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Bifurcation of Limit Cycles in Smooth and Non-smooth Dynamical Systems with Normal Form Computation

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A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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(Thesis format: Integrated Article)

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Abstract

This thesis contains two parts. In the first part, we investigate bifurcation of limit cycles around a singular point in planar cubic systems and quadratic switching systems. For planar cubic systems, we study cubic perturbations of a quadratic Hamiltonian system and obtain 10 small-amplitude limit cycles bifurcating from an elementary center, for which up to 5th-order Melnikov functions are used. Moreover, we prove the existence of 12 small-amplitude limit cycles around a singular point in a cubic system by computing focus values. For quadratic switching system, we develop a recursive algorithm for computing Lyapunov constants. With this efficient algorithm, we obtain a complete classification of the center conditions for a switching Bautin system. Moreover, we construct a concrete example of switching system to obtain 10 small-amplitude limit cycles bifurcating from a center.

In the second part, we derive two explicit, computationally efficient, recursive formulae for computing the normal forms, center manifolds and nonlinear transformations for general n-dimensional systems, associated with Hopf and semisimple singularities, respectively. Based on the formulae, we develop Maple programs, which are very convenient for an end-user who only needs to prepare an input file and then execute the program to "automatically" generate the results. Several examples are presented to demonstrate the computational efficiency of the algorithms. In addition, we show that a simple 3-dimensional quadratic vector field can have 7 small-amplitude limit cycles, bifurcating from a Hopf singular point. This result is surprisingly higher than the Bautin's result for quadratic planar vector fields which can only have 3 small-amplitude limit cycles around an elementary focus or an elementary center.

Keywords: planar cubic system, Hamiltonian system, Hilbert's 16th problem, higher-order Melnikov function, center, limit cycle, bifurcation, focus value, normal form

Co-Authorship Statement

The article versions of Chapters 2 through 7 are co-authored with Pei Yu. The four papers based on Chapters 3, 5, 6 and 7 have been published in journals, Chapter 2 has been submitted for publication, and Chapter 4 will be submitted for publication.

Pei Yu provided guidance through all the works, and revised the final drafts. Pei Yu also computed the focus values for Chapters 2, 3 and 7, and contributed the first drafts for Chapters 3 and 7.

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Chapter 1

Introduction

1.1 Overview of differential dynamical systems

A dynamical system can be considered as a fixed deterministic "rule", which describes the time dependence of a point in a geometrical space. For a dynamical system, a point, given by a real vector, represents a state. The "rule" is usually given in the form of differential equations or difference equations. At any given time a dynamical system has a state, and for a given time interval only one future state follows from the current state. Systems described by differential equations are called continuous dynamical systems, or differential dynamical systems, and ones described by difference equations are called discrete dynamical systems.

The concept of a dynamical system was developed from Newtonian mechanics in the seventeenth century. At that time, many great scientists like Galileo Galilei, Edme Mariotte and Robert Hooke attacked mathematical problems in this area [1, Chapter 21]. Since then dynamical systems theory has been being animate. Today, dynamical systems theory has been applied extensively in a wide range of research areas, including automatic control, space technology, celestial mechanics, biology, medical science, economics and so on. For example, the mathematical models that describe the motion of a mass on a vibrating spring, the flow of electric current in a simple series circuit and the number of fish each spring in a lake, etc. Besides differential dynamical systems, there are some other research fields of dynamical systems, including topological dynamical systems, symbolic dynamical systems, ergodic theory and complex dynamical systems and so on, see [2].

For some mathematical models described by differential equations, we may obtain the exact expression of the solutions. In this case, for any given initial condition all future dynamical behaviors of the solution can be determined. But there are many important problems, especially nonlinear ones, which are too complicated to be solved. How do we study the dynamical behaviors of such problems? Poincaré's work "On curves defined by differential equations"[3] showed us that the properties of the solutions could be studied and determined directly from the differential equations, even if they can not be solved in terms of known functions. This result initiated a new research branch of mathematics called qualitative theory of differential equations.

Bifurcation is one of main research objects in the qualitative analysis of differential equations. Generally speaking, bifurcation is the changes of topological structure of the orbits

in a dynamical system. In the real world, there are various of bifurcation phenomena that have been discovered and described in the natural science. In 1883, Osborne Reynolds found that the flow of fluid in pipes transitioned from laminar flow to turbulent flow when the velocity of the flow is increased above a certain threshold. Other bifurcation phenomena include the buckling of the Euler rod, the appearance of Taylor vortices and the onset of oscillations in an electric circuit and so on. All these phenomena have a common feature: A specific physical parameter crosses a threshold, which forces the system to the new organization of states that differs considerably from that observed before. With the advancement of computer science, symbolic computation has a great development, which provides us a more powerful tool to study and simulate all the bifurcation phenomena.

Bifurcations in differential systems can be divided into two principal classes: local bifurcations and global bifurcations. Local bifurcations includes the changes in the topological structure of obits around a singular point. Global bifurcations occur when changes are caused in the topological structure of "larger" invariant sets, like a family of periodic orbits, a homoclinic loop and a heteroclinic loop.

Bifurcation of limit cycles is one of the most popular topics in bifurcation theory and applications. A limit cycle is an isolated closed trajectory in the phase space of an autonomous differential system. A limit cycle corresponds to a periodic non-constant solution of the system. Limit cycles represent the simplest (after the steady states) type of behavior of a continuous time dynamical system, which can be obtained from bifurcations of an elementary center (like Hopf bifurcation), a compact family of periodic orbits (Poincaré bifurcation) or other closed loops consisting of a finite of saddle points and orbits as the connections, like a homoclinic loop (homoclinic bifurcation) or a heteroclinic loop (heteroclinic bifurcation).

In this thesis, we focus on the bifurcation of small-amplitude limit cycles in planar differential systems, higher-dimensional dynamical systems as well as switching systems.

1.1.1 Planar differential systems

A well-known problem about the bifurcation of limit cycles in planar differential systems is Hilbert's 16th problem [4], which is related to the following polynomial vector field,

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{1.1}$$

where P_n and Q_n are nth-degree polynomial functions. This problem was posed by Hilbert at the International Congress of Mathematicians in 1900. The second part of Hilbert's 16th problem is to find the upper bound of the number, called the Hilbert number, $H(n)$, of limit cycles in (1.1) and to study their distributions. The progress of solving the problem is very slow. After more than one century, it has not even been solved for the case of quadratic systems.

In early 1990s, Ilyashenko and Yakovenko [5], and Écalle [6] independently proved the finiteness of the number of limit cycles for any given planar polynomial vector fields. The existence of a finite uniform upper bound $H(n)$ for the number of limit cycles of planar polynomial vector fields of degree n remains unsolved for any $n > 1$. For $n = 2$, it was proved $H(2) \geq 4$ more than 30 years ago [7, 8]. Recently, this result was also obtained for near-integrable quadratic systems [9]. For cubic polynomial systems, many results have been

obtained on the lower bound of $H(n)$. Among them, the best one is $H(3) \geq 13$ [10, 11]. Note that the 13 limit cycles obtained in [10, 11] are distributed around several singular points. For more on the research of Hilbert's 16th problem see [12].

The local version of the second part of Hilbert's 16th problem is to find the maximum number, $M(n)$, of small-amplitude limit cycles bifurcating from an elementary center or an elementary focus in planar polynomial systems of degree n. Sometimes, it is also called the cyclicity problem. The center-focus problem is closely related to the cyclicity problem, which is to determine whether a singular point is a center or a focus in any planar systems. Both problems are very difficult and still open.

In the 1880s, Poincaré first gave a necessary and sufficient condition to have a center at a singular point with pure imaginary eigenvalues, for which there exists a local analytic non-zero first integral in the neighborhood of the singular point. In order to find the first integral, Poincaré created an algorithm for computing the so-called Poincaré-Lyapunov constants V_1, V_2, V_3, $\cdots$. The singular point becomes a center if and only if all the Poincaré-Lyapunov constants vanish. If the first nonzero Poincaré-Lyapunov constant is V_k, then the singular point is a (weak) focus, and at most k small-amplitude limit cycles can bifurcate from it [13].

In 1952, Bautin proved $M(2) = 3$ by computing focus values, and obtained all center conditions for general quadratic systems [14]. Moreover, it is well-known that quadratic systems with one center can be classified into four sub-systems: Hamiltonian system, Lotka-Volterra system, reversible system, and codimension-4 system, denoted by Q^H, Q_3^{LV}, Q^R and Q_4, respectively [15]. For $n = 3$, the center problem is a long way from being solved. Some center conditions were obtained only for particular cubic systems [16, 17]. For the cyclicity problem of cubic systems, there are results with examples in the literature. Among them, the maximal number of small-amplitude limit cycles around a singular point was eleven [18, 19, 20].

In order to help understand and attack Hilbert's 16th problem, Arnold posed the so-called weak Hilbert's 16th problem [21], namely, to find the upper bound of the number of zeros, $N(n, m)$, of the Melnikov function given by the integral

$$M(h) = \oint_{H(x,y)=h} q(x, y)dx - p(x, y)dy, \tag{1.2}$$

along closed orbits defined by $H(x, y) = h$ for $h \in (h_1, h_2)$, where $H(x, y)$, $p(x, y)$ and $q(x, y)$ are all polynomial functions of x and y, and $\deg(H(x, y)) = n$, $\deg(p(x, y)) = \deg(q(x, y)) = m = n - 1$. The weak Hilbert's 16th problem is closely related to the near-Hamiltonian system [22]:

$$\dot{x} = H_y(x, y) + \varepsilon p(x, y), \qquad \dot{y} = -H_x(x, y) + \varepsilon q(x, y), \tag{1.3}$$

where $0 < \varepsilon \ll 1$ represents a small perturbation, because the limit cycles of the above system bifurcating from the periodic orbits $H(x, y) = h$ for $h \in (h_1, h_2)$ correspond to the isolated zeros of the first non-zero Melnikov function $M_k(h)$. The first-order Melnikov function, $M_1(h)$, of system (1.3) is given by (1.2). Then if $M_1(h) \not\equiv 0$, the number of limit cycles bifurcating from orbits $H(x, y) = h$ can be determined by the number of isolated zeros of $M_1(h) = 0$.

For any n and m, the finiteness of $N(n, m)$ was proved independently by Khovansky [23] and Varchenko [24] in 1984. But the explicit expression of $N(n, m)$ has not been obtained. When n is fixed, it was proved that for the set of "good" $H(x, y)$ there exists a constant $c(H) <$

$+\infty$ such that $N(n,m) \leq \exp(c(H)m)$ [25, 26, 27]. For quadratic near-Hamiltonian systems, i.e. $n = 3$ and $m = 2$, all possible cases have been considered for the cubic Hamiltonians [28, 29, 30, 31, 32, 33, 34]. Putting all the results together we have $N(3,2) = 2$. For cases where n is fixed, lots of results were obtained for polynomials P and Q with arbitrary degree m, for instance, $m-1$ zeros for the Bogdanov-Takens Hamiltonian, $H = (x^2+y^2)/2 - x^3/3$ [35], and $[\frac{2(m-1)}{3}]$ for the Hamiltonian triangle $H = (x^2+y^2)/2 - x^3/3 + xy^2$ [36].

If $M_1(h) \equiv 0$, we need to consider higher-order Melnikov functions. In some cases, higher-order Melnikov functions can help obtain more limit cycles [37, 38, 39, 40]. But it is usually not easy to compute those higher-order Melnikov functions [41]. Only for some special polynomial Hamiltonian functions, there is a procedure presented by Françoise [42] for computing higher-order Melnikov functions through decompositions of one-forms. Its generalization can be found in [43, 44].

1.1.2 Switching differential systems

In recent years, more and more attention has been attracted to non-smooth dynamical systems whose functions on right-hand side are not differentiable or even not continuous. The basic methods of qualitative theory for such systems can be found in [45, 46].

One class of planar non-smooth systems is the so-called switching system, given in the form of

$$(\dot{x},\dot{y}) = \begin{cases} (f^+(x,y), g^+(x,y)), & \text{if } y > 0, \\ (f^-(x,y), g^-(x,y)), & \text{if } y < 0, \end{cases} \tag{1.4}$$

where $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ are analytic functions in x and y. In (1.4), the system defined for $y > 0$ is called the upper system, ant the system defined for $y < 0$ is called the lower system. A detailed discussion of the research on the dynamics of switching systems can be found in a survey article [47].

A great deal of work has been done to generalize the classical bifurcation theory and methods for smooth systems to non-smooth ones, for examples, the methods for Hopf bifurcation [48, 49, 50, 51, 52, 53, 54]. Poincaré map has been introduced into the study of Hopf bifurcation in switching systems [46, 49, 50], so that the corresponding Lyapunov constants can be defined. In [49], the authors developed a new method for computing the Lyapunov constants of switching systems, by applying a suitable decomposition of certain one-forms. The approach based on Lyapunov constant is an important tool for the study of the center problem and the cyclicity problem in switching systems, just like that for smooth systems. But it becomes much more complicated in switching systems.

As we mentioned above, a planar smooth system has a center at a singular point, if and only if there exists a local non-zero first integral near the singular point. For system (1.4), this statement obviously remains true if the singular point is not located on the x-axis. But if it is located on the x-axis, the singular point may not be a center even there exist first integrals for both the upper and lower systems in (1.4). In this case, it is also required that these two first integrals properly coincide on the x-axis. Conditions for a singular point to be a center have been obtained for some switching Kukles systems [49], switching Liénard systems [50] and a switching Bautin system [53].

The number of small-amplitude limit cycles bifurcating from a weak focus in switching

systems has been investigated in [51, 52, 53, 54, 55]. For linear switching systems, Han and Zhang proved that 2 small-amplitude limit cycles can appear near a focus [51]. Note that smooth linear systems can not produce limit cycles. The cyclicity problem for quadratic switching systems is much more difficult than that in smooth systems, and some results have been obtained only for some special systems [49, 52, 53]. It has been proved that quadratic switching systems can have at most 5 limit cycles near a singular point, when its lower system is linear [49]. The best result so far for the number of small-amplitude limit cycles in quadratic switching systems is 9 [52].

Because of various forms of non-smoothness, switching systems can exhibit more complex bifurcations that only non-smooth systems can have, such as border-collision bifurcation [56], grazing bifurcation [57, 58] and so on. These types of bifurcations will not be discussed in this thesis.

1.1.3 Higher-dimensional differential systems

For higher-dimensional differential systems, a rich variety of bifurcations may occur around a singular point, like Hopf, Hopf-zero, double-zero, double-Hopf and so on. A detailed study of some local bifurcations in higher-dimensional systems could be found in [59].

For the bifurcation of small-amplitude limit cycles in higher-dimensional systems, there are very few results in the literature. Over the last twenty years, several researchers have paid attention to a 3-dimensional competitive Lotka-Volterra model, and particularly studied the bifurcation of limit cycles [60, 61, 62, 63]. So far, the best result for this system is 4 limit cycles [63], which include 3 small-amplitude limit cycles and a large one.

Normal form theory plays an important role in studying local dynamical behaviors around a singular point. Its basic idea is to introduce a near-identity nonlinear transformation into a given differential system and to get a simpler one, which keeps the topological structure of the original system around the singular point [59, 64, 65]. When the Jacobian matrix evaluated at a singular point of higher-dimensional differential systems has eigenvalues with zero real part, center manifold theory is always applied together with normal form theory. It allows us to determine the dynamical behaviors by studying the flows on the center manifold, which has less dimension than the original system.

In order to find more small-amplitude limit cycles around a singular point, we usually need to compute higher-order normal forms. But even with the help of computer algebra systems, such as Maple, Mathematica, Matlab and so on, it is still not easy to obtain higher-order normal forms since considerably more computer memory and computational time are demanded as the order of normal forms increases.

Lots of methods have been developed for computing normal forms and equivalent quantities (focus values or Poincaré-Lyapunov constants), including the time-averaging method [59, 66], Poincaré method [67, 68], the perturbation technique combined with multiple time scales [69]. Recently, researchers have also paid attention to further reduction of the (conventional or classical) normal forms, called the simplest normal forms (e.g. see [70, 71, 72, 73]). The computation of the parametric simplest normal forms [72, 73] is much more complex and difficult, which will not be discussed in this thesis. For Hopf bifurcation of two-dimensional systems, a recursive formula was presented for the computation of Poincaré-Lyapunov constants in terms of the coefficients of the original system [68]. For

general n-dimensional differential systems, the method of multiple time scales was combined with a perturbation technique to obtain normal forms for a number of different singularities such as Hopf [74], Hopf-zero [75], double Hopf [76, 77] and so on. Yu [78] developed a unified procedure for computing the center manifold, the normal form of a differential system and the corresponding nonlinear transformation.

1.2 Preliminaries

In this section, we shall give a brief introduction to displacement function and normal form theory, which play a critical role in this thesis.

Consider an n-dimensional differential system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\delta}),\ \boldsymbol{x} \in \mathbf{R}^n,\ \boldsymbol{\delta} \in \mathbf{R}^m, \tag{1.5}$$

where the dot represents differentiation with respect to time t, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\delta})$ is an analytic function in the region $D \subset \mathbf{R}^n$, and $\boldsymbol{\delta}$ is a parameter vector.

According to the theorem of existence and uniqueness of solutions, for any point $\boldsymbol{x}_0 \in D$ system (1.5) has a unique solution $\boldsymbol{x} = \boldsymbol{\varphi}(t, \boldsymbol{x}_0, \boldsymbol{\delta})$ satisfying $\boldsymbol{\varphi}(0, \boldsymbol{x}_0, \boldsymbol{\delta}) = \boldsymbol{x}_0$. If $\boldsymbol{\varphi}(t, \boldsymbol{x}_0, \boldsymbol{\delta}) \equiv \boldsymbol{x}_0$, then point $\boldsymbol{x}_0$ is called a singular point (singularity) of system (1.5), otherwise a regular point. The curve defined by $\boldsymbol{x} = \boldsymbol{\varphi}(t, \boldsymbol{x}_0, \boldsymbol{\delta})$ is call the orbit of (1.5) through $\boldsymbol{x}_0$. An orbit is called a periodic orbit if it is closed. A limit cycle is an isolated periodic orbit, i.e. there are no other periodic orbits in a neighborhood of it.

The topological structure of orbits near a regular point is simple: the family of orbits in a small neighborhood of a regular point is topologically equivalent to the family of parallel lines [79]. For a singular point, the situation is complicated.

Without loss of generality, suppose that the origin is a singular point of system (1.5). The linearization of system (1.5) at the origin is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \tag{1.6}$$

where $\boldsymbol{A}\boldsymbol{x}$ is the linear part of $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{A}$ is an $n \times n$ matrix. Then matrix $\boldsymbol{A}$ has eigenvalues λ_i, $i = 1, \cdots, n$, with zero, negative and positive real parts. The corresponding eigenvectors of matrix $\boldsymbol{A}$ span three sub-spaces of $\mathbf{R}^n$: center eigenspace E^c, stable eigenspace E^s and unstable eigenspace E^u, and $\mathbf{R}^n = E^c \bigoplus E^s \bigoplus E^u$. Obviously, each eigenspace E^c, E^s or E^u is invariant for the linearized system (1.6). All orbits of system (1.6) in eigenspace E^s (E^u) tend to the origin as $t \to +\infty$ ($-\infty$), and the orbits in E^c are periodic or singular points.

According to Grobman [80] and Hartman [81], if all eigenvalues of system (1.6) at the origin have non-zero real part, then system (1.6) is said to be hyperbolic, and system (1.5) has the same topological structure of the orbits near the origin as system (1.6). This means that the changes in nonlinear parts of system (1.5) have no effects on the topological structure of the orbits around a singular point, if its corresponding linearized system is hyperbolic.

If system (1.6) involves eigenvalues with zero real part, then small-amplitude limit cycles could bifurcate from the origin under small perturbations. The cyclicity problem naturally arises here. In this situation, the topological structure of the orbits near the origin in system (1.5) is unstable, and various bifurcations may occur, depending on the number of zero

eigenvalues and pairs of purely imaginary eigenvalues. If a singular point $\boldsymbol{x}_{\boldsymbol{\delta}}$ of system (1.5) has eigenvalues with zero real part at $\boldsymbol{\delta} = \boldsymbol{\delta}_0$, then $(\boldsymbol{x}_{\boldsymbol{\delta}_0}, \boldsymbol{\delta}_0)$ is called a critical point, or a bifurcation point. The following is Hopf bifurcation theorem, which can also serve as the definition of Hopf bifurcation.

Theorem 1.2.1 ([82, Section 11.2]) *Let $J(\boldsymbol{\delta})$ be the Jacobian of system (1.5) evaluated at a singular point $\boldsymbol{x}_{\boldsymbol{\delta}}$ of it. Suppose that $J(\boldsymbol{\delta}_0)$ has a simple pair of purely imaginary eigenvalues and no other eigenvalues with zero real part. A Hopf bifurcation arises when these two eigenvalues cross the imaginary axis because of a variation of $\boldsymbol{\delta}$ around $\boldsymbol{\delta}_0$.*

In order to study bifurcation problems, the critical point usually needs to be determined. The eigenvalues can be obtained from the characteristic polynomial of the Jacobian $J(\boldsymbol{\delta})$ as

$$p_n(\lambda) = \det(\lambda I - J(\boldsymbol{\delta})) = \lambda^n + a_1(\boldsymbol{\delta})\lambda^{n-1} + \cdots + a_n(\boldsymbol{\delta}). \tag{1.7}$$

To find the critical point of a Hopf bifurcation, we do not have to solve the eigenvalues from the above polynomial. Instead, we use the Hurwitz criteria [83], Δ_i, $i = 1, \cdots, n-1$, of the polynomial $p_n(\lambda)$. The theorem is given below.

Theorem 1.2.2 ([83]) *Suppose that $J(\boldsymbol{\delta})$ is the Jacobian of system (1.5) evaluated at a singular point $\boldsymbol{x}_{\boldsymbol{\delta}}$ of it. Let (1.7) hold. The necessary and sufficient condition for system (1.5) to have a Hopf bifurcation at $\boldsymbol{x}_{\boldsymbol{\delta}}$ without eigenvalues having positive real part is $\Delta_i > 0$ for $i = 1, 2, \cdots, n-2$, $\Delta_{n-1} = 0$ and $a_n > 0$.*

If matrix $J(\boldsymbol{\delta})$ has n different eigenvectors, which is equivalent to $J(\boldsymbol{\delta})$ being diagonalizable, then we say that the singular point $\boldsymbol{x}_{\boldsymbol{\delta}}$ in system (1.5) is semisimple.

The next two subsections are devoted to two important tools for bifurcation of small-amplitude limit cycles, displacement function and the method of normal forms associated with semisimple singular points.

1.2.1 Displacement function and Melnikov functions

Consider a planar analytic differential system with a parameter vector of the form

$$\dot{x} = f(x, y, \boldsymbol{\delta}), \quad \dot{y} = g(x, y, \boldsymbol{\delta}), \tag{1.8}$$

where $\boldsymbol{\delta} \in \mathbf{R}^m$, $m \geq 1$, and $f(0, 0, \boldsymbol{\delta}) = g(0, 0, \boldsymbol{\delta}) = 0$. Then the origin is a singular point of system (1.8).

The linearization of system (1.8) at the origin is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{\partial(f, g)}{\partial(x, y)}\Big|_{(0,0)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.9}$$

with the characteristic polynomial $p_2(\lambda) = \lambda^2 + a_1\lambda + a_2$, where $a_1 = -(A_{11} + A_{22})$, $a_2 = A_{11}A_{22} - A_{21}A_{12}$. Denote by λ_1 and λ_2 its two eigenvalues. Then the origin of system (1.9) is a

$$\begin{cases} \text{saddle point}, & \text{if } \lambda_1\lambda_2 < 0, \\ \text{node}, & \text{if } \lambda_1\lambda_2 > 0,\ \mathrm{Im}(\lambda_1) = 0, \\ \text{focus}, & \text{if } \mathrm{Re}(\lambda_1) \neq 0,\ \mathrm{Im}(\lambda_1) \neq 0, \\ \text{center}, & \text{if } \mathrm{Re}(\lambda_1) = 0,\ \mathrm{Im}(\lambda_1) \neq 0, \\ \text{degenerate point}, & \text{if } \lambda_1\lambda_2 = 0. \end{cases}$$

Therefore, we have a center point at the origin with a pair of purely imaginary eigenvalues, when $a_1 = 0$ and $a_2 > 0$. In this case, the origin in system (1.8) could be an elementary center or an elementary focus, and the center-focus problem arises here. Under proper conditions of $\boldsymbol{\delta}$, small-amplitude limit cycles can bifurcate from the origin.

Assume system (1.9) has an elementary center at the origin for $\boldsymbol{\delta} = \boldsymbol{\delta}_0$. Without loss of generality, we may further assume that system (1.9) has a focus at the origin for any $0 < |\boldsymbol{\delta} - \boldsymbol{\delta}_0| \ll 1$. Under a proper linear transformation, system (1.8) becomes

$$\begin{aligned} \dot{x} &= \alpha(\boldsymbol{\delta})x - \beta(\boldsymbol{\delta})y + F(x, y, \boldsymbol{\delta}), \\ \dot{y} &= \beta(\boldsymbol{\delta})x + \alpha(\boldsymbol{\delta})y + G(x, y, \boldsymbol{\delta}), \end{aligned} \tag{1.10}$$

where $|\boldsymbol{\delta}-\boldsymbol{\delta}_0| \ll 1, \alpha(\boldsymbol{\delta}_0) = 0, \beta(\boldsymbol{\delta}_0) > 0$ and $F,\ G = O(|x, y|^2)$. Letting $x = r\cos(\theta), y = r\sin(\theta)$, system (1.10) is transformed into

$$\dot{r} = \alpha(\boldsymbol{\delta})r + \cos(\theta)F + \sin(\theta)G, \quad \dot{\theta} = \beta(\boldsymbol{\delta}) + (\cos(\theta)G - \sin(\theta)F)/r.$$

Then eliminating time t yields

$$\frac{dr}{d\theta} = \frac{\alpha(\boldsymbol{\delta})r + \cos(\theta)F + \sin(\theta)G}{\beta(\boldsymbol{\delta}) + (\cos(\theta)G - \sin(\theta)F)/r} = R_1(\theta)r + R_2(\theta)r^2 + R_3(\theta)r^3 + R_4(\theta)r^4 + \cdots, \tag{1.11}$$

where $R_i(\theta)$ is a polynomial function of $\sin(\theta)$ and $\cos(\theta)$.

Suppose $r(\theta, \rho)$ is the solution of equation (1.11) satisfying $r(0, \rho) = \rho$. Then for system (1.10) we define the Poincaré map as $\mathcal{P}(\rho) = r(2\pi, \rho)$, $0 < \rho \ll 1$ (see Figure 1.1). The function $d(\rho) = \mathcal{P}(\rho) - \rho$, is called the displacement function of system (1.10). It is easy to see that $d(\rho) \equiv 0$ for $0 < \rho \ll 1$ if and only if the origin is a center of system (1.10). If $d(\rho) < 0$ (> 0) for $0 < \rho \ll 1$, then we say the origin is a stable (unstable) focus.

Obviously, if and only if ρ_0 is an isolated zero of the function $d(\rho)$, i.e. $\mathcal{P}(\rho_0) = \rho_0$, then system (1.10) has a small-amplitude limit cycle near the origin passing through point $(\rho_0, 0)$. So the number of isolated zeros of $d(\rho) = 0$, $0 < \rho \ll 1$, corresponds to the number of small-amplitude limit cycles near the origin in system (1.10).

The displacement function $d(\rho)$ can be written as a power series in ρ:

$$d(\rho) = v_1\rho + v_2\rho^2 + v_3\rho^3 + \cdots,\ 0 < \rho \ll 1. \tag{1.12}$$

It is easy to see that if $v_k = 0$, $k = 1, 2, 3, \cdots$, then the origin is a center.

Theorem 1.2.3 [84, Chapter 2] *Let (1.12) hold for system (1.10). If for some $k \geq 1$ it holds that*

$$d(\rho) = v_k\rho^k + O(\rho^{k+1}), \quad v_k \neq 0, \tag{1.13}$$

then k is odd, and the origin is a stable (unstable) focus of system (1.10) if $v_k < 0$ (> 0).

Definition 1.4.1 Let (1.13) hold with $k = 2i + 1$, $i \geq 0$. We call the origin a focus of order i, and v_k the ith order focus value. Sometimes v_k is also called the ith order Lyapunov constant.

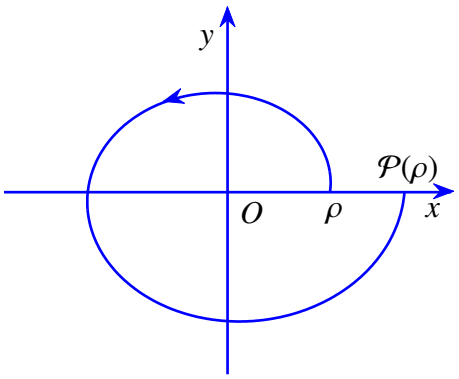

Figure 1.1: The Poincaré map for system (1.10).

Theorem 1.2.4 [84, Chapter 2] *(i) If system (1.10) has a kth-order focus at the origin ($k \geq 1$) for $\boldsymbol{\delta} = \boldsymbol{\delta}_0$, then it has at most k limit cycles near the origin for $\boldsymbol{\delta}$ near $\boldsymbol{\delta}_0$. Moreover, k limit cycles can appear near the origin if*

$$\operatorname{rank}\Big(\frac{\partial(v_1, v_3, \cdots, v_{2k-1})}{\partial(\delta_1, \delta_2, \cdots, \delta_m)}(\boldsymbol{\delta}_0)\Big) = k.$$

(ii) If system (1.10) is analytic and satisfies

$$v_{2j+1} = O(|v_1, v_3, \cdots, v_{2k+1}|), \quad j \geq k+1$$

for some $k \geq 1$, then for any given $N > 0$ system (1.10) has at most k limit cycles near the origin for $|v_1| + |v_2| + \cdots + |v_{2k+1}| < N$.

One basic approach to compute the focus values of system (1.10) is to substitute the power series of the solution $r(\theta, \rho)$ given in ρ,

$$r(\theta, \rho) = r_1(\theta)\rho + r_2(\theta)\rho^2 + r_3(\theta)\rho^3 + \cdots,$$

into equation (1.11) and then compare the coefficients of the terms in ρ with the same power. For $r_1(\theta)$, we get $r_1'(\theta) = R_1(\theta)r_1(\theta)$, or

$$r_1(\theta) = e^{\frac{\alpha(\delta)}{\beta(\delta)}\theta}, \quad \text{since } R_1(\theta) = \frac{\alpha(\boldsymbol{\delta})}{\beta(\boldsymbol{\delta})} \text{ from (1.11).}$$

Then $v_1 = e^{2\pi\frac{\alpha(\delta)}{\beta(\delta)}} - 1$. When $v_1 = 0$, i.e. $\alpha(\boldsymbol{\delta}) = 0$, we can easily obtain

$$r_1 = 1, \quad r_2(\theta) = \int_0^\theta R_2(\theta)d\theta, \quad r_3(\theta) = \int_0^\theta (R_3(\theta) + 2r_2(\theta)R_2(\theta))d\theta, \ \cdots. \tag{1.14}$$

Obviously, $v_i = r_i(2\pi)$, $i > 1$.

Usually, we do not use (1.14) to study bifurcation of small-amplitude limit cycles in system (1.10) since (1.14) involves integration of trigonometric functions, which demands lots of computational time, especially when higher-order focus values are needed. On the other hand, computation of normal forms or Poincaré-Lyapuvnov constants only contains algebraic computations, which can be easily implemented in a computer using a computer algebra system. Recently, researchers paid more and more attention to the computation of (1.14), because Poincaré map and displacement function can be also applied to Hopf bifurcations of switching systems.

The displacement function can be also expressed in terms of Melnikov functions for near-Hamiltonian systems in the form of

$$\dot{x} = H_y + \varepsilon p(x, y, \varepsilon, \boldsymbol{\delta}), \quad \dot{y} = -H_x + \varepsilon q(x, y, \varepsilon, \boldsymbol{\delta}), \tag{1.15}$$

where $H(x, y)$, $p(x, y, \varepsilon, \boldsymbol{\delta})$ and $q(x, y, \varepsilon, \boldsymbol{\delta})$ are C^∞ functions, and ε is small. For $\varepsilon = 0$ system (1.15) becomes

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \tag{1.16}$$

which is a Hamiltonian system.

Suppose system (1.16) has a family of periodic orbits given by Hamiltonian levels,

$$\gamma_h : \ H(x, y) = h, \ h \in (h_0, h_1)$$

and a center at the origin, denoted by γ_{h_0}, such that $\gamma_h \to \gamma_{h_0}$ as $h \to h_0^+$. Using the Hamiltonian level $H = h$, $h \in (h_0, h_1)$, to parameterize the positive x-axis near the origin, we can express the Poincaré map $\mathcal{P}$ in terms of h, ε and $\boldsymbol{\delta}$. Thus, the corresponding displacement function $d(h, \varepsilon, \boldsymbol{\delta}) = \mathcal{P}(h, \varepsilon, \boldsymbol{\delta}) - h$ can be written as

$$d(h, \varepsilon, \boldsymbol{\delta}) = \varepsilon M_1(h, \boldsymbol{\delta}) + \varepsilon^2 M_2(h, \boldsymbol{\delta}) + \varepsilon^3 M_3(h, \boldsymbol{\delta}) + \cdots, \tag{1.17}$$

where

$$M_1(h, \boldsymbol{\delta}) = \oint_{\gamma_h} (q dx - p dy)|_{\varepsilon=0},$$

is called (the first-order) Melnikov function, and $M_i(h, \boldsymbol{\delta})$, $i > 1$, is called ith-order (higher-order) Melnikov function.

Then the number of limit cycles bifurcating from the periodic orbits γ_h, $h \in (h_0, h_1)$, in system (1.15) could be determined by the number of zeros of the first non-zero Melnikov function $M_k(h, \boldsymbol{\delta})$ in (1.17). The zeros near $h = h_0$ correspond to the limit cycles near the center γ_{h_0}. For the first-order Melnikov function, $M_1(h, \boldsymbol{\delta})$, we have the following two theorems [84].

Theorem 1.2.5 *For system (1.15) we have*

(i) The Meknikov function $M_1(h, \boldsymbol{\delta})$ is of class C^∞ in $h \geq h_0$ at $h = h_0$. If the functions H, p and q are analytic in x and y, then $M_1(h, \boldsymbol{\delta})$ is also analytic in h at $h = h_0$.

(ii) If there exists a function $B_k(\boldsymbol{\delta}_0) \neq 0$ such that

$$M_1(h, \boldsymbol{\delta}) = B_k(\boldsymbol{\delta}_0) h^{k+1} + O(h^{k+2}) \quad \text{for } 0 < h - h_0 \ll 1,$$

then there exist $\varepsilon > 0$ and a neighborhood V of γ_{h_0} such that (1.15) has at most k limit cycles in V for $0 < \varepsilon < \varepsilon_0$ and $\boldsymbol{\delta}$ near $\boldsymbol{\delta}_0$.

In many cases (polynomial systems for example), the first-order Melnikov function may be written in the form of

$$M_1(h, \boldsymbol{\delta}) = \sum_{j=1}^{k} \delta_j I_j(h), \quad k \geq 2. \tag{1.18}$$

Theorem 1.2.6 *Let (1.18) hold. Suppose $W(h)$, $h - h_0 \ll 1$, is the Wronskian of the functions $I_1(h), I_2(h), \cdots, I_k(h)$. If $W(h_0) \neq 0$, then system (1.15) has at most $k-1$ limit cycles near γ_{h_0}.*

When $M_1(h) \equiv 0$, we need to consider higher-order Melnikov functions. Although higher-order Melnikov functions can be easily expressed as iterated integrals, it is not easy to compute them. In [42], Françoise developed a procedure to compute higher-order Melnikov functions. Assume that for system (1.15) H, p and q are polynomials in x and y, and H satisfies the so-called $*$-condition: for any polynomial one-form ω,

$$\text{if } \oint_{\gamma_h} \omega \equiv 0, \text{ then } \omega = rdH + dR,$$

where r and R are polynomials. We further assume that p and q can be expressed in the form of

$$\begin{aligned} p(x, y, \varepsilon, \boldsymbol{\delta}) &= p_1(x, y) + \varepsilon p_2(x, y) + \varepsilon^2 p_3(x, y) + \cdots, \\ q(x, y, \varepsilon, \boldsymbol{\delta}) &= q_1(x, y) + \varepsilon q_2(x, y) + \varepsilon^2 q_3(x, y) + \cdots. \end{aligned} \tag{1.19}$$

Then, Françoise's procedure [42] gives

Theorem 1.2.7 *Under the conditions (1.19), assume that for some $k \geq 2$, $M_1(h) = M_2(h) = \cdots = M_{k-1}(h) \equiv 0$ in (1.17). Let $\omega_j = q_j dx - p_j dy$, $j = 1, 2, \cdots, k$. Then*

$$M_k(h) = \oint_{\gamma_h} \Phi_k,$$

where

$$\Phi_1 = \omega_1, \quad \Phi_m = \omega_m + \sum_{i+j=m} r_i \omega_j, \ 2 \leq m \leq k,$$

and the functions r_i, $1 \leq i \leq k-1$ are determined successively from the representation $\Phi_i = dR_i + r_i dH$.

1.2.2 Normal form theory

Let us re-consider system (1.10) with $\alpha = 0$. Then system (1.10) has a Hopf singularity at the origin. For any positive integer m there exist two positive polynomials $Q_1(u, v)$ and $Q_2(u, v)$ of degree $2m + 1$ with

$$Q_1(u, v) = u + O(|u, v|^2), \quad Q_2(u, v) = v + O(|u, v|^2),$$

such that through transformations $x = Q_1(u, v)$ and $y = Q_2(u, v)$, system (1.10) can be transformed into

$$\begin{aligned} \dot{u} &= -\beta v + \sum_{j=1}^{m} (a_j u - b_j v)(u^2 + v^2)^j + O(|u, v|^{2m+2}), \\ \dot{v} &= \beta u + \sum_{j=1}^{m} (b_j u + a_j v)(u^2 + v^2)^j + O(|u, v|^{2m+2}), \end{aligned} \tag{1.20}$$

which is called the normal form of system (1.10) of order $2m + 1$. In polar coordinates, the normal form (1.20) can be rewritten as

$$\begin{aligned} \dot{r} &= a_1 r^3 + a_2 r^5 + \cdots + a_m r^{2m+1} + O(r^{2m+2}), \\ \dot{\theta} &= \beta + b_1 r^2 + b_2 r^4 + \cdots + b_m r^{2m} + O(r^{2m+1}). \end{aligned} \tag{1.21}$$

There is an equivalence relation between v_{2k+1} in (1.12) and a_k in (1.21): $a_k = 0 \Leftrightarrow v_{2k+1} = 0$ if $a_j = 0$, $v_{2j+1} = 0$, $1 \le j \le k - 1$. Then for system (1.10), a_k's can be used to study the center-focus problem as well as the cyclicity problem. So a_k is also called the kth order focus value of system (1.10) in the literature.

Note that the normal form (1.21) is derived from planar differential systems. Combined with center manifold theory, normal form theory can also be applied to higher-dimensional differential systems, to simplify the study of dynamic behaviors of orbits around a singular point. For Hopf singular points, the normal form of a higher-dimensional system still keeps the form of (1.21). Next, we will give a brief introduction to the center manifold theory and normal form theory in n-dimensional systems.

Without loss of generality, any n-dimensional analytic system can be written in the following form

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + O(|\boldsymbol{x}^T, \boldsymbol{y}^T, \boldsymbol{z}^T|^2), \\ \dot{\boldsymbol{y}} &= B\boldsymbol{y} + O(|\boldsymbol{x}^T, \boldsymbol{y}^T, \boldsymbol{z}^T|^2), \\ \dot{\boldsymbol{z}} &= C\boldsymbol{z} + O(|\boldsymbol{x}^T, \boldsymbol{y}^T, \boldsymbol{z}^T|^2), \end{aligned} \tag{1.22}$$

where $(\boldsymbol{x}^T, \boldsymbol{y}^T, \boldsymbol{z}^T)^T \in \mathbf{R}^n$, and the eigenvalues of matrices A, B and C have zero, negative and positive real parts, respectively. Then the corresponding eigenvectors of matrix $\mathrm{diag}(A, B, C)$ span three sub-spaces of $\mathbf{R}^n$: center eigenspace E^c, stable eigenspace E^s and unstable eigenspace E^u, and $\mathbf{R}^n = E^c \bigoplus E^s \bigoplus E^u$. According to center manifold theorem [59], there exist a center manifold W^c, a stable manifold W^s and a unstable manifold W^u, which are tangent to E^c, E^s and E^u at the origin O, respectively, see Figure 1.2.

For real applications, we usually assume that system (1.22) does not have the unstable manifold W^u. If the origin is a semisimple singularity, then system (1.22) becomes

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y}), \\ \dot{\boldsymbol{y}} &= B\boldsymbol{y} + \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y}), \end{aligned} \tag{1.23}$$

where A and B are diagonal matrices, $\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{y})$ are analytic functions starting at least from second-order terms.

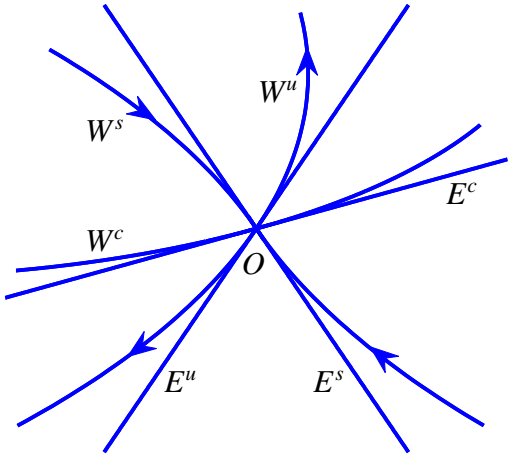

Figure 1.2: The stable, unstable and center manifolds for system (1.22).

Since the center manifold W^c is tangent to E^c at the origin, W^c can be expressed in terms of the variables of E^c in the neighborhood of the origin. Assume the center manifold W^c of system (1.23) is given by

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}), \quad |\boldsymbol{x}| \ll 1.$$

Then $\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0}$, $D\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0}$, and $\boldsymbol{h}(\boldsymbol{x})$ satisfies

$$B\boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x})) = D\boldsymbol{h}(\boldsymbol{x})(A\boldsymbol{x} + \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x}))).$$

Expanding the above equation in Taylor series and balancing the coefficients of corresponding terms, we can get the power series of $\boldsymbol{h}(\boldsymbol{x})$ up to any order we wish to obtain. In [85], the authors presented a method to compute center manifolds to any order, and showed how to analyze the errors of the analytical approximations.

Thus, system (1.23) restricted on the center manifold W^c is described by

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x})). \tag{1.24}$$

Further, to simplify system (1.24), we introduce a nonlinear transformation,

$$\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{u}) = \boldsymbol{u} + O(|\boldsymbol{u}|^2), \tag{1.25}$$

where $\boldsymbol{Q}(\boldsymbol{u})$ is a polynomial in $\boldsymbol{u}$ of degree $2m + 1$, resulting in a simple system,

$$\dot{\boldsymbol{u}} = A\boldsymbol{u} + \boldsymbol{C}(\boldsymbol{u}) + O(|\boldsymbol{u}|^{2m+2}). \tag{1.26}$$

where $\boldsymbol{C}(\boldsymbol{u})$ is a polynomial in $\boldsymbol{u}$ of degree $2m+1$, and $\boldsymbol{C}(\boldsymbol{u}) = O(|\boldsymbol{u}|^2)$. System (1.26) is obtained by eliminating as many terms as we can through the transformation (1.25), and it is called the normal form of system (1.23) of order $2m + 1$. The terms retained in the normal form (1.26) are called resonant terms.

1.3 Outline and Contributions

1.3.1 Outline of thesis

The thesis is outlined as follows.

Chapter 2 contains two parts. In the first part, we shall show that the result given in [18], which claims the existence of 11 small-amplitude limit cycles around a singular point in a particular cubic polynomial vector field by the second order Melnikov function, is not correct. Mistakes made in [18] leading to the erroneous conclusion have been identified. In fact, only 9 small-amplitude limit cycles can be obtained from this example after the mistakes are corrected, which agrees with the result obtained later by using the method of normal form computation [86].

In the second part of Chapter 2, we consider a cubic near-Hamiltonian system in the form

$$\dot{x} = y + a_1xy + a_2y^2 + \varepsilon P(x, y, \varepsilon), \quad \dot{y} = -x + x^2 - a_1/2y^2 + \varepsilon Q(x, y, \varepsilon), \tag{1.27}$$

where P and Q are cubic polynomials in x and y with coefficients depending analytically on the small parameter ε. It is proved that 10 small-amplitude limit cycles can bifurcate from an elementary center in system (1.27), for which up to 5th-order Melnikov functions are used. This demonstrates a good example in applying higher-order Melnikov functions to study bifurcation of limit cycles.

In Chapter 3, we consider two previously developed cubic systems in [19, 20], which have been proved to exhibit 11 small-amplitude limit cycles. Applying a different method, we not only prove the existence of the 11 limit cycles, but also show that one of the systems given by

$$\begin{aligned}\dot{x} &= 10x(8axy - 3x^2 - 9x - 12y^2 - 6),\\ \dot{y} &= 24a - 16ax + 90y + 15xy - 16axy^2 + 60y^3,\end{aligned}$$

can actually have 12 small-amplitude limit cycles around a singular point under suitable cubic perturbations. So $M(3) \geq 12$. This is the best result so far obtained in cubic planar vector fields around a singular point.

Chapter 4 is concerned with quadratic polynomial switching systems. A computationally efficient algorithm for computing Lyapunov constants of switching systems is developed. We apply this algorithm to the following switching Bautin system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y - a_3x^2 + (a_5 + a_2)xy + (a_6 + a_3)y^2 \\ x + \delta y + a_2x^2 + (a_4 - a_3)xy - a_2y^2 \end{pmatrix}, & y > 0, \\ \begin{pmatrix} \delta x - y - b_3x^2 + (b_5 + b_2)xy + (b_6 + b_3)y^2 \\ x + \delta y + b_2x^2 + (b_4 - b_3)xy - b_2y^2 \end{pmatrix}, & y < 0, \end{cases} \tag{1.28}$$

and determine its all possible center conditions by computing resultants of Lyapunov constants when $a_6b_6 = 0$. In addition, we prove that a switching Bautin system can have at least 10 small-amplitude limit cycles around a singular point.

Chapters 5 and 6 are devoted to computation of normal forms. Explicit, recursive formulae are presented for simultaneously computing the normal forms, center manifolds and nonlinear transformations for general n-dimensional systems, associated with Hopf and

semisimple singularities, respectively. Maple programs are developed based on the analytical formulae for Hopf bifurcation and semisimple cases with center manifolds of any dimension. The computational efficiency of the algorithm is demonstrated by examples.

There are very few results in the literature about bifurcation of limit cycles in higher-dimensional differential systems. As an application of our developed algorithm for computing normal forms, in Chapter 7, we study a simple 3-dimensional differential system, given by

$$\begin{aligned} \dot{x}_1 &= \alpha\, x_1 + x_2 + f_1(x_1, x_2, x_3), \\ \dot{x}_2 &= -x_1 + \alpha\, x_2 + f_2(x_1, x_2, x_3), \\ \dot{x}_3 &= -\beta\, x_3 + f_3(x_1, x_2, x_3), \end{aligned} \tag{1.29}$$

where α and $\beta > 0$ are parameters, and f_i's are quadratic polynomials. We obtain 7 small-amplitude limit cycles around the origin in system (1.29).

1.3.2 Contributions of thesis

This thesis contains contributions in both theoretical development and applications, mainly in three aspects.

(i) Obtained 10 small-amplitude limit cycles in a quadratic Hamiltonian system under cubic polynomials perturbation by using higher-order Melnikov functions, up to 5th order. This is an excellent example in demonstrating the powerful method of using higher-order Melnikov function method. The result of 10 small-amplitude limit cycle is a record in this direction of research. See Chapter 2.

(ii) The research of the candidate is focused on one special type of non-smooth dynamical systems called switching system, which has different definitions of continuous vector field in the two different regions divided by a straight line. Switching systems have recently been extensively studied by researchers from many different areas. Switching systems also serve as a rich source of open problems, see [47]. The particular problem addressed in this thesis is: under what condition does a quadratic polynomial switching system have a center? This is a fundamental problem and a complete answer is given in this system, and a better result on the number of limit cycles has been obtained. See Chapter 4.

(iii) Developed the computationally efficient algorithms and Maple programs for computing the center manifolds and normal forms of general n-dimensional systems, associated with Hopf singularity and semisimple singularity. These algorithms improved the computational efficiency of the existing algorithms, and the user-friendly Maple programs can be easily applied by those engineers and applied scientists to solve real world problems. See Chapters 5 and 6.

Bibliography

[1] M. Kline, Mathematical Thought from Ancient to Modern Times Vol. 2, Oxford University Press, New York, 1986.

[2] M. Brin, G. Stuck, Introduction to dynamical systems, Cambridge University Press, Cambridge, 2002.

[3] H. Poincaré, Mémoire sur les courbes définies par une équation différentielle, Chapter XI (Cycles limites), Joural de Mathématiques, 3rd series 7 (1881) 375–422; 8 (1882) 251–296.

[4] D. Hilbert, Mathematical problems, (M. Newton, Transl.) Bull. Amer. Math. 8 (1902) 437–479.

[5] Yu. S. Ilyashenko, S. Yakovenko, Finitely smooth normal forms of local families of diffeomorphismes and vector fields, Russ. Math. Surv. 46 (1991) 3–19.

[6] J. Écalle, Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac (French), Hermann, Paris, 1992.

[7] S. Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica, 23 (1980) 153–158.

[8] L. S. Chen, M. S. Wang, The relative position, and the number, of limit cycles of a quadratic differential system, Acta. Math. Sinica, 22 (1979) 751–758.

[9] P. Yu, M. Han, Four limit cycles from perturbing quadratic integrable systems by quadratic polynomials, Int. J. Bifurcation & Chaos, 22(10) (2012) 1250254 (28 pages).

[10] J. Li, Y. Liu, New results on the study of Z_q-equivariant planar polynomial vector fields, Qual. Theory Dyn. Syst., 9(1-2) (2010) 167–219.

[11] C. Li, C. Liu, J. Yang, A cubic system with thirteen limit cycles, J. Diff. Eqns. 246 (2009) 3609–3619.

[12] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Int. J. Bifurcation Chaos 13 (2003) 47-106.

[13] T. R. Blows, N. G. Lloyd, The number of limit cycles of certain polynomial differential equations, Proc. Roy. Soc. Edinburgh Sect. A 98 (1984) 215–239.

[14] N. Bautin, On the number of limit cycles appearing from an equilibrium point of the focus or center type under varying coefficients, Matem. Sb. 30 (1952) 181–196.

[15] H. Żołądek, Quadratic systems with center and their perturbations, J. Diff. Eqns. 109 (1994) 223–273.

[16] N. G. Lloyd, J. M. Pearson, Computing centre conditions for certain cubic systems, J. Comput. Appl. Math. 40(3) (1992) 323–336.

[17] H. Żołądek, Remarks on The classification of reversible cubic system with centers, Topol. Methods Nonlinear Anal. 8 (1996) 335–342.

[18] H. Żołądek, Eleven small limit cycles in a cubic vector field, Nonlinearity 8 (1995) 843–860.

[19] C. Christopher, Estimating limit cycle bifurcation from centers, in: D. Wang, Z. Zheng (Eds.), Trends in Mathematics: Differential Equations with Symbolic Computation, Birkhäuser Verlag Switzerland (2005) 23–35.

[20] Yu. L. Bondar, A. Sadovskii, On a Żołądek theorem, Differential Equations 44(2) (2008) 274–277.

[21] V. I. Arnold, Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields, Funct. Anal. Appl. 11 (1977) 85–92.

[22] M. Han, Bifurcation of limit cycles of planar systems, Handbook of Differential Equations, Ordinary Differential Equations, Vol. 3 (Eds. A. Canada, P. Drabek and A. Fonda), Elsevier, 2006.

[23] A. G. Khovansky, Real analytic manifolds with finiteness properties and complex Abelian integrals, Funct. Anal. Appl. 18 (1984) 119–128.

[24] A. N. Varchenko, An estimate of the number of zeros of an Abelian integral depending on a parameter and limiting cycles, Funct. Anal. Appl. 18 (1984) 98–108.

[25] D. Novikov, S. Yakovenko, Simple exponential estimate for the number of real zeros of complete Abelian integrals, Ann. Inst. Fourier 45 (1995) 897–927.

[26] Yu. S. Ilyashenko, S. Yakovenko, Double exponential estimate for the number of zeros of complete Abelian integrals and rational envelopes of linear ordinary differential equations with an irreducible monodromy group, Invent. Math. 121 (1995) 613–647.

[27] Yu. S. Ilyashenko, S. Yakovenko, Counting real zeros of analytic functions satisfying linear ordinary differential equations, J. Diff. Eqns. 126 (1996) 613–650.

[28] I. D. Iliev, The cyclicity of the period annulus of the quadratic Hamiltonian triangle, J. Diff. Eqns. 128 (1996) 309–326.

[29] Y. Zhao, Z. Liang, G. Liu, The cyclicity of period annulus of the quadratic Hamiltonian systems with non-Morsean point, J. Diff. Eqns. 162 (1996) 199–233.

[30] Y. Zhao, S. Zhu, Perturbations of the non-generic quadratic Hamiltonian vector fields with hyperbolic segment, Bull. Sci. Math. 125(2) (2001) 109–138.

[31] S.-N. Chow, C. Li, Y. Yi, The cyclicity of period annulus of degenerate quadratic Hamiltonian system with elliptic segment, Erg. Th. Dyn. Syst. 22(4) (2002) 1233–1261.

[32] E. Horozov, I. D. Iliev, On the number of limit cycles in perturbations of quadratic Hamiltonian systems, Proc. London Math. Soc. 69 (1994) 198–224.

[33] L. Gavrilov, The infinitesimal 16th Hilbert problem in the quadratic case, Invent. Math. 143 (2001) 449–497.

[34] Z. Zhang, C. Li, On the number of limit cycles of a class of quadratic Hamiltonian systems under quadratic perturbations, Adv. Math. 26(5) (1997) 445–460.

[35] G. S. Petrov, Elliptic integrals and their non-oscillatorness (Russian), Funct. Anal. Appl. 20 (1986) 46–49.

[36] L. Gavrilov, Nonoscillation of elliptic integrals related to cubic polynomials with symmetry of order three, Bull. Lond. Math. Soc. 30 (1998) 267–273.

[37] I. D. Iliev, High-order Melnikov functions for degenerate cubic Hamiltonians, Adv. Diff. Eqns. 1(4) (1996) 689–708.

[38] I. D. Iliev, On second order bifurcations of limit cycles, J. London Math. Soc. 58 (1998) 353–366.

[39] I. D. Iliev, On the limit cycles available from polynomial perturbations of the Bogdanov-Takens Hamiltonian, Israel J. Math. 115 (2000) 269–284.

[40] A. Buică, A. Gasull, J. Yang, The third order Melnikov function of a quadratic center under quadratic perturbations, J. Math. Anal. Appl. 331 (2007) 443–454.

[41] A. Jebrane, H. Zołądek, A note on higher order Melnikov functions, Qual. Theory Dyn. Syst. 6 (2) (2005) 273–287.

[42] J.-P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergod. Th. Dynam. Sys. 16 (1996) 87–96.

[43] A. Jebrane, P. Mardešić, M. Pelletier, A note on a generalization of Françoise's algorithm for calculating higher order Melnikov functions, Bull. Sci. Math. 128(9) (2004) 749–760.

[44] J.-P. Françoise, M. Pelletier, Iterated integrals, Gelfand-Leray residue and first return map, J. Dyn. Control Syst. 12(3) (2006) 357–369.

[45] A. F. Filippov, Differential Equation with Discontinuous Right-Hand Sides, Kluwer Academic, Netherlands, 1988.

[46] M. Kunze, Non-Smooth Dynamical Systems, Springer-Verlag, Berlin, 2000.

[47] O. Makarenkov, J. S. Lamb, Dynamics and bifurcations of nonsmooth systems: A survey, Physica D: Nonlinear Phenomena 241 (2012) 1826–1844.

[48] B. Coll, A. Gasull, R. Prohens, Degenerate Hopf bifurcation in discontinuous planar systems, J. Math. Anal. Appl. 253 (2001) 671–690.

[49] A. Gasull, J. Torregrosa, Center-focus problem for discontinuous planar differential equations, Int. J. Bifurcation Chaos 13 (2003) 1755–1765.

[50] X. Liu, M. Han, Hopf bifurcation for non-smooth Liénard systems, Int. J. Bifurcation Chaos 19 (7) (2009) 2401–2415.

[51] M. Han, W. Zhang, On Hopf bifurcation in non-smooth planar systems, J. Diff. Eqns. 248 (2010) 2399–2416.

[52] X. Chen, Z. Du, Limit cycles bifurcate from centers of discontinuous quadratic systems, Comput. Math. Appl. 59 (2010) 3836–3848.

[53] X. Chen, W. Zhang, Isochronicity of centers in a switching Bautin system, J. Diff. Eqns. 252 (2012) 2877–2899.

[54] J. Llibre, A. C. Mereu, Limit cycles for discontinuous quadratic differential systems with two zones, J. Math. Anal. Appl. 413(2) (2014) 763–775.

[55] Y. Tian, M. Han, Hopf bifurcation for two types of Liénard systems, J. Diff. Eqns. 251(4-5) (2011) 834–859.

[56] H. E. Nusse, J. A. Yorke, Border-collision bifurcations including "period two to period three" for piecewise smooth systems, Physica D 57 (1992) 39–57.

[57] M. D. Bernardo, C. J. Budd, A. R. Champneys, Grazing, skipping and sliding: analysis of the nonsmooth dynamics of the DC/DC buck converter, Nonlinearity 11 (1998) 859–890.

[58] C. J. Budd, Non-smooth dynamical systems and the grazing bifurcation, in: Nonlinear Mathematics and its Applications, Cambridge University Press, Cambridge, 1996 219–235.

[59] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, 4th edn, Springer-Verlag, New York, 1983.

[60] J. Hofbauer, J. W.-H. So, Multiple limit cycles for three dimensional Lotka-Volterra equations, Appl. Math. Lett. 7 (1994) 65–70.

[61] Z. Lu, Y. Luo, Two limit cycles in three-dimensional Lotka-Volterra systems, Comput. Math. Appl. 44 (2002) 51–66.

[62] M. Gyllenberg, P. Yan, Y. Wang, A 3-D competitive Lotka-Volterra system with three limit cycles: A falsification of a conjecture by Hofbauer and So, Appl. Math. Lett. 19 (2006) 1–7.

[63] M. Gyllenberg, P. Yan, Four limit cycles for a three-dimensional competitive Lotka-Volterra system with a heteroclinic cycle, Comput. Math. Appl. 58 (2009) 649–669.

[64] A. H. Nayfeh, Mehtods of normal forms, John Wiley, New York, 1993.

[65] S. N. Chow, C. Z. Li, D. Wang, Normal forms and bifurcations of planar vector fields, Cambridge University Press, Cambridge, 1994.

[66] J. A. Sanders, F. Verhulst, J. Murdock, Averaging methods in nonlinear dynamical systems, 2nd edn, Springer-Verlag, New York, 2007.

[67] H. Poincaré, Les methodes Nouvelles la Necanique Celeste, Gauthier-Villars, Paries, 3 vols. Goroff, D.L. (Trans.) New Methods of Celestial Mechanics, American Institute of Physics, 1993.

[68] J. Giné, X. Santallusia, On the Poincaré-Lyapunov constants and the Poincaré series, Appl. Math. (Warsaw) 28(1) (2001) 17–30.

[69] M. Han, P. Yu, Normal forms, Melnikov functions and bifurcations of limit cycles, Springer, London, 2012.

[70] P. Yu, Simplest normal forms of Hopf and generalized Hopf bifurcations, Int. J. Bifurcation Chaos 9(10) (1999) 1917-1939.

[71] P. Yu, Y. Yuan, An efficient method for computing the simplest normal forms of vector fields, Int. J. Bifurcation Chaos 13(1) (2003) 19-46.

[72] P. Yu, A. Y. T. Leung, The simplest normal form of Hopf bifurcation, Nonlinearity 16(1) (2003) 277-300.

[73] M. Gazor, P. Yu, Spectral sequences and parametric normal forms, J. Diff. Eqns. 252 (2012) 1003-1031.

[74] P. Yu, Computation of the normal forms via a perturbation technique, J. Sound Vib. 211(1) (1998) 19-38.

[75] P. Yu, A. Y. T. Leung, A perturbation method for computing the simplest normal forms of dynamical systems, J. Sound Vib. 261(1) (2003) 123-151.

[76] P. Yu, Symbolic computation of normal forms for resonant double Hopf bifurcations using multiple time scales, J. Sound Vib. 247(4) (2001) 615-632.

[77] P. Yu, Analysis on double Hopf bifurcation using computer algebra with the aid of multiple scales, Nonlinear Dynamics, 27 (2002) 19-53.

[78] P. Yu, A simple and efficient method for computing center manifold and normal forms associated with semisimple cases, Dyn. Contin. Discrete Impuls. Syst. Ser. B: Appl. Algorithms 10(2) (2003) 273-286.

[79] Z. Zhang, D. Ding, W. Huang, Z. Dong, Qualitative theory of differential equations (Trans.), American Mathematical Society, Providence, 1992.

[80] D. M. Grobman, Homeomorphism of systems of differential equations (Russian), Dokl. Akad. Nuak SSSR 128 (1959) 880–881.

[81] P. Hartman, On the local linearization of differential equations, Proc. Amer. Math. Soc. 14 (1963) 568–573.

[82] J. Hale, H. Koçak, Dynamics and bifurcations, Springer-Verlag, New York, 1991.

[83] P. Yu, Closed-Form conditions of bifurcation points for general differential equations, Int. J. Bifurcation Chaos 15(4) (2005) 1467–1483.

[84] M. Han, Bifurcation theory of limit cycles, Science Press, Beijing, 2013.

[85] R. M. Corless, X. Liu, K. O. Geddes, Computation of center manifolds, ORCCA Technical Report TR-00-15 (2000).

[86] P. Yu, M. Han, A study on Żołądek's example, J. Appl. Anal. Comput. 1 (2011) 143–153.

Chapter 2

Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system

2.1 Introduction

The well-known Hilbert's 16th problem [1] has been studied for more than one century, and the research on this problem is still very active today. To be more specific, consider the following planar system:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{2.1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent nth-degree polynomials of x and y. The second part of Hilbert's 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that system (2.1) can have. The progress in the solution of the problem is very slow. Even the simplest case $n = 2$ has not been completely solved, though in the early 1990's, [2] and [3] proved that $H(n)$ is finite for any given planar polynomial vector fields. For general quadratic polynomial systems, the best result is $H(2) \geq 4$, obtained more than 30 years ago [4, 5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [7, 8]. Note that the 13 limit cycles obtained in [7, 8] are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems. A comprehensive review on the study of Hilbert's 16th problem can be found in the survey article [9].

In order to help understand and attack Hilbert's 16th problem, the so called weakened Hilbert's 16th problem was posed by Arnold [10]. The problem is to ask for the maximal number of isolated zeros of the Abelian integral or Melnikov function:

$$M(h, \delta) = \oint_{H(x,y)=h} Q(x, y)\, dx - P(x, y)\, dy, \tag{2.2}$$

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where $H(x,y)$, $P(x,y)$ and $Q(x,y)$ are all real polynomials of x and y, and the level curves $H(x,y) = h$ represent at least a family of closed orbits for $h \in (h_1, h_2)$, and δ denotes the parameters (or coefficients) involved in Q and P. The weakened Hilbert's 16th problem itself is a very important and interesting problem, closely related to the study of limit cycles in the following near-Hamiltonian system [11]:

$$\dot{x} = H_y(x,y) + \varepsilon\, P(x,\, y), \qquad \dot{y} = -\, H_x(x,y) + \varepsilon\, Q(x,\, y), \tag{2.3}$$

where $0 < \varepsilon \ll 1$ is a small perturbation. Studying the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the first-order Melnikov function $M(h,\delta)$, given in (2.2).

When we focus on the maximum number of small-amplitude limit cycles, $M(n)$, bifurcating from an elementary center or an elementary focus, one of the best-known results is $M(2) = 3$, which was proved by Bautin in 1952 [12]. For $n = 3$, several results have been obtained (e.g. see [13, 14, 15]). Among them, in 1995 Żołądek [13] first constructed a rational Darboux integral to show the existence of 11 small-amplitude limit cycles in a cubic vector field, which was considered the best and was cited by many researchers in this field. The rational Darboux integral proposed by Żołądek [13] is given by

$$H_0 = \frac{f_1^5}{f_2^4} = \frac{(x^4 + 4x^2 + 4y)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^4}, \tag{2.4}$$

which in turn generates the dynamical system in the form of

$$\begin{aligned} \dot{x} &= x^3 + xy + 5x/2 + a, \\ \dot{y} &= -ax^3 + 6x^2y - 3x^2 + 4y^2 + 2y - 2ax, \end{aligned} \tag{2.5}$$

with the integrating factor $M = 20 f_1^4 f_2^{-5}$.

For $a < -2^{5/4}$, system (2.5) has a center $C_0 = (-a/2, -a^2/4 - 1/2)$ and five (real or complex) critical points $(r, -r^2 - 5/2 - a/r)$, where r satisfies the polynomial equation $r^5 - 10r - 4a = 0$. In addition, system (2.5) has a saddle point and a non-elementary critical point at infinity. Let $h_0 = H_0(C_0) = -2/a$. Around C_0, there exists a family of periodic orbits given by $\gamma_h : H_0(x,y) = h$, $0 < h - h_0 \ll 1$. γ_h approaches C_0 as $h \to h_0^+$.

Recently, Yu & Han [14] applied a different method to system (2.5) and only got 9 small-amplitude limit cycles. This difference motivated us to reconsider the Żołądek's example and find that the result in [13] is not correct. In the next section, we shall present a detailed analysis on the Żołądek's example and point out where the mistakes were made in the paper [13].

In the second part, we will present an example to demonstrate the use of higher-order Melnikov functions to obtain 10 small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with 3rd-degree polynomial functions. In general, a perturbed quadratic Hamiltonian system can be described by

$$\begin{aligned} \dot{x} &= y + a_1 xy + a_2 y^2 + \varepsilon P(x,y,\varepsilon), \\ \dot{y} &= -x + x^2 - \frac{1}{2} a_1 y^2 + \varepsilon Q(x,y,\varepsilon), \end{aligned} \tag{2.6}$$

where P and Q are nth-degree polynomials of x and y with coefficients depending analytically on the small parameter ε. When $\varepsilon = 0$, system (2.6) has a cubic Hamiltonian,

$$H(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{3}x^3 + \frac{1}{2}a_1xy^2 + \frac{1}{3}a_2y^3, \tag{2.7}$$

and its parameters a_1 and a_2 take values from the set,

$$\Omega = \left\{-1 \le a_1 \le 2,\ 0 \le a_2 \le (1-\frac{a_1}{2})\sqrt{1+a_1}\right\}.$$

The Hamiltonian given in (2.7) is actually the so-called normal form [16] for all quadratic Hamiltonian systems which have a center at the origin. There exists a family of closed ovals around the origin given by $\Gamma_h : H(x,y) = h$, $h \in (0, \frac{1}{6})$.

For any $h \in (0, \frac{1}{6})$ the displacement function $d(h,\epsilon)$ of system (2.6) has a representation

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots, \tag{2.8}$$

where $M_i(h)$ is called the ith-order Melnikov function, particularly the higher-order Melnikov functions if $i \ge 2$. Then, we may determine the number of the limit cycles of system (2.6) emerging from the closed ovals $\{\Gamma_h\}$ by studying the zeros of the first non-vanishing Melnikov function $M_i(h)$ in $h \in (0, \frac{1}{6})$.

Suppose $M_1(h) \not\equiv 0$ in (2.8). Denote $Z(n)$ the sharp upper bound of the number of zeros of $M_1(h)$. Gavrilov [17] proved $Z(2) = 2$ for the Hamiltonian H with four distinct critical values (in a complex domain). Horozov & Iliev [18] obtained a linear estimate $Z(n) \le 5(n+3)$. Also, some sharp upper bounds were given for some particular cubic Hamiltonian: $n-1$ for the Bogdanov-Takens Hamiltonian, $H = \frac{1}{2}(x^2+y^2) - \frac{1}{3}x^3$ (see [19]), and $[\frac{2}{3}(n-1)]$ for the Hamiltonian triangle, $H = \frac{1}{2}(x^2+y^2) - \frac{1}{3}x^3 + xy^2$ (see [20]).

Moreover, for the Bogdanov-Takens Hamiltonian, there are some results on the upper bound of the number of zeros of the first nonvanishing higher-order Melnikov functions $M_k(h)$. Li & Zhang [21] got a sharp upper bound $2n-2$ if n is even or $2n-3$ if n is odd for $k = 2$. Iliev [22] obtained a sharp upper bound $3n-4$ for $k = 3$, by applying the Françoise's procedure [23] for computing higher-order Melnikov functions. The higher-order Melnikov functions can be also easily expressed via iterated integrals, which will be seen in the next section.

In this chapter, we study the number of small-amplitude limit cycles in (2.6) bifurcating from the origin, using higher-order Melnikov functions. Hereafter we suppose P and Q are cubic polynomials with the following forms,

$$\begin{aligned} P(x,y,\varepsilon) &= \sum_{m=1}^{\infty} \varepsilon^{m-1} P_m(x,y) \quad \text{with } P_m(x,y) = \sum_{i+j=1}^{3} a_{ijm}x^iy^j, \\ Q(x,y,\varepsilon) &= \sum_{m=1}^{\infty} \varepsilon^{m-1} Q_m(x,y) \quad \text{with } Q_m(x,y) = \sum_{i+j=1}^{3} b_{ijm}x^iy^j. \end{aligned} \tag{2.9}$$

Our main result is given below, and its proof will be given in Section 2.4.

Theorem 2.1.1 *Let the functions P and Q in (2.6) be given by (2.9). Then system* (2.6) *can have* $[\frac{4k}{3}]+4$ *small-amplitude limit cycles around the origin, when* $M_k(h)$ *is the first non-vanishing Melnikov function in* (2.8)*,* $1 \le k \le 5$.

Remark 1. It follows from Theorem 2.1.1 that 10 small-amplitude limit cycles exist in the vicinity of the origin of system (2.6) when $k = 5$, i.e., $M(3) \ge 10$.

The rest of the chapter is organized as follow. In the next section, we consider the Żołądek's example [13], and show that the result given in [13] is not correct. In Section 2.3, we present some preliminary results for polynomial one-forms with respect to the Hamiltonian (2.7). Then, in Section 2.4 by choosing special forms for the polynomials P and Q without loss of generality, we prove Theorem 2.1.1. Finally, conclusion is drawn in Section 2.5.

2.2 Żołądek's example

In this section, we consider the Żołądek's example, described by (2.4) and (2.5), and briefly describe the method used in [13]. Suppose the perturbed system of (2.5) is described by

$$\begin{aligned} \dot{x} &= M^{-1}H_{0y} + \varepsilon p(x, y, \varepsilon), \\ \dot{y} &= -M^{-1}H_{0x} + \varepsilon q(x, y, \varepsilon), \end{aligned} \tag{2.10}$$

where $p(x, y, \varepsilon)$ and $q(x, y, \varepsilon)$ are polynomials of x and y with coefficients depending analytically on the small parameter ε and $\max(\deg(p), \deg(q)) \le 3$.

Let S be a section transversal to the closed orbit γ_h. Using $H_0 = h$ as a parameter, $0 < h - h_0 \ll 1$, we define the Poincaré map $\mathcal{P}(h, \varepsilon)$ of system (2.10), and thus the corresponding displacement function, $d(h, \varepsilon) = \mathcal{P}(h, \varepsilon) - h$ has the form

$$d(h, \varepsilon) = \varepsilon \int_{L(h,\varepsilon)} M(qdx - pdy) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + O(\varepsilon^3), \tag{2.11}$$

where $L(h, \varepsilon)$ is a trajectory of the perturbed system (2.10). We can use the first non-vanishing Melnikov function $M_k(h)$ in (2.11) to investigate the number of the limit cycles around the center C_0. Generally, the zeros of $M_k(h)$ correspond to the limit cycles of system (2.10).

Let $\varpi = qdx - pdy$, $\deg(\varpi) = \max(\deg(p), \deg(q))$. Then, the first-order Melnikov function $M_1(h)$ can be expressed in the form of

$$M_1(h) = \oint_{\gamma_h} M\varpi|_{\varepsilon=0} = h \oint_{\gamma_h} \frac{\varpi}{f_1 f_2}\Big|_{\varepsilon=0}.$$

When $M_1(h) \equiv 0$, we may use an iterated integral to express the second-order Melnikov function $M_2(h)$. The first integral of system (2.10) can be approximated as $H_\varepsilon = H_0 - \varepsilon H_1$, where the function $H_1(B)$ is the integral $H_1(B) = \int_A^B M\varpi|_{\varepsilon=0}$, computing along the γ_h, with $A = \gamma_h \bigcap S$ and $B \in \gamma_h$. Thus, for system (2.10) we have the second-order Melnikov function, given by

$$M_2(h) = \frac{d}{d\varepsilon}\Big(\int_{H_\varepsilon = h} M\varpi \Big)\Big|_{\varepsilon=0}. \tag{2.12}$$

Suppose that the polynomials p and q are expanded as

$$p(x,y,\varepsilon) = p_1(x,y) + \varepsilon p_2(x,y) + O(\varepsilon^2),$$
$$q(x,y,\varepsilon) = q_1(x,y) + \varepsilon q_2(x,y) + O(\varepsilon^2).$$

Further, let $\varpi_i = q_i dx - p_i dy$, $i = 1, 2$. Then (2.12) can be rewritten as

$$M_2(h) = \frac{d}{d\varepsilon}\Big(\int_{H_\varepsilon=h} M\varpi_1\Big)\Big|_{\varepsilon=0} + \oint_{\gamma_h} M\varpi_2 = \oint_{\gamma_h} \frac{d(M\varpi_1)}{dH_0} H_1 + h\oint_{\gamma_h} \frac{\varpi_2}{f_1 f_2}, \tag{2.13}$$

where $\frac{d(M\varpi_1)}{dH_0}$ is the Gelfand-Leray form (see [24]).

In [13], the author studied small-amplitude limit cycles of system (2.10), bifurcating from the center C_0, by using the second-order Melnikov function $M_2(h)$. More precisely, for (2.13), the author chose twelve Abelian integrals $I_{\omega_i}(h) = \oint_{\gamma_h} \omega_i/(f_1 f_2)$, $i = 1, \ldots, 12$, where one-forms ω_i are given as follows:

$$\omega_k = x^{k-1}dx,\ k = 1,2,3,4, \quad \omega_5 = (18x^2 + 18y + 5)dx, \quad \omega_6 = xydx,$$
$$\omega_7 = x^2ydx, \quad \omega_8 = xy^2dx, \quad \omega_9 = y^3dx, \quad \omega_{10} = xy^2dy, \quad \omega_{11} = y^3dy,$$
$$\omega_{12} = y^2(5 - 3x^2)dx + xy(x^2 + 1)dy.$$

Then, by showing the independency of the integrals $I_{\omega_i}(h)$, the author claimed that 11 small-amplitude limit cycles could bifurcate from the center C_0 after suitable cubic perturbations.

Later, system (2.10) was re-investigated by using the method of focus values computation [14]. Based on the computation of ε-order and ε^2-order focus values, the authors of [14] showed that system (2.10) has at most 9 small-amplitude limit cycles bifurcating from the center C_0. This obvious difference raises a question: which conclusion is correct? If the result of 11 limit cycles is not correct, then what possible mistakes were made in the article [13]? In the following, we will answer these questions.

In [13], a key part in the proof of the existence of 11 limit cycles is the lemma in Section 5.1, which states that the eleven integrals, $I_{\omega_j}(h)$, $j = 1, \ldots, 11$, form a basis of the linear space of integrals $I_\omega(h) = \oint_{\gamma_h} \omega/(f_1 f_2)$, $\deg(\omega) \le 3$. We will show that this is not true. Firstly, we find the relation $aI_{\omega_4}(h) - I_{\omega_5}(h) = 0$, showing that $I_{\omega_4}(h)$ and $I_{\omega_5}(h)$ are linearly dependent. This can be seen from the proof of the lemma [13], where the author obtained nine one-forms η_j, $j = 1, \ldots, 9$ such that $I_{\eta_j}(h) = 0$, where

$$\eta_1 = (x^3 + 2x)dx + dy,$$
$$\eta_2 = (-3ax^2 + 12xy - 6x - 2a)dx - (3x^2 + y + 5/2)dy,$$
$$\eta_3 = (6x^2 + 8y + 2)dx - xdy,$$
$$\eta_4 = (-3ax^3 + 12x^2y - 6x^2 - 2ax)dx - (2x^3 - a)dy,$$
$$\eta_5 = (ax^3 + 3x^2 + 4y^2 + 2ax)dx - xydy,$$
$$\eta_6 = (-ax^3 + 6x^2y - 3x^2 + 4y^2 + 2y - 2ax)dx - (x^3 + xy + 5x/2 + a)dy,$$
$$\eta_7 = (3ax^2y - 12xy^2 + 6xy + 2ay)dx - (3x^2y - ax^3 - 3x^2 + 3y^2 - y/2 - 2ax)dy,$$
$$\eta_8 = (-5x^3 - 7xy + x/2 + a)dx + x^2dy,$$
$$\eta_9 = (21xy/2 - 7xy^2 + ay)dx + (2x^2y - 3x^2/2 + ax + y)dy.$$

It is easy to show that $(3a\eta_1-5\eta_3-\eta_4)/2-\eta_5+\eta_6 = a\omega_4-\omega_5$, which yields $aI_{\omega_4}(h)-I_{\omega_5}(h) = 0$. Secondly, we find another one-form η_{10}, given by

$$\eta_{10} = [-\tfrac{29}{3}ax^3 - \tfrac{8}{3}y^3 - (2a^2 - \tfrac{5}{2})x^2 - 9axy + 6y^2 + \tfrac{13}{6}ax + a^2]dx + xy^2dy,$$

such that the integral $I_{\eta_{10}}(h)$ vanishes near h_0. Therefore, based on these ten one-forms η_j, $1 \le j \le 10$, we can remove $I_{\omega_5}(h)$ from the basis without adding another integral. Thus, the number of integrals in the basis claimed in [13] should be one less.

Next, consider the integral $I_{\omega_{12}}(h)$, where $\omega_{12} = y^2(5-3x^2)dx+xy(x^2+1)dy$, which was used in [13] when the second-order Melnikov function was considered. Obviously, $\deg(\omega_{12}) = 4$. In Remark 7 of [13], the author showed that $I_{\omega_{12}}(h)$ could appear in the second-order Melnikov function $M_2(h)$ of system (2.10), under a suitable perturbation.

However, $I_{\omega_{12}}(h)$ has no contribution to generate small-amplitude limit cycles in the vicinity of C_0, since for x^2y^2dx and x^3ydy in ω_{12}, we can show that there are two one-forms ξ_1 and ξ_2, given by

$$\begin{aligned}
\xi_1 =& [x^2y^2 - \frac{1}{36}a(a^2+144)x^3 + (a^2+3)x^2y + \frac{11}{6}axy^2 - \frac{4}{9}y^3 + \frac{1}{24}(3a^4-28a^2+46)x^2 \\
&+ \frac{1}{6}a(6a^2-7)xy + \frac{1}{6}(3a^2+4)y^2 + \frac{1}{72}a(9a^4+36a^2+77)x + \frac{1}{12}(3a^4-10a^2+12)y \\
&+ \frac{1}{288}(9a^6+36a^4+100a^2+80)]dx, \\
\xi_2 =& [\frac{1}{48}a(a^2-60)x^3 - \frac{3}{2}(a^2+4)x^2y - \frac{19}{4}axy^2 + \frac{4}{3}y^3 - \frac{1}{16}(3a^4+70a^2-16)x^2 \\
&- \frac{1}{2}a(3a^2-4)xy - (3a^2+2)y^2 + \frac{1}{48}a(3a^4-36a^2+137)x - \frac{1}{16}(15a^4+2a^2+48)y \\
&+ \frac{1}{32}a^4 + \frac{1}{48}a^2 - \frac{5}{6}]dx + [x^3y + \frac{1}{4}(a^2+2)x^3 + \frac{3}{2}ax^2y + \frac{3}{8}a(a^2+2)x^2 + \frac{3}{4}a^2xy \\
&+ \frac{3}{16}a^2(a^2+2)x + \frac{1}{8}a^3y + \frac{1}{32}a^3(a^2+2)]dy,
\end{aligned}$$

satisfying $I_{\xi_1}(h) = I_{\xi_2}(h) = 0$ for $0 < h - h_0 \ll 1$. This implies that $I_{\omega_{12}}(h)$ can be expressed as a linear combination of integrals $I_\omega(h)$, $\deg(\omega) \le 3$, for h near h_0.

Summarizing the above results shows that $I_{\omega_5}(h)$ and $I_{\omega_{12}}(h)$ can be removed from the basis, since they can be expressed as linear combinations of the other elements in the basis. Thereby, now there are only ten of the integrals chosen in [13] left. This clearly indicates that at most 9 (not 11) small-amplitude limit cycles may appear in the vicinity of the center C_0.

In [13], having obtained the twelve integrals $I_{\omega_j}(h)$, $1 \le j \le 12$, in order to show the existence of 11 limit cycles, the author tried to prove the independency of the twelve integrals. In the first place, the author showed that $I_{\omega_{11}}(h)$ is independent of the other integrals $I_{\omega_j}(h)$, $j \ne 11$, by considering the behavior of the integrals at $h = 1$. In order to prove the independency of the remaining integrals $I_{\omega_j}(h)$, $j \ne 11$, the author considered the integrals I_ω as functions of two variables $h \in \mathbf{C}$ and $a \in \mathbf{C}$. With prolonging $I_\omega(h,a)$ to the point $a = 0$, the author used the independency of the integrals, $I_{\omega_j}(h,a)$, $j \ne 11$, for a close to 0, to determine their

independence for generic $a \in \mathbf{C}$. The closed orbit γ_h has the form

$$\begin{aligned}\gamma_h &= \left\{(x,y) : x = \epsilon e^{i\theta},\ y = -\frac{1}{2} + \frac{\epsilon}{5}e^{-i\theta} + O(\epsilon^2),\ \theta \in [0, 2\pi]\right\} \\ \epsilon &= (-h)^{-1/8}2^{-5/8},\end{aligned} \tag{2.14}$$

for $a = 0$ and h is close to the critical value $h_0 = -2/a = \infty$.

In addition, the author introduced new variables u and v in the following form,

$$u = \frac{1}{H_0^{1/4}x}, \quad H_0 v^4 = 1 + \frac{4}{x^2} + \frac{4y}{x^4}.$$

Let $K_{i,j} = \int_{\delta_h} u^i v^j dv$, where δ_h is the image of the closed orbit γ_h defined in (2.14) under the change of variables. In Lemma 3 of [13], the integrals $I_{\omega_j}(h, 0)$ and some partial derivatives with respect to a, for h close to h_0, are expressed in terms of h and $K_{i,j}$. Using these expressions and eleven independent functions: $1, \tau - 1, (\tau - 1)/\tau, g_{i,j} = h^{1/4}K_{i,j}(h)$, $(i, j) = (-1, 0), (-1, -1)$, $(-2, 0), (-2, -1), (1, 0), (1, -1)$ and $h^{1/2}K_{-2,2}, h^{1/4}K_{-3,2}$, the author claimed the independency of integrals $I_{\omega_j}(h, a)$, $j \neq 11$ for a close to 0. Especially, the expressions for I_{ω_5} and its derivatives with respect to a in Lemma 3 were given by

$$\begin{aligned}&I_{\omega_5}(h, 0) = 0, \\ &\frac{\partial I_{\omega_5}}{\partial a}(h, 0) = I_{\omega_4}(h, 0), \\ &\frac{\partial}{\partial a}\Big(\frac{\partial I_{\omega_5}}{\partial a} - I_{\omega_4}\Big)(h, 0) = \frac{9}{56}h^{1/2}K_{-2,2} + \sum_{j=-2}^{0} \alpha_j h^{-1/2}K_{-2,j}.\end{aligned} \tag{2.15}$$

Based on the third equation of (2.15), the author claimed the independence of I_{ω_5} from other integrals. But this is not correct since we have already shown that $aI_{\omega_4}(h) - I_{\omega_5}(h) = 0$, implying that the third equation of (2.15) should be replaced by

$$\frac{\partial}{\partial a}\Big(\frac{\partial I_{\omega_5}}{\partial a} - I_{\omega_4} - a\frac{\partial I_{\omega_4}}{\partial a}\Big)(h, 0) \equiv 0,$$

which is the correct second-order derivative of the function $F(h, a) = I_{\omega_5}(h) - aI_{\omega_4}(h)$. For the integral $I_{\omega_{12}}(h)$, the author made a similar error in the proof of its independence from other integrals.

2.3 Cubic Hamiltonians with cubic perturbations

In order to prove Theorem 2.1.1, we need some preliminary results for cubic Hamiltonians with cubic perturbations. Let $\omega_{ij} = x^i y^j dx$ and $\sigma_{ij} = x^i y^j dy$.

Lemma 2.3.1 *For the cubic Hamiltonian given in* (2.7)*, the following identities hold.*

$$
\begin{aligned}
(a)\ \sigma_{ij} &= \frac{1}{j+1}\, d(x^i y^{j+1}) - \frac{i}{j+1}\omega_{i-1,j+1};\\
(b)\ \omega_{ij} &= \omega_{i-1,j} + \frac{j-2i+4}{2j+4}a_1\omega_{i-2,j+2} - \frac{i-2}{j+2}\omega_{i-3,j+2} - \frac{i-2}{j+3}a_2\omega_{i-3,j+3}\\
&\quad - x^{i-2}y^j dH + d\Big(\frac{1}{j+2}x^{i-2}y^{j+2} + \frac{a_1}{j+2}x^{i-1}y^{j+2} + \frac{a_2}{j+3}x^{i-2}y^{j+3}\Big),\ i \geq 2;\\
(c)\ \omega_{0,j} &= \frac{3j}{a_2(j+1)}\Big[H\omega_{0,j-3} - \frac{1}{6}\omega_{1,j-3} - \frac{a_1(j-3)+6j-2}{12(j-1)}\omega_{0,j-1} - \frac{a_1(j+1)}{3(j-1)}\omega_{1,j-1}\\
&\quad + r_{0,j}(x,y)dH + dR_{0,j}(x,y)\Big],\ j \geq 3;
\end{aligned}
$$

$$
\begin{aligned}
(d)\ \omega_{1,j} &= \frac{3j}{a_2(j+2)}\Big[H\omega_{1,j-3} - \frac{(j+2)a_1^2}{6(j+1)}\omega_{0,j+1} + \frac{a_2}{6j}\omega_{0,j} - \frac{a_1(5j+3)+6j+2}{12(j-1)}\omega_{1,j-1}\\
&\quad - \frac{a_1 j - 3a_1 - 2}{12(j-1)}\omega_{0,j-1} - \frac{1}{6}\omega_{1,j-3} + r_{1,j}(x,y)dH + dR_{1,j}(x,y)\Big],\ j \geq 3;
\end{aligned}
$$

where $r_{i,j}(x,y)$ and $R_{i,j}(x,y)$ are polynomials in x and y with degrees $i+j-2$ and $i+j+1$, respectively.

Proof A direct calculation using integration by parts yields formula (a). From the Hamiltonian, we have the equation $\frac{1}{3}x^3 = \frac{1}{2}(x^2+y^2) + \frac{1}{2}a_1xy^2 + \frac{1}{3}a_2y^3 - H$, giving

$$x^2dx = xdx + ydy + \frac{a_1}{2}y^2dx + a_1xydy + a_2y^2dy - dH,$$

which yields

$$\omega_{i,j} = \omega_{i-1,j} + \sigma_{i-2,j+1} + \frac{a_1}{2}\omega_{i-2,j+2} + a_1\sigma_{i-1,j+1} + a_2\sigma_{i-2,j+2} - x^{i-2}y^j dH,\ i \geq 2. \tag{2.16}$$

Then, combining (2.16) with the formula (a) results in the formula (b).

Similarly, the equation $\frac{1}{3}a_2y^3 = H - \frac{1}{2}(x^2+y^2) + \frac{1}{3}x^3 - \frac{1}{2}a_1xy^2$ generates

$$\frac{1}{3}a_2\omega_{i,j} = H\omega_{i,j-3} - \frac{1}{2}\omega_{i+2,j-3} - \frac{1}{2}\omega_{i,j-1} + \frac{1}{3}\omega_{i+3,j-3} - \frac{1}{2}a_1\omega_{i+1,j-1},\ j \geq 3. \tag{2.17}$$

Finally, the formulas (c) and (d) follow the formula (b) and (2.17).

From Lemma 2.3.1, we know that any polynomial one-form ω, $\deg(\omega) = m$, can be expressed in the form of

$$\omega = r(x,y)dH + dR(x,y) + \sum_{i=0,1}\sum_{j=0}^{m-i}\alpha_{i,j}\omega_{i,j}.$$

The next lemma shows that there also exist some relationships among the one-forms $\omega_{i,j}$, $i = 0, 1$.

Lemma 2.3.2 *For any non-negative integer $m \bmod 3 \neq 2$, there exist $\beta_{i,j,m}$, $\widetilde{r}_m(x,y)$ and $\widetilde{R}_m(x,y)$ satisfying the following identity*

$$\sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j,m}\omega_{i,j} = \widetilde{r}_m(x,y)dH + d\widetilde{R}_m(x,y), \tag{2.18}$$

where $\widetilde{R}_m(x,y)$ and $\widetilde{r}_m(x,y)$ are polynomials of degrees $m+1$ and $m-1$ in x and y, respectively; and $\beta_{i,j,m}$ are polynomials in a_1 and a_2, with $\beta_{0,0,0} = \beta_{1,0,1} = 1$, $\beta_{0,1,1} = 0$, and

$$\begin{aligned}\beta_{0,m+3,m+3} =& \frac{m+4}{3(m+3)}(a_2\beta_{0,m,m} + \frac{a_1^2}{2}\beta_{1,m-1,m}),\\ \beta_{1,m+2,m+3} =& \frac{m+4}{3(m+2)}(a_1\beta_{0,m,m} + a_2\beta_{1,m-1,m}),\end{aligned} \tag{2.19}$$

if $\beta_{1,-1,0}$ is defined as $\beta_{1,-1,0} = 0$.

Proof We use the method of mathematical induction to prove this lemma. It is easy to see that the conclusion is true for $m = 0, 1$. Now, suppose (2.18) holds for $m \bmod 3 \neq 2$. Then, we prove that (2.18) also holds for $m+3$. Multiplying (2.18) by H on both sides yields

$$\sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j,m}H\omega_{i,j} = H\widetilde{r}_m dH + Hd\widetilde{R}_m. \tag{2.20}$$

The right-hand side of (2.20) can be rewritten as

$$H\widetilde{r}_m dH + Hd\widetilde{R}_m = (H\widetilde{r}_m - \widetilde{R}_m)dH + d(H\widetilde{R}_m). \tag{2.21}$$

For the left-hand side of (2.20), it follows from the formulas (c) and (d) in Lemma 2.3.1 that

$$\begin{aligned}&H\omega_{i,j} = \xi_{i,j+3} + \eta_{i,j+3},\ i+j<m,\\ &H\omega_{0,m} = \frac{a_2(m+4)}{3(m+3)}\omega_{0,m+3} + \frac{a_1(m+4)}{3(m+2)}\omega_{1,m+2} + \eta_{0,m+3},\\ &H\omega_{1,m-1} = \frac{a_1^2(m+4)}{6(m+3)}\omega_{0,m+3} + \frac{a_2(m+4)}{3(m+2)}\omega_{1,m+2} + \eta_{1,m+2},\ m>0,\end{aligned} \tag{2.22}$$

where $\eta_{i,j} = r_{i,j}dH + dR_{i,j}$, and $\xi_{i,j}$ is a one-form with $\deg(\xi_{i,j}) \le i+j$. Then, substituting (2.22) into the left-hand side of (2.20) yields

$$\begin{aligned}\sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j}H\omega_{i,j} =& \frac{m+4}{3(m+3)}(a_2\beta_{0,m,m} + \frac{a_1^2}{2}\beta_{1,m-1,m})\omega_{0,m+3}\\ &+ \frac{m+4}{3(m+2)}(a_1\beta_{0,m,m} + a_2\beta_{1,m-1,m})\omega_{1,m+2}\\ &+ \sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j}(\xi_{i,j+3} + \eta_{i,j+3}).\end{aligned} \tag{2.23}$$

Finally, combining (2.23) with (2.20) and (2.21) shows that the conclusion is also true for $m+3$.

The proof of the lemma is complete.

Noting that $\beta_{0,0,0} = \beta_{1,0,1} = 1$, $\beta_{1,-1,0} = \beta_{0,1,1} = 0$, we know from (2.19) that $\beta_{k,m-k,m}$ in Lemma 2.3.2 are polynomials in a_1 and a_2 with positive coefficients for $m \bmod 3 = k$, $k < 2$. Thus, $\omega_{k,m-k}$ can be expressed in terms of other one-forms $\omega_{i,j}$, $i + j \le m$ and $r_m dH + dR_m$. This gives the following lemma.

Lemma 2.3.3 *Any polynomial one-form ω of degree m can be expressed as*

$$\omega = r(x, y)dH + dR(x, y) + \sum_{i=0,1} \sum_{j \bmod 3 \neq 0}^{1 \le j \le m-i} \alpha_{ij}\omega_{ij}, \tag{2.24}$$

where $R(x, y)$ and $r(x, y)$ are polynomials of degrees $m + 1$ and $m - 1$ in x and y, respectively.

Now, it follows from (2.24) that

$$M(h) = \oint_{\Gamma_h} \omega = \sum_{i=0,1} \sum_{j \bmod 3 \neq 0}^{1 \le j \le m-i} \alpha_{ij} \oint_{\Gamma_h} \omega_{ij}, \tag{2.25}$$

that is, any Melnikov function $M(h) = \oint_{\Gamma_h} \omega$, $\deg(\omega) = m$, can be expressed as a linear combination of integrals $I_{ij}(h) = \oint_{\Gamma_h} \omega_{ij}$, $i = 0, 1$, $j \bmod 3 \neq 0$. A reasonable expectation is that the integrals $I_{i,j}(h)$ form a basis for the linear space of Melnikov functions $M(h) = \oint_{\Gamma_h} \omega$. Actually, it will be seen in the next section that the space of Melnikov functions $M(h)$ could be Chebyshev with accuracy at least 2. So the number of limit cycles in system (2.6) is not determined by the number of elements in the basis. Further, the coefficients $\alpha_{i,j}$ in (2.25) could become very complicated when $M(h)$ is a higher-order Melnikov function of system (2.6). In this case, it is really not easy to prove the independency of $\alpha_{i,j}$s, which is the second big obstacle in the use of the independency of the integrals $I_{i,j}(h)$ to determine the number of limit cycles.

To overcome the above mentioned difficulty, we turn to an alternative, which decreases the complexity in computing $M(h)$ by (2.24), but it still does not solve the problem of independency of basis. Let $\omega_j = Q_j(x, y)dx - P_j(x, y)dy$. Then, for higher-order Melnikov functions of system (2.6), we have the following result.

Lemma 2.3.4 *(cf. [22, 23]) Let* (2.9) *hold. Assume that for some $k \ge 2$, system* (2.6) *has*

$$M_m(h) = \oint_{\Gamma_h} r_m dH + dR_m \equiv 0, \ 1 \le m \le k - 1. \tag{2.26}$$

Then,

$$\begin{aligned} M_k(h) &= \oint_{\Gamma_h} (\omega_k + \sum_{i+j=k} r_i\omega_j), \\ r_m dH + dR_m &= \omega_m + \sum_{i+j=m} r_i\omega_j, \ 1 \le m \le k - 1. \end{aligned} \tag{2.27}$$

Proof We prove this lemma by using the method of mathematical induction. First, write system (2.6) in the Pfaffian form,

$$dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots = 0. \tag{2.28}$$

Multiplying (2.28) by $1 + \varepsilon r_1 + \ldots + \varepsilon^{k-1} r_{k-1}$ and combing the like terms yield

$$\begin{aligned} &dH + \varepsilon(r_1 dH - \omega_1) + \varepsilon^2(r_2 dH - r_1\omega_1 - \omega_2) + \cdots \\ &\quad + \varepsilon^k(-r_{k-1}\omega_1 - \cdots - r_1\omega_{k-1} - \omega_k) + O(\varepsilon^{k+1}) = 0, \end{aligned}$$

which, by using (2.27), can be written as

$$dH - \varepsilon dR_1 - \cdots - \varepsilon^{k-1} dR_{k-1} - \varepsilon^k(r_{k-1}\omega_1 + \cdots + r_1\omega_{k-1} + \omega_k) + O(\varepsilon^{k+1}) = 0.$$

Then, we integrate the above equation along the phase curve γ from point A to point B, which are used to define the first return map. Note that

$$d(h, \varepsilon) = \int_\gamma dH = H(B) - H(A) = O(|A - B|)$$

and

$$\Big| \int_\gamma (\varepsilon dR_1 + \varepsilon^2 dR_2 + \cdots + \varepsilon^{k-1} dR_{k-1}) \Big| = \varepsilon\, O(|A - B|).$$

In addition, it follows from (2.8) and (2.26) that $d(h, \varepsilon) = O(\varepsilon^k)$. Therefore, $|A - B| = O(\varepsilon^k)$ and we finally obtain

$$d(h, \varepsilon) = \varepsilon^k \int_\gamma (r_{k-1}\omega_1 + \cdots + r_1\omega_{k-1} + \omega_k) + O(\varepsilon^{k+1}),$$

which yields

$$M_k(h) = \oint_{\Gamma_h} (\omega_k + \sum_{i+j=k} r_i\omega_j).$$

The proof is finished.

2.4 Proof of Theorem 2.1.1

Now we are ready to prove our main result – Theorem 2.1.1.

Proof We return to system (2.6) with $P(x, y)$ and $Q(x, y)$ defined in (2.9), and want to use higher-order Melnikov functions to prove the existence of 10 small-amplitude limit cycles around the origin.

Due to the difficulty in the proof of independency of basis, we use the computation of focus values to prove the theorem. However, the computation becomes very demanding or almost impossible for computing higher-order focus values if all the coefficients are retained in the computation, and in fact many terms are not necessarily needed. Thus, before computing the focus values of system (2.6), without loss of generality, we want to simplify this system, by

choosing a group of coefficients a_{ijm}, b_{ijm} in the polynomials $P(x,y)$ and $Q(x,y)$, which does not affect the number of limit cycles bifurcating from the origin.

In the following, we shall show how to choose a group of coefficients which are necessary for the first non-vanishing Melnikov function $M_k(h)$ in (2.8). Based on the results presented in the previous section (in particular, Lemmas 2.3.1, 2.3.3 and 2.3.4), we provide an algorithm as follows.

Consider $M_1(h)$ in system (2.6), we know $M_1(h) = \oint_{\Gamma_h} \omega_1$. Using Lemma 2.3.3, we have

$$\omega_1 = Q_1 dx - P_1 dy = \sum_{i=0}^{1}\sum_{j=1}^{2} \alpha_{ij1} x^i y^j dx + r_1 dH + dR_1, \tag{2.29}$$

with $r_1 = -(b_{211} + 3a_{301})y$. Then,

$$M_1(h) = \oint_{\Gamma_h} (\alpha_{011} y dx + \alpha_{111} xy dx + \alpha_{021} y^2 dx + \alpha_{121} xy^2 dx).$$

It is seen that $M_1(h)$ depends on α_{ij1}, $i = 0,1$, $j = 1,2$. So only four coefficients in the polynomials $P_1(x,y)$ and $Q_1(x,y)$ are needed in order to keep α_{ij1}, $i = 0,1$, $j = 1,2$ being independent without decreasing the number of zeros of $M_1(h)$. We choose these four coefficients as b_{ij1}, $i = 0,1$, $j = 1,2$. (Certainly, this is not a unique choice.) Then, we have polynomials

$$P_1(x,y) = 0, \quad Q_1(x,y) = b_{011}x + b_{111}xy + b_{021}y^2 + b_{121}xy^2. \tag{2.30}$$

Next, let us consider $M_2(h)$ when $M_1(h) = \oint_{\Gamma_h} r_1 dH + dR_1 \equiv 0$, i.e., all $\alpha_{ij1} = 0$ in (2.29). Lemma 2.3.4 gives $M_2(h) = \oint_{\Gamma_h} \widetilde{\omega}_2$, where $\widetilde{\omega}_2 = \omega_2 + r_1\omega_1$. Thus, by using Lemma 2.3.3, we obtain

$$\widetilde{\omega}_2 = \sum_{i=0}^{1}\sum_{j=1}^{2} \alpha_{ij2} x^i y^j dx + \alpha_{042} y^4 dx + r_2 dH + dR_2,$$

which shows that $M_2(h)$ depends on α_{ij2}, $i = 0,1$, $j = 1,2$ and α_{042}. Obviously, the coefficient α_{042} is derived from $r_1\omega_1$ by Lemma 2.3.3 because the one-form y^4dx of degree 4 comes from $r_1\omega_1$. For ε-order perturbations, b_{ij1}, $i = 0,1$ $j = 1,2$ are needed to get all $\alpha_{ij1} = 0$ in (2.29). For r_1 we may simply take $b_{211} = 1$ and $a_{301} = 0$, yielding $r_1 = -y$. We also see that the one-form y^4dx can be derived from x^3ydx by using the formula (b) in Lemma 2.3.1. Hence, we may choose b_{301} for α_{042} so that $b_{301}x^3ydx$ could appear in $r_1\omega_1$. For α_{ij2}, $i = 0,1$, $j = 1,2$, by an argument similar to that for $M_1(h)$, we choose b_{012}, b_{112}, b_{022} and b_{122}. Hence, we obtain the following polynomials,

$$\begin{aligned} &P_1(x,y) = 0, \quad Q_1(x,y) = b_{011}x + b_{111}xy + b_{021}y^2 + b_{121}xy^2 + b_{301}x^3 + x^2y, \\ &P_2(x,y) = 0, \quad Q_2(x,y) = b_{012}x + b_{112}xy + b_{022}y^2 + b_{122}xy^2. \end{aligned} \tag{2.31}$$

Following the above procedure, we can choose the coefficients for $M_3(h)$, and so on. We list the polynomials for up to $M_5(h)$ in the following (the detailed arguments are omitted here

for brevity)

$$\begin{aligned}
&P_j(x,y) = a_{21j}x^2y + a_{12j}xy^2,\ j = 1,2,3, \quad P_4(x,y) = P_5(x,y) = 0,\\
&Q_1(x,y) = b_{011}y+b_{111}xy+b_{021}y^2+b_{121}xy^2+b_{301}x^3+b_{031}y^3+b_{211}x^2y,\\
&Q_2(x,y) = b_{012}y + b_{112}xy + b_{022}y^2 + b_{122}xy^2 + b_{302}x^3 + b_{032}y^3,\\
&Q_3(x,y) = b_{013}y + b_{113}xy + b_{023}y^2 + b_{123}xy^2 + b_{303}x^3,\\
&Q_4(x,y) = b_{014}y + b_{114}xy + b_{024}y^2 + b_{124}xy^2 + b_{304}x^3,\\
&Q_5(x,y) = b_{015}y + b_{115}xy + b_{025}y^2 + b_{125}xy^2.
\end{aligned} \tag{2.32}$$

Here, the difficult part is to compute the functions r_i, $i = 1,2,3,4$ in $\widetilde{\omega}_i$.

Having determined the coefficients we need in P and Q of system (2.6), we now use the computation of focus values to prove the existence of 10 small-amplitude limit cycles. We compute the focus values up to ε^5 order as follows:

$$V = \sum_{i=0}^{5} \varepsilon^i V_i, \quad \text{where} \quad V_i = \{v_{i0}, v_{i1}, v_{i2}, \cdots\}. \tag{2.33}$$

We call v_{ij} the jth ε^i-order focus value of system (2.6), and note that $v_{0j} = 0$, $j = 0,1,2,\ldots$ since at $\varepsilon = 0$ system (2.6) is a Hamiltonian system. The computation of V_i is equivalent to the computation of ith-order Melnikov function $M_i(h)$. But the computation of focus values is much easier than that of the higher-order Melnikov functions. The disadvantage of the focus value computation is that conditions obtained from the first few focus values can not be used to prove an infinite number of focus values to equal zero. But this can be easily verified by the above formulas $\widetilde{\omega}_i$.

The focus values v_{ij} can be obtained by using many different symbolic programs (e.g., the Maple program developed in [25]). Firstly, note that $v_{i0} = \frac{1}{2}b_{01i}$, $i = 1,2,\ldots$. In order to execute the Maple program, set $b_{01i} = 0$, $i = 1,2,\ldots$. In addition, set $b_{211} = 1$. Now, we start from V_1 and obtain

$$v_{11} = \tfrac{1}{8}(a_{121} + 3b_{031} + b_{111} - \tfrac{1}{2}a_1b_{111} - 2a_2b_{021} + 1).$$

Setting $v_{11} = 0$ yields $b_{031} = (\frac{1}{2}a_1b_{111} + 2a_2b_{021} - a_{121} - b_{111} - 1)/3$. Further, setting $v_{12} = 0$ results in

$$b_{121} = a_1b_{021} - a_{211} + \frac{1}{4a_2(5a_1-2)}(3a_1^2 + 20a_2^2 + 4a_1 - 20)(b_{111}+1).$$

Then, we have

$$\begin{aligned}
v_{13} &= \frac{35}{3072(5a_1-2)}(b_{111}+1)(a_1^3 - 3a_1^2 + 4 - 4a_2^2)F_{11},\\
v_{14} &= \frac{-7}{73728(+5a_1-2)}(b_{111}+1)(a_1^3 - 3a_1^2 + 4 - 4a_2^2)F_{12},\\
v_{15} &= \frac{-7}{84934656(+5a_1-2)}(b_{111}+1)(a_1^3 - 3a_1^2 + 4 - 4a_2^2)F_{13},
\end{aligned}$$

where

$$
\begin{aligned}
F_{11} &= 3a_1^2 + 12a_1 - 4 - 4a_2^2,\\
F_{12} &= 27a_1^4 - 90a_1^3 - 1308a_1^2 + 1608a_1 - 256 + (420a_1^2 + 1608a_1 - 1376 - 256a_2^2)a_2^2,\\
F_{13} &= 19683a_1^6 + 343116a_1^5 - 124524a_1^4 - 6168672a_1^3 + 7612368a_1^2 + 1585344a_1 - 1071424\\
&\quad +4[3(140715a_1^4 + 622536a_1^3 + 39880a_1^2 - 1689568a_1 + 421808)\\
&\quad -(404508a_1^2 - 396336a_1 + 267856a_2^2 - 1265424)a_2^2]a_2^2.
\end{aligned}
$$

It is easy to see that setting $b_{111} = -1$ results in $v_{13} = v_{14} = v_{15} = \cdots = 0$, as discussed above. In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin, we have to use the coefficients a_1 and a_2 to solve $F_{11} = F_{12} = 0$ (i.e., $v_{13} = v_{14} = 0$). If the solution of $F_{11} = F_{12} = 0$ yields $F_{13} \neq 0$, i.e., we have parameter values such that $v_{10} = v_{11} = \cdots = v_{14} = 0$, but $v_{15} \neq 0$, then we obtain 5 small-amplitude limit cycles by properly perturbing b_{011}, b_{031}, b_{021}, a_1 and a_2, respectively. In fact, by using the Groebner basis reduction procedure, we can reduce F_{12} and F_{13} to

$$
\begin{aligned}
\tilde{F}_{12} &= F_{12}|_{F_{11}=0} = 18(a_1 + 2)(11a_1^3 + 46a_1^2 - 84a_1 + 24),\\
\tilde{F}_{13} &= F_{13}|_{F_{11}=\tilde{F}_{12}=0} = -\frac{179712}{121}(a_1 + 2)(3073a_1^2 - 5272a_1 + 1500) \neq 0.
\end{aligned}
$$

In fact, solving the system of two equations, $F_{11} = F_{12} = 0$ (or $F_{11} = \tilde{F}_{12} = 0$) we obtain the solutions for a_1 as follows:

$$
\begin{aligned}
&a_1 = a_{11}^i, \quad a_2 = a_{21}^i = \pm\tfrac{1}{2}\sqrt{3(a_{11}^i)^2 + 12a_{11}^i - 4}, \ \ i = 1, 2, 3, \ \text{ for which}\\
&a_{11}^1 = -5.61185383\cdots, \quad a_{11}^2 = 0.36507058\cdots, \quad a_{11}^3 = 1.06496506\cdots,
\end{aligned}
\tag{2.34}
$$

where the second number '1' in the subscripts of a_{11}^i and a_{21}^i denotes the solutions corresponding to the first-order Melnikov function, i.e, $k = 1$. Note that $a_1 = -2$ is not a solution of $F_{11} = 0$. Further, we obtain

$$
\det\left(\frac{\partial(F_{11}, F_{12})}{\partial(a_1, a_2)}\right)_{F_{11}=\tilde{F}_{12}=0} = 576a_2(a_1 + 1)(11a_1^2 + 40a_1 - 36) \neq 0,
$$

which can be easily verified by directly substituting the solutions given in (2.34) into the above determinant.

Summarizing the above results we can conclude that based on the ε^1-order focus values (equivalently based on the first-order Melnikov function $M_1(h)$) we obtain at most 5 small-amplitude limit cycles around the origin.

Now let $b_{111} = -1$, then $b_{121} = a_1 b_{021} - 1$ and $b_{031} = -\frac{1}{3}(a_{121} + \frac{1}{2}a_1 - 2a_2 b_{021})$, under which all ε^1-order focus values vanish, or equivalently, the first-order Melnikov function $M_1(h) \equiv 0$. Then, one uses the ε^2-order focus values to solve the polynomial equations $v_{21} = v_{22} = v_{23} = 0$, yielding the solutions for b_{032}, b_{122} and b_{112}. Under these solutions, we further obtain

$$
\begin{aligned}
v_{24} &= -\ F_{20}F_{21}/(3a_1^2 + 12a_1 - 4 - 4a_2^2)/36864,\\
v_{25} &= \quad F_{20}F_{22}/(3a_1^2 + 12a_1 - 4 - 4a_2^2)/31850496,\\
v_{26} &= 11F_{20}F_{23}/(3a_1^2 + 12a_1 - 4 - 4a_2^2)/107297229312,
\end{aligned}
$$

for which we have applied the Groebner basis reduction procedure to obtain

$$\begin{aligned}
F_{20} &= [2(3a_1^3-4a_2^2)b_{021}-3(a_1^3-4a_2^2)b_{301}-6a_1^2a_{211}+4a_2a_{121}-4a_1a_2b_{211}]b_{211}\\
&\quad +12a_1(a_1a_{121}-a_2a_{211})b_{021},\\
F_{21} &= 81a_1^4-648a_1^3-648a_1^2+1632a_1-880-(504a_1^2-1632a_1-1696+880a_2^2)a_2^2,\\
\tilde{F}_{22} &= F_{22}|_{F_{21}=0}\\
&= 1408[243a_1^3-522a_1^2+5172a_1+6664+(1053a_1^2-2424a_1-5572+1300a_2^2)a_2^2]a_2^2\\
&\quad -50688(63a_1^3+56a_1^2-148a_1+80),\\
\tilde{F}_{23} &= F_{23}|_{F_{21}=\tilde{F}_{22}=0}\\
&= 72(675121644a_1^3+475639745a_1^2-1491227668a_1+849702020)\\
&\quad +\{3893155245a_1^3+22056197796a_1^2-131201934348a_1-117343356608\\
&\quad +20[303274623a_1+3083354476-26(55458a_1-130879)a_2^2]a_2^2\}a_2^2\neq 0,
\end{aligned}$$

Similarly, we obtain the following solutions satisfying $F_{21}=\tilde{F}_{22}=0$:

$$\begin{aligned}
&a_1=a_{12}^i,\quad i=1,2,\ldots,7,\\
&a_2=a_{22}^i=\sqrt{\tfrac{10179a_1^6-81864a_1^5-179172a_1^4+204992a_1^3-32496a_1^2-124032a_1+66880}{4(5109a_1^4+12076a_1^3-75936a_1^2-167664a_1+48944)}},\ (a_1=a_{12}^i),\\
&\text{where}\\
&a_{12}^1=-2.43192492\cdots,\ a_{12}^2=0.12148877\cdots,\ a_{12}^3=0.23963547\cdots,\\
&a_{12}^4=\ \ 0.89471272\cdots,\ a_{12}^5=1.60031174\cdots,\ a_{12}^6=7.33752703\cdots,\\
&a_{12}^7=\ 10.40950390\cdots.
\end{aligned}\tag{2.35}$$

In addition, we can show that for the above solutions the following determinant is non-zero,

$$\begin{aligned}
&\det\left(\frac{\partial(F_{21},F_{22})}{\partial(a_1,a_2)}\right)_{F_{21}=\tilde{F}_{22}=0}\\
&=\frac{360448}{351}a_2\{36(1571445a_1^3+860083a_1^2-3207848a_1+1911580)\\
&\qquad +a_2^2[4977612a_1^3+24045705a_1^2-138196596a_1-132836684\\
&\qquad\quad +20a_2^2(-119877a_1+2945227+169a_2^2(459a_1+1799))]\}\neq 0.
\end{aligned}$$

The above results show that we have parameter values such that $v_{20}=v_{21}=\cdots=v_{25}=0$, but $v_{26}\neq 0$. Then, taking proper perturbations on the coefficients b_{012}, b_{032}, b_{122}, b_{112}, a_1 and a_2 yields 6 small-amplitude limit cycles around the origin of system (2.6) when the ε^2-order focus values (or the second-order Melnikov function $M_2(h)$) are used.

In order to get more limit cycles, we let $F_{20}=0$ and solve this equation for b_{301}, yielding all the ε^2-order focus values $v_{2j}=0$. Under the conditions obtained above, we then use the ε^3-order focus values v_{3j} to determine the number of small-amplitude limit cycles. Similarly, we may linearly solve the polynomial equations $v_{31}=v_{32}=v_{33}=v_{34}=0$ for the coefficients b_{023}, b_{123}, b_{113} and b_{302}. After this, no coefficients can be solved linearly. So we solve a_{211} from the equation, $v_{35}=0$, which is quadratic about a_{211}, to obtain two solutions $a_{211}^{\pm}$. We choose $a_{211}=a_{211}^{+}$ and then v_{36}, v_{37} and v_{38} are simplified to

$$v_{36}=-624\,F_{30}\,F_{31},\quad v_{37}=-1248\,F_{30}\,F_{32},\quad v_{38}=-208\,F_{30}\,F_{33},$$

where F_{30} is a lengthy irrational function, and we further apply the Groebner reduction procedure to F_{32} and F_{33} to obtain

$$\begin{aligned}
F_{31} =& 405a_1^4 + 6264a_1^3 + 6264a_1^2 - 5664a_1 + 1360 - 8(99a_1^2 + 708a_1 + 524 - 170a_2^2)a_2^2,\\
\tilde{F}_{32} =& F_{32}|_{F_{31}=0}\\
=& 4(261117a_1^3 + 307422a_1^2 - 260532a_1 + 60680) - [9(1035a_1^3 + 13266a_1^2\\
&+ 111492a_1 + 84376 + 5(513a_1^2 - 4824a_1 - 57156 + 2660a_2^2)a_2^2]a_2^2,\\
\tilde{F}_{33} =& F_{32}|_{F_{31}=\tilde{F}_{32}=0}\\
=& 4(152348063679a_1^3 + 175217936814a_1^2 - 151386504684a_1\\
&+ 35757329960) + \{7428338685a_1^3 - 38896637238a_1^2\\
&- 568264627476a_1 - 439876872808 - 20[714254595a_1 - 6998804702\\
&- 380(11970a_1 + 132193)a_2^2]a_2^2\}a_2^2 \neq 0.
\end{aligned}$$

Solving $F_{31} = \tilde{F}_{32} = 0$ yields

$$\begin{aligned}
&a_1 = a_{13} = 0.01871627\cdots,\\
&a_2 = a_{23} = \pm\sqrt{\frac{99a_{13}^2+708a_{13}+524-12\sqrt{1104+8496a_{13}+504a_{13}^2-2724a_{13}^3-171a_{13}^4}}{340}}.
\end{aligned} \tag{2.36}$$

Further, we have

$$\det\left(\frac{\partial(F_{31}, F_{32})}{\partial(a_1, a_2)}\right)_{(a_1,a_2)=(a_{13},a_{23})} = -0.1124026367\cdots\times 10^{10} \neq 0.$$

This, together with the above results, suggests that we may have parameter values such that $v_{3i} = 0$, $i = 0, 1, 2, \ldots, 7$, $v_{38} \neq 0$, and so the system could have at most 8 small-amplitude limit cycles. Then, properly applying perturbations on the coefficients, b_{013}, b_{023}, b_{123}, b_{113}, b_{302}, a_{211}, a_1 and a_2 yields 8 limit cycles,

Now, we want all ε^3-order focus values to vanish (i.e., $M_3(h) \equiv 0$). This can be achieved by solving the coefficient a_{121} from a polynomial equation. Having obtained the conditions for which all the ε^1-, ε^2- and ε^3-order focus values vanish, we now use the ε^4-order focus values to linearly solve for b_{024}, b_{124}, b_{114}, b_{303}, a_{212} and a_{122} one by one from the equations $v_{41} = v_{42} = v_{43} = v_{44} = v_{45} = v_{46} = 0$. Then, the higher-order focus values are given by

$$v_{47} = \frac{13}{1179648}F_{40}F_{41}, \quad v_{48} = \frac{-13}{127401984}F_{40}F_{41}, \quad v_{49} = \frac{13}{244611809280}F_{40}F_{41},$$

where F_{40} is a common factor, and F_{41}, F_{42} and F_{43} are functions of a_1 and a_2, given by

$$\begin{aligned}
F_{41} =& 37179a_1^8 - 524880a_1^7 - 4747248a_1^6 - 12436416a_1^5 + 7737120a_1^4 + 13042944a_1^3\\
&- 17299200a_1^2 + 6945792a_1 - 578816 - 16a_2^2\{12393a_1^6 - 802548a_1^4 - 102708a_1^5\\
&- 1317600a_1^3 - 40464a_1^2 + 232128a_1 + 144704 - 2a_2^2[3(11475a_1^4 - 35496a_1^3\\
&- 271896a_1^2 - 38688a_1 + 129712) + 18088a_2^2(3a_1^2 + 12a_1 - 4 - a_2^2)]\},
\end{aligned}$$

$$
\begin{aligned}
F_{42} =& 2676888a_1^{10} - 52205877a_1^9 - 223716978a_1^8 + 3206239200a_1^6 + 200795760a_1^7 \\
&- 5054946912a_1^5 - 3905952192a_1^4 + 10386531072a_1^3 - 7205736960a_1^2 \\
&+ 2022961920a_1 - 144704000 + 2a_2^2\{780759a_1^8 - 9325368a_1^7 - 641496672a_1^6 \\
&- 2909890656a_1^5 + 558977760a_1^4 + 5294374272a_1^3 - 2824768512a_1^2 \\
&+ 1366334976a_1 - 486784256 - 16a_2^2[945999a_1^6 - 2537325a_1^5 - 136313118a_1^4 \\
&- 397028520a_1^3 + 244645056a_1^2 + 201986928a_1 - 100792544 - 2a_2^2(4716225a_1^4 \\
&- 23135436a_1^3 - 141272064a_1^2 + 42697968a_1 + 50396272 \\
&+ 9044a_2^2(3495a_1 + 810a_1^2 - 1682 - 250a_2^2))]\}, \\
F_{43} =& 5(1366216713a_1^{12} - 33939081972a_1^{11} + 44272893168a_1^{10} + 493387025040a_1^9 \\
&- 628994298672a_1^8 - 14032675198080a_1^7 + 18889326323712a_1^6 \\
&+ 18007656030720a_1^5 - 42717415378176a_1^4 + 28085352201216a_1^3 \\
&- 6517758455808a_1^2 - 261522960384a_1 + 82636402688) - 16a_2^2\{3(3255393240a_1^{10} \\
&- 44439681807a_1^9 - 361043394498a_1^8 - 1113177716208a_1^7 - 333653885856a_1^6 \\
&+ 4370955883488a_1^5 + 418550391360a_1^4 - 5262495843072a_1^3 + 6238547740160a_1^2 \\
&- 2585249949440a_1 + 164748398080) - a_2^2[3(17368810155a_1^8 - 138413665080a_1^7 \\
&- 974515821120a_1^6 - 2142258103008a_1^5 - 1380949222176a_1^4 + 8851316920704a_1^3 \\
&+ 2793260427776a_1^2 - 2239742773760a_1 - 792175055104) - 16a_2^2(2(7413637185a_1^6 \\
&- 9049012605a_1^5 - 244787495850a_1^4 - 257911746696a_1^3 + 922435001664a_1^2 \\
&+ 209975885040a_1 - 361224302752) + a_2^2(3(9653815755a_1^4 + 43625458140a_1^3 \\
&- 8316724720a_1^2 - 161578121840a_1 + 49510940944) - 180880a_2^2(26034a_1^2 \\
&- 28239a_1 - 170778 + 8923a_2^2)))]\}.
\end{aligned}
$$

Similarly, we obtain the solutions of a_1 and a_2 for $F_{41} = F_{42} = 0$, but $F_{43} \neq 0$ are given as follows:

$$
\begin{aligned}
&a_1 = a_{14}^i,\ a_2 = \pm a_{24}^i = \pm a_2(a_{14}^i), \quad i = 1, 2, \ldots, 6, \quad \text{where} \\
&a_{14}^1 = -4.58252393\cdots,\ a_{14}^2 = -1.72294798\cdots,\ a_{14}^3 = -0.21827689\cdots, \\
&a_{14}^4 = -0.09420293\cdots,\ a_{14}^5 = \ \ 0.14811742\cdots,\ a_{14}^6 = \ \ 1.45012903\cdots,
\end{aligned} \tag{2.37}
$$

and $a_2(.)$ denotes a rational function of the variable, which satisfy $F_{43} \neq 0$ and

$$
\det\left(\frac{\partial(F_{41}, F_{42})}{\partial(a_1, a_2)}\right)_{F_{41}=\tilde{F}_{42}=0} \neq 0.
$$

This suggests that with the ε^4-order focus values, we can obtain 9 small-amplitude limit cycles by properly perturbing the coefficients, b_{014}, b_{024}, b_{124}, b_{114}, b_{303}, a_{212}, a_{122}, a_1 and a_2.

Finally, in order to have all the ε^4-order focus values to become zero, we let $b_{021} = -\frac{2a_2}{a_1^2}$. Then, we have the following simplified conditions, under which all the ε^1-, ε^2-, ε^3-, ε^4-order

focus values equal zero.

$$b_{121} = b_{112} = b_{301} = 0, \quad b_{031} = -\frac{1}{10a_1^2}(a_1^3 + 8a_2^2), \quad b_{111} = -1,$$

$$b_{032} = \frac{12}{25}a_2 b_{022} - \frac{2}{125a_2a_1^5(a_1^3 - 2a_2^2)}\{(5a_1 + 31)a_1^6 - 2a_2^2[(8a_1 + 13)a_1^3 + 4(a_1 + 14)a_2^2]\},$$

$$b_{122} = \frac{3}{5}a_1 b_{022} - \frac{2}{25a_1^4(a_1^3 - 2a_2^2)}\{(2a_1 - 2)a_1^6 - a_2^2((17a_1 - 50)a_1^3 + 4(a_1 + 38)a_2^2)\},$$

$$b_{113} = b_{302}, \quad a_{211} = -\frac{2a_2}{a_1}, \quad a_{121} = -\frac{a_1^3 + 8a_2^2}{5a_1^2}, \quad b_{114} = b_{303}, \quad b_{021} = -\frac{2a_2}{a_1^2},$$

$$b_{023} = \frac{1}{2a_2}a_{123} - \frac{3 + a_1}{10a_2}b_{022} - \frac{a_1 - 2}{25a_2a_1^5(a_1^3 - 2a_2^2)}\{2a_1^6 + a_2^2[(3a_1 - 22)a_1^3 + 4(a_1 + 14)a_2^2]\},$$

$$\begin{aligned} b_{123} = {} & -a_{213} + \frac{a_1}{2a_2}a_{123} - \frac{a_1^2}{50a_2}[5(a_1 + 3)a_1^3 + 4(a_1 - 34)a_2^2]b_{022} \\ & - \frac{1}{125a_2a_1^7(a_1^3 - 2a_2^2)}\{10(a_1 - 2)a_1^9 + a_2^2[(5a_1^2 - 142a_1 + 208)a_1^6 \\ & \qquad - 8a_2^2((a_1^2 - 118a_1 + 96)a_1^3 + 8a_2^2(a_1 - 1)(a_1 + 21))]\}, \end{aligned}$$

$$b_{302} = \frac{2}{5}b_{022} + \frac{4}{25a_1^5(a_1^3 - 2a_2^2)}\{2a_1^6 + a_2^2[(3a_1 - 22)a_1^3 + 4(a_1 + 14)a_2^2]\},$$

$$a_{212} = \frac{2a_1}{5}b_{022} - \frac{2}{25a_1^4(a_1^3 - 2a_2^2)}\{(3a_1 + 7)a_1^6 - a_2^2[(3a_1 + 20)a_1^3 - 24(a_1 + 3)a_2^2]\},$$

$$a_{122} = \frac{14a_2}{25}b_{022} + \frac{2a_2}{125a_1^5(a_1^3 - 2a_2^2)}\{(15a_1 - 32)a_1^6 - 4a_2^2[(12a_1 - 43)a_1^3 + 6(a_1 + 14)a_2^2]\},$$

$$\begin{aligned} b_{024} = {} & \frac{a_1 - 2}{2a_1a_2}a_{213} - \frac{1}{20a_2^2(a_1^3 - 4a_2^2)}[(3a_1 - 1)a_1^3 - 4(9a_1 - 13)a_2^2]a_{123} - \frac{4a_1^2(a_1 - 2)}{25(a_1^3 - 4a_2^2)}b_{022}^2 \\ & + \frac{1}{500a_1^3a_2^2(a_1^3 - 2a_2^2)(a_1^3 - 4a_2^2)}\{(15a_1^2 + 40a_1 - 15)a_1^9 - 2a_2^2[(113a_1^2 + 366a_1 \\ & - 609)a_1^6 - 4a_2^2((16a_1^2 + 571a_1 - 781)a_1^3 - 8a_2^2(3a_1^2 + 102a_1 - 166))]\}b_{022} \\ & + \frac{2}{2500a_1^8a_2^2(a_1^3 - 2a_2^2)(a_1^3 - 4a_2^2)}\{(30a_1^2 - 70a_1 + 20)a_1^{12} - a_2^2[(65a_1^3 + 857a_1^2 \\ & - 864a_1 - 1420)a_1^9 - 8a_2^2((9a_1^3 + 849a_1^2 - 314a_1 - 1640)a_1^6 - 2a_2^2((18a_1^3 \\ & + 1759a_1^2 - 740a_1 - 2500)a_1^3 - 8a_2^2(11a_1^3 + 207a_1^2 - 108a_1 + 100)))]\}, \end{aligned}$$

$$\begin{aligned} b_{303} = {} & -\frac{2}{a_1}a_{213} + \frac{3(a_1^3 - 12a_2^2)}{5a_2(a_1^3 - 4a_2^2)}a_{123} + \frac{16a_1^2a_2}{25(a_1^3 - 4a_2^2)}b_{022}^2 - \frac{1}{125a_1^3a_2(a_1^3 - 4a_2^2)(a_1^3 - 2a_2^2)} \\ & \times\{15(a_1 + 3)a_1^9 - 2a_2^2[(113a_1 + 417)a_1^6 - 16a_2^2((4a_1 + 107)a_1^3 \\ & \qquad - 2(3a_1 + 83)a_2^2)]\}b_{022} - \frac{2}{625a_1^8a_2(a_1^3 - 4a_2^2)(a_1^3 - 2a_2^2)}\{30(a_1 - 2)a_1^{12} \\ & - a_2^2[(65a_1^2 + 1062a_1 - 40)a_1^9 - 8a_2^2((9a_1^2 + 967a_1 + 370)a_1^6 \\ & - 4a_2^2((9a_1^2 + 935a_1 + 450)a_1^3 - 4a_2^2(11a_1^2 + 204a_1 - 50)))]\}, \end{aligned}$$

$$
\begin{aligned}
b_{124} = {} & \frac{a_1^3 - 2a_1^2 - 4a_2^2}{2a_1^2 a_2}\, a_{213} - \frac{125}{2500a_1^2a_2^2(a_1^3 - 4a_2^2)}\{(3a_1 - 1)a_1^6 - 4a_2^2[(12a_1 - 5)a_1^3 \\
& - 4(9a_1 + 14)a_2^2]\}a_{123} - \frac{2}{25(a_1^3 - 4a_2^2)}[(2a_1 + 1)a_1^3 - 4(2a_1 + 7)a_2^2]b_{022}^2 \\
& + \frac{1}{500a_1^5a_2^2(a_1^3 - 2a_2^2)(a_1^3 - 4a_2^2)}\{5(3a_1^2 + 8a_1 - 3)a_1^{12} - 2a_2^2[(53a_1^2 + 436a_1 - 459)a_1^9 \\
& + 4a_2^2((26a_1^2 - 1141a_1 + 624)a_1^6 - 4a_2^2((53a_1^2 - 758a_1 + 474)a_1^3 \\
& - 16a_2^2(a_1^2 - 48a_1 + 21)))]\}b_{022} + \frac{1}{1250a_1^9a_2^2(a_1^3 - 2a_2^2)(a_1^3 - 4a_2^2)}\{10(3a_1^2 - 7a_1 + 2)a_1^{14} \\
& - a_2^2[(65a_1^3 + 617a_1^2 - 904a_1 - 2140)a_1^{11} - 4a_2^2((218a_1^3 + 1205a_1^2 + 994a_1 \\
& - 5000)a_1^8 - 4a_2^2((156a_1^3 + 1731a_1^2 + 934a_1 - 4140)a_1^5 \\
& - 4a_2^2((15a_1^3 + 1392a_1^2 + 54a_1 + 20)a_1^2 - 48a_2^2(a_1^2 + 20a_1 - 56))))]\}.
\end{aligned}
$$

Under the above conditions, we use the ε^5-order focus values to find 10 small-amplitude limit cycles. Linearly solving the seven polynomial equations, $v_{51} = v_{52} = \cdots = v_{57} = 0$ one by one for the seven coefficients, b_{025}, b_{125}, b_{115}, b_{304}, a_{213}, a_{123} and b_{022}. Then, v_{58}, v_{59} and v_{510} are given in terms of a_1 and a_2:

$$
v_{58} = \frac{187}{6193152000}F_{50}F_{51}, \quad v_{59} = \frac{-187}{990904320000}F_{50}F_{52}, \quad v_{510} = \frac{17}{11890851840000}F_{50}F_{53},
$$

where the common factor F_{50} is a rational function of a_1 and a_2, and F_{5i}, $i = 1, 2, 3$ are polynomials of a_1 and a_2, with degrees 6, 7 and 8 with respect to a_2^2, respectively. F_{51} and F_{52} are given blow (F_{53} is omitted here).

$$
\begin{aligned}
F_{51} = {} & 3365793a_1^{12} + 60938568a_1^{11} - 774250488a_1^{10} + 1966200480a_1^9 + 13136171760a_1^8 \\
& - 8029124352a_1^7 - 42401159424a_1^6 + 61639418880a_1^5 + 11348709120a_1^4 \\
& - 85053265920a_1^3 + 68653025280a_1^2 - 20425531392a_1 + 2343047168 \\
& - 8a_2^2\{3(1620567a_1^{10} + 26340228a_1^9 - 214842132a_1^8 + 216250560a_1^7 + 2573086176a_1^6 \\
& + 131414400a_1^5 - 4093628544a_1^4 + 1881934848a_1^3 + 1137593088a_1^2 - 1275165696a_1 \\
& + 718412800) - 2a_2^2[3(10180485a_1^8 + 153299952a_1^7 - 674144208a_1^6 - 353045952a_1^5 \\
& + 4636649952a_1^4 + 880277760a_1^3 - 3232210176a_1^2 + 170572800a_1 + 1300940032) \\
& + 16a_2^2(7853517a_1^6 + 134834868a_1^5 - 120423348a_1^4 - 748001952a_1^3 + 215457840a_1^2 \\
& + 31982400a_1 - 434094272 - 133a_2^2(3(14175a_1^4 - 72216a_1^3 - 415512a_1^2 - 299616a_1 \\
& - 611344) - 8a_2^2(1215a_1^2 - 74988a_1 - 63300 + 8602a_2^2)))]\}, \\
F_{52} = {} & 595745361a_1^{14} + 9456106860a_1^{13} - 180495550692a_1^{12} + 866884039776a_1^{11} \\
& + 1125517505040a_1^{10} - 7989977121984a_1^9 + 3366147119040a_1^8 + 34380042236928a_1^7 \\
& - 59273145771264a_1^6 + 7717979427840a_1^5 + 76097098183680a_1^4 \\
& - 94586831216640a_1^3 + 49990295040000a_1^2 - 12029752197120a_1 \\
& + 1171523584000 - 4a_2^2\{908390133a_1^{12} + 9845436600a_1^{11} - 161757046008a_1^{10} \\
& + 687515327712a_1^9 - 956879159760a_1^8 - 5927821906176a_1^7 + 11861554007808a_1^6
\end{aligned}
$$

$$
\begin{aligned}
&+ 5082805251072a_1^5 - 21066056398080a_1^4 + 16917933938688a_1^3 - 4446179260416a_1^2\\
&- 2461820755968a_1 + 1818858868736 - 4a_2^2\,[3(680430375a_1^{10} + 5494977468a_1^9\\
&- 93511317348a_1^8 + 342554370624a_1^7 - 417148084512a_1^6 - 2601812555136a_1^5\\
&+ 3432229497216a_1^4 + 1662404330496a_1^3 - 2256035345664a_1^2 - 180750713856a_1\\
&+ 499346197504) + 4a_2^2(6850884663a_1^8 + 97499706480a_1^7 - 620311388976a_1^6\\
&+ 219072133440a_1^5 + 3577077122976a_1^4 - 2324658546432a_1^3 - 1344157539072a_1^2\\
&+ 1227803667456a_1 - 445085841664 + 4a_2^2(4026591891a_1^6 + 79808495508a_1^5\\
&- 88999950204a_1^4 + 243954873936a_1^2 - 494926244640a_1^3 - 33890758848a_1\\
&- 111271460416 - 532a_2^2(3(1871505a_1^4 - 8465592a_1^3 - 82349256a_1^2 - 24101472a_1\\
&- 58663792) - 4a_2^2(248955a_1^2 - 22082340a_1 + 2150500a_2^2 - 13355108))))]\}.
\end{aligned}
$$

It can be shown that there are in a total 12 real solutions for (a_1, a_2) such that $F_{51} = F_{52} = 0$, but $F_{53} \neq 0$, as follows:

$$
\begin{aligned}
&a_1 = a_{15}^i,\ a_2 = \pm a_{25}^i = \pm a_2(a_{15}^i) \quad i = 1, 2, \ldots, 6, \quad \text{where}\\
&a_{15}^1 = -2.39560267\cdots,\ a_{15}^2 = -1.53681619\cdots,\ a_{15}^3 = -0.38249860\cdots,\\
&a_{15}^4 = -0.19575710\cdots,\ a_{15}^5 = \ \ 0.05960015\cdots,\ a_{15}^6 = \ \ 0.29402249\cdots,
\end{aligned}
\tag{2.38}
$$

and $a_2(.)$ denotes a rational function of the variable, which satisfy $F_{53} \neq 0$ and

$$
\det\left(\frac{\partial(F_{51}, F_{52})}{\partial(a_1, a_2)}\right)_{F_{51}=F_{52}=0} \neq 0,
$$

implying that we can apply perturbations on the 10 parameters, b_{015}, b_{025}, b_{125}, b_{115}, b_{304}, a_{213}, a_{123}, b_{022}, a_1 and a_2 to obtain 10 small-amplitude limit cycles around the origin.

Finally, we need to check the critical values given in equations (2.34), (2.35), (2.36), (2.37) and (2.38) are properly distributed in the bifurcation diagram in terms of the parameters a_1 and a_2 with the Hamiltonian function $H(x, y)$ given in (2.7). See Figure 1 in [16] in terms of the parameters a and b with the Hamiltonian function $H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3$. For convenience, we define the following points in the a_1-a_2 plane:

$$
\begin{aligned}
k = 1:&\ P_1 = (\ \ 0.3650705869\ldots,\ 0.4417795388\ldots)\\
k = 2:&\ P_2 = (\ \ 0.1214887712\ldots,\ 0.6855794168\ldots)\\
&\ P_3 = (\ \ 0.8947127237\ldots,\ 0.3648137316\ldots)\\
k = 3:&\ P_4 = (\ \ 0.0187162703\ldots,\ 0.5708409903\ldots)\\
k = 4:&\ P_5 = (-0.0942029335\ldots,\ 0.6741464973\ldots)\\
&\ P_6 = (\ \ 0.1481174260\ldots,\ 0.2303270018\ldots)\\
k = 5:&\ P_7 = (-0.1957571086\ldots,\ 0.7336772199\ldots)\\
&\ P_8 = (\ \ 0.0596001501\ldots,\ 0.4237619510\ldots),
\end{aligned}
$$

where the number k denotes the order of Melnikov function. Note that all of the points satisfy the conditions $-1 \leq a_1 \leq 2$ and $0 \leq a_2 \leq (1 - a_1/2)\sqrt{1 + a_1}$, that is, inside the curve defined by

$$
a_2^2 = \left(1 - \frac{a_1}{2}\right)^2 (1 + a_1),
$$

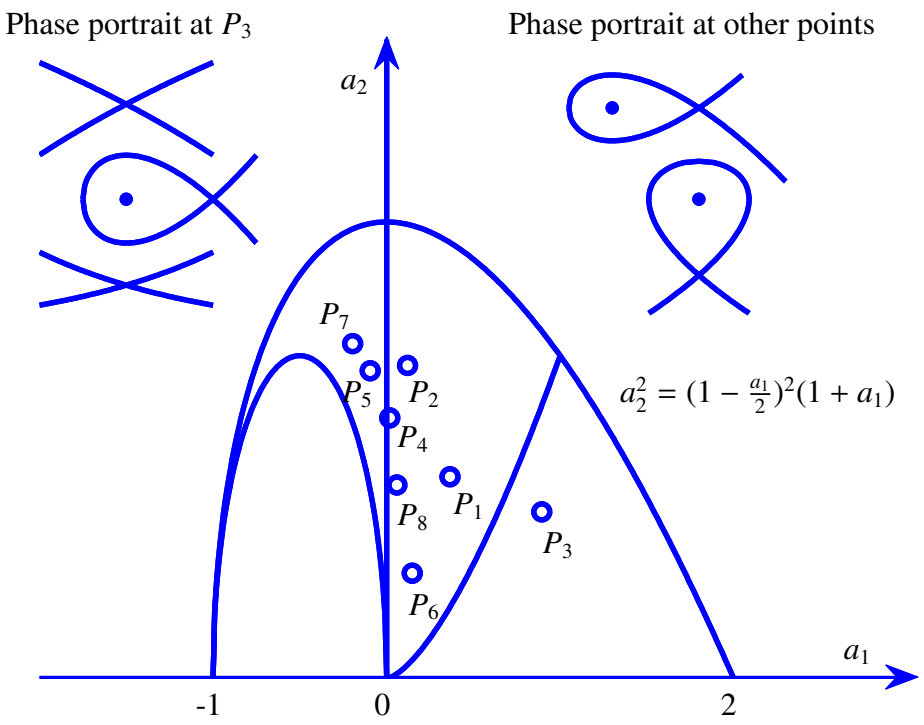

Figure 2.1: Distribution of points P_i and their corresponding phase portraits.

as shown in Figure 2.1. But it should be noted that there are other points outside the curve (not shown in this figure) which are also solutions. For each k, there exist proper Hamiltonian functions for which the conclusion in Theorem 2.1.1 holds. It has been seen from our solution procedures that $a_2 = 0$ is not allowed, and none of the above cases is degenerate. In particular, the degenerate case, defined by $a_1^3 = 2a_2^2$, does not belong to our parameter values. The corresponding phase portraits for the eight sets of parameter values (8 points P_i) are also sketched in Figure 2.1.

The above results indeed show that by using the kth-order Melnikov function M_k, we may obtain $[\frac{4k}{3}]+4$ number small-amplitude limit cycles bifurcating from the origin of system (2.6).

2.5 Conclusion

In this chapter, we have shown that the result of 11 small-amplitude limit cycles found in [13] is wrong, and proved that there are nine limit cycles when the two mistakes are corrected. Further, we have given an example of 10 small-amplitude limit cycles obtained by perturbing a quadratic Hamiltonian system. This demonstrates how to use higher-order Melnikov functions combined with the method of focus value computation to obtain more limit cycles.

Bibliography

[1] D. Hilbert, Mathematical problems, (M. Newton, Transl.) Bull. Amer. Math. 8 (1902) 437–479.

[2] Yu. S. Ilyashenko, S. Yakovenko, Finitely smooth normal forms of local families of diffeomorphismes and vector fields, Russ. Math. Surv. 46 (1991) 3–19.

[3] J. Écalle, Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac (French), Hermann, Paris, 1992.

[4] S. Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica, 23 (1980) 153–158.

[5] L. Chen, M. Wang, The relative position, and the number, of limit cycles of a quadratic differential system, Acta. Math. Sinica, 22 (1979) 751–758.

[6] P. Yu, M. Han, Four limit cycles from perturbing quadratic integrable systems by quadratic polynomials, Int. J. Bifurcation & Chaos, 22(10) (2012) 1250254 (28 pages).

[7] J. Li, Y. Liu, New results on the study of Z_q-equivariant planar polynomial vector fields, Qualitative Theory of Dynamical Systems, 9(1-2) (2010) 167–219.

[8] C. Li, C. Liu, J. Yang, A cubic system with thirteen limit cycles, J. Diff. Eqs. 246 (2009) 3609–3619.

[9] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Int. J. Bifurcation and Chaos 13 (2003) 47-106.

[10] V. I. Arnold, Geometric Methods in the Theory of Ordinary Differential Equations, Springer-Verlag, New York, 1983.

[11] M. Han, Bifurcation of limit cycles of planar systems, Handbook of Differential Equations, Ordinary Differential Equations, Vol. 3 (Eds. A. Canada, P. Drabek and A. Fonda), Elsevier, 2006.

[12] N. Bautin, On the number of limit cycles appearing from an equilibrium point of the focus or center type under varying coefficients, Matem. Sb. 30 (1952) 181–196.

[13] H. Żołądek, Eleven small limit cycles in a cubic vector field, Nonlinearity 8 (1995) 843–860.

[14] P. Yu, M. Han, A study on Żołądek's example, Journal of Applied Analysis and Computation 1 (2011) 143–153.

[15] E. M. James, N. G. Lloyd, A cubic system with eight small-amplitude limit cycles, IMA, J. of Appl. Math. 47 (1991) 163–171.

[16] E. Horozov, I. D. Iliev, On the number of limit cycles in perturbations of quadratic Hamiltonian systems, Proc. Lond. Math. Soc. 69 (1994) 198–224.

[17] L. Gavrilov, The infinitesimal 16th Hilbert problem in the quadratic case, Invent. math. 143 (2001) 449–497.

[18] E. Horozov, I. D. Iliev, Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians, Nonlinearity 11 (1998) 1521–1537.

[19] G. S. Petrov, Elliptic integrals and their non-oscillatorness, Funct. Anal. Appl. 20 (1986) 46–49 (in Russian).

[20] L. Gavrilov, Nonoscillation of elliptic integrals related to cubic polynomials with symmetry of order three, Bull. Lond. Math. Soc. 30 (1998) 267–273.

[21] B. Li, Z. Zhang, A note on a result of G. S. Petrov about the weakened 16th Hilbert problem, Journal of Mathematical Analysis and Applications 190 (1995) 489–516.

[22] I. D. Iliev, On the limit cycles available from polynomial perturbations of the Bogdanov-Takens Hamiltonian, Israel J. of Math. 115 (2000) 269–284.

[23] J. P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergodic Theory and Dynam. Syst. 16 (1996) 87–96.

[24] V. I. Arnold, A. N. Varchenko, S.M. Gusein-Zade, Singularities of Differentiable Maps, v. 2, Monographs in Mathematics, Birkhäuser, Boston, 1988.

[25] P. Yu, Computation of normal forms via a perturbation technique, J. Sound and Vib. 211(1) (1998) 19–38.

Chapter 3

Twelve limit cycles around a singular point in a planar cubic-degree polynomial system

3.1 Introduction

Studying bifurcation of limit cycles in planar polynomial systems is the second part of the well-known Hilbert's 16th problem [1]. The progress in the solution of the problem is very slow. It has not even solved the simplest quadratic systems after more than one century since the problem was posed by Hilbert at the Paris conference of the International Congress of Mathematicians in 1900. More precisely, the second part of Hilbert's 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that the following system,

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{3.1}$$

can have, where $P_n(x, y)$ and $Q_n(x, y)$ represent n^{th}-degree polynomials of x and y. In early 1990's, Ilyashenko and Yakovenko [2], and Écalle [3] independently proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, the best result is $H(2) \geq 4$, obtained more than 30 years ago [4, 5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [7, 8]. Note that the 13 limit cycles obtained in [7, 8] are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [9].

In order to help understand and attack Hilbert's 16th problem the so called weak Hilbert's 16th problem was posed by Arnold [10], which is closely related to the so-called

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near-Hamiltonian system [11]:

$$\dot{x} = H_y(x, y) + \varepsilon\, p_n(x,\ y), \qquad \dot{y} = -\,H_x(x, y) + \varepsilon\, q_n(x,\ y), \tag{3.2}$$

where $H(x, y)$, $p_n(x, y)$ and $q_n(x, y)$ are all polynomial functions of x and y, and $0 < \varepsilon \ll 1$ is a small perturbation. Investigating the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the (first-order) Melnikov function, given as an integral

$$M(h, \delta) = \oint_{H(x,y)=h} q_n(x, y)\, dx - p_n(x, y)\, dy, \tag{3.3}$$

along closed orbits $H(x, y) = h$ for $h \in (h_1, h_2)$, where δ represents the parameters (or coefficients) involved in the polynomial functions q_n and p_n.

When we focus on the maximum number of small-amplitude limit cycles, $M(n)$, bifurcating from an elementary center or an elementary focus, one of the best-known results is $M(2) = 3$, which was solved by Bautin in 1952 [12]. For $n = 3$, a number of results have been obtained. Around an elemental focus, James and Lloyd [13] considered a particular class of cubic systems to obtain 8 limit cycles in 1991, and the systems were reinvestigated couple of years later by Ning *et al.* [14] to find another solution of 8 limit cycles. Yu and Corless [15] constructed a cubic system and combined symbolic and numerical computations to show 9 limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [16]. Another cubic system was also recently constructed by Lloyd and Pearson [17] to show 9 limit cycles with purely symbolic computation.

On the other hand, around a center, there are also few results obtained in the past two decades. Żołądek studied classification of cubic centers and listed 17 cases for reversible centers and 35 cases for Darboux centers [18, 19]. In 1995, Żołądek [20] first proposed a rational Darboux integral,

$$H_0 = \frac{f_1^5}{f_2^4} = \frac{(x^4 + 4x^2 + 4y)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^4}, \tag{3.4}$$

and used it to prove the existence of 11 small-amplitude limit cycles around a center. This result was extensively cited by many researchers in this area. After more than ten years, another two cubic systems are constructed to show 11 limit cycles [21, 22]. Recently, the system defined by (3.4) was reinvestigated by Yu and Han with the method of focus value computation, who only obtained 9 limit cycles [23]. This obvious difference motivated a further investigation on this problem. Very recently, Tian and Yu [24] have proved that the 11 limit cycles obtained by Żołądek [20] are not correct, and the mistakes leading to the erroneous result have been identified.

In this chapter, we will consider the two cubic systems proposed by Christopher [21], and Bondar and Sadovskii [22]. The first system discussed in [21] is determined by a Darboux first integral,

$$H_1 = \frac{(xy^2 + x + 1)^5}{x^3(xy^5 + \frac{5}{2}xy^3 + \frac{5}{2}y^3 + \frac{15}{8}xy + \frac{15}{4}y + a)^2}, \tag{3.5}$$

where a is a parameter, from which we obtain the dynamical system,

$$\begin{aligned} \dot{x} &= 10x(8axy - 3x^2 - 9x - 12y^2 - 6), \\ \dot{y} &= 24a - 16ax + 90y + 15xy - 16axy^2 + 60y^3. \end{aligned} \tag{3.6}$$

System (3.6) has an equilibrium point, given by

$$x_e = \frac{6(8a^2+25)}{32a^2-75}, \quad y_e = \frac{70\,a}{32a^2-75}. \tag{3.7}$$

Shifting the equilibrium point (x_e, y_e) to the origin and setting $a = 2$ finally yields the system:

$$\begin{aligned} \dot{x} &= -10(342+53x)(289x-2112y+159x^2-848xy+636y^2), \\ \dot{y} &= -605788x+988380y-432745xy+755568y^2-89888xy^2+168540y^3, \end{aligned} \tag{3.8}$$

which has been used in [21] to show 11 small-amplitude limit cycles around the origin (i.e., around the equilibrium point (x_e, y_e)).

The second system given in [22] is described by

$$\begin{aligned} \dot{x} &= y\,[1-2r(3r^2+5)x+(r^2+3)(3r^2+1)^2x^2], \\ \dot{y} &= -x(1-8rx)[1-3r(r^2+3)x]+2[2(3r^2-1)-r(r^2+3)(15r^2-7)x]\,xy \\ &\quad -[r(r^2+11)-(r^2+3)(3r^4+22r^2-1)x]\,y^2+2r(r^2+3)(r^2-1)y^3, \end{aligned} \tag{3.9}$$

where r is a parameter. It can be shown that the origin of system (3.9) is a center [22].

To find the small-amplitude limit cycles bifurcating from the origin of the systems (3.8) and (3.9), in general we may apply perturbations to the systems and then compute the Melnikov functions around the loops defined by the first integral $H(x, y) = h$. For system (3.8), we may use H_1, while for system (3.9), we need to find the first integral, which is not an easy job. Even we have these H functions, it is difficult to compute the Melnikov functions. Therefore, we turn to using focus value computation to analyze bifurcation of limit cycles around the origin. Suppose the focus values around the origin of the system (3.8) or (3.9) are given in the following form:

$$V = \sum_{i\geq 0} \varepsilon^i V_i, \quad \text{where} \quad V_i = \{v_{i0}, v_{i1}, v_{i2}, \cdots\}, \tag{3.10}$$

where ε is a small perturbation parameter. We call v_{ij} the jth ε^i-order focus value of the system, and note that $v_{0j} = 0,\ j = 0, 1, 2, \ldots$ since the origin is a center of these two systems. We use $M(n)$ to denote the number of limit cycles bifurcating from a singular point, where n is the order of the system.

The rest of the chapter is organized as follow. In the next section, we use the method of focus value computation to show that there are 11 small-amplitude limit cycles around the origin of the system (3.8) and (3.9). In Section 3.3, we use system (3.6) with the free coefficient a to prove that there exist 12 small-amplitude limit cycles around the origin. Conclusion is drawn in Section 3.4.

3.2 11 limit cycles in systems (3.8) and (3.9)

In this section, we will use the method of focus value computation to show that the systems (3.8) and (3.9) can have 11 small-amplitude limit cycles bifurcating from the origin, i.e., $M(3) \geq 11$. Firstly, we consider system (3.8) and have the following result.

Theorem 3.2.1 *System (3.8) can have* 11 *small-amplitude limit cycles bifurcating from the origin by proper cubic perturbation.*

Proof Adding perturbations to the non-perturbed system, we have two choices: either to the original system (3.6) (with $a = 2$) or after the shifting the equilibrium (3.7) to the origin plus a linear transformation applied such that the Jacobian of the resulting system is in Jordan canonical form. These two choices are equivalent, giving the same result on the number of limit cycles, but the latter is simpler. Therefore, we take the second choice. We first apply a linear transformation and a time scaling, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{bmatrix} 1 & 0 \\ \frac{289}{2112} & \frac{12\sqrt{23602332855}}{7223040} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \to \frac{t}{12\sqrt{23602332855}},$$

into system (3.8) to obtain

$$\begin{aligned}
\dot{x} =& y - \frac{384833 \cdot 12\sqrt{23602332855}}{6969827487744}\, x^2 + \frac{5022227}{10593792}\, xy - \frac{53 \cdot 12\sqrt{23602332855}}{1271255040}\, y^2 \\
&- \frac{20396149 \cdot 12\sqrt{23602332855}}{2383681000808448}\, x^3 + \frac{1047757}{21187584}\, x^2 y - \frac{2809 \cdot 12\sqrt{23602332855}}{434769223680}\, xy^2 \\
&+ \varepsilon\, p_3(x, y) \equiv f_1(x, y) + \varepsilon\, p_3(x, y), \\
\dot{y} =& -x - \frac{15317}{371712}x^2 - \frac{154813\,12\sqrt{23602332855}}{757589944320}xy + \frac{5149003}{35312640}y^2 + \frac{4490071331}{774425276416}x^3 \\
&- \frac{448925953\,12\sqrt{23602332855}}{23836810008084480}\, x^2 y + \frac{165731}{47083520}\, xy^2 + \frac{2809\,12\sqrt{23602332855}}{869538447360}\, y^3 \\
&+ \varepsilon\, q_3(x, y) \equiv f_2(x, y) + \varepsilon\, q_3(x, y),
\end{aligned} \tag{3.11}$$

where the linear part of the unperturbed system is now in the Jordan canonical form, and the cubic polynomial perturbations have been added, given in the general form:

$$\begin{aligned}
p_3(x, y) &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
q_3(x, y) &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.
\end{aligned} \tag{3.12}$$

To make the origin of the system be an elementary center, it requires that $a_{10} + b_{01} = 0$, or $b_{01} = -a_{10}$. To further simplify the analysis, introducing another linear transformation and a time scaling, given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{bmatrix} \varepsilon\, a_{10} & 1 + \varepsilon\, a_{01} \\ -1 + \varepsilon\, b_{10} & -\varepsilon\, a_{10} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \to \frac{t}{\omega_c},$$

where $\omega_c = \sqrt{1 + \varepsilon(a_{01} - b_{10}) - \varepsilon^2(a_{10}^2 + a_{01}b_{10})}$, into system (3.11) to obtain

$$\begin{aligned}
\dot{x} &= f_1(x, y) + \varepsilon\, \tilde{p}(x, y), \\
\dot{y} &= f_2(x, y) + \varepsilon\, \tilde{q}(x, y),
\end{aligned} \tag{3.13}$$

where higher-order ε terms have been dropped since system (3.11) only has first-order ε terms

added, f_1 and f_2 are given in (3.11), and $\tilde{p}$ and $\tilde{q}$ are given below,

$$\begin{aligned}
\tilde{p} =& \Big[a_{20} - \frac{5022227}{10593792}a_{10} + \frac{1937634155}{247808\sqrt{23602332855}}(a_{01} - b_{10})\Big]x^2 \\
&+ \Big[a_{11} + \frac{53\sqrt{23602332855}}{52968960}a_{10} - \frac{5022227}{10593792}a_{01}\Big]xy \\
&+ \Big[a_{02} + \frac{4389205759}{743424\sqrt{23602332855}}(3a_{01} + b_{10})\Big]y^2 \\
&+ \Big[a_{30} - \frac{1047757}{21187584}a_{10} + \frac{5404979485}{4460544\sqrt{23602332855}}(a_{01} - b_{10})\Big]x^3 \\
&+ \Big[a_{21} + \frac{2809\sqrt{23602332855}}{18115384320}a_{10} - \frac{1047757}{21187584}a_{01}\Big]x^2y \\
&+ \Big[a_{12} + \frac{232627905227}{254251008\sqrt{23602332855}}(3a_{01} + b_{10})\Big]xy^2 + a_{03}y^3, \\
\tilde{q} =& \Big[b_{20} + \frac{5233613631}{123904\sqrt{23602332855}}a_{10} - \frac{15317}{371712}b_{10}\Big]x^2 \\
&+ \Big[b_{11} + \frac{4832063}{26484480}a_{10}\frac{3585623893}{123904\sqrt{23602332855}}(a_{01} - b_{10})\Big]xy \\
&+ \Big[b_{02} - \frac{53\sqrt{23602332855}}{105937920}a_{10} - \frac{5149003}{35312640}a_{01}\Big]y^2 \\
&+ \Big[b_{30} + \frac{4327705513}{1486848\sqrt{23602332855}}a_{10} + \frac{4490071331}{774425276416}b_{10}\Big]x^3 \\
&+ \Big[b_{21} + \frac{8985991}{211875840}a_{10} + \frac{448925953\sqrt{23602332855}}{3972801668014080}(a_{01} + b_{10})\Big]x^2y \\
&+ \Big[b_{12} - \frac{2809\sqrt{23602332855}}{14492307456}a_{10} - \frac{165731}{47083520}a_{01}\Big]xy^2 \\
&+ \Big[b_{03} - \frac{2809\sqrt{23602332855}}{144923074560}(3a_{01} + b_{10})\Big]y^3.
\end{aligned} \tag{3.14}$$

Now, based on system (3.13), we use focus value computation (e.g., using the Maple program developed in [25]) to obtain the ε-order focus values v_{1j}, $j = 1, 2, \ldots$, all of them are linear functions of a_{ij} and b_{ij}. For example, v_{11} is given by

$$\begin{aligned}
v_{11} =& \frac{8776116145175287}{1132248475384012800}a_{10} - \frac{215539887437\sqrt{23602332855}}{2264496950768025600}a_{01} - \frac{4187}{60192}a_{20} \\
&+ \frac{24857\sqrt{23602332855}}{171005581440}a_{11} - \frac{57611}{601920}a_{02} - \frac{440628167\sqrt{23602332855}}{13242672226713600}b_{10} + \frac{159}{12160}b_{11} \\
&- \frac{2597\sqrt{23602332855}}{5500179520}b_{20} - \frac{371\sqrt{23602332855}}{2044631952}b_{02} + \frac{1}{8}(a_{12} + b_{21} + 3a_{30} + 3b_{03}).
\end{aligned}$$

Thus, we can use any one of the parameters, say b_{03}, to solve $v_{11} = 0$ to obtain b_{03} expressed in terms of other parameters. Similarly, we can solve the equation $v_{12} = 0$ for b_{12}, $v_{13} = 0$ for b_{21}, $v_{14} = 0$ for b_{30}, $v_{15} = 0$ for b_{02}, $v_{16} = 0$ for b_{11}, $v_{17} = 0$ for b_{20}, $v_{18} = 0$ for b_{10}, $v_{19} = 0$ for a_{30},

$v_{110} = 0$ for a_{21}, respectively. Finally, under theses solutions, the 11th- and 12th-order focus values are given by $v_{111} = c_{11} F_{10}$ and $v_{112} = c_{12} F_{10}$, where c_{11} and c_{12} are some constants, and the common factor F_{10} is a function of a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{12} and a_{03}, given by

$$\begin{aligned} F_{10} = {} & 691234068956115\sqrt{23602332855}\,a_{10} + 77703544185425357945\,a_{01} \\ & -2095858834724574\sqrt{23602332855}\,a_{20} - 302178196047882421530\,a_{11} \\ & +3804860775858270\sqrt{23602332855}\,a_{02} \\ & -42463100922992640\sqrt{23602332855}\,a_{12} \\ & -21814740024914038118400\,a_{03}. \end{aligned}$$

This shows that the best result we can obtain is $v_{1j} = 0$, $j = 0, 1, 2, \ldots, 10$ but $v_{111} \neq 0$, implying that system (3.11) can have at most 11 small-amplitude limit cycles bifurcating from the origin. The above function F_{10} also implies that besides the b_{ij} parameters, only two a_{ij} parameters are used to solve the focus values. In other words, we may leave one free parameter, say a_{10} which can be used to scale the focus values, and set all other parameters zero, $a_{01} = a_{20} = a_{11} = a_{02} = a_{12} = a_{03} = 0$. Certainly, one can choose other possible combinations of the parameters a_{ij} and b_{ij} to show the same result. Thus, without loss of generality, we may assume $0 < v_{111} \ll 1$.

Finally, taking small perturbations in backward order on a_{21} for v_{110}, on a_{30} for v_{19}, on b_{10} for v_{18}, on b_{20} for v_{17}, on b_{11} for v_{16}, on b_{02} for v_{15}, on b_{30} for v_{14}, on b_{21} for v_{13}, on b_{12} for v_{12}, on b_{03} for v_{11}, and on b_{01} for v_{10} so that

$$v_{1j} v_{1(j+1)} < 0, \quad |v_{1j}| \ll |v_{1(j+1)}| \quad \text{for } j = 0, 1, 2, \ldots, 10. \tag{3.15}$$

This shows that there exist 11 small-amplitude limit cycles around the origin of system (3.8).

Next, we consider system (3.9) and have the following result.

Theorem 3.2.2 *System (3.9) can have* 11 *small-amplitude limit cycles bifurcating from the origin by proper cubic perturbation.*

Proof According to [22], the perturbations added to system (3.9) are given by

$$\begin{aligned} \bar{p}_3(x, y) &= a_0\, x + a_1\, x^2 + a_2\, xy + a_3 y^2 + a_4\, x^3 + a_5\, x^2 y + a_6\, xy^2, \\ \bar{q}_3(x, y) &= a_0\, y + a_7\, x^2 + a_8\, xy + a_9 y^2 + a_{10}\, x^3 + a_{11}\, x^2 y, \end{aligned} \tag{3.16}$$

and so the perturbed system (3.9) becomes

$$\begin{aligned} \dot{x} &= \tilde{f}_1(x, y) + \varepsilon\, \bar{p}_{x,y}, \\ \dot{y} &= \tilde{f}_2(x, y) + \varepsilon\, \bar{q}_{x,y}, \end{aligned} \tag{3.17}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are given in (3.9). First note that the zero-order ε-order focus value $v_{10} = a_0$. Letting $a_0 = 0$ and then executing the Maple program for computing the focus values [25] to obtain

$$v_{11} = \frac{1}{8}[4r(3r^2 + 11)a_1 + 8r(r^2 + 4)a_3 + 3a_4 + a_6 + 4(3r^2 - 1)(a_7 + a_9) + 2r(r^2 + 3)a_8 + a_{11}],$$

$$
\begin{aligned}
v_{12} =& \frac{-1}{96}[4r(729r^8 + 7816r^6 + 26110r^4 + 27656r^2 + 89)a_1 - 24r^2(3r^2 + 1)(r^2 + 3)^2 a_2 \\
&+ 8r(245r^8 + 2773r^6 + 9669r^4 + 10643r^2 - 2)a_3 + (709r^6 + 5107r^4 + 7227r^2 + 45)a_4 \\
&- 4r(3r^4 + 58r^2 - 13)a_5 + (215r^6 + 1809r^4 + 3177r^2 + 15)a_6 \\
&+ 4(759r^8 + 4548r^6 - 2596r^2 + 5022r^4 + 11)a_7 + 20r(3r^4 - 22r^2 + 3)a_{10} \\
&+ 2r(r^2 + 3)(263r^6 + 1601r^4 + 2585r^2 - 1)a_8 + (263r^6 + 1553r^4 + 2649r^2 - 17)a_{11} \\
&+ 4(735r^8 + 4744r^6 + 5778r^4 - 2744r^2 - 1)a_9],
\end{aligned}
$$

$\vdots$

Similarly, we can linearly solve the equations $v_{1i} = 0$ for a_i, $i = 1, 2, \ldots, 10$ one by one. Then, the 11th and 12th-order focus values are obtained as $v_{111} = 2340 \frac{F_{20N}(r)}{F_{20D}(r)} F_{21}(r)$ and $v_{112} = -120 \frac{F_{20N}(r)}{F_{20D}(r)} F_{22}(r)$, where F_{20N}, F_{20D}, F_{21} and F_{22} are all polynomials of r, given by

$$
\begin{aligned}
F_{20N} =& \, a_{11} r^{12}(r^2 - 1)^{11}(r^2 + 3)^8(3r^2 + 1)^8(3r^6 - 31r^4 - 19r^2 - 1), \\
F_{20D} =& \, 1323377757r^{34} + 26996002137r^{32} + 37965987792r^{30} \\
&- 3268655299296r^{28} - 16760339520156r^{26} + 12963886884900r^{24} \\
&+ 38642548810176r^{22} + 3147430151568r^{20} + 6604961423494r^{18} \\
&+ 38723884663134r^{16} + 6683800535760r^{14} - 3528405585600r^{12} \\
&- 58333647916r^{10} + 237931831540r^8 + 32335049504r^6 \\
&+ 6091835792r^4 + 715493989r^2 + 26239665, \\
F_{21} =& \, 63r^{12} + 868r^{10} + 1407r^8 - 1232r^6 - 803r^4 + 588r^2 + 133, \\
F_{22} =& \, 355761r^{18} + 9537262r^{16} + 77877345r^{14} + 179927041r^{12} \\
&+ 41987927r^{10} - 169741431r^8 - 36416005r^6 + 63798403r^4 \\
&+ 14431452r^2 + 383173.
\end{aligned}
$$

In order to obtain maximal number of limit cycles, we may solve $F_{21} = 0$ for r. However, unfortunately, $F_{21} = 0$ has no real solutions for r. Thus, the best result we can get is $v_{1i} = 0$, $i = 0, 1, 2, \ldots, 10$, but $v_{111} \neq 0$, implying that system (3.17) can have at most 11 small-amplitude limit cycles bifurcating from the origin. Again, without loss of generality, we may use the parameter a_{11} to scale the focus values such that $0 < v_{111} \ll 1$. Further, by perturbing, in backward order, on a_i for v_{1i}, $i = 10, 9, \ldots, 0$ such that the relations given in (3.15) hold, which implies that system (3.17) exhibits 11 small-amplitude limit cycles around the origin.

3.3 12 limit cycles in system (3.6)

Now we return to system (3.6) and let the parameter a be free to vary. In this case, we have the following result.

Theorem 3.3.1 *System (3.6) can have* 12 *small-amplitude limit cycles bifurcating from the center* (x_e, y_e) *given in (3.7) by proper cubic perturbation with a properly chosen value of a, i.e.,* $M(3) \geq 12$.

In order to prove Theorem 3.3.1, we need a lemma. Consider the following generally perturbed system:

$$\begin{aligned} \dot{x} &= P(x, y, \delta_{\mathbf{1}}) + \varepsilon\, p(x, y, \delta_{\mathbf{2}}), \\ \dot{y} &= Q(x, y, \delta_{\mathbf{1}}) + \varepsilon\, q(x, y, \delta_{\mathbf{2}}), \end{aligned} \tag{3.18}$$

where P, Q, p and q are polynomials of x and y. Suppose that the vector parameter $\delta_{\mathbf{1}}$ involved in P and Q is m dimension, while the vector parameter $\delta_{\mathbf{2}}$ involved in p and q is l dimension. It is further assumed that when $\varepsilon = 0$ system (3.18) is integrable. In Hamiltonian case, $P = H_y$, $Q = -H_x$, where H is a Hamiltonian. If it is not a Hamiltonian system one needs to multiply the integrating factor in order to use Melnikov function. This increases complexity of computation. While with the method of focus value computation, it does not need to find the integrating factor and thus greatly simplifies the computation. For system (3.18), we have the following lemma.

Lemma 3.3.2 *By properly choosing the parameters $\delta_{\mathbf{1}}$ and $\delta_{\mathbf{2}}$ in system (3.18), k small-amplitude limit cycles exist around the origin of the system, satisfying $k \le m + l$. The exact number of the limit cycles depends upon how many parameters can be chosen independently to solve the focus value equations (or to determine the zeros of Melnikov functions).*

The proof can follow the proof for Theorem 3 in [26].

We first consider ε-order focus values (equivalent to first-order Melnikov function), and then consider ε^2-order focus values. We will show that using ε^2-order focus values does not increase the number of limit cycles.

3.3.1 Based on ε-order focus values

Proof Following the procedure in the proof for Theorem 3.2.1, for system (3.6), we first shift the equilibrium point defined in (3.7) to the origin, and then apply a liner transformation with a proper time scaling such that the Jacobian of the resulting system evaluated at the origin is in the form of $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. The time scaling is taken as $t \to \frac{t}{12\omega_c}$, where ω_c is given by

$$\omega_c = \sqrt{(8a^2 + 25)(32a^2 - 75)(16384a^6 - 14400a^4 + 165000a^2 + 84375)}. \tag{3.19}$$

Then, adding the ε-order perturbation terms $p_3(x, y)$ and $q_3(x, y)$, given by

$$\begin{aligned} p_3(x, y) &= a_{101}x + a_{011}y + a_{201}x^2 + a_{111}xy + a_{021}y^2 + a_{301}x^3 + a_{211}x^2y + a_{121}xy^2 + a_{031}y^3, \\ q_3(x, y) &= b_{101}x + b_{011}y + b_{201}x^2 + b_{111}xy + b_{021}y^2 + b_{301}x^3 + b_{211}x^2y + b_{121}xy^2 + b_{031}y^3, \end{aligned} \tag{3.20}$$

into system (3.6), resulting a new system. Here, note that we add one more sub-index "1" to explicitly indicate the ε-order perturbation in order to distinct from the the ε^2-order perturbation considered in the next subsection. Further, under the condition $b_{011} = -a_{101}$, we apply another linear transformation and a second time scaling, given below,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{bmatrix} 1 & 0 \\ \frac{-\varepsilon a_{101}}{1+\varepsilon a_{011}} & \frac{\omega_c}{1+\varepsilon a_{011}} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \to \frac{t}{\omega_c^*},$$

where $\omega_c^* = \sqrt{1+\varepsilon(a_{011}-b_{101})-\varepsilon^2(a_{101}^2+a_{011}b_{101})}$, and obtain the final system, given by

$$\begin{aligned}
\dot{x} =& -\frac{\sqrt{5}C_a}{768a^2(4a^2-5)^2\omega_c}x^2+\frac{(32a^2-75)A_a}{384B_a}xy-\frac{\sqrt{5}(32a^2-75)\omega_c}{3840B_a}y^2-\frac{\sqrt{5}(32a^2-75)C_a}{4608B_a\omega_c}x^3\\
&+\frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}x^2y-\frac{\sqrt{5}(32a^2-75)^2\omega_c}{23040(8a^2+25)B_a}xy^2\\
&+\varepsilon\Big\{a_{201}x^2+a_{111}xy+a_{021}y^2+a_{301}x^3+a_{211}x^2y+a_{121}xy^2+a_{031}y^3\\
&+\Big[-\frac{(32a^2-75)A_a}{384B_a}a_{101}+\frac{\sqrt{5}(8a^2+25)C_a}{1536B_a\omega_c}(a_{011}-b_{101})\Big]x^2\\
&+\Big[\frac{\sqrt{5}(32a^2-75)\omega_c}{1920B_a}a_{101}-\frac{(32a^2-75)A_a}{384B_a}a_{011}\Big]xy+\frac{\sqrt{5}(32a^2-75)\omega_c}{7680B_a}(3a_{011}+b_{101})y^2\\
&+\Big[-\frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}a_{101}+\frac{\sqrt{5}(32a^2-75)C_a}{9216B_a\omega_c}(a_{011}-b_{101})\Big]x^3\\
&+\Big[-\frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}a_{011}+\frac{\sqrt{5}(32a^2-75)^2\omega_c}{11520(8a^2+25)B_a}a_{101}\Big]x^2y\\
&+\frac{\sqrt{5}(32a^2-75)^2\omega_c}{46080(8a^2+25)B_a}(3a_{011}+b_{101})xy^2\Big\},
\end{aligned}$$

$$\begin{aligned}
\dot{y} =& -\frac{(16a^2+225)D_a}{768B_a}x^2-\frac{\sqrt{5}(4288a^4-1200a^2+3375)D_a^2}{1920B_a\omega_c}xy+\frac{\sqrt{5}(32a^2-75)^2\omega_c}{46080(8a^2+25)B_a}y^3\\
&-\frac{(2048a^6-30400a^4+23000a^2-28125)D_a}{3840(8a^2+25)B_a}y^2-\frac{7\sqrt{5}(16a^2+15)(16a^2+225)C_a}{46080(32a^2+9)B_a\omega_c}x^2y\\
&+\frac{(16a^2+225)(3328a^4-8320a^2-3375)D_a^3}{3072(8a^2+25)B_a\omega_c^2}x^3-\frac{(32a^2-75)^2(512a^4-1040a^2-5625)}{46080B_a}xy^2\\
&+\varepsilon\Big\{b_{201}x^2+b_{111}xy+b_{021}y^2+b_{301}x^3+b_{211}x^2y+b_{121}xy^2+b_{031}y^3\\
&+\Big[-\frac{5(16a^2+225)D_a}{3840B_a}b_{101}+\frac{\sqrt{5}(1152a^4+2720a^2+3375)D_a^2}{1280B_a\omega_c}a_{101}\Big]x^2\\
&+\Big[\frac{(6144a^6-17120a^4-28125)D_a}{960(8a^2+25)B_a}a_{101}+\frac{\sqrt{5}(4288a^4-1200a^2+3375)D_a^2}{3840B_a\omega_c)}(a_{011}-b_{101})\Big]xy\\
&+\Big[+\frac{(32a^2-75)(2048a^6-30400a^4+23000a^2-28125)}{3840B_a}a_{011}-\frac{\sqrt{5}(32a^2-75)\omega_c}{3840B_a}a_{101}\Big]y^2\\
&+\Big[\frac{5(8a^2+25)(16a^2+225)(3328a^4-8320a^2-3375)C_a}{15360(32a^2+9)B_a\omega_c^2}b_{101}\\
&-\frac{\sqrt{5}(2816a^4-16000a^2-10125)C_a}{15360(32a^2+9)B_a\omega_c}a_{101}\Big]x^3-\frac{\sqrt{5}(32a^2-75)^2\omega_c}{92160(8a^2+25)B_a}(3a_{011}+b_{101})y^3\\
&+\Big[\frac{7(256a^4-400a^2-1125)D_a^2}{23040(8a^2+25)^2B_a}a_{101}+\frac{7\sqrt{5}(16a^2+15)(16a^2+225)C_a}{92160(32a^2+9)B_a\omega_c}(a_{011}-b_{101})\Big]x^2y\\
&+\Big[\frac{(512a^4-1040a^2-5625)D_a^2}{46080(8a^2+25)^2B_a}a_{011}-\frac{\sqrt{5}(32a^2-75)^2\omega_c}{9216(8a^2+25)B_a}a_{101}\Big]xy^2\Big\},
\end{aligned}$$

where

$$\begin{aligned} &A_a = 2048a^6 - 768a^4 - 4600a^2 - 5625,\ B_a = a^2(4a^2-5)^2(8a^2+25),\\ &C_a = (8a^2+25)(32a^2+9)(32a^2-75)^3,\ D_a = (8a^2+25)(32a^2-75), \end{aligned} \tag{3.21}$$

and ω_c is given in (3.19). Now applying the method of focus value computation (e.g., executing the Maple program in [25]) yields $v_{1j},\ j = 1, 2, \cdots$. For example,

$$\begin{aligned} v_{11} =& \Big[\frac{(32a^2-75)^3(805306368a^{12}+3501916160a^{10}-8707840000a^8-6744000000a^6)}{460800a^2(4a^2-5)^2(8a^2+25)\omega_c^2}\\ &+\frac{(32a^2-75)^3(-55188750000a^4-33328125000a^2-83056640625)}{460800a^2(4a^2-5)^2(8a^2+25)\omega_c^2}\Big]a_{101}\\ &-\frac{7\sqrt{5}(32a^2-75)^3(111616a^6+270400a^4-232500a^2+421875)}{921600B_a\omega_c}a_{011}\\ &-\frac{(32a^2-75)^2}{96(4a^2-5)D_a}\Big[(16a^2+15)a_{201}+\frac{128a^2+575}{10}a_{021}+\frac{32a^2-425}{20}b_{111}\Big]\\ &-\frac{7\sqrt{5}(32a^2-75)^2}{48(4a^2-5)\omega_c}\Big[(2a^2+15)b_{021}+\frac{21(8a^2+25)}{20}b_{201}-\frac{128a^2+225}{40}a_{111}\Big]\\ &-\frac{21\sqrt{5}(32a^2-75)^3(1024a^4+100a^2+5625)}{921600a^2(4a^2-5)^2\omega_c}b_{101}+\frac{1}{8}[a_{121}+b_{211}+3(a_{301}+b_{031})]. \end{aligned}$$

Similarly, we linearly solve the polynomial equations one by one for $v_{11} = 0$ using b_{031}, for $v_{12} = 0$ using b_{121}, for $v_{13} = 0$ using b_{211}, for $v_{14} = 0$ using b_{301}, for $v_{15} = 0$ using b_{021}, for $v_{16} = 0$ using b_{111}, for $v_{17} = 0$ using b_{201}, for $v_{18} = 0$ using b_{101}, for $v_{19} = 0$ using a_{301}, for $v_{110} = 0$ using a_{211}, and then obtain

$$v_{111} = 12103\,\frac{F_{30N}}{F_{30D}}\,F_{31}, \qquad v_{112} = -931\,\frac{F_{30N}}{F_{30D}}\,F_{32}, \tag{3.22}$$

where F_{30D} is a 62^{th}-degree polynomial of a, and F_{30N} is given by

$$\begin{aligned} F_{30N} =& \frac{15(32a^2-75)^{32}(2048a^6-3200a^4-13500a^2+5625)}{a^{12}(4a^2-5)^{11}(8a^2+25)^{11}\omega_c^2}a_{101}\\ &+\frac{\sqrt{5}(16a^2+15)(32a^2-75)^{33}(2048a^6+7360a^4-55300a^2+16875)}{a^{12}(4a^2-5)^{11}(8a^2+25)^{10}\omega_c^3}a_{011}\\ &-\frac{6(32a^2-75)^{31}(17408a^6-40640a^4-67500a^2+28125)}{a^{12}(4a^2-5)^{11}(8a^2+25)^{10}\omega_c^2}a_{201}\\ &-\frac{6\sqrt{5}(32a^2-75)^{32}(16384a^8+109056a^6-558080a^4-52500a^2+84375)}{a^{12}(4a^2-5)^{11}(8a^2+25)^9\omega_c^3}a_{111}\\ &+30(32a^2-75)^{32}\Big[\frac{67108864a^{14}-6291456a^{12}-2059927552a^{10}+5214044160a^8}{a^{12}(4a^2-5)^{11}(8a^2+25)^9\omega_c^4}\\ &+\frac{-809408000a^6-7945500000a^4+1307812500a^2+474609375}{a^{12}(4a^2-5)^{11}(8a^2+25)^9\omega_c^4}\Big]a_{021}\\ &-\frac{23040(32a^2-75)^{31}(16a^2+15)(2048a^6+7360a^4-55300a^2+16875)}{a^{10}(4a^2-5)^9(8a^2+25)^8\omega_c^4}a_{121} \end{aligned}$$

$$-\Big[\frac{7680\sqrt{5}(32a^2-75)^{32}(44171264a^{10}+194867200a^8-849510400a^6)}{a^{10}(4a^2-5)^9(8a^2+25)^8\omega_c^5}$$
$$+\frac{7680\sqrt{5}(32a^2-75)^{32}(-1291260000a^4+47362500a^2+170859375)}{a^{10}(4a^2-5)^9(8a^2+25)^8\omega_c^5}\Big]a_{031}.$$

The lengthy expressions for F_{31} and F_{32} are given in Appendix A. It can be shown by using the Groebner basis reduction procedure that $F_{32}|_{F_{31}=0} \neq 0$ and $F_{30D}|_{F_{31}=0} \neq 0$. $F_{31} = 0$ has three real solutions for a^2 (up to 1000 digit points, but only list 50 digits here):

$$\begin{aligned} a^2 = \quad & 4.0800973527117710361029748439520196435462690445802 1\cdots, \\ & 55.4186330411036726081995166213765432023414346476832 1\cdots, \\ & 244.1815793145813410974746372740248973294610865416749 8\cdots, \end{aligned}$$

and all of them satisfy $\omega_c > 0$ (see the expression of ω_c given in (3.19)). So there are in a total six solutions. Taking the positive value of the second solution for a:

$$a = 7.4443692171401381002446239854626739506334765029627 2\cdots$$

and setting the non-used parameters $a_{011} = a_{201} = a_{111} = a_{021} = a_{121} = a_{031} = 0$ and $a_{101} = 1$, we obtain the critical parameter values:

$$\begin{aligned} b_{031} = \quad & -0.191661521454980893552025483577979467511324959358 48\cdots, \\ b_{121} = \quad & 0.114170845140145931442029506980872092367466869694 14\cdots, \\ b_{211} = \quad & -1.214408622533951644841934342617175470309044515750 14\cdots, \\ b_{301} = \quad & 0.133158827400165161862956872786096941328246958613 67\cdots, \\ b_{021} = \quad & 0.396311897494278190438086156791046423479127035802 86\cdots, \\ b_{111} = \quad & -5.329849265403488838708417542426806455728427016025 86\cdots, \\ b_{201} = \quad & -0.124805818175797143086573571209620815365877703959 55\cdots, \\ b_{101} = \quad & -19.016174390164447456646092211889214210206502558930 27\cdots, \\ a_{301} = \quad & -0.340391032692486934415793098160485175262969072123 22\cdots, \\ a_{211} = \quad & -0.154485210131592450232648114586829225169937112827 15\cdots, \end{aligned}$$

under which the focus values become

$$\begin{aligned} & v_{11} = -0.6\times10^{-999}, \quad v_{12} = 0.4\times10^{-999}, \quad v_{13} = -0.6\times10^{-999}, \\ & v_{14} = 0.12\times10^{-998}, \quad v_{15} = -0.13\times10^{-997}, \quad v_{16} = 0.8\times10^{-997}, \\ & v_{17} = 0.4\times10^{-996}, \quad v_{18} = 0.8\times10^{-995}, \quad v_{19} = -0.2\times10^{-994}, \\ & v_{110} = 0.16\times10^{-992}, \quad v_{111} = -0.5\times10^{-991}, \\ & v_{112} = 0.34448628111762061551098308016 4\cdots\times10^{-18}. \end{aligned}$$

Therefore, we can take perturbations in backward order on a for v_{111}, on a_{211} for v_{110}, on a_{301} for v_{19}, on b_{101} for v_{18}, on b_{201} for v_{17}, on b_{111} for v_{16}, on b_{021} for v_{15}, on b_{301} for v_{14}, on b_{211} for v_{13}, on b_{121} for v_{12}, on b_{031} for v_{11}, on b_{011} for v_{10}, to obtain 12 small-amplitude limit cycles bifurcating from the origin.

3.3.2 Based on ε^2-order focus values

Proof In order to show that higher-order Melnikov functions will not generate more limit cycles, equivalently, here we use the ε^2-order focus values to prove this. To achieve this, we change the perturbation to include the ε^2-order perturbation, given as follows:

$$\begin{aligned} p_3(x,y) &= a_{101}x + a_{011}y + a_{201}x^2 + a_{111}xy + a_{021}y^2 + a_{301}x^3 + a_{211}x^2y + a_{121}xy^2 + a_{031}y^3 \\ &\quad + \varepsilon[a_{102}x + a_{012}y + a_{202}x^2 + a_{112}xy + a_{022}y^2 + a_{302}x^3 + a_{212}x^2y + a_{122}xy^2 + a_{032}y^3], \\ q_3(x,y) &= b_{101}x + b_{011}y + b_{201}x^2 + b_{111}xy + b_{021}y^2 + b_{301}x^3 + b_{211}x^2y + b_{121}xy^2 + b_{031}y^3 \\ &\quad + \varepsilon[b_{102}x + b_{012}y + b_{202}x^2 + b_{112}xy + b_{022}y^2 + b_{302}x^3 + b_{212}x^2y + b_{122}xy^2 + b_{032}y^3]. \end{aligned} \tag{3.23}$$

In order to make the origin an elementary center, we have $b_{011} = -a_{101}$ and $b_{012} = -a_{102}$. Also, in order to have all ε-order focus values to vanish, we may solve a_{121} from the equation $F_{30N} = 0$. Then, all the solutions for the ε-order perturbation parameters are obtained, as given in Appendix A. Now, we use these parameter expressions to simplify the ε^2-order focus values $v_{2j},\ j = 1, 2, \cdots$ and then linearly solve the polynomial equations one by one for $v_{21} = 0$ using b_{032}, for $v_{22} = 0$ using b_{122}, for $v_{23} = 0$ using b_{212}, for $v_{24} = 0$ using b_{302}, for $v_{25} = 0$ using b_{022}, for $v_{26} = 0$ using b_{112}, for $v_{27} = 0$ using b_{202}, for $v_{28} = 0$ using b_{102}, for $v_{29} = 0$ using a_{302}, for $v_{210} = 0$ using a_{212}. Finally, we obtain

$$\begin{aligned} v_{211} &= -\,146381281540702208\,\frac{F_{40N}}{5\,F_{30D}}\,F_{31}, \\ v_{212} &= 180161577280864256\,\frac{F_{40N}}{5\,F_{30D}}\,F_{32}, \end{aligned} \tag{3.24}$$

where the common factor F_{40N} is a function of the unused ε-order perturbation parameters, a_{101}, a_{011}, a_{201}, a_{111}, a_{021}, a_{031} and ε^2-order perturbation parameters, a_{102}, a_{012}, a_{202}, a_{112}, a_{022}, a_{122}, a_{032}, while the polynomial functions, F_{30D}, F_{31} and F_{32} are exactly the same as that obtained from using the ε-order focus values, see equation (3.22). Therefore, the best result we can have is $v_{2j} = 0,\ j = 0, 1, 2, \ldots, 11$, but $v_{212} \neq 0$, indicating that using ε^2-order perturbations still gives 12 small-amplitude limit cycles bifurcating from the origin. This suggests that using higher-order perturbations may do not increase the number of limit cycles.

3.4 Conclusion

In this chapter, we have applied the method of focus value computation to confirm the results of 11 small-amplitude limit cycles around a singular point in two existing systems in the literature. Further, we used one of the two systems with a free parameter to obtain 12 small-amplitude limit cycles. This is the best result so far obtained in cubic planar vector fields around a singular point.

Appendix A

In this appendix, we list the expressions for F_{31}, F_{32} in equation (3.22), as well as the solutions for the ε-order perturbation parameters under which all the ε-order focus values vanish.

$$
\begin{aligned}
F_{31} = {} & \frac{10000267165920440481912705903 49312}{124556484375} a^{52} \\
& + \frac{819686924091392323696339765486944256}{124556484375} a^{50} \\
& + \frac{631565347915542732579532606804066304}{332150625} a^{48} \\
& - \frac{144491654217902126140674110413441335296}{199290375} a^{46} \\
& - \frac{115507069376852987648285501307291172864}{39858075} a^{44} \\
& - \frac{673280660493096644103991475399592771 5840}{531441} a^{42} \\
& - \frac{867565289921678101848586263625952395 26400}{177147} a^{40} \\
& + \frac{163554126127002314837288798161362485 2480000}{59049} a^{38} \\
& + \frac{702352279936011559337765108502589603 84000000}{59049} a^{36} \\
& + \frac{376899976798232659047065576520743321600000000}{6561} a^{34} \\
& + \frac{171684034961379014743720051976420800000000000}{243} a^{32} \\
& - \frac{259690447131535430519473691236445425000000000}{2187} a^{30} \\
& + \frac{229840243178947065714422636653200209228515625 00}{243} a^{28} \\
& + \frac{908740714146641997798947934773427473907470703125}{324} a^{26} \\
& + \frac{412523334522378068655113329690490665452194213867 1875}{165888} a^{24} \\
& + \frac{107567819827193027781148287212115671421527862548828125}{1179648} a^{22} \\
& + \frac{284999570580602343108427299249127665312290191650390625}{50331648} a^{20} \\
& - \frac{704195633244279596986967502655697730006158351898193359375}{536870912} a^{18} \\
& - \frac{410853279732262501367963542595769453933392465114593505859375}{68719476736} a^{16} \\
& - \frac{162503920119659528483773412550437751518928632140159606933593 75}{1099511627776} a^{14} \\
& - \frac{834793107195537567694252283677102759177903644740581512451171875}{35184372088832} a^{12} \\
& - \frac{730105308842955775332536779788738344186434987932443618774414 0625}{281474976710656} a^{10} \\
& - \frac{139272038204268938862630537578825353778379096183925867080688476 5625}{72057594037927936} a^{8} \\
& - \frac{109078434917642227843697157987188279676210513571277260780334472 65625}{1152921504606846976} a^{6} \\
& - \frac{513479447745103566713857455417718176332764414837583899497985839 84375}{18446744073709551616} a^{4} \\
& - \frac{120902616645152499727691279661925752150591506506316363811492919921875}{295147905179352825856} a^{2} \\
& - \frac{664110802995276876556908533007016713727693968394305557012557983 3984375}{30223145490365729367654 4},
\end{aligned}
$$

$$
\begin{aligned}
F_{32} = {} & \frac{5801794936047356569857584023552864689324032}{84075626953125} a^{66} \\
& + \frac{4871211304076925410039730660578079195954937856}{84075626953125} a^{64} \\
& + \frac{9811561701770134762901213332237710167822984 8064}{5605041796875} a^{62} \\
& - \frac{1962468978666360643254373266776181402601021610 3936}{3363025078125} a^{60} \\
& - \frac{2125659190521304240539622185463297620029911492 1984}{134521003125} a^{58}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{19526367742432237170886155241477929055543595997593 6}{1793613375}a^{56}\\
&-\frac{70119435977099141985002957362217236522946784930037 76}{1076168025}a^{54}\\
&+\frac{57721050631981993916080642512611169178003890252021 76}{43046721}a^{52}\\
&+\frac{70992038737144757058356540939663171263951096564940800}{4782969}a^{50}\\
&+\frac{10429634905075021491277172839820714249329922183004160000}{14348907}a^{48}\\
&+\frac{83853150271215686752220226312264946195301195055104000000}{4782969}a^{46}\\
&+\frac{87437962180467579496457634960502924312130761523200000000}{531441}a^{44}\\
&+\frac{58267868815877273416473310969755954338574198272000000000 0}{531441}a^{42}\\
&+\frac{21292488896396114083846506545169895680528187655000000000000}{531441}a^{40}\\
&+\frac{47240009367317348656168824744785223517153830430859375000000}{59049}a^{38}\\
&+\frac{10919491620426678750599007836144315153629494616546630859375}{1458}a^{36}\\
&+\frac{89140128376927347670942144404721861811295305640422821044921875}{2519424}a^{34}\\
&+\frac{62969126642842688457530575225310461176272569859724044799804 6875}{11943936}a^{32}\\
&-\frac{13018640272043676580361762632811892601512660120578050613403320 3125}{382205952}a^{30}\\
&-\frac{11916027311634601269526718143543915503838673970840746164321899414 0625}{48922361856}a^{28}\\
&-\frac{16584764394152739829047427930053869434191112509710341691970825195 3125}{21743271936}a^{26}\\
&-\frac{24407664203960220715836313385831611983950578039440028369426727294921875}{1855425871872}a^{24}\\
&-\frac{26822982831292963241441867136595001776015586686010472476482391357421875}{3298534883328}a^{22}\\
&+\frac{12372396834293391235862811740374168819805034460103837773203849792480468 75}{70368744177664}a^{20}\\
&+\frac{72533241103424724463702284290789792227269813989958493039011955261230468 75}{140737488355328}a^{18}\\
&+\frac{41543234734081356752073807144962506852273937433254759525880217552185058593 75}{72057594037927936}a^{16}\\
&+\frac{42329477774405611120670414032163925242228041885811762767843902111053466796875}{230584300921369 3952}a^{14}\\
&-\frac{98721422022667387245449475883289345581801747508007338183233514428138732910156 25}{295147905179352825856}a^{12}\\
&-\frac{30759339192605941087292459791259656325044229693090301225311122834682464599609375}{590295810358705651712}a^{10}\\
&-\frac{10517418339169072658281117260237305664446979541714023298482061363756656646728515625}{3022314549036572936765 44}a^{8}\\
&-\frac{14619852069348093811561760308887976067989345860892882456028019078075885772705078125}{1208925819614629174706176}a^{6}\\
&-\frac{92786188852033362909653020826728595199811021118030396337417187169194222149658203125}{4835703278458516698824704}a^{4}\\
&-\frac{39287985409566836038657261638272866880065746708616813975822879001498222351074218 75}{3868562622766813359059763 2}a^{2}\\
&+\frac{34145931423536452686251205334784599587659265074179870680382009595632553100585937 5}{6189700196426901374495621 12},
\end{aligned}
$$

$$
\begin{aligned}
b_{031} = &-\frac{(32a^2-75)(2048a^6-3200a^4-13500a^2+5625)\omega_c^2}{3072(8a^2+25)^2(16a^2+15)B_aE_a}a_{101}-\frac{\sqrt{5}(32a^2-75)^2\omega_c}{46080(8a^2+25)B_a}a_{011}\\
&+\frac{(17408a^6-40640a^4-67500a^2+28125)\omega_c^2}{7680(8a^2+25)(16a^2+15)B_aE_a}a_{201}\\
&+\frac{\sqrt{5}(32a^2-75)(16384a^8+109056a^6-558080a^4-52500a^2+84375)\omega_c}{7680(16a^2+15)B_aE_a}a_{111}\\
&-\frac{32a^2-75}{1536(16a^2+15)}\Big[\frac{67108864a^{14}-6291456a^{12}-2059927552a^{10}+5214044160a^8}{B_aE_a}\\
&\qquad-\frac{809408000a^6+7945500000a^4-1307812500a^2-474609375}{B_aE_a}\Big]a_{021}\\
&-\frac{\sqrt{5}}{15}\Big[\frac{603979776a^14+304087040a^12-19492044800a^10+53643008000a^8}{(16a^2+15)E_a\omega_c}\\
&\qquad-\frac{7845600000a^6+155209500000a^4+3585937500a^2-21357421875}{(16a^2+15)E_a\omega_c}\Big]a_{031},
\end{aligned}
$$

$$
\begin{aligned}
b_{121} = {} & \frac{\sqrt{5}\omega_c}{15360}\Big[\frac{14680064a^{12}+195297280a^{10}-106700800a^8-1193728000a^6+2031300000a^4}{(32a^2-75)(8a^2+25)^2(16a^2+15)B_aE_a} \\
& \quad + \frac{1746562500a^2-791015625}{(32a^2-75)(8a^2+25)^2(16a^2+15)B_aE_a}\Big]a_{101} + \frac{(32a^2-75)^2(512a^4-1040a^2-5625)}{46080B_a}a_{011} \\
& - \frac{\sqrt{5}\omega_c}{38400}\Big[\frac{22020096a^{12}+985661440a^{10}-81203200a^8-4414080000a^6}{(8a^2+25)(16a^2+15)B_aE_a} \\
& \quad + \frac{10975500000a^4+8732812500a^2-3955078125}{(8a^2+25)(16a^2+15)B_aE_a}\Big]a_{201} \\
& - \frac{(32a^2-75)(8a^2+25)}{7680(16a^2+15)}\Big[\frac{16777216a^{12}+17301504a^{10}-299581440a^8+495603200a^6}{B_aE_a} \\
& \quad + \frac{1821720000a^4+761062500a^2-474609375}{B_aE_a}\Big]a_{111} \\
& - \frac{\sqrt{5}(8a^2+25)(32a^2+9)(32a^2-75)^2}{16a^2+15}\Big[\frac{22020096a^{12}+985661440a^{10}-81203200a^8-4414080000a^6}{E_a} \\
& \quad + \frac{10975500000a^4+8732812500a^2-3955078125}{E_a}\Big]a_{021} \\
& + \frac{(32a^2-75)}{3(16a^2+15)}\Big[\frac{19058917376a^{16}-78454456320a^{14}-480496844800a^{12}+4597039104000a^{10}+3557792000000a^8}{E_a\omega_c^2} \\
& \quad + \frac{-45933048000000a^6-40921650000000a^4+5299804687500a^2+4805419921875}{E_a\omega_c^2}\Big]a_{031}
\end{aligned}
$$

$$
\begin{aligned}
b_{211} = {} & -\frac{35(32a^2-75)}{46080(8a^2+25)}\Big[\frac{1073741824a^{16}+10536091648a^{14}-14659092480a^{12}-130029977600a^{10}}{(16a^2+15)B_aE_a} \\
& + \frac{206808064000a^8+250732800000a^6-557125500000a^4-176554687500a^2+106787109375}{(16a^2+15)B_aE_a}\Big]a_{101} \\
& + \frac{7\sqrt{5}(16a^2+15)(16a^2+225)(32a^2-75)^3}{(46080a^2(4a^2-5)^2\omega_c}a_{011} + \frac{7}{7680}\Big[\frac{536870912a^{16}+17012097024a^{14}}{(16a^2+15)B_aE_a} \\
& \quad + \frac{536870912a^{16}+17012097024a^{14}+13662945280a^{12}-259778150400a^{10}+308864000000a^8}{7680(16a^2+15)B_aE_a} \\
& \quad + \frac{317592000000a^6-914512500000a^4-294257812500a^2+177978515625}{7680(16a^2+15)B_aE_a}\Big]a_{201} \\
& - \frac{\sqrt{5}(8a^2+25)^2(32a^2-75)^2}{7680(16a^2+15)}\Big[\frac{4194304a^{12}-52297728a^{10}-169410560a^8-1519603200a^6}{B_aE_a\omega_c} \\
& \quad - \frac{884160000a^4+420187500a^2-284765625}{B_aE_a\omega_c}\Big]a_{111} \\
& + \frac{7(8a^2+25)^3(32a^2-75)^2}{1536(16a^2+15)}\Big[\frac{32078036992a^{14}-48473571328a^{12}+40654602240a^{10}}{B_aE_a\omega_c^2} \\
& \quad + \frac{151542988800a^8-581091840000a^6-411075000000a^4-7973437500a^2+64072265625}{B_aE_a\omega_c^2}\Big]a_{021} \\
& - \frac{\sqrt{5}(8a^2+25)(32a^2-75)}{3(16a^2+15)}\Big[\frac{3848290697216a^{20}+12685185908736a^{18}-98385621155840a^{16}}{E_a\omega_c^3} \\
& \quad + \frac{46400222003200a^{14}+874754277376000a^{12}-4888711833600000a^{10}-2960110272000000a^8}{E_a\omega_c^3} \\
& \quad + \frac{183037554000000000a^6+10748143125000000ja^4-2100858398437500a^2-1081219482421875}{E_a\omega_c^3}\Big]a_{031},
\end{aligned}
$$

$$
\begin{aligned}
b_{301} = {} & \frac{\sqrt{5}(32a^2-75)}{3072(16a^2+15)}\Big[\frac{240518168576a^{18}+1523102777344a^{16}-12233610362880a^{14}-120369184768000a^{12}}{B_aE_a\omega_c} \\
& \quad + \frac{151498719232000a^{10}-200093644800000a^8-99326640000000a^6}{B_aE_a\omega_c} \\
& \quad + \frac{274416187500000a^4+27052734375000a^2-24027099609375}{B_aE_a\omega_c}\Big]a_{101} \\
& - \frac{(8a^2+25)(32a^2-75)^3(16a^2+225)(3328a^4-8320a^2-3375)}{3072a^2(4a^2-5)^2\omega_c^2}a_{011} \\
& - \frac{\sqrt{5}(8a^2+25)}{2560(16*a^2+15)}\Big[\frac{223338299392a^{18}+4358586499072a^{16}-16686451261440a^{14}}{B_aE_a\omega_c} \\
& \quad - \frac{80077180108800a^{12}-390446694400000a^{10}+410136384000000a^8+223834800000000a^6}{B_aE_a\omega_c} \\
& \quad + \frac{574332187500000a^4-25866210937500a^2-40045166015625}{B_aE_a\omega_c}\Big]a_{201} \\
& - \frac{(8a^2+25)^3(32a^2-75)^2}{512(16a^2+15)}\Big[\frac{21646802944a^{12}-72642068480a^{10}+179301990400a^8}{B_aE_a\omega_c^2} \\
& \quad - \frac{429881280000a^6+500337000000a^4+1898437500a^2-64072265625}{B_aE_a\omega_c^2}\Big]a_{111} \\
& - \frac{\sqrt{5}(8a^2+25)^3(32a^2-75)^2A_a}{512(16a^2+15)}\Big[\frac{21646802944a^{12}-72642068480a^{10}+179301990400a^8}{B_aE_a\omega_c^3} \\
& \quad - \frac{429881280000a^6+500337000000a^4+1898437500a^2-64072265625}{B_aE_a\omega_c^3}\Big]a_{021}
\end{aligned}
$$

$$+\frac{5(8a^2+25)^2(32a^2-75)}{16a^2+15}\Big[\frac{123145302310912a^{22}+178533200560128a^{20}-3617892545331200a^{18}}{E_a\omega_c^4}$$
$$+\frac{2980885141913600a^{16}+21154374352896000a^{14}-149298441420800000a^{12}}{E_a\omega_c^4}$$
$$+\frac{381401047552000000a^{10}+160875573600000000a^8-794835045000000000a^6}{E_a\omega_c^4}$$
$$-\frac{302276812500000000a^4-83774487304687500a^2-27030487060546875}{E_a\omega_c^4}\Big]a_{031},$$

$$b_{021}=\frac{\sqrt{5}(135168a^6+227840a^4+432000a^2+646875)\omega_c}{80(8a^2+25)^2(16a^2+15)E_a}a_{101}$$
$$-\frac{\sqrt{5}(3792896a^8-7065600a^6+18484000a^4+18675000a^2-16171875)\omega_c}{200(4a^2-5)(8a^2+25)(16a^2+15)(32a^2-75)E_a}a_{201}$$
$$-\frac{524288a^{10}+9306112a^8-42483200a^6+189188000a^4+45975000a^2-73828125}{40(4a^2-5)(16a^2+15)E_a}a_{111}$$
$$-\frac{\sqrt{5}(8a^2+25)(32a^2-75)(32a^2+9)(3792896a^8-7065600a^6+18484000a^4+18675000a^2-16171875)}{40(4a^2-5)(16a^2+15)E_a\omega_c}a_{021}$$
$$-\frac{1536(10616832a^{10}+14177280a^8-174312000a^6-6425000a^4+555703125a^2+284765625)B_a}{(16a^2+15)E_a\omega_c^2}a_{031},$$

$$b_{111}=-\frac{24117248a^{10}+129826816a^8-414392320a^6-28512000a^4-342450000a^2-436640625}{24(8a^2+25)(16a^2+15)E_a}a_{101}$$
$$+\Big[\frac{140509184a^{12}+632684544a^{10}-3217612800a^8+4179008000a^6-5355000000a^4}{20(4a^2-5)(16a^2+15)(32a^2-75)E_a}$$
$$-\frac{4865625000a^2-3955078125}{20(4a^2-5)(16a^2+15)(32a^2-75)E_a}\Big]a_{201}$$
$$+\frac{147\sqrt{5}(8a^2+25)^3(32a^2-75)(3328a^4+5400a^2-3375)}{20(4a^2-5)(16a^2+15)E_a\omega_c}a_{111}$$
$$+\frac{147(8a^2+25)^3(32a^2-75)(3328a^4+5400a^2-3375)A_a}{4(4a^2-5)(16a^2+15)E_a\omega_c^2}a_{021}$$
$$-\frac{1024\sqrt{5}(8a^2+25)B_a}{16*a^2+15}\Big[\frac{1543503872a^{14}+6099828736a^{12}-36852367360a^{10}-19248000000a^8}{E_a\omega_c^3}$$
$$-\frac{104343000000a^6-135160875000a^4-352571484375a^2-192216796875}{E_a\omega_c^3}\Big]a_{031},$$

$$b_{201}=-\frac{\sqrt{5}}{48}\Big[\frac{7516192768a^{14}+108095602688a^{12}+242027069440a^{10}-491692032000a^8}{(8a^2+25)(16a^2+15)E_a\omega_c}$$
$$-\frac{764928000000a^6+1623960000000ja^4+3909515625000a^2+2456103515625}{(8a^2+25)(16a^2+15)E_a\omega_c}\Big]a_{101}$$
$$+\frac{\sqrt{5}}{40}\Big[\frac{17179869184a^{16}+389936054272a^{14}+3816816640000a^{12}+19906969600000a^{10}+15888230400000a^8}{(4a^2-5)(16a^2+15)(32a^2-75)E_a\omega_c}$$
$$-\frac{69810300000000a^6+27306562500000a^4-49485937500000a^2-2669677734375}{(4a^2-5)(16a^2+15)(32a^2-75)E_a\omega_c}a_{201}$$
$$+\frac{(32a^2-75)(8a^2+25)^2}{8(4a^2-5)(16a^2+15)}\Big[\frac{67108864a^{12}+2878603264a^{10}+6557491200a^8+89104320000a^6}{E_a\omega_c^2}$$
$$+\frac{149073300000a^4+1468125000a^2-55529296875}{E_a\omega_c^2}\Big]a_{111}$$
$$+\frac{\sqrt{5}(8a^2+25)^2(32a^2-75)}{8(4a^2-5)(16a^2+15)}\Big[\frac{3580928983040a^{16}+9305952616448a^{14}+196061429760000a^{12}}{E_a\omega_c^3}$$
$$+\frac{78571315200000a^{10}-47351270400000a^8-1235033100000000a^6}{E_a\omega_c^3}$$
$$+\frac{-1516122562500000a^4+152634375000000a^2+408460693359375}{E_a\omega_c^3}\Big]a_{021}$$
$$-\frac{2560a^2(8a^2+25)^2(4a^2-5)^2}{16a^2+15}\Big[\frac{481036337152a^{18}+5655398187008a^{16}+16872028241920a^{14}}{E_a\omega_c^4}$$
$$-\frac{51098320896000a^{12}+274010419200000a^{10}+661372152000000a^8+403593750000000a^6}{E_a\omega_c^4}$$
$$+\frac{1894665937500000a^4+2928636474609375a^2+1081219482421875}{E_a\omega_c^4}\Big]a_{031},$$

$$b_{101}=-a_{011}-\frac{2\sqrt{5}}{5}\Big[\frac{34359738368a^{18}+36238786560a^{16}+443023360000a^{14}+2845179904000a^{12}}{(8a^2+25)(16a^2+15)(32a^2-75)E_a\omega_c}$$
$$-\frac{128199680000000a^{10}+17001600000000a^8-25578000000000a^6}{(8a^2+25)(16a^2+15)(32a^2-75)E_a\omega_c}$$
$$+\frac{4680000000000a^4+200126953125000a^2-13348388671875}{(8a^2+25)(16a^2+15)(32a^2-75)E_a\omega_c}\Big]a_{101}$$

$$+\frac{96\sqrt{5}a^2(4a^2-5)}{5(16a^2+15)}\Big[\frac{536870912a^{14}+96468992a^{12}+6935347200a^{10}+48501760000a^8}{(32a^2-75)^2E_a\omega_c}$$
$$-\frac{193125600000a^6+70267500000a^4-171843750000a^2+11865234375}{(32a^2-75)^2E_a\omega_c}\Big]a_{201}$$
$$-\frac{672a^2(4a^2-5)(8a^2+25)^2(2326528a^8-41164800a^6-31260000a^4-8775000a^2+18984375)}{(16a^2+15)E_a\omega_c^2}a_{111}$$
$$-\frac{96\sqrt{5}a^2(4a^2-5)(8a^2+25)^2}{16a^2+15}\Big[\frac{17179869184a^{16}+74625056768a^{14}-982492643328a^{12}+1010963251200a^{10}}{E_a\omega_c^3}$$
$$-\frac{3191193600000a^8-1906850400000a^6-7689307500000a^4-635343750000a^2+1601806640625}{E_a\omega_c^3}\Big]a_{021}$$
$$-\frac{491520B_a^2}{16a^2+15}\Big[\frac{515396075552a^{16}+428691423232a^{14}-470936453120a^{12}-4153282560000a^{10}}{(32a^2-75)E_a\omega_c^4}$$
$$-\frac{6479961600000a^8+10387608000000a^6-15258600000000a^4-25445390625000a^2-10144775390625}{(32a^2-75)E_a\omega_c^4}\Big]a_{031},$$

$$a_{301}=-\frac{(32a^2-75)^2}{4608(8a^2+25)}\Big[\frac{67108864a^{14}-115343360a^{12}-2115502080a^{10}+7158784000a^8}{(16a^2+15)B_aE_a}$$
$$-\frac{298880000a^6+16416300000a^4-2303437500a^2-1423828125}{(16a^2+15)B_aE_a}\Big]a_{101}$$
$$+\frac{\sqrt{5}(32a^2+9)(32a^2-75)^4}{(4608a^2(4a^2-5)^2\omega_c}a_{011}+\frac{(32a^2-75)}{768(16a^2+15)}\Big[\frac{67108864a^{14}-6291456a^{12}-2059927552a^{10}}{768(16a^2+15)B_aE_a}$$
$$+\frac{5214044160a^8-809408000a^6-7945500000a^4+1307812500a^2+474609375}{768(16a^2+15)B_aE_a}\Big]a_{201}$$
$$-\frac{\sqrt{5}(8a^2+25)^2(32a^2+9)(32a^2-75)^3(16384a^8+109056a^6-558080a^4-52500a^2+84375)}{768(16a^2+15)B_aE_a\omega_c}a_{111}$$
$$-\frac{5(8a^2+25)^3(32a^2+9)^2(32a^2-75)^4(17408a^6-40640a^4-67500a^2+28125)}{768(16a^2+15)B_aE_a\omega_c^2}a_{02}$$
$$-\frac{\sqrt{5}(8a^2+25)(32a^2-75)^3}{3(16a^2+15)}\Big[\frac{2147483648a^{16}+28689039360a^{14}+83844136960a^{12}-49192140800a^{10}}{E_a\omega_c^3}$$
$$-\frac{1668308480000a^8+2393026080000a^6+137097000000a^4-376776562500a^2-64072265625}{E_a\omega_c^3}\Big]a_{031},$$

$$a_{211}=-\frac{\sqrt{5}(32a^2-75)^2(136a^2+75)(2048a^6-3200a^4-13500a^2+5625)\omega_c}{768(8a^2+25)^2(16a^2+15)B_aE_a}a_{101}$$
$$-\frac{(32a^2-75)^2(128a^4-176a^2-225)}{2304B_a}a_{011}$$
$$+\frac{\sqrt{5}(32a^2-75)(136a^2+75)(17408a^6-40640a^4-67500a^2+28125)\omega_c}{1920(8a^2+25)(16a^2+15)B_aE_a}a_{201}$$
$$+\frac{(8a^2+25)(32a^2-75)}{384(16a^2+15)}\Big[\frac{4194304a^{12}+4325376a^{10}-110247936a^8+128967680a^6}{B_aE_a}$$
$$+\frac{92424000a^4+7762500a^2-18984375}{B_aE_a}\Big]a_{111}$$
$$+\frac{\sqrt{5}(8a^2+25)(32a^2+9)(32a^2-75)^3(136a^2+75)(17408a^6-40640a^4-67500a^2+28125)}{384(16a^2+15)B_aE_a\omega_c}a_{021}$$
$$+\frac{5(32a^2-75)^2}{3(16a^2+15)}\Big[\frac{335544320a^{14}+7241465856a^{12}+17690132480a^{10}-85375795200a^8}{E_a\omega_c^2}$$
$$-\frac{246324832000a^6+96446400000a^4-39479062500a^2-12814453125}{E_a\omega_c^2}\Big]a_{031},$$

$$a_{121}=\frac{(32a^2-75)(2048a^6-3200a^4-13500a^2+5625)\omega_c^2}{1536(8a^2+25)^2(16a^2+15)B_aE_a}a_{101}+\frac{\sqrt{5}(32a^2-75)^2\omega_c}{23040(8a^2+25)B_a}a_{011}$$
$$-\frac{(17408a^6-40640a^4-67500a^2+28125)\omega_c^2}{3840(8a^2+25)(16a^2+15)B_aE_a}a_{201}$$
$$-\frac{\sqrt{5}(32a^2-75)(16384a^8+109056a^6-558080a^4-52500a^2+84375)\omega_c}{3840(16a^2+15)B_aE_a}a_{111}$$
$$+\frac{(32a^2-75)}{768(16a^2+15)}\Big[\frac{67108864a^{14}-6291456a^{12}-2059927552a^{10}+5214044160a^8}{B_aE_a}$$
$$-\frac{809408000a^6+7945500000a^4-1307812500a^2-474609375}{B_aE_a}\Big]a_{021}-\frac{\sqrt{5}(32a^2-75)}{3(16a^2+15)}$$
$$\times\Big[\frac{44171264a^{10}+194867200a^8-849510400a^6-1291260000a^4+47362500a^2+170859375)}{E_a\omega_c}\Big]a_{031}.$$

where ω_c is given in (3.19), A_a, B_a and C_a are given in (3.21), and $E_a = 2048a^6 + 7360a^4 - 55300a^2 + 16875$.

Bibliography

[1] D. Hilbert, Mathematical problems, (M. Newton, Transl.) Bull. Amer. Math. 8 (1902) 437–479.

[2] Yu. S. Ilyashenko, S. Yakovenko, Finitely smooth normal forms of local families of diffeomorphismes and vector fields, Russ. Math. Surv. 46 (1991) 3–19.

[3] J. Écalle, Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac (French), Hermann, Paris, 1992.

[4] S. Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica, 23 (1980) 153–158.

[5] L.S. Chen, M. S. Wang, The relative position, and the number, of limit cycles of a quadratic differential system, Acta. Math. Sinica, 22 (1979) 751–758.

[6] P. Yu, M. Han, M. Four limit cycles from perturbing quadratic integrable systems by quadratic polynomials, Int. J. Bifurcation & Chaos, 22(10) (2012) 1250254 (28 pages).

[7] J. Li, Y. Liu, New results on the study of Z_q-equivariant planar polynomial vector fields, Qualitative Theory of Dynamical Systems, 9(1-2) (2010) 167–219.

[8] C. Li, C. Liu, J. Yang, A cubic system with thirteen limit cycles, J. Diff. Eqs. 246 (2009) 3609–3619.

[9] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Int. J. Bifurcation and Chaos 13 (2003) 47-106.

[10] V. I. Arnold, Geometric Methods in the Theory of Ordinary Differential Equations, Springer-Verlag, New York, 1983.

[11] M. Han, Bifurcation of limit cycles of planar systems, Handbook of Differential Equations, Ordinary Differential Equations, Vol. 3 (Eds. A. Canada, P. Drabek and A. Fonda), Elsevier, 2006.

[12] N. Bautin, On the number of limit cycles appearing from an equilibrium point of the focus or center type under varying coefficients, Matem. Sb. 30 (1952) 181–196.

[13] E. M. James, N.G. Lloyd, A cubic system with eight small-amplitude limit cycles, IMA, J. of Appl. Math. 47 (1991) 163–171.

[14] S. Ning, S. Ma, K. H. Kwek, Z. Zheng, A cubic system with eight small-amplitude limit cycles, Appl. Math. Lett. 7(4) (1994) 23–27.

[15] P. Yu, R. Corless, Symbolic computation of limit cycles associated with Hilbert's 16th problem, Communications in Nonlinear Science and Numerical Simulation, 14(12) (2009) 4041–4056.

[16] C. Chen, R. Corless, M. Maza, P. Yu, Y. Zhang, A modular regular chains method and its application to dynamical systems, Int. J. Bifurcation and Chaos 23(9) (2013) 1350154 (21 pages).

[17] N. Lloyd, J. Pearson, A cubic differential system with nine limit cycles, Journal of Applied Analysis and Computation 2 (2012) 293–304.

[18] H. Żołądek, The classification of reversible cubic system with centers, Topol. Methods Nonlinear Anal. 4 (1994) 79–136.

[19] H. Żołądek, Remarks on "The classification of reversible cubic system with centers", Topol. Methods Nonlinear Anal. 8 (1996) 335–342.

[20] H. Żołądek, Eleven small limit cycles in a cubic vector field, Nonlinearity 8 (1995) 843–860.

[21] C. Christopher, Estimating limit cycle bifurcation from centers, in Trends in Mathematics: *Differential Equations with Symbolic Computation*, pp. 23–35, Birkhauser, 2006.

[22] Yu. L. Bondar, A. P. Sadovskii, On a Żołądek theorem, Differential Equations 44(2) (2008) 274–277.

[23] P. Yu, M. Han, A study on Żołądek's example, Journal of Applied Analysis and Computation, 1, (2011), 143–153.

[24] Y. Tian, P. Yu, Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system (submitted for publication).

[25] P. Yu, Computation of normal forms via a perturbation technique, J. Sound and Vib. 211(1) (1998) 19–38.

[26] M. Han, J. Yang, P. Yu, Hopf Bifurcations for near-Hamiltonian Systems, Int. J. Bifurcation and Chaos 19(12) (2009) 4117–4130.

Chapter 4

Center conditions in a switching Bautin system

4.1 Introduction

Many problems arising in science and engineering are modeled by dynamical systems whose vector fields (i.e. the right-hand sides of the equations) are not continuous or not differentiable. These systems are indistinctly called discontinuous or non-smooth systems. A full discussion on this subject can be found in the classical books [1, 2].

During the past few decades, increasing interest has been attracted to the qualitative analysis of non-smooth systems, because non-smooth systems describe some real problems more accuratly and display rich complex dynamical phenomena, which must not be disregarded in applications, for instance the squealing noise in car brakes [3, 4], or the absence of a thermal equilibrium in gases modeled by scattering billiards [5, 6]. Because of various forms of non-smoothness, non-smooth systems can exhibit not only the classical bifurcations, like Hopf bifurcation, homoclinic bifurcation, but also more complicated bifurcations that only non-smooth systems can have, such as border-collision bifurcation [7, 8, 9], grazing bifurcation [10, 11] and so on. A great deal of work has been done to generalize the classical bifurcation methods for smooth systems to non-smooth ones, see for instance [12, 13, 14, 15, 16, 17].

One class of planar non-smooth dynamical systems is the so-called switching system, which has different definitions of the continuous vector fields in two different regions divided by a line (or a curve). Our attention in this chapter is focused on switching systems, given in the form of

$$(\dot{x},\dot{y}) = \begin{cases} (\delta x - y + f^{+}(x,y,\boldsymbol{\mu}), x + \delta y + g^{+}(x,y,\boldsymbol{\mu})), & \text{if } y > 0, \\ (\delta x - y + f^{-}(x,y,\boldsymbol{\mu}), x + \delta y + g^{-}(x,y,\boldsymbol{\mu})), & \text{if } y < 0, \end{cases} \tag{4.1}$$

where $\boldsymbol{\mu} \in \mathbf{R}^m$ is a parameter vector and $\delta = \mu_1$, $f^{\pm}(x,y,\boldsymbol{\mu})$ and $g^{\pm}(x,y,\boldsymbol{\mu})$ are analytic functions in x and y starting at least from second-order terms. Obviously, the origin is an equilibrium of system (4.1). There are two systems in (4.1): the system defined in the upper half-plane for $y > 0$ is called the upper system, and the system defined in the lower half-plane for $y < 0$ is called the lower system.

Many contributions have been made to the study of Hopf bifurcation in switching systems, see for example [12, 13, 16, 18, 19, 20]. As in the study of smooth dynamical systems, the center problem, determining the center conditions of a singular point being a center, and the cyclicity problem, finding the maximal number of small-amplitude limit cycles around a singular point, are fundamental in the analysis of Hopf bifurcation in switching systems. These two problems in switching systems can be investigated by computing the Lyapunov constants [12, 15, 16]. Gasull and Torregrosa [12] applied a suitable decomposition of certain one-forms and developed a new method for computing the Lyapunov constants of switching systems.

For the center problem, it is well-known that a singular point is a center in smooth systems if and only if there exists a local first integral around the singular point. However, the situation is quite complicated in switching systems. The origin of system (4.1) can be a center even if it is not a center of either the upper system or the lower system. On the other hand, if the origin is a center for both the upper system and the lower system of (4.1), one can not ensure that system (4.1) has a center at the origin. It also requires that their first integrals of the upper and lower systems coincide on the line $y = 0$. So far, some center conditions have been obtained for some switching Kukles systems [12], switching Liénard systems [13, 18] and switching Bautin systems [16].

For planar smooth polynomial dynamical systems, linear systems can not produce limit cycles, and quadratic systems can have at most 3 small-amplitude limit cycles around a singular point [21]. For cubic systems, it is only proved that 12 small-amplitude limit cycles can appear around an elementary center [22]. With the same degrees, switching polynomial systems can exhibit more limit cycles. For example, Han and Zhang [20] proved that 2 limit cycles can appear near a focus in linear switching systems. Without loss of generality, quadratic switching systems can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y - a_3x^2 + (a_5 + a_2)xy + (a_6 + a_3)y^2 \\ x + \delta y + a_2x^2 + (a_4 - a_3)xy + (a_1 - a_2)y^2 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} \delta x - y - b_3x^2 + (b_5 + b_2)xy + (b_6 + b_3)y^2 \\ x + \delta y + b_2x^2 + (b_4 - b_3)xy + (b_1 - b_2)y^2 \end{pmatrix}, & \text{if } y < 0. \end{cases} \tag{4.2}$$

The number of small-amplitude limit cycles bifurcating from a focus in system (4.2) was investigated in [12, 15, 16, 17]. Among them, it was showed in [12] that system (4.2) can have at most 5 small limit cycles when its lower system is linear. In [15], 9 small limit cycles were obtained from a concrete example of switching Bautin systems through perturbations.

In this chapter, we develop a recursive procedure to compute the Lyapunov constants of system (4.1), which only involves algebraic computations. We then apply the method to discuss the following switching Bautin system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y - a_3x^2 + (a_5 + a_2)xy + (a_6 + a_3)y^2 \\ x + \delta y + a_2x^2 + (a_4 - a_3)xy - a_2y^2 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} \delta x - y - b_3x^2 + (b_5 + b_2)xy + (b_6 + b_3)y^2 \\ x + \delta y + b_2x^2 + (b_4 - b_3)xy - b_2y^2 \end{pmatrix}, & \text{if } y < 0. \end{cases} \tag{4.3}$$

For system (4.3) we obtain a complete center classification under the condition $a_6b_6 = 0$. Moreover, we introduce perturbations into system (4.2) with an elementary center, and get 10 small-amplitude limit cycles.

Denote by $\mathcal{E}$ the interchange of parameters $(a_2, a_3, a_4, a_5, a_6) \leftrightarrow (b_2, -b_3, -b_4, b_4, b_5, -b_6)$. Note that by the change of variables $(x, y, t) \to (x, -y, -t)$, the upper system and the lower system in (4.3) exchange their equations, which can be derived equivalently by the interchange $\mathcal{E}$ in (4.3).

Theorem 4.1.1 *Let $a_6 b_6 \neq 0$. Then system (4.3) has a center at the origin if and only if one of the following conditions or the corresponding one under the interchange of parameters $\mathcal{E}$ holds:*

$$
\begin{aligned}
\text{I}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_2 = a_5 = b_2 b_3 = 0,\\
\text{II}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_2 - a_5 = (b_2 - a_2)(b_2 + 2a_2) = b_3 = a_4 - 3a_3 = 0,\\
\text{III}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_2 = a_5,\ b_2 = 0,\ a_4 - 3a_3 = b_4 b_3 - 2a_5^2 = 0,\\
\text{IV}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_2 = a_5 = b_2,\ a_4 - 3a_3 = b_4 - 3b_3 = 0,\\
\text{V}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = b_3 = a_3 a_4 - b_2(a_5 + b_2) = a_2 = 0,\\
\text{VI}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = b_3 = a_3 + a_4 = 3a_2 + a_5 = (a_2 - b_2)(2a_2 - b_2) = 0,\\
\text{VII}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = a_3 = a_5 = b_3 = 0,\\
\text{VIII}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = b_3 = a_3 = (b_2 - a_2)(a_2 + b_2 + a_5) = 0,\\
\text{IX}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = b_3 b_4 - a_3 a_4 = b_2 = a_2 = 0,\\
\text{X}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = b_4 - a_4 = b_2 - a_2 = b_3 - a_3 = 0,\\
\text{XI}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = b_4 + a_4 = b_2 - a_2 = b_3 + a_3 = 0,\\
\text{XII}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = 9b_3 b_4 + 2a_5^2 = a_4 + a_3 = 3a_2 + b_5 = b_2 = 0,\\
\text{XIII}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = b_4 + b_3 = a_4 + a_3 = 3a_2 + a_5 = a_2 - b_2 = 0,\\
\text{XIV}&: \delta = b_6 = 0,\ b_5 = a_5,\ a_6 = a_2 + b_2 - a_5 = 0,\ (2b_2 - a_2)a_4^2 = (3a_2 - 4b_2)^2 a_5,\\
&\quad (2a_2 - b_2)b_4^2 = (3b_2 - 4a_2)^2 a_5,\ (2b_2 - a_2)a_3^2 = (b_2 - a_2)^2 a_5 = (2a_2 - b_2)b_3^2.
\end{aligned}
$$

Note that through the perturbations of parameters in conditions I–XIV, we can get small-amplitude limit cycles bifurcating from the origin of system (4.3). It is important to determine the maximal number of small-amplitude limit cycles bifurcating from a center. Obviously, by adding the extra condition, $a_1 = b_1 = 0$, to the conditions I–XIV in Theorem 4.1.1, the origin is still a center of system (4.2).

Regarding the number of small-amplitude limit cycles in system (4.2), we obtain the following new result, which is the best so far for quadratic switching systems.

Theorem 4.1.2 *For system (4.2) under the condition* X *with $a_1 = b_1 = 0$, 10 limit cycles can appear near the origin under small perturbations.*

The proofs for the above two theorems will be given later in section 4.4.

4.2 Preliminary

Let $r^+(\theta, \rho)$ and $r^-(\theta, \rho)$ be the solutions of the upper and lower systems of (4.1) in polar coordinates, respectively, with $r^+(0, \rho) = \rho$ and $r^-(\pi, \rho) = \rho$. Then through the positive

half-return map $\mathcal{P}^+(\rho) = r^+(\pi, \rho)$ and the negative half-return map $\mathcal{P}^-(\rho) = r^-(2\pi, \rho)$, we can define the Poincaré map $\mathcal{P}(\rho) = \mathcal{P}^-(\mathcal{P}^+(\rho))$, see Figure 4.1. Suppose the displacement function $d(\rho) = \mathcal{P}(\rho) - \rho$ can be expanded as

$$d(\rho) = V_1\rho + V_2\rho^2 + V_3\rho^3 + \cdots, \tag{4.4}$$

where V_k is called the kth-order Lyapunov constant of the switching system (4.1). It is easy to see that the origin is a center of system (4.1) if and only if $d(\rho) \equiv 0$ for $0 < \rho \ll 1$, which means that all the Lyapunov constants in (4.4) vanish. The isolated zeros of $d(\rho) = 0$ near $\rho = 0$ correspond to the limit cycles around the origin. It is easy to get $V_1 = e^{2\delta\pi} - 1$ since $\mathcal{P}^{\pm}(\rho) = e^{\delta\pi}\rho + O(\rho^2)$. Thus, $V_1 = 0$ if and only if $\delta = 0$. It is well known that for the first nonzero Lyapunov constants V_k in a smooth system, k must be an odd number. While if V_k is the first nonzero term in (4.4), k could be any positive integer. Because of this small difference, the theorem used to determine the number of limit cycles by Lyapunov constants should take some corresponding changes. We have the following lemma, which is based on Theorem 2.3.2 in [23]. The proof is omitted here.

Lemma 4.2.1 *Assume that there exists a sequence of Lyapunov constants of system (4.1), V_{i_0}, V_{i_1}, $\cdots$, V_{i_k}, such that $V_j = O(|V_{i_0}, \cdots, V_{i_l}|)$ for any $i_l < j < i_{l+1}$, where $i_0 = 1$. If for system (4.1) at the critical point $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, $V_{i_0} = V_{i_1} = \cdots = V_{i_k-1} = 0$, $V_{i_k} \neq 0$, and*

$$rank\left(\frac{\partial(V_{i_0}, V_{i_1}, \cdots, V_{i_{k-1}})}{\partial(\mu_1, \mu_2, \cdots, \mu_m)}(\boldsymbol{\mu}_0)\right) = k,$$

then k limit cycles can appear near the origin of system (4.1) for $|\boldsymbol{\mu} - \boldsymbol{\mu}_0|$ small.

Based on Lemma 4.2.1, we remark that the expressions in this chapter for V_k, $k = 2, 3, \ldots$, are obtained by taking into account $V_1 = V_2 = \ldots = V_{k-1} = 0$. Then for any $i_l < j < i_{l+1}$, $V_j = O(|V_{i_0}, \cdots, V_{i_l}|)$ in Lemma 4.2.1 becomes $V_j \equiv 0$.

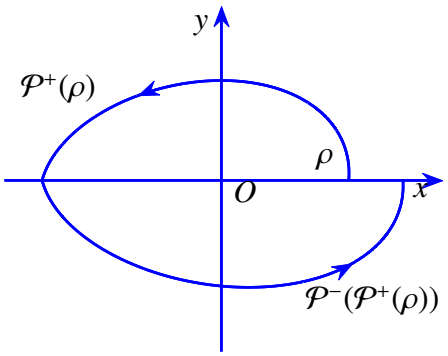

Figure 4.1: Poincaré map of (4.1).

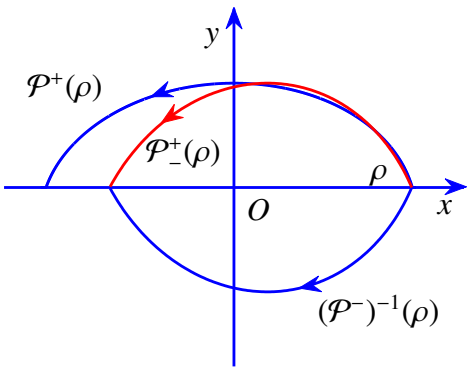

Figure 4.2: Half-return maps $\mathcal{P}^+$ and $(\mathcal{P}^-)^{-1}$.

From now on, we assume that $\delta = 0$ in system (4.1) and so $V_1 = 0$. It is very difficult to compute the remaining Lyapunov constants by using (4.4), since it involves the composition

of two maps $\mathcal{P}^+(\rho)$ and $\mathcal{P}^-(\rho)$. To simplify the computation of Lyapunov constants, a new function was introduced [12],

$$\mathcal{P}^+(\rho) - (\mathcal{P}^-)^{-1}(\rho) = W_1\rho + W_2\rho^2 + W_3\rho^3 + \cdots, \tag{4.5}$$

where $(\mathcal{P}^-)^{-1}(\rho)$ is the inverse map of $\mathcal{P}^-(\rho)$. For $(\mathcal{P}^-)^{-1}(\rho)$, we have $(\mathcal{P}^-)^{-1}(\rho) = \mathcal{P}^+_-(\rho)$, where $\mathcal{P}^+_-(\rho)$ is the positive half-return map of the system obtained from the lower system with the change of variables $(x, y, t) \to (x, -y, -t)$ (see Figure 4.2). Thus, to get (4.5) we only need to compute two positive half-return maps $\mathcal{P}^+(\rho)$ and $\mathcal{P}^+_-(\rho)$. It is proved that for (4.4) and (4.5), $V_k \neq 0$, $V_j = 0$, $1 \leq j \leq k-1$ is equivalent to $W_k \neq 0$, $W_j = 0$, $1 \leq j \leq k-1$. In Section 4.3, we shall present a new method to compute W_ks in (4.5). Because of the equivalence of V_k and W_k, we still use V_k instead of W_k in the rest of the chapter.

Note that any Lyapunov constant V_k is a polynomial in terms of the coefficients of system (4.1). Thus, having obtained the Lyapunov constants, we need to solve a sysem of multivariate polynomial equations, and find the center conditions. We shall use the Maple built-in command "resultant" to solve these polynomial equations and find their common zeros.

Denote by $\mathbf{R}[x_1, x_2, \cdots, x_r]$ the polynomial ring of multivariate polynomials in x_1, x_2, $\cdots$, x_r with coefficients in $\mathbf{R}$. Let

$$\begin{aligned} p(x_1, x_2, \cdots, x_r) &= \sum_{i=0}^{m} p_i(x_1, \cdots, x_{r-1}) x_r^i, \\ q(x_1, x_2, \cdots, x_r) &= \sum_{i=0}^{n} q_i(x_1, \cdots, x_{r-1}) x_r^i, \end{aligned} \tag{4.6}$$

be two polynomials in $\mathbf{R}[x_1, x_2, \cdots, x_r]$ of respective positive degrees m and n in x_r. We call the following matrix the Sylvester matrix of p and q with respect to x_r,

$$\mathrm{Syl}(p, q, x_r) = \begin{pmatrix} p_m & p_{m-1} & \cdots & p_0 & & & \\ & p_m & p_{m-1} & \cdots & p_0 & & \\ & & \ddots & \ddots & & \ddots & \\ & & & p_m & p_{m-1} & \cdots & p_0 \\ q_n & q_{n-1} & \cdots & q_0 & & & \\ & q_n & q_{n-1} & \cdots & q_0 & & \\ & & \ddots & \ddots & & \ddots & \\ & & & q_n & q_{n-1} & \cdots & q_0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n \\ \\ \\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m \end{matrix},$$

whose determinant is called the resultant of p and q with respect to x_r, denoted by $\mathrm{Res}(p, q, x_r)$. We have the following lemma.

Lemma 4.2.2 [24, Chapter 7] *Consider two multivariate polynomials $p(x_1, x_2, \cdots, x_r)$ and $q(x_1, x_2, \cdots, x_r)$ in $\mathbf{R}[x_1, x_2, \cdots, x_r]$ given by (4.6). Let $\mathrm{Res}(p, q, x_r) = h(x_1, \cdots, x_{r-1})$. Then*

1. If real vector $<\alpha_1, \alpha_2, \cdots, \alpha_r> \in \mathbf{R}^r$ is a common zero of equations $p(x_1, x_2, \cdots, x_r) = 0$ and $q(x_1, x_2, \cdots, x_r) = 0$, then $h(\alpha_1, \cdots, \alpha_{r-1}) = 0$.

2. Conversely, if $h(\alpha_1, \cdots, \alpha_{r-1}) = 0$, then at least one of the following four conditions hold:

(a) $p_m(\alpha_1, \cdots, \alpha_{r-1}) = \cdots = p_0(\alpha_1, \cdots, \alpha_{r-1}) = 0$, *or*
(b) $q_n(\alpha_1, \cdots, \alpha_{r-1}) = \cdots = q_0(\alpha_1, \cdots, \alpha_{r-1}) = 0$, *or*
(c) $p_m(\alpha_1, \cdots, \alpha_{r-1}) = q_n(\alpha_1, \cdots, \alpha_{r-1}) = 0$, *or*
(d) for some $\alpha_r \in \mathbf{R}$, $< \alpha_1, \cdots, \alpha_r >$ *is a common zero of both* $p(x_1, \cdots, x_r)$ *and* $q(x_1, \cdots, x_r)$.

From the first statement of Lemma 4.2.2, we know that if the resultant h does not have zeros on the region $D \subset \mathbf{R}^{r-1}$, then polynomials p and q do not have zeros in $D \times \mathbf{R}$. According to the second statement, in order to solve $p = q = 0$, we first find the zeros of $h = 0$, and then substitute them into p and q to solve for x_r. In this way, no zeros should be missed. For m multivariate polynomials with m variables, we can apply the command "resultant" repeatedly. For instance, take $m = 3$. To solve $F_j(x_1, x_2, x_3) = 0$, $j = 1, 2, 3$, suppose we compute $\mathrm{Res}(F_1, F_j, x_1)$ to obtain $\mathrm{Res}(F_1, F_j, x_1) = F_a(x_2, x_3)E_j(x_2, x_3)$, $j = 2, 3$. Then, we need to find the solutions for $F_a(x_2, x_3) = 0$ and $E_2(x_2, x_3) = E_3(x_2, x_3) = 0$. For $E_2 = E_3 = 0$, we can use resultant again, like solving $\mathrm{Res}(E_2, E_3, x_2) = 0$.

4.3 Computation of Lyapunov constants

In this section, we consider a differential system of the form,

$$\dot{x} = -y + \sum_{i=2}^{+\infty} P_i(x, y), \quad \dot{y} = x + \sum_{i=2}^{+\infty} Q_i(x, y), \tag{4.7}$$

where $P_i(x, y)$ and $Q_i(x, y)$ are homogeneous polynomials in x and y of degree i. Obviously, the origin is a Hopf singular point of system (4.7). Introducing the transformation $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into (4.7) yields

$$\begin{aligned} \dot{r} &= \sum_{i=2}^{+\infty} (\cos(\theta)P_i + \sin(\theta)Q_i) = \sum_{i=2}^{+\infty} A_i(\theta)r^i, \\ \dot{\theta} &= 1 + \sum_{i=2}^{+\infty} (\cos(\theta)Q_i - \sin(\theta)P_i)/r = 1 + \sum_{i=2}^{+\infty} B_i(\theta)r^{i-1}, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} A_i(\theta) &= \cos(\theta)P_i(\cos(\theta), \sin(\theta)) + \sin(\theta)Q_i(\cos(\theta), \sin(\theta)), \\ B_i(\theta) &= \cos(\theta)Q_i(\cos(\theta), \sin(\theta)) - \sin(\theta)P_i(\cos(\theta), \sin(\theta)). \end{aligned} \tag{4.9}$$

Let $r(\theta, \rho)$ be the solution of system (4.8) with $r(0, \rho) = \rho$. Suppose that $r(\theta, \rho)$ can be expressed as the power series of ρ in the form of

$$r(\theta, \rho) = r_1(\theta)\rho + r_2(\theta)\rho^2 + r_3(\theta)\rho^3 + \cdots, \ |\rho| \ll 1, \tag{4.10}$$

where $r_1(0) = 1$, $r_i(0) = 0$, $i \geq 2$. Then, we have the positive half-return map of system (4.7), given by

$$\mathcal{P}^+(\rho) = r(\pi, \rho) = r_1(\pi)\rho + r_2(\pi)\rho^2 + r_3(\pi)\rho^3 + \cdots, \ |\rho| \ll 1.$$

Hence, we need to compute $r_j(\theta)$ in order to get Lyapunov constants.

Eliminating the time t from (4.8) we have

$$\frac{dr}{d\theta} = \frac{\sum_{i=2}^{+\infty} A_i(\theta) r^i}{1 + \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1}}, \tag{4.11}$$

which can be rewritten in the power series of r as

$$\frac{dr}{d\theta} = R_2(\theta) r^2 + R_3(\theta) r^3 + R_4(\theta) r^4 + \cdots, \tag{4.12}$$

where $R_i(\theta)$ is a polynomial in $\sin(\theta)$ and $\cos(\theta)$.

Lemma 4.3.1 *For system (4.11), let (4.9) and (4.12) hold. Then* $\deg(R_i(\theta), \{\sin(\theta), \cos(\theta)\}) = 3(i-1)$ *and* $R_i(\theta)$ *is odd* (*even*) *in* $\sin(\theta)$ *and* $\cos(\theta)$ *if* i *is even* (*odd*).

Proof It follows from (4.9) that $A_i(\theta)$ and $B_i(\theta)$ are homogenous polynomials of $\sin(\theta)$ and $\cos(\theta)$ of degree $i+1$. Also note that

$$\frac{1}{1 + \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1}} = 1 + \sum_{j=1}^{+\infty} \Big(- \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1} \Big)^j = 1 + \sum_{i=1}^{+\infty} \widetilde{B}_i(\theta) r^i, \ |r| \ll 1.$$

Thus, $\widetilde{B}_i(\theta) r^i$ is a linear combination of products of $B_2 r, B_3 r^2, \cdots, B_{i+1} r^i$. Suppose that $\widetilde{B}_i(\theta) = \sum B_{i_1} B_{i_2} \cdots B_{i_m}$. Then $\sum_{j=1}^{m} (i_j - 1) = i$. Since $i_j \geq 2$, the largest value for m should be i. Further, we have

$$\deg(B_{i_1} B_{i_2} \cdots B_{i_m}, \{\sin(\theta), \cos(\theta)\}) = \sum_{j=1}^{m} (i_j + 1) = i + 2m \leq 3i. \tag{4.13}$$

Therefore, $\deg(\widetilde{B}_i, \{\sin(\theta), \cos(\theta)\}) = 3i$, and from (4.13), we see $\widetilde{R}_i(\theta)$ is odd (even) in $\sin(\theta)$ and $\cos(\theta)$ if i is odd (even).

Clearly, we have

$$\frac{\sum_{i=2}^{+\infty} A_i(\theta) r^i}{1 + \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1}} = \Big(\sum_{i=2}^{+\infty} A_i(\theta) r^i \Big) \Big(1 + \sum_{i=1}^{+\infty} \widetilde{B}_i(\theta) r^i \Big)$$

Combining the above equation with (4.11) and (4.12) yields $R_i(\theta) = \sum_{j=2}^{i-1} A_j(\theta) \widetilde{B}_{i-j}(\theta) + A_i(\theta)$. Finally, taking into account that $A_j(\theta)$ is a homogeneous polynomial in $\sin(\theta)$ and $\cos(\theta)$ of degree $j+1$ for any $j \geq 2$, the proof is complete.

Further, assume that $r^j(\theta, \rho) = \sum_{i=j}^{+\infty} r_{j,i}(\theta) \rho^i$ for any $j \geq 2$. Substituting equation (4.10) into system (4.12) and comparing the coefficients yields $r_1'(\theta) = 0$ and

$$r_i'(\theta) = R_i(\theta) + R_{i-1}(\theta) r_{i-1,i}(\theta) + \cdots + R_2(\theta) r_{2,i}(\theta), \ i \geq 2. \tag{4.14}$$

It is easy to get $r_1(\theta) = 1$, $r_2(\theta) = \int_0^\theta R_2(\theta) d\theta$ and

$$r_3(\theta) = \int_0^\theta (R_3(\theta) + 2R_2(\theta) r_2(\theta)) d\theta = \int_0^\theta R_3(\theta) d\theta + r_2^2(\theta).$$

But computation of $r_i(\theta)$ becomes more and more involved by direct integration, as i grows. To overcome this difficulty, we present a new method to compute $r_i(\theta)$, which is closely related the proof of the following theorem.

Theorem 4.3.2 *Suppose $r(\theta,\rho)$ is the solution of system (4.7) with $r(0,\rho)=\rho$, and let (4.10) hold. Then for any $i\geq 1$, we have*

$$r_i(\theta)=\sum_{j=1}^{3i-3}(S_{i,j}(\theta)\sin^j(\theta)+C_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta))+C_{i,0}(\theta), \tag{4.15}$$

where $S_{i,j}(\theta)$ and $C_{i,j}(\theta)$ are polynomials in θ.

Proof We apply the method of mathematical induction to prove this lemma. It is easy to see that the conclusion is true for $i=1$, since $r_1(\theta)=1$. Now, suppose (4.15) holds for $i-1$. Then, we prove that (4.15) also holds for i.

Firstly, we need to prove that for any $2\leq j\leq i-1$, $\deg(r_{j,i}(\theta),\{\sin(\theta),\cos(\theta)\})=3(i-j)$. Note that

$$r^j(\theta,\rho)=\rho^j(1+r_2(\theta)\rho+r_3(\theta)\rho^2+\cdots)^j=\rho^j(1+r_{j,j+1}(\theta)\rho+r_{j,j+2}(\theta)\rho^2+\cdots).$$

Thus, $r_{j,i}(\theta)\rho^{i-j}$ should be a linear combination of products of $r_k(\theta)\rho^{k-1}$, $2\leq k\leq i-1$. Suppose that $r_{j,i}(\theta)=\sum r_{i_1}r_{i_2}\cdots r_{i_n}$, where $i_k\leq i-1$, $k=1,\ldots,n$. Then $\sum_{k=1}^{n}(i_k-1)=i-j$. Since $\deg(r_{i_k}(\theta),\{\sin(\theta),\cos(\theta)\})=3(i_k-1)$, we have

$$\deg(r_{j,i}(\theta),\{\sin(\theta),\cos(\theta)\})=\max(\sum_{k=1}^{n}3(i_k-1))=3(i-j).$$

From Lemma 4.3.1, we know $\deg(R_j(\theta),\{\sin(\theta),\cos(\theta)\})=3(j-1)$. Then, the right hand-side of equation (4.14) has degree $3(i-1)$ in $\sin(\theta)$ and $\cos(\theta)$. Applying $\sin^2(\theta)+\cos^2(\theta)=1$ to equation (4.14) and decreasing the degree in $\cos(\theta)$ gives

$$r_i'(\theta)=\sum_{j=1}^{3i-3}(T_{i,j}(\theta)\sin^j(\theta)+D_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta))+D_{i,0}(\theta), \tag{4.16}$$

where $T_{i,j}(\theta)$ and $D_{i,j}(\theta)$ are polynomials in θ. Then,

$$r_i(\theta)=\sum_{j=1}^{3i-3}\Big(\int_0^\theta T_{i,j}(\theta)\sin^j(\theta)d\theta+\int_0^\theta D_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta)d\theta\Big)+\int_0^\theta D_{i,0}(\theta)d\theta,$$

On the other hand, for any polynomial $f(\theta)$ and number j we have

$$\int f(\theta)\sin^j(\theta)\cos(\theta)d\theta=\frac{1}{j+1}f(\theta)\sin^{j+1}(\theta)-\frac{1}{j+1}\int f'(\theta)\sin^{j+1}(\theta)d\theta, \tag{4.17}$$

and

$$\begin{aligned}
&\int f(\theta)\sin^{j+1}(\theta)d\theta=\int f(\theta)\sin^j(\theta)d(-\cos(\theta))\\
=&-f(\theta)\sin^j(\theta)\cos(\theta)+\int f'(\theta)\sin^j(\theta)\cos(\theta)d\theta+j\int f(\theta)\sin^{j-1}(\theta)\cos^2(\theta)d\theta\\
=&-f(\theta)\sin^j(\theta)\cos(\theta)+\frac{1}{j+1}f'(\theta)\sin^{j+1}(\theta)-\frac{1}{j+1}\int f''(\theta)\sin^{j+1}(\theta)d\theta\\
&+j\int f(\theta)\sin^{j-1}(\theta)d\theta-j\int f(\theta)\sin^{j+1}(\theta)d\theta.
\end{aligned}$$

Hence,

$$\begin{aligned}\int f(\theta)\sin^{j+1}(\theta)d\theta = &-\frac{1}{j+1}f(\theta)\sin^{j}(\theta)\cos(\theta)+\frac{1}{(j+1)^2}f'(\theta)\sin^{j+1}(\theta)\\ &-\frac{1}{(j+1)^2}\int f''(\theta)\sin^{j+1}(\theta)d\theta+\frac{j}{j+1}\int f(\theta)\sin^{j-1}(\theta)d\theta.\end{aligned} \tag{4.18}$$

It follows from equations (4.17) and (4.18) that the conclusion is true for i, and thus the proof is complete.

In the above proof, the precedure of computing $r_i(\theta)$ is present: (1) to compute $r_{j,i}(\theta)$, $2 \le j \le i-1$; (2) to substitute $r_{j,i}(\theta)$ into (4.14), and to apply $\cos^2(\theta) = 1-\sin^2(\theta)$ to get (4.16); (3) for any j in descending order, to use (4.17) and (4.18) repeatedly to compute $\int_0^\theta T_{i,j}(\theta)\sin^j(\theta)d\theta$ and $\int_0^\theta D_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta)d\theta$ by decreasing the degrees of polynomials $T_{i,j}(\theta)$ and $D_{i,j}(\theta)$; and (4) to compute $\int_0^\theta D_{i,0}(\theta)d\theta$.

4.4 Proofs of Theorems 4.1.1 and 4.1.2

Now, we are ready to prove Theorems 4.1.1 and 4.1.2.

Proof of Theorem 4.1.1 Without loss of any generality, we suppose $b_6 = 0$ since $a_6b_6 = 0$. Denote by $C(\mathcal{E})$ the condition which is obtained from the condition C with the interchange of variables $\mathcal{E}$.

For system (4.3), we have $\delta = 0$ due to $V_1 = 0$, as we have discussed before. From the second Lyapunov constant $V_2 = \frac{2}{3}(a_5 - b_5)$, we solove $V_2 = 0$ to get $b_5 = a_5$. Then, we obtain $V_3 = -\frac{\pi}{8}(a_2 - a_5)a_6$.

First, we assume $a_6 \neq 0$. Then, $V_3 = 0$ yields $a_2 = a_5$. Further, by linearly solving $V_4 = 0$ for b_4, we have

$$b_4 = \frac{1}{a_5b_3}[2a_5^3 - b_2a_5^2 - (3a_3^2 - a_3a_4 + 6a_3a_6 - 2a_4a_6 + b_2^2)a_5 + 3b_2b_3^2], \quad a_5b_3 \neq 0. \tag{4.19}$$

In case $a_5 = 0$, we have $V_4 = \frac{2}{5}b_2b_3^2$, which yields the center condition I by solving $V_4 = 0$. If $a_5 \neq 0$ and $b_3 = 0$, we obtain

$$a_4 = \frac{1}{a_3 + 2a_6}(3a_3^2 + 6a_3a_6 - 2a_5^2 + a_5b_2 + b_2^2), \tag{4.20}$$

by solving $V_4 = 0$ when $a_3 + 2a_6 \neq 0$. Under the condition (4.20), V_5 is given by

$$V_5 = \frac{\pi a_5a_6}{48(a_3+2a_6)^2}(b_2+2a_5)(b_2-a_5)(5a_3a_6 + 2a_5^2 - a_5b_2 + 10a_6^2 - b_2^2).$$

From $V_5 = 0$, we have condition II if $(b_2 + 2a_5)(b_2 - a_5) = 0$, or another equation

$$a_3 = -\frac{1}{5a_6}(2a_5^2 - a_5b_2 + 10a_6^2 - b_2^2).$$

When the above equation holds, V_6 and V_7 are given by $V_6 = \frac{2a_5}{875a_6^2}F_{11}$ and $V_7 = \frac{\pi a_5 a_6}{64}F_{12}$, where

$$\begin{aligned}F_{11} =& -3b_2^6 - 9a_5b_2^5 + (9a_5^2 + 30a_6^2)b_2^4 + (33a_5^3 + 60a_5a_6^2)b_2^3 - (18a_5^4 + 90a_5^2a_6^2 - 50a_6^4)b_2^2\\ &- (36a_5^5 + 120a_5^3a_6^2 - 50a_5a_6^4)b_2 + 24a_5^6 + 120a_5^4a_6^2 - 350a_5^2a_6^4,\\ F_{12} =& b_2^4 + 2a_5b_2^3 - (3a_5^2 + 5a_6^2)b_2^2 - (4a_5^3 + 5a_5a_6^2)b_2 + 4a_5^4 + 35a_5^2a_6^2.\end{aligned}$$

Then $\mathrm{Res}(F_{11}, F_{12}, b_2) = 244140625a_6^{12}a_5^8(9a_5^2 + 40a_6^2)^2 \neq 0$ since $a_5a_6 \neq 0$, which means V_6 and V_7 do not have common solutions.

If $b_3 = a_3 + 2a_6 = 0$, we have

$$\begin{aligned}V_4 &= -\frac{2a_5}{15}(b_2 + 2a_5)(b_2 - a_5), \quad V_5 = -\frac{\pi a_5a_6}{48}(a_4 + 6a_6)(a_4 + a_6),\\ V_6 &= \frac{4a_5a_6}{315}(a_4 + 6a_6)[2(a_4 + a_6)(a_4 - 4a_6) + 9a_5^2].\end{aligned}$$

Thus, $V_4 = V_5 = V_6 = 0$ yields $(b_2 + 2a_5)(b_2 - a_5) = a_4 + 6a_6 = 0$, which are clearly included in the condition II.

When (4.19) holds, we obtain

$$V_5 = -\frac{\pi a_5a_6}{48}(3a_3 - a_4)(3a_3 - a_4 + 5a_6). \tag{4.21}$$

Taking $a_4 = 3a_3$ yields $V_5 = 0$ and $V_6 = -\frac{2b_3^2}{21}b_2(a_5 - b_2)(4a_5 + 3b_2)$. Setting $V_6 = 0$ yields $b_2(a_5 - b_2) = 0$, which gives the conditions III and IV, or $b_2 = -\frac{4}{3}a_5$ which results in $V_7 \equiv 0$ but $V_8 = \frac{448}{2187}a_5^5b_3^2 \neq 0$ since $a_5b_3 \neq 0$.

For (4.21), if $a_4 = 3a_3 + 5a_6$, we have $V_5 = 0$ and

$$B_3 = \frac{3a_5a_6}{b_2D_{21}}(3a_3^3 + 12a_3^2a_6 + 10a_3a_6^2 + 2a_5^2a_6 - 4a_6^3), \quad b_2D_{21} \neq 0, \tag{4.22}$$

obtained by linearly solving $V_6 = 0$, where $B_3 = b_3^2$ and $D_{21} = 9a_3a_6 + 4a_5^2 - a_5b_2 + 18a_6^2 - 3b_2^2$. If $b_2 = 0$, we have $V_6 = \frac{2a_5a_6}{7}F_{21}$ and $V_7 = \frac{25\pi a_5a_6^3}{64}F_{22}$, where

$$F_{21} = 3a_3^3 + 12a_3^2a_6 + 10a_3a_6^2 + 2a_5^2a_6 - 4a_6^3, \quad F_{22} = a_3^2 + 3a_3a_6 + a_5^2 + 2a_6^2.$$

Then, $\mathrm{Res}(F_{21}, F_{22}, b_2) = a_5^4(9a_5^2 + 40a_6^2) \neq 0$, which means that there are no center conditions in this case. If $D_{21} = 0$, we have $a_3 = -\frac{1}{9a_6}(4a_5^2 - a_5b_2 + 18a_6^2 - 3b_2^2)$, and $V_6 = -\frac{2a_5}{1701a_6^2}F_{23}$, $V_7 = \frac{25\pi a_5a_6}{5184}F_{24}$, where F_{23} and F_{24} are polynomials in a_5, a_6 and b_2. Similarly, it can be easily shown that the two equations, $V_6 = V_7 = 0$, do not have solutions by verifying $\mathrm{Res}(F_{23}, F_{24}, b_2) \neq 0$.

Now suppose (4.22) holds, we have $V_7 = \frac{25\pi a_5a_6^3}{64}F_{31}$, $V_8 = \frac{2a_5a_6}{27D_{21}}(F_{32}a_3 + D_{31})$, $V_9 \equiv 0$ and $V_{10} = \frac{2a_5a_6}{18711D_{21}^2}F_{33}$ with

$$\begin{aligned}F_{31} =& a_3^2 + 3a_3a_6 + a_5^2 + 2a_6^2,\\ F_{32} =& a_6[-a_5^6 + b_2a_5^5 - (17a_6^2 + 18b_2^2)a_5^4 - (13a_6^2b_2 - 9b_2^3)a_5^3\\ &- (30a_6^4 + 60a_6^2b_2^2 - 9b_2^4)a_5^2 + (-30a_6^4b_2 + 90a_6^2b_2^3)a_5 + 90a_6^2b_2^4],\\ D_{31} =& 16a_5^6 + 14b_2a_5^5 + (16a_6^2 + 24b_2^2)a_5^4 + (14a_6^2b_2 - 27b_2^3)a_5^3\\ &+ (15a_6^4 + 30a_6^2b_2^2 - 27b_2^4)a_5^2 + (15a_6^4b_2 - 45a_6^2b_2^3)a_5 - 45a_6^2b_2^4.\end{aligned}$$

If $D_{31} = 0$, it follows from $V_8 = 0$ that $F_{32} = 0$. Note that D_{31} and F_{32} are homogeneous polynomials in a_6, a_5 and b_2. Thus, by a variable scaling: $a_5 \to a_5a_6$ and $b_2 \to b_2a_6$, we can eliminate a_6. Without loss of any generality, we take $a_6 = 1$, and then obtain $\mathrm{Res}(F_{32}, D_{31}, b_2) = -21870a_5^{24}(3862879a_5^6 + 35074080a_5^4 + 92750400a_5^2 + 50112000) \neq 0$ for nonzero a_5. This indicates that there are no zeros for the equations: $D_{31} = F_{32} = 0$. If $D_{31} \neq 0$, we have $a_3 = -\frac{F_{32}}{D_{31}}$, and $F_{31} = \frac{a_5^4}{D_{31}^2}\widetilde{F}_{31}$, $F_{33} = \frac{a_5^4a_6}{D_{31}^8}\widetilde{F}_{33}$, where $\widetilde{F}_{31}$ and $\widetilde{F}_{33}$ are homogeneous polynomials in a_6, a_5 and b_2. Similarly, by verifying $\mathrm{Res}(\widetilde{F}_{31}, \widetilde{F}_{33}, b_2) \neq 0$, we conclude that $V_7 = V_{10} = 0$ do not have zeros when $a_5a_6 \neq 0$.

Now we consider the case $a_6 = 0$, for which $V_3 = 0$, and get

$$b_4 = \frac{1}{a_5b_3}[(a_2 - b_2)a_5^2 + (a_2^2 + a_3a_4 - b_2^2)a_5 - 3a_2a_3^2 + 3b_2b_3^2], \quad a_5b_3 \neq 0, \tag{4.23}$$

by solving $V_4 = 0$. If $b_3 = 0$, $V_4 = 0$ yields $a_4 = -\frac{1}{a_3a_5}[a_5a_2^2 - (3a_3^2 + a_5^2)a_2 - b_2a_5^2 - b_2^2a_5]$ provided $a_3a_5 \neq 0$. Further, we have $V_5 \equiv 0$, $V_6 = -\frac{2a_3^2}{105}a_2F_{41}$, $V_7 \equiv 0$, and $V_8 = -\frac{2a_3^2}{2835a_5}a_2F_{42}$, where

$$\begin{aligned}
F_{41} &= 15a_2^2 + 5a_2a_5 - 2a_5^2 - 9a_5b_2 - 9b_2^2,\\
F_{42} &= 315a_5a_2^4 + (675a_3^2 + 315a_5^2)a_2^3 + (225a_3^2a_5 + 1890a_5^3 + 225a_5^2b_2 + 225a_5b_2^2)a_2^2\\
&\quad - (90a_3^2a_5^2 + 405a_3^2a_5b_2 + 405a_3^2b_2^2 - 602a_5^4 - 54a_5^3b_2 - 54a_5^2b_2^2)a_2\\
&\quad - 248a_5^5 - 1176a_5^4b_2 - 1446a_5^3b_2^2 - 540a_5^2b_2^3 - 270a_5b_2^4.
\end{aligned}$$

Obviously, $a_2 = 0$ is a solution of $V_6 = V_8 = 0$, resulting in condition V. For $F_{41} = F_{42} = 0$, we have

$$\begin{aligned}
\mathrm{Res}(F_{41}, F_{42}, a_2) = 2700a_5^2\,&(a_5 + 3b_2)^2(2a_5 + 3b_2)^2(b_2 - a_5)(b_2 + 2a_5)\\
&\times (29a_5^2 + 108a_5b_2 + 108b_2^2).
\end{aligned}$$

Solving $F_{41} = F_{42} = \mathrm{Res}(F_{41}, F_{42}, a_2) = 0$, we obtain condition VI, derived from $(a_5 + 3b_2)(2a_5 + 3b_2) = 0$, and other center conditions derived from $(b_2 - a_5)(b_2 + 2a_5) = 0$ are already included in II. If $b_3 = a_3 = 0$, $V_4 = \frac{2}{15}a_5(a_2 - b_2)(a_2 + a_5 + b_2)$. Solving $V_4 = 0$ we get VII and VIII. If $b_3 = a_5 = 0$, center conditions obtained from $V_4 = 0$ are included in I or VII, where $V_4 = -\frac{2}{5}a_2a_3^2$. If $b_3 \neq 0$ and $a_5 = 0$, we get $b_2 = \frac{1}{b_3^2}a_2a_3^2$ from $V_4 = 0$. Then $V_5 \equiv 0$ and $V_6 = \frac{2}{35b_3^4}a_2a_3^2(8a_2^2a_3^4 - 8a_2^2b_3^4 - 3a_3a_4b_3^4 + 3b_3^5b_4)$. When $a_2a_3 = 0$, we get subcases of I and I($\mathcal{E}$). Otherwise, we linearly solve $V_6 = 0$ using b_4, for which $V_7 \equiv 0$, and further obtain

$$V_8 = \frac{2a_2^3a_3^2}{945b_3^8}(a_3^2 - b_3^2)[-75a_3b_3^4(a_3^2 + b_3^2)a_4 + 105a_3^4(a_3^2 + b_3^2)a_2^2 - 95b_3^4(a_3^2 + b_3^2)a_2^2 + 21a_3^2b_3^6].$$

When $a_3^2 - b_3^2 = 0$, we get subcases of X and XI. Otherwise, we linearly solve $V_8 = 0$ using a_4, and get

$$\begin{aligned}
V_{10} &= -\frac{4a_2^3a_3^2(a_3^2 - b_3^2)}{37125b_3^{12}(a_3^2 + b_3^2)}F_{43}, \quad V_{12} = -\frac{4a_2^3a_3^2(a_3^2 - b_3^2)}{4021875b_3^{16}(a_3^2 + b_3^2)^2}F_{44},\\
V_{14} &= -\frac{4a_2^3a_3^2(a_3^2 - b_3^2)}{4524609375b_3^{20}(a_3^2 + b_3^2)^3}F_{45}, \quad V_9 = V_{11} = V_{13} \equiv 0,
\end{aligned} \tag{4.24}$$

where F_{43}, F_{44} and F_{45} are homogeneous polynomials in a_3, a_2 and b_3. Thus, without loss of generality, we take $b_3 = 1$ and obtain

$$\begin{aligned}
F_{43} &= 150a_2^4a_3^{12} + 300a_2^4a_3^{10} - (50a_2^4 - 465a_2^2)a_3^8 - (400a_2^4 - 1240a_2^2)a_3^6 - (50a_2^4 - 1240a_2^2 \\
&\quad + 48)a_3^4 + (300a_2^4 + 465a_2^2)a_3^2 + 150a_2^4, \\
F_{44} &= 35250a_2^6a_3^{18} + 105750a_2^6a_3^{16} + (90250a_2^6 + 111675a_2^4)a_3^{14} - (11250a_2^6 - 403975a_2^4)a_3^{12} \\
&\quad - (62000a_2^6 - 675150a_2^4 + 4335a_2^2)a_3^{10} - (62000a_2^6 - 765700a_2^4 - 4165a_2^2)a_3^8 \\
&\quad - (11250a_2^6 - 675150a_2^4 - 4165a_2^2 + 2352)a_3^6 + (90250a_2^6 + 403975a_2^4 - 4335a_2^2)a_3^4 \\
&\quad + (105750a_2^6 + 111675a_2^4)a_3^2 + 35250a_2^6, \\
F_{45} &= 64267500a_2^8a_3^{24} + 257070000a_2^8a_3^{22} + (374412500a_2^8 + 209282250a_2^6)a_3^{20} \\
&\quad + (212300000a_2^8 + 963353000a_2^6)a_3^{18} - (5995000a_2^8 + 2057488500a_2^6 + 9179400a_2^4)a_3^{16} \\
&\quad - (57200000a_2^8 - 2903448125a_2^6 - 82855050a_2^4)a_3^{14} - (41030000a_2^8 - 3282832625a_2^6 \\
&\quad - 209672700a_2^4 + 8036550a_2^2)a_3^{12} - (57200000a_2^8 + 3282832625a_2^6 + 271994100a_2^4 \\
&\quad - 11441925a_2^2)a_3^{10} - (5995000a_2^8 + 2903448125a_2^6 + 209672700a_2^4 - 11441925a_2^2 \\
&\quad - 889056)a_3^8 + (212300000a_2^8 + 2057488500a_2^6 + 82855050a_2^4 - 8036550a_2^2)a_3^6 \\
&\quad + (374412500a_2^8 + 963353000a_2^6 + 9179400a_2^4)a_3^4 + (257070000a_2^8 + 209282250a_2^6)a_3^2 \\
&\quad + 64267500a_2^8,
\end{aligned}$$

from which we have

$$\begin{aligned}
\mathrm{Res}(F_{43}, F_{44}, a_3) &= 4.7937764808221021667122840881347656 25 \times 10^{56} a_2^{80} E_c^2 E_{41}^2, \\
\mathrm{Res}(F_{43}, F_{45}, a_3) &= 1.4812312887061701793811607785755768418312072753906 25 \times 10^{77} \\
&\quad \times a_2^{104} E_c^2 E_{42}^2,
\end{aligned}$$

where $E_c = (5a_2^2 + 1)^2 + 5a_2^2 \neq 0$, and E_{41} and E_{42} are polynomials in a_2 of degrees 16 and 24, respectively, satisfying $\mathrm{Res}(E_{41}, E_{42}, a_2) \neq 0$. Therefore, there are no solutions for $V_{10} = V_{12} = V_{14} = 0$ in (4.24).

With (4.23) holding, we get $V_5 \equiv 0$ and further obtain

$$\begin{aligned}
a_4 = \frac{1}{9a_5a_3E_0}[&(-2a_2a_3^2 + 2b_2b_3^2)a_5^3 + (-4a_2^2a_3^2 + 9a_2b_2b_3^2 - 5b_2^2b_3^2)a_5^2 \\
&+ (6a_2^3a_3^2 + 9a_2^2b_2b_3^2 - 15b_2^3b_3^2)a_5 + 27a_2^2a_3^4 - 27a_2a_3^2b_2b_3^2],
\end{aligned} \tag{4.25}$$

by solving $V_6 = 0$ provided $a_5a_3E_0 \neq 0$, where $E_0 = a_2a_3^2 - b_2b_3^2$. If $a_3E_0 = 0$, we obtain conditions IX–XI, V($\mathcal{E}$), VI($\mathcal{E}$) and a subcase of II($\mathcal{E}$). Here, we omit the details of the discussion for $a_5a_3E_0 \neq 0$, since it is similar to the case $a_5b_3 = 0$ for $V_4 = 0$.

When (4.23) and (4.25) hold, we have $a_3a_5b_3E_0 \neq 0$, $V_7 = V_9 = V_{11} = V_{13} = V_{15} \equiv 0$ and

$$\begin{aligned}
V_8 &= \frac{2F_1}{1701a_5E_0}, \quad V_{10} = \frac{2F_2}{56133a_5^2E_0^2}, \quad V_{12} = \frac{2F_3}{19702683a_5^3E_0^3}, \\
V_{14} &= \frac{2F_4}{6206345145a_5^4E_0^4}, \quad V_{16} = \frac{2F_5}{949570807185a_5^5E_0^5},
\end{aligned}$$

where F_j, $1 \le j \le 5$, is a homogeneous polynomial in a_2, a_3, a_5, b_2, b_3, and also a polynomial of a_3^2 and b_3^2. Taking $a_5 = 1$, $A_3 = a_3^2$ and $B_3 = b_3^2$, we have

$$\begin{aligned}
F_1 =& a_2^2[135b_2(3a_2+3b_2+1)(a_2-b_2)B_3 + 2a_2(a_2-1)(3a_2+1)(6a_2+1)]A_3^2\\
&+ a_2b_2[135b_2(3a_2+3b_2+1)(a_2-b_2)B_3 - 189a_2^4 + 450a_2^2b_2^2 - 189b_2^4 - 189a_2^3\\
&\quad + 171a_2^2b_2 + 171a_2b_2^2 - 189b_2^3 - 48a_2^2 + 64a_2b_2 - 48b_2^2 - 2a_2 - 2b_2)]B_3A_3\\
&+ 2b_2^3(b_2-1)(3b_2+1)(6b_2+1)B_3^2,\\
F_2 =& 63[135b_2(3a_2+3b_2+1)(a_2-b_2)B_3 + 2a_2(a_2-1)(3a_2+1)(6a_2+1)]a_2^4A_3^4\\
&- a_2^3[8505b_2^2(3a_2+3b_2+1)(a_2-b_2)B_3^2 - 18b_2(2898a_2^4 - 3150a_2^2b_2^2 + 2421a_2^3\\
&\quad - 1050a_2^2b_2 - 1245a_2b_2^2 + 5387a_2^2 - 415a_2b_2 - 4860b_2^2 + 1634a_2 - 1620b_2)B_3\\
&\quad - 2a_2(a_2-1)(3a_2+1)(672a_2^3 + 416a_2^2 + 1349a_2 + 216)]A_3^3\\
&- a_2^2b_2B_3[8505b_2^2(3a_2+3b_2+1)(a_2-b_2)B_3^2 + 18b_2(a_2-b_2)(3024a_2^3 + 3024a_2^2b_2\\
&\quad + 3024a_2b_2^2 + 3024b_2^3 + 2358a_2^2 + 2553a_2b_2 + 2358b_2^2 + 10191a_2 + 10191b_2\\
&\quad + 3247)B_3 - 16821a_2^6 + 40635a_2^4b_2^2 - 6615a_2^2b_2^4 - 5103b_2^6 - 26397a_2^5\\
&\quad + 15495a_2^4b_2 + 28218a_2^3b_2^2 - 7182a_2^2b_2^3 - 2205a_2b_2^4 - 8505b_2^5 - 47076a_2^4\\
&\quad + 10706a_2^3b_2 + 99634a_2^2b_2^2 - 2394a_2b_2^3 - 45612b_2^4 - 47564a_2^3 + 38500a_2^2b_2\\
&\quad + 36100a_2b_2^2 - 41832b_2^3 - 13986a_2^2 + 13724a_2b_2 - 10424b_2^2 - 864a_2 - 432b_2]A_3^2\\
&+ a_2b_2^2B_3^2[8505b_2^2(3a_2+3b_2+1)(a_2-b_2)B_3^2 + 18b_2(3150a_2^2b_2^2 - 2898b_2^4\\
&\quad + 1245a_2^2b_2 + 1050a_2b_2^2 - 2421b_2^3 + 4860a_2^2 + 415a_2b_2 - 5387b_2^2 + 1620a_2\\
&\quad - 1634b_2)B_3 - 5103a_2^6 - 6615a_2^4b_2^2 + 40635a_2^2b_2^4 - 16821b_2^6 - 8505a_2^5\\
&\quad - 2205a_2^4b_2 - 7182a_2^3b_2^2 + 28218a_2^2b_2^3 + 15495a_2b_2^4 - 26397b_2^5 - 45612a_2^4\\
&\quad - 2394a_2^3b_2 + 99634a_2^2b_2^2 + 10706a_2b_2^3 - 47076b_2^4 - 41832a_2^3 + 36100a_2^2b_2\\
&\quad + 38500a_2b_2^2 - 47564b_2^3 - 10424a_2^2 + 13724a_2b_2 - 13986b_2^2 - 432a_2 - 864b_2]A_3\\
&- 2b_2^4(3b_2+1)(b_2-1)(378B_3b_2^2 + 672b_2^3 + 63B_3b_2 + 416b_2^2 + 1349b_2 + 216)B_3^3.
\end{aligned}$$

The other three lengthy polynomials F_3, F_4 and F_5 are omitted here for brevity. In order to solve $F_1 = F_2 = F_3 = F_4 = F_5 = 0$, we compute the following resultants

$$\begin{aligned}
&\mathrm{Res}(F_1, F_2, A_3) = -5292F_aE_1, && \mathrm{Res}(F_1, F_3, A_3) = -47628F_aF_bE_2,\\
&\mathrm{Res}(F_1, F_4, A_3) = -7001316F_aF_b^2E_3, && \mathrm{Res}(F_1, F_5, A_3) = -7001316F_aF_b^3E_4,
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
F_a =& B_3^6a_2^8b_2^7(3b_2+1)(b_2-1)(3a_2+3b_2+1)^2(a_2-b_2)^6\\
&\times(405B_3a_2^2b_2 - 405B_3b_2^3 + 36a_2^4 + 135B_3a_2b_2 - 135B_3b_2^2 - 18a_2^3 - 16a_2^2 - 2a_2),\\
F_b =& B_3^2a_2^4b_2^2(3a_2+3b_2+1)(a_2-b_2)^2,
\end{aligned}$$

and E_j, $1 \le j \le 4$, is a polynomial of B_3, a_2 and b_2. Note that F_a contains all the common factors of resultants $\mathrm{Res}(F_1, F_j, A_3)$, $j = 2, 3, 4, 5$. Then conditions III($\mathcal{E}$), XII, XIII and XIII($\mathcal{E}$) can be easily obtained if $a_2b_2(a_2 - b_2) = 0$, For example, taking $a_2 = 0$ we have $F_1 = 2b_2^3(b_2 -$

$1)(3b_2+1)(6b_2+1)B_3^2$, and then $b_2 \neq 0$ since $E_0 \neq 0$. We can get condtion III($\mathcal{E}$) if $b_2 = 1$ and condition XII if $b_2 = -\frac{1}{3}$. If $b_2 = -\frac{1}{6}$, then $F_2 = -\frac{49}{139968}B_3^3 < 0$. Note that $E_0 \neq 0$, $A_3 > 0$ and $B_3 > 0$ from (4.23) and (4.25). The rest factors in F_a can not lead to new center conditions. Here, we only present the details for the case $b_2 - 1 = 0$ with $a_2b_2(a_2 - b_2) \neq 0$. Similar procedures can be applied to other cases.

Assume $a_2b_2(a_2 - b_2) \neq 0$. When $b_2 = 1$, we have

$$\begin{aligned} F_1 &= a_2A_3(a_2-1)G_1, \quad F_2 = a_2A_3(a_2-1)G_2,\\ F_3 &= a_2A_3(a_2-1)G_3, \quad F_4 = a_2A_3(a_2-1)G_4, \end{aligned} \tag{4.27}$$

where G_j, $1 \leq j \leq 4$, is a polynomial in a_2, A_3 and B_3. For (4.27), $a_2 \neq 1$ since $b_2 = 1$ and $a_2 - b_2 \neq 0$. Then

$$\begin{aligned} \mathrm{Res}(G_1, G_2, B_3) &= -714420A_3^2a_2^3(1+3a_2)(3a_2+4)^3(a_2-1)^2G_5,\\ \mathrm{Res}(G_1, G_3, B_3) &= \quad 173604060A_3^3a_2^4(1+3a_2)(3a_2+4)^4(a_2-1)^3G_6,\\ \mathrm{Res}(G_1, G_4, B_3) &= -2067103542420A_3^4a_2^5(1+3a_2)(3a_2+4)^5(a_2-1)^4G_7, \end{aligned}$$

where G_j, $5 \leq j \leq 7$, is a polynomial in a_2 and A_3. We consider the common factor $(1 + 3a_2)(3a_2 + 4) = 0$ firstly. If $a_2 = -\frac{1}{3}$, then $G_1 = -B_3I_{11}$ and $G_2 = \frac{1}{9}B_3I_{12}$, where $I_{11} = 135A_3 + 405B_3 + 328$ and

$$\begin{aligned} I_{12} =& 229635B_3^3 + (76545A_3 + 1499310)B_3^2 - (25515A_3^2 - 758700A_3 - 1052136)B_3\\ &- 8505A_3^3 + 86310A_3^2 + 256312A_3, \end{aligned}$$

satisfying $\mathrm{Res}(I_{11}, I_{12}, B_3) = 14696640(1080A_3 + 42107) > 0$. Thus, there are no solutions for $G_1 = G_2 = G_3 = G_4 = 0$ if $a_2 = -\frac{1}{3}$. If $a_2 = -\frac{4}{3}$, then $G_1 = \frac{56}{3}(4A_3 + 3B_3) > 0$. Next we consider $G_5 = G_6 = G_7 = 0$, and have

$$\begin{aligned} \mathrm{Res}(G_5, G_6, A_3) =& 2789427520800a_2^8(a_2-8)(3a_2+5)(3a_2-8)(3a_2+4)^2G_aG_{81}G_{82},\\ \mathrm{Res}(G_5, G_7, A_3) =& -376572715308000a_2^{12}(a_2-8)(3a_2+5)(3a_2-8)(3a_2+4)^3G_aG_{91}G_{92}, \end{aligned}$$

where $G_a = 486a_2^4 + 945a_2^3 - 1227a_2^2 - 2779a_2 - 428$, and G_{j1} and G_{j2}, $j = 8, 9$ are polynomials in a_2. If $a_2 = 8$, we have $G_5 = 12544(9A_3 + 196)I_{21}$ and $G_6 = 39337984(9A_3 + 196)I_{22}$, where $I_{21} = 244111680A_3 - 240599605103$ and

$$\begin{aligned} I_{22} =& 672679027641600A_3^3 + 95795633236828680A_3^2 - 2828944931798009164 77A_3\\ &- 4376089823211446777789, \end{aligned}$$

satisfying $\mathrm{Res}(I_{21}, I_{22}, A_3) \neq 0$. Thus, there are no solutions for $G_5 = G_6 = G_7 = 0$ if $a_2 = 8$. In a similar way, it can be proved that no solutions exist for $G_5 = G_6 = G_7 = 0$ if $(3a_2 + 5)(3a_2 - 8)(3a_2 + 4) = 0$. If $G_a = 0$, we compute $I_{31} = \mathrm{Res}(G_a, G_5, a_2)$ and $I_{32} = \mathrm{Res}(G_a, G_6, a_2)$, with $\mathrm{Res}(I_{31}, I_{32}, A_3) \neq 0$. Moreover, we get $\mathrm{Res}(G_{8i}, G_{9j}, a_2) \neq 0$ for $i, j = 1, 2$, and then there are no solutions for $G_{8i} = G_{9j} = 0$. Therefore, if $b_2 = 1$, no zeros exist for $F_1 = F_2 = F_3 = F_4 = 0$.

For (4.26), we consider $E_1 = E_2 = E_3 = E_4 = 0$ with $a_5A_3E_0F_a \neq 0$, and get

$$\begin{aligned} \mathrm{Res}(E_1, E_2, B_3) &= -3050238993994800F_cE_5,\\ \mathrm{Res}(E_1, E_3, B_3) &= -900567811781994726000F_cF_dE_6,\\ \mathrm{Res}(E_1, E_4, B_3) &= -3876649133536003979359344600000F_cF_d^2E_7, \end{aligned} \tag{4.28}$$

where

$$
\begin{aligned}
F_c &= a_2^3 b_2^{19}(3a_2+3b_2+1)^2(3a_2+1)^2(a_2-1)^2(3b_2+2+3a_2)(a_2+b_2-1)E_aE_bE_cE_d,\\
F_d &= a_2 b_2^9(3a_2+3b_2+1)(3a_2+1)(a_2-1),\\
E_a &= a_2^2-(7b_2+1)a_2+b_2^2-b_2, \quad E_b = 3a_2^2-(6b_2+2)a_2+3b_2^2-2b_2-1,\\
E_c &= 486a_2^4+(486b_2+459)a_2^3-(1134b_2^2+207b_2-114)a_2^2\\
&\quad -(1134b_2^3+1323b_2^2+321b_2+1)a_2-189b_2^3-189b_2^2-48b_2-2,\\
E_d &= (1134b_2+189)a_2^3+(1134b_2^2+1323b_2+189)a_2^2\\
&\quad -(486b_2^3-207b_2^2-321b_2-48)a_2-486b_2^4-459b_2^3-114b_2^2+b_2+2.
\end{aligned}
$$

F_c contains all the common factors of resultants $\mathrm{Res}(E_1, E_j, B_3)$, $j = 2, 3, 4$.

If $a_2 + b_2 - 1 = 0$, we have $a_2 = -b_2 + 1$ and

$$E_1 = 2b_2 I_a I_{41}, \quad E_2 = 4b_2^2 I_a I_{42}, \quad E_3 = 8b_2^3 I_a I_{43},$$

where $I_a = (3b_2-2)B_3+(2b_2-1)^2$. Then $I_a = 0$ yields $E_1 = E_2 = E_3 = 0$, i.e., $\mathrm{Res}(F_1, F_j, A_3) = 0$, $j = 2, 3, 4$, when $a_2 = -b_2 + 1$. We substitute $a_2 = -b_2 + 1$ into F_j, and get $\widetilde{F}_j$, $j = 1, 2, 3$. Next, we need to solve $I_a = \widetilde{F}_1 = \widetilde{F}_2 = \widetilde{F}_3 = 0$. We have

$$
\begin{aligned}
&\mathrm{Res}(I_a, \widetilde{F}_1, B_3) = 2b_2(b_2-1)I_b I_{51}, \quad \mathrm{Res}(I_a, \widetilde{F}_2, B_3) = 2b_2(b_2-1)I_b I_{52},\\
&\mathrm{Res}(I_a, \widetilde{F}_3, B_3) = 2b_2(b_2-1)I_b I_{53}, \quad \text{with } I_b = (3b_2-1)A_3-(2b_2-1)^2.
\end{aligned}
$$

Note that $b_2(b_2 - 1) \neq 0$ since $F_a \neq 0$. If $I_a = I_b = 0$, then $\widetilde{F}_1 = \widetilde{F}_2 = \widetilde{F}_3 = 0$, and we have condition XIV. For $I_{51} = I_{52} = I_{53} = 0$, we have

$$
\begin{aligned}
\mathrm{Res}(I_{51}, I_{52}, A_3) &= \quad 7b_2^3(3b_2+1)(b_2-1)^3(3b_2-2)^5(2b_2-1)^6 J_a J_1,\\
\mathrm{Res}(I_{51}, I_{53}, A_3) &= -21b_2^4(3b_2+1)(b_2-1)^5(3b_2-2)^8(2b_2-1)^9 J_a J_2,
\end{aligned}
$$

where $(3b_2 + 1)(b_2 - 1) \neq 0$, and

$$
\begin{aligned}
J_a &= 18b_2^3+651b_2^2-748b_2+214,\\
J_1 &= 605304b_2^4-2895060b_2^3+2555877b_2^2-373639b_2-148730,\\
J_2 &= 378882563472b_2^{10}-29071087999056b_2^9+1809686685986 10b_2^8-499455418644927b_2^7\\
&\quad +1319463134471394b_2^6-2296405188740916b_2^5+2157213472303974b_2^4\\
&\quad -1020839133559269b_2^3+181189011015338b_2^2+20664548818076b_2\\
&\quad -8460097956280.
\end{aligned}
$$

If $3b_2 - 2 = 0$, then $I_a = \frac{1}{9} \neq 0$. If $2b_2 - 1 = 0$, then $I_a = -\frac{1}{2}B_3 \neq 0$. Moreover, we get $\mathrm{Res}(J_a, I_{51}, b_2) \neq 0$ and $\mathrm{Res}(J_1, J_2, b_2) \neq 0$. Thus, there are no solutions for $I_{51} = I_{52} = I_{53} = 0$. For $I_{41} = I_{42} = I_{43} = 0$, we have

$$
\begin{aligned}
&\mathrm{Res}(I_{41}, I_{42}, B_3) = 180754903347840 00b_2^8(b_2-1)^2(3b_2-4)^2 I_c I_d J_3,\\
&\mathrm{Res}(I_{41}, I_{43}, B_3) = 106733962877866041600 00b_2^{12}(b_2-1)^3(3b_2-4)^3 I_c I_d J_4,
\end{aligned}
$$

where $I_c = 186b_2^2 + 235b_2 - 528$, $I_d = 186b_2^2 - 607b_2 - 107$, and J_3 and J_4 are polynomials of b_2. It is easy to prove that there are no solutions for $I_{41} = I_{42} = I_{43} = 0$.

For the other factors in F_c, using similar procedures, we can show that no more center conditions exist, and thus the details are omitted. Since E_js in (4.28), $j = 5, 6, 7$, are polynomials in a_2 and b_2, it is straightforward to prove that $E_5 = E_6 = E_7 = 0$ can not result in more center conditions. It should be pointed out that although the computations are straightforward, it is really time-consuming and memory demanding.

Now we prove the sufficiency for the center conditions I-XIV by computing their corresponding first integrals. We shall not discuss all the cases one by one. Actually, most of cases belong to three special types of systems. We use the following notation: for any C = I, ..., XIV, C^+ denotes the upper system of system (4.3) under the condition C, C^- the lower system of (4.3) under the condition C.

Firstly, a quadratic Hamiltonian system is given by

$$\dot{x} = -y - Ax^2 + 2Bxy + (C + A)y^2, \quad \dot{y} = x + Bx^2 + 2Axy - By^2$$

with the Hamilonian $H = \frac{1}{2}(x^2 + y^2) + \frac{1}{3}Bx^3 + Ax^2y - Bxy^2 - \frac{1}{3}(A + C)y^3$. Under the conditions I–XIV, the upper systems of II^+–IV^+ are Hamiltonian systems. The general form for I^+ and I^- ($b_2 = 0$) is given by

$$\dot{x} = -y + Ax^2 + By^2, \quad \dot{y} = x + Cxy,$$

with the first integral

$$H = (Cy + 1)^{-2\frac{A}{C}} \left[\frac{1}{2}x^2 + \frac{B}{2(A - C)}y^2 - \frac{A - B - C}{(A - C)(2A - C)}y - \frac{A - B - C}{2A(A - C)(2A - C)} \right],$$

if $C(A - C)(2A - C) \neq 0$; or

$$\begin{aligned} & H = e^{-2Ay}\Big(\frac{1}{2}x^2 + \frac{B}{2A}y^2 - \frac{A - B}{2A^2}y - \frac{A - B}{4A^3}\Big), \quad \text{if } C = 0,\ A \neq 0, \\ \text{or} \quad & H = -\frac{C^3x^2 + B + C}{2(Cy + 1)^2} + \frac{2B + C}{Cy + 1} + B\ln(Cy + 1), \quad \text{if } C \neq 0,\ A = C, \\ \text{or} \quad & H = -\frac{4A^3x^2 + 2A + B}{8A^3(2Ay + 1)} - \frac{A + B}{4A^3}\ln(2Ay + 1) + \frac{By}{4A^2}, \quad \text{if } C \neq 0,\ C = 2A. \end{aligned}$$

Systems VI^+, XII^+ ($b_3 \neq 0$) and $XIII^\pm$ can be written in the form,

$$\dot{x} = -y - Ax^2 + 2Bxy + Ay^2, \quad \dot{y} = x - Bx^2 - 2Axy + By^2,$$

with the first integral

$$H = \frac{4A^2x^2 + 4A^2y^2 - 2Ay - 2Bx + 1}{2Ay + 2Bx - 1}.$$

All the remaining upper systems and lower systems except $X^\pm$, $XI^\pm$ and $XIV^\pm$ can be written in the form,

$$\dot{x} = -y + Axy, \quad \dot{y} = x + Bx^2 + Cxy - By^2, \tag{4.29}$$

with the first integral

$$H = (-Ax+1)^{2B\omega}(Bx+\frac{C}{2}y-\frac{\omega}{2}y+1)^{(\omega+C)A}(Bx+\frac{C}{2}y+\frac{\omega}{2}y+1)^{(\omega-C)A},$$

if $AB(\omega^2-C^2)\omega \neq 0$, where $\omega = \sqrt{4AB+4B^2+C^2}$. When $B=0$, system (4.29) has the first integral

$$\begin{aligned} & H = \frac{1}{2}y^2 - \frac{1}{A^2}(Ax+\ln(1-Ax)), \quad \text{if } A\neq 0, C=0, \\ \text{or} \quad & H = \frac{1}{C^2}(Cy-\ln(Cy+1)) - \frac{1}{A^2}(Ax+\ln(1-Ax)), \quad \text{if } AC\neq 0, \\ \text{or} \quad & H = \frac{1}{C^2}(Cy-\ln(Cy+1)) + \frac{1}{2}x^2, \quad \text{if } A=0,\ C\neq 0. \end{aligned}$$

When $B \neq 0$, we have

$$H = 4\ln(2Bx+Cy+2) - \frac{16B^2}{4B^2+C^2}\ln(|(4B^2+C^2)x+4B|) + \frac{8(Bx+1)}{2Bx+Cy+2}, \ \text{if } \omega = 0.$$

For $B\omega \neq 0$, we obtain the first integral

$$\begin{aligned} & H = \frac{C-\omega}{2B^2\omega}\ln(2Bx+Cy+\omega y+2) - \frac{C+\omega}{2B^2\omega}\ln(2Bx+Cy-\omega y+2) + \frac{1}{B}x, \ \text{if } A=0, \\ \text{or} \quad & H = -\frac{1}{C^2}\ln(Bx+Cy+1) + \frac{B^2+C^2}{B^2C^2}\ln(Bx+1) + \frac{B^2y+C}{B^2C(Bx+1)}, \ \text{if } \omega^2 = C^2. \end{aligned}$$

For the center condition XIV, we have the following first integrals,

$$\begin{aligned} H^{\pm} = & \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{a_5}{3}x^3 \mp \frac{a_5(a_5-2b_2)}{\alpha_{\pm}}x^2y - (a_5-b_2)xy^2 \mp \frac{\alpha_{\pm}}{3}y^3 + \frac{b_2(a_5-b_2)}{4}x^4 \\ & \mp \frac{a_5b_2(a_5-2b_2)}{\alpha_{\pm}}x^3y - \frac{3a_5b_2(\alpha_{\pm}^2-b_2^2)}{2\alpha_{\pm}^2}x^2y^2 \pm \frac{a_5b_2(a_5-b_2)}{\alpha_{\pm}}xy^3 - \frac{a_5^2b_2(a_5-2b_2)}{4\alpha_{\pm}^2}y^4, \end{aligned}$$

where $\alpha_+ = \sqrt{-a_5^2+3a_5b_2} \neq 0$, $\alpha_- = \sqrt{2a_5^2-3a_5b_2} \neq 0$. XIV$^{\pm}(\alpha_+=0)$ and XIV$^{\pm}(\alpha_-=0)$ are in the form of I$^{\pm}$. Under the center condition X, system (4.3) is smooth, and has a center at the origin. Under the center condition XI, system (4.3) is symmetric with respect to the x-axis.

Therefore, for the fourteen center conditions we have obtained the first integrals $H^+(x,y)$ and $H^-(x,y)$ for the upper system and the lower system in (4.3) near the origin. More specifically, for any center conditions I, ..., XIV, either both $H^+(x,0)$ and $H^-(x,0)$ are even functions, or $H^+(x,0) \equiv H^-(x,0)$, or $H^+(x,0) = H^+(\rho,0)$ and $H^-(x,0) = H^-(\rho,0)$ have common zeros $x(\rho)$ satisfying $x(\rho) \to 0^-$ as $\rho \to 0^+$.

Proof of Theorem 4.1.2 For system (4.2), we add perturbations as $a_k = a_k + \varepsilon p_k$ and $b_k = b_k + \varepsilon q_k$, $k = 1, \cdots, 6$, and $\delta = \varepsilon p_0$, where $0 < \varepsilon \ll 1$. Then we get $V_1 = e^{2p_0\pi\varepsilon} - 1$. Taking $p_0 = 0$, we get $V_1 = 0$, and then compute the Lyapunov constants, which are polynomials of ε. To prove the existence of 10 small-amplitude limit cycles, we need to solve the ε-order

Lyapunov constants, i.e. the coefficient $V_{k,1}$ of ε in kth-order Lyapunov constant V_k for all $k > 1$.

First, we get

$$V_{2,1} = \frac{2}{3}(2p_1 + p_5 - 2q_1 - q_5).$$

Setting $p_5 = -2p_1 + 2q_1 + q_5$ yields $V_{2,1} = 0$ and then we obatin

$$V_{3,1} = -\frac{\pi}{8}[(a_4 - 3)(p_1 + q_1) + (1 - a_5)(p_6 + q_6)].$$

Letting

$$p_6 = -\frac{1}{1 - a_5}[(a_4 - 3)(p_1 + q_1) + (1 - a_5)q_6],$$

results in $V_{3,1} = 0$. Similarly, we can linearly solve the polynomial equations one by one, for $V_{4,1} = 0$ using p_4, for $V_{5,1} = 0$ using q_1, for $V_{6,1} = 0$ using p_2, for $V_{8,1} = 0$ using p_3, for $V_{10,1} = 0$ using q_6, ($V_{7,1} = V_{9,1} \equiv 0$) and then obtain

$$V_{12,1} = -\frac{32p_1}{125E_0}F_aF_b, \quad V_{14,1} = -\frac{32p_1}{73125E_0}F_aF_c, \quad V_{11,1} = V_{13,1} \equiv 0,$$

where

$$\begin{aligned}
F_a =& -(a_4 - a_5 - 2)(a_4^2a_5 + a_4a_5^2 - 4a_4a_5 - 2a_5^2 - 3a_4 + a_5 + 10),\\
F_b =& 94623744a_4^{14}a_5^6 + 930466816a_4^{15}a_5^4 + 615054336a_4^{14}a_5^5 - 275404800a_4^{13}a_5^6\\
&+ 1342162944a_4^{12}a_5^7 + 2270969856a_4^{16}a_5^2 + 5488177152a_4^{15}a_5^3 - 2424275968a_4^{14}a_5^4\\
&+ 10977847296a_4^{13}a_5^5 + 9213454848a_4^{12}a_5^6 + 924797952a_4^{11}a_5^7 + 70958592a_4^{10}a_5^8 + \cdots,\\
F_c =& 3643883520a_4^{16}a_5^6 + 703622160384a_4^{14}a_5^8 + 35831521280a_4^{17}a_5^4 + 23685242880a_4^{16}a_5^5\\
&+ 7044986537984a_4^{15}a_5^6 + 4776306345984a_4^{14}a_5^7 - 2047910092800a_4^{13}a_5^8\\
&+ 9980323651584a_4^{12}a_5^9 + 87453204480a_4^{18}a_5^2 + 211345244160a_4^{17}a_5^3\\
&+ 18137210559488a_4^{16}a_5^4 + 43606528505856a_4^{15}a_5^5 - 16936192867328a_4^{14}a_5^6\\
&+ 83213921538048a_4^{13}a_5^7 + 70628108476416a_4^{12}a_5^8 + 6876797571072a_4^{11}a_5^9\\
&+ 527648090112a_4^{10}a_5^{10} + \cdots,\\
E_0 =& 436926698208a_4^7a_5^7 + 4296445865712a_4^8a_5^5 + 318301099644528a_4^7a_5^6\\
&+ 436926698208a_4^6a_5^7 - 314587222709760a_4^5a_5^8 + 10486240756992a_4^9a_5^3\\
&+ 3127375663540128a_4^8a_5^4 + 2056104220650480a_4^7a_5^5 - 4828695405245712a_4^6a_5^6\\
&- 1589648559755256a_4^5a_5^7 + 509238066761424a_4^4a_5^8 - 520052002542072a_4^3a_5^9\cdots.
\end{aligned}$$

By solving $F_b = F_c = 0$, we obtain a solution pair,

$$\begin{aligned}
a_4 =& \quad 5.994346337166853568266496631271437869140312765303 87\cdots,\\
a_5 =& -8.148612683168578697071817451613251454431803398883 16\cdots,
\end{aligned} \tag{4.30}$$

which satisfy

$$\det\left(\frac{\partial(V_{12,1}, V_{14,1})}{\partial(a_4, a_5)}\right) \approx -49.555 \neq 0.$$

Setting the non-used parameters $q_2 = q_3 = q_4 = q_5 = 0$, and $p_1 = 1$, we obtain the following critical parameter values

$$\begin{aligned}
p_2 &= \ \ 0.300021284283343813154400993721237512940813436278 97\cdots,\\
p_3 &= \ \ 0.822063216146157520485250818420483278038893873957 65\cdots,\\
p_4 &= 15.592924677967975865716338821082721574817904320638 44\cdots,\\
p_6 &= -4.624289330635562968874393696336406298900002789328 71\cdots,\\
q_6 &= \ \ 4.624289330635562968874393696336406298900002789328 71\cdots,\\
p_5 &= -4, \quad q_1 = -1,
\end{aligned} \tag{4.31}$$

under which the Lyaponov constants become

$$\begin{aligned}
&V_{2,1} = 0, \quad V_{3,1} = 1.0\times 10^{-998}, \quad V_{4,1} = 0.6\times 10^{-996}, \quad V_{5,1} = -0.7\times 10^{-995},\\
&V_{6,1} = 1.0\times 10^{-994}, \quad V_{7,1} = -0.8\times 10^{-993}, \quad V_{8,1} = 0.6\times 10^{-992}, \quad V_{9,1} = -0.6\times 10^{-991},\\
&V_{10,1} = 0.5\times 10^{-990}, \quad V_{11,1} = -0.3\times 10^{-989}, \quad V_{12,1} = 0.5\times 10^{-988}, \quad V_{13,1} = -0.2\times 10^{-987},\\
&V_{14,1} = 0.3\times 10^{-986}, \quad V_{15,1} = -0.4\times 10^{-985}, \quad V_{16,1} = 13.3.
\end{aligned}$$

Then with (4.30) and (4.31) holding, we have $V_{j,1} \approx 0$, $j = 2,\ldots,14$, and $V_{16,1} \neq 0$. Therefore, we can take perturbations in backward order on a_5 for $V_{14,1}$, on a_4 for $V_{12,1}$, on q_6 for $V_{10,1}$, on p_3 for $V_{8,1}$, on p_2 for $V_{6,1}$, on q_1 for $V_{5,1}$, on p_4 for $V_{4,1}$, on p_6 for $V_{3,1}$, on p_5 for $V_{2,1}$, on p_0 for V_1, to obtain 10 small-amplitude limit cycles bifurcating from the origin.

4.5 Conclusion

In this chapter, we have studied planar switching systems, in particular, a switching Bautin system. We have developed a computationally efficient algorithm to compute the Lyapunov constants for planar switching systems. With the help of this algorithm and Maple built-in command 'resultant', we present, with rigorous proof, a complete classification on the center problem for the Bautin switching system under the condition $a_6 b_6 = 0$. Moreover, we have selected one of the center conditions to construct a special integrable system and then perturbed this system to obtain 10 small-amplitude limit cycles, which improves the existing result. Future work includes the classification of the center problem under the condition $a_6 b_6 \neq 0$, and obtaining possible more limit cycles.

Bibliography

[1] A. F. Filippov, Differential Equation with Discontinuous Right-Hand Sides, Kluwer Academic, Netherlands, 1988.

[2] M. Kunze, Non-Smooth Dynamical Systems, Springer-Verlag, Berlin, 2000.

[3] R. A. Ibrahim, Friction-Induced Vibration, Chatter, Squeal, and Chaos C Part II: Dynamics and Modeling, ASME Applied Mechanics Reviews 47(7) (1994) 227–253.

[4] J. Badertscher K. A. Cunefare, A. A. Ferri, Braking impact of normal dither signals, Journal of Vibration and Acoustics 129(1) (2007) 17–23.

[5] A. Kaplan, N. Friedman, M. Andersen, N. Davidson, Observation of Islands of Stability in SoftWall Atom-Optics Billiards, Physical Review Letters 87(27) (2001) 27410114.

[6] T. Kinoshita, T. Wenger, D. S. Weiss, A quantum Newton's cradle, Nature 440 (2006) 900–903.

[7] H. E. Nusse, J. A. Yorke, Border-collision bifurcations including "period two to period three" for piecewise smooth systems, Physica D 57 (1992) 39–57.

[8] Y. Zou, T. Kupper, W.-J. Beyn, Generalized Hopf bifurcation for planar Filippov systems continuous at the origin. J. Nonlinear Sci. 16(2) (2006) 159–177.

[9] D. J. W. Simpson, J. D. Meiss, Andronov-Hopf bifurcations in planar, piecewise-smooth, continuous flows. Phys. Lett. A 371(3) (2007) 213–220.

[10] C. J. Budd, Non-smooth dynamical systems and the grazing bifurcation, in: Nonlinear Mathematics and its Applications, Cambridge Univ. Press, Cambridge, 1996 219–235.

[11] M. D. Bernardo, C. J. Budd, A. R. Champneys, Grazing, skipping and sliding: analysis of the nonsmooth dynamics of the DC/DC buck converter, Nonlinearity 11 (1998) 859–890.

[12] A. Gasull, J. Torregrosa, Center-focus problem for discontinuous planar differential equations, International Journal of Bifurcation and Chaos, 13(7) (2003) 1755–1765.

[13] X. Liu, M. Han, Hopf bifurcation for non-smooth Liénard systems, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 19 (7) (2009) 2401–2415.

[14] Y. Tian, M. Han Hopf bifurcation for two types of Liénard systems, J. Differential Equations 251(4-5) (2011) 834–859.

[15] X. Chen, Z. Du, Limit cycles bifurcate from centers of discontinuous quadratic systems, Comput. Math. Appl. 59 (2010) 3836–3848.

[16] X. Chen, W. Zhang, Isochronicity of centers in a switching Bautin system, J. Differential Equations 252 (2012) 2877–2899.

[17] J. Llibre, A. C. Mereu, Limit cycles for discontinuous quadratic differential systems with two zones, J. Math. Anal. Appl. 413(2) (2014) 763–775.

[18] B. Coll, R. Prohens, A. Gasull, The center problem for discontinuous Liénard differential equation, Internat. J. Bifur. Chaos 9 (1999) 1751–1761.

[19] B. Coll, A. Gasull, R. Prohens, Degenerate Hopf bifurcation in discontinuous planar systems, J. Math. Anal. Appl. 253 (2001) 671–690.

[20] M. Han, W. Zhang, On Hopf bifurcation in non-smooth planar systems, J. Differential Equations 248 (2010) 2399–2416.

[21] N. Bautin, On the number of limit cycles appearing from an equilibrium point of the focus or center type under varying coefficients, Matem. Sb. 30 (1952) 181–196.

[22] P. Yu, Y. Tian, Twelve limit cycles around a singular point in a planar cubic-degree polynomial system, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 2690–2705.

[23] M. Han, Bifurcation theory of limit cycles, Science Press, Beijing, 2013.

[24] B. Mishra, Algorithmic algebra, Springer-Verlag, New York 1993.

Chapter 5

An explicit recursive formula for computing the normal form and center manifold of general n-dimensional differential systems associated with Hopf bifurcation

5.1 Introduction

Two of the useful tools in the study of stability and bifurcations near singular points are center manifold theory and normal form theory. The center manifold theory can be applied to reduce the dimension of the state spaces which need to be considered when some eigenvalues of the linearization have zero real part. The basic idea of normal form theory is to transform the original system to a simpler one which keeps the topological structure of the original system around the singular point. Most developments in this direction for the past three decades can be found in [1, 2, 3].

Since computation of normal forms is very involved and time consuming, in particular, for higher-order normal forms, computer algebra systems such as Maple, Mathematica, must be used. Several efficient methodologies for computing normal forms have been developed in the past decade (e.g. see [4, 5, 6]). Recently, researchers have also paid attention to computation of the simplest normal forms (e.g. see [7, 8, 9, 10]). Yu *et al.* applied the method of multiple time scales combined with a perturbation technique to obtain normal forms of differential equations for a number of different singularities such as Hopf [4], Hopf-zero [11], double Hopf [12, 13], etc. This method does not need solving differential equations, nor involve integration, but only needs algebraic calculations, which greatly facilitates implementation using computer algebra systems such as Maple. For Hopf bifurcation, this method only requires solving two-dimensional matrix systems for any higher-order normal forms of general n-dimensional systems. Giné and Santallusia [6] obtained a recursive formula for the Poincaré-Lyapunov constants of Hopf bifurcation for general two-dimensional systems,

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which can be computed recursively in terms of the coefficients of the original system. Yu [5] computed the center manifold of differential equations with a proper nonlinear transformation which is incorporated with normal form computation to develop a unified procedure for computing normal forms of general n-dimensional systems. The formulas developed in [4, 5] are in recursive format, but not explicit, which may involve some repetitive computations, and so may demand more memory in a computer to obtain higher-order normal forms or focus values. Since practical problems often have Hopf bifurcation in high-dimensional systems, and thus the recursive formula in [6], which only computes focus values, can not be directly applied to such systems. Moreover, for Hopf bifurcation, one may also need to determine the frequency of motion, implying that normal form, rather than just focus values, need be computed.

In this chapter, based on the result of Yu [5], we will develop an efficient method to compute the normal form for Hopf bifurcation in general n-dimensional dynamical systems. We shall present explicit recursive formulas for simultaneously computing the center manifold and normal form of a given general system, which is the first time available in the literature.

5.2 Main result

Consider a system of differential equations of the form,

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{G}(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathbf{R}^n, \quad \boldsymbol{G}(\boldsymbol{y}) : \mathbf{R}^n \to \mathbf{R}^n, \tag{5.1}$$

where $\boldsymbol{G}(\mathbf{0}) = \mathbf{0}$, $D_y\boldsymbol{G}(\mathbf{0}) = \mathbf{0}$, and it is assumed, without loss of generality, that the matrix A has eigenvalues $i,\ -i,\ \lambda_1,\ \cdots,\ \lambda_{k_1}, \lambda_{k_1+1},\ \bar{\lambda}_{k_1+1}, \cdots, \lambda_{k_1+k_2},\ \bar{\lambda}_{k_1+k_2}$. Here $\lambda_1,\ \cdots, \lambda_{k_1}$ are non-zero real numbers, and $\lambda_{k_1+1},\ \cdots,\ \lambda_{k_1+k_2}$ are complex numbers with non-zero real part, and $2 + k_1 + 2k_2 = n$.

Then, there exists a linear transformation,

$$\boldsymbol{y} = T\boldsymbol{x},$$

such that (5.1) can be transformed into

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \tag{5.2}$$

with $x_2 = \bar{x}_1$, where

$$J = \mathrm{diag}(i, -i, \lambda_1, \cdots, \lambda_{k_1+k_2}, \bar{\lambda}_{k_1+1}, \cdots, \bar{\lambda}_{k_1+k_2}),$$
$$\boldsymbol{f}(\boldsymbol{x}) = \sum_{m\geq 2} \boldsymbol{f}_m(\boldsymbol{x}) = \sum_{m\geq 2} \sum_{\widetilde{m}} \boldsymbol{a}_{\widetilde{m}} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

and $\widetilde{m} \triangleq m_1 m_2 \cdots m_n$, denoting a choice of the values of $m_1, m_2, \cdots, m_n$ which satisfies $\sum_{j=1}^{n} m_j = m$ with $m_j \geq 0$.

Let $\boldsymbol{x} = (x_1, \bar{x}_1, \boldsymbol{x}_r^T)^T$ and $J = \mathrm{diag}(i, -i, J_r)$. Then, Eq.(5.2) can be written as

$$\begin{aligned} \dot{x}_1 &= ix_1 + f_1(x_1, \bar{x}_1, \boldsymbol{x}_r), \\ \dot{\bar{x}}_1 &= -i\bar{x}_1 + \bar{f}_1(x_1, \bar{x}_1, \boldsymbol{x}_r), \\ \dot{\boldsymbol{x}}_r &= J_r\boldsymbol{x}_r + \boldsymbol{f}_r(x_1, \bar{x}_1, \boldsymbol{x}_r). \end{aligned} \tag{5.3}$$

Note that the second equation of (5.3) is a complex conjugate of the first equation.

The center manifold of (5.3) may be defined in the form of

$$\boldsymbol{x}_r = \boldsymbol{H}(x_1, \bar{x}_1) = \sum_{m\geq 2} \boldsymbol{H}_m(x_1, \bar{x}_1) \text{ with } \boldsymbol{H}_m(x_1, \bar{x}_1) = \sum_{j=0}^{m} \boldsymbol{h}_j^m x_1^j \bar{x}_1^{m-j}, \tag{5.4}$$

which satisfies

$$\boldsymbol{H}_{x_1}(x_1, \bar{x}_1)\dot{x}_1 + \boldsymbol{H}_{\bar{x}_1}(x_1, \bar{x}_1)\dot{\bar{x}}_1 = J_r \boldsymbol{H}(x_1, \bar{x}_1) + \boldsymbol{f}_r(x_1, \bar{x}_1, \boldsymbol{H}(x_1, \bar{x}_1)).$$

Then, the differential equations describing the dynamics on the center manifold are given by

$$\begin{aligned} \dot{x}_1 &= ix_1 + f_1(x_1, \bar{x}_1, \boldsymbol{H}(x_1, \bar{x}_1)), \\ \dot{\bar{x}}_1 &= -i\bar{x}_1 + \bar{f}_1(x_1, \bar{x}_1, \boldsymbol{H}(x_1, \bar{x}_1)). \end{aligned} \tag{5.5}$$

Next, introduce the transformation, given by

$$x_1 = u + Q(u, \bar{u}) = u + \sum_{m\geq 2} Q_m(u, \bar{u}) \equiv q(u, \bar{u})$$

with $Q_m(u, \bar{u}) = \sum_{j=0}^{m} q_j^m u^j \bar{u}^{m-j}$, into (5.5) to obtain the normal form,

$$\dot{u} = iu + C(u, \bar{u}) \text{ where } C(u, \bar{u}) = \sum_{m\geq 1} a_m u^{m+1} \bar{u}^m. \tag{5.6}$$

Let $\boldsymbol{h}(u, \bar{u}) = \boldsymbol{H}(q(u, \bar{u}), \bar{q}(u, \bar{u}))$ and

$$F_1(u, \bar{u}) = f_1(q(u, \bar{u}), \bar{q}(u, \bar{u}), \boldsymbol{h}(u, \bar{u})), \quad \boldsymbol{F}_r(u, \bar{u}) = \boldsymbol{f}_r(q(u, \bar{u}), \bar{q}(u, \bar{u}), \boldsymbol{h}(u, \bar{u})).$$

Then we have the following equations

$$\begin{aligned} &\begin{pmatrix} Q_u(u, \bar{u}) & Q_{\bar{u}}(u, \bar{u}) \\ \boldsymbol{h}_u(u, \bar{u}) & \boldsymbol{h}_{\bar{u}}(u, \bar{u}) \end{pmatrix} \begin{pmatrix} iu \\ -i\bar{u} \end{pmatrix} - \begin{pmatrix} iQ(u, \bar{u}) \\ J_r \boldsymbol{h}(u, \bar{u}) \end{pmatrix} \\ =&\begin{pmatrix} F_1(u, \bar{u}) \\ \boldsymbol{F}_r(u, \bar{u}) \end{pmatrix} - \begin{pmatrix} Q_u(u, \bar{u}) & Q_{\bar{u}}(u, \bar{u}) \\ \boldsymbol{h}_u(u, \bar{u}) & \boldsymbol{h}_{\bar{u}}(u, \bar{u}) \end{pmatrix} \begin{pmatrix} C(u, \bar{u}) \\ \bar{C}(u, \bar{u}) \end{pmatrix} - \begin{pmatrix} C(u, \bar{u}) \\ 0 \end{pmatrix}. \end{aligned} \tag{5.7}$$

Solving (5.7) order by order, we obtain the center manifold and the normal form as well as the associated nonlinear transformation.

Suppose for $k \geq 0$,

$$\begin{aligned} q^k(u, \bar{u}) &= \sum_{m=k}^{\infty} \sum_{j=0}^{m} q_j^{k,m} u^j \bar{u}^{m-j}, \\ \boldsymbol{h}^k(u, \bar{u}) &= \sum_{m=2k}^{\infty} \sum_{j=0}^{m} \widetilde{\boldsymbol{h}}_j^{k,m} u^j \bar{u}^{m-j}. \end{aligned} \tag{5.8}$$

We have the following result.

Theorem 5.2.1 *For the differential system* (5.3)*, the recursive formulas for the coefficients of the center manifold* (5.4) *and the normal form* (5.6) *are given as follows: for* $s \geq 2$, $0 \leq j \leq s$,

(1) *if* s *is even, then*

$$i(2j-s-1)q_j^s = a_j^s - C_{1,j}^s, \quad i(2j-s)\widetilde{\boldsymbol{h}}_j^s - J_r\widetilde{\boldsymbol{h}}_j^s = \boldsymbol{b}_j^s - \boldsymbol{C}_{r,j}^s;$$

(2) *if* s *is odd, then*

$$i(2j-s-1)q_j^s = a_j^s - C_{1,j}^s, \text{ for } j \neq \frac{s+1}{2}, \quad a_{\frac{s-1}{2}} = a_{\frac{s+1}{2}}^s - C_{1,\frac{s+1}{2}}^s,$$

$$i(2j-s)\widetilde{\boldsymbol{h}}_j^s - J_r\widetilde{\boldsymbol{h}}_j^s = \boldsymbol{b}_j^s - \boldsymbol{C}_{r,j}^s;$$

where $\widetilde{\boldsymbol{h}}_j^{m,s} = (\widetilde{h}_{j,1}^{m,s}, \widetilde{h}_{j,2}^{m,s}, \cdots, \widetilde{h}_{j,n-2}^{m,s})^T$ *and* $\boldsymbol{a}_j^s = (a_j^s, \bar{a}_j^s, \boldsymbol{b}_j^{sT})^T$,

$$\begin{aligned}
\boldsymbol{a}_j^s &= \sum_{m=2}^{s} \sum_{\widetilde{m}} \sum_{\widetilde{s}} \sum_{\widetilde{j}} \boldsymbol{a}_{\widetilde{m}} q_{j_1}^{m_1,s_1} \bar{q}_{s_2-j_2}^{m_2,s_2} \widetilde{h}_{j_3,1}^{m_3,s_3} \widetilde{h}_{j_4,2}^{m_4,s_4} \cdots \widetilde{h}_{j_{n-1},n-3}^{m_{n-1},s_{n-1}} \widetilde{h}_{j_n,n-2}^{m_n,s_n} \\
\widetilde{\boldsymbol{h}}_j^s &= \boldsymbol{h}_j^s + \sum_{m=2}^{s-1} \sum_{k=0}^{m} \sum_{l_1=k}^{s+k-m} \sum_{j_1=\max(0,l_1+j-s)}^{\min(l_1,j)} \boldsymbol{h}_k^m q_{j_1}^{k,l_1} \bar{q}_{s+j_1-l_1-j}^{m-k,s-l_1}, \\
C_{1,j}^s &= \sum_{m=1}^{\min([\frac{s-2}{2}],j,s-j)} ((j-m)a_m + (s-j-m)\bar{a}_m) q_{j-m}^{s-2m}, \\
\boldsymbol{C}_{r,j}^s &= \sum_{m=1}^{\min([\frac{s-2}{2}],j,s-j)} ((j-m)a_m + (s-j-m)\bar{a}_m) \widetilde{\boldsymbol{h}}_{j-m}^{s-2m}, \\
q_j^{m,s} &= \sum_{l=m-1}^{s-1} \sum_{j_1=\max(0,l+j-s)}^{\min(l,j)} q_{j_1}^{m-1,l} q_{j-j_1}^{s-l}, \ m \geq 2, \\
\widetilde{\boldsymbol{h}}_j^{m,s} &= \sum_{l=m-1}^{s-1} \sum_{j_1=\max(0,l+j-s)}^{\min(l,j)} \widetilde{\boldsymbol{h}}_{j_1}^{m-1,l} \widetilde{\boldsymbol{h}}_{j-j_1}^{s-l}, \ m \geq 2.
\end{aligned}$$

Proof For any given integer $s \geq 2$, suppose that we have obtained q_j^n and $\boldsymbol{h}_j^n$ for $n < s$, $0 \leq j \leq n$ and a_m, $m \leq [\frac{s-2}{2}]$. Now, we want to derive the formulas for q_j^s and $\boldsymbol{h}_j^s$ for $0 \leq j \leq s$ and $a_{[\frac{s-1}{2}]}$. We divide the proof in three steps, which can also serve as guidelines for developing programs using a computer algebra system.

Step 1. Denote

$$\boldsymbol{H}(x_1, \bar{x}_1) = \boldsymbol{h}(u, \bar{u}) = \sum_{m=2}^{s} \sum_{k=0}^{m} \widetilde{\boldsymbol{h}}_k^m u^k \bar{u}^{m-k} + \mathrm{o}(|u, \bar{u}|^s).$$

In this step, we derive the formula for $\widetilde{\boldsymbol{h}}_k^s$, $0 \leq k \leq s$. First of all, we need to compute

$x_1^k = q^k(u,\bar{u})$, where $2 \le k \le s$. Since $q^k(u,\bar{u}) = q(u,\bar{u})q^{k-1}(u,\bar{u})$, we have

$$\begin{aligned} q^k(u,\bar{u}) =& \Big(\sum_{m=1}^{s}\sum_{j=0}^{m} q_j^m u^j \bar{u}^{m-j} + \mathrm{o}(|u,\bar{u}|^s)\Big)\Big(\sum_{m=k-1}^{s}\sum_{j=0}^{m} q_j^{k-1,m} u^j \bar{u}^{m-j} + \mathrm{o}(|u,\bar{u}|^s)\Big) \\ =& \sum_{m=k}^{s}\sum_{j=0}^{m}\sum_{l=k-1}^{m-1}\sum_{j_1=\max(0,l+j-m)}^{\min(l,j)} q_{j_1}^{k-1,l} q_{j-j_1}^{m-l} u^j \bar{u}^{m-j} + \mathrm{o}(|u,\bar{u}|^s). \end{aligned}$$

Then, for $2 \le k \le s$, $0 \le j \le s$, we obtain

$$q_j^{k,s} = \sum_{l=k-1}^{s-1}\sum_{j_1=\max(0,l+j-s)}^{\min(l,j)} q_{j_1}^{k-1,l} q_{j-j_1}^{s-l}.$$

For $2 \le m \le s$,

$$\begin{aligned} \boldsymbol{H}_m(x_1,\bar{x}_1) =& \sum_{k=0}^{m} \boldsymbol{h}_k^m x_1^k \bar{x}_1^{m-k} = \sum_{k=0}^{m} \boldsymbol{h}_k^m q^k(u,\bar{u})\bar{q}^{m-k}(u,\bar{u}) \\ =& \sum_{k=0}^{m} \boldsymbol{h}_k^m \Big(\sum_{l=k}^{s}\sum_{j=0}^{l} q_j^{k,l} u^j \bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big)\Big(\sum_{l=m-k}^{s}\sum_{j=0}^{l} \bar{q}_j^{m-k,l} \bar{u}^j u^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big) \\ =& \sum_{k=0}^{m} \boldsymbol{h}_k^m \Big(\sum_{l=m}^{s}\sum_{j=0}^{l}\sum_{l_1=k}^{l+k-m}\sum_{j_1=\max(0,l_1+j-l)}^{\min(l_1,j)} q_{j_1}^{k,l_1} \bar{q}_{l+j_1-l_1-j}^{m-k,l-l_1} u^j \bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big) \\ =& \sum_{l=m}^{s}\sum_{j=0}^{l}\sum_{k=0}^{m}\sum_{l_1=k}^{l+k-m}\sum_{j_1=\max(0,l_1+j-l)}^{\min(l_1,j)} \boldsymbol{h}_k^m q_{j_1}^{k,l_1} \bar{q}_{l+j_1-l_1-j}^{m-k,l-l_1} u^j \bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s), \end{aligned}$$

where $q_j^{0,0} = 1$ and $q_j^{0,l} = 0$ if $l \ge 1$. In particular,

$$\boldsymbol{H}_s(x_1,\bar{x}_1) = \sum_{j=0}^{s} \boldsymbol{h}_j^s u^j \bar{u}^{s-j} + \mathrm{o}(|u,\bar{u}|^s).$$

Since $\boldsymbol{h}(u,\bar{u}) = \boldsymbol{H}(x_1,\bar{x}_1) = \sum_{m\ge 2} \boldsymbol{H}_m(x_1,\bar{x}_1)$ and $\boldsymbol{H}_m(q(u,\bar{u}),\bar{q}(u,\bar{u})) = \mathrm{O}(|u,\bar{u}|^m)$, we obtain

$$\widetilde{\boldsymbol{h}}_j^s = \boldsymbol{h}_j^s + \sum_{m=2}^{s-1}\sum_{k=0}^{m}\sum_{l_1=k}^{s+k-m}\sum_{j_1=\max(0,l_1+j-s)}^{\min(l_1,j)} \boldsymbol{h}_k^m q_{j_1}^{k,l_1} \bar{q}_{s+j_1-l_1-j}^{m-k,s-l_1}.$$

Step 2. Denote $\boldsymbol{F}(u,\bar{u}) = (F_1(u,\bar{u}), \bar{F}_1(u,\bar{u}), \boldsymbol{F}_r^T(u,\bar{u}))^T$,

$$\boldsymbol{F}(u,\bar{u}) = \sum_{m=2}^{s}\sum_{j=0}^{m} \boldsymbol{a}_j^m u^j \bar{u}^{m-j} + \mathrm{o}(|u,\bar{u}|^s). \tag{5.9}$$

In this step, we derive the formula for $\boldsymbol{a}_j^s$, $0 \le j \le s$.

Let $\boldsymbol{h}^k(u,\bar{u}) = \sum\limits_{m=2k}^{s}\sum\limits_{j=0}^{m}\widetilde{\boldsymbol{h}}_j^{k,m}u^j\bar{u}^{m-j} + \mathrm{o}(|u,\bar{u}|^s)$, $k \geq 0$, where $\widetilde{\boldsymbol{h}}_j^{1,m} = \widetilde{\boldsymbol{h}}_j^{m}$, $\widetilde{\boldsymbol{h}}_0^{0,0} = 1$ and $\widetilde{\boldsymbol{h}}_j^{0,m} = 0$ if $m \geq 1$. Using the same method for computing $q_j^{k,s}$, $s \geq 2k$, $0 \leq j \leq s$, we have

$$\widetilde{\boldsymbol{h}}_j^{k,s} = \sum_{l=2k-2}^{s-2}\sum_{j_1=\max(0,j+l-s)}^{\min(l,j)}\widetilde{\boldsymbol{h}}_{j_1}^{k-1,l}\widetilde{\boldsymbol{h}}_{j-j_1}^{s-l}.$$

Let $\widetilde{\boldsymbol{h}}_j^{m,l} = (\widetilde{h}_{j,1}^{m,l}, \widetilde{h}_{j,2}^{m,l}, \cdots, \widetilde{h}_{j,n-2}^{m,l})^T$. For $2 \leq m \leq s$, substituting $q(u,\bar{u})$ and $\widetilde{\boldsymbol{h}}(u,\bar{u})$ into $\boldsymbol{f}_m(\boldsymbol{x})$ yields

$$\begin{aligned}
\boldsymbol{f}_m(\boldsymbol{x}) &= \sum_{\widetilde{m}}\boldsymbol{a}_{\widetilde{m}}x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} = \sum_{\widetilde{m}}\boldsymbol{a}_{\widetilde{m}}q^{m_1}(u,\bar{u})\bar{q}^{m_2}(u,\bar{u})h_1^{m_3}(u,\bar{u})\cdots h_{n-2}^{m_n}(u,\bar{u})\\
&= \sum_{\widetilde{m}}\boldsymbol{a}_{\widetilde{m}}\Big(\sum_{l=m_1}^{s}\sum_{j=0}^{l}q_j^{m_1,l}u^j\bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big)\Big(\sum_{l=m_2}^{s}\sum_{j=0}^{l}\bar{q}_j^{m_2,l}\bar{u}^j u^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big)\\
&\qquad\Big(\sum_{l=2m_3}^{s}\sum_{j=0}^{l}\widetilde{h}_{j,1}^{m_3,l}u^j\bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big)\cdots\Big(\sum_{l=2m_n}^{s}\sum_{j=0}^{l}\widetilde{h}_{j,n-2}^{m_n,l}u^j\bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big)\\
&= \sum_{\widetilde{m}}\boldsymbol{a}_{\widetilde{m}}\Big(\sum_{l=m}^{s}\sum_{j=0}^{l}\sum_{l_1=m_1}^{l}\sum_{l_2=m_2}^{l}\sum_{l_2=2m_3}^{l}\cdots\sum_{l_n=2m_n}^{l}\sum_{j_1=0}^{l_1}\sum_{j_2=0}^{l_2}\sum_{j_3=0}^{l_3}\cdots\sum_{j_n=0}^{l_n}\\
&\qquad q_{j_1}^{m_1,l_1}\bar{q}_{j_2}^{m_2,l_2}\widetilde{h}_{j_3,1}^{m_3,l_3}\widetilde{h}_{j_4,2}^{m_4,l_4}\cdots\widetilde{h}_{j_{n-1},n-3}^{m_{n-1},l_{n-1}}\widetilde{h}_{j_n,n-2}^{m_n,l_n}u^j\bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s)\Big)\\
&= \sum_{l=m}^{s}\sum_{j=0}^{l}\sum_{\widetilde{m}}\sum_{\widetilde{l}}\sum_{\widetilde{j}}\boldsymbol{a}_{\widetilde{m}}q_{j_1}^{m_1,l_1}\bar{q}_{j_2}^{m_2,l_2}\widetilde{h}_{j_3,1}^{m_3,l_3}\widetilde{h}_{j_4,2}^{m_4,l_4}\cdots\widetilde{h}_{j_{n-1},n-3}^{m_{n-1},l_{n-1}}\widetilde{h}_{j_n,n-2}^{m_n,l_n}u^j\bar{u}^{l-j} + \mathrm{o}(|u,\bar{u}|^s)
\end{aligned}$$

Since $\boldsymbol{F}(u,\bar{u}) = \boldsymbol{f}(\boldsymbol{x}) = \sum\limits_{m\geq 2}\boldsymbol{f}_m(\boldsymbol{x})$, for $0 \leq j \leq s$, we consequently obtain

$$\boldsymbol{a}_j^s = \sum_{m=2}^{s}\sum_{\widetilde{m}}\sum_{\widetilde{s}}\sum_{\widetilde{j}}\boldsymbol{a}_{\widetilde{m}}q_{j_1}^{m_1,s_1}\bar{q}_{j_2}^{m_2,s_2}\widetilde{h}_{j_3,1}^{m_3,s_3}\widetilde{h}_{j_4,2}^{m_4,s_4}\cdots\widetilde{h}_{j_{n-1},n-3}^{m_{n-1},s_{n-1}}\widetilde{h}_{j_n,n-2}^{m_n,s_n},$$

where $0 \leq j_k \leq s_k$, for any $1 \leq k \leq n$, and the index $\widetilde{s}$ satisfies that

$$s_k\left\{\begin{array}{ll} = 0 & \\ \geq m_k & \text{for } k = 1,\ 2 \\ \geq 2m_k & \text{for } 3 \leq k \leq n \end{array}\right\}\ \begin{array}{l}\text{if } m_k = 0,\\ \text{if } m_k \neq 0.\end{array}$$

Step 3. Denote

$$\left(\begin{array}{cc} Q_u(u,\bar{u}) & Q_{\bar{u}}(u,\bar{u}) \\ \boldsymbol{h}_u(u,\bar{u}) & \boldsymbol{h}_{\bar{u}}(u,\bar{u}) \end{array}\right)\left(\begin{array}{c} C(u,\bar{u}) \\ \bar{C}(u,\bar{u}) \end{array}\right) = \left(\begin{array}{c} \sum\limits_{m=4}^{s}\sum\limits_{j=1}^{m}C_{1,j}^m u^j\bar{u}^{m-j} + \mathrm{o}(|u,\bar{u}|^s) \\ \sum\limits_{m=4}^{s}\sum\limits_{j=1}^{m}\boldsymbol{C}_{r,j}^m u^j\bar{u}^{m-j} + \mathrm{o}(|u,\bar{u}|^s) \end{array}\right). \tag{5.10}$$

In this step, we derive the formulas for $C_{1,j}^s$ and $\boldsymbol{C}_{r,j}^s$, $0 \leq j \leq s$.

Note that

$$\begin{aligned}
&Q_u(u,\bar u)C(u,\bar u)+Q_{\bar u}(u,\bar u)\bar C(u,\bar u)\\
=&(\sum_{m=2}^{s}\sum_{j=0}^{m} jq_j^m u^{j-1}\bar u^{m-j}+\mathrm{o}(|u,\bar u|^{s-1}))(\sum_{m=1}^{[\frac{s-1}{2}]} a_m u^{m+1}\bar u^m+\mathrm{o}(|u,\bar u|^s))\\
&+(\sum_{m=2}^{s}\sum_{j=0}^{m}(m-j)q_j^m u^j\bar u^{m-j-1}+\mathrm{o}(|u,\bar u|^{s-1}))(\sum_{m=1}^{[\frac{s-1}{2}]}\bar a_m\bar u^{m+1}u^m+\mathrm{o}(|u,\bar u|^s))\\
=&\sum_{l=4}^{s}\sum_{j=2}^{l}\sum_{m=1}^{\min([\frac{l-2}{2}],j,l-j)} a_m(j-m)q_{j-m}^{l-2m}u^j\bar u^{l-j}\\
&+\sum_{l=4}^{s}\sum_{j=1}^{l}\sum_{m=1}^{\min([\frac{l-2}{2}],j,l-j)}\bar a_m(l-m-j)q_{j-m}^{l-2m}u^j\bar u^{l-j}+\mathrm{o}(|u,\bar u|^s).
\end{aligned}$$

For the first term in the last expression above, if $j=1$, then $m=1$ and $j-m=0$. So we obtain

$$\begin{aligned}
&Q_u(u,\bar u)C(u,\bar u)+Q_{\bar u}(u,\bar u)\bar C(u,\bar u)\\
=&\sum_{l=4}^{s}\sum_{j=1}^{l}\sum_{m=1}^{\min([\frac{l-2}{2}],j,l-j)}((j-m)a_m+(l-j-m)\bar a_m)q_{j-m}^{l-2m}u^j\bar u^{l-j}+\mathrm{o}(|u,\bar u|^s).
\end{aligned}$$

Therefore, for $s\geq 4$, $0\leq j\leq s$, comparing the above equation with (5.10) we have

$$C_{1,j}^s=\sum_{m=1}^{\min([\frac{s-2}{2}],j,s-j)}((j-m)a_m+(s-j-m)\bar a_m)q_{j-m}^{s-2m}.$$

Similarly,

$$\boldsymbol{C}_{r,j}^s=\sum_{m=1}^{\min([\frac{s-2}{2}],j,s-j)}((j-m)a_m+(s-j-m)\bar a_m)\widetilde{\boldsymbol{h}}_{j-m}^{s-2m}.$$

Finally, from the left-hand side of (5.7), we obtain

$$\begin{aligned}
&iu\boldsymbol{h}_u(u.\bar u)-i\bar u\boldsymbol{h}_{\bar u}(u,\bar u)-J_r\boldsymbol{h}(u,\bar u)\\
=&iu\sum_{m=2}^{s}\sum_{j=0}^{m}j\widetilde{\boldsymbol{h}}_j^m u^{j-1}\bar u^{m-j}-i\bar u\sum_{m=2}^{s}\sum_{j=0}^{m}(m-j)\widetilde{\boldsymbol{h}}_j^m u^j\bar u^{m-j-1}-J_r\sum_{m=2}^{s}\sum_{j=0}^{m}\widetilde{\boldsymbol{h}}_j^m u^j\bar u^{m-j}+\mathrm{o}(|u,\bar u|^s)\\
=&\sum_{m=2}^{s}\sum_{j=0}^{m}(i(2j-m)\widetilde{\boldsymbol{h}}_j^m-J_r\widetilde{\boldsymbol{h}}_j^m)u^j\bar u^{m-j}+\mathrm{o}(|u,\bar u|^s),
\end{aligned}\tag{5.11}$$

and similarly,

$$iuQ_u(u.\bar u)-i\bar uQ_{\bar u}(u,\bar u)-iQ(u,\bar u)=\sum_{m=2}^{s}\sum_{j=0}^{m}i(2j-m-1)q_j^m u^j\bar u^{m-j}+\mathrm{o}(|u,\bar u|^s).\tag{5.12}$$

Substituting (5.6), (5.9), (5.10), (5.11) and (5.12) into (5.7) completes the proof of Theorem 5.2.1.

The Maple program developed using the above formulas is given in Appendix B for the convenience of readers.

5.3 Application

In this section, we present two examples to demonstrate the computational efficiency of the method developed in the previous section. We apply the obtained formulas to compute the normal forms for these two examples, and compare the computational efficiency with existing programs. The Maple program developed in this chapter (see the source code in Appendix B) and the Maple program given in [4] are executed on a desktop machine with CPU 3.4 GHZ and 32 G RAM memory for a comparison. We have tested a number of systems and found that in general (in particular, more terms involved in the system) the method and program developed in this chapter are better than the perturbation method as well as the program developed in [4]. Only in some special cases is the situation reversed.

In the following, for the first example, we show how to use the normal form to determine the maximum number of small amplitude limit cycles bifurcating from a focus point, as well as the maximum number of critical periods of periodic solutions in the neighborhood of the critical point. For the second example, we focus on the comparison of computational efficiency with existing programs, and show that the recursive formulas and Maple program obtained in this chapter are superior than other methods.

5.3.1 A 5-dimensional dynamical system

The first example is a general 5-dimensional dynamical system involving a number of constant parameters, given by

$$\begin{aligned}
\dot{x}_1 &= a_0\, x_1 + x_2 + a_1\, x_1^3 + a_2\, x_1^2\, x_2,\\
\dot{x}_2 &= -x_1 + a_0\, x_2 + a_3\, x_2^3 + a_4\, x_3^2 + a_5\, x_4\, x_5,\\
\dot{x}_3 &= -x_3 + a_6\, x_1\, x_2,\\
\dot{x}_4 &= -x_4 + x_5 + a_7\, x_1^2,\\
\dot{x}_5 &= -x_4 - x_5 + a_8\, x_2^2,
\end{aligned} \tag{5.13}$$

where the $a_i,\ i = 0, 1, 2, \ldots, 8$ are real numbers. The system has an equilibrium at the origin, and its linear part is in the Jordan canonical form, with eigenvalues, $a_0 \pm i$, -1 and $-1 \pm i$, indicating that the origin undergoes a Hopf bifurcation at the critical point $a_0 = 0$. For system (5.13), we compute the normal form up to 17th order, given in polar coordinates as follows:

$$\begin{aligned}
\dot{r} &= r\,(v_0 + v_2\, r^2 + \cdots + v_{16}\, r^{16}) + \cdots,\\
\dot{\theta} &= 1 + t_0 + t_2\, r^2 + \cdots + t_{16}\, r^{16} + \cdots,
\end{aligned} \tag{5.14}$$

where r and θ represent the amplitude and phase of motion, respectively; v_{2k} is usually called the kth order focus value. The coefficients v_0 and t_0 are obtained from the linear analysis. For system (5.13), $v_0 = a_0$ and $t_0 = 0$. The first equation of (5.14) can be used to determine

bifurcation of limit cycles near the origin and their stability, while the second equation of (5.14) can be used to determine the frequency of the limit cycles. The coefficients obtained from the output of the computer program are

$$
\begin{aligned}
v_2 &= \tfrac{1}{8}\,a_1,\\
v_4 &= -\tfrac{1}{32}\,a_1\,a_2 - \tfrac{1}{100}\,a_4a_6\,(a_7+3\,a_8),\\
v_6 &= \tfrac{1}{226304000}\,a_1\,[359125\,a_1^2+5442125\,a_2^2-16\,a_4a_6\,(35509\,a_7+74168\,a_8)]\\
&\quad -\tfrac{1}{2176000}\,a_5\,[2\,a_5\,(10075\,a_7^4+11603\,a_7^3\,a_8-5643\,a_7^2\,a_8^2-4423\,a_7\,a_8^3+8300\,a_8^4)\\
&\qquad\qquad -a_3a_6(7413a_7^2+2664a_7a_8+1891a_8^2)]\\
&\quad +\tfrac{1}{14144000}\,a_6[2210\,a_3^2\,a_6-a_2\,a_4(105928\,a_7+22921\,a_8)],\\
&\;\;\vdots\\
v_{16} &= \tfrac{14747267412298226915175888345459026 83}{44467543404263778624567705600000000000000}\,a_2\,a_3\,a_4^2\,a_5\,a_6^3\,a_7^3\,a_8+\cdots,\\
t_2 &= \tfrac{3}{8}\,a_2,\\
t_4 &= -\tfrac{11}{256}\,a_1^2-\tfrac{51}{256}\,a_2^2-\tfrac{1}{160}\,a_4\,a_6\,(a_7+a_8),\\
t_6 &= \tfrac{1}{45260800}\,a_1[2845375\,a_1\,a_2-16\,a_4\,a_6\,(16738\,a_7+101\,a_8)]\\
&\quad +\tfrac{1}{45260800}\,a_2[7839975\,a_2^2-16\,a_4\,a_6\,(11566\,a_7-7399\,a_8)]-\tfrac{3}{1280}\,a_3^2\,a_6^2\\
&\quad -\tfrac{1}{10880000}\,a_5[40\,a_5\,(4800\,a_7^4+1708\,a_7^3\,a_8-5713\,a_7^2\,a_8^2+1907\,a_7\,a_8^3-820\,a_8^4)\\
&\qquad\qquad -a_3\,a_6\,(121649\,a_7^2+9416\,a_7\,a_8-77889\,a_8^2)],\\
&\;\;\vdots\\
t_{16} &= -\tfrac{1123901661383495502683149717755444 97279}{64033262502139841219377496064000000000000}\,a_2\,a_3\,a_4^2\,a_5\,a_6^3\,a_7^3\,a_8+\cdots,
\end{aligned}
\tag{5.15}
$$

where the lengthy expressions for v_8, t_8, etc. are omitted here for brevity.

To determine the maximum number of small amplitude limit cycles bifurcating from the origin, one may solve the polynomial equations $v_2 = v_4 = \cdots = 0$ for the parameters a_i. Suppose one can obtain $v_0 = a_0 = 0$, $v_2(a_i) = v_4(a_i) = \cdots = v_{2(k-1)}(a_i) = 0$, but $v_{2k}(a_i) \neq 0$, then one can conclude that at most k limit cycles may bifurcate from the origin. Moreover, by properly perturbing the parameters a_i, one can obtain k small amplitude limit cycles in the vicinity of the origin.

Now, suppose under certain conditions the origin of system (5.13), restricted to the center manifold, becomes a center, we can then study the critical periods of the periodical solutions around the origin. The procedure is similar to that of finding the maximum number of limit cycles, as described as follows. Let

$$
h = r^2 > 0 \qquad \text{and} \qquad p(h) = t_2\,h + t_4\,h^2 + \cdots + t_{2k}\,h^k + \cdots. \tag{5.16}
$$

Then, the second equation of (5.14) can be written as $d\theta = (\,1 + p(h)\,)\,dt$. Let the period of motion be $T(h)$. Then, integrating this equation on both sides from 0 to $2\,\pi$ yields $2\,\pi = (\,1 + p(h)\,)\,T(h)$, which in turn results in

$$
T(h) = \frac{2\,\pi}{1 + p(h)} \qquad \text{for} \quad 0 < h \ll 1 \qquad (\text{and so } 1 + p(h) \approx 1). \tag{5.17}
$$

Now, the local critical periods are determined by $T'(h) = \frac{-2\pi p'(h)}{(1+p(h))^2} = 0$. Thus, for $0 < h \ll 1$ (meaning that we consider small limit cycles), the local critical periods are determined by

$$p'(h) = t_2 + 2\,t_4\,h + \cdots + k\,t_{2k}\,h^{k-1} + \cdots = 0. \tag{5.18}$$

Then, similar to the above discussion in determining the maximum number of limit cycles, we can find the sufficient conditions for the polynomial $p'(h)$ to have maximal number of zeros. If $t_2 = t_4 = \cdots = t_{2(k-1)} = 0$, but $t_{2k} \neq 0$, then equation $p'(h) = 0$ can have at most $k-1$ real roots. Hence, $t_2, t_4, \cdots, t_{2(k-1)}$ can be perturbed to have $k-1$ real roots, and thus system (5.13) can have $k-1$ critical periods.

Next, we use (5.15) to determine the maximum number of small amplitude limit cycles bifurcation from the origin of system (5.13). To find the critical parameter values, letting $a_1 = 0$ yields $v_2 = 0$. Then, setting $a_8 = -\frac{4}{3}\,a_7$, we have $v_4 = 0$. With these parameter values, solving $v_6 = 0$ for a_4 yields

$$a_4 = \frac{13}{2441880\,a_2\,a_6\,a_7}\,[5508\,a_3^2\,a_6^2 + a_5\,a_7^2(117009\,a_3\,a_6 - 689750\,a_5\,a_7^2)], \quad (a_2\,a_6\,a_7 \neq 0).$$

Further solving $v_8 = 0$ for a_2 results in

$$a_2 = \left\{\frac{231361\left[9a_3a_6(612a_3a_6+13001a_5a_7^2)\left(5508a_3^2a_6^2+a_5a_7^2(117009a_3a_6-1379500a_5a_7^2)\right)+475755062500a_5^4a_7^8\right]}{167580\left[\,299559536604\,a_3^2a_6^2-a_5a_7^2(206823894568700a_5a_7^2+4779756148743a_3a_6)\,\right]}\right\}^{1/3},$$

and then v_{10} and v_{12} are simplified to

$$\begin{aligned}
v_{10} =& -\frac{481\left[5508a_3^2a_6^2-a_5a_7^2(689750a_5a_7^2-117009a_3a_6)\right]}{299559536604a_3^2a_6^2-a_5a_7^2(206823894568700a_5a_7^2+4779756148743a_3a_6)} \\
&\times\{1311714011917175715037915 2\,a_3^4\,a_6^4 \\
&\quad - a_5\,a_7^2[396452109512292474434225 6808\,a_3^3\,a_6^3 \\
&\qquad - a_5\,a_7^2(5023732926867851939154239 3913\,a_3^2\,a_6^2 \\
&\qquad\quad + 10\,a_5\,a_7^2\,(28729541880842908054014 6201219\,a_3\,a_6 \\
&\qquad\qquad + 5091707242938707252742537 63100\,a_5\,a_7^2))]\},
\end{aligned}$$

$$\begin{aligned}
v_{12} =& -\frac{6015386\left[5508a_3^2a_6^2-a_5a_7^2(689750a_5a_7^2-117009a_3a_6)\right]^2}{343\left[299559536604a_3^2a_6^2-a_5a_7^2(206823894568700a_5a_7^2+4779756148743a_3a_6)\right]} \\
&\times\{4369100288210241965165567410201975915584\,a_3^6a_6^6 \\
&\quad + a_5\,a_7^2[4871446732562495676518695709477639977790 4\,a_3^5\,a_6^5 \\
&\qquad - a_5\,a_7^2(3845973325800546252799748753542377356118852\,a_3^4a_6^4 \\
&\qquad\quad + a_5a_7^2[1032135628011591680005562644820239825227385 83\,a_3^3a_6^3 \\
&\qquad\qquad + 4a_5a_7^2(1970250745184090404872572101422063216780018 63a_3^2a_6^2 \\
&\qquad\qquad\quad - 25a_5a_7^2(1655865146522467778793242469018245647102 1097a_3a_6 \\
&\qquad\qquad\qquad + 1964142636187940853096040778176251299555195 00a_5a_7^2))])]\}.
\end{aligned}$$

It can be shown that besides the common factor in v_{10} and v_{12}, the only possible parameter values for $v_{10} = v_{12} = 0$ are $a_5\,a_7 = a_3\,a_6 = 0$, which are obviously not allowed. This suggests

that there exist parameter values such that $v_i = 0$, $i = 0, 2, \ldots, 10$, but $v_{12} \neq 0$. Therefore, for system (5.13), we can at most have 6 small amplitude limit cycles bifurcating from the origin. To find the parameter values such that $v_{10} = 0$, we may set $a_5 = a_6 = a_7 = 1$ and then solve an equation from $v_{10} = 0$ for a_3, yielding 4 real solutions. Choosing one of them, we have the following set of critical values:

$$a_5 = a_6 = a_7 = 1, \qquad a_0 = a_1 = 0, \quad a_8 = -\tfrac{4}{3},$$
$$a_3 = -1.8401519905\cdots, \quad a_2 = -0.1765825495\cdots, \quad a_4 = 26.7243747795\cdots,$$

under which $v_i = 0, i = 0, 2, \ldots, 10$, $v_{12} = 0.0573817846\cdots$. Then making proper perturbations in backwards order, on a_3 for v_{10}, on a_2 for v_8, on a_4 for v_6, on a_8 for v_4, and then on a_1 for v_2, and finally on a_0 for v_0 such that

$$0 < v_0 \ll -v_2 \ll v_4 \ll -v_6 \ll v_8 \ll -v_{10} \ll v_{12},$$

leading to 6 small limit cycles.

Next, we consider critical periods of periodic solutions near the origin. To do this, we first need to find the conditions under which the origin is a center, restricted to the center manifold. There are a number of such conditions. Here, we consider one satisfying

$$a_1 = a_5 = a_6 = 0, \tag{5.19}$$

under which

$$t_2 = \tfrac{3}{8}\,a_2, \quad t_4 = -\tfrac{51}{256}\,a_2^2, \quad t_6 = \tfrac{1419}{8192}\,a_2^3, \quad t_8 = -\tfrac{47505}{262144}\,a_2^4, \quad t_{10} = \tfrac{438825}{2097152}\,a_2^5, \quad \cdots$$

Therefore, under the condition (5.19), system (5.13) does not have critical periods near the origin; it is either monotonically decreasing (increasing) for $a_2 > 0$ ($a_2 < 0$). When $a_2 = 0$, the origin is a isochronous center.

5.3.2 A 3-dimensional competitive Lotka-Volterra system

In this section, we consider a 3-dimensional competitive Lotka-Volterra system, described by the following differential equations:

$$\dot{x}_i = x_i\Big(r_i - \sum_{j=1}^{3} a_{ij}x_j\Big), \quad i = 1,\ 2,\ 3, \tag{5.20}$$

where x_i represents the population of ith species, and the coefficients take positive real values, $r_i > 0$, $a_{ij} > 0$, $i, j = 1, 2, 3$. Over the last twenty years, a number of articles concerning about bifurcation of limit cycles for system (5.20) have been published (e.g., see [14, 15, 16, 17]). Particularly, for system (5.20) four limit cycles were found by Gyllenberg and Yan [17], using appropriate parameter values. These four limit cycles include three small amplitude limit cycles, proved by using focus value computation, and one large limit cycle, shown by constructing a heteroclinic cycle. In this section, we consider the Hopf bifurcation emerging from an interior singular point and use the normal form (or focus values) to study the maximum number of limit cycles bifurcating from this point.

It is noted that system (5.20) has a total of 12 parameters. Since we are interested in the limit cycles bifurcating from an interior equilibrium solution of system (5.20), we, without loss of generality, may assume that $E = (1, 1, 1)$ is the equilibrium solution, which yields $r_i = \sum_{j=1}^{3} a_{ij}$ and reduce the number of parameters to 9. Taking the translation $x_i \to x_i + 1$ such that the equilibrium solution is moved to the origin, we have

$$\dot{x}_i = -(1 + x_i)\Big(\sum_{j=1}^{3} a_{ij}x_j\Big). \tag{5.21}$$

The Jacobian of system (5.21) at $x = 0$ is the matrix $A = (-a_{ij})$, which has the characteristic polynomial $P(\lambda) = \lambda^3 - T\lambda^2 + M\lambda - D$, where

$$\begin{aligned} T &= -(a_{11} + a_{22} + a_{33}), \\ M &= a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}, \\ D &= \det(A). \end{aligned} \tag{5.22}$$

When $TM = D$ and $M > 0$, there exist a pair of purely imaginary eigenvalues $\pm i\sqrt{M}$ and a negative eigenvalue T, and Hopf bifurcation occurs.

It needs a parameter, say a_{31}, to satisfy $TM = D$. Moreover, one may apply a time scaling to set $M = 1$, using a parameter, say a_{32}. Finally, we may use a parameterization so that $a_{33} = 1$. Solving equations $TM = D$ and $M = 1$ yields

$$\begin{aligned} a_{31} = \ & (a_{11}^2a_{22}a_{23} + a_{11}^2a_{23} - a_{11}a_{13}a_{22}a_{21} - a_{11}a_{21}a_{13} \\ & -a_{11}a_{23}a_{21}a_{12} + a_{13}a_{21}^2a_{12} - a_{13}a_{22}a_{21} + a_{21}a_{13} + a_{22}a_{23} \\ & +a_{23} + a_{23}a_{21}a_{12})/(a_{23}^2a_{12} - a_{13}^2a_{21} - a_{13}a_{22}a_{23} + a_{23}a_{11}a_{13}), \\ a_{32} = \ & (-a_{13} - a_{11}a_{13} - a_{12}a_{23} + a_{11}a_{22}a_{12}a_{23} - a_{21}a_{12}^2a_{23} \\ & -a_{13}a_{22}^2 - a_{13}a_{11}a_{22}^2 - a_{13}a_{21}a_{12} + a_{22}a_{12}a_{23} + a_{11}a_{12}a_{23} \\ & +a_{22}a_{21}a_{12}a_{13})/(a_{23}^2a_{12} - a_{13}^2a_{21} - a_{13}a_{22}a_{23} + a_{23}a_{11}a_{13}). \end{aligned}$$

Thus, only 6 parameters a_{ij}, $i = 1, 2$, $j = 1, 2, 3$ are left for determining the focus values. In general, we might be able to find the parameter values satisfying $v_1 = v_2 = \cdots = v_6 = 0$, but $v_7 \neq 0$, and thus 7 small limit cycles may be found in the vicinity of the equilibrium solution. If, in addition, there exists an additional large limit cycle near the heteroclinic loop, then the maximum number of limit cycles becomes 8.

To apply the Maple program, we first need to put the linear part of system (5.20) in Jordan canonical form. To achieve this, introducing the linear variable transformation $x \to \mathfrak{T}x$, where

$$\mathfrak{T} = \begin{pmatrix} a_{13}a_{22} - a_{12}a_{23} & a_{13} & a_{13}(a_{11} + 1) + a_{12}a_{23} \\ a_{11}a_{23} - a_{13}a_{21} & a_{23} & a_{23}(a_{22} + 1) + a_{13}a_{21} \\ a_{12}a_{21} - a_{11}a_{22} + 1 & -(a_{11} + a_{22}) & (a_{11} + 1)(a_{22} + 1) - a_{12}a_{21} \end{pmatrix}$$

into system (5.21) yields the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 + q_1(x), \\ \dot{x}_2 &= -x_1 + q_2(x), \\ \dot{x}_3 &= T\,x_3 + q_3(x), \end{aligned} \tag{5.23}$$

where $T = -(a_{11} + a_{22} + 1)$ and $q_i(x)$, $i = 1,\ 2,\ 3$ are quadratic homogenenous polynomials, given in the form of

$$q_i = b_{i200}x_1^2 - b_{i200}x_2^2 + b_{i002}x_3^2 + b_{i110}x_1x_2 + b_{i102}x_1x_3 + b_{i011}x_2x_3, \quad i = 1, 2, 3,$$

where the coefficients b_{ijkl}'s are lengthy expressions in terms of the original 6 coefficients a_{ij}, $i = 1, 2$; $j = 1, 2, 3$. We have executed the Maple program developed in this chapter and that given in [4] on the desktop machine to obtain the following results. For the Maple program given in [4], it took 251 minutes CPU time and 12.35 GB Ram memory to get the focus values up to 3rd order; while for the program developed in this chapter, it only took 31 minutes CPU time and 3.86 GB Ram memory to get the focus values up to 4th order. This clearly shows that the recursive formulas derived and Maple program developed in this chapter are more computationally efficient for higher-order normal forms than that given in [4], though the program in [4] was proved computationally efficient, in particular, for lower-order normal forms. In order to get higher-order focus values of system (5.23), we need a more powerful machine with higher memory.

The first focus value obtained from the computer output is

$$v_1 = \tfrac{1}{8(4+T^2)}\{2[(b_{2011} - b_{1101})T - 2(b_{1011} + b_{2101})]b_{3200} - [(b_{1011} + b_{2101})T + 2(b_{2011} - b_{1101})]b_{3110}\}.$$

The focus values starting from the second one have very long expressions. The number of terms in each of the focus values are given below:

Focus value	Number of terms
v_1	8
v_2	1036
v_3	24088
v_4	261401

It can be seen that the number of terms increases very rapidly as the order of the focus values increases. Moreover, when the original parameters a_{ij} are substituted into these expression, they even have more terms. Thus, finding possible values of the 6 parameters $a_{ij} > 0$, $i = 1, 2$; $j = 1, 2, 3$ such that $v_j = 0$, $j = 1, 2, \ldots, 6$, but $v_7 \neq 0$, is very difficult and challenging. It not only needs power computer systems (high speed with large memory), but also needs efficient polynomial solvers implemented with a computer algebra system such as Maple.

5.4 Conclusion

In this chapter, we have derived explicit recursive formulas for computing normal forms and center manifold of general n-dimensional dynamical systems associated with Hopf bifurcation. Maple program has also been developed, which is convenient in application. Two examples are presented to show that the method and program developed in this chapter are computationally efficient.

Appendix B

This symbolic Maple script is developed on the basis of the formulas in Theorem 5.2.1, which can be used to find the normal forms of Hopf bifurcations of general n-dimensional systems. Here, the input is for the second example in the section of application.

```
with(LinearAlgebra):
M1  := 1:                              #  No. of non-zero real eigenvalues
M2  := 0:                              #  No. of complex conjugate eigenvalues
N   := 2:                              #  the highest degree of the vector field
Ord := 13:
M   := 2 + M1 + M2*2:
L   := 1 + M1 + M2:
f[1]:= x[2]+b1200*x[1]^2-b1200*x[2]^2+b1002*x[3]^2
           +b1110*x[1]*x[2]+b1101*x[1]*x[3]+b1011*x[2]*x[3]:
f[2]:=-x[1]+b2200*x[1]^2-b2200*x[2]^2+b2002*x[3]^2
           +b2110*x[1]*x[2]+b2101*x[1]*x[3]+b2011*x[2]*x[3]:
f[3]:=-L*x[3]+b3200*x[1]^2-b3200*x[2]^2+b3002*x[3]^2
           +b3110*x[1]*x[2]+b3101*x[1]*x[3]+b3011*x[2]*x[3]:
x[1]:= v[1]+v[L+1]:
x[2]:= I*(v[1]-v[L+1]):
f[1]:= simplify(f[1]-I*f[2])/2:
IEf[1]:= diff(f[1],v[1]):
j := 3:
for n from 2 to M1+1 do
    x[j]  := v[n]:
    f[n]  := simplify(f[j]):
    IEf[n]:= diff(f[n],v[n]):
    j     := j+1:
od:
k := L+2:
for n from M1+2 to L do
    x[j]  := v[n]+v[k]:
    x[j+1]:= I*(v[n]-v[k]):
    f[n]  := simplify(f[j]-I*f[j+1])/2:
    IEf[n]:= diff(f[n],v[n]):
    j := j+2:
    k := k+1:
od:
for j to L do
    for k from 1 to M do
        IEf[j] := subs(v[k]=0,IEf[j]):
    od:
    REf[j] := subs(I=0,IEf[j]):
    IEf[j] := (IEf[j]-REf[j])/I:
od:
SizeIndex := Array(1..L,2..N):
Mr := seq(1,i=1..M):
for m from 2 to N do
    i    := 1:
    Mv   := [seq(0,j=1..M)]:
    temp := 1:
    Mv[1]:= m+1:
```

```
    Mv[2]:= -1:
    while i < M do
          Mv[i+1] := 1+Mv[i+1]:
          Mv[1] := Mv[i]-1:
          if i<>1 then Mv[i]:=0: fi:
          if Mv[i+1]=1 then temp:=temp+1: fi:
          if Mv[1]=0 then temp:=temp-1: i := i+1:
          else i:=1: fi:
          Mcv := [[seq(0,j=1..temp+2)],[seq(0,j=1..temp+2)]]:
          Mcv[1,1] := temp:
          Mcv[2,1] := m+add(Mv[n],n=2..L)+add(Mv[n],n=L+2..M):
          j := 2:
          for k from i to M do
              if Mv[k]<>0 then
                 Mcv[1,j] := k:
                 Mcv[2,j] := Mv[k]:
                 j := j+1:
              fi:
          od:
          for j from 1 to L do
              coef := f[j]:
              for k to M do
                   if Mv[k]=0 then
                      coef := subs(v[k]=0,coef):
                   else
                      coef := subs(v[k]=0,diff(coef,`$`(v[k],Mv[k])))
                                          /factorial(Mv[k]):
                   fi:
               od:
               if coef<>0 then
                  SizeIndex[j,m] := SizeIndex[j,m]+1:
                  Mcv[1,-1] := subs(I=0,coef):
                  Mcv[2,-1] := subs(I=1,coef-Mcv[1,-1]):
                  Index[j,m,SizeIndex[j,m]] := Mcv:
                  Mr:=seq(max(Mr[n],Mv[n]),n=1..M):
               fi:
          od:
    od:
od:
Mr := [max(Mr[1],Mr[1+L]),seq(Mr[n],n=2..M1+1),seq(max(Mr[n],Mr[n+M2+1]),n=M1+2..L)]:
Rh[1,1,1,1] := 1:
Ih[1,1,1,1] := 0:
Rh[1,1,1,0] := 0:
Ih[1,1,1,0] := 0:
si := 1/2:
for s from 2 to Ord do
print(`order =`,s):
    si := si+1/2:
    Ms := min(Mr[1],s):
    for k from 2 to Ms do
        for i from 0 to s do
            Ih[1,k,s,i] := 0:
            Rh[1,k,s,i] := 0:
            for m from k-1 to s-1 do
```

```
            for l from max(0,i+m-s) to min(m,i) do
                Rh[1,k,s,i] := Rh[1,k,s,i] + Rh[1,k-1,m,l]*Rh[1,1,s-m,i-l]
                                           - Ih[1,k-1,m,l]*Ih[1,1,s-m,i-l]:
                Ih[1,k,s,i] := Ih[1,k,s,i] + Ih[1,k-1,m,l]*Rh[1,1,s-m,i-l]
                                           + Rh[1,k-1,m,l]*Ih[1,1,s-m,i-l]:
            od:
        od:
    od:
od:
for j from 2 to L do
    Ms := min(Mr[j],si):
    for k from 2 to Ms do
        for i from 0 to s do
            Ih[j,k,s,i] := 0:
            Rh[j,k,s,i] := 0:
            for m from 2*k-2 to s-2 do
                for l from max(0,i+m-s) to min(m,i) do
                    Rh[j,k,s,i] := Rh[j,k,s,i] + Rh[j,k-1,m,l]*Rh[j,1,s-m,i-l]
                                               - Ih[j,k-1,m,l]*Ih[j,1,s-m,i-l]:
                    Ih[j,k,s,i] := Ih[j,k,s,i] + Ih[j,k-1,m,l]*Rh[j,1,s-m,i-l]
                                               + Rh[j,k-1,m,l]*Ih[j,1,s-m,i-l]:
                od:
            od:
        od:
   od:
od:
for j from 1 to L do
    for k from 0 to si do
        if s=Ord then k:=iquo(s+1,2): fi:
        if k<si then nk:=2: else nk := 1: fi:
        temp := min(s-k, k, si-1):
        sk := k:
        for t from 1 to nk do
            Ra[t] := 0:
            Ia[t] := 0:
            for m from 1 to temp do
                Ra[t] := Ra[t] - (s-2*m)*Ren[m]*Rh[j,1,s-2*m,sk-m]
                               + (2*sk-s)*Imn[m]*Ih[j,1,s-2*m,sk-m]:
                Ia[t] := Ia[t] - (2*sk-s)*Imn[m]*Rh[j,1,s-2*m,sk-m]
                               - (s-2*m)*Ren[m]*Ih[j,1,s-2*m,sk-m]:
            od:
            sk := s-sk:
        od:
        for m from 2 to min(s,N) do
            Size := SizeIndex[j,m]:
            for i from 1 to Size do
                Mv := Index[j,m,i]:
                Nonzero := Mv[1,1]:
                Sleft := s-Mv[2,1]:
                if Sleft>=0 then
                   NIs:= binomial(Sleft+Nonzero-1,Sleft):
                   S  := Vector(Nonzero):
                   for l to NIs do
                       if l=1 then
```

```
   S[1]:= Sleft:
   p   := 1:
else
   S[p+1]:= S[p+1]+1:
   S[1]  := S[p]-1:
   if p <> 1 then
      S[p] := 0:
   fi:
   if S[1]=0 then
      p := p+1:
   else
      p := 1:
   fi:
fi:
for r from 1 to Nonzero do
    if Mv[1,r+1]=1 or Mv[1,r+1]=L+1 then
       Sv[r] := S[r]+Mv[2,r+1]:
    else
       Sv[r] := S[r]+2*Mv[2,r+1]:
    fi:
od:
Sv[r]:= k:
Ks   := k+1:
q    := Nonzero:
Kv[q]:= -1:
while q <= Nonzero do
      Kv[q]:= Kv[q]+1:
      Ks   := Ks-1:
      temp := Ks:
      qq   := 1:
      while temp >= Sv[qq] do
            Kv[qq] := Sv[qq]:
            temp := temp-Sv[qq]:
            qq := qq+1:
      od:
      Kv[qq] := temp:
      for n from qq+1 to q-1 do
          Kv[n] := 0:
      od:
      for t from 1 to nk do
          Rei := Mv[1,-1]:
          Imi := Mv[2,-1]:
          for n from 2 to Nonzero+1 do
           teR := Rei:
           if Mv[1,n]<L+1 then
             Rei := teR*Rh[Mv[1,n],Mv[2,n],Sv[n-1],Kv[n-1]]
                   -Imi*Ih[Mv[1,n],Mv[2,n],Sv[n-1],Kv[n-1]]:
             Imi := teR*Ih[Mv[1,n],Mv[2,n],Sv[n-1],Kv[n-1]]
                   +Imi*Rh[Mv[1,n],Mv[2,n],Sv[n-1],Kv[n-1]]:
             Kv[n-1] := Sv[n-1]-Kv[n-1]:
           else
             Kv[n-1] := Sv[n-1]-Kv[n-1]:
             if Mv[1,n]=L+1 then
                T := 1:
```

```
                                else
                                   T := Mv[1,n]-M2-1:
                                fi:
                                Rei := teR*Rh[T,Mv[2,n],Sv[n-1],Kv[n-1]]
                                      +Imi*Ih[T,Mv[2,n],Sv[n-1],Kv[n-1]]:
                                Imi := Imi*Rh[T,Mv[2,n],Sv[n-1],Kv[n-1]]
                                      -teR*Ih[T,Mv[2,n],Sv[n-1],Kv[n-1]]:
                             fi:
                            od:
                            Ra[t] := Ra[t]+Rei:
                            Ia[t] := Ia[t]+Imi:
                        od:
                        if t=2 then
                           for n from 1 to Nonzero do
                               Kv[n] := Sv[n]-Kv[n]:
                           od:
                        fi:
                        if Nonzero=1 or k = 0 then
                           break:
                        fi:
                        if qq>1 then
                           q := qq:
                           Ks:= Ks-temp:
                        else
                           if Kv[1]=0 then
                              Ks:= Kv[q]:
                              q := q+1:
                           else
                              Ks:= Kv[1]:
                              q := 2:
                           fi:
                           while Sv[q]=Kv[q] do
                                 Ks:= Ks+Kv[q]:
                                 q := q+1:
                           od:
                        fi:
                    od:
                od:
            fi:
        od:
        for t from 1 to nk do
            Ra[t] := factor(Ra[t]):
            Ia[t] := factor(Ia[t]):
        od:
    od:
    for t from 1 to nk do
        if  j=1 then
            if  k=(s+1)/2 then
                Ren[k-1] := Ra[t]:
                Imn[k-1] := Ia[t]:
                Rh[1,1,s,k] := 0:
                Ih[1,1,s,k] := 0:
            else
                Rh[1,1,s,k] := Ia[t]/(2*k-s-1):
```

```
                    Ih[1,1,s,k] := -Ra[t]/(2*k-s-1):
                fi:
            else
                temp := 2*k-s-IEf[j]:
                Rh[j,1,s,k] := (-REf[j]*Ra[t]+temp*Ia[t])/(REf[j]^2+temp^2):
                Ih[j,1,s,k] := (-REf[j]*Ia[t]-temp*Ra[t])/(REf[j]^2+temp^2):
            fi:
            k := s-k:
        od:
    od:
    if s=Ord then
       break:
    fi:
od:
od:
save Ren, Imn, `output`:
```

Bibliography

[1] J. Guchenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (4th edition), New York, Springer-Verlag 1993.

[2] A. H. Nayfeh, Mehtods of Normal Forms, New York, John Wiley 1993.

[3] S. N. Chow, C. Z. Li, D. Wang, Normal forms and Bifurcations of Planar Vector Fields, Cambridge, Cambridge University Press 1994.

[4] P. Yu, Computation of the normal forms via a perturbation technique, J. Sound Sound Vibration 211(1) (1998) 19-38.

[5] P. Yu, A simple and efficient method for computing center manifold and normal forms associated with semisimple cases, Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications and Algorithms 10(1-3) (2003) 273-286.

[6] J. Giné, X. Santallusia, On the Poincaré-Lyapunov constants and the Poincaré series, Appl. Math. (Warsaw) 28(1) (2001) 17-30.

[7] P. Yu, Simplest normal forms of Hopf and generalized Hopf bifurcations, Int. J. Bifurcation and Chaos 9(10) (1999) 1917-1939.

[8] P. Yu, Y. Yuan, An efficient method for computing the simplest normal forms of vector fields, Int. J. Bifurcation Chaos 13(1) (2003) 19-46.

[9] P. Yu, A. Y. T. Leung, The simplest normal form of Hopf bifurcation, Nonlinearity 16(1) (2003) 277-300.

[10] M. Gazor, P. Yu, Spectral sequences and parametric normal forms, J. Diff. Equs, 252 (2012) 1003-1031.

[11] P. Yu, A. Y. T. Leung, A perturbation method for computing the simplest normal forms of dynamical systems, J. Sound, Vib., 261(1) (2003) 123-151.

[12] P. Yu, Symbolic computation of normal forms for resonant double Hopf bifurcations using multiple time scales, J. Sound, Vib., 247(4) (2001) 615-632.

[13] P. Yu, Analysis on double Hopf bifurcation using computer algebra with the aid of multiple scales, Nonlinear Dynamics, 27 (2002) 19-53.

[14] J. Hofbauer, J.W.-H. So, Multiple limit cycles for three dimensional Lotka-Volterra Equations, Appl. Math. Lett. 7 (1994) 65-70.

[15] Z. Lu, Y. Luo, Two limit cycles in three-dimensional Lotka-Volterra systems, Comput. Math. Appl. 44 (2002) 51-66.

[16] M. Gyllenberg, P. Yan, Y. Wang, A 3-D competitive Lotka-Volterra system with three limit cycles A falsification of a conjecture by Hofbauer and So, Appl. Math. Lett. 19 (2006) 1-7.

[17] M. Gyllenberg, P. Yan, Four limit cycles for a three-dimensional competitive Lotka-Volterra system with a heteroclinic cycle, Comput. Math. Appl. 58 (2009) 649-669.

Chapter 6

Computing the normal forms associated with semisimple cases

6.1 Introduction

Normal form theory has been used for several decades as one of the important tools in simplifying the study of nonlinear differential systems. Its basic idea is to introduce a near-identity transformation into a given differential system to eliminate as many of nonlinear terms as possible, which are usually called non-resonant terms. The terms retained in the resulting system are normal form terms, called resonant terms. Since normal forms keep the fundamental dynamical characteristics of the original system in the vicinity of a singular point, it can be used to study the local bifurcations and properties of stability/instability for the original system. There are various books which have extensive discussions on normal form theory, for example, see [1, 2, 3]. More recent progress can be found in the article [4].

For higher-dimensional dynamical systems, normal form theory is usually applied together with center manifold theory, see [5, 6, 7, 8, 9]. If the Jacobian matrix of a differential system evaluated at a singular point contains eigenvalues with zero real part and non-zero real part, then center manifold theory should be considered in the study of the local dynamics of the system, and the dimension of the center manifold is equal to the number of eigenvalues with zero real part. Center manifold theory plays an important role in simplifying the analysis of local dynamical behavior of nonlinear differential systems near a singular point, because it allows us to determine the behavior by studying the flow on a lower dimensional manifold.

Several computer algebra systems such as Maple, Mathematica, Macsyma, etc., have been widely used for the computation of normal forms. Even with the help of these computer algebra systems, it is still not easy to obtain higher-order normal forms since considerably more computer memory and computational time are demanded as the order of normal forms increases. Therefore, in the past two decades, various methods have been developed to compute normal forms for general n-dimensional differential systems. However, many methods are not computationally efficient because lots of unnecessary computations are involved, for example, see [6, 10, 11]. To be precise, in order to get an expression for the kth-order normal form computation, $(k-1)$th-order normal forms, center manifolds and

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near-identity transformation are substituted into the original system. Thus, besides the kth-order terms, the obtained expression also contains lower-order ($< k$) and higher-order ($> k$) terms, which are not desirable for efficient computation. To overcome this problem, Yu [7, 12] developed a recursive formula for computing the coefficients of normal forms and center manifolds, which avoid those lower-order ($< k$) and higher-order ($> k$) terms in the kth-order computation. However, these formulas are not given in explicit recursive expressions and may be not so efficient in computation. For general planar systems, [13] obtained an explicit recursive formula for computing Poincaré-Lyapunov constants, and the computation based on this formula is efficient.

In this chapter, we consider general n-dimensional differential systems associated with semisimple cases, i.e., the Jacobian matrix of the linearized system evaluated at a singular point can be transformed into a diagonal Jordan canonical form. Around semisimple singularities, a rich variety of bifurcations, such as Hopf, double-zero, Hopf-zero, double-Hopf, etc., may occur. A detailed study for some types of these bifurcations can be found in [14, Chapter 7] by applying normal form theory to simplifying the systems. Particularly, for some special bifurcations like Hopf-zero, double-Hopf without resonance, the normal forms are symmetric with respect to rotation in the direction associated with the imaginary eigenvalues. In this case, the normal forms can be decoupled, and the systems are further simplified. Many methods have been developed and used to compute the normal forms of systems with semisimple singularities, not only for the particular cases like Hopf [9, 12, 13], Hopf-zero [15] and double-Hopf [16, 17], but also for general semisimple cases involving center manifold [6, 7]. In order to provide a good algorithm to compute the normal forms of general cases, in this chapter we will develop a computationally efficient method and a Maple program without restriction on the dimension of the center manifold. This chapter is an extension of our recent work [9], which focuses on general differential systems associated with Hopf bifurcation.

In the next section, an explicit, computationally efficient, recursive formula is derived for computing the normal forms and center manifolds of dynamical systems associated with semisimple singularities. The explicit formula is given in terms of the system coefficients of the original differential system, which is easily used for developing a Maple program. In Section 6.3, several examples are presented to demonstrate the computational efficiency of the method and the Maple program. Finally, conclusion is drawn in Section 6.4.

6.2 Main result

Consider a system of differential equations in the general form,

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{G}(\boldsymbol{y}), \quad \boldsymbol{y} \in \boldsymbol{R}^n, \quad \boldsymbol{G}(\boldsymbol{y}) : \boldsymbol{R}^n \to \boldsymbol{R}^n, \tag{6.1}$$

where the dot represents differentiation with respect to time, t, the matrix A is diagonalizable, $\boldsymbol{G}(\boldsymbol{0}) = \boldsymbol{0}$ and $D_y\boldsymbol{G}(\boldsymbol{0}) = \boldsymbol{0}$. Denote by λ_i, $i = 1, \cdots, n$, the eigenvalues of A. Without loss of generality, it is assumed that there are only k eigenvalues λ_j, $j = 1, \cdots, k$, having zero real part, implying that system (6.1) has a k-dimensional center manifold.

Then, through a proper linear transformation, system (6.1) can be transformed into

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \tag{6.2}$$

where J is a diagonal matrix, and $\boldsymbol{f}(\boldsymbol{x})$ is expanded as

$$\boldsymbol{f}(\boldsymbol{x}) = \sum_{m\geq 2} \boldsymbol{f}_m(\boldsymbol{x}), \quad \text{where} \quad \boldsymbol{f}_m(\boldsymbol{x}) = \sum_{\{m(n)\}} \boldsymbol{f}_{m(n)} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

and $m(n)$ denotes a vector $(m_1, m_2, \cdots, m_n)$ of n nonnegative integers, which satisfies $\sum_{j=1}^{n} m_j = m$.

Suppose that the matrix J has the form $J = \mathrm{diag}(J_o, J_r)$, where

$$J_o = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k), \quad J_r = \mathrm{diag}(\lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_n).$$

Let $\boldsymbol{x} = (\boldsymbol{x}_o^T, \boldsymbol{x}_r^T)^T$, where $\boldsymbol{x}_o = (x_1, x_2, \cdots, x_k)^T$ and $\boldsymbol{x}_r = (x_{k+1}, x_{k+2}, \cdots, x_n)^T$. Then, system (6.2) can be written as

$$\begin{aligned} \dot{\boldsymbol{x}}_o &= J_o \boldsymbol{x}_o + \boldsymbol{f}_o(\boldsymbol{x}_o, \boldsymbol{x}_r), \\ \dot{\boldsymbol{x}}_r &= J_r \boldsymbol{x}_r + \boldsymbol{f}_r(\boldsymbol{x}_o, \boldsymbol{x}_r). \end{aligned} \tag{6.3}$$

The center manifold of (6.3) may be defined as $\boldsymbol{x}_r = \boldsymbol{H}(\boldsymbol{x}_o)$, which satisfies $\boldsymbol{H}(\boldsymbol{0}) = \boldsymbol{0}$, $D\boldsymbol{H}(\boldsymbol{0}) = \boldsymbol{0}$. Then, the differential equation describing the dynamics on the center manifold is given by

$$\dot{\boldsymbol{x}}_o = J_o \boldsymbol{x}_o + \boldsymbol{f}_o(\boldsymbol{x}_o, \boldsymbol{H}(\boldsymbol{x}_o)). \tag{6.4}$$

Next, introduce a near-identity nonlinear transformation, given by

$$\boldsymbol{x}_o = \boldsymbol{u} + \boldsymbol{Q}(\boldsymbol{u}) = \boldsymbol{u} + \sum_{m\geq 2} \sum_{\{m(k)\}} \boldsymbol{q}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k} \equiv \boldsymbol{q}(\boldsymbol{u}), \tag{6.5}$$

into (6.4) to obtain the normal form,

$$\dot{\boldsymbol{u}} = J_o \boldsymbol{u} + \boldsymbol{C}(\boldsymbol{u}), \quad \text{where} \quad \boldsymbol{C}(\boldsymbol{u}) = \sum_{m\geq 2} \sum_{\{m(k)\}} \boldsymbol{c}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k}. \tag{6.6}$$

Now the center manifold can be expressed in the new variable $\boldsymbol{u}$, as follows:

$$\boldsymbol{x}_r = \boldsymbol{H}(\boldsymbol{q}(\boldsymbol{u})) = \sum_{m\geq 2} \sum_{\{m(k)\}} \boldsymbol{h}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k} \equiv \boldsymbol{h}(\boldsymbol{u}). \tag{6.7}$$

Combining the above steps yields the following equations

$$D_{\boldsymbol{u}} \begin{pmatrix} \boldsymbol{Q}(\boldsymbol{u}) \\ \boldsymbol{h}(\boldsymbol{u}) \end{pmatrix} J_o \boldsymbol{u} - \begin{pmatrix} J_o \boldsymbol{Q}(\boldsymbol{u}) \\ J_r \boldsymbol{h}(\boldsymbol{u}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}_o(\boldsymbol{u}) \\ \boldsymbol{F}_r(\boldsymbol{u}) \end{pmatrix} - D_{\boldsymbol{u}} \begin{pmatrix} \boldsymbol{Q}(\boldsymbol{u}) \\ \boldsymbol{h}(\boldsymbol{u}) \end{pmatrix} \boldsymbol{C}(\boldsymbol{u}) - \begin{pmatrix} \boldsymbol{C}(\boldsymbol{u}) \\ 0 \end{pmatrix}, \tag{6.8}$$

where $\boldsymbol{F}_o(\boldsymbol{u}) = \boldsymbol{f}_o(\boldsymbol{q}(\boldsymbol{u}), \boldsymbol{h}(\boldsymbol{u}))$, $\boldsymbol{F}_r(\boldsymbol{u}) = \boldsymbol{f}_r(\boldsymbol{q}(\boldsymbol{u}), \boldsymbol{h}(\boldsymbol{u}))$. Comparing the coefficients on both sides of (6.8), we obtain the recursive formulas for the coefficients of the center manifold and the normal form as well as the associated nonlinear transformation.

For convenience, we first introduce some notations. Suppose the powers of $\boldsymbol{q}(\boldsymbol{u})$ and $\boldsymbol{h}(\boldsymbol{u})$ can be expressed, for $j \geq 0$, as

$$\begin{aligned} \boldsymbol{q}^j(\boldsymbol{u}) &= \sum_{m=j}^{\infty} \sum_{\{m(k)\}} \boldsymbol{q}^j_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k}, \\ \boldsymbol{h}^j(\boldsymbol{u}) &= \sum_{m=2j}^{\infty} \sum_{\{m(k)\}} \boldsymbol{h}^j_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k}. \end{aligned} \tag{6.9}$$

We have the following main result.

Theorem 6.2.1 *For any fixed $s(k)$, $s \geq 2$, let $\Lambda = \sum_{i=1}^{k} \lambda_i s_i$. Then the recursive formulas for the coefficients of the nonlinear transformation* (6.5)*, the normal form* (6.6) *and the center manifold* (6.7) *of system* (6.3)*, i.e., $\boldsymbol{q}_{s(k)}$, $\boldsymbol{c}_{s(k)}$ and $\boldsymbol{h}_{s(k)}$, are given below.*

(1) *For $\boldsymbol{q}_{s(k)}$ and $\boldsymbol{c}_{s(k)}$, if $\Lambda - \lambda_j = 0$, $j = 1, \cdots, k$, then*

$$q_{s(k),j} = 0, \quad c_{s(k),j} = a_{s(k),j} - b_{s(k),j},$$

otherwise,

$$q_{s(k),j} = (a_{s(k),j} - b_{s(k),j})/(\Lambda - \lambda_j), \quad c_{s(k),j} = 0.$$

(2) *For $\boldsymbol{h}_{s(k)}$, we have*

$$h_{s(k),j-k} = (a_{s(k),j} - b_{s(k),j})/(\Lambda - \lambda_j), \ j = k+1, \cdots, n;$$

where

$$\begin{aligned}
\boldsymbol{a}_{s(k)} &= \sum_{m=2}^{s} \sum_{\{m(n)\}} \sum_{\{j(n)\}}^{j=s} \sum_{\{j_1(k)\}} \sum_{\{j_2(k)\}} \cdots \sum_{\{j_n(k)\}} \boldsymbol{f}_{m(n)} q_{j_1(k),1}^{m_1} \cdots q_{j_k(k),k}^{m_k} h_{j_{k+1}(k),1}^{m_{k+1}} \cdots h_{j_n(k),n-k}^{m_n}, \\
\boldsymbol{b}_{s(k)} &= \sum_{i=1}^{k} \sum_{l=2}^{s-1} \sum_{\{l(k)\}} (s_i + 1 - l_i) \begin{pmatrix} \boldsymbol{q}_{s(k)-l(k)+e_i(k)} \\ \boldsymbol{h}_{s(k)-l(k)+e_i(k)} \end{pmatrix} c_{l(k),i}, \\
\boldsymbol{q}_{s(k)}^{j} &= \sum_{l=j-1}^{s-1} \sum_{l(k) \leq s(k)} \boldsymbol{q}_{l(k)}^{j-1} \boldsymbol{q}_{s(k)-l(k)}, \\
\boldsymbol{h}_{s(k)}^{j} &= \sum_{l=2j-2}^{s-2} \sum_{l(k) \leq s(k)} \boldsymbol{h}_{l(k)}^{j-1} \boldsymbol{h}_{s(k)-l(k)}.
\end{aligned}$$

Proof For any given integer $s \geq 2$, suppose that we have obtained $\boldsymbol{q}_{m(k)}$, $\boldsymbol{h}_{m(k)}$ and $\boldsymbol{c}_{m(k)}$ for $m < s$. Now, we want to derive the formulas for $\boldsymbol{q}_{s(k)}$, $\boldsymbol{h}_{s(k)}$ and $\boldsymbol{c}_{s(k)}$. We divide the proof in three steps, which can also be served as the guidelines for developing programs using a computer algebra system.

Step 1. First of all, we need to compute all the coefficients of terms with degree s for $\boldsymbol{x}_o^j = \boldsymbol{q}^j(\boldsymbol{u})$, $2 \leq j \leq s$. Since $\boldsymbol{q}^j(\boldsymbol{u}) = \boldsymbol{q}(\boldsymbol{u})\boldsymbol{q}^{j-1}(\boldsymbol{u})$, we have

$$\begin{aligned}
\boldsymbol{q}^j(\boldsymbol{u}) =& \Big(\sum_{m=1}^{\infty} \sum_{\{m(k)\}} \boldsymbol{q}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k}\Big)\Big(\sum_{m=j-1}^{\infty} \sum_{\{m(k)\}} \boldsymbol{q}_{m(k)}^{j-1} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k}\Big) \\
=& \sum_{m=j}^{s} \sum_{\{m(k)\}} \sum_{l=j-1}^{m-1} \sum_{l(k) \leq m(k)} \boldsymbol{q}_{l(k)}^{j-1} \boldsymbol{q}_{m(k)-l(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k} + \mathrm{o}(|\boldsymbol{u}|^s),
\end{aligned}$$

where $l(k) \leq m(k)$ means $l_i \leq m_i$ for $i = 1, \cdots, k$. Then, we obtain

$$\boldsymbol{q}_{s(k)}^{j} = \sum_{l=j-1}^{s-1} \sum_{l(k) \leq s(k)} \boldsymbol{q}_{l(k)}^{j-1} \boldsymbol{q}_{s(k)-l(k)}, \quad 2 \leq j \leq s.$$

Similarly, for $\boldsymbol{x}_r^j = \boldsymbol{h}^j(\boldsymbol{u})$, we have

$$\boldsymbol{h}^j_{s(k)} = \sum_{l=2j-2}^{s-2} \sum_{l(k)\le s(k)} \boldsymbol{h}^{j-1}_{l(k)} \boldsymbol{h}_{s(k)-l(k)}, \quad 2 \le j \le s.$$

Step 2. Denote

$$\begin{pmatrix} \boldsymbol{F}_o(\boldsymbol{u}) \\ \boldsymbol{F}_r(\boldsymbol{u}) \end{pmatrix} = \sum_{m=2}^{s} \sum_{m(k)} \boldsymbol{a}_{\{m(k)\}} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k} + \mathrm{o}(|\boldsymbol{u}|^s). \tag{6.10}$$

In this step, we derive the formula for $\boldsymbol{a}_{s(k)}$. Let $\boldsymbol{q}^m_{l(k)} = (q^m_{l(k),1}, q^m_{l(k),2}, \cdots, q^m_{l(k),k})^T$ and $\boldsymbol{h}^m_{l(k)} = (h^m_{l(k),1}, h^m_{l(k),2}, \cdots, h^m_{l(k),n-k})^T$. For $2 \le m \le s$, substituting $\boldsymbol{q}(\boldsymbol{u})$ and $\boldsymbol{h}(\boldsymbol{u})$ into $\boldsymbol{f}_m(\boldsymbol{x})$ yields

$$\begin{aligned}
\boldsymbol{f}_m(\boldsymbol{x}) &= \sum_{\{m(n)\}} \boldsymbol{f}_{m(n)} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} = \sum_{\{m(n)\}} \boldsymbol{f}_{m(n)} \prod_{i=1}^{k} q_i^{m_i}(\boldsymbol{u}) \prod_{i=1}^{n-k} h_i^{m_{k+i}}(\boldsymbol{u}) \\
&= \sum_{\{m(n)\}} \boldsymbol{f}_{m(n)} \prod_{i=1}^{k} \Big(\sum_{l=m_i}^{\infty} \sum_{\{l(k)\}} q^{m_i}_{l(k),i} u_1^{l_1} u_2^{l_2} \cdots u_k^{l_k} \Big) \prod_{i=1}^{n-k} \Big(\sum_{l=2m_{k+i}}^{\infty} \sum_{\{l(k)\}} h^{m_{k+i}}_{l(k),i} u_1^{l_1} u_2^{l_2} \cdots u_k^{l_k} \Big) \\
&= \sum_{\{m(n)\}} \boldsymbol{f}_{m(n)} \Big(\sum_{l=m}^{s} \sum_{\{l(k)\}} \sum_{j_1=m_1}^{l} \cdots \sum_{j_k=m_k}^{l} \sum_{j_{k+1}=2m_{k+1}}^{l} \cdots \sum_{j_n=2m_n}^{l} \sum_{\{j_1(k)\}} \sum_{\{j_2(k)\}} \cdots \sum_{\{j_n(k)\}} \\
&\qquad q^{m_1}_{j_1(k),1} q^{m_2}_{j_2(k),2} \cdots q^{m_k}_{j_k(k),k} h^{m_{k+1}}_{j_{k+1}(k),1} h^{m_{k+2}}_{j_{k+2}(k),2} \cdots h^{m_n}_{j_n(k),n-k} u_1^{l_1} u_2^{l_2} \cdots u_k^{l_k} + \mathrm{o}(|\boldsymbol{u}|^s) \Big) \\
&= \sum_{l=m}^{s} \sum_{\{l(k)\}} \sum_{\{m(n)\}} \sum_{\{j(n)\}}^{j=l} \sum_{\{j_1(k)\}} \sum_{\{j_2(k)\}} \cdots \sum_{\{j_n(k)\}} \\
&\qquad \boldsymbol{f}_{m(n)} q^{m_1}_{j_1(k),1} \cdots q^{m_k}_{j_k(k),k} h^{m_{k+1}}_{j_{k+1}(k),1} \cdots h^{m_n}_{j_n(k),n-k} u_1^{l_1} u_2^{l_2} \cdots u_k^{l_k} + \mathrm{o}(|\boldsymbol{u}|^s),
\end{aligned}$$

where $\sum_{i=1}^{n} j_i(k) = l(k)$.

Since $\boldsymbol{f}(\boldsymbol{x}) = \sum_{m\ge 2} \boldsymbol{f}_m(\boldsymbol{x})$, we consequently obtain

$$\boldsymbol{a}_{s(k)} = \sum_{m=2}^{s} \sum_{\{m(n)\}} \sum_{\{j(n)\}}^{j=s} \sum_{\{j_1(k)\}} \sum_{\{j_2(k)\}} \cdots \sum_{\{j_n(k)\}} \boldsymbol{f}_{m(n)} q^{m_1}_{j_1(k),1} \cdots q^{m_k}_{j_k(k),k} h^{m_{k+1}}_{j_{k+1}(k),1} \cdots h^{m_n}_{j_n(k),n-k},$$

where the vector $j(n)$ satisfies

$$j_i \begin{cases} = 0 & \text{if } m_i = 0, \\ \left.\begin{array}{ll} \ge m_i & \text{for } 1 \le i \le k \\ \ge 2m_i & \text{for } k+1 \le i \le n \end{array}\right\} & \text{if } m_i \ne 0. \end{cases}$$

Step 3. Denote

$$D_{\boldsymbol{u}} \begin{pmatrix} \boldsymbol{Q}(\boldsymbol{u}) \\ \boldsymbol{h}(\boldsymbol{u}) \end{pmatrix} \boldsymbol{C}(\boldsymbol{u}) = \sum_{m=3}^{s} \sum_{\{m(k)\}} \boldsymbol{b}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k} + \mathrm{o}(|\boldsymbol{u}|^s). \tag{6.11}$$

In this step, we derive the formula for $\boldsymbol{b}_{s(k)}$. Note that

$$
\begin{aligned}
&D_{\boldsymbol{u}}\begin{pmatrix}\boldsymbol{Q}(\boldsymbol{u})\\ \boldsymbol{h}(\boldsymbol{u})\end{pmatrix}\boldsymbol{C}(\boldsymbol{u}) = \sum_{i=1}^{k}\begin{pmatrix}\boldsymbol{Q}_{u_i}(\boldsymbol{u})\\ \boldsymbol{h}_{u_i}(\boldsymbol{u})\end{pmatrix}C_i(\boldsymbol{u})\\
&= \sum_{i=1}^{k}\Big(\sum_{m=2}\sum_{\{m(k)\}} m_i\begin{pmatrix}\boldsymbol{q}_{m(k)}\\ \boldsymbol{h}_{m(k)}\end{pmatrix}u_1^{m_1}u_2^{m_2}\cdots u_k^{m_k}u_i^{-1}\Big)\Big(\sum_{m=2}\sum_{\{m(k)\}} c_{m(k),i}u_1^{m_1}u_2^{m_2}\cdots u_k^{m_k}\Big)\\
&= \sum_{m=3}^{s}\sum_{\{m(k)\}}\sum_{i=1}^{k}\sum_{l=2}^{m-1}\sum_{l}(m_i+1-l_i)\begin{pmatrix}\boldsymbol{q}_{m(k)-l(k)+e_i(k)}\\ \boldsymbol{h}_{m(k)-l(k)+e_i(k)}\end{pmatrix}c_{l(k),i}u_1^{m_1}u_2^{m_2}\cdots u_k^{m_k},
\end{aligned}
$$

where $e_i(k)$ is a unit vector with a 1 in the ith place. Therefore, comparing the above equation with (6.11) we have

$$
\boldsymbol{b}_{s(k)} = \sum_{i=1}^{k}\sum_{l=2}^{s-1}\sum_{\{l(k)\}}(s_i+1-l_i)\begin{pmatrix}\boldsymbol{q}_{s(k)-l(k)+e_i(k)}\\ \boldsymbol{h}_{s(k)-l(k)+e_i(k)}\end{pmatrix}c_{l(k),i}.
$$

Finally, from the left-hand side of (6.8), we obtain

$$
\begin{aligned}
&D_{\boldsymbol{u}}\boldsymbol{Q}(\boldsymbol{u})J_o\boldsymbol{u} - J_o\boldsymbol{Q}(\boldsymbol{u}) = \sum_{i=1}^{k}\lambda_i u_i\boldsymbol{Q}_{u_i} - J_o\boldsymbol{Q}(\boldsymbol{u})\\
&= \sum_{i=1}^{k}\sum_{m=2}\sum_{\{m(k)\}}\lambda_i m_i\boldsymbol{q}_{m(k)}u_1^{m_1}u_2^{m_2}\cdots u_k^{m_k} - \sum_{m=2}\sum_{\{m(k)\}}J_o\boldsymbol{q}_{m(k)}u_1^{m_1}u_2^{m_2}\cdots u_k^{m_k}\\
&= \sum_{m=2}^{s}\sum_{\{m(k)\}}\Big(\sum_{i=1}^{k}\lambda_i m_i I_k - J_o\Big)\boldsymbol{q}_{m(k)}u_1^{m_1}u_2^{m_2}\cdots u_k^{m_k} + \mathrm{o}(|\boldsymbol{u}|^s),
\end{aligned}
\tag{6.12}
$$

and similarly,

$$
\begin{aligned}
&D_{\boldsymbol{u}}\boldsymbol{h}(\boldsymbol{u})J_o\boldsymbol{u} - J_r\boldsymbol{h}(\boldsymbol{u})\\
&= \sum_{m=2}^{s}\sum_{\{m(k)\}}\Big(\sum_{i=1}^{k}\lambda_i m_i I_{n-k} - J_r\Big)\boldsymbol{h}_{m(k)}u_1^{m_1}u_2^{m_2}\cdots u_k^{m_k} + \mathrm{o}(|\boldsymbol{u}|^s).
\end{aligned}
\tag{6.13}
$$

Substituting (6.6) and (6.10)-(6.13) into (6.8) and comparing the coefficients of the same order results in the formulas in Theorem 6.2.1, and we thus complete the proof.

The source code of the Maple program developed using the formulas in Theorem 6.2.1 is given in Appendix C for the convenience of readers.

6.3 Application

In this section, we present several examples to demonstrate the applicability and the computational efficiency of the Maple program (see the source code in Appendix C) developed in this chapter. We show three examples associated with Hopf, Hopf-zero and

double Hopf singularities, and compute their normal forms and center manifolds, as well as the corresponding nonlinear transformations. We have tested a number of systems for comparing the algorithm developed in this chapter with that given in [6]. It is shown that for most cases the method developed in this chapter is better than that given in [6]. Only in some special cases, the situation is reversed. The program given in [6] can only deal with the cases where the dimension of the center manifold is less than seven. All the Maple programs are executed on a desktop machine with CPU 3.4 GHZ and 32G RAM memory to generate the normal forms as needed.

Example 1. We consider a 5-dimensional system:

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1^2 - x_1x_3 + x_5^2,\\
\dot{x}_2 &= -x_1 + x_2^2 + x_1x_4 + x_2^3,\\
\dot{x}_3 &= -x_3 + x_1^2,\\
\dot{x}_4 &= -x_4 + x_5 + x_1^2 + x_4x_5,\\
\dot{x}_5 &= -x_4 - x_5 + x_2^2 - 2x_4^2.
\end{aligned} \tag{6.14}$$

The Jacobian matrix of this system evaluated at the origin has eigenvalues $\pm i$, -1 and $-1 \pm i$. So the origin is a Hopf singularity and system (6.14) has a 2-dimensional center manifold. The normal form given in polar coordinates up to 5th order is given as follows:

$$\begin{aligned}
\dot{r} =& \frac{3}{40}r^3 - \frac{25633}{102000}r^5 - \frac{163441769}{2663424000}r^7 + \cdots,\\
\dot{\theta} =& 1 - \frac{7}{12}r^2 + \frac{6692923}{14688000}r^4 - \frac{47098141289}{299635200000}r^6 + \cdots.
\end{aligned} \tag{6.15}$$

The lengthy expressions for the center manifold and nonlinear transformation are omitted here for brevity.

Remark 1. The coefficients of the terms r^3 and r^5, etc., in the first equation of (6.15) are called the first, second, etc., *focus values*. In general, the normal form of system (6.3), given in polar coordinates, is in the form of

$$\begin{aligned}
\dot{r} =& r\,(v_0 + v_1\,r^2 + v_2\,r^4 + \cdots v_k\,r^{2k} + \cdots),\\
\dot{\theta} =& 1 + t_0 + t_1\,r^2 + t_2\,r^4 + \cdots t_k\,r^{2k} + \cdots,
\end{aligned} \tag{6.16}$$

where v_k is called the kth-order focus value, which is a function of the system parameters of (6.3). Small limit cycles bifurcating from the origin and their stability can be determined from the first equation of (6.16). The second equation of (6.16) can be used to determine the frequency of the bifurcating periodic motion (limit cycle).

Example 2. The second example is a 6-dimensional differential system, described by

$$\begin{aligned}
\dot{x}_1 &= -x_1^2 + 2x_1x_2 + 3x_1x_4 - x_1x_5 - x_2^2 + x_2x_4,\\
\dot{x}_2 &= x_3 - x_1^2 + 2x_1x_3 + 8x_1x_4 + x_3x_5,\\
\dot{x}_3 &= -x_2 - x_3^2 + 3x_1x_6 - x_3x_4 - 6x_4^2 - x_4x_6 + 2x_5^2,\\
\dot{x}_4 &= -x_4 - x_1^2 + 2x_1x_2 + 3x_1x_4 - x_1x_5 - x_2^2,\\
\dot{x}_5 &= -x_5 + x_6 - 7x_1^2 + 2x_1x_3 + 3x_1x_6 - x_3x_4 - x_4x_6,\\
\dot{x}_6 &= -x_5 - x_6 + x_1x_4 - 5x_3^2 + x_3x_5 - 4x_4^2 + x_5^2.
\end{aligned} \tag{6.17}$$

This system has a singular point at the origin, with its Jacobian matrix evaluated at the origin having three eigenvalues, 0 and $\pm i$, with zero real part, and three eigenvalues, -1 and $-1 \pm i$, with negative real part, implying that system (6.17) contains a 3-dimensional center manifold associated with a Hopf-zero singularity at the origin. Executing our Maple program gives the normal form (in cylindrical coordinates) up to 5th order,

$$\begin{aligned}
\dot{y} =& -y^2 - \frac{1}{2}r^2 + \frac{1}{2}y^3 - \frac{5}{4}yr^2 + \frac{59}{4}y^4 - \frac{259}{40}y^2r^2 + \frac{1}{36}r^4 \\
&+ 84y^5 + \frac{18509}{400}y^3r^2 + \frac{11483}{4800}yr^4 + \cdots, \\
\dot{r} =& \frac{29}{10}y^2r + \frac{9}{40}r^3 - \frac{1171}{25}y^3r - \frac{1371}{200}yr^3 \\
&- \frac{19331}{80}y^4r - \frac{263299}{2250}y^2r^3 - \frac{576761}{1224000}r^5 + \cdots, \\
\dot{\theta} =& 1 + y - \frac{61}{20}y^2 - \frac{163}{240}r^2 + \frac{4501}{200}y^3 - \frac{1357}{800}yr^2 \\
&+ \frac{4579}{160}y^4 + \frac{123833}{2250}y^2r^2 - \frac{102206489}{58752000}r^4 + \cdots.
\end{aligned}$$

Example 3. The last example is a 7-dimensional differential system,

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1^3 - x_1^2x_5 + x_1^2x_7, \\
\dot{x}_2 &= -x_1 - 2x_1x_3^2, \\
\dot{x}_3 &= \sqrt{2}x_4 + x_1^2x_3 - 4x_5^3, \\
\dot{x}_4 &= -\sqrt{2}x_3, \\
\dot{x}_5 &= -x_5 + (x_1 - x_5)^2, \\
\dot{x}_6 &= -x_6 + x_7 + (x_1 - x_4)^2, \\
\dot{x}_7 &= -x_6 - x_7 + (x_2 - x_6)^2,
\end{aligned} \tag{6.18}$$

whose Jacobian matrix evaluated at the origin has eigenvalues $\pm i$, $\pm\sqrt{2}i$, -1 and $-1 \pm i$, and four of them have zero real part. So the center manifold of system (6.18) is four dimensional. System (6.18) was studied by [6] and the normal form in polar coordinates up to 5th order was also given. We executed the Maple programs developed in this chapter as well as that given in [6] on the desktop machine. We have found that the Maple program given in [6] failed when it was executing to find the 9th-order normal form, because Maple was unable to allocate enough memory to complete the computation. While the program developed in this chapter only took 122 seconds and 13938 MB memory to finish the 9th-order normal form computation. The normal form up to 7th order given in polar coordinates is listed below.

$$\begin{aligned}
\dot{r}_1 =& \frac{3}{8}r_1^3 + \frac{157}{1360}r_1^5 - \frac{9}{40}r_1^3r_2^2 - \frac{428923841}{3847168000}r_1^7 - \frac{433291}{832320}r_1^5r_2^2 - \frac{612973}{8921600}r_1^3r_2^4 + \cdots, \\
\dot{\theta}_1 =& 1 + \frac{1}{2}r_2^2 - \frac{5543}{21760}r_1^4 - \frac{3}{80}r_1^2r_2^2 - \frac{1}{16}r_2^4 \\
&- \frac{888039}{9617920}r_1^6 + \frac{1744833}{5178880}r_1^4r_2^2 - \frac{1448249}{93676800}r_1^2r_2^4 + \frac{3}{32}r_2^6 + \cdots,
\end{aligned}$$

$$\dot{r}_2 = \frac{1}{4}r_1^2r_2^2 - \frac{1}{16}r_1^2r_2^3 + \frac{10213}{348160}r_1^6r_2 - \frac{3457}{446080}r_1^4r_2^3 + \frac{27}{256}r_1^2r_2^5 + \cdots,$$
$$\dot{\theta}_2 = \sqrt{2} - \frac{1}{32}\sqrt{2}r_1^4 + \frac{125}{89216}\sqrt{2}r_1^4r_2^2 + \cdots.$$

6.4 Conclusion

In this chapter, we have derived an explicit, recursive formula for computing the normal forms, center manifolds and nonlinear transformations for general n-dimensional systems, associated with semisimple singularities. A Maple program is also developed on the basis of the formula, which is very convenient for practical applicants who may not be familiar with normal form theory. It only needs users to prepare an input file and the Maple program will be "automatically" executed to generate the desired result. Three examples are presented to show the applicability of the new method and new program, and in particular, one of the examples demonstrates the advantage of the new method over the existing methods and programs.

Appendix C

In this appendix, for the convenience of readers, we list the symbolic Maple program developed in this chapter using the recursive formulas in Theorem 6.2.1, which can be used for computing the normal forms of general n-dimensional systems associated with semisimple cases. The input here takes the third example in the section of application.

```
with(LinearAlgebra):
M1      :=   0:          #  No. of zero eigenvalues
M2      :=   2:          #  No. of pairs of purely imaginary eigenvalues
M3      :=   1:          #  No. of non-zero real eigenvalues
M4      :=   1:          #  No. of pairs of complex conjugate eigenvalues
N       :=   3:          #  Highest order in the system
Ord     :=   5:
Mc      :=   M1 + 2*M2:
M       :=   Mc + M3 + 2*M4:
L       :=   M1 + M2 + M3 + M4:
f[1] :=   x[2] +  x[1]^3 - x[1]^2*x[5] + x[1]^2*x[7]:
f[2] := - x[1] - 2*x[1]*x[3]^2:
f[3] :=   sqrt(2)*x[4] + x[1]^2*x[3] - 4*x[5]^3:
f[4] := - sqrt(2)*x[3]:
f[5] := - x[5] + (x[1] - x[5])^2:
f[6] := - x[6] + x[7] + (x[1] - x[4])^2:
f[7] := - x[6] - x[7] + (x[2] - x[6])^2:
L3seq := proc()
    global l12,S3,p:
    if l12 = 0 then
       S3[p+1] := S3[p+1]+1: l12 := S3[p]-1:
       S3[p] := 0: p := max(0,sign(-l12))*p+1:
    else S3[1] := S3[1]+1: l12 := l12-1: fi:
end:
L3product := proc(sl,sr,q2r,q2i)
    local l3rmx,qpmx,qpr,qpi,ctpo,l12,l12r,p,pr,ctl,ctr,ctp,l3,l3r,sb,
```

```
          sp,S3,S3r,i,temp:
    l3rmx := binomial(sr+Mc-2,Mc-2): ctpo := 1:
    qpmx := binomial(sl+sr+Mc-1,Mc-1):
    qpr := Array(1..qpmx):  qpi := Array(1..qpmx):
    S3 := [seq(0,i=1..Mc-1)]:
    p := 1: ctl := 1: l12 := sl:
    for l3 to binomial(sl+Mc-2,Mc-2) do
        S3r := [seq(0,i=1..Mc-1)]:
        pr := 1: ctr := 1: l12r := sr: ctp := ctpo:
        for l3r to l3rmx do
            for i from ctp to ctp+l12+l12r do
                sb := max(0,i-ctp-l12): sp := min(l12r,i-ctp):
                qpr[i] := qpr[i]+add(q2r[ctr+j]*q1r[ctl+i-ctp-j]
                                -q2i[j+ctr]*q1i[ctl+i-ctp-j],j=sb..sp):
                qpi[i] := qpi[i]+add(q2r[ctr+j]*q1i[ctl+i-ctp-j]
                                +q2i[j+ctr]*q1r[ctl+i-ctp-j],j=sb..sp):
            od:
            ctp := i: ctr := ctr+l12r+1:
            if l12r = 0 then
               ctp := ctp-binomial(S3r[pr]+1,2)-S3r[pr]*l12:
               temp := l12+S3[1]:
               for i from 2 to pr do
                   ctp := ctp+binomial(temp+i,i+1)
                             -binomial(temp+S3r[pr]+i-1,i+1):
                   temp := temp+S3[i]:
               od:
               ctp := ctp+binomial(temp+S3r[pr]+i-1,i):
               S3r[pr+1] := S3r[pr+1]+1: l12r := S3r[pr]-1:
               S3r[pr] := 0: pr := max(0,sign(-l12r))*pr+1:
            else S3r[1] := S3r[1]+1: l12r := l12r-1: fi:
        od:
        ctl := ctl+l12+1:
        if l12 = 0 then
           ctpo := ctpo+binomial(sr+p+1,p+1):
           S3[p+1] := S3[p+1]+1: l12 := S3[p]-1:
           S3[p] := 0: p := max(0,sign(-l12))*p+1:
        else ctpo := ctpo+l12+sr+1: S3[1] := S3[1]+1: l12 := l12-1:  fi:
    od:
    return [qpr,qpi]:
end:
for i to M1 do x[i] := v[i]: od:
j := M1+1: k := L+1:
for i from M1+1 to M1+M2 do
    x[j] := (v[i]+v[k])/2:
    x[j+1] := I*(v[i]-v[k])/2:
    f[i] := simplify(f[j]-I*f[j+1]):
    j := j+2: k := k+1:
od:
for i from M1+M2+1 to L-M4 do
    x[j] := v[i]:
    f[i] := simplify(f[j]):
    j := j+1:
od:
for i from L-M4+1 to L do
```

```
    x[j] := (v[i]+v[k])/2:
    x[j+1] := I*(v[i]-v[k])/2:
    f[i] := simplify(f[j]-I*f[j+1]):
    j := j+2: k := k+1:
od:
for j to L do
    f[j] := simplify(f[j]):
    IEf[j] := diff(f[j],v[j]):
    for k to M do IEf[j] := subs(v[k]=0,IEf[j]): od:
    REf[j] := subs(I=0,IEf[j]):
    IEf[j] := subs(I=1,IEf[j]-REf[j]):
od:
Qd := [seq(1,j=1..M1+M2),seq(2,j=1..M3+M4),seq(1,j=1..M2),seq(2,j=1..M4)]:
Qc := [seq(j,j=1..M1),seq(L+j,j=1..M2),seq(M1+M2+j,j=1..M3),
       seq(M-M4+j,j=1..M4),seq(M1+j,j=1..M2),seq(L-M4+j,j=1..M4)]:
Qb := [seq(j,j=1..M1),seq(seq(M1+i*M2+j,i=0..1),j=1..M2)]:
SizeIndex := Array(1..2*N): Mr := [seq(1,i=1..L)]:
vecf := Vector([seq(f[j],j=1..L)]):
for m from 2 to N do
    Ml := [m+1,-1,seq(0,i=1..M-2)]: i := 1:
    while Ml[M] <> m do
          Ml[i+1] := 1 + Ml[i+1]: Ml[1] := Ml[i]-1:
          if i <> 1 then Ml[i] := 0: fi:
          if Ml[1] = 0 then i := i+1: else i := 1: fi:
          Mlc := Ml: ji := 0:
          for l to 2 do
              coef[l] := vecf:  cterm := 1:
              for k to M do
                   coef[l] := coeff(coef[l],v[k],Mlc[k]):
                   cterm := cterm*v[k]^Mlc[k]:
              od:
              if coef[l] = 0 then coef[l] := Vector(L): fi:
              vecf := vecf-cterm*coef[l]:
              if Norm(coef[l],2) <> 0 then
                 ji := ji+1:
                 if ji = 1 then
                    Mlc := [seq(Mlc[Qc[k]],k=1..M)]:
                    mlmx := max(Mlc-Ml):
                    if mlmx = 0 then l := l+1: fi:
                 fi:
              else l := l+1: fi:
          od:
          if ji > 0 then
             Mr := [seq(max(Mr[n],Ml[n]),n=1..L)]:
             qdg := m+add(Ml[n],n=M1+M2+1..L)+add(Ml[n],n=L+M2+1..M):
             jr := 0: jc := 0:
             for k from i to M do
                 if Ml[k] <> 0 then
                    if k < M1+1 or (k < L-M4 and k > M1+M2) then
                       jr := jr+1: j := -jr:
                    else jc := jc+1: j := jc: fi:
                    Kvt[j] := Ml[k]: Ivt[j] := k: Qvt[j] := Qd[k]*Ml[k]:
                 fi:
             od:
```

```
            Kv := [seq(Kvt[j],j=1..jc),seq(Kvt[-j],j=1..jr)]:
            Iv := [seq(Ivt[j],j=1..jc),seq(Ivt[-j],j=1..jr)]:
            Qv := [seq(Qvt[j],j=1..jc),seq(Qvt[-j],j=1..jr)]:
            SizeIndex[qdg] := SizeIndex[qdg]+1:
            N := max(N,qdg): sqdg := SizeIndex[qdg]:
            Index[qdg,sqdg] := [Kv,Iv,Qv,jc,jr,ji]:
        fi:
        for l to ji do
            eql := []:
            for k to L do
                if coef[l][k] <> 0 then eql := [op(eql),k]: fi:
            od:
            coefi := [seq(coef[l][eql[k]],k=1..nops(eql))]:
            coefr := subs(I=0,coefi):
            coefi := subs(I=1,coefi-coefr):
            sqdgn := (-1)^(l-1)*sqdg:
            Coef[qdg,sqdgn] := [eql,coefr,coefi]:
        od:
    od:
od:
for j to M do
    Ih[j,1,1] := Array(1..Mc): Rh[j,1,1] := Array(1..Mc):
od:
for j to M1 do Rh[j,1,1][j] := 1: od:
for j to M2 do
    Rh[M1+j,1,1][M1+2*j-1] := 1:
    Rh[L+j,1,1][M1+2*j] := 1:
od:
for s from 2 to Ord do
print(`order=`,s):
    smx := binomial(s+Mc-1,Mc-1):
    Ku := [seq(min(Mr[j],s),j=1..L)]:
    for j to L do
        for k from 2 to Ku[j] do
            Rh[j,k,s] := Array(1..smx):  Ih[j,k,s] := Array(1..smx):
        od:
    od:
    for sl to s-1 do
        l12 := sl: sr := s-sl: l3rmx := binomial(sr+Mc-2,Mc-2):
        S3 := [seq(0,i=1..Mc-1)]: p := 1: ctl := 1: ctpo := 1:
        for l3 to binomial(sl+Mc-2,Mc-2) do
            for j to L do
                Lslr[j] := [seq(Rh[j,1,sl][i],i=ctl..ctl+l12)]:
                Lsli[j] := [seq(Ih[j,1,sl][i],i=ctl..ctl+l12)]:
            od:
            S3r := [seq(0,i=1..Mc-1)]:
            pr := 1: ctr := 1: l12r := sr: ctp := ctpo:
            for l3r to l3rmx do
                for l to L do
                    for k to min(Ku[l]-1,sr) do
                        Lsrr := [seq(Rh[l,k,sr][i],i=ctr..ctr+l12r)]:
                        Lsri := [seq(Ih[l,k,sr][i],i=ctr..ctr+l12r)]:
                        for i from ctp to ctp+l12+l12r do
                            sb := max(0,i-ctp-l12): sp := min(l12r,i-ctp):
```

```
                        Rh[l,k+1,s][i] := Rh[l,k+1,s][i]
                             +add(Lsrr[j+1]*Lslr[l][i-ctp+1-j]
                             -Lsri[j+1]*Lsli[l][i-ctp+1-j], j=sb..sp):
                        Ih[l,k+1,s][i] := Ih[l,k+1,s][i]
                             +add(Lsri[j+1]*Lslr[l][i-ctp+1-j]
                             +Lsrr[j+1]*Lsli[l][i-ctp+1-j], j=sb..sp):
                    od:
                od:
            od:
            ctp := i: ctr := ctr+l12r+1:
            if l12r = 0 then
               ctp := ctp-binomial(S3r[pr]+1,2)-S3r[pr]*l12:
               temp := l12+S3[1]:
               for i from 2 to pr do
                   ctp := ctp+binomial(temp+i,1+i)
                            -binomial(temp+S3r[pr]+i-1,1+i):
                   temp := temp+S3[i]:
               od:
               ctp := ctp+binomial(temp+S3r[pr]+i-1,i):
               S3r[pr+1] := S3r[pr+1]+1: l12r := S3r[pr]-1:
               S3r[pr] := 0: pr := max(0,sign(-l12r))*pr+1:
            else S3r[1] := S3r[1]+1: l12r := l12r-1: fi:
        od:
        ctpo := ctpo+binomial(sr+l12+p+max(0,sign(-l12)),sr+l12):
        ctl := ctl+l12+1: L3seq():
    od:
od:
Tt := Array([seq(j,j=1..smx)]):
Lm := M1:
for L5t from 2*M2-2 by -2 to 0 do
    S5 := [seq(0,j=1..L5t+1)]:
    ct := 1: l14 := s: p := 1:
    for l5 to binomial(s+L5t,L5t) do
        for lm2 from 0 to iquo(l14-1,2) do
            ct := ct+binomial(l14+Lm,Lm)-binomial(l14-lm2-1+Lm,Lm):
            dml := binomial(l14+Lm,Lm+1):
            for lm1 from l14-lm2-1 by -1 to 0 do
                lmmx := binomial(lm1+Lm-1,Lm-1):
                dmcm := dml-binomial(lm1+lm2+Lm,Lm+1):
                for j from ct to ct+lmmx-1 do
                    temp := Tt[j]: Tt[j] := Tt[j+dmcm]:
                    Tt[j+dmcm] := temp:
                od: ct := ct+lmmx:
            od: l14 := l14-1:
        od:
        ct := ct+binomial(l14+Lm+1,Lm+1):
        l14 := l14+lm2-1:
        if l14 = 0 then
           l5 := l5+p: ct := ct+p: S5[p+1] := S5[p+1]+1: l14 := S5[p]:
           S5[p] := 0: p := max(0,sign(1-l14)*p)+1:
        else S5[1] := S5[1]+1: fi:
    od: Lm := Lm+2:
od:
for j from 1 to M2 do
```

```
    for k from 2 to Ku[M1+j] do
        Rh[L+j,k,s] := Array([seq(Rh[M1+j,k,s][Tt[i]],i=1..smx)]):
        Ih[L+j,k,s] := Array([seq(-Ih[M1+j,k,s][Tt[i]],i=1..smx)]):
    od:
od:
for j from 1 to M4 do
    for k from 2 to Ku[L-M4+j] do
        Rh[M-M4+j,k,s] := Array([seq(Rh[L-M4+j,k,s][Tt[i]],i=1..smx)]):
        Ih[M-M4+j,k,s] := Array([seq(-Ih[L-M4+j,k,s][Tt[i]],i=1..smx)]):
    od:
od:
T[s] := copy(Tt):
if s = Ord then L := M1+M2: fi:
for j to L do Rht[j] := Array(1..smx): Iht[j] := Array(1..smx): od:
for m from 2 to min(s,N) do
    sm := s-m:
    for mi to SizeIndex[m] do
        Kv := Index[m,mi][1]: Iv := Index[m,mi][2]: Qv := Index[m,mi][3]:
        jc := Index[m,mi][4]: jr := Index[m,mi][5]: ji := Index[m,mi][6]:
        slg := jc+jr: l3mx := binomial(sm+slg-1,slg-1):
        l12 := sm: p := 1: S3 := [seq(0,i=1..slg+1)]:
        for l3 to l3mx do
            Sv := [l12+Qv[1],seq(S3[j]+Qv[j+1],j=1..slg-1)]:
            q1r := copy(Rh[Iv[1],Kv[1],Sv[1]]):
            q1i := copy(Ih[Iv[1],Kv[1],Sv[1]]):
            sl := Sv[1]:
            for j from 2 to jc do
                qp := L3product(sl,Sv[j],
                      Rh[Iv[j],Kv[j],Sv[j]],Ih[Iv[j],Kv[j],Sv[j]]):
                q1r := copy(qp[1]): q1i := copy(qp[2]): sl := sl+Sv[j]:
            od:
            slmx := binomial(sl+Mc-1,Mc-1):
            if ji = 2 then
               if jc > 1 then
                  q3r := Array([seq(q1r[T[sl][i]],i=1..slmx)]):
                  q3i := Array([seq(-q1i[T[sl][i]],i=1..slmx)]):
               else ivc := Qc[Iv[1]]:
                  q3r := copy(Rh[ivc,Kv[1],Sv[1]]):
                  q3i := copy(Ih[ivc,Kv[1],Sv[1]]):
               fi:
            fi:
            for i to ji do
                slc := sl:
                for j from max(jc,1)+1 to slg do
                    qp := L3product(slc,Sv[j],
                          Rh[Iv[j],Kv[j],Sv[j]],Ih[Iv[j],Kv[j],Sv[j]]):
                    q1r := copy(qp[1]): q1i := copy(qp[2]):
                    slc := slc+Sv[j]:
                od:
                lfa := Coef[m,(-1)^(i-1)*mi]:
                for l to nops(lfa[1]) do
                    jl := lfa[1,l]:
                    if jl > L then break: fi:
                    Rht[jl] := Array([seq(Rht[jl][j]+lfa[2,l]*q1r[j]
```

```
                                        -lfa[3,l]*q1i[j],j=1..smx)]):
                    Iht[jl] := Array([seq(Iht[jl][j]+lfa[2,l]*q1i[j]
                                        +lfa[3,l]*q1r[j],j=1..smx)]):
                od:
                if ji = 2 then q1r := copy(q3r): q1i := copy(q3i): fi:
            od: L3seq():
        od:
    od:
od:
for sl from 2 to s-1 do
    l12 := sl: sr := s-sl: l3rmx := binomial(sr+Mc-2,Mc-2):
    S3 := [seq(0,i=1..Mc-1)]: ctpo := 1: p := 1: ctl := 1:
    for l3 to binomial(sl+Mc-2,Mc-2) do
        for j to Mc do
            Lslr[j] := [seq(Ren[j,sl][i],i=ctl..ctl+l12)]:
            Lsli[j] := [seq(Imn[j,sl][i],i=ctl..ctl+l12)]:
        od:
        S3r := [seq(0,i=1..Mc-1)]:
        l12r := sr: ctp := ctpo: pr := 1: ctr := 1:
        for l3r to l3rmx do
            for j to L do
                for wri to Mc do
                    jw := Qb[wri]:
                    Lsrr := [seq(dRh[j,sr+1,wri][i],i=ctr..ctr+l12r)]:
                    Lsri := [seq(dIh[j,sr+1,wri][i],i=ctr..ctr+l12r)]:
                    for jl to l12+l12r+1 do
                        sb := max(1,jl-l12): sp := min(l12r+1,jl):
                        Lsrt[wri][jl] := add(Lsrr[i]*Lslr[jw][jl+1-i]
                                -Lsri[i]*Lsli[jw][jl+1-i],i=sb..sp):
                        Lsit[wri][jl] := add(Lsrr[i]*Lsli[jw][jl+1-i]
                                +Lsri[i]*Lslr[jw][jl+1-i],i=sb..sp):
                    od:
                od:
                for i from ctp to ctp+l12+l12r do
                    Rht[j][i] := Rht[j][i]-add(Lsrt[wri][i-ctp+1],wri=1..Mc):
                    Iht[j][i] := Iht[j][i]-add(Lsit[wri][i-ctp+1],wri=1..Mc):
                od:
            od:
            ctp := i: ctr := ctr+l12r+1:
            if l12r = 0 then
               ctp := ctp-binomial(S3r[pr]+1,2)-S3r[pr]*l12:
               temp := l12+S3[1]:
               for i from 2 to pr do
                   ctp := ctp+binomial(temp+i,1+i)
                             -binomial(temp+S3r[pr]+i-1,1+i):
                   temp := temp+S3[i]:
               od:
               ctp := ctp+binomial(temp+S3r[pr]+i-1,i):
               S3r[pr+1] := S3r[pr+1]+1: l12r := S3r[pr]-1:
               S3r[pr] := 0: pr := max(0,sign(-l12r))*pr+1:
            else S3r[1] := S3r[1]+1: l12r := l12r-1: fi:
        od:
        ctpo := ctpo+binomial(sr+l12+p+max(0,sign(-l12)),sr+l12):
        ctl := ctl+l12+1: L3seq():
```

```
    od:
od:
lic := Array(1..smx):
S3 := [seq(0,i=1..Mc)]: p := 1: l12 := s:
for l5 to smx do
    S5 := [l12,op(S3)]:
    lic[l5] := add(IEf[i]*(S5[2*i-M1-1]-S5[2*i-M1]),i=M1+1..M1+M2):
    L3seq():
od:
for j to M1+M2 do
    Ren[j,s] := Array(1..smx): Imn[j,s] := Array(1..smx):
    Rh[j,1,s] := Array(1..smx): Ih[j,1,s] := Array(1..smx):
    Iy := -IEf[j]:
    for l5 to smx do
        Il := Iy+lic[l5]:
        if Il <> 0 then
           Rh[j,1,s][l5] := factor(Iht[j][l5]/Il):
           Ih[j,1,s][l5] := -factor(Rht[j][l5]/Il):
        else Ren[j,s][l5] := factor(Rht[j][l5]):
        Imn[j,s][l5] := factor(Iht[j][l5]): fi:
    od:
od:
if s < Ord then
   for j from M1+M2+1 to L do
       Rh[j,1,s] := Array(1..smx): Ih[j,1,s] := Array(1..smx):
       Ry := -REf[j]: Iy := -IEf[j]:
       for l5 to smx do
           Il := Iy+lic[l5]: temp := Ry*Ry+Il*Il:
           Rh[j,1,s][l5] := factor((Rht[j][l5]*Ry+Iht[j][l5]*Il)/temp):
           Ih[j,1,s][l5] := factor((Iht[j][l5]*Ry-Rht[j][l5]*Il)/temp):
       od:
   od:
   for j from M1+1 to M1+M2 do
       Ren[M2+j,s] := Array([seq(Ren[j,s][Tt[i]],i=1..smx)]):
       Imn[M2+j,s] := Array([seq(-Imn[j,s][Tt[i]],i=1..smx)]):
       Rh[L-M1+j,1,s] := Array([seq(Rh[j,1,s][Tt[i]],i=1..smx)]):
       Ih[L-M1+j,1,s] := Array([seq(-Ih[j,1,s][Tt[i]],i=1..smx)]):
   od:
   for j from L-M4+1 to L do
       Rh[M2+M4+j,1,s] := Array([seq(Rh[j,1,s][Tt[i]],i=1..smx)]):
       Ih[M2+M4+j,1,s] := Array([seq(-Ih[j,1,s][Tt[i]],i=1..smx)]):
   od:
   qdemx := binomial(s+Mc-2,Mc-1):
   for wri to Mc do
       for j to L do
           dRh[j,s,wri] := Array(1..qdemx):
           dIh[j,s,wri] := Array(1..qdemx):
       od:
       temp := Mc-wri:
       Si1 := [seq(0,j=1..temp+2)]:
       lsimx := binomial(s+temp,temp);
       l1i := s: kst := 1: oml := 0: po := 1:
       for lsi from 1 to lsimx do
          if wri > 1 then
```

```
                oml := oml+binomial(l1i+wri-2,wri-2):
                for li from 1 to l1i do
                    limx := binomial(l1i-li+wri-2,wri-2):
                    for j from kst to kst+limx-1 do
                        for jl to L do
                            dRh[jl,s,wri][j] := li*Rh[jl,1,s][j+oml]:
                            dIh[jl,s,wri][j] := li*Ih[jl,1,s][j+oml]:
                        od:
                   od: kst := kst+limx:
                od:
             else
                for jl to L do
                    dRh[jl,s,wri][kst] := l1i*Rh[jl,1,s][kst+oml]:
                    dIh[jl,s,wri][kst] := l1i*Ih[jl,1,s][kst+oml]:
                od: kst := kst+1:
             fi:
             if l1i = 1 then
                oml := oml+po: Si1[po+1] := Si1[po+1]+1: l1i := Si1[po]:
                Si1[po] := 0: lsi := lsi+po: po := max(0,sign(1-l1i)*po)+1:
             else Si1[1] := Si1[1]+1: l1i := l1i-1: fi:
          od:
       od:
    fi:
od:
ZC := [seq(0,j=1..M1)]:
RC := [seq(0,j=1..M2)]:
IC := [seq(IEf[M1+j],j=1..M2)]:
for s from 2 to Ord do
    l12 := s: p := 1: l3mx := binomial(s+Mc-1,Mc-1):
    S3 := [seq(0,i=1..Mc)]:
    for l3 to l3mx do
        Sl := [l12,op(S3)]: term := 1:
        for j from 1 to M1 do term := term*y[j]^Sl[j]: od:
        thetan := 0:
        for j from M1+1 to M1+M2 do
            term := term*r[j-M1]^(Sl[2*j-M1-1]+Sl[2*j-M1]):
            thetan := thetan+theta[j-M1]*(Sl[2*j-M1-1]-Sl[2*j-M1]):
        od:
        for j from 1 to M1 do
            ZC[j] := ZC[j]+term*(factor(Ren[j,s][l3])*cos(thetan)
                                 -factor(Imn[j,s][l3])*sin(thetan)):
        od:
        for j from 1 to M2 do
            RC[j] := RC[j]+term*(factor(Ren[j+M1,s][l3])*cos(thetan-theta[j])
                                 -factor(Imn[j+M1,s][l3])*sin(thetan-theta[j])):
            IC[j] := IC[j]+term/r[j]*(factor(Ren[j+M1,s][l3])*sin(thetan-theta[j])
                                      +factor(Imn[j+M1,s][l3])*cos(thetan-theta[j])):
        od:
        L3seq():
    od:
od:
for i from 1 to M1 do
    ZC[i] := combine(ZC[i],trig): print("y",i,ZC[i]):
od:
```

```
for i from 1 to M2 do
    RC[i] := combine(RC[i],trig): print("r",i,RC[i]):
    IC[i] := combine(IC[i],trig): print("theta",i,IC[i]):
od:
save M1,M2,ZC,RC,IC, output:
```

Bibliography

[1] S-N. Chow, C. Li, D. Wang, Normal forms and bifurcations of planar vector fields, Cambridge, Cambridge University Press (1994).

[2] P. Kahn, Y. Zarmi, Nonlinear dynamics: exploration through normal forms, New York, John Wiley Sons (1998).

[3] M. Han, P. Yu, Normal Forms, Melnikov Functions and Bifurcations of Limit Cycles, London, Springer (2012).

[4] L. Stolovitch, Progress in normal form theory, Nonlinearity 22 (2009) R77–R99.

[5] J. Carr, Applications of center manifold theory, New York, Springer-Verlag (1981).

[6] Q. Bi, P. Yu, Symbolic computation of normal forms for semi-simple cases, J. Comput. Appl. Math. 102 (1999) 195–220.

[7] P. Yu, A simple and efficient method for computing center manifold and normal forms associated with semisimple cases, Dyn. of Contin. Discr. Impul. Syst. Series B: Appl. Algorithm 10 (2003) 273–286.

[8] Y. Kuznetsov, Practical computation of normal forms on center manifolds at degenerate Bogdanov-Takens bifurcations, Int. J. Bifurcation and Chaos 15(11) 2005 3535–3546.

[9] Y. Tian, P. Yu, An explicit recursive formula for computing the normal form and center manifold of n-dimensional differential systems associated with Hopf bifurcation. Int. J. Bifurcation and Chaos 2013; 23(6):1350104 (18 pages).

[10] A. Y. T. Leung, An algorithm for higher order Hopf normal form, J. Shock Vib. 2 (1995) 307–319.

[11] W. Zhang, K. Huseyin, M. Ye, On the computation of the coefficients associated with high order normal forms, J. Sound Vib. 232 (2000) 525–540.

[12] P. Yu, Computation of the normal forms via a perturbation technique, J. Sound Vib. 211(1) (1998) 19–38.

[13] J. Giné, X. Santallusia, On the Poincaré-Lyapunov constants and the Poincaré series, Appl. Math. (Warsaw) 28(1) (2001) 17–30.

[14] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, New York, Springer-Verlag; (1983).

[15] M. Gazor, F. Mokhtari, A. Sanders, Normal forms for Hopf-Zero singularities with nonconservative nonlinear part, J. Differential Equations 254 (2013) 1571–1581.

[16] Q. Zhang, A. Leung, Normal form of double Hopf bifurcation in forced oscillators, J. Sound Vib. 231(4) (2000) 1057–1069.

[17] P. Yu, Symbolic computation of normal forms for resonant double Hopf bifurcations using multiple time scales, J. Sound Vib. 247(4) (2001) 615–632.

Chapter 7

Seven limit cycles around a focus point in a simple 3-dimensional quadratic vector field

7.1 Introduction

Limit cycle theory has been playing a very important role in the study of dynamical behavior of nonlinear systems, emerging from many physical and engineering models, and recently even from financial systems and social system. In mathematics, for a two-dimensional phase space, a limit cycle is a closed trajectory in the phase space having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity. Higher-dimensional vector fields often exhibit limit cycles which may co-exits with more complex dynamical behaviors such as chaos.

The study of limit cycles was initiated by [1]. He built a new branch of mathematics, called "qualitative theory of differential equations", and introduced the concept of limit cycles. Later, in the past more than 100 years, the development of limit cycle theory was perhaps motivated by the well-known Hilbert's 16th problem. The second part of this problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that planar polynomial systems of degree n can have. In early 1990's, [2] and [3] proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, the best result is 4 with $(3, 1)$ distribution, obtained more than 30 years ago [4, 5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$ is still open. In other words, the finiteness problem remains unsolved even for quadratic polynomial systems. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [7, 8]. Note that the 13 limit cycles are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems.

Suppose we consider Hilbert's 16th problem with limit cycles bifurcating from isolated fixed points, then the question becomes studying degenerate Hopf bifurcations, giving rise to

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weak (fine) focus points. This local problem has been completely solved only for generic quadratic systems [9], which can have 3 limit cycles in the vicinity of such a singular point. For cubic systems, [10] obtained a formal construction, via symbolic computation, of a special cubic system with 8 limit cycles. In 2009, Yu and Corless [11] showed the existence of 9 limit cycles with the help of a numerical method for another special cubic system. Recently, this special system was reconsidered using purely symbolic computation with the regular chains method to confirm the existence of 9 limit cycles, and find all the possible real solutions [12].

Due to the importance of limit cycle theory and frequently appearing in higher-dimensional dynamical systems, we want to study bifurcation of limit cycles in higher-dimensional vector fields. In this chapter, particular attention will be focused on 3-dimensional systems with a Hopf singular point. We would like to investigate what is the maximal number of limit cycles which may exist in the vicinity of a singular point of 3-dimensional systems. This is certainly a very challenge problem. There are very few results in the literature. Over the last twenty years, a 3-dimensional competitive Lotka-Volterra model has been studied extensively. The model is described by a 3-dimensional differential system:

$$\dot{x}_i = x_i\Big(b_i - \sum_{j=1}^{3} a_{ij}x_j\Big), \quad i = 1,\ 2,\ 3, \tag{7.1}$$

where the dot indicates differentiation with respect to time, t, x_i represents the population of ith species, and the coefficients take positive real values, $b_i > 0$, $a_{ij} > 0$, $i,\ j = 1, 2, 3$. This is a special case of general 3-dimensional quadratic systems. In the past two decades, several researchers have paid attention to system (7.1) and particularly studied bifurcation of limit cycles (e.g., see [13, 14, 15, 16]). So far, the best result is 4 limit cycles, obtained by [16], using appropriate parameter values. These 4 limit cycles include 3 small-amplitude limit cycles, proved by using focus value computation, and one large limit cycle, shown by constructing a heteroclinic cycle. Recently, Tian and Yu revisited this problem [17] and showed that this system might have maximal 8 limit cycles, but it is very difficult to prove using the existing methodology.

In this chapter, we turn to consider general 3-dimensional quadratic system, given by

$$\begin{aligned} \dot{x}_1 &= \alpha\, x_1 + x_2 + f_1(x_1, x_2, x_3), \\ \dot{x}_2 &= -x_1 + \alpha\, x_2 + f_2(x_1, x_2, x_3), \\ \dot{x}_3 &= -\beta\, x_3 + f_3(x_1, x_2, x_3), \end{aligned} \tag{7.2}$$

where α and $\beta > 0$ are parameters, and f_i's are quadratic polynomials. This system has a Hopf singularity at the origin when $\alpha = 0$. For general quadratic polynomials f_i and $\beta \neq 1$, the highest order of the focus value obtained from a desktop machine with CPU 3.4 GHZ and 32G RAM memory is 4. Moreover, even just solving these four polynomial equations is not an easy job. Therefore, we make a number of simplifications in (7.2) so that we can manage to obtain higher-order focus values, at least up to 8th order, and then try to apply the modular regular chains [12] to obtain 7 limit cycles in the vicinity of the origin. Compared to the Bautin's result for quadratic planar vector fields which can only have 3 small-amplitude limit cycles around a focus or center, this result is quite surprising. The description of the simple 3-dimensional quadratic vector filed and proof of the existence of 7 limit cycles around the origin will be given in the next section.

7.2 Main result

We start from the general 3-dimensional quadratic systems (7.2). Without loss of generality, the system can be written in the following form, with its linear part in Jordan canonical form,

$$\begin{aligned}
\dot{x}_1 &= \alpha\, x_1 + x_2 + a_{11}x_1^2 + (2b_{11} + a_{12})x_1x_2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{13}x_1x_3 + a_{23}x_2x_3,\\
\dot{x}_2 &= -x_1 + \alpha\, x_2 + b_{11}x_1^2 + (2a_{11} + b_{12})x_1x_2 - b_{11}x_2^2 + b_{33}x_3^2 + b_{13}x_1x_3 + b_{23}x_2x_3,\\
\dot{x}_3 &= -\beta\, x_3 + c_{11}x_1^2 + c_{12}x_1x_2 + c_{22}x_2^2 + c_{33}x_3^2 + c_{13}x_1x_3 + c_{23}x_2x_3,
\end{aligned} \tag{7.3}$$

where α, $\beta > 0$ and a_{ij}, b_{ij}, c_{ij} are parameters, and the formula in Bautin's equation [Bautin, 1952] has been used in the first two equations of (7.3), which can be achieved by a proper rotation around the x_3 axis. It is easy to see that the origin is an equilibrium point for any values of parameters, and a Hopf bifurcation occurs from the origin when α crosses the critical value $\alpha = \alpha_c = 0$.

Thus, we can use the formulas and Maple program developed in [17] to compute the normal form, which can then be used to determine small-amplitude limit cycles bifurcating from the origin. It is obvious that the zero-order focus value $v_0 = \alpha$, and at the critical point: $\alpha = \alpha_c = 0$, $v_0 = 0$. Then under the condition $\alpha = \alpha_c = 0$, the Maple program is executed on the desktop machine to obtain the focus values $v_1, v_2, \ldots$. It should be noted that for the general system (7.2), the computation of the higher-order normal form is very time consuming and memory demanding. Moreover, even we can obtain higher-order normal forms by using the Maple program, it is almost impossible to find the solutions of the multivariate polynomial system of focus values. Thus, in order to simplify the computation, we make some simplifications. First, we suppose $b_{11} \neq 0$ and $c_{12} \neq 0$. Then, we can use parameter scaling and state variable scaling in (7.2) so that $b_{11} = c_{12} = 1$. In order to make the computation of focus values manageable, we further set $a_{13} = a_{23} = a_{33} = b_{13} = b_{23} = b_{12} = c_{11} = c_{22} = c_{23} = 0$ and $\beta = 1$, resulting the following simple 3-dimensional quadratic system,

$$\begin{aligned}
\dot{x}_1 &= x_2 + a_{11}x_1^2 + (2 + a_{12})x_1x_2 + a_{22}x_2^2,\\
\dot{x}_2 &= -x_1 + x_1^2 + 2a_{11}x_1x_2 - x_2^2 + b_{33}x_3^2,\\
\dot{x}_3 &= -x_3 + x_1x_2 + c_{33}x_3^2 + c_{13}x_1x_3.
\end{aligned} \tag{7.4}$$

This is perhaps the simplest 3-dimensional quadratic system since it has only one coupling coefficient b_{33} between the first two equations and the third equation. When $b_{33} = 0$, the first two equations are decoupled from the third equation, and the problem becomes finding the limit cycles of the planar system, described by the first two equations of (7.4), and it is easy to show that this planar system has three small limit cycles around the origin, as expected. In fact, when $b_{33} = 0$, we can use the Maple program to find the first focus value v_1, given by $v_1 = -\frac{1}{8}\, a_{12}(a_{11} + a_{22})$. Letting $a_{12} = 0$ yields $v_1 = 0$ and then executing the Maple program produces $v_2 = -\frac{1}{12}a_{11}(a_{11} + a_{22})(a_{11} + 5a_{22})$. Further, letting $a_{11} = -5a_{22}$ results in $v_2 = 0$ and finally executing the Maple program yields

$$v_3 = 25a_{22}^3(1 - 3a_{22}^2), \quad v_4 = \frac{140}{9}a_{22}^3(1 - 3a_{22}^2)(7 - 38a_{22}^2), \quad v_5 = \cdots,$$

and all the v_i's contain the factor $a_{22}(1 - 3a_{22}^2)$, clearly indicating that maximal three small-amplitude limit cycles can be obtained around the origin when $b_{33} = 0$.

Now, suppose $b_{33} \neq 0$. We have the following main result.

Theorem 7.2.1 *Suppose the parameters, a_{11}, a_{12}, a_{22}, b_{33}, c_{33} and c_{13}, in system (7.3) are arbitrary non-zero constants. Then system (7.3) can have at least 7 small-amplitude limit cycles around the origin.*

In order to prove Theorem 7.2.1, we need the following lemma [18].

Lemma 7.2.2 *Suppose the focus values obtained from a general dynamical system are functions of k independent system parameters $p_1, p_2, \ldots, p_k$. Further, assume that at a critical point, p_c defined by $(p_1, p_2, \ldots, p_k) = (p_{1c}, p_{2c}, \ldots, p_{kc})$, the focus values satisfy*

$$v_j(p_c) = 0, \quad j = 0, 1, \ldots, k-1, \;\; v_k(p_c) \neq 0;$$

and

$$\det\Big[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(p_1, p_2, \ldots, p_k)}\Big]_{p_c} \neq 0.$$

Then, proper perturbations can be made to the parameters $p_1, p_2, \ldots, p_k$ around the critical point p_c to generate k small-amplitude limit cycles in the vicinity of the Hopf critical point.

Proof By using the Maple program [17], we can obtain the first seven focus values in terms of the system coefficients:

$$\begin{aligned} v_1 &= v_1(a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33}),\\ v_2 &= v_2(a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33}),\\ &\vdots\\ v_7 &= v_7(a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33}), \end{aligned} \tag{7.5}$$

and via them we can estimate the number of small-amplitude limit cycles around the origin, which are embedded in the center manifold (which is also obtained from the Maple program), described by

$$\begin{aligned} x_3 = {} & \frac{1}{5}(x_1^2 + x_1x_2 - x_2^2) - \frac{1}{5}(2\,a_{11} - c_{13} + 1)x_1^3 - \frac{1}{5}(3\,a_{11} + 2\,a_{12} - c_{13} + 2)x_1^2x_2\\ & + \frac{1}{5}(4\,a_{11} - a_{12} - 2\,a_{22} - c_{13} - 1)x_1x_2^2 - \frac{1}{5}(a_{22} + 2)x_2^3 - \frac{1}{5}\Big(c_{13} - \frac{1}{5}c_{33} + 2a_{11}c_{13} - c_{13}^2\Big)x_1^4\\ & - \frac{1}{5}\Big(2c_{13} - \frac{2}{5}c_{33} + 3a_{11}c_{13} + 2a_{12}c_{13} - c_{13}^2\Big)x_1^3x_2 + \frac{1}{5}\Big(-\frac{2}{5}c_{33} - 2c_{13} - c_{13}a_{22}\Big)x_1x_2^3\\ & - \frac{1}{5}\Big(c_{13} + \frac{1}{5}c_{33} - 4a_{11}c_{13} + a_{12}c_{13} + 2a_{22}c_{13} + c_{13}^2\Big)x_1^2x_2^2 + \frac{1}{25}c_{33}x_2^4 + \cdots \end{aligned} \tag{7.6}$$

It is obvious to see from (7.6) that the center manifold near the origin is approximated by a *hyperbolic parabolid*, as shown in Figure 7.1.

To obtain the maximal number of small-amplitude limit cycles bifurcating from the origin, we solve the parameters a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33} from the six polynomial equations $v_1 = v_2 = \cdots = v_6 = 0$. Alternatively, we may solve these six polynomial equations one by one, with one parameter at each time. We start from the first focus value v_1, which is the same as that for the case $b_{33} = 0$, i.e., $v_1 = -a_{12}(a_{11} + a_{22})/8$. Letting

$$a_{22} = -\,a_{11} \tag{7.7}$$

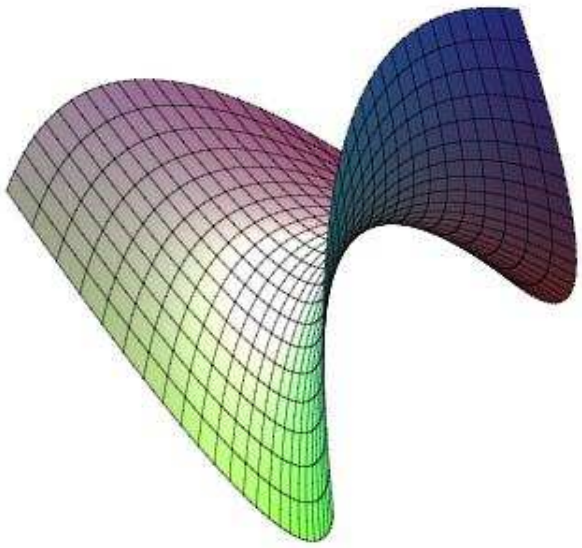

Figure 7.1: The second-order approximation of the center manifold described by (7.6).

yields $v_1 = 0$, and then executing the Maple program we have

$$v_2 = \frac{1}{1200}\, b_{33}(a_{12} + 3c_{13} + 18a_{11} + 10).$$

Setting

$$a_{12} = -(3c_{13} + 18a_{11} + 10) \tag{7.8}$$

results in $v_2 = 0$ and then executing the Maple program gives

$$\begin{aligned} v_3 = \frac{b_{33}}{272000}[&- 187b_{33} + (695c_{13} + 2070a_{11} - 790)c_{33} + (92290a_{11}^2 + 74582a_{11} + 15384)c_{13} \\ &+ (3342a_{11} - 45)c_{13}^2 - 666c_{13}^3 + 228172a_{11}^3 + 220428a_{11}^2 + 57028a_{11} + 2020]. \end{aligned}$$

Thus, we may solve for b_{33} from the equation $v_3 = 0$ to obtain

$$\begin{aligned} b_{33} = \frac{1}{187}[&(695c_{13} + 2070a_{11} - 790)c_{33} + (92290a_{11}^2 + 74582a_{11} + 15384)c_{13} \\ &+ (3342a_{11} - 45)c_{13}^2 - 666c_{13}^3 + 228172a_{11}^3 + 220428a_{11}^2 + 57028a_{11} + 2020]. \end{aligned} \tag{7.9}$$

Now, under the conditions given in (7.7)-(7.9), we have $v_1 = v_2 = v_3 = 0$, and further execute the Maple program to obtain

$$v_4 = \frac{F_0F_1}{1483804608000}, \qquad v_6 = \frac{F_0F_3}{161342710997621312839680000000},$$
$$v_5 = \frac{F_0F_2}{16015652769093120000}, \quad v_7 = \frac{F_0F_4}{15831592173930593701080760320000000000},$$

where

$$
\begin{aligned}
F_0 =\,& 5(414a_{11} + 139c_{13} - 158)c_{33} + (15384 + 74582a_{11} + 92290a_{11}^2)c_{13} \\
&+ 3(1114a_{11} - 15)c_{13}^2 - 666c_{13}^3 + 4(505 + 14257a_{11} + 55107a_{11}^2 + 57043a_{11}^3), \\
F_1 =\,& 4(4078333c_{13} + 14139153a_{11} - 2787647)c_{33}^2 \\
&+ 2[2(373041446a_{11}^2 + 500749565a_{11} + 111353261)c_{13} - (98445579a_{11} + 52751465)c_{13}^2 \\
&\quad - 16677015c_{13}^3 + 4(856767634a_{11}^3 + 967186323a_{11}^2 + 181154724a_{11} - 11713817)]c_{33} \\
&+ 29601792c_{13}^5 + 4(5370668262a_{11}^3 + 4204559671a_{11}^2 + 34389389a_{11} - 252727446)c_{13}^2 \\
&+ 9(86303536a_{11} + 25802705)c_{13}^4 + 6(882002754a_{11}^2 + 437785405a_{11} + 14608506)c_{13}^3 \\
&+ 8(13079993487a_{11}^4 + 24728477022a_{11}^3 + 15158560637a_{11}^2 + 3657980072a_{11} \\
&+ 287618390)c_{13} + 16(16499286495a_{11}^5 + 37837627784a_{11}^4 + 29685004857a_{11}^3 \\
&+ 9784662107a_{11}^2 + 1218699212a_{11} + 2203887), \\
F_2 =\,& -350064(144902698c_{13} + 366576733a_{11} - 142697703)c_{33}^3 \\
&+ 4[132287548607492c_{13}^3 + (1587431095048589a_{11} + 371135538912053)c_{13}^2 \\
&\quad + 2(1985595066771294a_{11}^2 + 332568076619189a_{11} - 167986412567563)c_{13} \\
&\quad + 4(832557288593889a_{11}^3 + 343973807357514a_{11}^2 - 129381390471427a_{11} \\
&\quad - 22505970396248)]c_{33}^2 \\
&- 4[213711424998672c_{13}^5 + 9(311807824390159a_{11} + 148318218680889)c_{13}^4 \\
&\quad - (1180924979804906a_{11}^2 + 5452054185589939a_{11} + 352322443563555)c_{13}^3 \\
&\quad - 4(42788477935364425a_{11}^3 + 6736358721517659 7a_{11}^2 + 24044781463740170a_{11} \\
&\quad + 2314226039107142)c_{13}^2 - 4(27591776267013391a_{11} + 17364671526832319 9a_{11}^4 \\
&\quad + 321001980070626737a_{11}^3 + 162732256648660003a_{11}^2 + 879991334108382)c_{13} \\
&\quad - 8(116707360013076057a_{11}^5 + 257051474918548297a_{11}^4 \\
&\quad + 171010776019582988a_{11}^3 + 42694054908035380a_{11}^2 \\
&\quad + 2402049457670883a_{11} - 250745335036453)]c_{33} \\
&+ 630912865266000c_{13}^7 + 9(2964712669290231a_{11} + 782392810580879)c_{13}^6 \\
&+ 6(73541596962729056a_{11}^2 + 38427258756511039a_{11} + 3606485585632344)c_{13}^5 \\
&+ 12(322411353706968259a_{11}^3 + 280026912237356567a_{11}^2 + 58525009028373887a_{11} \\
&+ 366306838658389)c_{13}^4 + 8(2731040674800736927a_{11}^4 + 381028989622808549 8a_{11}^3 \\
&+ 168865916911063594 0a_{11}^2 + 23965700960368200 9a_{11} - 731762510421390)c_{13}^3 \\
&+ 16(5442580446106842105a_{11}^5 + 113322997591777000 61a_{11}^4 \\
&+ 84353696734727401 99a_{11}^3 + 27683833478323993 42a_{11}^2 \\
&+ 37248928649081613 2a_{11} + 11783278257257817)c_{13}^2 \\
&+ 32(639742617239577155 4a_{11}^6 + 17484633288012047816a_{11}^5 \\
&+ 17729148621051130479a_{11}^4 + 86451419954342028 21a_{11}^3 \\
&+ 21016330857274692 05a_{11}^2 + 225196488860266879a_{11} \\
&+ 6078244989652798)c_{13} + 64(33356355498591042 92a_{11}^7
\end{aligned}
$$

$$
\begin{aligned}
&+ 1100497789631047707 1a_{11}^6 + 13758987325223239501a_{11}^5 \\
&+ 8649165820899777993a_{11}^4 + 2937263004217804984a_{11}^3 \\
&+ 518248503098891291a_{11}^2 + 38482349044241143a_{11} + 320434896140845), \\
F_3 = {} & 15197445120(1295313405565c_{13} + 4841990370990a_{11} - 974241807866)c_{33}^4 \\
&- 1664[247525665110477726 43c_{13}^3 + 2(81606533150962260337a_{11} \\
&+ 206630065005504758855)c_{13}^2 + 4(237848699207660471 40a_{11}^2 \\
&+ 456247687686341245921a_{11} - 3206719088129442123)c_{13} \\
&- 4(718015199110540951179a_{11}^3 + 164083276954411890501a_{11}^2 \\
&+ 4971877139175690141 61a_{11} + 116882338879114660895)]c_{33}^3 \\
&+ 4[135516201497462967471977c_{13}^5 \\
&+ (36525991142858375366326 77a_{11} + 13721188098394513881587 19)c_{13}^4 \\
&+ 8(45216944742492029841543 40a_{11}^2 + 34223576241410892731210 95a_{11} \\
&+ 55177282368291253511912 7)c_{13}^3 + 8(23001454127003267087701629a_{11}^3 \\
&+ 22632657768411737154856217a_{11}^2 + 41754495825771155781890 71a_{11} \\
&- 14459260316912917915206 1)c_{13}^2 + 16(25197277762204780005004103a_{11}^4 \\
&+ 23591855696103476578045814a_{11}^3 - 13399914896322960893938 48a_{11}^2 \\
&- 33682859604798515760602 86a_{11} - 42370929378293079454247 1)c_{13} \\
&+ 16(23833261557825482972806401a_{11}^5 + 23605088260502022606031673a_{11}^4 \\
&- 19292583598888132669773 58a_{11}^3 - 38528036393694380886513 66a_{11}^2 \\
&- 27335478633901824916446 7a_{11} + 72135305352486339433005)]c_{33}^2 \\
&- [70895937834694655576981 4c_{13}^7 + 6(16645548091775106391666 37a_{11} \\
&+ 14685495731921427531075 27)c_{13}^6 - 4(10142472888611771695397006a_{11}^2 \\
&- 80933517020719683217325 15a_{11} - 87976781085040671163936 11)c_{13}^5 \\
&- 8(27031813819030943621116 4404a_{11}^3 + 37486691473068725003069 4433a_{11}^2 \\
&+ 94857726635504393832071 718a_{11} + 16117145012249539126494 61)c_{13}^4 \\
&- 32(72702088741666074841141 7149a_{11}^4 + 14958402296224911172846 75129a_{11}^3 \\
&+ 84284965997531826681655 8199a_{11}^2 + 17562002014427476516149 6311a_{11} \\
&+ 11157942239036269527525 724)c_{13}^3 - 32(38150350765002455136569 04529a_{11}^5 \\
&+ 95076055464643027640411 40761a_{11}^4 + 73681677299082924452873 43410a_{11}^3 \\
&+ 24176902601664551144481 21506a_{11}^2 + 32792492408838628907088 0621a_{11} \\
&+ 12481403710220223481605 845)c_{13}^2 - 64(46233659648813868282065 41934a_{11}^6 \\
&+ 12873274210089905549397 428201a_{11}^5 + 11847721961389382224450 162225a_{11}^4 \\
&+ 48487499758367080417555 41250a_{11}^3 + 87967975884427191016923 1560a_{11}^2 \\
&+ 47359781290947171771497 733a_{11} - 19936551324775302597635 43)c_{13} \\
&- 128(21826883177049102161017 82722a_{11}^7 + 66503676301302239483386 76851a_{11}^6 \\
&+ 70555235799221108399315 49040a_{11}^5 + 35319625646501529844234 97445a_{11}^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 876158343574723264099830730a_{11}^3 + 91867822030408479430764349a_{11}^2 \\
&+ 93074881175678687047012a_{11} - 40453067608089466698904 5)]c_{33} \\
&+ 523779830429499928376064c_{13}^9 \\
&+ 9(47118785257588309619084 64a_{11} + 99934535087776690197814 1)c_{13}^8 \\
&+ 6(22587645272901583535634223 4a_{11}^2 + 11084652388505635794997753 3a_{11} \\
&+ 10253305691653840091798898)c_{13}^7 \\
&+ 12(18333452648194751659366277 78a_{11}^3 + 15659710645566005371100671 63a_{11}^2 \\
&+ 36639709161878615338784017 1a_{11} + 16051590214691783803149352)c_{13}^6 \\
&+ 8(26086580583552734554940584951a_{11}^4 + 33532903248572391712065771052a_{11}^3 \\
&+ 14238018150683934981297945709a_{11}^2 + 20745402243934510690611085 26a_{11} \\
&+ 27861686142403760251746546)c_{13}^5 \\
&+ 16(78548659777326308787336671659a_{11}^5 + 14113836350245398399469229 7539a_{11}^4 \\
&+ 92813909971704479331226525567a_{11}^3 + 26717320570403893049039935251a_{11}^2 \\
&+ 28901066749252255659991922 18a_{11} + 14630542556546467684436742)c_{13}^4 \\
&+ 32(15805646263323878580315104916 8a_{11}^6 + 37926101370434927233922419665 5a_{11}^5 \\
&+ 35115533484958103203371261159 0a_{11}^4 + 15922658864025402101951685761 0a_{11}^3 \\
&+ 36006057259210021488914610840a_{11}^2 + 34434273344205748939214418 71a_{11} \\
&+ 53348711666072696845705930)c_{13}^3 \\
&+ 64(21393448923166882173672610577 6a_{11}^7 + 65386020699644131165185872 9543a_{11}^6 \\
&+ 78793091276694133518088019706 9a_{11}^5 + 48978960538788931358666266171 0a_{11}^4 \\
&+ 16837281048588316725092794097 0a_{11}^3 + 31052460989227661080767094867a_{11}^2 \\
&+ 25951215812668872102202323 61a_{11} + 47677864066744036593911688)c_{13}^2 \\
&+ 128(17579518054145487832799432362 7a_{11}^8 + 64606939217750657718980970 4414a_{11}^7 \\
&+ 94397463297482392662851552342 7a_{11}^6 + 73081151690641207924249747 1216a_{11}^5 \\
&+ 32872157613087211829632963 4775a_{11}^4 + 86861820610588450810846403654a_{11}^3 \\
&+ 12669250522915877666530948173a_{11}^2 + 82831797235306395296959892 4a_{11} \\
&+ 84304855935841548102003 02)c_{13} \\
&+ 256(65489078114939885475729056623a_{11}^9 + 27620778389608102302663159544 0a_{11}^8 \\
&+ 46714855000761922831605077835 1a_{11}^7 + 42747496455237050956127271965 1a_{11}^6 \\
&+ 23526151864815831382248111374 5a_{11}^5 + 80473535394907792966346350991a_{11}^4 \\
&+ 16803980816127953612017002885a_{11}^3 + 19679237822908975278973209 37a_{11}^2 \\
&+ 10025604451707805864349665 2a_{11} + 23573817148144886900184 5).
\end{aligned}
$$

Now in order to obtain limit cycles bifurcating from the origin (the Hopf critical point) as many as possible, we need to find critical parameter values of a_{11}, c_{13} and c_{33} such that $v_4 = v_5 = v_6 = 0$ (i.e. $F_1 = F_2 = F_3 = 0$), but $v_7 \neq 0$. In this case, we can conclude that there exist at most 7 small-amplitude limit cycles bifurcating from the origin. Then, proper

perturbations may be applied to the four parameters to generate 7 small-amplitude limit cycles, or we can apply Lemma 7.2.2 to prove the existence of 7 limit cycles. Since we set $\alpha = 0$ to get $v_0 = 0$, $a_{22} = -a_{11}$ to get $v_1 = 0$, $a_{12} = -(18a_{11} + 3c_{13} + c_{23} + 10)$ to obtain $v_2 = 0$, and $b_{33} = \frac{1}{187}[(2070a_{11} + 695c_{13} - 95c_{23} - 790)c_{33} + \cdots]$ (given in (7.9)) to obtain $v_3 = 0$, perturbations on b_{33}, a_{12}, a_{22} and α can be made one by one. Thus, we only need to consider $v_4 = v_5 = v_6 = 0$, i.e. $F_1 = F_2 = F_3 = 0$, but $v_7 \neq 0$, at some critical values a_{11c}, c_{13c}, c_{33c}, and further

$$\det\Big[\frac{\partial(v_4, v_5, v_6)}{\partial(a_{11}, c_{13}, c_{33})}\Big]_{(a_{11c}, c_{13c}, c_{33c})} \neq 0.$$

To find the critical values a_{11c}, c_{13c}, c_{33c} such that $F_1 = F_2 = F_3 = 0$, we apply the *Regular Chain* method [12]. We use (7.7)-(7.9) to simplify v_4 to v_6 to obtain polynomial equations $F_1 = F_2 = F_3 = 0$. Then execute the Maple program (see [12]) on the same desktop machine to obtain the following results by using the modular regular chains method: the formulas of c_{13} and c_{33} expressed in terms of a_{11},

$$c_{13} = -\frac{c_{13N}(a_{11})}{12\, c_{13D}(a_{11})}, \qquad c_{33} = -\frac{c_{33N}(a_{11})}{N\, c_{33D}(a_{11})}, \tag{7.10}$$

where N is an integer, and $c_{13N}(a_{11})$, $c_{13D}(a_{11})$, $c_{33N}(a_{11})$ and $c_{33D}(a_{11})$ are 156th-degree polynomials of a_{11}; and a resultant equation, given by a 157th-degree polynomial $g(a_{11}) = 0$, which in turn gives a total of 19 real solutions. We solve a_{11} from this polynomial equation up to 1000 digit points, with the results listed below (only show the first 50 digits).

$$\begin{aligned}
a_{11}^1 &= -4.1127688849570565462470880834507887321139624950346 0\cdots,\\
a_{11}^2 &= -1.8201094286625800457709061186837160599876497379435 6\cdots,\\
a_{11}^3 &= -0.7644031138740396821992995384296758977158111423261 5\cdots,\\
a_{11}^4 &= -0.7541052064646377658988697454759772906867385168099 3\cdots,\\
a_{11}^5 &= -0.4606193413136485705555028641335219090656498937712 8\cdots,\\
a_{11}^6 &= -0.4475477209087094247603501104369576395078955907563 2\cdots,\\
a_{11}^7 &= -0.3818793791821958449634381322824693062741932217779 8\cdots,\\
a_{11}^8 &= -0.3142892028016016046952533690328926060081710383347 0\cdots,\\
a_{11}^9 &= -0.2831472983077952988221377398814878448626151748851 3\cdots,\\
a_{11}^{10} &= -0.1333083851557641359211914794711978576128397538804 4\cdots,\\
a_{11}^{11} &= -0.0286180334615408319218564891222443446892681697479 9\cdots,\\
a_{11}^{12} &= -0.0112961888335329994069642435639453007595924638122 8\cdots,\\
a_{11}^{13} &= \ 0.0000326186210328566732007587389168562977380249346 5\cdots,\\
a_{11}^{14} &= \ 0.0155796576032488273409965388850140359268047772240 9\cdots,\\
a_{11}^{15} &= \ 0.0262993672534860992692158098076824247078286868545 9\cdots,\\
a_{11}^{16} &= \ 0.0467422435646149345078632889447006098740314643835 2\cdots,\\
a_{11}^{17} &= \ 0.5603227592680635727058855605711671790604459278385 9\cdots,\\
a_{11}^{18} &= \ 5.3843891890342750418559445419479757303790206470580 2\cdots,\\
a_{11}^{19} &= 26.0149217370477450884359579396365354777780754732027 4\cdots.
\end{aligned}$$

We take $a_{11} = a_{11}^7$, which yields

$$c_{13} = -0.4126110281660668528891423244321370200465034827854 4\cdots,$$
$$c_{33} = -0.3316057668231894998764328671969248895736996189656 0\cdots,$$

and use (7.7)-(7.9) to obtain $a_{22} = -a_{11}^7$ and

$$a_{12} = -1.8883380902222742319906866456191414269250115596400 0\cdots,$$
$$b_{33} = -0.1467933934957948872226691228249372076600121812701 9\cdots.$$

For these critical parameter values, the focus values become

$$v_1 = 0.0, \quad v_2 = -0.1 \times 10^{-1000}, \quad v_3 = -0.6847 \times 10^{-1000},$$
$$v_4 = -0.1321925631038378675602206874299722253538021993100 4\cdots \times 10^{-942},$$
$$v_5 = -0.3176241835860192653330069567992326109935234300925 7\cdots \times 10^{-942},$$
$$v_6 = -0.4695093576878567609492717209832578285633176321022 1\cdots \times 10^{-942},$$
$$v_7 = -0.8377633908144676526279575146980829087246908580442 5\cdots \times 10^{-5}.$$

The errors are due to numerical computation in the final step of solving the 157th-degree polynomial of a_{11}. In fact, we can perform the interval computation in Maple to identity the interval for each of parameters up to any accuracy, which proves that there exist solutions such that $v_1 = v_2 = \cdots = v_6 = 0$, but $v_7 \neq 0$. Therefore, we can conclude that there exist at most 7 small-amplitude limit cycles around the origin. Moreover, a direct calculation shows that

$$\det\Big[\frac{\partial(v_4, v_5, v_6)}{\partial(a_{11}, c_{13}, c_{33})}\Big]_{(a_{11c}, c_{13c}, c_{33c})} \approx -0.00000000333723796304 \neq 0,$$

implying that there exist 7 small-amplitude limit cycles around the origin.

7.3 Conclusion

In this chapter, we have shown that a simple 3-dimensional quadratic vector field can exhibit 7 small-amplitude limit cycles in the vicinity of a Hopf critical point. The method of normal forms is applied to compute the focus values associated with Hopf bifurcation, while the modular regular chains method is used to solve higher-degree multivariate polynomial equations. This result may be further improved in future by developing more powerful computational tools.

Bibliography

[1] H. Poincaré, Mémoire sur les courbes définies par une équation différentielle.I, II, J. Math. Pures Appl. 7(3) (1881) 375-422; 8 (1882) 251-296; Sur les courbes définies par les équations différentielles. III, IV, ibid. 1(4) (1885) 167-224; 2 (1886) 155–217.

[2] Yu. S. Ilyashenko, S. Yakovenko, Finitely smooth normal forms of local families of diffeomorphismes and vector fields, Russ. Math. Surv. 46 (1991) 3–19.

[3] J. Écalle, Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac (French). Hermann, Paris, 1992.

[4] S. Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica 23 (1980) 153–158.

[5] L. S. Chen, M. S. Wang, The relative position, and the number, of limit cycles of a quadratic differential system, Acta. Math. Sinica 22 (1979) 751–758.

[6] P. Yu, M. Han, Four limit cycles from perturbing quadratic integrable systems by quadratic polynomials, Int. J. Bifurcation Chaos 22(10) (2012) 1250254 (28 pages).

[7] J. Li, Y. Liu, New results on the study of Z_q-equivariant planar polynomial vector fields, Qualitative Theory of Dynamical Systems 9(1-2) (2010) 167–219.

[8] C. Li, C. Liu, J. Yang, A cubic system with thirteen limit cycles, J. Diff. Eqs. 246 (2009) 3609–3619.

[9] N. N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sbornik (N.S.) 30(72) (1952) 181–196.

[10] E. James, N. Lloyd, A cubic system with eight small-amplitude limit cycles, IMA journal of applied mathematics 47 (1991) 163–171.

[11] P. Yu, R. Corless, Symbolic computation of limit cycles associated with Hilbert's 16th problem, Communications in Nonlinear Science and Numerical Simulation 14 (2009) 4041–4056.

[12] C. Chen, R. Corless, M. Maza, P. Yu, Y. Zhang, A modular regular chains method and its application to dynamical systems, Int. J. Bifurcation and Chaos 23(9) (2013) 1350154 (21 pages).

[13] J. Hofbauer, J. W.-H. So, Multiple limit cycles for three dimensional Lotka-Volterra Equations, Appl. Math. Lett. 7 (1994) 65–70.

[14] Z. Lu, Y. Luo, Two limit cycles in three-dimensional Lotka-Volterra systems, Comput. Math. Appl. 44 (2002) 51–66.

[15] M. Gyllenberg, P. Yan, Y. Wang, A 3-D competitive Lotka-Volterra system with three limit cycles: A falsification of a conjecture by Hofbauer and So, Appl. Math. Lett. 19 (2006) 1–7.

[16] M. Gyllenberg, P. Yan, Four limit cycles for a three-dimensional competitive Lotka-Volterra system with a heteroclinic cycle, Comput. Math. Appl. 58 (2009) 649–669.

[17] Y. Tian, P. Yu, An explicit recursive formula for computing the normal form and center manifold of n-dimensional differential systems associated with Hopf bifurcation, Int. J. Bifurcation and Chaos 23(6) (2013) 1350104 (18 pages).

[18] P. Yu, M. Han, Small limit cycles bifurcating from fine focus points in cubic order Z_2-equivariant vector fields, Chaos Solitons Fractals 24 (2005) 329–348.

Chapter 8

Conclusion and future work

8.1 Conclusion

In this thesis, bifurcation of limit cycles is investigated for smooth and non-smooth systems by computing Melnikov functions or focus values. In particular, we obtain new results about the lower bounds on the maximal number of small-amplitude limit cycles bifurcating from a center in planar cubic polynomial vector fields and in switching Bautin systems. A new algorithm for computing Lyapunov constants has been developed for switching systems. Moreover, two efficient and recursive formulae are derived for computing the normal forms and center manifolds of high-dimensional systems associated with Hopf and semisimple singularities, respectively.

In Chapter 2, it is showed that the cubic system in [1] can not have 11 small-amplitude limit cycles near the origin, and proved that there are actually only 9 limit cycles when (first-order) Melnikov function or the second-order Melnikov function is used. Further, a quadratic Hamiltonian system is perturbed by cubic polynomial functions for the study of bifurcation of limit cycles by using higher-order Melnikov functions and focus values computation. Decomposition of one-forms is introduced to express higher-order Melnikov functions and simplify the computation of focus values. It is showed that the cyclicityies of the system for the first five Melnikov functions M_k are given by $[\frac{4k}{3}] + 4$, $k = 1, \ldots, 5$, and thus 10 small-amplitude limit cycles are obtained around the origin for $k = 5$. This demonstrates an efficient approach of using higher-order Melnikov functions combined with the method of focus value computation to obtain more limit cycles.

Two existing systems taken from [2, 3] are re-investigated in Chapter 3, by applying the method of focus value computation. We have not only confirmed the existence of 11 small-amplitude limit cycles around a center in these two systems, but also obtained 12 small-amplitude limit cycles from one of the two systems by using a free parameter from the unperturbed system. This is the best result so far obtained from cubic planar vector fields around a singular point.

Chapter 4 is devoted to the study of Hopf bifurcation in switching systems. A new method with an efficient algorithm has been developed for computing Lyapunov constants, and then applied to study bifurcation of limit cycles in a switching Bautin system. A complete classification on the conditions of a singular point being a center in this Bautin system is

obtained. Further, an example of switching systems is constructed to show the existence of 10 small-amplitude limit cycles bifurcating from a center. This is a new lower bound of the maximal number of small-amplitude limit cycles near a singular point in quadratic switching systems.

In Chapter 5, explicit recursive formulae are derived for computing the normal forms and center manifolds of general n-dimensional dynamical systems associated with Hopf bifurcation. A Maple program is developed based on these formulae, which is very convenient in real applications. Two examples, including a 3-dimensional competitive Lotka-Volterra system, are present to show that the method and the Maple program are computationally efficient.

In Chapter 6, the method and formulae developed in Chapter 5 for Hopf bifurcation are generalized to general n-dimensional systems, associated with semisimple singularities. A Maple program is also developed on the basis of the formulae, which is very convenient for practical applicants who may not be familiar with normal form theory. Three examples are present to show the applicability of the new method and new program, and in particular, one of the examples demonstrates the advantage of the new method over the existing methods and programs.

In Chapter 7, we have applied the normal form and the Maple program developed in Chapter 5 to compute the focus values for a simple 3-dimensional quadratic vector field associated with a Hopf singularity. It is shown that 7 small-amplitude limit cycles can exist in the vicinity of a Hopf critical point. The modular regular chains method is applied to solve higher-degree multivariate polynomial equations. This result may be further improved in future by developing more powerful computational tools.

8.2 Future work

For future works, there exist many interesting but also challenging problems that remain open and are worth exploring.

In Chapter 2, a general quadratic Hamiltonian system is perturbed by cubic polynomial functions, to obtain the maximal number of small-amplitude limit cycles bifurcating from the origin for the first five Melnikov functions by the method of computing focus values. Because of the complexity of solving focus values, it has not been possible to get more than 10 limit cycles. Thus, our first question is: can we find a way to overcome the difficulty and obtain more limit cycles? We may prove the independency of Abelian integrals and then determine the number of limit cycles. On the other hand, the limit cycles bifurcating from the origin should be finite. Next question is: can we give an upper bound for the number of small-amplitude limit cycles?

For switching systems, we have used approximation of the solutions to compute Lyapunov constants. We may develop a new algorithm to compute the normal forms of switching systems, and then apply the normal forms to calculate the Lyapunov constants, just like that for smooth systems. This way, the computation may be more systematic and efficient. Note that in Chapter 4 we only obtain the center conditions for the Bautin system under the condition $a_6 b_6 = 0$. Future study will be focused on the case $a_6 b_6 \neq 0$. Also note that the existence of 10 small-amplitude limit cycles in a perturbed switching Bautin system is proved by using only first-

order Lyapunov constants. It is interesting to investigate how many limit cycles which can be obtained under cubic perturbations, by using higher-order Lyapunov constants.

Bibliography

[1] H. Żołądek, Eleven small limit cycles in a cubic vector field, Nonlinearity 8 (1995) 843–860.

[2] C. Christopher, Estimating limit cycle bifurcation from centers, in Trends in Mathematics: Differential Equations with Symbolic Computation, Birkhauser Verlag Switzerland (2005) 23–35.

[3] Yu. L. Bondar', A. P. Sadovskii, On a Żołądek theorem, Differential Equations 44(2) (2008) 274–277.

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(3) **Yun Tian** and Pei Yu, Seven limit cycles around a focus point in a simple 3-dimensional quadratic vector field, *Int. J. Bifurcation and Chaos*, **24**(6) 2014 1450083 (10 pages).

(4) Pei Yu and **Yun Tian**, Twelve limit cycles around a singular point in a planar cubic-degree polynomial system, *Communications in Nonlinear Science and Numerical Simulation*, **19**(8) 2014 pp.2690-2705.

(5) **Yun Tian** and Pei Yu, An explicit recursive formula for computing the normal forms associated with semisimple cases, *Communications in Nonlinear Science and Numerical Simulation*, **19**(7) 2014 pp.2294-2308.

(6) **Yun Tian** and Pei Yu, An explicit recursive formula for computing the normal form and center manifold of general n-dimensional differential systems associated with Hopf bifurcation, *Int. J. Bifurcation and Chaos*, **23**(6) 2013 (18 pages).

(7) Maoan Han, **Yun Tian** and Pei Yu, Small-amplitude limit cycles of polynomial Liénard systems, *Science China Mathematics*, **56**(8) 2013 pp.1543-1556.

(8) Dongmei Yan and **Yun Tian**, Hopf cyclicity for a Liénard system, *Journal of Zhejiang University* (Science Edition), **38**(1) 2011 pp.10-18.

(9) **Yun Tian** and Maoan Han, Hopf bifurcation for two types of Liénard systems, *J. Differential Equations*, **251**(4-5) 2011 pp.834-859.