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Ghost number of group algebras

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Abstract

The generating hypothesis for the stable module category of a finite group is the statement that if a map in the thick subcategory generated by the trivial representation induces the zero map in Tate cohomology, then it is stably trivial. It is known that the generating hypothesis fails for most groups. Generalizing work done for p-groups, we define the ghost number of a group algebra, which is a natural number that measures the degree to which the generating hypothesis fails. We describe a close relationship between ghost numbers and Auslander-Reiten triangles, with many results stated for a general projective class in a general triangulated category. We then compute ghost numbers and bounds on ghost numbers for many families of p-groups. For non-p-groups, we introduce two other closely related invariants, the simple ghost number, which considers maps which are stably trivial when composed with any map from a simple module, and the strong ghost number, which considers maps which are ghosts after restriction to every subgroup of G. We produce the first computations of the ghost number for non-p-groups. We prove that there are close relationships between the three invariants, and make computations of the new invariants for many families of groups. We also discuss how computational algebra can be applied to calculate the ghost number.

Keywords: Tate cohomology, stable module category, generating hypothesis, ghost map, GAP.

Co-Authorship

Chapters 2 and 3 of this thesis are based on two consecutive papers [23, 24] that study the ghost numbers of p-groups and non-p-groups, respectively. These are collaborated works with Dan Christensen. I have done most of the work to find the results and wrote down the first draft of both papers. Christensen has helped to improve the style of the papers, especially the introductions, to make them better for submission. The other chapters of the thesis are written solely by myself.

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Contents

Chapter 1

Introduction

In this thesis, we study the ghost number of group algebras for finite groups. Briefly speaking, the ghost number of a group algebra measures the failure of the *generating hypothesis* on the stable module category. This is motivated by the famous conjecture by Peter Freyd in stable homotopy theory, which states that if a map between two compact spectra is sent to zero by the stable homotopy group functor, then the map is null homotopic. The conjecture is referred to as the generating hypothesis and is still an open question. In [25], where Freyd made this conjecture, he also showed that it has many interesting consequences. For example, if the generating hypothesis holds for spectra, then the stable homotopy group functor is fully-faithful on compact spectra. The generating hypothesis can be generalised to a triangulated category, and has been studied in various cases, such as the derived category of a ring R and the stable module category of a group algebra kG. For the stable module category, it is known that the generating hypothesis fails for most groups (see Theorem 1.3.1). Hence we continue the study to see how badly the generating hypothesis can fail, and this is measured by the ghost number of the group algebra, which is the subject of the thesis.

This thesis is divided into five chapters. Chapter 1 is an introduction chapter where we introduce the background of the subject and summarize the results on ghost numbers of group algebras. Chapters 2 and 3 are based on two consecutive papers that study the ghost numbers of p-groups and non-p-groups, respectively. They contain both theoretical and computational results. Chapter 4 focuses on applying computational algebra to study the group algebra. We present some improved code for GAP [26] to work on modular representations and provide examples of computations. The last chapter is a conclusion chapter that briefly summarizes the results of the thesis and the relation between the chapters.

1.1 The generating hypothesis and its generalisation

In this section, we discuss the generalisation of the generating hypothesis to a triangulated category.

In homotopy theory, homotopy groups play a central role. They detect whether a nice space, such as a CW-complex, is contractible. Recall that the generating hypothesis for the stable homotopy category of spectra is the conjecture that if a map between two compact spectra is sent to zero by the stable homotopy group functor, then the map is null homotopic. Note that the stable homotopy category is a triangulated category. Hence we can apply the ideas from homotopy theory to other areas by generalisation from the stable homotopy category to a triangulated category.

Definition 1.1.1. A **triangulated category** is an additive category T together with a translation functor Σ and a class $\triangle$ of (distinguished) triangles $X \to Y \to Z \to \Sigma X$ in T, such that Σ is a self-equivalence of T and $\triangle$ satisfies the following axioms:

TR1 For each $X \in \mathsf{T}$, $X \xrightarrow{id} X \to 0 \to \Sigma X$ is a triangle;

for each map $f : X \to Y$ in T, there exists a triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$; and

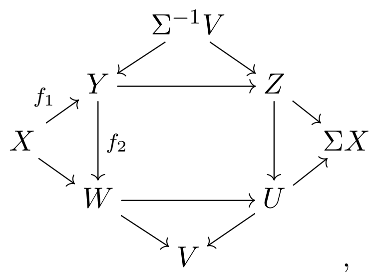
the triangles are closed under isomorphisms.

TR2 If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle, then $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ and $\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} h} X \xrightarrow{f} Y \xrightarrow{g} Z$ are triangles too.

TR3 Given a commutative square $\beta \circ f = f' \circ \alpha$ in T, complete the maps f and g into triangles. Then there exists a map γ making the following diagram into a map between triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'. \end{array}$$

TR4 (The octahedral axiom) Let $X \xrightarrow{f_1} Y \xrightarrow{f_2} W$ be maps in T. Complete the maps f_1 and f_2 into triangles. Then there exists a commutative octahedron in T:



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such that $X \to W \to U \to \Sigma X$ and $\Sigma^{-1}V \to Z \to U \to V$ are also triangles.

If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle in T, then Z is called the **cofibre** of f and X is called the **fibre** of g.

For example, the derived category $D(R)$ for a ring R is a triangulated category. The stable module category $\mathsf{StMod}(kG)$ for a group algebra kG is also a triangulated category. We will give more details on $\mathsf{StMod}(kG)$ in Section 1.2.

In general, let T be a triangulated category, and let $\mathbb{S}$ be a set of distinguished objects in T. We write $[-,-]$ for hom-sets in T. Then the set of functors $[S,-]_*$ with $S \in \mathbb{S}$ is analogous to the stable homotopy group functor in the sense that

$$\text{if } [S,M]_* = 0 \text{ for all } S \in \mathbb{S} \text{ and } M \in \mathsf{Loc}\langle\mathbb{S}\rangle \text{ , then } M = 0.$$

Here $\mathsf{Loc}\langle\mathbb{S}\rangle$ is the localising subcategory of T generated by $\mathbb{S}$. A full subcategory S of T is **localising** if it is closed under suspension, retracts, triangles, and arbitrary sums. The localising subcategory generated by $\mathbb{S}$ is the smallest localising subcategory that contains $\mathbb{S}$, and is denoted by $\mathsf{Loc}\langle\mathbb{S}\rangle$. If we do not require S to be closed under arbitrary sums, then S is said to be **thick**. The thick subcategory generated by $\mathbb{S}$ is defined similarly and denoted by $\mathsf{Thick}\langle\mathbb{S}\rangle$. We say that T satisfies **the generating hypothesis with respect to** $\mathbb{S}$ if the functors $[S,-]_*$ are faithful on $\mathsf{Thick}\langle\mathbb{S}\rangle$ for all $S \in \mathbb{S}$.

Note that if $\mathbb{S}$ consists of finitely many compact objects in T, then

$$\mathsf{Thick}\langle\mathbb{S}\rangle = \mathsf{Loc}\langle\mathbb{S}\rangle \cap \text{compact objects in } \mathsf{T}.$$

In general, an object X in a triangulated category T is **compact** if the canonial map $\oplus[X, C_i] \to [X, \oplus C_i]$ is an isomorphism for all coproducts in T. For example, finite

CW-complex are compact in spectra and perfect complex are compact in the derived category of a ring.

We will introduce the stable module category $\mathsf{StMod}(kG)$ of the group algebra kG in Section 1.2.2, and show that it is a triangulated category. Hence, taking $\mathbb{S}$ to consist of the trivial representation k, we can state the generating hypothesis on $\mathsf{StMod}(kG)$ in this setting.

One can also consider the global generating hypothesis with respect to $\mathbb{S}$ on T, i.e., the statement that the functors $[S, -]_*$ are faithful on $\mathsf{Loc}\langle\mathbb{S}\rangle$ for all $S \in \mathbb{S}$. This is studied by Hovey and Lockridge in the derived category of a ring spectrum E, and they show that the global generating hypothesis puts very strong constraints on E. The interested reader is referred to [27].

1.2 The generating hypothesis on the stable module category

In Section 1.2.1, we review some basic facts in representation theory and define the group cohomology and the Tate cohomology. Then, in Section 1.2.2, we describe the stable module category and state the generating hypothesis on it.

1.2.1 Background

We begin with the basic concepts in representation theory.

Let G be a finite group and k be a field. We define the group algebra kG to be the algebra over k, whose underlying space is the vector space generated by elements in G. Then a general element in kG is of the form $\sum a_g \cdot g$ with $a_g \in k$ and $g \in G$. We can multiply two basis elements g and h using the multiplication in G and extend this linearly to a general element. This defines the multiplication in kG, and if u is the unit in G, then $1 \cdot u$ is the unit in kG. With an abuse of notation, we write 1 for the unit in kG. Then the kG-modules are exactly the representations of G over k, and the study of representation theory is the same as the study of modules over group algebras.

We write $\mathsf{Mod}(kG)$ for the category of kG-modules and $\mathsf{mod}(kG)$ for its full subcategory of finitely-generated modules. Then we make the following definitions for kG-modules.

Definition 1.2.1. Let G be a group and k be a field. A kG-module M is said to be **irreducible** or **simple** if it has no proper submodules. A kG-module M is **completely reducible** or **semisimple** if every submodule of M splits off as a summand. The group algebra kG is **semisimple** if every kG-module M is semisimple.

Maschke's Theorem provides a criterion for when the group algebra kG is semisimple:

Theorem 1.2.2 (Maschke)**.** *Let G a finite group and k be a field. The group algebra kG is semisimple if and only if the characteristic of k does not divide the order of the group G.*

If kG is semisimple and k is algebraically closed, then the representations of G over k can be described by the character table. It provides a complete list of all irreducible summands of the free module kG, or, more precisely, the character functions of these irreducible modules. Note that since kG is semisimple, these are all the irreducible modules, and every module in $\mathsf{mod}(kG)$ is a sum of the irreducible modules.

The following is an example of the character table of the group S_3:

	$\{1\}$	$\{3\}$	$\{2\}$
S_3	id	(12)	(123)
$\chi_{trivial}$	1	1	1
χ_{sign}	1	-1	1
χ_2	2	0	-1

When kG is not semisimple, we know that there are submodules that do not split off as summands. For example, consider the central element $\Sigma_{g\in G} g$ in kG. It generates a two-sided ideal of kG of dimension one and, since $char(k) \mid |G|$ by Maschke's Theorem, the ideal is nilpotent. It follows that the ideal, considered as a submodule of kG, is not a direct summand. A kG-module M is said to be **indecomposable** if it has no proper summands. Note that if kG is semisimple, then indecomposable modules are the same as irreducible modules. Now we can state the Krull-Schmidt property of $\mathsf{mod}(kG)$. It is a very important feature of the module category over a group algebra.

Theorem 1.2.3 (Krull-Schmidt). *Let G be a group and k be a field. Let M be a finitely-generated kG-module. Suppose that $M = M_1 \oplus \cdots \oplus M_k$ and $M = N_1 \oplus \cdots \oplus N_l$ are two decompositions of M into indecomposable summands. Then $k = l$, and we can reorder the summands N_i so that $M_i \cong N_i$ for each i.*

The non-semisimple case is much more complicated than the semisimple case because there are non-trivial extensions between simple modules. Hence we study the homological properties of the group algebra kG when it is not semisimple. We still need a lemma before we give the definition of Tate cohomology.

Lemma 1.2.4. *Projective modules and injective modules coincide in* $\mathsf{mod}(kG)$.

Proof. We have a non-degenerate quadratic form

$$kG \times kG \longrightarrow k$$

that sends a pair $(g, g') \in kG$ to $\delta(g, g')$. This gives us an isomorphism $kG \cong kG^*$. Hence kG is an injective module over itself and the lemma follows. □

Now let G be a finite group and k be a field whose characteristic divides the order of G. We define the group cohomology and Tate cohomology of a kG-module M.

Definition 1.2.5. Let G be a finite group and k be a field. Let

$$P_* : \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0$$

be a projective resolution of the trivial representation k. The n-th **group cohomology** $H^n(G, M)$ of M is defined to be the n-th cohomology of the chain complex $\mathrm{Hom}(P_*, M)$ for $n \geqslant 0$.

If, instead of a projective resolution, we take a complete resolution

$$T_* : \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\partial_0} P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

of k, that is, a doubly infinite exact sequence of projective modules such that $\mathrm{im}(\partial_0) = k$, then the n-th **Tate cohomology** $\widehat{H}^n(G, M)$ of M is defined to be the n-th cohomology of the chain complex $\mathrm{Hom}(T_*, M)$ for $n \in \mathbb{Z}$. We can also replace the trivial module k by an arbitrary kG-module L and compute the resolutions P_* and T_* of L. The cohomology of the chain complexes $\mathrm{Hom}(P_*, M)$ and $\mathrm{Hom}(T_*, M)$ of M are denoted by $\mathrm{Ext}^n(L, M)$ and $\widehat{\mathrm{Ext}}^n(L, M)$, respectively.

1.2.2 The stable module category

We define the stable module category in this section and go over its basic properties. In particular, we show that the Tate cohomology functor is represented by the trivial kG-module k in $\mathsf{StMod}(kG)$. Then we state the generating hypothesis and its variations on the stable module category.

Let G be a finite group and k be a field. Note that a projective kG-module has trivial reduced group cohomology and Tate cohomology. Hence, by Maschke's Theorem, there is no cohomology when $char(k) \nmid |G|$. Thus we assume that the characteristic of k divides the order of G and focus on non-projective modules. For M and N in $\mathsf{StMod}(kG)$, we write $\mathrm{PHom}(M, N)$ for the subspace of $\mathrm{Hom}(M, N)$ that consists of maps between M and N that factor through a projective module. The stable module category $\mathsf{StMod}(kG)$ is a quotient category of the module category $\mathsf{Mod}(kG)$, the hom-sets being the quotient

$$\underline{\mathrm{Hom}}(M, N) = \mathrm{Hom}(M, N)/\mathrm{PHom}(M, N).$$

We write $\mathsf{stmod}(kG)$ for the full subcategory of finitely-generated modules in $\mathsf{StMod}(kG)$. It will follow that two modules M and N are isomorphic in the stable category if and only if there exists P and Q projective, such that $M \oplus P \cong N \oplus Q$. In particular, projective modules are isomorphic to zero in the stable category. By the Krull-Schmidt property, two finitely-generated modules M and N are isomorphic in $\mathsf{stmod}(kG)$ if and only if they have the same projective-free summands.

The module category $\mathsf{Mod}(kG)$ is abelian, but there is no reason that $\mathsf{StMod}(kG)$ is abelian. However, we can show that $\mathsf{StMod}(kG)$ is a triangulated category. We define the desuspension and suspension functors first.

Definition 1.2.6 (Desuspension and suspension)**.** Let M be a kG-module. The **desuspension** of M, denoted by ΩM, is the kernel in the short exact sequence

$$\Omega M \to P \xrightarrow{\epsilon} M,$$

where ϵ is a surjection from a projective module P to M. Dually, the **suspension** of M, denoted by ΣM, or $\Omega^{-1}M$, is the cokernel in the short exact sequence

$$M \xrightarrow{\epsilon} P \to \Sigma M,$$

where ϵ is an injection from M to an injective module P.

Desuspensions and suspensions are well-defined in the stable module category by Schanuel's Lemma. It is clear that they are inverses of each other.

Lemma 1.2.7 (Schanuel's Lemma)**.** *Let R be a ring, and let M be an R-module. Let $P \xrightarrow{\epsilon} M$ and $Q \xrightarrow{\theta} M$ be projective covers of M in $\mathsf{Mod}(R)$. Then $\ker\epsilon \oplus Q \cong \ker\theta \oplus P$.*

Remark 1.2.8. For M and N in $\mathsf{Mod}(kG)$, the tensor product $M \otimes N$ can be viewed as a kG-module via the diagonal action. One can show that if P is a projective module, then $P \otimes N$ is also projective. Since the functor $- \otimes N$ preserves short exact sequences in $\mathsf{Mod}(kG)$, it follows that we have a stable isomorphism $\Omega k \otimes M \cong \Omega M$. Here k is the trivial kG-module.

Now we can show that the cohomology groups are represented by hom-sets in $\mathsf{StMod}(kG)$.

Theorem 1.2.9. *Let G be a finite group and k be a field whose characteristic divides the order of G. Then, for M and L in $\mathsf{StMod}(kG)$ and $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\widehat{\mathrm{Ext}}^n(L, M) \cong \underline{\mathrm{Hom}}(L, \Sigma^n M) \cong \underline{\mathrm{Hom}}(\Omega^n L, M).$$

In particular, Tate cohomology is represented by the trivial representation k. □

By usual homological algebra, $\widehat{\mathrm{Ext}}^1(L, M)$ is equivalent to the isomorphism classes of extensions between L and M. Then, by the theorem, given a short exact sequence $M \to N \to L$, there is a corresponding map in $\underline{\mathrm{Hom}}(L, \Sigma M)$. More precisely, it is the connecting map $\delta : L \to \Sigma M$ in the following Snake-Lemma diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & 0 & \longrightarrow & L & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & = & P & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Sigma M & \longrightarrow & \Sigma N & \longrightarrow & 0 & & ,
\end{array}$$

where we choose P to be projective and the maps $M \to P$ and $N \to P$ to be injective. Then we use this to define a triangle

$$M \to N \to L \xrightarrow{\delta} \Sigma M$$

in $\mathsf{StMod}(kG)$. Note that $N \to L \oplus P \to \Sigma M$ is also a short exact sequence, and this corresponds the rotated triangle

$$N \to L \xrightarrow{\delta} \Sigma M \to \Sigma N.$$

Then, to compute the cofibre of a map $f : M \to N$ in $\mathsf{StMod}(kG)$, we replace f by an injection f' that is stably isomorphic to it, and then $\mathrm{coker}(f')$ is the cofibre of f.

Note that there is also a multiplication structure on the Tate cohomology $\widehat{H}^n(G, k)$. It is easy to describe the algebra structure of Tate cohomology using the natural isomorphism $\widehat{H}^n(G, k) \cong \underline{\mathrm{Hom}}(\Omega^n k, k)$. For $\zeta \in \underline{\mathrm{Hom}}(\Omega^m(M), N)$ and $\gamma \in \underline{\mathrm{Hom}}(\Omega^n(L), M)$, we define

$$\zeta\gamma := \zeta \circ \Omega^m(\gamma) \in \underline{\mathrm{Hom}}(\Omega^{m+n}(L), N).$$

This makes $\widehat{H}^*(G, k)$ into a graded commutative algebra. Similarly, we can define the multiplication on the group cohomology $H^*(G, k)$, and it is well-known [5, Section 4.2] that $H^*(G, k)$ is finitely generated. On the other hand, $\widehat{H}^*(G, k)$ is finitely generated if and only if the cohomology is periodic, i.e., $\Omega^n k \cong k$ for some $n \geqslant 0$.

We end this section with the generating hypothesis on the stable module category. Note that the Tate cohomology functor $\widehat{H}^*(G, -)$ on $\mathsf{StMod}(kG)$ plays an analogous role to the stable homotopy functor, and it is represented by the trivial representation k. Hence we state the generating hypothesis on $\mathsf{StMod}(kG)$ as follows:

The generating hypothesis holds on $\mathsf{StMod}(kG)$ if the Tate cohomology functor $\widehat{H}^*(G, -)$ is faithful on $\mathsf{Thick}\langle k \rangle$.

We call a map in the kernel of the Tate cohomology functor a **ghost**. Then the generating hypothesis (with respect to the trivial representation k) is the statement that every ghost in $\mathsf{Thick}\langle k \rangle$ is stably-trivial. It is important that we restrict to $\mathsf{Thick}\langle k \rangle$ here. In general, $\mathsf{Thick}\langle k \rangle$ can be a proper subcategory of $\mathsf{stmod}(kG)$, the category of finitely-generated kG-modules. And whenever this is the case, there exists a non-projective kG-module M, whose Tate cohomology is trivial. Thus the identity map on M is a stably non-trivial ghost in $\mathsf{stmod}(kG)$, but it is not in $\mathsf{Thick}\langle k \rangle$. Restricting to $\mathsf{Thick}\langle k \rangle$ prevents this from happening. On the other hand, we can consider the generating hypothesis with respect to all simple modules. Since the simple modules generate the stable module category, there is no need to restrict to $\mathsf{Thick}\langle k \rangle$. A map $M \to N$ is called a **simple ghost**, if the composite of maps $X \to M \to N$ is stably-trivial for any X that is a suspension of some simple module, and the **simple generating hypothesis** is the statement that every simple ghost in $\mathsf{stmod}(kG)$ is stably-trivial. If G is a p-group, then k is the only simple module and the simple generating hypothesis is equivalent to the generating hypothesis. One can also consider the **strong ghosts** in $\mathsf{StMod}(kG)$, which are the maps whose restrictions to any subgroup are still ghosts. Simple ghosts and strong ghosts are studied in Chapter 3.

1.3 Background and literature review

In this section, we review the previous work in the study of the generating hypothesis and ghost numbers for group algebras. We begin in Section 1.3.1 with the results on the generating hypothesis on $\mathsf{StMod}(kG)$ and review some techniques used in the proof. Then we introduce projective classes in Section 1.3.2, and using this idea, we define ghost numbers in Section 1.3.3. Finally, we introduce strong ghosts and the strong generating hypothesis in Section 1.3.4.

1.3.1 The generating hypothesis

In a series of papers [9, 16, 18, 20], it is proved that the generating hypothesis holds in $\mathsf{StMod}(kG)$ if and only if the Sylow p-subgroup P of G is C_2 or C_3.

Theorem 1.3.1 (Benson, Carlson, Chebolu, Christensen and Mináč [9, 16, 18, 20])**.** *Let G be a finite group and k be a field whose characteristic p divides the order of G. The generating hypothesis holds for $\mathsf{StMod}(kG)$ if and only if the Sylow p-subgroup of G is C_2 or C_3.* □

We review the techniques used to prove the theorem. Note that they are quite different for p-groups and non-p-groups.

1.3.1.1 The induction technique

Let H be a subgroup of G, and let k be a field. It is well known that the induction functor is both left and right adjoint to the restriction functor:

$$\uparrow_H^G : \mathsf{stmod}(kH) \leftrightarrows \mathsf{stmod}(kG) : \downarrow_H^G.$$

Since the restriction of the trivial module is also trivial, then, by the adjunction

$$\underline{\mathrm{Hom}}(\Omega^n k, f\uparrow_H^G) \cong \underline{\mathrm{Hom}}(\Omega^n (k\downarrow_H^G), f),$$

we see that if f is a ghost in $\mathsf{stmod}(kH)$, then $f\uparrow_H^G$ is a ghost in $\mathsf{stmod}(kG)$.

For a p-group G, since $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(kG)$, the induction technique becomes very useful. In particular, since the induction functor is faithful, it follows that if G is p-group and the generating hypothesis fails for a subgroup H, then the generating hypothesis fails for G. Hence, the work to disprove the generating hypothesis for a p-group can be reduced to the study of small groups.

However, the induction technique does not apply in general if G is not a p-group, since the image of a map induced up might not be in $\mathsf{Thick}\langle k\rangle$.

1.3.1.2 Auslander-Reiten triangles.

Auslander-Reiten triangles are used to disprove the generating hypothesis for a general finite group. In general, a triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ in a triangulated category is called an **Auslander-Reiten triangle**, if

1. $\gamma \neq 0$,

2. any map $X \to Y'$ that is not split monic factors through α,

3. any map $Y' \to Z$ that is not split epic factors through β.

Krause constructed Auslander-Reiten triangles in triangulated categories via Brown representability, and pointed out that this could be a counterexample to the generating hypothesis if the beginning-term X in the triangle is compact [30]. In the stable module category, Auslander-Reiten triangles have the form

$$\Omega^2 M \to H \to M \xrightarrow{\gamma} \Omega M,$$

with M being indecomposable and finitely-generated. One can show that if $M \in \mathsf{Thick}\langle k \rangle$ is indecomposable and not isomorphic to $\Sigma^n k$ for any $n \in \mathbb{Z}$, then γ is a stably-non-trivial ghost in $\mathsf{Thick}\langle k \rangle$. In [16], such a module M is proved to exist if the Sylow p-subgroup of G is not C_2 or C_3, hence the generating hypothesis fails in $\mathsf{StMod}(kG)$ in this case.

1.3.2 Projective classes and the universal ghost

We introduce the idea of a projective class in this section. It is used throughout our study of the generating hypothesis and ghost numbers.

Definition 1.3.2. Let T be a triangulated category. A **projective class** in T consists of a class $\mathcal{P}$ of objects of T and a class $\mathcal{I}$ of morphisms of T such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{I}$,

(ii) $\mathcal{I}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$,

(iii) for each X in T, there is a triangle $P \to X \to Y \to \Sigma P$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{I}$.

Note that the class $\mathcal{P}$ is closed under arbitrary sums and retracts in T and that the class $\mathcal{I}$ is an ideal in T. Also note that the map $X \to Y$ satisfying the third condition in the definition is a (weakly) universal map out of X in $\mathcal{I}$. It is zero if and only if X is a retract of P, and then $X \in \mathcal{P}$ and every map out of X in $\mathcal{I}$ is zero.

Given a projective class $(\mathcal{P}, \mathcal{I})$, there is a sequence of **derived projective classes** $(\mathcal{P}_n, \mathcal{I}^n)$ [21]. The ideal $\mathcal{I}^n$ consists of all n-fold composites of maps in $\mathcal{I}$, and X is in $\mathcal{P}_n$ if and only if it is a retract of an object M such that M sits inside a triangle $P \to M \to Q \to \Sigma P$ with $P \in \mathcal{P}_1 = \mathcal{P}$ and $Q \in \mathcal{P}_{n-1}$. For $n = 0$, we set $\mathcal{P}_0$ to consist of all zero objects and $\mathcal{I}^0$ to consist of all maps in T. The sequence $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \cdots$ provides a filtration of the localising subcategory generated by $\mathcal{P}$.

In $\mathsf{StMod}(kG)$, the ghosts form an ideal of a projective class $(\mathcal{F}, \mathcal{G})$, called the ghost projective class. Here $\mathcal{G}$ consists of all ghosts in $\mathsf{StMod}(kG)$ and $\mathcal{F}$ is generated by the trivial representation k by sums, retracts and suspensions. By assembling together the maps $\Omega^{n_i} k \to M$ that represent the generators of $\widehat{H}^*(G, M)$, we form a map $\oplus \Omega^{n_i} k \to M$ that is surjective on Tate cohomology. Hence the cofibre of $\oplus \Omega^{n_i} k \to M$ is a universal ghost and $(\mathcal{F}, \mathcal{G})$ is a projective class in $\mathsf{StMod}(kG)$. Note that if the cohomology is periodic, then $\oplus \Omega^{n_i} k$ can be chosen to be a finite sum and the universal ghost can be constructed within $\mathsf{stmod}(kG)$. Note that if $\mathcal{G}$ is the zero ideal, then the generating hypothesis holds. In general, the smallest integer n such that $\mathcal{G}^n$ becomes zero provides a measurement of the failure of the generating hypothesis. We make this precise in the next section.

1.3.3 Ghost numbers

Since the generating hypothesis fails for $\mathsf{StMod}(kG)$ for most groups, we define the ghost number to measure the degree of its failure. For $M \in \mathsf{Thick}\langle k \rangle$, the **ghost length** of M is the smallest integer n, such that every composite $M \to M_1 \to \cdots \to M_n$ of n ghosts in $\mathsf{Thick}\langle k \rangle$ is stably-trivial, and the **ghost number** of $\mathsf{StMod}(kG)$ is the upper bound of the ghost lengths of modules in $\mathsf{Thick}\langle k \rangle$. With this terminology, the generating hypothesis holds on $\mathsf{StMod}(kG)$ if and only if the ghost number is 1.

Another closely related invariant, the generating number, is introduced similarly. For $M \in \mathsf{Thick}\langle k \rangle$, the **generating length** of M is the smallest integer n, such that every composite $M \to M_1 \to \cdots \to M_n$ of n ghosts in $\mathsf{StMod}(kG)$ is stably-trivial, and the **generating number** of $\mathsf{StMod}(kG)$ is the upper bound of the generating lengths of modules in $\mathsf{Thick}\langle k \rangle$. Clearly, the generating length is greater than or equal to the ghost length, and the same holds for the generating number and the ghost number.

Note that the generating length has better formal properties than the ghost length. Indeed, since $(\mathcal{F}_n, \mathcal{G}^n)$ is a projective class in $\mathsf{StMod}(kG)$, the generating length of M

equals the smallest integer n such that $M \in \mathcal{F}_n \backslash \mathcal{F}_{n-1}$. See Section 2.3.4 for discussion on ghost length and generating length.

Computations of and bounds on ghost numbers and generating numbers of cyclic p-groups, abelian p-groups, and the quaternion group Q_8 are given in [19]. It is also proved that the ghost number of a p-group G is always finite. The idea is to use the radical sequence of a module, which allows us to build up the module from k within finitely many steps, and the number of steps is universally bounded by the radical length of kG. It follows that the generating number, hence the ghost number of a p-group, is finite.

1.3.4 Strong ghosts

Another variation of the generating hypothesis is to consider strong ghosts. A map in $\mathsf{StMod}(kG)$ is called a **strong ghost** if its restriction to any subgroup H of G is a ghost. It follows from the results in [17] that every strong ghost in $\mathsf{stmod}(kG)$ is stably-trivial if and only if the Sylow p-subgroup P of G is C_2, C_3, or C_4.

1.4 Results of the thesis

In this section, we summarize the main results of the thesis by chapters.

1.4.1 Ghost numbers of p-groups

We continue the study of the ghost number of a p-group in Chapter 2 (which is based on [23]), improving on the results in [19]. We provide general bounds on ghost numbers as well as computations of ghost numbers of various p-groups.

1.4.1.1 General lower bounds on the ghost number of a p-group

For an Auslander-Reiten triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ in $\mathsf{Thick}\langle k \rangle$, we show that if Z has generating length n, then γ is an $(n-1)$-fold ghost, i.e., the map γ is the longest possible composite of ghosts out Z (Proposition 2.3.6). This suggests that we can factorize γ as a composite of ghosts in $\mathsf{Thick}\langle k \rangle$ to find a lower bound on the ghost number.

For a p-group P, we consider the module $M = M_{\lfloor p/2 \rfloor}\uparrow_{C_p}^P$, where C_p is a cyclic subgroup of order p, and $M_{\lfloor p/2 \rfloor}$ is the indecomposable C_p-module of radical length $\lfloor p/2 \rfloor$. We find that the ghost length of M is equal to its radical length. As a result, we have Corollary 2.4.17, which says that, for a p-group P,

$$\text{the ghost number of } kP \geqslant 1/3 \text{ the radical length of } kP,$$

and Proposition 2.4.33, which says that if P has order p^r,

$$\text{the ghost number of } kP \geqslant (r-1)(p-1)+1.$$

The inequalities recover that, for a p-group P, the generating hypothesis holds only if $P = C_2$ or C_3. The results also apply to the quaternion group Q_8 (Proposition 2.4.13) and the dihedral groups D_{4q} of order $4q$ (Corollary 2.4.18), and improve the lower bounds on their ghost numbers shown in [19].

1.4.1.2 The ghost numbers of D_{4q} and $C_3 \times C_3$

We give an upper bound for the ghost number of a dihedral 2-group D_{4q}, with the aid of the classification theorem [4, Section 4.11]. This upper bound coincides with the lower bound we get in Corollary 2.4.18, so we can compute that (Corollary 2.4.25):

$$\text{the ghost number of } kD_{4q} = q+1.$$

The classification of representations of the Kronecker Quiver is also applied to the study of the ghost number of $C_3 \times C_3$, and we have Theorem 2.4.28:

$$\text{the ghost number of } k(C_3 \times C_3) = 3.$$

1.4.2 Ghost numbers of non-p-groups

In Chapter 3 (which is based on [24]), we generalise the study of ghost numbers to arbitrary finite groups. We show that the ghost number is finite if $\mathsf{Thick}\langle k \rangle = \mathsf{stmod}(B_0)$, and compute the ghost numbers of various examples in this case. We also study simple ghosts and strong ghosts in Chapter 3.

1.4.2.1 Finiteness and lower bound of the ghost number of a non-p-group

When the group G is a non-p-group, we prove finiteness under the assumption that $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$. Here $\mathsf{stmod}(B_0)$ is the full subcategory of $\mathsf{stmod}(kG)$ consisting of modules in the principal block B_0. It follows from the assumption that all the simple modules in the principal block have finite ghost lengths. Hence, given $M \in \mathsf{stmod}(B_0)$, the semisimple modules that appear in the radical sequence of M have finite ghost lengths, and, like the p-group case, there is a *universal* upper bound for the ghost lengths of $M \in \mathsf{stmod}(B_0)$. Hence the ghost number of kG is finite in this case (Theorem 3.4.7).

Now let e_0 be the principal block idempotent of kG. Left multiplication by e_0 provides a natural projection of $\mathsf{stmod}(kG)$ onto $\mathsf{stmod}(B_0)$. Assuming $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$ again, the image of the functor $e_0(-)$ lands inside $\mathsf{Thick}\langle k\rangle$. We prove that the composite of functors $e_0(-\uparrow) : \mathsf{stmod}(kP) \to \mathsf{stmod}(B_0)$ is faithful, where P is a Sylow p-subgroup of G. Since the functor $e_0(-\uparrow)$ preserves ghosts, we get a lower bound for the ghost number of kG in this case (Proposition 3.4.10):

If $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$, then

$$\text{the ghost number of } kG \geqslant \text{the ghost number of } kP.$$

1.4.2.2 Examples of ghost numbers of non-p-groups

Recall that a map in $\mathsf{StMod}(kG)$ is a simple ghost if it is stably-trivial on any map from the suspensions of the simple modules. By comparing ghosts with simple ghosts, we get information about the ghost number. In the following examples:

1. $G = A \times B$ is a direct product, with Sylow p-subgroup A (Corollary 3.4.2),
2. the Sylow p-subgroup P of G is cyclic and normal (Theorem 3.5.5), and
3. the dihedral group D_{2ql} of order $2ql$ with l odd (Corollary 3.4.14),

we show that the simple modules in the principal block are suspensions of k. It follows that $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$ and, by Proposition 3.4.10, the ghost number of kG is greater than or equal to the ghost number of kP. On the other hand, we show that ghosts and simple ghosts coincide in the principal block, so the *simple ghost number* provides an upper bound for the ghost number. Hence in these examples,

$$\text{the ghost number of } kG = \text{the ghost number of } kP.$$

1.4.2.3 Simple ghosts and the simple generating hypothesis

We have already seen that we need to consider simple ghosts in certain examples.

In Section 3.3.1, we show that if the Sylow p-subgroup P of G is normal, then the simple ghost number of kP is equal to the ghost number of kG (Theorem 3.3.2).

In Section 3.5.2, we prove that the simple generating hypothesis holds for the group $SL(2, p)$ at any prime p (Theorem 3.5.9). This is an interesting result because the generating hypothesis fails for its Sylow p-subgroup, the cyclic group of order p, when $p \geqslant 5$.

1.4.2.4 Strong ghost numbers

Recall that a map $M \to N$ is a strong ghost in $\mathsf{StMod}(kG)$ if its restriction to any subgroup H of G is a ghost. Observe that the map $M \to N$ is a strong ghost if and only if the composite of maps $X \to M \to N$ is stably-trivial for any X that is a suspension of the module $k{\uparrow}_H^G$ for some subgroup H of G. Such test objects generate $\mathsf{StMod}(kG)$. Hence, for $M \in \mathsf{stmod}(kG)$, we define its **strong ghost length** to be the smallest integer n, such that every composite of n strong ghosts in $\mathsf{stmod}(kG)$ out of M is stably-trivial. The **strong ghost number** of kG can be defined similarly as its ghost number and simple ghost number. Unlike ghosts or simple ghosts, both the restriction and the induction functors preserve strong ghosts, so the strong ghost number of kG equals that of kP (Proposition 3.6.4), where P is the Sylow p-subgroup of G.

For cyclic p-groups other than C_2, C_3, and C_4, whose strong ghost numbers are 1, we compute in Theorem 3.6.6 that

$$\text{strong ghost number of } kG = \lceil \frac{p+1}{2} \rceil, \text{when } |G| \neq 2, 3, \text{ or } 4.$$

Combining this with our earlier results about dihedral groups, we get an upper bound for the strong ghost number of dihedral groups in Theorem 3.6.7:

$$\text{strong ghost number of } kD_{4q} \leqslant 3.$$

The computation of strong ghost numbers suggests that the concept of a strong ghost is much stronger than a ghost.

1.4.3 Computations with GAP

In Chapter 4, we show how to apply GAP to the study of the group algebra. GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory [26]. The GAP package 'reps' has been developed to handle group representations in positive characteristic. The overall structure of the reps package was designed and most of it is written by Peter Webb , who is also the maintainer. Contributions were made by Dan Christensen, Roland Loetscher, Robert Hank, Bryan Simpkins, Brad Froehle and others.

We have improved the code used in GAP to compute the universal ghost and ghost length. Recall that the universal ghost is the cofibre of a map that is surjective on Tate cohomology, and to compute the cofibre of a map, we need to replace it by an injection. The new `ReplaceWithInj` function is faster than the previous version and uses less memory. We explain the idea of the function and how to implement it in Section 4.4.1, and we show that the code has the resulting cofibre as small as possible. We also discuss the `Simple` function in the same section, which is used in the `ReplaceWithInj` function. Given an indecomposable projective module P, it computes the corresponding simple module of P. The functions are presented in pseudo-code.

The new function makes computations of cofibres and universal ghosts more efficient. This allows us to apply the idea of a universal ghost to find an upper bound of the ghost length. More precisely, given $M \in \mathsf{stmod}(kG)$, we can compute the n-fold universal ghost out M, and if the map is stably-trivial, then the ghost number of M is at most n. We implement this in Section 4.4.2. Note that the Tate cohomology is in general not finitely-generated, so we can only compute *unstable universal ghosts* within a certain range, i.e., maps that are stably-trivial on maps from $\Sigma^i k$, with $l \leqslant i \leqslant m$ for some integers l and m. Nevertheless, if the n-fold unstable universal ghost out of M is stably-trivial, then n is still an upper bound of the ghost length of M. And we can enlarge the range $[l, m]$, and get a decreasing sequence of upper bounds, whose limit is exactly the ghost length (Proposition 4.3.2). In the case when G has periodic cohomology, then the Tate cohomology is finitely-generated and the computation of the ghost length becomes a finite process.

Then we apply the functions to compute various examples. We make some computations with the group $S_3 \times C_3$, the first example where $\mathsf{Thick}\langle k \rangle \neq \mathsf{stmod}(B_0)$. We also present some computations with the group Q_8 and C_9 in Section 4.5.

Chapter 2

Ghost numbers of group algebras

2.1 Introduction

In modular representation theory, the Tate cohomology functor plays a central role, analogous to the role that the homotopy groups play in homotopy theory. Thus it is natural to study the *kernel* of Tate cohomology, that is, the collection of maps which induce the zero map in Tate cohomology. These maps are called **ghosts**, and are the topic of the present paper.

Let G be a finite group, and let k be a field whose characteristic p divides the order of G. We write $\mathsf{StMod}(kG)$ for the stable module category of kG, the triangulated category formed from the module category by killing the projectives, $\mathsf{stmod}(kG)$ for the full subcategory of finitely generated modules, and $\mathsf{Thick}\langle k\rangle$ for the thick subcategory generated by the trivial representation, a full subcategory of $\mathsf{stmod}(kG)$. (See Section 2.2 for complete definitions and further background.)

The generating hypothesis (GH) for the stable module category is the statement that if a map in $\mathsf{Thick}\langle k\rangle$ induces the zero map in Tate cohomology, then it is stably trivial. Using the terminology of the first paragraph, this is equivalent to saying that all ghosts in $\mathsf{Thick}\langle k\rangle$ are trivial. This problem is motivated by Freyd's famous conjecture in homotopy theory [25], which is still open.

By work of Benson, Carlson, Chebolu, Christensen and Mináč (Theorem 2.2.1 below), it is known that the generating hypothesis fails for most groups. The extent to which it fails is measured by the **ghost number** of kG, which is the smallest number n such that every composite of n ghosts in $\mathsf{Thick}\langle k\rangle$ is stably trivial. With this terminology, the generating hypothesis is the statement that the ghost number is one. The ghost number was studied for p-groups in [19], but even for p-groups it was found to be difficult to calculate, and in most cases only crude bounds are known. It is a long-term goal to understand whether this invariant has a simple description in terms of other invariants of kG.

In this chapter we develop new techniques for the study of ghost numbers and use them to make new computations in many cases. For example, we make the first computations of the ghost numbers of group algebras of wild representation type at an odd prime ($k(C_3 \times C_3)$ and others mentioned in the detailed summary below) as well as the first computations of the ghost numbers of non-abelian group algebras (the dihedral 2-groups). We also give many new bounds on ghost numbers, including lower bounds, which are generally difficult to come by. As one example, we show that the ghost number is always at least one-third of the radical length, the first general lower bound we are aware of. Our work includes results which are quite general, in some cases applying to any projective class in any triangulated category.

Chapter 3 builds on the work here in order to compute the ghost numbers of non-p-groups. For example, using the results on dihedral 2-groups, we are able to compute the ghost number of an arbitrary dihedral group at the prime 2.

We now give a summary of the contents of the paper. We begin in Section 2.2.1 by reviewing the stable module category. In Section 2.2.2 we recall the statement of the generating hypothesis in this situation and state the result of Benson, Carlson, Chebolu, Christensen and Mináč that says that the GH fails unless the Sylow p-subgroup of G is C_2 or C_3. The ghost number, which measures the degree to which the GH fails, is best studied using the idea of a projective class, so we introduce projective classes and their associated invariants in Section 2.2.3. Briefly, a projective class consists of a collection $\mathcal{P}$ of objects (thought of as "projective" building blocks) and an ideal $\mathcal{I}$ of morphisms (the maps invisible to the objects in $\mathcal{P}$) satisfying some axioms.

In Section 2.3 we present a variety of new results, many of which hold for arbitrary projective classes in arbitrary triangulated categories. For example, in Section 2.3.1, we give new bounds on the length of an object in a triangle in terms of the lengths of the other two objects and the filtration of the connecting homomorphism in the powers of the ideal. Then, in Section 2.3.2, we show that the connecting map $\gamma : Z \to \Sigma X$ in an Auslander-Reiten triangle, which we call the **almost zero map**, has a remarkable property: if $(\mathcal{P}, \mathcal{I})$ is any projective class such that there is a nonzero map from Z in $\mathcal{I}^k$, then γ is in $\mathcal{I}^k$. So the almost zero map is in some sense a universal example of a non-zero map from Z. We specialize to the case of the stable module category in Section 2.3.3, where we show that the **heart** of an indecomposable module M (the fibre of the almost zero map) has length which differs by at most one from M, with respect to any projective class. We also show that this is true for any summand of the heart, by showing that the lengths of the domain and codomain of any irreducible map differ by at most one. We finish Section 2.3 with Section 2.3.4, which describes the extent to which our results hold for the ghost length, the invariant used in defining the ghost number.

Section 2.4 contains detailed computational results on the ghost numbers of p-groups. We begin by recalling some background results in Section 2.4.1, such as the fact that the ghost number of kG is less than the nilpotency index of the Jacobson radical, as well as the fact that multiplication by $x-1$, where x is a central element of G, is always a ghost. In Section 2.4.2 we show that the generating length invariant is in a precise sense a stabilized version of the socle length, and show that if these are equal for a module M, the same is true for $\mathrm{rad}(M)$ and $M/\mathrm{soc}(M)$. This follows from a general result involving nested unstable projective classes in a triangulated category. We begin our computations in Section 2.4.3, where we study the ghost numbers of abelian p-groups. The main result here is an improved lower bound on the ghost number. This follows from a result giving a lower bound on the ghost length of induced modules for general p-groups. We also compute the exact ghost length of many modules over abelian p-groups. In Section 2.4.4 we show that the ghost number for the quaternion group Q_8 is 3 or 4, improving the existing lower bound by 1. In Section 2.4.5, we compute the ghost length and generating length of certain modules induced up from a cyclic normal subgroup of a p-group, generalizing the technique used for Q_8. This is used in the same section to show that the ghost number and the radical length are within a factor of three of each other for any p-group. More precisely, we show that $(\mathrm{rad\,len}\ kG)/3 \leqslant \mathrm{ghost\,num}\ kG < \mathrm{rad\,len}\ kG$ for p odd, the first general lower bound we are aware of. For $p = 2$, the factor of 3 is replaced with a factor of 2. We also use the induction result in Section 2.4.6, where we show that the ghost number of the dihedral 2-group D_{4q} of order $4q$ is exactly $q+1$. This is the longest section of the paper. That the ghost length is at least $q+1$ follows immediately from the induction result of the previous section, but that it is no more than $q+1$ requires using the classification of kD_{4q}-modules. In Section 2.4.7 we show that the ghost number of $k(C_3 \times C_3)$ is exactly 3. While $k(C_3 \times C_3)$-modules are not classifiable, we make use of the fact that certain quotients can be classified. Our argument also shows that the ghost number of the group algebra $k(C_{p^r} \times C_{p^s})$, for $p^r, p^s > 2$, is at most $p^r + p^s - 3$. It follows that the ghost number of $k(C_3 \times C_{3^s})$ is 3^s and that the ghost number of $k(C_4 \times C_{2^s})$ is $2^s + 1$. We end the paper with Section 2.4.8, in which we give complete lists of the group algebras of p-groups with ghost numbers 1, 2 or 3, with the possible exception of kQ_8. We also prove that for each prime p there are gaps in the possible ghost numbers that can occur, and state a conjecture related to this.

2.2 The generating hypothesis and the ghost projective class

In this section, we recall background material which provides context to our results and which we use in our proofs.

2.2.1 The stable module category

Here we recall the basics of the stable module category. A good reference is [14].

Let G be a finite group, and let k be a field whose characteristic p divides the order of G. The **stable module category** $\mathsf{StMod}(kG)$ is a quotient category of the category $\mathsf{Mod}(kG)$ of left kG-modules by the ideal of maps that factor through a projective. Thus the objects of $\mathsf{StMod}(kG)$ are left kG-modules and the hom-sets are $\underline{\mathrm{Hom}}(M,N) = [M,N] := \mathrm{Hom}(M,N)/\mathrm{PHom}(M,N)$, where $\mathrm{PHom}(M,N)$ denotes the stably trivial maps, i.e., those that factor through a projective module. Two modules M and N are isomorphic in the stable module category if and only if they have the same projective-free summands. In particular, projective modules are isomorphic to zero in the stable module category. We write $\mathsf{stmod}(kG)$ for the full subcategory of finitely generated kG-modules in $\mathsf{StMod}(kG)$. (More precisely, we include all modules which are *stably isomorphic* to finitely generated kG-modules.)

The stable module category is a triangulated category. The desuspension ΩM of a module M is the kernel of any surjection $P \to M$ with P projective. This is well-defined in the stable module category by Schanuel's Lemma [14, Prop. 4.2], and we write $\tilde{\Omega} M$ for the projective-free summand of ΩM.

The group algebra kG is injective as a module over itself. In particular, this implies that projective modules and injective modules coincide in $\mathsf{mod}(kG)$. The suspension ΣN of a module N is defined to be the cokernel of any injection $N \to P$ with P injective. We will often write $\Omega^{-1} N$ for ΣN since Ω and Σ are inverse functors up to natural isomorphism.

Write k for the trivial representation and $\mathsf{Thick}\langle k \rangle$ for the thick subcategory generated by k, the smallest full triangulated subcategory of $\mathsf{StMod}(kG)$ that is closed under retracts and contains k. This is in fact a full subcategory of $\mathsf{stmod}(kG)$, and plays a central role in our formulation of the generating hypothesis. The localizing category generated by k, denoted $\mathsf{Loc}\langle k\rangle$, is the smallest full triangulated subcategory of $\mathsf{StMod}(kG)$ that is closed under arbitrary coproducts and retracts and contains k.

2.2.2 The generating hypothesis

An important feature of the stable module category is that the Tate cohomology of a kG-module M is representable, i.e., we have a canonical isomorphism $\hat{H}^n(G, M) \cong [\Omega^n k, M]$.

We say that the **generating hypothesis (GH)** holds for the stable module category $\mathsf{StMod}(kG)$ if and only if the Tate cohomology functor $\hat{H}^*(G, -)$ restricted to $\mathsf{Thick}\langle k\rangle$ is faithful. It has been shown that the GH fails for most group algebras [9, 16, 18, 20].

Theorem 2.2.1 (Benson, Carlson, Chebolu, Christensen and Mináč)**.** *Let G be a finite group, and let k be a field whose characteristic p divides the order of G. Then the GH holds for $\mathsf{StMod}(kG)$ if and only if the Sylow p-subgroup P of G is either C_2 or C_3.*

It is worth pointing out here why we restrict to $\mathsf{Thick}\langle k\rangle$. It is known that whenever the thick subcategory is not all of $\mathsf{stmod}(kG)$, there are non-projective modules whose Tate cohomology is zero. The identity map on such a module is sent to zero by $\hat{H}^*(G, -)$, so the GH would be trivially false if we included such modules. Restricting to $\mathsf{Thick}\langle k\rangle$ prevents this from happening. In general, the stable module category is generated by the simple modules as a triangulated category. For a p-group G, the trivial representation k is the only simple module, so we have that $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(kG)$ in this case.

We call a map in $\mathsf{StMod}(kG)$ that is in the kernel of the Tate cohomology functor a **ghost**. Thus the GH is the statement that all ghosts in $\mathsf{Thick}\langle k\rangle$ are stably trivial. When the GH fails, the vanishing of composites of ghosts gives a measure of the failure and leads to invariants of modules and of kG. This is formalized in the idea of a projective class.

2.2.3 The ghost projective class

Definition 2.2.2. Let $\mathbb{T}$ be a triangulated category. A **projective class** in $\mathbb{T}$ consists of a class $\mathcal{P}$ of objects of $\mathbb{T}$ and a class $\mathcal{I}$ of morphisms of $\mathbb{T}$ such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{I}$,

(ii) $\mathcal{I}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$.

(iii) for each X in $\mathbb{T}$, there is a cofibre sequence $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{I}$.

In this paper, we make the additional assumption that the projective class is **stable**, that is, that $\mathcal{P}$ (or equivalently $\mathcal{I}$) is closed under suspension and desuspension. With slight alterations, most of our results remain true without this assumption, but the extra bookkeeping complicates the arguments. The one exception is that in Section 2.4.2 we make use of an unstable projective class.

Remark 2.2.3. It follows from the definition that $\mathcal{P}$ is closed under arbitrary coproducts and retracts, and $\mathcal{I}$ is an ideal.

We write $\mathcal{G}$ for the ideal of ghosts in the stable module category, and $\mathcal{F}$ for all retracts of direct sums of suspensions of k in $\mathsf{StMod}(kG)$. For a module $M \in \mathsf{StMod}(kG)$, since $\hat{H}^n(G, M) \cong [\Omega^n k, M]$, we can form a map $\oplus \Omega^i k \to M$ that is surjective on Tate cohomology by assembling sufficiently many homogeneous elements in $\hat{H}^*(G, M)$. Completing this map into a triangle in $\mathsf{StMod}(kG)$

$$\Omega U_M \to \oplus \Omega^i k \to M \xrightarrow{\phi_M} U_M, \tag{2.2.1}$$

we get a ghost $\phi_M : M \to U_M$. The map ϕ_M is a (weakly) universal ghost in the sense that every ghost out of M factors though it, but the factorization is not necessarily unique. It follows easily that $(\mathcal{F}, \mathcal{G})$ forms a projective class in $\mathsf{StMod}(kG)$. This is called the **ghost projective class**.

While the ghost projective class is the focus of this paper, some of our results apply to any projective class, so we mention two other examples at this point: The **simple ghost projective class** is the projective class whose projectives are generated by all simple objects, and it was proposed for study in [12] as a way to avoid focusing on $\mathsf{Thick}\langle k \rangle$. And the **strong ghost projective class** is the projective class whose ideal consists of the maps which are ghosts under restriction to every subgroup. (See [17] for more on this topic.)

For any projective class $(\mathcal{P}, \mathcal{I})$, there is a sequence of **derived projective classes** $(\mathcal{P}_n, \mathcal{I}^n)$ [21]. The ideal $\mathcal{I}^n$ consists of all n-fold composites of maps in $\mathcal{I}$, and X is in $\mathcal{P}_n$ if and only if it is a retract of an object M that sits inside a cofibre sequence $P \to M \to Q$ with $P \in \mathcal{P}_1 = \mathcal{P}$ and $Q \in \mathcal{P}_{n-1}$. For $n = 0$, we let $\mathcal{P}_0$ consist of all zero objects and $\mathcal{I}^0$ consist of all maps in $\mathbb{T}$. The **length** $\mathrm{len}_{\mathcal{P}}(X)$ of an object X of $\mathbb{T}$ with respect to $(\mathcal{P}, \mathcal{I})$ is the smallest n such that X is in $\mathcal{P}_n$, if this exists. The fact that each pair $(\mathcal{P}_n, \mathcal{I}^n)$ is a projective class implies that the length of X is equal to the smallest n such that every map in $\mathcal{I}^n$ with domain X is trivial.

The length of a module M with respect to the ghost projective class is called the **generating length** of M, and this exists when M is in $\mathsf{Thick}\langle k\rangle$. But since we are interested in the collection $\mathcal{G}_t$ of ghosts in $\mathsf{Thick}\langle k\rangle$, we also get another invariant. We describe both invariants, and the associated invariants of kG, in the following definition, generalizing the definition given in [19] for p-groups.

Definition 2.2.4.

- The **generating length** $\mathrm{gel}(M)$ of $M \in \mathsf{Thick}\langle k\rangle$ is the smallest n such that $M \in \mathcal{F}_n$. That is, $\mathrm{gel}(M) = \mathrm{len}_{\mathcal{F}}(M)$.
- The **ghost length** $\mathrm{gl}(M)$ of $M \in \mathsf{Thick}\langle k\rangle$ is the smallest integer n such that every map in $(\mathcal{G}_t)^n$ with domain M is trivial.
- The **generating number** of kG is the least upper bound of the generating lengths of modules in $\mathsf{Thick}\langle k\rangle$.
- The **ghost number** of kG is the least upper bound of the ghost lengths of modules in $\mathsf{Thick}\langle k\rangle$.

With this terminology, the generating hypothesis is the statement that the ghost number of kG is 1.

Let M be in $\mathsf{Thick}\langle k\rangle$. Since each $(\mathcal{F}_n, \mathcal{G}^n)$ is a projective class and $(\mathcal{G}_t)^n \subseteq \mathcal{G}^n$, it follows that

$$\mathrm{gl}(M) \leqslant \mathrm{gel}(M)$$

and therefore that

$$\text{ghost number of } kG \leqslant \text{generating number of } kG.$$

When G has periodic Tate cohomology, the coproduct in (2.2.1) can be taken to be finite, and it follows that the ghost projective class restricts to a projective class in $\mathsf{Thick}\langle k\rangle$ [19]. This implies that equality holds in this case. We don't know whether equality holds in general, except for the trivial observation that $M \cong 0$ if and only $\mathrm{gel}(M) = 0$ if and only if $\mathrm{gl}(M) = 0$ and the less trivial fact that $\mathrm{gel}(M) = 1$ if and only if $\mathrm{gl}(M) = 1$ (see Corollary 2.3.7 or [16]). Thus the GH is equivalent to the generating number of kG being 1. See Remark 2.3.13 for further discussion of whether ghost length equals generating length.

2.3 Auslander-Reiten triangles and generating lengths

In this section, we explain how Auslander-Reiten triangles (in short, A-R triangles) provide examples of ghosts, and, more generally, of non-trivial maps in $\mathcal{I}^n$ for n as large as possible, for any projective class $(\mathcal{P}, \mathcal{I})$. This extends the work of [16], where these triangles are called "almost split sequences." Because we have in mind applications to other projective classes, in this section we state many of our results for a general projective class in a general triangulated category.

In Section 2.3.1, we give results about the relationship between the lengths of the objects in a triangle when one of the maps is in a power $\mathcal{I}^m$ of the ideal. In Section 2.3.2, we recall A-R triangles and prove that the third map in an A-R triangle is the longest possible non-trivial composite of maps in $\mathcal{I}$ with the given domain. In Section 2.3.3, we apply these results to the study of lengths in the stable module category, and also show a close relationship between lengths and irreducible maps. Finally, in Section 2.3.4 we explain the extent to which our results on generating length are true for ghost length.

2.3.1 Relations between the lengths of objects in a triangle

Consider a projective class $(\mathcal{P}, \mathcal{I})$ in a triangulated category $\mathbb{T}$. Let

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

be a triangle in $\mathbb{T}$, where X, Y and Z have finite lengths k, n and l, respectively. We know that $n \leqslant k + l$ [21]. Rotating the triangle, we also get $l \leqslant n + k$ and $k \leqslant n + l$.

Here we show that when γ is in $\mathcal{I}^m$, one can refine these inequalities by subtracting m from l. Our methods also show that $n \geqslant m$. Note that $\mathcal{I}^0$ consists of all maps in $\mathbb{T}$.

Lemma 2.3.1. *Let $(\mathcal{P}, \mathcal{I})$ be a projective class in a triangulated category $\mathbb{T}$, and let*

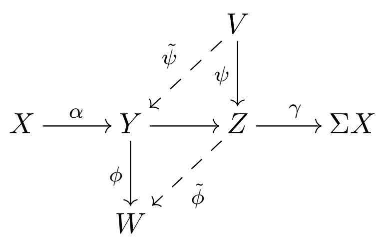
$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

be a triangle in $\mathbb{T}$, where X, Y and Z have finite lengths k, n and l, respectively, and $\gamma \in \mathcal{I}^m$ with $m \leqslant l$. Then

$$\text{len}_{\mathcal{P}}(Y) = n \leqslant \max(k - m + l, l).$$

Note that if $m \geqslant l$, then γ must be zero, and so the restriction to $m \leqslant l$ is natural. When $m = l$, the triangle splits, and the lemma says that $n \leqslant \max(k, l)$.

Proof. Let $n' = \max(k, m)$, and let $\phi : Y \to W$ be in $\mathcal{I}^{n'}$. Then $\phi \circ \alpha$ is zero (since $n' \geqslant k$), so ϕ factors through a map $\tilde{\phi} : Z \to W$. We claim that $\tilde{\phi}$ is in $\mathcal{I}^m$. Consider the diagram



with $\psi : V \to Z$ being any map from an object $V \in \mathcal{P}_m$. Now $\gamma \in \mathcal{I}^m$, so $\gamma \circ \psi$ is zero, and ψ factors through some map $\tilde{\psi} : V \to Y$. Hence $\tilde{\phi} \circ \psi = \phi \circ \tilde{\psi}$ is zero (since $n' \geqslant m$), and the claim follows. If $g : W \to W'$ is in $\mathcal{I}^{l-m}$, then $g \circ \tilde{\phi}$ is zero because Z has length l. Then $g \circ \phi$ is zero, meaning that the length of Y is at most $n' + l - m$. □

Lemma 2.3.2. *Let $(\mathcal{P}, \mathcal{I})$ be a projective class in a triangulated category $\mathbb{T}$, and let*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

be a triangle in $\mathbb{T}$, where X, Y and Z have finite lengths k, n and l, respectively, and $\gamma \in \mathcal{I}^m$ with $m \leqslant l$. Then

$$\text{len}_{\mathcal{P}}(Y) = n \geqslant \max(k - l + m, m).$$

When $m = l$, this says that $n \geqslant \max(k, l)$, so the two lemmas together recover the fact that when the triangle splits, $n = \max(k, l)$.

Proof. We prove that the length of Y is at least $k - l + m$. The other inequality can be proved similarly.

Consider a map $\phi : X \to W$ in $\mathcal{I}^{l-m}$. Since $\phi \circ \Sigma^{-1}\gamma$ is in $\mathcal{I}^l$ and has domain $\Sigma^{-1}Z$ of length l, it is zero and ϕ factors through a map $\tilde{\phi} : Y \to W$:

$$\begin{array}{ccccccc} \Sigma^{-1}Z & \xrightarrow{\Sigma^{-1}\gamma} & X & \longrightarrow & Y & \longrightarrow & Z \\ & & {\scriptstyle\phi}\big\downarrow & \swarrow {\scriptstyle\tilde{\phi}} & & & \\ & & W & & & & \end{array} .$$

Let $g : W \to W'$ be in $\mathcal{I}^n$. Then $g \circ \tilde{\phi}$ is zero because Y has length n, hence any map in $\mathcal{I}^{n+l-m}$ with domain X is zero. This implies that $k \leqslant n + l - m$, i.e., that $n \geqslant k - l + m$. □

2.3.2 Auslander-Reiten triangles give composites of ghosts

We begin by recalling the definition.

Definition 2.3.3. Let $\mathbb{T}$ be a triangulated category. A triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is called an **Auslander-Reiten triangle**, if

(a) $\gamma \neq 0$,

(b) any map $X \to Y'$ that is not split monic factors through α,

(c) any map $Y' \to Z$ that is not split epic factors through β.

A map α that is not split monic and satisfies (b) is said to be **left almost split**. Dually, a map β that is not split epic and satisfies (c) is said to be **right almost split**.

We know that Auslander-Reiten triangles exist in great generality.

Theorem 2.3.4 (Krause, [30])**.** *Let $\mathbb{T}$ be a triangulated category with all small coproducts, and suppose that all cohomological functors are representable. Let Z be a compact object in $\mathbb{T}$ with local endomorphism ring. Then there exists an Auslander-Reiten triangle*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X.$$

The triangle is unique up to a non-canonical isomorphism. □

Remark 2.3.5. Let β be the second map in the A-R triangle above. One can show that, for any endomorphism g of Y with $\beta g = \beta$, the map g is an isomorphism (see [30]). We say that the map β is **right minimal** in this case. Dually, the first map α in an A-R triangle is **left minimal**. A map β that is right almost split sits inside an Auslander-Reiten triangle if and only if it is right minimal [30].

For convenience, we call the map γ here the **almost zero map** with domain Z. It is unique up to an automorphism of ΣX. The following proposition follows from the definitions and the earlier lemmas.

Proposition 2.3.6. *Suppose that* $(\mathcal{P}, \mathcal{I})$ *is a projective class on a triangulated category* $\mathbb{T}$*, and that*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

is a distinguished triangle with β *right almost split. If* Z *has finite length* l *and* X *has finite length* k *with respect to* $(\mathcal{P}, \mathcal{I})$*, then the third map* γ *is in* $\mathcal{I}^{l-1}$*, and*

$$\begin{aligned} k-1 \leqslant \mathrm{len}_{\mathcal{P}}(Y) &\leqslant k+1, \text{ if } k \geqslant l; \\ l-1 \leqslant \mathrm{len}_{\mathcal{P}}(Y) &\leqslant l, \qquad \text{ if } k \leqslant l-1. \end{aligned}$$

For any summand S *of* Y*,* $\mathrm{len}_{\mathcal{P}}(S) \leqslant \max(k+1, l)$*.*

Proof. We test γ on all objects W in $\mathcal{P}_{l-1}$. Because Z has larger length than W, a map $\phi : W \to Z$ cannot be split epic, so it factors through β. Hence $\gamma \circ \phi$ is zero, which implies that $\gamma \in \mathcal{I}^{l-1}$.

The inequalities follow from Lemmas 2.3.1 and Lemma 2.3.2, with $m = l - 1$. The statement about the summand S follows immediately. □

Note in particular that for any A-R triangle, the almost zero map γ is an example of a non-zero map in the largest possible power of the ideal, for *any* projective class.

In the case when $\mathbb{T}$ is $\mathsf{StMod}(kG)$ with G being a p-group, we know that ghosts and *dual ghosts* coincide [19]. Hence γ non-zero implies that $k \geqslant l$, and so we are in the first case of Proposition 2.3.6.

In the next section, we develop these ideas further.

2.3.3 Auslander-Reiten triangles, irreducible maps and lengths

The category $\mathsf{StMod}(kG)$ satisfies the hypotheses on $\mathbb{T}$ in Theorem 2.3.4, and its compact objects are precisely those in $\mathsf{stmod}(kG)$. For projective-free $M \in \mathsf{stmod}(kG)$, the stable endomorphism ring $\underline{\mathrm{End}}(M)$ being local is equivalent to M being indecomposable. In this case, the Auslander-Reiten triangle has the form [4, 4.12.8]

$$\Omega^2 M \xrightarrow{\alpha} H(M) \xrightarrow{\beta} M \xrightarrow{\gamma} \Omega M.$$

The module $H(M)$ is called the **heart** of M, and the triangle shows that it is also in $\mathsf{stmod}(kG)$.

The general theory we have set up in the last two sections applies to an A-R triangle for any projective class $(\mathcal{P}, \mathcal{I})$ on $\mathsf{StMod}(kG)$. As a special case of Proposition 2.3.6, using that $k = l$ in this case, we get

Corollary 2.3.7. *Let G be a finite group, let k be a field whose characteristic divides the order of G, and let $(\mathcal{P}, \mathcal{I})$ be a projective class on $\mathsf{StMod}(kG)$. Consider the Auslander-Reiten triangle $\Omega^2 M \xrightarrow{\alpha} H(M) \xrightarrow{\beta} M \xrightarrow{\gamma} \Omega M$ for some indecomposable non-projective module M in $\mathsf{stmod}(kG)$ with finite length l with respect to $(\mathcal{P}, \mathcal{I})$. Then*

$$\mathrm{len}_{\mathcal{P}}(M) - 1 \leqslant \mathrm{len}_{\mathcal{P}}(H(M)) \leqslant \mathrm{len}_{\mathcal{P}}(M) + 1,$$

and γ is a non-trivial map in $\mathcal{I}^{l-1}$. □

As above, we emphasize again that the same map $\gamma : M \to \Omega M$ provides a map in $\mathcal{I}^n$ with n maximal for *any* projective class $(\mathcal{P}, \mathcal{I})$. Put another way, γ is in the intersection of all projective class ideals that contain a non-trivial map from M.

Remark 2.3.8. One might hope that the heart $H(M)$ always has larger generating length than M when $\mathrm{gel}(M)$ is less than the generating number of kG, but unfortunately this is not true in general. For example, take $G = C_5 \times C_5$ and $M = k{\uparrow}_{C_5}^{G}$. One can compute that $\mathrm{gel}(M) = \mathrm{gel}(H(M)) = 5$, while the generating number of kG is at least 6 (Theorem 2.4.9).

Let S be an indecomposable non-projective summand of $H(M)$. Then, clearly, $\mathrm{len}_{\mathcal{P}}(S) \leqslant \mathrm{len}_{\mathcal{P}}(H(M)) \leqslant \mathrm{len}_{\mathcal{P}}(M) + 1$. We will show below that $\mathrm{len}_{\mathcal{P}}(M) - 1 \leqslant \mathrm{len}_{\mathcal{P}}(S)$ because of the right minimality of the map β.

We first need the notion of irreducible map.

Definition 2.3.9. Let G be a finite group, and let k be a field whose characteristic divides the order of G. A map $\lambda : M \to N$ in $\mathsf{StMod}(kG)$ is said to be **irreducible** if it is not split monic or split epic, and for any factorization $\lambda = \nu \circ \mu$, either μ is split monic or ν is split epic.

Irreducible maps are closely related to Auslander-Reiten triangles:

Proposition 2.3.10 (Auslander and Reiten [2])**.** *Let M and N be indecomposable non-projective modules in $\mathsf{stmod}(kG)$. Then a map $f : M \to N$ is irreducible if and only if the following equivalent conditions are satisfied:*

(a) M is a summand of $H(N)$ and f is the composite $M \to H(N) \xrightarrow{\beta} N$.

(b) N is a summand of $\Omega^{-2}H(M)$ and f is the composite $M \xrightarrow{\Omega^{-2}\alpha} \Omega^{-2}H(M) \to N$. □

Combining Corollary 2.3.7 and Proposition 2.3.10, one can prove

Corollary 2.3.11. *Let $f : M \to N$ be an irreducible map with M and N non-projective indecomposables in $\mathsf{stmod}(kG)$, and let $(\mathcal{P}, \mathcal{I})$ be a projective class on $\mathsf{StMod}(kG)$. If M and N have finite lengths with respect to $(\mathcal{P}, \mathcal{I})$, then*

$$\mathrm{len}_{\mathcal{P}}(M) - 1 \leqslant \mathrm{len}_{\mathcal{P}}(N) \leqslant \mathrm{len}_{\mathcal{P}}(M) + 1.$$

In particular, for M indecomposable and S any summand of $H(M)$, we have

$$\mathrm{len}_{\mathcal{P}}(M) - 1 \leqslant \mathrm{len}_{\mathcal{P}}(S) \leqslant \mathrm{len}_{\mathcal{P}}(M) + 1.$$ □

2.3.4 Ghost lengths

The results of Sections 2.3.1 to 2.3.3 apply to the generating length of a module in $\mathsf{StMod}(kG)$, since generating length is the length with respect to the ghost projective class. When kG has periodic cohomology, there is a projective class on $\mathsf{Thick}\langle k \rangle$ whose ideal is $\mathcal{G}_t$, and ghost length is the length with respect to this projective class. In general, we don't know whether ghost length is a length with respect to a projective class, but we can still prove the analogue of half of Corollary 2.3.7:

Proposition 2.3.12. *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Consider the Auslander-Reiten triangle $\Omega^2 M \to H(M) \to M \to$*

ΩM *for some indecomposable module* M *in* $\mathsf{Thick}\langle k\rangle$*. Then the following holds:*

$$\operatorname{gl}(M) - 1 \leqslant \operatorname{gl}(H(M))$$

Proof. We mimic the proof of Lemma 2.3.2. Suppose that $\operatorname{gl}(H(M)) = l - 1$. We must prove that $\operatorname{gl}(M) \leqslant l$. Since $\operatorname{gl}(M) = \operatorname{gl}(\Omega^2 M)$, it suffices to show that any map $\phi : \Omega^2 M \to N$ in $(\mathcal{G}_t)^l$ is stably trivial, where $\mathcal{G}_t$ consists of ghosts between objects in $\mathsf{Thick}\langle k\rangle$. Write ϕ as $\phi_2\phi_1$, where ϕ_1 is in $\mathcal{G}_t$ and ϕ_2 is in $(\mathcal{G}_t)^{l-1}$. Then, by Proposition 2.3.6, the composite $\phi_1\,\Omega\gamma$ is stably trivial, so ϕ_1 factors through $H(M)$:

$$\begin{array}{ccccccccc}
\Omega M & \xrightarrow{\Omega\gamma} & \Omega^2 M & \longrightarrow & H(M) & \longrightarrow & M & \xrightarrow{\gamma} & \Omega M \\
 & & \downarrow{\scriptstyle \phi_1} & \swarrow{\scriptstyle \psi} & & & & & \\
 & & W & & & & & & \\
 & & \downarrow{\scriptstyle \phi_2} & & & & & & \\
 & & N. & & & & & &
\end{array}$$

Now since $\operatorname{gl}(H(M)) = l-1$, the composite $\phi_2\psi$ is stably trivial and so ϕ is stably trivial as well. □

The analogue of the other half of Corollary 2.3.7 would say that $\operatorname{gl}(H(M)) \leqslant \operatorname{gl}(M) + 1$, and we don't know whether this is true.

Remark 2.3.13. A related question is whether the generating length and ghost length always agree. We know of no counterexamples. However, Corollary 2.3.7 implies that the longest composite of ghosts starting from a given module M in $\mathsf{Thick}\langle k\rangle$ can always be attained by a map in $(\mathcal{G}^m)_t$, the intersection of $\mathcal{G}^m$ and $\mathsf{Thick}\langle k\rangle$. Thus if $(\mathcal{G}_t)^m = (\mathcal{G}^m)_t$, then the ghost length and generating length agree. Note that a related statement for the objects of $\mathcal{P}$, i.e., that $(\mathcal{P}^c)_n = (\mathcal{P}_n)^c$, where the superscript c means to take the intersection with the compact objects, is known to be true [13, 2.2.4].

2.4 Ghost numbers of p-groups

In this section we study finite p-groups, using the fact that $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(kG)$. We begin in Section 2.4.1 by recalling several results that we will use. In Section 2.4.2 we show that the generating length invariant is a stabilized version of the socle length, and give a result that shows that if these are equal for a module M, the same is true for rad(M) and $M/\mathrm{soc}(M)$. Then we give new computations of bounds on ghost numbers for various p-groups: abelian p-groups in Section 2.4.3, the quaternion group Q_8 in Section 2.4.4, dihedral 2-groups in Section 2.4.6, and the groups $C_{p^r} \times C_{p^s}$ in Section 2.4.7. In several cases we determine the ghost number completely, such as for D_{4q}, $C_3 \times C_{3^s}$ and $C_4 \times C_{2^s}$. In Section 2.4.5, we compute the ghost length and generating length of certain modules induced up from a cyclic normal subgroup. This is used in the same section to show that the ghost number and the radical length are within a factor of three of each other for any p-group. It is also used in Section 2.4.6 in the computation of the ghost number of kD_{4q} and in Section 2.4.8, where we classify group algebras with small ghost number and put constraints on which ghost numbers can occur.

When we write "p-group", we always mean "finite p-group".

2.4.1 Background

We recall the following theorem, and then explain the terminology and give an idea of the proof.

Theorem 2.4.1 (Chebolu, Christensen and Mináč [19])**.** *Let G be a p-group, and let k be a field of characteristic p. Then the generating length of a kG-module M is at most its radical length, and the following inequalities hold:*

$$\textit{ghost number of } kG \leqslant \textit{generating number of } kG < \textit{nilpotency index of } J(kG) \leqslant |G|.$$

In particular, the ghost number of kG is finite in this case. □

Let G be any finite group, and let k be a field whose characteristic divides the order of G. Let $J = J(kG)$ be the Jacobson radical of kG, i.e., the largest nilpotent ideal of kG. The nilpotency index of $J(kG)$ is the smallest integer m such that $J^m = 0$, and for any module M, we have a radical series

$$M = \mathrm{rad}^0(M) \supseteq \mathrm{rad}^1(M) \supseteq \mathrm{rad}^2(M) \supseteq \cdots \supseteq 0,$$

with $\mathrm{rad}^n(M) = J^n M$, and a socle series

$$0 = \mathrm{soc}^0(M) \subseteq \mathrm{soc}^1(M) \subseteq \mathrm{soc}^2(M) \subseteq \cdots \subseteq M,$$

with $\mathrm{soc}^n(M)$ consisting of the elements of M annihilated by J^n. The radical length of M is the smallest integer n such that $\mathrm{rad}^n(M) = 0$. This is equal to the socle length of M, the smallest integer m such that $\mathrm{soc}^m(M) = M$. The successive quotients in the sequences are direct sums of simple modules.

If G is a p-group, then each quotient is a direct sum of k's, so the generating length of a module M is less than or equal to its radical length. Note that the nilpotency index of $J(kG)$ is exactly the radical length of kG, and if M is a projective-free kG-module, it always has smaller radical length than kG. The theorem then follows.

The following lemma is proved by studying Tate cohomology in degrees 0 and -1. We write $\mathrm{rad}(M)$ for $\mathrm{rad}^1(M)$ and $\mathrm{soc}(M)$ for $\mathrm{soc}^1(M)$.

Lemma 2.4.2 (Chebolu, Christensen and Mináč [19])**.** *Let G be a p-group, and let k be a field of characteristic p. Let $f : M \to N$ be a map in $\mathsf{Mod}(kG)$ between projective-free modules M and N. Then:*

(a) $\mathrm{soc}(M) \subseteq \ker(f)$ *iff* $[k, f] = 0$.

(b) $\mathrm{im}(f) \subseteq \mathrm{rad}(N)$ *iff* $[\Omega^{-1}k, f] = 0$.

In particular, if f represents a ghost in the stable category, then both inclusions hold. □

As a corollary, we get

Corollary 2.4.3 (Chebolu, Christensen and Mináč [19])**.** *Let G be a p-group, and let k be a field of characteristic p. Let $f : M \to N$ be a map in $\mathsf{Mod}(kG)$ between projective-free modules M and N. If f is an l-fold ghost, then:*

(a) $\mathrm{soc}^l(M) \subseteq \ker(f)$.

(b) $\mathrm{im}(f) \subseteq \mathrm{rad}^l(N)$. □

The next lemma provides ghosts with a particular form.

Lemma 2.4.4 (Benson, Chebolu, Christensen and Mináč [9])**.** *Let G be a p-group, and let k be a field of characteristic p. Let $x \in G$ be a central element. Then left multiplication by $x - 1$ on a kG-module M is a ghost.* □

Note that in general there are ghosts not of this form. Nevertheless these ghosts work well for abelian groups in providing lower bounds for ghost numbers (see Section 2.4.3). It is not hard to check that if G is a cyclic p-group with generator g, then $g-1$ is a universal ghost.

2.4.2 Generating and socle lengths

We now show that the generating length is a stabilized version of the socle length. In this section we allow our projective classes to be unstable, that is, we don't assume that the projectives are closed under suspension and desuspension.

Let G be a p-group, let k be a field of characteristic p, and let M be a kG-module. Note that $\mathrm{soc}(M)$ contains exactly the image of maps from k. So, when we build up M in a socle sequence in Theorem 2.4.1, we are only using maps from k, not all suspensions of k. This suggests that we consider the unstable projective class generated by k in $\mathsf{StMod}(kG)$. We will show that the length with respect to this projective class is exactly the socle length for projective-free modules in $\mathsf{stmod}(kG)$.

Note that the regular representation kG is the only indecomposable projective kG-module, and $\mathrm{soc}(kG) \cong k$ is its unique minimal left submodule. Thus any map $kG \to M$ in $\mathsf{Mod}(kG)$ with M projective-free has $\mathrm{soc}(kG)$ in its kernel, since the map cannot be injective. It follows that a map $\oplus k \to M$ in $\mathsf{Mod}(kG)$ with M projective-free is stably trivial if and only if it is the zero map. For finitely generated modules, a similar argument shows that the same is true for a map $M \to \oplus k$ in $\mathsf{mod}(kG)$ with M projective-free.

Proposition 2.4.5. *Let G be a p-group, and let k be a field of characteristic p. Let $(\mathcal{P}, \mathcal{I})$ be the unstable projective class in $\mathsf{StMod}(kG)$ generated by k. Then a map $f : M \to N$ between projective-free objects M and N is in $\mathcal{I}$ if and only if it is represented by a map f such that $\mathrm{soc}(M) \subseteq \ker(f)$. Hence, if M is finitely-generated and projective-free, the length of M with respect to $(\mathcal{P}, \mathcal{I})$ is exactly its socle length.*

Proof. That $f \in \mathcal{I}$ is equivalent to $\mathrm{soc}(M) \subseteq \ker(f)$ is Lemma 2.4.2 (a).

Now let M be projective-free. Then $M \to M/\mathrm{soc}(M)$ is a universal map in $\mathcal{I}$. It follows that $M \to M/\mathrm{soc}^k(M)$ is universal in $\mathcal{I}^k$. If M has socle length n, then $M \in \mathcal{P}^n$ and $M \to M/\mathrm{soc}^{n-1}(M)$ is non-zero. If further M is finitely-generated, then the universal map $M \to M/\mathrm{soc}^{n-1}(M) \cong \oplus k$ is stably non-trivial, by the remarks preceding this proposition. Thus M has length n with respect to $(\mathcal{P}, \mathcal{I})$. ☐

Note that the stable projective class generated by k in $\mathsf{StMod}(kG)$ is exactly the ghost projective class. Thus the generating length is indeed the socle length stabilized and is generally less than or equal to the socle length. We have also recovered Theorem 2.4.1 from this observation. In Section 2.4.5, we are going to prove that the generating number of kG is within a factor of 3 of the socle length of kG.

Here we show that if the generating length of a module $M \in \mathsf{StMod}(kG)$ happens to equal its socle length (see, for example, Proposition 2.4.10 and Theorem 2.4.15), then the same holds for $\mathrm{rad}(M)$ and $M/\mathrm{soc}(M)$, a result that we will use in Section 2.4.6 when studying dihedral groups.

Proposition 2.4.6. *Let k be a field of characteristic p, and let G be a p-group. Assume that $M \in \mathsf{StMod}(kG)$ has generating length equal to its radical length. Then* $\mathrm{gel}(M/\mathrm{soc}(M)) = \mathrm{gel}(M) - 1$*, and similarly* $\mathrm{gel}(\mathrm{rad}(M)) = \mathrm{gel}(M) - 1$.

Proof. Since the generating length of M is strictly less than the nilpotency index of $J(kG)$, M is projective-free. The proposition is then a special case of the following more general lemma. □

Lemma 2.4.7. *Let $\mathbb{T}$ be a triangulated category, and let $(\mathcal{P},\mathcal{I})$ and $(\mathcal{P}',\mathcal{I}')$ be (possibly unstable) projective classes on $\mathbb{T}$ such that $\mathcal{P}' \subseteq \mathcal{P}$. Suppose that $M \in \mathbb{T}$ has $\mathrm{len}_{\mathcal{P}'}(M) = \mathrm{len}_{\mathcal{P}}(M) = m$ and that there exist $L \in \mathcal{P}'_{m-n}$ and $N \in \mathcal{P}'_n$ with a triangle*

$$L \to M \to N.$$

Then

$$\mathrm{len}_{\mathcal{P}'}(L) = \mathrm{len}_{\mathcal{P}}(L) = m - n, \text{ and } \mathrm{len}_{\mathcal{P}'}(N) = \mathrm{len}_{\mathcal{P}}(N) = n.$$

Proof. We have that $\mathrm{len}_{\mathcal{P}'}(L) \leqslant m - n$ and $\mathrm{len}_{\mathcal{P}'}(N) \leqslant n$. But $\mathrm{len}_{\mathcal{P}'}(L) + \mathrm{len}_{\mathcal{P}'}(N) \geqslant m = (m-n) + n$, so the equalities follow for $(\mathcal{P}',\mathcal{I}')$. Since $\mathcal{P}' \subseteq \mathcal{P}$, the same results hold for $(\mathcal{P},\mathcal{I})$ too. □

Intuitively, this easy fact says that when $\mathrm{len}_{\mathcal{P}'}(M) = \mathrm{len}_{\mathcal{P}}(M)$, the related object L can be built from $\mathcal{P}'$ as efficiently as it can be built from $\mathcal{P}$. It applies to generating lengths and socle lengths.

We now provide examples of computations of ghost numbers of certain groups, improving on results in [19].

2.4.3 Ghost numbers of abelian p-groups

We first prove a general proposition. It generalizes [9, Lemma 2.3] and [19, Prop. 5.10].

Proposition 2.4.8. *Let k be a field of characteristic p, and let H be a non-trivial subgroup of a p-group G. Assume that there exists a central element x in G. Let l be the smallest positive integer such that $x^l \in H$. Suppose that $M \in \mathsf{StMod}(kH)$ has generating length $m \geqslant 1$. Then $\mathrm{gel}(M{\uparrow}^G) \geqslant \mathrm{gel}(M) + (l-1)$, and*

$$\textit{generating number of } kG \geqslant \textit{generating number of } kH + (l-1).$$

Suppose that $M \in \mathsf{stmod}(kH)$ has ghost length $n \geqslant 1$. Then $\mathrm{gl}(M{\uparrow}^G) \geqslant \mathrm{gl}(M) + (l-1)$, and

$$\textit{ghost number of } kG \geqslant \textit{ghost number of } kH + (l-1).$$

Proof. For brevity, we write $\downarrow$ for $\downarrow^G_H$ and $\uparrow$ for $\uparrow^G_H$. Let $f : M \to N$ be a non-trivial $(m-1)$-fold ghost in $\mathsf{StMod}(kH)$. We will show that $(x-1)^{l-1} \circ f{\uparrow}$ is stably non-trivial. Since ghosts induce up to ghosts and $x-1$ is a ghost, it follows that there exists a non-trivial composite of $(m-1)+(l-1)$ ghosts in $\mathsf{StMod}(kG)$.

Consider the map $M \xrightarrow{i} M{\uparrow\downarrow} \xrightarrow{f{\uparrow\downarrow}} N{\uparrow\downarrow} \xrightarrow{(x-1)^{l-1}\downarrow} N{\uparrow\downarrow} \xrightarrow{r} N$, where i and r are the natural maps. To be more explicit, $M{\uparrow}^G_H = kG \otimes_H M$, $i(\alpha) = 1 \otimes \alpha$ and $r(g \otimes \alpha) = g\alpha$ if $g \in H$ and is zero otherwise. By naturality of the inclusion, the composite equals $M \xrightarrow{f} N \xrightarrow{i} N{\uparrow\downarrow} \xrightarrow{(x-1)^{l-1}\downarrow} N{\uparrow\downarrow} \xrightarrow{r} N$. Since $x^i \notin H$ for $i \leqslant l-1$, the map $N \xrightarrow{i} N{\uparrow\downarrow} \xrightarrow{(x-1)^{l-1}\downarrow} N{\uparrow\downarrow} \xrightarrow{r} N$ is simply multiplication by $(-1)^{l-1}$, an isomorphism. Since N is stably non-zero, it follows that $(x-1)^{l-1}{\downarrow} \circ f{\uparrow\downarrow}$ and therefore $(x-1)^{l-1} \circ f{\uparrow}$ are stably non-trivial.

The result on ghost length and ghost number can be proved similarly by replacing $\mathsf{StMod}(kG)$ with $\mathsf{stmod}(kG)$. □

We can apply this proposition to abelian groups.

Theorem 2.4.9. *Let k be a field of characteristic p, and let $A = C_{p^r} \times C_{p^{r_1}} \times \cdots \times C_{p^{r_l}}$ be an abelian p-group. Then*

$$m - p^r + \left\lceil \frac{p^r - 1}{2} \right\rceil \leqslant \textit{ghost number of } kA \leqslant \textit{generating number of } kA \leqslant m - 1,$$

where m is the nilpotency index of $J(kA)$, and p^r is the order of the smallest cyclic summand.

When the prime p is greater than 2, the result here improves on that in [19], where the lower bound for the ghost number of kA is given by $m - p^r + p^{r-1} = m - p^r + \lceil (p^r - 1)/p \rceil$.

Note that since

$$m = 1 + (p^r - 1) + (p^{r_1} - 1) + \cdots + (p^{r_l} - 1),$$

our lower bound can also be written as

$$\left\lceil \frac{p^r - 1}{2} \right\rceil + (p^{r_1} - 1) + \cdots + (p^{r_l} - 1).$$

Also note that when A is cyclic, we have $m = p^r$, and the lower bound $d = \lceil \frac{p^r - 1}{2} \rceil$ here is exactly the ghost number of A [19, Thm. 5.4].

Proof. Let g be a generator of C_{p^r}, and let g_i be a generator of $C_{p^{r_i}}$, $i = 1, 2, \cdots, l$. Write $d = \lceil \frac{p^r - 1}{2} \rceil$. By the proof of [19, Prop. 5.3], kC_{p^r} has ghost number d. We can now apply Proposition 2.4.8 by successively including the summands $C_{p^{r_i}}$ to obtain

$$\text{ghost number of } kA \geqslant d + (p^{r_1} - 1) + \cdots + (p^{r_l} - 1).$$

The other inequalities are from Theorem 2.4.1. □

Proposition 2.4.8 allows us to make this explicit. Let $M = N\uparrow_{C_{p^r}}^{A}$, with $N = kC_{p^r}/(g-1)^d$. Note that $(g-1)^{d-1}$ is a stably non-trivial $(d-1)$-fold ghost on N in $\mathsf{stmod}(kC_{p^r})$ and, since A is abelian, the self map $(g-1)\uparrow_{C_{p^r}}^{A}$ on M is simply left multiplication by $g-1$. Hence we have a particular form for the non-trivial $(m-p^r+d-1)$-fold ghost on M:

$$\theta = (g-1)^{d-1}(g_1 - 1)^{p^{r_1}-1} \cdots (g_l - 1)^{p^{r_l}-1}.$$

More generally, we have the following result.

Proposition 2.4.10. *Let k be a field of characteristic p, let $A = C_{p^{r_1}} \times C_{p^{r_2}} \times \cdots \times C_{p^{r_l}}$ be an abelian p-group, and let M_i be an indecomposable $C_{p^{r_i}}$-module of dimension n_i for each i. Then the A-module $M = M_1 \otimes \cdots \otimes M_l$ has radical length $1+(n_1-1)+\cdots+(n_l-1)$. If $n_i \leqslant \frac{p^{r_i}}{2}$ for some i, then the generating length of M equals its radical length.*

Before proving the proposition, we state the following lemma.

Lemma 2.4.11 ([29, Theorem 1.2])**.** *Let G be a p-group, and let k be a field of characteristic p. Then the elements $h-1$ with $h \neq 1$ form a basis for* $\mathrm{rad}(kG)$*. It follows that the products $(h_1-1)\cdots(h_n-1)$ with $h_i \neq 1$ span* $\mathrm{rad}^n(kG)$*.* □

Note that it suffices to consider generators of the group G when we generate $\mathrm{rad}^n kG$ as a sub-module. We can now compute the radical length of the module M and prove the proposition.

Proof of Proposition. Let g_i be a generator of $C_{p^{r_i}}$. Then the various g_i-1 with $1 \leqslant i \leqslant l$ generate $\mathrm{rad}(kG)$. We regard M_i as the quotient $kC_{p^{r_i}}/(g_i-1)^{n_i}$, so the elements $(g_i-1)^j$ with $0 \leqslant j \leqslant n_i - 1$ form a basis of M_i. Now let $m = (n_1-1)+\cdots+(n_l-1)$. Since any $(m+1)$-fold product of the elements $g_i - 1$ has to be zero in M, $\mathrm{rad}^{m+1}(M) = 0$. On the other hand, the element $(g_1-1)^{n_1-1} \otimes \cdots \otimes (g_l-1)^{n_l-1} \in M$ is non-zero and spans $\mathrm{rad}^m(M)$. It follows that the radical length of M is $m+1$.

To prove the last statement, without loss of generality we can assume that $n_1 \leqslant \frac{p^{r_1}}{2}$. We then consider the restriction of M to $H = C_{p^{r_1}}$. Note that we have a vector space isomorphism

$$M{\downarrow_H} \cong \bigoplus_{i_2=0}^{n_2-1} \cdots \bigoplus_{i_l=0}^{n_l-1} M_1.$$

Since G acts componentwise, this is actually an isomorphism of kH-modules, and we have kH-maps $i : M_1 \to M{\downarrow_H}$ sending α to $\alpha \otimes 1 \otimes \cdots \otimes 1$ and $r : M{\downarrow_H} \to M_1$ sending $\alpha \otimes (g_2-1)^{i_2} \otimes \cdots \otimes (g_l-1)^{i_l}$ to $(-1)^{i_2+\cdots+i_l}\alpha$ for $0 \leqslant i_k \leqslant n_k - 1$.

We can form the m-fold ghost $f = (g_1-1)^{n_1-1}\cdots(g_l-1)^{n_l-1}$ on M. And one can check that $r \circ f{\downarrow_H} \circ i$ is $\pm(g_1-1)^{n_1-1}$ on M_1, which is stably non-trivial. Hence f is stably non-trivial and the ghost length of M is at least $m+1$. Since this is also the radical length of M, we have $\mathrm{gl}(M) = \mathrm{gel}(M) = m+1$. □

Remark 2.4.12. We don't know which of the lower bound and upper bound better approximates the ghost number in general, but we suspect that the lower bound is better. We show in Section 2.4.7 that the upper bound can be refined by 1 for rank 2 abelian p-groups $C_{p^r} \times C_{p^s}$, with $p^r, p^s \geqslant 3$. In particular, the lower bound we have here is the exact ghost number for the group $C_3 \times C_3$.

2.4.4 Ghost number of the quaternion group Q_8

In this section, we study the quaternion group $Q_8 = \langle \epsilon, i, j \mid \epsilon^2 = 1,\ i^2 = j^2 = (ij)^2 = \epsilon \rangle$ over a field k of characteristic 2. It has been shown in [19] that the ghost number of kQ_8 is 2, 3, or 4.

Proposition 2.4.13. *Let k be a field of characteristic 2. Then there exists a stably non-trivial double ghost in* $\mathsf{stmod}(kQ_8)$*. Hence*

$$3 \leqslant \textit{ghost number of } kQ_8 \leqslant \textit{generating number of } kQ_8 \leqslant 4.$$

Proof. We have a quotient map from Q_8 to the Klein four group V that identifies ϵ with 1. We also write i and j for the generators of V. The rank one free kV-module can be viewed as a kQ_8-module, and we write kV for it. It has radical length 3, and we will show that it admits a stably non-trivial double ghost, hence $\mathrm{gl}(kV) = \mathrm{gel}(kV) = 3$.

Right multiplication R_{i+1} on kV by $i+1$ is a left kQ_8-map, and we claim that it is a ghost. To see this, consider the short exact sequence

$$0 \to kV \xrightarrow{i} kQ_8 \to kV \to 0$$

of left kQ_8-modules, where the kernel kV is generated by $\epsilon + 1$ in kQ_8. It follows from this sequence that $\Omega kV = kV$ and that $\Omega R_{i+1} = R_{i+1}$.

Thus to show that R_{i+1} is a ghost, we just need to check that it is stably trivial on maps from k. Multiplication by $i+1$ kills the socle of kV, which is generated by $1 + i + j + ij$, so this follows from Lemma 2.4.2(a).

Next we show that there is a non-trivial double ghost. For any map $f : kQ_8 \to kV$, the composite fi is zero, since $\epsilon + 1$ acts trivially on kV. Thus a kQ_8-map $kV \to kV$ is stably trivial if and only if it is zero, As a result, multiplication by $(i+1)(j+1)$ on kV is stably non-trivial, and we get the desired double ghost.

It follows that the ghost number of kQ_8 is at least 3. The nilpotency index of $J(kQ_8)$ is 5, so the generating number of kQ_8 is at most 4. □

Remark 2.4.14. The map $R_{(i+1)(j+1)} = R_{1+i+j+ij} : kV \to kV$ constructed in the proof is in fact the almost zero map with domain kV in $\mathsf{stmod}(kQ_8)$. To see this, we consider the inclusion $\mathrm{rad}(kV) \to kV$. Since this map is not split-epi, its composition with the almost zero map $\gamma : kV \to kV$ factors through a projective module P. But P is also injective, thus we can change γ by a map factoring through P to ensure that $\mathrm{rad}(kV) \subseteq \ker(\gamma)$. Since $kV/\mathrm{rad}(kV) \cong \mathrm{soc}(kV) \cong k$ and $\mathrm{soc}(kV)$ is generated by the element $1+i+j+ij$, it must be that $R_{1+i+j+ij}$ is the almost zero map (up to a scalar factor). This gives another proof that this map is stably non-trivial.

In the next section, we generalize the technique used here.

2.4.5 p-groups with cyclic normal subgroups

In Section 2.4.3, we produced ghosts using left multiplication by $x-1$ for abelian groups. More generally, in Lemma 2.4.4, we saw that left multiplication by $x-1$ for x a central element produces a ghost. For a non-central element, in order to produce a left module map, one must consider *right* multiplication, when this makes sense, and indeed we used this technique in Section 2.4.4 to produce ghosts for Q_8. However, it is not always true that right multiplication by $x-1$ produces ghosts. Generalizing the known examples, we show that if M is induced up from a cyclic normal subgroup, then right multiplication by $x-1$ on M is well-defined and is a ghost.

Theorem 2.4.15. *Let C_{p^r} be a cyclic normal subgroup of a p-group G, and let k be a field of characteristic p. Let M_n be an indecomposable kC_{p^r}-module of dimension n, and write $M = M_n{\uparrow}^G$. Then, for each $x \in G$, one can define the right multiplication map R_{x-1} on M and it is a ghost. Moreover, if $n \leqslant \lceil \frac{p^r-1}{2} \rceil$, then $\mathrm{gl}(M) = \mathrm{gel}(M) =$ rad len M.*

Note that for $n = 1$, we have $M \cong kH{\downarrow}_G$, where $H = G/C_{p^r}$ and the restriction is taken along the quotient map. Thus the ghosts in the previous section are examples of this construction.

Proof. Let g be a generator of C_{p^r}. We can identify M_n with the left submodule of kC_{p^r} generated by $(g-1)^{p^r-n}$, and so we have a short exact sequence of kC_{p^r}-modules:

$$0 \to M_n \to kC_{p^r} \to M_{p^r-n} \to 0,$$

where M_{p^r-n} is an indecomposable kC_{p^r}-module of dimension $p^r - n$. Inducing up, we get

$$0 \to M_n{\uparrow}^G \xrightarrow{i} kG \xrightarrow{p} M_{p^r-n}{\uparrow}^G \to 0. \tag{2.4.1}$$

The inclusion i identifies $M = M_n\!\uparrow^G$ with the left submodule of kG generated by $(g-1)^{p^r-n}$. Since $C_{p^r} \leqslant G$ is normal, this submodule is actually a sub-bimodule. Thus the right multiplication map $R_{x-1} : M \to M$ is well-defined and is a left kG-module map, for each $x \in G$. We must show that it is a ghost.

Since (2.4.1) is in fact a short exact sequence of bimodules, R_{x-1} is two-periodic as a left kG-map, so it suffices to check that R_{x-1} is left stably-trivial on maps from k and $\Omega^{-1}k$. By Lemma 2.4.2, this is equivalent to $\mathrm{soc}_L(M) \subseteq \ker(R_{x-1})$ and $\mathrm{im}(R_{x-1}) \subseteq \mathrm{rad}_L(M)$, where we use subscripts to indicate left and right socles and radicals. Clearly, $\mathrm{soc}_R(M) \subseteq \ker(R_{x-1})$ and $\mathrm{im}(R_{x-1}) \subseteq \mathrm{rad}_R(M)$. Now $\mathrm{soc}_L(kG) = \mathrm{soc}_R(kG) \cong k$, so $\mathrm{soc}_L(M) = \mathrm{soc}_R(M) \cong k$, which gives the first inclusion. And one can also show that $\mathrm{rad}_L(M) = \mathrm{rad}_R(M)$, which gives the second inclusion.

To prove the last claim, let $n \leqslant \lceil \frac{p^r-1}{2} \rceil$ and assume that $\mathrm{rad\,len}\ M = l$. We want to construct an $(l-1)$-fold ghost. Note that $\mathrm{soc}_L(M) = \mathrm{soc}_R(M) = \mathrm{rad}_R^{l-1}(M) = M(g_1 - 1)\cdots(g_{l-1}-1)$ for some $g_1, \ldots, g_{l-1}$ in G, so the $(l-1)$-fold ghost $f := R_{g_{l-1}-1} \circ \cdots \circ R_{g_1-1}$ takes M onto its socle. For any map $h : kG \to M$, the composite hi is zero, since the image of i is generated by $(g-1)^{p^r-n}$ which acts trivially on M since $n \leqslant p^r - n$. Thus a map $M \to M$ is stably trivial if and only if it is zero, and so our $(l-1)$-fold ghost f is stably non-trivial. Thus $l \leqslant \mathrm{gl}(M) \leqslant \mathrm{gel}(M) \leqslant \mathrm{rad\,len}\ (M) = l$, and we are done. $\square$

Remark 2.4.16. As in Remark 2.4.14, we can also see that f is non-trivial using the theory of Auslander-Reiten triangles. There is a canonical inclusion j of M into $M_{p^r-n}\!\uparrow^G = \Omega M$ induced from the kC_{p^r}-map $M_n \to M_{p^r-n}$, and one can show that the composite jf is exactly the almost zero map out of M.

Note that any p-group G has a non-trivial center, hence a cyclic normal subgroup C_p. Applying the theorem to the short exact sequence of groups $C_p \to G \to H$, we get

Corollary 2.4.17. *Let G be a p-group, and let k be a field of characteristic p. Then*

$$\frac{1}{2}\mathit{rad\,len}\ kG \leqslant \mathit{ghost\,num}\ kG \leqslant \mathit{gen\,num}\ kG < \mathit{rad\,len}\ kG,$$

when p is even, and

$$\frac{1}{3}\mathit{rad\,len}\ kG \leqslant \mathit{ghost\,num}\ kG \leqslant \mathit{gen\,num}\ kG < \mathit{rad\,len}\ kG,$$

when p is odd.

Proof. Choose a cyclic normal subgroup C_p of G, and let $M = M_n{\uparrow}^G$, where M_n is an indecomposable kC_p-module of dimension $n = \lceil \frac{p-1}{2} \rceil$. Since $\operatorname{rad\,len} M = \operatorname{gl}(M) \leqslant$ ghost num kG, we only need to show that $2(\operatorname{rad\,len} M) \geqslant \operatorname{rad\,len} kG$ for p even and $3(\operatorname{rad\,len} M) \geqslant \operatorname{rad\,len} kG$ for p odd. By (2.4.1), we know that

$$\operatorname{rad\,len} M + \operatorname{rad\,len} M_{p-n}{\uparrow}^G \geqslant \operatorname{rad\,len} kG.$$

For p even, $p - n = n$, and so the result follows.

For p odd, $p - n = n + 1$. We will show that $2(\operatorname{rad\,len} M) \geqslant \operatorname{rad\,len} M_{n+1}{\uparrow}^G$, and the corollary will follow. There is a short exact sequence

$$0 \to M \to M_{n+1}{\uparrow}^G \to M_1{\uparrow}^G \to 0,$$

induced up from C_p-maps, and one sees that $M_1{\uparrow}^G$ is a submodule of M again by inducing up the C_p-map $k \to M_n$. It follows that

$$2(\operatorname{rad\,len} M) \geqslant \operatorname{rad\,len} M + \operatorname{rad\,len} M_1{\uparrow}^G \geqslant \operatorname{rad\,len} M_{n+1}{\uparrow}^G,$$

and we are done. □

We expect that for odd primes, the lower bound can be improved to an expression that is generically close to $(\operatorname{rad\,len} kG)/2$.

2.4.6 Ghost numbers of dihedral 2-groups

Our next goal is to study the dihedral 2-groups. We will show that the ghost number and generating number of kD_{4q} are both $q + 1$. Here we write D_{4q} for the dihedral 2-group of order $4q$, with q a power of 2:

$$D_{4q} = \langle x, y \mid x^2 = y^2 = 1, (xy)^q = (yx)^q \rangle.$$

It has a normal cyclic subgroup C_{2q}, generated by $g = xy$.

Since kC_{2q} has ghost number q, which is realized by the ghost length of $M = kC_{2q}/(g-1)^q$ [19, Prop. 5.3], the ghost length of $N = M{\uparrow}_{C_{2q}}^{D_{4q}}$ is at least q in $\mathsf{stmod}(kD_{4q})$. By Theorem 2.4.15, we actually have $\operatorname{gl}(N) = \operatorname{gel}(N) = \operatorname{rad\,len} N$. Note that $(xy)^q \in$

D_{4q} is central of order 2 and that $M \cong k\uparrow_{C_2}^{C_{2q}}$, hence $N = M\uparrow_{C_{2q}}^{D_{4q}} \cong k\uparrow_{C_2}^{D_{4q}} \cong kD_{2q}\downarrow_{D_{4q}}^{D_{2q}}$, where the restriction is along the quotient map in the short exact sequence $C_2 \to D_{4q} \to D_{2q}$. It is not hard to see that the radical length of kD_{2q} is $q+1$ (see Remark 2.4.20) and that its q-th radical is generated by $((y-1)(x-1))^{\frac{q}{2}} = ((x-1)(y-1))^{\frac{q}{2}}$ (which makes sense for $q=1$ since we have identified $x=y$ in that case). Thus we have proved the following consequence of Theorem 2.4.15:

Corollary 2.4.18. *Let k be a field of characteristic 2. Then the ghost number of kD_{4q} is at least $q+1$. In fact,* $\mathrm{gl}(N) = \mathrm{gel}(N) = q+1$, *where* $N = k\uparrow_{C_2}^{D_{4q}}$. □

The proof of Theorem 2.4.15 shows that an explicit q-fold ghost $N \to N$ is given by $R_{((x-1)(y-1))^{\frac{q}{2}}}$.

To get upper bounds for the generating numbers of dihedral 2-groups, we need classification theorems [4].

Let $\Lambda = k\langle X, Y\rangle/(X^2, Y^2)$ be the quotient of the free algebra on two non-commuting variables. In kD_{4q}, writing $X = x-1$ and $Y = y-1$, one can show that $(XY)^r - (YX)^r = (xy)^r - (yx)^r$ for r a power of 2, and so $kD_{4q} \cong \Lambda/((XY)^q - (YX)^q)$ [4, Lemma 4.11.1].

In the isomorphism $kD_{4q} \cong \Lambda/((XY)^q - (YX)^q)$, we have implicitly assumed that the characteristic of k is 2. However, for the classification we describe below, k can have any characteristic, and we apply it in this generality in the next section.

Λ-modules are classifiable. Let $\mathcal{W}$ be the set of words in the **direct letters** a and b and the **inverse letters** a^{-1} and b^{-1}, such that a and a^{-1} are always followed by b or b^{-1} and vice versa, together with the "**zero length word**" 1.

Given $C = l_1 \cdots l_n \in \mathcal{W}$, where each l_i is a direct or inverse letter, let $M(C)$ be the vector space over k with basis $z_0, \ldots, z_n$ on which Λ acts according to the schema

$$kz_0 \overset{l_1}{\longleftarrow} kz_1 \overset{l_2}{\longleftarrow} kz_2 \cdots kz_{n-1} \overset{l_n}{\longleftarrow} kz_n,$$

with X acting via a and Y acting via b. For example, if $C = ab^{-1}a^{-1}$, then the schema is

$$kz_0 \overset{a}{\leftarrow} kz_1 \overset{b}{\rightarrow} kz_2 \overset{a}{\rightarrow} kz_3$$

and the module $M(ab^{-1}a^{-1})$ is given by

$$X \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with the matrices acting on row vectors on the right. Such a module is called a **module of the first kind**. Clearly, $M(C) \cong M(C^{-1})$, where C^{-1} reverses the order of the letters in C and inverts each letter.

Let $C = l_1 \cdots l_n$ be a word in $\mathcal{W}$ of even non-zero length that is not a power of a smaller word, and let V be a vector space with an indecomposable automorphism ϕ on it. An automorphism is indecomposable if its rational canonical form has only one block, and the block corresponds to a power of an irreducible polynomial over k. Let $M(C, \phi)$ be the vector space $\oplus_{i=0}^{n-1} V_i$, with $V_i \cong V$, and let Λ act on $M(C, \phi)$ via the schema

$$V_0 \xleftarrow{l_1=\phi} V_1 \xleftarrow{l_2=id} V_2 \longleftarrow \cdots \longleftarrow V_{n-2} \xleftarrow{l_{n-1}=id} V_{n-1}\,, \qquad V_0 \xrightarrow{l_n=id} V_{n-1}$$

Such a module is called a **module of the second kind**. It is clear that $M(C, \phi) \cong M(C^{-1}, \phi^{-1})$. And if C' differs from C by a cyclic permutation, say $l_1 \cdots l_n \mapsto l_n l_1 \cdots l_{n-1}$, then $M(C, \phi) \cong M(C', \phi)$. Moreover, if V' is another vector space with an indecomposable automorphism ϕ', and $V \cong V'$ via an isomorphism that commutes with ϕ and ϕ', then $M(C, \phi) \cong M(C', \phi')$.

Theorem 2.4.19 ([4, Section 4.11])**.** *For any field k, the above provides a complete list of all indecomposable Λ-modules, up to isomorphism. One of these modules has $(XY)^q - (YX)^q$ in its kernel if and only if one of the following holds:*

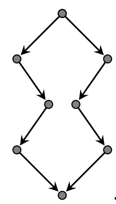
(a) The module is of the first kind and the corresponding word does not contain $(ab)^q$, $(ba)^q$, or their inverses.

(b) The module is of the second kind and no power of the corresponding word contains $(ab)^q$, $(ba)^q$, or their inverses.

(c) The module is $M((ab)^q(ba)^{-q}, id)$. It is a module of the second kind and is the projective indecomposable module for the algebra $\Lambda/((XY)^q - (YX)^q)$.

Thus, when k has characteristic 2, a complete list of indecomposable kD_{4q}-modules, up to isomorphism, consists of the Λ-modules satisfying one of these three conditions. □

Remark 2.4.20. The identification $kD_{4q} \cong \Lambda/((XY)^q - (YX)^q)$ yields that $kD_{4q} = M((ab)^q(ba)^{-q}, id)$. It is not hard to see from the schema of $M((ab)^q(ba)^{-q}, id)$ that it has radical length $2q+1$. Here is an illustration for $q = 2$:



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The module $N = k\uparrow_{C_2}^{D_{4q}} = kD_{4q} \otimes_{kC_2} k$ is the quotient of kD_{4q} where we identify $(xy)^q$ with 1, in other words, $(xy)^{\frac{q}{2}} = (yx)^{\frac{q}{2}}$, for $q > 1$. This is equivalent to $(XY)^{\frac{q}{2}} = (YX)^{\frac{q}{2}}$. Hence $N = M((ab)^{\frac{q}{2}}(ba)^{-\frac{q}{2}}, id)$ and it follows that N has radical length $q+1$.

We want to prove that the generating number of kD_{4q} does not exceed $q+1$. Note that when $q = 1$, the dihedral group D_4 is just $C_2 \times C_2$, and the claim follows from Theorem 2.4.9, so we assume that $q \geqslant 2$ from now on unless otherwise stated.

Now let M be an indecomposable kD_{4q}-module. By Theorem 2.4.19, it corresponds to a word satisfying one of the conditions (a), (b) or (c). Then $\mathrm{soc}(M)$ contains the submodule spanned by the vector spaces at positions of the form $b^{-1}a$ or $a^{-1}b$ (interpreted cyclically if M is of the second kind). Such a position exists if M is of the second kind since the condition that the word is not a power of a smaller word forces the word to contain both direct and inverse letters. However, such positions are removed in $M/\mathrm{soc}(M)$, so the indecomposable summands of $M/\mathrm{soc}(M)$ are of the first kind and correspond to words not containing $b^{-1}a$ or $a^{-1}b$.

Similarly, the indecomposable summands of $\mathrm{rad}(M)$ are of the first kind and correspond to words not containing ba^{-1} or ab^{-1}. It follows that the indecomposable summands of $\mathrm{rad}(M/\mathrm{soc}(M))$ are of the first kind and correspond to words not containing $b^{-1}a$, $a^{-1}b$, ba^{-1} or ab^{-1}. Thus the words must consist entirely of direct or inverse letters. But since $M(C) \cong M(C^{-1})$, we can assume that the words only contain direct letters. By (a), the possible words are $(ab)^{q-1}a$, $(ba)^{q-1}b$, or subwords of these. And we can prove

Lemma 2.4.21. *Let M be a kD_{4q}-module of the first kind, with $q \geqslant 2$. If M corresponds to a word that only contains direct letters, then its generating length is less than or equal to q.*

Proof. We are going to show that

$$\mathrm{gel}(M((ab)^r a)) \leqslant q \quad \text{and} \quad \mathrm{gel}(M((ab)^r)) \leqslant q$$

for $0 \leqslant r \leqslant q-1$, the case of words starting with b being similar.

Since D_{4q} is a 2-group, the generating length of a module is always no more than its radical length, hence its dimension. So, for any word C, $\mathrm{gel}(M(C)) \leqslant \dim M(C) = |C|+1$, where $|C|$ denotes the number of letters in C. Thus we are done if $r \leqslant q/2-1$.

To handle $r \geqslant q/2$, we temporarily introduce the following notation for modules with symmetry under reflection when exchanging X with Y. For a word u, write u' for the inverse word with all as and bs exchanged, so for example $(ab^{-1}ab)' = a^{-1}b^{-1}ab^{-1}$. Write $M'(u)$ for $M(uu')$ and $M'(u,\phi)$ for $M(uu',\phi)$. Then $kD_{4q} = M'((ab)^q, id)$, and one can see that $\tilde{\Omega}k = M'((b^{-1}a^{-1})^{q-1}b^{-1})$ and $\tilde{\Omega}^{-2}k = M'((ab)^{q-1}ab^{-1})$. It follows that we have short exact sequences

$$0 \to k \to \tilde{\Omega}^{-2}k \to M((ab)^{q-1}a) \oplus M((ba)^{q-1}b) \to 0$$

and

$$0 \to k \to \tilde{\Omega}k \to M((ab)^{q-1}) \oplus M((ba)^{q-1}) \to 0.$$

Since $q \geqslant 2$, one sees that $\mathrm{gel}(M((ab)^{q-1}a)) = \mathrm{gel}(M((ab)^{q-1})) \leqslant 2$, which handles the case $r = q-1$.

Now for $r \leqslant q-2$, $M((ab)^r a)$ and $M((ab)^r)$ embed in $M((ab)^{q-1})$. Thus their ghost lengths are no more than the codimension plus two, and one can check that this is no more than q when $r \geqslant q/2$. □

In general, for a p-group G and a kG-module M, we know that $M/\mathrm{rad}(M)$ and $\mathrm{soc}(M)$ are sums of trivial modules. Thus $\mathrm{rad}(M)$ is the fibre of a map $M \to \oplus k$ and $M/\mathrm{soc}(M)$ is the cofibre of a map $\oplus k \to M$. So

$$\mathrm{gel}(M) \leqslant \mathrm{gel}(\mathrm{rad}(M)) + 1 \quad \text{and} \quad \mathrm{gel}(M) \leqslant \mathrm{gel}(M/\mathrm{soc}(M)) + 1.$$

Hence

$$\mathrm{gel}(M) \leqslant \mathrm{gel}(\mathrm{rad}(M/\mathrm{soc}(M))) + 2,$$

and so by Lemma 2.4.21 and the discussion preceding it, the generating number of kD_{4q} does not exceed $q+2$. This is one more than the correct answer. We will show in Proposition 2.4.26 that the module $M((ab)^{\frac{q}{2}-1}a)$ has length q, so we can't improve this bound by improving Lemma 2.4.21.

We will have to be a bit more clever in the construction to get the exact generating number. The above process takes two steps to produce a module $\mathrm{rad}(M/\mathrm{soc}(M))$ whose summands involve only direct letters, by removing "top" and "bottom" elements. We next show that we can add top elements instead of removing them, with the same effect, and as a result we will be able to do both steps at the same time.

Lemma 2.4.22. *Let M be a non-projective indecomposable module, with corresponding word C. There exists a short exact sequence*

$$0 \to M \to M' \to \oplus k \to 0,$$

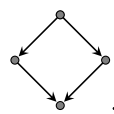
where the indecomposable summands of M' are of the first kind and correspond to words that contain no ab^{-1} or ba^{-1}.

Proof. First suppose that M is of the first kind. If C contains no ab^{-1} or ba^{-1}, we simply set M' to be M. Otherwise, assume for example that C contains ab^{-1} and factor the word C as L_1L_2, with L_1 ending with a and L_2 starting with b^{-1}. Write z for the basis element of $M(C)$ corresponding to the vertex connecting L_1 with L_2, and write z_i for the corresponding basis element in $M(L_i)$, $i = 1, 2$. Then we have a short exact sequence $M \to M(L_1) \oplus M(L_2) \to k$, where the first map takes z to $z_1 - z_2$ and does the natural thing on the other basis elements, and the second map takes z_1 and z_2 to 1 in k and the other basis elements to 0. More generally, we can write $C = L_1L_2 \cdots L_n$, broken at the spots $a^{-1}b$ and $b^{-1}a$, and set $M' = \oplus M(L_i)$.

Now suppose that $M = M(C, \phi)$ is of the second kind, where $\phi : V \to V$ is an indecomposable automorphism. We can assume that $C = a^{-1}L\,b$ up to inverse and cyclic permutation. Fix a basis $v_1, \ldots, v_n$ of V, where $n = \dim(V)$. Let $M'' = \oplus_{i=1}^{n} M_i$, with each $M_i = M(C)$. We write w_i and z_i for the basis elements in M_i corresponding to the beginning and end of the word C. Then we have a short exact sequence $M(C, \phi) \to M'' \to V$, where the first map sends v_i to $\phi(w_i) - z_i$ for the first vertex and does the natural thing on the other vertices, and the second map sends w_i to v_i, z_i to $\phi(v_i)$ and the other basis elements to 0. Here we regard V as a module with trivial action. Repeating the process for a module of the first kind, we get a short exact sequence $M'' \to M' \to \oplus k$. It is not hard to see that the cokernel of the composite $M \to M'' \to M'$ also has a trivial action, and we are done. □

Note that the short exact sequence is represented by a map $\oplus \Omega k \to M$, and this makes it possible to combine it with a map $\oplus k \to M$.

Example 2.4.23. We illustrate an example for $q = 2$. Write kV for the module $M(a^{-1}b^{-1}ab, id_k)$:



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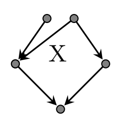
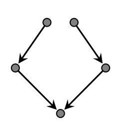
We begin by defining a cofibre sequence

$$\Omega k \to kV \to M(a^{-1}b^{-1}ab) \to k.$$

To see what the maps are, first consider the module



which has kV as a codimension 1 submodule. We can choose a basis so that this becomes $M' = M(a^{-1}b^{-1}ab)$



and the map $M' \to k$ takes both top points to k and has kernel kV. Then M' corresponds to a word that does not contain ba^{-1} or ab^{-1}, and the summands of $M'/\mathrm{soc}(M') \cong M(a) \oplus M(b)$ correspond to words that only contain direct letters. Note that the map from k to $\mathrm{soc}(M')$ factors through $kV \to M'$, so we can combine the two steps to get a cofibre sequence

$$\Omega k \oplus k \to kV \to M(a) \oplus M(b) \to k \oplus \Sigma k.$$

By Lemma 2.4.21, the generating length of the third term is at most q, which is 2 in our case.

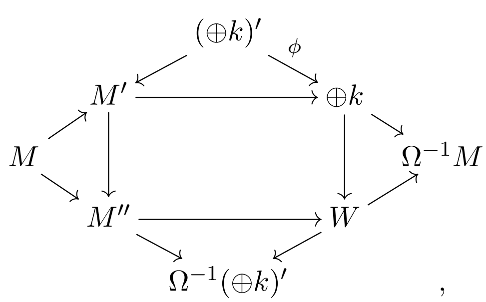
Now we are ready to prove

Theorem 2.4.24. *Let k be a field of characteristic 2. Then the generating number of kD_{4q} is at most $q+1$, for all $q \geqslant 1$.*

Proof. The case when $q = 1$ is dealt with in Theorem 2.4.9, so we prove the theorem for $q \geqslant 2$.

Let M be a non-projective indecomposable module, with corresponding word C. In the short exact sequence $M \to M' \to \oplus k$ from Lemma 2.4.22, the indecomposable summands of M' correspond to words that contain no ab^{-1} or ba^{-1}. Hence the indecomposable summands of $M'' = M'/\mathrm{soc}(M')$ correspond to words of direct letters, and $\mathrm{gel}(M'') \leqslant q$.

We can form the octahedron



,

where $(\oplus k)'$ is $\mathrm{soc}(M')$.

The proof will be finished once we show that $\mathrm{gel}(W) = 1$. Here W is the cofibre of a map ϕ between direct sums of trivial modules. Such a map is the sum of an identity map and a zero map. Hence W is a direct sum of trivial modules k and the modules $\Omega^{-1}k$, so $\mathrm{gel}(W) = 1$. □

Corollary 2.4.25. *Let k be a field of characteristic 2. Then the ghost number and generating number of kD_{4q} are $q+1$ for all $q \geqslant 1$.* □

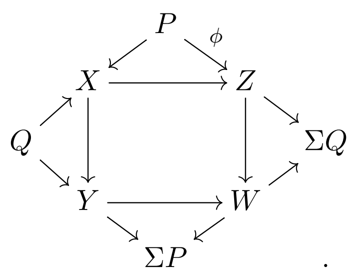
We now summarize and generalize the idea in the proof of the Theorem. Suppose that we start building an object Q from P, Y and Z by first using a triangle

$$P \to X \to Y \to \Sigma P$$

and then using a triangle

$$Q \to X \to Z \to \Sigma Q.$$

Then we can form the octahedron



Assume that P has length m, Y has length n, and Z has length l. Then the length of Q does not exceed $m+n+l$. Indeed, $n + \text{len}(W)$ bounds the length of Q. For example, if ϕ is in $\mathcal{I}^s$ for some positive integer, we have $\text{len}(W) \leqslant m + l - s$ by Lemma 2.3.1. Or, if $\phi = 0$, then $W \cong Z \oplus \Sigma P$ and the two steps can be combined. This is analogous to the fact in topology that when a second cell is attached to a CW-complex without touching a first cell, then they can be attached to the complex at the same time.

We finish this section by computing the generating lengths of the modules $M(ab)^r$ and $M((ab)^r a)$, with $r \leqslant q/2 - 1$. Note that there is a category automorphism on $\mathsf{StMod}(kD_{4q})$ induced by the group automorphism on D_{4q} that exchanges x and y. It exchanges the a's and b's in the word which an indecomposable module corresponds to and preserves the ghost projective class. As a result,

$$\text{gel}(M((ab)^r)) = \text{gel}(M((ba)^r)) \text{ and } \text{gel}(M((ab)^r a)) = \text{gel}(M((ba)^r b))$$

for D_{4q}-modules with $0 \leqslant r \leqslant q - 1$.

Recall from Corollary 2.4.18 that the module $M = kD_{2q}$ in $\mathsf{StMod}(kD_{4q})$ has its generating length equal to its radical length $q+1$. By Proposition 2.4.6, $\text{gel}(\text{rad}(M/\text{soc}(M)))$ $= \text{gel}(M) - 2 = q - 1$. Note that $M = M((ab)^{l+1}(a^{-1}b^{-1})^{l+1}, id)$, where $l = q/2 - 1$, so $\text{rad}(M/\text{soc}(M)) \cong M((ab)^l) \oplus M((ba)^l)$. Then, since exchanging a's and b's preserves the generating length,

$$\mathrm{gel}(M((ab)^l)) = \mathrm{gel}(M((ba)^l)) = q - 1.$$

It follows that

$$\mathrm{gel}(M((ab)^r) = 2r + 1 \text{ if } r \leqslant l, \text{ and}$$

$$\mathrm{gel}(M((ab)^r a) = 2(r + 1) \text{ if } r \leqslant l - 1.$$

We need to be a bit trickier to handle the module $M((ab)^l a)$.

Proposition 2.4.26. *The kD_{4q}-module $M((ab)^l a)$ has generating length q, where $l = q/2 - 1$.*

Proof. We have a triangle

$$\Sigma k \oplus k \to M \to M((ab)^l a) \oplus M((ba)^l b),$$

where the map $\Sigma k \to M$ is a surjection.

Hence $\mathrm{gel}(M((ab)^l a) \oplus M((ba)^l b)) \geqslant q$. Since its radical length is q, this must be an equality. Then, using the symmetry again,

$$\mathrm{gel}(M(ab)^l a) = \mathrm{gel}(M(ba)^l b) = q. \qquad \square$$

2.4.7 Ghost number of $C_{p^r} \times C_{p^s}$

Let $G = C_{p^r} \times C_{p^s}$. In this section we show that

$$\text{the ghost number of } kG \leqslant \text{the generating number of } kG \leqslant p^r + p^s - 3$$

and give the exact result when p^r is 3 or 4. Note that a general upper bound for the generating number for a p-group is given by the radical length of kG minus 1 (Theorem 2.4.1). This gives $p^r + p^s - 2$ for the group $C_{p^r} \times C_{p^s}$, and our result refines this upper bound by 1. To keep the indices simple, we give a detailed proof for the group $C_3 \times C_3$ at the prime 3, and we indicate how to modify the proof to cover the general case. We are going to show that the composite of any three ghosts is stably trivial for the group $C_3 \times C_3$, using Theorem 2.4.19.

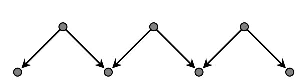
Here is an overview of our strategy. Given a finitely generated projective-free module N with radical length n and an l-fold ghost $g : N \to N_1$ in $\mathsf{Mod}(kG)$, where N_1 is an arbitrary projective-free module, we can form the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\quad g \quad} & N_1 \\ \Big\downarrow{\scriptstyle p_1} & & \Big\uparrow{\scriptstyle h} \\ N/\mathrm{rad}^{n-l}(N) & \xrightarrow{p_2} & N/\mathrm{soc}^l(N). \end{array}$$

The l-fold ghost g factors through $N/\mathrm{soc}^l(N)$ by Corollary 2.4.3, and the canonical projection $N \to N/\mathrm{soc}^l(N)$ factors through $N/\mathrm{rad}^{n-l}(N)$ because $\mathrm{rad}^{n-l}(N) \subseteq \mathrm{soc}^l(N)$. If we have a good control over the modules $N/\mathrm{rad}^{n-l}(N)$ or $N/\mathrm{soc}^l(N)$, we can factorize a long composite of ghosts as an l-fold ghost $g : N \to N_1$ followed by another composite of ghosts $f : N_1 \to N_2$, and check whether f is stably trivial on $N/\mathrm{rad}^{n-l}(N)$ or $N/\mathrm{soc}^l(N)$. For example, we can take l to be $n-1$, so that $N/\mathrm{rad}(N)$ is a sum of trivial modules. Hence, if the map f is a ghost, the composite $f \circ g$ is stably trivial, and so we have reproved that the generating length of N is at most its radical length n (Theorem 2.4.1). If we want to improve the bound, we need to choose l smaller. We will take $l = n-2$.

The relevance of Theorem 2.4.19 is that there is an isomorphism $k(C_{p^r} \times C_{p^s}) \cong k[X,Y]/(X^{p^r}, Y^{p^s})$, where $X = x-1$ and $Y = y-1$, and x and y are the generators of the cyclic summands. Under this isomorphism, $\mathrm{rad}(k(C_{p^r} \times C_{p^s})) \cong (X,Y)$ and $\mathrm{rad}^2(k(C_{p^r} \times C_{p^s})) \cong (X^2, XY, Y^2)$. Therefore $k(C_{p^r} \times C_{p^s})/\mathrm{rad}^2(k(C_{p^r} \times C_{p^s})) \cong \Lambda'$, where $\Lambda' = \Lambda/(XY, YX) \cong k[X,Y]/(X^2, Y^2, XY)$ and $\Lambda = k\langle X, Y\rangle/(X^2, Y^2)$ is the ring from Section 2.4.6. Thus when M is a $k(C_{p^r} \times C_{p^s})$-module, $M/\mathrm{rad}^2(M)$ will be a Λ'-module. Up to isomorphism, the indecomposable Λ'-modules biject with the Λ-modules of Theorem 2.4.19 satisfying conditions (a) or (b) for $q = 1$. Condition (c) is excluded by the requirement that XY be in the kernel.

Our proof will use this classification, so we will make it more explicit. A module satisfying condition (a) is of the first kind. If it has odd dimension, it is either the trivial module k; the module $M((b^{-1}a)^n)$ for some positive integer n, which we say has shape "W"; or the module $M((ab^{-1})^n)$ for some positive integer n, which we say has shape "M". For example, the "M" module $M((ab^{-1})^3)$ looks like



A module of the first kind with even dimension is one of the above with one end removed.

One can check that a module satisfying condition (b) of Theorem 2.4.19 corresponds to the word $b^{-1}a$, up to inverse and cyclic permutation. Recall that the additional data one needs to specify are a vector space V with an indecomposable automorphism ϕ. Since ϕ is indecomposable, one can choose a basis $\{v_1, v_2, \dots, v_m\}$ for V such that $\phi(v_i) = v_{i+1}$ for $i < m$. Thus we can view such a module as a quotient of an "M" module, with a relation that identifies the right bottom basis element with a linear combination of the other bottom basis elements, as specified by $\phi(v_m)$.

We point out that this is very similar to the classification of kV-modules given in [4, Theorem 4.3.3], where k has characteristic 2.

Recall that the radical length of $k(C_{p^r} \times C_{p^s})$ is $p^r + p^s - 1$. If N is projective-free, then its radical length n is at most $p^r + p^s - 2$, so we pick $l = p^r + p^s - 4$. Note that $N/\mathrm{rad}^2(N)$ and $N/\mathrm{soc}^l(N)$ are naturally Λ'-modules. And we have the following lemma, which helps describe summands of $N/\mathrm{soc}^l(N)$.

Lemma 2.4.27. *Let $G = C_{p^r} \times C_{p^s}$ be an abelian p-group of rank 2 with generators x and y, respectively, and let k be a field of characteristic p. Write $X = x-1$ and $Y = y-1$ in kG, and let $l = p^r + p^s - 4$. Suppose M is a kG-module containing elements z_0, z_2, and z_4 such that $Yz_0 - Xz_2$ and $Yz_2 - Xz_4$ are in $\mathrm{soc}^l(M)$. If $p^s \geqslant 3$, then Xz_0 and Xz_2 are in $\mathrm{soc}^l(M)$. Similarly, if $p^r \geqslant 3$, then Yz_2 and Yz_4 are in $\mathrm{soc}^l(M)$.*

Intuitively, this is saying that we cannot have a "W"-shape in the module $M/\mathrm{soc}^l(M)$. In particular, only k, $M(ab^{-1})$ and $M((ab^{-1})^2)$ can appear as indecomposable summands of $M/\mathrm{soc}^l(M)$ if M is projective-free and $p^r, p^s \geqslant 3$. Note that to exclude a module like $M(a)$, one takes $z_2 = z_4 = 0$, so the "W" isn't visible in this case.

Proof. Assume that $p^s \geqslant 3$. To show that $Xz_0 \in \mathrm{soc}^l(M)$, we need to show that it is killed by $\mathrm{rad}^l(kG)$, which is generated by $X^{p^r-1}Y^{p^s-3}$, $X^{p^r-2}Y^{p^r-2}$ and $X^{p^r-3}Y^{p^s-1}$ (where the last one is omitted if $p^r = 2$). We compute

$$X^{p^r-1}Y^{p^s-3}Xz_0 = X^{p^r}Y^{p^s-3}z_0 = 0,$$
$$X^{p^r-2}Y^{p^r-2}Xz_0 = X^{p^r-1}Y^{p^s-3}Yz_0 = X^{p^r}Y^{p^s-3}z_2 = 0,$$

and

$$X^{p^r-3}Y^{p^s-1}Xz_0 = X^{p^r-2}Y^{p^s-2}Yz_0 = X^{p^r-1}Y^{p^s-3}Yz_2 = X^{p^r}Y^{p^s-3}z_4 = 0,$$

where we have made used of fact that $Yz_0 - Xz_2$ and $Yz_2 - Xz_4$ are killed by the generators. Hence $Xz_0 \in \mathrm{soc}^l(M)$. Similarly,

$$X^{p^r-1}Y^{p^s-3}Xz_2 = 0,$$
$$X^{p^r-2}Y^{p^s-2}Xz_2 = X^{p^r-1}Y^{p^s-3}Yz_2 = X^{p^r}Y^{p^r-3}z_4 = 0,$$

and

$$X^{p^r-3}Y^{p^s-1}Xz_2 = X^{p^r-3}Y^{p^s-1}Yz_0 = 0.$$

Hence $Xz_2 \in \mathrm{soc}^l(M)$. The other case is symmetrical. □

We are now ready to prove the main theorem.

Theorem 2.4.28. *Let $G = C_3 \times C_3$ with generators x and y, respectively, and let k be a field of characteristic 3. Then the ghost number of kG is 3.*

Proof. Theorem 2.4.9 gives a lower bound of 3, so it suffices to show that the composite of any three ghosts in $\mathsf{Mod}(kG)$ out of a finitely-generated module is stably trivial. As we have explained, we consider the diagram

$$\begin{array}{ccccccc} N & \xrightarrow{g_1} & N_1 & \xrightarrow{g_2} & N_2 & \xrightarrow{g_3} & N_3 \\ \downarrow{\scriptstyle p_1} & & & & \uparrow{\scriptstyle h} & & \\ N/\mathrm{rad}^2(N) & & \xrightarrow{p_2} & & N/\mathrm{soc}^2(N), & & \end{array}$$

where g_1, g_2, and g_3 are ghosts in $\mathsf{Mod}(kG)$ and N, N_1, N_2, and N_3 are projective-free. Note that this diagram commutes in the module category. We will show that the composite $g_3 \circ h \circ p_2$ is stably trivial, by restricting to each indecomposable summand M of $N/\mathrm{rad}^2(N)$. We divide the summands M into four cases, and write j for the inclusion map $M \to N/\mathrm{rad}^2(N)$.

Case 1: M is not of the form k, $M(ab^{-1})$ or $M((ab^{-1})^2)$.

We claim that $\mathrm{soc}(M) \subseteq \ker(p_2 \circ j)$, hence $p_2 \circ j$ factors through a sum of trivial modules. Therefore, since g_3 is a ghost, the composite $g_3 \circ h \circ p_2 \circ j$ is stably trivial. We actually show that $p_1^{-1}j((\mathrm{soc}(M))) \subseteq \mathrm{soc}^2(N)$, which suffices, since p_2 kills $\mathrm{soc}^2(N)$. Observe using the classification that since M is not k, $M(ab^{-1})$ or $M((ab^{-1})^2)$, the elements $X(z_0)$, $X(z_2)$, $Y(z_2)$ and $Y(z_4)$ span $\mathrm{soc}(M)$ as z_0, z_2, and z_4 vary over elements satisfying $Y(z_0) = X(z_2)$ and $Y(z_2) = X(z_4)$. Suppose that we have $s \in p_1^{-1}j(\mathrm{soc}(M))$, say $p_1(s) = j(X(z_0))$ for some $z_0 \in M$ satisfying the above relations. Since p_1 is surjective, we have $\tilde{z_0}$, $\tilde{z_2}$, and $\tilde{z_4} \in N$ that project to $j(z_0)$, $j(z_2)$, and $j(z_4)$, respectively. Then $p_1(Y(\tilde{z_0})) = p_1(X(\tilde{z_2}))$ and $p_1(Y(\tilde{z_2})) = p_1(Y(\tilde{z_4}))$. Since N is projective-free, its radical length is at most 4, hence $\mathrm{rad}^2(N) \subseteq \mathrm{soc}^2(N)$. Now we can apply Lemma 2.4.27 and see that $X(\tilde{z_0}) \in \mathrm{soc}^2(N)$. It follows that $s \in \mathrm{soc}^2(N)$ because $p_1(s) = p_1(X(\tilde{z_0}))$. The other cases when $p_1(s) = j(Xz_2)$, $j(Yz_2)$, or $j(Yz_4)$ are similar.

Case 2: $M = M(ab^{-1})$.

The map p_1 is surjective, so g_3hp_2 has its image in $\mathrm{rad}^3(N_3)$, using Corollary 2.4.3 and the fact that the diagram commutes in $\mathsf{Mod}(kG)$. M has a basis $\{z, Xz, Yz\}$ for some z and the map g_3hp_2 sends z to an element of the form $X^2Yw_1 + XY^2w_2$. After restriction to M, g_3hp_2 factors through the injective module which is free on two generators v_1 and v_2 via the maps sending z to $X^2Yv_1 + XY^2v_2$, v_1 to w_1 and v_2 to w_2. Thus g_3hp_2 is stably trivial on M.

Case 3: $M = M((ab^{-1})^2)$.

The module $M((ab^{-1})^2)$ has schema $kz_0 \xleftarrow{X} kz_1 \xrightarrow{Y} kz_2 \xleftarrow{X} kz_3 \xrightarrow{Y} kz_4$. By considering the injective hull of $M((ab^{-1})^2)$, which is free on three generators, we see that a map out of it is stably trivial if it sends z_1 to $XY^2w_1 + X^2Yw_2$ and z_3 to $XY^2w_2 + X^2Yw_3$ for some elements w_1, w_2, and w_3. This is equivalent to z_1 being sent to $X\alpha$ and z_3 being sent to $Y\alpha$ for some α in the 2^{nd} radical.

To prove that this is the case, we form the following diagram:

$$\begin{array}{ccccccccc}
 & & \tilde{\Omega}^{-2}k & \overset{f}{\dashrightarrow} & N & \xrightarrow{g_1} & N_1 & \xrightarrow{g_2} & N_2 & \xrightarrow{g_3} & N_3\,. \\
 & & & & \downarrow{\scriptstyle p_1} & & & \nearrow{\scriptstyle hp_2} & & & \\
M((ab^{-1})^2) & \xrightarrow{j} & & & N/\mathrm{rad}^2(N) & & & & & &
\end{array}$$

Writing $g = g_3 \circ g_2 \circ g_1$, we will show below that we can choose $\tilde{z_1}$ and $\tilde{z_3}$ in N with

$$g(\tilde{z_1}) = g_3 h p_2 j(z_1), \quad g(\tilde{z_3}) = g_3 h p_2 j(z_3), \quad \text{and} \quad Y\tilde{z_1} = X\tilde{z_3}.$$

Since $\tilde{\Omega}^{-2}k$ is the free module on two generators u_1 and u_2 subject to the relation $Yu_1 = Xu_2$, the last displayed equality allows us to construct the dotted map f, by sending the generators to $\tilde{z_1}$ and $\tilde{z_3}$, respectively. We will now show that

$$g(\tilde{z_1}) = X\alpha \quad \text{and} \quad g(\tilde{z_3}) = Y\alpha$$

for some $\alpha \in \text{rad}^2(N_3)$. Since g_1 is a ghost, the composite $g_1 f$ is stably trivial. It follows that, modulo $\text{soc}^2(N_1)$, $g_1(\tilde{z_1}) = X\alpha'$ and $g_1(\tilde{z_3}) = Y\alpha'$ for some $\alpha' \in N_1$. Since g_3g_2 is a double ghost, it kills $\text{soc}^2(N_1)$ and takes α' into $\text{rad}^2(N_3)$. Hence we can set $\alpha = g_3g_2(\alpha')$.

We still need to pick the $\tilde{z_1}$ and $\tilde{z_3}$. First choose $\tilde{z_1}'$ and $\tilde{z_3}'$ in N that project to $j(z_1)$ and $j(z_3)$ in $M((ab^{-1})^2)$, respectively. The difference $Y\tilde{z_1}' - X\tilde{z_3}'$ is in $\text{rad}^2(N)$, say $Y\tilde{z_1}' - X\tilde{z_3}' = Y\beta - X\gamma$ for some β and $\gamma \in \text{rad}(N)$. We set $\tilde{z_1} = \tilde{z_1}' - \beta$ and $\tilde{z_3} = \tilde{z_3}' - \gamma$ so that $Y\tilde{z_1} = X\tilde{z_3}$. By Corollary 2.4.3, $g(\beta) = g(\gamma) = 0$, hence

$$g(\tilde{z_1}) = g(\tilde{z_1}') = g_3 h p_2 j(z_1) \quad \text{and} \quad g(\tilde{z_3}) = g(\tilde{z_3}') = g_3 h p_2 j(z_3).$$

Case 4: $M = k$ is trivial.

Then clearly $g_3 \circ h \circ p_2$ is stably trivial when restricted to M, since g_3 is a ghost. □

Since we don't require the modules N_1, N_2, and N_3 to be finitely-generated in the proof, we have actually proved a stronger result, a bound for the generating number, giving:

Corollary 2.4.29. *Let k be a field of characteristic 3. Then the generating number of $k(C_3 \times C_3)$ is 3.* □

Remark 2.4.30. The arguments in this section go through for the group $G = C_{p^r} \times C_{p^s}$ with $2 < p^r \leqslant p^s$, and we get that the generating number of kG is less than or equal to $p^r + p^s - 3$. Theorem 2.4.9 gives a lower bound of $\left\lceil \frac{p^r-1}{2} \right\rceil + p^s - 1$. In particular, if $p^r = 3$, the ghost number of kG is p^s, and if $p^r = 4$, the ghost number of kG is $p^s + 1$.

We now indicate the modifications needed in the proof of the general case. Instead of g_2 being a ghost, we take it to be a $(p^r + p^s - 5)$-fold ghost. Then the map h has domain $N/\mathrm{soc}^{p^r+p^s-4}(N)$. In Case 1, one checks that $p_1^{-1}j(\mathrm{soc}(M)) \subseteq \mathrm{soc}^{p^r+p^s-4}(N)$. In Case 2, the map g_3hp_2 sends $z \in M(ab^{-1})$ to an element of the form $X^{p^r-1}Y^{p^s-2}w_1 + X^{p^r-2}Y^{p^s-1}w_2$. In Case 3, a map out of $M((ab^{-1})^2)$ is stably trivial if it sends z_1 to $X\alpha$ and z_3 to $Y\alpha$ for some α in the $(p^r + p^s - 4)^{th}$ radical. Case 4 is unchanged.

2.4.8 Possible ghost numbers for group algebras

In this Section, we classify group algebras with certain small ghost numbers, and also put constraints on which ghost numbers can occur. Whenever we write kG, k can be any field whose characteristic divides the order of G.

In [19] it is shown that the abelian groups G such that the ghost number of kG is 2 are C_4, $C_2 \times C_2$ and C_5. The results of the previous section and Theorem 2.4.9 give a complete list of abelian p-groups of ghost number 3:

Proposition 2.4.31. *Let G be an abelian p-group. Then the ghost number of kG is 3 if and only if G is C_7, $C_3 \times C_3$, or $C_2 \times C_2 \times C_2$ if and only if the generating number of kG is 3.* □

Below we will extend this to non-abelian p-groups, with one ambiguous group. We first recall a consequence of Jennings' formula which will also be useful in studying the gaps in the possible ghost numbers.

Lemma 2.4.32 ([4, Thm. 3.14.6])**.** *Let k be a field of characteristic p. If G is a group of order p^r, then*

$$\begin{aligned} \text{nilpotency index of } J(k(C_p^r)) &\leqslant \text{nilpotency index of } J(kG) \\ &\leqslant \text{nilpotency index of } J(k(C_{p^r})). \end{aligned}$$

Note that the nilpotency index of $J(k(C_p^r))$ is $r(p-1)+1$.

Proposition 2.4.33. *Let k be a field of characteristic p. If G is a group of order p^r, then the ghost number of kG is at least $(r-1)(p-1)+1$.*

Proof. The group G has a quotient H of order p^{r-1}. By Theorem 2.4.15, $\mathrm{rad\,len}\ (kH)$ is a lower bound for the ghost number of kG. Now by the previous lemma, $\mathrm{rad\,len}\ (kH) \geqslant (r-1)(p-1)+1$, so we are done. □

Theorem 2.4.34. *The following is a complete list of the p-groups G such that kG has the specified ghost number:*

1: the abelian groups C_2 and C_3;

2: the abelian groups C_4, $C_2 \times C_2$ and C_5;

3: the abelian groups C_7, $C_3 \times C_3$ and $C_2 \times C_2 \times C_2$, the dihedral group D_8 of order 8, and possibly the quaternion group Q_8, which has ghost number 3 or 4.

In each case, except possibly for Q_8, the generating number equals the ghost number.

Proof. The case of ghost number 1 is the main result of [9].

A non-abelian p-group must have order p^r for $r \geqslant 3$, so by Proposition 2.4.33 it must have ghost number at least 3. Thus a p-group of ghost number 2 must be abelian, and this case is proved in [19].

The only ways for $(r-1)(p-1)+1$ to equal 3 are $p^r = 8$ or 9. The non-abelian groups of order 8 are D_8 and Q_8, which are discussed in Corollary 2.4.25, Theorem 2.4.28 and Corollary 2.4.29, and there are no non-abelian groups of order 9. The abelian case is Proposition 2.4.31. □

Next we observe that, for a fixed prime p, not all positive integers can be the ghost number of some p-group. For example, since the generating hypothesis fails for $p > 3$, the number 1 cannot be the ghost number of a p-group with $p > 3$. On the other hand, the elementary abelian 2-group of rank l has ghost number $l-1$, so every positive integer can be a ghost number at the prime 2. Here is a result giving gaps in the possible ghost numbers at odd primes.

Theorem 2.4.35. *Let p be an odd prime, and let k be a field of characteristic p. Write $(l_1, l_2, l_3, \cdots)$ for the increasing sequence of integers that are ghost numbers of the group algebras kG, with G being a p-group. Then $l_1 = \frac{p-1}{2}$,*

$$\frac{3(p-1)}{2} \leqslant l_2 = \text{ghost number of } C_p \times C_p \leqslant 2p-3,$$

and $\min(\frac{p^2-1}{2}, 2p-1) \leqslant l_3$.

Proof. We know that the ghost number of C_p is $\frac{p-1}{2}$ and that of C_{p^2} is $\frac{p^2-1}{2}$ [19, Thm. 5.4]. And the ghost number of $C_p \times C_p$ is constrained by Theorems 2.4.9 and Remark 2.4.30:

$$\frac{3(p-1)}{2} \leqslant \text{ghost number of } C_p \times C_p \leqslant 2p-3.$$

By Proposition 2.4.33, the groups of order p^r with $r \geqslant 3$ have ghost numbers at least $2p-1$. Comparing these numbers, we get

$$\frac{p-1}{2} \leqslant 2p-3 \leqslant \min(\frac{p^2-1}{2}, 2p-1),$$

and the theorem follows. □

Thus one sees that for large primes there are large gaps in the sequence of possible ghost numbers.

Observe that when $p \geqslant 5$,

$$\text{the ghost number of } k(C_p^3) \leqslant 3p-3 \leqslant \frac{p^2-1}{2} = \text{the ghost number of } kC_{p^2},$$

where the first inequality uses Theorem 2.4.1. And by Theorem 2.4.9 and Proposition 2.4.33, the ghost number of $k(C_p^r)$ is no more than the ghost number of any p-group with larger size. We conjecture that this is also true for groups of the same size, which would imply that l_3 is the ghost number of $k(C_p^3)$ when $p \geqslant 5$. The following conjecture should be viewed as the stabilized version of Lemma 2.4.32.

Conjecture 2.4.36. *Let k be a field of characteristic p. If G is a p-group of order p^r, then*

$$\textit{ghost number of } k(C_p^r) \leqslant \textit{ghost number of } kG \leqslant \textit{ghost number of } k(C_{p^r}).$$

Chapter 3

Ghost numbers of group algebras II

3.1 Introduction

In this paper, we study several closely related invariants of a group algebra kG, where G is a finite group, and k is a field whose characteristic p divides the order of G. To describe these invariants, we work in the stable module category $\mathsf{StMod}(kG)$, which is the triangulated category formed from the category of kG-modules by killing the maps that factor through a projective. A map f in $\mathsf{StMod}(kG)$ is called a **ghost** if it induces the zero map in Tate cohomology, or equivalently, if $\underline{\mathrm{Hom}}(\Omega^i k, f) = 0$ for each $i \in \mathbb{Z}$. Our most basic invariant is the **ghost number** of kG, which is the smallest n such that every composite of n ghosts in $\mathsf{Thick}\langle k \rangle$ is zero. Here $\mathsf{Thick}\langle k \rangle$ denotes the thick subcategory generated by the trivial module. When there are no non-trivial ghosts in $\mathsf{Thick}\langle k \rangle$ (so $n = 1$), we say that the **generating hypothesis** holds for kG. This is motivated the Freyd's generating hypothesis in stable homotopy theory [25], which is still an open question. In a series of papers [9, 16, 18, 20] (with a minor correction made below), it has been shown that the generating hypothesis holds for kG if and only if the Sylow p-subgroup of G is C_2 or C_3. However, computing the ghost number in cases where it is larger than one has proven to be difficult. Some preliminary work was done in [19], where the ghost numbers of cyclic p-groups were computed, and various upper and lower bounds were obtained in other cases. Substantial progress was made in our previous paper [23], where we computed the ghost numbers of $k(C_3 \times C_3)$ and other algebras of wild representation type, as well as the ghost numbers of dihedral 2-groups, the first non-abelian computations.

In this chapter, we extend the past work in two different ways. Our initial motivation was to produce the first computations of ghost numbers for non-p-groups. For a p-group, $\mathsf{Thick}\langle k \rangle$ coincides with $\mathsf{stmod}(kG)$, the full subcategory of finitely generated modules, which allows one to use induction from a subgroup to produce ghosts in $\mathsf{Thick}\langle k \rangle$. But for a general p-group, $\mathsf{Thick}\langle k \rangle$ is usually a proper subcategory of $\mathsf{stmod}(kG)$, which makes things more delicate. Nevertheless, we obtain a variety of exact computations of ghost numbers in this setting, e.g., for all dihedral groups at all primes, as well as new bounds. One of our new techniques is to produce ghosts for kG by inducing up a ghost from a subgroup and then projecting onto the principal block. We show that this composite is faithful, and so when $\mathsf{Thick}\langle k \rangle$ coincides with the principal block of $\mathsf{stmod}(kG)$, we are able to use this technique to study the ghost number of kG. As an example, we prove that the ghost number is finite in this situation. Our main results on ghost numbers are described in the detailed summary below.

Our work on non-p-groups led us to realize the importance of another invariant in this setting, which is the **simple ghost number**, a concept suggested in [12]. A **simple ghost** is a map f such that $\underline{\mathrm{Hom}}(\Omega^i S, f) = 0$ for each simple module S and each $i \in \mathbb{Z}$, and the **simple ghost number** of kG is the smallest n such that every composite of n simple ghosts in $\mathsf{stmod}(kG)$ is trivial. The point here is that $\mathsf{stmod}(kG)$ is the thick subcategory generated by the simple modules, so this is exactly analogous to the ghost number, with the trivial module k replaced by the set of all simple modules. Moreover, for a p-group, k is the only simple module, so the two notions coincide. In turns out that there is a close relationship between the simple ghost number of kG and the ghost number of kP, where P is a Sylow p-subgroup of G, and by studying both invariants at once we can make many more computations. Again, these are described in the detailed summary below.

One of the most important techniques in our work is the use of induction and restriction, which brings us to the third and final invariant that we study in this paper. A **strong ghost** is a map f whose restriction to every subgroup is a ghost, or equivalently, such that $\underline{\mathrm{Hom}}(\Omega^i k{\uparrow}_H^G, f) = 0$ for each subgroup H of G and each $i \in \mathbb{Z}$. The **strong ghost number** of kG is the smallest n such that every composite of n strong ghosts in $\mathsf{stmod}(kG)$ is trivial. This follows the same pattern as above, since $\mathsf{stmod}(kG)$ is the thick subcategory generated by the test objects $k{\uparrow}_H^G$. Unlike the other invariants, one can show that the strong ghost number of kG equals the strong ghost number of kP, and so it suffices to study p-groups. Below we summarize our computations of and bounds on strong ghost numbers.

The overall organization of the paper is as follows. In Section 3.2, we introduce general concepts that will be of use in the rest of the paper and recall some background material on modular representation theory. Sections 3.3, 3.4 and 3.5 study both the ghost number and the simple ghost number, and are distinguished by the assumptions placed on the group: In Section 3.3, we assume that the Sylow p-subgroup of G is normal. In Section 3.4, we assume that $\mathsf{Thick}\langle k\rangle$ coincides with the principal block. And in Section 3.5, we assume that the Sylow p-subgroup is cyclic. Finally, in Section 3.6, we study the strong ghost number.

Note that there is some overlap in the assumptions made in Sections 3.3, 3.4 and 3.5. For example, in Section 3.4.1 we study groups whose Sylow p-subgroup is a direct factor, and these groups satisfy the assumptions of Sections 3.3 and 3.4. This includes the case of p-groups. And in Section 3.5.1, we study groups with a cyclic normal Sylow p-subgroup, and these satisfy the assumptions of all three sections. In general, the assumptions made are independent, except that Sunil Chebolu and Jan Mináč have an unpublished proof that when the Sylow p-subgroup is cyclic, $\mathsf{Thick}\langle k\rangle$ coincides with the principal block. (This may be one of those results that is "known to the experts".)

We now summarize the main results of each section in more detail. In Section 3.2.1, working in a general triangulated category, we define the Freyd length and Freyd number with respect to a set $\mathbb{P}$ of test objects. The Freyd number generalizes the ghost number, simple ghost number and strong ghost number defined above. We also recall the closely related concept of length with respect to a projective class, and we prove general results about both of these invariants. In Section 3.2.2, we recall the basics of the stable module category, and in Section 3.2.3 we formally introduce ghosts and simple ghosts, specializing the Freyd length and Freyd number to these two situations.

In Section 3.3 we assume that our group G has a normal Sylow p-subgroup P. Under this assumption, in Section 3.3.1 we show that a map in $\mathsf{StMod}(kG)$ is a simple ghost if and only if its restriction to P is a ghost, and show that the simple ghost number of kG is equal to the ghost number of kP. It follows that when P is normal, the simple generating hypothesis holds if and only if P is C_2 or C_3. (We don't have a characterization of when the simple generating hypothesis holds in general, but we do know that it does not depend only on the Sylow p-subgroup. See Section 3.5.2.) In Section 3.3.2, we apply this result to the group A_4 at the prime 2, deducing that the simple ghost number is 2 and that the ghost number is between 2 and 4. We also give an example of a ghost for A_4 whose restriction to the Sylow p-subgroup is not a ghost.

In Section 3.4, we focus on groups whose principal block is generated by k in the sense that $\mathsf{stmod}(B_0) = \mathsf{Thick}\langle k\rangle$ (or, equivalently, $\mathsf{StMod}(B_0) = \mathsf{Loc}\langle k\rangle$). We show that this holds when the Sylow p-subgroup P is a direct factor, in Section 3.4.1, using a result that shows that there is an equivalence between $\mathsf{stmod}(kP)$ and $\mathsf{Thick}_G\langle k\rangle$. This last result corrects an error in [20]; see the comments after Theorem 3.4.1. In Section 3.4.2, we show that if $\mathsf{stmod}(B_0) = \mathsf{Thick}\langle k\rangle$, then the ghost number of kG is finite. We prove this by using a comparison to the simple ghost number, which is finite for any G. We conjecture that the ghost number is finite for general G. This is related to a question proposed in [8]. (See Remark 3.4.8.) Still assuming that the principal block is generated by k, we show that the ghost number of kG is greater than or equal to the ghost number of kP, by first showing that the composite of inducing up from P to G followed by projection onto the principal block is faithful. In Section 3.4.3, working at the prime 2, we show that for a dihedral group D_{2ql} of order $2ql$, with q a power of 2 and l odd, the principal block is generated by k and the ghost number of D_{2ql} is equal to the ghost number of the Sylow 2-subgroup D_{2q}, which was shown to be $\lfloor \frac{q}{2} + 1 \rfloor$ in [23]. By computing the simple ghost lengths of modules in non-principal blocks, we are also able to show that the simple ghost number of D_{2ql} is again $\lfloor \frac{q}{2} + 1 \rfloor$.

Section 3.5 studies the case when the Sylow p-subgroup P is cyclic. In Section 3.5.1, we assume that P is cyclic and normal, and show that every simple module in the principal block is a suspension of the trivial module. It follows that $\mathsf{stmod}(B_0) = \mathsf{Thick}\langle k \rangle$ and that a map in $\mathsf{Thick}\langle k \rangle$ is a ghost if and only if it is a simple ghost. Thus the simple ghost number of kG, the ghost number of kG and the ghost number of kP are all equal. Since P is a cyclic p-group, its ghost number is known [19]. In particular, this allows us to compute the ghost numbers of the dihedral groups at an odd prime. Combined with the results above, this completes the computation of the ghost numbers of the dihedral groups, at any prime. The group $SL(2,p)$ has a cyclic Sylow p-subgroup P, but it is not normal. By studying the normalizer L of P and applying the results of Section 3.5.1 to L, we show in Section 3.5.2 that the simple generating hypothesis holds for $SL(2,p)$ over a field k of characteristic p. Along the way, we find that there is an equivalence $\mathsf{stmod}(kG) \to \mathsf{stmod}(kL)$, but that the simple generating hypothesis does *not* hold for kL.

In Section 3.6 we study strong ghosts. We begin in Section 3.6.1 by showing that the strong ghost number of a group algebra kG equals the strong ghost number of kP, where P is a Sylow p-subgroup of G. Then we compute the strong ghost numbers of cyclic p-groups in Section 3.6.2. Finally, in Section 3.6.3, we show that the strong ghost number of a dihedral 2-group D_{4q} is between 2 and 3, with the upper bound being the non-trivial result.

3.2 Background

In this section, we provide background material that will be used throughout the paper. In Section 3.2.1, we define invariants of a triangulated category T which depend on a set $\mathbb{P}$ of test objects, and prove general results about these invariants. In Section 3.2.2, we recall some background results about the stable module category of a finite group. In Section 3.2.3, we apply the general theory to two sets of test objects in the stable module category of a group, giving rise to invariants called the ghost number and the simple ghost number.

3.2.1 The generating hypothesis and related invariants

We begin this section by stating the generating hypothesis with respect to a set of objects in a triangulated category and defining invariants, the Freyd length and the length, which measure the degree to which the generating hypothesis fails. Motivated by this, we recall the definition of a projective class. Then, working in a general triangulated category, we study the relationship between the lengths (and Freyd lengths) of an object with respect to different projective classes. We also compare lengths in different categories by using the pullback projective class.

Let T be a triangulated category, and let $\mathbb{P}$ be a set of objects in T. The thick subcategory generated by $\mathbb{P}$, denoted $\mathsf{Thick}\langle\mathbb{P}\rangle$, is the smallest full triangulated subcategory of T that is closed under retracts and contains $\mathbb{P}$. It is easy to see that $\mathbb{P}$ detects zero objects in $\mathsf{Thick}\langle\mathbb{P}\rangle$, i.e., if $M \in \mathsf{Thick}\langle\mathbb{P}\rangle$ and $[\Sigma^i\mathbb{P}, M] = 0$ for all $P \in \mathbb{P}$ and $i \in \mathbb{Z}$, then $M \cong 0$. Here we write $[-,-]$ for the hom-sets in T.

The **generating hypothesis** for the set of test objects $\mathbb{P}$ is the statement that $\mathbb{P}$ detects trivial *maps* in $\mathsf{Thick}\langle\mathbb{P}\rangle$, i.e., if f is a map in $\mathsf{Thick}\langle\mathbb{P}\rangle$ and $[\Sigma^i P, f] = 0$ for all $P \in \mathbb{P}$ and $i \in \mathbb{Z}$, then f is the zero map [12].

When the generating hypothesis for $\mathbb{P}$ fails, there is a natural invariant which measures the degree to which it fails. Let $\mathcal{I}$ denote the class of maps such that $[\Sigma^i P, f] = 0$ for all $P \in \mathbb{P}$ and $i \in \mathbb{Z}$, and write $\mathcal{I}_t$ for such maps in $\mathsf{Thick}\langle\mathbb{P}\rangle$. The **Freyd length** $\mathrm{len}^{\mathrm{F}}_{\mathbb{P}}(X)$ of an object X in $\mathsf{Thick}\langle\mathbb{P}\rangle$ with respect to $\mathbb{P}$ is the smallest number n such that every composite $X \to X_1 \to \cdots \to X_n$ of n maps in $\mathcal{I}_t$ is zero. The **Freyd number** of T with respect to $\mathbb{P}$ is the least upper bound of the Freyd lengths of the objects in $\mathsf{Thick}\langle\mathbb{P}\rangle$. With this terminology, the generating hypothesis holds for $\mathbb{P}$ if and only if the Freyd number of T with respect to $\mathbb{P}$ is 1.

It turns out to be fruitful to consider a related invariant, where none of the objects are required to lie in $\mathsf{Thick}\langle\mathbb{P}\rangle$. The **length** $\mathrm{len}_{\mathbb{P}}(X)$ of an object X in T with respect to $\mathbb{P}$ is the smallest number n such that every composite $X \to X_1 \to \cdots \to X_n$ of n maps in $\mathcal{I}$ is zero, if this exists (which is the case when $X \in \mathsf{Thick}\langle\mathbb{P}\rangle$). This is clearly at least as big as the Freyd length, but has better formal properties which make it easier to work with. These properties are best expressed in terms of the *projective class* generated by $\mathbb{P}$. To motivate the definition, note that $\langle\mathbb{P}\rangle$ detects the same maps in T as $\mathbb{P}$ does, where $\langle\mathbb{P}\rangle$ denotes the closure of $\mathbb{P}$ under retracts, sums, suspensions and desuspensions. Moreover, it is easy to show ([21]) that $\mathcal{P} := \langle\mathbb{P}\rangle$ and $\mathcal{I}$ determine each other in the sense of the following definition:

Definition 3.2.1. Let T be a triangulated category. A **projective class** in T consists of a class $\mathcal{P}$ of objects of T and a class $\mathcal{I}$ of morphisms of T such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{I}$,

(ii) $\mathcal{I}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$.

(iii) for each X in T, there is a triangle $P \to X \to Y \to \Sigma P$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{I}$.

Our main examples will be projective classes of the form $(\langle\mathbb{P}\rangle, \mathcal{I})$, which we call the **(stable) projective class generated by** $\mathbb{P}$.

Given a projective class $(\mathcal{P}, \mathcal{I})$, there is a sequence of **derived projective classes** $(\mathcal{P}_n, \mathcal{I}^n)$ [21]. The ideal $\mathcal{I}^n$ consists of all n-fold composites of maps in $\mathcal{I}$, and X is in $\mathcal{P}_n$ if and only if it is a retract of an object M that sits inside a triangle $P \to M \to Q \to \Sigma P$ with $P \in \mathcal{P}_1 = \mathcal{P}$ and $Q \in \mathcal{P}_{n-1}$. For $n = 0$, we let $\mathcal{P}_0$ consist of all zero objects and $\mathcal{I}^0$ consist of all maps in T.

Extending the definition above to any projective class, we define the **length** $\mathrm{len}_{\mathcal{P}}(X)$ of an object X in T with respect to $(\mathcal{P}, \mathcal{I})$ to be the smallest number n such that every map in $\mathcal{I}^n$ with domain X is trivial. The fact that each pair $(\mathcal{P}_n, \mathcal{I}^n)$ is a projective class implies that the length of X is equal to the smallest n such that $X \in \mathcal{P}_n$. When $\mathcal{P} = \langle\mathbb{P}\rangle$, we write $\mathrm{len}_{\mathbb{P}}(X)$ as above.

We note that different sets of objects can generate the same projective class but different thick subcategories, so the Freyd length depends on the choice of generating set $\mathbb{P}$, not just on the projective class $\langle\mathbb{P}\rangle$ it generates.

The following lemma is a direct consequence of the definition of a projective class. This idea is used in comparing the ghost length and the simple ghost length of a module.

Lemma 3.2.2. *Let T be a triangulated category, and let $(\mathcal{P}, \mathcal{I})$ and $(\mathcal{Q}, \mathcal{J})$ be projective classes on T. Then we have the following relationships:*

- *If M has finite length with respect to $(\mathcal{P}, \mathcal{I})$, then*
$$\operatorname{len}_{\mathcal{P}_n}(M) = \left\lceil \frac{\operatorname{len}_{\mathcal{P}}(M)}{n} \right\rceil.$$

- *If $\mathcal{Q} \subseteq \mathcal{P}$, then*
$$\operatorname{len}_{\mathcal{P}}(M) \leqslant \operatorname{len}_{\mathcal{Q}}(M).$$

- *If $\mathcal{Q} \subseteq \mathcal{P}_n$, then*
$$\operatorname{len}_{\mathcal{P}}(M) \leqslant n \operatorname{len}_{\mathcal{P}_n}(M) \leqslant n \operatorname{len}_{\mathcal{Q}}(M).$$

Proof. To show $\operatorname{len}_{\mathcal{P}_n}(M) = \left\lceil \frac{\operatorname{len}_{\mathcal{P}}(M)}{n} \right\rceil$, we actually need to prove two inequalities:

$$\operatorname{len}_{\mathcal{P}_n}(M) \leqslant \left\lceil \frac{\operatorname{len}_{\mathcal{P}}(M)}{n} \right\rceil \quad \text{and} \quad \operatorname{len}_{\mathcal{P}}(M) \leqslant n \operatorname{len}_{\mathcal{P}_n}(M). \tag{3.2.1}$$

For the second inequality, let $\operatorname{len}_{\mathcal{P}_n}(M) = m$. Then $M \in (\mathcal{P}_n)_m \subseteq \mathcal{P}_{mn}$, which means that $\operatorname{len}_{\mathcal{P}}(M) \leqslant mn$. Equivalently, we can prove the inequality using the inclusion $\mathcal{I}^{mn} \subseteq (\mathcal{I}^n)^m$, i.e., if every m-fold composite of n-fold composites of maps in $\mathcal{I}$ out of M is trivial, then every mn-fold composite of maps in $\mathcal{I}$ out of M is trivial.

Using the inclusions the other way, i.e., $(\mathcal{P}_n)_m \supseteq \mathcal{P}_{mn}$ and $\mathcal{I}^{mn} \subseteq (\mathcal{I}^n)^m$, one can prove that $\operatorname{len}_{\mathcal{P}_n}(M) \leqslant \lceil \operatorname{len}_{\mathcal{P}}(M)/n \rceil$.

The other inequalities in the lemma follow with similar proofs. □

The analog of Lemma 3.2.2 for Freyd lengths is a bit more subtle because of the need to take into account the appropriate thick subcategories. For example, if $(\langle \mathbb{P} \rangle, \mathcal{I})$ and $(\langle \mathbb{Q} \rangle, \mathcal{J})$ are projective classes and $\mathbb{Q} \subseteq \mathbb{P}$, then clearly $\mathcal{I} \subseteq \mathcal{J}$. But the inclusion $\mathsf{Thick}\langle \mathbb{Q} \rangle \subseteq \mathsf{Thick}\langle \mathbb{P} \rangle$ goes in the other direction, so in general there is no inclusion between $\mathcal{I}_t$ and $\mathcal{J}_t$.

Nevertheless, if we include assumptions which control the thick subcategories, then most of the results go through. We simply work with $(\mathcal{I}_t)^n$ instead of $\mathcal{I}^n$. However, one difference is that we only have an inclusion $(\mathcal{I}_t)^{mn} \subseteq ((\mathcal{I}^n)_t)^m$, rather than an equality, and as a result, we lose the first inequality from equation (3.2.1). In the next lemma, we give a result which we will use later.

Lemma 3.2.3. *Let* T *be a triangulated category, and let* $(\langle \mathbb{P} \rangle, \mathcal{I})$ *and* $(\langle \mathbb{Q} \rangle, \mathcal{J})$ *be projective classes on* T *generated by sets* $\mathbb{P}$ *and* $\mathbb{Q}$*. If* $\mathbb{P} \subseteq \mathsf{Thick}\langle \mathbb{Q} \rangle$*,* $\mathbb{Q} \subseteq \langle \mathbb{P} \rangle_n$ *and* $M \in \mathsf{Thick}\langle \mathbb{P} \rangle$*, then*

$$\operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M) \leqslant n \operatorname{len}^{\mathrm{F}}_{\mathbb{Q}}(M).$$

Proof. Let $m = \operatorname{len}^{\mathrm{F}}_{\mathbb{Q}}(M)$. We must show that any composite $M = M_0 \to M_1 \to \cdots \to M_{mn}$ of maps in $\mathcal{I}$ with the M_i in $\mathsf{Thick}\langle \mathbb{P} \rangle$ is zero. The inclusion $\mathbb{P} \subseteq \mathsf{Thick}\langle \mathbb{Q} \rangle$ tells us that $\mathsf{Thick}\langle \mathbb{P} \rangle \subseteq \mathsf{Thick}\langle \mathbb{Q} \rangle$, so these maps are in $\mathsf{Thick}\langle \mathbb{Q} \rangle$. The inclusion $\mathbb{Q} \subseteq \langle \mathbb{P} \rangle_n$ tells us that $\mathcal{I}^n \subseteq \mathcal{J}$. Thus the above composite is an m-fold composite of maps in $\mathcal{J} \cap \mathsf{Thick}\langle \mathbb{Q} \rangle$, and so is zero by the definition of m. $\square$

Consider a triangle

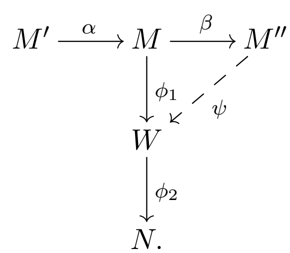
$$M' \longrightarrow M \longrightarrow M'' \longrightarrow \Sigma M'$$

in T. We know that $\operatorname{len}(M) \leqslant \operatorname{len}(M') + \operatorname{len}(M'')$ by [21, Note 3.6]. We will prove the analog for Freyd lengths.

Lemma 3.2.4. *Let* T *be a triangulated category with a set of test objects* $\mathbb{P}$ *and* $\mathcal{I}_t$ *be the class of maps in* $\mathsf{Thick}\langle \mathbb{P} \rangle$ *that are trivial on* $\mathbb{P}$*. Let* $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to \Sigma M'$ *be a triangle in* T*. If* M' *and* M'' *have finite Freyd lengths, then*

$$\operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M) \leqslant \operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M') + \operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M'').$$

Proof. Let $n = \operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M')$ and $l = \operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M'')$. We want to show that any map $\phi : M \to N$ in $(\mathcal{I}_t)^{n+l}$ is trivial. Write ϕ as $\phi_2\phi_1$, where ϕ_1 is in $(\mathcal{I}_t)^n$ and ϕ_2 is in $(\mathcal{I}_t)^l$. Then, since $\operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M') = n$, the composite $\phi_1\alpha$ is stably trivial and ϕ_1 factors through M'':



Now since $\operatorname{len}^{\mathrm{F}}_{\mathbb{P}}(M'') = l$, the composite $\phi_2\psi$ is trivial and so ϕ is trivial as well. $\square$

Now we explain how to compare lengths in different categories, using the pullback projective class.

Definition 3.2.5. Let $U : \mathsf{T} \to \mathsf{S}$ be a triangulated functor between triangulated categories, together with a left adjoint $F : \mathsf{S} \to \mathsf{T}$ that is also triangulated, and let $(\mathcal{P}, \mathcal{I})$ be a projective class on S. We define

$$\mathcal{I}' := \{M \to N \text{ in } \mathsf{T} \text{ such that } UM \to UN \text{ is in } \mathcal{I}\} = U^{-1}(\mathcal{I}).$$

Then $\mathcal{I}'$ forms the ideal of a projective class on T with relative projectives

$$\mathcal{P}' = \{\text{retracts of } FP \text{ for } P \text{ in } \mathcal{P}\} = \langle F(\mathcal{P})\rangle.$$

The projective class $(\mathcal{P}', \mathcal{I}')$ on T is called the **pullback** of $(\mathcal{P}, \mathcal{I})$ along the right adjoint U [22]. It is the projective class on T generated by the class of objects $F(\mathcal{P})$.

One readily sees that the following relationships hold, since F sends $\mathcal{P}$ into $\mathcal{P}'$ and U sends $\mathcal{I}'$ into $\mathcal{I}$.

Lemma 3.2.6. *Suppose we are in the above situation and that $M \in \mathsf{S}$ and $N \in \mathsf{T}$. Then*

$$\operatorname{len}_{\mathcal{P}}(M) \geqslant \operatorname{len}_{\mathcal{P}'}(FM),$$

and, if the functor U is faithful,

$$\operatorname{len}_{\mathcal{P}'}(N) \leqslant \operatorname{len}_{\mathcal{P}}(UN). \qquad \square$$

3.2.2 The stable module category

Let G be a finite group, and let k be a field whose characteristic p divides the order of G. The stable module category $\mathsf{StMod}(kG)$ is a quotient category of the module category $\mathsf{Mod}(kG)$. For kG-modules M and N, the hom-set $\underline{\operatorname{Hom}}(M, N)$ in $\mathsf{StMod}(kG)$ is the quotient $\operatorname{Hom}(M, N)/\operatorname{PHom}(M, N)$, where $\operatorname{PHom}(M, N)$ consists of the maps that factor through a projective module. Then $\mathsf{StMod}(kG)$ is a triangulated category with triangles coming from short exact sequences in $\mathsf{Mod}(kG)$. Two modules M and N are said to be **stably isomorphic** if they are isomorphic in the stable module category, and this holds if and only if their projective-free summands are isomorphic as kG-modules.

We use the symbol $\cong$ for isomorphism as kG-modules, unless otherwise stated. The desuspension ΩM of a module M is defined to be the kernel in any short exact sequence

$$0 \longrightarrow \Omega M \longrightarrow Q \longrightarrow M \longrightarrow 0,$$

where Q is a projective kG-module. Note that ΩM is well-defined in the stable module category, and we denote by $\widetilde{\Omega} M$ the projective-free summand of ΩM. We write $\mathsf{stmod}(kG)$ for the full subcategory of finitely generated modules in $\mathsf{StMod}(kG)$. (More precisely, we include all modules which are stably isomorphic to finitely generated kG-modules.) We refer to [14] for more background on $\mathsf{StMod}(kG)$.

Now let P be a Sylow p-subgroup of G. We consider the adjunction

$$\uparrow^G : \mathsf{StMod}(kP) \rightleftarrows \mathsf{StMod}(kG) : \mathrm{res} = \downarrow_P,$$

with $\uparrow^G$ as a left adjoint. We quote the following important facts in modular representation theory for further use:

Lemma 3.2.7 ([4])**.** *Let G be a finite group, let k be a field whose characteristic p divides the order of G, and let P be a Sylow subgroup of G. Then the following hold:*

(i) The restriction functor $\downarrow_P : \mathsf{StMod}(kG) \to \mathsf{StMod}(kP)$ is faithful.

(ii) Each kG-module M is a summand of the module $M\downarrow_P\uparrow^G$.

(iii) A kG-module Q is projective if and only if its restriction $Q\downarrow_P$ is projective. □

Theorem 3.2.8 (Mackey's Theorem [4])**.** *Let L and H be subgroups of G, and let V be a kH-module. Then*

$$(V\uparrow_H^G)\downarrow_L \;\cong \bigoplus_{s\in L\backslash G/H} (sV)\downarrow_{L\cap sHs^{-1}}\uparrow^L.$$

Here $sV = s \otimes V$ is the corresponding $k(sHs^{-1})$-module for $s \in G$, and the sum is taken over the double coset representatives. □

3.2.3 Ghost lengths and simple ghost lengths in $\mathsf{StMod}(kG)$

By the **generating hypothesis** on $\mathsf{StMod}(kG)$, we mean the generating hypothesis with respect to the set $\{k\}$ containing only the trivial module. Since Tate cohomology is represented by k, the associated ideal $\mathcal{G}$ consists of **ghosts** in $\mathsf{StMod}(kG)$, i.e., maps which induce the zero map in Tate cohomology. Thus the generating hypothesis is the statement that there are no non-trivial maps in $\mathsf{Thick}\langle k\rangle$ which induce the zero map in Tate cohomology. The projective class $(\mathcal{F}, \mathcal{G})$ generated by k has $\mathcal{F} = \langle k\rangle$, summands of direct sums of suspensions and desuspensions of k. We call $(\mathcal{F}, \mathcal{G})$ the **ghost projective class**. When we need to indicate the dependence on the group, we write $(\mathcal{F}^G, \mathcal{G}_G)$.

For a module $M \in \mathsf{Thick}\langle k\rangle$, its **ghost length** $\mathrm{gl}(M)$ is defined to be its Freyd length with respect to $\{k\}$, and the **ghost number** of kG is the Freyd number of $\mathsf{StMod}(kG)$ with respect to $\{k\}$. With this terminology, the generating hypothesis is the statement that the ghost number of kG is 1.

Since the restriction functor preserves the trivial module, we can induce up a (non-trivial) ghost from a subgroup of G to get a (non-trivial) ghost of G. This provides a very convenient tool when we study p-groups. However, the inducing up technique has limited use for a general finite group, since the ghosts, when induced up, do not always land in $\mathsf{Thick}\langle k\rangle$, which is often smaller than $\mathsf{stmod}(kG)$.

In general, the stable module category is generated by the set $\mathbb{S}$ of simple modules. This suggests that we examine the projective class $(\mathcal{S}, \mathrm{s}\mathcal{G})$ generated by $\mathbb{S}$, which we call the **simple ghost projective class**, and compare it to the ghost projective class. Here $\mathcal{S} = \langle\mathbb{S}\rangle$, and the maps in $\mathrm{s}\mathcal{G}$ are called **simple ghosts**. The **simple generating hypothesis** for kG is the generating hypothesis with respect to $\mathbb{S}$. The Freyd length (respectively number) with respect to $\mathbb{S}$ will be called the **simple ghost length** (respectively **number**). For $M \in \mathsf{stmod}(kG)$, the simple ghost length is denoted by $\mathrm{sgl}(M) = \mathrm{len}^{\mathrm{F}}_{\mathbb{S}}(M)$. Note that while the ghost length is only defined for $M \in \mathsf{Thick}\langle k\rangle$, the simple ghost length is defined for all $M \in \mathsf{stmod}(kG)$ since $\mathsf{stmod}(kG) = \mathsf{Thick}\langle\mathbb{S}\rangle$. If G is a p-group, then $\mathbb{S} = \{k\}$, so the simple ghost projective class and the ghost projective class coincide.

Remark 3.2.9. The radical series of a kG-module M gives a construction of M using simple modules, showing that $\mathrm{len}_{\mathbb{S}}(M)$ is at most the radical length of M, since the pair $(\mathcal{S}_n, \mathrm{s}\mathcal{G}^n)$ is a projective class, as described in Section 3.2.1. Therefore,

$$\mathrm{sgl}(M) = \mathrm{len}^{\mathrm{F}}_{\mathbb{S}}(M) \leqslant \mathrm{len}_{\mathbb{S}}(M) \leqslant \mathrm{rad\,len}\ (M) \leqslant \mathrm{rad\,len}\ (kG).$$

This shows that the simple ghost number of kG is finite. In particular, for P a p-group, the ghost number of kP is finite. In Conjecture 3.4.9 we assert that the ghost number of kG is always finite, but this is an open question.

In the last section of the paper, we will study another projective class on $\mathsf{StMod}(kG)$, which is called the strong ghost projective class.

3.3 Groups with normal Sylow p-subgroups

In this section, we assume that our group G has a normal Sylow p-subgroup P. Under this assumption, in Section 3.3.1 we show that the simple ghost number of kG is equal to the ghost number of kP. In Section 3.3.2, we apply this result to the group A_4 at the prime 2, deducing that the simple ghost number is 2 and that the ghost number is between 2 and 4.

3.3.1 The simple projective class as a pullback

In this section, we show that the simple ghost projective class on $\mathsf{StMod}(kG)$ is the pullback of the ghost projective class on $\mathsf{StMod}(kP)$, under the assumption that the Sylow p-subgroup P is normal in G. Then we show that simple ghost lengths in $\mathsf{StMod}(kG)$ are the same as ghost lengths in $\mathsf{StMod}(kP)$. The main result of this section should be viewed as the stabilised version of the next lemma:

Lemma 3.3.1 ([1, Lemma 5.8])**.** *Let k be a field of characteristic p, and let G be a finite group with a normal Sylow p-subgroup P. Let M be a kG-module. Then $\operatorname{rad}(M){\downarrow_P} = \operatorname{rad}(M{\downarrow_P})$. It follows that the radical sequence of M coincides with that of $M{\downarrow_P}$. In particular, M is semisimple if and only if $M{\downarrow_P}$ is.* □

We write $(\langle \mathcal{F}^P{\uparrow^G}\rangle, \operatorname{res}^{-1}(\mathcal{G}_P))$ for the pullback of $(\mathcal{F}^P, \mathcal{G}_P)$ along the restriction functor. Then, by Lemma 3.2.7(ii), we have $\mathcal{F}^G \subseteq \langle \mathcal{F}^P{\uparrow^G}\rangle$. Equivalently, $\operatorname{res}^{-1}(\mathcal{G}_P) \subseteq \mathcal{G}_G$, i.e., if a map in $\mathsf{StMod}(kG)$ restricts to a ghost in $\mathsf{StMod}(kP)$, then it is a ghost. (Note that we write ${\downarrow_P}$ for the restriction functor except when considering preimages, in which case we write res^{-1}.) We can describe $\operatorname{res}^{-1}(\mathcal{G}_P)$ more precisely when P is normal in G.

Theorem 3.3.2. *Let k be a field of characteristic p, and let G be a finite group with a normal Sylow p-subgroup P. Then the projective classes $(\mathcal{S}, \mathrm{s}\mathcal{G})$ and $(\langle \mathcal{F}^P{\uparrow^G}\rangle, \operatorname{res}^{-1}(\mathcal{G}_P))$ on $\mathsf{StMod}(kG)$ coincide, and for $M \in \mathsf{stmod}(kG)$ and $L \in \mathsf{stmod}(kP)$, we have*

$$\operatorname{sgl}(M) = \operatorname{gl}(M{\downarrow_P}) \quad \text{and} \quad \operatorname{gl}(L) = \operatorname{sgl}(L{\uparrow^G}).$$

Hence

$$\textit{simple ghost number of } kG = \textit{ghost number of } kP.$$

In particular, the simple generating hypothesis holds for kG if and only if $P \cong C_2$ or $P \cong C_3$.

The first claim of the theorem is saying that a map in $\mathsf{StMod}(kG)$ is a simple ghost if and only if its restriction to P is a ghost.

Proof. We first show that both functors $\uparrow^G$ and $\mathrm{res} = \downarrow_P$ preserve the test objects. The containment $\mathrm{res}(\mathcal{S}) \subseteq \mathcal{F}^P$ follows directly from Lemma 3.3.1. To see that $\langle \mathcal{F}^P\uparrow^G \rangle \subseteq \mathcal{S}$, by Lemma 3.3.1 it suffices to check that $k\uparrow^G\downarrow_P \cong \oplus k$, and this is true by Mackey's theorem (Theorem 3.2.8). Finally, by Lemma 3.2.7(ii), we have inclusions $\mathcal{S} \subseteq \langle \mathrm{res}(\mathcal{S})\uparrow^G \rangle \subseteq \langle \mathcal{F}^P\uparrow^G \rangle$, hence $\mathcal{S} = \langle \mathcal{F}^P\uparrow^G \rangle$. It follows immediately that $\mathrm{s}\mathcal{G} = \mathrm{res}^{-1}(\mathcal{G}_P)$, and so $\mathrm{s}\mathcal{G}\downarrow_P \subseteq \mathcal{G}_P$. Note that we also have that $\mathcal{G}_P\uparrow^G \subseteq \mathrm{s}\mathcal{G}$, using that $\mathrm{res}(\mathcal{S}) \subseteq \mathcal{F}^P$ and that $\uparrow^G$ is right adjoint to restriction.

We now prove that $\mathrm{sgl}(L\uparrow^G) = \mathrm{gl}(L)$, with the other equality following similarly. Since the induction functor takes a non-trivial ghost in $\mathsf{stmod}(kP)$ into a non-trivial simple ghost in $\mathsf{stmod}(kG)$, we get $\mathrm{sgl}(L\uparrow^G) \geqslant \mathrm{gl}(L)$ for $L \in \mathsf{stmod}(kP)$.

To show that $\mathrm{sgl}(L\uparrow^G) \leqslant \mathrm{gl}(L)$, we claim that the natural isomorphism $\alpha : \underline{\mathrm{Hom}}_G(L\uparrow^G, M) \to \underline{\mathrm{Hom}}_P(L, M\downarrow_P)$ takes simple ghosts to ghosts. Indeed, if $g : L\uparrow^G \to M$ is a simple ghost, then the morphism $\alpha(g)$ is the composite $L \xrightarrow{\eta} L\uparrow^G\downarrow_P \xrightarrow{g\downarrow_P} M\downarrow_P$, and is a ghost. It follows that $\mathrm{sgl}(L\uparrow^G) \leqslant \mathrm{gl}(L)$. □

Remark 3.3.3. One can also consider the *unstable* projective classes generated by the simple modules on $\mathsf{StMod}(kG)$ and $\mathsf{StMod}(kP)$. We write $(\mathcal{S}_u, \mathrm{s}\mathcal{G}_u)$ for the unstable projective class generated by the simple modules on $\mathsf{StMod}(kG)$ and $(\mathcal{F}_u, \mathcal{G}_u)$ for the unstable projective class generated by the trivial module on $\mathsf{StMod}(kP)$. Here $\mathcal{S}_u$ consists of retracts of direct sums of simple modules in $\mathsf{StMod}(kG)$ and $\mathcal{F}_u$ consists of direct sums of the trivial module in $\mathsf{StMod}(kP)$.

For a projective-free kP-module L, the radical length of L is exactly the length with respect to the projective class $(\mathcal{F}_u, \mathcal{G}_u)$. And Lemma 3.3.1 says that if M is a projective-free kG-module and L is a projective-free kP-module, then

$$\mathrm{len}_{\mathcal{S}_u}(M) = \mathrm{rad\,len}\ (M\downarrow_P) \quad \text{and} \quad \mathrm{rad\,len}\ (L) = \mathrm{len}_{\mathcal{S}_u}(L\uparrow^G).$$

Moreover, the projective classes $(\mathcal{S}_u, \mathrm{s}\mathcal{G}_u)$ and $(\langle \mathcal{F}_u{\uparrow}^G \rangle, \mathrm{res}^{-1}(\mathcal{G}_u))$ are the same on $\mathsf{StMod}(kG)$. Hence we see that Theorem 3.3.2 and Lemma 3.3.1 are stable and unstable versions of each other.

Remark 3.3.4. When the Sylow p-subgroup is not normal, there is no obvious relationship between the simple ghost number of G and the ghost number of its Sylow p-subgroup, or between their radical lengths. See Section 3.5.2 for more discussion.

3.3.2 The group A_4 at the prime 2

In this section, we show that in general the restriction functor from a finite group G to a Sylow p-subgroup P does not preserve ghosts. We also compute the simple ghost number of kA_4 at the prime 2 and give bounds on its ghost number.

Let G be A_4, the alternating group on 4 letters, and set $p = 2$, so $P = V$, the Klein four group, is normal in A_4. It is known that $\mathsf{Thick}_{A_4}\langle k \rangle = \mathsf{stmod}(kA_4)$ [20]. For convenience, we assume that k contains a third root of unity ζ, i.e., $\mathbb{F}_4 \subseteq k$. Then $k{\uparrow}_V^G \cong k \oplus k_\zeta \oplus k_{\bar{\zeta}}$. Here k_ζ is the one-dimensional module with the cyclic permutation (123) acting as ζ and elements of even order acting as the identity, and similarly for $k_{\bar{\zeta}}$. Note that by Lemmas 3.2.7(ii) and 3.3.1, these are all the simple kA_4-modules, i.e., $\mathbb{S} = \{k, k_\zeta, k_{\bar{\zeta}}\}$. By Theorem 3.3.2, a map restricts to a ghost in $\mathsf{stmod}(kV)$ if and only if it is a simple ghost in $\mathsf{stmod}(kA_4)$. Since $k_\zeta \not\cong \widetilde{\Omega}^i k$ for all $i \in \mathbb{Z}$, the class of kA_4-modules $\mathcal{F} = \langle k \rangle$ is strictly contained in $\mathcal{S} = \langle \mathbb{S} \rangle$, or equivalently, simple ghosts are strictly contained in ghosts. Therefore, there exists a ghost in $\mathsf{stmod}(kA_4)$ which does not restrict to a ghost in $\mathsf{stmod}(kP)$.

For a specific example, we consider the connecting map $\gamma : k_\zeta \to \Omega k_\zeta$ in the Auslander-Reiten triangle [4, Section 4.12]

$$\Omega^2 k_\zeta \longrightarrow E \longrightarrow k_\zeta \xrightarrow{\gamma} \Omega k_\zeta$$

associated to the simple module k_ζ. Since γ is stably non-trivial, it is not a simple ghost. But since $k_\zeta \notin \mathcal{F}$, the map is a ghost, by [16, Theorem 2.1].

We now compute the simple ghost number of kA_4 and give bounds on the ghost number. We are able to get an upper bound for the ghost number of A_4, since the simple modules have bounded ghost lengths.

Proposition 3.3.5. *Let k be a field of characteristic 2. Assume that k contains a third root of unity ζ. Then*

$$\textit{simple ghost number of } kA_4 = \textit{ghost number of } kV = 2,$$

and

$$2 \leqslant \textit{ghost number of } kA_4 \leqslant 4.$$

Proof. By Theorem 3.3.2, the simple ghost number of kA_4 is equal to the ghost number of kV, which is known to be 2 (see [19]).

Since $\mathsf{stmod}(kA_4) = \mathsf{Thick}\langle k\rangle$ and every simple ghost is a ghost, the ghost number of kA_4 is at least 2. On the other hand, there is a short exact sequence

$$\widetilde{\Omega}^2 k \to \widetilde{\Omega} k_\zeta \oplus \widetilde{\Omega} k_{\bar{\zeta}} \to k$$

in $\mathsf{mod}(kA_4)$ (see [4, Section 4.17]). It follows that $\mathbb{S} \subseteq \mathcal{F}_2$. Thus, by Lemma 3.2.3, the ghost number of kA_4 is at most twice the simple ghost number. □

Note that $\mathsf{stmod}(kA_4) = \mathsf{Thick}\langle k\rangle$. In the next section, we prove finiteness under a weaker hypothesis.

3.4 Groups whose principal block is generated by k

In this section, we further our study of the ghost number of a group algebra kG by making use of the fact that the thick subcategory $\mathsf{Thick}\langle k\rangle$ generated by k is contained in $\mathsf{stmod}(B_0)$, where B_0 is the principal block of kG, and $\mathsf{stmod}(B_0)$ consists of modules in $\mathsf{stmod}(kG)$ whose projective-free summands are in the principal block B_0. The reader is referred to [1] and [4] for background on block theory.

We focus on the case in which $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$. In Section 3.4.1, we show that this holds when the Sylow p-subgroup A is a direct factor. In this situation, we prove that $\mathsf{stmod}(kA)$ is equivalent to $\mathsf{Thick}_G\langle k\rangle$, and use these results to show that the ghost numbers of kA and kG agree.

In Section 3.4.2, we show that when $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$, the ghost number of kG is finite. The finiteness of the ghost number remains an open question without this hypothesis. We also show that in general the composite of functors

$$e_0(-\uparrow^G) : \mathsf{StMod}(kP) \to \mathsf{StMod}(B_0)$$

is faithful, where P is a Sylow p-subgroup of G and e_0 is the principal idempotent, which allows us to prove that the ghost number of kG is at least as large as the ghost number of kP when $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$. We quote Theorem 3.4.11 which provides conditions equivalent to $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$.

Finally, in Section 3.4.3, we use this material to compute the ghost numbers of the dihedral groups at the prime 2. In addition, we give a block decomposition of each dihedral group and compute its simple ghost number.

3.4.1 Direct products

In this section, we study the ghost number of certain direct products, making a slight correction to a result in [20]. Let k be a field of characteristic p, and $G = A \times B$ with A being a p-group, and the order of B being coprime to p. (That is, A is the Sylow p-subgroup of G.) Write $i : A \to A \times B$ for the inclusion of A into G and $\pi : A \times B \to A$ for the projection onto A. Then $\pi i = \mathrm{id}_A$.

We will prove the following result. Recall that for a class $\mathbb{P}$ of objects, $\mathsf{Loc}\langle\mathbb{P}\rangle$ denotes the localizing category generated by $\mathbb{P}$, i.e., the smallest full triangulated subcategory that is closed under arbitrary coproducts and retracts and contains $\mathbb{P}$.

Theorem 3.4.1. *Let k be a field of characteristic p, and let $G = A\times B$ with A a p-group and the order of B coprime to p. Then the projection $\pi\colon G \to A$ induces a triangulated functor $\pi^*\colon \mathsf{StMod}(kA) \to \mathsf{StMod}(kG)$ that preserves the trivial representation k, and it restricts to triangulated equivalences*

$$\pi^* : \mathsf{StMod}(kA) \to \mathsf{Loc}_G\langle k\rangle$$

and

$$\pi^* : \mathsf{stmod}(kA) \to \mathsf{Thick}_G\langle k\rangle.$$

The inverse functors are the restriction functors. Moreover, the image of π^* consists of the kG-modules whose projective-free summands have trivial B-actions.

This theorem corrects the statement of Lemma 4.2 in [20], which has $\mathsf{StMod}(kG)$ in place of $\mathsf{Loc}_G\langle k\rangle$ and $\mathsf{stmod}(kG)$ in place of $\mathsf{Thick}_G\langle k\rangle$. That statement is false whenever B is non-trivial. The problem is that the restriction functor $\mathsf{StMod}(kG) \to \mathsf{StMod}(kA)$ is not full. For example, writing kB for the kG-module on which A acts trivially, note that $kB \cong k{\uparrow}_A^G$. Then one can see that the dimension of $\underline{\mathrm{Hom}}_G(kB, kB)$ is $|B|$, while the dimension of $\underline{\mathrm{Hom}}_A(kB{\downarrow}_A, kB{\downarrow}_A)$ is $|B|^2$. The correction is simply to restrict attention to $\mathsf{Loc}_G\langle k\rangle$. The uses of Lemma 4.2 in [20] can be replaced with the above theorem and the fact that $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$ (Corollary 3.4.4 below), so all of the main results of [20] are correct.

Proof of Theorem. We first note that the functor $\pi^* : \mathsf{Mod}(kA) \to \mathsf{Mod}(kG)$ induced by $\pi : G \to A$ passes down to the stable module categories. To prove this, it suffices to show that if P is a projective kA-module, then π^*P is projective. Since $\pi i = \mathrm{id}$, the restriction of π^*P to A is P, and since A is the Sylow p-subgroup of G, it follows from Lemma 3.2.7(iii) that π^*P is projective. It is easy to see that the functor $\pi^* \colon \mathsf{StMod}(kA) \to \mathsf{StMod}(kG)$ is triangulated and preserves coproducts and the trivial representation.

Let $\mathrm{im}(\pi^*)$ be the essential image of π^* in $\mathsf{StMod}(kG)$. The modules in $\mathrm{im}(\pi^*)$ are exactly those whose projective-free summands have trivial B-actions. It follows that π^* is full and that $\mathrm{im}(\pi^*)$ is closed under coproducts. Since $i^*\pi^* = \mathrm{id}$, the functor π^* is also faithful. Thus π^* induces a triangulated equivalence between $\mathsf{StMod}(kA)$ and $\mathrm{im}(\pi^*)$. Because $\mathsf{StMod}(kA) = \mathsf{Loc}_A\langle k\rangle$ and π^* is triangulated, we get that $\mathrm{im}(\pi^*)$ is contained in $\mathsf{Loc}_G\langle k\rangle$ and that $\mathrm{im}(\pi^*)$ is triangulated. Hence $\mathrm{im}(\pi^*) = \mathsf{Loc}_G\langle k\rangle$, and we get the triangulated equivalence $\pi^* : \mathsf{StMod}(kA) \to \mathsf{Loc}_G\langle k\rangle$. Clearly, the restriction functor $i^* : \mathsf{Loc}_G\langle k\rangle \to \mathsf{StMod}(kA)$ on the localizing subcategory generated by k is inverse to π^*. Restricting to compact objects, we get the equivalence $\pi^* : \mathsf{stmod}(kA) \to \mathsf{Thick}_G\langle k\rangle$, since $\mathsf{Thick}\langle k\rangle$ consists of exactly the compact objects in $\mathsf{Loc}\langle k\rangle$ by [32, Lemma 2.2]. □

As a corollary, we can compute the ghost number of kG.

Corollary 3.4.2. *In the same set-up as above, the following holds:*

$$\textit{ghost number of } kG = \textit{ghost number of } kA,$$

In particular, the generating hypothesis holds for kG if and only if it holds for kA if and only if A is C_2 or C_3. □

To show that $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$, we compute the principal block idempotent using the next formula:

Theorem 3.4.3 ([31, Theorem 1])**.** *Let k be a field of characteristic p, let G be a finite group, and let $e_0 = \sum \epsilon_g g$ be the principal block idempotent in kG with each ϵ_g in k. Then*

$$\epsilon_g = |\{(u,s) \in G_p \times G_{p'} \mid us = g\}| \; |G_{p'}|^{-1}$$

for a p-regular element $g \in G$, and $\epsilon_g = 0$ if g is not p-regular. □

We say that g is p**-regular** if its order is not divisible by p; otherwise it is said to be p**-singular**; an exception is that the identity element 1 is both p-regular and p-singular. We write G_p for the set of p-singular elements and $G_{p'}$ for the set of p-regular elements.

Corollary 3.4.4. *Let k be a field of characteristic p. Let $G = A \times B$, with A being the Sylow p-subgroup of G. Then $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$ and $\mathsf{Loc}_G\langle k\rangle = \mathsf{StMod}(B_0)$.*

The conditions $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$ and $\mathsf{Loc}_G\langle k\rangle = \mathsf{StMod}(B_0)$ are equivalent by Theorem 3.4.11.

Proof. We compute that the principal idempotent e_0 is $\frac{1}{|B|}(\sum_{b\in B} b)$, using Theorem 3.4.3. Since $be_0 = e_0$ for each $b \in B$, the projective-free modules in $\mathsf{stmod}(B_0)$ and $\mathsf{StMod}(B_0)$ all have trivial B actions. Thus, by Theorem 3.4.1, the claim follows. □

One can also prove the corollary using Theorem 3.4.11.

Note that the only simple module in $\mathsf{stmod}(B_0)$ is the trivial module k in this case. Indeed, since $A \leqslant G$ is normal, a simple module S has trivial A-action (Lemma 3.3.1); and if S is in $\mathsf{stmod}(B_0)$, then it has trivial B-action too, by Theorem 3.4.1. Hence S is the trivial module k.

Remark 3.4.5. One can check that the algebra map $kA \to k(A \times B) \xrightarrow{e_0} e_0(k(A \times B))$ is an isomorphism. It induces the equivalence $\mathsf{stmod}(B_0) \to \mathsf{stmod}(kA)$ with inverse π^*. This also explains why we need to shrink the domain of the functor i^* to get an equivalence.

We combine the discussion in Section 3.3.1 and the results of this section in the next proposition.

Proposition 3.4.6. *Let k be a field of characteristic p. Let $G = A \times B$, with A being the Sylow p-subgroup of G. Then, for $M \in \mathsf{stmod}(B_0)$,*

$$\mathrm{gl}(M) = \mathrm{sgl}(M) = \mathrm{gl}(M{\downarrow_A}),$$

and for $N \in \mathsf{stmod}(kA)$,

$$\mathrm{sgl}(N{\uparrow}) = \mathrm{gl}(N) = \mathrm{gl}(e_0(N{\uparrow})) = \mathrm{sgl}(e_0(N{\uparrow})).$$

Proof. Since the trivial module k is the only simple module in $\mathsf{stmod}(B_0)$, $\mathrm{gl}(M) = \mathrm{sgl}(M)$ for $M \in \mathsf{stmod}(B_0)$. The equalities $\mathrm{sgl}(M) = \mathrm{gl}(M{\downarrow_A})$ and $\mathrm{sgl}(N{\uparrow}) = \mathrm{gl}(N)$ are from Theorem 3.3.2. That $\mathrm{gl}(N) = \mathrm{gl}(e_0(N{\uparrow}))$ for $N \in \mathsf{stmod}(kA)$ is a result of Theorem 3.4.1, as one checks that the functor $e_0(-{\uparrow})$ is isomorphic to the equivalence $\pi^* : \mathsf{stmod}(kA) \to \mathsf{Thick}_G\langle k\rangle$. The last equality is a special case of the first. □

Note that one can't expect $\mathrm{gl}(N) = \mathrm{gl}(e_0(N{\uparrow}))$ for groups that aren't direct products, even when $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$. For example, this fails for A_4, using the discussion in Section 3.3.2 and the fact that $e_0 = 1$ in this case.

3.4.2 Finiteness of the ghost number and a lower bound

Let G be a finite group, let k be a field whose characteristic p divides the order of G, and let P be a Sylow p-subgroup of G. In this section, assuming that $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$, we prove that the ghost number of kG is finite (Theorem 3.4.7) and is greater than or equal to the ghost number of kP (Proposition 3.4.10).

Theorem 3.4.7. *Let k be a field of characteristic p, and let G be a finite group with Sylow p-subgroup P. Suppose that $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$. Then the ghost number of kG is finite.*

In particular, the theorem holds for any p-group G, where $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0) = \mathsf{stmod}(kG)$, recovering [19, Theorem 4.7]. Our proof follows the approach used in Proposition 3.3.5 for the alternating group A_4.

Proof. Recall that the simple ghost number of kG is finite (Remark 3.2.9). It then follows that, since there are no non-zero maps between different blocks, the Freyd number of

kG with respect to $\mathbb{Q} := \mathbb{S} \cap B_0$, the set of simple modules in the principle block, is finite. On the other hand, since $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$ and $\mathbb{Q}$ is a finite set, we have $\mathbb{Q} \subseteq \mathcal{F}_n = \langle \mathbb{P}\rangle_n$ for some n, where $\mathbb{P} = \{k\}$. It then follows from Lemma 3.2.3 that the ghost number of kG is bounded above by n times the Freyd number of kG with respect to $\mathbb{Q}$, and thus is finite. □

We call the Freyd number of kG with respect to $\mathbb{Q}$ the **simple ghost number of** B_0.

Remark 3.4.8. Note that each $M \in \mathsf{Thick}\langle k\rangle$ has finite ghost length. But we need to find an universal upper bound to prove finiteness of the ghost number. One idea is to look at the radical sequence as was done for p-groups in [19]. When $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$, the simple modules that can appear in the radical sequence for $M \in \mathsf{Thick}_G\langle k\rangle$ all have finite ghost lengths. However, whether the ghost number is finite when $\mathsf{Thick}_G\langle k\rangle \neq \mathsf{stmod}(B_0)$ remains open, since we cannot answer the following question proposed in [8]: does there exist a simple module in the principal block with vanishing Tate cohomology? Indeed, if there exists a simple module in $\mathsf{stmod}(B_0)$ but not in $\mathsf{Thick}_G\langle k\rangle$, and its Tate cohomology does not vanish, then it can appear in the radical sequence of a module $M \in \mathsf{Thick}_G\langle k\rangle$. Hence the proof here does not apply to the case where $\mathsf{Thick}_G\langle k\rangle \neq \mathsf{stmod}(B_0)$.

We state the question in the general case as a conjecture:

Conjecture 3.4.9. *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Then the ghost number of kG is finite.*

Now we determine a lower bound for the ghost number of kG. Note that for a group G with subgroup H, the induction functor sends ghosts to ghosts and is faithful. However, induction does not preserve $\mathsf{Thick}\langle k\rangle$ in general, so this technique is of limited use in computing the ghost number of G. To try to remedy this, we can consider the composite $e_0(-{\uparrow^G})$ of induction with projection onto the principal block. This will provide us with a ghost in $\mathsf{Thick}_G\langle k\rangle$ if we assume that $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$.

Note that we have adjunctions

$$\uparrow^G : \mathsf{stmod}(kH) \rightleftarrows \mathsf{stmod}(kG) : \downarrow_H \quad \text{and} \quad e_0(-) : \mathsf{stmod}(kG) \rightleftarrows \mathsf{stmod}(B_0) : j,$$

where j denotes the inclusion. We show that the composite $e_0(-{\uparrow^G})$ is faithful in the case where H is a Sylow p-subgroup.

Proposition 3.4.10. *Let k be a field of characteristic p, and let G be a finite group with Sylow p-subgroup P. Then the functor*

$$e_0(-\uparrow^G) : \mathsf{stmod}(kP) \to \mathsf{stmod}(B_0)$$

is faithful. In particular, if $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$, then

$$\textit{ghost number of } kG \geqslant \textit{ghost number of } kP.$$

We don't know of a counterexample to the last inequality.

Proof. It suffices to show that the unit map

$$\begin{array}{ccc} M & \longrightarrow & j(e_0(M\uparrow^G))\downarrow_P \\ m & \longmapsto & e_0 \otimes m \end{array}$$

of the composite adjunction is split monic.

It is well known that $\uparrow^G$ is both left and right adjoint to $\downarrow_P$, with unit map $\eta : M \to M\uparrow^G\downarrow_P$ sending m to $1 \otimes m$, and counit map $\epsilon : M\uparrow^G\downarrow_P \to M$ sending $g \otimes m$ to gm if $g \in P$ and to 0 if $g \notin P$.

The unit map for the adjunction $e_0(-) : \mathsf{stmod}(kG) \rightleftarrows \mathsf{stmod}(B_0) : j$ is the natural projection $N \to j(e_0N)$ by left multiplication by e_0. Since the stable module category $\mathsf{stmod}(kG)$ decomposes into blocks, it is easy to check that $e_0(-)$ is also right adjoint to j, with counit the natural inclusion $j(e_0N) \to N$.

The composite

$$M \to j(e_0(M\uparrow^G))\downarrow_P \to M\uparrow^G\downarrow_P \xrightarrow{\epsilon} M$$

sends m to $\epsilon(e_0 \otimes m)$. We show that it is an isomorphism. Since P is a p-subgroup of G and the only possible non-zero coefficient ϵ_h for $h \in P$ is $\epsilon_1 = |G_{p'}|^{-1}$ by Theorem 3.4.3, one sees that $\epsilon(e_0 \otimes m) = \epsilon_1 m$. But ϵ_1 is invertible in k, so the composite is an isomorphism. It follows that $M \to (e_0(M\uparrow^G))\downarrow_P$ is split monic and the functor $e_0(-\uparrow^G)$ is faithful.

It is clear that the composite $e_0(-\uparrow^G)$ preserves ghosts. Hence $\mathrm{gl}(e_0(L\uparrow^G)) \geqslant \mathrm{gl}(L)$ for $L \in \mathsf{stmod}(kP)$, and the ghost number of kG is greater than or equal to the ghost number of kP. $\square$

We quote the next theorem to end this section. It provides conditions for checking whether $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$. Recall that a finite group is said to be p**-nilpotent** if $G_{p'}$, the set of p-regular elements of G, forms a subgroup.

Theorem 3.4.11 ([6, Theorem 1.4]). *Let G be a finite group, and let k be a field of characteristic p. Then $\mathsf{Thick}\langle k\rangle = \mathsf{stmod}(B_0)$ if and only if $\mathsf{Loc}\langle k\rangle = \mathsf{StMod}(B_0)$ if and only if the centralizer of every element of order p is p-nilpotent.* □

3.4.3 Dihedral groups at the prime 2

Let $G = D_{2ql}$ be a dihedral group of order $2ql$, where q is a power of 2 and l is odd, with presentation $D_{2ql} = \langle x, y \mid x^{ql} = y^2 = (xy)^2 = 1\rangle$. Let k be a field of characteristic 2. In this section, we will determine the ghost number and simple ghost number of kD_{2ql} by analyzing the blocks. (See Theorem 3.5.7 for the ghost number of kD_{2ql} at an odd prime.)

We can compute the principal block idempotent of kD_{2ql} using Theorem 3.4.3 and the fact that $l = 1$ in k.

Lemma 3.4.12. *The 2-regular elements of D_{2ql} are exactly those in the subgroup $C_l = \langle x^q\rangle$. The principal idempotent is $e_0 = 1 + x^q + x^{2q} + \cdots + x^{(l-1)q}$.* □

We regard D_{2q} as the subgroup of D_{2ql} generated by x^l and y, and so we have a natural unital algebra map $\alpha : kD_{2q} \to kD_{2ql} \to e_0kD_{2ql}$. Note that D_{2q} is a Sylow 2-subgroup of D_{2ql}.

Lemma 3.4.13. *The algebra map $\alpha : kD_{2q} \to e_0kD_{2ql}$ is an isomorphism.*

Proof. As an algebra, e_0kD_{2ql} is generated by e_0x and e_0y. Clearly, $e_0y = \alpha(y)$ is in the image of α. And since l is odd and $e_0x^q = e_0$, we see that $e_0x = e_0x^{kl}$ for some integer k. Hence the map α is surjective. Since e_0kD_{2ql} is projective as a kD_{2ql}-module, its dimension is at least $2q$, which equals the dimension of kD_{2q}, so α has to be an isomorphism. □

As a corollary, we can compute the ghost number of kD_{2ql}.

Corollary 3.4.14. *The thick subcategory generated by k is the same as the principal block,*

$$\mathsf{Thick}_{D_{2ql}}\langle k\rangle = \mathsf{stmod}(e_0 kD_{2ql}),$$

and the ghost number of kD_{2ql} is $\lfloor \frac{q}{2}+1\rfloor$.

Proof. Since α is an isomorphism, it induces an equivalence

$$\mathsf{stmod}(kD_{2q}) \to \mathsf{stmod}(e_0 kD_{2ql})$$

that sends M to $e_0(M\uparrow_{D_{2q}}^{D_{2ql}})$. The first statement follows from the facts that this equivalence sends k to k and that $\mathsf{Thick}_{D_{2q}}\langle k\rangle = \mathsf{stmod}(kD_{2q})$. It also follows that

$$\text{the ghost number of } kD_{2ql} = \text{the ghost number of } kD_{2q}.$$

The second statement then follows from [23, Corollary 4.25], which shows that the ghost number of kD_{2q} is $\lfloor \frac{q}{2}+1\rfloor$. □

So, in this case, the lower bound given by Proposition 3.4.10 is an equality.

We next consider the simple ghost number of kD_{2ql}.

Remark 3.4.15. Note that the only simple module in the principal block is k, by Lemma 3.4.13. Also, the inverse to the equivalence $\mathsf{stmod}(kD_{2q}) \to \mathsf{stmod}(e_0 kD_{2ql})$ is given by restriction. It follows that, for $M \in \mathsf{stmod}(e_0 kD_{2ql})$, we have

$$\mathrm{sgl}(M) = \mathrm{gl}(M) = \mathrm{gl}(M\downarrow_{D_{2q}}).$$

To compute the simple ghost number of kD_{2ql}, it remains to consider the non-principal blocks. From now on, we assume that k contains an l-th primitive root of unity ζ. Let C_{ql} be the cyclic subgroup of D_{2ql} generated by x. We will show that inducing up is fully-faithful on each non-principal block, using the following lemmas.

It is not hard to compute the idempotent decomposition of 1 in kC_{ql}.

Lemma 3.4.16. *The identity $1 \in kC_{ql}$ has an decomposition into orthogonal primitive idempotents:*

$$1 = \sum_{i=0}^{l-1} e_i, \ \textit{with } e_i = \sum_{j=0}^{l-1} (\zeta^i x^q)^j.$$

The block corresponding to e_i has exactly one simple module k_i, the one-dimensional module on which x^q acts as ζ^{l-i}.

Proof. It is easy to check that the e_i's are orthogonal and idempotent, and that $e_i k_i = k_i$. It is well known that the k_i's are a complete list of simple kC_{ql}-modules, so it follows that the idempotents are primitive. □

Since conjugation by y in D_{2ql} takes e_0 to e_0 and e_i to e_{l-i} for $i > 0$, we can deduce the idempotent decomposition for kD_{2ql}.

Lemma 3.4.17. *The identity $1 \in kD_{2ql}$ has a decomposition into orthogonal primitive central idempotents:*

$$1 = e_0 + \sum_{i=1}^{\frac{l-1}{2}} e_i', \text{ with } e_i' = e_i + e_{l-i} = \sum_{j=0}^{l-1} (\zeta^i x^q)^j + \sum_{j=0}^{l-1} (\zeta^{l-i} x^q)^j.$$

Moreover, the block corresponding to e_i' has exactly one simple module, namely $S_i := k_i{\uparrow}_{C_{ql}}^{D_{2ql}}$. It follows that $\mathsf{stmod}(e_i' kD_{2ql}) = \mathsf{Thick}_{D_{2ql}}\langle S_i \rangle$.

Proof. Clearly, the e_i''s are orthogonal central idempotents. They are primitive since there are exactly $(l+1)/2$ simple kD_{2ql}-modules [1, Theorem 3.2]. It follows that there is exactly one simple module in each block.

Define S_i to be $k_i{\uparrow}_{C_{ql}}^{D_{2ql}} = kD_{2ql} \otimes_{C_{ql}} k_i$, where k_i is the simple kC_{ql}-module defined in Lemma 3.4.16. With respect to the basis $\{1 \otimes 1,\, y \otimes 1\}$ of $kD_{2ql} \otimes_{C_{ql}} k_i$, it is easy to check that S_i is represented using the following matrices:

$$x^l \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x^q \mapsto \begin{bmatrix} \zeta^{l-i} & 0 \\ 0 & \zeta^i \end{bmatrix}, \quad \textit{and} \quad y \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And from this representation, one sees quickly that $S_i{\downarrow}_{C_{ql}} = k_i \oplus k_{l-i}$. The action of y on S_i exchanges k_i and k_{l-i}, hence, as kD_{2ql}-modules, both k_i and k_{l-i} generate the whole module S_i. Thus S_i is a simple module. It is also clear that the module S_i is in the block $e_i' kD_{2ql}$, and so $\mathsf{stmod}(e_i' kD_{2ql}) = \mathsf{Thick}_{D_{2ql}}\langle S_i \rangle$. □

We next provide a list of all the indecomposable kC_{ql}-modules. The result can be found in [1, p. 14, 34]. Recall that for each $1 \leqslant n \leqslant q$ there is a unique indecomposable

kC_q-module M_n of radical length n, and that these are all of the indecomposable kC_q-modules.

Lemma 3.4.18 ([1]). *The modules $e_i(M_n{\uparrow}^{C_{ql}})$, for $1 \leqslant n \leqslant q$ and $0 \leqslant i < l$, are a complete list of the indecomposable kC_{ql}-modules.* □

Now we can show that the induction functor induces an equivalence between the non-principal blocks of kC_{ql} and kD_{2ql}.

Proposition 3.4.19. *For $i \neq 0$, let $B_i = e_i kC_{ql}$ be a non-principal block of kC_{ql}. Then the composite of functors*

$$\mathsf{stmod}(B_i) \longrightarrow \mathsf{stmod}(kC_{ql}) \xrightarrow{\uparrow_{C_{ql}}^{D_{2ql}}} \mathsf{stmod}(kD_{2ql})$$

is fully-faithful, hence induces an equivalence $\mathsf{stmod}(B_i) \to \mathsf{stmod}(e_i' kD_{2ql})$.

Proof. We begin by showing that $\uparrow_{C_{ql}}^{D_{2ql}}$ is fully-faithful when restricted to $\mathsf{stmod}(B_i)$. Let $M := e_i(M_n{\uparrow}^{C_{ql}})$ be one of the indecomposable kC_{ql}-modules described in Lemma 3.4.18, and write $N := M_n{\uparrow}^{C_{ql}}$. Using Mackey's Theorem, we have $M{\uparrow}^{D_{2ql}}{\downarrow}_{C_{ql}} \cong e_i(N) \oplus y(e_i(N)) = e_i(N) \oplus e_{l-i}(N)$, and the natural map $M \xrightarrow{\eta} M{\uparrow}{\downarrow} \cong e_i(N) \oplus e_{l-i}(N)$ is an isomorphism onto $e_i(N)$.

Because $\uparrow$ is left adjoint to $\downarrow$, the following diagram commutes

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{C_{ql}}(M, M) & \xrightarrow{\uparrow} & \underline{\mathrm{Hom}}_{D_{2ql}}(M{\uparrow}, M{\uparrow}) \\ & \searrow^{\eta_*} & \downarrow \cong \\ & & \underline{\mathrm{Hom}}_{C_{ql}}(M, M{\uparrow}{\downarrow}). \end{array}$$

By the discussion in the previous paragraph, η_* is an isomorphism, and so $\uparrow$ is as well. Since this is true for every indecomposable in $\mathsf{stmod}(B_i)$, it follows that the induction functor is fully-faithful when restricted to $\mathsf{stmod}(B_i)$, and induces a triangulated equivalence between $\mathsf{stmod}(B_i)$ and its essential image. Since $\mathsf{stmod}(B_i) = \mathsf{Thick}_{C_{ql}}\langle k_i \rangle$ (Lemma 3.4.16) and $k_i{\uparrow} = S_i$, the essential image of $\mathsf{stmod}(B_i)$ is $\mathsf{stmod}(e_i' kD_{2ql}) = \mathsf{Thick}_{D_{2ql}}\langle S_i \rangle$ (Lemma 3.4.17), and the claim follows. □

Remark 3.4.20. Note that the inverse of the equivalence is given by the composite of restriction and then projection onto the block $e_i kC_{ql}$.

We can now compute the simple ghost number of kD_{2ql}.

Theorem 3.4.21. *For $M \in \mathsf{stmod}(e'_i kD_{2ql})$ with $i \neq 0$, we have*

$$\mathrm{sgl}(M) = \mathrm{sgl}(M{\downarrow_{C_{ql}}}) = \mathrm{gl}(M{\downarrow_{C_q}}).$$

For $M \in \mathsf{stmod}(e_0 kD_{2ql})$, we have

$$\mathrm{sgl}(M) = \mathrm{gl}(M{\downarrow_{D_{2q}}}).$$

Hence the simple ghost number of kD_{2ql} = the ghost number of $kD_{2ql} = \lfloor \frac{q}{2} + 1 \rfloor$.

Proof. We have equivalences

$$\mathsf{stmod}(B_i) \to \mathsf{stmod}(e'_i kD_{2ql}) \text{ and } \mathsf{stmod}(kD_{2q}) \to \mathsf{stmod}(e_0 kD_{2ql}).$$

The equivalences preserve simple modules, hence radical lengths and simple ghost lengths. Then, for $M \in \mathsf{stmod}(e'_i kD_{2ql})$, we have $\mathrm{sgl}(M) = \mathrm{sgl}(e_i(M{\downarrow_{C_{ql}}})) = \mathrm{sgl}(e_{l-i}(M{\downarrow_{C_{ql}}}))$ by Proposition 3.4.19 and Remark 3.4.20. Since $M{\downarrow_{C_{ql}}} = e_i(M{\downarrow_{C_{ql}}}) \oplus e_{l-i}(M{\downarrow_{C_{ql}}})$, it follows that

$$\mathrm{sgl}(M) = \mathrm{sgl}(M{\downarrow_{C_{ql}}}).$$

And by Theorem 3.3.2, $\mathrm{sgl}(M{\downarrow_{C_{ql}}}) = \mathrm{gl}(M{\downarrow_{C_q}})$.

For $M \in \mathsf{stmod}(e_0 kD_{2ql})$, we have seen in Remark 3.4.15 that

$$\mathrm{sgl}(M) = \mathrm{gl}(M) = \mathrm{gl}(M{\downarrow_{D_{2q}}}).$$

Since the ghost number of C_q is $\lfloor q/2 \rfloor$ (Lemma 3.6.5), and the ghost number of D_{2ql} is $\lfloor \frac{q}{2} + 1 \rfloor$ [23, Corollary 4.25], it follows that the simple ghost length is maximized by $\mathrm{sgl}(M)$ for some $M \in \mathsf{stmod}(e_0 kD_{2ql})$, and that the simple ghost number of kD_{2ql} equals its ghost number. □

3.5 Groups with cyclic Sylow p-subgroups

We consider a group G with a cyclic Sylow p-subgroup P in this section. When the Sylow p-subgroup is normal, we know from Section 3.3.1 that simple ghost lengths can be computed by restricting to P. We show in Section 3.5.1 that, when P is also cyclic, the simple ghost length of a module in the principal block is equal to its ghost length and that the finitely generated modules in the principal block are exactly those in $\mathsf{Thick}\langle k\rangle$. We use this to compute the ghost numbers of dihedral groups at odd primes. In Section 3.5.2, we study the group $SL(2,p)$ at the prime p, which has a cyclic Sylow p-subgroup which is not normal. Nevertheless, by restricting to the normalizer L of P, we are able to show that the simple generating hypothesis holds for $SL(2,p)$ for any p, even though it fails for L and P when $p > 3$.

3.5.1 The case of a cyclic normal Sylow p-subgroup

Let k be a field of characteristic p, and let G be a finite group with cyclic Sylow p-subgroup C_{p^r}. We assume that k is algebraically closed and that C_{p^r} is normal in G.

Since $P \leqslant G$ is normal, Theorem 3.3.2 says that

$$\mathrm{sgl}(M) = \mathrm{gl}(M{\downarrow_P})$$

for $M \in \mathsf{stmod}(kG)$. In this section, using that P is in addition cyclic, we are going to show that

$$\mathrm{sgl}(M) = \mathrm{gl}(M)$$

for $M \in \mathsf{stmod}(B_0)$, as we found for direct products in Proposition 3.4.6.

Our approach is as follows. We will show that all simple modules in the principal block $\mathsf{StMod}(B_0)$ are suspensions of the trivial module k. Hence the simple ghost projective class and the ghost projective class coincide when both are pulled back to $\mathsf{StMod}(B_0)$. It then follows that $\mathsf{Thick}\langle k\rangle$ equals $\mathsf{stmod}(B_0)$ and that for M in $\mathsf{stmod}(B_0)$, its ghost length is the same as its simple ghost length.

We say that a kG-module M is **uniserial** if the successive quotients in the radical sequence associated to M are simple. Note that this is equivalent to the successive quotients in the socle sequence associated to M being simple.

An important fact about the representations of G when its Sylow p-subgroup is normal and cyclic is that the indecomposable modules are uniserial:

Theorem 3.5.1 ([1, pp. 42–43])**.** *Let G be a finite group, and let k be a field of characteristic p. Assume that the Sylow p-subgroup P of G is normal and cyclic. Then there are* *finitely many indecomposable kG-modules. Every indecomposable module M is uniserial and is characterised by its radical length and the simple module $M/\mathrm{rad}(M)$.* □

Recall that in general there is a bijection between indecomposable projective kG-modules and simple kG-modules given by the assignment that sends a projective module Q to its radical quotient $Q/\mathrm{rad}(Q)$ [1, Theorem 5.3]. The inverse sends a simple module to its projective cover, i.e. the unique indecomposable projective module that surjects onto it. Also note that for a projective kG-module Q, we have an isomorphism $Q/\mathrm{rad}(Q) \cong \mathrm{soc}(Q)$ [1, Theorem 6.6].

When $P = C_{p^r} \leqslant G$ is cyclic and normal, we can say more.

Lemma 3.5.2. *Let G be a finite group with cyclic normal Sylow p-subgroup $P = C_{p^r}$, let k be a field of characteristic p, and let Q be the projective cover of the trivial module k. If S is a simple module, then $Q \otimes S$ is its projective cover.*

Proof. First note that $Q \otimes M$ is projective for any kG-module M [1, Lemma 7.4], so $Q \otimes S$ is projective.

To see that $Q \otimes S$ is indecomposable, first note that $S \cong k \otimes S \subseteq \mathrm{soc}(Q \otimes S)$. Since $Q \otimes S$ is projective, so is its restriction to P, by Lemma 3.2.7(iii). Since projective kP-modules are free, this restriction must have rank $\dim S$ and socle $k^{\oplus \dim S}$. Then, by Lemma 3.3.1, the dimension of $\mathrm{soc}(Q \otimes S)$ must also be $\dim S$, and so we actually have $S \cong \mathrm{soc}(Q \otimes S)$. Thus $Q \otimes S$ is indecomposable.

We have seen that $Q \otimes S$ is an indecomposable projective, and it comes with a surjection onto S, so it must be the projective cover of S. □

We continue to write Q for the projective cover of the trivial module k. The proof above shows that $Q{\downarrow_{kC_{p^r}}} = kC_{p^r}$. It follows that the radical layers of Q are all 1-dimensional. Now, let W be the 1-dimensional simple module $\mathrm{rad}(Q)/\mathrm{rad}^2(Q)$. We show that $W \cong \widetilde{\Omega}^2 k$.

Lemma 3.5.3. *The module W is isomorphic to the double desuspension of the trivial module k, i.e., $W \cong \widetilde{\Omega}^2 k$. Moreover, each composition factor of Q is a tensor power $W^{\otimes n}$ of the module W.*

Proof. The map $Q \otimes W \to W$ lifts through the quotient map $\pi : \mathrm{rad}(Q) \to W$ and gives a map $f : Q \otimes W \to \mathrm{rad}(Q)$. Since $\ker(\pi) = \mathrm{rad}^2(Q)$ is the unique maximal submodule of $\mathrm{rad}(Q)$, the map f is surjective. As we saw for Q, the radical layers of $Q \otimes W$ are also all 1-dimensional, so Q and $Q \otimes W$ have the same radical length. Since the radical length of $\mathrm{rad}(Q)$ is one less than that of $Q \otimes W$, the composite

$$W \to Q \otimes W \xrightarrow{f} \mathrm{rad}(Q)$$

is zero, where the map $W \to Q \otimes W$ is the inclusion of the last radical (which equals the socle) of $Q \otimes W$. By comparing dimensions, one sees that this is a short exact sequence. And since $\mathrm{rad}(Q) \cong \widetilde{\Omega} k$, we have $W \cong \widetilde{\Omega}^2 k$.

To see that the composition factors of Q are $W^{\otimes n}$, first note that

$$\mathrm{rad}^n(Q)/\mathrm{rad}^{n+1}(Q) \cong \mathrm{rad}^{n-1}(Q \otimes W)/\mathrm{rad}^n(Q \otimes W)$$

for $1 \leqslant n \leqslant p^r - 1$. We get these isomorphisms by comparing the radical layers along the surjective map $f : Q \otimes W \to \mathrm{rad}(Q)$, using that both Q and $Q \otimes W$ have 1-dimensional layers. On the other hand, $\mathrm{rad}^n(Q \otimes W) \cong \mathrm{rad}^n(Q) \otimes W$, since tensoring with W preserves the radical layers. Thus

$$\mathrm{rad}^{n-1}(Q \otimes W)/\mathrm{rad}^n(Q \otimes W) \cong (\mathrm{rad}^{n-1}(Q)/\mathrm{rad}^n(Q)) \otimes W.$$

Combining the two displayed isomorphisms and using that $W = \mathrm{rad}(Q)/\mathrm{rad}^2(Q)$, it follows inductively that

$$\mathrm{rad}^n(Q)/\mathrm{rad}^{n+1}(Q) \cong W^{\otimes n}. \qquad \square$$

Note that $M \otimes W \cong \widetilde{\Omega}^2 M$ for any module M. In particular, $W^{\otimes n} \cong \widetilde{\Omega}^{2n} k$. Also note that, more generally, the indecomposable projective module $Q \otimes S$ is uniserial with composition factors $W^{\otimes n} \otimes S$. Thus the following lemma, together with Lemma 3.5.3, implies that the modules $W^{\otimes n}$ are all the simple modules in $\mathsf{StMod}(B_0)$. (See also [1, Exercise 13.3].) Thus the simple modules in $\mathsf{StMod}(B_0)$ are all in $\mathcal{F}$, and so simple ghosts and ghosts agree in the principal block.

Lemma 3.5.4 ([1, Proposition 13.3])**.** *Let k be a field of characteristic p, and let G be a finite group with a cyclic normal Sylow p-subgroup C_{p^r}. Then two simple modules S and T are in the same block if and only if there exists a sequence of simple modules*

$$S = S_1, S_2, \cdots, S_m = T$$

such that S_i and S_{i+1} are composition factors of an indecomposable projective kG-module, for $1 \leqslant i < m$. □

We can use the above observations to compute the ghost number of kG.

Theorem 3.5.5. *Let k be a field of characteristic p, and let G be a finite group with a cyclic normal Sylow p-subgroup C_{p^r}. Then $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$, and a map in $\mathsf{Thick}_G\langle k\rangle$ is a ghost if and only if its restriction to $\mathsf{stmod}(kC_{p^r})$ is a ghost. As a result,*

$$\text{ghost number of } kG = \text{ghost number of } kC_{p^r} = \lfloor p^r/2 \rfloor.$$

Moreover, let M be a uniserial kG-module of radical length l in $\mathsf{Thick}_G\langle k\rangle$. Then

$$\mathrm{gl}(M) = \mathrm{sgl}(M) = \min(l,\, p^r - l).$$

In particular, using the natural terminology, the ghost number of kG is equal to the simple ghost number of B_0.

Proof. Since the simple modules in the principal block are contained in $\mathcal{F}$, the pullback of the simple ghost projective class to $\mathsf{StMod}(B_0)$ coincides with the pullback of the ghost projective class to $\mathsf{StMod}(B_0)$. It follows that $\mathsf{Thick}_G\langle k\rangle = \mathsf{stmod}(B_0)$, and $\mathrm{gl}(M) = \mathrm{sgl}(M)$ for a module M in $\mathsf{stmod}(B_0)$. Since $P \leqslant G$ is normal, $\mathrm{sgl}(M) = \mathrm{gl}(M{\downarrow_P})$ for $M \in \mathsf{stmod}(kG)$, by Theorem 3.3.2. Hence

$$\mathrm{gl}(M) = \mathrm{sgl}(M) = \mathrm{gl}(M{\downarrow_P}),$$

and we can compute the ghost lengths in kG by restricting to kC_{p^r}. The ghost lengths in kC_{p^r} are computed in [19] (summarized in Lemma 3.6.5 below). □

Remark 3.5.6. We give a concrete description of the module W [1, Exercise 5.3]. Let x be a generator of the cyclic group C_{p^r}. Then the one-dimensional module W is given by the group homomorphism that sends $g \in G$ to $\overline{\alpha(g)} \in k^\times$, where $\alpha(g)$ is the integer such that $gxg^{-1} = x^{\alpha(g)}$ and $\overline{\alpha(g)}$ is its image under the canonical map $\mathbb{Z} \to k$. If we further compose this map with the self map on $k^\times$ that takes α to α^n, we get the module $W^{\otimes n}$. Since $\overline{\alpha(g)}$ lands in $\mathbb{F}_p \subseteq k$, we always have $W^{\otimes(p-1)} = k$.

Let M be a non-projective uniserial module with radical length $l \geqslant 2$. We give an explicit construction of a (weakly) universal simple ghost out of M. Let W^* be the dual of W, so $W \otimes W^* \cong k$. We have

$$M/\mathrm{rad}(M) \cong (\mathrm{rad}(M)/\mathrm{rad}^2(M)) \otimes W^* \cong \mathrm{rad}(M \otimes W^*)/\mathrm{rad}^2(M \otimes W^*). \qquad (3.5.1)$$

To see that the first isomorphism holds, note that it holds for the module Q, hence for the modules $Q \otimes S$ with S simple. Since M is a quotient of one of the uniserial modules $Q \otimes S$, the isomorphism holds for M too. Recall by Theorem 3.3.2 that a map f is simple ghost if and only if its restriction to a Sylow p-subgroup is a ghost. And for a p-group P, we know that a ghost $g : M \to N$ has $\mathrm{im}(g) \subseteq \mathrm{rad}(N)$ and $\mathrm{soc}(M) \subseteq \ker(g)$ by [19, Corollary 2.6]. Hence we consider the short exact sequences

$$0 \to \mathrm{soc}(M) \to M \xrightarrow{\pi} M/\mathrm{soc}(M) \to 0$$

and

$$0 \to \mathrm{rad}(M \otimes W^*) \xrightarrow{i} M \otimes W^* \to M \otimes W^*/\mathrm{rad}(M \otimes W^*).$$

Equation (3.5.1) implies that $M/\mathrm{soc}(M) \cong \mathrm{rad}(M \otimes W^*)$. Now let g be the composite $M \xrightarrow{\pi} M/\mathrm{soc}(M) \cong \mathrm{rad}(M \otimes W^*) \xrightarrow{i} M \otimes W^*$. Then $\mathrm{im}(g) \subseteq \mathrm{rad}(N)$ and $\mathrm{soc}(M) \subseteq \ker(g)$. By Lemma 3.3.1, the inclusions still hold when restricted to the normal Sylow p-subgroup C_{p^r}. Since $\Omega^2 k \cong k$ in $\mathsf{stmod}(kC_{p^r})$, the proof of [19, Proposition 2.1] shows that $g\downarrow_{C_{p^r}}$ is a ghost. So by Theorem 3.3.2, the map g is a simple ghost. One can check that the fibre of g is $\mathrm{soc}(M) \oplus \Omega(M \otimes W^*/\mathrm{rad}(M \otimes W^*))$. Thus g is a weakly universal simple ghost. This process can be iterated, producing composites $M \to M \otimes (W^*)^n$ of n simple ghosts which are nonzero for $n < \mathrm{sgl}(M)$. If M is in the principal block, then these simple ghosts are ghosts, and so we have exhibited the ghosts predicted by Theorem 3.5.5.

Theorem 3.5.7. *Let D_{2ql} be a dihedral group, with q a power of 2 and l odd. Let k be a field of characteristic p which divides $2ql$. If p is odd, then the ghost number of kD_{2ql} is $\lfloor p^r/2 \rfloor$, where p^r is the p-primary part of l. If p is even, then the ghost number of kD_{2ql} is $\lfloor q/2 + 1 \rfloor$.*

Proof. If p is odd, then its Sylow p-group is cyclic and normal, so its ghost number is given by Theorem 3.5.5. If p is even, then its ghost number was computed in Corollary 3.4.14. □

3.5.2 The simple generating hypothesis for the group $SL(2,p)$

In this section, we show that the simple generating hypothesis holds for kG, where G is the group $SL(2,p)$ of order $p(p-1)(p+1)$ and k is a field of characteristic p. Background on representations of $SL(2,p)$ can be found in [1, p. 14, 75]. We will also need to know about representations of the normalizer $N(P)$ of P in $SL(2,p)$, which illustrates the results of Section 3.5.1.

We let $P \leqslant G$ consist of all elements of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. P has order p and is a Sylow p-subgroup of G. Let $L = N(P)$ be the normalizer of P in G. It consists of the elements of the form $\begin{pmatrix} a & 0 \\ c' & 1/a \end{pmatrix}$.

For $i \in \mathbb{Z}$, consider the one-dimensional simple module S_i of L given by the group map $L \to k^\times$ that sends $\begin{pmatrix} a & 0 \\ c' & 1/a \end{pmatrix}$ to a^i. Note that $S_0 = k$ is the trivial representation. Clearly, $S_i \cong S_j$ if and only if $i \equiv j$ (modulo $p-1$) and $S_i \otimes S_j \cong S_{i+j}$. These are all of the simple kL-modules, since there can be at most $p-1$ non-isomorphic indecomposable projective kL-modules.

Applying the discussion in Section 3.5.1 to the group L, one obtains a kL-module $W \cong \widetilde{\Omega}^2 k$. By Remark 3.5.6, one can check that $W \cong S_{-2}$. It follows that kL has two blocks, with the module S_i in the principal block if and only if i is even. Moreover, $S_{-2i} \cong W^{\otimes i} \cong \widetilde{\Omega}^{2i} k$, using Lemma 3.5.3, so all of the simple modules in the principal block are suspensions of the trivial module k. By Theorems 3.3.2 and 3.5.5, the simple ghost number of kL, the ghost number of kL and the ghost number of kP are all equal to $\lfloor p/2 \rfloor$.

We will show below that the simple ghost number of kG is actually 1, which is surprising since the simple generating hypothesis fails for its subgroups P and L when $p > 3$. It is even more surprising in view of the next result, which shows that $\mathsf{stmod}(kG)$ and $\mathsf{stmod}(kL)$ are equivalent.

The Sylow p-subgroup P is cyclic of order p. Thus it is a trivial intersection subgroup of G ($gPg^{-1} \cap P$ is either P or trivial), and we have an equivalence between $\mathsf{stmod}(kG)$ and $\mathsf{stmod}(kL)$ by restriction and inducing up:

Theorem 3.5.8 ([1, Theorems 10.1, 10.3])**.** *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Let P be a Sylow p-subgroup of G and let*

$L = N(P)$ be the normalizer of P in G. Assume that P is a trivial intersection subgroup of G. Then the restriction functor

$$\mathsf{stmod}(kG) \to \mathsf{stmod}(kL)$$

is an equivalence, with inverse given by the inducing up functor. □

Note that the equivalence preserves the trivial representation k both ways, so the ghost number of kG equals that of kL, which is $\lfloor p/2 \rfloor$, by Theorem 3.5.5. But the equivalence does not preserve simple modules.

By Theorem 3.5.8, to study the simple ghost number of $\mathsf{StMod}(kG)$, it is equivalent to study the pullback projective class of $(\mathcal{S}, \mathrm{s}\mathcal{G})$ on $\mathsf{StMod}(kL)$, i.e. the projective class on $\mathsf{StMod}(kL)$ generated by the modules $S{\downarrow_L}$, for S a simple kG-module. We are going to show that this projective class contains all finitely-generated modules. It will then follow that the simple generating hypothesis holds for kG.

Theorem 3.5.9. *Let $G = SL(2,p)$. Every module in $\mathsf{stmod}(kG)$ is a direct sum of suspensions of simple modules. In particular, the simple generating hypothesis holds for kG.*

Note that despite the equivalence of Theorem 3.5.8, we already observed that the simple generating hypothesis does not hold for kL unless $p \leqslant 3$.

Proof. By the remarks immediately preceding the theorem, it suffices to show that the modules $S{\downarrow_L}$, with S a simple module in $\mathsf{stmod}(kG)$, generate everything in $\mathsf{stmod}(kL)$ under direct sums, suspensions and retracts.

By Theorem 3.5.1, the indecomposable kL-modules are $M_{i,j}$, for $1 \leqslant i \leqslant p$ and $0 \leqslant j \leqslant p-2$, where $M_{i,j}$ has radical length i and radical quotient $M/\mathrm{rad}(M) \cong S_j$. It thus suffices to show that each module $M_{i,j}$ is a suspension of some $S{\downarrow_L}$. For convenience, in the following we will interpret the subscript j modulo $p-1$.

There are p simple kG-modules [1, p. 14], and we write $V_1, \ldots, V_p$ for their restrictions to L. The kL-module V_i is uniserial of radical length i, with radical quotient $V_i/\mathrm{rad}(V_i) \cong S_{i-1}$ [1, p. 76], so $V_i = M_{i,i-1}$. Note that the module V_1 is trivial and the module V_p is projective. The case $p = 2$ follows immediately, since $L = C_2$, and $M_{1,0} = V_1 \cong k$

and $M_{2,0} = M_{2,1} = V_2 \cong kC_2$ are the only two indecomposable kL-modules. Thus we assume that p is odd.

Recall that $W \cong \widetilde{\Omega}^2(k)$, hence $- \otimes W$ is isomorphic to the functor $\Omega^2(-)$ on $\mathsf{stmod}(kL)$. Since $- \otimes W$ preserves radical lengths and shifts the simple module S_j to S_{j-2}, we have a stable isomorphism $\Omega^{2k} V_i \cong M_{i,i-1-2k}$ for $k \in \mathbb{Z}$. This gives all modules $M_{i,j}$ where $i + j$ is odd.

To get the modules $M_{i,j}$ with $i + j$ even and $1 \leqslant i < p$, note that $V_p \otimes S_{p-i-1}$ is the projective cover of V_{p-i}. It follows that $\widetilde{\Omega} V_{p-i}$ has radical length i and radical quotient S_{i-2}, i.e., $\widetilde{\Omega} V_{p-i} \cong M_{i,i-2}$. Then we can apply Ω^{2k} again to obtain the modules $M_{i,j}$ where $i + j$ is even. □

In general, for which groups the simple generating hypothesis holds remains open.

3.6 Strong ghosts

In Section 3.6.1, we motivate and define strong ghosts and show that the strong ghost number of a group algebra kG equals the strong ghost number of kP, where P is a Sylow p-subgroup of G. In Section 3.6.2, we compute the strong ghost numbers of cyclic p-groups. In Section 3.6.3, we show that the strong ghost number of a dihedral 2-group D_{4q} is between 2 and 3.

3.6.1 The strong ghost projective class

If H is a subgroup of a finite group G, then it is rare for the restriction functor from G to H to preserve ghosts. For example, we saw in Section 3.3.2 that restriction from the group A_4 to its Sylow p-subgroup P does not preserve ghosts. As another example, if G is a p-group and $N \leqslant G$ is any normal subgroup, then the restriction from G to N does not preserve ghosts, since $k{\uparrow}_N^G$ is indecomposable [1, Theorem 8.8] and is not a suspension of k. Strong ghosts, which were introduced in [17], will by definition restrict to ghosts.

Definition 3.6.1. Let G be a finite group, and let k be a field whose characteristic divides the order of G. A map in $\mathsf{StMod}(kG)$ is called a **strong ghost** if its restriction to $\mathsf{StMod}(kH)$ is a ghost for every subgroup H of G.

It follows immediately that the restriction of a strong ghost to any subgroup is again a strong ghost.

In [17], Carlson, Chebolu and Mináč study strong ghosts in $\mathsf{Thick}\langle k\rangle$, but their results imply the following theorem, which says that most groups admit strong ghosts in $\mathsf{stmod}(kG)$:

Theorem 3.6.2 (Carlson, Chebolu and Mináč [17])**.** *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Then every strong ghost in $\mathsf{stmod}(kG)$ is stably trivial if and only if the Sylow p-subgroup of G is C_2, C_3, or C_4.*

Note that in passing from ghosts to strong ghosts, we only get one more p-group, namely C_4, where all strong ghosts are stably trivial.

We next observe that strong ghosts form an ideal of a projective class and use this in further study of strong ghosts.

Let H be a subgroup of G. We know that the restriction functor

$$\downarrow_H : \mathsf{StMod}(kG) \to \mathsf{StMod}(kH)$$

is both left and right adjoint to the induction functor

$$\uparrow^G : \mathsf{StMod}(kH) \to \mathsf{StMod}(kG).$$

The pullback (see Definition 3.2.5) of the ghost projective class along the restriction functor consists of maps in $\mathsf{StMod}(kG)$ which restrict to ghosts in $\mathsf{StMod}(kH)$. The intersection of such ideals when H ranges over all subgroups of G consists of exactly the strong ghosts and again forms an ideal of a projective class: the relative projectives are obtained from modules of the form $k\uparrow_H^G$ by closing under suspensions, desuspensions, direct sums and retracts. This is the **strong ghost projective class** on $\mathsf{StMod}(kG)$ and is denoted by $(\mathrm{st}\mathcal{F}, \mathrm{st}\mathcal{G})$. (In the terminology of [21], it is the *meet* of the pullbacks.)

Note that we can set $\mathbb{P} = \{k\uparrow_H^G \mid H \text{ is a subgroup of } G\}$ in $\mathsf{StMod}(kG)$, and this generates exactly the strong ghost projective class. Since every kG-module M is a summand of $M\downarrow_P\uparrow_P^G$, where P is a Sylow p-subgroup of G, and induction is a triangulated functor, we have that $\mathsf{Thick}_G\langle\mathbb{P}\rangle = \mathsf{stmod}(kG)$. Hence, using the terminology in Section 3.2.1, Theorem 3.6.2 is the statement that the generating hypothesis with respect to $\mathbb{P}$ holds in $\mathsf{StMod}(kG)$ if and only if the Sylow p-subgroup of G is C_2, C_3, or C_4.

For $M \in \mathsf{stmod}(kG)$, we define the **strong ghost length** of M, denoted by $\mathrm{stgl}(M)$, to be the Freyd length with respect to $\mathbb{P}$, i.e., $\mathrm{stgl}(M) = \mathrm{len}^{\mathrm{F}}_{\mathbb{P}}(M)$. The **strong ghost number** of kG is defined to be the Freyd number of $\mathsf{StMod}(kG)$ with respect to $\mathbb{P}$.

One can show that strong ghosts induce up to strong ghosts by proving the dual statement, i.e., that relative projectives restrict to relative projectives. This follows from Mackey's Theorem (Theorem 3.2.8) and the observation that $s(\Omega^n_H k) \cong \Omega^n_{sHs^{-1}} k$ [17]. Since the induction functor is always faithful, one obtains the following result:

Proposition 3.6.3 (Carlson, Chebolu and Mináč [17])**.** *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Let H be a subgroup of G. If g is a stably non-trivial strong ghost in $\mathsf{StMod}(kH)$, then $g{\uparrow}^G$ is a stably non-trivial strong ghost in $\mathsf{StMod}(kG)$.* □

Next, we prove that the induction functor preserves strong ghost lengths.

Proposition 3.6.4. *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Let H be a subgroup of G. Then for any M in $\mathsf{stmod}(kH)$, $\mathrm{stgl}(M{\uparrow}^G) = \mathrm{stgl}(M)$, and so the strong ghost number of kG is at least as big as the strong ghost number of kH. Moreover, if P is a Sylow p-subgroup of G, then*

$$\textit{strong ghost number of } kP = \textit{strong ghost number of } kG.$$

Proof. The proof is essentially the same as the proof of Theorem 3.3.2. By Proposition 3.6.3, we have $\mathrm{stgl}(M{\uparrow}^G) \geqslant \mathrm{stgl}(M)$. Conversely, since the natural isomorphism $\alpha : \underline{\mathrm{Hom}}_G(M{\uparrow}^G, L) \to \underline{\mathrm{Hom}}_H(M, L{\downarrow}_H)$ preserves strong ghosts, $\mathrm{stgl}(M{\uparrow}^G) \leqslant \mathrm{stgl}(M)$.

When P is a Sylow p-subgroup of G, the restriction functor is faithful by Lemma 3.2.7(i). The last equality follows. □

3.6.2 Strong ghost numbers of cyclic p-groups

We study the strong ghost numbers of cyclic p-groups in this section. Our result suggests that the notion of a strong ghost is much stronger than that of a ghost.

We first review ghost lengths in $\mathsf{stmod}(kC_{p^r})$, following [19, Section 5.1].

Lemma 3.6.5. *Let $G = C_{p^r}$ be a cyclic group of order p^r with generator g, let k be a field of characteristic p, and let M_n be the indecomposable kC_{p^r}-module of radical length*

n. Then the self map $g-1$ on M_n is a weakly universal ghost, i.e., any ghost with domain M_n factors through $g-1$. Moreover $\mathrm{gl}(M_n) = \min(n, p^r - n)$ and the ghost number of kG is $\lfloor p^r/2 \rfloor$.

Proof. That the map $g-1$ is a ghost is proved in [9, Lemma 2.2]. It is weakly universal, since it fits into a triangle

$$k \oplus \Sigma k \longrightarrow M_n \xrightarrow{g-1} M_n \longrightarrow k \oplus \Sigma k.$$

The l-fold composite $(g-1)^l$ on M_n is stably trivial if and only if $l \geqslant \min(n, p^r - n)$ (see [19, Propositions 5.2, 5.3]). Hence $\mathrm{gl}(M_n) = \min(n, p^r - n)$. Since all indecomposables are of this form, the ghost number of kG is $\lfloor p^r/2 \rfloor$. □

Theorem 3.6.6. *Let $G = C_{p^r}$ be a cyclic group of order p^r, let k be a field of characteristic p, and let M_n be the indecomposable kC_{p^r}-module of radical length $n < p^r$. Writing $N = \min(n, p^r - n) = \mathrm{gl}(M_n)$, we have the following:*

(i) If $N \leqslant p^{r-1}$, then

$$\mathrm{stgl}(M_n) = \begin{cases} 1, & \text{if } N \mid p^r, \\ 2, & \text{otherwise.} \end{cases}$$

(ii) If $N > p^{r-1}$, then

$$\mathrm{stgl}(M_n) = \left\lceil \frac{N}{p^{r-1}} \right\rceil = \left\lceil \frac{\mathrm{gl}(M_n)}{p^{r-1}} \right\rceil.$$

It follows that

$$\text{strong ghost number of } kG = \begin{cases} \left\lceil \dfrac{p+1}{2} \right\rceil, & \text{if } p = 2 \text{ and } r \geqslant 3, \text{ or } p \text{ is odd and } r \geqslant 2, \\ \left\lceil \dfrac{p-1}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. We divide the proof into three cases:

Case 1: We first determine the indecomposable modules in $\mathrm{st}\mathcal{F}$, i.e., those of strong ghost length 1. The set $\mathrm{st}\mathcal{F}$ is generated by $\mathbb{P} = \{k\!\uparrow_{C_{p^j}}^{C_{p^r}} = M_{p^{r-j}} \mid 1 \leqslant j \leqslant r\}$, and so an indecomposable module M_n is in $\mathbb{P}$ if and only if $n \mid p^r$. Since $\mathrm{st}\mathcal{F}$ also contains the suspensions of modules in $\mathbb{P}$ and $\Sigma M_n \cong M_{p^r - n}$, it follows that $\mathrm{stgl}(M_n) = 1$ if and only if $n \mid p^r$ or $(p^r - n) \mid p^r$, i.e., $N \mid p^r$.

This implies that $\mathbb{P} \subseteq \mathcal{F}_{p^{r-1}}$, or equivalently, that $\mathcal{G}^{p^{r-1}} \subseteq \mathrm{st}\mathcal{G}$, which will be useful below.

Case 2: For $N < p^{r-1}$, we show that M_n is contained in $\mathrm{st}\mathcal{F}_2$. Indeed, for such n we have a triangle

$$M_n \oplus \Sigma M_n \longrightarrow M_{p^{r-1}} \xrightarrow{(g-1)^N} M_{p^{r-1}} \longrightarrow M_n \oplus \Sigma M_n,$$

where g is a generator of C_{p^r}. Hence $M_n \in \mathrm{st}\mathcal{F}_2$ and $\mathrm{stgl}(M_n) \leqslant 2$, completing the proof of (i).

Case 3: We compute the strong ghost length of M_n for $N > p^{r-1}$. By the previous observation, the self map $(g-1)^{p^{r-1}}$ on M_n is a strong ghost. This map fits into the triangle

$$M_{p^{r-1}} \oplus \Sigma M_{p^{r-1}} \longrightarrow M_n \xrightarrow{(g-1)^{p^{r-1}}} M_n \longrightarrow M_{p^{r-1}} \oplus \Sigma M_{p^{r-1}},$$

with fibre in $\mathrm{st}\mathcal{F}$, so it is a weakly universal strong ghost. By Lemma 3.6.5, its jth power is stably trivial if and only if $jp^{r-1} \geqslant N = \mathrm{gl}(M_n)$. The equality in (ii) then follows.

The calculation of the strong ghost number follows from these results:

When $p = 2$, the ghost number of C_{2^r} is 2^{r-1}, hence all C_{2^r}-modules are dealt with in (i), and the strong ghost number of C_{2^r} is 2 provided $r \geqslant 3$, and 1 otherwise.

When p is odd, the modules in (ii) dominate. The strong ghost length is maximized when $N = (p^r - 1)/2$ (the ghost number of C_{p^r}) and is

$$\left\lceil \frac{p^r - 1}{2p^{r-1}} \right\rceil = \left\lceil \frac{p - \frac{1}{p^{r-1}}}{2} \right\rceil,$$

which simplifies to the desired expressions. □

3.6.3 Strong ghost numbers of dihedral 2-groups

In this section we find an upper bound for the strong ghost number of a dihedral 2-group, using the result from the previous section on the strong ghost numbers of cyclic p-groups.

We write D_{4q} for the dihedral 2-group of order $4q$, with q a power of 2:

$$D_{4q} = \langle x, y \mid x^2 = y^2 = 1,\, (xy)^q = (yx)^q \rangle.$$

It has a normal cyclic subgroup C_{2q}, generated by $g = xy$. We prove the following theorem on the strong ghost number of D_{4q}:

Theorem 3.6.7. *Let D_{4q} be the dihedral 2-group of order $4q$, with $q = 2^r$ and $r \geqslant 1$. Then*

$$2 \leqslant \textit{the strong ghost number of } kD_{4q} \leqslant 3.$$

Recall that the strong generating hypothesis fails for kD_{4q} by Theorem 3.6.2, so the strong ghost number of D_{4q} is at least 2 for $r \geqslant 0$. When $r = 0$, so $D_{4q} \cong C_2 \times C_2$, the strong ghost number is 2. Since, for a p-group, the strong ghost length is bounded above by the ghost length, and by [19, Corollary 5.13], the ghost number of $C_2 \times C_2$ is also 2.

Our goal will be to prove the upper bound. We will make use of the notation from [4] (see also [23, Section 4.6]), where the indecomposable kD_{4q}-modules are written using words in the letters a and b. By the proof of [23, Theorem 4.24], every non-projective indecomposable kD_{4q}-module M sits in a triangle $\Omega W \to M \to M'' \to W$, where M'' is a sum of modules of the form $M((ab)^s)$ and $M((ab)^s a)$, for $0 \leqslant s < q$ (and the same, with a and b reversed), and W is a sum of suspensions and desuspensions of the trivial module. Thus, by Lemma 3.2.4, it will suffice to show that the modules $M((ab)^s)$ and $M((ab)^s a)$ have strong ghost length at most 2.

Proof of Theorem. By the discussion above, it suffices to show that

$$\operatorname{stgl}(M((ab)^s)) \leqslant 2 \quad \text{and} \quad \operatorname{stgl}(M((ab)^s a)) \leqslant 2$$

for $0 \leqslant s < q$.

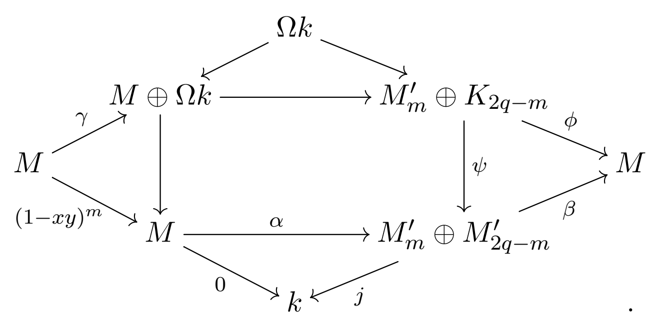
It will be convenient to make the following notational convention: when we write $(ab)^{\frac{m}{2}}$, we mean $aba\cdots$ with m letters in total. For example, $(ab)^{\frac{5}{2}} = ababa$. In addition, $(ba)^{-\frac{m}{2}}$ denotes $((ba)^{\frac{m}{2}})^{-1}$, so $(ba)^{-\frac{5}{2}} = b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$. Let $M = M((ab)^{\frac{q}{2}}(ba)^{-\frac{q}{2}}, \mathrm{id})$ $\cong k\!\uparrow_{C_2}^{D_{4q}} \cong M_q\!\uparrow_{C_{2q}}^{D_{4q}}$, which has strong ghost number 1. Similarly, for $0 \leqslant m \leqslant q-1$, we write M'_m for the module $M((ab)^{\frac{m}{2}}(ba)^{-\frac{m}{2}}, \mathrm{id})$, which has $2m$ letters in total. Then $M'_m \cong M_m\!\uparrow_{C_{2q}}^{D_{4q}}$, as one can check that $(1-xy)^m(yxyx\cdots) = XYXY\cdots - YXYX\cdots$

(m factors in each expression) by induction. Thus, by Proposition 3.6.4, M'_m has strong ghost length at most 2, since M_m does. Inducing up the triangle $M_q \xrightarrow{(1-g)^m} M_q \to M_m \oplus M_{2q-m} \to M_q$ in $\mathsf{stmod}(kC_{2q})$, we get the triangle

$$M \xrightarrow{(1-xy)^m} M \xrightarrow{\alpha} M'_m \oplus M'_{2q-m} \xrightarrow{\beta} M. \tag{3.6.1}$$

Let $j : M'_m \oplus M'_{2q-m} \to k$ be zero on M'_m and non-zero on M'_{2q-m}. Then the composite $j\alpha$ is stably trivial. One can check this fact by looking at the adjoint of $j\alpha$. Similarly, let $i : k \to M'_m \oplus M'_{2q-m}$ be zero on M'_m and non-zero on M'_{2q-m}. The composite βi is stably trivial as well.

The kernel of the non-zero map $M'_{2q-m} \to k$ is $M((ab)^{q-\frac{m+1}{2}}(ba)^{\frac{m+1}{2}-q})$, which we denote K_{2q-m}. We then form the octahedron

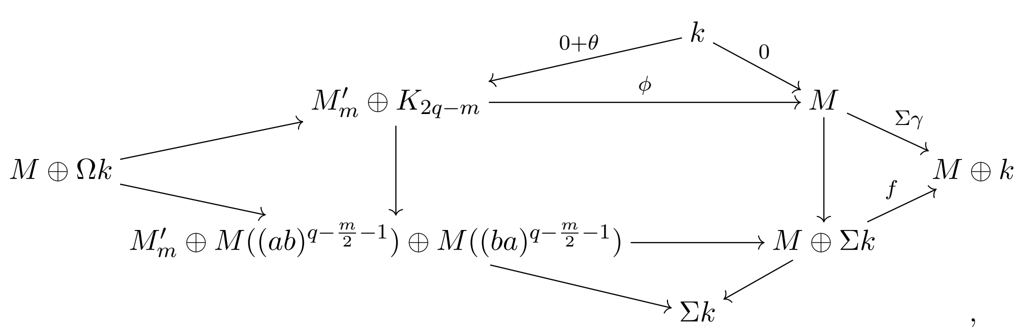


(3.6.2)

We can use K_{2q-m} to build the module $M((ab)^{q-\frac{m}{2}-1})$, using the triangle

$$k \xrightarrow{\theta} K_{2q-m} \longrightarrow M((ab)^{q-\frac{m}{2}-1}) \oplus M((ba)^{q-\frac{m}{2}-1}) \longrightarrow \Sigma k.$$

Now consider the map $0+\theta : k \to M'_m \oplus K_{2q-m}$. Since $\phi(0+\theta) = \beta\psi(0+\theta) = \beta i$ is stably trivial, we get another octahedron



(3.6.3)

and the triangle

$$M \oplus \Omega k \longrightarrow M'_m \oplus M((ab)^{q-\frac{m}{2}-1}) \oplus M((ba)^{q-\frac{m}{2}-1}) \longrightarrow M \oplus \Sigma k \xrightarrow{f} M \oplus k$$

shows that $\text{stgl}(M((ab)^s))$ and $\text{stgl}(M((ab)^{s-1}a)) \leqslant 2$ for $\frac{q-1}{2} \leqslant s \leqslant q-1$.

Note that in order to get $M((ab)^{q-1})$, we set $m = 0$, hence the map $(1 - xy)^m$ is the identity on M and $M((ab)^{q-1})$ is a summand of the cofibre of the non-trivial map $k \to \Omega k$.

To construct the modules $M((ab)^s)$ and $M((ab)^s a)$ for s small, we first suspend the map γ to get a triangle

$$M \xrightarrow{\Sigma\gamma} M \oplus k \longrightarrow M'_{2q-m} \oplus M((ba)^{-\frac{m}{2}}(ab)^{\frac{m}{2}}) \xrightarrow{\Sigma\phi} M.$$

Then we have

$$\Omega k \xrightarrow{\theta'} M((ba)^{-\frac{m}{2}}(ab)^{\frac{m}{2}}) \longrightarrow M((ba)^{\frac{m}{2}}) \oplus M((ab)^{\frac{m}{2}}) \longrightarrow k,$$

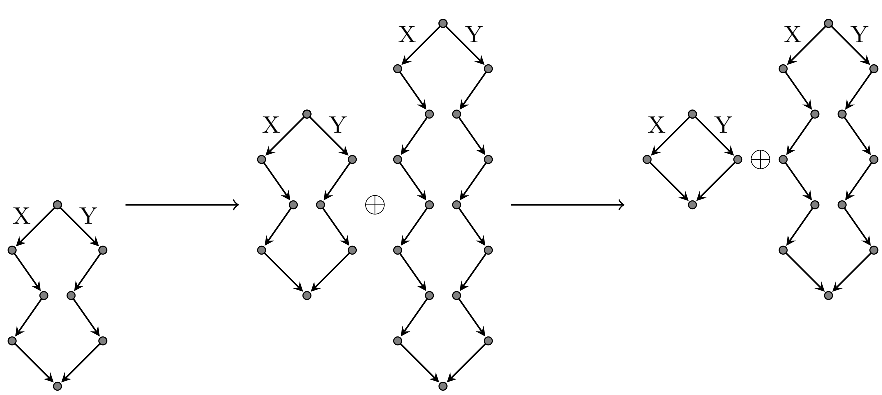
with $\Sigma\phi(0 + \theta')$ stably trivial and we get a triangle similar to the one above:

$$M \oplus \Omega k \xrightarrow{g} M \oplus k \longrightarrow M((ab)^{\frac{m}{2}}) \oplus M((ba)^{\frac{m}{2}}) \oplus M'_{2q-m} \longrightarrow M \oplus k.$$

Thus $\text{stgl}(M((ab)^s))$ and $\text{stgl}(M((ab)^{s-1}a)) \leqslant 2$ for $1 \leqslant s \leqslant \frac{q-1}{2}$.

The two remaining cases are $M((ab)^0) = k$ and $M((ab)^{q-1}a) \cong k\uparrow_{\langle 1,y\rangle}^{D_{4q}}$, both of which have strong ghost length 1, so we are done. $\square$

We illustrate the triangles in the first octahedron as follows, taking $q = 4$ and $m = 2$. The triangle (3.6.1) corresponds to a short exact sequence



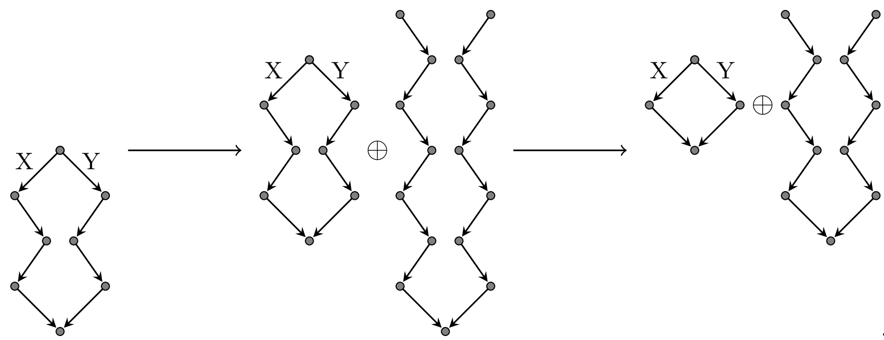
,

where a free module kD_{16} has been included. In these diagrams, the downward-left arrows indicate the action of $X = x - 1$ and the downward-right arrows indicate the action of $Y = y - 1$.

And the triangle

$$M \xrightarrow{\gamma} M \oplus \Omega k \longrightarrow M_2' \oplus M((ab)^2ab^{-1}(ba)^{-2}) \xrightarrow{\phi} M$$

appearing in (3.6.2) and (3.6.3) has Ωk in place of the free summand and corresponds to a short exact sequence



One can check that the map $0 + \theta : k \to M_2' \oplus M((ab)^2ab^{-1}(ba)^{-2})$ factors through the middle term.

Chapter 4

Computations with GAP

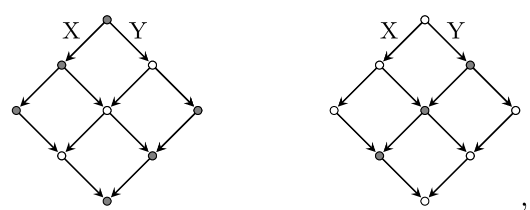
4.1 Introduction

In this chapter, we discuss how to apply GAP to compute ghost lengths for some examples. GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory [26]. The GAP package 'reps' handles group representations in positive characteristic. Its overall structure was designed and most of it written by Peter Webb, who is also the maintainer. Contributions were made by Dan Christensen, Roland Loetscher, Robert Hank, Bryan Simpkins, Brad Froehle and others.

4.2 The group $S_3 \times C_3$ at the prime 3

We begin with a motivating example. Let $G = C_3 \times S_3$, and let k be a field of characteristic 3. We write x for a generator of C_3, $y = (123)$ for an element of order 3 in S_3 and $z = (12)$ for an element of order 2 in S_3. Thus G is a group on three generators x, y, and z subject to the relations $x^3 = y^3 = z^2 = 1$, $xy = yx$, $xz = zx$, and $yz = zy^2$.

There are two simple kG-modules k and ϵ. Here k is the trivial representation and ϵ is a 1-dimensional module with z acting as -1. They correspond to the indecomposable projective modules:



where we use a solid dot for k and a circle for ϵ. The arrows down-left indicate the action of $X = 1 - x$, and the arrows down-right indicate the action of $Y = y - y^2$. Note that $Xz = zX$, while $Yz = -zY$. The correspondence is a result of the following lemma.

Lemma 4.2.1. *[4, Theorem 1.6.3] Let P be an indecomposable projective kG-module. Then $P/\mathrm{rad}(P)$ is simple and $P/\mathrm{rad}(P) \cong \mathrm{soc}(P)$.* □

With an abuse of notation, we write ϵ for its restrictions to $C_3 \times C_2$ and S_3. Since the principal idempotent of kG is 1 [31], both k and ϵ are in the principal block. However, ϵ is not in $\mathsf{Thick}_G\langle k\rangle$. Indeed, by restricting to $C_3 \times C_2$, one sees easily that ϵ is not in the principal block of $k(C_3 \times C_2)$, hence cannot be in $\mathsf{Thick}_{C_3\times C_2}\langle k\rangle$. Since restriction is triangulated, it follows that $\epsilon \notin \mathsf{Thick}_G\langle k\rangle$.

More generally, we know that there are only 6 indecomposable $k(C_3 \times C_2)$-modules:

It is clear that the first three modules are in $\mathsf{Thick}_{C_3\times C_2}\langle k\rangle$. We know that ϵ is not in $\mathsf{Thick}_{C_3\times C_2}\langle k\rangle$, and the fifth module is isomorphic to $\Omega\epsilon$ in $\mathsf{stmod}(k(C_3 \times C_2))$, hence is not in $\mathsf{Thick}_{C_3\times C_2}\langle k\rangle$ either. The last module is projective as a $k(C_3 \times C_2)$-module, hence is in $\mathsf{Thick}_{C_3\times C_2}\langle k\rangle$.

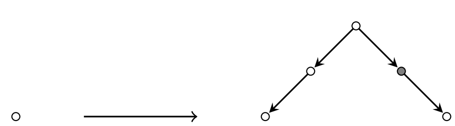
Since the restriction functor is triangulated, we deduce the following proposition.

Proposition 4.2.2. *Let M be a kG-module. If M is in $\mathsf{Thick}\langle k\rangle$, then the modules*

cannot be summands of $M{\downarrow}_{C_3\times C_2}$.

Conversely, we can view the $k(C_3\times C_2)$-modules as kG-modules with trivial y-action. Again, it is easy to see that the first three modules listed above are in $\mathsf{Thick}_G\langle k\rangle$. One also sees that the three-dimensional modules are induced up from the subgroup S_3, as $k{\uparrow}^G$ and $\epsilon{\uparrow}^G$. Since $\Omega^2 k \cong \epsilon$ in $\mathsf{stmod}(kS_3)$, the last module is a double suspension of the third one in $\mathsf{stmod}(kG)$, hence is in $\mathsf{Thick}_G\langle k\rangle$. It then follows that the other two modules are not in $\mathsf{Thick}_G\langle k\rangle$, and we conjecture that the converse of the proposition is also true.

For example, we consider the cokernel M of a non-zero map f



that sends ϵ to the difference of the bottom elements. By Proposition 4.2.2, the domain and codomain of f are not in $\mathsf{Thick}_G\langle k\rangle$. Nevertheless, we expect M to be in $\mathsf{Thick}_G\langle k\rangle$. Note that this is equivalent to showing that M has finite generating length.

4.3 A computational method to calculate the generating length

Before we show how to apply computational methods to calculate the generating length, we need to introduce some notation. Let T be a triangulated category and $\mathbb{P}$ be a finite set of compact objects in T. Recall that $\langle\mathbb{P}\rangle$ denotes the closure of $\mathbb{P}$ under retracts, direct sums, suspensions, and desuspensions, and this constitutes part of a projective class $(\langle\mathbb{P}\rangle, \mathcal{I})$. Then we can inductively define $\langle\mathbb{P}\rangle_1 = \langle\mathbb{P}\rangle$ and $\langle\mathbb{P}\rangle_n$ to consist of the objects X that are retracts of some object M such that M sits in a triangle $P \to M \to Q$ with $P \in \langle\mathbb{P}\rangle$ and $Q \in \langle\mathbb{P}\rangle_{n-1}$. Now we set $\langle\mathbb{P}\rangle^c$ to be the closure of $\mathbb{P}$ under retracts, finite direct sums, suspensions, and desuspensions, and define $\langle\mathbb{P}\rangle^c_n$ in the same way as $\langle\mathbb{P}\rangle_n$, with $\langle\mathbb{P}\rangle$ replaced by $\langle\mathbb{P}\rangle^c$. Writing T^c for the collection of compact objects in T, it is not hard to see that $\langle\mathbb{P}\rangle^c = \langle\mathbb{P}\rangle \cap \mathsf{T}^c$. More generally, we have

Lemma 4.3.1. *[13, Proposition 2.2.4] Let T be a triangulated category and let $\mathbb{P}$ be a set of compact objects in T. With the notion described above,*

$$\langle\mathbb{P}\rangle^c_n = \langle\mathbb{P}\rangle_n \cap \mathsf{T}^c.$$

In particular, $\mathsf{Thick}\langle\mathbb{P}\rangle = \mathsf{Loc}\langle\mathbb{P}\rangle \cap \mathsf{T}^c$. □

We have chosen the notation to be consistent with that in Chapter 3. It is slightly different than that of [13]. Note that we have a filtration of $\mathsf{Thick}\langle\mathbb{P}\rangle$ by

$$\langle\mathbb{P}\rangle^c \subseteq \langle\mathbb{P}\rangle^c_2 \subseteq \cdots \langle\mathbb{P}\rangle^c_n \subseteq \cdots \subseteq \mathsf{Thick}\langle\mathbb{P}\rangle.$$

Now consider $\mathbb{P} = \{k\}$ in $\mathsf{StMod}(kG)$. We write $\mathbb{P}(-m, m)$ for the set $\{\Sigma^i k \mid -m \leqslant i \leqslant m\}$ of finitely many suspensions of k contained in $\langle\mathbb{P}\rangle$. Recall that $\langle\mathbb{P}(-m, m)\rangle$ is part of a projective class. Given $M \in \mathsf{Thick}\langle k\rangle$, we write $\mathrm{gel}_m(M)$ for the length of M with respect to $\mathbb{P}(-m, m)$. Since $\mathbb{P}(-m, m) \subseteq \mathbb{P}(-m-1, m+1) \subseteq \cdots \subseteq \langle\mathbb{P}\rangle$, we get a decreasing sequence greater than or equal to $\mathrm{gel}(M)$:

$$\mathrm{gel}_m(M) \geqslant \mathrm{gel}_{m+1}(M) \geqslant \cdots \geqslant \mathrm{gel}(M).$$

Moreover, since $M \in \langle\mathbb{P}\rangle^c_n$ for some positive integer n, there are only finitely many spheres $\Sigma^{n_i} k$ needed to built up M in n steps. Hence there exists an integer m, such that $M \in \langle\mathbb{P}(-m, m)\rangle^c_n$, and, as a result of Lemma 4.3.1, we get

Proposition 4.3.2. *Let G be a finite group and k be a field whose characteristic divides the order of G. Let M be a module in $\mathsf{Thick}\langle k\rangle$. Then* $\operatorname{gel}(M) = \lim_{n\to\infty} \operatorname{gel}_m(M)$. $\square$

Since $(\operatorname{gel}_m(M))$ is a sequence of integers, we have $\operatorname{gel}_m(M) = \operatorname{gel}(M)$ for m large. Using the formal property of a projective class, we are going to show that, for each integer n, $\operatorname{gel}_n(M)$ can be computed by a finite process. In particular, if the cohomology of kG has periodicity n, then $\operatorname{gel}(M) = \operatorname{gel}_n(M)$ and the computation of the generating length of M is a finite process. We recall the following lemma on the basic property of a projective class.

Lemma 4.3.3. *Let T be a triangulated category, and $(\mathcal{P}, \mathcal{I})$ be a (possibly unstable) projective class on T. Let M be an object in T. Then the following are equivalent:*

1. *M is in $\mathcal{P}_n$.*
2. *Every n-fold composite of maps in $\mathcal{I}$ out of M is zero.*
3. *The n-fold composite of universal maps in $\mathcal{I}$ out of M is zero.*

To implement this idea, we first compute the (unstable) universal ghost $f : M \to N$ in the range $[-m, m]$. Since there are only finitely many suspensions of k needed, this is a finite computation. If f is stably trivial, then we know that M actually has generating length 1. Otherwise, we can make a recursive call to compute the universal ghost out of N, and test whether the composite of the universal ghosts is stably trivial. Finally, the first integer n such that n-fold composite of universal ghosts out of M is stably trivial is the generating length of M in the range $[-m, m]$. We present the method in pseudo-code:

```
GhostLengthHelper = function with inputs: a map f from M to N,
                                          an integer n
    g = universal ghost g from N to L
    if f composed with g is stably trivial then
        return f and n
    return GhostLengthHelper(f composed with g, n+1)

GhostLength of M = GhostLengthHelper(the identity map on M, 1)
```

Example 4.3.4. With the help of the `GhostLength` function, we can compute that the four dimensional module M we considered in Section 4.2 has $\operatorname{gel}_3(M) = 3$, and so

$\mathrm{gl}(M) \leqslant \mathrm{gel}(M) \leqslant 3$. Now we show that

$$\mathrm{gl}(M) = \mathrm{gel}(M) = 3.$$

To compute the lower bound, we consider left multiplication by the central element $1-x$ on M. Restricting to $C_3 \times C_3$, we know that $1-x$ is a ghost and $(1-x)^2$ is stably non-trivial. Then, by Theorem 3.3.2, $1-x$ is a simple ghost, hence a ghost, on M. Since the restriction functor to the Sylow p-subgroup is faithful by Lemma 3.2.7, and $1-x$ is a map in $\mathsf{Thick}_G\langle k\rangle$, the ghost length of M is at least 3. Note that it follows directly from Theorem 3.3.2 that the simple ghost length of M is 3, but this does not guarantee that the simple ghosts are in $\mathsf{Thick}_G\langle k\rangle$.

Remark 4.3.5. We remark here that there is not a universal choice of N such that $\mathrm{gel}_N(M) = \mathrm{gel}(M)$ for all $M \in \mathsf{Thick}\langle k\rangle$. Indeed, if the group cohomology is not periodic, then $\mathrm{gel}_N(\Omega^n k) = \mathrm{gel}(\Omega^n k)$ if and only if $N \geqslant |n|$, and the number N can grow infinitely large. Note that the numbers $\mathrm{gel}_n(M)$ give upper bounds of the ghost length of M. Hence if a lower bound of the ghost length of M is known, we can hope to get the exact answer for the ghost length of M. It would also be interesting to know whether there is a way to compute lower bounds for the ghost length which converge to the correct answer.

4.4 The ReplaceWithInj function and related functions

We have improved the GAP code used in the reps package to compute the universal ghost and ghost length. We introduce the `ReplaceWithInj` function in this section, which is essential for computing the universal ghost. We also show the relation of `ReplaceWithInj` with other functions.

4.4.1 The ReplaceWithInj function and the Simple function

Recall that the universal ghost is the cofibre of a map that is surjective on Tate cohomology, and computing the cofibre depends on a function that replaces a map by an injection that is stably equivalent to it. For simplicity, we write $f+g$ for the map $M \to N \oplus P$, where $f : M \to N$ and $g : M \to P$ are maps out of M. If P is projective, then the maps f and $f+g$ are stably equivalent. Now let $\{P_i\}$ be the set of non-isomorphic indecomposable projective kG-modules, and let $\mathcal{B}_i$ be a basis for $\mathrm{Hom}(M, P_i)$. Observe that the natural map

$$\alpha : M \to \oplus_i(\oplus_{g \in \mathcal{B}_i} P_i)$$

is injective. Then for any map $f : M \to N$, the map $f + \alpha$ is a replacement of f by an injection. But in this way, we will have added more maps than we need to f. For example, we don't need the maps g with $\ker(f + g) = \ker(f)$. And we can do better than this. We need a lemma before we state the condition that we will put on g.

Lemma 4.4.1. *Let $f : M \to N$ be a map in $\mathsf{mod}(kG)$. Then the map f is injective if and only if, for any simple module S, the map*

$$\mathrm{Hom}(S, f) : \mathrm{Hom}(S, M) \to \mathrm{Hom}(S, N)$$

is injective.

Proof. Since $\ker(\mathrm{Hom}(S, f)) \cong \mathrm{Hom}(S, \ker(f))$, the map f being injective implies that $\mathrm{Hom}(S, f)$ is injective for any $S \in \mathsf{mod}(kG)$. Conversely, if $\mathrm{Hom}(S, \ker(f)) = 0$ for all simple modules, then, since the simple modules generate the module category, $\ker(f) = 0$ and f is injective. □

It follows from the lemma that we only need to add to f those maps g that shrink $\ker(\mathrm{Hom}(S, f))$. More precisely, let P be an indecomposable projective module, and let g be a map from M to P. Then, to decide whether we want to replace f by $f + g$, it suffices to check the condition

$$\ker(\mathrm{Hom}(S, f + g)) \subsetneq \ker(\mathrm{Hom}(S, f)), \tag{4.4.1}$$

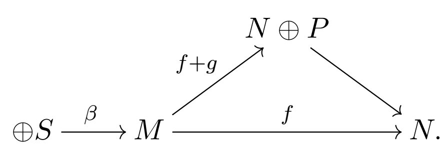
where $S = P/\mathrm{rad}(P)$ is the corresponding simple module of P. Indeed, if $S' \ncong S$ is another simple module, then $\mathrm{Hom}(S', P) = 0$, and since $\ker(\mathrm{Hom}(S, f + g)) = \ker(\mathrm{Hom}(S, f)) \cap \ker(\mathrm{Hom}(S, g))$, there is no need to test g on S'. It follows that we can work with one simple module at a time. Note that if we have replaced f by $f' = f + g$, then we can replace the condition in Equation 4.4.1 by $\ker(\mathrm{Hom}(S, f' + g)) \subsetneq \ker(\mathrm{Hom}(S, f'))$, and if $\{g_1, g_2, \ldots, g_l\}$ is a basis for $\mathrm{Hom}(M, P)$, then

$$\ker(\mathrm{Hom}(S, \sum_{i=1}^{l}(g_i))) = \ker(\mathrm{Hom}(S, \alpha)) = 0,$$

where $\alpha : M \to \oplus_i(\oplus_{g\in\mathcal{B}_i} P_i)$ is the injection we started with. Hence, for each indecomposable projective module P, we can use the following peudo-code to produce a replacement f such that $\ker(\mathrm{Hom}(S, f)) = 0$ for $S = P/\mathrm{rad}(P)$.

```
f = a given map from M to N
P = an indecomposable projective module
S = the corresponding simple module of P
for g in a basis for Hom(M, P)
    if ker(Hom(S, f+g)) is strictly contained in ker(Hom(S, f)) then
        replace f by f+g
        continue the loop over g until ker(Hom(S, f)) = 0
return f
```

Then, by Lemma 4.4.1, we can loop the preceding process over all indecomposable projective modules and produce a replacement by an injection. But we still need to describe how to determine whether $\ker(\mathrm{Hom}(S, f + g)) \subsetneq \ker(\mathrm{Hom}(S, f))$. This is done by a rank computation. We form the map $\beta : \oplus S \to M$, where the sum ranges over a basis for $\mathrm{Hom}(S, M)$. Then we compare the dimensions of $\mathrm{im}((f + g) \circ \beta)$ and $\mathrm{im}(f \circ \beta)$ in the diagram



It is clear that $\mathrm{rank}((f + g) \circ \beta) \geqslant \mathrm{rank}(f \circ \beta)$. Since $\oplus S$ is semi-simple, the equality holds if and only if $\ker(\mathrm{Hom}(S, f + g)) = \ker(\mathrm{Hom}(S, f))$. In other words, the following conditions are equivalent:

1. $\ker(\mathrm{Hom}(S, f + g)) \subsetneq \ker(\mathrm{Hom}(S, f))$,
2. $\mathrm{rank}((f + g) \circ \beta) > \mathrm{rank}(f \circ \beta)$.

Note that $\mathrm{rank}(f \circ \beta)$ is at most $\mathrm{rank}(\beta)$, and this is equivalent to $\ker(\mathrm{Hom}(S, f)) = 0$, so we can break out the loop over the basis for $\mathrm{Hom}(M, P)$ when $\mathrm{rank}(f \circ \beta) = \mathrm{rank}(\beta)$. We can also check at the same time whether f is injective or not and, if yes, we return f to avoid the extra loop over the other projective modules. To conclude the discussion, we display the function "ReplaceWithInj" in the following pseudo-code:

```
f = a given map from M to N
if Rank(f) == dimension of M then   % f is injective
    return N and f
L = list of non-isomorphic indecomposable projectives
for P in L
    S = the corresponding simple module of P
    b = map from a sum of S to M, ranging over a basis for Hom(S, M)
    r = Rank(f composed with b)
    rankb = Rank(b)
    if r !== rankb then
        % r not maximal, so need to loop over a basis for Hom(M, P)
        for g in a basis for Hom(M, P)
            newf = f+g
            newr = Rank(newf composed with b)
            if newr > r then
                f = newf
                r = newr
                N = direct sum of N and P
            if r == rankb then         % r is maximal
                if Rank(f) == dimension of M then
                    return N and f
                break out of the loop over the basis for Hom(M, P)
```

Remark 4.4.2. We remark here that the code we just presented actually produces an optimal answer. That is, the replacement we produce is always minimal, unless the map f itself contains a stably-trivial summand, in which case we need to exclude the summand. To see that the process is optimal, observe first that $\ker((f+g)\circ\beta) \subseteq \ker(f\circ\beta)$ is the kernel of the composite

$$\ker(f \circ \beta) \to \oplus S \xrightarrow{\beta} M \xrightarrow{g} P.$$

Since $\ker(f \circ \beta)$ is a direct sum of copies of the simple module S and P is the corresponding projective module, the image of this composite is either zero or isomorphic to S. It follows that, when we replace f by $f + g$, we always have

$$\operatorname{rank}((f + g) \circ \beta) = \operatorname{rank}(f \circ \beta) + \dim(S).$$

Thus, to replace a map $f : M \to N$ by an injection, we need to add exactly

$$\frac{\text{rank}(\beta) - \text{rank}(f \circ \beta)}{\dim(S)}$$

copies of the projective module P to N, as our code will do. Since this number is independent of the choice of a basis for $\text{Hom}(M, P)$, our code is optimal.

Note that the new code we introduced depends on a decomposition function to find all indecomposable projective modules and, for each indecomposable projective module, we need to find the corresponding simple module. We describe how to do these now.

It follows from Lemma 4.2.1 that there is a self map on P

$$f : P \to P/\text{rad}(P) \cong \text{soc}(P) \to P,$$

with $\text{im}(f) \cong S$, and we can compute the image of *all* self maps on P to find S as the image whose dimension is the smallest. But this is not very efficient. So we consider $M = \text{im}(f)$, the image of an arbitrary self map f on P. Then M also satisfies the condition that $M/\text{rad}(M) \cong \text{soc}(M) \cong S$, being both a submodule and a quotient module of P. Hence, we can replace P by M to work with a smaller hom-set, and find S as the image of a self map on M. To implement this, we can loop over all self maps f on P and compute $M = \text{im}(f)$. Then, if M is a proper submodule of P, we replace P by M and make a recursive call and compute the images of self maps on M. The recursion will end with a module S that has no proper submodules. In other words, S is simple. Note that if $\text{Hom}(M, M)$ has dimension 1, then the map $M \to M/\text{rad}(M) \cong \text{soc}(M) \to M$ is an isomorphism, hence M is simple, and we can return M in this case. Then, assuming that P is an indecomposable projective module, we can find the corresponding simple module S using the following pseudo-code:

```
Simple = a function with one input P
    M = P
    hom = Hom(M, M)
    if hom has dimension 1 then
        return M
    for all maps f in hom
        if 0 < Rank(f) < dimension of M then
            M = im(f)
            return Simple(M)
    return M
```

Remark 4.4.3. Note that not every simple module S has $\dim(\mathrm{Hom}(M, M)) = 1$ when the field k is small, so, in general, we have to search over all self maps on M. Also note that, for an arbitrary module M, $\dim(\mathrm{Hom}(M, M)) = 1$ does not imply that M is simple. For a counterexample, take $G = S_3$, the symmetric group on three letters and consider the two dimensional module $M = \widetilde{\Omega}k$, where the condition $M/\mathrm{rad}(M) \cong \mathrm{soc}(M)$ fails. But for the modules M that arises in the algorithm, the condition always holds.

4.4.2 Other functions related to ReplaceWithInj

We show in this section how the `ReplaceWithInj` function can be used in other functions.

1. `Cofibre` and `Suspension`.

   With the `ReplaceWithInj` function, we can compute the cofibre of a map f. In particular, replacing the zero map out of M, we get an injection of M into a projective module, and it cofibre is the suspension of M. Since the `ReplaceWithInj` function provides an optimal answer, the suspension of M we get is projective-free. `Cofibre` is also essential in the `GhostLength` function, where we need to compute universal ghosts.

2. `CreateRandomModule`.

   We can create random modules in $\mathsf{Thick}\langle k\rangle$ using cofibres. The function `CreateRandomModule` takes a random map $f : P \to Q$ between random modules P and Q that are sums of suspensions of k and computes that cofibre R_1. Note that R_1 has generating length at most 2. Iterating the process n-times, we can build up a module R_n of length at most $n+1$. Note that the function depends on the number of summands that we allow in each step and the number of steps n that we take.

3. `IsStablyTrivial`.

   Let $f : M \to P$ be an injection of M into a projective module. Then since P is also injective, every map from M to a projective module factors through f. Hence `ReplaceWithInj` provides an algorithm to detect whether a map $g : M \to N$ is stably-trivial or not, by checking whether it factors through f.

4. `ReplaceWithSurj`, `Fibre` and `Desuspension`.

   Note that the pseudo-code we present in `ReplaceWithInj` is dualizable, so we can write the dual functions `ReplaceWithSurj`, `Fibre` and `Desuspension`.

4.5 More examples

We give more examples of computations in this section.

4.5.1 Comparing new code with old code

We begin with an easy computation of suspensions of the trivial representation for the alternating group A_4 and the field $GF(4)$ to compare the different versions of the `Suspension` function. We iterate `Suspension` to compute $\Sigma^{14}(k)$ and test the time used.

$\Sigma^{14}(k)$	Dimension	Time
new function	29	0.7 s
old function without decomposing	37	13.2 s
old function with decomposing	29	45.4 s

It is clear from the table that our new function gives the optimal answer with less time. The old `ReplaceWithSurj` adds a free module to the codomain, so it may produce some projective summands in the cofibre. In the example, it raises the dimension of $\Sigma^{14}k$ by 8. To get the optimal answer, we ask GAP to compute the projective-free summand, but this has taken much more time.

4.5.2 Computations in C_9 and Q_8

We test our code for the cyclic group C_9 of order 9 with $k = GF(3)$ and the quaternion group Q_8 of order 8 with $k = GF(2)$. Note that the cohomology of C_9 has periodicity 2 and that the cohomology of Q_8 has periodicity 4. Also note that the generating number of kC_9 is 4 and that the generating number of kQ_8 is 3 or 4. In the examples, we create modules using the `CreateRandomModule` function, and keep the cofibres R_n with $n \geqslant 3$, so that R_n can have lengths greater than or equal to 4. Then we compute their generating lengths.

For the group C_9, we first take $n = 4$ and record the dimensions and lengths of R_3 and R_4. We performed 6 trials and get

n	3	4	3	4	3	4	3	4	3	4	3	4
Dimension	17	22	30	29	17	8	22	15	7	15	7	16
Length	1	2	2	3	1	1	2	2	3	4	2	2

The process seldom produces a module that achieves that generating number.

But if we take $n = 17$, then we have created some kC_9-modules of length 4:

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Dimension	22	14	20	19	11	11	11	12	11	19	18	18	8	16	9
Length	2	2	3	4	4	4	4	4	3	3	3	3	3	3	2

It is interesting to note that the lengths can decrease as we take more steps to build up the modules.

For the group Q_8, we are looking for some module that has length 4. It would then follow that the generating number of Q_8 should be 4. We have tried to build up kQ_8-modules with n up to 100, but in all the examples, there are no kQ_8-modules of length 4, strongly suggesting that the generating number of kQ_8 is 3.

Conjecture 4.5.1. *Let $G = Q_8$ and k be a field of characteristic 2. Then*

$$\textit{generating number of } kQ_8 = 3.$$

For evidence, here is the result when we built up kQ_8-modules with $n = 9$. We allowed up to 20 summands in each step to build up the modules.

n	3	4	5	6	7	8	9
Dimension	39	33	61	57	55	63	55
Length	3	3	2	3	3	3	3

Chapter 5

Conclusion

We provide a brief summary of the thesis in this chapter and describe the relation between the chapters.

The thesis focuses on the study of the stable module category $\mathsf{StMod}(kG)$. Since $\mathsf{StMod}(kG)$ is a triangulated category, we can generalize the generating hypothesis from the stable homotopy category of spectra to $\mathsf{StMod}(kG)$. In $\mathsf{StMod}(kG)$, the generating hypothesis is the statement that the Tate cohomology functor $\widehat{H}^n(G, -)$ is faithful on the subcategory $\mathsf{Thick}\langle k \rangle$. In a series of papers [9, 16, 18, 20], Benson, Carlson, Chebolu, Christensen and Mináč proved that the generating hypothesis holds in $\mathsf{StMod}(kG)$ if and only if the Sylow p-subgroup P of G is C_2 or C_3. Since the generating hypothesis fails in $\mathsf{StMod}(kG)$ in most cases, we study the ghost number of kG, which measures the failure of the generating hypothesis in $\mathsf{StMod}(kG)$. It is the smallest integer n such that every n-fold composite of ghosts in $\mathsf{Thick}\langle k \rangle$ is stably trivial. This is first studied in [19] for a p-group, where it is shown that the ghost number of kG is always finite in this case. There are also some computations of and bounds on ghost numbers given in [19]. The ghost number is best described using the idea of a projective class [21], and this has been used throughout the thesis. The notation and background that we need here are introduced in Chapter 1. It also contains a literature review of the previous work in Section 1.3 and a summary of the main results of the thesis in Section 1.4.

In Chapter 2, which is based on [23], we continue the study of the ghost number of a group algebra. We have improved on the results in [19] and provided new computations for p-groups (See Section 2.4). And in general, we have proved that, for p-groups, the ghost number and the radical length of kG are within a constant factor of each other

(Corollary 2.4.17). More precisely, let G be a p-group, and let k be a field of characteristic p. Then

$$\frac{1}{3}\,\text{rad len}\ kG \leqslant \text{ghost num}\ kG \leqslant \text{gen num}\ kG < \text{rad len}\ kG.$$

Note that the trivial module k is the only simple module when G is a p-group, so the induction technique is very useful in our study of p-groups. This fails in the case of an arbitrary finite group. On the other hand, we have proved results on Auslander-Reiten triangles that apply to a general projective class in a triangulated category in Section 2.3.2. For example, we consider the simple ghosts and the strong ghosts in Chapter 3, where the results apply.

In Chapter 3, which is based on [24], we generalize the study of ghost numbers to arbitrary finite groups. In general, since $\mathsf{Thick}\langle k\rangle \neq \mathsf{StMod}(kG)$, a module induced up from a subgroup might not be in $\mathsf{Thick}\langle k\rangle$ and the induction technique fails. Hence we consider the projection onto the principal block B_0 of kG. Under the assumption that $\mathsf{Thick}\langle k\rangle = \mathsf{StMod}(B_0)$, we show in Section 3.4.2 that

$$\text{ghost number of } kP \leqslant \text{ghost number of } kG,$$

with P being a Sylow p-subgroup of G, and that

$$\text{ghost number of } kG \text{ is finite.}$$

Examples of computations of ghost numbers are given in Sections 3.4 and 3.5. We have also introduced the simple ghost number in Section 3.2.3 and the strong ghost number in Section 3.6, and we show that they are closely related to the ghost number.

In Chapter 4, we apply computational algebra to the study of ghost numbers. We introduce a method to compute the generating number in Section 4.3, and then we describe the improved GAP code in the reps package to compute universal ghosts and ghost length in Section 4.4. We have made computations for the group $S_3 \times C_3$ at the prime 3, the first example where $\mathsf{Thick}\langle k\rangle \neq \mathsf{StMod}(B_0)$ (See Example 4.3.4). And for the quaternion group Q_8 of order 8, we have experimental data that suggests Conjecture 4.5.1, which says that

$$\text{generating number of } kQ_8 = 3.$$

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