general covariance matrix $\sum$. They expressed the pdf in a series form. In this section and the next, we derive the distribution for two sample correlation matrices by using the same approach used by Ali, Fraser, and Lee.

From now onwards, we assume that $\underset{\sim}{\mu} = 0$. In this section we also assume that the variances of all the marginal distributions are equal. Then $\sum = \sigma^2 P$ for some real number $\sigma > 0$. Note that P is the correlation matrix for the vector variable X. The MLE for $\rho_{ij}$, $i < j$, are

$$r_{ij} = pS_{ij}(x)/S(\underset{\sim}{x})^2,$$

where

$$S_{ij}(\underset{\sim}{x}) = \sum x_{ik}x_{jk}, \quad 1 \leq i < j \leq p,$$

and

$$S(\underset{\sim}{x})^2 = \sum_{i=1}^{p} S_{ii}(\underset{\sim}{x}).$$

Our aim is to derive the distribution for $r_{ij}$, $i < j$, for general covariance matrix, by using the method of marginal likelihood analysis. To do this, we associate the sample with the following conditional structural model:

$$\begin{cases} x = \theta \underset{\sim}{e} \\ f(\underset{\sim}{e}:\rho)d\underset{\sim}{e} = (2\pi)^{-np/2}|P|^{-n/2}et_r\{-\frac{1}{2}e'P^{-1}e\}de \end{cases}$$

with additional quantity P, where

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{p1} & \cdots & x_{pn} \end{pmatrix}, \quad \underset{\sim}{e} = \begin{pmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & & \\ e_{p1} & \cdots & e_{pn} \end{pmatrix}$$

and $\theta$ is a diagonal matrix of p-order with all its diagonal elements equal to $\sigma(>0)$. For a transformation variable, we take

$$[\underset{\sim}{x}] = \begin{pmatrix} S(\underset{\sim}{x}) & & 0 \\ & \ddots & \\ 0 & & S(\underset{\sim}{x}) \end{pmatrix}$$

The reference of the orbit $G\underset{\sim}{x}$ is

$$D = [\underset{\sim}{e}]^{-1}\underset{\sim}{e}$$

$$= \begin{pmatrix} e_{11}/S(\underset{\sim}{e}) & \cdots & e_{1n}/S(\underset{\sim}{e}) \\ \vdots & & \\ e_{p1}/S(\underset{\sim}{e}) & \cdots & e_{pn}/S(\underset{\sim}{e}) \end{pmatrix} .$$

Note that DD' is symmetric and is given by

$$DD' = (S_{ij}(\underset{\sim}{e})/S(\underset{\sim}{e})^2)$$

$$= \begin{pmatrix} t_1^2 & & & * \\ r_p/p & t_2^2 & & \\ \vdots & & & \\ r_{1p}/p & r_{2p}/p & \cdots & t_p^2 \end{pmatrix}$$

$$= R^*, \text{ say},$$

where $t_i^2 = S_{ii}(\underset{\sim}{e})/S(\underset{\sim}{e})^2$, $i = 1, 2, \ldots, p$. It is clear that $\sum_{i=1}^{p} t_i^2 = 1$. The conditional distribution for $[\underset{\sim}{e}]$, given the orbit $\underset{\sim}{D}$, is

$$f[\underset{\sim}{e}]:P)d[\underset{\sim}{e}]$$
$$= K_P(D)\bar{f}([\underset{\sim}{e}]D:P)S(\underset{\sim}{e})^{pn} \frac{d[e]}{S(\underset{\sim}{e})}$$

$$= K_P(D)[|P|^{n/2}(2\pi)^{np/2}]^{-1}\text{etr}\left\{-\tfrac{1}{2}D'[\underset{\sim}{e}]'P^{-1}[\underset{\sim}{e}]D\right\}S(\underset{\sim}{e})^{pn-1}dS(\underset{\sim}{e})$$

$$= K_\rho(d)[|P|^{n/2}(2\pi)^{np/2}]^{-1}\text{etr}\left\{-\tfrac{1}{2}P^{-1}[\underset{\sim}{e}]DD'[\underset{\sim}{e}]'\right\}S(\underset{\sim}{e})^{pn-1}dS(\underset{\sim}{e}) .$$

Hence if $P^{-1} = (\rho^{ij})$, then

$$P^{-1}[\underset{\sim}{e}]DD'[\underset{\sim}{e}]' = \begin{pmatrix} \rho^{11} & \cdots & \rho^{1p} \\ \vdots & & \\ \rho^{p1} & \cdots & \rho^{pp} \end{pmatrix} \begin{pmatrix} S(\underset{\sim}{e}) & & 0 \\ & \ddots & \\ 0 & & S(\underset{\sim}{e}) \end{pmatrix} \begin{pmatrix} t_1^2 & & & * \\ r_{12}/p & t_2^2 & & \\ r_{1p}/p & r_p/p & \cdots & t_p^2 \end{pmatrix} \begin{pmatrix} S(\underset{\sim}{e}) & & 0 \\ \vdots & & \\ 0 & & S(\underset{\sim}{e}) \end{pmatrix}$$

$$= \begin{bmatrix} S(\underset{\sim}{e})^2\left\{p^{-1}\sum_{j=2}^{p} r_{j1}\rho^{j1}+\rho^{11}t_1^2\right\} & & * \\ & \ddots & \\ & S(\underset{\sim}{e})^2\left\{p^{-1}\sum_{j\neq i}^{p} r_{ji}\rho^{ji}+\rho^{ii}t_i^2\right\} & \\ & & \ddots \\ * & & S(\underset{\sim}{e})^2\left\{p^{-1}\sum_{j=1}^{p-1} r_{jp}\rho^{jp}+\rho^{pp}t_p^2\right\} \end{bmatrix} .$$

Therefore

$$tr\left\{\frac{1}{2}P^{-1}[\underset{\sim}{e}]DD'[\underset{\sim}{e}]'\right\} = \frac{1}{2}S(\underset{\sim}{e})^2[p^{-1}\sum_{i=1}^{p}(\sum_{j\neq i}^{p} r_{ji}\rho^{ji}+\rho^{ii}t_i^2)]$$

$$=\alpha^2 S(\underset{\sim}{e})^2$$

where $\alpha^2 = \frac{1}{2}$ times the expression in the bracket. The normalizing constant factor $K_P(D)$ is

$$K_P(D)^{-1} = \int_0^\infty [|P|^{n/2}(2\pi)^{np/2}]^{-1}\exp\left\{-\alpha^2 S(\underset{\sim}{e})^2\right\}S(\underset{\sim}{e})^{pn-1}dS(\underset{\sim}{e})$$

$$= \left\{2|P|^{n/2}(2\pi)^{np/2}\right\}^{-1}\int_0^\infty \exp\{-\alpha^2 y\}^{\frac{pn}{2}-1}dy$$

$$= \left\{2|P|^{n/2}(2\pi)^{np/2}\right\}^{-1}\Gamma(\frac{pn}{2})(\alpha^2)^{-pn/2} .$$

Therefore the marginal likelihood function for P, based on the orbit $\underset{\sim}{D}$, is

$$L^*(D:P) = K_I(D)/K_P(D)$$
$$= |P|^{-n/2}(\alpha_1^2)^{-pn/2} ,$$

which is expression as a ratio relative to that P = I, the p×p identity matrix and where $\alpha_1^2 = \alpha^2/\{P^{-1}\sum_{i=1}^{p} r_{ji}+t_i^2)\}$. The marginal likelihood function for P depends only on DD' as a function of $\underset{\sim}{D}$. Hence the pdf for R* for general correlation matrix P (and hence for general covariance matrix $\Sigma = \sigma^2 P$) can be obtained from the pdf h(R*:I)dR* for P = I, by modulating by the marginal likelihood function L*(D:P):

$$h(R^*:P)dR^* = L^*(D:P)h(R^*:I)dR.$$

<u>Distribution for R* for P = I</u>: For $\Sigma$ = I, it is well-known that the random matrix $(S_{ij}(\underset{\sim}{x})$ (See Anderson (1966)) has a Wishart distribution whose pdf is given by

$$(4.5.1)\qquad K|S_{ij}(\underset{\sim}{x})|^{(n-p-1)/2}\exp\{-\tfrac{1}{2}\sum_{i=1}^{p} S_{ii}(x)\}$$

where

$$K^{-1} = 2^{np/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\{(n+1-i)/2\}.$$

Consider now the substitutions:

$$\begin{cases} t_p'^2 = \sum_{i=1}^{p} S_{ii}(\underset{\sim}{x}) \\ S_{ii}(\underset{\sim}{x}) = t_i^2 t_p'^2, \quad i = 1, 2, \ldots, (p-1) \\ S_{ij}(\underset{\sim}{x}) = S_{ij}(\underset{\sim}{x}), \quad 1 \le i < j \le p \end{cases}$$

whose jacobian is unity. Further, we note that the jacobian of the following substitutions:

$$t_p'^2 = t_p'^2$$

$$S_{ii}(\underset{\sim}{x}) = t_i^2 t_p'^2, \; i = 1, 2, \ldots, (p-1)$$

$$S_{ij}(\underset{\sim}{x}) = p^{-1} r_{ij} t_p^2, \; 1 \leq i < j \leq p \; ,$$

is the absolute of the determinant of the following matrix:

$$
\begin{array}{c}
 \\
t_1^2 \\ \vdots \\ t_{p-1}^2 \\ t_p'^2 \\ r_{12} \\ \vdots \\ r_{1p} \\ \vdots \\ r_{(p-1)p}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
S_{11} & \cdots & S_{(p-1)(p-1)} & t_p' & S_{12} & \cdots & S_{1p} & \cdots & S_{(p-1)p}
\end{array} \\
\left(\begin{array}{ccc|c|c|c|c}
t_p'^2 & & 0 & & * & & * \\
 & \ddots & & & & & \\
 & & t_p'^2 & & & & \\
t_1^2 & \cdots & t_{p-1}^2 & 1 & & & \\
\hline
 & & & & p^{-1}t_p'^2 & & * \\
 & 0 & & & \ddots & & \\
 & & & & p^{-1}t_p'^2 & & \\
\hline
 & & & & & & \\
\hline
 & 0 & & & 0 & & p^{-1}t_p'^2
\end{array}\right)
\end{array}
$$

which is equal to

$$p^{-p(p-1)/2} t_p'^{2(p-1)+p(p-1)} = p^{-p(p-1)/2} t_p'^{p^2+p-1} .$$

Therefore the joint pdf for R* and $t_p'^2$ is

$$K \begin{vmatrix} t_1^2 t_p'^2 & & & * \\ p^{-1} r_{1p} t_p'^2 & t_2^2 t_p'^2 & & \\ \vdots & & & \\ p^{-1} r_{1p} t_p'^2 & p^{-1} r_{2p} t_p'^2 & \cdots & t_p'^2 (1 - \sum_{i=1}^{p-1} t_i^2) \end{vmatrix}^{(n-p-1)/2} .$$

$$\cdot \exp\{- \frac{1}{2} t_p'^2\} p^{-p(p-1)/2} t_p'^{p^2+p-1} dR^* dt_p'^2$$

$$= K|R^*|^{(n-p-1)/2} \exp\{- \frac{1}{2} t_p'^2\} (t_p'^2)^{(p^2+p-1)/2 + \frac{1}{2} p(n-p-1)} p^{-p(p-1)/2} .$$

$$\cdot dR^* dt_p'^2 .$$

Integrating out $t_p'^2$ over the range $(0,\infty)$, we obtain the pdf for R*:

$$K p^{-p(p-1)/2} 2^{(pn+1)/2} \Gamma((pn+1)/2) |R^*|^{(n-p-1)/2} dR^*$$

since

$$\int_0^\infty \exp\{-\frac{1}{2} t_p'^2\}(t_p'^2)^{-\frac{1}{2}(pn-1)} dt_p'^2 = \Gamma(\frac{pn+1}{2})2^{(pn+1)/2}.$$

Therefore the pdf R*, for general correlation matrix P, is

$$h(R^*:P)dR^* = L^*(D:P)h(R^*:I)dR$$

$$= K^*|P|^{-n/2}|R^*|^{(n-p-1)/2}\alpha_1^{-pn}dR^*$$

where

$$K^* = K\cdot p^{-p(p-1)/2}\cdot 2^{(pn+1)/2}\Gamma((pn+1)/2).$$

The desired distribution for $r_{ij}$ can be obtained from the pdf for R* by integrating out $t_i^2$, i = 1, 2, ..., (p-1).

4.6. DISTRIBUTION OF A SAMPLE CORRELATION MATRIX II: Let us have the same sample of the last section. We further assume here that the variances of α, ($1 \leq \alpha < p$), of the marginal distributions equal $\sigma_1^2$ and the rest of them equal $\sigma_1^2(\neq\sigma_1^2)$. Then we can, without loss of generality, assume that $\sum = DPD'$ where D is a diagonal matrix of the form

$$D = \operatorname{diag}(\overbrace{\sigma_1,\ldots,\sigma_1}^{\alpha}, \overbrace{\sigma_2,\ldots,\sigma_2}^{p-\alpha}) \tag{4.6.1}$$

with $\sigma_1 \neq \sigma_2$ and $\sigma_1,\sigma_2 > 0$. The MLE for $\rho_{ij}$, $1 \leq i < j \leq p$, are:

$$r_{ij} = \begin{cases} \alpha S_{ij}(\underset{\sim}{x})/S_1(\underset{\sim}{x})^2, & 1 \leq i < j \leq \alpha \\ \beta S_{ij}(\underset{\sim}{x})/S_2(\underset{\sim}{x})^2, & \alpha < i < j \leq p \\ (\alpha\beta)^{1/2}S_{ij}(\underset{\sim}{x})/[S_1(\underset{\sim}{x})S_2(\underset{\sim}{x})], & 1 \leq i \leq \alpha < j \leq p \end{cases}$$

where $\beta = p - \alpha$, $S_1(\underset{\sim}{x})^2 = \sum_{i=1}^{\alpha} S_{ii}(\underset{\sim}{x})^2$ and $S_2(\underset{\sim}{x})^2 = \sum_{l=\alpha+1}^{p} S_{ii}(\underset{\sim}{x})^2$.

Our object is to derive the distribution for $r_{ij}$'s by using the method of likelihood modulation. To do this, we consider the following associated conditional model:

$$\begin{cases} \underset{\sim}{x} = \theta \underset{\sim}{e} \\ f(\underset{\sim}{e})d\underset{\sim}{e} = (2\pi)^{-pn/2}|P|^{-n/2}\text{etr} -\left\{\frac{1}{2}\, e'P^{-1}\underset{\sim}{e}\right\}d\underset{\sim}{e} \end{cases}$$

where $\theta$ is p-order diagonal matrix of the form (4.6.1). We take

$$[\underset{\sim}{x}] = \text{diag}(\overbrace{S_1(\underset{\sim}{x}) \;\ldots\; S_1(\underset{\sim}{x})}^{\alpha} \;\; \overbrace{S_2(\underset{\sim}{x}) \;\ldots\; S_2(\underset{\sim}{x})}^{p-\alpha})$$

as our transformation variable. The reference point D of the orbit $G\underset{\sim}{x}$ is

$$D = \begin{pmatrix} e_{11}/S_1(\underset{\sim}{e}) & \cdots & e_{1n}/S_1(\underset{\sim}{e}) \\ \vdots & & \\ e_{\alpha 1}/S_1(\underset{\sim}{e}) & \cdots & e_{\alpha n}/S_1(\underset{\sim}{e}) \\ e_{(\alpha+1)1}/S_2(\underset{\sim}{e}) & \cdots & e_{(\alpha+1)n}/S_2(\underset{\sim}{e}) \\ \vdots & & \\ e_{p1}/S(\underset{\sim}{e}) & \cdots & e_{pn}/S_2(\underset{\sim}{e}) \end{pmatrix}$$

We note that $R^* = DD'$ is a symmetric matrix:

$R^* =$

$$\begin{pmatrix} S_{11}(\underset{\sim}{e})/S_1(\underset{\sim}{e})^2 & & & & * \\ \vdots & & & & \\ S_{\alpha 1}(\underset{\sim}{e})/S_1(\underset{\sim}{e})^2 & \cdots & S_{\alpha\alpha}(\underset{\sim}{e})/S_1(\underset{\sim}{e})^2 & & \\ S_{(\alpha+1)1}(\underset{\sim}{e})/[S_1(\underset{\sim}{e})S_2(\underset{\sim}{e})] & \cdots & S_{(\alpha+1)\alpha}(\underset{\sim}{e})/[S_1(\underset{\sim}{e})S_2(\underset{\sim}{e})] & S_{(\alpha+1)(\alpha+1)}(\underset{\sim}{e})/S_2(\underset{\sim}{e})^2 & \\ \vdots & & & & \\ S_{p1}(\underset{\sim}{e})/[S_1(\underset{\sim}{e})S_2(\underset{\sim}{e})] & \cdots & S_{p\alpha}(\underset{\sim}{e})/[S_1(\underset{\sim}{e})S_2(\underset{\sim}{e})] & S_{p(\alpha+1)}(\underset{\sim}{e})/S_2(\underset{\sim}{e})^2 & \\ & & & \cdots S_{pp}(\underset{\sim}{e})/S_2(\underset{\sim}{e})^2 & \end{pmatrix}$$

$$= \begin{pmatrix} t_1^2 & & & * & \\ \vdots & & & & \\ \alpha^{-1} r_{1\alpha} & \cdots & t_\alpha^2 & & \\ (\alpha\beta)^{-1/2} r_{1(\alpha+1)} & \cdots & (\alpha\beta)^{-1/2} r_{\alpha(\alpha+1)} & t_{(\alpha+1)}^2 & \\ \vdots & & & & \\ (\alpha\beta)^{-1/2} r_{1p} & \cdots & (\alpha\beta)^{-1/2} r_{\alpha p} & \beta^{-1} r_{(\alpha+1)p} \;\cdots & t_p^2 \end{pmatrix}$$

The conditional distribution for $[\underset{\sim}{e}]$, given the orbit D, is

$$K_p(D) f([\underset{\sim}{e}]D:P) S_1(\underset{\sim}{e})^{\alpha n-1} S_2(\underset{\sim}{e})^{\beta n-1} dS_1(\underset{\sim}{e}) dS_2(\underset{\sim}{e})$$

$$= K_p(D) K \operatorname{etr}\left\{-\frac{1}{2} D'[\underset{\sim}{e}]' P^{-1}[\underset{\sim}{e}]D\right\} S_1(\underset{\sim}{e})^{\alpha n-1} S_2(\underset{\sim}{e})^{\beta n-1} dS_1(\underset{\sim}{e}) dS_2(\underset{\sim}{e})$$

$$= K_p(D) K \operatorname{etr}\left\{-\frac{1}{2} P^{-1}[\underset{\sim}{e}]DD'[\underset{\sim}{e}]\right\} S_1(\underset{\sim}{e})^{\alpha n-1} S_2(\underset{\sim}{e})^{\beta n-1} dS_1(\underset{\sim}{e}) dS_2(\underset{\sim}{e}).$$

Note that $[e]DD'[\underset{\sim}{e}]'$ is symmetric and is equal to

$$\left(\begin{array}{ccc|ccc} S_1(\underset{\sim}{e})^2 t_1^2 & & * & & * & \\ \vdots & & & & & \\ S_1(\underset{\sim}{e})^2 r_{1\alpha}/\alpha & \cdots & S_1(\underset{\sim}{e})^2 t_\alpha^2 & & & \\ \hline S_1(\underset{\sim}{e})S_2(\underset{\sim}{e}) r_{1(\alpha+1)}/(\alpha\beta)^{-1} & \cdots & & S_2(\underset{\sim}{e})^2 t_{\alpha+1}^2 & & \\ \vdots & & & & & \\ S_1(\underset{\sim}{e})S_2(\underset{\sim}{e}) r_{1p}/(\alpha\beta)^{-1} & & \cdots & S_2^2(\underset{\sim}{e})^2 r_{(\alpha+1)p}/\beta & \cdots & t_p^2 \end{array}\right)$$

where $t_\alpha^2 = (1 - \sum_{i=1}^{\alpha-1} t_i^2)$ and $t_p^2 = (1 - \sum_{i=\alpha+1}^{p-1} t_i^2)$. So if

$P^{-1} = (\rho^{ij})$, we have

$$P^{-1}[\underset{\sim}{e}]DD'[\underset{\sim}{e}]'$$

$$= \left(\begin{array}{l} (\rho^{11} t_1^2 S_1(\underset{\sim}{e})^2 + S_1(\underset{\sim}{e})^2 \sum_{j=1}^{\alpha} \rho^{1j} r_{j1}/\alpha + S_1(\underset{\sim}{e})S_2(\underset{\sim}{e}) \sum_{j=\alpha+1}^{p} \rho^{1j} r_{j1}/(\alpha\beta)^{-1/2}) \quad * \\ \quad \ddots \\ \quad ((S_1(\underset{\sim}{e})^2 \sum_{j=1}^{\alpha-1} \rho^{\alpha j} r_{j\alpha}/\alpha + \rho^{\alpha\alpha} t_\alpha^2 S_1(\underset{\sim}{e})^2 + \\ \qquad S_1(\underset{\sim}{e})S_2(\underset{\sim}{e}) \sum_{j=\alpha+1}^{p} \rho^{\alpha j} r_{j\alpha}/(\alpha\beta)^{-1/2}) \\ \qquad (S_1(\underset{\sim}{e})S_2(\underset{\sim}{e}) \sum_{j=1}^{\alpha} \rho^{(\alpha+1)j} r_{(\alpha+1)j}/(\alpha\beta)^{-1/2} + \\ \qquad\quad \rho^{(\alpha+1)(\alpha+1)} t_{(\alpha+1)}^2 S_2(\underset{\sim}{e}) + S_2(\underset{\sim}{e}) \sum_{j=\alpha+2}^{p} \rho^{(\alpha+1)j} r_{(\alpha+1)j}/\beta) \\ \qquad\qquad \ddots \\ \qquad\quad (S_1(\underset{\sim}{e})S_2(\underset{\sim}{e}) \sum_{j=1}^{\alpha} \rho^{pj} r_{pj}/(\alpha\beta)^{-1/2} + \\ \qquad\qquad S_2(\underset{\sim}{e})^2 \sum_{j=\alpha+1}^{p-1} \rho^{pj} r_{pj}/\beta + \rho^{pp} S(\underset{\sim}{e})^2 t_p^2) \end{array}\right)$$

Therefore $\text{etr}\{-\frac{1}{2}P^{-1}[\underset{\sim}{e}]DD'[\underset{\sim}{e}]\}$

$$= \exp\{-(AS_1(\underset{\sim}{e})^2 + BS_2(\underset{\sim}{e}) + CS_1(\underset{\sim}{e})S_2(\underset{\sim}{e}))\}$$

where

$$A = \frac{1}{2}\left(\sum_{i=1}^{\alpha} \rho^{ii}t_i^2 + 2\sum_{1\leq i<j\leq\alpha} \rho^{ij}r_{ij}/\alpha\right),$$

$$B = \frac{1}{2}\left(\sum_{i=\alpha+1}^{p} \rho^{ii}t_i^2 + 2\sum_{\alpha<i<j\leq p} \rho^{ij}r_{ij}/\beta\right),$$

and

$$C = \frac{1}{2}\sum_{1\leq i\leq\alpha<j\leq p} 2\rho^{ij}r_{ji}/(\alpha\beta)^{1/2} \quad .$$

The normalizing constant factor $K_p(D)$ is

$$K_p(D)^{-1} = K\int_0^\infty\int_0^\infty \exp\{-[AS_1(\underset{\sim}{e})^2+BS_2(\underset{\sim}{e})^2+CS_1(\underset{\sim}{e})S_2(\underset{\sim}{e})\}\cdot$$

$$S_1(\underset{\sim}{e})^{\alpha n-1}S_2(\underset{\sim}{e})^{\beta n-1}dS_1(\underset{\sim}{e})dS_2(\underset{\sim}{e})$$

$$= K\int_0^\infty\int_0^\infty \exp\{-AS_1(\underset{\sim}{e})^2-BS_2(\underset{\sim}{e})^2\}\left[\sum_{i=0}^{\alpha}\frac{(-C)^i}{i!}S_1(\underset{\sim}{e})^iS_2(\underset{\sim}{e})^i\right]\cdot$$

$$S_1(\underset{\sim}{e})^{\alpha n-1}S_2(\underset{\sim}{e})^{\beta n-1}dS_1(\underset{\sim}{e})dS_2(\underset{\sim}{e})$$

$$=K\int_0^\infty\int_0^\infty \sum_{i=1}^{\alpha}\frac{(-C)^i}{i!}\exp\{-AS_1(\underset{\sim}{e})^2-BS_2(\underset{\sim}{e})^2\}S_1(\underset{\sim}{e})^{\alpha n+i-1}S_2(\underset{\sim}{e})^{\beta n+i-1}$$

$$dS_1(e)dS_2(e)$$

$$= \frac{K}{4}\sum_{i=1}^{\alpha}\frac{(-C)^i}{i!}\frac{\Gamma\{(\alpha n+i)/2\}}{A^{(\alpha n+i)/2}}\cdot\frac{\{\Gamma(\beta n+i)/2\}}{B^{(\beta n+i)/2}} \quad .$$

For $P = I$, we have $A = B = 1/2$, $C = 0$, and

$$K_I(D)^{-1} = \frac{K}{4}\left(\frac{1}{2}\right)^{-pn/2}\Gamma\left(\frac{\alpha n}{2}\right)\Gamma\left(\frac{\beta n}{2}\right) \quad .$$

Therefore the marginal likelihood function for P, based on D is (expressed as a ratio relative to that for P = I)

$$L^*(D:P) = K_I(D)/K_P(D)$$

$$= \left\{2^{-pn/2}\Gamma\left(\frac{\alpha n}{2}\right)\Gamma\left(\frac{\beta n}{2}\right)\right\}^{-1} \sum_{i=o}^{\infty} \frac{(-C)^i}{i!} \frac{\Gamma((\alpha n+i)/2)\Gamma((\beta n+i)/2)}{A^{(\alpha n+i)/2}\, B^{(\beta n+i)/2}} .$$

Distribution for $R^*$ when $P = I$: The pdf of $(S_{ij})$ is given by (4.5.1). The substitutions

$$\begin{cases} S_1(\underset{\sim}{e})^2 = \sum_{i=1}^{\alpha} S_{ii}(\underset{\sim}{e}) \\ S_2(\underset{\sim}{e})^2 = \sum_{i=\alpha+1}^{p} S_{ii}(\underset{\sim}{e}) \\ S_{ii}(\underset{\sim}{e}) = S_{ii}(\underset{\sim}{e}),\ i = 1, 2, \ldots, (\alpha-1),(\alpha+1),\ldots, (p-1), \\ S_{ij}(\underset{\sim}{e}) = S_{ij}(\underset{\sim}{e}),\ 1 \le i < j \le p \end{cases}$$

has jacobian equal unity. Further, we consider a second set of substitutions:

$$\begin{cases} S_i(\underset{\sim}{e})^2 = S_i(\underset{\sim}{e})^2, \quad i = 1, 2, \\ S_{ii}(\underset{\sim}{e}) = \begin{cases} t_i^2 S_1(\underset{\sim}{e})^2,\ i = 1, 2, \ldots, (\alpha-1) \\ t_i^2 S_2(\underset{\sim}{e})^2,\ i = (\alpha+1), \ldots, (p-1) \end{cases} \\ S_{ij}(\underset{\sim}{e}) = \begin{cases} r_{ij} S_1(\underset{\sim}{e})^2/\alpha, \quad 1 \le i < j \le \alpha \\ r_{ij} S_2(\underset{\sim}{e})^2/\beta, \quad \alpha < i < j \le p \\ r_{ij} S_1(\underset{\sim}{e}) S_2(\underset{\sim}{e})/(\alpha\beta)^{1/2},\ 1 \le i \le \alpha < j \le p. \end{cases} \end{cases}$$

The jacobian of the substitutions can be found in a similar manner as before:

| | $S^2_{11} \cdots S^2_{(\alpha-1)(\alpha-1)}$ | $S^2_{(\alpha+1)(\alpha+1)} \cdots S^2_{(p-1)(p-1)}$ | $S^2_1 \quad S^2_2$ | $S_{12} \cdots S_{1\alpha}$ | $S_{1(\alpha+1)} \cdots S_{1p}$ | $\cdots$ | $S_{(p-1)p}$ |
|---|---|---|---|---|---|---|---|
| $t^2_1$ ⋮ $t^2_{\alpha-1}$ | $S^2_1 \cdot \cdot \cdot S^2_1$ | 0 | 0 | 0 | 0 | | 0 |
| $t^2_{\alpha+1}$ ⋮ $t^2_{p-1}$ | | $S^2_2 \cdot \cdot \cdot S^2_2$ | 0 | 0 | 0 | | 0 |
| $S^2_1$ $S^2_2$ | $t^2_1 \cdots t^2_{\alpha-1}$ | $t^2_{\alpha+1} \cdots t^2_{p-1}$ | 1 1 | * | * | | * |
| $r_{12}$ ⋮ $r_{1\alpha}$ | 0 | 0 | 0 | $S^2_1 \cdot \cdot \cdot S^2_1$ | 0 | | 0 |
| $r_{1(\alpha+1)}$ ⋮ $r_{1p}$ | 0 | 0 | 0 | 0 | $S^2_1 S_2/(\alpha\beta)^{1/2} \cdot \cdot \cdot S_1 S_2/(\alpha\beta)^{1/2}$ | | 0 |
| ⋮ | | | | | | | |
| $r_{(p-1)p}$ | 0 | 0 | 0 | 0 | 0 | | $S^2_2/\beta$ |

where $S_i = S_i(\underset{\sim}{e})$, $i=1,2$ and $S_{ij} = S_{ij}(\underset{\sim}{e})$, $i \leq i < j \leq p$.

The required jacobian is therefore the product of the absolute of the determinate of the diagonal submatrices of the above matrix. That is

$$J = S_1(\underset{\sim}{e})^{2(\alpha-1)} S_2(\underset{\sim}{e})^{2(\beta-1)} (\alpha^{-1}S_1(\underset{\sim}{e})^2)^{\alpha(\alpha-1)/2} (\beta^{-1}S_2(\underset{\sim}{e})^2)^{\beta(\beta-1)/2} \cdot$$

$$[(\alpha\beta)^{1/2}(S_1(\underset{\sim}{e})\, S_2(\underset{\sim}{e}))]^{\alpha\beta}$$

$$= \alpha^{-\{\alpha(\alpha-1)/2+\alpha\beta/2\}} \beta^{-\{\beta(\beta-1)/2+\alpha\beta/2\}} S_1(\underset{\sim}{e})^{2(\alpha-1)+\alpha(\alpha-1)+\alpha\beta} \cdot$$

$$S_2(\underset{\sim}{e})^{2(\beta-1)+\beta(\beta-1)+\alpha\beta} .$$

Therefore the joint distribution for $R^*$ and $S_1(\underset{\sim}{e})$ and $S_2(\underset{\sim}{e})$, is

$$KJ|R^*[e][e]'|^{(n-p-1)/2} \exp\{-\tfrac{1}{2}(S_1(\underset{\sim}{e})^2+S_2(\underset{\sim}{e})^2)\} dR^* dS_1(\underset{\sim}{e})^2 dS_2(\underset{\sim}{e})^2 .$$

Hence the pdf for $R^*$ when $P = I$ is

$$K|R^*|^{(n-p-1)/2} \alpha^{-\alpha(p-1)/2} \beta^{-\beta(p-1)/2} \exp\{-\tfrac{1}{2}(S_1(\underset{\sim}{e})^2 + S_2(\underset{\sim}{e})^2)\} S_1(\underset{\sim}{e})^a S_2(\underset{\sim}{e})^b dR^* \cdot$$

$$dS_1(\underset{\sim}{e})^2 dS_2(\underset{\sim}{e})^2$$

where

$$a = 2(\alpha-1) + \alpha(\alpha-1) + \alpha\beta + (n-p-1)$$

$$= n\alpha - 2,$$

and

$$b = 2(\beta-1) + \beta(\beta-1) + \alpha\beta + (n-p-1)$$

$$= n\beta - 2.$$

Integrating out $S_1(\underset{\sim}{e})^2$ and $S_2(\underset{\sim}{e})^2$ over the region $(0,\infty)\times(0,\infty)$, we obtain the pdf for R*

$$h(R^*:I)dR^* = \int_0^\infty\int_0^\infty K|R^*|^{(n-p-1)/2}\alpha^{-\alpha(p-1)/2}\beta^{-\beta(p-1)/2} \cdot$$

$$\exp\{-\tfrac{1}{2}(S_1(\underset{\sim}{e})^2+S_2(\underset{\sim}{e})^2)\}S_1(\underset{\sim}{e})^{n\alpha-2}S_2(\underset{\sim}{e})^{n\beta-2}dS_1(\underset{\sim}{e})^2dS_2(\underset{\sim}{e})^2dR^*$$

$$= K^*|R^*|^{(n-p-1)/2}$$

where

$$K^* = K\alpha^{-\alpha(p-1)}\beta^{-\beta(p-1)/2}\, 2^{\frac{np}{2}-2} \cdot \Gamma(\tfrac{n}{2}-1)\Gamma(\tfrac{n}{2}-1).$$

Therefore the general distribution for R*, for general correlation matrix P, is

$$h(R^*:P)dR^* = L^*(D:P)h(R^*:I)dR^*$$

$$= \frac{K^*}{2^{pn/2}\Gamma(\frac{\alpha n}{2})\Gamma(\frac{\beta n}{2})}|R^*|^{(n-p-1)/2}\sum_{i=o}^{\infty}\frac{(-C)^i}{i!}\frac{(\frac{\alpha n+i}{2})\;(\frac{\beta n+i}{2})}{A^{(\alpha n+i)/2}B^{(\beta n+i)/2}}\,dR^*.$$

The distribution for $r_{ij}$, for general correlation matrix, can thus be obtained from the pdf for R* above by integrating out the $t_i(\underset{\sim}{e})$'S.

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