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On The Possibility Structure Of Physical Systems

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ON THE POSSIBILITY STRUCTURE OF PHYSICAL SYSTEMS

by

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ABSTRACT

This dissertation is a contribution to the theory of theories. Specifically, it concerns the interpretation of principle theories as opposed to constructive theories. The distinction is due to Einstein, and is basically this: In the case of constructive theories the idea is to reduce a wide class of diverse systems to component systems of a particular kind. The existence claims to which theories of this type have led are well known. (The molecular hypothesis of the kinetic theory of thermodynamic systems is an example.) Classical discussions of the reality of theoretical concepts have focussed on constructive theories. Principle theories have a different aim. These theories introduce abstract structural constraints which events are held to satisfy. Theories of space-time structure provide the most accessible illustration of principle theories. In this work the concept of a principle theory is extended to include theories of logical structure.

Interpretations of principle theories show in what fundamental respects they are related to preceding theories. For example, the special theory of relativity represents the transition from Newtonian mechanics to Maxwell's electrodynamics as involving a modification of the structure of space-time. It is in this sense that the special theory of relativity is an interpretation of classical electrodynamics.

Classical mechanics and quantum mechanics are represented as principle theories of logical structure. (Theories of

this type are called "statistical theories" or sometimes, "phase space theories".) The logical structure of a physical system is understood as imposing the most general constraint on the occurrence and non-occurrence of events.

The logical structures of quantum mechanics include the Boolean algebras of classical mechanics. Such structures represent the possibility structure of events, that is, roughly speaking, they represent the way in which the properties of a physical system hang together. The quantum theory has shown that significantly different assumptions may be made concerning this structure.

Chapter II formulates a general concept of completeness applicable to statistical theories. The analysis arises naturally from the consideration of Gleason's theorem and its corollaries and depends on the notion of a proper extension of a statistical theory. Extensions are defined relative to a category of algebraic structures representing the phase spaces of the theory and a suitable concept of statistical state. Basically, complete statistical theories have no proper extensions. This notion of completeness is a mathematical property of a certain class of algebraic structures rather than a metamathematical one. There exists an important model-theoretic connection between completeness and the formal theory of this class of structures; this is explained in Chapter I. But the concept of completeness does not depend on this connection.

A consequence of this analysis is that classical mechanics and quantum mechanics are complete in exactly the same sense. In neither theory do there exist extensions in the category of algebraic structures associated with their phase spaces. As principle theories, classical mechanics and quantum mechanics specify different kinds of constraints on the possible events open to a physical system, and each theory is complete relative to the category of algebraic structures defined.

The characteristic feature of the logical structures of quantum mechanics is the failure of semi-simplicity, which always holds in Boolean algebras. Chapter III examines the theory of truth for phase space theories. If the discussion of this chapter is correct, semi-simplicity is irrelevant to the classical concept of truth as correspondence, though it is decidedly not irrelevant to the classical theory of logical structure. This means that the concept of truth is the same in both classical and quantum mechanics; in particular, both theories are bivalent.

PREFACE

This dissertation consists of three papers which were prepared for separate publication. They appear here without modification. With the exception of Chapter II, each paper is largely self-contained. This has led to a slight overlap in the mathematical exposition of Chapters I and III.

Chapter I, "The Interpretation of Quantum Mechanics", was written in collaboration with Jeffrey Bub.

Numbers of works cited refer to the References following each chapter. The Bibliography is a cumulative list of works cited.

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I wish to thank Jeffrey Bub and Hilary Putnam for introducing me to the issues dealt with in this thesis.

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Chapter I

The Interpretation of Quantum Mechanics

> ...no solution of the problem is possible as long as in adherence to the tendencies of Huyghens and Mach one disregards the structure of the world.
>
> Hermann Weyl ([21] p. 105)

0. Introduction

The quantum theory is interpreted in the technical (semantical) sense. By an interpretation of quantum mechanics we mean something much less precise. As a rough approximation, an interpretation of a theory should show in what fundamental respects the theory is related to preceding theories. In the case of the quantum theory this means understanding the transition from classical mechanics to elementary (i.e. non-relativistic) quantum mechanics.

To begin with, we distinguish between two types of physical theory: 'principle' theories and 'constructive' theories.[1] The difference is basically this: In the case of constructive theories the idea is to reduce a wide class of diverse systems to component systems of a particular kind. The existence claims to which theories of this type have led are well known - especially in the case of the molecular hypothesis of the kinetic theory. The classical discussions of the reality of theoretical concepts have focussed on theories of this type. Principle theories have a different aim. These theories introduce abstract, structural constraints which events are held to satisfy.[2]

In this paper, classical and quantum mechanics are represented as a particular type of principle theory. We call theories of this type 'phase space theories' or 'theories of logical structure' since the type of structural constraint they introduce concerns the logical structure of events, and this is given by the phase space of the theory.[3] The logical structure of a physical system is understood as imposing the most general constraint on the occurrence and non-occurrence of events.

In Section 1 we consider the concept of an abstract structural constraint in the more familiar context of space-time theories. This motivates the preliminary discussion, in Section 2, of our concept of logical structure.

Section 3 presents an elementary characterization of the imbeddability relations between the phase space structures of classical and quantum mechanics. We mean this in the technical sense, i.e., in terms of the validity of elementary or first-order propositional formulae. The characterization depends on Kochen and Specker's crucial Theorem 4.[4]

As is well-known, the set $\underline{S}$ of statistical states of the quantum theory does not contain any states that are dispersion free. In Section 4 we discuss this in the light of Kochen and Specker's Theorem 1. In this section, we also compare the quantum theory with classical statistical mechanics. We conclude this section by relating our interpretation of the quantum theory in terms of logical structure to the original proposal of Birkhoff and von Neumann [2].

The mathematical discussion of this paper is based on Kochen and Specker's concept of a partial Boolean algebra. In Section 5, we consider two alternative representations of Hilbert space: orthomodular partially ordered sets (posets) and orthomodular lattices.

1. Space-Time Theories

The following discussion is based on the formulations of Anderson [1] and Trautman [18,19]. However the central idea of this section, viz. the distinction between coordinate transformations and symmetries, originates with Weyl. (See, e.g. [21], Section 13.)

Denote by E the set of all possible histories of physical systems, and by F the subset of physical histories that may actually occur; i.e., the histories in F are dynamically possible or allowed by the physical laws. The structural constraints of a space-time theory may be understood in terms of the concept of a symmetry. By this we mean an automorphism of E on E which carries F into F. Symmetries thus preserve the laws of motion, which determine the subset F of E. The space-time structure of a physical theory is given by the invariants of its symmetry group.

It is clearly possible to restrict F on the assumption that a particular symmetry group obtains; we are then restricting the character of physical laws on the basis of space-time structure. For example, the transition from Newtonian mechanics to special relativity consists in a modification of the symmetry group from the inhomogeneous Galilean group to the Poincaré group (inhomogeneous Lorentz group). The two fundamental invariants of the classical symmetry group, the Euclidian metric, and absolute time, are dropped altogether. Space-time structure is determined by the Maxwell-Lorentz theory, and the classical laws are modified: new law of motion for rapidly moving mass points.

A theory of space-time structure, if correct, tells us something perfectly objective about the world. There is an important misconception of the relation of relativity principles to the role of coordinates in physics which suggests that at least the choice of space-time <u>metric</u> is largely, if not wholly, a matter of convention or descriptive simplicity. We discuss this matter here since it anticipates issues which will arise later in connection with the quantum theory.

To see what is involved, it suffices to consider the principle of general relativity. To begin with, we distinguish this principle from the principle of general covariance. As we use the term (cf. Anderson [1], Section 4.2), generally covariant theories have the property that the transform of a solution of an equation is the solution of the transformed equation for 'arbitrary' coordinate transformations. That is to say, general covariance requires that the coordinates should not occur essentially in the formulation of physical laws. Roughly speaking, the possibility of a generally covariant formulation of physical laws is mainly a mathematical development, which was initiated by Minkowski.

In going from special to general relativity, the symmetry group is enlarged to include all diffeomorphisms, i.e. all maps which preserve the topological and differential structure of space-time. In the general theory metrical structures do not occur among the invariants of the space-time symmetry group. This is expressed by saying that the metric is not an absolute element of the general theory, but a dynamical variable which appears as a component in the histories in F. The local validity of the special theory requires that for infinitesimal regions of space-time the metric must assume (flat) Minkowskian values.

The important point to notice in the transition from special to general relativity theory is that it concerns the symmetry group, not merely the covariance group of the theory. In contrast to general relativity, the principle of general covariance is compatible with the existence of a symmetry group which is properly included in the set of all diffeomorphisms. For example, in a generally covariant formulation of special relativity, we may replace the coordinates by their curvilinear transforms. But this leaves invariant the Minkowski tensor which represents an absolute element of the special theory. In general relativity, there are no absolute metrical elements.

So long as these two principles are not kept clearly distinct, the generalization of a relativity principle, together with the corresponding change in space-time structure which this induces, will appear to be a purely formal development. This is because it seems plausible to view a change in the covariance group of a theory as largely a matter of mathematical convenience. But even if this were true, it would be irrelevant to the interpretation of relativity principles; since they concern the symmetry group, not the covariance group of the theory. Similarly, hypotheses concerning the metric depend on relativity principles, and thus on the symmetry group, not the covariance group of the theory. But the character of the symmetry group, and therefore the metrical structure of space-time, is independent of how we describe the dynamically possible histories in E. So even if it were possible to show that the choice of covariance group is conventional, nothing would follow concerning the choice of symmetry group.

(Notice, even the claim that the covariance group is conventional cannot be completely correct, since the requirement of general covariance restricts the class of mathematical objects which may represent physical magnitudes, and, to this extent, restricts the actual content of the theory. On this see Trautman's discussion.)

2. Phase Space Theories[5]

The fundamental problem for a phase space theory is the representation problem. It is required to find a phase space structure and a probability algorithm which correctly represents the totality of all possible events associated with a certain class of physical systems. In classical particle mechanics an event is represented by a point in a subset Ω of $6\underline{N}$ dimensional Euclidian space, where the $6\underline{N}$-tuple $(q_1, \ldots, q_{3\underline{N}}, p_1, \ldots, p_{3\underline{N}})$ of real numbers denotes the coordinates of position and momentum of the $\underline{N}$ components. In quantum mechanics an event is represented by a ray in a separable Hilbert space $\mathcal{H}$. In this section, we confine our attention to phase space structures.

Consider the set of all intervals of the real line, $\underline{R}$, half-open (on the right). The Borel subsets of $\underline{R}$ are the sets contained in the σ-ring generated by this set. A theoretical proposition about the system asserts that the value of a physical magnitude lies in one of these intervals. In the case of basic propositions (i.e., basic theoretical propositions) the intervals are atoms in the field of Borel subsets of $\underline{R}$.

We may imagine that the propositions of a phase space theory express the result of ideal - i.e., non-interfering - measurements. In the case of basic propositions, the measurements are also infinitely precise.

Now consider the system at a particular instant. The greatest lower bound $\Lambda_i\{a_i\}$, $i \varepsilon I$ (I is just some index set) of the set of all the basic propositions true of the system at that instant is called an atom or atomic proposition. Each atomic proposition determines an ultrafilter in the algebra of theoretical propositions.

Notice, theoretical propositions are algebraic objects and the structure of theoretical propositions is an algebraic structure of a certain kind. For example, in the commutative algebra $\underline{R}^{\Omega}$, theoretical propositions are associated with the characteristic functions of the Borel subsets of Ω; in the set of self-adjoint operators on a separable Hilbert space, theoretical propositions correspond to the projection operators. We adopt the following notational convention: $\underline{T}$ denotes the algebra of theoretical propositions of an arbitrary phase space theory; $\underline{C}$, the algebra of propositions of classical mechanics, and $\underline{Q}$, of quantum mechanics.

The phase space of the theory provides an alternative way of viewing this structure in terms of the topology of the space. For example, in the case of classical mechanics, the points in Ω correspond one-to-one with maximally consistent sets of theoretical propositions, i.e., with ultrafilters in the Boolean algebra of theoretical propositions. Now let $\underline{S}(\underline{C})$ denote the Stone space of $\underline{C}$ (the set of all ultrafilters in $\underline{C}$). The Stone isomorphism $h: \underline{C} \rightarrow \underline{S}(\underline{C})$ which maps a theoretical proposition onto the set of ultrafilters which contain it preserves the structure of $\underline{C}$. Because of the correspondence between the points of Ω and the ultrafilters of $\underline{S}(\underline{C})$, we may replace the Stone space of $\underline{C}$ by Ω. Then Ω is the Boolean space of $\underline{C}$; and h is an isomorphism of $\underline{C}$ onto the perfect and reduced field $F(\Omega)$ of simultaneously open and closed subsets of Ω. Under this mapping the image of a consistent set of propositions (i.e., a proper filter in $\underline{C}$) is a non-empty closed subset of Ω. An ultrafilter in $\underline{C}$ corresponds to a singleton subset $\{\omega\}$ of Ω. The unit filter in $\underline{C}$ is associated with the whole space, and the dual of the unit filter, the zero ideal, with the empty set.

In classical mechanics, atomic events in the history of a system are represented by the points in Ω, or the ultrafilters in $\underline{C}$, so that the algebra $\mathcal{B}$ of all possible events associated with a physical system is a Boolean algebra.

The theoretical propositions of quantum mechanics form a partial Boolean algebra. This structure may be viewed as a collection $\underline{Q} = \{\underline{Q}_i\}_{i\varepsilon I}$ of Boolean algebras such that for every $i, j\varepsilon I$ there is a $k\varepsilon I$ such that $\underline{Q}_i \cap \underline{Q}_j = \underline{Q}_k$; and if $a_1, \ldots, a_n$ are elements of $\underline{Q} = \cup_{i\varepsilon I}\{\underline{Q}_i\}$ such that any two of them lie in a common $\underline{Q}_i$, then there is a $k\varepsilon I$ such that $a_1, \ldots, a_n \varepsilon \underline{Q}_k$. $\underline{Q}$ is a partial Boolean algebra if we restrict the algebraic operations to elements in $\underline{Q}$ which lie in a common Boolean algebra $\underline{Q}_i$. For the quantum theory, $\underline{Q}$ is taken to be isomorphic to the partial Boolean algebra of linear subspaces of a suitable Hilbert space. A partial Boolean algebra may be pictured as 'built up' from its maximal Boolean subalgebras. In terms of this representation, the phase space of the quantum theory is just the isomorphic collection of Boolean spaces corresponding to the Boolean algebras $\underline{Q}_i$. Just as in classical mechanics, an atomic event is represented by an ultrafilter in Q or the image of this filter in the collection of Boolean spaces corresponding to Q, so that the algebra A of all possible events is a particular type of partial Boolean algebra.

For our purposes, it is sufficient to distinguish between Boolean and non-Boolean systems of events. The logical structure of an individual event is Boolean or non-Boolean according to whether the physical system to which it belongs is or is not Boolean. The distinction depends on a reflexive and symmetric binary relation of compatibility. Let $\hat{A}$ denote the set of all possible atomic events which a phase space theory associates with a physical system. If the relation of compatibility is transitive in $\hat{A}$, the system is Boolean. This is the case in classical mechanics. The quantum theory generalizes the logical structures of classical physics by introducing a relation of compatibility which is not transitive in $\hat{A}$. This leads to a class of event structures which differ strongly from classical logical structures in the sense that they are not even imbeddable into a Boolean algebra.

This distinction between classical and non-classical logical structures does not coincide with the distinction between classical and non-classical formal logics. That is, non-imbeddability into a Boolean algebra is a necessary condition for the logical structure of a system of events to be considered non-classical. But the non-classical logics usually considered in the literature determine classical logical structures. Each formal logic is associated with a characteristic algebra: the Lindenbaum-Tarski algebra of the logic.

This coincides with the logical structure determined by the formal logic. The Pseudo-Boolean algebras[6] associated with Intuitionist logic and the modal systems of Lewis coincide with distributive lattices. So the theorem of MacNiele [14] applies: for each such algebra, there is a Boolean imbedding. And by a result of Peremans [15], the imbedding is constructive.

The fact that there exist strongly different theoretical conceptions of the logical structure of a physical system indicates that this is as objective a component of the world as the events themselves. At least this is the major consideration in favor of realism elsewhere in science. If it is maintained that logical structure is conventional it must be possible to show that there is something which the question of logical structure does not share with other theoretical issues which would justify such an interpretation. For example, it is generally required that conventions be dispensible. Hence, if the choice of non-Boolean logical structure were conventional, it should be possible to reformulate the theory without this choice. But the logical structure of the quantum theory does not have this character.

Notice, in this connection, that we maintain a sharp distinction between logical structure in the sense of the phase space structure and the syntax and semantics of the formal language L in which the propositions of T are reconstructed. The choice of phase space is directly related to the representation problem, and therefore, to the quantum theory. The syntax and semantics of L raises a completely different set of problems. The events which the propositions of T describe are represented in a certain algebraic structure. This is given by the phase space which the theory associates with the physical system considered. Now on any reconstruction of the theory, this structure is retained, whatever the choice of L. Thus the syntax and semantics of L - i.e., logical structure in the conventional sense - is not theoretically important.

There is another respect in which our approach differs from accounts which use the concept of a theoretical proposition: Usually the concept of a theoretical proposition is introduced in order to identify a phase space theory with the pair (T,S) consisting of the system T of its theoretical propositions, and set S of its statistical states. Now the properties asserted by theoretical propositions are elementary or first-order properties, so they are expressible in an elementary language. But most theoretically interesting properties are not even general first-order properties; i.e., properties P such that a structure has P if and only if it is a model of some (possibly infinite) set Σ of first order sentences. For example, the property of being a Euclidean space is not a general first order property. This is true of other space-time properties as well. Since, in our interpretation, such properties are an essential component of principle theories, the identification of a theory with a first-order reconstruction of the system of its theoretical propositions is not justified.

Finally, it is necessary to consider the objection that the concept of logical structure introduced here involves an unjustifiable extension of 'logic'. Insofar as this is not a completely verbal issue, it overlooks several important considerations.

(A) It is possible to characterize the difference between the classical and quantum mechanical phase spaces in terms of the validity and refutability of classical tautologies, the so-called 'propositions of logic'. The concept of validity employed in this characterization is a generalization of the classical concept of validity in a straight-forward sense. Both of these points will be explained in detail in the discussion of Kochen and Specker's work.

(B) The phase space structures with which we are concerned are Boolean algebras or generalizations of Boolean algebras. From a mathematical point of view, classical propositional logic is essentially a Boolean algebra when equivalent sentences are 'identified'. There is also the well-known equivalence of the representation theory of Boolean algebras with the metatheory of classical logic.

(C) The identification of logic, or logical structure with the syntax and semantics of formal languages is by no means a necessary delimitation of the subject but is due to a particular point of view: viz., Formalism. Thus, in part, this paper may be viewed as a rejection of Formalism as an adequate theory of the application of logic in physics.

3. Validity and Imbeddability

This section is essentially an exposition and clarification of the work of Kochen and Specker. For the most part, we adopt their notation and terminology.

A <u>partial algebra</u> over a field <u>K</u> is a set <u>A</u> with a reflexive and symmetric binary relation $\leftrightarrow$ (termed 'compatibility'), closed under the operations of addition and multiplication, which are defined only from $\leftrightarrow$ to <u>A</u>, and the operation of scalar multiplication from <u>K</u> × <u>A</u> to <u>A</u>. That is:

(i) $\leftrightarrow \subseteq A \times A$

(ii) every element of <u>A</u> is compatible with itself

(iii) if a is compatible with b, then b is compatible with a, for all a, bε<u>A</u>

(iv) if any a, b, cε<u>A</u> are mutually compatible, then $(a+b)\leftrightarrow c$, $ab\leftrightarrow c$, and $\lambda a\leftrightarrow b$ for all $\lambda\varepsilon$<u>K</u>.

In addition, there is a unit element 1 which is compatible with every element of <u>A</u>, and if a,b,c are mutually compatible, then the values of the polynomials in a,b,c form a commutative algebra over the field <u>K</u>.

A partial algebra over the field $\underline{Z}_2$ of two elements is termed a <u>partial Boolean algebra</u>. The Boolean operations $\wedge$, v and ' may be defined in terms of the ring operations in the usual way:

$$a \wedge b = ab$$

$$a \vee b = a + b - ab$$

$$a' = 1 - a$$

If a,b,c are mutually compatible, then the values of the polynomials in a,b,c form a Boolean algebra.

Clearly, if *B* is a set of mutually compatible elements in a partial algebra A, then *B* generates a commutative sub-algebra in A; and in the case of a partial Boolean algebra A, *B* generates a Boolean sub-algebra in A. Just as the set of idempotent elements of a commutative algebra forms a Boolean algebra, so the set of idempotents of a partial algebra forms a partial Boolean algebra. A partial Boolean algebra may also be defined directly in terms of the Boolean operations $\wedge, \vee$ and $'$. A partial Boolean algebra associated with a Hilbert space may be regarded as a partially ordered set with a reflexive and symmetric relation of compatibility, such that each maximal compatible subset is a Boolean algebra.

We restrict the discussion now to partial *Boolean* algebras. A *homomorphism*, h, between two partial Boolean algebras, A and A', is a map $h: A \to A'$ which preserves the algebraic operations; i.e. for all compatible $a, b \in A$:

$$h(a) \leftrightarrow h(b)$$

$$h(a+b) = h(a) + h(b)$$

$$h(ab) = h(a)h(b)$$

$$h(1) = 1.$$

A homomorphism is an *imbedding* if it is one-to-one, and into. A *weak imbedding* is a homomorphism which is an imbedding on Boolean sub-algebras of A. More precisely, a homomorphism, h, of A into A' is a weak imbedding if $h(a) \neq h(b)$ whenever $a \leftrightarrow b$ and $a \neq b$ in A.

A necessary and sufficient condition for the imbeddability of a partial Boolean algebra A into a Boolean algebra B, is that for every pair of distinct elements a, bεA there exists a homomorphism h:A$\to$$\underline{Z}_2$ which separates them in $\underline{Z}_2$, i.e. such that h(a) $\neq$ h(b) in $\underline{Z}_2$. This is Kochen and Specker's Theorem 0. The result depends on the semi-simplicity property of Boolean algebras, i.e. essentially, the homomorphism or ultrafilter theorem.

The counterpart of Theorem 0 for weak imbeddability is the following: A necessary and sufficient condition for the weak imbeddability of a partial Boolean algebra A into a Boolean algebra B is that for every non-zero element aεA there exists a homomorphism h:A$\to$$\underline{Z}_2$ such that h(a) $\neq$ 0.

A propositional or Boolean function $\phi(x_1,\ldots,x_n)$ may be regarded as a polynomial over $\underline{Z}_2$. To say that a particular propositional function, e.g. the function

$$x_1 \wedge (x_2 \wedge x_3) \equiv (x_1 \wedge x_2) \wedge x_3$$

is a classical tautology, is to say that every substitution of elements from a Boolean algebra B for the variables x_1,x_2,x_3 yields the unit element in B. We seek a generalization of this classical notion of validity to include substitutions from partial Boolean algebras. Kochen and Specker propose that a propositional function such as the above is valid in a partial Boolean algebra A if every 'meaningful' substitution of elements from A yields the unit element in A. A 'meaningful' substitution is one which satisfies the compatibility relations; otherwise the partial operations are undefined in A. In this particular case, we require that the elements a_1,a_2,a_3 of A substituted for x_1,x_2,x_3 satisfy the conditions:

$$a_2 \leftrightarrow a_3$$

$$a_1 \leftrightarrow a_2$$

$$a_1 \leftrightarrow a_2 \wedge a_3$$

$$a_1 \wedge a_2 \leftrightarrow a_3$$

$$a_1 \wedge (a_2 \wedge a_3) \leftrightarrow (a_1 \wedge a_2) \wedge a_3.$$

This notion is formalized in the following definition:

Let $a = \langle a_1, \ldots, a_n \rangle$ be an element in A^n, the n-fold Cartesian product $A \times \ldots \times A$ of the partial Boolean algebra A. The domain, $\underline{D}_\phi$, in A of a propositional function $\phi(x_1, \ldots, x_n)$ is defined recursively, together with a recursive definition of a map ϕ^* (corresponding to ϕ) from $\underline{D}_\phi$ into A, as follows:

(1) If ϕ is the polynomial 1, then $\underline{D}_\phi = A^n$ and $\phi^*(a) = 1$.

(2) If ϕ is the polynomial $x_i (i=1,\ldots,n)$, then $\underline{D}_\phi = A^n$, and $\phi^*(a) = a_i$.

(3) If $\phi = \psi \otimes \chi$ (where $\otimes$ is either + or .), then $\underline{D}_\phi$ consists of those sequences a which belong to the intersection of the domains of ψ and χ (i.e. $a \varepsilon \underline{D}_\psi \cap \underline{D}_\chi$), and also satisfy the compatibility condition $\psi^*(a) \leftrightarrow \leftrightarrow \chi^*(a)$. The map $\phi^*(a)$ is defined by $\phi^*(a) = \psi^*(a) \otimes \chi^*(a)$.

The definition of the domain of a propositional function in a given partial Boolean algebra A serves to make precise the notion of a 'meaningful' substitution, while the map ϕ^* defines the value of the polynomial in A for each such substitution.

The statement that the identity

$$\phi(x_1,\ldots,x_n) = 1$$

holds in A is to be understood in the sense that

$$\phi^*(a) = 1$$

for all $a\varepsilon \underline{D}_\phi$.

The statement that the identity

$$\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n)$$

holds in A is to be understood in the sense that

$$\phi^*(a) = \psi^*(a)$$

for all $a\varepsilon \underline{D}_\phi \cap \underline{D}_\psi$.

Now, the generalized definition of validity is this: A propositional function $\phi(x_1,\ldots,x_n)$[i.e. a Boolean function - a polynomial over $\underline{Z}_2$] is valid in the <u>partial Boolean algebra</u> A if the identity $\phi = 1$ holds in A.

ϕ is refutable in A if for some $a\varepsilon \underline{D}_\phi$, $\phi^*(a) = 0$ in A.

ϕ is logically valid in the generalized sense, i.e. Q-valid, if ϕ is valid in every partial Boolean algebra A. If the choice of A is restricted to Boolean algebras, this definition of validity coincides with the usual definition: the set of valid propositional formulae is just the set of classical tautologies. Thus the recursive definition of the domain of a propositional function coupled with the recursive definition of the map ϕ^* just generalizes the classical, Boolean interpretation of ϕ.

It is important to appreciate the distinction between the validity of a Boolean function

$$\psi \equiv \chi$$

in a partial Boolean algebra A, and the holding of the identity

$\psi = \chi$

in A. To say that $\psi \equiv \chi$ is valid in A is to say that

$(\psi \equiv \chi) = 1$

in A, i.e. writing $\phi = (\psi \equiv \chi)$, we require that

$\phi^*(a) = 1$

for every sequence $a \varepsilon \underline{D}_\psi \cap \underline{D}_\chi$ satisfying the additional compatibility condition

$\psi^*(a) \leftrightarrow \chi^*(a)$.

But for the identity $\psi \equiv \chi$ to hold in A, we require that $\psi^*(a) = \chi^*(a)$ for <u>every</u> sequence $a \varepsilon \underline{D}_\psi \cap \underline{D}_\chi$, not only those sequences satisfying the additional compatibility condition $\psi^*(a) \leftrightarrow \chi^*(a)$. Thus, the set of admissable sequences $a \varepsilon A^n$ is <u>smaller</u> in the case of the validity of the biconditional than in the case of the identity. If the identity holds in A, then certainly the biconditional is valid in A, but the converse is not in general true. The validity of the biconditional amounts to the holding of the identity for the restricted set of sequences which satisfy the compatibility condition $\psi^*(a) \leftrightarrow \chi^*(a)$.

For example, let $\phi = (\psi \equiv \chi)$ be the classical tautology:

$x_1 \wedge (x_2 \vee x_3) \equiv (x_1 \wedge x_3) \vee (x_1 \wedge x_3)$.

ϕ is not only valid in every partial Boolean algebra, it is also the case that the identity

$x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$

holds in every A. For if $a = \langle a_1, a_2, a_3 \rangle \varepsilon \underline{D}_\phi$,

$a_2 \leftrightarrow a_3$

$a_1 \leftrightarrow a_2$

$a_1 \leftrightarrow a_3$

But then a_1, a_2, a_3, generate a Boolean algebra. It follows that

$$a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3),$$

and hence

$$a_1 \wedge (a_2 \vee a_3) \leftrightarrow (a_1 \wedge a_2) \vee (a_1 \wedge a_3).$$

Thus, every sequence $a \epsilon \underline{D}_\psi \cap \underline{D}_\chi$ automatically satisfies the compatibility condition $\psi^*(a) \leftrightarrow \chi^*(a)$.

In the case of a partial Boolean algebra A imbeddable into a Boolean algebra, the validity of the biconditional $\psi \equiv \chi$ in $\underline{Z}_2$ (i.e. the classical tautologousness of the biconditional) entails the holding of the identity $\psi = \chi$ in A. Thus, in the case of imbeddability (and only in this case):

$\psi \equiv \chi$ is valid in $\underline{Z}_2$

is equivalent to

$\psi = \chi$ holds in A.

This is a consequence of Kochen and Specker's Theorem 4, to which we now turn.

Kochen and Specker's Theorem 4 establishes an elementary condition for the imbeddability of a partial Boolean algebra into a Boolean algebra. This clarifies the relationship between the validity of classical tautologies in a partial Boolean algebra A and the imbeddability of A into a Boolean algebra. The statement of the theorem is as follows:

(1) A necessary and sufficient condition for the imbeddability of a partial Boolean algebra A into a Boolean algebra is the holding of the corresponding identity $\psi = \chi$ in A for every classical tautology of the form $\psi \equiv \chi$.

(2) A necessary and sufficient condition for the <u>weak imbeddability</u> of a partial Boolean algebra A into a Boolean algebra is the validity in A of every classical tautology.

(3) A necessary and sufficient condition for the existence of a <u>homomorphism</u> from a partial Boolean algebra A into a Boolean algebra is the irrefutability in A of every classical tautology.

The first part of the theorem states that A is imbeddable into a Boolean algebra if and only if, for every Boolean function of the form $\psi\equiv\chi$ which is valid in $\underline{Z}_2$ (i.e. for which the identity $(\psi\equiv\chi) = 1$ holds in $\underline{Z}_2$), $\psi=\chi$ is valid in A.

If A is imbeddable into a Boolean algebra and ϕ is a propositional formula not valid in A, i.e.

$$\phi^*(a) \neq 1$$

for some $a\varepsilon\underline{D}_\phi$ in A, then by Theorem 0 there is a homomorphism onto $\underline{Z}_2$ such that

$$h(\phi^*(a)) \neq h(1)$$

i.e.

$$\phi^*(h(a)) \neq 1 \text{ in } \underline{Z}_2,$$

hence ϕ is not valid in $\underline{Z}_2$. Thus if A is imbeddable into a Boolean algebra, <u>all</u> classical tautologies are valid in A.

(If we assume that the theorem holds, this may be proved directly, as follows: Suppose that for every tautology of the form $\phi\equiv\psi$, $\phi=\psi$ is valid in A. Now let χ be a classical tautology. Then

$$\chi\equiv 1$$

is a classical tautology, where 1 is the constant Boolean function. It follows that

$$\chi=1$$

holds in A, i.e. that χ is valid in A.)

The difference between weak imbeddability and (strong) imbeddability for the set of functions valid in A is just this: In the case of weak imbeddability, all the classical tautologies are valid in A (and in general there are also functions valid in A which are not classical tautologies). In the case of (strong) imbeddability, all the classical tautologies are valid in A. There may also be functions valid in A which are not classical tautologies. But here we know in addition that if $\psi\equiv\chi$ is a classical tautology, then $\psi=\chi$ holds in A.

Thus, for weak imbeddability, if $\psi\equiv\chi$ is a classical tautology (i.e. if $\psi\equiv\chi$ is valid in $\underline{Z}_2$), we know that $\psi\equiv\chi$ is valid in A (by the second part of the theorem), but we cannot conclude that $\psi=\chi$ holds in A. In the case of (strong) imbeddability, this inference is legitimate, i.e. from the validity of a biconditional in $\underline{Z}_2$, we may infer that the corresponding <u>identity</u> holds in A. This means that in the case of imbeddability we may infer the holding of the identity

$$\psi=\chi$$

from the validity of the biconditional; i.e. from

$$(\psi\equiv\chi) = 1 \text{ in } A,$$

whenever $\psi\equiv\chi$ is a classical tautology, as well as the converse (which follows immediately from the definition of validity for an identity).

Notice that we cannot conclude that <u>only</u> the classical tautologies are valid in A if A is imbeddable into a Boolean algebra; it does not follow that if A is imbeddable, and ϕ is valid in A, then ϕ is valid in $\underline{Z}_2$. For, to say that ϕ is valid in A is to say that

$$\phi^*(a) = 1 \text{ in } A$$

for every $a \in \underline{D}_\phi$ in A, and to say that ϕ is valid in $\underline{Z}_2$ is to say that

$$\phi^*(a) = 1 \text{ in } \underline{Z}_2$$

for every $a \in \underline{D}_\phi$ in $\underline{Z}_2$. But the imbedding into $\mathcal{B}$ may only use a proper subset of the sequences in $\underline{Z}_2^\omega$ associated with the elements of $\mathcal{B}$. We can conclude that all and <u>only</u> the classical tautologies are valid in A only if the imbedding is an isomorphism.

We give an exposition here only of the proof of the first part of the theorem.

The necessity of the condition is relatively easy to prove. We are required to show that the holding of the corresponding identity $\psi = \chi$ in A for every classical tautology of the form $\psi \equiv \chi$ is a necessary condition for the imbeddability of A into a Boolean algebra. In other words, we are required to show that if A is imbeddable, then for every biconditional $\psi \equiv \chi$ which is a classical tautology (i.e. which is valid in $\underline{Z}_2$), the corresponding identity $\psi = \chi$ holds in A.

Suppose A is imbeddable into a Boolean algebra, and that $\psi \equiv \chi$ is a classical tautology. We must show that this entails that the identity $\psi = \chi$ holds in A. We show this by proving that

$\psi \neq \chi$ in A

leads to a contradiction.

If $\psi \neq \chi$ in A, then for some $a \varepsilon \underline{D}_\psi \cap \underline{D}_\chi$:

$\psi^*(a) \neq \chi^*(a)$.

Now, by Theorem 0, since A is imbeddable into a Boolean algebra, for each b, $c \varepsilon A (b \neq c)$ there exists a homomorphism h: $A \to \underline{Z}_2$ such that

$h(b) \neq h(c)$

and so there exists a homomorphism h: $A \to \underline{Z}_2$ such that:

$h(\psi^*(a)) \neq h(\chi^*(a))$

or

$\psi^*(h(a_1),\ldots,h(a_n)) \neq \chi^*(h(a_1),\ldots,h(a_n))$.

In other words, $\langle h(a_1),\ldots,h(a_n)\rangle$ is an admissable sequence in $\underline{Z}_2^n$ such that

$\psi^*(h(a_1),\ldots,h(a_n)) \neq \chi^*(h(a_1),\ldots,h(a_n))$.

This means that $\psi \neq \chi$ in $\underline{Z}_2$, and so the biconditional $\psi \equiv \chi$ is not valid in $\underline{Z}_2$, i.e. $\psi \equiv \chi$ is not a classical tautology, contrary to our original assumption.

(Notice, it would not in general be permissable to infer the non-validity of the biconditional $\psi \equiv \chi$ from the fact that the identity $\psi = \chi$ failed to hold in a partial Boolean algebra. This inference is, however, obviously legitimate in $\underline{Z}_2$.)

To prove the sufficiency of the condition, we must show that the holding of the corresponding identity $\psi = \chi$ in A for every classical tautology entails the imbeddability of A

into a Boolean Algebra. Kochen and Specker prove the contrapositive; if A is not imbeddable into a Boolean algebra, then there exists a classical tautology $\psi \equiv \chi$ such that for some $a \epsilon D_{\phi} \cap D_{\psi}$, $\psi = \chi$ is not valid in A.

Let $\underline{K}_1$ denote the set of positive statements from the diagram of A, i.e. the sentences formulated in some first-order language L which describe all equations of the form $\alpha + \beta = \gamma$ or $\xi\eta = \zeta$ which subsist among elements of A.

Let $\underline{K}_2$ be the set of sentences formulated in L describing the class of Boolean algebras.

Write $\underline{K} = \underline{K}_1 \cup \underline{K}_2$.

It is very important to bear in mind throughout the following that $\underline{K}$ is a subset of the sentences in L, the first-order language in which the Boolean axioms are formulated, and in which relations of the form: $\alpha + \beta = \gamma$ and $\xi\eta = \zeta$ which subsist among elements of A are formulated.

Now, the models of the set of sentences $\underline{K}_1$ are all homomorphic images of A. Hence, the class of all models of $\underline{K}$ comprises all homomorphic images of A which are Boolean algebras.

If A is not imbeddable into a Boolean algebra, then, by Theorem 0, there exists a pair of elements $a, b \epsilon A$ such that <u>no homomorphism onto $\underline{Z}_2$ will separate them</u>. That is, a and b are two distinct elements in A which are identified by every homomorphism onto $\underline{Z}_2$.

If a and b are not separated by any homomorphism onto $\underline{Z}_2$, then they cannot be separated by a homomorphism into any

Boolean algebra (by the homomorphism theorem, or the semi-simplicity property of Boolean algebras). That is to say, a and b are identified in every model of $\underline{K}$ (since the models of $\underline{K}$ are just a class of Boolean algebras, viz. those which are homomorphic images of A).

Thus:

$$\underline{K} \models_{h(A)} a = b$$

(where $h(A)$ is the class of Boolean algebras which are homomorphic images of A). Note that $a=b$ is to be understood here as a sentence in the first-order language L.

By the completeness of L, we have:

$$\underline{K} \models a=b.$$

By the (syntactic) compactness of L, $a=b$ follows from a finite subset $\underline{L}$ of $\underline{K}_1$, where

$$\underline{L} = \{\alpha_j+\beta_j = \gamma_j, \xi_k\eta_k = \zeta_k \mid 1 \le j \le n, 1 \le k \le m\}.$$

Hence:

$$\underline{K}_2 \cup \underline{L} \models a=b$$

or

$$\underline{K}_2, \{\wedge_{j,k}\underline{L}\} \vdash a=b$$

where

$$\wedge_{j,k}\underline{L}$$

is a finite conjunction of sentences of L. Hence by the deduction theorem for L:

$$\underline{K}_2 \vdash \wedge_{j,k}\underline{L} \rightarrow a=b.$$

Clearly, $\wedge_{j,k}\underline{L}$ is logically equivalent to the conjunction

$$\wedge_{j,k}\{\alpha_j+\beta_j+\gamma_j = 0,\ \xi_k\eta_k+\zeta_k = 0 \mid 1 \le j \le n,\ 1 \le k \le m\}$$

of sentences of L, which is logically equivalent to the sentence:

$$\vee_{j,k}\{\alpha_j+\beta_j+\gamma_j,\ \xi_k\eta_k+\zeta_k \mid 1\le j\le n,\ 1\le k\le m\} = 0$$

where the sign $\vee$ is to be understood as the supremum or least upper bound in the partial Boolean algebra A. That is,

$$\vee_{j,k}\{\alpha_j+\beta_j+\gamma_j,\ \xi_k\eta_k+\zeta_k \mid 1\le j\le n,\ 1\le k\le m\}$$

denotes an element in A: the least upper bound of all the elements of the form $\alpha_j+\beta_j+\gamma_j$, and $\xi_k\eta_k+\zeta_k$. The sentence asserting that the least upper bound of all these elements is zero is equivalent to the conjunction of all the sentences asserting separately that $\alpha_j+\beta_j+\gamma_j = 0 (1\le j\le n)$ and $\xi_k\eta_k+\zeta_k = 0 (1\le k\le m)$.

It follows immediately that:

$$\underline{K}_2 \vdash (\vee_{j,k}\{\alpha_j+\beta_j+\gamma_j,\ \xi_k\eta_k+\zeta_k \mid 1\le j\le n,\ 1\le k\le m\} = 0) \rightarrow a=b.$$

Write:

$$\rho(\alpha_1,\ldots,\zeta_m) \stackrel{\text{def}}{=} \vee_{j,k}\{\alpha_j+\beta_j+\gamma_j,\ \xi_k\eta_k+\zeta_k \mid 1\le j\le n,\ 1\le k\le m\}.$$

Then:

$$\underline{K}_2 \vdash \rho(\alpha_1,\ldots,\zeta_m) = 0 \rightarrow a=b.$$

Since the constants $\alpha_1,\ldots,\zeta_m$, a,b do not occur in $\underline{K}_2$, they may be replaced by variables $x_1,\ldots,x_m,x$, y to obtain:

$$\underline{K}_2 \vdash \rho(x_1,\ldots,x_m) = 0 \rightarrow x=y.$$

We have now shown that the conditional:

$$\rho(x_1,\ldots,x_m) = 0 \rightarrow x=y,$$

which is to be understood as a formula in L, is valid in all Boolean algebras.

Let

ψ denote $x \rightarrow \rho$

χ denote $y \rightarrow \rho$

i.e. ψ and χ are Boolean functions, explicitly:

ψ is the function $1 - \chi + \rho - (1-x)\rho$

χ is the function $1 - y + \rho - (1-y)\rho$.

Since:

$\rho = 0 \rightarrow x=y$

is valid in all Boolean algebras, it follows that the identity

$\psi=\chi$

holds in $\underline{Z}_2$, i.e.

$\psi^*(a) = \chi^*(a)$

for every sequence $a \varepsilon \underline{D}_\psi \cap \underline{D}_\chi$. For, suppose under some substitution for the variables $x_1,\ldots,x_m$, that $\rho^*(a) = 0$. Then because $\rho=0 \rightarrow x=y$ is valid in $\underline{Z}_2$, we have:

$\psi^*(a) = \chi^*(a)$,

If, under some substitution $\rho^*(a) = 1$, we have:

$\psi^*(a) = 1 = \chi^*(a)$.

Since the identity

$\psi=\chi$

holds in $\underline{Z}_2$, it follows that

$\psi\equiv\chi$

is valid in $\underline{Z}_2$, i.e. that $\psi\equiv\chi$ is a <u>classical tautology</u>. But $\psi=\chi$ <u>does not hold in A</u>. For, substituting the sequence $\langle \alpha_1,\ldots,\zeta_m, a,b\rangle$ of elements from A for the variables $(x_1,\ldots, x_m,x,y)$ yields the value $1-a$ for ψ and $1-b$ for χ. That is, under this valuation for ψ and χ in A, we have

$\psi^* = 1-a$

and

$\chi^* = 1-b$

with $a\neq b$.

Thus, in the case of a partial Boolean algebra A for which there is no Boolean imbedding, there is a biconditional, $\psi \equiv \chi$, which is a classical tautology, and a sequence $<\alpha_1, \ldots, \zeta_m, a, b> \in \underline{D}_\psi \cap \underline{D}_\chi$ under which the corresponding identity, $\psi = \chi$, does not hold in A. This proves the theorem.

4. The Basic Problem

The set $\underline{S}$ of statistical states of quantum mechanics does not contain states which are dispersion free. This is the property of the quantum theory which generates the problem of 'interpretation': i.e., the problem is to understand the absence of dispersion free states.

A statistical state $\psi \varepsilon \underline{S}$ is a map $\psi : \underline{T} \rightarrow [0,1]$ such that $\psi(1) = 1$ and $\psi(\vee_i \{a_i\}) = \Sigma_i \psi(a_i)$ if $\{a_i\}$ is a disjoint sequence. That is, a statistical state is a generalized probability assignment to the theoretical propositions of the theory which satisfies the usual conditions for a probability measure on each maximal compatible subset of the partial Boolean algebra $\underline{T}$. The probability algorithm of a phase space theory is a function $\underline{P}$ which assigns to each magnitude $\underline{A}$ and each state ψ, a probability measure on $\underline{R}$. $\underline{P}_{A\psi}(U)$ denotes the probability that in the state ψ the value of the magnitude $\underline{A}$ lies in $\underline{U}$. For dispersion free states, the probability assigned to each magnitude reduces to an atomic measure concentrated on the value of $\underline{A}$. It is not difficult to show that ψ is dispersion free if and only if ψ is a homomorphism of $\underline{T}$ onto $\underline{Z}_2$. (See e.g., Gudder [8] for a proof.)

In classical mechanics the algebra $\mathcal{B}$ of events is a Boolean algebra, so there is a one-to-one correspondence between atomic events $a \varepsilon \hat{\underline{B}}$ and two-valued homomorphisms on $\mathcal{B}$. This property is preserved in the algebra of theoretical propositions. Thus when $\mathcal{B}$ is a Boolean algebra, each atomic event determines a two-valued homomorphism, and hence, a dispersion free state, on $\underline{T}$.

Each event in $\hat{\underline{B}}$ is represented bi-uniquely by a point $\omega\epsilon\Omega$, since the field of subsets of Ω is perfect and reduced. Each point $\omega\epsilon\Omega$ determines a two-valued homomorphism on $F(\Omega)$, and therefore, a dispersion free state on $\underline{C}$. Conversely, since $\underline{C}$ is isomorphic to $F(\Omega)$, $\psi \circ h^{-1}$, where h is the Stone isomorphism, is a statistical state on Ω.

A dispersion free state on Ω may be replaced by the point $\omega\epsilon\Omega$ which determines it. A <u>classical mechanical state</u> is just the phase point which determines ψ when ψ is dispersion free. Thus a classical mechanical state corresponds to an atomic proposition in $\underline{C}$. Because $\underline{C}$ is a Boolean algebra, each atom in $\underline{C}$ determines a homomorphism onto $\underline{Z}_2$, and hence a dispersion free statistical state.

In a Boolean algebra, each $a\epsilon\hat{\underline{B}}$ determines a maximal proper filter $\underline{F}$ in B; similarly, in a partial Boolean algebra each $a\epsilon\hat{A}$, determines a maximal proper filter $\underline{F}$ in A. Notice, in case A is a Boolean algebra, each maximal proper filter in A may be used to define a homomorphism onto $\underline{Z}_2$ by the condition, $h(a) = 1$, if $a\epsilon\underline{F}$ and $h(a) = 0$ if $a\notin\underline{F}$. This possibility depends on the distributivity of Boolean algebras, for this implies that maximal proper filters in A are prime filters.[8] If the ultrafilters in A are not prime, we may have $a\notin\underline{F}$ and $a'\notin\underline{F}$, but $1\epsilon\underline{F}$, and therefore, $a \vee a'\epsilon\underline{F}$. Hence, the correspondence between ultrafilters in A and two-valued homomorphisms on A breaks down. This means that the correspondence between atomic propositions (or events) and two-valued homomorphisms breaks down.

Notice, even if the ultrafilters in A are not prime, A may be the homomorphic image of a Boolean algebra. If this is the case, there exist homomorphisms $h:A\to\underline{Z}_2$, however, these are not in general determined by maximal proper filters in A.

In general, event structures which determine dispersion free states on the associated algebra of theoretical propositions include all those that may be mapped homomorphically into a Boolean algebra, since, by the homomorphism theorem, every Boolean algebra admits homomorphisms onto $\underline{Z}_2$. All the event structures isomorphic to or containing the partial Boolean algebra $\underline{B}(H_3)$ of linear subspaces of a three dimensional Hilbert space[9] fall outside of this class. By Kochen and Specker's Theorem 1, for each such A there are no homomorphisms onto $\underline{Z}_2$, hence, no two-valued homomorphisms on the partial Boolean algebra $\underline{Q}$ of theoretical propositions associated with A. Because of the equivalence between two-valued homomorphisms and dispersion free states, there are no dispersion free states on $\underline{Q}$.

Thus the absence of dispersion free states on $\underline{Q}$ is a direct consequence of the fact that A is a particular type of partial Boolean algebra, just as the existence of dispersion free states on $\underline{C}$ is a consequence of fact that B is a Boolean algebra.[10]

The fact that there are no homomorphisms $h:\underline{Q}\to\underline{Z}_2$ must be sharply distinguished from the question of the bivalence of the language L in which the theoretical propositions are formulated. It is trivially possible to make bivalent assignments of truth values to the propositions of $\underline{Q}$, and therefore, to the corresponding sentences of L: Let every proposition associated with an event in an ultrafilter in A be true, and every proposition associated with an event outside the filter, false. Since the ultrafilters in A are not prime filters, the bivalent assignment of truth values to L is not induced by a homomorphism of A onto $\underline{Z}_2$. But since the homomorphism theorem is equivalent to Stone's representation theorem, this is to be expected, if A is strongly non-Boolean.

In the view advanced here, events $b_i \varepsilon A$ incompatible with an event $a \varepsilon A$ are excluded by the logical structure of the system; the chief advantage of the bivalent truth value assignment defined above is that it makes this fact explicit. This definition has the consequence that it is a sufficient but not a necessary condition for the truth of a disjunction that one of the disjuncts be true. For example, the propositional formula

$$x_i + x_j + x_k - x_i x_j x_k$$

in L is true whenever it is interpreted over three mutually compatible events $a_i, a_j, a_k \varepsilon \underline{\hat{A}}$ since

$$a_i \vee a_j \vee a_k = 1.$$

Now the trivial event (i.e., 1) is compatible with every event and is a member of every filter. But it does not follow that exactly one of every triple of mutually compatible atomic propositions is true.

It may be objected that such a truth value assignment is 'unintuitive'. But this is surely a pseudo-problem. For if the quantum theory is assumed, the models $\underline{M}$ of L are isomorphic to $\underline{B}(H_3)$. Therefore the ultrafilters in $\underline{M}$ are not prime filters. Thus, so far as the event structures are concerned, this property is preserved in any reconstruction of the theory. The objection is interesting only if it is coupled with a classical solution to the representation problem. But this problem is left untouched by the choice of L.

In the remainder of this section we compare the quantum theory with classical statistical mechanics, since for this theory it is also true that dispersion free states are not theoretically fundamental. This is done in two stages. We begin by considering why dispersion free states are not theoretically important in statistical mechanics. Next, we examine the sense in which the description of statistical mechanics is incomplete relative to the Newtonian description. We conclude this section with some remarks on the interpretation of Birkhoff and von Neumann.

Classical mechanics and classical statistical mechanics share the same phase space as well as the same dynamics. Thus for statistical mechanics a dispersion free state is determined by the classical mechanical state of the system. Just as in classical mechanics, physical magnitudes $\underline{A}\varepsilon\mathcal{O}$ are associated with functions in $\underline{R}^{\Omega} = \{f_A: \Omega\rightarrow\underline{R}\}$ from Ω into the Borel subsets of $\underline{R}$. Each $\underline{A}$ is associated with a family of subsets of Ω by the inverse image

$$f_A^{-1}(U) = \{\omega | f_A(\omega)\varepsilon U\}$$

of the Borel subsets of R under the map corresponding to A. In statistical mechanics Ω is also a sample space, so that a point $\omega\varepsilon\Omega$ is also interpreted as the phase point of a sample system in an ensemble of similar systems.

Now for a certain class K of regions of the sample space, there are macroscopic magnitudes, i.e. properties of the ensemble, with values concentrated on a small subset of R.[11] These magnitudes obey the phenomenological equations of classical thermodynamics. For some $\Lambda\varepsilon K$, and any distribution of phase points in Λ, the values of each macroscopic magnitude remain concentrated on a small subset of R. This is to be expected; for the basic Newtonian laws are symmetric with respect to time, hence the law of motion of an individual system is time-symmetric. But the phenomenological laws are irreversible; if the macroscopic magnitudes were not independent of the precise location of the phase point, the macroscopic laws would be reversible, not irreversible. Therefore the theoretical unimportance of the classical mechanical state is a necessary condition for the successful application of statistical mechanics to thermodynamic systems.[12]

The application of the laws of Newtonian physics to a thermodynamic system requires too fine a specification of the classical mechanical state. The slightest discrepancy amplifies very rapidly and renders the initial specification theoretically useless. Because of this difficulty, we forgo a complete description in terms of the classical mechanical state in favor of an incomplete description involving a proper subset $\underline{S}'$ of the statistical states $\psi:F(\Omega) \to [0,1]$. Relative to this description, there exist magnitudes $\underline{A}$, $\underline{B}\epsilon\mathcal{O}$ such that

$$\underline{P}_{A\psi}(U) = \underline{P}_{B\psi}(U)$$

for every Borel set $U\subset\underline{R}$ and every $\psi\epsilon\underline{S}'$. $\underline{A}$ and $\underline{B}$ are equivalent with respect to the set $\underline{S}'$ of statistical states. Therefore, $\underline{C}$ contains theoretical propositions of the form

$$f_A(\omega)\epsilon U$$

and

$$f_B(\omega)\epsilon U$$

which are equivalent with respect to $\underline{S}'$. By extending $\underline{S}'$ to $\underline{S}$ it is possible to distinguish these magnitudes together with the theoretical propositions corresponding to them. Thus, the statistical description in terms of $\underline{S}'$ is incomplete relative to the classical description in the sense that the extension to $\underline{S}$ leads to an imbedding of this description into the classical description. For this reason, the absence of dispersion free states may be taken to mean that, relative to classical physics, the description of statistical mechanics is based on incomplete knowledge of the exact classical state of the system.

Now in the case of quantum mechanics, the absence of dispersion free states cannot be understood in this way. For if $\mathcal{A}$ is a partial Boolean algebra of events corresponding to a Hilbert space of at least three dimensions, there is no imbedding of $\mathcal{A}$ into a Boolean algebra; hence, there is no imbedding of the partial Boolean algebra Q of theoretical propositions associated with A into a classical (i.e. Boolean) description. This follows from Kochen and Specker's Theorem 1, because the existence of two-valued homomorphisms is a necessary condition for the imbeddability of Q into a Boolean algebra. There even exist finite subalgebras D of Q which are not imbeddable into a Boolean algebra. D contains pairs of propositions such that $a \neq b$ but $h(a) = h(b)$ for every homomorphism $h: D \to Z_2$. If D is weakly imbeddable, a and b will have to be incompatible. This follows immediately from the definitions of strong and weak imbeddability, or more precisely, from Kochen and Specker's Theorem 0 and its counterpart for weak imbeddings.

It is often suggested that the quantum theory is more vague than classical physics in the sense that there are distinctions which can be made in classical physics which are ill-defined in the quantum theory. The opposite is the case -- there are finer distinctions possible in the quantum mechanical case than in the classical case. The a and b above are in a sense distinguishable quantum mechanically, but not classically, i.e.[13] not in terms of homomorphisms onto Z_2. Intuitively, there exist completely symmetric but distinct elements in a non-imbeddable A.

Though Birkhoff and von Neumann recognize that the algebra of theoretical propositions of quantum mechanics is not a Boolean algebra, they do not consider the possibility of imbedding Q into a Boolean algebra, and thus, into a classical description. This is due to their conception of the role of the classical mechanical state in statistical mechanics.

In their view there are basically two reasons for ignoring the exact classical description in statistical mechanics: first, it is 'convenient' to do so, and secondly, knowledge of the phase point requires a degree of precision which it is impossible to obtain experimentally. While this is certainly true, it is not an analysis of the relationship between the two theories. Statistical mechanics can ignore the classical state because it deals with irreversible processes, and these must be independent of the exact phase point of the system. This account is combined with a particularly naive confusion of reference with evidence. Birkhoff and von Neumann are thus led to the view that it is meaningless to suppose that the system is always in a state corresponding to a point in Ω; i.e., the exact classical description is not relevant to statistical mechanics on largely independent, epistemological grounds. But blurring the distinction between the world, and our knowledge of the world, makes it impossible to distinguish a Boolean description based on incomplete knowledge from a complete non-Boolean description. Since the concept of the exact state of a system is considered fundamentally incoherent, the non-Boolean character of Q may merely express the impossibility of knowing the complete classical state. Because of this unclarity, Birkhoff and von Neumann have only succeeded in reformulating the orthodox interpretation.[14] Though their discussion is couched in terms of the logical structure of quantum propositions it suffers from all the ambiguities of the conventional view.

5. Alternative Representations

In this section we consider alternative representations of the Hilbert space structure of quantum mechanics, viz.: orthomodular posets, and orthomodular lattices.[15] It is clear that the partial Boolean algebra of subspaces of a Hilbert space may be extended to an orthomodular poset by simply defining the order relation in each maximal Boolean sub-algebra in the usual way; and even to an orthomodular lattice by defining g.l.b. and l.u.b. for incompatible elements. Thus the mathematical differences are not essential.

In the lattice and poset approaches there are basically two properties that are held to distinguish Q from C: non-distributivity, and the existence of incompatible pairs. Neither corresponds to non-imbeddability. Because of this, interpretations based on these representations suffer from the same ambiguity as the view of Birkhoff and von Neumann.

The orthomodular lattice[16] H_2 of linear subspaces of a two dimensional Hilbert space is isomorphic to the lattice of subspaces through a point in ordinary two-dimensional Euclidian space. In this representation, compatibility corresponds to orthogonality, i.e. two linear subspaces are compatible if and only if they are orthogonal in the sense of elementary geometry. (Thus $a \leftrightarrow b$, if a is a subspace of b.) Joins, meets, and complements correspond to spans, intersections, and orthocomplements. The unit of the lattice is the whole space, and the zero is associated with the zero-dimensional subspace or origin. It is obvious from Kochen and Specker's Theorem 0 that the partial Boolean algebra $\underline{B}(H_2)$ associated with H_2 is imbeddable into a Boolean algebra.

Since $B(H_2)$ is imbeddable into a Boolean algebra, the distributive law

$$a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$$

of classical logic holds. In fact the identity holds in $\underline{B}(H_2)$. This is not peculiar to $B(H_2)$, for the distributive identity is $\underline{Q}$-valued, i.e. valid in all partial Boolean algebras.[17] Clearly, non-distributivity, in the sense of failure of the distributive law of classical logic, depends on the definition of validity. More interestingly: even when the propositional (i.e. Boolean) functions in L are interpreted as lattice polynomials, the failure of the distributive law does not correspond to non-imbeddability.

Similarly, there are clearly incompatible elements in H_2; yet H_2 may be imbedded into a Boolean algebra. So the existence of incompatible pairs must be distinguished from non-imbeddability.

The work of Zierler and Schlesinger [22] shows that there always exists a map $h: A \to B$ from an orthomodular poset A into a Boolean algebra which preserves the ordering and orthocomplementation. That is,

(i) if $a \leq b$, then $h(a) \leq h(b)$

(ii) $h(a') = h(a)'$.

The map is also monomorphic so that

(iii) if $h(a) \leq h(b)$, then $a \leq b$.

Notice it does not follow that such a map preserves lattice meets and joins, for although the ordering of elements above and below a and b is preserved in the image h(A) of A in B, there may be an element of B smaller (in B) than the image of any element above a and b in A, so that this element would qualify as h(a) $\vee$ h(b) and not h(a$\vee$b). That is, for any $x \epsilon A$, such that $a \vee b \leq x$ in A,

$$h(a \vee b) \leq h(x),$$

but the smallest element above h(a) and h(b) might be an element in B which is not the image of any element in A.

Zierler and Schlesinger show further that there does not in general exist a map satisfying conditions (i)-(iii) which also preserves the lattice operations for compatible elements. Now this is already clear from the work of Kochen and Specker, since independently of the question of the preservation of order, meets and joins cannot be preserved in the case of partial Boolean algebras associated with Hilbert spaces of three or more dimensions. On any representation, what is fundamental about the non-Boolean structures of quantum mechanics is that they are not imbeddable into a Boolean algebra, and this depends on the fact that $\leftrightarrow$ is not transitive in $\underline{\hat{A}}$. In this sense, the order structure is redundant.

(Notice non-transitivity of $\leftrightarrow$ in $\underline{\hat{A}}$ is not the same as non-transitivity of $\leftrightarrow$ in A. For every element in A is compatible with the unit. Thus if compatibility is transitive in A, then every element is compatible with every other: $a \leftrightarrow 1$ and $1 \leftrightarrow b$ implies $a \leftrightarrow b$. That is, there are no incompatible pairs. Conversely if $\leftrightarrow = \underline{A} \times \underline{A}$, $\leftrightarrow$ is obviously transitive.

Thus transitivity of $\leftrightarrow$ in A is equivalent to $\leftrightarrow = \underline{A} \times \underline{A}$.)

One final point: It seems natural to understand the generalization of the order relation in a Boolean algebra as a generalized implication. However this leads to difficulties. By a result of Fây [6], implication cannot be defined as in classical logic by

$$a \Rightarrow b \text{ if and only if } a' \vee b = 1.$$

For in an orthomodular poset or orthomodular lattice, if the relation

$$a' \vee b = 1$$

is transitive, the poset or lattice is a Boolean algebra. For this reason it has been argued (e.g. by Gudder and Greechie [9]) that the transition from classical to quantum mechanics is not properly concerned with logic. But in view of Kochen and Specker's Theorem 4, this is obviously a purely verbal issue.

6. Conclusion

It remains to be shown how the discussion of this paper leads to a solution of the 'measurement problem'. This subject, together with a complete discussion of the role of probabilities on non-Boolean event structures, will be dealt with in a separate paper. The present paper clarifies what is required of an interpretation of the quantum theory: The problem is to explain the transition from classical mechanics to quantum mechanics, given that the set S of statistical states of the quantum theory does not contain dispersion free states. It also explains the sense in which quantum mechanics and classical mechanics are theories of the world's logical structure. This, in conjunction with Theorems 1 and 4 of Kochen and Specker, completely solves the problem of interpretation. Clarification of the problem of hidden variables, in the sense of the importance of Gleason's Theorem and its corollaries, is immediate: such results have the character of completeness theorems for the logical structures of quantum mechanics. The whole discussion rests on the distinction between logical structure in the sense of the syntax and semantics of a formal language, and the logical structure of events. This distinction is completely analogous to the one drawn in Section 1 between coordinate transformations and symmetries. Only the first component of each pair involves conventional elements. Logical structure and space-time symmetries are objective structural properties of the world.

FOOTNOTES

1. This distinction is suggested by Einstein [4]. We have retained his terminology. Cf. also [5], pp. 53ff.

2. There are theories which are clearly both constructive and principle theories, e.g. classical statistical mechanics.

3. Notice that in Bub [3], 'phase space theory' denotes a <u>classical</u> phase space theory. We extend the use of the term here to include any theory in which the concept of logical structure occurs explicitly.

4. This theorem is proved in [13]. Unless otherwise indicated, all references to Kochen and Specker, are to this paper.

5. This paper assumes some acquaintance with the representation theory of Boolean algebras. See, e.g. Sikorski [17], Chapter I.

6. For a characterization of these algebras, see Rasiowa and Sikorski [16].

7. Kochen and Specker use the term 'commeasurability' to refer to this relation, clearly suggesting that the relation should be understood in terms of simultaneous measurability. This is at least misleading, since the simultaneous measurability of two magnitudes is a <u>consequence</u> of the fact that they are compatible; but compatibility is not operationally definable in terms of simultaneous measurability. [Cf. the discussion below (i.e. Section 4) of Birkhoff and von Neumann.]

8. See Rasiowa and Sikorski [16], Chapter I, Section 9, for a discussion of this point.

9. For definiteness, we restrict the discussion of this section to partial Boolean algebras of this class.

10. In general, for a Hilbert space of three or more dimensions, all possible statistical states on the partial Boolean algebra of linear subspaces are generated by the statistical operators according to the algorithm of the quantum theory. That is to say, the probability algorithm of the theory generates <u>all</u> possible statistical states on $\underline{Q}$. (This is essentially the content of Gleason's theorem [7].) Yet the set $\underline{S}$ of statistical states does not contain states which are dispersion free. So, by the equivalence between two-valued homomorphisms and dispersion free states, an extension of the theory which recovers the correspondence between events and two-valued homomorphisms does not exist.

11. $\mathcal{H}$ is the class of Borel subsets of Ω modulo Borel sets of Lebesgue measure zero. This class is identical with the class of Lebesgue measurable subsets of Ω modulo sets of Lebesgue measure zero (see e.g. Halmos [10], Section 15).

12. This has the character of a randomness assumption. It is also a sufficient condition for applying statistical mechanics to thermodynamic systems. For a thorough discussion see van Kampen [20], Chapter 1.

13. By Stone's epresentation theorem, every pair of distinct elements in a Boolean algebra must be distinguishable by a homomorphism onto $\underline{Z}_2$.

14. In its original form, Heisenberg's interpretation was compatible with the existence of a classical mechanical state. This assumption was later rejected by Heisenberg and Bohr and replaced by the thesis that an atomic system cannot be significantly described independently of a measurement process.

15. Birkhoff and von Neumann assume an orthocomplemented, modular lattice. This assumes more structure than an orthomodular lattice. (See Jauch [11], Chapter 5, Section 6, for a more detailed discussion of this point.) For our purposes, the difference is not important, and everything said concerning orthomodular lattices may be extended to the lattice of Birkhoff and von Neumann.

16. For simplicity of exposition we restrict the discussion to lattices.

17. Cf. Section 3, above. For a generalization, see Kochen and Specker [12], Section 6.

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Chapter II

Fundamental Statistical Theories

0. Introduction.

Einstein argued that since quantum mechanics is not a fundamental theory, it cannot be regarded as in any sense final. (See especially [3] and [1], on which this discussion is based.) The concept of a fundamental statistical theory may be roughly explained as follows. Let us suppose that a certain class K of physical systems may be known with complete precision. For any system of this class it is possible to completely specify its type and its state. Both pieces of information--the possible types of system, and their possible states--are theory relative. What is assumed is that for any S in K there is no extra-theoretical limit on the amount of information obtainable concerning S. I shall say that a statistical theory is fundamental if it is based on a maximal amount of information concerning the systems of K. That is to say, for a fundamental theory, the degree of imprecision of our knowledge may be ignored, since the theory is supposed to hold even when this is made arbitrarily small. By contrast, a statistical theory which is not fundamental is explicitly designed to take account of the case where, for whatever reasons, a maximal amount of information is not available.

For example, in classical statistical mechanics, the theoretically important states are characterized by some positive dispersion. In this case the dispersion is easily explained in terms of the incompleteness of our knowledge of the exact phase point of the system. The fundamental theory is given by classical mechanics which represents the time evolution of the phase point of the system, and the possible phase points are in one-to-one correspondence with dispersion free states; these are the pure statistical states of classical mechanics.

The pure statistical states of the quantum theory are not dispersion free. In this sense, the theory is <u>significantly</u> statistical. The problem with which this paper is concerned is, Under what conditions is a significantly statistical theory correctly regarded as fundamental?

In the case of atomic systems, the response to this question favored by many physicists consists in denying that any theory can be fundamental in the sense just outlined. (Cf. e.g. Pauli's letters 115 and 116 to Born in [1] as well as the subsequent commentary by Born.) Knowledge of the systems dealt with by the quantum theory is essentially incomplete in the sense that any predictively adequate theory must accept the existence of a significant restriction on what can be known concerning this class of systems.

Beginning with Heisenberg's γ-ray microscope thought experiment, there is a long series of quasi-physical arguments aimed at making this view plausible. All of these arguments appeal to the operational incompatibility of direct measurements of certain pairs of physical magnitudes. This is quite irrelevant as Einstein showed. His argument may be briefly reconstructed as follows.

Two systems S_1 and S_2 are coupled if there exist magnitudes A_i^1 and A_j^2 of S_1 and S_2(respectively) such that the probability that $A_i^1 = \lambda_i$ is 1 (0) if and only if the probability that $A_j^2 = \lambda_j$ is 0(1). (It is a theorem that any two quantum mechanical systems which interact and then separate are coupled in this sense. A classic example is given by a pair of spin-½ particles in the singlet spin state.) In any theory admitting the existence of coupled systems, it is unnecessary to interact directly with S_1, say, in order to determine the value of the magnitude A_i^1; it suffices to measure the magnitude A_j^2 of S_2. Since the systems are spacially separated, this cannot possibly affect S_1.

The fact that quantum mechanics admits the existence of coupled systems means that the theory does not support the usual (operationist) interpretation of the statistical character of the theory. The idea that our knowledge is essentially incomplete assumes that a direct measurement of all magnitudes is not possible. This of course may well be true. The difficulty is that direct measurements are not necessary for determining the values of the A_i^1; moreover, this fact is a consequence of the quantum theory. Einstein made a definitive contribution to this phase of the problem by showing that the rejection of verificationism removes any methodological objection to fundamental theories of atomic systems.

The solution developed in this paper is that a statistical theory is fundamental only if it is complete; moreover the quantum theory is complete. Hence, this analysis removes a further objection to regarding quantum mechanics as a fundamental theory. Clearly the major problem with this approach is that completeness is apparently inconsistent with the significantly statistical character of the quantum theory.

The account of completeness presented in Section 1 is based on a critical analysis of Bub's very important work, "On the Completeness of Quantum Mechanics". This paper assumes familiarity with the concepts and theorems of [7].

1. The Completeness of Statistical Theories.

Let us consider the statistical theory of a fixed system S. Such a theory consists of an algebra M of physical magnitudes (it is assumed that M is at least a partial algebra) together with a set S of statistical states. Elements $\psi \in S$ assign probabilities to ranges of values of magnitudes in M: for each $A \in M$, and $\psi \in S$, $P_{A,\psi}(U)$ denotes the probability that the value of the magnitude A lies in the (measurable) subset U of Real numbers. ($P_{A,\psi}: F(R) \to [0,1]$ is the distribution function of the magnitude A determined by ψ.)

The two algebraic structures of relevance to this discussion are: M is the partial algebra R^{Ω} of real valued functions on a classical phase space. M is the partial algebra N(H) of self-adjoint operators on a separable Hilbert space. These correspond, respectively, to classical and quantum mechanics. Compatibility is interpreted as commutativity: For A_1, A_2 in M, $A_1 \leftrightarrow A_2$ if and only if $A_1A_2 = A_2A_1$. It is a theorem that A_1 and A_2 commute if and only if there exists a $C \in M$ and Borel functions $g_i: R \to R$ $(i = 1,2)$ such that $A_i = g_i(C)$. Linear sums and products of compatible magnitudes may be defined by the linear sum and product of the associated functions g_i. Explicitly:

$$\lambda_1 A_1 + \lambda_2 A_2 = (\lambda_1 g_1 + \lambda_2 g_2)(C)$$

$(\lambda_1, \lambda_2 \in R)$ and

$$A_1 A_2 = (g_1 g_2)(C).$$

The subalgebra of idempotent elements of M form a partial Boolean algebra L. In the case of R^{Ω} this is the subalgebra B of characteristic functions on Ω. This is isomorphic to the field F(Ω) of (measurable) subsets of Ω. For N(H), this is the partial Boolean algebra N of projection operators. N is isomorphic to the partial Boolean algebra B(H) of closed linear subspaces of H. [Let ∪,∩,' denote the span, intersection, and orthocomplement of the subspaces in H. Then for a,b, in H, a↔b if there exist mutually orthogonal subspaces a_1, b_1, c such that $a = a_1 \cup c$ and $b = b_1 \cup c$. The operations ∪,∩ are restricted to ↔. H is the unit of B(H) and {0} is the 0 of B(H).]

Physical properties are introduced in terms of the magnitudes A in M as follows. Take a real number λ in the range of A. Then A = λ (this is read "the value of A is λ") is a property of S. More generally, given a subset U of R and magnitude A, A∈U -- the value of A lies in U -- represents a property of the system S.

Von Neumann ([9] Ch. III. 5) observed that every property of S is represented by an idempotent magnitude in L. This is simply seen in the case of classical mechanics. Let Ω be a subset of n-dimensional Euclidian space. An elementary event in the history of S is represented by a point in Ω. As is well-known, an event ω is associated with a pure statistical state of classical mechanics: the 2-valued measure on F(Ω) determined by ω. Now let f_A be a real valued function in R^{Ω} representing the magnitude A. A property $A \varepsilon U$ holds for S if and only if S is in a state ω such that $f_A(\omega) \varepsilon U$. Let Γ denote the subset $f_A^{-1}(U)$ of Ω. It is clear that S has the property $A \varepsilon U$ if and only if the state ω of S lies in Γ. The property $A \varepsilon U$ is said to be associated with Γ. In general there are many properties $A_i \varepsilon U_j$ such that $f_{A_i}^{-1}(U_j) = \Gamma$ for some $U_j \varepsilon R$. Now let a be the characteristic function of Γ, and let P be a property associated with Γ. By the correspondence between properties of S and subsets of Ω, it follows that every property P is represented by the characteristic function a, in the sense that P holds if and only if S is in a state ω such that $a(\omega) = 1$. Since a is two valued this is equivalent to $a(\omega) \neq 0$.

The situation in quantum mechanics is exactly analogous. Elementary events, represented by rays K in H, are associated with pure statistical states. In quantum mechanics statistical states are given by measures on the closed linear subspaces of H. The pure state associated with K is determined by taking the square of the norm of the projection of a unit vector lying in K onto each subspace of H. Since there is a one-to-one correspondence which associates each projection operator with the subspace which is its range, this determines a probability measure on N. Recall atoms in B are characteristic functions of singleton subsets $\{\omega\}$ of $F(\Omega)$. N is also atomic. An atom in N is a projection operator onto a one-dimensional subspace of H. Thus in each theory, there is a one-to-one correspondence between elementary events and atoms in L.

To summarize: Every magnitude may be replaced by a set of properties, and every property corresponds to a two-valued quantity, i.e. to an idempotent magnitude. (For this reason it suffices to consider the algebra L of idempotent magnitudes.) The correspondence is not one-to-one since very many properties are associated with the same subset of Ω (or subspace of H), and therefore, represented by the same idempotent magnitude. There is a one-one correspondence between idempotent magnitudes and equivalence classes of properties represented by the same idempotent in L. An idempotent may therefore be thought of as the equivalence class of properties it represents.

For any statistical theory, we may distinguish two ways of viewing the algebra of idempotent magnitudes. First, L may be regarded as an abstract property of physical magnitudes: in this case, the idempotents in M are simply postulated as having the structure L. The algebraic structure introduced in this way is termed the logical space L_1 of a statistical theory. (Bub [2] p. 45.) The characteristic feature of L_1 is that its introduction is independent of statistical considerations.

There is another way of viewing the algebraic structure of a statistical theory: Let M be the set of physical magnitudes. No algebraic structure is assumed for M. Rather an algebraic structure is defined in terms of the distribution functions $P_{A,\psi}$ of the magnitudes $A \epsilon M$ by writing $A_1 \leftrightarrow A_2$

if there is a $C \in M$ and functions g_i: $R \to R$ ($i = 1,2$) such that

$$P_{A_i,\psi}(U) = P_{C,\psi}g_i^{-1}(U)$$

for all $U \subseteq R$ and $\psi \in \underline{S}$. I.e. the magnitudes A_1 and A_2 are compatible if they are statistically equivalent to $g_i(C)$. The definition of the partial operations is given in terms of the associated functions as before. The resulting structure $\underline{M}_2$ is determined by the set $\underline{S}$ of statistical states of the theory. The subalgebra of idempotents of $\underline{M}_2$ is termed ([2] p. 45) the logical space $\underline{L}_2$; it is distinguished from $\underline{L}_1$ by its dependence on the relation of statistical equivalence.

It is important to recognize that the elements of $\underline{L}_1$ and $\underline{L}_2$ are the same. Each logical space consists of the subset of idempotent magnitudes in $\underline{M}$ or the equivalence classes of properties which they represent. Since $\underline{L}_1$ and $\underline{L}_2$ are independently specified, they may be structurally different. Here it is essential to be very clear: Although the relation of statistical equivalence is obviously an equivalence relation on $\underline{L}_1$, it is not necessarily compatible with the structure of $\underline{L}_1$, so that $\underline{L}_1$ and $\underline{L}_2$ may not even be homomorphic.

Within this framework, Bub has proposed a general criterion for the completeness of statistical theories: A statistical theory is said to be complete if and only if the logical spaces $\underline{L}_1$ and $\underline{L}_2$ are isomorphic ([2] p. 45). The problem of demonstrating the isomorphism of the two logical spaces is the completeness problem for a statistical theory. Gleason's theorem [4] establishes that all generalized probability measures on B(H) are given by the statistical algorithm of quantum mechanics. (Here it is necessary to assume that H is at least three dimensional.) Gleason's theorem is regarded as having solved the completeness problem for quantum mechanics in this sense.

Now the first point to notice is that the isomorphism condition is, in most cases, automatically satisfied. So far as any actually proposed statistical theory is concerned, $\underline{L}_1$ is defined as $\underline{L}_2$. But if this is the case, the completeness problem is trivial. It might be argued that introducing $\underline{L}_1$ in this way obscures the fact that it is always possible to construct a statistical theory where the isomorphism condition does not hold. A construction of this type amounts to a reinterpretation of the statistical theory; in the case of the quantum theory, hidden variable theories may be viewed as reinterpretations in this sense. It would appear that the isomorphism condition is intended to exclude a reinterpretation based on a structurally different $\underline{L}_1$ on the ground that such a theory is incomplete. This suggests that the isomorphism condition is of the greatest importance when considering theories of this type. What is unclear is that hidden variable reinterpretations are unsatisfactory because they are incomplete. I will return to this question in a moment. At this point I want to examine the case of theories for which the relation of statistical equivalence is compatible with the operations and relations of $\underline{L}_1$.

A set of statistical states on $\underline{L}_1$ is full if and only if it is order-determining. I.e. if $\psi(a) \leq \psi(b)$ for all $\psi \varepsilon \underline{S}$ implies $a \leq b$, a, b in $\underline{L}_1$. Clearly, if statistical equivalence is compatible with $\underline{L}_1$, then $\underline{L}_1$ and $\underline{L}_2$ are isomorphic if and only if $\underline{S}$ is a full set of statistical states.

It is not trivial that $\underline{L}_1$ has a full set of states. There even exist partial Boolean algebras which admit no states. (See Greechie and Gudder [5] Sect. 7 for a discussion and references.) The difficulty is that even if it can be shown that $\underline{S}$ is full on $\underline{L}_1$, and hence that $\underline{L}_1$ and $\underline{L}_2$ are isomorphic, the existence of possible extensions of the theory would remain an open question. An extension is defined as follows: Let $h: \underline{L}_1 \to \underline{L}_1'$ be a homomorphism between the two partial Boolean algebras $\underline{L}_1$ and $\underline{L}_1'$. Let $\underline{S}$ and $\underline{S}'$ denote the associated sets of statistical states. Then $\psi' \varepsilon \underline{S}'$ is an extension of $\psi \varepsilon \underline{S}$ if $\psi = \psi' \circ h$. $\underline{S}'$ is an extension of $\underline{S}$ if every $\psi \varepsilon \underline{S}$ has an extension in $\underline{S}'$. The statistical theory $(\underline{L}_1', \underline{S}')$ is an extension of the theory $(\underline{L}_1, \underline{S})$ if $\underline{S}'$ is an extension of $\underline{S}$ and h is an imbedding. The extension is proper if for some $\psi' \varepsilon \underline{S}'$, $\psi' | h[\underline{L}_1] \neq \psi$.

Now Gleason's theorem excludes certain extensions of the quantum theory. For logical spaces represented by B(H), Gleason's theorem is equivalent to the general result that the quantum theory has no proper extensions in the category of partial Boolean algebras. A Boolean extension may be defined as an extension for which $\underline{L}_1'$ is a Boolean algebra.

Theorems 0 and 1 of [7] imply that the quantum theory has no Boolean extensions. This is a similar (but weaker) result concerning a sub-category of the category of partial Boolean algebras.

It is claimed that the isomorphism condition fully explicates the sense in which Gleason's theorem may be regarded as a completeness theorem for quantum mechanics. But the existence of extensions is simply ignored by this condition, since isomorphism concerns only the structures $\underline{L}_1$ and $\underline{L}_2$. *A priori*, there is no reason to expect that completeness in this sense is incompatible with the existence of imbeddings of $\underline{L}_1$, leading to extensions of the theory. This is a defect, since an analysis of completeness in quantum mechanics should capture the full scope of the problem solved by Gleason.

This is perhaps more clearly a difficulty for an analysis which explicitly treats completeness as the isomorphism of two *compatible* logical spaces. I say this because if Bub intends completeness to apply only to those statistical theories for which statistical equivalence is compatible with the structure of $\underline{L}_1$, then the whole motivation for regarding $\underline{L}_1$ and $\underline{L}_2$ as independent logical spaces becomes obscure. This distinction seems basic to the whole analysis. The difficulty is that it is not especially relevant to at least one phase of the completeness problem; it fails to provide an explication of the mathematical problem solved by Gleason's theorem and its corrollaries. To put this slightly differently: the characterization of $\underline{L}_1$ and $\underline{L}_2$ leads to two independent logical spaces; the isomorphism condition establishes a connection between the two structures. The problem is, how are we to regard this connection?

Notice that $\underline{L}_2$ is essentially the logical space determined by infinitely precise measurements performed on infinitely many copies of S. $\underline{L}_2$ is explained by the assumption of an underlying logical structure $\underline{L}_1$. In other words, $\underline{L}_2$ forms the ideal evidential basis for the hypothesis that the algebraic structure of L is given by $\underline{L}_1$, and the isomorphism condition is just the simplest restriction on the structure of $\underline{L}_1$ which is consistent with the structure determined by $\underline{L}_2$. The important point which the distinction between $\underline{L}_1$ and $\underline{L}_2$ clarifies is that $\underline{L}_1$ represents a structural assumption which occurs quite explicitly in the derivation of the (ideal) statistical predictions of the theory. The problem of completeness raises a further question, viz., Does the algebraic structure determined by the isomorphism condition occur essentially, or is it possible to regard it as merely a property of a particular formulation of the theory?

The discussion of Gleason's theorem suggests that a statistical theory ($\underline{L}_1$, $\underline{S}$) is complete if it has no proper extensions. Completeness in this sense is compatible with the fact that the quantum theory does not contain dispersion-free states: If the logical space $\underline{L}_1$ were imbeddable into a Boolean algebra, quantum mechanics would be an incomplete theory, since by the fundamental property of Boolean algebras, $\underline{L}_1$ must admit two-valued measures. But there exist non-Boolean logical spaces, represented by B(H). For this class of logical spaces, Gleason's theorem implies that the quantum theory is complete, even though $\underline{S}$ does not contain two-valued measures.

This analysis assumes that completeness is relative to a category of algebraic structures. It might be objected that this amounts to an unnecessary weakening of the concept of completeness. Intuitively it may seem that a statistical theory is complete only if it contains all statistical states. In other words, completeness should depend only on the membership of $\underline{S}$, so that a statistical theory is incomplete if it fails to include dispersion free states.

The first point to notice is that this suggestion only appears to make completeness independent of some underlying notion of algebraic structure. If a theory must contain dispersion free states, the only complete theories are those for which $\underline{L}_1$ may be homomorphically mapped into a Boolean algebra. (This is basically a consequence of Theorem 0 of [7].) The inclusion of additional states therefore limits the class of logical structures compatible with the completeness of a statistical theory to the category of partial Boolean algebras which are homomorphically related to a Boolean algebra. (See [7] Sect. 5 for a definition of this concept.) Every logical space of a complete theory is represented by an object of this category. From the point of view of possible extensions of a statistical theory, this modification results in a mathematically weaker concept of completeness. The category of partial Boolean algebras which are homomorphically related to a Boolean algebra is a proper sub-category of the category of all partial Boolean algebras. So the completeness problem is limited to showing that there are no extensions in this sub-category.

Secondly, this objection has a certain initial plausibility when it is implicitly assumed that the non-existence of two-valued homomorphisms has the same significance in both imbeddable and non-imbeddable structures. In the case where $\underline{L}_1$ is imbeddable into a Boolean algebra, distinctions are possible which are obscured by the absence of dispersion free states. E.g. this is true of the non-atomic lattice of idempotent macroscopic magnitudes of classical statistical mechanics. Now the case of a non-imbeddable $\underline{L}_1$ is very different. N contains pairs of idempotents such that $a \neq b$ but $h(a) = h(b)$ for every homomorphism $h: N \to Z_2$. Thus a and b are distinguishable quantum mechanically, but not classically, i.e. not in terms of a homomorphism onto Z_2 (= the two element Boolean algebra).

To summarize this discussion of [2]: The distinction between $\underline{L}_1$ and $\underline{L}_2$ clarifies the fact that the introduction of an algebra of idempotents represents an additional explanatory assumption of a statistical theory. This structure occurs essentially in determining the set of statistical states of the theory. In the case of quantum mechanics, the nature of this assumption was first made completely explicit by von Neumann. Completeness means that $\underline{L}_1$ has no proper extensions. The quantum theory is complete with respect to the category of partial Boolean algebras. This is the content of Gleason's theorem. By Stone's representation theorem (or more accurately, by the representation theorem for Boolean σ-algebras of Loomis and Sikorski), classical mechanics is complete relative to the category of Boolean algebras. (Since this makes the theory of states on a Boolean algebra a sub-theory of the theory of measures on an arbitrary field of sets.) It is in this sense -- the same sense in both cases -- that quantum mechanics and classical mechanics are complete statistical theories.

One final remark concerning the distinction between $\underline{L}_1$ and $\underline{L}_2$: There is a similarly named, but otherwise very different distinction given by van Fraassen. In [8] elements of L_1 are theoretical statements; they correspond to what I called properties. For van Fraassen L_1 is a set of objects which is not necessarily related to the algebra of idempotent magnitudes of the theory. Elements of L_2 are statements of the form: "The value of A is <u>certainly</u> in U" or "The value of A lies in U with probability $\lambda \epsilon [0,1]$". Van Fraassen's L_1 and L_2 are distinguished by their elements; L_1 and L_2 consist of essentially different statements. Moreover no structure is assumed for L_1. Two algebraic structures are imposed on the statements of L_1. (Cf. [8] Sections II.4 and II.5.) Each of these is based on the set $\underline{S}$ of statistical states of the theory, and therefore correspond to logical spaces of the type $\underline{L}_2$.

For van Fraassen, logical structures occur only at the level of the evidential basis of a statistical theory, and never as explanatory principles. This is a distortion of what actually occurs in either classical or quantum mechanics. In both cases the algebraic structure introduced by $\underline{L}_1$ occurs explicitly and essentially in the assignment of probabilities to the ranges of values of the magnitudes of the theory. This is the principle introduced by quantum mechanics which leads to the significantly statistical character of the theory. The fact that $\underline{L}_1$ is not imbeddable ~~in~~to a logical space which admits two-valued measures shows that the statistical character of the theory is an essential component of this assumption and supports the view that quantum mechanics is a fundamental theory.

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Chapter III

The Possibility Structure of Physical Systems

0. Introduction.

This paper develops the logical interpretation of non-relativistic quantum mechanics initially proposed by Hilary Putnam [11]. (See also [1] and [2].) The main features of this interpretation are briefly summarized in this introduction.

Certain physical theories postulate abstract structural constraints which events are held to satisfy. Such theories are termed "principle theories". Interpretations of principle theories aim to explain their relation to the theories they replace. Interpretations are therefore concerned with the nature of the transitions between theories.

Theories of space-time structure provide the most accessible illustration of principle theories. For example, Newtonian mechanics in the absence of gravitation represents the 4-dimensional geometry of space-time by the inhomogeneous Galilean group, which acts transitively in the class of free motions, i.e. the inhomogeneous Galilean group is the symmetry group of the free motions: it is a subgroup of the symmetry group of every mechanical system, and the largest such subgroup. Einstein's special principle of relativity is the hypothesis that the symmetry group of the free motions is the Poincaré group. The transition from the Galilean group to the Poincaré group is associated with a corresponding modification in space-time structure. The absolute time and Euclidean metric of Newtonian mechanics are dropped altogether, and the metrical relations of space-time are determined by the Minkowski tensor.

The special theory of relativity represents the transition from Newtonian Mechanics to Maxwell's electrodynamics as involving a modification of the structure of space-time. In this sense, the special theory may be regarded as an interpretation of classical electrodynamics.

Theoretical transitions in the class of space-time theories suggest an analogous approach to the interpretation of quantum mechanics. In this view, classical and quantum mechanics are represented as a particular type of principle theory. I call theories of this type "theories of logical structure" (or sometimes "phase space theories"), since the type of structural constraint they introduce concerns the logical structure of events and this is given by the algebra of idempotent magnitudes of the theory. The logical structure of a physical system imposes the most general kind of constraint on the occurrence and non-occurrence of events. The event structures of classical mechanics are essentially Boolean algebras. The logical structure of a quantum mechanical system is represented by the partial Boolean algebra of subspaces of a Hilbert space. In general, this is not imbeddable into a Boolean algebra.

The mathematical investigations of Kochen and Specker [9] lead to a general concept of completeness applicable to phase space theories. The explication depends on the notion of a proper extension of a phase space theory. Extensions are defined relative to a category of algebraic structures (representing the phase spaces of the theory) and a suitable concept of statistical state: Let A denote the partial Boolean algebra of idempotents, S the set of statistical states ψ on A. Suppose there is an imbedding ϕ carrying A into A' such that for every $\psi \varepsilon \underline{S}$, $\psi = \psi' \circ \phi$, where ψ' is a statistical state on A'. Then the theory (A', $\underline{S}$') is an extension of the theory (A,$\underline{S}$). The extension is proper if **for some** $\psi' \varepsilon S'$, $\psi' | \phi[A] \neq \psi$. Complete phase space theories have no proper extensions.

A proper extension of a phase space theory must not be confused with the more usual notion of a proper extension of a formal theory. Besides trivializing the notion, this would imply that completeness is a property of the theory's formalization. This, however, is not the case. The relevant notion of completeness is a mathematical property of a certain class of algebraic structures rather than a metamathematical one. There exists an important connection between completeness and the formal theory of this class of structures, but the concept of completeness does not depend on this connection.

A great deal of unclarity has surrounded the problem of completeness in quantum mechanics. An important consequence of this analysis is that classical mechanics and quantum mechanics are complete in exactly the same sense. In neither theory do there exist extensions in the category of algebraic structures associated with their respective phase spaces. As principle theories, classical mechanics and quantum mechanics specify different kinds of constraints on the possible events open to a physical system, i.e. they determine different possibility structures of events, and each theory is complete relative to the category of algebraic structures defined.

Finally, the approach to phase space theories outlined here has interesting consequences for the nature of logical truth. The logical structures of quantum mechanics include the Boolean algebras of classical mechanics. Such structures represent the possibility structure of events, that is, roughly speaking, they represent the way in which the properties of a physical system hang together. The quantum theory has shown that significantly different assumptions may be made concerning this structure. Now classical propositional validity is essentially validity in the category B of general Boolean algebras. The choice of Boolean algebras, has an empirical justification in classical mechanics, for the magnitudes of this theory form a commutative algebra and therefore the subalgebra of idempotent magnitudes form a Boolean algebra.[9] When viewed in this way, the justification of classical validity is intimately bound up with the logical structures postulated by classical mechanics. The quantum theory extends this class of structures to include all partial Boolean algebras of a certain type. The Boolean imbeddability properties of these structures have a model-theoretic characterization in terms of the validity of classical tautologies. Now a consequence of the work of Kochen and Specker is that there exists a classical tautology which is quantum mechanically refutable (i.e. refutable, in a partial Boolean algebra of the quantum theory). In this sense, classical logic is false, and the truth of logic, an empirical question.

[One remark on the mathematical exposition: All qualifications regarding measurability; viz., the restriction to Borel functions, Borel subsets, and Boolean σ-algebras, have been omitted. This merely means that the exposition is not as general as it might be.]

1. Preliminary Notions:

A partial algebra over a field K is a set A with a reflexive and symmetric binary relation $\leftrightarrow$ (termed "compatibility") such that A is closed under the operation of scalar multiplication from $K \times A$ to A, and the operations of addition and multiplication defined from $\leftrightarrow$ to A. That is:

(i) $\leftrightarrow \subseteq A \times A$

(ii) every element of A is compatible with itself

(iii) if a is compatible with b, then b is compatible with a, for all $a, b \in A$

(iv) If any $a,b,c \in A$ are mutually compatible, then $(a+b) \leftrightarrow c$, $ab \leftrightarrow c$, and $\lambda a \leftrightarrow b$ for all $\lambda \in K$.

In addition, there is a unit element 1 which is compatible with every element of A, and if a,b,c are mutually compatible, then the values of the polynomials in a,b,c form a commutative algebra over the field K.

A partial algebra over the field Z_2 of two elements is termed a <u>partial Boolean algebra</u>. The Boolean operations $\wedge$, $\vee$ and $'$ may be defined in terms of the ring operations in the usual way:

$$a \wedge b = ab$$

$$a \vee b = a+b-ab$$

$$a' = 1-a.$$

If a,b,c are mutually compatible, then the values of the polynomials in a,b,c, form a Boolean algebra.

Clearly, if B is a set of mutually compatible elements in a partial algebra A, then B generates a commutative sub-algebra in A; and in the case of a partial Boolean algebra

A, B generates a Boolean sub-algebra in A. Just as the set of idempotent elements of a commutative algebra forms a Boolean algebra, so the set of idempotents of a partial algebra forms a partial Boolean algebra.

We shall be mainly concerned with partial Boolean algebras. A homomorphism, h, between two partial Boolean algebras, A and A', is a map $h:A \rightarrow A'$ which preserves the algebraic operations, i.e. for all compatible $a, b \in A$:

$$h(a) \leftrightarrow h(b)$$

$$h(a+b) = h(a) + h(b)$$

$$h(ab) = h(a)h(b)$$

$$h(1) = 1$$

A homomorphism is an imbedding if it is one-to-one.

A weak imbedding is a homomorphism which is an imbedding on Boolean sub-algebras of A. More precisely, a homomorphism, h, of A into A' is a weak imbedding if $h(a) \neq h(b)$ whenever $a \leftrightarrow b$ and $a \neq b$ in A. So that in the case of a weak imbedding, incompatible elements may be mapped onto the same element.

An algebra is simple if its only proper filter is the unit filter {1}. Z_2 is the only simple Boolean algebra. A necessary and sufficient condition for the imbeddability of a partial Boolean algebra A into a Boolean algebra B, is that for every pair of distinct elements $a, b \in A$ there exists a homomorphism $h:A \rightarrow Z_2$ which separates them in Z_2, i.e. such that $h(a) \neq h(b)$ in Z_2. This is Kochen and Specker's Theorem 0. The result depends on the semi-simplicity property of Boolean algebras, i.e. the fact that every Boolean algebra is imbeddable into a direct union of the simple Boolean algebra Z_2.

The direct union of a family $\{B_i\}_{i\varepsilon I}$ of Boolean algebras is defined on the set of all sequences $\{a_i\}_{i\varepsilon I}$ of elements of the B_i. The operations are defined point-wise, i.e.

$$\{a_i\}'_{i\varepsilon I} = \{a'_i\}_{i\varepsilon I}$$

$$\{a_i\}_{i\varepsilon I} \vee \{b_i\}_{i\varepsilon I} = \{a_i \vee b_i\}_{i\varepsilon I}$$

$$\{a_i\}_{i\varepsilon I} \wedge \{b_i\}_{i\varepsilon I} = \{a_i \wedge b_i\}_{i\varepsilon I}.$$

The direct <u>product</u> of the B_i is essentially the closure of the direct union under the operation of forming isomorphic images.

Semi-simplicity is equivalent to the homomorphism theorem: Every Boolean algebra admits a two-valued homomorphism, i.e. a homomorphism onto Z_2.

The semi-simplicity property and the homomorphism theorem are alternative formulations of Stone's representation theorem and the ultrafilter theorem (respectively). This is a consequence of the fact that in every Boolean algebra there is a natural one-to-one correspondence between ultrafilters and two-valued homomorphisms. Let S be the Stone space of a Boolean algebra B. (S is the set of all ultrafilters in B). Let $\underline{P}(S)$ denote the Boolean algebra of all subsets of S. Replacing ultrafilters by two-valued homomorphisms and subsets of S by the sequence of values of their characteristic functions yields Z_2^S -- the direct union of Z_2 to the power of S -- from $\underline{P}(S)$. In this context, the Stone isomorphism becomes the imbedding $k:B\to Z_2^S$ given by

$$k(a) = \{h_t(a)\}_{t\in S}.$$

The mathematical connection of these ideas to logic arises in the following way. Propositional formulae are regarded as Boolean polynomials in a suitable first-order language $\underline{L}$. Realizations of $\underline{L}$ are objects in the category $\mathcal{B}$ of general Boolean algebras. A formula $\phi(x_1,\ldots,x_n)$ is classically valid (C-valid) if, for any B in $\mathcal{B}$ every substitution of elements for the variables $x_1,\ldots,x_n$ yields the unit of B. If ϕ is a propositional formula not valid in B, i.e. if $\phi(a) \neq 1$ for some $a = (a_1,\ldots,a_n)$ in B^n, then $\phi(k(a)) \neq 1$ in Z_2^S, where $k(a) = (k(a_1),\ldots,k(a_n))$, so that ϕ is refutable in Z_2. Hence, by semi-simplicity, classical validity is equivalent to tautologousness, i.e. validity in Z_2.

Now extend the class of realizations of $\underline{L}$ to the category of partial Boolean algebras. Validity in a partial Boolean algebra N depends on the domain of a propositional formula. $\phi(x_1,\ldots,x_n)$ is valid in N if every substitution of elements from the domain of ϕ yields the unit of N. The concept of the domain of a propositional formula may be simply explained by an example.

Let $\phi = \psi \equiv \chi$ be the propositional formula

$$x_1 \wedge (x_2 \vee x_3) \equiv (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$$

The domain of ϕ is the set of all elements $a = (a_1, a_2, a_3)$ of N such that:

$$a_1 \leftrightarrow a_2$$
$$a_1 \leftrightarrow a_3$$
$$a_2 \leftrightarrow a_3$$
$$a_1 \leftrightarrow (a_2 \vee a_3)$$
$$(a_1 \wedge a_2) \leftrightarrow (a_1 \wedge a_3)$$
$$a_1 \wedge (a_2 \vee a_3) \leftrightarrow (a_1 \wedge a_2) \vee (a_2 \wedge a_3),$$

for only then will the operations appearing in ϕ be defined in N.

It follows from the first three compatibilities that any three elements in the domain of ϕ generate a Boolean algebra, and hence, satisfy the distributive law. Hence ϕ is valid in all partial Boolean algebras. [Notice also that the last three compatibilities are therefore redundant.]

The generalized definition of propositional validity is: A propositional formula is Q-valid if it is valid in all partial Boolean algebras. The notion of Q-validity is formalized in [8].

The fundamental model-theoretic result in this field is:

(i) A partial Boolean algebra N is imbeddable into a Boolean algebra iff for every classical tautology of the form $\phi \equiv \psi$ the corresponding identity $\phi = \psi$ is valid in N; i.e. $\phi(a) = \psi(a)$ holds for all a in the intersection of the domains of ϕ and ψ.

(ii) N is weakly imbeddable into a Boolean algebra iff every classical tautology is valid in N.

(iii) There is a homomorphism from N into a Boolean algebra iff every classical tautology is not refutable in N. [This is Theorem 4 of [9]. (See [2] for an exposition of the proof of this theorem.)]

Notice that both C-validity and Q-validity have been defined algebraically, as the validity of propositional formulae in certain algebraic categories. In the case of classical validity, this definition differs sharply from more usual characterizations. Because of the equivalence of validity in B and validity in Z_2, classical propositional validity is defined as validity in the two-element <u>matrix</u> or <u>truth-table</u> $<\{0,1\},\{1\}, v, ->$, where $\{1\}$ is the set of designated elements and v and - have their well-known matrix definitions. $\phi(x_1,\ldots,x_n)$ is a classical tautology if it yields the designated value 1 for all substitutions of the elements 0 and 1 for the variables $x_1,\ldots,x_n$.

The transition to Q-validity is greatly simplified when classical validity is understood algebraically. But there is another reason for replacing the matrix definition. First, it should be clear that a definition of classical validity is not merely a stipulation. Rather, one concept (or group of concepts) proved very fruitful in initiating the modern mathematical development of logic, and a definition of validity should provide some explication of this concept. The matrix definition is misleading since it ignores the connection of classical propositional logic with the theory of Boolean algebras. This connection is important for the whole development of the subject. For example, on the matrix definition it is trivial that classical propositional validity is effective. But when classical validity is defined algebraically, effectiveness depends on semi-simplicity, which is decidedly non-trivial. Hence the algebraic definition suggests that effectiveness did not play a major role in the initial formulation of mathematical logic, and in fact, considerations of this type actually occur much later: viz., when the scope of logic came to be drawn in terms of the distinction between syntax and semantics.

2. Partial Boolean Algebras and Orthomodular Posets.

Every partial Boolean algebra is isomorphic to a collection $\underline{L} = \{L_i;\ i\varepsilon I\}$ of Boolean algebras satisfying the following conditions:

(i) The L_i have a common 1.

(ii) If $a \varepsilon L_i \cap L_j$, then $a^{\perp_i} = a^{\perp_j}$ ($\perp_i$ = ortho-complemention relative to L_i, etc.)

(iii) If $a, b \varepsilon L_i \cap L_j$, then $a \wedge_i b = a \wedge_j b$.

(iv) Let $L = \cup\{L_i: i\varepsilon I\}$. Given any a,b,c in L such that any two of them lie in a common L_i, there exists an L_j such that $a,b,c \varepsilon L_j$.

Families of Boolean algebras satisfying (i) - (iv) are called logical structures. (Notice, the first three conditions simply insure that in any logical structure the operation of taking the intersection of two algebras makes sense.) The partial Boolean algebra associated with a logical structure is defined on $L = \cup\underline{L}$: $a \leftrightarrow b$ iff there is an L_i containing a,b. The 1 of L is the common 1 of all the L_i. $a^{\perp} = a^{\perp_i}$ for some L_i. If $a \leftrightarrow b$, then $a \wedge b = a \wedge_i b$ for some $i\varepsilon I$. (The zero and join of L are thought of as being defined in the usual way.)

A partial Boolean algebra is said to be transitive if $a \leq b$, i.e. $a \wedge b = a$, and $b \leq c$ implies $a \leftrightarrow c$, in which case $a \leq c$. Logical structures associated with transitive partial Boolean algebras satisfy the further condition:

(v) If $a \leq_i b$ and $b \leq_j c$, there is an L_k such that $a,b,c\varepsilon L_k$.

Orthomodular posets are perhaps more familiar in the present context. They are structures $P = \langle P, \leq, \vee, \wedge, \perp, 0, 1\rangle$ where $\leq$ is a partial order on P and $\perp$ is an orthocomplementation; 1 and 0 are greatest and least elements, and $\vee, \wedge$ are the l.u.b. and g.l.b. with respect to $\leq$. P is weakly modular: $a \vee b$ exists whenever a is orthogonal to b (i.e. $a \leq b^{\perp}$), and if $a \leq b$, then $a \vee (b \wedge a^{\perp}) = b$. In any orthomodular poset it is possible to define a relation C of compatibility: a C b if there exist mutually orthogonal elements a_1, b_1, c such that $a = a_1 \vee c$ and $b = b_1 \vee c$. (I.e., a and b are compatible if they are orthogonal except for an overlap.)

The representation theory for orthomudular posets has been established by Finch [3] (Theorems 1.1 and 3.1). The logical structures considered by Finch differ from those associated with transitive partial Boolean algebras with respect to condition (iv). In [3] this is replaced by the weaker

(iv') Suppose $a \leq_i b^{\perp_i}$ for some $a, b \in L_i$. If $a \leq_j c^{\perp_j}$ and $b \leq_k c^{\perp_k}$ then there is an $m \in I$ such that $a, b, c \in L_m$. That is, in an orthomodular poset we may have that $a, b \in L_i$, $a, c \in L_k$, and $b, c \in L_k$, but there is no L_m containing a,b,c.

A <u>compatible</u> orthomodular poset is one which satisfies the condition: (a v b) C c whenever a,b,c are pairwise compatible. This is a necessary and sufficient condition for every set of mutually compatible elements to be contained in a Boolean subalgebra of P.

There is a very simple connection between orthomodular posets and partial Boolean algebras: Every compatible ortho-modular poset is a transitive partial Boolean algebra and conversely.

I have presented the connection between orthomodular posets and partial Boolean algebras in terms of their representation theory. A direct proof of these remarks has been given by Gudder in [6]. His formulation is based on the notion of an <u>associative</u> partial Boolean algebra: Suppose $a \leftrightarrow b$ and $b \leftrightarrow c$, $a,b,c \varepsilon L$. Then L is associative, if $(a \wedge b) \leftrightarrow c$ iff $a \leftrightarrow (b \wedge c)$, and hence $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. (By a lemma of Gudder and Schelp ([7] Lemma 3.3) a partial Boolean algebra is associative if and only if it is transitive.) Gudder shows that every associative partial Boolean algebra is a compatible orthomodular poset, and conversely. (Theorems 2.3 and 2.4.)

The partial Boolean algebra B(H) of closed linear subspaces of a separable Hilbert space is a transitive partial Boolean algebra. The partial ordering is given by the subspace relation, and the operations of meet, join and orthocomplement are represented by the intersection, span and orthogonal complement of subspaces. The zero of B(H) is the 0-dimensional subspace, and the unit is the whole space H. The definition of $\leftrightarrow$ is $a \leftrightarrow b$, if aCb, i.e. if there are mutually orthogonal subspaces a_1, b_1, c such that $b = b_1 \vee c$, and $a = a_1 \vee c$. Equivalently, the subspaces of H form a compatible orthomodular poset.

To sum up: The representation of H as an orthomodular poset is based on its order structure since it is the fact that H is partially ordered which is retained in the general concept of an orthomodular poset. The concept of a partial Boolean algebra is based on the compatibility structure of H since it preserves the fact that every triple of pairwise compatible subspaces is contained in a Boolean subalgebra of H.

3. Applications to the Problem of Hidden Variables.

Partial Boolean algebras were introduced by Kochen and Specker in connection with the problem of hidden variables. This is characterized as the problem of imbedding the non-commutative partial algebra of self-adjoint operators on H into the commutative algebra of real valued functions on a classical probability space. Their first theorem shows that there are no two-valued homomorphisms on a finite subalgebra of the partial Boolean algebra $B(E^3)$ of lines through a point in ordinary three dimensional Euclidian space. It follows from this that there are no two-valued homomorphisms on B(H). (It is necessary to assume that the dimension of H be at least three.) Hence, by Theorem 0 there is no imbedding of B(H) into a Boolean algebra. But the partial Boolean algebra of subspaces of H is isomorphic to the subalgebra of idempotent operators on H. Hence the partial algebra of physical magnitudes is not imbeddable into a commutative algebra.

By the equivalence of two-valued homomorphisms and two-valued probability measures (or dispersion free states), the absence of two-valued homomorphisms is an immediate corrollary to Gleason's theorem. An independent proof of this corrollary was first given by Bell. (See [1] p. 69f, for an exposition of the proof.) Kochen and Specker's proof differs from Bell's since it does not depend on the denseness of the unit sphere in H.

There is an interesting reformulation of the imbedding problem in terms of the concept of a logical structure (suggested by a paper of Maczynski [10]). This actually characterizes the problem of finding a weak imbedding of a partial Boolean algebra into a Boolean algebra. The problem may be further generalized by the concept of a homomorphic relation, but we shall not consider this here (see [9] Sect. 5). The interest of the reformulation from our point of view is that it clarifies an important distinction, viz. the distinction between a Boolean theory based on the idempotent magnitudes of quantum mechanics and a Boolean representation of the algebra of idempotent magnitudes.

Let $\underline{N} = \{B_i : i\varepsilon I\}$ denote the logical structure associated with B(H) for a three dimensional Hilbert space. $\{f_i^j : B_i \to B_j,\ i,\ j\varepsilon I\}$ is the set of inclusion homomorphisms from B_i into B_j. f_i^i = the identity map. Notice that $f_j^k \circ f_i^j = f_i^k$, since $\underline{N}$ is a logical structure. Write $B_i \leq B_j$ if there is an f_i^j. A Boolean representation of $\underline{N}$ is a pair $(C, \{h_i\}_{i\varepsilon I})$ where C is a non-degenerate Boolean algebra and each $h_i : B_i \to C$ is an imbedding carrying B_i into C such that the following conditions are satisfied:

(i) $\underset{i\varepsilon I}{\cup} h_i[B_i]$ generates C.

(ii) If $B_i \leq B_j$, then $h_i = h_j | f_i^j[B_i]$.

(iii) Given any $(C', \{h_i'\}_{i\varepsilon I})$ satisfying (i) and (ii) there is a unique homomorphism $h: C \to C'$ such that $h \circ h_i = h_i'$.

The presence of condition (ii) implies that a Boolean representation of $\underline{N}$ is a weak imbedding of B(H) into C.

2 OF/DE 2

Suppose that $(\underline{F}, \{k_i\}_{i\varepsilon I})$ is the direct product of the B_i. That is to say, take the Cartesian product $X = \prod_{i\varepsilon I} X_i$ of the spaces X_i. Associate each $Y\varepsilon\underline{F}(X_i)$ with a subset of X by the mapping

$$Y \rightarrow \{x\varepsilon X | x_i \varepsilon Y\}.$$

The field product $\underline{F}(X)$ of the $\underline{F}(X_i)$ is the field generated by the union of the images of the $\underline{F}(X_i)$ under this correspondence. $\underline{F}(X)$ contains an isomorphic copy of each $\underline{F}(X_i)$, and hence, an isomorphic image of each B_i.

Condition (ii) distinguishes a Boolean representation of the B_i from their direct product. To see this consider an element a in $B_i \leq B_j$. a is mapped by k_i onto the set of points in X whose i-th coordinate is an ultrafilter in B_i containing a. The image of $a = f_i^j(a)$ under k_j is the set of points whose j-th coordinate is an ultrafilter in B_j containing a. It is clear that in general $k_i(a) \neq k_j(a)$.

Given the direct product, we may always obtain a structure satisfying condition (ii), but this is not necessarily a Boolean representation. To do this we proceed as follows. Let I be the ideal in $\underline{F}$ generated by all elements of the form: $k_i(a) - k_j(f_i^j(a)) \cup k_j(f_i^j(a)) - k_i(a)$. For X, Y in $\underline{F}$ define $X \sim Y$ if $X-Y \cup Y-X \varepsilon I$. $\underline{F}/I$ is the set of equivalence classes of elements of $\underline{F}$ under the relation $\sim$. Let $\phi : \underline{F} \rightarrow \underline{F}/I$ be the canonical homomorphism from $\underline{F}$ into the quotient algebra $\underline{F}/I$. Write $k_i' = \phi \circ k_i$. Then $(\underline{F}/I, \{k_i'\}_{i\varepsilon I})$ is the <u>direct limit</u> of $\underline{N}$. The direct limit is a Boolean representation if, and only if, it is non-degenerate.

It is always possible to construct a Boolean theory based on the idempotent magnitudes in the B_i simply by taking the direct product of the logical structure. This corresponds to Kochen and Specker's trivial hidden variable construction. Such a theory is excluded by condition (ii), for this condition makes the problem an imbedding problem. Rejecting this characterization of the problem is therefore equivalent to weakening the notion of a Boolean representation of a logical structure. But this overlooks the fact that exactly the same condition occurs in classical mechanics since, of the two representations, $(\underline{F}, \{k_i\}_{i\varepsilon I})$ and $(\underline{F}/I, \{k_i'\}_{i\varepsilon I})$, classical mechanics uses the direct limit, not the direct product. Hence, condition (ii), or equivalently, the condition of weak imbeddability, can hardly be regarded as an *ad hoc* restriction, arbitrarily introduced to exclude classical extensions of the quantum theory.

Orthomodular posets are mainly associated with the "quantum logic" formulation of quantum mechanics. This represents a new axiomatic approach to the theory which aims at generalizing von Neumann's presentation in terms of Hilbert space. The principal question here is: "To what extent can von Neumann's formulation be recovered without explicitly using the concept of Hilbert Space". This is a mathematical investigation, motivated by mainly mathematical considerations. In this latter respect, it differs, not only from Kochen and Specker's investigations, but from von Neumann's as well.

Before von Neumann's treatise, the relationship between wave mechanics and matrix mechanics was obscure. For example, Dirac and Jordan viewed the similarity between the two theories in terms of a "correspondence" between the "points" over which matrices in matrix mechanics and differential operators in wave mechanics are defined -- an idea which could not be consistently maintained as von Neumann showed. Von Neumann's formalization of these two theories in terms of Hilbert space was based on the observation that the algebraic structure of physical magnitudes is the same in both matrix and wave mechanics. This structure is represented by the non-commutative algebra of self-adjoint operators on a separable Hilbert space. The Hilbert space formalization is thus motivated by a question concerning the relationship between these two theories, and von Neumann presented the definitive clarification of the precise respect in which wave mechanics and matrix mechanics are equivalent.

In the quantum logic approach, the problem of hidden variables consists in showing that in any acceptable generalization of the theory there are no dispersion free states. Now the difficulties with the notion of an <u>acceptable</u> generalization are obvious enough. However equally serious problems arise in connection with the notion of a <u>generalization</u> of quantum mechanics. For example, why should a generalization in any way preserve the algebraic structure of the theory? For this may be an inessential feature of the theory's formulation, and therefore, not properly part of any generalization of the theory. This is the criticism usually urged against hidden variable theorems.

These and similar difficulties are undercut by Kochen and Specker's formulation of the problem as an imbedding problem. First, this restricts the issue to the relationship between two _given_ theories, viz., quantum mechanics and classical mechanics. The question is basically this: Is quantum mechanics a complete statistical theory? Or is it statistical in the sense of classical statistical mechanics? In this case the absence of dispersion free states is the result of incomplete knowledge. Secondly, it completes the program initiated by von Neumann: In _Mathematical Foundations_ it was shown that the peculiarity of wave mechanics and matrix mechanics consists in the algebraic structure of their physical magnitudes. The solution to the imbedding problem shows that this algebraic structure occurs essentially, and therefore cannot be regarded as a property of the theory's formulation.

4. Idempotents as Propositions.

To begin with, let us consider the relationship between the properties of a physical system and quantities taking values in R. Let Ω be a subset of n-dimensional Euclidian space. In classical particle mechanics a physical quantity is represented by a function in R^{Ω}. Take a quantity A and real number λ in the range of $f_A \varepsilon R^{\Omega}$. Then $f_A = \lambda$ (this is read "the value of f_A is λ") represents a property of the physical system S. More generally, given A and a subset U of R, $f_A \varepsilon$ U (the value of f_A lies in U) represents a property of S.

To define the holding of a property we shall require the notion of a state of S. This must not be confused with a statistical state. A state is simply an event in the history of S. In classical mechanics an event is represented by a point ω in Ω. As is well known, an event represented by ω is associated with a pure statistical state: the two-valued measure on F(Ω) determined by ω; but the concept of an event is not a statistical concept.

The property $f_A \varepsilon$ U holds for S if and only if S is in a state such that $f_A(\omega) \varepsilon$ U. Notice, in von Neumann's terminology ([13] p. 249) $f_A \varepsilon$ U is then said to be a property of the <u>state</u> ω. This is misleading, as will be shown in connection with quantum mechanics.

Let us now consider the representation of properties of S by idempotent magnitudes. Let Γ denote the subset $f_A^{-1}(U)$ of Ω. It is clear that S has the property $f_A \varepsilon$ U if and only if the state ω of S lies in Γ. The property $f_A \in U$ is said to be associated with Γ. In general there are many properties $f_{A_i} \in U_j$ such that $f_{A_i}^{-1}(U_j) = \Gamma$ for some $U_j \subseteq R$. Now let χ_Γ be the characteristic function of Γ, and let P be a property associated with Γ. By the correspondence between properties of S and subsets of Ω, it follows that every property P is represented by the characteristic function χ_Γ, in the sense that P holds if and only if S is in a state ω such that $\chi_\Gamma(\omega) = 1$. Since χ_Γ is two valued this is equivalent to $\chi_\Gamma(\omega) \neq 0$. This is the formulation employed by von Neumann ([13] Ch. III. 5).

Now the characteristic functions of the subsets of Ω are just the subset of idempotent elements in R^Ω. The idempotents form a Boolean ring with unit $1 = \chi_\Omega$, i.e. a Boolean algebra. To simplify the notation, let B denote the Boolean subalgebra of idempotents in R^Ω. Elements of B will be denoted by a,b,c... . Whenever lattice operations are used, it is assumed that they have been defined in terms of the ring operations in the usual way.

To summarize, we have seen that every magnitude may be replaced by a set of properties, and that every property corresponds to a two valued quantity, i.e. idempotent magnitude in B. The correspondence is not one-to-one, since very many properties are associated with the same subset of Ω, and therefore represented by the same idempotent magnitude. Since a is also a magnitude in R^Ω, a = 1, (or a $\neq$ 0) is also a property of S just as $f_A = \lambda$ is a property of S. Here the property $a = 1$ represents a whole class of properties of the form $f_A \in U$.

That idempotents may be regarded as propositions is justified by the following considerations. For each a in B we may define: a is <u>true</u> (i.e. $a = 1$, $a \neq 0$) if and only if S is in a state ω such that $a(\omega) = 1$. (Since a is two valued, this is equivalent to $a(\omega) \neq 0$.) I.e. a is true if and only if S has any (and therefore, <u>all</u>) of the properties represented by a. Also,

a' is true iff $1 - a(\omega) = 1$

$a \vee b$ is true iff $a(\omega) + b(\omega) - a(\omega)b(\omega) = 1$,

so that the lattice operations ' and v in B represent the operations of negation and disjunction of classical logic. In a similar way the other propositional connectives may be identified with the corresponding Boolean operations.

[Note, strictly speaking B should be regarded as the Lindenbaum-Tarski algebra of a suitable formal language <u>L</u>, and truth then defined for sentences for <u>L</u>. For our purposes, this would introduce an unnecessary complication.]

The distinction drawn between states and statistical states is expressed notationally as follows. Let $a \in B$. If I write $\hat{\omega}(a)$, this denotes the probability of the proposition a in the pure state determined by $\bar{\omega}$. But $a(\omega)$ denotes the truth value of a for the state ω. In other words, if I write $\omega(a)$, i.e. if ω appears as a function, then it denotes a probability measure; but when ω appears as an argument, it denotes a state.

Notice, in classical mechanics we always have

$\omega(a) = a(\omega)$.

This depends on the fact that the pure states, like the propositions are two valued. For this reason it is possible to define the truth of a proposition a in B by writing

a is true iff S is in a state ω such that $\omega(a) = 1$.

a' is true iff $1 - \omega(a) = 1$.

a v b is true iff $\omega(a) + \omega(b) - \omega(a)\omega(b) = 1$.

This definition is formally equivalent to the one given earlier. But the conception of truth which underlies the two definitions is very different. This definition identifies truth with probability equal to 1, so that a proposition is true only if the statistical state of S assigns it probability 1. That is to say, for a proposition a to be true, the system must be in a pure <u>statistical</u> state ω such that $\omega(a) = 1$.

In classical mechanics each ω in Ω determines a two-valued homomorphism $h: B \rightarrow Z_2$. Each h corresponds biuniquely to a two-valued measure ω on $F(\Omega)$. Substituting h for the probability measure determined by ω yields a formally equivalent definition of truth. This definition is related to the one given in van Fraassen [12] as follows. A possible world is a classical truth value assignment to the propositions of B. Each such world is represented by a two-valued homomorphism. A proposition is true only if it is true in some possible world, i.e. only if $h(a) = 1$, for some $h: B \rightarrow Z_2$. This is a necessary condition for the truth of a. (a is true if this truth value assignment is determined by the actual world.) The mistake underlying the identification of truth with probability equal to 1 is transparent. It is less obvious that exactly the same misconception underlies this definition.

First, notice that if a possible world is represented by a state or event (rather than a two-valued homomorphism), the resulting definition coincides with the one given here, i.e. with $a(\omega) = 1$. It is not the concept of a possible world which poses a problem, but the interpretation of a possible world as a two-valued homomorphism. For this has the consequence that the truth of an atomic proposition is defined only if the truth or falsity of every <u>other</u> proposition is also specified. But this is certainly not required by the correspondence theory of truth. To put this very simply, on any reasonable definition, the truth of the proposition (a) "It is raining" depends, not on the truth or falsity of every other proposition, but just on the state of the weather. This is independent of whether or not our knowledge of the truth of (a) depends on our knowledge of the truth of other propositions. It is this simple insight which is preserved in the correspondence theory, and which is given up when, in the <u>definition</u> of the truth of (a), it is required that $h(a) = 1$.

We may summarize this discussion of the definition of "truth" in classical mechanics. Classically a state is associated with a two-valued measure and a two-valued homomorphism. Because the Boolean algebra of subsets of Ω

is perfect and reduced, this association is biunique. There are therefore three formally equivalent definitions of "a is true": A) $a(\omega) = 1$, B) $\omega(a) = 1$ and C) $h(a) = 1$. In the first definition the truth of a is exclusively determined by the state of S: A proposition a is true if and only if the system has (any) one of the properties represented by a. Each of the other two definitions imposes an additional requirement on the truth of a. In the case of B, a must have a probability equal to 1. If the definition is $h(a) = 1$, then the truth of a requires that the truth or falsity of every other proposition in B is also given. The formal equivalence, which holds classically, does not mean that A, B, and C are explications of equivalent conceptions of truth. The conception underlying B is remarkably like the pragmatic theory of truth, while C is similar to the doctrine of internal relations of the coherence theory. Both conceptions of truth confuse the meaning or reference of the proposition "a is true" with considerations that are strictly evidential in character.

Thus far we have concentrated on classical mechanics, but the situation in quantum mechanics is exactly analogous. The idempotent magnitudes, i.e. the propositions a in the partial Boolean algebra N are projection operators acting on a suitable Hilbert space H. A state or event of a system S is represented by a (unit) ray K in H. Each event K is associated with a pure statistical state. In quantum mechanics, statistical states are given by measures on the closed linear subspaces of H. The pure state associated with K is determined by taking the square of the norm of the projection of a unit vector lying in K onto each subspace of H. Since by Theorem 12 of von Neumann [13], there is a one-to-one correspondence which associates each projection operator with the subspace in H which is its range, this determines a probability measure on N. Recall, in classical mechanics a statistical state is a measure on the field $F(\Omega)$ of subsets of Ω. There the correspondence between subsets and propositions is trivial. Note, atoms in B are the characteristic functions associated with singleton subsets $\{\omega\}$ of $F(\Omega)$. N is also atomic. An atom in N is a projection operator onto a one-dimensional subspace of H. Thus in each theory there is a one-to-one correspondence between events and atomic propositions.

Quantum mechanics is probabilistic in the sense that the set S of statistical states does not include two-valued measures. This means that the expectation (or average) value is never dispersion free for all magnitudes -- even in the case of pure statistical states. But exactly the same magnitudes, and therefore, exactly the same propositions, occur in both classical and quantum mechanics; that is, the propositions of both B and N make assertions about properties of physical systems, not ensembles of such systems. In quantum mechanics there are no statistical states which determine a probability of 1 or 0 for every a in N, and which may therefore be regarded as two-valued truth value assignments to the propositions of N. Although the statistical state determines the probability of every a in N, the corresponding event does not determine the truth values of all propositions.

Suppose that the event represented by K determines the truth or falsity of a. Then the situation is exactly as described for classical mechanics, and the truth of a' and a v b is defined by writing

a' is true iff $1 - a(K) = 1$

a v b is true iff $a(K) + b(K) - a(K)b(K) = 1$.

That is, the partial operations ', v of N represent the logical operations of negation and disjunction of classical logic. (Recall that in a partial Boolean algebra a v b exists iff a and b are compatible.)

Of course, the identity

$a(K) = K(a)$

cannot hold in quantum mechanics. It is possible to make it hold by identifying the truth value of a in the state K with the probability assigned to a by the statistical state determined by K, i.e. by stipulating that $a(K) = K(a)$. But there is no more justification for identifying truth with probability in quantum mechanics, than there is in classical mechanics.

It follows from what has been said so far that an event does not determine all properties of S. Here it is important to be extremely clear. Let us consider a specific property P. Then there are states K such that K does not determine whether P holds or fails to hold. But P is a property of S and it is always determinate whether or not P holds for S: The answer to this question is contained in S, though not in every event in the history of S. The determinateness of the holding of P is completely independent of whether or not the holding of P is determined by every state of S. I regard this claim as central to the logical interpretation of quantum mechanics.

This is obscure if it is assumed that P is a property of a state rather than a property of S and that S has only a single state K. It is true that there is a single <u>statistical</u> state which describes S. The uniqueness of the statistical state of S is required by quantum mechanics, as it is by classical mechanics. The claim that S has a unique statistical state must not be confused with a certain interpretation of mixed states. If S is represented by the mixed state

$$W = \sum_i w_i K_i$$

then this is understood to mean that S is in the pure state K_i with weight w_i. But we may also have

$$W = \sum_i w'_i K'_i$$

with $w_i \neq w'_i$ and $K_i \neq K'_i$. That is, there is no theoretical basis for singling out any one representation of W. But of course this is completely independent of whether or not there is any basis for supposing S to be in a unique statistical state. The point is simply this: There is a unique representation of the statistical state of S in the theory, but there is no unique representation of the mixture W in terms of pure states.

In quantum mechanics a system S has a single statistical state. But S has many states -- enough to determine the truth value of each a in N. In both classical mechanics and quantum mechanics, every a in N or B is true or false, and if a is false, than a' is true. This is a consequence of the fact that both algebras are idempotent, hence every a in N or B takes only the values 0 (false) and 1 (true). But B is also semi-simple, and therefore admits two-valued measures. The correspondence between two-valued measures and events means that each event determines a two-valued truth value assignment to all the propositions of B. This is to be contrasted with N which is not semi-simple (Theorem 1 of [9]), hence there is no extension of N which recovers the correspondence between events and two-valued measures.

The structure of N makes it necessary to distinguish an event from a possible world since, intuitively, a possible world should determine all properties of S. But S has many more properties than are represented by the propositions whose truth is determined by *K*. Hence *K* cannot be regarded as a possible world in this sense. The situation is rather as follows: In classical mechanics a single point ω in Ω represents a possible world, since a single event determines all properties of S. In quantum mechanics, a possible world is represented only by the whole logical space N, since no one event determines all properties of S. This suggests that the usefulness of the concept of a possible world is restricted to the classical case.

In quantum mechanics a pure statistical state of S is significantly probabilistic in the sense that pure states are not degenerate statistical states as in classical mechanics. But unlike classical statistical mechanics, this does not arise from an incompleteness in the theory, since it is impossible to introduce two-valued measures given the logical structure of N. In classical statistical mechanics the degenerate statistical states, i.e. the two-valued measures, are recoverable by imbedding the algebra of idempotent macroscopic magnitudes into an atomic Boolean algebra. But N is atomic, and there are no (proper) extensions of N; in particular there are no Boolean extensions.

So far it has been shown that the classical concepts of truth and logical structure are largely independent, and that the concept of truth is the same in both classical and quantum mechanics. In particular, any formal language L, which is based on N, i.e. for which N is the Lindenbaum-Tarski Algebra, is bivalent. L differs from a classical language in the following respect. A sentence ϕ of L corresponds to an element a of the logical structure N. This is exactly as in classical logic. But unlike the classical case, a is never associated with a two-valued homomorphism on N. To put this slightly differently: A Boolean representation of the fact that every sentence ϕ is true or false requires that the corresponding a in B is representable by the sequence k(a) of its truth values under all possible truth value assignments. Semi-simplicity insures that this is always the case. Now every sentence ϕ of L is true or false, and only true or false. For L is bivalent if N is idempotent, i.e. if for every a in N, $a \wedge a = a$. But the failure of semi-simplicity means that there is no Boolean representation of this fact. Whether every proposition a in N (and hence every sentence ϕ in L) is true or false depends exclusively on a. The bivalence of L is independent of how the corresponding propositions in N are interrelated. This is obviously not true if bivalence is represented by semi-simplicity.

It has been maintained (e.g. by van Fraassen [12] and Friedmann & Glymour [4])that there is a problem with applying the classical concept of truth to elements of N. This problem does not arise for propositions in B or in a maximal Boolean subalgebra B_i of N, since B and B_i each admit two-valued homomorphisms. If the discussion of this section is correct, semi-simplicity is irrelevant to the classical concept of truth, so that new "semantic analyses" of quantum mechanics are completely ancillary.

5. "Anomolies".

I shall now briefly consider the bearing of this discussion on the paradoxes or anomolies of quantum mechanics. (A complete discussion is contained in a forthcoming paper with Jeffrey Bub.) It seems likely that all of the paradoxes have essentially the following form. There exist two statistical states, one pure, the other a mixture, which are the same for a given class of idempotents, but which are generally distinct. The pure state correctly characterizes the system (i.e. it is confirmed experimentally). Such a state generates a collection of propositions asserting the probability that the system has a certain property. A mixed state gives the probability that the system is in a given pure state, and from this the probability that S has a certain property is inferred. For example suppose we are interested in a property P but that the pure state of S is neither 0 nor 1 for the proposition a in N representing P. One might suppose that S is really characterized by a mixture, so that S is in a pure state K with a certain weight w such that K assigns a a probability of 1 or 0. But this is excluded theoretically (by the fact that pure states are not reducible to mixtures) as well as experimentally.

The situation just described arises in the two slit experiment. (This is the only case I shall explicitly consider.) Recall that the statistical state of the electron is experimentally determined by examining the diffraction pattern which appears on the emulsion opposite the screen after very many electrons have been emitted by the source.

The electrons are emitted one at a time and at intervals of any length. The statistical state K associated with both slits open is not the weighted sum of the pure statistical states K_{α_i} which assign a probability of 1 to the propositions a_i: "The electron passes through slit i" (i = 1,2). The pure state exhibts interference, while the mixed state does not. Now the fact that the statistical state is not the mixture but another pure state is not paradoxical. By itself this is no more puzzling than the relativistic replacement of the classical addition theorem for relative velocities. It is clear that an anomoly arises only if there is some reason to suppose that the statistical state must be represented by a mixture of the K_{α_i}.

The basic idea seems to be that if the probability of a_i is neither 1 nor 0 in the state K, then the electron is in some sense indeterminate with respect to the property $a_i = 1$ (or $a_i \neq 0$). This means that neither a_1 nor a_2 is true or false. On this basis it is often suggested that quantum mechanics requires some thorough-going revision in our space-time concepts: the two slit experiment is anomolous given our classical and relativistic conceptions of space-time. But how seriously this suggestion should be considered is unclear since it has never been seriously developed. In any case it overlooks the fact that quantum mechanics, like classical mechanics and relativity, assumes that the symmetry group of all physical systems is a subgroup of the manifold mapping group, and that therefore, the theory makes precisely the same continuity and differentiability assumptions as these theories. Hence on this suggestion, quantum mechanics is a fundamentally incoherent theory. But the theoretical and experimental success of quantum mechanics is simply too great for this conclusion to be seriously considered.

That the whole problem is misconceived, is immediate from the analysis of this paper. The statistical state of the electron is indeed the pure state. This state exhibits interference, and is not to be replaced by a mixture. The pure state like all of the statistical states of the theory, is significantly probabilistic: i.e. it assigns a probability which is not dispersion free on very many propositions in N. In particular, it is not 0 or 1 for a_i. But each a_i is true or false. Moreover, this holds for every proposition in N. A difficulty arises when one attempts to express this fact by a simultaneous truth value assignment to all propositions in N, since this is possible only if N is imbeddable into a Boolean algebra.

More generally, quantum mechanics is indeterministic in the sense that the pure statistical states of the theory are not degenerate measures concentrated on 0 and 1 as in classical mechanics; hence the maximal amount of information concerning a physical system is significantly probabilistic. This arises from the fact that certain properties are independent, as are the idempotents which represent them. More exactly, the properties are <u>strongly independent</u> in the sense that they are related to other properties in a way which excludes their being related to each other. This is the significance of incompatibility.

Indeterminism in this sense must not be confused with the very different concept of indeterminateness. The theory is indeterministic or significantly statistical in the sense that the pure states take values in the open interval (0,1). The thesis that the theory is indeterminate holds that there are properties P such that P neither holds nor fails to hold of S. This is not implied by indeterminism nor is it in any way required by the view that the theory provides a maximal amount of information concerning the system. Rather, indeterminateness is suggested by essentially two mistaken ideas. The first of these is that incompatible propositions are somehow inconsistent. But just the opposite is the case: since a pair of propositions are incompatible, they cannot be inconsistent.

The second mistake concerns the conception of the truth of atomic propositions. According to the correspondence theory of truth an atomic proposition a in N is true if and only if the properties represented by a hold of S. This definition is independent of the semi-simplicity of N, so that at any given instant exactly one proposition in each $B_i \subset N$ is true of S. The essential point is that the absence of a simultaneous truth value assignment does not imply that the properties of S which, on the account given here, are supposed to obtain, cannot obtain simultaneously. The opposite view rests on a simple equivocation. A simultaneous truth value assignment is a two-valued homomorphism. That simultaneous truth value assignments do not exist is a fact about the structure of N which has nothing to do with what occurs simultaneously. The properties of S which obtain simultaneously include incompatible properties. But their logical structure excludes the existence of a two-valued homomorphism, and hence, of a simultaneous truth value assignment to the corresponding idempotents. At any one instant the system is characterized by exactly one class of properties from each $B_i \subset N$, and all such classes of properties obtain at the same time. But there is no Boolean representation of this fact.

To summarize this part of the discussion: There are two different accounts of indeterminism which are historically important. The first, which apparently goes back to Aristotle, rejects bivalence: A theory is indeterministic if it assumes that there are propositions whose truth value is indeterminate. The second, represented by the quantum theory, retains bivalence while rejecting semi-simplicity. An indeterministic theory is then characterized by the absence of two-valued homomorphisms, and therefore, of two-valued measures. The coherence of indeterminateness seems to rest on the Aristotelian metaphysic of act and potency. But nothing of this sort is required by the indeterminism of quantum mechanics. This form of indeterminism implies that there is no Boolean representation of the properties obtaining at a given time;

yet for any property P it is completely determinate whether or not P holds.

The anomoloüs character of the two slit experiment depends on the assumption that the system is indeterminate with respect to the property $a_i = 1$, if the statistical state is $K \neq K_{\alpha_i}$. This assumes that the usual notions of truth and falsity make sense only in the case of propositions which form a Boolean algebra. This view lies at the basis of both the Copenhagen and hidden variable interpretations of the quantum theory. According to the Copenhagen interpretation, at each instant the system "projects" exactly one maximal Boolean subalgebra of N; in a hidden variable interpretation all Boolean subalgebras are represented but their structure is Boolean. The difference is that the Copenhagen interpretation is willing to consider systems with properties corresponding to at most <u>one</u> maximal Boolean subalgebra in the logical structure associated with N. Now the implicit restriction of $\{B_i: i \in I\}$ to maximal Boolean subalgebras has an interesting consequence. If B_i, B_j are maximal Boolean subalgebras, then $B_i \leq B_j$ implies $B_i = B_j$, i.e. the logical structure $\{B_i: i \in M\}$ consisting of all maximal Boolean subalgebras is totally unordered by $\leq$. In this case condition (ii) of Section 3 is trivially satisfied, so that the direct limit of the B_i coincides with the direct product, which is a Boolean representation of $\{B_i: i \in M\}$.

Thus both the Copenhagen and hidden variable interpretations are committed to representations of $\{B_i: i \in M\}$

which are semi-simple: In the hidden-variable interpretation this requires dropping condition (ii), while in the Copenhagen interpretation this is accomplished by replacing $\{B_i: i \in I\}$ with $\{B_i: i \in M\}$. In effect both views fail to consider the possibility that the properties of physical systems exemplify the logical structure of N. I.e. both views completely fail as accounts of the significance of the transition from classical mechanics to quantum mechanics.

It should be remarked that the hidden variable interpretation is at least coherent. The idea that a system projects a single B_i at each instant has the absurd consequence that the system must somehow anticipate the decision of the experimenter to measure a magnitude associated with B_i. This consequence is avoidable only if measurements are regarded as a theoretically opaque subclass of physical interactions. It is a strange comment on current investigations in foundations of physics that both possibilities are seriously considered and widely entertained.

[The possibility of a classical theory of the maximal magnitudes was noted by Wiener and Siegel ([14], Appendix), and independently, by Gudder [5], who proved this theorem in a more general context. The discussion here follows the very elegant presentation of Maczynski.]

In conclusion, I wish to compare the discussion in Section 4 with von Neumann [13] Ch. III. 5, "Projections as Propositions". <u>Prima facie</u> there is only the slightest difference between the conception of truth presented here and the one implicitly assumed by von Neumann. But the difference has important consequences. Adopting von Neumann's approach, one is led to propose an additional class of non-unitary time-transformations. This is the content of the projection postulate, which may be simply explained as follows: Let us assume that a system S is in a pure state represented by a unit ray K_ψ in H (the Hilbert space associated with S). Now suppose we find that the value of a magnitude A is a_i. The value a_i is associated with a ray K_{α_i} in H. K_{α_i} represents the pure statistical state which assigns the property $A = a_i$ a probability equal to one. The projection postulate requires that the statistical state of S undergo a transition: $K_\psi \rightarrow K_{\alpha_i}$ which is generally discontinuous. Since the dynamics of the theory considers only continuous transitions, the projection postulate represents an additional hypothesis.

Essentially the same idea underlies the characterization of the probabilities of quantum mechanics as "transition" probabilities. The probability $|(\psi,\alpha_i)|^2$ which K_ψ assigns to $A = a_i$ is not the probability that the property $A = a_i$ obtains, but rather, the probability that it will obtain. This requires that S undergo a transition from the state K_ψ to K_{α_i}. The problem of finding a theoretical account for such stochastic transitions has come to be known as the "measurement problem", since transitions of this type are supposed to occur whenever a measurement is performed. The projection postulate is simply a precise characterization of this class of transitions.

If the analysis of this paper is correct, the projection postulate results from a misconception of the logical structure of the theory: there is nothing about the logical structure which requires that a proposition is true only in certain statistical states. The projection postulate is therefore quite clearly ancillary to an understanding of the theory. Similarly, since the measurement problem requires the occurence of transitions of the kind described by the projection postulate, it follow that this cannot be a real difficulty for the theory.

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