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# Numerical Solutions For Fluid Flow In Rectangular Cavities

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# NUMERICAL SOLUTIONS FOR FLUID FLOW IN RECTANGULAR CAVITIES

by

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# ABSTRACT

This thesis concerns itself with obtaining numerical solutions for viscous incompressible fluid flow in rectangular cavities where the fluid is driven by a uniform velocity at the top. In the past, this problem was formulated in terms of the stream function and vorticity and numerical solutions could only be obtained for Reynolds numbers up to 400.

To solve the equations numerically relaxation techniques are applied, reasons for the numerical instabilities for higher Reynolds numbers are given, and means of overcoming these are suggested. The superiority of the explicit time-dependent method, as opposed to the implicit, with respect to computing time for this particular problem is pointed out. A stability analysis for the computational procedure is carried out for both the implicit and explicit method. An expression is derived for the maximum allowable time step for the explicit method. It is also shown that the implicit method is stable for all time steps, Reynolds numbers and mesh size. In both cases the analysis is extended to include

all non-linear terms in the equations. The explicit method is then used to obtain solutions for Reynolds numbers up to 2000.

In order to obtain numerical solutions for the three-dimensional case, the problem must be formulated in terms of the velocities and pressure. The difficulties associated with this have not been analysed in the past. It is pointed out that, in solving the Navier-Stokes equations in terms of the velocities and pressure, use of the straightforward method of the 5-point difference formulation does not insure that the continuity equation will be satisfied at future times. Reasons for these discrepancies are derived and it is shown that, in order to overcome these errors in the continuity equation, the Navier-Stokes equations must be written in a special form and the Marker-In-Cell method must be applied for numerical computation.

Finally, it having been established that solutions can be obtained in terms of velocities and pressures in the two-dimensional case, the numerical solution for the three-dimensional time-dependent fluid flow in a rectangular basin is considered. The FORTRAN IV computer program for this is included as an appendix.

## ACKNOWLEDGMENTS

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# CHAPTER I

## INTRODUCTION

Although the basic equations governing the flow of viscous incompressible fluids have been known for more than a century (see early references in Rosenhead 1963), the analytic difficulties of this fourth-order system, and particularly the fact that the equations are strongly non-linear, have prevented the discovery of any truly general method of solution. Those exact solutions as are known correspond either to very simple geometries or to limiting values of the parameters involved.

No detailed discussion of the general properties of the equations or of the various methods of obtaining approximate solutions, will be given here. This subject has been amply and lucidly treated in many standard references (see, for example, Rosenhead 1963; Schlichting 1968). The fact remains, however, that in spite of many significant developments, the solutions to a large class of problems are still unavailable.

The recent availability of large-scale digital computers should make it possible to solve many problems by a direct numerical process. The numerical solution to these problems should be helpful in the understanding of the nature of the solutions of the Navier-Stokes equations. One of the leaders in this field (Goldstein 1960, page 132) has said:

> "As the Reynolds number increases, there are considerable difficulties associated even with the numerical computation; if sufficient effort were available, these might, by no means, prove insuperable. At present, the number of numerical solutions available is still regrettably small. Such numerical solutions would not, by themselves, give us the mathematical insight we should like into the nature of the information contained in the Navier-Stokes equations; but it would be helpful to have a large number of such solutions, especially if they were presented in an easily assimilable form, say graphically and with the aid of cinematograph film."

Although serious work on such numerical solutions began some years ago (see references in Thom and Apelt 1961), it is only within the past decade that significant progress has been made, and, even today, there are many difficulties associated with the attempt to formulate any general approach to such problems.

It is the purpose of the present thesis to examine in some detail a relatively simple class of fluid flow problems, namely those occuring in closed regions, and to outline methods by which many of these difficulties may be overcome.

## 1.1 The Choice of Problem

The treatment of fluid flow problems involving a wake region, or more generally any fluid flow past and around a bluff obstacle, involves great difficulties ultimately associated with the instability of the flow, the asymmetric shedding of vortices behind the obstacle, and the resulting highly complicated nature of the wake region. Such problems, of course, have recently been the subject of keen interest (Fromm 1963). However, problems not involving such a wake region are still of interest and importance and many aspects of the solutions are as yet not completely understood. For example, convergence and stability of the numerical procedure itself has been obtained only recently for other than relatively low Reynolds numbers (Kawaguti 1961; Burggraf 1966;

Pearson 1967, for Reynolds numbers of up to 64, 400 and 1500 respectively), and, in any case, examination is warranted to determine an efficient numerical process for finding the solutions for these and higher Reynolds numbers.

The particular problem of two-dimensional flow in a closed rectangular cavity, driven for example by a shear force on the top surface, not only exhibits these difficulties while avoiding the involvement of a wake region, but has the additional advantage of having been studied experimentally for a relatively large range of Reynolds numbers (Mills 1965; Pan and Acrivos 1967, for up to 1000 and 4000 respectively).

This problem also is interesting in that, conceptually, it would seem that an extension to the corresponding three-dimensional problem would be reasonably straightforward. Analytically, however, this is not the case, because the relatively simple formulation available for two-dimensional problems does not extend to the three-dimensional case. In two dimensions, the problem may be formulated in terms of the scalar "Stream Function", and a set of two second-order equations can be used, the only difficulty being involved with stability for high Reynolds numbers and overall computer time. For three dimensions, however, such a formulation is no longer available and the govering equations must be written directly in terms of velocity components. If, however, this velocity formulation is used in the two-dimensional problem, one encounters new difficulties. In particular, the velocities calculated for the next time increment will not satisfy the continuity equation and thus, violating the condition that the fluid remains incompressible in time.

The above three reasons would, therefore, seem to indicate that a contribution to our understanding of such flow problems can be made by studying the flow in a two or three-dimensional cavity driven at a uniform speed by such a shear force.

## 1.2 The Governing Equations

The derivation of the equations of motion of a compressible fluid is given in detail in Schlichting (1968, chapter III and IV.) Here we shall simply state the results for the case of two-dimensional, non-steady, incompressible flow in the x,y-plane and proceed to put these equations in non-dimensional form.

The equations are (Schlichting, p. 63):

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}.\nabla\underline{V} + \frac{1}{\rho}\nabla P = \nu\nabla^2\underline{V}$$

and

$$\nabla.\underline{V} = 0$$

where

$$\underline{V} = (u,v),$$

the velocity vector with components u in the x-direction and v in the y-direction.

$P$ is the pressure

$\nabla$ is $i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$

$\nabla^2$ is $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\rho$ is the density

To put equation (1.1) in a non-dimensional form, we define the new variables

$$x^* = \frac{x}{a} \quad , \quad y^* = \frac{y}{a}$$

$$p^* = \frac{P}{\rho U^2} \quad , \quad t^* = \frac{tU}{a}$$

$$\underline{V}^* = \frac{\underline{V}}{U} \quad , \quad \nabla^* = a\nabla$$

where

a is some standard reference length, in our case, it will be the width of the cavity.

U is the free-stream velocity

Substituting into equation (1.1), we obtain:

$$\frac{U^2}{a}\frac{\partial \underline{V}^*}{\partial t^*} + \frac{U^2}{a}\underline{V}^*.\nabla^*\underline{V}^* + \frac{U^2\rho}{a\rho}\nabla^* p^* = \frac{\nu U}{a^2}\nabla^{2*}\underline{V}^*$$

Finally we divide through by $\frac{U^2}{a}$ and drop the superscripts *; this will give the equation in non-dimensional form

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}.\nabla\underline{V} + \nabla P = \frac{\nu}{Ua}\nabla^2\underline{V} = \frac{1}{R}\nabla^2\underline{V} \tag{1.2}$$

R is known as the Reynolds number and can be interpreted as the ratio

$$R = \frac{\text{Inertial Force}}{\text{Friction Force}} = \frac{Ua}{\nu}$$

An alternative formulation of (1.2) can be obtained using the so-called "Stream Function" $\psi$, defined by

$$u = \frac{\partial\psi}{\partial y} \; ; \; v = -\frac{\partial\psi}{\partial x}$$

which, when substituted into (1.2), will give

$$\frac{\partial\nabla^2\psi}{\partial t} - \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = \frac{1}{R}\nabla^4\psi. \tag{1.3}$$

Let us now introduce the vector of vorticity defined by

$$\underline{\omega} = \nabla x\underline{V}$$

which, for two-dimensional flow, reduces to

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

which reduces, when substituting for u and v, to

$$\nabla^2\psi = -\omega. \qquad (1.4)$$

Substituting into (1.3), one obtains

$$\frac{\partial\omega}{\partial t} - \frac{\partial(\psi,\omega)}{\partial(x,y)} = \frac{1}{R}\nabla^2\omega. \qquad (1.5)$$

Equation (1.5) combined with (1.4), describes the motion of fluid flow with respect to the stream function and vorticity.

## 1.3 The Finite Difference Equations

The subject of finite difference is amply treated in various references (e.g. Hartree 1958, chapter IV). Here we shall only write down the necessary equations for use in this thesis.

To begin with, when a function f is continuous, single-valued and has continuous derivatives,we can expand this function in terms of a Taylor series as:

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{6}h^3f''' + O(h^4) \qquad (1.4)$$

and also

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f''' + O(h^4) \qquad (1.5)$$

Subtracting (1.5) from (1.4), and rearranging results in

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + 0(h^2) \tag{1.6}$$

Adding equation (1.4) and (1.5) gives

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + 0(h^4)$$

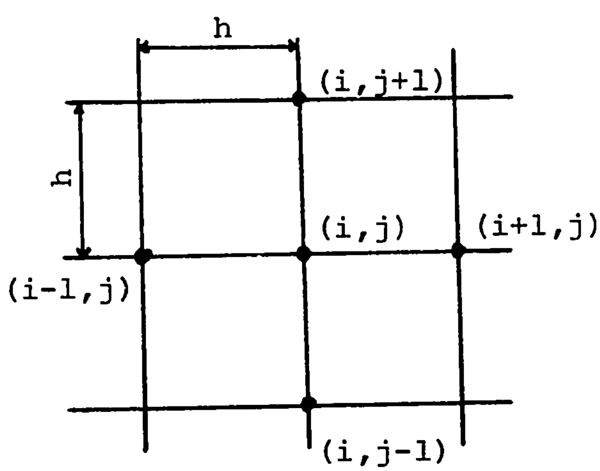
from this we obtain

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + 0(h^2) \tag{1.7}$$

Equations (1.6) and (1.7) are central difference approximations to the first and second order differential terms with truncation error of order $h^2$. $f'(x)$ can also be approximated with a forward difference, that is from (1.4)

$$f'(x) = \frac{f(x+h) - f(x)}{h} + 0(h). \tag{1.8}$$

Throughout the thesis, these formulations of finite differences will be used. We now only need to show the grid configuration used for two-dimensional problems, namely



and, therefore, using (1.7), we have, for example, for

$$\nabla^2 P = \frac{P_{i,j+1} + P_{i,j-1} + P_{i+1,j} + P_{i-1,j} - 4\,P_{i,j}}{h^2} + 0(h^2). \tag{1.8}$$

To apply finite difference approximation to the square cavity, we divide it into a total of MxN square cells of size

$$h = \frac{1}{M} = \frac{1}{N} .$$

The quantities i and j in (1.8) range from 0 to M and 0 to N respectively. Now it is true that, in order to achieve higher accuracy to the differential equation, either a better difference approximation (see Hartree 1958, chapter IV) must be used and/or M and N must be made sufficiently large. In this thesis, the results to be established need only use the above difference formulations.

## 1.4 Summary of the Thesis

In chapter II, the various methods that have been used to solve two-dimensional problems in terms of the stream function formulation, are reviewed and examined. The reasons that instabilities result from particular numerical procedures are pointed out, and suggestions are made on how these difficulties may be overcome. In particular, similarities between the methods used, to solve time-dependent and steady-state problems, are examined in detail.

In chapter III is presented a new analysis of stability of the explicit time-dependent equation and, for completeness, an analysis of the corresponding implicit method is also presented.

In chapter IV are presented the results of experimental computer runs, designed to test and verify the predictions of chapter III. In addition, the methods thereby found to be most efficient are used to solve for a large range of Reynolds numbers (up to R=2000) the simple two-dimensional problem. A discussion of the degree of fineness of the grid necessary for accurate solutions is also included.

The study of the formulation in terms of velocity components begins in chapter V, with an examination of the reasons why the continuity equation will not be satisfied if straightforward methods are applied.

Chapter VI is devoted to a new formulation in terms of the velocity and pressure, which allows the velocities to be calculated in such a way that the continuity condition remains satisfied throughout time. The inconsistencies found in chapter V for the straightforward method are found no longer to occur.

In chapter VII, the same formulation that is proven successful for two-dimensional problems is applied to the three-dimensional time-dependent case. It is established that the only further difficulties encountered are those implied by computer size and time required for solutions. The computer program, written in FORTRAN IV, for the three-dimensional case is presented as an appendix.

CHAPTER II

# STEADY-STATE FLOW IN RECTANGULAR CAVITIES

This chapter deals with the numerical solution for the steady-state two-dimensional flow in a rectangular cavity, driven by a uniform velocity (say, a moving belt) on the upper boundary. The problem has been studied numerically by Kawaguti (1961) and by Burggraf (1966), who obtained solutions for Reynolds numbers up to 64 and 400 respectively. Convergence could not be obtained for larger values. Experimental studies of this problem have also been carried out (Pan and Acrivos 1967) for Reynolds numbers up to 4000.

In this chapter, we review the methods used by the above and the results obtained. Also, we argue that instabilities had to occur for high Reynolds numbers for the methods that were used.

## 2.1 Formulation

We consider a fixed square cavity described by $0 \leq x \leq 1$, $0 \leq y \leq 1$. On the top surface we impose a constant tangential velocity of unity to the right; all other tangential and normal velocities are set equal to zero.

The boundary conditions for the non-dimensional Navier-Stokes equations are then:

$$\left.\begin{array}{ll} \psi_y(x,1) = 1 \ , & \psi_x(x,1) = 0 \\ \psi_y(x,0) = \psi_x(x,0) = 0 & \end{array}\right\} \qquad (2.1a)$$

$$\left.\begin{array}{l} \psi_x(0,y) = \psi_y(0,y) = 0 \\ \psi_x(1,y) = \psi_y(1,y) = 0 \end{array}\right\} \qquad (2.1b)$$

Pictorially, this is represented by Figure 2.1.

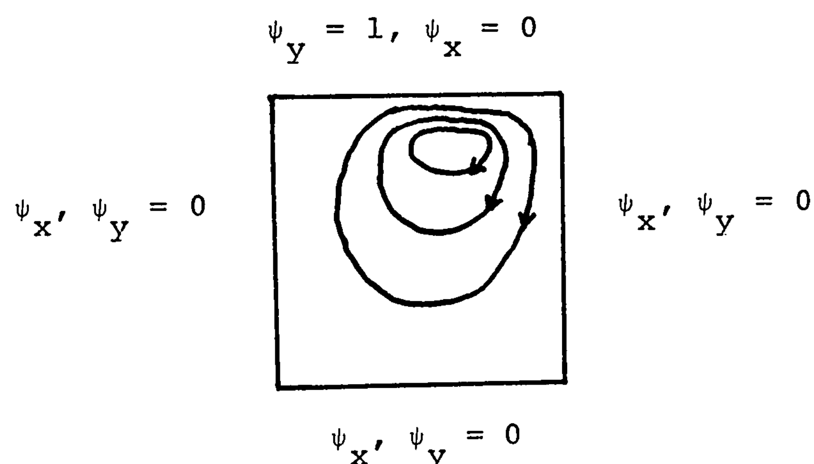



Figure 2.1 Boundary Conditions

For completeness, we repeat here the two-dimensional Navier-Stokes equations (Schlichting 1968) for an incompressible fluid restricted to steady-state flow. They are in non-dimensional form,

$$\nabla^2\omega = -R\,\frac{\partial(\psi,\omega)}{\partial(x,y)} \qquad (2.2)$$

$$\nabla^2\psi = -\omega \qquad (2.3)$$

Here and elsewhere, $\omega$ and $\psi$ are the familiar non-dimensional vorticity and stream function respectively, and R is the non-dimensional Reynolds number.

## 2.2 Finite Difference Approximation and Computation

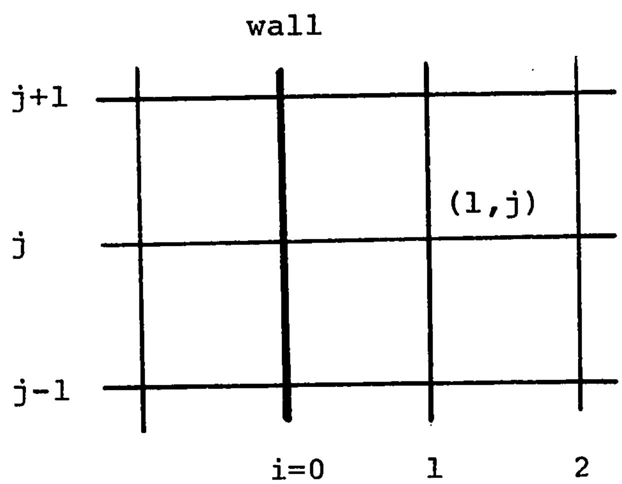
The usual way (Thom and Apelt 1961, p. 124 ff.) to solve equations (2.2) and (2.3) is to represent them in a finite difference form and solve the two sets of equations by some relaxation process. The difference approximation and notation used, was established in chapter I. We begin by replacing (2.2) and (2.3) by

$$\omega_{i,j+1} + \omega_{i,j-1} + \omega_{i+1,j} + \omega_{i-1,j} - 4\,\omega_{i,j}$$
$$= -\frac{R}{4}\left[\left(\psi_{i+1,j} - \psi_{i-1,j}\right)\left(\omega_{i,j+1} - \omega_{i,j-1}\right) - \left(\psi_{i,j+1} - \psi_{i,j-1}\right)\left(\omega_{i+1,j} - \omega_{i-1,j}\right)\right] \qquad (2.4)$$

$$\psi_{i,j+1} + \psi_{i,j-1} + \psi_{i+1,j} + \psi_{i-1,j} - 4\,\psi_{i,j} + h^2\,\omega_{i,j} = 0. \qquad (2.5)$$

For equation (2.5), we know both $\psi$ and $\frac{\partial\psi}{\partial n}$, the derivative of $\psi$ normal to the boundary, at all points on the boundary. Only one of these conditions can be imposed on (2.5) without over-determining the solution.

The solution is obtained more quickly, and the problem is simpler to formulate (Forsythe and Wasow 1960, p. 376), if Dirichlet conditions ($\psi$ specified on the boundaries) are used. The known values of $\psi_n$ cannot, of course, be ignored, but are incorporated into the solution of (2.4). To specify boundary conditions for $\omega$, one resorts (Thom and Apelt 1961, p. 125) to a Taylor expansion at the boundary. We illustrate this for the left wall.




Let us represent $\psi_{1,j}$ by a Taylor series around the boundary points (0,j)

$$\psi_{1,j} = \psi_{0,j} - h\left(\frac{\partial\psi}{\partial x}\right)_{i=0} + \frac{h^2}{2}\left(\frac{\partial^2\psi}{\partial x^2}\right)_{i=0} + 0(h^3). \qquad (2.6)$$

But, by the given boundary condition

$$\left(\frac{\partial\psi}{\partial x}\right)_{i=0} = 0$$

and, since $\psi_{0,j} = 0$ then on the left wall, using

$$\nabla^2\psi = -\omega \quad \text{and} \quad \left(\frac{\partial^2\psi}{\partial y^2}\right)_{i=0} = 0,$$

we have

$$\left(\frac{\partial^2\psi}{\partial x^2}\right)_{i=0} = -\omega \ .$$

Substituting this into (2.6), we obtain

$$\omega_{0,j} = \frac{-2}{h^2}\,\psi_{1,j}.$$

Similarly for the right wall

$$\omega_{M,j} = \frac{-2}{h^2}\,\psi_{M-1,j} \ ,$$

for the bottom wall (2.7)

$$\omega_{i,0} = \frac{-2}{h^2}\,\psi_{i,1} \ ,$$

and for the top wall

$$\omega_{i,N} = \frac{-2}{h} - \frac{2}{h^2}\,\psi_{i,N-1} \ .$$

For maximum speed of convergence, successive overrelaxation (Young 1954 ) is ordinarily used. To apply this, we rewrite the finite difference equations as

$$\omega_{i,j}^{n+1} = \omega_{i,j}^{n} + \frac{\alpha}{4}\Big[\omega_{i,j+1}^{n} + \omega_{i,j-1}^{n+1} + \omega_{i+1,j}^{n} + \omega_{i-1,j}^{n+1} - 4\,\omega_{i,j}^{n}\Big]$$
$$+ \frac{\alpha R}{16}\Big[\Big(\psi_{i+1,j} - \psi_{i-1,j}\Big)\Big(\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n+1}\Big)$$
$$- \Big(\psi_{i,j+1} - \psi_{i,j-1}\Big)\Big(\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n+1}\Big)\Big] \qquad (2.8)$$

and

$$\psi_{i,j}^{n+1} = \psi_{i,j}^{n} + \frac{\alpha}{4}\Big[\psi_{i,j+1}^{n} + \psi_{i,j-1}^{n+1} + \psi_{i+1,j}^{n} + \psi_{i-1,j}^{n+1}$$
$$- 4\,\psi_{i,j}^{n} + h^2\,\omega_{i,j}^{n}\Big] \qquad (2.9)$$

where $\alpha$ is the relaxation factor and n indicates the particular stage of the iteration procedure.

In the relaxation process, the error of the current approximation of the function to be solved at a grid point is calculated and the old value of the function then is changed in proportion to this error. In many cases, it is desirable to make this change larger than actually required, to liquidate the error (residual) at each point. It is said that we "over-relax" the error and, in this case, $\alpha>1$. For $\alpha<1$, the term under-relaxation is applied.

Whenever the new values of the function $\omega$ or $\psi$ are used as soon as they become available in the iteration procedure, and if $\alpha > 1$, we have "successive" overrelaxation. If the field is swept completely, using only values of the function from the previously completed sweep, we have "simulteneous" relaxation.

The speed of convergence to the solution depends on the size of the relaxation factor. Naturally enough, since it so directly controls the speed of convergence, the optimum size of this relaxation factor $\alpha$ has been the subject of much investigation. As usual, however, such explicit and complete answers as exist, are for linear problems. For the two-dimensional Laplace equation in rectangular coordinates, for example, the situation is understood fairly well; Frankel (1950) showed that, as the number of grid points, given by M and N increases, the optimum value of $\alpha$ behaves like

$$\alpha_{opt} \rightarrow 2 - \frac{2\pi}{\sqrt{2}}\left[\frac{1}{M^2} + \frac{1}{N^2}\right]^{\frac{1}{2}}. \qquad (2.10)$$

For non-linear equations, especially for the Navier-Stokes equations, only slight progress has been made; generally higher Reynolds numbers require smaller values of $\alpha$. A detailed discussion is given later in this chapter and in chapter III.

In the computational procedure, equations (2.8) and (2.9) are coupled together by the boundary conditions for the vorticity equation. Each equation is iterated until the sum of the residuals at each point is within a specified tolerance. The stream function is then substituted into the difference approximation for

$$\nabla^4\psi = -R \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} \quad . \qquad (2.11)$$

and the sum of the absolute values of the residuals of the finite difference equation to (2.11) at each point is taken as a means to test the convergence of the difference form of the Navier-Stokes equation for the steady state.

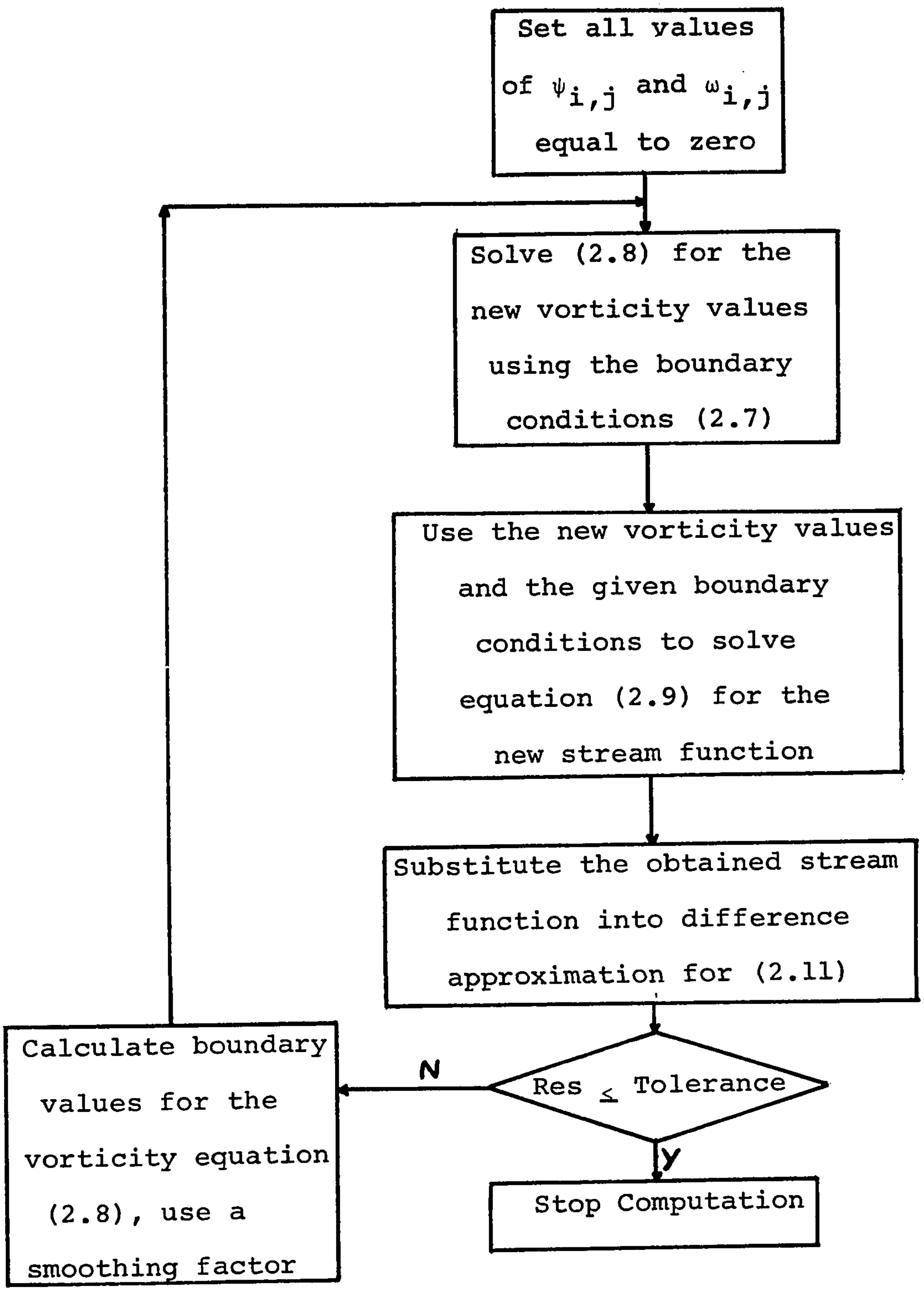
The total residual is given by

$$\text{Res} = \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} |\text{Res}_{i,j}| \qquad (2.12)$$

and, whenever Res becomes less than some preset tolerance, computation is stopped and the values of $\psi$ and $\omega$ so far obtained, are taken as the steady state solution.

A single application whereby each of the separate second-order equations is solved in turn by successive overrelaxation, and the substitution of the stream function into the difference form of the fourth order equation, is referred to as a "flip-flop". This is indicated by the loop in the flowchart.

Flowchart for Flip-Flop Method



## 2.3 Discussion of Experimental Results

In examining the computational procedure, we must consider separately two different convergence problems:

(1) Convergence of the successive overrelaxation method for each second order equation;

(2) Convergence of the flip-flop method.

The linear Poisson equation presents no problem; however, for the non-linear equation, the relaxation factor must be reduced for larger Reynolds numbers (Burggraf 1966). For example, it was found by experiments that convergence can be achieved for

$R = 10.0 \qquad \alpha = 1.65$

$R = 90.0 \qquad \alpha = 1.45$

It is not possible to predict theoretically the relaxation factor for higher Reynolds numbers[1].

---

[1] This is true for the successive overrelaxation method. A theoretical analysis to predict $\alpha$ for the simulteneous relaxation will be shown in chapter III.

The values must be obtained experimentally. One way of obtaining $\alpha$ for large R and assuring convergence, is to assume a value of $\alpha$, say 1.5, and then reduce $\alpha$ during the iteration as required.

The effectiveness of this method is known to be improved by limiting the number of times the $\omega$ field is iterated ("swept"), rather than continuing to sweep until some tolerance is reached before considering the other equation. Mills (1965) pointed out that for R=100, a numerical solution can be successfully obtained if the $\omega$ field is only swept twice for each flip-flop. In fact, one can show that the solution will invariably diverge for large R if (2.8) is swept too many times at each time step; the eigenvalues of the iteration matrix increase with R and $\alpha$.

This problem need not arise with the linear Poisson equation (2.9), the convergence of which depends only on the mesh size and relaxation factor. In fact, it is often profitable in practice to sweep (2.8) exactly once and (2.9) many times for each flip-flop. The (divergent) iteration matrix of (2.8) is then made part of a product with several corresponding matrices of (2.9), and the eigenvalues of the result are kept safely less than unity, while a reasonably-sized relaxation factor for (2.8) may still be used.

To avoid instabilities in the flip-flop, it is necessary to introduce a smoothing parameter so that the new values of the vorticity at the k+1 first flip-flop are taken to be weighted averages of the $k^{th}$ flip-flop and the values obtained from (2.7).

It is found, when using the boundary values (2.7) without smoothing, oscillation of the vorticity on the boundary occurs. This results in the divergence of the flip-flop method. Thus, if the $k^{th}$ flip-flop has just been completed the new vorticity boundary values, say for the left wall, are:

$$\omega_{0,j}^{k+1} = \omega_{0,j}^{k} + S\left[\frac{-2}{h^2}\psi_{1,j}^{k} - \omega_{0,j}^{k}\right] \qquad (2.13)$$

where S represents the smoothing factor. Similarly, it is found that for Reynolds numbers greater than 50, all values of $\psi$ must be smoothened from one flip-flop to the other. Usually, something of the order of $\frac{3}{4}$ of the old value of $\psi$ and $\frac{1}{4}$ of the new is used in the computation of the vorticity equation (Pearson 1965). Some typical values for various Reynolds numbers are shown in Table 2.1.

Table 2.1

Reynolds Numbers and Smoothing Factors for

20x20 grid, h=.05

| Reynolds Number | Smoothing Factor |
|---|---|
| 60 | .04 |
| 80 | .03 |
| 100 | .02 |
| 120 | .01 |
| 140 | .01 |

For each case shown in Table 2.1, increasing the smoothing factor by .01 resulted in divergence of the flip-flop, although the iteration procedure for the non-linear Poisson equation converged. Maximum speed of convergence is achieved by using the largest smoothing factor possible.

## 2.4 Convergence of the Iteration Method

It is possible to present an analysis for the single second-order equation in terms of a so-called "iteration" matrix. Any successive approximation scheme of this kind may be represented by considering the unknown variables as components of a vector, and the algorithm as a matrix multiplying this vector. Thus, we could write

$$A\phi = f$$

where A is a matrix (Fox 1962) and $\phi$ is a vector of the unknown values of $\psi$ or $\omega$ and f represents the boundary values.

The error after n steps, say $e^{(n)} = \phi - \phi^{(n)}$ will then satisfy the equation.

$$e^{(n+1)} = C^n e^{(n)}$$

where C is a matrix which will depend on A and f.

Convergence is assured if the absolute value or modulus of each eigenvalue of the matrix is less than unity. Let us now show how such an analysis applies to our problem.

As an illustration, let us write equation (2.4) as

$$\omega_{i,j+1}\, a_{i,j} + \omega_{i,j-1}\, b_{i,j} + \omega_{i+1,j}\, c_{i,j} + \omega_{i-1,j}\, d_{i,j} - 4\,\omega_{i,j} = 0 \qquad (2.14)$$

where

$$a_{i,j} = 1 + \frac{R}{4}\left(\psi_{i+1,j} - \psi_{i-1,j}\right)$$

$$b_{i,j} = 1 - \frac{R}{4}\left(\psi_{i+1,j} - \psi_{i-1,j}\right)$$

$$c_{i,j} = 1 - \frac{R}{4}\left(\psi_{i,j+1} - \psi_{i,j-1}\right)$$

$$d_{i,j} = 1 + \frac{R}{4}\left(\psi_{i,j+1} - \psi_{i,j-1}\right)$$

The grid is represented by Figure 2.2.

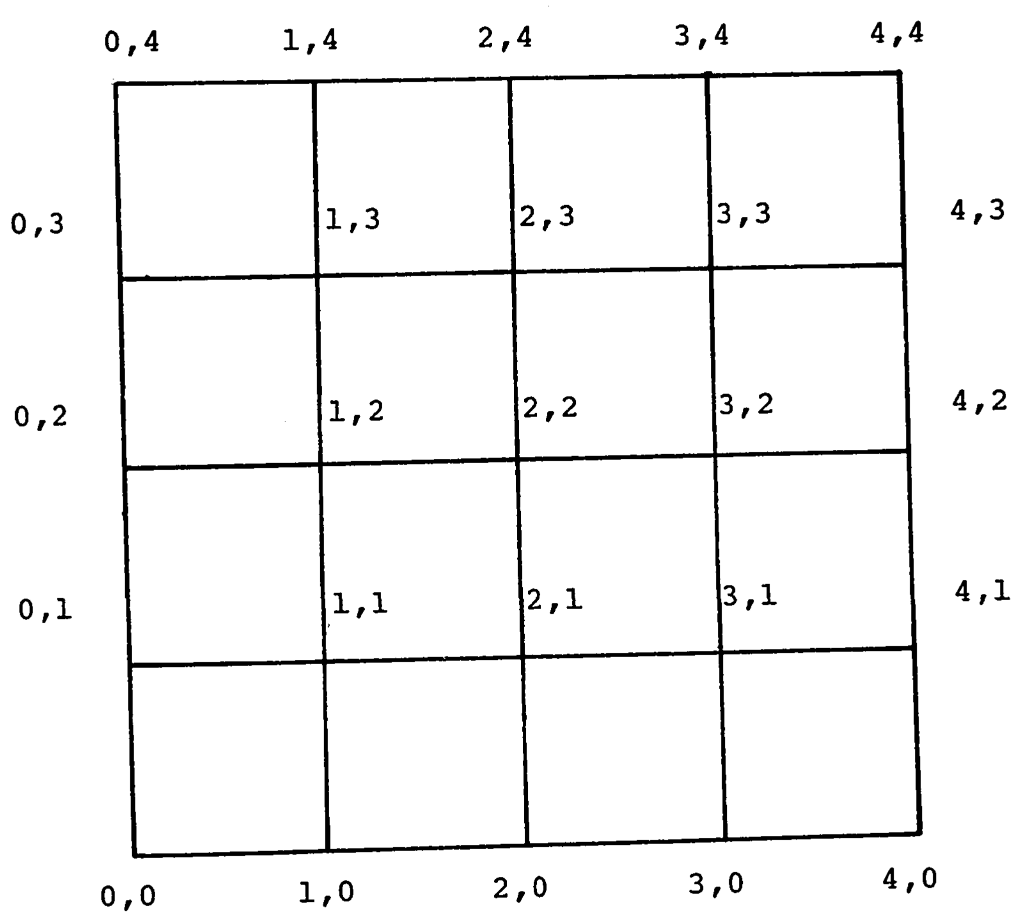



Figure 2.2
4x4 Sample Grid

The equations to describe all points are given in (2.15).

Points

$$(1,1) \quad -4\omega_{1,1} + a_{1,1}\omega_{1,2} + c_{1,1}\omega_{2,1} = -b_{1,1}\omega_{1,0} - d_{1,1}\omega_{0,1}$$

$$(1,2) \quad b_{1,2}\omega_{1,1} - 4\,\omega_{1,2} + a_{1,2}\omega_{1,3} + c_{1,2}\omega_{2,2} = -c_{1,2}\omega_{0,2}$$

$$(1,3) \quad b_{1,3}\omega_{1,2} - 4\,\omega_{1,3} + c_{1,3}\omega_{2,3} = -a_{1,3}\omega_{1,4} - d_{1,3}\omega_{0,3}$$

$$(2,1) \quad d_{2,1}\omega_{1,1} - 4\,\omega_{2,1} + a_{2,1}\omega_{2,2} + c_{2,1}\omega_{3,1} = -b_{2,1}\omega_{2,0}$$

$$(2,2) \quad d_{2,2}\omega_{1,2} + b_{2,2}\omega_{2,1} - 4\,\omega_{2,2} + a_{2,2}\omega_{2,3} + c_{2,2}\omega_{3,2} = 0$$

$$(2,3) \quad d_{2,3}\omega_{1,3} + b_{2,3}\omega_{2,2} + 4\,\omega_{2,3} + c_{2,3}\omega_{3,3} = -a_{2,3}\omega_{2,4}$$

$$(3,1) \quad d_{3,1}\omega_{2,1} - 4\,\omega_{3,1} + a_{3,1}\omega_{3,2} = -c_{3,1}\omega_{4,1} - b_{3,1}\omega_{3,0}$$

$$(3,2) \quad d_{3,2}\omega_{2,2} + b_{3,2}\omega_{3,1} - 4\,\omega_{3,2} + a_{3,2}\omega_{3,3} = -c_{3,2}\omega_{4,2}$$

$$(3,3) \quad d_{3,3}\omega_{2,3} + b_{3,3}\omega_{3,2} - 4\,\omega_{3,3} = -a_{3,3}\omega_{3,4} - c_{3,3}\omega_{4,3}$$

(2.15)

In matrix form this is

$$\begin{pmatrix} -4 & a_{1,1} & 0 & c_{1,1} & 0 & 0 & 0 & 0 & 0 \\ b & -4 & a_{1,2} & 0 & c_{1,2} & 0 & 0 & 0 & 0 \\ 0 & b_{1,3} & -4 & 0 & 0 & c_{1,3} & 0 & 0 & 0 \\ d_{2,1} & 0 & 0 & -4 & a_{2,1} & 0 & c_{2,1} & 0 & 0 \\ 0 & d_{2,2} & 0 & b_{2,2} & -4 & a_{2,2} & 0 & c_{2,2} & 0 \\ 0 & 0 & d_{2,3} & 0 & b_{2,3} & -4 & 0 & 0 & c_{2,3} \\ 0 & 0 & 0 & d_{3,1} & 0 & 0 & -4 & a_{3,1} & 0 \\ 0 & 0 & 0 & 0 & d_{3,2} & 0 & b_{3,2} & -4 & a_{3,2} \\ 0 & 0 & 0 & 0 & 0 & d_{3,3} & 0 & b_{3,3} & -4 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{1,2} \\ \omega_{1,3} \\ \omega_{2,1} \\ \omega_{2,2} \\ \omega_{2,3} \\ \omega_{3,1} \\ \omega_{3,2} \\ \omega_{3,3} \end{pmatrix} = \begin{pmatrix} -b_{1,}\omega_{1,0}-d_{1,1}\omega_{0,1} \\ -c_{1,2}\omega_{0,2} \\ -a_{1,3}\omega_{1,4}-d_{1,3}\omega_{0,3} \\ -b_{2,1}\omega_{2,0} \\ 0 \\ -a_{2,3}\omega_{2,4} \\ -b_{3,1}\omega_{3,0}-c_{3,1}\omega_{4,1} \\ -c_{3,2}\omega_{4,2} \\ -a_{3,3}\omega_{3,4}-c_{3,3}\omega_{4,3} \end{pmatrix} \tag{2.16}$$

which can be written as

$$A\omega = f$$

In order to obtain the largest eigenvalue for the successive overrelaxation method, equation (2.16) must be put into the form of (2.8).

Let us partition (Young 1954; Fox 1962) the matrix A as follows (the following, in fact, holds for the NxN case).

$$A = B - L - U$$

where

B is a diagonal matrix

L is a lower triangular matrix

U is an upper triangular matrix

then

$$(B-L-U)\ \omega = f$$

or

$$B\omega = (L + U)\ \omega + f\ .$$

Rearranging (2.14), we have

$$\underbrace{\omega^{n+1}_{i,j-1} b_{i,j} + \omega^{n+1}_{i-1,j} d_{i,j}}_{\text{generates } -L} + \underbrace{\omega^{n}_{i,j+1} a_{i,j} + \omega^{n}_{i+1,j} c_{i,j}}_{\text{generates } -U} = \underbrace{4\ \omega_{i,j}}_{\text{gives } B}$$

This set of equations in matrix form is

$$B\omega^{n+1} = L\omega^{n+1} + U\omega^{n} + f$$

where n indicates the stage of the iteration procedure. Therefore,

$$\omega^{n+1} = B^{-1}(L\omega^{n+1} + U\omega^{n} + f)$$

To get this into the form of (2.8), add and subtract $\omega^{n}$ on the right hand side, incorporating the relaxation factor, we then have

$$\omega^{n+1} = \omega^{n} + \alpha\left[B^{-1}(L\omega^{n+1} + U\omega^{n} + f) - \omega^{n}\right] \qquad (2.17)$$

(2.17) is the matrix form for the vorticity equation (2.8). Rearrangement of (2.17) gives

$$\omega^{n+1} = \left[I-\alpha B^{-1}L\right]^{-1}\left\{\left[(1-\alpha)I+\alpha B^{-1}U\right]\omega^{n}+\alpha B^{-1}f\right\} \qquad (2.18)$$

where I is a unit matrix.

Define the error after n+1 iterations as

$$e^{n+1} = \omega^{n+1} - \omega_0 \qquad (2.19)$$

where $\omega_0$ is the steady state solution. Substitution of (2.19) into (2.18) gives

$$e^{n+1} = \left[I-\alpha B^{-1}L\right]^{-1}\left[(1-\alpha)I+\alpha B^{-1}U\right]e^{n} \qquad (2.20)$$

The criteria for convergence of the iteration procedure is that the modulus of the largest eigenvalue of

$$\left[I-\alpha B^{-1}L\right]^{-1} \left[(1-\alpha)I+\alpha B^{-1}U\right]$$

must be less than unity.

For grids of sizes of approximately up to 10x10, the eigenvalues of the iteration matrix can be calculated directly, using existing computer programs. For larger grids, one cannot calculate the eigenvalues directly, unless a very large computer is available; and even this has its limit. Since only the largest eigenvalue is required to establish convergence or not, the following process (Fox 1964) may be used for larger grids.

Pick any arbitrary vector

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \omega_k \end{pmatrix}$$

It is known that, if the eigenvalues are distinct, the corresponding eigenvectors form a complete system, that is, for any vector $\omega$

$$\omega = c_1\omega_1 + c_2\omega_2 + \dots + c_k\omega_k$$

where the $\omega_i$'s are the eigenvectors and the c's are constants. Let us multiply both sides by the matrix A; then

$$A\omega = \sum_{i=1}^{k} c_i A\omega_i = \sum_{i=1}^{k} c_i\lambda_i\omega_i$$

and in general

$$A^n\omega = \sum_{i=1}^{k} c_i\lambda_i^n\omega_i$$

Now, if $|\lambda_1|>|\lambda_i|$ for i=2,3...k, then as $n\to\infty$

$$A^n\omega \simeq c_1\lambda_1^n\omega_1$$

If we take successive ratios, we have

$$\frac{A^{n+1}\omega}{A^n\omega} \simeq \frac{c_1\lambda_1^{n+1}\omega_1}{c_1\lambda_1^n\omega_1} = \lambda_1.$$

This method was used to obtain the largest eigenvalue for a 10x10 grid. It was found that for a 20x20 grid, the largest eigenvalue was complex. Figure 2.3 gives the relationship between the relaxation factor used and the corresponding maximum eigenvalue.

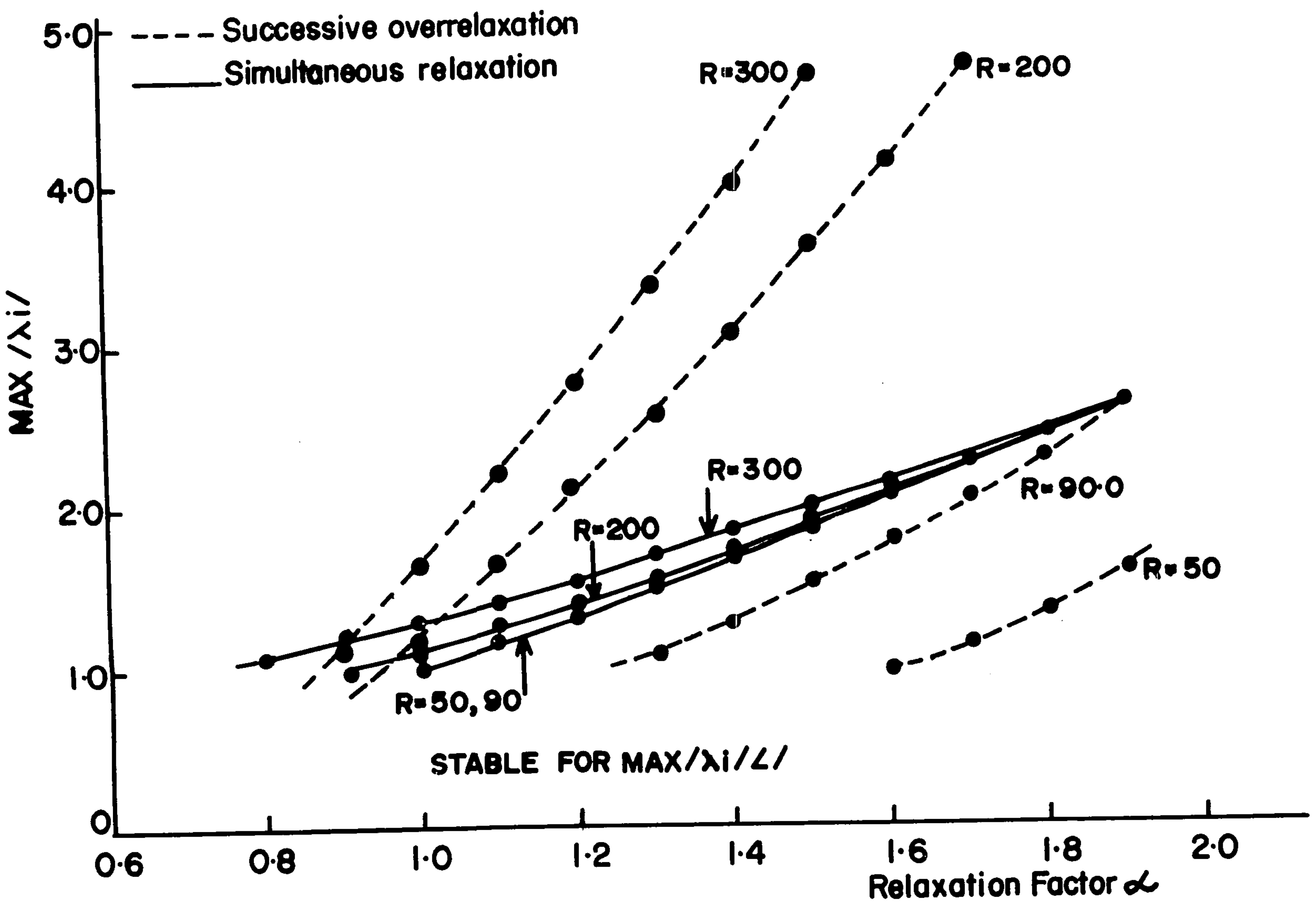



Figure 2·3 Eigenvalues for h=0·1

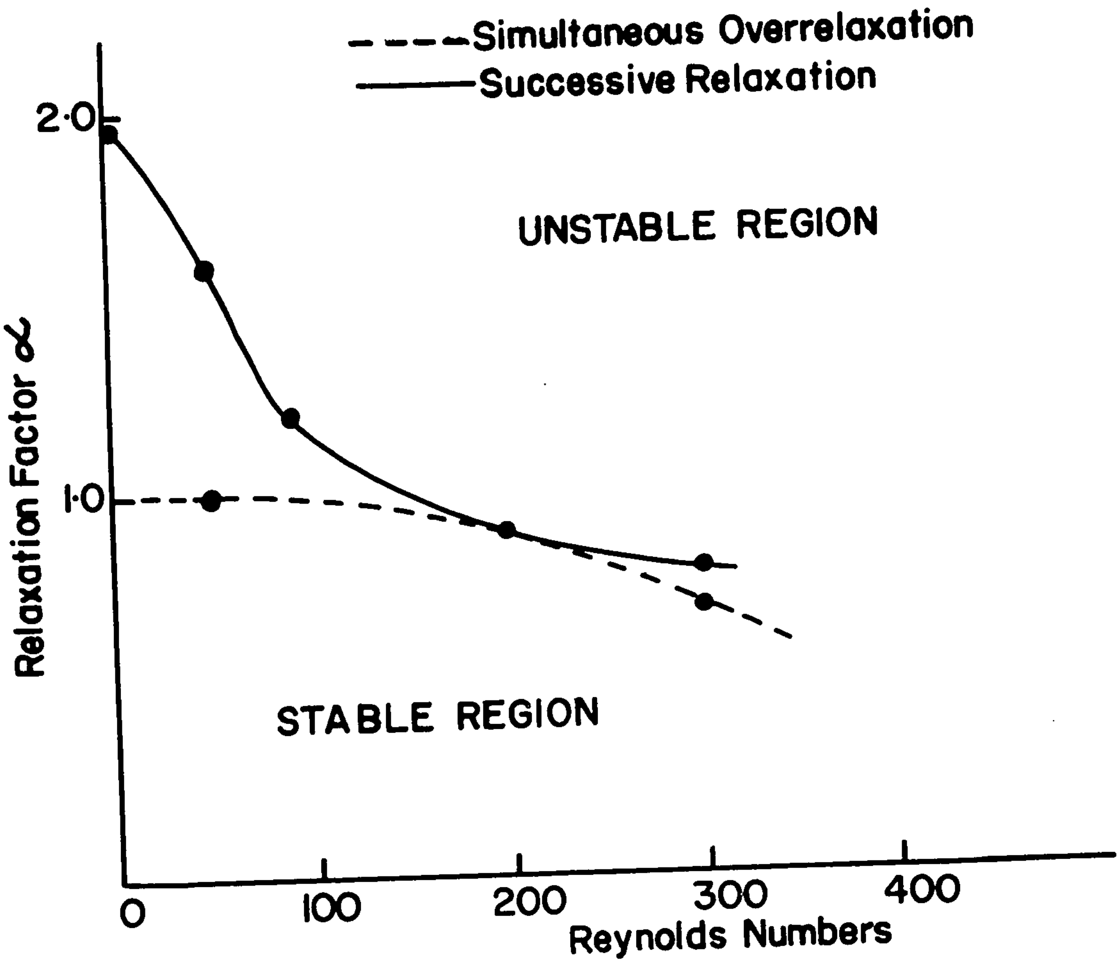



Figure 2·4 Stability Region for h=·1

For the simultaneous relaxation method (Fox 1962), using similar arguments, we obtain $e^{n+1} = B^{-1}(L+U)e^{n}$ and again for convergence the modulus of the largest eigenvalue of $B^{-1}(L+U)$ must be less than unity. In all cases, the relaxation factor for the simultaneous relaxation method must be $\alpha \leq 1$.

Figure 2.4 shows the stability region of the iteration processes for a 10x10 grid versus Reynolds numbers.

## 2.5 The Time-Dependent Equation

The problem may also be solved by finding the steady-state solution of the time-dependent equations

$$\frac{\partial \omega}{\partial t} = \frac{\partial(\psi,\omega)}{\partial(x,y)} + \frac{1}{R}\nabla^2\omega \tag{2.21a}$$

and

$$\nabla^2\psi = -\omega \tag{2.21b}$$

where the boundary conditions are those given earlier in this chapter, and the initial conditions are

$$\psi = \omega = 0 \quad \text{in} \quad 0 \leq x \leq 1 \quad t \leq 0 \tag{2.22}$$

Once the steady-state solution has been reached, the system (2.21) should have the same solution as the problem studied earlier in the chapter; here we simply wish to point out similarities between the two approaches.

To begin with, the stability of the linear case of (2.21a), namely

$$\frac{\partial \omega}{\partial t} = \frac{1}{R} \nabla^2 \omega$$

is well understood. For example, it is known (see e.g. Richtmyer and Morton 1967, p. 189) that stability will be assured if

$$\frac{\Delta t}{Rh^2} \leq \frac{1}{4} \quad ; \qquad (2.23)$$

this is for the simplest explicit finite difference equation. Other forms can be devised, including explicit forms forms (DuFort and Frankel 1953), which have better stability properties than the straight forward method, however, it was recognized by DuFort and Frankel themselves, and pointed out by Pearson (1965) experimentally, that the truncation error is larger for this method than for the simple explicit form.

The non-linear explicit time-dependent difference equation of the simplest kind is equivalent to the simulteneous relaxation form of the steady-state problem. For (2.21a), the simplest formulation is

$$\omega_{i,j}^{t+\Delta t} = \omega_{i,j}^{t} + \frac{\Delta t}{4h^2}\Big[\omega_{i,j+1}^{t} + \omega_{i,j-1}^{t} + \omega_{i+1,j}^{t} + \omega_{i-1,j}^{t} - 4\,\omega_{i,j}^{t}\Big]$$

$$+ \frac{\Delta t}{4h^2}\Big[\Big(\psi_{i+1,j}^{t} - \psi_{i-1,j}^{t}\Big)\Big(\omega_{i,j+1}^{t} - \omega_{i,j-1}^{t}\Big) - \Big(\psi_{i,j+1}^{t} - \psi_{i,j-1}^{t}\Big)\Big(\omega_{i+1,j}^{t} - \omega_{i-1,j}^{t}\Big)\Big] \qquad (2.24)$$

while for the steady-state problem, for simultaneous relaxation, one writes

$$\omega_{i,j}^{n+1} = \omega_{i,j}^{n} + \frac{\alpha}{4}\Big[\omega_{i,j+1}^{n} + \omega_{i,j-1}^{n} + \omega_{i+1,j}^{n} + \omega_{i-1,j}^{n} - 4\,\omega_{i,j}^{n}\Big]$$

$$+ \frac{\alpha R}{16}\Big[\Big(\psi_{i+1,j} - \psi_{i-1,j}\Big)\Big(\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n}\Big) - \Big(\psi_{i,j+1} - \psi_{i,j-1}\Big)\Big(\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n}\Big)\Big] \qquad (2.25)$$

The stability criteria for the two equations are, of course, idential if, and only if,

$$\frac{\Delta t}{Rh^2} = \frac{\alpha}{4} .$$

For small Reynolds numbers (2.23) infers that we are limited by $\alpha \leq 1$ for the stability of the steady-state problem, if simultaneous relaxation is to be used; simultaneous overrelaxation is inherently unstable.

For larger Reynolds numbers, stability of the time dependent equations can also be analysed for both explicit and implicit formulation, and it is the subject of the next chapter.

## CHAPTER III

## STABILITY ANALYSIS OF THE EXPLICIT TWO-DIMENSIONAL TIME-DEPENDENT NAVIER-STOKES EQUATION.

In the past, only the linear case of the time-dependent Navier-Stokes equations, that is to say, the case in which the convection terms are omitted, has been subjected to a strict convergence analysis. As we pointed out in the previous chapter, for that case the stability criteria is

$$\frac{\Delta t}{Rh^2} \leq \frac{1}{4} .$$

For the nonlinear case, Thom and Apelt (1961, p. 136) have obtained a criteria for convergence, based on the "mesh Reynolds number", which indicates the size of the mesh required for convergence (see chapter IV).

For explicit limitations on the size of the time step, however, the best available criteria seems to be that of Fromm (1963). His requirement, in summary, is that

$$\Delta t \leq \text{minimum}\left[\frac{Rh^2}{4}, \frac{h}{|u| + |v|}\right]$$

and is derived from an argument, similar to that beginning this chapter. It is our purpose to examine this criteria in more detail. We will find a limitation for the time step for the explicit method at any stage of the problem. This will lead to a method whereby the computer can be programmed to adjust its own time step in a highly efficient way.

For completeness, we include a summary of the demonstration that an appropriate implicit time-dependent method is stable in the present context, for all values of the time step; however, we shall show in chapter IV that this is not necessarily the most important criteria for practical problem-solving on available machines.

## 3.1 Stability Analysis of the Explicit Form

Let us write the finite difference equation of the explicit two-dimensional vorticity equation as:

$$\omega_{i,j} = \omega_{i,j} + \frac{\Delta t}{Rh^2}\left[\omega_{i,j+1} + \omega_{i,j-1} + \omega_{i+1,j} + \omega_{i-1,j} - 4\omega_{i,j}\right]$$
$$- \frac{\Delta t}{2h}\left[v_{i,j}\left(\omega_{i,j+1} - \omega_{i,j-1}\right) + u_{i,j}\left(\omega_{i+1,j} - \omega_{i-1,j}\right)\right] \qquad (3.1)$$

where $$v_{i,j} = \frac{-1}{2h}\left(\psi_{i+1,j} - \psi_{i-1,j}\right)$$

$$u_{i,j} = \frac{1}{2h}\left(\psi_{i,j+1} - \psi_{i,j-1}\right)$$

and the time t is to be understood when no superscript is written.

The stability of this scheme will be analysed by a procedure very similar to that used by O'Brien, Hyman, and Kaplan (1951) and Fromm (1963).

We assume that there exists an error at each point, defined by

$$\varepsilon_{i,j} = \omega_{i,j} - \omega^{0}_{i,j}$$

where $\omega^{0}_{i,j}$ is some reference level of the vorticity which can be taken as the steady-state solution.

Substituting into equation (3.1), we have:

$$\varepsilon^{t+\Delta t}_{i,j} = \varepsilon_{i,j} + \frac{\Delta t}{Rh^2}\left[\varepsilon_{i,j+1} + \varepsilon_{i,j-1} + \varepsilon_{i+1,j} + \varepsilon_{i-1,j} - 4\varepsilon_{i,j}\right]$$
$$- \frac{\Delta t}{2h}\left[v_{i,j}\left(\varepsilon_{i,j+1} - \varepsilon_{i,j-1}\right) + u_{i,j}\left(\varepsilon_{i+1,j} - \varepsilon_{i-1,j}\right)\right]$$

(3.2)

The error on the boundary is zero and we may thus expand the error difference equation in a finite double series of orthogonal functions. The choice of frequencies of the harmonics must be such that the sum reduces to the correct boundary conditions. Let us choose

$$\varepsilon_{i,j} = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \varepsilon_0 r^n e^{\kappa k_1 i} e^{\kappa k_2 j}$$

where

$$k_1 = \frac{(2p+1)\pi}{2}$$

$$k_2 = \frac{(2q+1)\pi}{2}$$

r is the "growth factor" of the error, n is the number of time steps, $\varepsilon_0$ is some reference level, may be taken as one,

$\kappa$ is $\sqrt{-1}$

Substituting into equation (3.2), examining each term in the series, and dividing through by

$$\varepsilon_0 r^n e^{\kappa k_1 i} e^{\kappa k_2 j}$$

one obtains

$$r = 1 + \frac{\Delta t}{Rh^2}\left[e^{\kappa k_2} + e^{-\kappa k_2} + e^{\kappa k_1} + e^{-\kappa k_1} - 4\right]$$

$$- \frac{\Delta t}{2h}\left[v_{i,j}\left(e^{\kappa k_2} - e^{-\kappa k_2}\right) + u_{i,j}\left(e^{\kappa k_1} - e^{-\kappa k_1}\right)\right].$$

Now
$$\cos k_1 = \frac{e^{\kappa k_1} + e^{-\kappa k_1}}{2}$$

$$\sin k_1 = \frac{e^{\kappa k_1} - e^{\kappa k_1}}{2\kappa}$$

so that

$$r = 1 + \frac{\Delta t}{Rh^2}\left[2 \cos k_2 + 2 \cos k_1 - 4\right] - \frac{\kappa \Delta t}{2h}\left[2\, v_{i,j} \sin k_2 + 2\, u_{i,j} \sin k_1\right]$$

which becomes

$$r = 1 - \frac{4\Delta t}{Rh^2}\left[\sin^2 \frac{k_1}{2} + \sin^2 \frac{k_2}{2}\right] - \frac{\kappa \Delta t}{h}\left[v_{i,j} \sin k_2 + u_{i,j} \sin k_1\right] \qquad (3.3)$$

This is a complex number and the condition for convergence requires that the modulus of r be less than unity. So we insist that

$$1 \geq |r|^2 = \left(1 - \frac{4\Delta t}{Rh^2}\left[\sin^2\frac{k_1}{2} + \sin^2\frac{k_2}{2}\right]\right)^2 + \left(\frac{\Delta t}{h}\left[v_{i,j} \sin k_2 + u_{i,j} \sin k_1\right]\right)^2 \geq 0$$

or

$$1 \geq 1 + \frac{16\Delta t^2}{R^2 h^4}\left(\sin^2 \frac{k_1}{2} + \sin^2 \frac{k_2}{2}\right)^2 - \frac{8\Delta t}{Rh^2}\left[\sin^2 \frac{k_1}{2} + \sin^2 \frac{k_2}{2}\right] + \left(\frac{\Delta t}{h}\right)^2\left(v_{i,j} \sin k_2 + u_{i,j} \sin k_1\right)^2 \geq 0.$$

Let $$a = \sin^2\frac{k_1}{2} + \sin^2\frac{k_2}{2} \tag{3.4}$$

and $$b = v_{i,j}\sin k_2 + u_{i,j}\sin k_1 \tag{3.5}$$

then the above requirement is simply that

$$1 \geq |r|^2 = 1 + \frac{16\Delta t^2}{R^2h^4}a^2 - \frac{8\Delta t}{Rh^2}a + \left(\frac{\Delta t}{h}\right)^2 b^2 \geq 0$$

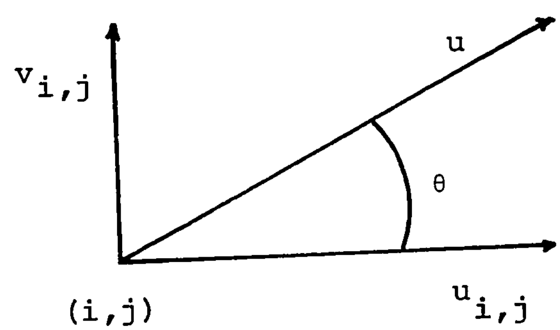
The right hand inequality is trivially satisfied; the only requirement is then that

$$\frac{\Delta t}{Rh^2} \leq \frac{1}{2a + \frac{R^2h^2b^2}{8a}} \tag{3.6}$$

[The linear criteria is easily recaptured if we set b=0, however, this will be improved upon in a moment.]

Our problem is now to find a theoretical minimum. As it turns out, this will be too strict in the sense that this minimum is not actually attained anywhere in the grid, and that in practice, $\Delta t$ need not be chosen quite this small.

We assume that the convection velocity at any one grid point is fixed



and set $u_{i,j} = u \cos\theta$

$v_{i,j} = u \sin\theta$

substituting into (3.5), we obtain

$$b = u\ (\sin\theta \sin k_2 + \cos\theta \sin k_1)$$

thus
$$\frac{\Delta t}{Rh^2} \leq \frac{1}{2a + \frac{R^2h^2u^2c^2}{8a}} \tag{3.7}$$

where

$$c = \sin\theta \sin k_2 + \cos\theta \sin k_1$$

The problem now is to minimize, since this will give us maximum $\Delta t$ allowable, the right hand side of (3.7) over $k_1$, $k_2$ and $\theta$ as a function of R, u and h.

We let
$$Q = 2a + \frac{R^2h^2u^2c^2}{8a} \tag{3.8}$$

and for maximum Q, the three necessary conditions are:

$$\frac{\partial Q}{\partial k_1} = \left[1 - \frac{R^2h^2u^2c^2}{16a^2}\right]\sin k_1 + \frac{R^2h^2u^2c^2}{4a} \cos\theta \cos k_1 = 0 \tag{3.9}$$

$$\frac{\partial Q}{\partial k_2} = \left[1 - \frac{R^2h^2u^2c^2}{16a^2}\right]\sin k_2 + \frac{R^2h^2u^2c^2}{4a} \sin\theta \cos k_2 = 0 \tag{3.10}$$

$$\frac{\partial Q}{\partial \theta} = \frac{R^2h^2u^2c^2}{4a}\left[\cos\theta \ \sin k_2 - \sin\theta \ \sin k_1\right] = 0 \qquad (3.11)$$

Equation (3.11) implies that

$$\tan\theta = \frac{\sin k_2}{\sin k_1} \qquad (3.12)$$

or $c = 0$ ;

However, $c = 0$ would imply that $\sin k_1 = \sin k_2 = 0$ giving a maximum value of $Q = 4$.

This is a maximum for the linear case.

Equations (3.9) and (3.10) combined give rise to

$$\frac{\sin k_2}{\sin k_1} = \tan\theta \frac{\cos k_2}{\cos k_1} \qquad (3.13)$$

Since both (3.12) and (3.13) must hold, we infer that

$\cos k_1 = \cos k_2$ and, therefore, $\theta = 45^\circ$ or $225^\circ$.

Similarly, we may have $k_1 = 2\pi - k_2$ which gives $\theta = 135^\circ$ or $315^\circ$.

This indicates that the maximum value of Q will occur at velocities with direction of $45^\circ$ in any one of the four quadrants. With no loss in generality, we may consider

$$\theta = 45^\circ \text{ and } k_1 = k_2$$

Now we have

$$Q = 2\left[\sin^2\frac{k_1}{2} + \sin^2\frac{k_2}{2}\right] + \frac{R^2h^2u^2\left(\sin\theta\ \sin k_2 + \cos\theta\ \sin k_1\right)^2}{8\left(\sin^2\frac{k_1}{2} + \sin^2\frac{k_2}{2}\right)}$$

which can be reduced to

$$Q = 4\ \sin^2\frac{k_1}{2} + \frac{R^2h^2u^2\ \sin^2 k_1}{8\ \sin^2\frac{k_1}{2}}$$

or

$$Q = 4 + \frac{R^2h^2u^2\ \ 4\ \sin^2\frac{k_1}{2}\cos^2\frac{k_1}{2}}{8\ \sin^2\ \frac{k_1}{2}} - 4\cos^2\frac{k_1}{2}$$

Therefore,

$$Q = 4 + \left[\frac{R^2h^2u^2}{2} - 4\right]\cos^2\frac{k_1}{2} \qquad (3.14)$$

Equation (3.14) shows that the maximum value of Q is

$$Q_{max} = \begin{cases} 4 & \text{if } R^2h^2u^2 \le 8 \\ \dfrac{R^2h^2u^2}{2} & \text{if } R^2h^2u^2 > 8. \end{cases}$$

In terms of $\Delta t$, sufficient stability criteria for the explicit equation (3.1) are, therefore,

$$\frac{\Delta t}{Rh^2} \le \frac{1}{4} \quad ; \quad R^2h^2u^2 \le 8 \qquad (3.15)$$

$$\frac{\Delta t}{Rh^2} \le \frac{2}{R^2h^2u^2} \quad ; \quad R^2h^2u^2 > 8 \qquad (3.16)$$

where u is the maximum $\left[v_{i,j}^2 + u_{i,j}^2\right]^{\frac{1}{2}}$ for $|u_{i,j}| = |v_{i,j}|$ for all mesh points (i,j).

In practice, the maximum velocity u may not occur at a 45° inclined to the horizontal axis. Therefore, using equation (3.16) may result in a smaller time step than it is actually necessary to use. A way to get around this is to maximize Q only with respect to $k_1$, $k_2$ and not $\theta$. In this case, we have $u_{i,j}$ and $v_{i,j}$ in the equation for Q and, thus, for maximum Q, the conditions are:

$$\frac{\partial Q}{\partial k_1} = 2 \sin\frac{k_1}{2} \cos\frac{k_1}{2} + \frac{R^2h^2}{8}\left[\frac{2abu_{i,j} \cos k_1}{a^2} - \frac{b^2 \sin\frac{k_1}{2}\cos\frac{k_1}{2}}{a^2}\right] = 0$$

and

$$\frac{\partial Q}{\partial k_2} = 2 \sin\frac{k_2}{2} \cos\frac{k_2}{2} + \frac{R^2h^2}{8}\left[\frac{2abv_{i,j} \cos k_2}{a^2} - \frac{b^2 \sin\frac{k_2}{2}\cos\frac{k_2}{2}}{a^2}\right] = 0$$

This will give the relation:

$$\frac{\tan k_1}{\tan k_2} = \frac{v_{i,j}}{u_{i,j}}$$

and, therefore, we can calculate $k_2$ from

$$k_2 = \tan^{-1}\left[\frac{u_{i,j}}{v_{i,j}} \tan k_1\right] \qquad (3.17)$$

Equation (3.17) and (3.6) may be used to obtain the experimental time step, as this will give, in most cases, a larger value for $\Delta t$ than from

$$\frac{\Delta t}{Rh^2} \leq \frac{2}{R^2h^2u^2}$$

However, this may be offset by the extra computations involved to solve for $k_2$ in equation (3.17). In general, one assumes values for $k_1$ and using the velocities at the grids point to be considered, the value of $k_2$ is calculated. In this fashion, one searches for the maximum Q over all the grids. More will be said about this in chapter IV.

3.2 Stability Analysis of the Implicit Form

We shall restrict ourselves to the Crank-Nicolson (1947) method, where each value of the stream function and vorticity on every mesh point, aside from $\frac{\omega^{t+\Delta t}-\omega^t}{\Delta t}$,

is replaced by

$$\omega^t_{i,j} = \frac{1}{2}\left(\omega^t_{i,j} + \omega^{t+\Delta t}_{i,j}\right).$$

The linear case of the implicit method is stable for all time steps (Richtmyer and Morton 1967, p. 189). Pearson (1965) used this method and indicated that experiments have shown it to be stable for the nonlinear case; even for high Reynolds numbers. It is the purpose of this section to show that the Crank-Nicholson implicit difference form of the Navier-Stokes equation is stable for all time steps and Reynolds numbers.

Let us write equation (3.1) in the implicit Crank-Nicholson form

$$\omega_{i,j} = \omega_{i,j} + \frac{\Delta t}{Rh^2}\Big[\omega^{t+\Delta t}_{i,j+1} + \omega^{t+\Delta t}_{i,j} + \omega^{t+\Delta t}_{i+1,j} + \omega^{t+\Delta t}_{i-1,j}$$
$$- 4\,\omega^{t+\Delta t}_{i,j} + \omega^{t}_{i,j+1} + \omega^{t}_{i,j-1} + \omega^{t}_{i+1,j} + \omega^{t}_{i-1,j} - 4\omega^{t}_{i,j}\Big]$$
$$- \frac{\Delta t}{8h}\Big[\Big(v^{t+\Delta t}_{i,j} + v^{t}_{i,j}\Big)\Big(\omega^{t+\Delta t}_{i,j+1} + \omega^{t}_{i,j+1} - \omega^{t+\Delta t}_{i,j-1}$$
$$- \omega^{t}_{i,j-1}\Big) + \Big(u^{t+\Delta t}_{i,j} + u^{t}_{i,j}\Big)\Big(\omega^{t+\Delta t}_{i+1,j} + \omega^{t}_{i+1,j} - \omega^{t+\Delta t}_{i-1,j}$$
$$- \omega^{t}_{i-1,j}\Big)\Big] \qquad (3.18)$$

For the stability analysis of this method, we use again the procedure of O'Brien, Hyman and Kaplan (1951).

For convenience of notation replace

$$\Big(v^{t+\Delta t}_{i,j} + v^{t}_{i,j}\Big) \text{ by } 2\,v^{*}_{i,j}$$

and

$$\Big(u^{t+\Delta t}_{i,j} + u^{t}_{i,j}\Big) \text{ by } 2\,u^{*}_{i,j}$$

By similar arguments to the ones used for the stability analysis of the explicit method, we obtain

$$r = 1 + \frac{\Delta t}{2Rh^2}\Big[\cos k_1 + \cos k_2 - 4 + r\Big(\cos k_1 + \cos k_2 - 4\Big)\Big]$$
$$- \frac{\Delta t}{2h}\Big[r\,v^{*}_{i,j}\sin k_2 + r\,u^{*}_{i,j}\sin k_1 + v^{*}_{i,j}\sin k_2 + u^{*}_{i,j}\sin k_1\Big]$$

Solving for r, we find

$$r = \frac{1 - \frac{2\Delta t}{Rh^2}\left(\sin^2\frac{k_1}{2} + \sin^2\frac{k_2}{2}\right) - \frac{\Delta t \kappa}{2h}\left(v^*_{i,j}\sin k_2 + u^*_{i,j}\sin k_1\right)}{1 + \frac{2\Delta t}{Rh^2}\left(\sin^2\frac{k_1}{2} + \sin^2\frac{k_2}{2}\right) + \frac{\Delta t \kappa}{2h}\left(v^*_{i,j}\sin k_2 + u^*_{i,j}\sin k_1\right)} \qquad (3.19)$$

where $\kappa = \sqrt{-1}$

For equation (3.1) to be stable, we must have the condition that

$$|r| \leq 1.$$

Let $a = \frac{2\Delta t}{Rh^2}\left(\sin^2\frac{k_1}{2} + \sin^2\frac{k_2}{2}\right)$

$b = \frac{\Delta t}{2h}\left(v^*_{i,j}\sin k_2 + u_{i,j}\sin k_1\right)$

then $r = \frac{1-a-\kappa b}{1+a+\kappa b}$

or $r = \frac{1-a^2-b^2-\kappa 2b}{1+2a+a^2+b^2}$

thus $|r|^2 = \frac{(1-a^2-b^2)^2 + (2b)^2}{(1+2a+a^2+b^2)^2}$

and from this it follows the condition for $|r|^2 \leq 1$ is

$$4a(1+2a+a^2+b^2) \geq 0 \qquad (3.20)$$

Now a and $b^2$ are always positive for any value of the forward time step and, therefore, equation (3.20) will always hold. This implies that the iteration procedure for the time-dependent vorticity equation in the Crank-Nicolson finite difference form is stable for all values of $\Delta t$, R and h. Notice that, although our equation was in a sense linearized by substitution of average velocities, the conclusion still holds for arbitrary $u^*_{i,j}$ and $v^*_{i,j}$, and would appear to be general.

CHAPTER IV

# IMPLICIT VERSUS EXPLICIT METHOD AND NUMERICAL RESULTS

It has been shown that the Crank-Nicolson implicit method is stable for all time steps. It seems natural, in order to obtain the steady-state solution in the fewest time steps, this method should be adopted. However, we have to ask ourselves, does the implicit method save computing time? This may be the case in some problems (Pearson 1965), however, it warrants some investigation for the problem considered here, with respect to computer time used, solution for higher Reynolds numbers and accuracy of the method.

## 4.1 Formulation of the Implicit Method

The implicit finite difference equation can be solved in several ways. One would be to solve the set of simultaneous equations by a standard method such as Gaussian elimination. However, whenever the system of equations becomes too large, one has to revert to iterative procedures.

Since in chapter III, stability analysis was performed for the Crank-Nicolson method, we shall restrict ourselves to the latter case.

In order to use the successive overrelaxation procedure, let us rewrite equation (3.18) as

$$\omega_{i,j} = \beta \left\{ \omega_{i,j} + \frac{\Delta t}{2Rh^2} \left[ \begin{aligned} &\omega_{i,j+1}^{t+\Delta t} + \omega_{i,j-1}^{t+\Delta t} + \omega_{i+1,j}^{t+\Delta t} + \omega_{i-1,j}^{t+\Delta t} \\ &+ \omega_{i,j+1}^{t} + \omega_{i,j-1}^{t} + \omega_{i+1,j}^{t} + \omega_{i-1,j}^{t} \\ &- 4\,\omega_{i,j}^{t} \end{aligned} \right] \right.$$

$$- \frac{\Delta t}{8h}\left[\left(v_{i,j}^{t+\Delta t} + v_{i,j}^{t}\right)\left(\omega_{i,j+1}^{t+\Delta t} + \omega_{i,j+1}^{t} - \omega_{i,j-1}^{t+\Delta t}\right.\right.$$

$$\left.- \omega_{i,j-1}^{t}\right) + \left(u_{i,j}^{t+\Delta t} + u_{i,j}^{t}\right)\left(\omega_{i+1,j}^{t+\Delta t} + \omega_{i+1,j}^{t} - \omega_{i-1,j}^{t+\Delta t}\right.$$

$$\left.\left.\left.- \omega_{i-1,j}^{t}\right)\right]\right\} \qquad (4.1)$$

where

$$\beta = \frac{1}{1 + \dfrac{2\Delta t}{Rh^2}}$$

In the process of solving for $\omega_{i,j}^{t+\Delta t}$ all terms involving the superscript t, $v_{i,j}^{t+\Delta t}$ and $u_{i,j}^{t+\Delta t}$ are known. It was pointed out in the beginning of chapter II, that for the successive overrelaxation method, values on the grid are used in the iteration process as soon as they become available.

To express this in notational form, a superscript n or n+1 must be added to the vorticity terms involving t+Δt. For simplification of the notation, whenever the superscript with respect to time is missing, t+Δt is to be understood, that is

$$\omega_{i,j}^{n+1} \equiv \omega_{i,j}^{(t+\Delta t),\, n+1}$$

$$\omega_{i,j}^{n} \equiv \omega_{i,j}^{(t+\Delta t),n}$$

Adding and subtracting $\omega_{i,j}^{t+\Delta t}$ on the right hand side of (4.1), incorporating a relaxation factor and dropping all superscript t+Δt on the vorticity terms, we have for the iteration procedure:

$$\begin{aligned}
\omega_{i,j}^{n+1} = \omega_{i,j}^{n} + \alpha\beta\Bigg\{ & \omega_{i,j}^{t} - \frac{1}{\beta}\,\omega_{i,j}^{n} \\
& + \frac{\Delta t}{2Rh^2}\Big[\omega_{i,j+1}^{n} + \omega_{i,j-1}^{n+1} + \omega_{i+1,j}^{n} + \omega_{i-1,j}^{n+1} \\
& \quad + \omega_{i,j+1}^{t} + \omega_{i,j-1}^{t} + \omega_{i+1,j}^{t} + \omega_{i-1,j}^{t} - 4\,\omega_{i,j}^{t}\Big] \\
& - \frac{\Delta t}{8h}\Big[\Big(v_{i,j}^{t+\Delta t} + v_{i,j}^{t}\Big)\Big(\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n+1} + \omega_{i,j+1}^{t} \\
& \quad - \omega_{i,j-1}^{t}\Big)\Big(u_{i,j}^{t+\Delta t} + u_{i,j}^{t}\Big)\Big(\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n+1} \\
& \quad + \omega_{i+1,j}^{t} - \omega_{i-1,j}^{t}\Big)\Big]\Bigg\} \qquad (4.2)
\end{aligned}$$

where α again is the relaxation factor and n represents the number of sweeps over the mesh.

## 4.2 Computational Procedure for the Implicit Method

The computational steps outlined, are essentially the same as used by Pearson (1965). We shall only describe the steps involved in going from time t to $t+\Delta t$. In general, to consider one time step, one must obtain a first approximation for $\psi^{t+\Delta t}$ and $\omega^{t+\Delta t}$, and then follow the same procedure as outlined in the flip-flop method to obtain more accurate values of $\psi$ and $\omega$ at time $t+\Delta t$.

The steps may be summed up as follows:

1) Obtain a first approximation of $\omega_{i,j}^{t+\Delta t}$ at all interior points by using the explicit representation of the vorticity equation.

2) Calculate the first approximation of $\psi_{i,j}^{t+\Delta t}$ using $\nabla^2\psi = -\omega$. This is carried out using successive overrelaxation.

3) Calculate the boundary conditions for (4.2) by the same method as described in chapter II, i.e. for the left wall

$$\omega_{0,j} = \frac{-2}{h}\,\psi_{1,j}$$

where the $\psi$'s are the values obtained from step 3. A smoothing factor must be used and the proportion of old to new boundary value will again depend on the Reynolds number.

4) Calculate the new values of $\omega_{i,j}^{t+\Delta t}$, represented by $\omega_{i,j}^{n+1}$ in equation (4.2), using successive overrelaxation.

5) Take the values of the vorticity found in step 4 and check if it satisfies $\nabla^2\psi = -\omega$ within a specified tolerance. If yes, carry out step 8, else continue with step 6.

6) Solve for the new values of $\psi_{i,j}^{t+\Delta t}$, using $\nabla^2\psi = -\omega$ and the vorticity found in step 4.

7) Repeat the whole procedure, beginning at step 3.

8) At this point one time step is completed. Substitute the values of $\psi_{i,j}^{t+\Delta t}$, found in step 6 into the difference approximation for

$$\nabla^4\psi = -R \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)}$$

If the sum of the absolute error over all points is less than a specified tolerance, computation is stopped and the values of the vorticity and stream function found so far, are taken as the steady-state values. The absolute error over all points again is

$$\sum_{i=1}^{M-1} \sum_{j=1}^{N-1} |R_{i,j}| \leq TOL \qquad (4.3)$$

where the tolerance is specified, such that the average error per point is less than $1x10^{-5}$.

9) If we are not within the specified tolerance, one more time step is considered and we start again with step 1.

The explicit method, on the other hand, involves much simpler computational procedures and are considered next.

## 4.3 Computational Method for the Explicit Equation

The essential difference in terms of computation procedure is that for each time step the vorticity and stream function equation are only swept once. This can be done as we are only interested in the steady-state solution and, therefore, it is not necessary to solve $\nabla^2\psi = -\omega$ to a high degree of accuracy at each time step. As with the implicit method, the error in the difference equation to the fourth order differential equation is taken as a criteria to determine when steady-state is reached.

To begin with, set all values of $\psi$ and $\omega$ to zero. The steps for solving the equations are:

1) Calculate the boundary values for the vorticity equation, using (2.7).

2) Compute $\omega_{i,j}^{t+\Delta t}$, using the time dependent vorticity equation in explicit form. Here, $\omega_{i,j}^{t+\Delta t}$ is calculated directly.

3) Obtain $\psi^{t+\Delta t}$ by applying successive overrelaxation on $\nabla^2\psi = -\omega$. The field is swept only once.

4) Check if the stream function satisfies the difference equation to the fourth-order equation, that is

$$\sum_{i=1}^{M-1}\sum_{j=1}^{N-1} |R_{i,j}| \leq TOL$$

5) If the total residual is less than the specified tolerance, computation is stopped, else repeat from step 1.

## 4.4 Discussion of Results with Respect to Computer Time

For the comparison of the two methods, we shall concern ourselves only with the accuracy achieved for the solution to the difference equation after a certain number of time steps. With this criteria in mind, the implicit method is compared to the explicit with respect to computer time used.

First of all, we want to determine the optimum time steps for various Reynolds numbers for both methods. In order to perform the number of runs necessary, only a 10x10 grid; h=1, and R=90 and 300 were used. The computer times for the implicit method to achieve

$$\sum_{i=1}^{M-1} \sum_{j=1}^{N-1} |R_{i,j}| \leq .001$$

for a set of time steps is shown in figure 4.1. On the other hand, because of the speed of computation, the convergence of the explicit method was measured as a function of the number of time steps, which was taken to be 100. Figure 4.2 illustrates these results. The compute time on the IBM-7040

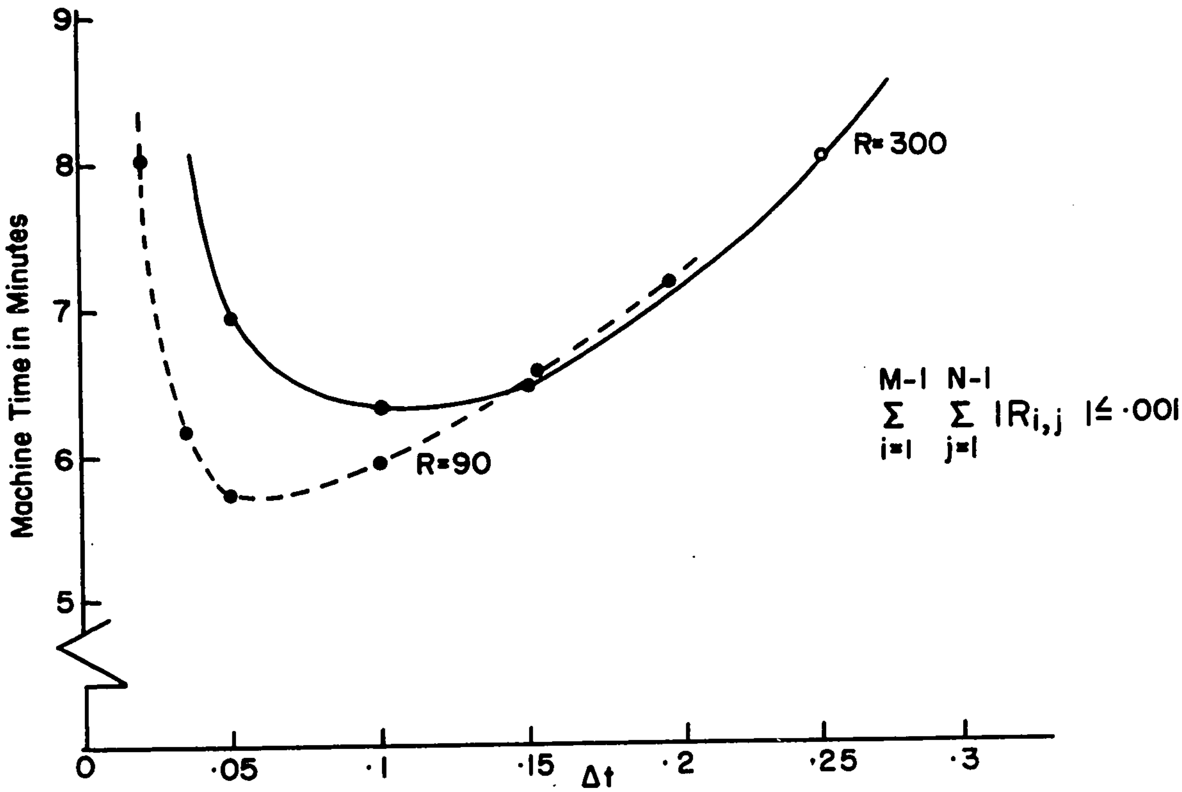



Figure 4·1 Computer Time for Implicit Method

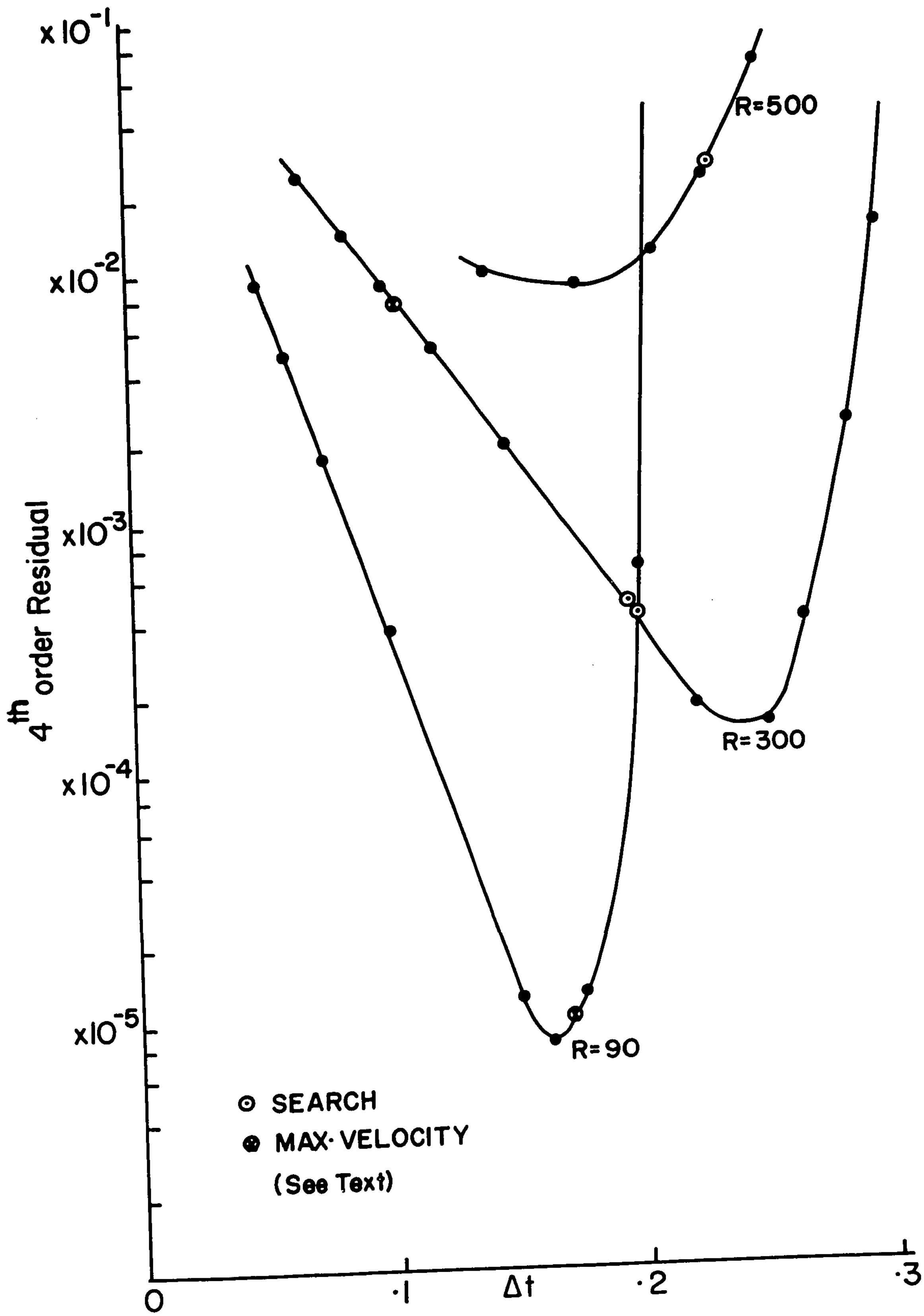



Figure 4·2 Explicit Method; Residuals after 100 Time Steps for the explicit method in each case, was about 1.7 minutes. In comparison, the best times for the implicit method are, by interpolation from figure 4.1, 5.7 and 6.3 minutes for Reynolds numbers of 90 and 300 respectively. Thus, for the implicit method, the computing time is several times that of the explicit.

In practice, we have no means to predict the optimum time step for the implicit method and, thus, will be faced with even greater computer times. Comparative runs were also obtained for a 20x20 grid and R=300.0, for which the results are shown in Table 4.1.

Table 4.1

R=300.0 20x20 grid, h=.05

| Method | $\Delta t$ | Time steps | Compute Time | $\sum\sum\lvert R_{i,j}\rvert$ |
|---|---|---|---|---|
| Implicit | .01 | 499 | 31 min | .048 |
| Implicit | .033 | 149 | 18.3 min | .046 |
| Implicit | .1 | 74 | 29 min | .033 |
| Explicit | .0555 | 100 | 3.2 min | .009 |

ComputerTimes for Implicit Versus Explicit Method for R=300

The size of the time step for the implicit method will determine the number of sweeps one must perform for each of the linear and non-linear equations. It will increase with larger time steps. However, one sweep of both equations is equivalent to one time step in the explicit method. This implies that, for each time step of the implicit method, several time steps of the explicit method can be performed; this is true in particular in the initial stages of computation. One can thus come to the conclusion that, although it is possible to use larger time steps for the implicit method, the explicit system is by far superior in terms of computation time.

Figure 4.2 shows the effects on the speed of convergence for various time steps for the explicit method. There is an optimum time step for every combination of Reynolds number and grid size, and it is smaller than the maximum permissible step and still ensure stability. The region between the optimum time step and the onset of instability can be considered as temporal stability. By this we mean the growth in the error is offset by the changes induced in the iteration matrix, due to an additional time step. In other words, considering a specific point, the iteration oscillates between stability and instability, however, stability is more dominant. A further increase in the size of the time step results in the domination of the unstable growth and thus divergence occurs.

The time step can be calculated, using the methods indicated in chapter III. Here again we have the two alternatives. One is to search for the maximum velocity over all the mesh points

$$u = \max\left[u_{i,j}^2 + v_{i,j}^2\right]$$

$$i=1,2,..M-1$$

$$j=1,2,..N-1$$

and choose

$$\Delta t = \min\left[\frac{Rh^2}{4}, \frac{1}{Ru^2}\right]. \qquad (4.4)$$

This results in time steps given by $\otimes$ on figure 4.2. As can be seen, they are much smaller than the maximum allowed. A reason for this is that the theory leading to (4.4) specifies the maximum u to be at $45^\circ$ to the horizontal, In practice, this is not the case, and, therefore, one should use the second alternative in which one searches for the maximum Q, taking into account the velocity components $u_{i,j}$ and $v_{i,j}$ in the calculations of the wave numbers $k_1$ or $k_2$. We let, for example, $k_1 = \frac{2p+1}{2}\pi$, $p = 0,1,\ldots M-1$ and calculate $k_2$ by

$$k_2 = \tan^{-1}\left[\frac{u_{i,j}}{v_{i,j}} \tan k_1\right] \qquad (4.5)$$

The values of $k_1$, $k_2$, $u_{i,j}$ and $v_{i,j}$ are then used to calculate Q, using equation (3.6). In this manner, one can search for the maximum Q over all mesh points. The time steps obtained by this method are indicated by ⊙ on figure 4.2 and are much improved over the ones obtained from (4.4).

The explicit method, for the problem considered, is far superior over the implicit method with respect to computing time. This is more pronounced when a finer grid is used, as one can illustrate by going from 10x10 to a 20x20 grid. In every case we were able to predict the time step close to the critical value which still will ensure stability for the explicit method. For these reasons, all computations were carried out using the explicit method.

## 4.5 Solutions for Higher Reynolds Numbers

We shall now present the steady-state solutions for Reynolds numbers of up to 2000. The solutions obtained for Reynolds numbers up to 400 agreed with Burggraf's (1966), and the pattern of the stream function is similar to the one observed in experiments (Pan and Acrivos 1966). Numerical results are shown for R=90, 200, 400, 1000 and 2000, in figures 4.5 to 4.9 respectively. The results were plotted on a CalComp plotter in the Computer Center of the University of Western Ontario. The computer plotting routines are not designed for contour plotting and, thus, the unsmoothened results are presented.

The size of the vortex in the lower downstream corner continues to increase up to Reynolds number of R=2000. This is in full agreement with the numerical results of Burggraf (1966) up to R=400. However, experimental work of Pan and Acrivos (1967) indicated that beyond a value of perhaps R=500, the size of this vortex again decreased; see figure 4.10. That the limit of this flow for very large R should consist of an inviscid core of constant vorticity was argued by Batchelor (1956), and for values as high as R=100000, Mills (1965a) observed such a flow experimentally.

However, for intermediate values of R, the comparison between experimental and numerical results is not impressive (Mills 1965), the experiment usually showing a larger central flow and small or non-existent back vortex. This discrepancy may be explained by the following argument.

The separation of flow occurs mainly because the kinetic energy is dissipated by viscosity within the boundary and, since turbulent boundary layer flow resists better the tendency of separation than laminar flow, it will stick better to the surface. As a result the corner vortex for higher Reynolds numbers will be smaller in experimental work.

Mills (1965) observed similar discrepancies between computed and experimental results for fluid flow in a rectangular cavity having a width to height ratio of 2:1. For R=100 he observed that a second vortex appears numerically which cannot be obtained by experiment. Mills argues that this discrepancy is due to skin friction of the fluid on the floor of the cavity and the surface tension tractions in the corner which will inhibit this weaker flow in the second vortex.

The only other work available in this subject appears to be a recent report by Greenspan (1968) in which the fluid appears not to separate from the wall for comparable Reynolds numbers. In this report, however, calculations have been carried out for very coarse grids only, and are not likely to be valid for high Reynolds numbers. A very fine mesh is needed for large values of R.

An approximate idea about the size of the grid can be derived following an argument presented by Thom and Apelt (1961, p. 136). They developed a criteria for the convergence of the Navier-Stokes equation as a function of the mesh Reynolds number, namely

$$\frac{q^2 n^2}{\nu^2} < 20 \qquad (4.6)$$

where q is the average resultant velocity through the portion of the mesh considered, 2n is the mesh size, $\nu$ is the kinematic viscosity, and $\frac{qn}{\nu}$ is the mesh Reynolds number.

In chapter I we have defined the Reynolds number as

$$R = \frac{Ua}{\nu}$$

Suppose we set U=q since, in the worst case, the average mesh velocity may be equal to U, thus

$$R = \frac{qa}{\nu}$$

From (4.6) we have

$$\frac{q^2 n^2}{\nu^2} \frac{a^2}{a^2} = \frac{R^2 n^2}{a^2} < 20,$$

but h=2n and a=1, therefore, $\frac{R^2 h^2}{4} < 20$ which gives

$$h < \frac{\sqrt{80}}{R} \simeq \frac{9}{R} \qquad (4.7)$$

Equation (4.7) will give h=.009 for R=1000 and h=.0045 for R=2000. In practice, a coarser grid than indicated can be used.

For $R \geq 1000$ and h=0.025, numerical oscillations of the stream function can be observed at the right top corner. Similar observations were pointed out by Bye (1966) and Simuni (1964) for lower Reynolds numbers and coarser grid. To eliminate these numerical oscillations, which have no reality for the differential equation, a finer grid must be used. It is sufficient to use h=.0125 for R=1000, but it could not be quite eliminated for R=2000. These results are illustrated in figure 4.3 and 4.4. In order to eliminate the oscillations for R=2000, an even finer grid than h=.0125 must be used. Due to the size of the computer available, the computation could not be carried out.

The total computing time for R=1000 and R=2000 was three and four hours respectively on the IBM-7040. For the first two or three hours a 40x40 grid was used, which then was used as the initial solution for a 80x80 grid. Computation was stopped when the average error per grid point was less than $5 \times 10^{-6}$. In comparison, the computer times quoted by Pearson (1967) to obtain solutions for the motion of a viscous fluid between two concentric rotating spheres for Reynolds numbers in the range of 1000 - 2000 on a mesh of 800 to 3200 points, using the implicit method, was between one to six hours on the IBM-7094 Mod. II. This same problem would take between four to twenty-four hours on the IBM-7040.

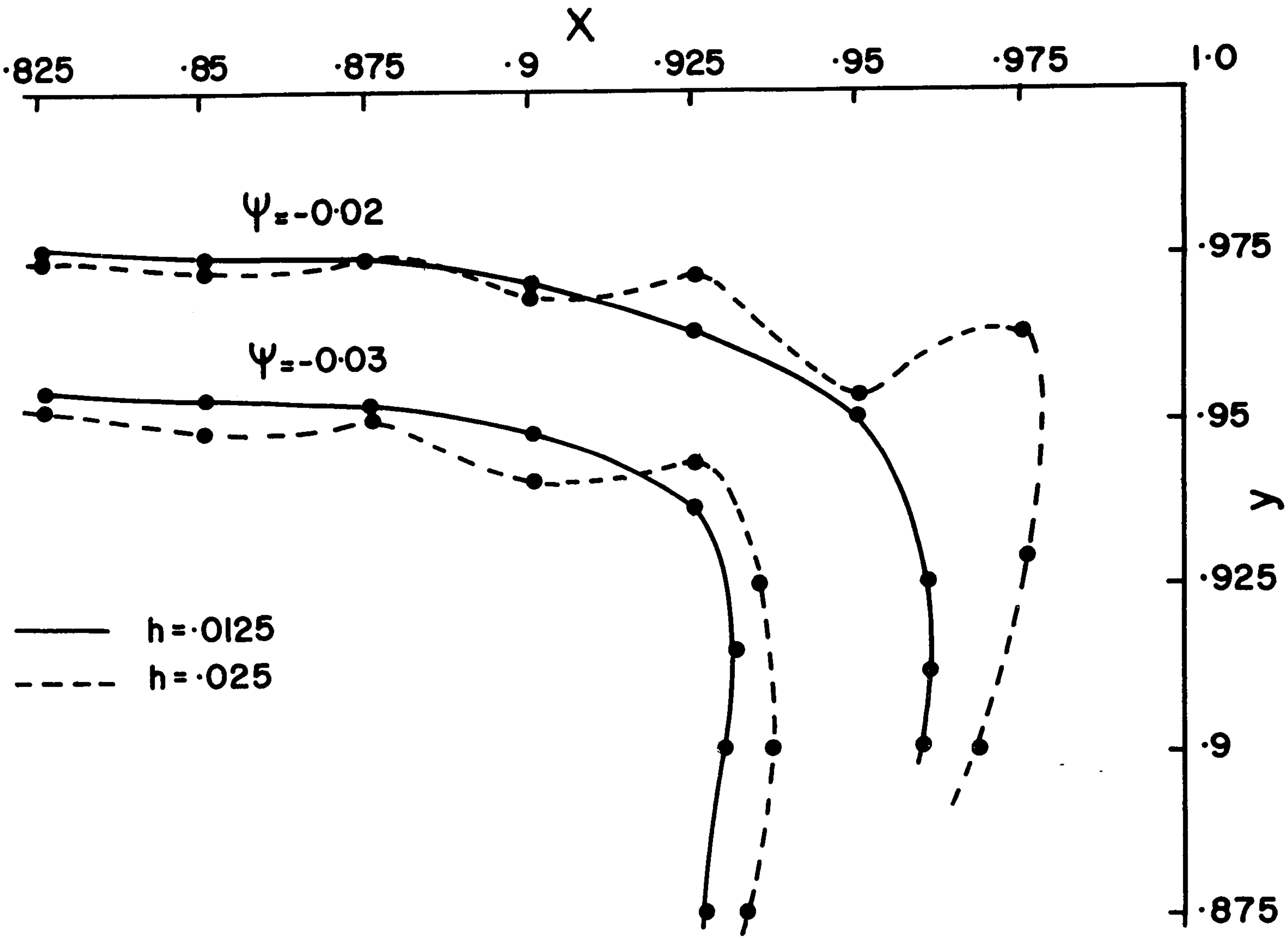



Figure 4·3 Stream Function, R=1000

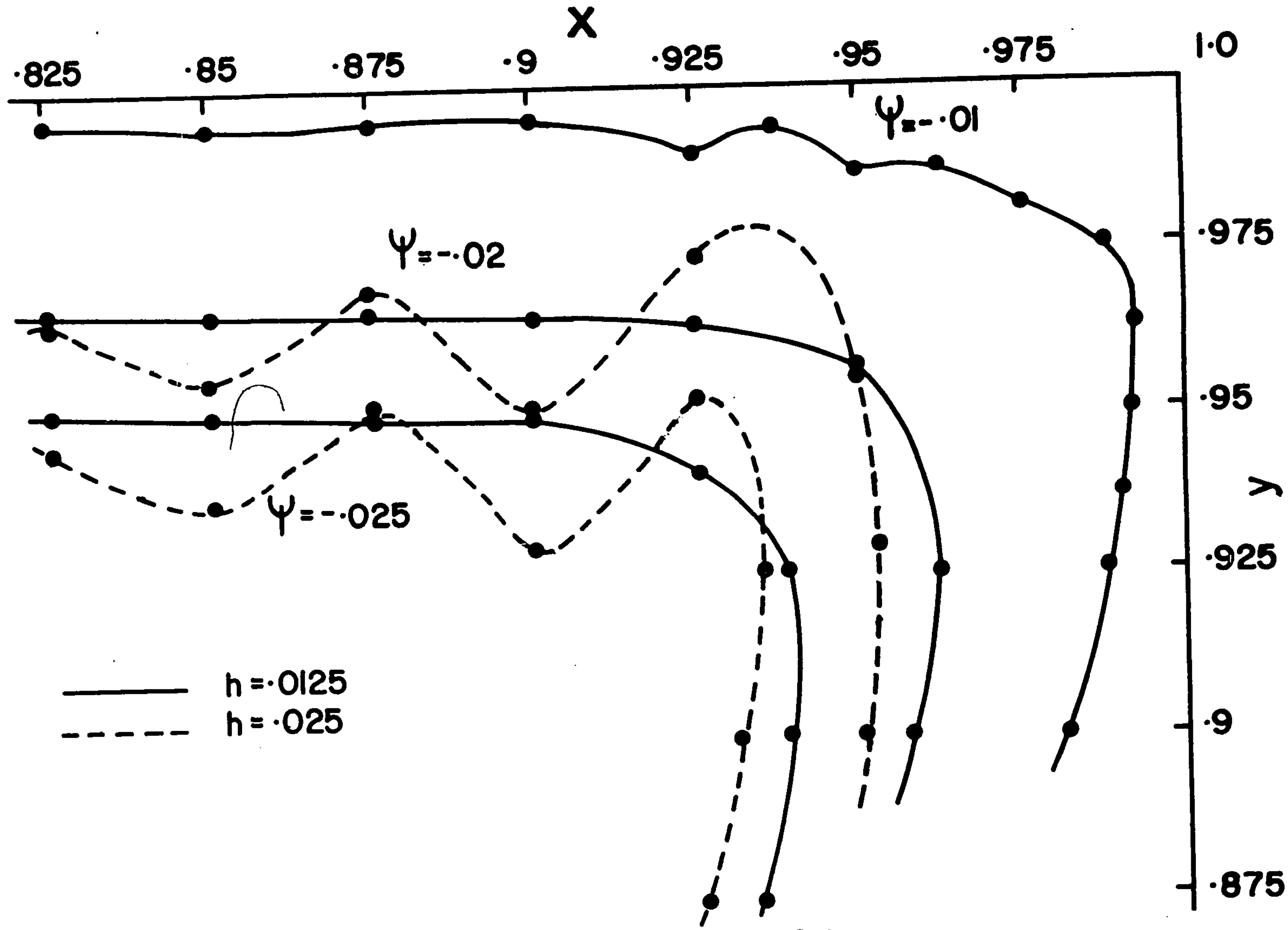



Figure 4·4 Stream Function, R=2000

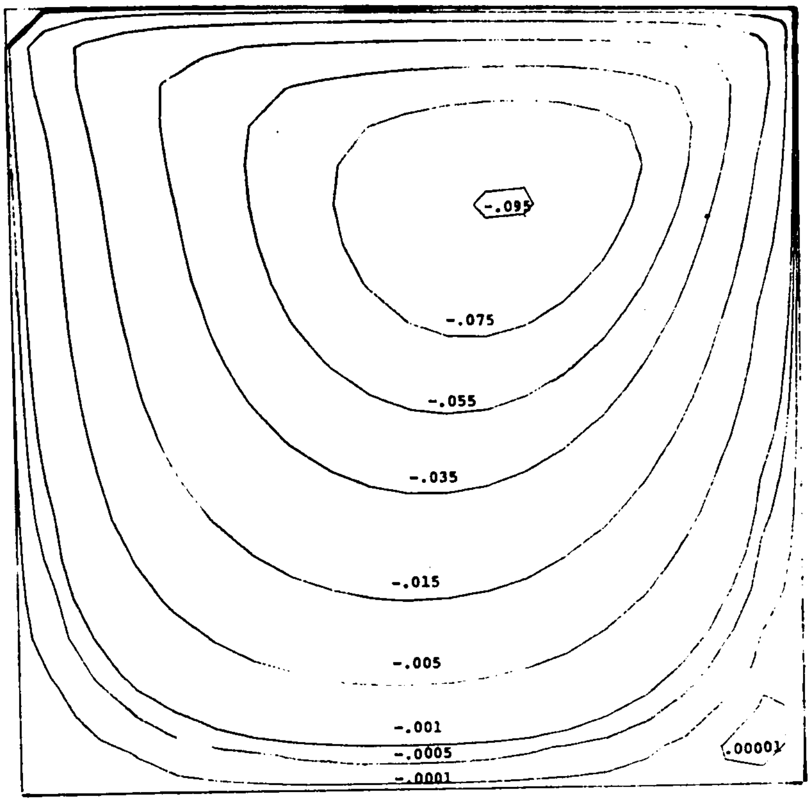



Figure 4.5  Stream Function Solution for R=90

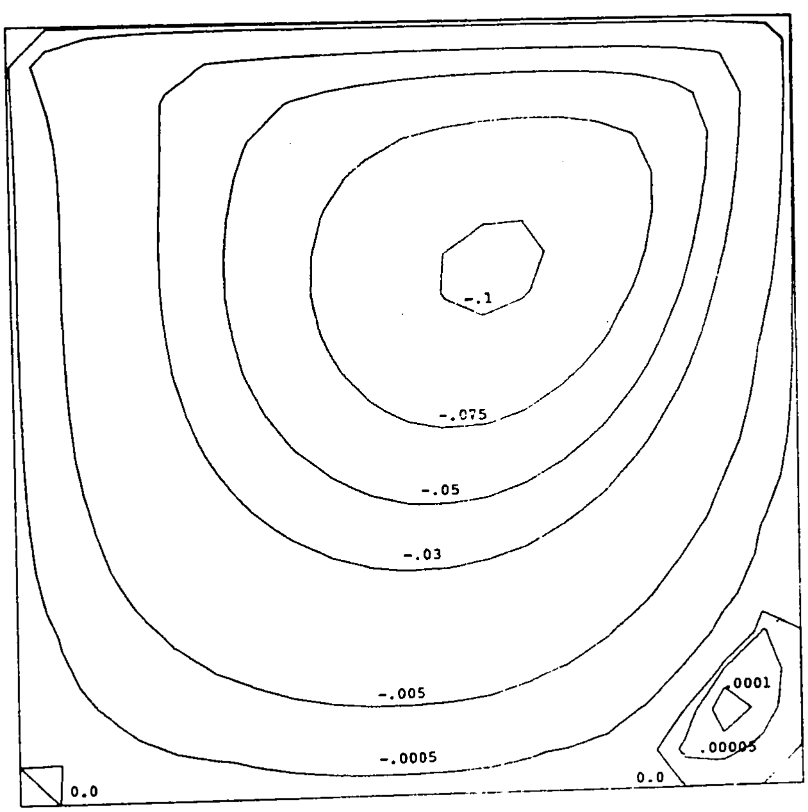



Figure 4.6  Stream Function Solution for R=200

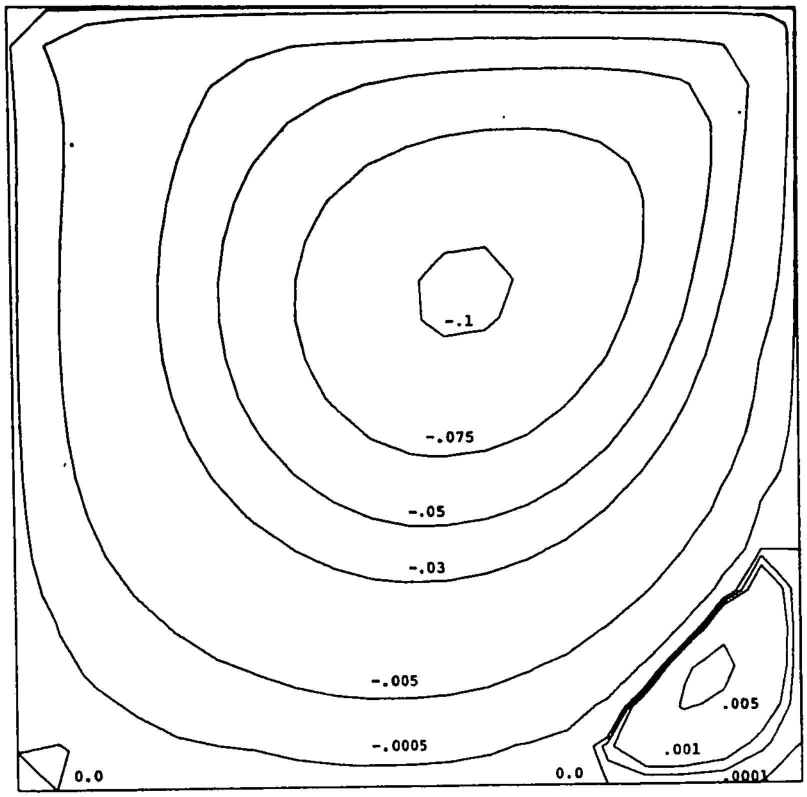



Figure 4.7 Stream Function Solution for R=400

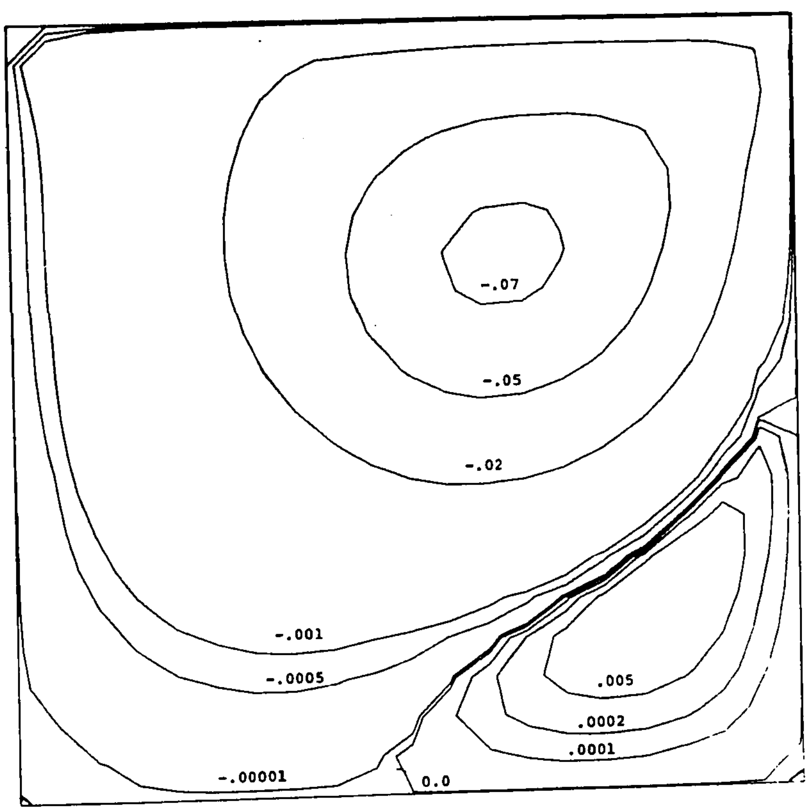



Figure 4.8 Stream Function Solution for R=1000

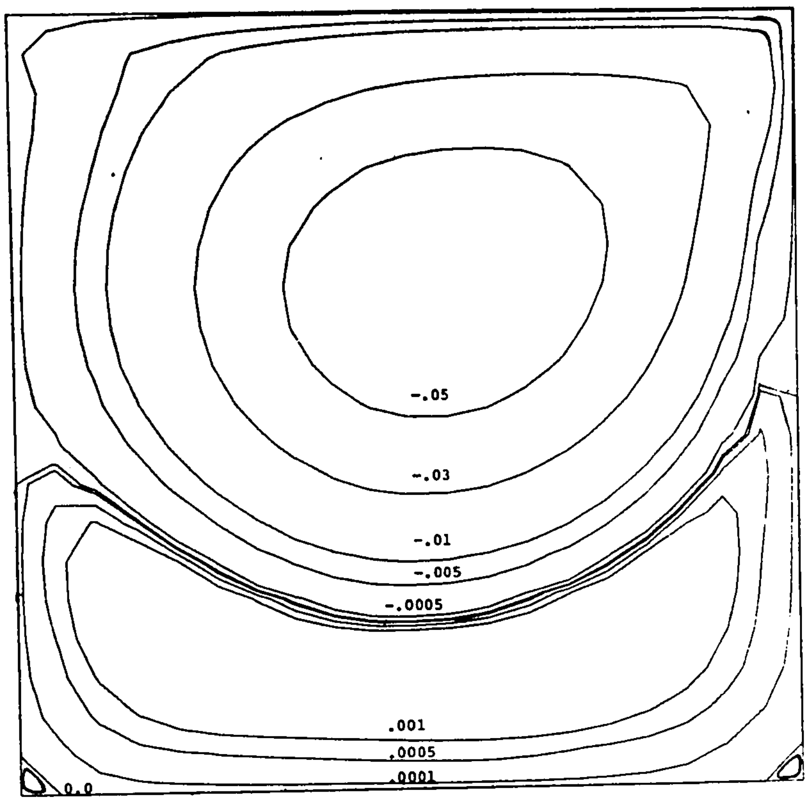



Figure 4.9 Stream Function Solution for R=2000

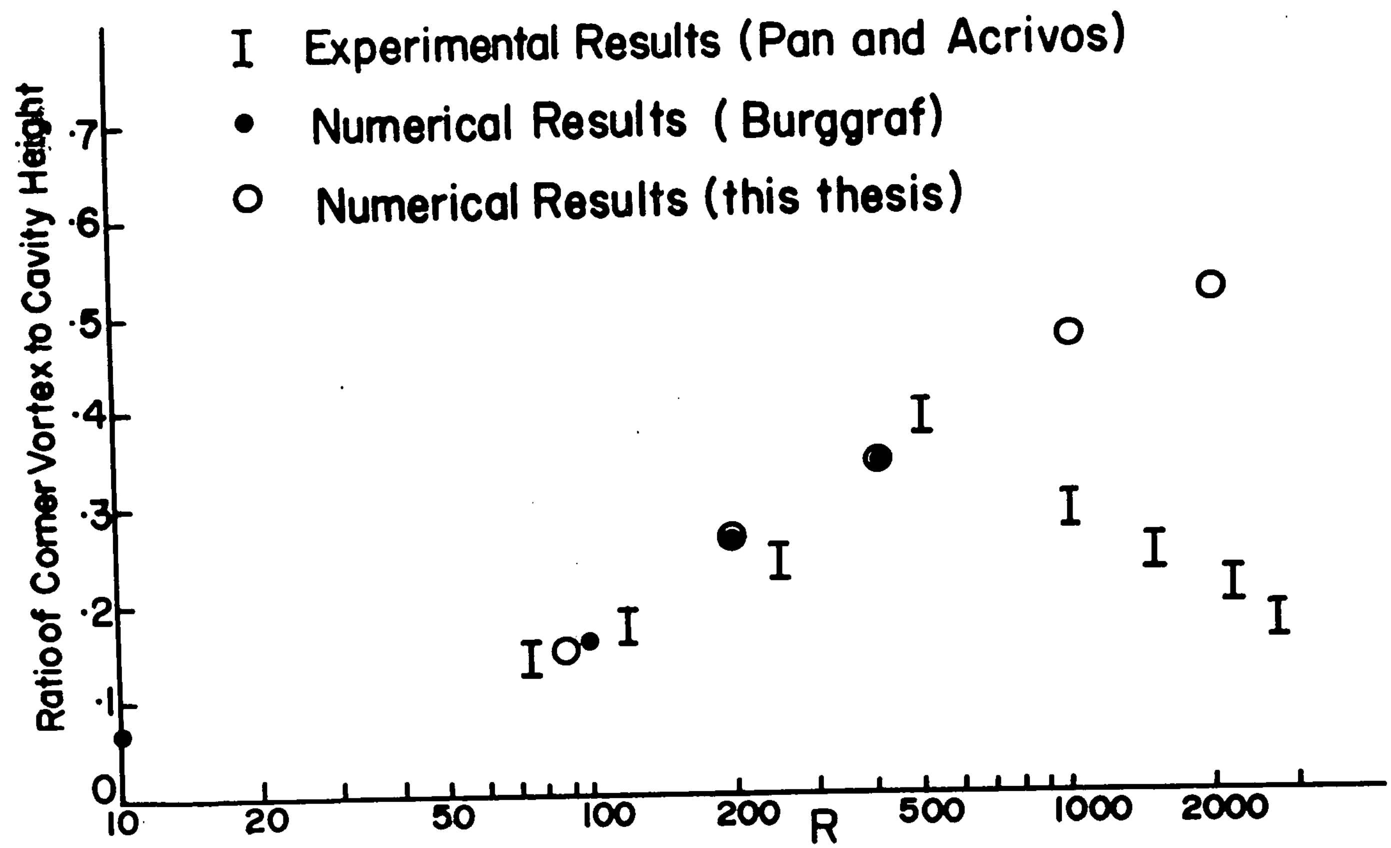



Figure 4·10 Ratio of Upstream Corner Vortex to Total Cavity Height versus Reynolds Number

# CHAPTER V

## INVESTIGATION OF THE SOLUTION TO THE NAVIER-STOKES EQUATION IN TERMS OF VELOCITIES AND PRESSURES

In the preceding chapters, we have shown how solutions may be obtained for problems of two-dimensional steady and non-steady flow, even for relatively large Reynolds numbers. The formulation has been in terms of the well-known stream function, and the only limitations on the calculations were those of straightforward computing power.

We are interested in finding a numerical method which will allow us to solve the three-dimensional time-dependent cavity problem. This problem was considered by Bye (1966). However, he assumed the sides to be far from the center and reduced the problem essentially to the two-dimensional case, which can be solved in terms of stream function and vorticity. Thus he obtained only the numerical solution for the circulation in a central vertical plane in the direction of the applied stress.

To consider the full three-dimensional problem, one is forced to use the original variables of the Navier-Stokes equations, namely the velocity components and the pressures.

A difficulty appears when the incompressible equations are attempted numerically, using velocity components, the velocities at later times turn out not to satisfy the continuity equation. This is perhaps not unreasonable, since the continuity requirement is here expressed as an independent requirement (as opposed to the stream function approach, where incompressibility is imbedded in the definition). Some mechanism must be found to make the divergence of the velocities self-zeroing.

One way might be to introduce a kind of artificial compressibility (or, for that matter, to use the full compressible equations), so that the bulk elasticity of the fluid would contribute to the future velocity pattern. This approach was tried extensively in the present study, but without success; it was found that values of such artificial parameters, smoothing factors and the like, which improved the continuity situation, made other aspects of the problem worse. In particular, when the velocity divergence was adjusted to be self-correcting, the Poisson equation for the pressure turned out to be improperly posed (in the sense that the condition on the Neumann boundary conditions, coming ultimately from Gauss' theorem, was violated and correct numerical solution was impossible).

A part of this argument has since appeared in the literature, and that author (Chorin 1967) is rather more optimistic about the prospects; however, he does not present complete evidence as to whether the method works in practice, and this aspect of the problem remains not completely answered.

Rather than pursue the above, we find that another aspect of the problem must be considered, which, perhaps, decreases the importance of such an "artificial compressibility". This is, that there is a fundamental inconsistency in the definition of the divergence of the velocity, if the finite-difference formulation is carried out in the usual way; this is true even for the two-dimensional u-v-P formulation. This inconsistency, we will show, implies that only under good fortune would we be able to solve the equations, using the simplest grid configuration, and thus a different configuration must be used, (previously formulated, but for a rather different reason; see Harlow and Welch 1965).

In essence, this inconsistency is easy to understand. The new velocities are predicted by formulae involving the terms

$$\frac{\partial u}{\partial t} + \ldots = - \frac{\partial P}{\partial x} \ldots .$$

or, in finite-difference form

$$u_{i,j}^{t+\Delta t} = u_{i,j}^{t} - \frac{\Delta t}{2h}\left[P_{i+1,j} - P_{i-1,j}\right] + \ldots .$$

so that the rate of change of the continuity $D=\nabla.v$ is given by

$$\left(\frac{\partial D}{\partial t}\right)_{i,j} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right)$$

$$\frac{\Delta t}{4h^2}\left[P_{i+2,j} - 2\,P_{i,j} + P_{i-2,j} + P_{i,j+2} - 2\,P_{i,j} + P_{i,j-2}\right] + \ldots ..$$

which constitutes an expanded and different form of the formulation of $\nabla^2 P$ from that used normally. We then argue that no immediate way to correct this situation is available if the velocities are positioned at the grid intersections.

In the present chapter, we examine in detail the nature of this inconsistency; in chapter VI we demonstrate what appears to be the simplest method of formulating the problem in two dimensions which will allow the continuity equation to be satisfied at $t+\Delta t$. The final chapter consists simply of a demonstration that, aside from the obvious problem of machine time, no further difficulties appear to arise in the three-dimensional problem.

## 5.1 Formulation of the Equations

One of the advantages of using the stream function approach, is that the pressure no longer appears explicitly, and the number of governing equations is reduced from three to two. However, for the more direct formulation, we lose this advantage and must use the three equations.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} = \frac{1}{R}\nabla^2 u \tag{5.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} = \frac{1}{R}\nabla^2 v \tag{5.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.3}$$

In the process of solving these equations, two difficulties arise immediately; first, there is no explicit expression for the pressure, and thus, some artificial way must be used to deal with this variable; second, the explicit time-derivatives of the velocities are separate from the continuity equation; and we have no assurance that velocities at a later time, predicted directly from (5.1) and (5.2), will any longer satisfy the continuity equation. This latter problem will occupy us for some time.

To obtain an explicit expression for the pressure at first appears to be a simple matter; one differentiates (5.1) and (5.2) with respect to x and y respectively and adds, bearing in mind that several terms vanish because of the continuity equation. The result is

$$\nabla^2 P + \frac{\partial u^2}{\partial x} + 2\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x} + \frac{\partial v^2}{\partial y} = 0. \tag{5.4}$$

One would expect to calculate the pressure from (5.4), then the new values of u and v from (5.1) and (5.2), a new pressure field, and so on. But, if this is carried out, using the straightforward grid, for which the velocities and pressures are calculated at the grid line intersection, the continuity equation fails to be satisfied at any time step. This chapter will point out the reasons why this is so.

Greenspan, Jain, Manohar, Noble and Sakuri (1964) suggested an alternate form of the Navier-Stokes equations in which they incorporated additional continuity terms uD and vD; where $D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, into equations (5.1) and (5.2) respectively. This will lead to

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial P}{\partial x} = \frac{1}{R}\nabla^2 u \tag{5.5}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial P}{\partial y} = \frac{1}{R}\nabla^2 v \tag{5.6}$$

and the pressure equation

$$\nabla^2 P + \frac{\partial^2 u^2}{\partial x^2} + 2\,\frac{\partial^2 uv}{\partial x \partial y} + \frac{\partial^2 v^2}{\partial y^2} = 0. \tag{5.7}$$

On the surface, for computational purposes, there seems to be no reason why one system of equation should be preferred over the other. However, Noble (1967) pointed out that evidence exists to indicate equations (5.5), (5.6) and (5.7) seem to give better results. There may be an argument that, incorporating the continuity equation into (5.5) and (5.6), will keep the accumulation of the error in the continuity equation at a sufficient low level at the next time step. In any case, in the presentation the equations used were the ones incorporating the extra continuity terms. It is not claimed that the results obtained in this chapter depend on which form of the Navier-Stokes equations is used.

The above is true, i.e. the form has no effect, provided that $D \equiv 0$. To be completely precise, we must note that never will D be exactly zero, and we must consider the equation

$$\begin{aligned} D^{t+\Delta t} &= D^t + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\Delta t \\ &= D^t + \Delta t\left(- \nabla^2 P + \ldots\right) \end{aligned}$$

so that, in fact, the value of $D^{t+\Delta t}$ depends on the residual tolerated in the solution of (5.7). To try to avoid all of these problems, we could leave in place all terms involving D, and write the three equations as

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} - uD + \frac{\partial P}{\partial x} = \frac{1}{R} \nabla^2 u \tag{5.8}$$

$$\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial v^2}{\partial y} - vD + \frac{\partial P}{\partial y} = \frac{1}{R} \nabla^2 v \tag{5.9}$$

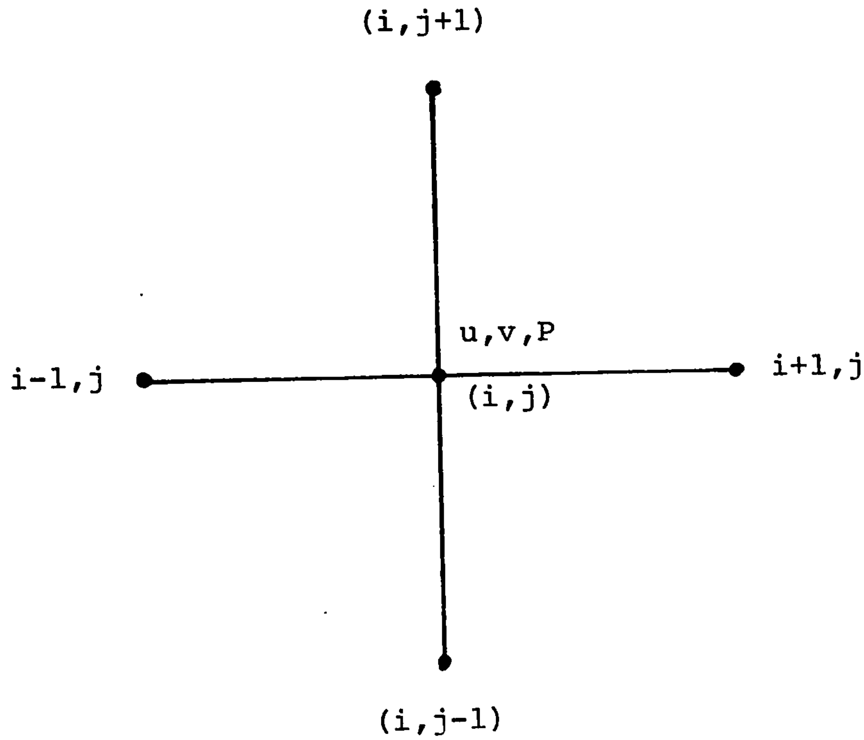
and

$$\frac{\partial D}{\partial t} + \frac{\partial^2 u^2}{\partial x^2} + 2\frac{\partial^2 uv}{\partial x \partial y} + \frac{\partial^2 v^2}{\partial y^2} - D\frac{\partial u}{\partial x} - u\frac{\partial D}{\partial x} - D\frac{\partial v}{\partial y} - v\frac{\partial D}{\partial y} + \nabla^2 P = \frac{1}{R} \nabla^2 D \tag{5.10}$$

Essentially, we have added and subtracted uD in (5.5) to give (5.8) and, since D ought to be identically zero, the two equations should in theory be the same. But the inclusion of the D terms in all equations relaxes the accuracy to which the pressure equation must be solved. This implies that fewer sweeps are necessary over the pressure field, and thus, will result in the saving of computing time. In practice, then, it does appear that there is a slight advantage to this latter formulation. (We shall see shortly that both are inconsistent).

## 5.2 Difference Equations and the Corresponding Boundary Conditions

We shall now present the difference formulation of equations (5.8) to (5.10). The ordinary 5-point finite difference grid formulation is used where u,v, and P are solved for at the grid line intersections.



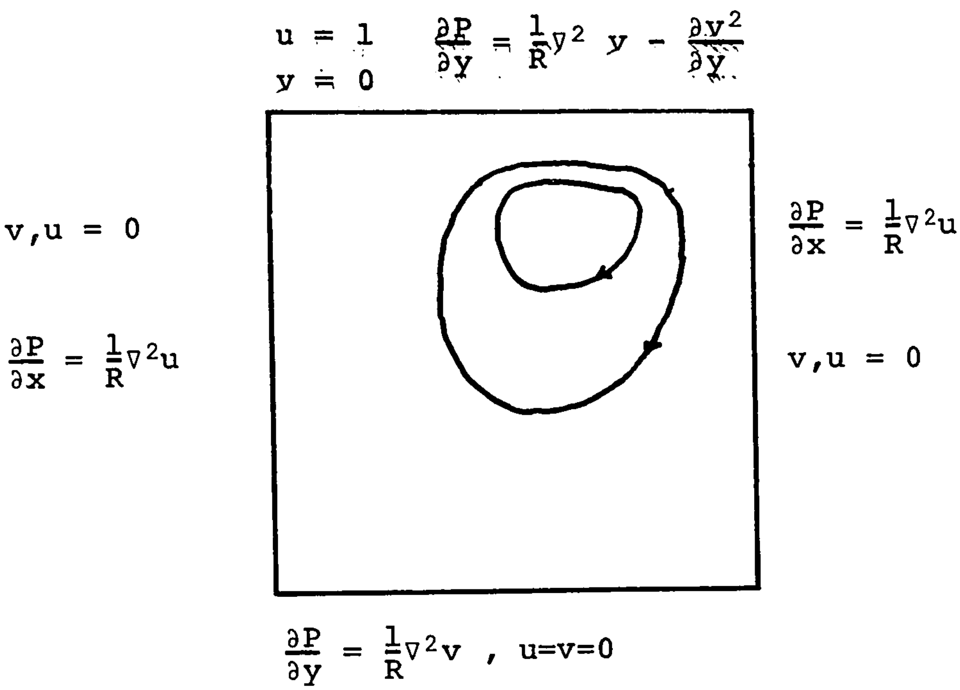

At first glance, it does not seem that there is any difficulty with the boundary conditions for the pressure equation. For example, from (5.8), we have for the left hand wall

$$\frac{\partial P}{\partial x} = \frac{1}{R}\nabla^2 u$$

since on the wall

$$\frac{\partial u}{\partial t} = \frac{\partial u^2}{\partial x} = uD = 0.$$

The remaining boundary conditions are shown in figure 5.1. On the top, the term $\frac{\partial v^2}{\partial y}$ must be included, to take care of the corner points.




Our point is to show that these boundary conditions, for the formulation used, will not allow us to solve the pressure equations.

Generally, to solve the equations (5.8) to (5.10) for each time step, the pressure equation is solved first using successive overrelaxation and the specified Neumann boundary conditions. The pressures so obtained are used to calculate the velocities which, in turn, are used to solve the pressure equations. The finite-difference equations used are

$$u_{i,j}^{t+\Delta t} = u_{i,j} + \frac{\Delta t}{Rh^2}\left[u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j} - 4u_{i,j}\right]$$

$$- \frac{\Delta t}{2h}\left[u_{i+1,j}^2 - u_{i-1,j}^2 + uv_{i,j+1} - uv_{i,j-1} + P_{i+1,j}\right.$$

$$\left. - P_{i-1,j}\right] + \Delta t u D_{i,j} \qquad (5.11)$$

$$v_{i,j}^{t+\Delta t} = v_{i,j} + \frac{\Delta t}{Rh^2}\Big[v_{i,j+1} + v_{i,j-1} + v_{i+1,j} + v_{i-1,j} - 4\,v_{i,j}\Big] - \frac{\Delta t}{2h}\Big[v^2_{i,j+1} - v^2_{i,j-1} + uv_{i+1,j} - uv_{i-1,j} + P_{i,j+1} - P_{i,j-1}\Big] + \Delta t u D_{i,j} \qquad (5{,}12)$$

$$P_{i,j+1} + P_{i,j-1} + P_{i+1,j} + P_{i-1,j} - 4\,P_{i,j} + \Phi_{i,j} = \frac{-h^2\, D_{i,j}^{t+\Delta t}}{\Delta t} \qquad (5.13)$$

where

$$\Phi_{i,j} = u^2_{i+1,j} - 2\,u^2_{i,j} + u^2_{i-1,j} + v^2_{i,j+1} - 2\,v^2_{i,j} + v^2_{i,j-1} + \frac{1}{2}\Big[uv_{i+1,j+1} - uv_{i+1,j-1} - uv_{i-1,j+1} + uv_{i-1,j-1}\Big] - \frac{1}{R}\Big[D_{i,j+1} + D_{i,j-1} + D_{i+1,j} + D_{i-1,j} - 4\,D_{i,j}\Big] - \frac{h}{2}\Big[D_{i,j}\left(u_{i,j+1} - u_{i,j-1}\right) + u_{i,j}\left(D_{i,j+1} - D_{i,j-1}\right) + D_{i,j}\left(v_{i+1,j} - v_{i-1,j}\right) + v_{i,j}\left(D_{i+1,j} - D_{i-1,j}\right)\Big] - \frac{h^2 D_{i,j}}{\Delta t}$$

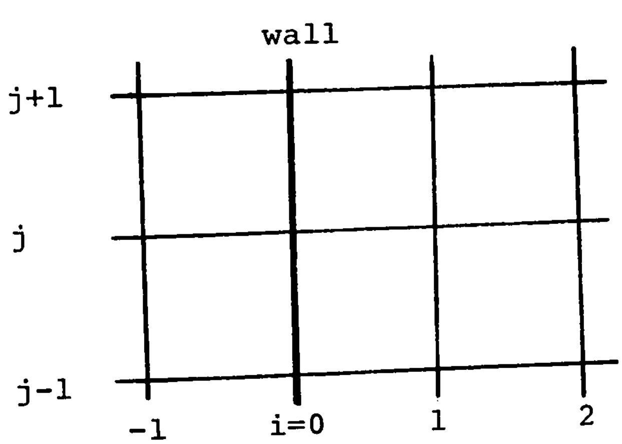
and

$$D_{i,j} = \frac{1}{2h}\Big[u_{i+1,j} - u_{i-1,j} + v_{i,j+1} - v_{i,j-1}\Big] .$$

Two problems arise with the boundary conditions for the pressure equation. First, we do not know the u velocity at the top corner and thus are unable to calculate $\nabla^2 v$ or $\nabla^2 u$. Second, the pressure on the boundary is required to calculate the velocities.

To get around these problems, it would be natural (although incorrect) to assume that the top corners are stagnation points and to use an extrapolation routine or forward difference method to calculate the pressures on the boundary. (The pressures might also be obtained by iterating directly for the values on the boundary; however, it is impossible to get an expression for the normal derivative of the pressures on the corner points and it is therefore required to guess the corner pressures. The results of several computer runs suggested it to be best to extrapolate for the boundary values).

A typical boundary condition would then be, taking the left wall as an example



$$\left(\frac{\partial P}{\partial x}\right)_{i=0} = \frac{1}{Rh^2}\left[u_{1,j} + u_{-1,j} + u_{0,j+1} + u_{0,j-1} - 4u_{0,j}\right]$$

On the wall $\frac{\partial v}{\partial y} = 0$, therefore, $\frac{\partial u}{\partial x} = 0$ which gives

$u_{1,j} = u_{-1,j}$ and, therefore

$$\left(\frac{\partial P}{\partial x}\right)_{i=0} = \frac{2}{Rh^2}\, u_{1,j}$$

The iteration procedure thus defined for the pressure did not converge. Regardless of the Reynolds number or the size of the mesh, the residual at each point of the grid approached a constant value and it did not change when further iterations were applied. At the same time, the continuity equation D could also not be satisfied. The fact that the error for the iteration procedure approached a constant at each point, that is, the impossibility of solving the pressure equation, suggests that the problem itself may be improperly posed. We note that for equation (5.10) Neumann boundary conditions are imposed, and, therefore, must satisfy a consistency condition coming ultimately from Gauss' theorem, namely that the integral of $\frac{\partial P}{\partial n}$ around the boundary must be equal to the area integral of $\nabla^2 P$.

Let us demand that $D^{t+\Delta t} \equiv 0$ in (5.10), then we can write

$$\nabla^2 P + \Phi = 0$$

where

$$\phi = \frac{\partial^2 u^2}{\partial x} + 2\frac{\partial uv}{\partial x \partial y} + \frac{\partial^2 v^2}{\partial y} - D\frac{\partial u}{\partial x} - u\frac{\partial D}{\partial y} - D\frac{\partial v}{\partial x} - v\frac{\partial D}{\partial y} - \frac{1}{R}\nabla^2 D - \frac{D}{\Delta t}.$$

We must have

$$\int_0^1\int_0^1 \nabla^2 P \, dx\, dy = \oint \frac{\partial P}{\partial n}\, ds$$

or

$$\oint \frac{\partial P}{\partial n}\, ds = -\int_0^1\int_0^1 \phi\, dx\, dy. \tag{5.14}$$

In order to find the proper boundary conditions, equation (5.14) must be computed not from (5.14), but directly from the equation. In discrete form (5.14) is

$$\sum_{i=0}^{M}\left[P_{i,0} - P_{i,1} + P_{i,N} - P_{i,N-1}\right] + \sum_{j=0}^{N}\left[P_{0,j} - P_{1,j} + P_{M,j} - P_{M-1,j}\right] = \sum_{i=1}^{M-1}\sum_{j=1}^{N-1} \Phi_{i,j} \tag{5.15}$$

where $\Phi_{i,j}$ is as defined previously. If we now carry out this summation process, we will be left on the left-hand side with values of P around the boundaries, in fact just that combination giving all normal derivatives. Thus we can pick out values that must be assumed, let us say by the boundary values of P, in terms of their immediate neighbours only. For the left-hand wall, for example, (5.15) leads to the requirement that

$$P_{0,j} = P_{1,j} - \frac{u_{2,j}}{2Rh} + u^2_{1,j} - \frac{h}{2\Delta t} u_{1,j} . \qquad (5.16)$$

Thus we have an explicit set of values of P on the boundary and (5.13) _must_ be well-posed.

A similar form to (5.16) can be obtained, for example, when differentiating (5.8) with respect to x and using the resulting equation to obtain the pressures on the boundary, namely

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 uv}{\partial x \partial y} + \frac{\partial^2 P}{\partial x^2} = \frac{1}{R} \nabla^2 \frac{\partial u}{\partial x}$$

which, for the left hand wall, becomes

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{R}\nabla^2 \frac{\partial u}{\partial x} - \frac{\partial^2 u^2}{\partial x^2} . \qquad (5.17)$$

In difference form, this is

$$P_{1,j} - 2 P_{0,j} + P_{-1,j} = - 2 u^2_{1,j} + \frac{1}{R}\left[\left(\frac{\partial u}{\partial x}\right)_{1,j} + \left(\frac{\partial u}{\partial x}\right)_{-1,j}\right]$$

and, substituting for $P_{-1,j}$, using

$$\left(\frac{\partial P}{\partial x}\right)_{i=0} = \frac{2}{Rh^2} u_{1,j}$$

we have

$$P_{0,j} = P_{1,j} + u^2_{1,j} - \frac{2}{Rh} u_{1,j} - \frac{1}{2R}\left[\left(\frac{\partial u}{\partial x}\right)_{1,j} + \left(\frac{\partial u}{\partial x}\right)_{-1,j}\right] . \qquad (5.18)$$

However, since $u_{1,j} = u_{-1,j}$, it can be assumed that

$$u_{-2,j} = (1+\varepsilon)\, u_{2,j}$$

where $|\varepsilon|$ is some small number, and thus, equation (5.18) reduces to

$$P_{0,j} = P_{1,j} + u^2_{1,j} - \frac{2}{Rh}\, u_{1,j} + \frac{\varepsilon u_{2,j}}{4Rh} . \qquad (5.19)$$

$\varepsilon$ can be obtained from equations (5.16) and (5.19) by comparing coefficients, which gives

$$\varepsilon = \frac{2u_{1,j}}{u_{2,j}} \left( \frac{4\Delta t - Rh^2}{\Delta t} \right) - 2 .$$

## 5.3 Investigation of the Numerical Error in the Continuity Equation

Although it is now possible to obtain a solution for the pressure equation at each time step, the continuity equation cannot be satisfied with the calculated velocities. This seems, at first, to be a contradiction, especially when considering that the residual error at each mesh point from the pressure equation is of the order

$$- \frac{h^2}{\Delta t} D^{t+\Delta t} .$$

and, if $h^2 = \Delta t$, then the residual error is equal to the error in the continuity equation for the next time step.

The only explanation for this must be that there is a discrepancy in the definition of the "continuity" itself.

To begin with, let us observe that the continuity equation can also be checked by substituting the velocities at time t+Δt, obtained from equations (5.11) and (5.12) into

$$D_{i,j}^{t+\Delta t} = \frac{1}{2h}\left[u_{i+1,j}^{t+\Delta t} - u_{i-1,j}^{t+\Delta t} + v_{i,j+1}^{t+\Delta t} - v_{i,j-1}^{t+\Delta t}\right] \qquad (5.20)$$

We shall now show that the terms obtained in substituting these velocities into (5.20) are not in agreement with the terms obtained from the pressure equation (5.13).

For convenience, ignore all terms involving D. This in no way will influence the results to be shown. The terms for

$$\frac{1}{2h}\left[u_{i+1,j}^{t+\Delta t} - u_{i-1,j}^{t+\Delta t}\right]$$

can directly be obtained from (5.11) and are:

$$u_{i+1,j}^{t+\Delta t} - u_{i-1,j}^{t+\Delta t} = u_{i+1,j} - u_{i-1,j} + \frac{\Delta t}{Rh^2}\Big[u_{i+2,j} - u_{i,j}$$

$$+ u_{i,j} - u_{i-2,j} + u_{i+1,j+1} - u_{i-1,j+1}$$

$$+ u_{i+1,j-1} - u_{i-1,j-1} - 4\left(u_{i+1,j} - u_{i-1,j}\right)\Big]$$

$$- \frac{\Delta t}{2h}\Big[u^2_{i+2,j} - 2\,u^2_{i,j} + u^2_{i-2,j} + uv_{i+1,j+1} -$$

$$- uv_{i-1,j+1} - uv_{i+1,j-1} + uv_{i-1,j-1}$$

$$+ P_{i+2,j} - 2 P_{i,j} + P_{i-2,j}\Big] \qquad (5.21)$$

similar for the $\frac{\partial v}{\partial x}$ term, we have

$$v_{i,j+1} - v_{i,j-1} = v_{i,j+1} - v_{i,j-1} + \frac{\Delta t}{Rh^2}\Big[v_{i+1,j+1}$$

$$- v_{i+1,j-1} + v_{i-1,j+1} - v_{i-1,j-1}$$

$$+ v_{i,j+2} - v_{i,j} - v_{i,j-2}$$

$$- 4\left(v_{i,j-1} - v_{i,j-1}\right)\Big]$$

$$- \frac{\Delta t}{2h}\Big[v^2_{i,j+2} - 2 v^2_{i,j} + v^2_{i,j-2}$$

$$+ uv_{i+1,j+1} - uv_{i+1,j-1} - uv_{i-1,j+1}$$

$$+ uv_{i-1,j-1} + P_{i,j+2} - 2 P_{i,j} + P_{i,j-2}\Big]$$

$$(5.22)$$

Adding these two equations will give $D^{t+\Delta t}$. Each term from the combined equation on the right hand side is now compared to the terms in the finite difference pressure equation. We note that terms involving the coefficient $\frac{\Delta t}{Rh^2}$ combine to give $\frac{\Delta t}{Rh^2}(\nabla D)_{i,j}$. The results are shown in tabular form.

| Differiential Term | Finite Difference Approximation: Terms from the Pressure Equation (5.13) | Finite Difference Approximation: Terms from the Summation of (5.21) + (5.22) |
|---|---|---|
| $\frac{\partial^2 u^2}{\partial x^2}$ | $\frac{1}{h^2}\left[u^2_{i+1,j} - 2u^2_{i,j} + u^2_{i-1,j}\right]$ | $\frac{1}{4h^2}\left[u^2_{i+2,j} - 2u^2_{i,j} + u^2_{i-2,j}\right]$ |
| $\frac{\partial^2 v^2}{\partial y^2}$ | $\frac{1}{h^2}\left[v^2_{i,j+1} - 2v^2_{i,j} + v^2_{i,j-1}\right]$ | $\frac{1}{4h^2}\left[v^2_{i,j+2} - 2v^2_{i,j} + v^2_{i,j-2}\right]$ |
| $\frac{\partial^2 uv}{\partial x \partial y}$ | $\frac{1}{4h^2}\left[uv_{i+1,j+1} - uv_{i+1,j-1} - uv_{i-1,j+1} + uv_{i-1,j-1}\right]$ | $\frac{1}{4h^2}\left[uv_{i+1,j+1} - uv_{i+1,j-1} - uv_{i-1,j+1} + uv_{i-1,j-1}\right]$ |
| $\nabla^2 P$ | $\frac{1}{h^2}\left[P_{i,j+1} + P_{i,j-1} + P_{i+1,j} \; P_{i-1,j} - 4P_{i,j}\right]$ | $\frac{1}{4h^2}\left[P_{i,j+2} + P_{i,j-2} + P_{i+2,j} + P_{i-2,j} - 4P_{i,j}\right]$ |

The differences in these terms are as a direct result in the two methods of approximating the second order derivatives. In one case, from the pressure equation, we have:

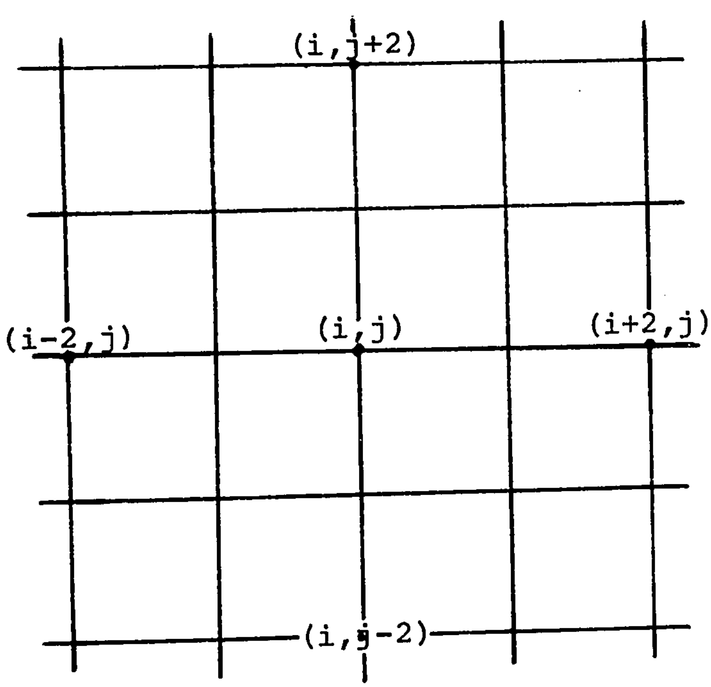
$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{h^2}\left[P_{i+1,j} - 2\ P_{i,j} + P_{i-1,j}\right] \tag{5.23}$$

whereas, for the terms coming from (5.21), we actually use

$$\frac{\partial}{\partial x}\ \frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(\frac{P_{i+1,j} - P_{i-1,j}}{2h}\right)$$

$$= \frac{P_{i+2,j} - 2\ P_{i,j} + P_{i-2,j}}{4h^2} \tag{5.24}$$

In the latter, an expanded grid is implied, that is, only every second point is used.



This is also true for $\frac{\partial^2 u^2}{\partial x^2}$ and $\frac{\partial^2 v^2}{\partial y^2}$. Our conclusion then is that direct calculation of the divergence of the velocity in two ways in the equations implies that, in one place, the ordinary calculations of $\nabla^2 P$ is used, and in another the expanded formulation (5.24).

The two different forms of the difference approximation to the second order equation are numerically equal when the pressure gradient is constant, or, perhaps, for some exceptional points. In most cases, however, the two terms will not be equal. It is because of this discrepancy, that the two ways of obtaining the continuity equations are incompatible. The same conclusion can be reached when using the Navier-Stokes equations in their original form, without the inclusion of the continuity terms. One could say that the continuity equation appears to be satisfied from the pressure equation (5.13); but when the stream function, which can be obtained using

$$u = \frac{\partial\psi}{\partial y} \quad \text{or} \quad v = -\frac{\partial\psi}{\partial x} ,$$

is substituted into the finite difference form of

$$\nabla^4\psi = -R\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}$$

it is found that the fourth order equation is not satisfied.

A slight improvement can be achieved if a finer grid is used. In the limit, as h becomes very small, the correct solution ought to be obtained, however, computationally, this is an impossibility. For this reason, this straightforward method of calculating the velocities directly is discarded. In the next chapter, an alternative approach is discussed.

## CHAPTER VI

## ALTERNATIVE METHOD TO SOLVE THE NAVIER-STOKES EQUATIONS IN TERMS OF VELOCITIES AND PRESSURES

So far, we have established that the velocites calculated at the intersection of two grid lines, do not necessarily satisfy the continuity equation throughout time. We are thus looking for a method which will be consistent in the apparent grid size used for the continuity equation.

Harlow and Welch (1965) described a new method to solve the Navier-Stokes equations in terms of velocities and pressures. In their system, the continuity equation is satisfied not at the grid line intersection, according to a kind of five-point formula, but rather over a single cell. This certainly reduces the effective grid size, more over, it allows us to escape the difficulties pointed out in chapter V with respect to the continuity equation.

In order to study a number of problems involving free boundaries, mixtures of fluids and so forth, Harlow and his co-workers have developed what they refer to variously as the "MAC", or Marker-And-Cell, or the "PIC", or Particle-In-Cell approach; (Fromm 1963; Harlow and Welch 1965; Harlow, Shannon and Welch 1965; Daly and Pracht 1968). This method involves calculating the pressure and velocity divergence at the center of a square cell, and the velocities themselves on its edges.

It will be shown in this chapter that the MAC method eliminates the inconsistencies inherent in the formulation of the Laplacian operation in the pressure equation. At the same time, we will establish the form the Navier-Stokes equation must be for numerical computation. Finally, some numerical results will be pointed out.

We shall employ the same methods as used in chapter V, to show these results. To begin with, let us define the new set-up of the grid configuration and the corresponding difference equations.

## 6.1 Grid Configuration and Difference Equations

In this new method by Harlow and Welch (1965), the pressure is calculated in the middle of each cell and the velocities on the sides of the cells. The notation and position of all the variables are shown below.

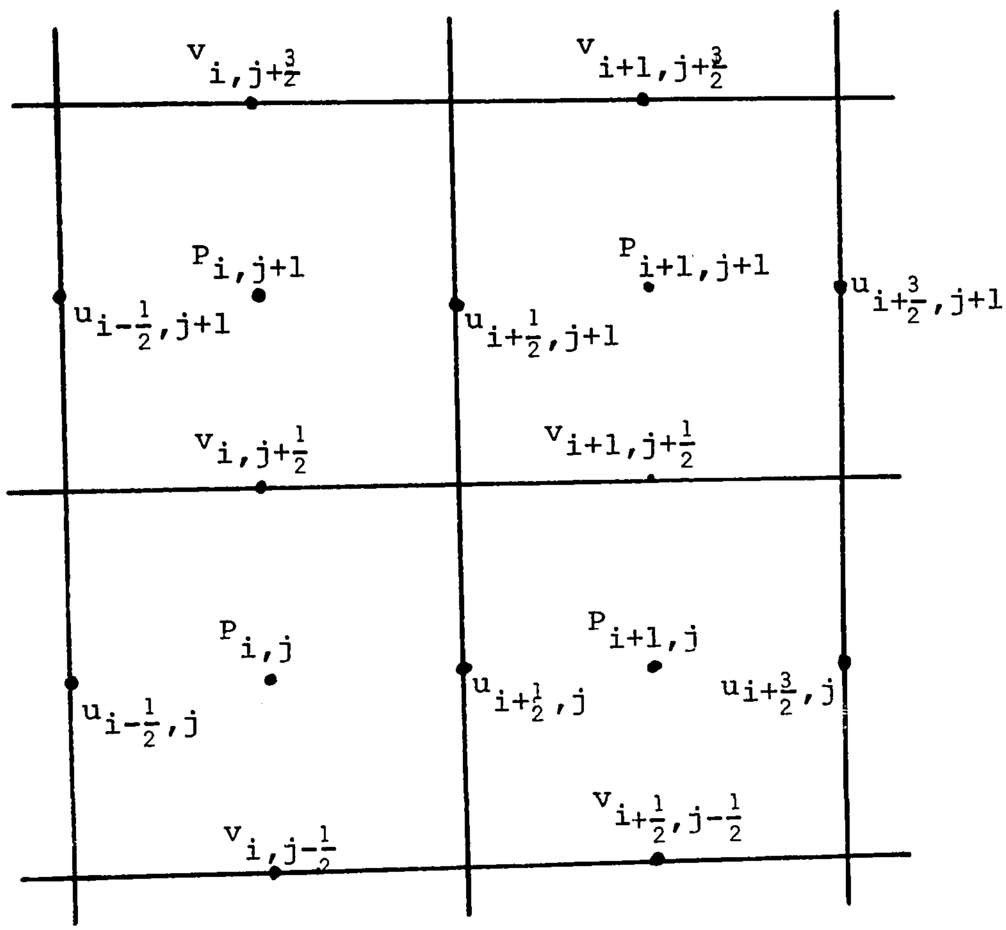



Figure 6.1 Notation and Position of Variables

For the purpose of discussion, let us recall

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial P}{\partial x} - uD = \frac{1}{R} \nabla^2 u \qquad (6.1)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial P}{\partial y} - vD = \frac{1}{R} \nabla^2 v \qquad (6.2)$$

and

$$\frac{\partial D}{\partial t} + \frac{\partial^2 u^2}{\partial x^2} + 2\frac{\partial^2 uv}{\partial x \partial y} + \frac{\partial^2 v^2}{\partial y^2} - D\frac{\partial u}{\partial x} - u\frac{\partial D}{\partial x}$$

$$- D\frac{\partial v}{\partial y} - v\frac{\partial D}{\partial y} + \nabla^2 P = \frac{1}{R} \nabla^2 D \qquad (6.3)$$

The difference approximations to these equations, using the MAC method in explicit form, are:

$$u^{t+\Delta t}_{i+\frac{1}{2},j} = u_{i+\frac{1}{2},j} + \frac{\Delta t}{Rh^2}\Big[u_{i+\frac{3}{2},j} + u_{i-\frac{1}{2},j} +$$

$$u_{i+\frac{1}{2},j+1} + u_{i+\frac{1}{2},j-1} - 4\, u_{i+\frac{1}{2},j}\Big] - \frac{\Delta t}{h}\Big[u^2_{i+1,j}$$

$$- u^2_{i,j} + uv_{i+\frac{1}{2},j+\frac{1}{2}} - uv_{i+\frac{1}{2},j-\frac{1}{2}} + P_{i+1,j} - P_{i,j}\Big]$$

$$+ \Delta t\ uD_{i+\frac{1}{2},j} \qquad (6.4)$$

$$v^{t+\Delta t}_{i,j+\frac{1}{2}} = v_{i,j+\frac{1}{2}} + \frac{\Delta t}{Rh^2}\Big[v_{i,j+\frac{3}{2}} + v_{i,j-\frac{1}{2}} + v_{i+1,j+\frac{1}{2}} + v_{i-1,j+\frac{1}{2}}$$

$$- 4\, v_{i,j+\frac{1}{2}}\Big] - \frac{\Delta t}{h}\Big[v^2_{i,j+1} - v^2_{i,j} + uv_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- uv_{i-\frac{1}{2},j+\frac{1}{2}} + P_{i,j+1} - P_{i,j}\Big] + \Delta t v\ D_{i,j+\frac{1}{2}} \qquad (6.5)$$

and

$$- \frac{h^2 D^t_{i,j}}{\Delta t} + P_{i,j+1} + P_{i,j-1} + P_{i+1,j} + P_{i-1,j} - 4\, P_{i,j}$$

$$+ u^2_{i+1,j} - 2\, u^2_{i,j} + u^2_{i-1,j} + v^2_{i,j+1} - 2\, v^2_{i,j} + v^2_{i,j-1}$$

$$+ 2 \left[ uv_{i+\frac{1}{2},j+\frac{1}{2}} - uv_{i+\frac{1}{2},j-\frac{1}{2}} - uv_{i-\frac{1}{2},j+\frac{1}{2}} + uv_{i-\frac{1}{2},j-\frac{1}{2}} \right]$$

$$- h \left[ u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j} \right] D_{i,j} - \frac{h}{2} \left[ D_{i+1,j} - D_{i-1,j} \right] u_{i,j}$$

$$- h \left[ v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}} \right] D_{i,j} - \frac{h}{2} \left[ D_{i,j+1} - D_{i,j-1} \right] v_{i,j}$$

$$- \frac{1}{R} \left[ D_{i,j+1} + D_{i,j-1} + D_{i+1,j} + D_{i-1,j} - 4\, D_{i,j} \right]$$

$$= \frac{-h^2 D^{t+\Delta t}}{\Delta t} \qquad (6.6)$$

Any value of u, v or D, which is not available at a designated point, is obtained by

$$u_{i,j} = \frac{1}{2} \left( u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j} \right)$$

$$v_{i,j} = \frac{1}{2} \left( v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}} \right)$$

$$D_{i,j+\frac{1}{2}} = \frac{1}{2} \left( D_{i,j+1} + D_{i,j} \right)$$

$$D_{i+\frac{1}{2},j} = \frac{1}{2} \left( D_{i+1,j} + D_{i,j} \right)$$

$$u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2} \left( u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j+1} \right)$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2} \left( v_{i+1,j+\frac{1}{2}} + v_{i,j+\frac{1}{2}} \right)$$

Again the continuity term on the right hand side of (6.6) will be proportional to the error at each mesh point when solving for the pressure. We now only have to show that the terms in the finite difference equation

$$D^{t+\Delta t} = \frac{1}{h}\left[ u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j} + v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}} \right] \qquad (6.7)$$

obtained when substituting the velocities from (6.4) and (6.5) into $D^{t+\Delta t}$ are the same as in the pressure equation (6.6). Already it has been shown that the critical terms are the second order partial differentials. For the moment, for illustrative purpose, let us ignore all terms involving D and take only that involving $\frac{\partial^2 P}{\partial x^2}$ and $\frac{\partial^2 P}{\partial y^2}$.

In the process of obtaining

$$\frac{1}{h}\left(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\right)$$

from (6.4) the pressure terms become

$$P_{i+1,j} - 2\,P_{i,j} + P_{i-1,j}\;.$$

A similar form holds for

$$\frac{1}{h}\left(v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}\right),$$

and thus

$$\nabla^2 P = \frac{1}{h^2}\left[P_{i,j+1} + P_{i,j-1} + P_{i+1,j} + P_{i-1,j} - 4\,P_{i,j}\right]\;.,$$

which is the same as in equation (6.6).

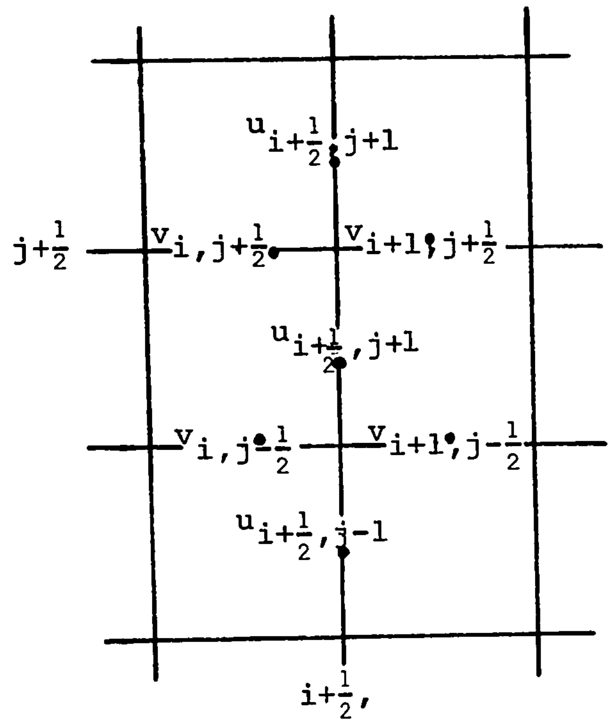
If this is carried out for all other terms not involving D, they all turn out to be identical to the terms in equation (6.6). Also at time t=0, D=0 and, if the pressure equation is solved accurately at each time step, the continuity equation over each mesh cell will be satisfied. This will ensure that all D terms in the Navier-Stokes equations contribute a negligible amount and can be ignored. Including the D terms does not really add complications to the program, but rather saves computer time.

Before discussing numerical results, we still have to consider the Navier-Stokes equations in their original form; this is, without incorporating continuity terms into the velocity equations. Immediately one can see that the pressure terms $\nabla^2 P$ will be in the proper format. Let us look at the term $v\frac{\partial u}{\partial y}$ at the point $(i+\frac{1}{2},j)$.

The difference approximation for this term is

$$v\frac{\partial u}{\partial y} = v_{i+\frac{1}{2},j}\left[\frac{u_{i+\frac{1}{2},j+1} - u_{i+\frac{1}{2},j-1}}{2h}\right].$$

However, $v_{i+\frac{1}{2},j}$ is not known at the grid point $(i+\frac{1}{2},j)$ and must be obtained by extrapolation. From the following configuration




$v_{i+\frac{1}{2},j}$ can be obtained in several ways, namely,

$$v_{i+\frac{1}{2},j} = \frac{1}{2}\left(v_{i+1,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}\right)$$

$$v_{i+\frac{1}{2},j} = \frac{1}{4}\left(v_{i+1,j+\frac{1}{2}} + v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}} + v_{i+1,j-\frac{1}{2}}\right)$$

$$v_{i+\frac{1}{2},j} = \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j-\frac{1}{2}}\right)$$

Numerically, these expressions for $v_{i+\frac{1}{2},j}$ are not equal and it is thus not possible to get a good value for the v-velocity as we know not which one to take. In fact, the second relation implies that

$$\nabla^2 v = 0 \quad \text{at the point} \quad (i+\tfrac{1}{2},j)$$

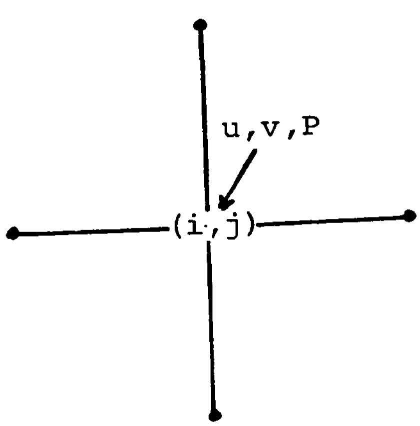
Similar difficulties arise for $u_{i,j+\frac{1}{2}}$. We are forced to the conclusion that the Navier-Stokes equations must not be used without incorporating the continuity terms into the velocity equations.

In summary then, the two systems of equations investigated are:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial x} - uD + \frac{\partial P}{\partial x} &= \frac{1}{R}\nabla^2 u \\ \frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} - vD + \frac{\partial P}{\partial y} &= \frac{1}{R}\nabla^2 v \end{aligned}\right\} \quad \text{I}$$

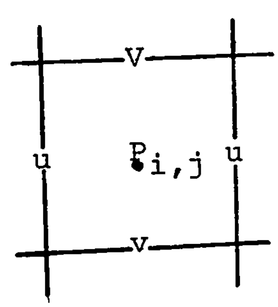
$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} &= \frac{1}{R}\nabla^2 u \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial y} + \frac{\partial P}{\partial y} &= \frac{1}{R}\nabla^2 v \end{aligned}\right\} \quad \text{II}$$

and the grid configurations, showing the position of the variables, are:




Method I

Ordinary




Method II

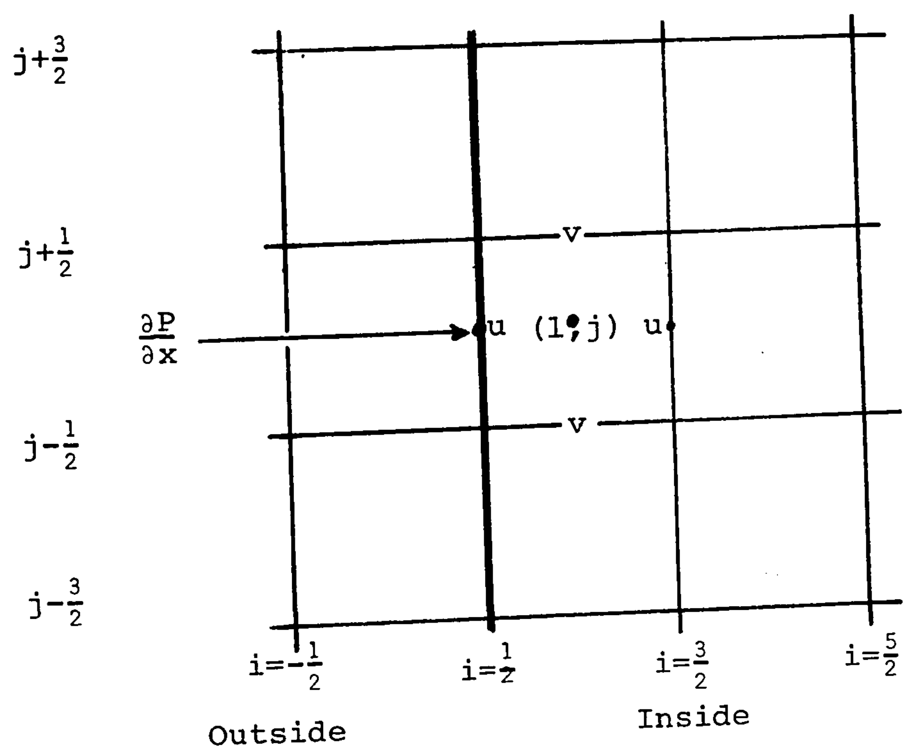
MAC

We have shown in chapter V that the continuity equation is inconsistent when using equations I or II with the ordinary grid. Equation II and method II require approximation of velocities which distort the physical system. At the same time, this combination introduces an inconsistency in the continuity equation with respect to the pressure and velocity equations. The only combination which avoids these difficulties is equation I and method II; the MAC method. For this, the terms in the continuity equation obtained from the pressure equation are the same as the terms obtained when substituting for the velocities directly. In table form, this is summed up by

| | Eqations I<br>Terms uD, etc. included | Equations II<br>Terms uD, etc. excluded |
|---|---|---|
| Grid I<br>Ordinary | cannot satisfy continuity equation, pressure boundary conditions inaccurate<br>Do not use | cannot satisfy continuity equation, pressure boundary conditions inaccurate<br>Do not use |
| Grid II<br>MAC | Acceptable | Approximations to velocities introduces unjustifyable velocity conditions within the cavity.<br>Do not use |

## 6.2 Boundary Conditions for the Pressure Equation

The boundary conditions for the pressure equation can be obtained from the velocity equation. No problem is associated with these conditions, except for the top corner cells. Before dealing with them, let us consider the left hand wall, and show how the boundary values are obtained for this typical boundary.




The u-velocity equation is

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} - uD + \frac{\partial P}{\partial x} = \frac{1}{R} \nabla^2 u \qquad (6.8)$$

and on the wall u=0, v=0 so that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} = 0$$

Then equation (6.8) reduces to

$$\frac{\partial P}{\partial x} = \frac{1}{R}\nabla^2 u$$

which in finite difference form is

$$\left(\frac{\partial P}{\partial x}\right)_{\frac{1}{2},j} = \frac{2}{Rh^2}\, u_{\frac{3}{2},j}. \qquad (6.9)$$

Similarly, for the bottom wall, we have

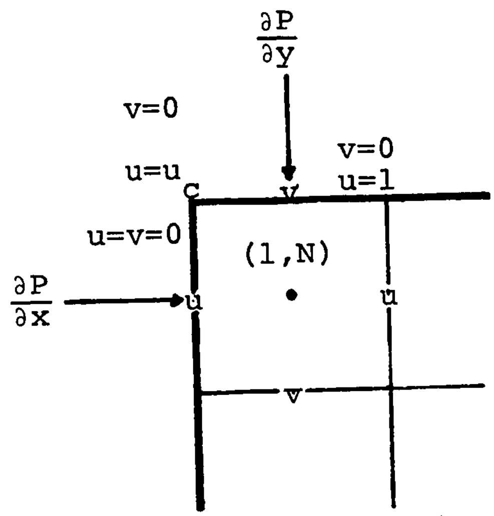
$$\left(\frac{\partial P}{\partial y}\right)_{i,\frac{1}{2}} = \frac{2}{Rh^2}\, v_{i,\frac{3}{2}} \qquad (6.10)$$

and similar expressions hold for the right and top walls.

The corner points need to be dealt with separately. The bottom corners present no problem, as we know the velocity at the corner is equal to zero and the boundary condition (6.9) and (6.10) can be applied. The top corners appear to present some difficulties. The horizontal velocity at the corners is not known and, at first, one could assume this to be a stagnation point and, thus, put u=v=0 at these points. Care must then be taken to incorporate $\frac{\partial u}{\partial x} \neq 0$ on the top boundary corner cells into the pressure boundary equation.

To be more general, let us assume a velocity of $u_c$, $0 \leq u_c \leq 1$, on the top corner points and then develop a general expression for the normal pressure derivatives.

For example, take the left top corner cell.




On the top boundary of the cell we have

$$\frac{\partial u}{\partial x} = \frac{1-u_c}{h} = -\frac{\partial v}{\partial y}$$

From the velocity equations, the boundary conditions then become

$$\frac{\partial P}{\partial x} = \frac{1}{R}\nabla^2 u$$

and in finite difference form

$$\left(\frac{\partial P}{\partial x}\right)_{\frac{1}{2},N} = \frac{1}{Rh^2}\left[u_{\frac{3}{2},N} + u_{-\frac{1}{2},N} + u_{\frac{1}{2},N-1} + u_{\frac{1}{2},N+1} - 4\ u_{\frac{1}{2},N}\right]$$

We know $u_{-\frac{1}{2},N} = u_{\frac{3}{2},N}$ and, using the averaging method defined earlier for velocities not lying on the required points, we have

$$u_c = \frac{u_{1/2,N} + u_{1/2,N+1}}{2}$$

from which

$$u_{\frac{1}{2},N+1} = 2\ u_c$$

and thus

$$\left(\frac{\partial P}{\partial x}\right)_{\frac{1}{2},N} = \frac{2}{Rh^2}\left[u_{\frac{3}{2},N} + u_c\right] \qquad (6.11)$$

On the top, the normal derivative is

$$\left(\frac{\partial P}{\partial y}\right)_{1,N+\frac{1}{2}} = \frac{1}{R}\ \nabla^2 v - \frac{\partial v^2}{\partial y}$$

which is approximated by

$$\frac{\partial P}{\partial y}_{1,N+\frac{1}{2}} = \frac{2}{Rh^2}\left[v_{1,N-\frac{1}{2}} - \left(1-u_c\right)\right] - \frac{1}{h}\left[\left(\frac{1}{2}\ v_{1,N-\frac{1}{2}} - \left(1-u_c\right)\right)^2 - \frac{1}{4}\left(v_{1,N-\frac{1}{2}}\right)^2\right] \qquad (6.12)$$

A similar expression can be obtained for the right top corner cell.

## 6.3 Discussion of Computing Procedure and Results

The pressure equation (6.6) is solved by successive overrelaxation, and using the Neumann boundary conditions specified.

Since we leave all D terms in the equation, it is not essential to solve the pressure equation to a high accuracy as it would be if they were omitted; the D terms have a correcting influence on the equations, which allows us to admit a greater tolerance criterion when solving the pressure equation. Therefore, the pressure equation can be iterated fewer times and this fact, in turn, saves computing time.

The steady-state was assumed to be reached when the pressure values did not change from one time step to the next. This meant that the error in the pressure equation was also small which, in turn, assured that the continuity equation was satisfied.

Computations were performed for Reynolds number 10, 90 and 200. The results obtained agreed with those from the stream function approach. The same numerical time steps computed for the explicit method in chapter IV could be used. In every case, numerical stability was observed.

It only remains to see the effect of the corner points on the solution. To establish this, several runs were performed for constant Reynolds number and mesh size and various top corner velocities $u_c$. In each case for different $u_c$'s, the solutions were identical. It appears that corner velocity has no effect on the solution, as long as the boundary value equations (6.11) and (6.12) are used. Care must be taken that the corner point is also considered when computing $\frac{\partial^2 v^2}{\partial y^2}$ in the pressure equation.

Once it was established that this method can be used to solve the two-dimensional time dependent Navier-Stokes equation, the method was directly applied to three-dimensional time dependent case. The final chapter deals with the results.

# CHAPTER VII

## THREE-DIMENSIONAL TIME-DEPENDENT FLOW IN A RECTANGULAR BASIN

The Navier-Stokes equation, written in terms of the velocities and pressure, as in chapter VI, can now be solved for three-dimensional time-dependent problems. It turns out to be only a matter of simple extension from the two-dimensional case. All arguments brought forward in the previous chapter still hold.

### 7.1 Problem Definition

For illustration, we shall take a rectangular (in fact a cubical) basin, which, on the top, is acted upon by a belt moving to the right with non-dimensional velocity of unity.

If w is the velocity in the z-direction, then the boundary conditions are:

$u=1,\ v=w=0$ on the top surface,

$u=v=w=0$ on all other boundaries,

$u=v=w=0$ at $t\leq 0$

Then the equations describing the system are, in non-dimensional form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} = \frac{1}{R}\,\nabla^2 u \qquad (7.1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial P}{\partial y} = \frac{1}{R}\,\nabla^2 v \qquad (7.2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} = \frac{1}{R}\,\nabla^2 w \qquad (7.3)$$

$$D \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (7.4)$$

where $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

Just as for the two-dimensional case, the equations must be written with additional continuity terms incorporated. Thus, adding and subtracting uD from (7.1), vD from (7.2) and wD from (7.3), we have, after rearranging,

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} + \frac{\partial P}{\partial x} - uD = \frac{1}{R}\,\nabla^2 u \qquad (7.5)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} + \frac{\partial P}{\partial y} - vD = \frac{1}{R}\,\nabla^2 v \qquad (7.6)$$

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} + \frac{\partial P}{\partial z} - wD = \frac{1}{R}\,\nabla^2 w \qquad (7.7)$$

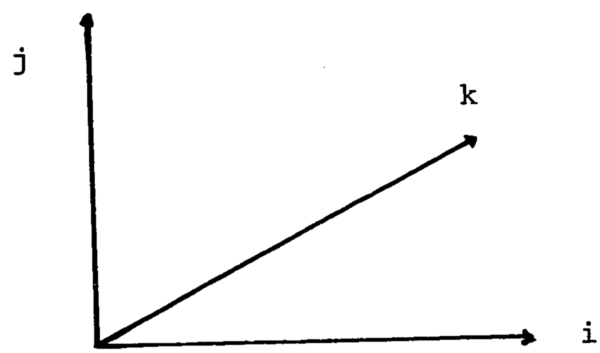
Differentiating (7.5), (7.6) and (7.7), with respect to x, y, z respectively and adding the resulting equations, we obtain for the Poisson equation for P

$$\frac{\partial D}{\partial t} + \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 v^2}{\partial y^2} + \frac{\partial^2 w^2}{\partial z^2} + 2\frac{\partial^2 uv}{\partial x \partial y} + \frac{\partial^2 uw}{\partial x \partial z} + \frac{\partial^2 vw}{\partial y \partial z}$$

$$- D\frac{\partial u}{\partial x} - u\frac{\partial D}{\partial x} - D\frac{\partial v}{\partial y} - v\frac{\partial D}{\partial y} - D\frac{\partial w}{\partial z} - w\frac{\partial D}{\partial z} + \nabla^2 P = \frac{1}{R}\nabla^2 D \qquad (7.8)$$

## 7.2 Notation and Difference Equations

For the finite difference equation in three-dimensions, the following notation will be used;




where (i,j,k) describe the x,y,z-directions respectively.

The basin is divided up into grid-cubes of dimensions h. The points at which the velocities are located are shown in figure (7.1).

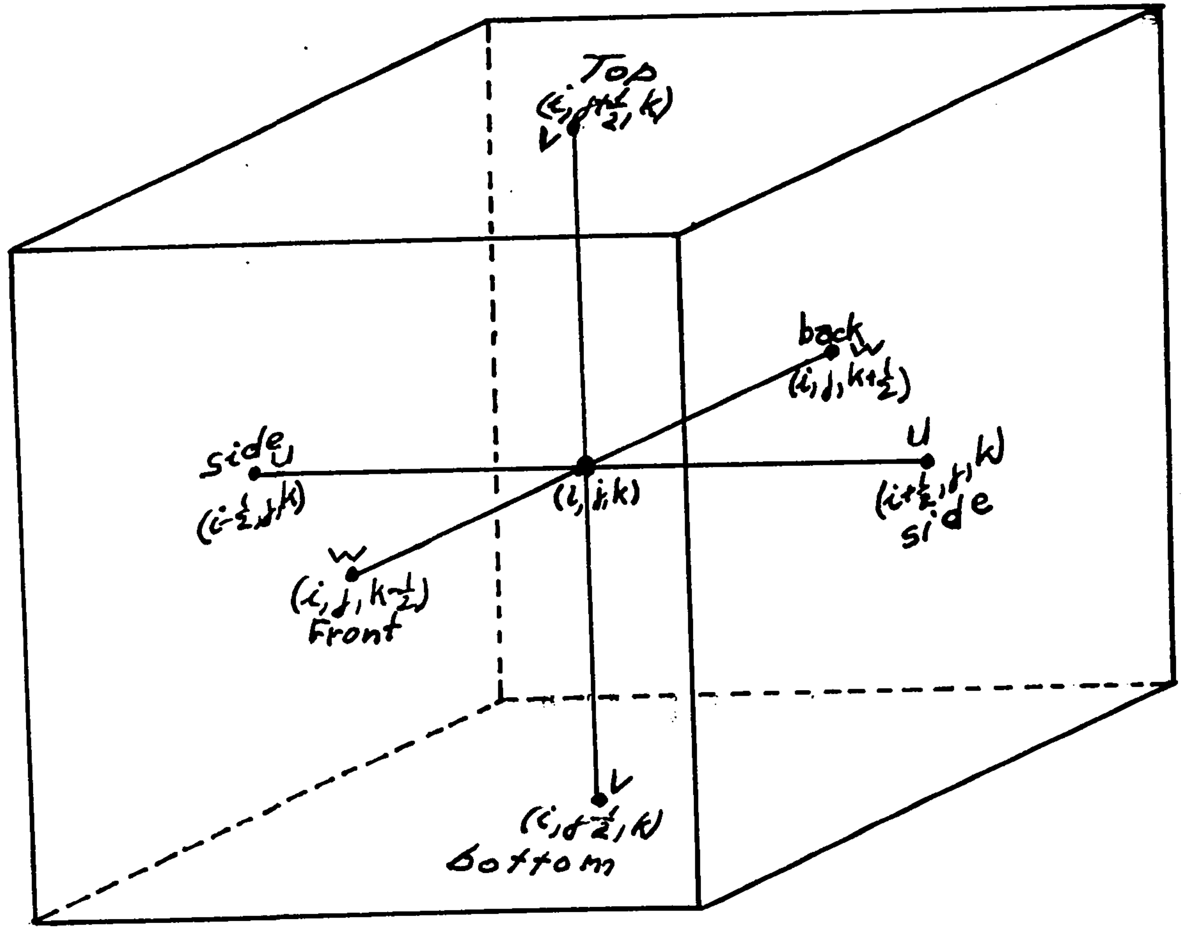



Figure 7.1 Notation and Position of Variables for a Grid-Cube

The continuity equation is satisfied over a grid-cube and can be written as:

$$D_{i,j,k} = \frac{1}{h}\Big[u_{i+\frac{1}{2},j,k} - u_{i-\frac{1}{2},j,k} + v_{i,j+\frac{1}{2},k} - v_{i,j-\frac{1}{2},k} + w_{i,j,k+\frac{1}{2}} - w_{i,j,k-\frac{1}{2}}\Big] \tag{7.9}$$

In finite difference form then, the Navier-Stokes equation in three-dimensions, are

$$\begin{aligned} u^{t+\Delta t}_{i+\frac{1}{2},j,k} = {} & u_{i+\frac{1}{2},j,k} + \frac{\Delta t}{Rh^2}\Big[u_{i+\frac{3}{2},j,k} + u_{i-\frac{1}{2},j,k} + u_{i+\frac{1}{2},j+1,k} \\ & + u_{i+\frac{1}{2},j-1,k} + u_{i+\frac{1}{2},j,k} + u_{i+\frac{1}{2},j,k-1} - 6\, u_{i+\frac{1}{2},j,k}\Big] \\ & - \frac{\Delta t}{h}\Big[u^2_{i+1,j,k} - u^2_{i,j,k} + uv_{i+\frac{1}{2},j+\frac{1}{2},k} \\ & - uv_{i+\frac{1}{2},j-\frac{1}{2},k} + uw_{i+\frac{1}{2},j,k+\frac{1}{2}} - uw_{i+\frac{1}{2},j,k-\frac{1}{2}} \\ & + P_{i+1,j,k} - P_{i,j,k}\Big] + \Delta t\; uD_{i+\frac{1}{2},j,k} \end{aligned} \tag{7.10}$$

$$\begin{aligned} v^{t+\Delta t}_{i,j+\frac{1}{2},k} = {} & v_{i,j+\frac{1}{2},k} + \frac{\Delta t}{Rh^2}\Big[v_{i,j+\frac{3}{2},k} + v_{i,j-\frac{1}{2},k} + v_{i+1,j+\frac{1}{2},k} \\ & + v_{i-1,j+\frac{1}{2},k} + v_{i,j+\frac{1}{2},k+1} + v_{i,j+\frac{1}{2},k-1} - 6\, v_{i,j+\frac{1}{2},k}\Big] \\ & - \frac{\Delta t}{h}\Big[v^2_{i,j+1,k} - v^2_{i,j,k} + uv_{i+\frac{1}{2},j+\frac{1}{2},k} - uv_{i-\frac{1}{2},j+\frac{1}{2},k} \\ & + vw_{i,j+\frac{1}{2},k+\frac{1}{2}} - vw_{i,j+\frac{1}{2},k-\frac{1}{2}} + P_{i,j+1,k} - P_{i,j,k}\Big] \\ & + \Delta t\; vD_{i,j+\frac{1}{2},k} \end{aligned} \tag{7.11}$$

$$w_{i,j,k+\frac{1}{2}} = w_{i,j,k+\frac{1}{2}} + \frac{\Delta t}{Rh^2}\Big[w_{i,j,k+\frac{3}{2}} + w_{i,j,k-\frac{1}{2}}$$

$$+ w_{i,j+1,k+\frac{1}{2}} + w_{i,j-1,k+\frac{1}{2}} + w_{i+1,j,k+\frac{1}{2}}$$

$$+ w_{i-1,j,k+\frac{1}{2}} - 6\, w_{i,j,k+\frac{1}{2}}\Big]$$

$$- \frac{\Delta t}{h}\Big[w^2_{i,j,k+1} - w^2_{i,j,k} + uw_{i+\frac{1}{2},j,k+\frac{1}{2}}$$

$$- uw_{i-\frac{1}{2},j,k+\frac{1}{2}} + vw_{i,j+\frac{1}{2},k+\frac{1}{2}} - vw_{i,j-\frac{1}{2},k+\frac{1}{2}}$$

$$+ P_{i,j,k+1} - P_{i,j,k}\Big] + \Delta t\; wD_{i,j,k+\frac{1}{2}} \qquad (7.12)$$

and

$$P_{i+1,j,k} + P_{i-1,j,k} + P_{i,j+1,k} + P_{i,j-1,k} + P_{i,j,k+1}$$

$$+ P_{i,j,k-1} - 6\, P_{i,j,k} + \Phi_{i,j} = -\frac{h^2}{\Delta t}\, D^{t+\Delta t}_{i,j} \qquad (7.13)$$

where

$$\Phi_{i,j} = u^2_{i+1,j,k} - 2\, u^2_{i,j,k} + u^2_{i-1,j,k} + v^2_{i,j+1,k}$$

$$- 2\, v^2_{i,j,k} + v^2_{i,j-1,k} + w^2_{i,j,k+1} - 2\, w^2_{i,j,k}$$

$$+ w^2_{i,j,k-1} + 2\Big[uv_{i+\frac{1}{2},j+\frac{1}{2},k} - uv_{i+\frac{1}{2},j-\frac{1}{2},k}$$

$$- uv_{i-\frac{1}{2},j+\frac{1}{2},k} + uv_{i-\frac{1}{2},j-\frac{1}{2},k} + uw_{i+\frac{1}{2},j,k+\frac{1}{2}}$$

$$- uw_{i+\frac{1}{2},j,k-\frac{1}{2}} - uw_{i-\frac{1}{2},j,k+\frac{1}{2}} + uw_{i-\frac{1}{2},j,k-\frac{1}{2}}$$

$$+ uw_{i,j+\frac{1}{2},k+\frac{1}{2}} - vw_{i,j+\frac{1}{2},k-\frac{1}{2}} - vw_{i,j-\frac{1}{2},k+\frac{1}{2}}$$

$$\left. + vw_{i,j-\frac{1}{2},k-\frac{1}{2}} \right] - \frac{h^2 D^t_{i,j}}{\Delta t}$$

$$- h\left(u_{i+\frac{1}{2},j,k} - u_{i-\frac{1}{2},j,k}\right) D_{i,j,k} - \frac{h}{2} u_{i,j,k} \left(D_{i+1,j,k}\right.$$

$$\left. - D_{i-1,j,k}\right) - h\left(v_{i,j+\frac{1}{2},k} - v_{i,j-\frac{1}{2},k}\right) D_{i,j,k}$$

$$- \frac{h}{2} v_{i,j,k} \left(D_{i,j+1,k} - D_{i,j-1,k}\right) - h\left(w_{i,j,k+\frac{1}{2}}\right.$$

$$\left. - w_{i,j,k-\frac{1}{2}}\right) D_{i,j,k} - \frac{h}{2} w_{i,j,k} \left(D_{i,j,k+1} - D_{i,j,k-1}\right.$$

$$- \frac{h^2}{R} (\nabla D)_{i,j} \qquad (7.14)$$

## 7.3 Discussion of Computational Procedure

The computational procedure for the three-dimensional problem is similar to the two-dimensional one. The pressure equation (7.13) is solved by successive overrelaxation. The relaxation factors taken were of the same magnitude as for the two-dimensional problems. The pressures so obtained for time t and the velocities at t are used to calculate the velocities at time $t+\Delta t$. Next the pressure equation was solved again, etc. Computation was stopped when the change in pressure from one time step to the next was within the required tolerance.

For small Reynolds numbers, the time step to use to assure stability, can be taken as an extension of the linear two-dimensional case, which will lead to

$$\Delta t \leq \frac{Rh^2}{6} .$$

The three-dimensional problem requires a great deal of computing time, even for a coarse grid. For example, for R=10, sixty time steps took twenty-five minutes of computer time. The average error on the grid points, for the pressure equation, was reduced to $1.5 \times 10^{-6}$ after sixty steps.

# APPENDIX

## COMPUTER PROGRAM FOR NUMERICAL SOLUTION FOR FLUID FLOW IN A RECTANGULAR BASIN

The following is the listing for the computer program to obtain numerical solutions for time-dependent three-dimensional incompressible viscous flow in a rectangular basin.

The program is written in FORTRAN-IV for the IBM-7040 computer having 32K, 36 bits word memory. The description of the program parameters are given at the beginning of the program listing.

```
C   THIS PROGRAM CALCULATES THE TIME DEPENDENT THREE-DIMENSIONAL FLUID
C   FLOW IN A RECTANGULAR BASIN IN TERMS OF THE VELOCITIES AND PRESSURES
C
C   SYMBOLS USED
C
C   U.V.W ARE THE VELOCITIES IN THE X.Y.Z-DIRECTION RESPECTIVELY
C   P   PRESSURE VALUES
C   PHI CONTAINS ALL REMAINING TERMS IN THE PRESSURE EQUATION
C   RES TOTAL RESIDUAL OF THE LINEAR PRESSURE EQUATION
C   CONT CONTINUITY EQUATION
C   PR  NORMAL PRESSURE DERIVATIVE ON THE RIGHT WALL
C   PL  NORMAL PRESSURE DERIVATIVE ON THE LEFT WALL
C   PT  NORMAL PRESSURE DERIVATIVE ON THE TOP WALL
C   PB  NORMAL PRESSURE DERIVATIVE ON THE BOTTOM WALL
C   PF  NORMAL PRESSURE DERIVATIVE ON THE FRONT WALL
C   PP  NORMAL PRESSURE DERIVATIVE ON THE BACK WALL
C   RFL RELAXATION FACTOR
C   M   GRID LINES IN THE X-DIRECTION
C   N   GRID LINES IN THE Y-DIRECTION
C   L   GRID LINES IN THE Z-DIRECTION
C   R   REYNOLDS NUMBER
C   DT  TIME STEP
C   H   MESH SIZE
C   TOL TOLERANCE TO WHICH THE POISSON EQUATION MUST BE SOLVED
C
C   THE PROGRAM IS DESIGNED FOR A 10*10*10 GRID SYSTEM AND R=10
C   THIS CAN BE CHANGED BY A CHANGE IN THE DIMENSION STATEMENTS
C   AND ADJUSTING THE VARIABLES M.N.L.R.DT.H.TOL
C
C
      DIMENSION U(12.12.12).V(12.12.12).W(12.12.12).C(12.12.12).
     1PHI(12.12.12).P(12.12.12).RES(12.12.12).CONT(12.12.12).
     2PR(12.12).PT(12.12).PL(12.12).PP(12.12).PB(12.12).PF(12.12)
      M=11
      N=11
      L=11
      H=0.1
      R=10.0
      DT=0.01
      N1=N-1
      M1=M-1
      L1=L-1
      RFL=1.65
      M2=M+1
      N2=N+1
      L2=L+1
      WRITE(6.8)
C
C INITIALIZATION OF U.V.W
C
      DO 10 I=1.M
      DO 10 J=1.N
```

```
      DO 10 K=1,L
      U(I,J,K)=0.0
      V(I,J,K)=0.0
   10 W(I,J,K)=0.0
      N3=N-2
      M3=M-2
      L3=L-2
      DO 20 J=2,N1
      DO 20 K=2,L
   20 U(M2,J,K)=1.0
      TOL=0.05
      DO 30 IT=1,60
      DO 15 I=1,M2
      DO 15 J=1,N2
      DO 15 K=1,L2
      PHI(I,J,K)=0.0
      C(I,J,K)=0.0
   15 RES(I,J,K)=0.0
      IF(IT.EQ.1) GO TO 851
      A=DT/(R*H*H)
C
C  COMPUTE U AT TIME T+DT
C
C  GENERAL  INTERIOR POINTS
C
      DO 40 I=3,M1
      DO 40 J=2,N1
      DO 40 K=3,L1
      RES (I,J,K)=U(I,J,K)+A*(U(I+1,J,K)+U(I-1,J,K)+U(I,J+1,K)
     1+U(I,J-1,K)+U(I,J,K+1)+U(I,J,K-1)-6.*U(I,J,K))
     2-DT/H*(0.25*((U(I,J+1,K)+U(I,J,K))**2-(U(I,J,K)+U(I,J-1,K))**2)
     3+0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J+1,K)+V(I,J,K))-(U(I,J,K)+
     4U(I-1,J,K))*(V(I-1,J+1,K)+V(I-1,J,K))+(U(I,J,K)+U(I,J,K+1))*
     5(W(I,J,K)+W(I,J+1,K))-(U(I,J,K)+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,
     6K-1)))+P(I,J+1,K)-P(I,J,K))+0.5*DT*U(I,J,K)*(CONT(I,J,K)
     7+CONT(I,J+1,K))
   40 CONTINUE
C
C  U FOR BOTTOM CELLS  FRONT EDGE
C
      I=2
      K=2
      DO 50 J=2,N1
      RES (I,J,K)=U(I,J,K)+A*(U(I+1,J,K)+U(I,J-1,K)+U(I,J+1,K)
     1+U(I,J,K+1)-8.0*U(I,J,K))
     2-DT/H*(0.25*(U(I,J,K)+U(I,J+1,K))**2-0.25*(U(I,J,K)+U(I,J-1,K))**2
     3+0.25*(U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))+0.25*(U(I,J,K)
     4+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))+P(I,J+1,K)-P(I,J,K))
     5+0.5*DT*U(I,J,K)*(CONT(I,J,K)+CONT(I,J+1,K))
   50 CONTINUE
C
C  U BOTTOM CELL BACK EDGE
C
      I=2
      K=L
```

```
      DO 60 J=2,N1
      RES(I,J,K)=U(I,J,K)+A*(U(I+1,J,K)+U(I,J-1,K)+U(I,J+1,K)
     1+U(I,J,K-1)-8.0*U(I,J,K))
     2-DT*H*(0.25*(U(I,J,K)+U(I,J+1,K))**2-0.25*(U(I,J,K)+U(I,J-1,K))**2
     3+0.25*(U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))-0.25*(U(I,J,K)
     4+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,K-1))+P(I,J+1,K)-P(I,J,K))
     5+0.5*DT* U(I,J,K)*(CONT(I,J,K)+CONT(I,J+1,K))
   60 CONTINUE
C
C  U REMAINING  BOTTOM CELLS
C
      I=2
      DO 70 J=2,N1
      DO 70 K=3,L1
      RES(I,J,K)=U(I,J,K)+A*(U(I+1,J,K)+U(I,J-1,K)+U(I,J+1,K)
     1+U(I,J,K-1)+U(I,J,K+1)-7.0*U(I,J,K))
     2-DT*H*(0.25*(U(I,J,K)+U(I,J+1,K))**2-0.25*(U(I,J,K)+U(I,J-1,K))**2
     3+0.25*(U(I+1,J,K)+U(I,J,K))*(V(I,J+1,K)+V(I,J,K))+0.25*(U(I,J,K)
     4+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-0.25*(U(I,J,K)+U(I,J,K-1))*
     5(W(I,J,K-1)+W(I,J+1,K-1))+P(I,J+1,K)-P(I,J,K))+0.5*DT*U(I,J,K)*
     6(CONT(I,J+1,K)+CONT(I,J,K))
   70 CONTINUE
C
C  U TOP CELLS FRONT EDGE
C
      I=M
      K=2
      DO 80 J=2,N1
      RES(I,J,K)=U(I,J,K)+A*(U(I-1,J,K)+U(I,J+1,K)+U(I,J-1,K)
     1+U(I,J,K+1)-8.0*U(I,J,K)+2.0)
     2-DT/H*(0.25*(U(I,J,K)+U(I,J+1,K))**2-0.25*(U(I,J,K)+U(I,J-1,K))**2
     3-0.25*(U(I,J,K)+U(I-1,J,K))*(V(M1,J,K)+V(M1,J+1,K))+0.25*(U(I,J,K)
     4+U(I,J,K+1))*(W(I  ,J,K)+W(I  ,J+1,K))+P(I,J+1,K)-P(I,J,K))
     5+0.5*DT*U(I,J,K)*(CONT(I,J,K)+CONT(I,J+1,K))
   80 CONTINUE
C
C  U TOP CELLS BACK EDGE
C
      I=M
      K=L
      DO 90 J=2,N1
      RES(I,J,K)=U(I,J,K)+A*(U(I-1,J,K)+U(I,J+1,K)+U(I,J-1,K)
     1+U(I,J,K-1)-8.0*U(I,J,K)+2.0)
     2-DT/H*(0.25*(U(I,J,K)+U(I,J+1,K))**2-0.25*(U(I,J,K)+U(I,J-1,K))**2
     3-0.25*(U(I,J,K)+U(I-1,J,K))*(V(M1,J,K)+V(M1,J+1,K))-0.25*(U(I,J,K)
     4+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,K-1))+P(I,J+1,K)-P(I,J,K))
     5+0.5*DT*U(I,J,K)*(CONT(I,J,K)+CONT(I,J+1,K))
   90 CONTINUE
C
C  U FOR REMAINING TOP CELLS
C
      I=M
      DO 110 J=2,N1
      DO 110 K=3,L1
      RES(I,J,K)=U(I,J,K)+A*(U(I-1,J,K)+U(I,J+1,K)+U(I,J-1,K)
```

```
     1+U(I.J.K+1)+U(I.J.K-1)-7.0*U(I.J.K)+2.0)
     2-DT/H*(0.25*((U(I.J+1.K)+U(I.J.K))**2-(U(I.J.K)+U(I.J-1.K))**2)+
     30.25*(-(U(I.J.K)+U(I-1.J.K))*(V(M1.J+1.K)+V(M1.J.K))+(U(I.J.K)+
     4U(I.J.K+1))*(W(I .J.K)+W(I .J+1.K))-(U(I.J.K)+U(I.J.K-1))*
     5(W(I.J.K-1)+W(I.J+1.K-1)))+P(I.J+1.K)-P(I.J.K))+0.5*DT*U(I.J.K)*
     6(CONT(I.J.K)+CONT(I.J+1.K))
  110 CONTINUE
C
C  U AT REMAINING FRONT WALL CELLS
C
      K=2
      DO 100 I=3.M1
      DO 100 J=2.N1
      RES(I.J.K)=U(I.J.K)+A*(U(I+1.J.K)+U(I-1.J.K)+U(I.J+1.K)
     1+U(I.J-1.K)+U(I.J.K+1)-7.0*U(I.J.K))
     2-DT/H*(0.25*((U(I.J.K)+U(I.J+1.K))**2-(U(I.J.K)+U(I.J-1.K))**2)
     3+0.25*((U(I+1.J.K)+U(I.J.K))*(V(I.J+1.K)+V(I.J.K))-(U(I.J.K)
     4+U(I-1.J.K))*(V(I-1.J+1.K)+V(I-1.J.K))+(U(I.J.K)+U(I.J.K+1))
     5*(W(I.J.K)+W(I.J+1.K)))+P(I.J+1.K)-P(I.J.K))+0.5*DT*U(I.J.K)
     6*(CONT(I.J.K)+CONT(I.J+1.K))
  100 CONTINUE
C
C  U AT REMAINING BACK WALL CELLS
C
      K=L
      DO 105 I=3.M1
      DO 105 J=2.N1
      RES(I.J.K)=U(I.J.K)+A*(U(I+1.J.K)+U(I-1.J.K)+U(I.J+1.K)
     1+U(I.J-1.K)+U(I.J.K-1)-7.0*U(I.J.K))
     2-DT/H*(0.25*((U(I.J.K)+U(I.J+1.K))**2-(U(I.J.K)+U(I.J-1.K))**2)
     3+0.25*((U(I+1.J.K)+U(I.J.K))*(V(I.J+1.K)+V(I.J.K))
     4-(U(I.J.K)+U(I-1.J.K))*(V(I-1.J+1.K)+V(I-1.J.K))-(U(I.J.K)
     5+U(I.J.K-1))*(W(I.J.K-1)+W(I.J+1.K-1)))+P(I.J+1.K)-P(I.J.K))
     6+0.5*DT*(CONT(I.J.K)+CONT(I.J+1.K))*U(I.J.K)
  105 CONTINUE
C
C  COMPUTE V AT T+DT AT ALL INTERIOR POINTS
C
      DO 120 I=2.M1
      DO 120 J=3.N1
      DO 120 K=3.L1
      C(I.J.K)=V(I.J.K)+A*(V(I.J.K+1)+V(I.J.K-1)+V(I.J+1.K)+V(I.J-1.K)
     1+V(I+1.J.K)+V(I-1.J.K)-6.*V(I.J.K))
     2-DT/H*(0.25*((V(I.J.K)+V(I+1.J.K))**2-(V(I.J.K)+V(I-1.J.K))**2)
     3+0.25*((U(I.J.K)+U(I+1.J.K))*(V(I.J.K)+V(I.J+1.K))-(U(I+1.J-1.K)
     4+U(I.J-1.K))*(V(I.J.K)+V(I.J-1.K))+(V(I.J.K)+V(I.J.K+1))*
     5(W(I.J.K)+W(I+1.J.K))-(V(I.J.K)+V(I.J.K-1))*(W(I.J.K-1)+W(I+1.J.
     6K-1)))+P(I+1.J.K)-P(I.J.K))+0.5*DT*(CONT(I+1.J.K)+CONT(I.J.K))*
     7V(I.J.K)
  120 CONTINUE
C
C  V AT LEFT HAND WALL FRONT EDGE
C
      J=2
      K=2
```

```
      DO 130 I=2,M1
      C(I,J,K)=V(I,J,K)+A*(V(I,J,K+1)+V(I,J+1,K)+V(I+1,J,K)+V(I-1,J,K)
     1-8.0*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*((U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))+(V(I,J,K)+
     4V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K)))+P(I+1,J,K)-P(I,J,K))+
     50.5*DT*V(I,J,K)*(CONT(I+1,J,K)+CONT(I,J,K))
  130 CONTINUE
C
C  V AT LEFT HAND WALL BACK EDGE
C
      J=2
      K=1
      DO 140 I=2,M1
      C(I,J,K)=V(I,J,K)+A*(V(I,J+1,K)+V(I+1,J,K)+V(I-1,J,K)+V(I,J,K-1)
     1-8.0*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*((U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))-(V(I,J,K)+
     4V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1)))+P(I+1,J,K)-P(I,J,K))
     5+0.5*DT*(CONT(I+1,J,K)+CONT(I,J,K))*V(I,J,K)
  140 CONTINUE
C
C  V AT LEFT HAND WALL,REMAINING POINTS
C
      J=2
      DO 150 I=2,M1
      DO 150 K=3,L1
      C(I,J,K)=V(I,J,K)+A*(V(I,J,K+1)+V(I,J,K-1)+V(I+1,J,K)+V(I-1,J,K)
     1+V(I,J+1,K)-7.*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*((U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))+(V(I,J,K)+
     4V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))-(V(I,J,K)+V(I,J,K-1))*
     5(W(I,J,K-1)+W(I+1,J,K-1)))+P(I+1,J,K)-P(I,J,K))+0.5*DT*V(I,J,K)*
     6(CONT(I,J,K)+CONT(I+1,J,K))
  150 CONTINUE
C
C  V AT RIGHT HAND WALL FRONT EDGE
C
      J=N
      K=2
      DO 160 I=2,M1
      C(I,J,K)=V(I,J,K)+A*(V(I,J-1,K)+V(I+1,J,K)+V(I-1,J,K)+V(I,J,K+1)
     1-8.0*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+(W(I,J,K)
     3+W(I+1,J,K))*
     4(V(I,J,K)+V(I,J,K+1)))+P(I+1,J,K)-P(I,J,K)          )+0.5*DT*V(I,J,
     5K)*(CONT(I,J,K)+CONT(I+1,J,K))
  160 CONTINUE
C
C  V AT RIGHT HAND WALL BACK EDGE
C
      J=N
      K=1
      DO 170 I=2,M1
```

```
      C(I,J,K)=V(I,J,K)+A*(V(I,J-1,K)+V(I+1,J,K)+V(I-1,J,K)+V(I,J,K-1)
     1-8.0*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))-
     4(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1)))+P(I+1,J,K)-
     5P(I,J,K))+0.5*DT*V(I,J,K)*(CONT(I,J,K)+CONT(I+1,J,K))
  170 CONTINUE
C
C   V AT RIGHT HAND WALL,REMAINING CELLS
C
      J=N
      DO 180 I=2,M1
      DO 180 K=3,L1
      C(I,J,K)=V(I,J,K)+A*(V(I,J,K+1)+V(I,J,K-1)+V(I+1,J,K)+V(I-1,J,K)
     1+V(I,J-1,K)-7.0*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+
     4(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))-(V(I,J,K)+V(I,J,K-1))
     5*(W(I,J,K-1)+W(I+1,J,K-1)))+P(I+1,J,K)-P(I,J,K))+0.5*DT*V(I,J,K)
     6*(CONT(I+1,J,K)+CONT(I,J,K))
  180 CONTINUE
C
C   V AT FRONT WALL REMAINING CELLS
C
      K=2
      DO 190 I=2,M1
      DO 190 J=3,N1
      C(I,J,K)=V(I,J,K)+A*(V(I,J+1,K)+V(I,J-1,K)+V(I+1,J,K)+V(I-1,J,K)
     1+V(I,J,K+1)-7.0*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*((U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))-(U(I+1,J-1,K)
     4+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+(V(I,J,K)+V(I,J,K+1))*
     5(W(I,J,K)+W(I+1,J,K)))+P(I+1,J,K)-P(I,J,K))+0.5*DT*V(I,J,K)*
     6(CONT(I+1,J,K)+CONT(I,J,K))
  190 CONTINUE
C
C   V AT BACK WALL REMAINING CELL
C
      K=L
      DO 200 I=2,M1
      DO 200 J=3,N1
      C(I,J,K)=V(I,J,K)+A*(V(I,J+1,K)+V(I,J-1,K)+V(I+1,J,K)+V(I-1,J,K)
     1+V(I,J,K-1)-7.0*V(I,J,K))
     2-DT/H*(0.25*((V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     3+0.25*((U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))-(U(I+1,J-1,K)
     4+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))-(V(I,J,K)+V(I,J,K-1))*
     5(W(I,J,K-1)+W(I+1,J,K-1)))+P(I+1,J,K)-P(I,J,K))+0.5*DT*V(I,J,K)
     6*(CONT(I,J,K)+CONT(I+1,J,K))
  200 CONTINUE
C
C   W AT ALL INTERIOR CELLS
C
      DO 210 I=3,M1
      DO 210 J=3,N1
      DO 210 K=2,L1
```

```
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J,K+1)+W(I,J,K-1)+W(I+1,J,K)
     1+W(I-1,J,K)+W(I,J+1,K)+W(I,J-1,K)-6.0*W(I,J,K))-
     2DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*((U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-
     4(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(V(I,J,K)
     5+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))-(V(I-1,J,K)+V(I-1,J,K+1))
     6*(W(I,J,K)+W(I-1,J,K)))+P(I,J,K+1)-P(I,J,K))+0.5*DT*W(I,J,K)*
     7(CONT(I,J,K)+CONT(I,J,K+1))
  210 CONTINUE
C
C  W AT RIGHT HAND WALL BOTTOM EDGE CELLS
C
      I=2
      J=N
      DO 220 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J-1,K)+W(I+1,J,K)+W(I,J,K+1)+W(I,J,K-1)
     1-8.0*W(I,J,K))
     2-DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*(-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+
     4(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K)))+P(I,J,K+1)
     5-P(I,J,K))+0.5*DT*W(I,J,K)*(CONT(I,J,K)+CONT(I,J,K+1))
  220 CONTINUE
C
C  W AT RIGHT HAND WALL TOP EDGE CELLS
C
      I=M
      J=N
      DO 230 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J-1,K)+W(I-1,J,K)+W(I,J,K+1)+W(I,J,K-1)
     1-8.0*W(I,J,K))
     2-DT*H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*(-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))
     4-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K)))+P(I,J,K+1)
     5-P(I,J,K))+0.5*DT*W(I,J,K)*(CONT(I,J,K)+CONT(I,J,K+1))
  230 CONTINUE
C
C  W AT LEFT HAND WALL BOTTOM EDGE CELLS
C
      I=2
      J=2
      DO 240 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J+1,K)+W(I+1,J,K)+W(I,J,K+1)+W(I,J,K-1)
     1-8.0*W(I,J,K))
     2-DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*((U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))+
     4(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K)))+P(I,J,K+1)
     5-P(I,J,K))+0.5*DT*W(I,J,K)*(CONT(I,J,K)+CONT(I,J,K+1))
  240 CONTINUE
C
C  W AT LEFT HAND WALL TOP EDGE CELLS
C
      I=M
      J=2
      DO 250 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J+1,K)+W(I-1,J,K)+W(I,J,K+1)+W(I,J,K-1)
```

```
     1-8.0*W(I,J,K))
     2-DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*((U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-
     4(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K)))+P(I,J,K+1)-
     5P(I,J,K))+0.5*DT*W(I,J,K)*(CONT(I,J,K)+CONT(I,J,K+1))
  250 CONTINUE
C
C  W AT REMAINING RIGHT WALL CELLS
C
      J=N
      DO 260 I=3,M1
      DO 260 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J-1,K)+W(I+1,J,K)+W(I-1,J,K)
     1+W(I,J,K+1)+W(I,J,K-1)-7.0*W(I,J,K))
     2-DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*(        -(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))
     4+(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))-(V(I-1,J,K)+
     5V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K)))+P(I,J,K+1)-P(I,J,K))
     6+0.5*DT*W(I,J,K)*(CONT(I,J,K)+CONT(I,J,K+1))
  260 CONTINUE
C
C  W AT REMAINING LEFT WALL CELLS
C
      J=2
      DO 270 I=3,M1
      DO 270 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J+1,K)+W(I+1,J,K)+W(I-1,J,K)
     1+W(I,J,K+1)+W(I,J,K-1)-7.0*W(I,J,K))
     2-DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*((U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))+(V(I,J,K)
     4+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))-(V(I-1,J,K)+V(I-1,J,K+1))
     5*(W(I,J,K)+W(I-1,J,K)))+P(I,J,K+1)-P(I,J,K))+0.5*DT*W(I,J,K)
     6*(CONT(I,J,K)+CONT(I,J,K+1))
  270 CONTINUE
C
C  W AT REMAINING BOTTOM WALL CELLS
C
      I=2
      DO 280 J=3,N1
      DO 280 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I,J+1,K)+W(I,J-1,K)+W(I+1,J,K)
     1+W(I,J,K+1)+W(I,J,K-1)-7.0*W(I,J,K))
     2-DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*((U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))
     4-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(V(I,J,K)
     5+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K)))+P(I,J,K+1)-P(I,J,K))
     6+0.5*DT*W(I,J,K)*(CONT(I,J,K)+CONT(I,J,K+1))
  280 CONTINUE
C
C  W AT REMAINING TOP WALL CELLS
C
      I=M
      DO 290 J=3,N1
      DO 290 K=2,L1
      PHI(I,J,K)=W(I,J,K)+A*(W(I-1,J,K)+W(I,J+1,K)+W(I,J-1,K)
```

```
     1+W(I,J,K+1)+W(I,J,K-1)-7.0*W(I,J,K))
     2-DT/H*(0.25*((W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+0.25*((U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J-1,K)
     4+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))-(V(I-1,J,K+1)+V(I-1,J,K))
     5*(W(I,J,K)+W(I-1,J,K)))+P(I,J,K+1)-P(I,J,K))+0.5*DT*W(I,J,K)
     6*(CONT(I,J,K)+CONT(I,J,K+1))
  290 CONTINUE
      DO 300 I=1,M
      DO 300 J=1,N
      DO 300 K=1,L
      U(I,J,K)=RES(I,J,K)
      V(I,J,K)=C(I,J,K)
  300 W(I,J,K)=PHI(I,J,K)
C
C  CALCULATION OF  CONTINUITY
C
      DO 350 I=2,M
      DO 350 J=2,N
      DO 350 K=2,L
      CONT(I,J,K)=(U(I,J,K)-U(I,J-1,K)+V(I,J,K)-V(I-1,J,K)+W(I,J,K)
     1-W(I,J,K-1))/H
  350 CONTINUE
      J=2
      JZ=N
      DO 360 I=2,M
      DO 360 K=2,L
      CONT(I,J-1,K)=-CONT(I,J,K)
  360 CONT(I,JZ+1,K)=-CONT(I,JZ,K)
      I=2
      IZ=M
      DO 370 J=2,N
      DO 370 K=2,L
      CONT(I-1,J,K)=-CONT(I,J,K)
  370 CONT(IZ+1,J,K)=-CONT(IZ,J,K)
      K=2
      KZ=L
      DO 380 I=2,M
      DO 380 J=2,N
      CONT(I,J,K-1)=-CONT(I,J,K)
  380 CONT(I,J,KZ+1)=-CONT(I,J,KZ)
C
C   COMPUTATION OF PRESSURE VALUES
C
C   FIRST CALCULATE PHI
C
  851 DO 400 I=3,M1
      DO 400 J=3,N1
      DO 400 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I,J,K)+V(I+1,J,K))**2-2.*
     2(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J,K)
     3+V(I-2,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))
     4**2+(W(I,J,K-1)+W(I,J,K-2))**2)
     5+2.*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     6-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1,K))-(U(I+1,J-1,K)
```

```
   7+ U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+(U(I,J-1,K)+U(I-1,J-1,K))
   8*(V(I-1,J,K)+V(I-1,J-1,K))+(U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+
   9W(I,J+1,K))))
 400 CONTINUE
     DO 410 I=3,M1
     DO 410 J=3,N1
     DO 410 K=3,L1
     PHI(I,J,K)=PHI(I,J,K)-((U(I,J,K)+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,
    1K-1))-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(U(I,J-1,K)
    2+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))+(V(I,J,K)+V(I,J,K+1))
    3*(W(I,J,K)+W(I+1,J,K))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,
    4J,K-1))-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))
    5+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1)))*0.50
 410 CONTINUE
     DO 420 I=2,M
     DO 420 J=2,N
     DO 420 K=2,L
     PHI(I,J,K)=PHI(I,J,K)-H*((U(I,J,K)-U(I,J-1,K))*CONT(I,J,K)
    1+0.25*(U(I,J,K)+U(I,J-1,K))*(CONT(I,J+1,K)-CONT(I,J-1,K))
    2+(V(I,J,K)-V(I-1,J,K))*CONT(I,J,K)+0.25*(V(I,J,K)+V(I-1,J,K))*
    3(CONT(I+1,J,K)-CONT(I-1,J,K))+(W(I,J,K)-W(I,J,K-1))*CONT(I,J,K)
    4+0.25*(W(I,J,K)+W(I,J,K-1))*(CONT(I,J,K+1)-CONT(I,J,K-1)))
    5-1./R*(CONT(I,J,K+1)+CONT(I,J,K-1)+CONT(I,J+1,K)+CONT(I,J-1,K)+
    6CONT(I+1,J,K)+CONT(I-1,J,K)-6.0*CONT(I,J,K))-H*H/DT*CONT(I,J,K)
 420 CONTINUE
C
C  PHI FOR BOTTOM WALL
C
      I=2
      DO 425 J=3,N1
      DO 425 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)
     2+V(I-1,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))
     3**2+(W(I,J,K-1)+W(I,J,K-2))**2)
     4+2.*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))-(U(I+1,
     5J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+(U(I,J,K)+U(I,J,K+1))
     6*(W(I,J,K)+W(I,J+1,K))-(U(I,J,K)+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,
     7K-1))-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(U(I,J-1,K)
     8+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))+(V(I,J,K)+V(I,J,K+1))
     9*(W(I,J,K)+W(I+1,J,K))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)-0.5*(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,
     1J,K-1))
  425 CONTINUE
C
C  PHI FOR BOTTOM FRONT EDGE
C
      I=2
      K=2
      DO 430 J=3,N1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)
     2+V(I-1,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     4-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+(U(I,J,K)+
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```
     5U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J-1,K)+U(I,J-1,K+1))
     6*(W(I,J,K)+W(I,J-1,K))+(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))
     7))+PHI(I,J,K)
  430 CONTINUE
C
C  PHI FOR BOTTOM BACK EDGE
C
      I=2
      K=L
      DO 440 J=3,N1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)
     2+V(I-1,J,K))**2+(W(I,J,K-1)+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))
     3**2) +2.0*(0.25*((U(I,J,K)+U(I+1,J,K))*(V(I,J,K)+V(I,J+1,K))
     4-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))-(U(I,J,K)+
     5U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,K-1))+(U(I,J-1,K)+U(I,J-1,K-1))
     6*(W(I,J,K-1)+W(I,J-1,K-1))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+
     7W(I+1,J,K-1))))+PHI(I,J,K)
  440 CONTINUE
C
C  PHI BOTTOM RIGHT EDGE
C
      I=2
      J=N
      DO 450 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)+
     2W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))**2+(W(I,J,K-1)+W(I,J,K-2))
     3**2)  +2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))
     4-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(U(I,J-1,K)+
     5U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))+(V(I,J,K)+V(I,J,K+1))
     6*(W(I,J,K)+W(I+1,J,K))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+
     7W(I+1,J,K-1))))+PHI(I,J,K)
  450 CONTINUE
C
C  PHI BOTTOM LEFT EDGE
C
      I=2
      J=2
      DO 460 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)+
     2W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))**2+(W(I,J,K-1)+W(I,J,K-2))
     3**2) +2.0*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     4+(U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J,K)+U(I,J,K-1))
     5*(W(I,J,K-1)+W(I,J+1,K-1))+(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,
     6J,K))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1))))+PHI(I,J,K)
  460 CONTINUE
C
C  PHI BOTTOM LEFT FRONT CORNER
C
      I=2
      J=2
      K=2
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
```

```
     1+(W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2+
     1(V(I+1,J,K)+V(I,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     2+2.0*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     3+(U(I,J,K+1)+U(I,J,K))*(W(I,J,K)+W(I,J+1,K))+(V(I,J,K)+V(I,J,K+1))
     4*(W(I,J,K)+W(I+1,J,K))))+PHI(I,J,K)
C
C   PHI BOTTOM RIGHT FRONT CORNER
C
      I=2
      J=N
      K=2
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I+1,J,K)+V(I,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)+
     2W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))
     4-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(V(I,J,K)+
     5V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))))+PHI(I,J,K)
C
C   PHI BOTTOM RIGHT BACK CORNER
C
      I=2
      J=N
      K=L
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I   ,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,
     2K-1)+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))
     4+(U(I,J-1,K)+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))-(V(I,J,K)
     5+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1))))+PHI(I,J,K)
C
C   PHI BOTTOM LEFT BACK CORNER
C
      I=2
      J=2
      K=L
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K-1)
     2+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.0*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     4-(U(I,J,K)+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,K-1))-(V(I,J,K)
     5+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1))))+PHI(I,J,K)
C
C   PHI LEFT HAND SIDE WALL
C
      J=2
      DO 470 I=3,M1
      DO 470 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J,K)
     2+V(I-2,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))
     3**2+(W(I,J,K-1)+W(I,J,K-2))**2) +2.0*(0.25*((U(I+1,J,K)+U(I,J,K))
     4*(V(I,J,K)+V(I,J+1,K))-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1
     5,K))+(U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J,K)+U(I,J,
     6K-1))*(W(I,J,K-1)+W(I,J+1,K-1))+(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)
     7+W(I+1,J,K))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1))
```

```
     1+(W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2+
     1(V(I+1,J,K)+V(I,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2)
     2+2.0*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     3+(U(I,J,K+1)+U(I,J,K))*(W(I,J,K)+W(I,J+1,K))+(V(I,J,K)+V(I,J,K+1))
     4*(W(I,J,K)+W(I+1,J,K))))+PHI(I,J,K)
C
C   PHI BOTTOM RIGHT FRONT CORNER
C
      I=2
      J=N
      K=2
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I+1,J,K)+V(I,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)+
     2W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))
     4-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(V(I,J,K)+
     5V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))))+PHI(I,J,K)
C
C   PHI BOTTOM RIGHT BACK CORNER
C
      I=2
      J=N
      K=L
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I   ,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,
     2K-1)+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))
     4+(U(I,J-1,K)+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))-(V(I,J,K)
     5+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1))))+PHI(I,J,K)
C
C   PHI BOTTOM LEFT BACK CORNER
C
      I=2
      J=2
      K=L
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I,J,K)+V(I+1,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K-1)
     2+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.0*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     4-(U(I,J,K)+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,K-1))-(V(I,J,K)
     5+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1))))+PHI(I,J,K)
C
C   PHI LEFT HAND SIDE WALL
C
      J=2
      DO 470 I=3,M1
      DO 470 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J,K)
     2+V(I-2,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))
     3**2+(W(I,J,K-1)+W(I,J,K-2))**2) +2.0*(0.25*((U(I+1,J,K)+U(I,J,K))
     4*(V(I,J,K)+V(I,J+1,K))-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1
     5,K))+(U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J,K)+U(I,J,
     6K-1))*(W(I,J,K-1)+W(I,J+1,K-1))+(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)
     7+W(I+1,J,K))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+W(I+1,J,K-1))
```

```
     8-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))
     9+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1))))+PHI(I,J,K)
  470 CONTINUE
C
C   PHI RIGHT HAND SIDE WALL
C
      J=N
      DO 480 I=3,M1
      DO 480 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J,
     2K)+V(I-2,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1)
     3)**2+(W(I,J,K-1)+W(I,J,K-2))**2)+2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1
     4,K))*(V(I,J,K)+V(I,J-1,K))+(U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)
     5+V(I-1,J-1,K))-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))
     6+(U(I,J-1,K)+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))+(V(I,J,K)+V(I
     7,J,K+1))*(W(I,J,K)+W(I+1,J,K))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)+
     8W(I+1,J,K-1))-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))
     9+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1))))+PHI(I,J,K)
  480 CONTINUE
C
C   PHI  BACK WALL
C
      K=L
      DO 485 I=3,M1
      DO 485 J=3,N1
      PHI(I,J,K)=0.25*((U(I,J+1,K)+U(I,J,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I+1,J,K)+V(I,J,K))**2-2.*(V(I,J
     2,K)+V(I-1,J,K))**2+(V(I-1,J,K)+V(I-2,J,K))**2+(W(I,J,K-1)+W(I,J,
     3K-2))**2-(W(I,J,K)+W(I,J,K-1))**2) +2.0*(0.25*((U(I+1,J,K)+U(I,J,
     4K))*(V(I,J,K)+V(I,J+1,K))-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,
     5J+1,K))-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+(U(I,J-1,
     6K)+U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))-(U(I,J,K)+U(I,J,K-1))*
     7(W(I,J,K-1)+W(I,J+1,K-1))+(U(I,J-1,K)+U(I,J-1,K-1))*(W(I,J,K-1)+
     8W(I,J-1,K-1))-(V(I,J,K)+V(I,J,K-1))* (W(I,J,K-1)+W(I+1,J,K-1))
     9+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1))))+PHI(I,J,K)
  485 CONTINUE
C
C   PHI FRONT WALL
C
      K=2
      DO 490 J=3,N1
      DO 490 I=3,M1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J
     2,K)+V(I-1,J,K))**2+(V(I-1,J,K)+V(I-2,J,K))**2+(W(I,J,K)+W(I,J,K+1)
     3 )**2-(W(I,J,K)+W(I,J,K-1))**2) +2.0*(0.25*((U(I+1,J,K)+U(I,J,K))
     4*(V(I,J,K)+V(I,J+1,K))-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1
     5,K))-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))+(U(I,J-1,K)+
     6U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))+(U(I,J,K)+U(I,J,K+1))*
     7(W(I,J,K)+W(I,J+1,K))-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,
     8K))+(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))
     9-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))))+PHI(I,J,K)
  490 CONTINUE
C
```

```
C  PHI FRONT WALL LEFT EDGE
C
      K=2
      J=2
      DO 500 I=3,M1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J,K)
     2+V(I-2,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2)
     3+2.0*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))-(U(I,J,K)
     4+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1,K))+(U(I,J,K)+U(I,J,K+1))*
     5(W(I,J,K)+W(I,J+1,K))+(V(I,J,K)+V(I,J,K+1))*(W(I,J,K)+W(I+1,J,K))
     6-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))))+PHI(I,J,K)
  500 CONTINUE
C
C  PHI FRONT WALL RIGHT EDGE
C
      K=2
      J=N
      DO 505 I=3,M1
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J
     2,K)+V(I-2,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))
     3**2) +2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))
     4+(U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))-(U(I,J-1,K)
     5+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+(V(I,J,K)+V(I,J,K+1))*
     6(W(I,J,K)+W(I+1,J,K))-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,
     7J,K))))+PHI(I,J,K)
  505 CONTINUE
C
C  PHI BACK WALL RIGHT EDGE
C
      K=L
      J=N
      DO 510 I=3,M1
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J
     2,K)+V(I-2,J,K))**2+(W(I,J,K-1)+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1)
     3)**2)+2.0*(0.25*(-(U(I+1,J-1,K)+U(I,J-1,K))*(V(I,J,K)+V(I,J-1,K))
     4+(U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))+(U(I,J-1,K)
     5+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))-(V(I,J,K)+V(I,J,K-1))*
     6(W(I,J,K-1)+W(I+1,J,K-1))+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+
     7W(I-1,J,K-1))))+PHI(I,J,K)
  510 CONTINUE
C
C  PHI BACK WALL LEFT EDGE
C
      K=L
      J=2
      DO 520 I=3,M1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I,J,K)+V(I+1,J,K))**2-2.*(V(I,J,K)+V(I-1,J,K))**2+(V(I-1,J,K)
     2+V(I-2,J,K))**2+(W(I,J,K-1)+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))
     3**2) +2.0*(0.25*((U(I+1,J,K)+U(I,J,K))*(V(I,J,K)+V(I,J+1,K))
     4-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1,K))-(U(I,J,K)+U(I,J,
     5K-1))*(W(I,J,K-1)+W(I,J+1,K-1))-(V(I,J,K)+V(I,J,K-1))*(W(I,J,K-1)
```

```
     6+W(I+1,J,K-1))+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1))
     7))+PHI(I,J,K)
  520 CONTINUE
C
C  PHI  TOP WALL CELLS
C
      I=M
      DO 530 J=3,N1
      DO 530 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K
     2)+V(I-1,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))
     3**2+(W(I,J,K-1)+W(I,J,K-2))**2)+2.0*(0.25*(-(U(I,J,K)+U(I-1,J,K))
     4*(V(I-1,J,K)+V(I-1,J+1,K))+(U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)+
     5V(I-1,J-1,K))+(U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J,K
     6)+U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,K-1))-(U(I,J-1,K)+U(I,J-1,K+1))*
     7(W(I,J,K)+W(I,J-1,K))+(U(I,J-1,K)+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,
     8J-1,K-1))-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))+(V(I-1,J
     9,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
  530 CONTINUE
C
C  PHI  TOP WALL FRONT EDGE
C
      I=M
      K=2
      DO 540 J=3,N1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,
     2K)+V(I-1,J,K))**2+(W(I,J,K)+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))
     3**2) +2.*(0.25*(-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1,K))
     4+(U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))+(U(I,J,K)+
     5U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J-1,K)+U(I,J-1,K+1))*
     6(W(I,J,K)+W(I,J-1,K))-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J
     7,K))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
  540 CONTINUE
C
C  PHI  FOR TOP WALL BACK EDGE
C
      I=M
      K=L
      DO 550 J=3,N1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-2.*(U(I,J,K)+U(I,J-1,K))
     1**2+(U(I,J-1,K)+U(I,J-2,K))**2+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K
     2)+V(I-1,J,K))**2+(W(I,J,K-1)+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))
     3**2) +2.*(0.25*(-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1,K))
     4+(U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))-(U(I,J,K)+
     5U(I,J,K-1))*(W(I,J,K-1)+W(I,J+1,K-1))+(U(I,J-1,K)+U(I,J-1,K-1))*
     6(W(I,J,K-1)+W(I,J-1,K-1))+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+
     7W(I-1,J,K-1))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
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```
635   550 CONTINUE
C
C  PHI  FOR TOP WALL LEFT EDGE
C
      I=M
      J=2
      DO 560 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)+
     2W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))**2+(W(I,J,K-1)+W(I,J,K-2))
     3**2) +2.0*(0.25*(-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1,K))
     4+(U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))-(U(I,J,K)+U(I,J,K-1))
     5*(W(I,J,K-1)+W(I,J+1,K-1))-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+
     6W(I-1,J,K))+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1))))
     7+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
  560 CONTINUE
C
C  PHI FOR TOP WALL RIGHT EDGE
C
      I=M
      J=N
      DO 570 K=3,L1
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)
     2+W(I,J,K+1))**2-2.*(W(I,J,K)+W(I,J,K-1))**2+(W(I,J,K-1)+W(I,J,
     3K-2))**2) +2.0*(0.25*((U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)+
     4V(I-1,J-1,K))-(U(I,J-1,K)+U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))+
     5(U(I,J-1,K)+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))-(V(I-1,J,K)+
     6V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))+(V(I-1,J,K)+V(I-1,J,K-1))
     7*(W(I,J,K-1)+W(I-1,J,K-1))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
  570 CONTINUE
C
C  PHI FOR LEFT TOP FRONT CORNER
C
      I=M
      J=2
      K=2
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)+
     2 W(I,J,K+1))**2) +2.0*(0.25*(-(U(I,J,K)+U(I-1,J,K))*(V(I-1,J,K)
     3+V(I-1,J+1,K))+(U(I,J,K)+U(I,J,K+1))*(W(I,J,K)+W(I,J+1,K))
     4-(V(I-1,J,K)+V(I-1,J,K+1))*(W(I,J,K)+W(I-1,J,K))))-0.25*(W(I,J,K)
     5+W(I,J,K-1))**2+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
C
C  PHI FOR RIGHT TOP FRONT CORNER
C
      I=M
      J=N
      K=2
```

```
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K)
     2+W(I,J,K+1))**2-(W(I,J,K)+W(I,J,K-1))**2) + 2.0*(0.25*((U(I,J-1,
     3K)+U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))-(U(I,J-1,K)+
     4U(I,J-1,K+1))*(W(I,J,K)+W(I,J-1,K))-(V(I-1,J,K)+V(I-1,J,K+1))
     5*(W(I,J,K)+W(I-1,J,K))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
C
C   PHI FOR RIGHT TOP BACK CORNER
C
      I=M
      J=N
      K=L
      PHI(I,J,K)=0.25*((U(I,J-1,K)+U(I,J-2,K))**2-(U(I,J,K)+U(I,J-1,K))
     1**2+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2
     2+(W(I,J,K-1)+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))**2) +
     32.0*(0.25*((U(I,J-1,K)+U(I-1,J-1,K))*(V(I-1,J,K)+V(I-1,J-1,K))
     4+(U(I,J-1,K)+U(I,J-1,K-1))*(W(I,J,K-1)+W(I,J-1,K-1))
     5+(V(I-1,J,K)+V(I-1,J,K-1))*(W(I,J,K-1)+W(I-1,J,K-1))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
C
C   PHI  FOR BACK LEFT TOP CORNER
C
      I=M
      J=2
      K=L
      PHI(I,J,K)=0.25*((U(I,J,K)+U(I,J+1,K))**2-(U(I,J,K)+U(I,J-1,K))**2
     1+(V(I-1,J,K)+V(I-2,J,K))**2-(V(I,J,K)+V(I-1,J,K))**2+(W(I,J,K-1)
     2+W(I,J,K-2))**2-(W(I,J,K)+W(I,J,K-1))**2) +2.*(0.25*(-(U(I,J,K)
     3+U(I-1,J,K))*(V(I-1,J,K)+V(I-1,J+1,K))-(U(I,J,K)+U(I,J,K-1))
     4*(W(I,J,K-1)+W(I,J+1,K-1))+(V(I-1,J,K)+V(I-1,J,K-1))*
     5(W(I,J,K-1)+W(I-1,J,K-1))))+PHI(I,J,K)
      PHI(I,J,K)=PHI(I,J,K)+(0.5*(V(I,J,K)+V(I-1,J,K))+U(M2,J-1,K)-
     1U(M2,J,K))**2-0.25*(V(I,J,K)+V(I-1,J,K))**2
C
C   COMPUTATION OF PRESSURE BOUNDARY VALUES
C
C   PB=BOTTOM,PT=TOP,PL=LEFT,PR=RIGHT,PF=FRONT,PP=BACK
C
      A=2./(R*H*H)
      DO 310 I=2,M
      DO 310 K=2,L
      PL(I,K)=A*U(I,2,K)
  310 PR(I,K)=A*U(I,N1,K)
      DO 320 J=2,N
      DO 320 I=2,M
      PF(I,J)=A*W(I,J,2)
  320 PP(I,J)=A*W(I,J,L1)
      DO 330 J=2,N
      DO 330 K=2,L
  330 PB(J,K)=A*V(2,J,K)
      DO 340 J=2,N
      DO 340 K=2,L
```

```
      PT(J,K)=A*(V(M1,J,K)-U(M2,J,K)+U(M2,J-1,K))
     1-((0.5*V(M1,J,K)-U(M2,J,K)+U(M2,J-1,K))**2-0.25*V(M1,J,K)**2)/H
  340 CONTINUE
C
C  PRESSURE CALCULATIONS
C
      K3=0
  850 DO 600 I=2,M
      DO 600 J=2,N
      DO 600 K=2,L
      IF(I.EQ.2.AND.J.EQ.2.AND.K.EQ.2) GO TO 610
      IF(I.EQ.2.AND.J.EQ.2.AND.K.EQ.L) GO TO 620
      IF(I.EQ.2.AND.J.EQ.N.AND.K.EQ.2) GO TO 630
      IF(I.EQ.2.AND.J.EQ.N.AND.K.EQ.L) GO TO 640
      IF(I.EQ.M.AND.J.EQ.2.AND.K.EQ.2) GO TO 650
      IF(I.EQ.M.AND.J.EQ.2.AND.K.EQ.L) GO TO 660
      IF(I.EQ.M.AND.J.EQ.N.AND.K.EQ.2) GO TO 670
      IF(I.EQ.M.AND.J.EQ.N.AND.K.EQ.L) GO TO 680
      IF(I.GT.2.AND.J.GT.2.AND.K.GT.2.AND.I.LT.M.AND.J.LT.N.AND.K.LT.L)
     1GO TO 605
      IF(I.EQ.2.AND.K.EQ.2) GO TO 690
      IF(I.EQ.2.AND.J.EQ.N) GO TO 700
      IF(I.EQ.2.AND.K.EQ.L) GO TO 710
      IF(I.EQ.2.AND.J.EQ.2) GO TO 720
      IF(I.EQ.M.AND.K.EQ.2) GO TO 730
      IF(I.EQ.M.AND.J.EQ.N) GO TO 740
      IF(I.EQ.M.AND.K.EQ.L) GO TO 750
      IF(I.EQ.M.AND.J.EQ.2) GO TO 760
      IF(J.EQ.2.AND.K.EQ.2) GO TO 762
      IF(J.EQ.2.AND.K.EQ.L) GO TO 764
      IF(J.EQ.N.AND.K.EQ.2) GO TO 766
      IF(J.EQ.N.AND.K.EQ.L) GO TO 768
      IF(I.EQ.2) GO TO 770
      IF(I.EQ.M) GO TO 780
      IF(J.EQ.2) GO TO 790
      IF(J.EQ.N) GO TO 800
      IF(K.EQ.2) GO TO 810
      IF(K.EQ.L) GO TO 820
  605 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I,J,K+1)+P(I,J,K-1)+P(I,J+1,K)+
     1P(I,J-1,K)+P(I+1,J,K)+P(I-1,J,K)-6.*P(I,J,K)+PHI(I,J,K))
      GO TO 600
  610 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I,J,K+1)+P(I+1,J,K)+P(I,J+1,K)-
     13.*P(I,J,K)-H*(PB(J,K)+PL(I,K)+PF(I,J))+PHI(I,J,K))
      GO TO 600
  620 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I,J,K-1)+P(I,J+1,K)+P(I+1,J,K)
     1-3.*P(I,J,K)+H*(PP(I,J)-PL(I,K)-PB(J,K))+PHI(I,J,K))
      GO TO 600
  630 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I,J-1,K)+P(I,J,K+1)
     1-3.0*P(I,J,K)+H*(PR(I,K)-PF(I,J)-PB(J,K))+PHI(I,J,K))
      GO TO 600
  640 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I,J-1,K)+P(I,J,K-1)
     1-3.0*P(I,J,K)+H*(PR(I,K)+PP(I,J)-PB(J,K))+PHI(I,J,K))
      GO TO 600
  650 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J+1,J)+P(I,J,K+1)
     1-3.0*P(I,J,K)+H*(PT(J,K)-PL(I,K)-PF(I,J))+PHI(I,J,K))
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      GO TO 600
  660 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J+1,K)+P(I,J,K-1)
     1-3.0*P(I,J,K)+H*(PT(J,K)+PP(I,J)-PL(I,K))+PHI(I,J,K))
      GO TO 600
  670 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J-1,K)+P(I,J,K+1)
     1-3.0*P(I,J,K)+H*(PT(J,K)+PR(I,K)-PF(I,J))+PHI(I,J,K))
      GO TO 600
  680 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J-1,K)+P(I,J,K-1)
     1-3.0*P(I,J,K)+H*(PT(J,K)+PR(I,K)+PP(I,J))+PHI(I,J,K))
      GO TO 600
  690 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I,J+1,K)+P(I,J-1,K)
     1+P(I,J,K+1)-4.0*P(I,J,K)-H*(PB(J,K)+PF(I,J))+PHI(I,J,K))
      GO TO 600
  700 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I,J-1,K)+P(I,J,K+1)
     1+P(I,J,K-1)-4.0*P(I,J,K)+H*(PR(I,K)-PB(J,K))+PHI(I,J,K))
      GO TO 600
  710 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I,J+1,K)+P(I,J-1,K)
     1+P(I,J,K-1)-4.0*P(I,J,K)+H*(PP(I,J)-PB(J,K))+PHI(I,J,K))
      GO TO 600
  720 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I,J+1,K)+P(I,J,K+1)
     1+P(I,J,K-1)-4.0*P(I,J,K)-H*(PL(I,K)+PB(J,K))+PHI(I,J,K))
      GO TO 600
  730 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J-1,K)+P(I,J+1,K)
     1+P(I,J,K+1)-4.0*P(I,J,K)+H*(PT(J,K)-PF(I,J))+PHI(I,J,K))
      GO TO 600
  740 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J-1,K)+P(I,J,K+1)
     1+P(I,J,K-1)-4.0*P(I,J,K)+H*(PT(J,K)+PR(I,K))+PHI(I,J,K))
      GO TO 600
  750 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J+1,K)+P(I,J-1,K)
     1+P(I,J,K-1)-4.0*P(I,J,K)+H*(PT(J,K)+PP(I,J))+PHI(I,J,K))
      GO TO 600
  760 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J+1,K)+P(I,J,K+1)
     1+P(I,J,K-1)-4.0*P(I,J,K)+H*(PT(J,K)-PL(I,K))+PHI(I,J,K))
      GO TO 600
  762 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J+1,K)+
     1P(I,J,K+1)-4.0*P(I,J,K)-H*(PL(I,K)+PF(I,J))+PHI(I,J,K))
      GO TO 600
  764 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J+1,K)+
     1P(I,J,K-1)-4.0*P(I,J,K)+H*(PP(I,J)-PL(I,K))+PHI(I,J,K))
      GO TO 600
  766 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J-1,K)+
     1P(I,J,K+1)-4.0*P(I,J,K)+H*(PR(I,K)-PF(I,J))+PHI(I,J,K))
      GO TO 600
  768 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J-1,K)+
     1P(I,J,K-1)-4.0*P(I,J,K)+H*(PR(I,K)+PP(I,J))+PHI(I,J,K))
      GO TO 600
  770 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I,J+1,K)+P(I,J-1,K)
     1+P(I,J,K+1)+P(I,J,K-1)-5.0*P(I,J,K)-H*PB(J,K)+PHI(I,J,K))
      GO TO 600
  780 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I-1,J,K)+P(I,J+1,K)+P(I,J-1,K)
     1+P(I,J,K+1)+P(I,J,K-1)-5.0*P(I,J,K)+H*PT(J,K)+PHI(I,J,K))
      GO TO 600
  790 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J+1,K)
     1+P(I,J,K+1)+P(I,J,K-1)-5.0*P(I,J,K)-H*PL(I,K)+PHI(I,J,K))
      GO TO 600
```

```
  800 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J-1,K)
     1+P(I,J,K+1)+P(I,J,K-1)-5.0*P(I,J,K)+H*PR(I,K)+PHI(I,J,K))
      GO TO 600
  810 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J+1,K)
     1+P(I,J-1,K)+P(I,J,K+1)-5.0*P(I,J,K)-H*PF(I,J)+PHI(I,J,K))
      GO TO 600
  820 P(I,J,K)=P(I,J,K)+REL/6.0*(P(I+1,J,K)+P(I-1,J,K)+P(I,J+1,K)
     1+P(I,J-1,K)+P(I,J,K-1)-5.0*P(I,J,K)+H*PP(I,J)+PHI(I,J,K))
  600 CONTINUE
      K3=K3+1
      IF(K3/05*05.NE.K3) GO TO 850
C
C  CALCULATION OF RESIDUALS
C
      TRES=0.0
      DO 830 I=3,M1
      DO 830 J=3,N1
      DO 830 K=3,L1
      Q=P(I+1,J,K)+P(I-1,J,K)+P(I,J+1,K)+P(I,J-1,K)+P(I,J,K+1)
     1+P(I,J,K-1)-6.0*P(I,J,K)+PHI(I,J,K)
      RES(I,J,K)=Q
      TRES=TRES+ABS(Q)
  830 CONTINUE
      IF(TRES.GT.TOL) GO TO 850
      IF(TRES.GT.1.0E04) STOP
      WRITE(6,5) TRES,K3,R,H,DT,LT
      IF(LT/20*20.NE.LT) GO TO 30
      WRITE(6,2)
      K=6
      I=M2+1
      DO 860 II=1,M2
      I=I-1
  860 WRITE(6,1)(U(I,J,K),J=1,N)
      WRITE(6,7)
      WRITE(6,3)
      K=6
      I=M+1
      DO 870 II=1,M
      I=I-1
  870 WRITE(6,1)(V(I,J,K),J=1,N2)
      WRITE(6,7)
      WRITE(6,4)
      J=9
      I=M2+1
      DO 880 II=1,M2
      I=I-1
  880 WRITE(6,1)(W(I,J,K),K=1,L2)
      WRITE(6,8)
      DO 5000 K=2,L
      DO 5000 I=2,M
 5000 WRITE(6,1) (CONT(I,J,K),J=2,N)
      WRITE(6,8)
   30 CONTINUE
      STOP
    1 FORMAT(5X,12F10.6)
```

```
2 FORMAT(1H1,10X,10HU-VELOCITY)
3 FORMAT(1H ,10X,10HV-VELOCITY)
4 FORMAT(1H ,10X,10HW-VELOCITY)
5 FORMAT(10X,6HT-RES=E13.6,6H ITER=,I4,6H REY.=F7.1,3H H=F6.3,4H DT=
 1F7.4,2X,10HTIME-STEP=,I5)
7 FORMAT(///)
8 FORMAT(1H1)
  END
```

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