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On The Relation Between The Abel-type And Borel-type Methods Of Summability

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ON THE RELATION BETWEEN THE ABEL-TYPE AND BOREL-TYPE METHODS OF SUMMABILITY

by

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ABSTRACT

The thesis is divided into five chapters. The first contains notations and definitions. The second contains a number of known results which give the relations between the Abel-type methods of summability and the relations between the Borel-type methods of summability. The others contain a number of theorems on the relations between an Abel-type and a Borel-type method of summability. In the third Chapter, the results that under certain conditions, a series which is summable by an ordinary, a strong and an absolute Abel-type method of summability is also summable by an ordinary, a strong and an absolute Borel-type method of summability respectively are given. In the last two chapters, it is proved that under a certain kind of Tauberian conditions, a series which is summable by an ordinary Abel-type method of summability is also summable by an ordinary, a strong or an absolute Borel-type method of summability.

The substance of Chapter III will appear in the Proceedings of the American Mathematical Society.

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CHAPTER I

INTRODUCTION

1.1. *INTRODUCTION.*

It is supposed throughout the thesis that σ, $a_n (n = 0, 1, \ldots)$ are arbitrary complex numbers, β is real, $\alpha > 0$, $\lambda > -1$ and N is a non-negative integer greater than $(1-\beta)/\alpha$.

The sequence $\{s_n\}$ is used as the associated sequence of the partial sums of the given series $\sum_{n=0}^{\infty} a_n$, that is,

$$s_n = \sum_{\gamma=0}^{n} a_\gamma , \quad n = 0, 1, \ldots \quad .$$

The symbol M is used throughout the thesis to denote a positive number, independent of the variables under consideration, but not necessarily having the same value at each occurence.

The theorems and lemmas in the thesis are numbered according to the section and chapter in which they occur, for example, Theorem 3.2.1 is the first theorem in section 2 of chapter 3.

1.2. *NOTATION.*

The following notation is used throughout the thesis:

$$E_n^\lambda = \binom{n+\lambda}{n} = \frac{(\lambda+1)(\lambda+2)\dots(\lambda+n)}{n!}, \quad n = 1, 2, \dots;$$

$$E_o^\lambda = 1 ;$$

$$E_n^\lambda = 0, \ n = -1, -2, \dots;$$

$$u_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} E_n^\lambda a_n \left(\frac{y}{1+y}\right)^n ;$$

$$\sigma_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} E_n^\lambda s_n \left(\frac{y}{1+y}\right)^n ;$$

$$U_\lambda(y) = \int_o^y \lambda u_\lambda(t)dt ;$$

$$a_{\alpha,\beta}(y) = \sum_{n=N}^{\infty} \frac{a_n y^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} ;$$

$$s_{\alpha,\beta}(y) = \sum_{n=N}^{\infty} \frac{s_n y^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} ;$$

$$A_{\alpha,\beta}(y) = \int_o^y e^{-t} a_{\alpha,\beta}(t)dt ;$$

$$S_{\alpha,\beta}(y) = \alpha e^{-y} s_{\alpha,\beta}(y) .$$

It is easily shown that, if $\sum_{n=o}^{\infty} E_n^\lambda s_n \left(\frac{y}{1+y}\right)^n$ is convergent for all $y > 0$, then

$$y\frac{d}{dy} \sigma_\lambda(y) = (\lambda+1)[\sigma_{\lambda+1}(y) - \sigma_\lambda(y)].$$

It is also easily shown that, if $\sum_{n=N}^{\infty} \frac{s_n y^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}$ is convergent for $y > 0$, then

$$\frac{d}{dy} A_{\alpha,\beta}(y) = e^{-y} a_{\alpha,\beta}(y);$$

$$\frac{d}{dy} S_{\alpha,\beta}(y) = S_{\alpha,\beta-1}(y) - S_{\alpha,\beta}(y);$$

$$\frac{d}{dy} A_{\alpha,\beta}(y) = A_{\alpha,\beta-1}(y) - A_{\alpha,\beta}(y);$$

$$a_{\alpha,\beta}(y) = s_{\alpha,\beta}(y) - s_{\alpha,\beta+\alpha}(y) .$$

1.3. *DEFINITIONS.*

1.3.1. *Summability Methods.*

A summability method is a function defined on a set of series of complex numbers to a set of complex numbers. If a summability method P assigns the value σ to the series $\sum_{n=o}^{\infty} a_n$, we say that $\sum_{n=o}^{\infty} a_n$ is P summable to the sum σ and write

$$\sum_{n=o}^{\infty} a_n = \sigma(P) .$$

We shall also say that the sequence $\{s_n\}$ is P-convergent to the limit σ and write

$$s_n \to \sigma(P) .$$

A method of summability P is said to be regular if $s_n \to \sigma(P)$ whenever the sequence $\{s_n\}$ converges to σ in the ordinary sense.

If every sequence convergent by the method P is also convergent by the method Q, we shall say that the method Q includes the method P, and write

$$P \subseteqq Q.$$

If $P \subseteqq Q$ and $Q \subseteqq P$, we say that the two methods P and Q are equivalent, and write

$$P \cong Q .$$

A summability method P is called left-translative if

$$s_n \to \sigma(P)$$

whenever

$$s_{n+1} \to \sigma(P) .$$

That P is called right-translative if

$$s_{n+1} \to \sigma(P)$$

whenever

$$s_n \to \sigma(P).$$

A summability method P is said to be translative if P is both left-translative and right-translative.

1.3.2. *Abel-type Methods.*

The Abel-type summability methods (A_λ) and (A'_λ) were introduced by Borwein [1] and [5], and are defined as follows:

The Abel-type summability method (A_λ)

If

$$(1-x)^{\lambda+1} \sum_{n=o}^{\infty} E_n^\lambda s_n x^n$$

is convergent for all x in the open interval (0, 1) and tends to a finite limit σ as $x \to 1-$, we say that the series $\sum_{n=o}^{\infty} a_n$ is summable (A_λ) to the sum σ, and write

$$\sum_{n=o}^{\infty} a_n = \sigma(A_\lambda) .$$

We also say that the sequence $\{s_n\}$ is (A_λ)-convergent to the limit σ and write

$$s_n \to \sigma(A_\lambda) .$$

Evidently, $s_n \to \sigma(A_\lambda)$ if and only if the series defining $\sigma_\lambda(y)$ is convergent for all $y > 0$ and $\sigma_\lambda(y)$ tends to σ as $y \to \infty$. The method (A_o) is the ordinary Abel method (A).

The Abel-type summability method (A'_λ)

If the series defining $u_\lambda(y)$ is convergent for all $y > 0$ and $U_\lambda(y)$ tends to a finite limit σ as $y \to \infty$, we say that the series $\sum_{n=o}^{\infty} a_n$

is summable (A'_λ) to the sum σ, and write

$$\sum_{n=0}^{\infty} a_n = \sigma(A'_\lambda) .$$

We also say that the sequence $\{s_n\}$ is (A'_λ)-convergent to the limit σ and write

$$s_n \to \sigma(A'_\lambda) .$$

1.3.3. *Strong Abel-type Methods.*

Strong Abel summability was introduced by Harrington and Hyslop [13]. Two definitions of strong Abel-type summability were given by Flett [11]. Strong Abel-type summability $[A_\lambda]$ has been investigated by Mishra who also introduced strong Abel-type summability with index [15]. Strong Abel-type summability $[A'_\lambda]$ has been studied by Rizvi in his doctoral thesis [20].

Strong Abel-type summability with index $p[A_\lambda]_p$

If

$$\int_0^y |\sigma_{\lambda+1}(t) - \sigma|^p \, dt = o(y) \quad (p > 0)$$

as $y \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ is summable $[A_\lambda]_p$ to the sum σ, and write

$$\sum_{n=0}^{\infty} a_n = \sigma[A_\lambda]_p .$$

We also say that the sequence $\{s_n\}$ is $[A_\lambda]_p$-convergent to the limit σ, and write

$$s_n \to \sigma[A_\lambda]_p .$$

Strong Abel-type summability with index p$[A'_\lambda]_p$

If $\lambda > 0$ and

$$\int_o^y |U_{\lambda+1}(t) - \sigma|^p \, dt = o(y) \quad (p > 0)$$

as $y \to \infty$, we say that the series $\sum_{n=o}^{\infty} a_n$ is summable $[A'_\lambda]_p$ to the sum σ, and write

$$\sum_{n=o}^{\infty} a_n = \sigma[A'_\lambda]_p \; .$$

We also say that the sequence $\{s_n\}$ is $[A'_\lambda]_p$-convergent to the limit σ, and write

$$s_n \to \sigma[A'_\lambda]_p \; .$$

For $p = 1$, the methods $[A_\lambda]_p$ and $[A'_\lambda]_p$ will be denoted by $[A_\lambda]$ and $[A'_\lambda]$, respectively.

1.3.4. *Absolute Abel-type Methods.*

Absolute Abel summability $|A|$ was first defined by Whittaker [22]. It has been subsequently investigated by various authors: For example, Flett [10] has given a generalization, and Mishra [16] has studied absolute Abel-type summability $|A_\lambda|$. Absolute Abel-type summability $|A'_\lambda|$ has been considered by Rizvi in his doctoral thesis [20].

Absolute Abel-type summability $|A_\lambda|$

If $\sigma_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit σ as $y \to \infty$, we say that the series $\sum_{n=o}^{\infty} a_n$ is summable $|A_\lambda|$ to the sum σ, and write

$$\sum_{n=o}^{\infty} a_n = \sigma|A_\lambda| .$$

We also say that the sequence $\{s_n\}$ is $|A_\lambda|$-convergent to the limit σ, and write

$$s_n \to \sigma|A_\lambda| .$$

When $\lambda = 0$, these reduce to Whittaker's $|A|$ summability [23].

Absolute Abel-type summability $|A'_\lambda|$

If $U_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit σ as $y \to \infty$, we say that the series $\sum_{n=o}^{\infty} a_n$ is summable $|A'_\lambda|$ to the sum σ, and write

$$\sum_{n=o}^{\infty} a_n = \sigma|A'_\lambda| .$$

We also say that the sequence $\{s_n\}$ is $|A'_\lambda|$-convergent to the limit σ, and write

$$s_n \to \sigma|A'_\lambda| .$$

1.3.5. *Borel-type Methods.*

The Borel-type summability methods (B,α,β) and (B',α,β) were introduced by Borwein [6] and are defined as follows:

The Borel-type summability method (B,α,β)

If the series defining $s_{\alpha,\beta}(y)$ is convergent for all $y \geqq 0$ and $S_{\alpha,\beta}(y)$ tends to a finite limit σ as $y \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is summable (B,α,β) to the sum σ, and write

$$s_n \to \sigma(B,\alpha,\beta) .$$

The $(B,1,1)$ method is the Borel exponential (B) method.

The Borel-type summability method (B',α,β)

Suppose from now on that $\sigma_N = \sigma - s_{N-1}$. If the series defining $a_{\alpha,\beta}(y)$ is convergent for all $y \geqq 0$ and $A_{\alpha,\beta}(y)$ tends to a finite limit σ_N as $y \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is summable (B',α,β) to the sum σ, and write

$$s_n \to \sigma(B',\alpha,\beta) .$$

We note that [3] the convergence of the series defining either $a_{\alpha,\beta}(y)$ or $s_{\alpha,\beta}(y)$ for all $y \geqq 0$ implies the convergence for all $y \geqq 0$, of the other series.

1.3.6. *Strong Borel-type Methods.*

The strong Borel-type summability methods $[B,\alpha,\beta]_p$ and $[B',\alpha,\beta]_p$ were introduced by Borwein and Shawyer [8] and are defined as follows:

The strong Borel-type summability method $[B,\alpha,\beta]_p$

If the series defining $s_{\alpha,\beta}(y)$ is convergent for all $y \geqq 0$ and

$$\int_0^y e^t |S_{\alpha,\beta-1}(t) - \sigma|^p \, dt = o(e^y) \quad (p > 0)$$

as $y \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is summable $[B,\alpha,\beta]_p$ to the sum σ, and write

$$s_n \to \sigma[B,\alpha,\beta]_p \, .$$

The strong Borel-type summability method $[B',\alpha,\beta]_p$

If the series defining $a_{\alpha,\beta}(y)$ is convergent for all $y \geq 0$ and

$$\int_0^y e^t |A_{\alpha,\beta-1}(t) - \sigma_N|^p \, dt = o(e^y) \quad (p > 0)$$

as $y \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is summable $[B',\alpha,\beta]_p$ to the sum σ, and write

$$s_n \to \sigma[B',\alpha,\beta]_p \, .$$

When $p = 1$, the methods $[B,\alpha,\beta]_p$ and $[B',\alpha,\beta]_p$ will be denoted by $[B,\alpha,\beta]$ and $[B',\alpha,\beta]$, respectively.

1.3.7. *Absolute Borel-type Methods.*

The ideas of absolute summability of Borel's method are due to Borel himself ([12] p.184). The absolute Borel-type summability methods $|B,\alpha,\beta|$ and $|B',\alpha,\beta|$ were introduced by Borwein and Shawyer [7] and are defined as follows:

Absolute Borel-type summability method $|B,\alpha,\beta|$

If $S_{\alpha,\beta}(y)$ is of bounded variation with respect to y in the range $[0, \infty)$, and tends to a finite limit σ as $y \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is summable $|B,\alpha,\beta|$ to the sum σ, and write

$$s_n \to \sigma|B,\alpha,\beta| .$$

Absolute Borel-type summability method $|B',\alpha,\beta|$

If $A_{\alpha,\beta}(y)$ is of bounded variation with respect to y in the range $[0, \infty)$, and tends to a finite limit σ as $y \to \infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ (or the sequence $\{s_n\}$) is summable $|B',\alpha,\beta|$ to the sum σ, and write

$$s_n \to \sigma|B',\alpha,\beta| .$$

We note that a function f(y) which is of bounded variation with respect to y in the range $[0, \infty)$, necessarily tends to a finite limit as $y \to \infty$.

(see Natanson [17] p. 239, corollary to Theorem 5)

CHAPTER II

PRELIMINARY RESULTS

2.1. *INTRODUCTION.*

In this chapter, first we prove a basic result which will play an important role in most of the theorems that are proved in the following three chapters. We also state without proof, certain known theorems which give inter-relations between the various Abel-type and Borel-type methods of summability.

2.2. *A BASIC RESULT.*

LEMMA 2.2.1.

For $\lambda > -1$, let $\nu_n = 0$, $n = 0,1,\ldots,N-1$ and

$$\nu_n = \frac{\Gamma(\alpha n+\beta+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n+\lambda+1)}, \quad n = N, N+1, \ldots,$$

and

$$J(t) = \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-u/t} u^\lambda s_{\alpha,\beta}(u)du, \quad t > 0 .$$

If $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$, then

$$J(t) = \alpha\left(\frac{t}{1+t}\right)^{\beta-1}\left(\frac{1+y}{1+t}\right)^{\lambda+1} \sigma_\lambda^*(y) ,$$

where

$$\sigma_\lambda^*(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} E_n^\lambda \nu_n s_n \left(\frac{y}{1+y}\right)^n ,$$

and t and y are related by

$$\left(\frac{t}{1+t}\right)^\alpha = \frac{y}{1+y} .$$

The following two lemmas are required for the proof of Lemma 2.2.1.

LEMMA 2.2.2.

The series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ if and only if the series $\sum_{n=0}^{\infty} s_n x^n$ converges for $|x| < 1$.

PROOF.

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$. Since $\sum_{n=0}^{\infty} x^n$ is convergent to $\frac{1}{1-x}$ for $|x| < 1$, we have

$$\sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} a_n x^n$$

$$= \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n .$$

Hence $\sum_{n=0}^{\infty} s_n x^n$ is convergent for $|x| < 1$.

Conversely, suppose that $\sum_{n=0}^{\infty} s_n x^n$ is convergent for $|x| < 1$.

$$\sum_{n=o}^{\infty} a_n x^n = \sum_{n=o}^{\infty} s_n x^n - \sum_{n=1}^{\infty} s_n x^{n+1} ,$$

and so $\sum_{n=o}^{\infty} a_n x^n$ is convergent for $|x| < 1$.

This completes the proof of the lemma.

LEMMA 2.2.3.

$$\sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \sim \frac{e^x}{\alpha} \quad \text{as} \quad x \to \infty.$$

This is due to Borwein (see [6] p. 130, with $\delta = \beta-1$).

PROOF OF LEMMA 2.2.1.

By hypothesis and in view of Lemma 2.2.2, we have

$$s_n = O\{(1+\delta)^{\alpha n}\} \quad \text{for all } \delta > 0.$$

It follows from Lemma 2.2.3 that

$$\sum_{n=N}^{\infty} \frac{|s_n| u^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \leq M(1+\delta)^{1-\beta} \sum_{n=N}^{\infty} \frac{[(1+\delta)u]^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}$$

$$\sim M(1+\delta)^{1-\beta} e^{(1+\delta)u}/\alpha$$

as $u \to \infty$. Choose δ such that $\delta t < 1$, then

$$\frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-u/t} u^\lambda \alpha e^{-u} \sum_{n=N}^\infty \frac{|s_n| u^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} du$$

$$\leq M \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} (1+\delta)^{1-\beta} \int_0^\infty e^{-u/t} u^\lambda e^{\delta u} du$$

$$= M(1-\delta t)^{-\lambda-1} ,$$

which is finite for each fixed t (>0).

Hence the inversion of the summation and the integration is legitimate and we have

$$J(t) = \frac{\alpha t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-u/t} e^{-u} \sum_{n=N}^\infty \frac{s_n u^{\alpha n+\beta-1+\lambda}}{\Gamma(\alpha n+\beta)} du$$

$$= \frac{\alpha t^{-\lambda-1}}{\Gamma(\lambda+1)} \sum_{n=N}^\infty \frac{s_n}{\Gamma(\alpha n+\beta)} \int_0^\infty e^{-(1+\frac{1}{t})u} u^{\alpha n+\beta+\lambda-1} du$$

$$= \frac{\alpha t^{-\lambda-1}}{\Gamma(\lambda+1)} \sum_{n=N}^\infty \frac{\Gamma(\alpha n+\beta+\lambda)}{\Gamma(\alpha n+\beta)} s_n \left(\frac{t}{1+t}\right)^{\alpha n+\beta+\lambda}$$

$$= \alpha\left(\frac{t}{1+t}\right)^{\beta-1}\left(\frac{1}{1+t}\right)^{\lambda+1} \sum_{n=N}^\infty E_n^\lambda \nu_n s_n \left(\frac{t}{1+t}\right)^{\alpha n} .$$

Let $\left(\frac{t}{1+t}\right)^\alpha = \frac{y}{1+y}$, we have

$$J(t) = \alpha\left(\frac{t}{1+t}\right)^{\beta-1} \left(\frac{1+y}{1+t}\right)^{\lambda+1} \frac{1}{(1+y)^{\lambda+1}} \sum_{n=N}^\infty E_n \nu_n s_n \left(\frac{y}{1+y}\right)^n$$

$$= \alpha\left(\frac{t}{1+t}\right)^{\beta-1} \left(\frac{1+y}{1+t}\right)^{\lambda+1} \sigma_\lambda^*(y) .$$

This completes the proof of Lemma 2.2.1.

2.3. *THEOREMS WHICH GIVE RELATIONS BETWEEN THE "A" METHODS AND BETWEEN THE "A'" METHODS.*

The following four known results for (A_λ) methods are all due to Borwein:

THEOREM 2.3.1.

The method (A_λ) *is regular for all* $\lambda > -1$;

(see [1] Theorem 1)

THEOREM 2.3.2.

$(A_\mu) \subseteq (A_\lambda)$ *for all* $\mu \geqq \lambda > -1$;

(see [1] Theorem 2)

THEOREM 2.3.3.

The method (A_λ) *is translative for all* $\lambda > -1$;

(see [1] Theorem 5)

THEOREM 2.3.4.

If $\lambda > -1$, a *is real,* $s_n \to \sigma(A_\lambda)$ *and* $(n+a)u_n = s_n (n = 0, 1, \ldots)$, *then*

$$u_n \to 0(A_\lambda) \quad .$$

(see [1] Lemma 4)

The following are known results of Rizvi:

THEOREM 2.3.5.

If $\lambda > -1$, a *is real,* $s_n \to \sigma|A_\lambda|$ *and* $(n+a)u_n = s_n$ $(n=0,1,\dots)$, *then*

$$u_n \to 0|A_\lambda| \ ;$$

(see [20] Lemma 2.2)

THEOREM 2.3.6.

For $\lambda > 0$ *and* $p \geqq 1$, $s_n \to \sigma[A'_\lambda]_p$ *if and only if* $s_n \to \sigma(A'_\lambda)$ *and*

$$\int_o^y \left|t \frac{d}{dt} U_\lambda(t)\right|^p dt = o(y)$$

as $y \to \infty$*;*

(see [20] Theorem 3.4 and Theorem 4.5)

THEOREM 2.3.7.

$|A'_\lambda| \subseteqq |A'_\mu|$ *for all* $\lambda \geqq \mu > 0$.

(see [20] Theorem 2.2)

The following results are due to Mishra:

THEOREM 2.3.8.

For $\lambda > -1$ *and* $p \geqq 1$, $s_n \to \sigma[A_\lambda]_p$ *if and only if* $s_n \to \sigma(A_\lambda)$ *and*

$$\int_o^y \left|t \frac{d}{dt} \sigma_\lambda(t)\right|^p dt = o(y)$$

as $y \to \infty$;

(see [15] Theorem 4)

THEOREM 2.3.9.

$|A_\lambda| \subseteqq |A_\mu|$ *for all* $\lambda \geqq \mu > -1$;

(see [16] Theorem 1)

THEOREM 2.3.10.

$|A_\lambda| \subseteqq [A_\lambda]$ *for all* $\lambda > -1$.

(see [16] Theorem 6)

2.4. *THEOREMS WHICH GIVE RELATIONS BETWEEN THE "A" METHODS AND THE "A'" METHODS.*

The following is a result due to Borwein:

THEOREM 2.4.1.

$(A'_\lambda) \cong (A_{\lambda-1})$ *for all* $\lambda > 0$.

(see [5] Theorem 2)

The following analogous results for strong and absolute Abel-type methods are due to Rizvi:

THEOREM 2.4.2.

For $\lambda > 0$ *and* $p \geqq 1$, $[A'_\lambda]_p \cong [A_{\lambda-1}]_p$;

(see [20] Theorem 3.7 and Theorem 4.7)

THEOREM 2.4.3.

For $\lambda > 0$, $|A'_\lambda| \cong |A_{\lambda-1}|$.

(see [20] Theorem 2.5)

2.5. *THEOREMS WHICH GIVE RELATIONS BETWEEN THE METHODS "B" AND BETWEEN THE METHODS "B'".*

The following known results are due to Borwein and Shawyer:

THEOREM 2.5.1.

(i) *If* $s_n \to \sigma|B,\alpha,\beta|$ *then* $s_n \to \sigma(B,\alpha,\beta)$;

(ii) *If* $s_n \to \sigma|B',\alpha,\beta|$ *then* $s_n \to \sigma(B',\alpha,\beta)$;

(see [7] Theorem 1)

THEOREM 2.5.2.

(i) *For* $p \geqq 1$, $s_n \to \sigma[B,\alpha,\beta]_p$ *if and only if* $s_n \to \sigma(B,\alpha,\beta)$ *and*

$$\int_0^y e^t |S'_{\alpha,\beta}(t)|^p \, dt = o(e^y)$$

as $y \to \infty$;

(ii) *For* $p \geqq 1$, $s_n \to \sigma[B',\alpha,\beta]_p$ *if and only if* $s_n \to \sigma(B',\alpha,\beta)$ *and*

$$\int_0^y e^t |A'_{\alpha,\beta}(t)|^p \, dt = o(e^y)$$

as $y \to \infty$.

(see [7] Theorem 11 and [8] Theorem 11, respectively)*

2.6. *THEOREMS WHICH GIVE RELATIONS BETWEEN THE "B" METHODS AND THE "B'" METHODS.*

The following is due to Borwein:

THEOREM 2.6.1.

$s_n \to \sigma(B',\alpha,\beta)$ *if and only if* $s_n \to \sigma(B,\alpha,\beta+1)$.

(see [4] Theorem 4)

The following analogous results for strong and absolute summability are due to Borwein and Shawyer:

THEOREM 2.6.2.

For $p \geqq 1$, $s_n \to \sigma[B',\alpha,\beta]_p$ *if and only if* $s_n \to \sigma[B,\alpha,\beta+1]_p$;

(see [7] Theorem 18 and [8] Theorem 18, respectively)*

THEOREM 2.6.3.

$s_n \to \sigma|B',\alpha,\beta|$ *if and only if* $s_n \to \sigma|B,\alpha,\beta+1|$.

(see [7] Theorem 17)

CHAPTER III

RELATIONS BETWEEN ABEL-TYPE AND BOREL-TYPE METHODS OF SUMMABILITY, I

3.1. *INTRODUCTION.*

It can be shown that a series summable by the Abel method is not necessarily summable by the Borel exponential method of summability, and that a series which is summable by the Borel exponential method is not, in general, summable by the Abel method of summability, but that under certain conditions, both methods will sum the same series to the same sum. We will discuss in this chapter that under certain conditions, a series which is summable by a Borel-type method is also summable by an Abel-type method of summability to the same sum. In other direction, that under certain conditions, a series which is summable by an Abel-type method is also summable by a Borel-type method of summability to the same sum, will be discussed in the next two chapters.

In 1931 Doetsch proved the following:

THEOREM A.

If (*i*) $s_n \to \sigma(B)$ *and*

(*ii*) $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

$$s_n \to \sigma(A)$$

(*see* [30] *Satz 1*)

In 1961 Włodarski proved the following generalization of Theorem A, that is

THEOREM B.

If (*i*) $s_n \to \sigma(B,\alpha,\beta)$ *and*

(*ii*) $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

$$s_n \to \sigma(A) .$$

(*see* [24] *Theorem 7*)

The last result was extended by Shawyer to absolute summability, that is

THEOREM C.

If (*i*) $s_n \to \sigma|B,\alpha,\beta|$ *and*

(*ii*) $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

$$s_n \to \sigma|A| .$$

(*see* [21] *Theorem 2*)

The main object of this chapter is to replace the Abel method by the more general Abel-type method and give result for ordinary, strong and absolute summability.

3.2. *STATEMENT OF THEOREMS.*

THEOREM 3.2.1.

If *(i)* $s_n \to \sigma(B,\alpha,\beta)$ *and*

(ii) $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

for all $\lambda > -1$

$$s_n \to \sigma(A_\lambda) .$$

We note that Theorem B is a special case that $\lambda = 0$.

THEOREM 3.2.2.

If *(i)* $s_n \to \sigma(B,\alpha,\beta)$ *and*

(ii) $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

for all $\lambda > -1$ *and* $p \geq 1$

$$s_n \to \sigma[A_\lambda]_p .$$

In view of Theorem 2.3.8, Theorem 3.2.1 is a special case of Theorem 3.2.2.

THEOREM 3.2.3.

If *(i)* $s_n \to \sigma[B,\alpha,\beta]_q$, $q \geq 1$ *and* *(ii)* $\sum_{n=0}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

for all $\lambda > -1$ *and* $p \geqq 1$

$$s_n \to \sigma[A_\lambda]_p .$$

HEOREM 3.2.4.

If *(i)* $s_n \to |B,\alpha,\beta|$ *and*

(ii) $\sum_{n=0}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

for all $\lambda > -1$

$$s_n \to \sigma|A_\lambda| .$$

We note that Theorem C is a special case that $\lambda = 0$.

The following are analogous results which give relations between the "B'" methods and the "A'" methods:

THEOREM 3.2.5.

If *(i)* $s_n \to \sigma(B',\alpha,\beta)$ *and*

(ii) $\sum_{n=0}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then*

for all $\lambda > 0$

$$s_n \to \sigma(A'_\lambda) .$$

THEOREM 3.2.6.

If *(i)* $s_n \to \sigma(B',\alpha,\beta)$ *and* *(ii)* $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$*, then*

for all $\lambda > 0$ *and* $p \geqq 1$

$$s_n \to \sigma[A'_\lambda]_p \ .$$

THEOREM 3.2.7.

If *(i)* $s_n \to \sigma[B',\alpha,\beta]_q$, $q \geqq 1$, *and*

(ii) $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$*, then*

for all $\lambda > 0$ *and* $p \geqq 1$

$$s_n \to [A'_\lambda]_p \ .$$

Finally, we have

THEOREM 3.2.8.

If *(i)* $s_n \to \sigma|B',\alpha,\beta|$ *and*

(ii) $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$*, then*

for all $\lambda > 0$

$$s_n \to \sigma|A'_\lambda| \ .$$

We may demonstrate these relations by the following diagram: under the conditions that $\sum_{n=o}^{\infty} a_n x^n$ is convergent for $|x| < 1$, $\lambda > 0$, $p \geqq 1$ and $q \geqq 1$,

TABLE I

$s_n \to \sigma\|B',\alpha,\beta\|$	$\Rightarrow$	$s_n \to \sigma\|A'_\lambda\|$
$\Downarrow$		$\Downarrow$
$s_n \to \sigma[B',\alpha,\beta]_q$	$\Rightarrow$	$s_n \to \sigma[A'_\lambda]_p$
$\Downarrow$		$\Downarrow$
$s_n \to \sigma(B',\alpha,\beta)$	$\Rightarrow$	$s_n \to \sigma(A'_\lambda)$
$\Uparrow$		$\Updownarrow$
$s_n \to \sigma(B,\alpha,\beta)$	$\Rightarrow$	$s_n \to \sigma(A_{\lambda-1})$
$\Uparrow$	$\searrow$	$\Uparrow$
$s_n \to \sigma[B,\alpha,\beta]_q$	$\Rightarrow$	$s_n \to \sigma[A_{\lambda-1}]_p$
$\Uparrow$		$\Uparrow$
$s_n \to \sigma\|B,\alpha,\beta\|$	$\Rightarrow$	$s_n \to \sigma\|A_{\lambda-1}\|$.

3.3. *PRELIMINARY RESULTS.*

The following lemmas are required for the proof of those theorems stated in the previous section:

LEMMA 3.3.1. (D. Borwein, Personal Communication)

Suppose that $-\infty \leq a < b \leq \infty$ *and Let*

$$F(w) = \int_a^b g(w,u)f(u)du$$

where $g(w,u)f(u) \in L(a,b)$ *for all* $w \geq k$.

If, for all u *in* (a,b)

$$\int_k^\infty |d_w g(w,u)| \leq h(u), \textit{ where } h(u)f(u) \in L(a,b),$$

then

$$\int_k^\infty |dF(w)| \leq \int_a^b h(u)|f(u)|du.$$

PROOF.

Let $k = w_o < w_1 < \ldots < w_m$, then for all m

$$\sum_{r=o}^{m-1} |F(w_{r+1}) - F(w_r)| \leq \int_a^b \sum_{r=o}^{m-1} |g(w_{r+1},u)-g(w_r,u)||f(u)|du$$

$$\leq \int_a^b \int_k^\infty |d_w g(w,u)||f(u)|du$$

$$\leq \int_a^b h(u)|f(u)|du \ .$$

Hence

$$\int_k^\infty |dF(u)| \leq \int_a^b h(u)|f(u)|du \ .$$

LEMMA 3.3.2.

Suppose that m *is a positive integer and that*

$$\mu_n = \prod_{r=1}^m \frac{b_r n+c_r}{d_r n+e_r} s_n \quad \textit{for} \quad n = 0, 1, \ldots$$

where b_r, c_r, d_r *and* e_r *are all real with* $\prod_{r=1}^m d_r \neq 0$ *and* $\prod_{r=1}^m (d_r n + e_r) \neq 0$ *for each* $n = 0, 1, \ldots$. *If* $s_n \to \sigma(A_\lambda)$, *then*

$$\mu_n \to \prod_{r=1}^m \frac{b_r}{d_r} \sigma(A_\lambda) \ \textit{where} \ \lambda > -1.$$

PROOF.

In view of Theorem 2.3.4, for each real number k, and for $\lambda>-1$

$$\frac{s_n}{n+k} \to O(A_\lambda)$$

It follows that

$$\frac{bn+c}{dn+e} s_n = \frac{b}{d} s_n + \frac{cd-be}{dn^2+de} s_n \to \frac{b}{d} \sigma(A_\lambda) ,$$

for all real b,c,d and e with $d \neq 0$ and $dn+e \neq 0$ for each $n = 0, 1, \ldots$.

The conclusion then follows immediately by repeatedly using the above result.

LEMMA 3.3.3.

Let $\lambda > -1$, and μ_n be defined as in Lemma 3.3.2. If $s_n \to \sigma|A_\lambda|$, then

$$\mu_n \to \prod_{r=1}^{m} \frac{b_r}{d_r} \sigma|A_\lambda| .$$

PROOF.

In view of Theorem 2.3.5, for each real number k and $\lambda > -1$

$$\frac{s_n}{n+k} \to O|A_\lambda|$$

The conclusion then follows by applying the same argument as in the proof of the previous lemma.

LEMMA 3.3.4.

For $\lambda > -1$. *Let*

$$I(t) = \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_o^\infty e^{-u/t} u^\lambda f(u)du, \text{ for all } t > 0.$$

If $f(u) \to \sigma$ *as* $u \to \infty$, *then* $I(t) \to \sigma$ *as* $t \to \infty$.

PROOF.

This is a special case of a standard result.

(see [12] Theorem 6)

LEMMA 3.3.5.

If $\lambda > -1$, *then the series* $\sum_{n=o}^{\infty} E_n^\lambda s_n x^n$ *is convergent for* $|x| < 1$ *if and only if the series* $\sum_{n=o}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$.

PROOF.

Since

$$E_n^\lambda \sim \frac{n^\lambda}{\Gamma(\lambda+1)}$$

as $n \to \infty$, we have that

$$\lim_{n\to\infty} |E_n^\lambda|^{1/n} = 1.$$

It follows that the radius of convergence of $\sum_{n=0}^{\infty} E_n^\lambda s_n x^n$ is equal to the radius of convergence of $\sum_{n=0}^{\infty} s_n x^n$.
By virtue of Lemma 2.2.2, the conclusion follows immediately.

LEMMA 3.3.6.

Let

$$\nu_n = \begin{cases} \dfrac{\Gamma(\alpha n+\beta+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n+\lambda+1)}, & \text{for } n = N, N+1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

If $s_n \to \sigma(B,\alpha,\beta)$ *and* $\sum_{n=0}^{\infty} a_n x^n$ *is convergent for* $|x| < 1$, *then for all* $\lambda > -1$

$$\nu_n s_n \to \alpha^\lambda \sigma(A_\lambda) .$$

PROOF.

Let $J(t)$ be defined as in Lemma 2.2.1, that is

$$J(t) = \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-u/t}\, u^\lambda S_{\alpha,\beta}(u)du, \; t > 0.$$

By Lemma 2.2.1

$$J(t) = \alpha\left(\frac{t}{1+t}\right)^{\beta-1}\left(\frac{1+y}{1+t}\right)^{\lambda+1} \sigma_\lambda^*(y) ,$$

where

$$\sigma_\lambda^*(y) = (1+y)^{-\lambda-1} \sum_{n=N}^{\infty} E_n^\lambda \nu_n s_n \left(\frac{y}{1+y}\right)^n ,$$

d t and y are related by

$$\left(\frac{t}{1+t}\right)^{\alpha} = \frac{y}{1+y} \ .$$

hypothesis, $S_{\alpha,\beta}(u) \to \sigma$ as $u \to \infty$, it follows from Lemma 3.3.4 hat

$$J(t) \to \sigma$$

s $t \to \infty$. Also, we have

$$\frac{1+y}{1+t} \to \frac{1}{\alpha}$$

as $y \to \infty$, and $t \to \infty$ if and only if $y \to \infty$.
Hence

$$\sigma_{\lambda}^{*}(y) \to \alpha^{\lambda}\sigma$$

as $y \to \infty$. Furthermore, it is easily shown that $\sum_{n=o}^{\infty} \nu_n s_n x^n$ is convergent for $|x| < 1$. Therefore, in view of Lemma 3.3.5 and by definition

$$\nu_n s_n \to \alpha^{\lambda}\sigma(A_{\lambda}) \ .$$

This completes the proof.

LEMMA 3.3.7.

Let ν_n be defined as in the previous lemma. For $\lambda > -1$,

f $s_n \to \sigma|B,\alpha,\beta|$, *then*

$$\nu_n s_n \to \alpha^{\lambda}\sigma|A_{\lambda}| \ .$$

ROOF.

In view of Lemma 2.2.1, we have

$$J(t) = \alpha\left(\frac{t}{1+t}\right)^{\beta-1}\left(\frac{1+y}{1+t}\right)^{\lambda+1} \sigma_{\lambda}^{*}(y) \ ,$$

write

$$\sigma_{\lambda}^{*}(y) = \frac{1}{\alpha} J(t)A_1(t)A_2(t), \text{ say,}$$

where

$$A_1(t) = \left(\frac{1+t}{t}\right)^{\beta-1} \text{ and } A_2(t) = \left(\frac{1+t}{1+y}\right)^{\lambda+1} \ .$$

For $h > 0$, it is easy to show that $A_1(t)$ and $A_2(t)$ are of bounded variation with respect to t in $[h, \infty)$.
For $t \geq h > 0$, we have

$$J(t) = \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^{\infty} e^{-u/t} u^{\lambda} S_{\alpha,\beta}(u)du$$

$$= \frac{1}{\Gamma(\lambda+1)} \int_0^{\infty} e^{-v} v^{\lambda} S_{\alpha,\beta}(tv)dv \ .$$

By assumption, $S_{\alpha,\beta}(u)$ is of bounded variation with respect to u in $[0, \infty)$, thus

$$\int_h^{\infty} |d_t S_{\alpha,\beta}(tv)| \leq \int_0^{\infty} |S'_{\alpha,\beta}(u)|du \leq M.$$

It follows from Lemma 3.3.1 that

$$\int_h^\infty |J'(t)|dt \leq M\int_0^\infty \left|\frac{1}{\Gamma(\lambda+1)} e^{-u}u^\lambda\right|du$$
$$= M .$$

Thus $J(t)$ is of bounded variation with respect to t in $[h, \infty)$, and so, $\sigma_\lambda^*(y)$ is of bounded variation with respect to y in the range $[g, \infty)$, where g, greater than zero, is dependent on h. Since $\frac{d}{dy}\sigma_\lambda^*(y)$ is continuous in $[0, g]$, it follows that $\sigma_\lambda^*(y)$ is of bounded variation with respect to y in $[0, \infty)$. In view of Theorem 2.5.1 (i), by assumption

$$s_n \to \sigma(B,\alpha,\beta) .$$

By virtue of the previous lemma, we have

$$\nu_n s_n \to \alpha^\lambda \sigma(A_\lambda) .$$

Thus by definition

$$\nu_n s_n \to \alpha^\lambda \sigma|A_\lambda| .$$

3.4. *PROOFS OF THE THEOREMS.*

3.4.1. *Proof of Theorem 3.2.1.*

In view of Theorem 2.3.2, we may assume that λ is an integer. By Lemma 3.3.6, we have

$$\nu_n s_n \to \alpha^\lambda \sigma(A_\lambda) ,$$

where

$$\nu_n = \frac{\Gamma(\alpha n+\beta+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n+\lambda+1)}$$

$$= \frac{(\alpha n+\beta+\lambda-1)(\alpha n+\beta+\lambda-2)\dots(\alpha n+\beta)}{(n+\lambda)(n+\lambda-1)\dots(n+1)}, \quad n \geq N.$$

By Lemma 3.3.2, we have

$$s_n = s_n \nu_n \frac{1}{\nu_n} \to \sigma(A_\lambda) .$$

This completes the proof.

3.4.2. *Proof of Theorem 3.2.2.*

In view of Theorem 2.3.8, by the previous theorem, it is sufficient to prove that

$$\int_0^x y \left| \frac{d}{dy} \sigma_\lambda(y) \right|^p dy = o(x)$$

as $x \to \infty$. Since

$$y \frac{d}{dy} \sigma_\lambda(y) = (\lambda+1)[\sigma_{\lambda+1}(y) - \sigma_\lambda(y)],$$

and by virtue of the previous theorem again, we have

$$\sigma_{\lambda+1}(y) - \sigma_\lambda(y) = o(1)$$

s $y \to \infty$. Thus for all $p \geq 1$, we have

$$y\left|\frac{d}{dy}\sigma_\lambda(y)\right|^p = o(1)$$

s $y \to \infty$. Hence

$$\int_o^x y\left|\frac{d}{dy}\sigma_\lambda(y)\right|^p dy = o(x)$$

as $x \to \infty$.

This completes the proof.

3.4.3. *Proof of Theorem 3.2.3.*

For all $q \geq 1$, in view of Theorem 2.5.2(i), we have, the assumption that

$$s_n \to \sigma[B,\alpha,\beta]_q$$

implies that

$$s_n \to \sigma(B,\alpha,\beta) .$$

It follows from Theorem 3.2.2 that for all $\lambda > -1$ and $p \geq 1$

$$s_n \to \sigma[A_\lambda]_p .$$

This completes the proof.

4.4. *Proof of Theorem 3.2.4.*

In view of Theorem 2.3.9, we may assume that λ is an nteger. By Lemma 3.3.7, we have

$$\nu_n s_n \to \alpha^\lambda \sigma |A_\lambda| ,$$

here

$$\nu_n = \frac{\Gamma(\alpha n+\beta+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n+\lambda+1)}$$

$$= \frac{(\alpha n+\beta+\lambda-1)(\alpha n+\beta+\lambda-2)\ldots(\alpha n+\beta)}{(n+\lambda)(n+\lambda-1)\ldots(n+1)} , \quad n \geq N.$$

By Lemma 3.3.3, we have

$$s_n = \nu_n s_n \frac{1}{\nu_n} \to \sigma |A_\lambda| .$$

This completes the proof.

3.4.5. *Proof of Theorem 3.2.5.*

In view of Theorem 2.6.1, by assumption

$$s_n \to \sigma(B,\alpha, \beta+1).$$

Since Theorem 3.2.1 is valid for all $\alpha > 0$ and all real β , we have for all $\lambda > -1$

$$s_n \to \sigma(A_\lambda) .$$

Theorem 2.4.1, we have

$$s_n \to \sigma(A'_{\lambda+1}) ,$$

r all $\lambda > -1$. It follows immediately that for all $\lambda > 0$

$$s_n \to \sigma(A'_{\lambda}) .$$

is completes the proof.

4.6. *Proof of Theorem 3.2.6.*

In view of Theorem 2.6.1, by assumption

$$s_n \to \sigma(B,\alpha,\beta+1).$$

ince Theorem 3.2.2 is valid for all $\alpha > 0$ and all real β, e have for all $\lambda > -1$ and all $p \geq 1$

$$s_n \to \sigma[A_\lambda]_p .$$

In view of Theorem 2.4.2, we have

$$s_n \to \sigma[A'_{\lambda+1}]_p ,$$

for all $\lambda > -1$. It follows immediately that for all $\lambda > 0$ and $p \geq 1$

$$s_n \to \sigma[A'_\lambda]_p \ .$$

is completes the proof.

.4.7. *Proof of Theorem 3.2.7.*

In view of Theorem 2.5.2(ii), for all $q \geq 1$, by assumption

$$s_n \to \sigma(B',\alpha,\beta).$$

y Theorem 3.2.6, we have for all $\lambda > 0$ and all $p \geq 1$

$$s_n \to \sigma[A'_\lambda]_p \ .$$

This completes the proof.

3.4.8. *Proof of Theorem 3.2.8.*

In view of Theorem 2.6.3, by assumption

$$s_n \to \sigma|B,\alpha,\beta+1| \ .$$

Since Theorem 3.4.4 is valid for all $\alpha > 0$ and all real β, it follows immediately that for all $\lambda > -1$

$$s_n \to \sigma|A_\lambda| \ .$$

Thus, in view of Theorem 2.4.3

$$s_n \to \sigma|A'_{\lambda+1}| ,$$

ere $\lambda > -1$. Hence for all $\lambda > 0$

$$s_n \to \sigma|A'_{\lambda}| .$$

his completes the proof.

CHAPTER IV

RELATIONS BETWEEN ABEL-TYPE AND BOREL-TYPE METHODS OF SUMMABILITY, II

4.1. *INTRODUCTION.*

In 1931, Doetsch proved the result that under a certain kind of Tauberian condition, a series which is summable by the Abel method is also summable by the Borel method to the same sum; that is

THEOREM A.

If *(i)* $s_n \to \sigma(A)$;

(ii) s_n *is real for each* $n = 0,1,\ldots,$ *and*

(iii) $\sum_{n=0}^{\infty} \frac{a_{n+1}x^n}{n!} = 0_L(\frac{e^x}{x})$ (*) , *then*

$$s_n \to \sigma(B).$$

(see [9] p. 405)

(*) $f(x) = 0_L(g(x))$ means that there is a positive constant M such that $f(x) \geq -Mg(x)$ for large x, where $g(x) > 0$ for all x.

The main object of this chapter is to replace the Abel thod and the Borel method by the more general Abel-type thod and the Borel-type method, respectively. The relations tween strong and absolute methods of summability are also vestigated.

In this chapter, we only consider methods A_λ for which ≥ 0. Negative values of λ will be considered in the next hapter. Because of the inclusion $A_\lambda \Rightarrow A_\mu$ whenever $\lambda \geq \mu > -1$ Theorem 2.3.2), we will state our theorems only for the case $= 0$ and note that they also hold whenever $\lambda \geq 0$.

It is supposed, for convenience, in this chapter, that $_n$ is real for each $n = 0, 1, \ldots$.

.2. *DEFINITIONS AND PRELIMINARY RESULTS.*

A real function $f(x)$, defined for $x > 0$, is said to be slowly decreasing if

$$\liminf \{f(x) - f(y)\} \geq 0$$

whenever

$$y \to \infty, \; x > y, \; x/y \to 1.$$

The following preliminary results are required for the proofs of the theorems stated in the subsequent section:

EMMA 4.2.1.

If (i) $I(y) = \int_0^\infty e^{-yt} d\alpha(t)$ *is convergent for all* $y > 0$;

(ii) $I(y) \to \sigma$ *as* $y \to 0$ *and*

(iii) $\alpha(t)$ *is slowly decreasing.*

en

$$\alpha(t) \to \sigma$$

as $t \to \infty$.

(see [12] Theorem 105)

LEMMA 4.2.2. ([12] p. 125)

If $f(x)$ *is differentiable and* $xf'(x) = O_L(1)$, *then* $f(x)$ *is slowly decreasing.*

PROOF.

We have

$$f(y) - f(x) = \int_x^y f'(t)dt$$

$$\geqq -M\log \frac{y}{x} \quad .$$

Hence, under the conditions that

$$x \to \infty,\ y > x \text{ and } y/x \to 1,$$

we have

$$\liminf \{f(y) - f(x)\} \geq 0.$$

Thus $f(x)$ is slowly decreasing by definition.

LEMMA 4.2.3.

If $f(x) = O_L(g(x))$, *then for large* x, *there is a positive constant* M *such that*

$$|f(x)| \leq f(x) + M\, g(x)\ .$$

PROOF.

If $f(x) \geq 0$, the result is obvious; if $f(x) < 0$, then

$$\begin{aligned} |f(x)| = -f(x) &\leq M\, g(x) \\ &= 2Mg(x) - Mg(x) \\ &\leq 2Mg(x) + f(x)\ . \end{aligned}$$

This completes the proof.

LEMMA 4.2.4.

If $f(x) = O_R(g(x))^{(*)}$, *then for large* x, *there is*

(*) $f(x) = O_R(g(x))$ means that there is a positive constant M such that $f(x) \leq Mg(x)$ for large x, where $g(x) > 0$ for all x.

ositive constant M such that

$$|f(x)| \leqq M\, g(x) - f(x) .$$

OOF.

If $f(x) \leqq 0$, the result is obvious; if $f(x) > 0$, then

$$\begin{aligned} |f(x)| &= f(x) \\ &\leqq Mg(x) \\ &= 2Mg(x) - Mg(x) \\ &\leqq Mg(x) - f(x). \end{aligned}$$

is completes the proof.

3. *STATEMENTS OF THEOREMS.*

EOREM 4.3.1.

If *(i)* $s_n \to \sigma(A)$ *and*

(ii) $S_{\alpha,\beta}(u)$ *is slowly decreasing, then*

$$s_n \to \sigma(B,\alpha,\beta) .$$

EMARK. In view of Theorem 2.3.2, we can replace (A) by (A_λ) the above theorem, where $\lambda \geqq 0$. This comment will apply o each of the following theorems.

THEOREM 4.3.2.

(I) *If* *(i)* $s_n \to \sigma(A)$ *and*

(ii) $uS'_{\alpha,\beta}(u) = O_L(1)$, *then*

$$s_n \to \sigma[B,\alpha,\beta]:$$

(II) *If* *(i)* $s_n \to \sigma(A)$ *and*

(ii) $uS'_{\alpha,\beta}(u) = O_R(1)$, *then*

$$s_n \to \sigma[B,\alpha,\beta].$$

Each of the other theorems in this section can be stated in parts corresponding to the parts of Theorem 4.3.2, we shall only state the parts corresponding to (I) in each case; the proofs of the other parts are obtained by using corresponding lemmas in the previous section.

THEOREM 4.3.3.

If *(i)* $s_n \to \sigma(A)$ *and*

(ii) $S'_{\alpha,\beta}(u) = O_L(\phi(u))$, *where*

$$u\phi(u) = O(1)$$

and

$$\int_k^\infty \phi(u)du < \infty \text{ for some } k > 0, \text{ then}$$

$$s_n \to \sigma|B,\alpha,\beta| .$$

THEOREM 4.3.4.

If (i) $s_n \to \sigma(A)$ *and*

(ii) $A_{\alpha,\beta}(u)$ *is slowly decreasing, then*

$$s_n \to \sigma(B',\alpha,\beta) .$$

THEOREM 4.3.5.

If (i) $s_n \to \sigma(A)$ *and*

(ii) $uA'_{\alpha,\beta}(u) = O_L(1)$, *then*

$$s_n \to \sigma[B',\alpha,\beta] .$$

THEOREM 4.3.6.

If (i) $s_n \to \sigma(A)$ *and*

(ii) $A'_{\alpha,\beta}(u) = C_L(\phi(u))$, *where*

$$u\phi(u) = O(1)$$

and

$$\int_k^\infty \phi(u)du < \infty \text{ for some } k > 0, \text{ then}$$

$$s_n \to \sigma|B',\alpha,\beta| .$$

We see from the above theorems that when more stringent conditions are imposed, stronger conclusions are obtained.

We denote condition (ii) of Theorem 4.3.3 and condition (ii) of Theorem 4.3.6 by C(S) and C(A), respectively; that is

C(S): $S'_{\alpha,\beta}(u) = O_L(\phi(u))$, where $u\phi(u) = O(1)$ and $\phi(u)$ is integrable over (k,∞) for some $k > 0$;

C(A): $A'_{\alpha,\beta}(u) = O_L(\phi(u))$, where $u\phi(u) = O(1)$ and $\phi(u)$ is integrable over (k,∞) for some $k > 0$.

A table showing the relations between the hypotheses and conclusions of the above theorems is given below.

TABLE II

For $\lambda \geqq 0$ and if a_n is real for each $n = 0, 1, \ldots$;

$$
\begin{array}{lcl}
s_n \to \sigma(A_\lambda) \text{ and } C(A) & \Rightarrow & s_n \to \sigma|B',\alpha,\beta|; \\
\Downarrow & & \Downarrow \\
s_n \to \sigma(A_\lambda) \text{ and } uA'_{\alpha,\beta}(u) = O_L(1) & \Rightarrow & s_n \to \sigma[B',\alpha,\beta]; \\
\Downarrow & & \Downarrow \\
s_n \to \sigma(A_\lambda) \text{ and } A_{\alpha,\beta}(u) \text{ is slowly decreasing} & \Rightarrow & s_n \to \sigma(B',\alpha,\beta); \\
 & & \Uparrow \\
s_n \to \sigma(A_\lambda) \text{ and } S_{\alpha,\beta}(u) \text{ is slowly decreasing} & \Rightarrow & s_n \to \sigma(B,\alpha,\beta); \\
\Uparrow & & \Uparrow \\
s_n \to \sigma(A_\lambda) \text{ and } uS'_{\alpha,\beta}(u) = O_L(1) & \Rightarrow & s_n \to \sigma[B,\alpha,\beta]; \\
\Uparrow & & \Uparrow \\
s_n \to \sigma(A_\lambda) \text{ and } C(S) & \Rightarrow & s_n \to \sigma|B,\alpha,\beta| .
\end{array}
$$

4.4. *PROOFS OF THE THEOREMS.*

4.4.1. *Proof of Theorem 4.3.1.*

Let

$$J(t) = \frac{1}{t}\int_0^\infty e^{-u/t} S_{\alpha,\beta}(u)\,du, \quad t > 0.$$

Since $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$, we know (as in the proof of Lemma 2.2.1) that

$$J(t) = \alpha\left(\frac{t}{1+t}\right)^{\beta-1}\left(\frac{1+y}{1+t}\right)\sigma_0^*(y),$$

where

$$\sigma_0^*(y) = (1+y)^{-1}\sum_{n=0}^{\infty} s_n\left(\frac{y}{1+y}\right)^n$$

$$= \sigma_0(y),$$

and t and y are related by

$$\left(\frac{t}{1+t}\right)^{\alpha} = \frac{y}{1+y} \quad .$$

We note that $t \to \infty$ if and only if $y \to \infty$, and that

$$\frac{1+y}{1+t} \to \frac{1}{\alpha}$$

as $t \to \infty$.

Thus

$$J(t) \to \sigma$$

as $t \to \infty$.

On the other hand, since

$$|S_{\alpha,\beta}(u)| \leq M_\varepsilon e^{\varepsilon u} \text{ for each } \varepsilon > 0, \text{ where } M_\varepsilon \text{ depends on } \varepsilon,$$

it follows that

$$J(t) = \int_0^\infty e^{-u/t} dS_{\alpha,\beta}(u) .$$

In view of Lemma 4.2.1, we have that

$$S_{\alpha,\beta}(u) \to \sigma$$

as $u \to \infty$.

This completes the proof.

4.4.2. *Proof of Theorem 4.3.2.*

(I)

In view of Lemma 4.2.2, condition (ii) implies that $S_{\alpha,\beta}(u)$ is slowly decreasing, so that by the previous theorem

$$S_{\alpha,\beta}(u) \to \sigma$$

as $u \to \infty$.

By virtue of Theorem 2.5.2 (i), it is sufficient to show that

$$\int_0^y e^t |S'_{\alpha,\beta}(t)|\,dt = o(e^y)$$

as $y \to \infty$.

In view of Lemma 4.2.3, condition (ii) implies that there is a positive constant M such that

$$|S'_{\alpha,\beta}(u)| \leq S'_{\alpha,\beta}(u) + \frac{M}{u} \quad \text{for large } u,$$

so that

$$\int_0^y e^t |S'_{\alpha,\beta}(t)|\,dt \leq \int_0^{y_0} e^t |S'_{\alpha,\beta}(t)|\,dt + \int_{y_0}^y e^t S'_{\alpha,\beta}(t)\,dt + \int_{y_0}^y \frac{Me^t}{t}\,dt$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

It is clear that

$$I_1 = o(e^y) \text{ and } I_3 = o(e^y)$$

as $y \to \infty$.

Also

$$|I_2| = |e^y S_{\alpha,\beta}(y) + M - \int_{y_0}^y e^t S_{\alpha,\beta}(t)\,dt|$$

$$\leq e^y |S_{\alpha,\beta}(y) - \sigma| + M + \int_{y_0}^y e^t |S_{\alpha,\beta}(t) - \sigma|\,dt$$

$$= o(e^y) \text{ as } y \to \infty.$$

Thus

$$\int_0^y e^t |S'_{\alpha,\beta}(t)| dt = o(e^y)$$

as $y \to \infty$.

This completes the proof of part (I).

(II)

Let

$$T_{\alpha,\beta}(u) = -S_{\alpha,\beta}(u) \text{ for each } u \geq 0,$$

then

$$uT'_{\alpha,\beta}(u) = O_L(1),$$

and so by Lemma 4.2.2, $T_{\alpha,\beta}(u)$ is slowly decreasing.

It is clear that

$$\int_0^\infty e^{-u/t} T_{\alpha,\beta}(u) du \to -\sigma$$

as $u \to \infty$.

It follows from Theorem 4.3.1 that

$$T_{\alpha,\beta}(u) \to -\sigma$$

as $u \to \infty$.

That is

$$S_{\alpha,\beta}(u) \to \sigma$$

as $u \to \infty$.

Furthermore

$$\int_o^y e^t |S'_{\alpha,\beta}(t)| dt = \int_o^y e^t |T'_{\alpha,\beta}(u)| dt,$$

and since $uT'_{\alpha,\beta}(u) = O_L(1)$, by applying the same argument as in the proof of part (I), we have that

$$\int_o^y e^t |T'_{\alpha,\beta}(t)| dt = o(e^y)$$

as $y \to \infty$.

Thus

$$\int_o^y e^t |S'_{\alpha,\beta}(t)| dy = o(e^y)$$

as $y \to \infty$.

Therefore

$$s_n \to \sigma[B,\alpha,\beta] \ .$$

This completes the proof of part (II).

4.4.3. *Proof of Theorem 4.3.3.*

The conditions

$$S'_{\alpha,\beta}(u) = O_L(\phi(u)) \text{ and } u\phi(u) = O(1)$$

together imply that

$$uS'_{\alpha,\beta}(u) = O_L(1).$$

By virtue of Theorem 4.3.2. (I) and Theorem 2.5.2 (i)

$$s_n \to \sigma(B,\alpha,\beta).$$

It is therefore sufficient to show that $S_{\alpha,\beta}(u)$ is of bounded variation with respect to u in the range $[0,\infty)$.
In view of Lemma 4.2.3, there is a positive constant M such that for all $u \geqq u_o \geqq k$,

$$|S'_{\alpha,\beta}(u)| \leqq S'_{\alpha,\beta}(u) + M\phi(u) .$$

Since

$$\int_k^\infty \phi(u)du < \infty,$$

it follows that

$$\int_{u_o}^\infty |S'_{\alpha,\beta}(u)|du \leqq \int_{u_o}^\infty S'_{\alpha,\beta}(u)du + M\int_{u_o}^\infty \phi(u)du$$
$$\leqq M.$$

Furthermore, $S'_{\alpha,\beta}(u)$ is continuous in $[0,u_o]$. Hence $S_{\alpha,\beta}(u)$ is of bounded variation with respect to u in the range $[0,\infty)$. This completes the proof.

4.4.4. *Proof of Theorem 4.3.4.*

Let

$$I(t) = \frac{1}{t}\int_o^\infty e^{-u/t}A_{\alpha,\beta}(u)du \ .$$

Since $\sum_{n=o}^{\infty} a_n x^n$ is convergent for $|x| < 1$, it follows that

$$a_n = O\{(1+\delta)^n\} \quad \text{for all } \delta > 0.$$

By applying the same argument as in the proof of Lemma 2.2.1, we have

$$A_{\alpha,\beta}(u) = O(e^{u\delta}) \ .$$

Hence

$$I(t) = \int_o^\infty e^{-u/t}dA_{\alpha,\beta}(u)$$

$$= \int_o^\infty e^{-u/t}e^{-u}\sum_{n=N}^{\infty}\frac{a_n u^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\,du$$

$$= \left(\frac{t}{1+t}\right)^\beta \sigma_o(y) - \left(\frac{t}{1+t}\right)^\beta\left(\frac{1}{1+y}\right)\sum_{n=o}^{N-1} s_n \left(\frac{y}{1+y}\right)^n ,$$

where t and y are related by

$$\left(\frac{t}{1+t}\right)^\alpha = \frac{y}{1+y} \ .$$

Since $s_n \to \sigma(A)$, it follows that

$$I(t) \to \sigma$$

as $t \to \infty$.

By virtue of Lemma 4.2.1, we have

$$A_{\alpha,\beta}(u) \to \sigma$$

as $u \to \infty$.

This completes the proof.

4.4.5. *Proof of Theorem 4.3.5.*

In view of Lemma 4.2.2, condition (ii) implies that $A_{\alpha,\beta}(u)$ is slowly decreasing, so that by the previous theorem

$$A_{\alpha,\beta}(u) \to \sigma$$

as $u \to \infty$.

By virtue of Theorem 2.5.2 (ii), it is sufficient to show that

$$\int_o^y e^t |A'_{\alpha,\beta}(u)| dt = o(e^y)$$

as $y \to \infty$.

In view of Lemma 4.2.3, condition (ii) implies that there is a positive constant M such that

$$|A'_{\alpha,\beta}(u)| \leq A'_{\alpha,\beta}(u) + \frac{M}{u} \quad \text{for large } u,$$

so that

$$\int_0^y e^t |A'_{\alpha,\beta}(t)| dt \leq \int_0^{y_0} e^t |A'_{\alpha,\beta}(t)| dt + \int_{y_0}^y e^t A'_{\alpha,\beta}(t) dt + M \int_{y_0}^y \frac{e^t}{t} dt$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

It is clear that

$$I_1 = o(e^y) \quad \text{and} \quad I_3 = o(e^y)$$

as $y \to \infty$.

Also

$$|I_2| = |e^y A_{\alpha,\beta}(y) + M - \int_{y_0}^y e^t A_{\alpha,\beta}(t) dt|$$

$$\leq e^y |A_{\alpha,\beta}(y) - \sigma| + M + \int_{y_0}^y e^t |A_{\alpha,\beta}(t) - \sigma| dt$$

$$= o(e^y) \text{ as } y \to \infty.$$

Thus

$$\int_0^y e^t |A'_{\alpha,\beta}(t)| dt = o(e^y)$$

as $y \to \infty$.

This completes the proof.

4.4.6. *Proof of Theorem 4.3.6.*

The conditions that

$$A'_{\alpha,\beta}(u) = O_L(\phi(u))$$

and

$$u\phi(u) = O(1)$$

imply that

$$uA'_{\alpha,\beta}(u) = O_L(1).$$

It follows from Theorem 4.3.5 and Theorem 2.5.2 (ii) that

$$s_n \to \sigma(B',\alpha,\beta).$$

Hence, it is sufficient to show that $A_{\alpha,\beta}(u)$ is of bounded variation with respect to u in the range $[0,\infty)$.
By virtue of Lemma 4.2.3, there is a positive constant M such that for $u \geqq u_o \geqq k$

$$|A'_{\alpha,\beta}(u)| \leqq A'_{\alpha,\beta}(u) + M\phi(u) .$$

Since $\phi(u)$ is integrable over $[u_o,\infty)$, it follows that

$$\int_{u_o}^{\infty} |A'_{\alpha,\beta}(u)|du < \infty .$$

Furthermore, $A'_{\alpha,\beta}(u)$ is continuous in $[0,u_o]$, so that $A_{\alpha,\beta}(u)$ is of bounded variation with respect to u in the range $[0,\infty)$.

This completes the proof.

CHAPTER V

RELATIONS BETWEEN ABEL-TYPE AND BOREL-TYPE METHODS OF SUMMABILITY, III

5.1. *INTRODUCTION.*

We have proved in the previous chapter some theorems which give relations between an Abel-type method (A_λ) (in the case that $\lambda \geqq 0$) and an ordinary, a strong or an absolute Borel-type method, respectively.

The object of this chapter is to prove analogous results for which $-1 < \lambda < 0$.

It is supposed for convenience, in this chapter, that a_n is real for each $n = 0, 1, \ldots$.

5.2. *DEFINITIONS AND PRELIMINARY RESULTS.*

Let

$$p_n \geqq 0, \; q_n \geqq 0, \; \sum_{\nu=0}^{\infty} p_\nu > 0 \quad \text{and}$$

$$\sum_{\nu=0}^{\infty} q_\nu > 0 \; (n = 0, 1, \ldots) .$$

Let

$$p(x) = \sum_{n=o}^{\infty} p_n x^n, \quad q(x) = \sum_{n=o}^{\infty} q_n x^n$$

and denote the radii of convergence of these power series by r_p and r_q respectively.

Let

$$p_s(x) = \frac{1}{p(x)} \sum_{n=o}^{\infty} p_n s_n x^n$$

and

$$q_s(x) = \frac{1}{q(x)} \sum_{n=o}^{\infty} q_n s_n x^n \quad .$$

If $r_p > 0$ and $\sum_{n=o}^{\infty} p_n s_n x^n$ is convergent in the open interval $(0, r_p)$, and if $p_s(x)$ tends to a finite limit σ as $x \to r_p-$, we write

$$s_n \to \sigma(P).$$

This defines the summability method (P); the method (Q), associated with the sequence $\{q_n\}$, is defined similarly.

If

$$p_n = E_n^{\lambda}, \quad n = 0, 1, \ldots,$$

then the summability method (P) is equivalent to the Abel-type method (A_λ) defined in the first chapter.

Let

$$p_n = \begin{cases} \dfrac{1}{\Gamma(\alpha n+\beta)}, & n \geqq N; \\ 0, & 0 \leqq n < N. \end{cases}$$

In view of Lemma 2.2.3,

$$\sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \sim \frac{e^x}{\alpha}$$

as $x \to \infty$. Substituting y for x^α and then writing x for y gives

$$\frac{1}{p(x)} \sim \alpha x^{(\beta-1)/\alpha} e^{-x^{\frac{1}{\alpha}}} .$$

Then setting $u = x^{\frac{1}{\alpha}}$, it follows that

$$\frac{1}{p(x)} \sum_{n=o}^{\infty} p_n s_n x^n \sim \alpha e^{-u} \sum_{n=N}^{\infty} \frac{s_n u^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} ,$$

so that the associated method (P) is equivalent to the Borel-type method (B,α,β) defined in chapter 1.

A sequence $\{\mu_n\}$ is said to be an m-sequence (moment sequence) if

$$\mu_n = \int_0^1 t^n d\chi(t) \quad (n = 0, 1, \ldots),$$

where $\chi(t)$ is a real function of bounded variation in the interval [0, 1]; if in addition

$$\mu_n \geqq \delta\int_0^1 t^n |d\chi^*(t)| \quad (0 < \delta \leqq 1,\ n = N, N+1, \ldots),$$

where $\chi^*(t)$, the associated normalized function of $\chi(t)$, is defined by

$$\chi^*(t) = \begin{cases} 0, & t = 0; \\ \frac{1}{2}\{\chi(t+) - (t-)\} - \chi(0), & 0 < t < 1; \\ \chi(1) - \chi(0), & t = 1, \end{cases}$$

we call the sequence $\{\mu_n\}$ an $\bar{m}$-sequence.

We note that ([23] p. 14 and [14] §247)

$$\int_0^1 t^n |d\chi(t)| \geqq \int_0^1 t^n |d\chi^*(t)| .$$

Hence a sequence $\{\mu_n\}$ such that

$$\mu_n = \int_0^1 t^n d\chi(t) \geqq \delta\int_0^1 t^n |d\chi(t)|$$

where $0 < \delta \leqq 1$, $n = N, N+1, \ldots$, is necessarily an $\bar{m}$-sequence.

The following preliminary results are required for the proofs of the theorems stated in the following section. The first three are all due to Borwein:

LEMMA 5.2.1.

If $0 < r_p < \infty$, *then a necessary and sufficient condition for* (P) *to be regular is that*

$$\sum_{n=o}^{\infty} p_n (r_p)^n = \infty.$$

(see [2] Theorem 1)

LEMMA 5.2.2.

If (i) g(s) *is an analytic function of* $s = b+ic$ *in the region* $b > b_o$ *such that, when* $b > b_o$ *and* $|s|$ *is large*

$$g(s) = C + O\left(\frac{1}{|s|}\right), \text{ where } C > 0;$$

(ii) g(b) *is real for* $b > b_o$ *and*

(iii) $k_n = g(n+\nu)$, *where* $\nu > 0$, $\nu - b_o > 0$, *then* $\{k_n\}$ *is an* $\bar{m}$*-sequence.*

(see [6] Lemma 3)

LEMMA 5.2.3.

If $p_n = \mu_n q_n$ $(n = 0,1,\ldots)$, *where* $\{\mu_n\}$ *is an* $\bar{m}$*-sequence, if* $r_p = r_q > 0$ *and* (P) *is regular, then*

$$(Q) \subseteq (P) .$$

(see [2] Theorem A')

LEMMA 5.2.4.

Suppose that $-1 < \lambda < 0$ *and*

$$\mu_n = \frac{\Gamma(\alpha n+\beta+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n+\lambda+1)} \quad (n = N, N+1, \ldots) .$$

Then

$$s_n \to \sigma(A_\lambda)$$

if and only if

$$\mu_n s_n \to \alpha^\lambda \sigma(A_\lambda) .$$

PROOF.

(i) Necessity.

Let $t_n = s_{n+N}$ $(n = 0, 1, \ldots)$. In view of Theorem 2.3.3, we have that

$$t_n \to \sigma(A_\lambda) .$$

Define

$$g(s) = \frac{\Gamma(\alpha s+\beta+\lambda)\Gamma(s+1)}{\Gamma(\alpha s+\beta)\Gamma(s+\lambda+1)} .$$

It is clear that $g(s)$ is an analytic function in the region $b > b_o$, where

$$b_o = \max \{-2,(-1-\beta-\lambda)/\alpha,\ (-1-\beta)/\alpha,\ -2-\lambda\}+1.$$

By Stirling's theorem, we know that, as $|s| \to \infty$

$$\Gamma(\alpha s+\beta) = \sqrt{2\pi}\ e^{-\alpha s}\ (\alpha s)^{\alpha s+\beta-\frac{1}{2}}\ (1+O(\tfrac{1}{|s|})\ ),$$

so that, as $|s| \to \infty$

$$g(s) = \alpha^{\lambda} + O(\tfrac{1}{|s|}) .$$

By Lemma 5.2.2,

$$\{k_n\} = \{\mu_{n+N}\} \text{ is an } \bar{m}\text{-sequence.}$$

Let $q_n = E_n^{\lambda}$, $p_n = q_n k_n$ $(n = 0, 1, \ldots)$.

Since $k_n \to \alpha^{\lambda}$ as $n \to \infty$ and $(Q) \cong (A_{\lambda})$ is regular, we have that

$$(1-x)^{\lambda+1} \sum_{n=o}^{\infty} E_n k_n x^n = (1-x)^{\lambda+1}\ p(x) \to \alpha^{\lambda}$$

as $x \to 1-$.

Now

$$p_n \sim (\alpha^\lambda n^\lambda)/\Gamma(\lambda+1)$$

and

$$\sum_{n=1}^{\infty} n^\lambda = \infty \quad \text{for all } \lambda > -1,$$

so by Lemma 5.2.1, (P) is regular.

Since $t_n \to \sigma(Q)$, it follows from Lemma 5.2.3 that

$$t_n \to \sigma(P),$$

that is

$$\frac{1}{p(x)} \sum_{n=o}^{\infty} E_n^\lambda k_n t_n x^n \to \sigma$$

as $x \to 1-$. Thus

$$(1-x)^{\lambda+1} \sum_{n=o}^{\infty} E_n^\lambda k_n t_n x^n \to \alpha^\lambda \sigma$$

as $x \to 1-$, which means that

$$k_n t_n \to \alpha^\lambda \sigma(A_\lambda) \ .$$

Again, the translativity of (A_λ) implies that

$$\mu_n s_n \to \alpha^\lambda \sigma(A_\lambda) .$$

This completes the proof of the necessity part.

(ii) Sufficiency.

This part can be proved by letting $f(s) = \{g(s)\}^{-1}$ and by applying the same argument as in the first part.

LEMMA 5.2.5.

If $f(u)$ *is a function defined on* $-\infty < u < \infty$ *such that*

$$\liminf\, [f(u) - f(t)] \geq 0$$

when $t \to \infty$ *and* $0 \leq u-t \to 0$, *then there are positive numbers* t_o, M_1 *and* M_2 *so that*

$$f(u) - f(t) \geq -M_1(u-t) - M_2$$

for all $u > t \geq t_o$.

(compare [18], p. 25 Lemma 3)

PROOF.

For any positive number M_2 there exists a t_o and a $\delta > 0$ such that

$$f(u) - f(t) \geqq -M_2 \text{ if } t \geqq t_o,\ 0 \leqq u-t < \delta .$$

Now, for any $u > t \geqq t_o$, let r be a positive integer such that

$$r\delta + t \leqq u + \delta < (r+1)\delta + t;$$

then

$$f(u) - f(t) = f(u) - f(r\delta+t) + f(r\delta+t)-\dots+f(\delta+t)-f(t)$$

$$\geqq -M_2(r+1).$$

Since $r\delta = r\delta+t-t \leqq u-t$, we have that

$$f(u) - f(t) \geqq -M_1(u-t) - M_2 ,$$

where $M_1 = M_2/\delta$.

LEMMA 5.2.6.

If $S_{\alpha,\beta}(u)$ *is slowly decreasing and*

$$s_n \to \sigma(A_\lambda) ,$$

then $S_{\alpha,\beta}(u)$ *is bounded for all* $u > 0$.

COMMENT. This result can be deduced from a general theorem due to Pitt ([19] p. 23 Theorem 10). However our proof of the special case is simpler than Pitt's and so we give it in full.

PROOF.

Let

$$f(t) = S_{\alpha,\beta}(e^t) \text{ for all } -\infty < t < \infty.$$

Since $t \to \infty$ and $0 \leqq v-t \to 0$ if and only if

$$e^v > e^t \to \infty \quad \text{and } e^{v-t} = e^v/e^t \to 1,$$

so that the function $f(t)$ satisfies the hypothesis of the previous lemma, and from which it follows that

$$f(v) - f(t) \geqq -M_1(v-t) - M_2 ,$$

for positive numbers M_1 and M_2 and for all $t \geqq t_o$.

Let $k(v) = \frac{1}{\Gamma(\lambda+1)} e^{-e^v}(e^v)^{\lambda+1}, -\infty < v < \infty$, then

$$\int_{-\infty}^{\infty} k(v)dv = 1.$$

Since for given $\epsilon > 0$, there are t_1 and $\delta > 0$ so that

$$f(v) - f(t) \geqq -\epsilon \text{ for } t \geqq t_1 \text{ and } 0 \leqq v-t < \delta,$$

$$f(t) - f(v) \geqq -\epsilon \text{ for } t \geqq t_1 \text{ and } 0 \leqq t-v < \delta,$$

we can choose an integer p such that

$$\frac{1}{2} < \int_{-\delta p}^{\delta p} k(v)dv = \gamma ,$$

and so obtain that

$$f(v) - f(t) \geqq -p\epsilon = -c \text{ for } t \geqq t_1 \text{ and } 0 \leqq v-t < \delta p,$$

$$f(t) - f(v) \geqq -p\epsilon = -c \text{ for } t \geqq t_1 \text{ and } 0 \leqq t-v < \delta p.$$

Let

$$g(t) = \int_{-\infty}^{\infty} k(v-t)f(v)dv, \quad -\infty < t < \infty, \text{ and } u = e^t; \text{ then}$$

$$g(t) = \frac{u^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^{\infty} e^{-v/u} v^{\lambda} s_{\alpha,\beta}(v)dv = \alpha\left(\frac{u}{1+u}\right)^{\beta-1}\left(\frac{1+y}{1+u}\right)^{\lambda+1}\sigma_{\lambda}^{*}(y),$$

where

$$\left(\frac{u}{1+u}\right)^{\alpha} = \frac{y}{1+y}, \text{ and } \sigma_{\lambda}^{*}(y) = \frac{1}{(1+y)^{\lambda+1}} \sum_{n=N}^{\infty} E_n^{\lambda} \frac{\Gamma(\alpha n+\beta+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n+\lambda+1)} s_n \left(\frac{y}{1+y}\right)^n .$$

Now $g(t) \to \alpha^{\lambda}\sigma$ as $y \to \infty$, by Lemma 5.2.4; and $g(t) \to 0$ as $t \to -\infty$, since $\alpha N+\beta-1 > 0$. Thus $g(t)$ is bounded for all t.

Let $\omega(t) = \inf_{v \leq t} f(v)$ and $\mu(t) = \sup_{v \leq t} f(v)$, then for $v \geq t \geq t_o$, it follows from $f(v)-f(t) \geq -M_1(v-t)-M_2$ that $\omega(v)-\omega(t) \geq -M_1(v-t)-M_2$. Now consider values of t for which $t' \geq \max(t_o, t_1) = T$, where $t' = t - \delta p$, we have that

$$g(t) = \int_{t-\delta p}^{t+\delta p} k(v-t)f(v)dv + \int_{-\infty}^{t-\delta p} k(v-t)f(v)dv + \int_{t+\delta p}^{\infty} k(v-t)f(v)dv$$

$$\geq (f(t')-c)\gamma + \omega(t') \int_{-\infty}^{t-\delta p} k(v-t)dv + f(t') \int_{t+\delta p}^{\infty} k(v-t)dv$$

$$+ \int_{t+\delta p}^{\infty} k(v-t)[f(v) - f(t')]dv$$

$$\geq f(t')\gamma + \omega(t')\left[1 - \int_{t-\delta p}^{t+\delta p} k(v-t)dv\right] - \int_{t+\delta p}^{\infty} k(v-t)M_1(v-t)dt-M$$

$$\geq f(t')\gamma + \omega(t')(1-\gamma) - M,$$

since

$$0 < \int_{t+\delta p}^{\infty} k(v-t)(v-t)dv = \int_{\delta p}^{\infty} k(v)vdv \leq \frac{1}{\Gamma(\lambda+1)} \int_{1}^{\infty} e^{-v} v^{\lambda} \log v \, dv$$

$$< \frac{1}{\Gamma(\lambda+1)} \int_{o}^{\infty} e^{-v} v^{\lambda+1} dv = \lambda+1 ,$$

so that

$$f(t) + \omega(t) \left(\frac{1-\gamma}{\gamma}\right) \leqq M.$$

Since $\omega(t)$ decreases, it follows that

$$\mu(t) + \omega(t) \left(\frac{1-\gamma}{\gamma}\right) \leqq M ,$$

for all $t \geqq T+\delta$. Also, for such t, we have that

$$g(t) = \int_{t-\delta p}^{t+\delta p} k(v-t)f(v)dv + \int_{-\infty}^{t-\delta p} k(v-t)f(v)dv + \int_{t+\delta p}^{\infty} k(v-t)f(v)dv$$

$$\leqq (f(t'')+c)\int_{t-\delta p}^{t+\delta p} k(v-t)dv + \mu(t'')\int_{-\infty}^{t-\delta p} k(v-t)dv - \left(\frac{1-\gamma}{\gamma}\right)\int_{t+\delta p}^{\infty} k(v-t)\omega(v)dv$$

$$+ M\int_{t+\delta p}^{\infty} k(v-t)dv$$

$$\leqq f(t'')\gamma - \omega(t'')\left(\frac{1-\gamma}{\gamma}\right)\left[\int_{-\infty}^{t-\delta p} k(v-t)dv + \int_{t+\delta p}^{\infty} k(v-t)dv\right]$$

$$- \left(\frac{1-\gamma}{\gamma}\right)\int_{t+\delta p}^{\infty} k(v-t)[\omega(v) - \omega(t)]dv + M$$

$$\leqq f(t'')\gamma - \omega(t'')\frac{(1-\gamma)^2}{\gamma} + M, \text{ where } t'' = t + \delta p,$$

so that

$$\omega(t) \left(\frac{1-\gamma}{\gamma}\right)^2 + M \leqq f(t)$$

for all $t \geqq T + \delta p$.

Now, if $\omega(t) \to -\infty$ as $t \to \infty$, then there exists a sequence of t_n such that

$$f(t_n) = \omega(t_n) \to -\infty .$$

Since

$$f(t) \geqq M + \left(\frac{1-\gamma}{\gamma}\right)^2 \omega(t) ,$$

we have that

$$\left(1 - \left(\frac{1-\gamma}{\gamma}\right)^2\right)\omega(t_n) \geqq M,$$

which is a contradiction since M is a constant and $\gamma > 1/2$. Thus $\omega(t)$ is bounded below, and it follows from the inequality,

$$M + \left(\frac{1-\gamma}{\gamma}\right)^2 \omega(t) \leqq f(t) \leqq M - \left(\frac{1-\gamma}{\gamma}\right) \omega(t),$$

that $f(t)$ is bounded for all $t \geqq T+\delta p$. Finally, since $f(t)$, is clearly bounded for all $t<T+\delta p$, $f(t)$ is bounded for all t. This completes the proof of the lemma.

LEMMA 5.2.7.

If (i) $g(u)$ *is integrable over* $(0,\infty)$ *and*

$$\int_0^\infty g(u)u^{-ix}du \neq 0 \text{ for any real } x;$$

(ii) f(u) *is bounded and slowly decreasing and*

(iii) $\frac{1}{t}\int_0^\infty g(\frac{u}{t})f(u)du \to \sigma\int_0^\infty g(u)du$

as $t \to \infty$, *then*

$$f(u) \to \sigma$$

as $u \to \infty$.

(see [12] p.296 Theorem 233)

5.3. *STATEMENTS OF THEOREMS.*

Suppose from now on that $-1 < \lambda < 0$.

THEOREM 5.3.1.

If *(i)* $s_n \to \sigma(A_\lambda)$ *and*

(ii) $S_{\alpha,\beta}(u)$ *is slowly decreasing, then*

$$s_n \to \sigma(B,\alpha,\beta).$$

THEOREM 5.3.2.

If *(i)* $s_n \to \sigma(A_\lambda)$ *and*

(ii) $uS'_{\alpha,\beta}(u) = O_L(1)$, *then*

$$s_n \to \sigma[B,\alpha,\beta] .$$

THEOREM 5.3.3.

If (i) $s_n \to \sigma(A_\lambda)$ *and*

(ii) $S'_{\alpha,\beta}(u) = O_L(\phi(u))$, *where*

$$u\phi(u) = O(1)$$

and

$$\int_k^\infty \phi(u)du < \infty \text{ for some } k > 0, \text{ then}$$

$$s_n \to \sigma|B,\alpha,\beta| .$$

COMMENT. At present, we can only prove the following three theorems in the special case that α is a (positive) integer; we conjecture that they will also be true for all $\alpha > 0$.

THEOREM 5.3.4.

If (i) $s_n \to \sigma(A_\lambda)$ *and*

(ii) $A_{\alpha,\beta}(u)$ *is slowly decreasing, where* α *is a (positive) integer, then*

$$s_n \to \sigma(B',\alpha,\beta) .$$

THEOREM 5.3.5.

If *(i)* $s_n \to \sigma(A_\lambda)$ *and*

(ii) $uA'_{\alpha,\beta}(u) = O_L(1)$, *where* α *is a (positive) integer, then*

$$s_n \to \sigma[B',\alpha,\beta].$$

THEOREM 5.3.6.

If *(i)* $s_n \to \sigma(A_\lambda)$ *and*

(ii) $A'_{\alpha,\beta}(u) = O_L(\phi(u))$, *where* α *is a (positive) integer and*

$$u\phi(u) = O(1)$$

with

$$\int_k^\infty \phi(u)du < \infty \textit{ for some } k > 0, \textit{ then}$$

$$s_n \to \sigma|B',\alpha,\beta| .$$

We denote, as in the previous chapter, the conditions $C(S):S'_{\alpha,\beta}(u) = O_L(\phi(u))$, where $u\phi(u) = O(1)$ and $\phi(u)$ is integrable over (k,∞) for some $k > 0$;

$C(A): A'_{\alpha,\beta}(u) = O_L(\phi(u))$, where $u\phi(u) = O(1)$ and $\phi(u)$ is integrable over (k,∞) for some $k > 0$.

The relations between the hypotheses and conclusions of the above theorems can be illustrated in the following table:

TABLE III

$s_n \to \sigma(A_\lambda)$ and	$C(S)$	$\Longrightarrow$	$s_n \to \sigma\lvert B,\alpha,\beta\rvert$
	$\Downarrow$		$\Downarrow$
$s_n \to \sigma(A_\lambda)$ and	$uS'_{\alpha,\beta}(u) = O_L(1)$	$\Longrightarrow$	$s_n \to \sigma[B,\alpha,\beta]$
	$\Downarrow$		$\Downarrow$
$s_n \to \sigma(A_\lambda)$ and	$S_{\alpha,\beta}(u)$ is slowly decreasing	$\Longrightarrow$	$s_n \to \sigma(B,\alpha,\beta)$
			$\Downarrow$
$s_n \to \sigma(A_\lambda)$ and	$A_{\alpha,\beta}(u)$ is slowly decreasing, α is integer	$\Longrightarrow$	$s_n \to \sigma(B',\alpha,\beta)$
	$\Uparrow$		$\Uparrow$
$s_n \to \sigma(A_\lambda)$ and	$uA'_{\alpha,\beta}(u) = O_L(1)$, α is integer	$\Longrightarrow$	$s_n \to \sigma[B',\alpha,\beta]$
	$\Uparrow$		$\Uparrow$
$s_n \to \sigma(A_\lambda)$ and	$C(A)$, α is integer	$\Longrightarrow$	$s_n \to \sigma\lvert B',\alpha,\beta\rvert$

5.4. *PROOFS OF THEOREMS.*

5.4.1. *Proof of Theorem 5.3.1.*

In view of Lemma 5.2.6, the hypotheses imply that $S_{\alpha,\beta}(u)$ is bounded.

Let

$$g(u) = e^{-u}u^{\lambda}/\Gamma(\lambda+1),\ u > 0,$$

$$\nu_n = \Gamma(\alpha n+\beta+\lambda)\Gamma(n+1)/\Gamma(\alpha n+\beta)\Gamma(n+\lambda+1),\ n = N,\ N+1,\ \ldots$$

and

$$J(t) = \frac{1}{t}\int_0^{\infty} g(\tfrac{u}{t})S_{\alpha,\beta}(u)du$$

$$= \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)}\int_0^{\infty} e^{-u/t}\ u^{\lambda}S_{\alpha,\beta}(u)du,\quad t > 0,$$

then

$$\int_0^{\infty} g(u)du = \frac{1}{\Gamma(\lambda+1)}\int_0^{\infty} e^{-u}u^{\lambda}du = 1.$$

In view of Lemma 2.2.1, we have

$$J(t) = \alpha\left(\frac{t}{1+t}\right)^{\beta-1}\left(\frac{1+y}{1+t}\right)^{\lambda+1}\sigma_{\lambda}^{*}(y),$$

where

$$\left(\frac{t}{1+t}\right)^{\alpha} = \frac{y}{1+y}$$

and

$$\sigma^{*}(y) = \frac{1}{(1+y)^{\lambda+1}}\sum_{n=N}^{\infty} E_n^{\lambda}\nu_n s_n\left(\frac{y}{1+y}\right)^n$$

which, by virtue of Lemma 5.2.4, tends to $\alpha^{\lambda}\sigma$ as $y \to \infty$. Also, we have that $t \to \infty$ if and only if $y \to \infty$ and that

$$\frac{1+y}{1+t} \to \frac{1}{\alpha}$$

as $t \to \infty$. Thus

$$\frac{1}{t}\int_{0}^{\infty} g\left(\frac{u}{t}\right)S_{\alpha,\beta}(u)du \to \sigma \int_{0}^{\infty} g(u)du$$

as $t \to \infty$. In view of Lemma 5.2.7, we have that

$$S_{\alpha,\beta}(u) \to \sigma$$

as $u \to \infty$.

This completes the proof.

5.4.2. *Proof of Theorem 5.3.2.*

In view of Lemma 4.2.2, hypothesis (ii) implies that $S_{\alpha,\beta}(u)$ is slowly decreasing. Hence by virtue of the previous theorem, we have that

$$s_n \to \sigma(B,\alpha,\beta).$$

By applying the same argument as in the proof of Theorem 4.3.2(I), we have

$$\int_0^y e^t |S'_{\alpha,\beta}(t)| dt = o(e^y)$$

as $y \to \infty$.

Therefore, by virtue of Theorem 2.5.2 (i), we obtain that

$$s_n \to \sigma[B,\alpha,\beta].$$

This completes the proof.

5.4.3. *Proof of Theorem 5.3.3.*

The conditions that

$$S'_{\alpha,\beta}(u) = O_L(\phi(u))$$

and

$$u\phi(u) = O(1)$$

imply that

$$uS'_{\alpha,\beta}(u) = O_L(1),$$

which, together with condition (i), implies, by the previous theorem, that

$$s_n \to \sigma(B,\alpha,\beta).$$

By applying the same argument as in the second half of the proof of Theorem 4.3.3, we can show that $S_{\alpha,\beta}(u)$ is of bounded variation with respect to u in the range $[0,\infty)$. Hence

$$s_n \to \sigma|B,\alpha,\beta|.$$

This completes the proof.

5.4.4. *Proof of Theorem 5.3.4.*

Let

$$I(t) = \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-u/t}\, u^\lambda A_{\alpha,\beta}(u)du .$$

Since

$$a_{\alpha,\beta}(u) = s_{\alpha,\beta}(u) - s_{\alpha,\beta+\alpha}(u),$$

$$= \sum_{\gamma=1}^{\alpha} [\frac{1}{\alpha}\, e^u s'_{\alpha,\beta+\gamma}(u)] ,$$

we have that

$$I(t) = \frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-u/t}\, u^\lambda \sum_{\gamma=1}^{\alpha} \frac{1}{\alpha}\, S_{\alpha,\beta+\gamma}(u)du.$$

Suppose that $s_n \to \sigma(A_\lambda)$. Then it can be shown, as in the proof of Theorem 5.3.1, for all $\alpha > 0$ and for all real β, that

$$\frac{t^{-\lambda-1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-u/t} \, u^\lambda S_{\alpha,\beta}(u)du \to \sigma$$

as $t \to \infty$.

Hence

$$I(t) \to \frac{1}{\alpha} \sum_{r=1}^{\alpha} \sigma = \sigma$$

as $t \to \infty$.

Let

$$g(u) = \frac{1}{\Gamma(\lambda+1)} \; e^{-u} u^\lambda, \; (u > 0) \; .$$

Then we see that

$$I(t) = \frac{1}{t} \int_0^\infty g(\tfrac{u}{t}) A_{\alpha,\beta}(u)du \to \sigma \int_0^\infty g(u)du$$

as $u \to \infty$.

That $A_{\alpha,\beta}(u)$ is bounded follows from Lemma 5.2.6. Therefore, by virtue of Lemma 5.2.7, we have that

$$A_{\alpha,\beta}(u) \to \sigma$$

as $u \to \infty$.

This completes the proof.

5.4.5. *Proof of Theorem 5.3.5.*

The proof of this theorem can be obtained by replacing "$S_{\alpha,\beta}$", "B" and Theorem 2.5.2 (i), in the proof of Theorem 5.3.2, by "$A_{\alpha,\beta}$", "B'" and Theorem 2.5.2 (ii), respectively.

5.4.6. *Proof of Theorem 5.3.6.*

The proof of this theorem can be obtained by replacing "$S_{\alpha,\beta}$", and "B", in the proof of Theorem 5.3.3, by "$A_{\alpha,\beta}$" and "B'", respectively.

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