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Optimal Natural Resource Exploitation By Open Economies

James John Mcrae

OPTIMAL NATURAL RESOURCE EXPLOITATION BY OPEN ECONOMIES

by

James John McRae

Department of Economics

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ABSTRACT

This dissertation is concerned with the economic problem experienced by countries who export products produced from non appropriated natural resource stocks. The problem generated is that today's decisions in regard to consumption and export of natural resource intensive products will be based upon a myopic view of the future availability of the resource stock, thus producing the result that future generations may be deprived of the basis of the comparative advantage. The myopic competitive economy, accepting at each point in time the available resource base as a parameter, will grope towards a steady state equilibrium by a series of static optimizations. However, the socially optimal route to steady state equilibrium must include the fact that for society as a whole the resource base is changing over time.

A relatively simple two sector model of trade is employed. The export good is a resource product commodity produced by combining a variable factor with the existing resource base, while the import competing good is produced by using the variable factor alone. It is assumed that the

variable factor is inelastically supplied, and that the terms of trade vary in a range such that the trade pattern does not change.

With this model, the socially optimal path is found by using variational methods to maximize a measure of social welfare subject to an equation showing how the resource base changes over time, and constraints obtained from the production and excess demand functions. It is demonstrated that the optimal steady state equilibrium is obtained by correctly assigning an initial social price for use of the resource mass, provided certain sufficient conditions are satisfied. The competitive economy, neglecting the effects its consumption and export of resource product have on the future size of the resource base, is shown to have an intertemporal path and steady state equilibrium which is not socially correct. The two intertemporal paths are illustrated geometrically in output space.

In comparing the two paths and associated steady state equilibriums, it is seen that the competitive economy overproduces the export good and underproduces the import competing product. It is, however, also possible that the market economy produces too little of both commodities, i.e., by allocating factors of production along the optimal intertemporal path, more of both goods can be obtained at the steady state equilibrium.

It is demonstrated that the joint imposition of a user tax on the resource base and an export tax on exports of resource product will produce optimality.

ACKNOWLEDGEMENTS

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Professor Charles Plourde, my chief advisor, was very helpful in both stimulating my interest in this area, and teaching me the necessary technique to help solve the problem. Professor James Melvin has been a source of knowledge indispensable throughout my graduate student career as well as on this dissertation. I would like to thank Professors R. Boyer and A. Bloomquist for helpful comments.

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CHAPTER I

A. Introduction

Only recently has there been a significant increase in understanding, both by the general public and by economic model builders, of the importance that natural resources have for the economic welfare of any country. This past lack of interest in the influence that natural resources can have in economic models is rather surprising considering the fact that natural resources can be viewed as being simply nature's capital as opposed to man-made capital. A country's stock of natural capital depends upon historical and political variables, but the theory of how to optimally use the natural-resource input once its size has been established does not differ significantly from the modern economic theory employed in analyzing man-made capital. Thus, we may speak of the act of investment in natural resources when present consumption of the resource is postponed in favor of higher future consumption, just as we speak of investment in man-made capital when present consumption is less than currently produced output.

One of the most important aspects in the theory of natural resources, is the tremendous export potential of the resource products produced from the existing raw natural resource base. It is reported[1] that between 1928 and 1955 crude petroleum exports alone accounted for over one half of the increase in exports from all non industrial countries. The importance of certain mineral product exports for countries or regions within countries should be obvious. For replenishable natural resources, exports of the resource products produced from forest stands and fish populations are important elements in the balance of payments of many countries.[2]

However, those countries whose export pattern is dominated by products relatively intensive in the use of some domestic natural resource are becoming increasingly more concerned over the volume and terms of such exports.[3] It seems that the concern is due to the fear that today's decisions in regard to consumption and export of natural resource intensive products are based upon a myopic view of the future availability of the resource stock. Thus, the present generation will make decisions on consumption and export volumes which may deprive future generations

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1 F. T. Moore (1961, p. 221).

2 J. Crutchfield (1963, p. 208) reports that the export of fish products is an important source of foreign exchange for Norway, Iceland and Peru.

3 A. Scott (1973, p. 259) gives a summary of how this concern has been expressed in Canada in regard to forest product exports.

of the basis of the comparative advantage. This dissertation attempts to capture this fact by making the assumption that at every point in time the resource exporting country is so myopic in regard to the future availability of the resource base that it always accepts the available stock as a parameter, neglecting to calculate the natural growth in the resource stock, or to subtract its own use of the resource in determining the stock available in the next period. Such an extreme situation would arise if the resource were only available in a non appropriated or common property format, and to a less extreme extent if the capital markets worked imperfectly (say) due to the extremely long nature of the investment period. This dissertation makes the first assumption that property rights to the resource cannot be allocated due to the physical problem in defining rights, as in the ocean fisheries problem, or due to the political problem of enforcing the boundaries once defined.

Expressed in terms of capital theory terminology, the problem is that due to the common property nature of the resource the capital markets do not signal information back to current decision makers on the effects of investment or disinvestment taking place in the resource stock. For the case of replenishable natural resources, harvesting less than the natural growth rate can be said to produce positive investment, i.e., the resource stock will be larger in the next period. Similarly, harvesting at a rate faster than the rate at which the resource is capable of regenerating itself produces positive disinvestment in the resource base. Viewed in this fashion, a non replenishable resource is merely a special case of a replenishable one for which harvesting always implies disinvestment. Correspondingly, the class of natural resource stocks considered will be the more general replenishable ones. Examples of the consumption goods produced are various ocean products such as processed fish, lobsters or shrimp, seal fur and whale derivatives, plus the output from forest stands such as timber, ashes, pulp, paper or cellulose.

The dissertation employs variational methods to demonstrate that a resource product exporting country myopically groping toward a steady state equilibrium by means of a series of static optimizations will produce an intertemporal path which is not the socially correct one. Specifically, it will be shown that a system of decentralized markets, which produce a myopic view of the future availability of the resource stock, will in steady state equilibrium yield one of two possible outcomes. The first is that the equilibrium may occur at a stationary production point at which too much resource product and too little of the import competing good is being produced.

The second possible situation is the interesting case in which too little of both commodities is being produced, i.e., by allocating factors of production along the optimal intertemporal path, more of both goods are potentially available at the steady state equilibrium. Finally, the policy tools which will force the myopic society to behave in socially optimal way are derived and discussed.

In regard to the open economy nature of the problem, it is assumed that for all terms of trade the trading pattern is such that the country possessing the raw natural resource exports resource product in exchange for imported manufactured goods for the rest of the world. The reason for this pattern of trade is assumed to be the standard Heckscher Ohlin one of differences in the relative factor endowments between the home and foreign countries. Thus, the technology of production, to be discussed later in this Chapter, and the taste pattern are the same between the home country and the foreign country so that the home country's comparative advantage in resource product production comes about solely because of its relatively greater abundance of resource mass. This comparative advantage in resource product production is assumed to remain unchanged over all time periods. Also, by assumption, trade with the outside world is strictly balanced at all time periods, thus there will be no international capital flows. Finally, it is assumed that there are no factor movements between countries, no transportation costs either in obtaining the natural resource product, or in moving it or the imported good to final consumption, and no specialization in production.

B. Specifications of the Economic and Biological Systems

We assume that the behavior of the replenishable natural resource in the absence of exploitation can be summarized by the differential equation[4]

$$\dot{X} = N(X) \tag{1-1}$$

where X is the aggregate mass of the raw natural resource measured in the same units as the produced resource product output. Thus, for example, X is measured in terms of pounds of fish products or board-feet of sawn timber. The particular function N which is relevant depends, of course, upon the individual resource and the natural environment in which the resource mass grows. However, for the two cases of fish populations and timber stands, the differential equation (1-1)

4 For the justification, see G.K. Goundry (1960, p.440), V.L. Smith (1968, p. 410) and J.P. Quirk and V.L. Smith (1969, p. 4).

is assumed continuous with continuous first and second derivatives with the following properties[5] for $X \geq 0$

1) two zero points $\underline{X}$ and $\bar{X}$ such that $N(\underline{X}) = N(\bar{X}) = 0$
2) a maximizer $\hat{X}$ where $N'(\hat{X}) = 0$
3) strict concavity for all X, i.e., $N''(X) < 0$

The graph of Equation (1-1) is given in Figure 1 which shows that the resource left in its natural state will continue to grow until the equilibrium resource mass $\bar{X}$ is reached. The other zero growth mass given by $\underline{X}$ is the point at which the population is too small to engage in growth, and thus will become extinct as time passes. This lower mass point $\underline{X}$ has been set at $\underline{X} = 0$ to preserve the strict concavity of the unexploited growth function. Finally, the mass $\hat{X}$ is the point of largest net growth in the resource. This mass is called the maximum sustained yield, because it defines the maximum sustainable harvest or cut which can take place.

Into the model depicting this natural resource environment, we may now introduce the functions summarizing the economic system. It is assumed that the home country is engaged in the production of only two commodities. The output from sector one is the resource product, and the output from sector two is a manufactured good. The total industry output for sectors one and two is the sum of outputs from individual firms within

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5 V. L. Smith (1968, p. 410).

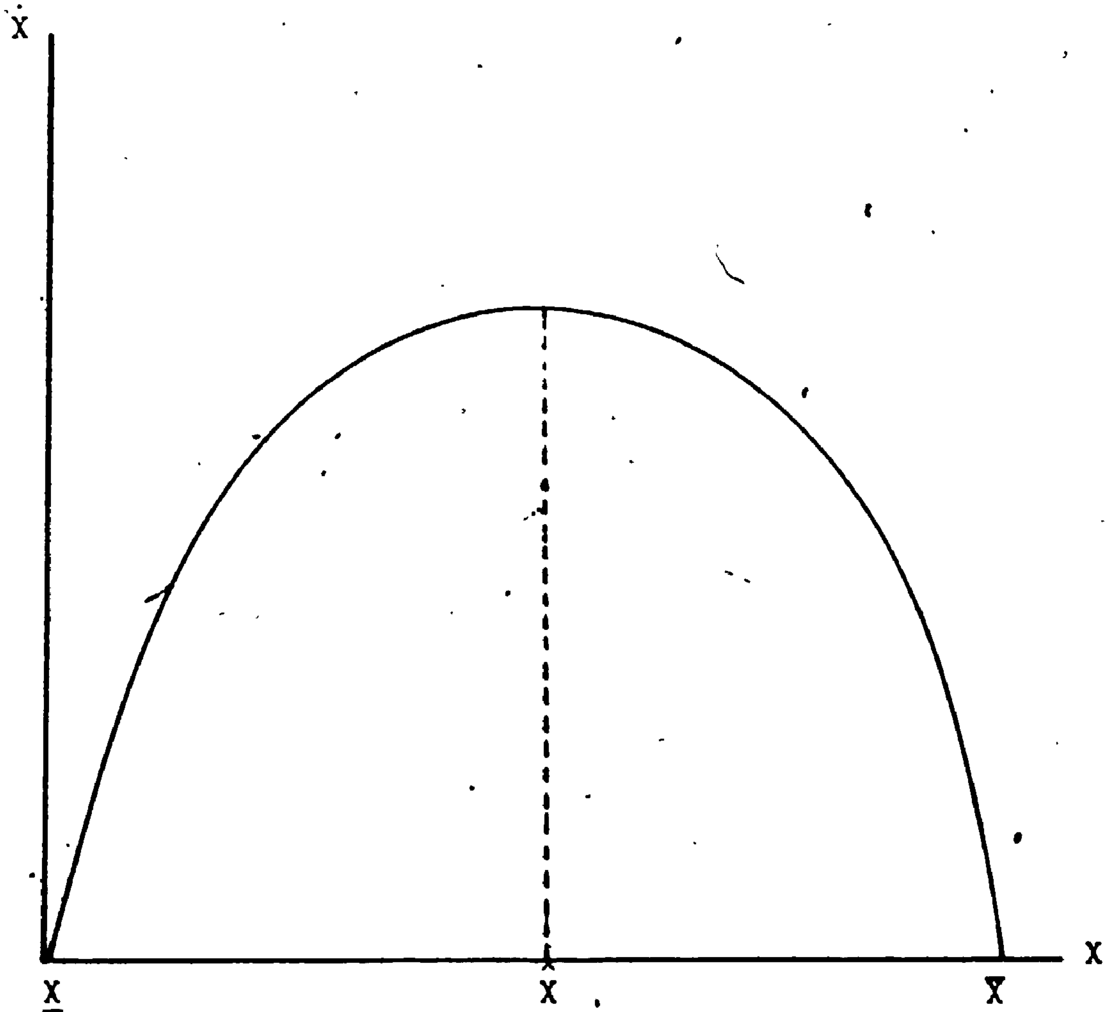

FIGURE 1

each of the sectors. Here it is assumed that there are n_1 identical firms in the resource product industry, and n_2 identical firms in the manufactured good industry. Thus, industry output for the resource product sector is given by n_1y_1, where y_1 is the output of resource product from one of the identical firms. Each of these firms has a production technology given by $G(\cdot)$, so that

$$(1\text{-}2) \qquad y_1 = G(l_1, X)$$

where l_1 is the amount of labor used per firm, and X is the existing stock of natural resource mass. The variable called labor may more correctly be thought of as labor combined with a fixed and non shiftable amount of physical capital. Reflecting the definition of X as an aggregate mass of the raw natural resource, it is assumed that all individual units within this mass have homogenous physical properties, as do all units of l_1. For convenience, the industry output n_1y_1 will be denoted by the upper case letter Y_1, and total industry labor input by the upper case L_1. Since all firms in this sector have identical technology $G(\cdot)$, the industry production function will be simply

$$(1\text{-}3) \qquad Y_1 = G(L_1, X)$$

The output y_2, obtained from one of the n_2 identical firms in sector two, is produced by a production technology

$F(\cdot)$ so that

(1-4) $y_2 = F(l_2)$

where l_2, thought of as labor combined with a fixed amount of physical capital, is the variable input employed per firm. Again, since each of the n_2 firms in this sector has the identical production technology $F(\cdot)$, the industry production function will be given by

(1-5) $Y_2 = F(L_2)$

where the upper case letters Y_2 and L_2 denote industry output and industry labor use respectively. The total economy labor supply is assumed to be fixed throughout the analysis. Thus,

(1-6) $\bar{L} = L_1 + L_2$

The following conditions are assumed about the production technologies:

1) both G and F are twice differentiable and have continuous second derivatives. The prime (') notation will be used to denote differentiation when the function contains only one argument. For functions with more than one variable, a numberical subscript will denote partial differentiation with respect to the correspondingly numbered argument.

2) the marginal products of labor in the two sectors,

written G_1 and F', and the marginal product of the resource stock in the resource product sector, written G_2, are all positive. The assumption that $G_2 > 0$ reflects the contact-technology[6] nature of the production functions (1-2) and (1-3). Thus, as the raw resource stock increases in size, the harvest or cut increases due to the fact that the resource input is more readily available.

3) the marginal products of both labor and resource stock diminish. Thus F'', G_{11} and G_{22} are all negative.

4) for both sectors, a zero input of either factor gives zero output. Thus $G(0,X) = G(L_1,0) = 0$, and $F(0) = 0$

In addition to being able to produce domestically both a manufactured good and the resource product, the home country may engage in trade with the rest of the world. As mentioned earlier, it is assumed that the exchange takes place at a world price ratio within a range such that the home country always exports the natural resource product and always imports the manufactured good. The supply and demand relationships for the rest of the world can be summarized by the offer curve

$$(1\text{-}7) \qquad Z_2 = \theta(Z_1)$$

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6 This term is due to J.P. Quirk and V.L. Smith (1969, p.3)

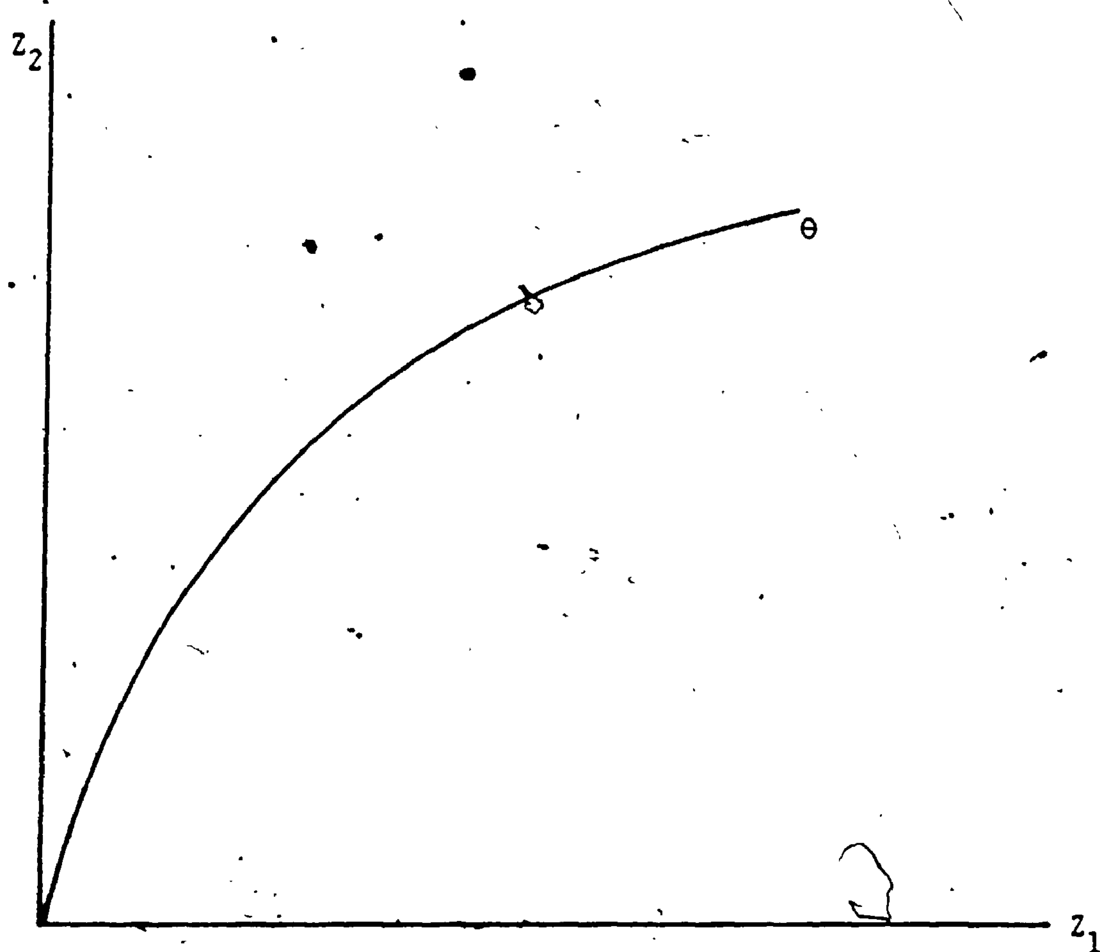

FIGURE 2

where Z_2 is a physical quantity of the manufactured good exported by the rest of the world and imported by the home country, while Z_1 is the physical quantity of resource product exported by the home country to the rest of the world. It is this ability to transform the raw resource stock not only into resource product for domestic consumption, but also into domestic consumption of the manufactured good by trade with the rest of the world that is of major concern to this dissertation.

The offer curve (1-7) illustrated in Figure 2. is assumed continuous with continuous first and second derivatives and with the following properties:

1) $\theta(0) = 0$

2) $\theta'(Z_1) > 0$

3) $\theta''(Z_1) < 0$

With this shape, the rest of the world offer curve is of greater than unitary elasticity throughout its range. Since it is well known that a less than unit elastic offer curve implies that the country's export good is Giffen, the curve as drawn rules out this possibility.

The equilibrium international terms of trade is determined where the home country's offer curve intersects this foreign offer curve. With this realized, it becomes clear that the export supply of resource product in the home country is sufficient in size to have some influence over the determination of the world price ratio. A special case of this situation which will be used in later chapters of this dissertation, is the small country assumption. Thus, assumption three above becomes $\Theta''(Z_1) = 0$, so that regardless of the home country's export decision, the world terms of trade do not change.

Interaction between the economic system given by equations (1-2) through (1-6), and the biological system given by (1-1) is provided by

$$(1\text{-}8) \qquad \dot{X} = N(X) - G(L_1, X)$$

Thus, in any time period, the economic system produces a flow of resource product from the existing stock of resource mass X. This harvest, or cut of resource product, is measured in the same aggregate biological mass units as the resource stock X, and hence, can be subtracted from the unexploited growth function N(X) just like the extraction of any other predator such as fire or disease. The result is that the net growth rate of the resource mass when the extraction of man is taken into account is given by equation (1-8). Since the raw resource is measured in terms of homogeneous biological mass units, no attention will be paid to the possibilities of selective harvesting, or to possible interactions between harvesting and the gross growth of the resource. A stationary equilibrium between the resource mass and the economic system clearly occurs when the extraction by man for the purposes of domestic consumption or international trade is just balanced by the gross growth of the resource stock, so that there occurs no net change in X.

C. Relevant Literature

The essential goal of this dissertation is to provide for a resource product exporting country a comparison between the intertemporal paths to stationary equilibrium produced by a market economy, composed of profit maximizing firms and utility maximizing households, and the optimal situation produced by a controlled economy. As such, it builds upon closed economy work by J.P. Quirk and V.L. Smith[7], C.G. Plourde[8], and V.L. Smith[9].

The models by Quirk and Smith, and Plourde are

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7 J.P. Quirk and V.L. Smith (1969, p. 1-15)

8 C.G. Plourde (1971)

9 V.L. Smith (forthcoming)

concerned with optimal exploitation through time of a replenishable natural resource in the context of a closed general equilibrium model. Together, they form the underlying basis for the material in Chapter III. While there are important differences in technique between the two models, both employ the Pontryagin Maximum Principle to maximize[10]

$$(1\text{-}9) \qquad \int_0^\infty [U(C_1) + V(C_2)]e^{-\rho t}\,dt$$

subject to the condition

$$(1\text{-}10) \qquad \dot{X} = N(X) - G(L_1,X)$$

and the constraints

$$(1\text{-}11) \qquad G(L_1,X) - C_1 \geqq 0$$

$$(1\text{-}12) \qquad F(L_2) - C_2 \geqq 0$$

$$(1\text{-}13) \qquad L_1 + L_2 = \bar{L}$$

where all variables are in the notation of this dissertation.

In attempting to analyze the convergence of the system towards steady state equilibrium, both models are hindered by ambiguity in determining stability properties of equilibria. In the Quirk and Smith model, stability is

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[10] This is the Plourde formulation of the objective function. Quirk and Smith do not employ separable utility.

shown to depend upon whether or not the resource product good is inferior with respect to increases in the resource stock, whether the marginal product of labor increases with the stock of raw natural resource, and the relationship between the marginal product of the resource mass and the percentage natural growth in the resource stock.[11] Plourde overcomes a similar difficulty by restricting the analysis in three ways. First, only resource good production functions which are Cobb-Douglas and homogenous to a degree less than unity are considered. Second, it is assumed that the marginal product of labor increases with the stock of natural resource. Finally, an assumption concerning the elasticity of marginal utility for the resource product good is made to produce the result that more labor is allocated to the resource product sector as the stock of raw resource increases.[12] This implies that it is efficient in terms of production and social consumption to produce less of the commodity from the second sector when resource stocks are increasing. Clearly, more resource product will then be produced due to the larger labor input associated with the increased resource stock.

The Plourde model is more detailed in analyzing

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11 J.P. Quirk and V. L. Smith (1969, p. 12).

12 In his unpublished dissertation, Plourde [1970 (a)] also considers the case where less labor is used in the resource product sector as the resource stock increases.

and discussing the stability of the steady state equilibrium produced, and its location in terms of the size of the stationary equilibrium resource stock. He shows that if an equilibrium exists it will be either saddle point stable or unstable. Hence, for the case of saddle point stability, optimal controls can be found to place the system on the optimal path. The location of the equilibrium in terms of the gross resource growth function is shown to be indeterminate without some restrictions on the social discount rate, production, utility and resource growth functions. With this result, the non-optimality of maximum sustained yield programs becomes obvious.

The Quirk and Smith model goes further by considering the problem, specific to fish resources, of interdependence between the growth rates of various species of fish. Also, they include a brief analysis of the case where capital stock, as well as the natural resource stock, enters as a state variable in the model. None of these problems will be considered in this dissertation.

A forthcoming paper by V. L. Smith provides a mathematical derivation of a production transformation curve similar to the one derived in Section A,1 of the next Chapter. This dissertation uses this curve as a locus of stationary equilibrium points towards which the market economy will gravitate in the long run. Smith continues by deriving a static production tax on resource product which will force the market economy to behave in a socially optimal fashion, once it has reached stationary equilibrium. Thus, it is entirely concerned with the question of how to keep the closed economy at an optimum once it has arrived there, and not with the problems involved in getting it to the optimum.

CHAPTER II

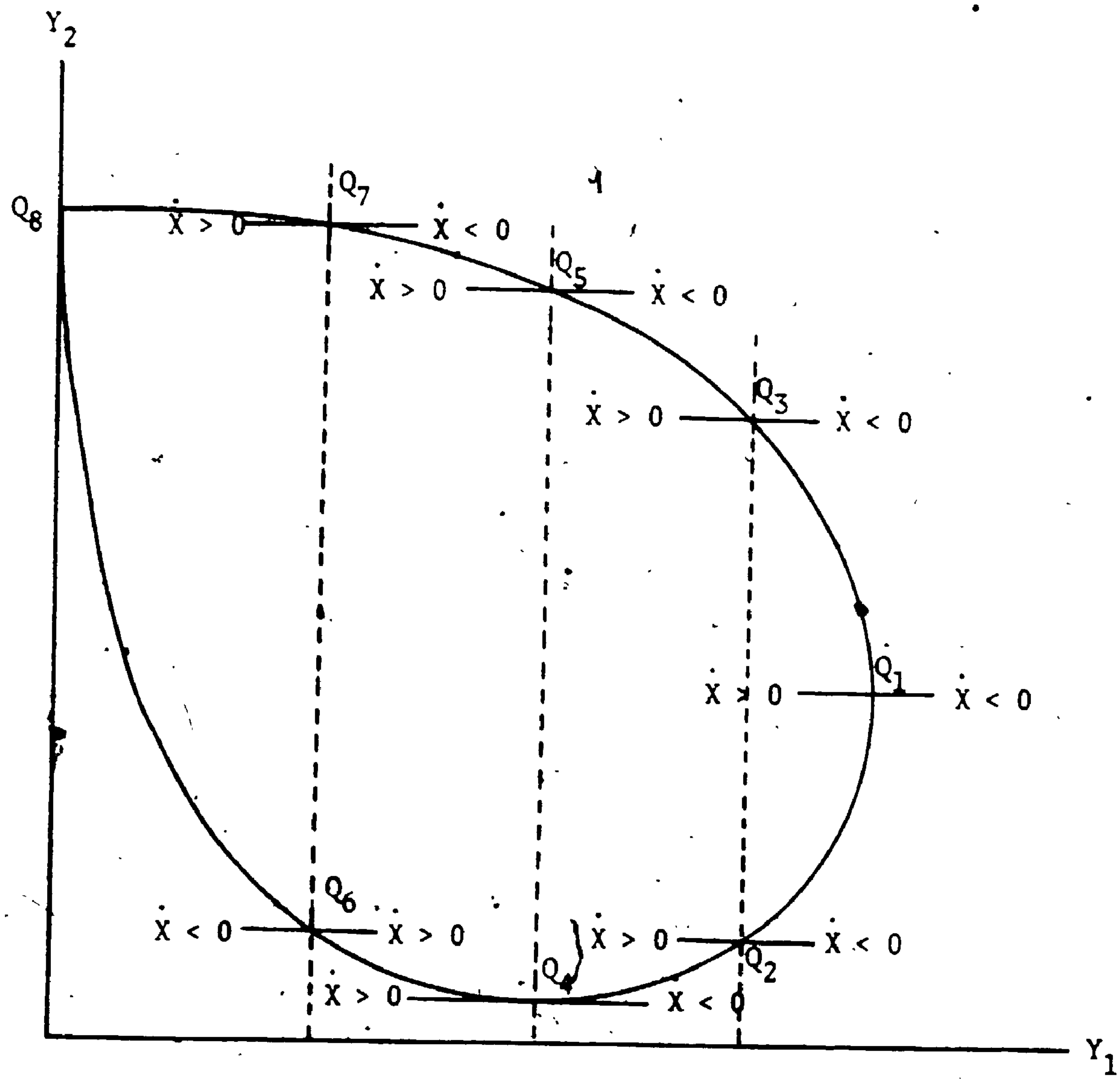
A. Derivation of the Transformation Curves

1. The Long Run

The long run will be defined as the period of time sufficiently long such that positive investment or disinvestment does not take place in the resource stock. Thus, the long run transformation curve is the locus of points in output space at which a stationary equilibrium occurs, and is defined by equations (1-3), (1-5), (1-6) and (1-8), with (1-8) set equal to zero. The long-run transformation curve resulting from these equations is given by Figure 3.

In order to gain an intuitive understanding of the properties of this transformation curve, consider the following geometric derivation.[1] Onto Figure 1 given in Section A of Chapter I, it is possible to plot the total product curve for production function (1-3), given various values of the labor input L_1. The point at which the total product curve intersects the unexploited growth function $\bar{N}(X)$ defines a resource product output level which is exactly equal to the gross

1. As mentioned in Chapter I, a similar curve, called the bionomic equilibrium transformation curve, has been derived mathematically by V.L. Smith (forthcoming). He does not, however, view it as a locus of stationary equilibrium points towards which the economy will gravitate, but as a socially correct transformation curve in a static analysis.



Y_2
Q_8
$\dot{X} > 0$
Q_7
$\dot{X} < 0$
$\dot{X} > 0$
Q_5
$\dot{X} < 0$
$\dot{X} > 0$
Q_3
$\dot{X} < 0$
$\dot{X} > 0$
Q_1
$\dot{X} < 0$
Q_6
$\dot{X} < 0$
$\dot{X} > 0$
$\dot{X} > 0$
$\dot{X} < 0$
$\dot{X} > 0$
Q_4
$\dot{X} < 0$
Q_2
Y_1

FIGURE 3

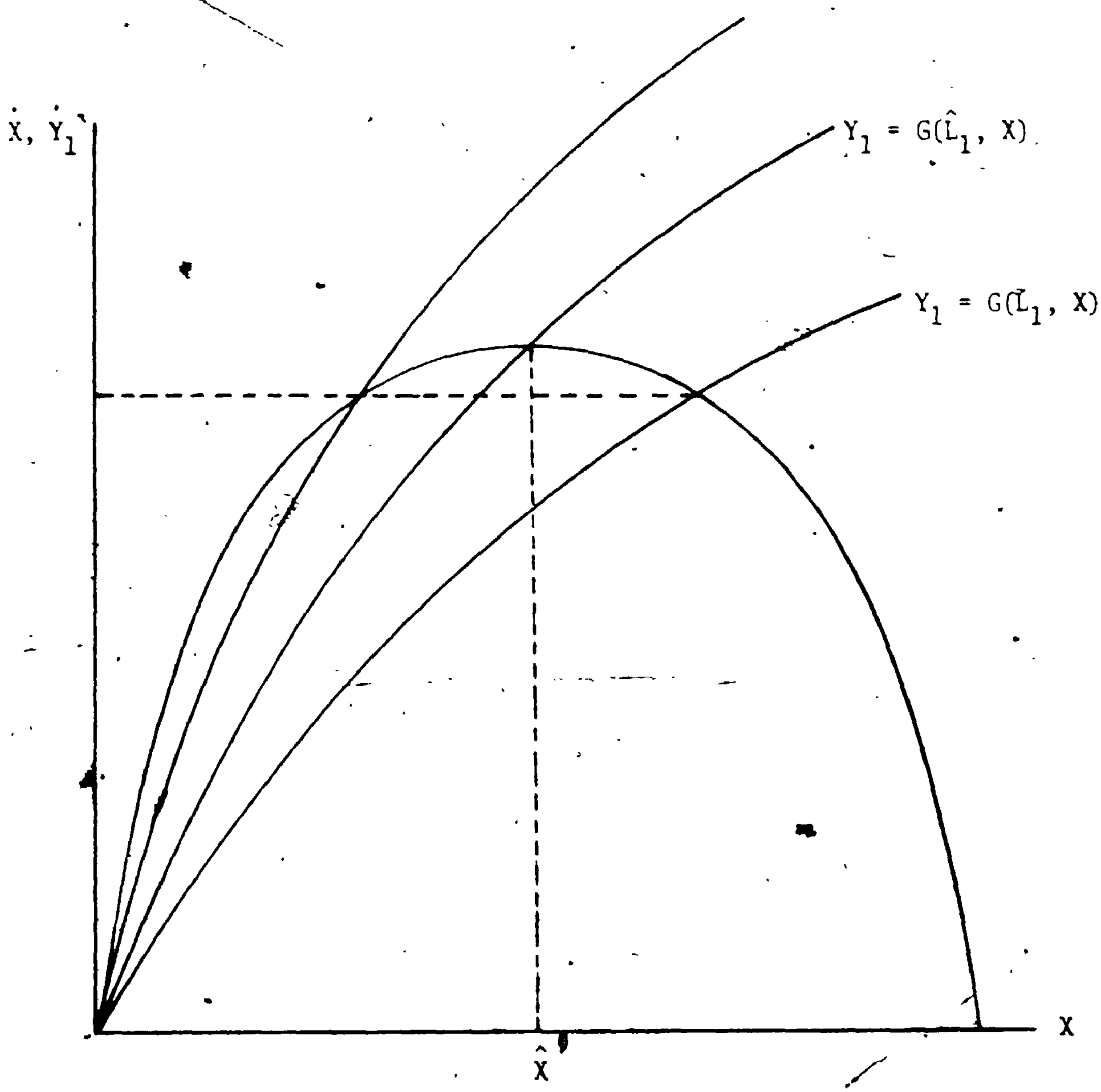


FIGURE 4

growth in the resource stock, thus leaving a stationary resource stock. For the unexploited resource stock growing at its maximum rate, a labor input $\hat{L}_1$ may be found such that the resource product produced matches exactly the growth in the resource, so that a stationary resource mass exists at $\hat{X}$. This situation is shown in Figure 4. At any point on this total resource product curve $G(\hat{L}_1, X)$ above the intersection point with $N(X)$, the produced resource product exceeds the growth in the resource stock giving positive net disinvestment ($\dot{X} < 0$) in nature's capital and thus, a shrinking resource stock. By similar analysis, any point on the resource product curve below the intersection point shows positive net investment ($\dot{X} > 0$) and thus, a growing resource stock. As mentioned earlier, this intersection point is the maximum sustained yield which has been much discussed in the fishing and forestry literature. With this known value of $\hat{L}_1$, it is possible to find L_2 by (1-6) and thus, to determine unique values for both Y_1 and Y_2. This point is shown as Q_1 in Figure 3. If for the Y_2 output level given by this point Q_1, the Y_1 output level is above that specified by Q_1, i.e., we are on the total resource product curve $G(\hat{L}_1, X)$ above the intersection point in Figure 4, there occurs positive disinvestment in the resource stock. For Y_1 output below that specified by Q_1, there exists positive net investment. These results are also shown in Figure 3.

If any other gross resource growth rate is chosen, there is no longer a unique resource mass at which a stationary equilibrium between the economic system and the biological system occurs. Thus, it is possible to obtain a stationary equilibrium by increasing L_1 from $\hat{L}_1$ or by decreasing it. Two possible points of resource mass equilibrium are then obtained, the first having more L_1 and a lower equilibrium resource stock than the second. By definition of a stationary equilibrium, and since the gross resource growth rate is the same, both equilibriums have the same Y_1 output level, lower than the one which generated point Q_1 in Figure 3. The result is that for this lower Y_1 value, points Q_2 and Q_3 are obtained in Figure 3.

Point Q_3 shows more Y_2 output than point Q_2, hence it must correspond to the stationary mass equilibrium point generated by the total resource product curve which uses the labor input $\tilde{L}_1$ in Figure 4. Since $\tilde{L}_1$ is less than $\hat{L}_1$, and since by (1-6) total labor supply is assumed to be fixed, the resulting larger available labor input in the manufactured good industry produces a Y_2 output greater than that obtained at Q_1. Point Q_2 is obviously generated by a total resource product curve which uses a labor input in excess of $\hat{L}_1$, thus leaving by (1-6) a lower available labor input for the manufactured good industry, and consequently , a lower Y_2 output level. Again, any point on either of these total resource product curves above its intersection point with N(X) implies $\dot{X} < 0$, and any point below the intersection point implies $\dot{X} > 0$. This net investment or disinvestment information is summarized on Figure 3 for the fixed Y_2 output levels around points Q_2 and Q_3. Finally, since point Q_3 is generated by a total resource product curve using fixed labor input $\tilde{L}_1$ less than $\hat{L}_1$, it corresponds to a stationary equilibrium which occurs on the $N' < 0$ side of the unexploited growth function. Obviously, point Q_2 corresponds to a stationary mass equilibrium which occurs with $N' > 0$.

The process can be repeated for successively lower gross resource growth rates, thus producing for each Y_1 level, two points on the transformation curve which diverge further and further from each other due to the fact that the stationary mass equilibriums are obtained by constantly increasing and decreasing the labor input from its previous level. For Y_1 levels on the newly generated total resource product curves above and below the intersection points, the motion of the resource stock is similar to that obtained at points Q_1, Q_2, Q_3. These results follow so long as the slope of the total resource product curve (G_2) exceeds the slope of the unexploited resource growth function (N') at points of stationary equilibrium.[2]

If, as lower and lower gross resource growth points are chosen in Figure 4, the stationary mass equilibriums occur with G_2 less than N', there results a change in the shape of the long run transformation curve. Now, for any gross resource growth rate, the two possible stationary equilibrium points will both be found on resource product curves obtained by lowering the L_1 input from its previous value. Thus, by (1-6), this leaves a higher labor supply input available to the manufactured good industry and hence, levels of Y_2 which both increase over their previous values. This is illustrated in Figure 3 by the fact that point Q_7 is above Q_5, and point Q_6 is above Q_4. Finally, point Q_7 shows a larger Y_2 output than Q_6, therefore it must be generated by the total resource product curve which uses the lower labor input value. Hence, point Q_7 corresponds to a stationary equilibrium which occurs on the $N' < 0$ side of the unexploited growth function. Point Q_6 is generated by the total resource product curve with the larger fixed labor input,

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2. In Smith's mathematical model, these are called respectively, the marginal technological productivity and the marginal biological productivity of the resource stock. See Smith (forthcoming).

thus giving intersection with N(X) on the N' > 0 side.

If the stationary mass equilibriums continue to be generated by total resource product curves for which G_2 is less than N' at the intersection point, the above analysis continues until the last possible gross growth rate is chosen. Now, for zero growth rate, the stationary mass equilibriums will occur at $\bar{X}$ and $\underline{X}$, and will obviously be produced by a total resource product curve which has zero labor input. Thus, by (1-6), all available labor goes into Y_2 production, generating point Q_8 in Figure 3. Point Q_4 is clearly the dividing point between stationary mass points which occur with $G_2 > N'$, i.e., distance $Q_8Q_1Q_4$, and points which occur with $G_2 < N'$, i.e. $Q_8Q_6Q_4$. The motion of the resource mass around this point Q_4 is derived in the same fashion as the other points.

The motion of the resource mass for stationary mass points on the distance $Q_8Q_6Q_4$ can be derived in the same way as for those points on the distance $Q_8Q_1Q_4$. As before, for each of the stationary equilibrium points in this distance, consider the resource product curve which generated the equilibrium. Now, with G_2 less than N' at the equilibrium point, Y_1 values on the product curve greater than the equilibrium value have positive investment in the resource stock. For Y_1 values less, i.e., below the intersection point with

$N(X)$, there occurs positive disinvestment and hence, a contracting resource stock. This information is summarized for disequilibrium Y_1 values around points Q_6 and Q_7 in Figure 3.

All points in the commodity space can be classified as to whether they have an increasing, decreasing or stationary resource mass. It is seen for any given Y_2 output level, that all points in the interior of set bounded by the long run transformation curve $Q_8Q_1Q_6Q_8$ show an increasing resource mass stock. Any point on the exterior of the set show a decreasing mass. Thus, it appears that stationary resource mass points along the distance $Q_8Q_6Q_4$ are unstable while those along the distance $Q_8Q_1Q_4$ are stable.

The long run transformation locus need not look exactly as portrayed in Figure 3, if stationary equilibria occurred with alternating values of G_2-N' as lower and lower gross growth rates are chosen. This is possible on the $N' > 0$ side of the unexploited growth function, if multiple equilibria occur between $G(L_1, X)$ and $N(X)$. The result would be that the long run transformation curve will have several positive and negative slopes along the distance $Q_1Q_4Q_8$ until point Q_8 is reached. Also, if all stationary resource mass equilibriums occur with $G_2 > N'$, points on the transformation curve will constantly diverge further and further from each other as lower Y_1 values are chosen. In the resulting transformation curve, the distance $Q_8Q_6Q_4$ would have a positive slope just like the distance Q_1Q_4.[3] Clearly, if all stationary equilibria on the $N' > 0$ side of $N(X)$ occur with $G_2 < N'$, distance Q_1Q_4 will have a negative slope just like the distance $Q_8Q_6Q_4$. Finally, the distance Q_1Q_4 may touch the Y_1 axis and "crawl" along it as lower and lower gross growth rates are chosen until point Q_4 is reached, if the total resource product curves which generate these equilibria use all the available labor supply. These are, however, merely special cases of the transformation curve drawn in Figure 3.

It is possible to summarize the information contained in the long run transformation curve as follows:

1) by definition of the long run, each point on the curve represents an output combination at which a stationary resource mass is obtained.

2) the output combinations which occur on the distance $Q_8Q_3Q_1$ are stationary equilibria from the $N' < 0$ side of the unexploited growth function. Point Q_1 corresponds to $N' = 0$.

3) output combinations which occur on the distance $Q_8Q_4Q_1$ are stationary equilibria from the $N' > 0$ side of $N(X)$

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3. Without deriving the transformation curve, Scott and Southey portray the curve with this shape. See A. Scott and C. Southey. (1969, p. 58).

4) the marginal technological productivity (G_2) exceeds the marginal biological productivity (N') of the resource stock at stationary equilibria for all output combinations on the distance $Q_8Q_1Q_4$. For distance $Q_8Q_6Q_4$, equilibrium resource stock values are obtained with $G_2 < N'$. Point Q_4 occurs when $G_2 = N'$.

2. The Short Run

For each of the short run periods which make up the long run, there exists a certain stock value of the raw natural resource. Thus, the transformation curve between the two outputs Y_1 and Y_2 is based upon equations (1-3), (1-5) and (1-6), where (1-8) has been dropped due to the definition of the short run. The resulting curve is the standard concave to the origin one of economic theory, drawn for a fixed and given value of the resource mass in Figure 5 as TT.[4]

A question of considerable importance for latter analysis is the comparative static result showing the effect on the short run transformation curve of an increase in the stock of the resource mass. Thus, from the equations determining

4. Appendix 1 gives a geometric derivation of this transformation curve.

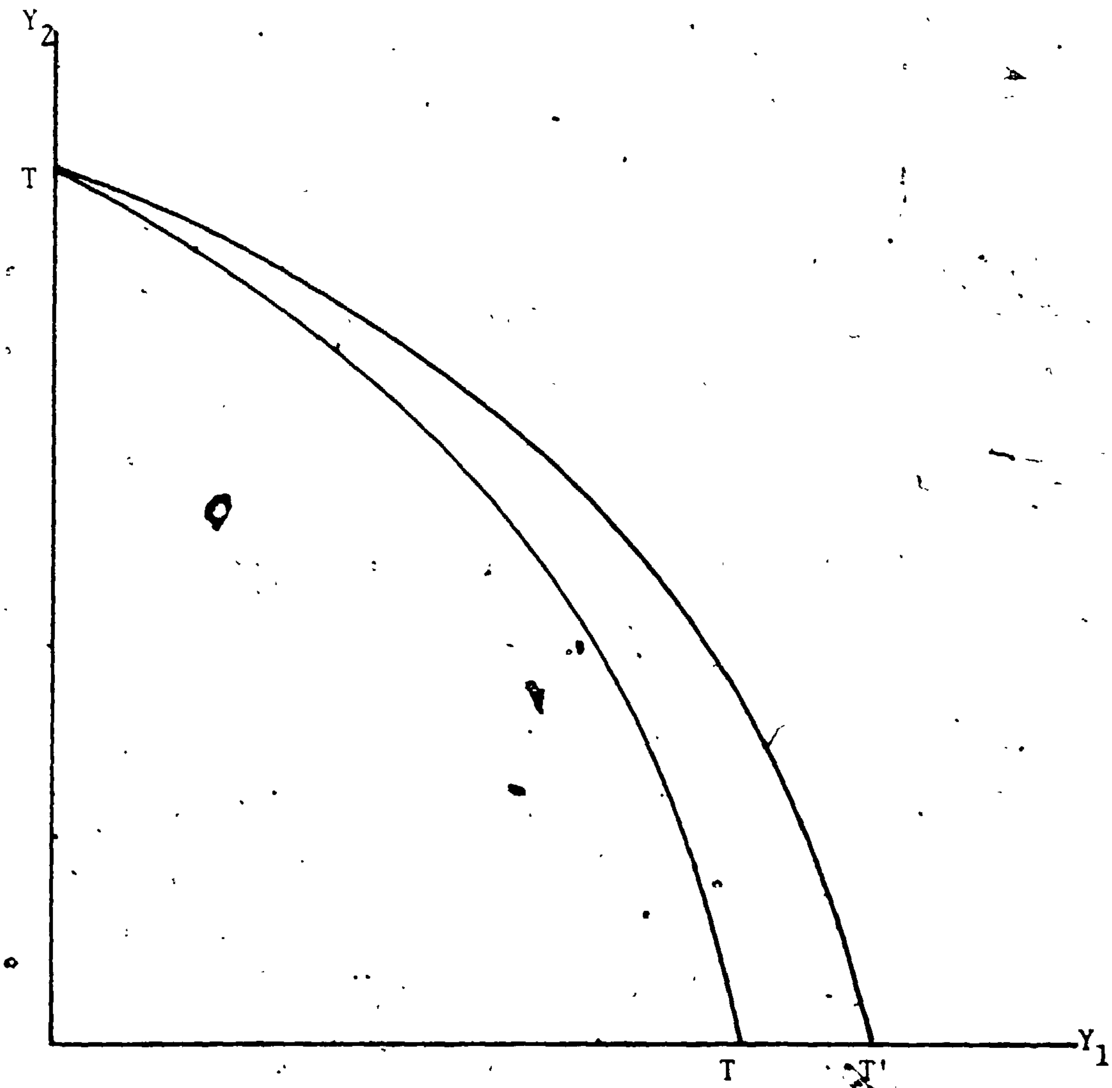

FIGURE 5

TT, an increase in X for a fixed level of Y_2 output produces the result.[5]

$$(2\text{-}1) \quad \left.\frac{dY_1}{dX}\right|_{Y_2 \text{ const}} = G_1\frac{dL_1}{dX} + G_2$$

With Y_2 output fixed, it follows from the labor constraint equation (1-6) that L_1 must also remain fixed. Thus the result (2-1) becomes

$$\left.\frac{dY_1}{dX}\right|_{Y_2 \text{ const}} = G_2$$

which must be positive.[6] This information is diagrammed in Figure 5 as the transformation curve TT' which shows that for each Y_2 output level, production of Y_1 increases when the X stock is increased.

The marginal rate of transformation for any one of these short run transformation curves may easily be found by totally differentiating (1-3), (1-5) and (1-6) to produce the result

$$(2\text{-}2) \quad \frac{dY_2}{dY_1} = \frac{-F'}{G_1}$$

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5. This result is also obtained by V.L. Smith (forthcoming).

6. Appendix 1 shows the geometry of this result.

B. Discussion

The intersection of the Home country's offer curve with the foreign country offer curve produces the equilibrium international price β, where β is the relative price of resource product in terms of the manufactured good. As discussed earlier, the price is such that the home country exports resource product in exchange for an imported manufactured good. With this information on the initial relative price of the resource product, and given the initial raw resource stock, a unique short run production point such as point K_1 in Figure 6 is obtained.[7]

It is known from the derivation and discussion of the long run transformation curve that all points in commodity space can be classified as to whether they have an increasing, decreasing or a stationary resource mass. Thus, for the production point K_1 in Figure 6, there occurs positive investment in the resource stock, because at this Y_2 output level the Y_1 production is less than the gross resource growth rate. The result is that in the next period there exists an increased available resource stock, from which resource product can be produced. By the result (2-1), the increased resource stock yields, for this Y_2 output, a larger amount of resource

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[7] See Footnote 1 in Chapter 3 and Appendix 4 for a discussion of the problems involved with the tangency between the price line and the transformation curve.

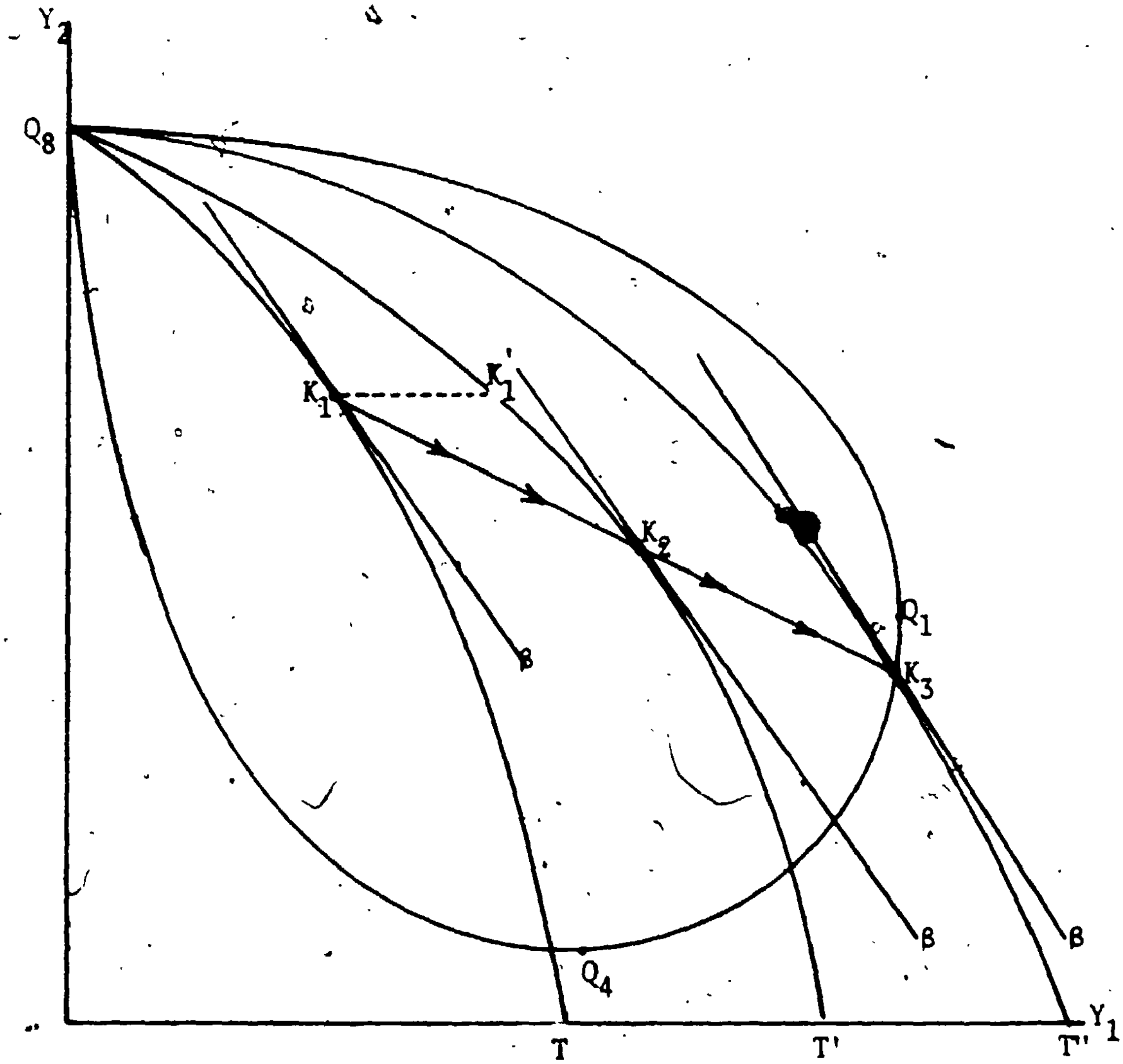

FIGURE 6

product K_1'.

Now for the special case of a linear foreign offer curve, the terms of trade β remain unchanged regardless of any change in the home country's export decision over time. This international price ratio produces the new unique production point K_2, which also shows a positive net investment in the resource stock and thus, a further expansion of the transformation curve in the next period. Positive investment in the resource stock will continue in this manner until the stable stationary resource mass point K_3 is reached at terms of trade β. Since the long run is simply a series of comparative static positions, the market economy on the basis of momentum generated by short run investment or disinvestment in the resourte stock produces a path $K_1K_2K_3$ to the long run stationary equilibrium point. Clearly, different initial resource stock and terms of trade values will produce other routes to stationary equilibrium. It seems that these different initial conditions will produce stable stationary equilibria which can occur anywhere on the distance $Q_8Q_1Q_4$.[8] The consumption points associated with the short run production points K_1, K_2 and K_3 are not

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8. While it is certainly possible that a stable equilibrium occur at the specialization point Q_8, corresponding to total depletion ($\underline{X}$), or total neglect of the resource mass ($\bar{X}$), this possibility has been ruled out in Chapter 1.

shown to avoid cluttering the diagram. Clearly, each consumption point will be determined where the prevailing international price ratio is just tangent to the highest community indifference curve. It is the difference between the consumption bundle and the production point which determines the size of the country's exports and imports.

For the more general case where the home country's export of resource product has some influence over the world price ratio, the series of comparative static production points is obtained in a similar manner. Now, depending whether this period's export decision is larger or smaller than the previous period's, the world price ratio β will change on route to the stationary equilibrium, producing a path different than the fixed terms of trade case shown in Figure 6. Regardless of whether or not the international price ratio is a parameter for the home country, the results are similar in the sense that given the initial conditions on resource stock and international relative prices, the market economy will generate a path which must lead to a stable stationary equilibrium. The values of Y_1 and Y_2 at this stationary point are the outputs produced in long run equilibrium.

A question of central importance to the dissertation is whether or not this path generated by the uncontrolled market process is socially optimal. To investigate the problem, the next Chapter will present the optimal version of the model, where it is assumed that allocation of the variable factor of production between the two industries, output and exports of resource product are directly and with certainty controlled by a central planning authority.

CHAPTER III

It was seen by the discussion surrounding Figure 6 in the last Chapter, that given the initial size of the resource stock and the international terms of trade, the market economy will produce a path which culminates in a stationary equilibrium resource stock. The questions raised at this juncture were the size of the stationary resource stock, i.e., the position of the stationary equilibrium on the long run transformation curve, and the social optimality of the market path produced. To investigate these questions, this Chapter will present an intertemporal general equilibrium model of the verbal and geometric discussion contained in Chapter II. Section A, by employing the Pontryagin Maximum Principle, will provide both a description of the socially optimal path to stationary equilibrium, and a discussion of the optimal steady state size of the resource stock. Section B will compare the results obtained in Section A for the socially optimal version of the model to the results which are derived for an uncontrolled economy composed of profit maximizing firms and utility maximizing households. Finally, this Section will investigate the policies which are available to create conditions such that the market economy obtains the same dynamic equilibrium path as the socially optimal version.

In regard to the question of the location of the steady state equilibrium, it will be seen by analyzing the sufficient conditions for stability that the equilibrium may occur anywhere on the unexploited resource function.

The intertemporal path to stationary equilibrium produced by the market economy will be shown to be non optimal for two reasons. The first is the production externality in the market solution caused by the common property nature of the raw resource stock. This produces a domestic distortion such that the relative price of the resource product facing consumers is not equal to the domestic marginal rate of transformation. In addition to this domestic distortion, there is also a foreign distortion due to the fact that the country possesses monopoly power to influence the international price of resource product. Thus, the marginal rate of transformation through trade will not be equal to the domestic marginal rate of substitution, due to the fact that the individual exporting units trade on the basis of the average not marginal terms of trade. In the special case where the home country lacks this influence over the determination of the world price ratio due to its position as a small exporter in a large world, the marginal terms of trade will equal the average and this foreign distortion disappears. In summary, the country is faced by a situation where the domestic marginal rate of transformation does not equal the domestic marginal rate of substitution, which in turn does not equal the foreign marginal rate of transformation. To correct these two distortions, the policy tools of a domestic tax and an export tariff can be applied. The domestic tax on firm's use of the raw natural resource mass will internalize the externality caused by the common property nature of the resource, thus giving equality between the marginal rate of substitution and the domestic marginal rate of transformation. An export tariff will restrict production of resource product to the point where the marginal rate of substitution is equal to the marginal foreign rate of transformation. The size of the required tax/tariff combination to force the market determined intertemporal path toward stationary equilibrium to be identical with the optimum path will be derived and discussed.

From Chapter I, the functions summarizing the production side of the domestic economy are given by equations (1-1) through (1-6). In this Chapter, it is necessary to assume that the production functions $G(\cdot)$ and $F(\cdot)$ are both identically homogeneous to a degree less than unity.[1] Trade with the rest of the world occurs along the foreign offer curve (1-7), and interaction between the economic and biological systems is given by (1-8). For both the optimal and market models to be developed in this Chapter, the objective is to maximize a welfare function $W(C_1, C_2)$, where welfare is defined by the discounted, infinite horizon utility integral

$$(3\text{-}1) \qquad W(C_1, C_2) = \int_0^\infty [U(C_1) + V(C_2)]e^{-\rho t}dt$$

where ρ is a constant positive social rate of discount employed to exponentially discount future utility, and C_1 and C_2 are the home country's consumption of sector one and sector two goods respectively. Following Plourde, the function has been assumed separable to obtain simplicity in analyzing the properties of the model. Finally, it is also assumed that the integral (3-1) converges, so that evaluation of alternative

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1 A sufficient condition required for stability later in this Chapter is that the production function $G(\cdot)$ be homogeneous to a degree less than unity. No conditions other than homogeneity are required for the technology $F(\cdot)$. However, H. Herberg and M. C. Kemp (1969, p. 413-414) have noted that when returns to scale differ between the two industries, the price line will cut the production possibility curve at the equilibrium point. To avoid the geometric difficulty this creates, it is assumed that the technology $F(\cdot)$ shows the identical degree of decreasing returns to scale. The assumption is entirely one of convenience, as the argument follows for equilibrium points characterized by non tangency between the price line and the production possibility curve. Herberg and Kemp's results are obtained on the basis of production functions with two arguments, but their proof can be applied to the production functions of this dissertation as shown in Appendix 4.

consumption programs is possible. The instantaneous utility functions $U(C_1)$ and $V(C_2)$ possess the following properties:

1) $U'(C_1) \to \infty$ as $C_1 \to 0$ and $V'(C_2) \to \infty$ as $C_2 \to 0$.

This assumption serves to keep utility positive.

2) $U'(C_1), V'(C_2) > 0$

3) $U''(C_1), V''(C_2) < 0$

Since it has been specified that for all terms of trade the trading pattern is such that the home country exports resource product and imports manufactured goods from the rest of the world, domestic consumption of resource product is given by

$$(3\text{-}2) \quad C_1 = Y_1 - Z_1$$

where the export of resource product (Z_1) is non negative. Likewise, the fact that the home country imports the non negative quantity (Z_2) of manufactured goods gives the definition

$$(3\text{-}3) \quad C_2 = Y_2 + Z_2$$

A. The Dynamic General Equilibrium Model: Optimal Version

1. The Necessary Conditions

Faced with the functions summarizing the economic and biological systems, plus the function showing how exports are transformed into imports through international trade, the planner must choose the time path for the control variables which will maximize social welfare. To be as general as possible, the production functions (1-3) and (1-5) for the two sectors can be thought of as weak inequalities. Thus, the amount of resource product harvested or cut is less than or equal to the amount technically available. The same applies for the manufactured output Y_2.

The controls available to the planning authority for the task of maximizing social welfare are the output levels Y_1 and Y_2, the variable factors of production L_1 and L_2, and Z_1, the amount of Y_1 production which is exported. From relations (3-2) and (3-3), it is clear that C_1 and C_2 may be used as controls instead of Y_1 and Y_2, where Z_1 determines a unique value of Z_2 by relation (1-7).

Mathematically, the problem is to maximize the criterion function

$$(3\text{-}1) \qquad \int_0^{\infty} [U(C_1) + V(C_2)] e^{-\rho t} dt$$

subject to the condition

$$(1\text{-}8) \qquad \dot{X} = N(X) - G(L_1, X)$$

the constraints

$$(3\text{-}4) \qquad G(L_1, X) - C_1 - Z_1 \geq 0$$

(3-5) $F(L_2) - C_2 + \Theta(Z_1) \geq 0$

(1-6) $\bar{L} - L_1 - L_2 = 0$

and the initial condition of the state variable

(3-6) $X = X_0$

The constraints (3-4) and (3-5) have been obtained by substituting (3-2) into the resource product production function to yield (3-4), and (3-3) and (1-7) into the manufactured good production function, to yield (3-5).

In this format, the problem can be solved by application of the Pontryagin Maximum Principle.[2] This Principle states that for choice of instruments $C_1^*(t)$, $C_2^*(t)$, $L_1^*(t)$, $L_2^*(t)$, $Z_1^*(t)$ which maximize (3-1), subject to an initial condition on the state variable (3-6) and equations (1-8), (3-4), (3-5), and (1-6) listed above, there exists an auxiliary variable P(t) and Lagrangian multipliers $q_1(t)$, $q_2(t)$ and $q_3(t)$ such that the instruments chosen will instantaneously maximize the current valued Hamiltonian subject to the constraints (3-4), (3-5) and (1-6). The current valued Hamiltonian is given by[3]

$$H = U(C_1) + V(C_2) + P[N(X) - G(L_1, X)]$$

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2. See K.J. Arrow and M. Kurz (1971, p.26-57)

3. See K.J. Arrow and M. Kurz (1971, p.47).

where the auxiliary variable P may be interpreted[4] as the rate at which the optimal value of the objective functional (3-1) changes when there is a small change in the raw resource mass from its initial value X_o. As such, P is the shadow price of the resource stock measured in terms of the present value of future utility foregone. It will be remembered that the utility derived from X comes from its ability to produce directly the consumption good C_1, and indirectly, the consumption good C_2 through trade with the rest of the world.

To interpret the term $P[N(X) - G(L_1,X)]$ in the Hamiltonian function, recall from the discussion of Chapter II that positive net investment is said to take place in nature's capital when $N(X) - G(L_1,X) > 0$, and positive net disinvestment when $N(X) - G(L_1,X) < 0$. Thus, the term $P[N(X) - G(L_1,X)]$ represents the present value of the increased potential to produce future utility when investment is positive, and is the present value of lost future utility when disinvestment in the resource stock takes place. The whole Hamiltonian H is the present value of total utility, composed of the direct effects contained in the instantaneous utility functions U and V, plus

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4. See M.D. Intriligator (1971, p.352). For a different interpretation of the auxiliary variable as an upper bound on the average rate of change of the objective functional when the initial state is allowed to change slightly, see D.W. Peterson (1973, p. 234-243).

the positive or negative indirect effects through changes in the size of the resource mass, where the resource mass changes due to investment or disinvestment.

In order to determine the necessary conditions for instantaneous maximization of the current value Hamiltonian subject to the constraints (3-4), (3-5) and (1-6), form the Lagrangian expression

$$L = H + q_1[G(L_1,X) - C_1 - Z_1] + q_2[F(L_2) - C_2 + \theta(Z_1)]$$
$$+ q_3[\bar{L} - L_1 - L_2]$$

and set the partial derivatives of the Lagrangian with respect to the instruments equal to zero. For an interior solution, evaluated at the optimal values of the controls, this produces the necessary conditions,

(3-7) $U'(C_1) = q_1$

(3-8) $V'(C_2) = q_2$

(3-9) $(q_1 - P)G_1 = q_2F' = q_3$

(3-10) $\frac{q_1}{q_2} = \theta'$

where all results have been rearranged to emphasize their economic content.

The Lagrangian multipliers q_1, q_2 and q_3 can be thought of as the change in the Lagrangian function from its optimal value with respect to a small relaxation in each of the constraints (3-4), (3-5) and (1-6) respectively. Thus q_1 and q_2 are the marginal evaluations or prices placed on outputs Y_1 and Y_2, while the multiplier q_3 is the implicit wage or rental rate for the variable factor of production labor. Conditions (3-7) and (3-8) state that the instantaneous marginal utility for each good should be equated to its price, while (3-9) states that the value of the marginal product in industry two equal the value of the marginal product in industry one, equal the implicit return to labor. Note that the implicit price of output in industry two is $(q_1 - P)$, thus, we are subtracting the cost of using nature's resource mass in determining the imputed price of a unit of Y_2. Reflecting the fact that we are dealing with an open economy which is free to exchange Y_1 for Y_2, equation (3-10) states that the implicit relative price of resource product equals the marginal rate at which resource product can be transformed into the manufactured good by means of trade with the rest of the world.

The Maximum Principle for infinite horizon discounted optimization problems also states[5] that for the optimal program

5. See K.J. Arrow and M. Kurz (1971, p. 48)

$C_1^*(t)$, $C_2^*(t)$, $L_1^*(t)$, $L_2^*(t)$, $Z_1^*(t)$, the auxiliary variable P(t) will have a continuous time profile given by

$$(3\text{-}11) \quad \dot{P} = \rho P - \frac{\partial L}{\partial X}\Big|_* = P[\rho - N' + G_2] - q_1 G_2$$

Together, the system of equations (1-8) and (3-7) through (3-11), plus the initial condition (3-6) on the state variable, form a complete dynamic system. The problem is to determine an initial value for the auxiliarly variable P, which when combined with the initial value on the state variable, produces the result that the time paths of (1-8) and (3-11) converge to stationary values.

2. Derivation of $L_1 = f(X,P)$ and $Z_1 = h(X,P)$

The problems in analyzing the steady state properties of the differential equations (1-8) and (3-11) can be seen by substituting the definition (3-2) into the necessary condition (3-7) to produce the result

$$(3\text{-}12) \quad q_1 = U'[G(L_1,X) - Z_1]$$

for the binding case where $Y_1 = G(L_1,X)$. By a similar substitution of definition (3-3) into the necessary condition (3-8), the result

(3-13) $q_2 = V'[F(L_2) + \Theta(Z_1)]$

is obtained, again for the case where $Y_2 = F(L_2)$.

Substituting these results into the differential equations (3-11) and (1-8), it is obvious that we have a system of two differential equations in five variables X, P, L_1, L_2 and Z_1. This number can be reduced to four by making use of the full employment equation (1-6) so that $L_2 = \bar{L} - L_1$. However, we desire that the number of variables be reduced to two in order to solve the system.

To make this reduction in the number of variables, substitute into the necessary condition (3-9) the results (3-12) and (3-13). After rearranging, this produces

$$(3\text{-}14) \quad \left\{U'[G(L_1,X) - Z_1] - P\right\} G_1(L_1,X) - V'[F(\bar{L} - L_1) + \Theta(Z_1)]\, F'(\bar{L} - L_1) = 0$$

Thus, we have an implicit relationship in the variables L_1, Z_1, X, P. Define $D(L_1, Z_1, X, P) = 0$ as this implicit function (3-14).

Now, from necessary condition (3-10), substitute the results (3-12) and (3-13) to obtain a second implicit relationship in L_1, Z_1 and X. Again rearranging, this produces

(3-15) $U'[G(L_1,X) - Z_1] - V'[F(\bar{L} - L_1) + \Theta(Z_1)]\ \Theta'(Z_1) = 0$

Define $B(L_1, Z_1, X) = 0$ as this second implicit relationship (3-15).

Given these two implicit relationships, it is possible to obtain

(3-16) $L_1 = f(X,P)$

and

(3-17) $Z_1 = h(X,P)$

where f and h are well defined, differentiable functions provided that the Jacobian determinant

$$J = \begin{vmatrix} D_1 & D_2 \\ B_1 & B_2 \end{vmatrix}$$

evaluated at the optimal values of the controls does not vanish. From the definitions of D and B given above, we may obtain the following results:

$$D_1 = U''G_1G_1 + (U'-P)\ G_{11} + V''F'F' + F''V'$$

$$D_2 = -U''G_1 - V''\Theta'F'$$

$$B_1 = U''G_1\Theta' + V''F'$$

$$B_2 = -U'' - V''\Theta'\Theta' - \Theta''V'$$

Given the specifications of functions in Chapter I, and the first part of this Chapter II, we have U', V', G_1, F' and Θ' all positive, while U", V", G_{11}, F" and Θ" are negative. As discussed earlier in Section A.1 of this Chapter, the magnitude $(q_1 - P)$ is simply the implicit price of resource product which must be positive if the output is not a free good. Since in equilibrium the necessary result $U'(C_1) = q_1$ holds, it may be stated that (U' - P) must also be positive.

With these specifications, the following signs are easily obtained:

$$D_1 < 0; \quad D_2 > 0; \quad B_1 < 0; \quad B_2 > 0$$

These signs produce the result that the Jacobian determinant cannot be signed unambiguously positive or negative. It must be assumed not to vanish in order that we are able to write the results (3-16) and (3-17).

3. Signs of f_1, f_2, h_1 and h_2

It is important for latter analysis to have information on the signs of the partial derivatives of f and h given in equations (3-16) and (3-17). From the Implicit Function Theorem[6], it is known that $\frac{\partial L_1}{\partial P}$, or f_2 will be given by

$$f_2 = \frac{-\begin{vmatrix} D_4 & D_2 \\ B_4 & B_2 \end{vmatrix}}{J}$$

From the definitions of the implicit functions, we may obtain the results

$$D_4 = -G_1$$

$$B_4 = 0$$

These signs produce the result that $f_2 < 0$ when $J < 0$, and $f_2 > 0$ when $J > 0$.

Turning now to the explicit relation given by (3-17), it is possible to obtain $\frac{\partial Z_1}{\partial P}$, or h_2, by

$$h_2 = \frac{-\begin{vmatrix} D_1 & D_4 \\ B_1 & B_4 \end{vmatrix}}{J}$$

With the information on the signs of these partial derivatives, it is possible to make the statement that $h_2 < 0$ when $J < 0$,

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6. W. Kaplan (1959, p. 90-96).

and $h_2 > 0$ when $J > 0$. Thus, both f_2 and h_2 are negative when J is negative, and positive for positive J.

This dissertation makes the assumption that f_2 is negative. See Appendix 2 for the implications of this assumption. Since a necessary and sufficient condition for negative f_2 is that $J < 0$, this result on the sign of the Jacobian may be used in determining the sign of h_2 as negative as well as the signs of $\frac{\partial L_1}{\partial X}$, or f_1 and $\frac{\partial Z_1}{\partial X}$, or h_1.

The sign of the partial derivative f_1 will be given by

$$f_1 = - \frac{\begin{vmatrix} D_3 & D_2 \\ B_3 & B_2 \end{vmatrix}}{J} .$$

Again, from the definitions of the implicit functions (3-14) and (3-15), it is easy to find

$$D_3 = U''G_2G_1 + (U' - P)G_{12}$$

$$B_3 = U''G_2$$

Following C. G. Plourde and V. L. Smith[7], we make the assumption that $G_{12} > 0$ and hence, obtain the results that $B_3 < 0$ and

7. C. G. Plourde (1971, p. 259) and V. L. Smith (forthcoming. This assumption states that the factors are complimentary in production.

$D_3 \gtreqless 0$. With these results, the sign of f_1 will be indeterminate due to the fact the D_3 cannot be signed.

As seen in Appendix 3, the result D_3 can be signed only by making specific assumptions regarding the relative desirability of the two consumption goods and the substitutability of the factors L_1 and X in production. The relative desirability of C_1 and C_2 is expressed by the elasticity of marginal utility for the resource product, and the elasticity of foregone marginal utility created in the instantaneous utility function U by exporting some of the resource product production instead of consuming it domestically. The measure of substitutability between the factors is captured by the elasticity of factor substitution .

Given this inability to sign D_3 in general, the strongest statement that can be made when it is remembered that $J < 0$, is that $f_1 > 0$ when $D_3 \geq 0$, and $f_1 \gtreqless 0$ when $D_3 < 0$.[8]

Finally, it is possible to obtain $\frac{\partial Z_1}{\partial X}$, or h_1 as

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8. The result that $f_1 = 0$ produces the situation discussed in Chapter II where Y_2 output remains constant as the resource stock increases. In this Chapter, concerned with more general results, the case of $f_1 = 0$ will be neglected.

$$h_1 = \frac{-\begin{vmatrix} D_1 & D_3 \\ B_1 & B_3 \end{vmatrix}}{J}$$

Again using the results from Appendix 2 that $J < 0$, it is possible to state that $h_1 > 0$ when $D_3 \geqq 0$, and $h_1 \gtrless 0$ when $D_3 < 0$.[9] The situation regarding the sign of h_1 is identical to those obtained for f_1, i.e., determinate only when $D_3 \geqq 0$ and indeterminate when $D_3 < 0$.

In summary, for the optimal program $C_1^*(t)$, $C_2^*(t)$, $L_1^*(t)$, $L_2^*(t)$, $Z_1^*(t)$ it was seen that both f_2 and h_2 are unambiguously negative. So as the social price or rental on the natural resource mass X increases, less of the variable factor of production L should be allocated to the natural resource product industry. This implies a decrease in Y_1 production and an increase in Y_2 output as labor, which is always fully employed, leaves the resource product industry and is employed in the manufactured goods sector. From the result that h_2 is also negative, it is seen that optimality requires a reduction in exports of resource product as the

9. The case of $h_1 = 0$ is also neglected.

social price increases. Thus, the home country's response to an increase in the social price of nature's capital is to save the resource for future domestic use by reducing production and export of resource product. Domestic consumption of resource product is protected by the simultaneous reduction in exports along with the reduction in production, although it is impossible to say whether the absolute size of domestic consumption will increase or decrease, because the relative sizes of these two effects are unknown. With the reduced export volume, there occurs a reduced volume of imports from the rest of the world. When it is remembered that domestic production of Y_2 is increased due to the transfer of L from the resource product sector to the manufactured good sector, it is clear that the percentage share of imports in sector two consumption has fallen, due to this result combined with the absolute decline in imports. Again, the absolute size of C_2 is unknown due to the unknown sizes of these offsetting effects. Optimal allocation through time thus requires that the economy become more self sufficient, and use less of the raw natural resource as the social price of X increases. These results are accomplished, while protecting the domestic consumption levels, by moving labor out of the Y_1 industry and into the Y_2 industry, and by reducing the level of exports.

For the results on the signs of f_1 and h_1, it was seen that both are positive when $D_3 \geqq 0$. Then as the resource mass X increases, more L_1 should be used in the production of resource product, and more resource product should be exported. Thus, production of Y_1 is increased due to the joint effects of a larger resource stock and a higher level of labor input. This implies, due to the fact that the labor is in fixed supply, a decrease in the amount of Y_2 produced domestically. However, the increased exports and resultant higher level of imports result in the replacement of domestically produced manufactured goods by imported ones. Due to the unknown sizes of these two effects, it is not known whether the absolute size of C_2 increases or decreases. The same is true for C_1 which is determined by the offsetting effects of a higher Y_1 production, but a higher level of exports. Optimal allocation through time for this case of $D_3 \geqq 0$ requires that the economy move toward specialization in the production of resource product, thus gaining a larger share of C_2 through imports.

Unfortunately, as shown in Appendix 3, without specification of the instantaneous utility functions U and V and the production function for resource product, it is impossible to be sure that $D_3 \geqq 0$. If $D_3 < 0$, it is possible that $f_1 \gtrless 0$ and $h_1 \gtrless 0$, thus giving three further cases of possible optimal behavior by the open economy.

The first is that $f_1 > 0$ while $h_1 < 0$. Now, as the resource stock increases, more L should be allocated to Y_1 production. When combined with the higher X stock value, this increased labor input produces a higher resource product output. With a fixed labor supply, the transfer of labor results in a fall in domestic Y_2 production. Simultaneous with these effects, there occurs a fall in exports and a resultant decline in imports to give the result that domestic consumption of resource product increases, while domestic consumption of the manufactured good declines.

A second possible situation is that both f_1 and h_1 are negative. Briefly, this implies that as the stock of the raw natural resource increases, less labor should be allocated to Y_1 production, thereby giving an increase in Y_2 output. Since X increases while L_1 decreases, it is unknown what happens to resource product output. With the information that exports and hence imports are reduced, it is seen that the absolute size of C_2 cannot be determined. The absolute size of C_1 is also unknown, due to the fact that the Y_1 production point is unspecified.

The last possible case is that $f_1 < 0$, while $h_1 > 0$. Now, along the optimal path, Y_2 output increases and it is unknown what happens to Y_1 production. To find the consumption points, note that exports increase producing an increase in imports, and hence, an increase in the absolute size of C_2. As in the situation above, it is impossible to say what happens to C_1.

These results on the optimal behavior of the open economy as the stock of the resource increases will be of considerable importance in the following Chapter. The next few Sections of this Chapter will be concerned with finding the steady state solution to the system.

4. Slope of $\dot{P} = 0$ and $\dot{X} = 0$ in Phase Space

The differential equations (1-8) and (3-11), plus the given initial condition (3-6) on the state variable, show how X and P vary optimally over time. To find a steady state solution for this system, we will obtain the slope of the curve of solutions for $\dot{P} = 0$ and $\dot{X} = 0$ in phase space (P,X).

Into the differential equations (1-8) and (3-11), we can substitute the explicit relation for L_1 and Z_1 given by (3-16) and (3-17) to obtain

$$(3\text{-}18) \qquad \dot{X} = N(X) - G[f(X,P),X]$$

$$(3\text{-}19) \qquad \dot{P} = P\left\{\rho - N'(X) + G_2[f(X,P),X]\right\} - U'\left\{G[f(X,P),X] - h(X,P)\right\} G_2[f(X,P),X]$$

where result (3-12) defining q_1 has been substituted into the differential equation (3-11).

Consider first the slope of the curve defined by $\dot{P} = 0$, i.e., dP/dX for $\dot{P} = 0$. Define the implicit function $R(X,P) = 0$ as relation (3-19) when $\dot{P} = 0$. To find dP/dX by the Implicit Function Theorem, we require that $R_p \neq 0$. From the definition of $R(X,P)$, it is possible to obtain

$$(3\text{-}20) \qquad R_p = \rho - N' + (P - U')G_{21}f_2 - G_2[U''G_1f_2 - U''h_2 - 1]$$

To determine the sign of R_p requires that we determine the signs of three separate terms.

a) <u>Sign $\rho - N'$</u>

To obtain the sign of this term when we are on the curve $\dot{P} = 0$, write equation (3-11) as

$$q_1G_2 = P[\rho - N' + G_2]$$

Divide by P to obtain

$$G_2\left[\frac{q_1 - P}{P}\right] = \rho - N'$$

From the discussion in Section A.2, we know that $(q_1 - P)$, the price placed on output Y_1, is positive. This follows from the result obtained by C.G. Plourde[10] that as long as there are production costs, the value placed on a unit of resource product (q_1) exceeds the value of a unit of unharvested resource stock (P). Then for resource stocks for which P is positive, it must follow that $\rho - N' > 0$, where $dX/dX = N'$ is the percent rate of growth in the unharvested resource stock, or the marginal biological productivity.[11] In preparation for the discussion of the next Chapter, which is concerned with the location of the stationary equilibrium on the long run transformation curve, note that this result does not preclude steady state value of X which occur on the $N' > 0$ side of the gross resource growth function unless $\rho = 0$.

b) Sign $(P-U')G_{21}f_2$

From the necessary condition (3-7), the result that $U' = q_1$ in equilibrium produces $(U'-P) > 0$, as discussed above. Thus, it follows that $(P-U') < 0$. Also, Section A.3

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10. C.G. Plourde (1971, p.262)

11. J.P. Quirk and V.L. Smith (1969, p.12) also obtain this result. They note that a necessary and sufficient condition for $P \geq 0$ is $\rho-N' > 0$.

made the assumption that the factors L_1 and X are complimentary in production, i.e., $G_{12} = G_{21} > 0$. Finally, for the result that $J < 0$, it was demonstrated that $f_2 < 0$. The result is that $(P-U')G_{21}f_2$ is positive.

c) Sign $U''G_1f_2 - U''h_2 - 1$

From Section A.3, we have that

$$f_2 = \frac{-[D_4B_2 - B_4D_2]}{J}$$

and

$$h_2 = \frac{-[D_1B_4 - B_1D_4]}{J}.$$

Substituting these into the expression $U''G_1f_2 - U''h_2 - 1$ will yield

$$(3\text{-}21) \quad \frac{-U''G_1[D_4B_2 - B_4D_2] + U''[D_1B_4 - B_1D]}{J} - 1$$

Noting that Section A.3 gives $D_4 = -G_1$ and $B_4 = 0$, we can rewrite (3-21) as

$$\frac{U''G_1G_1B_2 + U''B_1G_1}{J} - 1$$

We desire this expression to be negative. Since $J = D_1B_2 - B_1D_2 < 0$, this amounts to showing that

$$U''G_1G_1B_2 + U''B_1G_1 > D_1B_2 - B_1D_2$$

or

$$B_2[U''G_1G_1 - D_1] > B_1[-U''G_1 - D_2]$$

From the results obtained in Section A.2, we have

$$D_1 < U''G_1G_1$$

and

$$D_2 > -U''G_1$$

Recalling that $B_1 < 0$ and $B_2 > 0$, it is easily seen that both sides are positive and it is impossible to state that (3-21) is negative. Manipulation of the above provides no easily interpretable results. In order to sign R_p, it appears necessary to assume that

$$\frac{[U''G_1G_1 - D_1]}{[-U''G_1 - D_2]} < \frac{B_1}{B_2}$$

With this assumption, the whole term $U''G_2f_2 - U''h_2-1$ is negative. When multiplied by $-G_2$, the term in (3-20) becomes positive.

With this information on signs contained in the sub Sections (a), (b) and (c), it is possible to make the statement that $R_P > 0$, and hence, we may employ the Implicit Function Theorem to find dP/dX for $\dot{P} = 0$ as $-R_X/R_P$.

From the definition of $R(X,P)$ given in this Section, we can obtain

$$(3\text{-}22) \quad R_X = -PN'' + P[G_{21}f_1 + G_{22}] - U''G_2[G_1f_1 + G_2 - h_1]$$
$$-U'[G_{21}f_1 + G_{22}]$$

The sign of R_X depends upon the sign of f_1 and h_1 which were shown in Section A.3 to be ambiguous. There it was demonstrated that both f_1 and h_1 are positive when $D_2 \geq 0$ and indeterminate in sign for $D_3 < 0$. The four possible situations discussed in that Section are summarized here as:

1) $f_1 < 0$; $h_1 < 0$

2) $f_1 < 0$; $h_1 > 0$

3) $f_1 > 0$; $h_1 < 0$

4) $f_1 > 0$; $h_1 > 0$

Consider the possibility given by situation (1). Rearrange (3-22) into

$$R_X = -PN'' + [P-U'][G_{21}f_1+G_{22}] + U''G_2h_1 - U''G_2[G_1f_1+G_2] \qquad (3\text{-}23)$$

From the specification of functions given in Chapter I, we know that $N'' < 0$, $G_{22} < 0$ and $U'' < 0$. Combining this information with the result from this Section that $P > 0$ and $(P-U') < 0$, we may state that for situation (1) a sufficient condition for $R_X > 0$ is that $G_1f_1 + G_2 \geqq 0$. When it is remembered that $Y_1 = G[f(X,P),X]$, the total derivative with respect to X for fixed P is

$$\frac{\partial Y_1}{\partial X} = G_1f_1 + G_2 \geqq 0 \qquad (3\text{-}24)$$

The result obtained here is simply the general equilibrium equivalent of the result (2-1) obtained when output Y_2 is held constant. Now, along the optimal path where Y_2 output is free to vary as shown in Figure 6 of the last Chapter, the relationship between the resource stock and L_1 is no longer zero. This is demonstrated by the discussion concerning the sign of f_1 in Section A.3 of this Chapter. The sufficient condition being imposed on the optimal path is that as the resource stock increases, the economy's harvest of resource product must also increase, after adjusting L_1 to the new resource stock value, i.e., allowing Y_2 to change.

Now consider the sign pattern given by situation (2). Rearrange (3-22) once again, this time into

(3-25) $$R_X = -PN' + [P-U'] [G_{21}f_1 + G_{22}] - U'G_2[G_1f_1 + G_2-h_1]$$

Now a sufficient condition for $R_X > 0$ is that $G_1f_1 + G_2 - h_1 \geq 0$, or $\partial Y_1/\partial X \geq h_1$. In other words, as X increases in a situation where the home country is going to export more Y_1, it is no longer sufficient that the harvest of resource product increase. Now the Y_1 output must increase enough so that domestic consumption of resource product does not decline. Hence, we are imposing the condition that consumption of resource product cannot be socially inferior with respect to X. This same result is expressed by the weaker condition (3-24) when less Y_1 is exported as X increases.

Considering now situation (3), R_X will be positive provided that $G_{21}f_1 + G_{22}$ can be shown to be negative. By employing Euler's Theorem, it is possible to show that with $f_1 > 0$ we have $G_{21}f_1 + G_{22} < 0$ [12] for cases where the production function (1-3) is homogenous to a degree less than one. Since

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12. C.G. Plourde (1970(a), p.78-79) proves this for the closed version of the model.

the analysis has been restricted to resource production functions which are homogeneous to a degree less than one, this result follows.

Finally, for situation (4), equation (3-25) will be positive for homogeneous to a degree less than unity production functions provided that consumption of resource product is not socially inferior with respect to X as in situation (2).

In summary of all four possible situations regarding the signs of f_1 and h_1, it can be stated that $R_X > 0$ for the class of resource production functions under consideration, provided that $\partial Y_1/\partial X$ is non negative and exceed any increase in the export of resource product, even though some of the variable factor of production may be transferred to the Y_2 industry. The condition being imposed is that consumption of resource product must not be socially inferior with respect to X.

With the results that $R_P > 0$ and $R_X > 0$ for all possible signs on f_1 and h_1, it may be concluded that dP/dX for $\dot{P} = 0$ is negative. Thus the graph of R(X,P) in phase space (P,X) has a negative slope.

Consider now the sign of dP/dX defined by the points for which $\dot{X} = 0$. Define $E(X,P) = 0$ as relation (3-18) when $\dot{X} = 0$. As before, we wish to use implicit differentiation and as such require that $E_P \neq 0$. From the definition of $E(X,P)$ we can obtain

(3-26) $$E_P = -G_1 f_2 ,$$

which will be positive given the result obtained in Section A.3 that $f_2 < 0$. Thus, we may again employ the Implicit Function Theorem to find dP/dX for $\dot{X} = 0$ as $-E_X/E_P$. From the definition of $E(X,P)$, we may obtain

(3-27) $$E_X = N' - [G_1 f_1 + G_2]$$

and thus,

(3-28) $$\left.\frac{dP}{dX}\right|_{\dot{X} = 0} = \frac{[G_1 f_1 + G_2] - N'}{-G_1 f_2}$$

Since it is known from the discussion concerning the sign of R_X in this Section, that $\partial Y_1/\partial X = G_1 f_1 + G_2 \gtreqless 0$, and that $-G_1 f_2 > 0$, because $f_2 < 0$, it follows that the sign of dP/dX for $\dot{X} = 0$ will be the same as the sign of $\partial Y_1/\partial X - N'$. Thus, the positive or negative slope of the line $\dot{E}(X,P)$ in phase space for stationary biomass equilibriums is determined by the relationship which $\partial Y_1/\partial X$ bears to the marginal biological productivity N'.

In Section A.1 of Chapter II, it was noted that the shape of the long run transformation curve for stationary biomass points was determined by the relationship which the slope of the total resource product curve (G_2) bore to N' at points of stationary equilibrium. Along the resource product curve which generated the stationary equilibrium point, it follows by definition that L_1 is fixed at some level, and hence, so too is output level Y_2.[13] With Y_2 held constant, it was seen that $\partial Y_1/\partial X = G_2$, and thus, the two results on stationary biomass equilibrium are different only by the fact that the one for phase space analysis is concerned with the general equilibrium case where L_1 is free to adjust to the equilibrium biomass stock value.

For points of stationary biomass equilibrium which occur with resource masses on the $N' < 0$ side of the unexploited resource growth function, it follows from (3-28) that dP/dX for $\dot{X} = 0$ is positive. For resource masses with $N' > 0$, the sign of dP/dX for $\dot{X} = 0$ depends upon the relative sizes of the two terms N' and $\partial Y_1/\partial X$.

5. The Pontryagin Path

As suggested earlier, major interest is centered

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13. This follows from the full employment assumption.

around the long run properties of the model. In an effort to see these long run properties define (P^{∞}, X^{∞}) as the steady state equilibrium at which all motion ceases, i.e., $\dot{P} = \dot{X} = 0$. Also, define the <u>Pontryagin path</u>[14] as the system $[X(t), P(t), C_1^*(t), C_2^*(t), L_1^*(t), L_2^*(t), Z_1^*(t)]$ which satisfies the differential equations (3-18) and (3-19). If this path converges to the long run steady state equilibrium (P^{∞}, X^{∞}), it is optimal provided three conditions are met. The first is that the function H^O defined as

$$H^O = \max H \quad \text{evaluated at } C_1^*, C_2^*, L_1^*, L_2^*, Z_1^*$$

is concave in X for given P. The second is that the transversality conditions

$$\lim e^{-\rho t} P(t) \geqq 0 \quad \text{as } t \to \infty$$

$$\lim e^{-\rho t} P(t)X(t) = 0 \text{ as } t \to \infty$$

must be met. Finally, the third is that the differential equations (3-18) and (3-19) must be autonomous.

Clearly, this last condition is satisfied as time enters into (3-18) and (3-19) only as a subscript. Also, the transversality conditions are met because $X(t) \to X^{\infty}$ as $t \to \infty$.

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14. See K.J. Arrow and M. Kurz (1971, p. 57).

Concavity of the function H^{o} is discussed by C.G. Plourde[15] in a closed version of a similar model. With this information, the question is to determine the properties of a steady state solution, if one exists. In order to do this, consider the following trajectories in phase space.

6. The Trajectories in Phase Space

Section A.4 demonstrated $R_P > 0$, and that $R_X > 0$ for the class of production functions under consideration, provided resource product is not socially inferior in consumption with respect to X. With these results, it was concluded that dP/dX for $\dot{P} = 0$ is negative. To see the trajectories in the phase space (P,X), choose some arbitrary P^{o} value and determine the sign of $\partial\dot{P}/\partial X$. This has already been done in equations (3-22), (3-23) and (3-25), which showed under the sufficient conditions listed above that $R_X > 0$. Thus, above and to the right of $\dot{P} = 0$, we have $\dot{P} > 0$ and to the left and below, $\dot{P} < 0$ as shown in Figure 7.

The curve for $\dot{X} = 0$ was shown in Section A.4 to be positively sloped when $\partial Y_1/\partial X - N' > 0$, and negatively sloped when the opposite is true. Choose some arbitrary X^{o} value,

15. C.G. Plourde [1970(a), p.75-77]

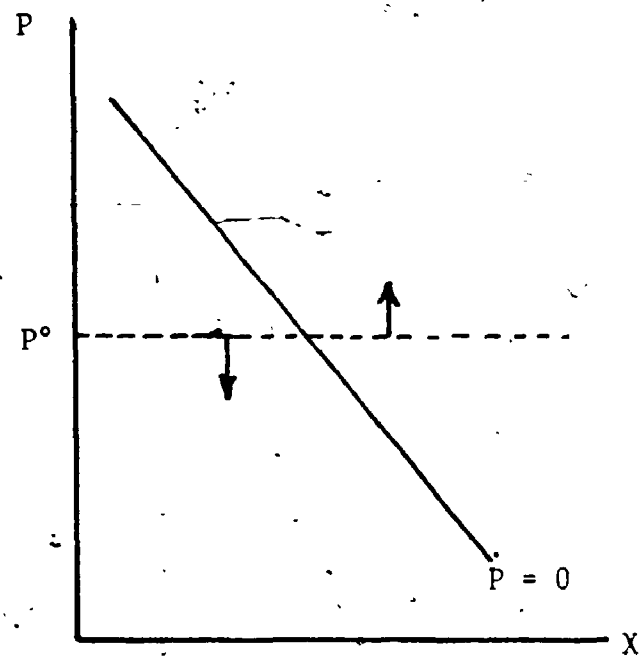

FIGURE 7

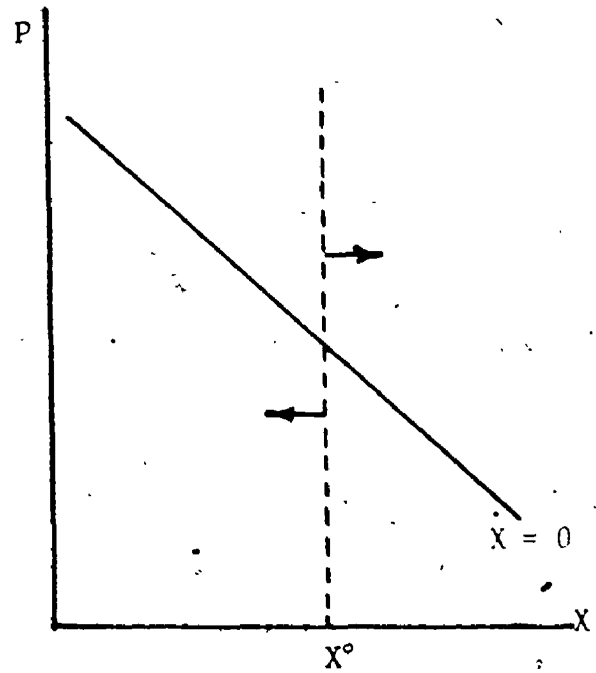

FIGURE 8

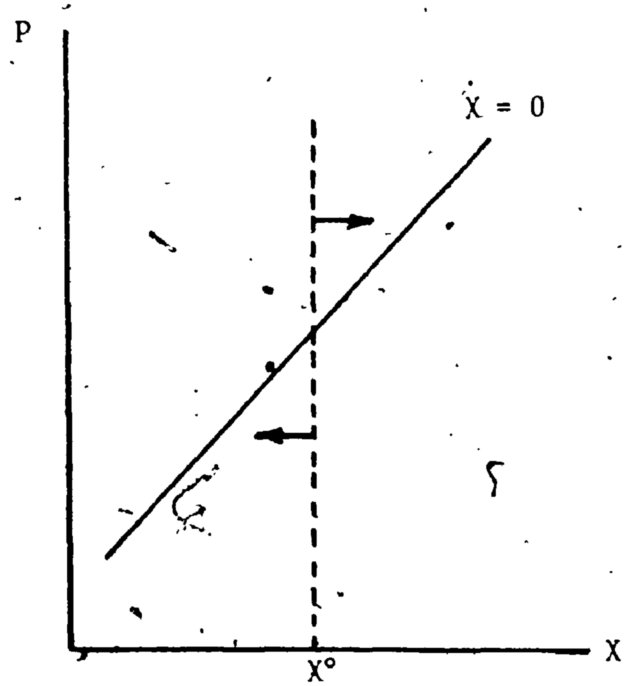

FIGURE 9

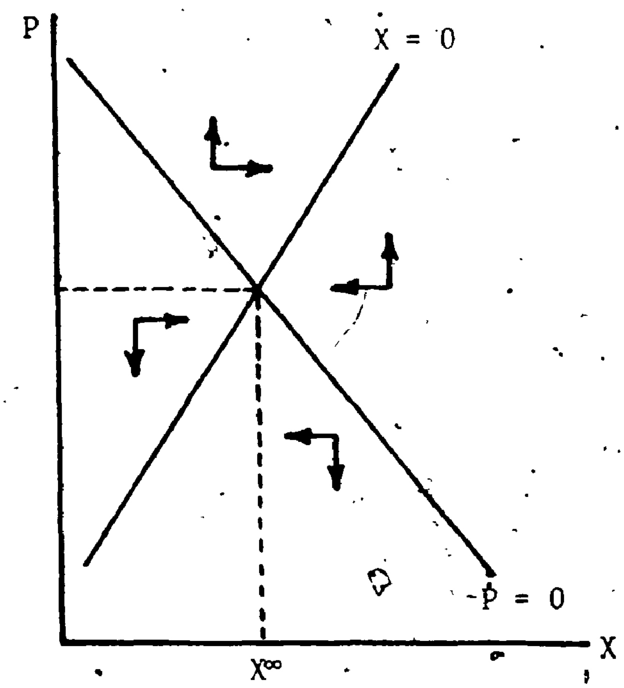

FIGURE 10

and determine the sign of $\partial \dot{X}/\partial P$. This is simply the result E_P given by equation (3-26) which shows that above $\dot{X} = 0$ we have $\dot{X} > 0$, and below, $\dot{X} < 0$. These results on the trajectories hold regardless of whether dP/dX for $\dot{X} = 0$ is positive or negative, because E_P is unambiguously positive for $J < 0$. The pictures of these motions are given by Figures 8 and 9.

Putting the curves for $\dot{P} = 0$ and $\dot{X} = 0$ together, we can observe the direction of motion of a point anywhere in phase space. These motions are demonstrated by arrows in Figures 10, 11 and 12 for the three possible intersections of the two curves. Thus in Figures 10 and 12, only a path which begins in regions II or IV will converge towards the steady state (P^{∞}, X^{∞}), while the motion in I and III is away from equilibrium. For Figure 11, the motion in all regions is away from equilibrium. Thus, the equilibrium (equilibria) if it (they) exist will either be saddle points or unstable. In the case of saddle points, an initial P can be found which will start the system towards (P^{∞}, X^{∞}). Subsequent P(t) values can be found which correspond to the eginvector of the negative eginvalue of the linearized system.

It follows from Figures 10, 11 and 12 that a necessary and sufficient condition for a saddle point equilibrium if it exists is that dP/dX for $\dot{X} = 0$ exceed dP/dX for $\dot{P} = 0$ at the

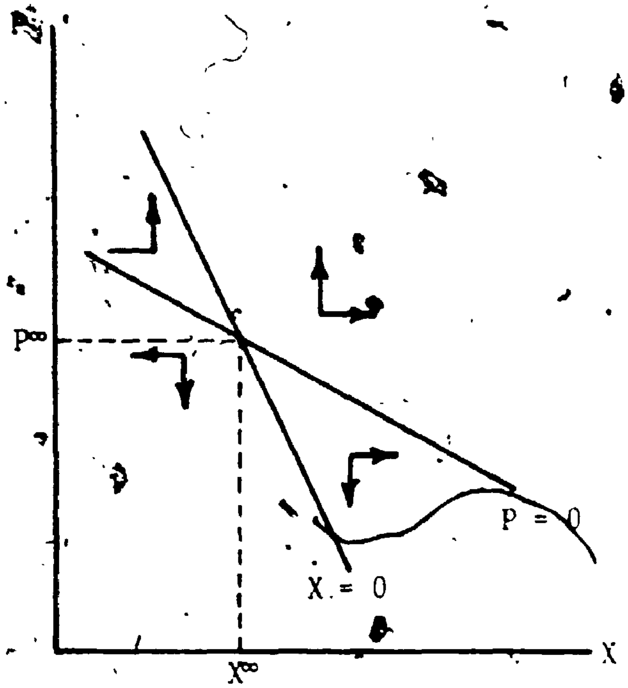

FIGURE 11

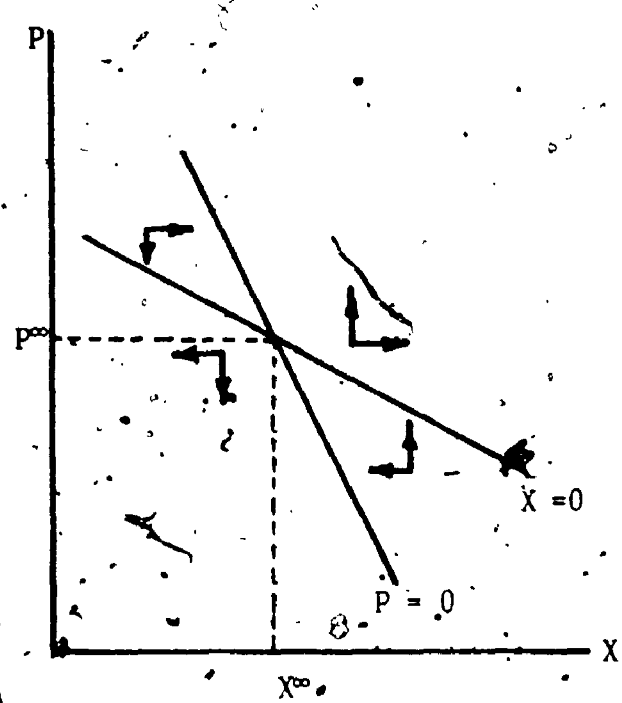

FIGURE 12

steady state equilibrium $\dot{P} = \dot{X} = 0$. From Section A.4, it is known that dP/dX for $\dot{X} = 0$ equals $-E_X/E_P$ and dP/dX for $\dot{P} = 0$ equals $-R_X/R_P$. Thus, the necessary and sufficient condition for a saddle point if it exists can be written as

$$(3\text{-}29) \qquad -\frac{E_X}{E_P} > -\frac{R_X}{R_P}$$

or

$$(3\text{-}30) \qquad E_X R_P < E_P R_X .$$

Since it is known that R_P, R_X and E_P are all positive, it is clear that a sufficient, though not necessary condition, to satisfy the result (3-30) is that $E_X < 0$. From (3-27), it is known that $E_X = N' - \partial Y_1/\partial X$, so that the sufficient condition becomes $\partial Y_1/\partial X > N'$ at the steady state equilibrium. As discussed earlier in this Chapter, this result is simply an equivalent statement to the one derived geometrically in Chapter II for situations where the Y_2 output was held constant. The geometry of Chapter II showed that a stable stationary biomass equilibrium will occur at resource masses for which $G_2 > N'$ at equilibrium. Now with L_1 free to adjust to new resource stock values, the sufficient condition for stability becomes the more general one that $\partial Y_1/\partial X > N'$.

In terms of where this optimal steady state equilibrium occurs on the unexploited natural mass function (1-1)[16] it is clear that either $N' < 0$, or some values of $N' > 0$ can satisfy the sufficient criteria for stability. This is the answer to the first of the questions which this Chapter has attempted to answer. It seems that a sufficient condition for a stable optimal stationary equilibrium is that the equilibrium, if it exists, occur with a resource stock mass such that the marginal biological productivity of the resource stock (N') be less than the marginal technolgical productivity $(\partial Y_1/\partial X)$, after adjusting L_1 to the new resource stock value. From this, it follows that maximum sustained yield programs which suggest a steady state resource mass at $\hat{X}$ are not necessarily optimal.

B. The Dynamic General Equilibrium Model: Competitive Version.

The second question of concern to this Chapter is the social optimality of the free market path to equilibrium. Here, due to the common property nature of the resource mass, each of the n_1 identical firms in the resource product industry

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16. For results in regard to this point in an industry model see V.L. Smith (1968, p.426-429), and the exchange of articles between R.F. Fullenbaum, E.W. Carson, F.W. Bell (1971, p.484-485), V.L. Smith (1971, p.490-491), R.F. Fullenbaum, E.W. Carson, F.W. Bell (1972, p.763-764) and V.L. Smith (1972, p.777-778). For results in the closed version of this model, see C.G. Plourde (1971, p.264).

considers the resource mass to be a parameter necessary to production, but beyond the firm's individual control. As a result, the price P placed on use of the resource mass is zero, and the equation (1-8) showing the net change over time in X is of no concern to the individual firms. Without this transition equation, the dynamic maximization problem simply becomes a series of static maximization problems indexed by time.[17] This Section will perform the intertemporal maximization for the competitive model and then compare the necessary conditions obtained to those obtained for the optimal version given in Section A.1.[18]

1. The Necessary Conditions

a) The Consumption Problem

Consider a model in which there are a fixed number n of identical consumers. The consumption problem faced by each of these consumers can be cast in the following format. From (3-2) and (3-3), it is possible to write

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17. M.D. Intriligator (1971, p.310)

18. The methodology of this Section borrows from J.P. Quirk and V.L. Smith (1969, p.5-8) and C.G. Plourde [1970(a), p.54-57].

(3-31) $c_1 = \bar{y}_1 - z_1$

(3-32) $c_2 = \bar{y}_2 + z_2$

where the lower case letters c_1, c_2, z_1 and z_2 are the per capita counterparts of the upper case ones used in Section A, and $\bar{y}_1$ and $\bar{y}_2$ are the per capita outputs of resource product and manufactured goods respectively. The lower case letters y_1 and y_2 without the bar notation are per firm output levels. It is assumed that the consumers know with certainty the domestic price of output from sector one, written p_1, the domestic price of output from sector two, written p_2, and the international price β of the exported resource product good in terms of the imported manufactured good. Also, consumers know exactly what their per capita incomes m will be over all time periods.

The problem faced by one of the n identical consumers in the open economy can be cast in the following format:

$$\max \int_0^\infty [U(c_1) + V(c_2)] e^{-\rho t} dt$$

subject to the budget constraint

$$p_1 c_1 + p_2 c_2 = m$$

and the international exchange condition

$$\beta z_1 - z_2 = 0$$

which states that the value of per capita exports must equal the value of per capita imports, where valuation takes place at the international price β, the relative price of resource product in terms of the manufactured good. As specified at the beginning of this Chapter, it is assumed that the objective function is increasing and strictly concave in its arguments.[19] By making use of the identities (3-31) and (3-32), it is possible to set up the current value Lagrangian for some period t.

$$L(c_1, c_2, \bar{y}_1, \bar{y}_2, \lambda_1, \lambda_2) = [U(c_1) + V(c_2)]$$
$$+ \lambda_1[m-p_1c_1-p_2c_2] + \lambda_2[\beta(\bar{y}_1-c_1) - (c_2-\bar{y}_2)]$$

The necessary condition for an interior solution at any t value is

(3-33) $$\frac{U'}{V'} = \beta$$

plus the constraints.

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19. See J.P. Quirk and V.L, Smith (1969, p.6) for a discussion of how this concavity assumption on the objective function fits in with neoclassical utility theory.

b) The Production Problem

Given the per firm production functions (1-2) and (1-4), it is possible to formulate the production problem faced by each of the firms in the two industries. It is assumed that producers know with certainty over all time periods the cost of the variable factor w, and the two output prices p_1 and p_2.

The production problem faced by one of the n_1 identical firms in the natural resource product industry can be formulated in the following manner:

$$\max \int_0^\infty [p_1 y_1 - w l_1] e^{-\rho t} dt$$

subject to

$$y_1 = G(l_1, \dot{X})$$

where the objective function is the firm's profits, discounted by the social rate of discount.

For one of the n_2 identical firms in the manufacturing goods industry, the problem is

$$\max \int_0^\infty [p_2 y_2 - w l_2] e^{-\rho t} dt$$

subject to

$$y_2 = F(l_2)$$

Substituting each constraint into its objective function, we may obtain the necessary conditions for an interior solution at each t as

(3-34) $p_1 G_1 - w = 0$

(3-35) $p_2 F' - w = 0$

These conditions are obviously stating that the value of the marginal product of the variable factor in each industry should be equated to its cost. These may be rearranged into the more convenient format

(3-36) $\dfrac{p_1}{p_2} = \dfrac{F'}{G_1}$

When the market economy neglects the effect that harvesting has on the existing stock of resource mass, the necessary conditions to be satisfied for each t are (3-33) and (3-36), which may be written as

(3-37) $\dfrac{p_1}{p_2} = \dfrac{U'}{V'} = \beta = \dfrac{F'}{G_1}$.

2. Comparison and Dynamic Policy

The necessary conditions for the optimal version given in Section A may be rearranged into the same format as (3-37) for the competitive model. By dividing (3-7) by (3-8), it is possible to produce

(3-38) $$\frac{U'}{V'} = \frac{q_1}{q_2}$$

Also, (3-9) may be rearranged into

(3-39) $$\frac{q_1}{q_2} = \frac{F'}{G_1}\left[\frac{q_1}{q_1 - P}\right]$$

When these two conditions are combined with the necessary condition (3-10) which states that the relative price of resource product for the home country is equal to the marginal terms of trade Θ', the result is

(3-40) $$\frac{q_1}{q_2} = \frac{U'}{V'} = \Theta' = \frac{F'}{G_1}\left[\frac{q_1}{q_1 - P}\right]$$

The marginal terms of trade Θ' and the international price ratio of resource product in terms of the manufactured good are connected by the relationship.[20]

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20. See H.G. Johnson (1958, p.58).

$$\Omega' = \beta\varepsilon$$

where ε is the elasticity of the foreign offer curve. Substituting this into (3-40) produces

(3-41) $$\frac{q_1}{q_2} = \frac{U'}{V'} = \beta\varepsilon = \frac{F'}{G_1}\left[\frac{q_1}{q_1 - P}\right]$$

The necessary conditions for the optimal version of the model are the two differential equation (1-8) and (3-11), plus the aggregated condition (3-41).

The necessary conditions defining the time path produced by the competitive version are not the same as the optimal conditions produced by the control version in two ways. The first difference arises due to the discrepancy between the marginal rate of transformation (MRT) for the competitive and control versions of the model. Chapter II showed that the MRT along a short run, fixed X transformation curve is given by the ratio F'/G_1. This then is the MRT for each short run period which occurs when the time path of the resource stock is not taken into account, i.e., the MRT which the competitive model, neglecting the influence of the harvesting decision on the stock of resource mass, would view as the correct one. However, when the changing resource mass caused by the harvesting decisions is taken into account in the optimization process the MRT for each period should be

$$\frac{F'}{G_I}\left[\frac{q_1}{q_1 - P}\right]$$

Since we are dealing with resource masses which have positive social prices, the term $q_1/q_1 - P$ must exceed unity thus, giving the result that the MRT along the socially optimal path will exceed that produced by the competitive model. Clearly, the MRT in the optimal model becomes identical with the MRT produced by the competitive version only when $P = 0$.

Thus, it may be stated that the first problem is caused by the external production externality attributable to the common property nature of the resource mass. This externality problem can be remedied by the well established[21] policy prescription that a charge equal to P(t) should be levied against the use of the resource mass. This will internalize into the profit calculations of each firm the user cost of nature's capital in the system.

The second difference between the two models is caused by the foreign distortion. By comparing the aggregated necessary conditions for the optimal version (3-41) and the competitive version (3-37), it is clear that the marginal

21. J.P. Quirk and V.L. Smith (1969, p.10) and C.G. Plourde [1970(a), p.56].

rate of transformation through trade (Θ') will be less than the international price ratio β due to the assumption that the foreign offer curve is characterized by $\varepsilon > 1$.

The classical policy prescription for this problem is, of course, the optimal tariff. Thus, when a country's export supply decision affects the international terms of trade, optimality requires the imposition of an export tax to restrict exports of resource product to the point where the marginal rate of transformation through trade is equal to the socially correct marginal rate of transformation in production. If the home country is sufficiently small, such that it has no influence over the international price of resource product, it follows that $\Theta' = \beta$. As such, the correct export tariff is zero.

With both the domestic distortion, caused by the common property nature of the resource mass, and the foreign distortion, caused by the home country's monopoly power in trade, it follows that the competitive model can be forced to behave like the optimal version if there is a simultaneous application of the user cost tax P(t), and the optimal tariff.[22] Together, they will interact to raise the market determined price ratio

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22. J. Bhagwati, V.K. Ramaswami, and T.N. Srinivasin (1969, p.1009) demonstrate that if only one of these corrective policy measures is applied, it is impossible to state whether it alone will increase welfare over the market situation.

p_1/p_2 to the optimal one q_1/q_2. As a final point, note that the user charge P(t) will clearly vary over time according to the result (3-11). Less clear is the result that the size of the optimal tariff will also vary. Comparison of (3-41) and (3-37) suggests that the difference between Θ' and β, at the optimum tariff point will be independent of time due to the assumption that ε at this point is fixed. However, it follows that if this point on the foreign offer curve at which welfare is maximized moves over time, then the size of the optimal tariff must also change. Due to the assumed dependence of the home country's taste pattern on time by the discount factor $e^{-\rho t}$, this will indeed be the case.

CHAPTER IV

The phase space pictures of Chapter III show how the resource mass (X) and its implicit price (P) vary optimally along the Pontryagin Path towards steady state equilibrium. This optimal path to a stationary resource mass may also be illustrated by means of the short and long run transformation curves derived in Chapter II and illustrated in Figure 6.

In Chapter II it was shown that given the initial relative price of resource product and the initial raw resource stock, a production point such as K_1 on the short run transformation curve TT can be found in Figure 6. Since each point in commodity space may be classified as to whether it has an increasing, decreasing or stationary resource mass, the route to stationary equilibrium produced by the market economy is easily obtained. Information on the socially optimal route for P given and fixed at its socially correct value given the existing X stock is provided by the signs of the partial derivatives f_1 and h_1. With information on these signs, the optimal route for production and consumption may easily be plotted on the same short and long run transformation curves that were used to plot the market path. To aid explanation, Section A will be concerned with

the special case in which the home country has no influence over the determination of the international price ratio, while Section B will consider the more general case in which exports of resource product do influence the world price ratio.

A. Constant Terms of Trade

For constant terms of trade, it follows that the foreign offer curve (Θ) must be linear, i.e., $\Theta'' = 0$. From Section A.3 of Chapter III

$$f_1 = D_3 B_2 - B_3 D_2$$

due to the fact that $J < 0$. Now, with $\Theta'' = 0$, D_3, B_3 and D_2 are unaffected as they do not depend upon Θ'', but B_2 becomes

$$B_2 = - U'' - V'' \Theta' \Theta'.$$

With these results, f_1 can be rearranged into

$$(4\text{-}1) \qquad f_1 = U'' G_2 V'' \Theta'(F' - G_1 \Theta') - U'' G_{12} (U' - P)$$
$$- G_{12} V'' \Theta' \Theta' (U' - P)$$

From the discussion of the signs given in Chapter III, it is clear that the last two terms are positive, and the first term's sign depends upon the sign of $F' - G_1 \Theta'$. Note, however, that F'/G_1 is simply the marginal rate of transformation given by result (2-2) for any fixed X, short run transformation curve. Also, Θ'

is the international price of resource product β for this special case of a linear foreign offer curve. For a competitive equilibrium, it clearly follows that $F'/G' = \Theta'$, and thus, the first term in (4-1) is zero, leaving f_1 unambiguously positive. Now, unlike the case discussed in Chapter III in which $\Theta'' < 0$, an increase in X stock produces the definite result that more labor should be allocated to the resource product industry. Thus, Y_1 output will rise due to the fact that more of both factors are employed, while Y_2 output will fall due to the fixed supply of labor assumption. This result, shown in Figure 13 as the movement from K_1 to K_2. is the well known Rybczynski Theorem. The complete movement to stationary mass equilibrium is shown by the production points K_1, K_2, K_3 in Figure 13. This will be the path produced by the market economy which at every point in time, i.e., for every short run transformation curve, equates the fixed relative international price of resource product to the competitive economy's marginal rate of transformation

The consumption points corresponding to this series of production points can easily be determined once the sign of h_1 is specified. Unfortunately, the specification of a linear foreign offer curve does not change the indeterminate nature of h_1 because Θ'' does not enter into D_1, D_3, B_1 or B_3 used in the determination of h_1. Taking the case where h_1 is positive produces the result that exports, and hence imports, will increase along the optimal path. Combining this information with the result that Y_1 output

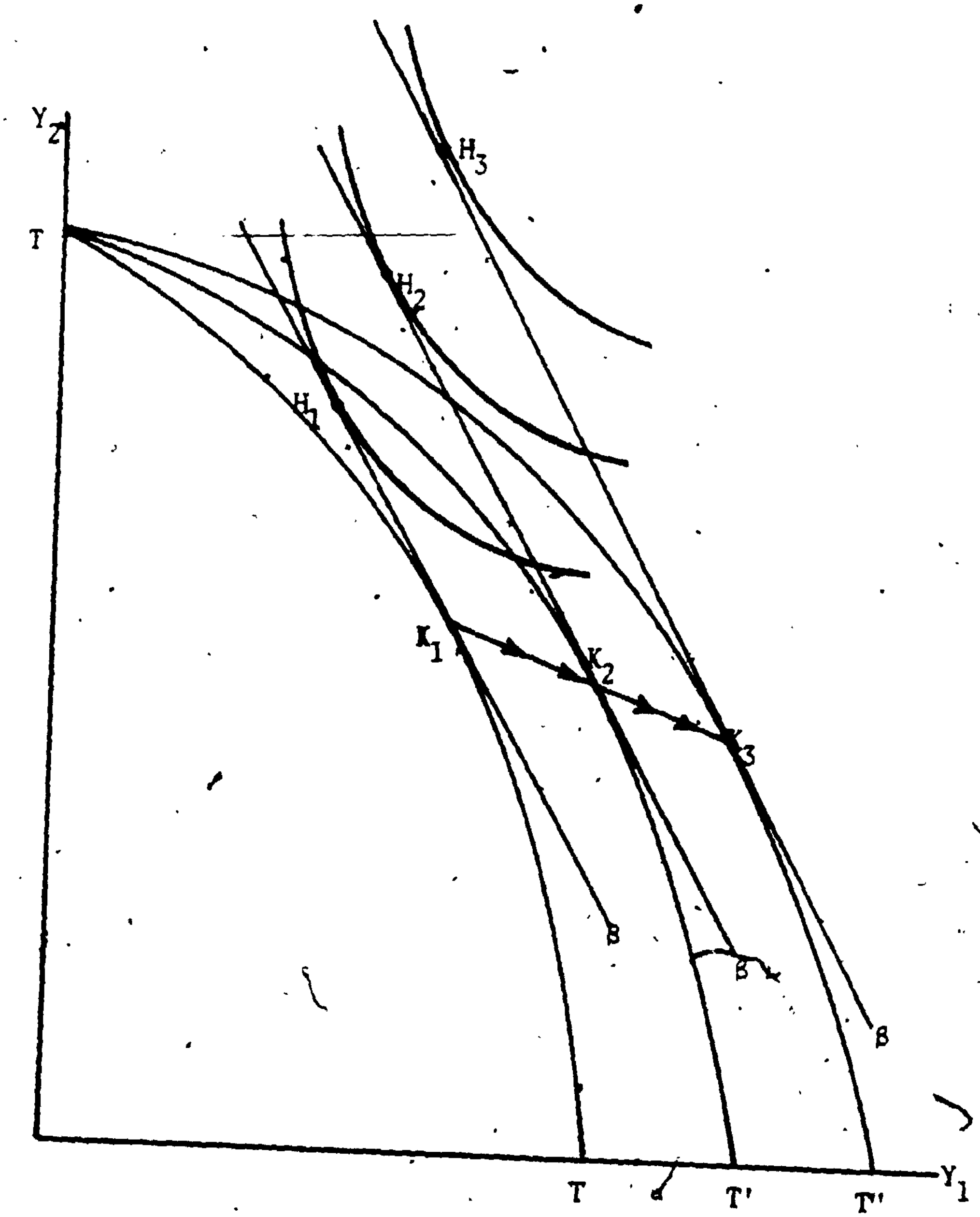

FIGURE 13

increases and Y_2 decreases, it follows that the absolute levels of consumption for each good are unknown. However, Section A.4 of Chapter III made clear that definite results on phase space motion require that consumption of resource product not be socially inferior with respect to changes in resource stock. Thus, C_1 must remain constant or increase over time. For the manufactured good, domestic production declines but imports increase. If it is assumed that consumption of sector two output is not inferior[1], it follows that C_2 must also increase as X increases. These consumption points are shown as H_1 and H_2 in Figure 13. With $h_1 < 0$, exports and imports decline along the optimal path giving the determinate results that C_1 rises absolutely and C_2 falls. The assumption that C_2 cannot be inferior renders this an impossible case.

The path for production points K_1, K_2 and K_3, and for consumption points H_1, H_2 and H_3 is produced on the basis that the market economy equates the observed marginal rate of transformation (F'/G') to the existing fixed price ratio Θ'. Comparison of results (3-37) and (3-41) clearly demonstrates that this policy will be non optimal, for the social MRT is different from the private one due to the fact that the resource mass has a positive implicit price (P). The common property nature of the resource stock causes the market economy to view P as zero. If a tax is levied on use

[1]This is an assumption imposed on the instantaneous utility functions U and V. It does not arise as a sufficient condition for stability.

of the resource mass, and set at the socially correct value given the existing X stock, there occurs an expansion in Y_2 production and a decrease in Y_1 production. Beginning from the initial position K_1 in Figure 14, the socially optimal production point after expansion in the resource stock is given by some point like S_1, showing more Y_2 output and less Y_1 output than the market economy point K_2.

This production tax drives a wedge between the competitive economy's MRT and the observed international and domestic price ratio θ'. Specifically, at the optimal production point, $F'/G' < \theta'$ as shown by point S_1. This result follows from the observation made in Section B.2 of Chapter III that with $P > 0$ the socially correct MRT will exceed the short run MRT equal to F'/G'. Thus, it follows that along the optimal production path, $F'/G' < \theta'$. With this result, it is clear that f_1 in equation (4-1) can no longer be unambiguously signed, i.e., the optimal production point S_1 in Figure 14 will show an increase in both outputs if $f_1 < 0$. Thus, as X expands less labor is used in the resource product industry, thereby making labor available to increase Y_2 production. The Y_1 output will also increase even though less L_1 is used with the higher resource stock because of the condition imposed in Section A.4 of the last Chapter that $\partial Y_1/\partial X > 0$.

The optimal series of short run production points, beginning from the initial point K_1 at which the production tax P has not yet been imposed, will be a series like K_1, S_1, S_2. The market

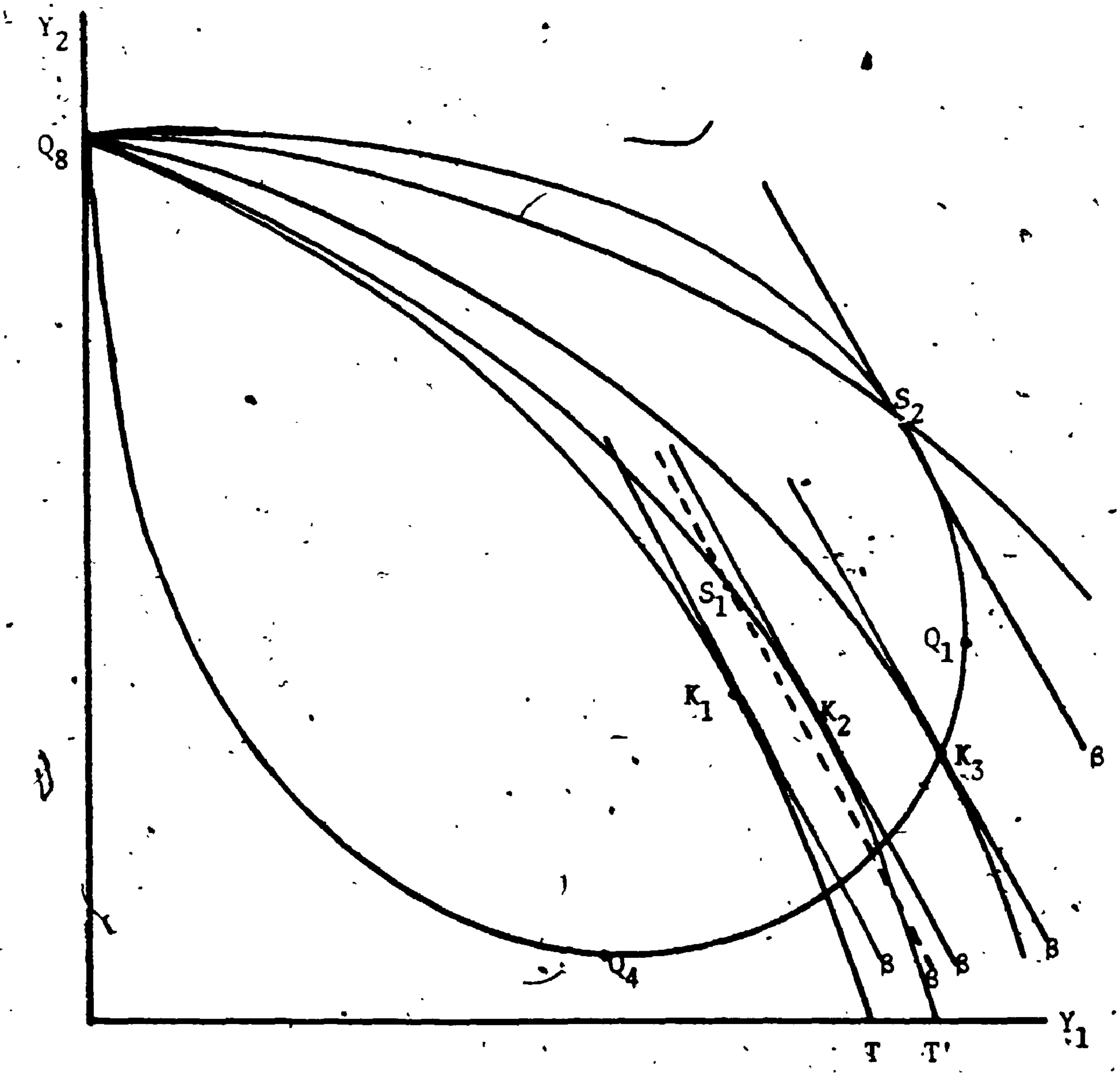

FIGURE 14

path will be the series K_1, K_2, K_3. Of course, the optimal series may, like the market series, show a decline in Y_2 output if f_1 in equation (4-1) remains positive when $F'/G_1 < \Theta'$. As in Figure 6 of the last Chapter, the consumption points have not been shown to avoid cluttering the diagram. However, C_1 and C_2 may easily be worked out for the four possible cases $f_1 > 0$, $h_1 > 0$; $f_1 > 0$, $h_1 < 0$; $f_1 < 0$, $h_1 > 0$; $f_1 < 0$; $h_1 < 0$ when it is remembered from Chapter III that $\partial Y_1/\partial X$ must not only be positive when $f_1 < 0$, but exceed any increase in exports so that consumption of resource product is not socially inferior.

The geometry of Figure 14 illustrates that with the imposition of a positive tax (P) on use of the resource mass, the optimal path to stationary equilibrium will lie above the market path. This is true regardless of whether f_1 is positive, as shown in Figure 14, or negative. In Section A.6 of the last Chapter, it was shown that a sufficient condition for stationary equilibrium is that $\partial Y_1/\partial X > N'$ at the steady state point. This result was shown to reduce to the result that $G_2 \gtrless N'$ when L_1 settles at its optimal stationary resource mass value. Combining this information on stability with the result from Chapter II that $G_2 > N'$ for all output combinations on the distance Q_8 Q_1 Q_4 in Figure 15, it follows that the optimal steady state equilibrium must occur somewhere on this distance Q_8 Q_1 Q_4. In terms of the unexploited natural mass function (1-1), it is clear that the steady state equilibrium

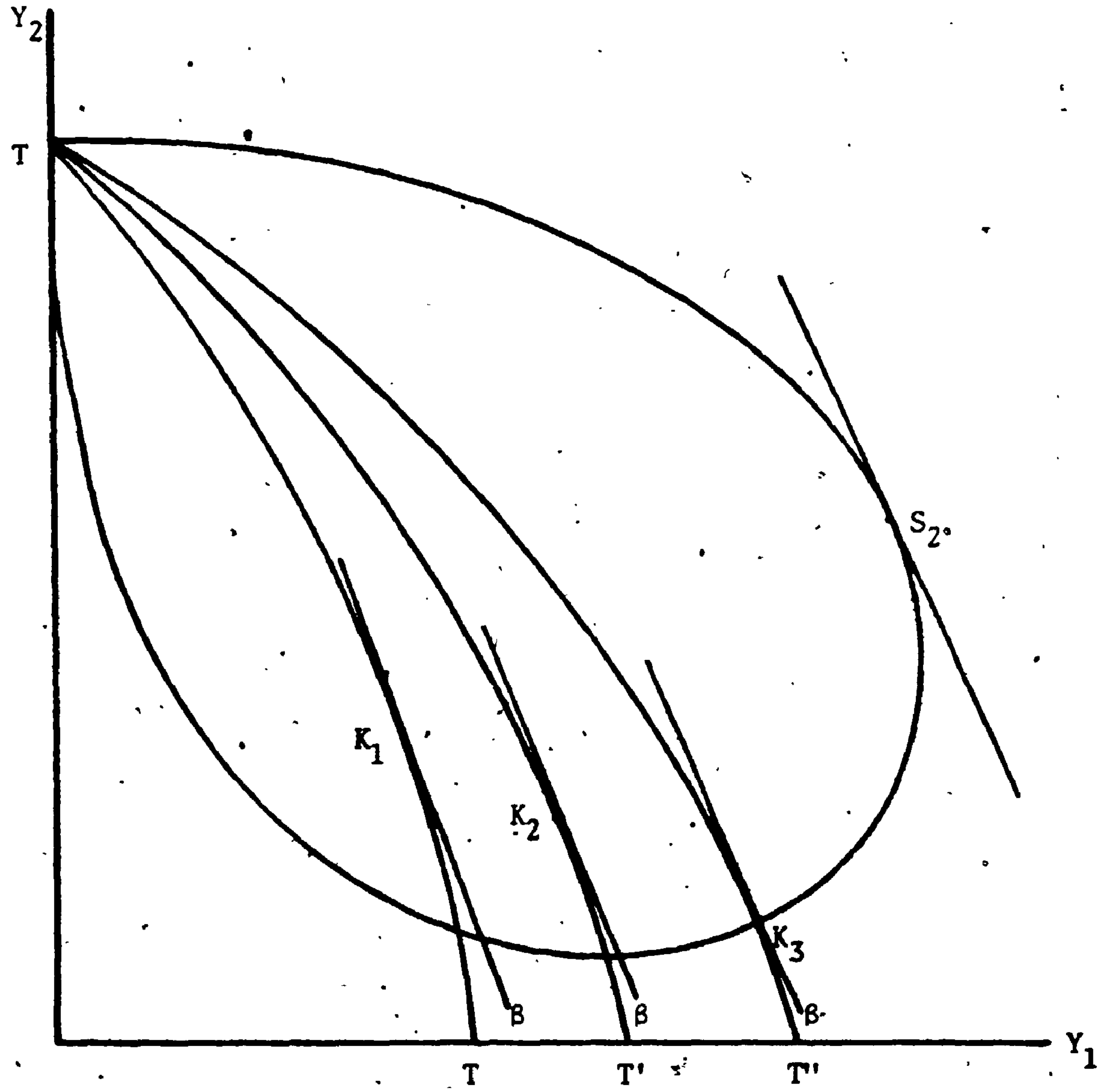

FIGURE 15

will corespond to resource masses for which $N' < 0$ if the equilibrium occurs on the distance $Q_8 Q_1$, and to $N' > 0$ if it occurs on $Q_1 Q_4$. Section A.4 of Chapter III showed that $\rho - N' > 0$ quarantees the result that $P > 0$. Thus, it follows that only if $\rho = 0$ will the optimal steady state equilibrium definitely take place on the distance $Q_8 Q_1$ of Figure 15 corresponding to $N' < 0$.[2] When the social rate of discount is required for the evaluation of alternative consumption paths in the dynamic model, it seems that the optimal stationary equilibrium may occur as shown in Figure 14 on the distance $Q_8 Q_1$, or on the distance $Q_1 Q_4$. However, given the fact that the international price ratio must also equal the marginal rate of substitution, it follows for convex to the origin indifference curves that the socially optimal equilibrium must take place on distance $Q_8 Q_1$.

For the optimal path to steady state equilibrium always above the market path, the socially correct equilibrium will have less Y_1 and more Y_2 production than the market equilibrium for the situation depicted in Figure 14. However, if the market path settles at a stationary equilibrium on the distance $Q_1 Q_4$ shown in Figure 15, the optimal stationary equilibrium may have more of both goods. The common sense

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[2]Thus in the stationary model by V. L. Smith (forthcoming), he obtains the result that the Pareto Optimal solution can only occur at a resource stock at which $N' < 0$.

of this position is that a high international price of resource product causes a large amount of labor to be allocated toward the Y_1 industry. This large labor allocation results in a large production of resource product output, and consequently, a small growth in the resource stock. For the situation in Figure 15 where this growth, or investment, in the resource stock is still positive, the path K_1, K_2, K_3 to stationary equilibrium will be obtained. More of both goods is potentially available in this case by taxing the resource product output, causing labor to move from the Y_1 industry to increase Y_2 output. With this lower L_1 input, the resource mass X is allowed to expand at its optimal rate, thus producing the steady state equilibrium S_2 in Figure 15. The point S_2 has more Y_1 output than K_3, because the increased available resource stock has more than compensated for the reduced use of L_1. This yields a Y_1 expansion, as well as the increased production of Y_2 caused by the higher available L_2 input.

B. Changing Terms of Trade

For the general case where the home country possesses some monopoly power in trade, it was shown in Section A.3 of Chapter III that f_1 and h_1 could be positive or negative along the optimal path. Thus, identical results to the ones obtained for the small country case follow with the exception that along the optimal path the optimal tariff as well as the production

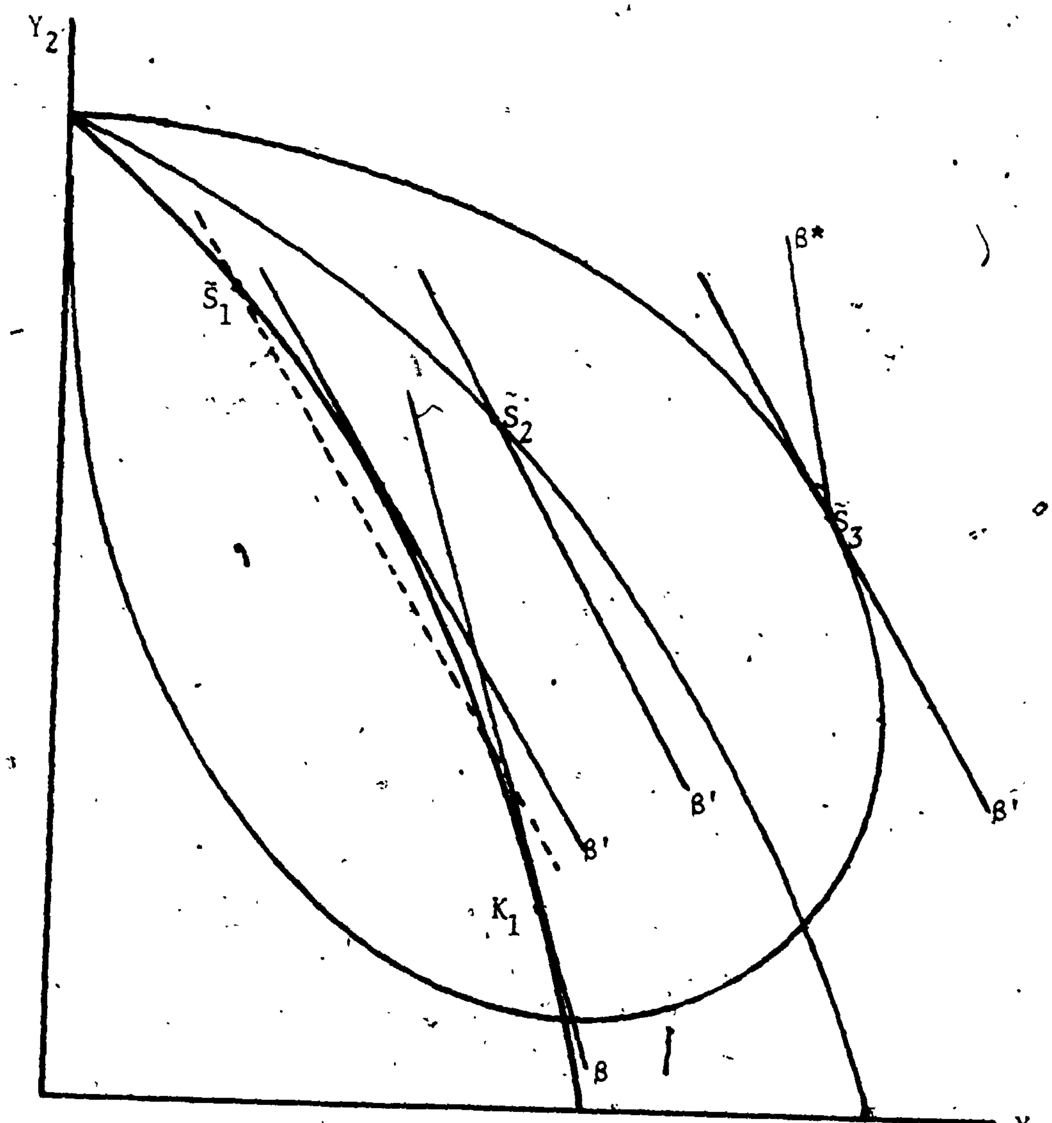

FIGURE 16

tax will be levied. The result is that the domestic relative price of resource product falls for producers and consumers due to the tariff, but in addition, the production tax drives a wedge between the MRT and this price ratio.

The results are illustrated in Figure 16. As before, beginning from the initial point K_1 with terms of trade β, the home country simultaneously imposes the optimal tariff and the optimal user tax which correspond to the given X stock value. The tariff distorts the domestic price ratio to β', while the production tax on resource mass use causes the production point to shift to $\tilde{S}_1$. At this production level of Y_1, it follows that $\dot{X} > 0$, so that the expanded resource stock generates the new production point $\tilde{S}_2$. Since f_1 may be positive or negative, $\tilde{S}_2$ may, as shown, have a lower level of Y_2 output, i.e. $f_1 < 0$, or have a higher level of Y_2 output than point $\tilde{S}_1$. On the basis of these taxes, the resource stock is allowed to expand at its socially optimal rate until the steady state equilibrium point $\tilde{S}_3$ is obtained. At this production point the terms of trade will be at their optimal level β^*. The consumption levels C_1 and C_2 along the optimal path may be calculated as before on the basis that C_1 cannot be inferior with respect to X for stability reasons, and C_2 is assumed also to be non-inferior for convenience. The final steady state consumption point will settle along β^*

at the point where the marginal rate of substitution is equal to the tariff distorted prices β'.

APPENDIX I

Geometric Derivation of a Short Run Transformation Curve

The short run transformation curve shown in Figure 5 of Chapter II may easily be derived by making use of Figure A1-1 of this Appendix. Quadrant II contains the total product curve for output of Y_2. The third quadrant contains a line with a slope of -1 which intersects the L_2 and L_1 axis at the full employment level $\bar{L}$. Finally, in quadrant IV is plotted the total product curve for output Y_1 on the basis of some fixed, short run X stock value.

A short run transformation curve corresponding to this existing resource mass level may easily be derived in quadrant I by the following method. If all available labor is used in the production of Y_2, it is clear that output level T may be produced. Alternatively, if all labor is used in Y_1 production, the output point T on the Y_1 axis can be obtained. If labor is allocated between the two industries in any other fashion, say point A in quadrant III, the output levels for Y_1 and Y_2 are easily calculated and plotted as point B in the first quadrant. Clearly, this same process can be repeated for all points along the full employment line in quadrant III. This generates the transformation curve TT.

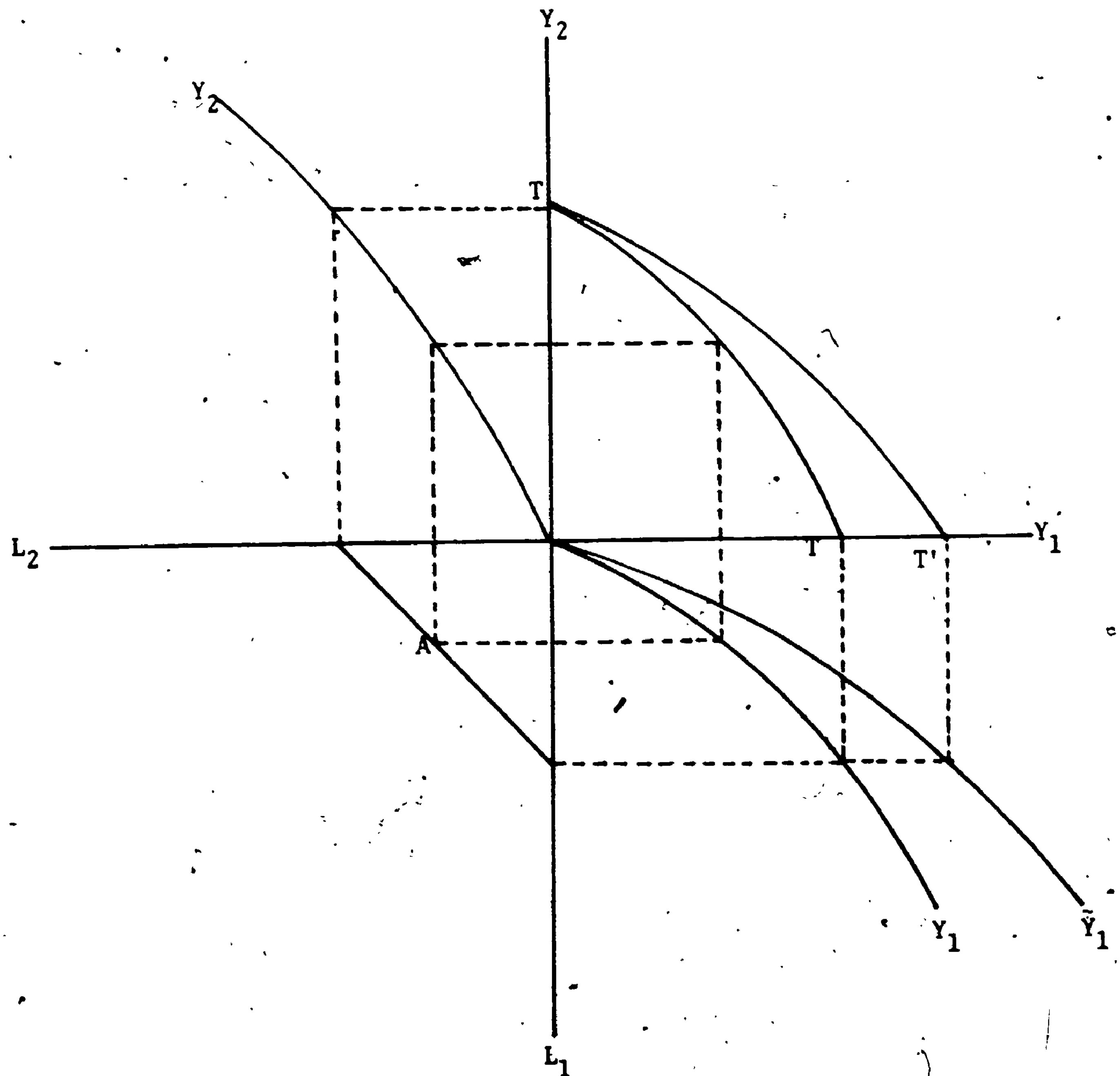

FIGURE A1-1

If the X stock increases in the next period, the transformation curve corresponding to this new level of resource mass may also be derived. This new higher X stock value produces the total product curve labelled Y_1 in quadrant IV. By repeating the same analysis, it is clear that if all available labor is used in the production of Y_2, the same point T is obtained on the derived transformation curve. Now, however, if all labor is used in the production of resource product, a higher output of Y_2 is obtained. This is shown by the point T' exceeding the old output point T. Again, any distribution of labor between the two extremes will generate the output levels plotted as the concave locus TT'.

APPENDIX 2

IMPLICATIONS OF $f_2 < 0$

Unlike the closed version of a similar model,[1] this open model does not, with the restrictions imposed, produce an unambiguous sign for f_2 i.e., as the social price of the unharvested resource increases is it optimal to allocate more or less of the variable factor to the resource product industry. The dissertation makes the assumption that as P increases less labour will be allocated to the Y_1 industry, thus the resource stock is saved for the future. With f_2 assumed negative, it follows immediately that J must also be negative.

The assumption that f_2 is positive implies the curious result that society should optimally allocate more of the variable factor towards harvesting the resource stock when its price increases. With this assumption it is difficult to derive the slopes of the curves in phase space.

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1. C. G. Plourde (1971, p. 259).

APPENDIX 3

Sign of D_3

From Section A.3 of Chapter III, the result D_3 is given by

$$U''G_2G_1 + (U'-P)G_{12}$$

Rearrange this into

$$\left[\frac{U''G_2G_1}{U'G_{12}} + 1 - \frac{P}{U'}\right] U'G_{12}$$

Multiply and divide the first term in the bracket by G, and note that for $q_1 > 0$ the constraint (3-4) is binding, i.e., $G(L_1X) = C_1+Z_1$. Also, by necessary condition (3-7) it is known that $q_1 = U'(C_1)$. Substituting these results yields

$$\left\{ \left[C_1 \frac{U''}{U'} + Z_1 \frac{U''}{U'}\right] \frac{G_2G_1}{G\,G_{12}} + 1 - \frac{P}{q_1}\right\} U'G_{12}$$

This result can be put in more manageable form by noting the following:

1) the elasticity of marginal utility for consumption of C_1, defined as a positive number, is given by $\mu = -C_1\frac{U''}{U'}$

2) the elasticity of foregone marginal utility created in the instantaneous utility function U by exporting

resource product is given by $\psi = -Z_1 \frac{U''}{U'}$. This too will be a positive number due to the fact that $\partial U/\partial Z_1 < 0$, and $\partial^2 U/\partial Z_1^2 > 0$ for $U(Y_1 - Z_1)$. The elasticity measured by this number is the utility opportunity cost of exporting some Y_1 which will presumably be recaptured by imports in the instantaneous utility function V.

3) the elasticity of factor substitution [1] in the resource product industry is given by $\sigma = \frac{G_2 G_1}{G\, G_{12}}$.

Substituting these produces the result

$$\left\{[-\mu - \psi]\,\sigma + 1 - \frac{P}{q_1}\right\} U'G_{12}$$

From this, it is clear that the sign of D_3 depends upon the sign of the large bracketed term. Since we are only interested in resource products which have positive social prices, i.e. $(q_1 - P) > 0$, it is clear that $1 - \frac{P}{q_1} > 0$. However, the sign of the term $[-\mu - \psi]\,\sigma$ remains unknown unless specific assumptions are made concerning the sizes of these elasticities.

1. The elasticity of factor substitution may be written in this form for homogeneous production functions.

APPENDIX 4

APPLICATION OF HERBERG AND KEMP'S PROOF

This Appendix will demonstrate that Herberg and Kemp's proof that the price line will cut the production possibility curve at the equilibrium point unless both industries have identically increasing or decreasing returns to scale can be applied to the production functions of this dissertation.

In terms of my functions and notation, assume two production functions

(A4-1) $$Y_1 = g_1(Y_1)\, G(L_1, X)$$

(A4-2) $$Y_2 = g_2(Y_2)\, F(L_2)$$

where the functions $g_1(Y_1)$ and $g_2(Y_2)$ capture the effects of scale economies on the output levels. Both g_1 and g_2 are positive functions with continuous first and second derivatives and elasticities given by

(A4-3) $$\omega_1 = \frac{Y_1}{g_1} \frac{dg_1}{dY_1}$$

and

(A4-4) $$\omega_2 = \frac{Y_2}{g_2} \frac{dg_2}{dY_2}$$

For decreasing returns to scale, ω_1 and ω_2 will be negative and for increasing returns to scale they will be positive. The explosive case of ω_1 or ω_2 greater than or equal to unity is ruled out. It is also assumed that the functions G(·) and F(·) are homogeneous of the first degree in their arguments.

In order to proceed with their analysis based on production functions with two arguments, Herberg and Kemp make three assumptions, all of which are satisfied by production functions of the type given in (A4-1) and (A4-2).

1. the marginal technical rate of subsitution for both industries is between the range $(0, \infty)$.

2. total factor supplies are inelastically available.

3. a technical assumption concerning the sign of a term as Y_1 and Y_2 approach zero. Simple inspection of this term[1] reveals that its sign is independent of the number of arguments.

With the aid of these assumptions, Herberg and Kemp

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[1] See Assumption B, H. Herberg and M. C. Kemp (1969, p. 405)

are able to produce the result[1]

(A4-5) $$\frac{dY_1}{dY_2} = -\left[\frac{g_1}{g_2}\right]\left[\frac{G_1}{F'}\right]\left[\frac{1-\omega_2}{1-\omega_1}\right]$$

Since all the assumptions are valid here, this result follows for the production functions used in this dissertation.

Following Herberg and Kemp, with perfect competition and incomplete specialization it follows that

(A4-5) $$P_1 = a_{1L}\, q_3 + a_{1X}\, P$$

(A4-6) $$P_2 = a_{2L}\, q_3$$

where the coefficient a_{1L} is the amount of labor used in the production of one unit of Y_1, and a_{1X} is the amount of X available to produce one unit of Y_1. The socially correct price corresponding to this X stock value is P. Finally, a_{2L} is the amount of labor required per unit of Y_2 production. By dividing (A4-6) by (A4-5) it is possible to form

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1 The demonstration involves establishing the properties of the production possibility set based on the "kernel" production functions $G(\cdot)$ $F(\cdot)$. Using the result that a continuous one-to-one mapping exists between this set and the production possibility set corresponding to the production functions (A4-1) and (A4-2), it is possible to establish dY_1/dY_2 by showing the changes undergone by the first set when being mapped into the latter set.

(A4-7) $$\frac{P_2}{P_1} = \frac{a_{2L}}{a_{1L} + a_{1X}\left[\frac{R}{q_3}\right]}$$

Noting that cost minimizing behavior in the Y_1 industry requires

(A4-8) $$\frac{P}{q_3} = \frac{G_2}{G_1}$$

it is possible to substitute this result into (A4-7) to yield

(A4-9) $$\frac{P_2}{P_1} = \frac{G_1\ a_{2L}}{G_1\ a_{1L} + G_2\ a_{1X}}$$

Remembering that the kernel production functions are homogeneous to the first degree, it follows that

(A4-10) $$g_1(Y_1)\ G(a_{1L}, a_{1X}) = 1$$

(A4-11) $$g_2(Y_2)\ F(a_{2L}) = 1$$

These results may be totally differentiated and rearranged to produce

(A4-12) $$\frac{a_{2L}}{G_1\ a_{1L} + G_2\ a_{1X}} = \frac{g_1}{g_2\ F'}$$

Substituting (A4-12) into (A4-9) gives

(A4-13) $$\frac{P_2}{P_1} = \frac{G_1\ g_1}{F'\ g_2}$$

Finally, substituting (A4-13) into (A4-5) produces

(A4-14) $$\frac{dY_1}{dY_2} = -\left[\frac{P_2}{P_1}\right]\left[\frac{1-\omega_2}{1-\omega_1}\right]$$

Thus, only if $\omega_1 = \omega_2$ will the price line be tangent to the production possibility curve at the equilibrium point.

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