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Decision Theory Based Models in Insurance and Beyond

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Graduate Program in Statistics and Actuarial Sciences
A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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by

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Abstract

Everyday, we make difficult choices under uncertainties. The decision making process becomes even more complicated when more agents get involved: one must consider their interactions and conflicts of interest because the final outcome is based not only on an agent's decision but on everybody's.

In insurance industry, companies try to avoid making large claim payments to policyholders (commonly known as insureds) by purchasing reinsurance policies from reinsurance companies (the reinsurer). Each policy details conditions upon which the reinsurer pays a share of the claim to the insurance company (also known as the cedent or the insurer). To reach an agreement, the interests of both parties must be taken into consideration. In this thesis, we will discuss approaches for constructing optimal reinsurance policies that are beneficial to both the insurer and the reinsurer.

Likewise, we have similar situations in real estate industry, where sellers and buyers negotiate contracts. For example, the difference between the seller's selling price and the buyer's budget (commonly referred to as the buyer's reservation price) affects the intensity of buyer arrivals, the bargaining process and its rate of success, as well as many other parameters. We will explore the likelihood of the buyer purchasing a property given factors such as the buyer's reservation price and the negotiated selling prices, which may or may not be dependent random variables.

Of course, these are just two illustrative scenarios that make our results more intuitive and better appreciated from the practical point of view, which has been a very important consideration throughout the thesis. Furthermore, it will be easily seen when reading the thesis that our developed and discussed methodologies can be adapted to numerous other scenarios, which may in turn require making certain adjustments to our results. Nevertheless, we are confident that the herein developed considerations are very general in nature and can already be used to facilitate decision making.

Keywords: Decision making, insurer, reinsurer, real estate, buyer, seller, reservation price, likelihood, Poisson process, premium calculation principle, risk measure, value-at-risk, conditional tail expectation, copula, order statistic, background risk, systematic risk.

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Chapter 1

Introduction

Life is full of choices. People make choices they feel good about (cf., e.g., Gilboa, 2011) by examining various factors such as uncertainties and the presence of competing decision makers, whose responses to different choices may be different. In our thesis, we discuss applications of decision theory to insurance and real estate industries. The similarity between these two areas is that there are at least two parties that negotiate contracts.

1.1 Methods and forms of reinsurance

In insurance industry, in order to cover all claim payments, companies transfer a part of the risk exposure to other parties. An example of such parties is a reinsurance company. By purchasing a reinsurance policy, the insurer enters into an agreement with the reinsurer to share the loss. There are two methods of reinsurance: facultative reinsurance and treaty reinsurance.

Definition 1.1.1 (cf., e.g., Bruggeman, 2010) Under facultative reinsurance, the primary insurer determines as to whether or not reinsurance is desirable, and the reinsurance company decides as to whether or not to accept or refuse any risk offered to her/him.

Definition 1.1.2 (cf., e.g., Bruggeman, 2010) Treaty reinsurance is a contract where the primary insurer has agreed to automatically cede the risks defined in the reinsurance contract to the reinsurer, and the reinsurer has agreed to accept that share of risks.

Under facultative reinsurance, negotiations take place separately for each insurance policy that is reinsured. Under treaty reinsurance, the insurer and the reinsurer negotiate a contract that includes multiple insurance policies issued by the insurer. For each insurance policy, the reinsurer covers a specified share of the claim. Under each method of reinsurance, there are two types: proportional reinsurance and non-proportional reinsurance.

Definition 1.1.3 (cf., e.g., Schwepcke and Arndt, 2004) In proportional reinsurance, the sums insured, the claims, and the original premiums are divided up proportionally between the direct insurer and the reinsurer.

One main form of proportional reinsurance is quota share.

Definition 1.1.4 (cf., e.g., Dror and Preker, 2002) Under a quota share reinsurance contract, the primary insurer cedes a fixed percentage of every exposure it insures within the class of business covered by the contract. The reinsurer receives a share of the premium (less a ceding commission) and pays the same percentage of each loss.

Non-proportional reinsurance is defined as follows:

Definition 1.1.5 (cf., e.g., Schwepcke and Arndt, 2004) In non-proportional reinsurance, the reinsurer insures her/his cedent against the economic loss the latter stands to suffer in the event of specific events occurring. The amount of indemnity paid to the cedent by the reinsurer depends entirely on the size of the loss. The reinsurer reimburses the part of the claim payment made by the direct insurer to her/his insured in excess of a specified amount and up to a prearranged limit.

One main form of non-proportional reinsurance is excess of loss.

Definition 1.1.6 (cf., e.g., Sundt, 1984) For excess of loss reinsurance, of each claim exceeding a fixed priority, which is also known as the retention, the reinsurer pays the exceeding amount, usually limited to a specified maximum (the policy limit).

Although there are other forms of reinsurance, for the sake of simplicity, in the thesis we only consider the excess of loss and quota share forms. For the rest of the thesis, unless otherwise stated, we denote the total claim amount by X, the insurer's share of the total amount by X_I, and the reinsurer's share by X_R. When an insurance policy issued by the insurer is reinsured in the form of excess of loss reinsurance, the insurer pays

$$X_I = \begin{cases} X & \text{if} \quad X \leq R, \\ R & \text{if} \quad X > R, \end{cases}$$

where $R > 0$ is the retention, and the reinsurer pays

$$X_R = \begin{cases} 0 & \text{if} \quad X \leq R, \\ X - R & \text{if} \quad X > R. \end{cases}$$

When an insurance policy issued by the insurer is reinsured in the form of quota share reinsurance, the insurer pays

$$X_I = \alpha X,$$

where $0 < \alpha < 1$, and the reinsurer pays the remaining part

$$X_R = (1 - \alpha)X.$$

1.2 Brief literature review on optimal reinsurance

Various studies related to optimal reinsurance have been reported in the literature. Given the reinsurer's pricing rule discussed by Lane (2000), and Kreps and Major (2001), Bu (2005) considers finding optimal reinsurance by maximizing the insurer's expected net income minus some function of its variance that accounts for the associated uncertainty. Daykin et al. (1994), Gajek and Zagrodny (2000), and Kaluszka (2001) construct an optimal reinsurance contract by minimizing the variance of the insurer's retained loss subject to the pricing rule of the reinsurance contract and the insurer's budget. Guerra and Centeno (2008) obtain an optimal reinsurance policy by maximizing the insurer's expected utility. Pesonen (1984), Goovaerts et al. (2001), Schmidli (2004), Gajek and Zagrodny (2004), and Liang and Guo (2007) take the insurer's survival probability into consideration. Cai and Tan (2007), Cai et al. (2008), and Tan et al. (2009) optimize a reinsurance contract under the value-at-risk and conditional tail expectation risk measures.

Although not mentioned as often, optimal reinsurance strategies that are beneficial to both the insurer and the reinsurer have also been reported. See, for example, Ignatov et al. (2004), and Dimitrova and Kaishev (2010). In Chapter 2, we will discuss in detail some of the optimal criteria in the literature.

1.3 An overview of Chapters 3 and 4

In Chapter 3, we introduce our first approach for finding optimal reinsurance that is beneficial to both the insurer and the reinsurer. Consider the following problem: The insurer underwrites an insurance policy with deductible $d > 0$. Then the total amount payable to the insureds is

$$(X-d)_+ = \begin{cases} 0 & \text{if} \quad X \leq d, \\ X-d & \text{if} \quad X > d. \end{cases}$$

Recall now the method of facultative reinsurance, in which negotiations between the insurer and the reinsurer take place for each insurance policy issued by the insurer. The insurer decides to purchase an excess of loss reinsurance policy with retention $R > 0$ and policy limit $L > 0$. When the claim size exceeds the reinsurer's policy limit L, a third party gets involved and covers the remaining amount. Let X_G be the third party's share of the claim. Then X_I, X_R, and X_G are given by

$$X_I = (X \wedge R) - (X \wedge d),$$

$$X_R = (X \wedge L) - (X \wedge R),$$

$$X_G = X - (X \wedge L),$$

where $\wedge$ means the minimum of the two values.

Note 1.3.1 Reinsurance companies themselves sometimes also need to purchase reinsurance. Hence, one example of the third party is another reinsurance company. In this case, the reinsurer (known as the retrocedent) passes on parts of the risk he/she has taken on from the direct insurer to another reinsurer (known as the retrocessionaire). For additional information, we refer to Schwepcke and Arndt (2004). Note that in this thesis, for the sake of simplicity, we assume that the insurer establishes reinsurance agreements with one reinsurer. In the real world, multiple reinsurers may be involved in the agreements. The reinsurer who sets the terms of the reinsurance contract is known as the lead reinsurer. The other reinsurers are known as the following reinsurers. When an insurance company collapses, special programs are in place to respond to unpaid claims of policyholders under policies issued by that insurance company. One example of such programs is the Property and Casualty Insurance Compensation Corporation (PACICC).

When the insurer does not share the claim with other parties, the variance of the amount payable to the insureds is

$$\mathbf{Var}[(X - d)_+].$$

When the insurer, the reinsurer, and the third party pay their shares of the claim, the variance of the amount payable to the insureds becomes the sum of the variances of X_I, X_R, and X_G, that is,

$$\mathbf{Var}[X_I] + \mathbf{Var}[X_R] + \mathbf{Var}[X_G].$$

The difference between $\mathbf{Var}[(X - d)_+]$ and $\mathbf{Var}[X_I] + \mathbf{Var}[X_R] + \mathbf{Var}[X_G]$ is called the variance reduction when the claim is shared. Based on this idea, we propose and explore an optimal criterion called the variance reduction approach. We then obtain an optimal reinsurance policy that is beneficial to the insurer, the reinsurer, and the third party.

In addition, in Chapter 3, we shall use the variance reduction approach to obtain an optimal reinsurance contract by the method of treaty reinsurance, in which multiple insurance policies issued by the insurer are reinsured. Excess of loss reinsurance is assumed with no policy limit, and no third party is involved. Two scenarios will be discussed in the cases when the claim size of each insurance policy can be independent or dependent on time. The dependent case will be discussed in detail in Chapter 4.

1.4 Value-at-risk and conditional tail expectation

Here we consider finding optimal reinsurance using risk measures. The value-at-risk (VaR) and the conditional tail expectation (CTE) are two of the most well known risk measures. We define the VaR as follows (cf., e.g., Denuit et al., 2005):

Definition 1.4.1 The VaR is the maximum amount of money that may be lost on a portfolio over a given period of time, with a given level of confidence. Specifically, the VaR at a given confidence level $1 - \alpha$ $(0 < \alpha < 1)$ over the considered time period is given by the smallest number $x \in \mathbb{R}$ such that the probability of a loss greater than x does not exceed α. The VaR of a random variable X at the confidence level $1 - \alpha$ is defined as

$$VaR_X(\alpha) = \inf\{x : \mathbf{P}(X > x) \leq \alpha\}. \tag{1.1}$$

The CTE is defined as follows (cf., e.g., Denuit et al., 2005):

Definition 1.4.2 The CTE is the expected value of X given that X exceeds a threshold value. Specifically, the CTE of X at the confidence level $1 - \alpha$ is defined as

$$CTE_X(\alpha) = \mathbf{E}[X \mid X \geq VaR_X(\alpha)]. \tag{1.2}$$

Note 1.4.3 The conditional tail expectation can be viewed as a weighted premium calculation principle when the weight function is an indicator function. For additional information, we refer to Furman and Landsman (2006), and Furman and Zitikis (2008). Various statistical inferential tools for the estimation and comparison of conditional tail expectations have been developed. For example, we refer to Brazauskas et al. (2008), and Necir et al. (2010). Properties that risk measures may satisfy include non-excessive loading, non-negative loading, translativity, constancy, subadditivity, comonotonic additivity, positive homogeneity, monotonicity, continuity with respect to convergence in distribution, and objectivity. Risk measures that satisfy translativity, positive homogeneity, subadditivity, and monotonicity are known as coherent risk measures. For additional information on these properties, we refer to Denuit et al. (2005). By finding the asymptotic distribution for the difference between empirical estimators of two risk measures, one can use non-parametric and parametric approaches. For additional information on these approaches, we refer to Jones and Zitikis (2005). For details on how empirical estimators of risk measures are obtained, we refer to Jones and Zitikis (2003, 2007). Jones et al. (2006) extend the empirical tests for the comparison of two risk measures by constructing tests for the equality of three or more risk measure values.

1.5 An overview of Chapter 5

In Chapter 5, we consider a CTE-based approach for constructing an optimal reinsurance contract. Under facultative reinsurance, the insurer purchases an excess of loss reinsurance policy with retention $R > 0$. Then the CTEs of X_I and X_R at a given confidence level $1-\alpha$ are defined as

$$CTE_{X_I}(\alpha, R) = \frac{1}{\alpha}\mathbf{E}[X_I \mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}]$$

and

$$CTE_{X_R}(\alpha, R) = \frac{1}{\alpha}\mathbf{E}[X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}],$$

respectively, where **1** denotes the indicator function. We have the equation

$$\begin{aligned}\mathbf{Cov}&[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}, X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] \\ &= \mathbf{E}[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] \\ &\quad - \mathbf{E}[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}]\mathbf{E}[X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}].\end{aligned}$$

In Chapter 5, we will explain why $\mathbf{Cov}[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}, X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}]$ is a special case of $\mathbf{Cov}[X_I, X_R]$, and we shall also show that an optimal reinsurance contract can be obtained when we maximize $\mathbf{Cov}[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}, X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}]$ with respect to R. To illustrate, we shall then find the optimal retention when X follows the exponential distribution and also the two-parameter Pareto distribution.

1.6 An overview of Chapter 6

In Chapter 6, we shall discuss a problem in the real estate industry: Consider a buyer who wants to purchase a property, and suppose that there are a number of similar properties on the market for sale. The buyer looks at the offers, one at a time, with a reservation price in mind. If the negotiated selling price is below the buyer's reservation price, the buyer purchases the property immediately and avoids the risk of losing the property to another potential buyer. If the negotiated selling price is above the buyer's reservation price, then the buyer moves on to the next offer. Unlike the case considered by Stigler (1962) and Gastwirth (1976), we assume that if the buyer passes on an offer, he/she does not have the option to go back and review it. This scenario is as realistic as the one of Stigler (1962) and Gastwirth (1976) because sellers and real estate agents often have multiple buying offers and, therefore, may not wait for one potential buyer's reply. The search ends when the buyer purchases a property with a negotiated selling price lower than her/his reservation price. If all of the properties under consideration are being sold at a price higher than the reservation price, then the buyer does not purchase a property.

In Chapter 6, the probability of the buyer purchasing a property will be formulated under various assumptions. We will start by assuming that a buyer's reservation price stays the same, and that the random variables of the negotiated selling prices are independent and identically distributed (i.i.d.). One or more of these assumptions will then be dropped. When random variables of the negotiated selling prices are dependent, we consider three ways to model their dependence: direct representation, copula representation, and background risk model. As an illustrative example, we shall calculate the probability of a buyer purchasing a condominium or a detached property in the London and St. Thomas area. The data are readily available in the London and St. Thomas Association of Realtors statistical report for the year 2012.

1.7 An overview of Chapter 7

In Chapter 7, we shall summarize the proposed methodologies and main results in this thesis. In addition, we shall also provide directions for future research.

Chapter 2

Optimal reinsurance: an overview

In this chapter, we recall and discuss some of the criteria that have been used in past studies for finding optimal reinsurance, including:

- Minimizing the variance of the insurer's retained loss (Section 2.1).
- Maximizing insurer's expected utility (Section 2.2).
- Finding an optimal reinsurance policy based on the insurer's survival probability (Section 2.3).
- Minimizing the CTE and VaR of the insurer's total cost (Section 2.4).
- Constructing an optimal reinsurance policy based on the probability of joint survival and the expected profit given joint survival (Section 2.5).

Of course, we have to note at the very outset that we will not be able to go into very detailed descriptions and discussions of the criteria, but we shall give a good flavour of the criteria.

2.1 Minimizing the variance of the insurer's retained loss

Kaluszka (2001) considers the following problem: The insurer pays a fixed premium amount for the reinsurance coverage. Three illustrative premium principles for determining the reinsurance premium p_R are provided by the following equations:

$$p_R = (1 + \theta)\mathbf{E}[X_R], \tag{2.1}$$

$$p_R = \mathbf{E}[X_R] + \theta\sqrt{\mathbf{Var}[X_R]}, \tag{2.2}$$

$$p_R = \mathbf{E}[X_R] + \theta\frac{\mathbf{Var}[X_R]}{\mathbf{E}[X_R]}, \tag{2.3}$$

where $\theta > 0$ is the premium loading coefficient. For the rest of this chapter, we denote the general reinsurance premium and the premium loading coefficient by p_R and θ, respectively.

Note 2.1.1 Three main methods are used to obtain premium principles: the ad hoc method, the characterization method, and the economic method. It is important to note that these methods are not mutually exclusive. For example, the Proportional Hazards Premium Principle can be derived using both the characterization method and the economic method. For additional information on the methods of developing premium principles, we refer to Young (2004). We also refer to Furman and Zitikis (2008, 2009), where a class of premium calculation principles, called weighted premiums and based on weighted loss distributions, has been discussed.

Kaluszka (2001) considers finding an optimal reinsurance contract by minimizing $\mathbf{Var}[X - X_R]$, which is the variance of the insurer's retained loss, subject to the value of reinsurance premium p_R. The variance $\mathbf{Var}[X - X_R]$ is minimized when the form of reinsurance is a combination of excess of loss and quota share, that is,

$$X_R = \alpha(X - R)_+,$$

where $0 < \alpha \leq 1$ and $R > 0$.

The problem is now to determine α and R. Recall two of the methods for finding p_R, which are given in equations (2.2) and (2.3). We rewrite $\mathbf{E}[X_R]$ as a function of p_R and $\sqrt{\mathbf{Var}[X_R]}$, that is, $f(p_R, \sqrt{\mathbf{Var}[X_R]})$. Kaluszka (2001) shows that if $\mathbf{E}[X] > f(p_R, \mathbf{Var}[X])$, then α and R exist and satisfy the following two equations:

$$-\mathbf{E}[(R - X)_+]f_2'(p_R, \alpha\mathbf{Var}[(X - R)_+]) = (1 - \alpha)\mathbf{Var}[(X - R)_+]$$

and

$$\alpha\mathbf{E}[(X - R)_+] = f(p_R, \alpha\mathbf{Var}[(X - R)_+]),$$

where

$$0 < R < \sup\{x : \mathbf{P}[X > x] > 0\}$$

and

$$f_2'(p_R, z) = \frac{\partial}{\partial z} f(p_R, z) \quad \text{for} \quad z > 0.$$

2.2 Maximizing insurer's expected utility

Suppose that the insurer's profit Π_I for a given period is

$$\Pi_I = p_I - p_R - (X - X_R),$$

where p_I is the premium that the insurer has collected from the insureds. Guerra and Centeno (2008) consider obtaining an optimal reinsurance policy by maximizing the expected utility function $\mathbf{E}_X[U(\Pi_I)]$, where $U(x) = -e^{-cx}$ for $x \in \mathbb{R}$ and $c > 0$. Let

$$G(z, \Omega(x)) = \int_0^\infty \exp\{-z(p_I - p_R - (x - \Omega(x)))\} f_X(x) \mathrm{d}x, \quad z > 0,$$

where f_X is the density function of X, and the function $\Omega(x)$ maps each possible value of the claims for a given period into the corresponding value refunded under the reinsurance policy for $x \geq 0$. By definition, the adjustment coefficient z_1 is the unique solution to $G(z_1, \Omega(x)) = 1$. When $\mathbf{E}_X[U(\Pi_I)] = -G(c, \Omega(x))$, Guerra and Centeno (2008) show that to maximize $\mathbf{E}_X[U(\Pi_I)]$, one must find the form of the function $\Omega(x)$ such that the adjustment coefficient is maximized.

Note 2.2.1 Here we provide some additional information on the expected utility theory (cf., e.g., Barbera et al., 2004). In economic theory, utility is usually understood as a numerical representation of a preference relation. Expected utility theory imposes a particular set of consistency conditions, which imply that the choice under uncertainty can be represented as the maximization of the expectation of the utility of consequences. According to the von Neumann and Morgenstern theory (cf., e.g., von Neumann and Morgenstern, 1953), utility functions of risk averters and risk seekers are concave and convex, respectively. Utility functions that have both concave and convex regions have been discussed as well. One example is the S-shaped utility function (cf., e.g., Broll et al., 2010).

2.3 Finding an optimal reinsurance policy based on the insurer's survival probability

Gajek and Zagrodny (2004) take the insurer's survival probability into consideration. During a considered period of time, let v be the value of the initial surplus. Furthermore, let p_I be the premium amount that the insurer has collected from the insureds. Finally, let p be the maximum amount that the insurer can spend on the reinsurance policy. Then the insurer's asset after purchasing the reinsurance policy is $v + p_I - p$. When $\mathbf{P}[X < v + p_I] = 1$, then the insurance company has enough assets to cover the claim amount. However, the insurer will seek help when $0 < \mathbf{P}[X < v + p_I] < 1$. Gajek and Zagrodny (2004) discuss two scenarios when $\mathbf{P}[X < v + p_I] < 1$, which we briefly review next.

The insurer needs to pay the premium $\pi_R(\mathbf{E}[\Omega_1(X)])$ for the reinsurance protection, where π_R is an increasing function and $\Omega_1(X)$ is the part of the total claim amount covered by the reinsurer. When $\pi_R(\mathbf{E}[(X - (v + p_I - p))_+]) \leq p$, the insurer can afford purchasing excess of loss reinsurance with retention $v + p_I - p$. The insurer's probability of ruin is then reduced to zero since the insurer is paying for the full reinsurance protection. Gajek and Zagrodny (2004) prove that in this case the optimal form of reinsurance is excess of loss with retention R^*, where R^* is the maximum value of $R \geq 0$ such that

$$\pi_R(\mathbf{E}[(X - R)_+]) \leq v + p_I - R.$$

When $\pi_R(\mathbf{E}[(X - (v + p_I - p))_+]) > p$, the insurer's probability of ruin is larger than zero. If the distribution of X is continuous and $\pi_R(x + (1 - F_X(v + p_I - p))(p - \pi_R(x))) \leq p$ for $0 \leq x \leq \pi_R^{-1}(p)$, then an optimal reinsurance policy is obtained when the reinsurer's share of the claim is written in the following form:

$$\Omega_1^*(x) = \begin{cases} 0 & \text{if } \; 0 \leq x \leq v + p_I - p, \\ x - v - p_I + p & \text{if } \; v + p_I - p < x \leq L, \\ 0 & \text{if } \; x > L, \end{cases}$$

where L is such that

$$p = \pi_R(\mathbf{E}[\Omega_1^*]).$$

2.4 Minimizing the CTE and VaR of the insurer's total cost

Cai and Tan (2007) discuss the construction of an optimal reinsurance contract in the excess of loss form under VaR and CTE risk measures: Suppose that the insurer purchases excess of loss reinsurance with retention $R > 0$. Then

$$VaR_{X_I}(\alpha, R) = \begin{cases} R & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) & \text{if} \quad R > S_X^{-1}(\alpha), \end{cases}$$

where S_X is the survival function of X and $0 < \alpha < S_X(0)$. Next, let T be the insurer's total cost in the presence of reinsurance, which is given by

$$T = X_I + \pi_R(\mathbf{E}[(X - R)_+]),$$

where π_R is defined in Section 2.3. Using the formula for the VaR of a random variable X at the confidence level $1 - \alpha$, which is given in equation (1.1), we have

$$VaR_T(\alpha, R) = \begin{cases} R + \pi_R(\mathbf{E}[(X - R)_+]) & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) + \pi_R(\mathbf{E}[(X - R)_+]) & \text{if} \quad R > S_X^{-1}(\alpha), \end{cases}$$

where $0 < \alpha < S_X(0)$. Cai and Tan (2007) consider minimizing $VaR_T(\alpha, R)$ with respect to R, that is,

$$\min_R VaR_T(\alpha, R).$$

They show that the optimal retention R_1^* exists if and only if

$$\alpha < \frac{1}{1+\theta} < S_X(0)$$

and

$$S_X^{-1}(\alpha) \geq S_X^{-1}\left(\frac{1}{1+\theta}\right) + \pi_R\left(\left(\frac{1}{1+\theta}\right)\right).$$

When R_1^* exists, it is given by

$$R_1^* = S_X^{-1}\left(\frac{1}{1+\theta}\right).$$

Next, using the formula for the CTE of X at the confidence level $1 - \alpha$, which is given in equation (1.2), we have

$$CTE_T(\alpha, R) = \begin{cases} R + \pi_R(\mathbf{E}[(X-R)_+]) & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) + \pi_R(\mathbf{E}[(X-R)_+]) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^{R} S_X(x)\mathrm{d}x & \text{if} \quad R > S_X^{-1}(\alpha). \end{cases}$$

Cai and Tan (2007) consider minimizing $CTE_T(\alpha, R)$ with respect to R, that is,

$$\min_R CTE_T(\alpha, R),$$

and show that the optimal retention R_2^* exists if and only if

$$0 < \alpha \leq \frac{1}{1+\theta} < S_X(0).$$

When R_2^* exists, it is given by

$$R_2^* = S_X^{-1}\left(\frac{1}{1+\theta}\right) \quad \text{for} \quad \alpha < \frac{1}{1+\theta},$$

$$R_2^* \geq S_X^{-1}\left(\frac{1}{1+\theta}\right) \quad \text{for} \quad \alpha = \frac{1}{1+\theta}.$$

2.5 An optimal criterion that is beneficial to both the insurer and the reinsurer

In this section, we review the work of Dimitrova and Kaishev (2010), in which an optimal reinsurance policy that is beneficial to both the insurer and the reinsurer is constructed. If the inter-claim times, say $\tau_1, \tau_2, \ldots$, are identically and exponentially distributed with parameter $\lambda > 0$, then the arrival time of the first claim is $T_1 = \tau_1$, the arrival time of the second claim is $T_2 = \tau_1 + \tau_2$, and so on. Let $N(t)$ be the number of claims that arrive up to and including the time t. Furthermore, let $X_1, X_2, \ldots$ be the claim sizes. Then for every $i \geq 1$, the insurer's share of the i^{th} claim is

$$X_{i,I} = (X_i \wedge R) + (X_i - L)_+,$$

where R is the retention and L is the policy limit ($0 < R < L$), and the reinsurer's share is

$$X_{i,R} = \min((X_i - R)_+, L - R).$$

The insurer and the reinsurer also share the premium income, which is given by

$$h(t) = h_I(t) + h_R(t),$$

where

- $h(t)$ is the aggregate premium income up to and including the time t;
- $h_I(t)$ is the insurer's share of the premium income up to and including the time t;
- $h_R(t)$ is the reinsurer's share up to and including the time t.

The functions h, h_I, and h_R are non-negative and non-decreasing. Consequently, the insurer's profit up to and including the time t is

$$\Pi_I(t) = h_I(t) - \sum_{i=1}^{N(t)} X_i^I,$$

and the reinsurer's profit up to and including the time t is

$$\Pi_R(t) = h_R(t) - \sum_{i=1}^{N(t)} X_i^R.$$

Let T_I be the insurer's time of ruin, and let T_R be the reinsurer's time of ruin. Dimitrova and Kaishev (2010) search for a common solution for both the insurer and the reinsurer by using the following optimal criterion: Suppose that $h(t) = h_I(t) + h_R(t)$, $h_I(t) = ah(t)$, and $h_R(t) = (1 - a)h(t)$ for $0 \leq a \leq 1$. Then the retention R^* and the policy limit L^* of the optimal reinsurance policy are such that $1 - \mathbf{P}[T_I > x, T_R > x]$ is minimized subject to

$$\frac{\mathbf{E}[\Pi_I(x) \mid T_I > x, T_R > x]}{\mathbf{E}[\Pi_R(x) \mid T_I > x, T_R > x]} = \frac{a}{1 - a},$$

where $x \geq 0$.

Chapter 3

The variance reduction approach

In this chapter, we develop optimal reinsurance by reducing the variance of the amount payable to the insureds. Consider the following problem assuming the method of facultative reinsurance: The insurer underwrites an insurance policy with deductible $d > 0$ and decides to acquire an excess of loss reinsurance policy from the reinsurer with retention $R > 0$ and policy limit $L > 0$, where $d < R < L$. When the claim size exceeds the policy limit L, the reinsurer seeks help from a third party to cover the remaining amount.

Although the decisions of all three parties affect the final outcome, we assume that the insurer does not negotiate directly with the third party. This assumption is reasonable since the insurer does not care how the reinsurer covers the amount exceeding the retention, as long as it is covered. Furthermore, we assume that the retention is decided between the insurer and the reinsurer, and that the policy limit is determined between the reinsurer and the third party.

Let X_G be the third party's share of the claim. Then we have

$$X_I = (X \wedge R) - (X \wedge d),$$

$$X_R = (X \wedge L) - (X \wedge R),$$

$$X_G = X - (X \wedge L).$$

The variance of the amount payable to the insureds is

$$\begin{aligned}
\mathbf{Var}[(X-d)_+] &= \mathbf{E}[(X-d)_+^2] - (\mathbf{E}[(X-d)_+])^2 \\
&= \mathbf{E}[(X-(X\wedge d))^2] - (\mathbf{E}[X-(X\wedge d)])^2 \\
&= \mathbf{E}[X^2] + \mathbf{E}[(X\wedge d)^2] - 2\mathbf{E}[X(X\wedge d)] - (\mathbf{E}[X])^2 - (\mathbf{E}[X\wedge d])^2 \\
&\quad + 2\mathbf{E}[X]\mathbf{E}[X\wedge d].
\end{aligned}$$

Similarly, the variance of the insurer's share of the claim is given by

$$\begin{aligned}
\mathbf{Var}[X_I] &= \mathbf{E}[X_I^2] - (\mathbf{E}[X_I])^2 \\
&= \mathbf{E}[((X\wedge R)-(X\wedge d))^2] - (\mathbf{E}[X\wedge R] - \mathbf{E}[X\wedge d])^2 \\
&= \mathbf{E}[(X\wedge R)^2] + \mathbf{E}[(X\wedge d)^2] - 2\mathbf{E}[(X\wedge R)(X\wedge d)] - (\mathbf{E}[X\wedge R])^2 \\
&\quad - (\mathbf{E}[X\wedge d])^2 + 2\mathbf{E}[X\wedge R]\mathbf{E}[X\wedge d].
\end{aligned}$$

The variance of the reinsurer's share of the claim is given by

$$\begin{aligned}
\mathbf{Var}[X_R] &= \mathbf{E}[X_R^2] - (\mathbf{E}[X_R])^2 \\
&= \mathbf{E}[((X\wedge L)-(X\wedge R))^2] - (\mathbf{E}[X\wedge L] - \mathbf{E}[X\wedge R])^2 \\
&= \mathbf{E}[(X\wedge L)^2] + \mathbf{E}[(X\wedge R)^2] - 2\mathbf{E}[(X\wedge L)(X\wedge R)] - (\mathbf{E}[X\wedge L])^2 \\
&\quad - (\mathbf{E}[X\wedge R])^2 + 2\mathbf{E}[X\wedge L]\mathbf{E}[X\wedge R].
\end{aligned}$$

Finally, the variance of the third party's share of the claim is given by

$$\begin{aligned}
\mathbf{Var}[X_G] &= \mathbf{E}[X_G^2] - (\mathbf{E}[X_G])^2 \\
&= \mathbf{E}[(X-(X\wedge L))^2] - [\mathbf{E}(X) - \mathbf{E}(X\wedge L)]^2 \\
&= \mathbf{E}[X^2] + \mathbf{E}[(X\wedge L)^2] - 2\mathbf{E}[X(X\wedge L)] - (\mathbf{E}[X])^2 - (\mathbf{E}[X\wedge L])^2 \\
&\quad + 2\mathbf{E}[X]\mathbf{E}[X\wedge L].
\end{aligned}$$

Next, we follow Borch (1974) and find an optimal criterion that is beneficial to the insurer, the reinsurer, and the third party. If the insurance company does not share the claim with other parties, then the variance of the amount payable to the insureds is $\mathbf{Var}[(X-d)_+]$. When the three parties share the claim, the variance of the amount payable to the insureds becomes $\mathbf{Var}[X_I] + \mathbf{Var}[X_R] + \mathbf{Var}[X_G]$. The difference between $\mathbf{Var}[(X-d)_+]$ and $\mathbf{Var}[X_I] + \mathbf{Var}[X_R] + \mathbf{Var}[X_G]$

is known as the variance reduction when the claim is shared. Our developed methodology is based on this idea. We call it the variance reduction approach. The rest of this chapter is organized as follows:

- In Section 3.1, we establish the optimal criterion as maximizing $\mathbf{Cov}[X_I, X_R]$ and/or $\mathbf{Cov}[X_R, X_G]$.
- In Section 3.2, we check if the correlation coefficient can be used as an alternative optimal criterion.
- In Section 3.3, we find the optimal retention R^* and/or the optimal policy limit L^* under various scenarios.
- In Section 3.4, we use the variance reduction approach assuming the method of treaty reinsurance. We find the optimal retention when the insurer and the reinsurer negotiate a reinsurance contract in the excess of loss form that includes multiple insurance policies issued by the insurer, and when the claim size of each insurance policy is independent of time.

3.1 Establishing the optimal criterion

When the insurer negotiates with the reinsurer to determine the retention R, the two parties share the amount

$$X_I + X_R = (X \wedge L) - (X \wedge d).$$

Without sharing, the variance of this amount is

$$\mathbf{Var}[X_I + X_R].$$

When the risk is shared between the insurer and the reinsurer, the variance becomes

$$\mathbf{Var}[X_I] + \mathbf{Var}[X_R].$$

We can write

$$\mathbf{Var}[X_I + X_R] = \mathbf{Var}[X_I] + \mathbf{Var}[X_R] + 2\mathbf{Cov}[X_I, X_R].$$

Our goal is to maximize the difference between $\mathbf{Var}[X_I + X_R]$ and $\mathbf{Var}[X_I] + \mathbf{Var}[X_R]$. To do so, we need to maximize the covariance $\mathbf{Cov}[X_I, X_R]$ with respect to the retention R.

When the reinsurer and the third party communicate to determine the policy limit L, they share the amount

$$X_R + X_G = X - (X \wedge R).$$

Without sharing, the variance of this amount is

$$\mathbf{Var}[X_R + X_G].$$

When the risk is shared between the reinsurer and the third party, the variance becomes

$$\mathbf{Var}[X_R] + \mathbf{Var}[X_G].$$

Similar to negotiations between the insurer and the reinsurer, we wish to maximize the difference between $\mathbf{Var}[X_R + X_G]$ and $\mathbf{Var}[X_R] + \mathbf{Var}[X_G]$. In other words, we need to maximize the covariance $\mathbf{Cov}[X_R, X_G]$ with respect to the policy limit L.

Next, we obtain expressions for the covariances $\mathbf{Cov}[X_I, X_R]$ and $\mathbf{Cov}[X_R, X_G]$.

Note 3.1.1 Covariance plays pivotal roles when solving a number of problems in actuarial science, statistics, economics, and finance. In our model, a closed-form expression for the covariance can be obtained. However, in many scenarios, this can be a challenging task. One possible solution to the task is the covariance decomposition. For details, we refer to Furman and Zitikis (2010). In addition to finding closed-form expressions, studies related to estimating the covariance of two random variables have also been reported in the literature. One example is the Grüss-type bound for the covariance of two transformed random variables (cf., e.g., Zitikis, 2009; Egozcue et al., 2010, 2011*b*).

Theorem 3.1.2 *Let the excess of loss reinsurance policy have retention $R > 0$ and policy limit $L > 0$. Then the covariances $\mathbf{Cov}[X_I, X_R]$ and $\mathbf{Cov}[X_R, X_G]$ are given by*

$$\mathbf{Cov}[X_I, X_R] = \int_d^R F_X(y)\mathrm{d}y \int_R^L S_X(y)\mathrm{d}y, \tag{3.1}$$

$$\mathbf{Cov}[X_R, X_G] = \int_R^L F_X(y)\mathrm{d}y \int_L^\infty S_X(y)\mathrm{d}y, \tag{3.2}$$

where F_X is the distribution function of X and S_X is the survival function of X.

Proof We begin with the covariance of X_I and X_R, which can be written as follows:

$$\mathbf{Cov}[X_I, X_R] = \mathbf{E}[X_I X_R] - \mathbf{E}[X_I]\mathbf{E}[X_R].$$

Recall that

$$X_I = (X \wedge R) - (X \wedge d),$$

$$X_R = (X \wedge L) - (X \wedge R),$$

$$X_G = X - (X \wedge L).$$

Using these representations, the covariance becomes

$$\begin{aligned}\mathbf{Cov}[X_I, X_R] &= \mathbf{E}[((X \wedge R) - (X \wedge d))((X \wedge L) - (X \wedge R))] - \mathbf{E}[(X \wedge R) - (X \wedge d)] \\ &\quad \times \mathbf{E}[(X \wedge L) - (X \wedge R)] \\ &= (\mathbf{E}[(X \wedge R)(X \wedge L)] - \mathbf{E}[X \wedge R]\mathbf{E}[X \wedge L]) \\ &\quad - (\mathbf{E}[(X \wedge d)(X \wedge L)] - \mathbf{E}[X \wedge d]\mathbf{E}[X \wedge L]) \\ &\quad - (\mathbf{E}[(X \wedge R)^2] - (\mathbf{E}[X \wedge R])^2) \\ &\quad + (\mathbf{E}[(X \wedge d)(X \wedge R)] - \mathbf{E}[X \wedge d]\mathbf{E}[X \wedge R]).\end{aligned}$$

Next, we write $\mathbf{Cov}[X_I, X_R]$ in terms of the distribution function of X and obtain

$$\begin{aligned}&\mathbf{Cov}[X_I, X_R] \\ &= \Big(R^2 - 2\int_0^R yF_X(y)\mathrm{d}y + R\int_0^L (1 - F_X(y))\mathrm{d}y - R\int_0^R (1 - F_X(y))\mathrm{d}y - \int_0^d (1 - F_X(y))\mathrm{d}y \\ &\times \int_0^L (1 - F_X(y))\mathrm{d}y\Big) - \Big(d^2 - 2\int_0^d yF_X(y)\mathrm{d}y + d\int_0^L (1 - F_X(y))\mathrm{d}y - d\int_0^d (1 - F_X(y))\mathrm{d}y \\ &- \int_0^d (1 - F_X(y))\mathrm{d}y \int_0^L (1 - F_X(y))\mathrm{d}y\Big) - \Big(R^2 - 2\int_0^R yF_X(y)\mathrm{d}y - \Big(\int_0^R (1 - F_X(y))\mathrm{d}y\Big)^2\Big) \\ &+ \Big(d^2 - 2\int_0^d yF_X(y)\mathrm{d}y + d\int_0^R (1 - F_X(y))\mathrm{d}y - d\int_0^d (1 - F_X(y))\mathrm{d}y - \int_0^d (1 - F_X(y))\mathrm{d}y \int_0^R (1 - F_X(y))\mathrm{d}y\Big).\end{aligned}$$

Consequently, we have

$$\mathbf{Cov}[X_I, X_R] = \int_d^R F_X(y)\mathrm{d}y \int_R^L S_X(y)\mathrm{d}y.$$

Similarly, we derive an expression for $\mathbf{Cov}[X_R, X_G]$. The result is provided in equation (3.2). This completes the proof of Theorem 3.1.2. □

We note from Theorem 3.1.2 that the covariances are non-negative. In other words, the difference between $\mathbf{Var}[X_I + X_R]$ and $\mathbf{Var}[X_I] + \mathbf{Var}[X_R]$ is positive, and the difference between $\mathbf{Var}[X_R + X_G]$ and $\mathbf{Var}[X_R] + \mathbf{Var}[X_G]$ is positive. This confirms the fact that the variance of the amount payable to the insureds is reduced when the claim is shared.

Note 3.1.3 Determining the sign of the covariance of two real-valued transformations of a random variable can be challenging in some cases. For additional information, we refer to Egozcue et al. (2011*a*).

Since the correlation coefficient is written in terms of the covariance, a natural question to ask is whether we can maximize the correlation coefficients $\mathbf{Corr}[X_I, X_R]$ and $\mathbf{Corr}[X_R, X_G]$ instead. In what follows, we use a numerical example to check if maximizing the correlation coefficient could be a suitable criterion for optimizing the reinsurance policy.

3.2 An illustrative example

We assume throughout this section that the total claim amount X follows a discrete distribution, and that only one insurance policy is issued by the insurer with no deductible, that is, $d = 0$. The insurance policy is then reinsured. The excess of loss reinsurance policy has retention R and policy limit L. We assume that the claim amount never exceeds the policy limit. Hence, no third party gets involved. Denote the optimal retention that maximizes $\mathbf{Cov}[X_I, X_R]$ by R_1, and the optimal retention that maximizes $\mathbf{Corr}[X_I, X_R]$ by R_2. Next, to check if R_1 and R_2 can be obtained, we present an illustrative example.

Suppose that for $0 < x_1 < x_2 \leq L$, the loss X is of the following form:

$$X = \begin{cases} x_1 & \text{with probability} \quad p_1, \\ x_2 & \text{with probability} \quad p_2, \end{cases} \tag{3.3}$$

where $p_1 + p_2 = 1$ and $p_1 > p_2$. Next, we shall derive an expression for $\mathbf{Cov}[X_I, X_R]$ and find R_1.

Theorem 3.2.1 *Suppose X follows the discrete distribution given by equation (3.3). Furthermore, let the retention of the excess of loss reinsurance policy be $R > 0$. Then the covariance of X_I and X_R is given by*

$$\mathbf{Cov}[X_I, X_R] = \begin{cases} 0 & \textit{when} \quad R \leq x_1, \\ (Rp_1 - x_1 p_1)(x_2 - R)p_2 & \textit{when} \quad x_1 < R \leq x_2, \\ 0 & \textit{when} \quad R > x_2. \end{cases} \tag{3.4}$$

Moreover, the optimal retention R_1 that maximizes $\mathbf{Cov}[X_I, X_R]$ is

$$R_1 = \frac{x_1 + x_2}{2}. \tag{3.5}$$

Proof Given distribution (3.3) of X, the covariance of X_I and X_R can be written as follows:

$$\begin{aligned} \mathbf{Cov}[X_I, X_R] = {} & (x_1 \wedge R)(x_1 - R)_+ p_1 + (x_2 \wedge R)(x_2 - R)_+ p_2 \\ & - ((x_1 \wedge R)p_1 + (x_2 \wedge R)p_2)((x_1 - R)_+ p_1 + (x_2 - R)_+ p_2). \end{aligned}$$

We now rewrite $\mathbf{Cov}[X_I, X_R]$ for $R \leq x_1$, $x_1 < R \leq x_2$, and $R > x_2$. When $R \leq x_1$, the covariance becomes

$$\begin{aligned} \mathbf{Cov}[X_I, X_R] &= R(x_1 - R)p_1 + R(x_2 - R)p_2 - (Rp_1 + Rp_2)((x_1 - R)p_1 + (x_2 - R)p_2) \\ &= R(1 - p_1 - p_2)((x_1 - R)p_1 + (x_2 - R)p_2). \end{aligned}$$

Since $p_1 + p_2 = 1$, we obtain

$$\mathbf{Cov}[X_I, X_R] = 0.$$

When $x_1 < R \leq x_2$, the covariance becomes

$$\begin{aligned} \mathbf{Cov}[X_I, X_R] &= R(x_2 - R)p_2 - (x_1 p_1 + Rp_2)(x_2 - R)p_2 \\ &= (Rp_1 - x_1 p_1)(x_2 - R)p_2. \end{aligned}$$

When $R > x_2$, we have $(x_1 - R)_+ = 0$ and $(x_2 - R)_+ = 0$. Hence, the covariance becomes

$$\mathbf{Cov}[X_I, X_R] = 0.$$

From these formulas, we see that the covariance is maximized when $x_1 < R \leq x_2$.

To obtain the retention that maximizes $\mathbf{Cov}[X_I, X_R]$, we first need to calculate the critical point(s) of $\mathbf{Cov}[X_I, X_R]$, and then find the critical point that maximizes $\mathbf{Cov}[X_I, X_R]$, which is the optimal retention. We begin by finding the critical point(s) of $\mathbf{Cov}[X_I, X_R]$ for $x_1 < R \leq x_2$.

Definition 3.2.2 (cf., e.g., Larson and Edwards, 2010) A critical, or stationary, point of a differentiable function is any value in its domain where its derivative is 0.

From this definition, we obtain the critical point(s) by differentiating $\mathbf{Cov}[X_I, X_R]$ with respect to R and equating the first derivative to zero, that is,

$$\frac{\partial}{\partial R}\mathbf{Cov}[X_I, X_R] = 0.$$

We then solve for R. The solution, which we denote by $\tilde{R}_1$, is given by

$$\tilde{R}_1 = \frac{x_1 + x_2}{2}.$$

Next, we show that this critical point $\tilde{R}_1$ maximizes $\mathbf{Cov}[X_I, X_R]$. This can be achieved using the second-derivative test.

Definition 3.2.3 (cf., e.g., Larson and Edwards, 2010) The second-derivative test is a criterion for determining whether a given critical point of a real function of one variable is a local maximum or a local minimum using the value of the second derivative at the point. The test states that if the function f is twice differentiable at a critical point x (i.e. $f'(x) = 0$), then we have the following three statements:

- If $f''(x) < 0$, then f has a local maximum at x.
- If $f''(x) > 0$, then f has a local minimum at x.
- If $f''(x) = 0$, then x is the inflection point.

Therefore, we need to show that the second derivative of $\mathbf{Cov}[X_I, X_R]$ with respect to R evaluated at $\tilde{R}_1$ is negative, that is,

$$\left(\frac{\partial}{\partial R}\right)^2 \mathbf{Cov}[X_I, X_R]\bigg|_{R=\tilde{R}_1} < 0.$$

The second derivative of $\mathbf{Cov}[X_I, X_R]$ with respect to R is given by

$$\left(\frac{\partial}{\partial R}\right)^2 \mathbf{Cov}[X_I, X_R] = -2p_1p_2.$$

Since $-2p_1p_2 < 0$, we conclude that R_1 maximizes $\mathbf{Cov}[X_I, X_R]$. This completes the proof of Theorem 3.2.1. □

To illustrate graphically, in Figure 3.1, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when $x_1 = 10$, $x_2 = 1000$, $p_1 = 0.9$, and $p_2 = 0.1$. Note that $\mathbf{Cov}[X_I, X_R]$ is maximized when $R = 505$.

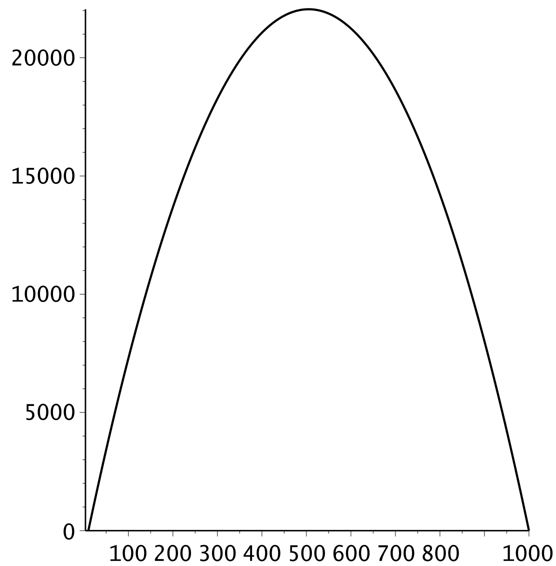

Figure 3.1: Retention R versus $\mathbf{Cov}[X_I, X_R]$ with $\mathbf{Cov}[X_I, X_R]$ given by equation (3.4).

Next, we find the retention that maximizes $\mathbf{Corr}[X_I, X_R]$.

Lemma 3.2.4 *Under conditions of Theorem 3.2.1, the correlation coefficient between X_I and X_R is given by*

$$\mathbf{Corr}[X_I, X_R] = 1 \quad \textit{for} \quad x_1 < R \le x_2,$$

and it is undefined for other values of R.

Proof The correlation coefficient between X_I and X_R can be written as follows:

$$\mathbf{Corr}[X_I, X_R] = \frac{\mathbf{Cov}[X_I, X_R]}{\sqrt{\mathbf{Var}(X_I)}\sqrt{\mathbf{Var}(X_R)}},$$

where

$$\mathbf{Var}[X_I] = ((x_1 \wedge R)^2 p_1 + (x_2 \wedge R)^2 p_2) - ((x_1 \wedge R)p_1 + (x_2 \wedge R)p_2)^2$$

and

$$\mathbf{Var}[X_R] = ((x_1 - R)_+^2 p_1 + (x_2 - R)_+^2 p_2) - ((x_1 - R)_+ p_1 + (x_2 - R)_+ p_2)^2.$$

Next, we rewrite $\mathbf{Corr}[X_I, X_R]$ for $R \leq x_1$, $x_1 < R \leq x_2$, and $R > x_2$. When $R \leq x_1$, we obtain

$$\begin{aligned}\mathbf{Var}[X_I] &= R^2 p_1 + R^2 p_2 - (Rp_1 + Rp_2)^2 \\ &= R(Rp_1 + Rp_2)(1 - p_1 - p_2).\end{aligned}$$

Since $p_1 + p_2 = 1$, we have that the variance $\mathbf{Var}[X_I] = 0$. Therefore, the correlation coefficient between X_I and X_R is undefined. When $x_1 < R \leq x_2$, we obtain

$$\begin{aligned}\mathbf{Var}[X_I] &= x_1^2 p_1 + R^2 p_2 - (x_1 p_1 + Rp_2)^2 \\ &= p_1 p_2 (x_1 - R)^2\end{aligned}$$

and

$$\begin{aligned}\mathbf{Var}[X_R] &= (x_2 - R)^2 p_2 - [(x_2 - R)p_2]^2 \\ &= p_1 p_2 (x_2 - R)^2.\end{aligned}$$

Then the correlation coefficient becomes

$$\begin{aligned}\mathbf{Corr}[X_I, X_R] &= \frac{(Rp_1 - x_1 p_1)(x_2 - R)p_2}{\sqrt{p_1 p_2 (x_1 - R)^2}\sqrt{p_1 p_2 (x_2 - R)^2}} \\ &= 1.\end{aligned}$$

When $R > x_2$, we have $(x_1 - R)_+ = 0$ and $(x_2 - R)_+ = 0$. Consequently,

$$\mathbf{Var}[X_R] = 0.$$

Therefore, the correlation coefficient between X_I and X_R is undefined. This completes the proof of Theorem 3.2.4. □

From this example, and the same conclusion actually holds if the binary distribution in (3.3) is replaced by any distribution that has any finite number of points, we note that the optimal retention R_1 that maximizes $\mathbf{Cov}[X_I, X_R]$ can be obtained. However, no optimal retention is obtained when we maximize $\mathbf{Corr}[X_I, X_R]$ with respect to the retention. Therefore, we conclude that maximizing the correlation coefficient is not a suitable criterion for finding an optimal reinsurance policy.

3.3 Covariance maximization in the presence of insurer, reinsurer, and a third party

We now construct an optimal reinsurance policy by maximizing $\mathbf{Cov}[X_I, X_R]$ and/or $\mathbf{Cov}[X_R, X_G]$. We assume that X follows a two-parameter Pareto distribution. Furthermore, we assume that the deductible d of the insurance policy issued by the insurer is fixed and known. The rest of this section is organized as follows:

- In Subsection 5.4.2, we discuss some properties of a two-parameter Pareto distribution.
- In Subsection 3.3.2, we obtain the optimal retention R^* that maximizes $\mathbf{Cov}[X_I, X_R]$ when the policy limit L is fixed and the retention R is still to be determined.
- In Subsection 3.3.3, we find the optimal policy limit L^* that maximizes $\mathbf{Cov}[X_R, X_G]$ when R is fixed and L is still to be determined.
- In Subsection 3.3.4, we find the optimal retention and the optimal policy limit when both R and L are still to be determined.

3.3.1 Two-parameter Pareto distribution

The probability density function (PDF) and the cumulative distribution function (CDF) of a two-parameter Pareto random variable with parameters $\alpha > 1$ and $\theta > 0$ are given by

$$f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \tag{3.6}$$

and

$$F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha, \tag{3.7}$$

respectively. The Pareto distribution is a heavy-tailed distribution.

Definition 3.3.1 (cf., e.g., Asmussen, 2003) Heavy-tailed distributions are probability distributions whose tails are not exponentially bounded, that is, they have heavier tails than the exponential distribution. In many applications it is the right tail of the distribution that is of interest, but a distribution may have a heavy left tail, or both tails may be heavy. The distribution of a random variable Y with cumulative distribution function F is said to have a heavy right tail if

$$\lim_{y\to\infty} e^{\lambda y}\mathbf{P}[Y > y] = \infty \quad \text{for all } \lambda > 0.$$

3.3.2 Finding the retention

In the next theorem, we find an optimal retention when the policy limit is already determined.

Theorem 3.3.2 *Suppose X follows a two-parameter Pareto distribution with parameters $\alpha > 1$ and $\theta > 0$. Let the deductible $d > 0$ of the insurance policy issued by the insurer be fixed. Furthermore, let the policy limit $L > 0$ of the reinsurance contract be fixed. Then the optimal retention R^* of the reinsurance contract that maximizes the covariance of X_I and X_R must satisfy the following two properties:*

$$\begin{aligned}
&\theta^{\alpha}(L+\theta)^{1-\alpha} - \theta^{2\alpha}(R^*+\theta)^{-\alpha}(L+\theta)^{1-\alpha} - \theta^{\alpha}(R^*+\theta)^{1-\alpha} + 2\,\theta^{2\alpha}(R^*+\theta)^{-2\alpha+1}\\
&- R^*\theta^{\alpha}(R^*+\theta)^{-\alpha} + R^*\theta^{\alpha}(R^*+\theta)^{-\alpha}\alpha + d\theta^{\alpha}(R^*+\theta)^{-\alpha} - d\theta^{\alpha}(R^*+\theta)^{-\alpha}\alpha\\
&- \theta^{2\alpha}(R^*+\theta)^{-\alpha}(d+\theta)^{1-\alpha} = 0
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\alpha\theta^{2\alpha}(d+\theta)^{\alpha-1}(R^*+\theta)^{\alpha-1} - 2\theta^{\alpha}(1-\alpha)(d+\theta)^{\alpha-1}(R^*+\theta)^{\alpha}(L+\theta)^{\alpha-1}\\
&+ 2(1-2\alpha)\theta^{2\alpha}(d+\theta)^{\alpha-1}(L+\theta)^{\alpha-1} + (\alpha(R^*-d)\theta^{\alpha}(1-\alpha)(d+\theta)^{\alpha-1}\\
&+ \alpha\theta^{2\alpha})(R^*+\theta)^{\alpha-1}(L+\theta)^{\alpha-1} < 0.
\end{aligned} \tag{3.9}$$

Proof Given the distribution of X, by differentiating $\mathbf{Cov}[X_I, X_R]$ with respect to R, we obtain

$$\begin{aligned}\frac{\partial}{\partial R}\mathbf{Cov}[X_I, X_R] = {} & \frac{\theta^\alpha}{1-\alpha}(L+\theta)^{-\alpha+1} - \frac{\theta^{2\alpha}}{1-\alpha}(R+\theta)^{-\alpha}(L+\theta)^{-\alpha+1} - \frac{\theta^\alpha}{1-\alpha}(R+\theta)^{-\alpha+1} \\ & + \frac{\theta^{2\alpha}}{1-\alpha}(R+\theta)^{-2\alpha+1} - (R-d)\left(\frac{\theta}{R+\theta}\right)^\alpha + \frac{\theta^{2\alpha}}{1-\alpha}(R+\theta)^{-2\alpha+1} \\ & - \frac{\theta^{2\alpha}}{1-\alpha}(R+\theta)^{-\alpha}(d+\theta)^{-\alpha+1}. \end{aligned} \tag{3.10}$$

Next, we set

$$\frac{\partial}{\partial R}\mathbf{Cov}[X_I, X_R] = 0,$$

and then solve for R. Using the "solve" feature of Maple, the solution, which we denote by $\tilde{R}$, must satisfy

$$\begin{aligned}&\theta^\alpha(L+\theta)^{1-\alpha} - \theta^{2\,\alpha}(\tilde{R}+\theta)^{-\alpha}(L+\theta)^{1-\alpha} - \theta^\alpha(\tilde{R}+\theta)^{1-\alpha} + 2\,\theta^{2\,\alpha}(\tilde{R}+\theta)^{-2\,\alpha+1} - \tilde{R}\theta^\alpha(\tilde{R}+\theta)^{-\alpha} \\ &+ \tilde{R}\theta^\alpha(\tilde{R}+\theta)^{-\alpha}\alpha + d\theta^\alpha(\tilde{R}+\theta)^{-\alpha} - d\theta^\alpha(\tilde{R}+\theta)^{-\alpha}\alpha - \theta^{2\,\alpha}(\tilde{R}+\theta)^{-\alpha}(d+\theta)^{1-\alpha} = 0,\end{aligned}$$

which is the first condition that the optimal retention must satisfy, reported as equation (3.8) above. The critical point $\tilde{R}$ maximizes $\mathbf{Cov}[X_I, X_R]$ if

$$\left(\frac{\partial}{\partial R}\right)^2 \mathbf{Cov}[X_I, X_R]\bigg|_{R=\tilde{R}} < 0. \tag{3.11}$$

The second derivative of $\mathbf{Cov}[X_I, X_R]$ with respect to R is given by

$$\begin{aligned}\left(\frac{\partial}{\partial R}\right)^2 \mathbf{Cov}[X_I, X_R] = {} & \frac{\alpha\theta^{2\alpha}}{(1-\alpha)(L+\theta)^{\alpha-1}(R+\theta)^{\alpha+1}} - \frac{2\theta^\alpha}{(R+\theta)^\alpha} + 2(1-2\alpha)\frac{\theta^{2\alpha}}{(1-\alpha)(R+\theta)^{2\alpha}} \\ & + \frac{\alpha(R-d)\theta^\alpha(1-\alpha)(d+\theta)^{\alpha-1} + \alpha\theta^{2\alpha}}{(1-\alpha)(d+\theta)^{\alpha-1}(R+\theta)^{\alpha+1}}.\end{aligned}$$

Condition (3.11) is satisfied if

$$\begin{aligned}&\alpha\theta^{2\alpha}(d+\theta)^{\alpha-1}(\tilde{R}+\theta)^{\alpha-1} - 2\theta^\alpha(1-\alpha)(d+\theta)^{\alpha-1}(\tilde{R}+\theta)^\alpha(L+\theta)^{\alpha-1} \\ &+ 2(1-2\alpha)\theta^{2\alpha}(d+\theta)^{\alpha-1}(L+\theta)^{\alpha-1} + (\alpha(\tilde{R}-d)\theta^\alpha(1-\alpha)(d+\theta)^{\alpha-1} \\ &+ \alpha\theta^{2\alpha})(\tilde{R}+\theta)^{\alpha-1}(L+\theta)^{\alpha-1} < 0,\end{aligned}$$

which is the second condition that the optimal retention must satisfy, and it is given in (3.9). This completes the proof of Theorem 3.3.2. □

To illustrate graphically, in Figure 3.2, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when $\alpha = 3$, $\theta = 100$, $d = 100$, and $L = 1000$. Note that $\mathbf{Cov}[X_I, X_R]$ is maximized when $R = 269.4882275$.

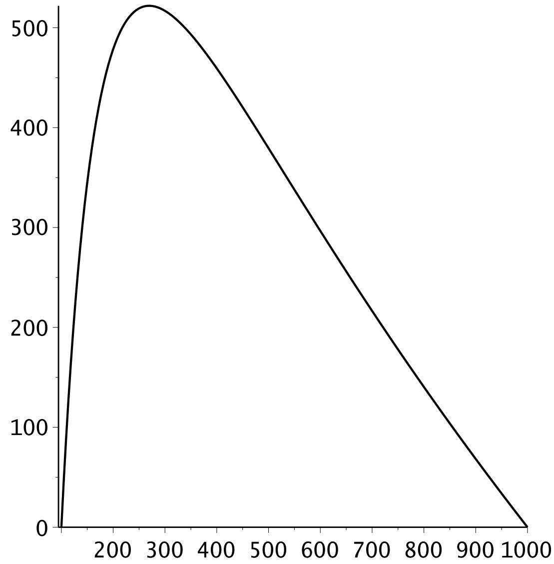

Figure 3.2: Retention R versus $\mathbf{Cov}[X_I, X_R]$ when L is fixed and known.

3.3.3 Finding the policy limit

In the next theorem, we find an optimal policy limit when the retention is already determined.

Theorem 3.3.3 *Suppose X follows the two-parameter Pareto distribution with parameters $\alpha > 1$ and $\theta > 0$. Let the deductible $d > 0$ of the insurance policy issued by the insurer be fixed. Furthermore, let the retention $R > 0$ of the reinsurance contract be fixed. Then the optimal policy limit L^* of the reinsurance contract that maximizes $\mathbf{Cov}[X_R, X_G]$ must satisfy the following two properties:*

$$- \theta^{\alpha} (L^* + \theta)^{1-\alpha} + 2\,\theta^{2\,\alpha} (L^* + \theta)^{-2\,\alpha+1} - L^*\theta^{\alpha} (L^* + \theta)^{-\alpha} + L^*\theta^{\alpha} (L^* + \theta)^{-\alpha}\,\alpha + R\theta^{\alpha} (L^* + \theta)^{-\alpha} - R\theta^{\alpha} (L^* + \theta)^{-\alpha}\,\alpha - \theta^{2\,\alpha} (L^* + \theta)^{-\alpha} (R + \theta)^{1-\alpha} = 0 \quad (3.12)$$

and

$$(\alpha - 1)\theta^{\alpha}(R + \theta)^{\alpha-1}(L^* + \theta)^{\alpha} + 2\theta^{2\alpha}(1 - 2\alpha)(R + \theta)^{\alpha-1} - \theta^{\alpha}(1 - \alpha)(R + \theta)^{\alpha-1}(L^* + \theta)^{\alpha} + \alpha(L^* - R)\theta^{\alpha}(1 - \alpha)(R + \theta)^{\alpha-1}(L^* + \theta)^{\alpha-1} + \alpha\theta(L^* + \theta)^{\alpha-1} < 0. \quad (3.13)$$

Proof Similar to the proof of Theorem 3.3.2, given the distribution of X, we first find the critical point(s) of $\mathbf{Cov}[X_R, X_G]$ by differentiating $\mathbf{Cov}[X_R, X_G]$ with respect to L. We have the equation

$$\frac{\partial}{\partial L}\mathbf{Cov}[X_R, X_G] = -\frac{\theta^\alpha}{(1-\alpha)(L+\theta)^{\alpha-1}} + \frac{\theta^{2\alpha}}{(1-\alpha)(L+\theta)^{2\alpha-1}} - L\left(\frac{\theta}{L+\theta}\right)^\alpha + \frac{\theta^{2\alpha}}{1-\alpha}(L+\theta)^{-2\alpha+1} + R\left(\frac{\theta}{L+\theta}\right)^\alpha - \frac{\theta}{1-\alpha}(R+\theta)^{-\alpha+1}(L+\theta)^{-\alpha}. \tag{3.14}$$

Next, we set

$$\frac{\partial}{\partial L}\mathbf{Cov}[X_R, X_G] = 0,$$

and solve for L. Using the “solve” feature of Maple, the solution, which we denote by $\tilde{L}$, must satisfy

$$-\theta^\alpha(\tilde{L}+\theta)^{1-\alpha} + 2\,\theta^{2\,\alpha}(\tilde{L}+\theta)^{-2\,\alpha+1} - \tilde{L}\theta^\alpha(\tilde{L}+\theta)^{-\alpha} + \tilde{L}\theta^\alpha(\tilde{L}+\theta)^{-\alpha}\alpha + R\theta^\alpha(\tilde{L}+\theta)^{-\alpha} - R\theta^\alpha(\tilde{L}+\theta)^{-\alpha}\alpha - \theta^{2\,\alpha}(\tilde{L}+\theta)^{-\alpha}(R+\theta)^{1-\alpha} = 0,$$

which is the first condition the optimal policy limit must satisfy, and it is reported as equation (3.12). The critical point $\tilde{L}$ maximizes $\mathbf{Cov}[X_R, X_G]$ if

$$\left(\frac{\partial}{\partial L}\right)^2 \mathbf{Cov}[X_R, X_G]\bigg|_{L=\tilde{L}} < 0. \tag{3.15}$$

The second derivative of $\mathbf{Cov}[X_R, X_G]$ with respect to L is given by

$$\left(\frac{\partial}{\partial L}\right)^2 \mathbf{Cov}[X_R, X_G] = -\frac{\theta^\alpha}{(L+\theta)^\alpha} + \frac{2\theta^{2\alpha}(1-2\alpha)}{(1-\alpha)(L+\theta)^{2\alpha}} - \frac{\theta^\alpha}{(L+\theta)^\alpha} + \frac{\alpha(L-R)\theta^\alpha}{(L+\theta)^{\alpha+1}} + \frac{\alpha\theta}{(1-\alpha)(R+\theta)^{\alpha-1}(L+\theta)^{\alpha+1}}.$$

Condition (3.15) is satisfied if

$$(\alpha-1)\theta^\alpha(R+\theta)^{\alpha-1}(\tilde{L}+\theta)^\alpha + 2\theta^{2\alpha}(1-2\alpha)(R+\theta)^{\alpha-1} - \theta^\alpha(1-\alpha)(R+\theta)^{\alpha-1}(\tilde{L}+\theta)^\alpha + \alpha(\tilde{L}-R)\theta^\alpha(1-\alpha)(R+\theta)^{\alpha-1}(\tilde{L}+\theta)^{\alpha-1} + \alpha\theta(\tilde{L}+\theta)^{\alpha-1} < 0,$$

which is the second condition that the optimal policy limit must satisfy, and it is given in (3.13).

This completes the proof of Theorem 3.3.3. □

To illustrate graphically, in Figure 3.3, we plot L versus $\mathbf{Cov}[X_R, X_G]$ when $\alpha = 3$, $\theta = 100$, $d = 100$, and $R = 900$. Note that $\mathbf{Cov}[X_R, X_G]$ is maximized when $L = 1900.500$.

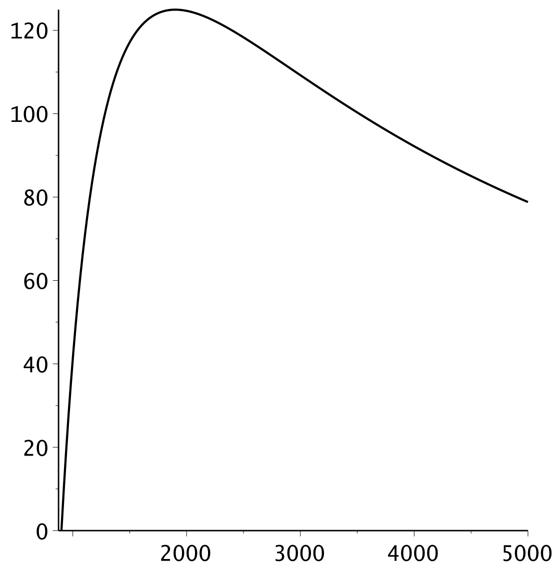

Figure 3.3: Policy limit L versus $\mathbf{Cov}[X_R, X_G]$ when R is fixed and known.

3.3.4 Finding the retention and the policy limit

When both R and L are still negotiable, we plot $\mathbf{Cov}[X_I, X_R]$ as a function of R and L in Figure 3.4, and $\mathbf{Cov}[X_R, X_G]$ as a function of R and L in Figure 3.5. We assume $\alpha = 3$, $\theta = 100$, and $d = 100$. We note that it is difficult to obtain closed-form expressions for R^* and L^* that maximize both $\mathbf{Cov}[X_I, X_R]$ and $\mathbf{Cov}[X_R, X_G]$. Hence, we shall use a decision-theory based approach, which looks particularly attractive from the practical point of view.

Recall from Theorem 3.3.2 that every time we change the value of L, a new optimal retention R^* that maximizes $\mathbf{Cov}[X_I, X_R]$ is obtained. In Figure 3.6, we plot L versus R^*, where R^* satisfies the two conditions for the optimal retention given in (3.8) and (3.9). Similarly, from Theorem 3.3.3, each time we change the value of R, a new optimal policy limit L^* that maximizes $\mathbf{Cov}[X_R, X_G]$ is obtained. In Figure 3.7, we plot R versus L^*, where L^* satisfies the two conditions for the optimal policy limit given in (3.12) and (3.13). Note that in Figure 3.6, the optimal retention R^* converges to 313.29 when L becomes large. However, in Figure 3.7, the optimal policy limit L^* is 729.53 when $R = 313.29$, which is considerably less than the value

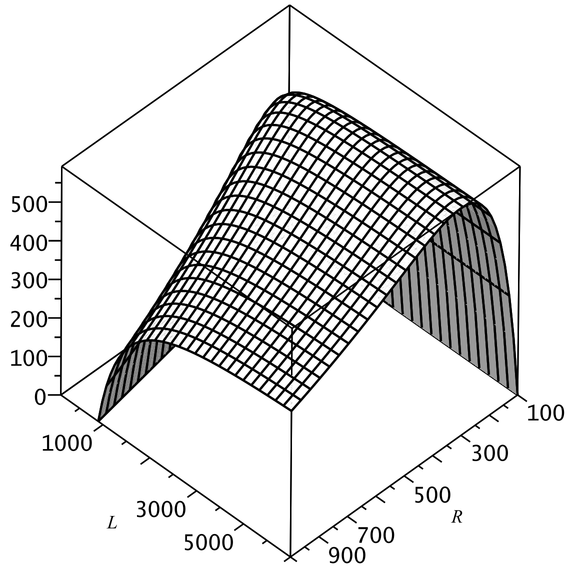

Figure 3.4: The covariance of X_I and X_R as a function of R and L.

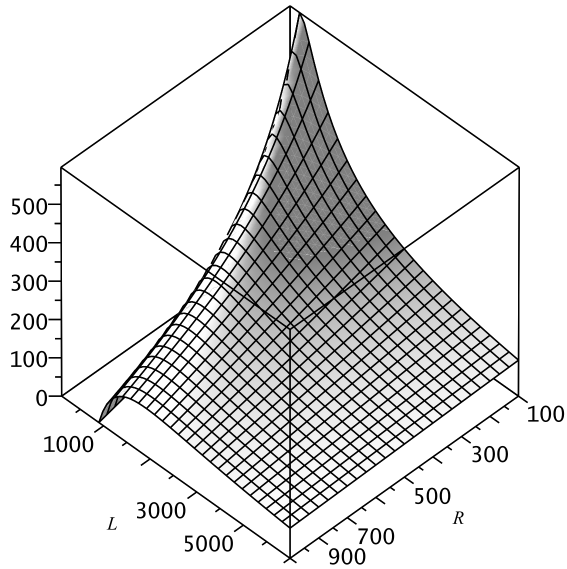

Figure 3.5: The covariance of X_R and X_G as a function of R and L.

of L corresponding to $R^* = 313.29$ in Figure 3.6. Next, we consider a negotiation process that helps the insurer and the reinsurer reach an agreement on R^* and L^*.

We begin the first round of negotiations by maximizing $\mathbf{Cov}[X_I, X_R]$ with respect to R. At this stage, we assume that the undecided policy limit is very large. The solution, which we denote by R^*, is 313.29 since we know from Figure 3.6 that the optimal retention converges to this value when the policy limit becomes large. The reinsurer then communicates with

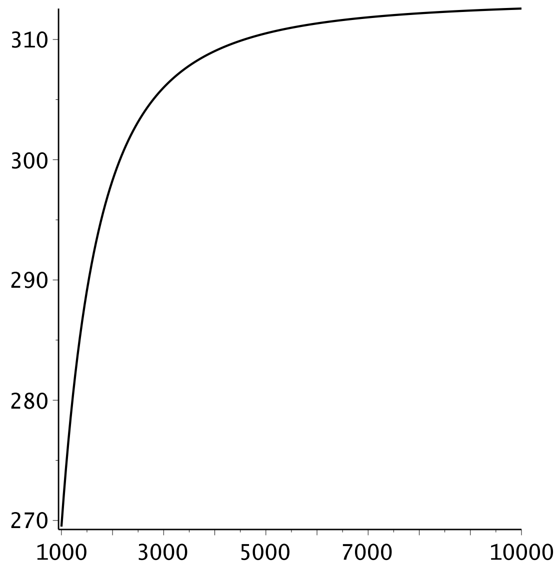

Figure 3.6: Policy limit L versus retention R^*.

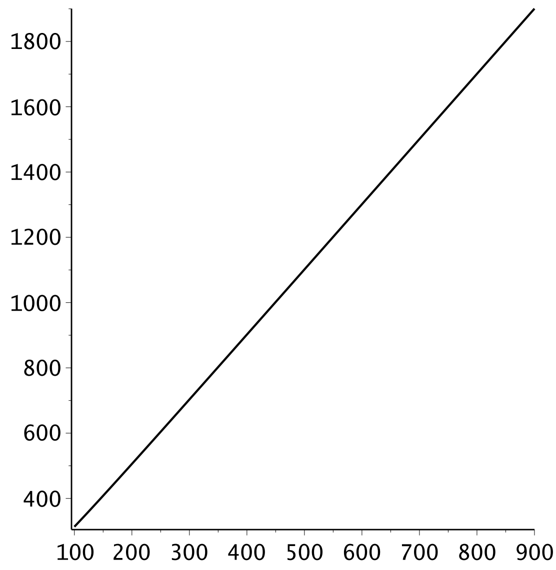

Figure 3.7: Retention R versus policy limit L^*.

the third party given the retention. By maximizing $\mathbf{Cov}[X_R, X_G]$ with respect to L given that $R = 313.29$, we obtain the solution L^*, which is equal to 729.53. This concludes the first round of negotiations. Next, the reinsurer provides the information about the new policy limit to the insurer and another round of negotiations begins between the insurer and the reinsurer. We maximize $\mathbf{Cov}[X_I, X_R]$ with respect to R given that the policy limit is $L = 729.53$. Then the new optimal retention is $R^* = 248.48$. Next, the reinsurer communicates with the third party to obtain a new optimal policy limit L^* that maximizes $\mathbf{Cov}[X_R, X_G]$ when $R = 248.48$. The new optimal policy limit is 601.133. This concludes the second round of negotiations. At the end of each round, we obtain a new pair of R^* and L^*. An agreement is reached when R^* and L^* both converge to a number. In Table 3.1, we present R^* and L^* for the first thirteen rounds of negotiations. Note that in the end, we obtain $R^* = 227.885$ and $L^* = 560.486$.

R^*	L^*
313.291	729.530
248.483	601.133
233.562	571.678
229.500	563.671
228.348	561.400
228.018	560.749
227.923	560.562
227.896	560.508
227.888	560.492
227.886	560.488
227.885	560.487
227.885	560.486
227.885	560.486

Table 3.1: Optimal R^* and L^* in the first thirteen rounds of negotiations when $\alpha = 3$, $\theta = 100$, and $d = 100$.

3.4 Covariance maximization: multiple policy claims

We have obtained an optimal reinsurance policy using the variance reduction approach under facultative reinsurance. What happens when we assume the method of treaty reinsurance?

Consider the following problem: During a given time period, the insurer issues a number of insurance policies. Let N be the number of insurance policies issued that require a claim payment from the insurer in the given time period. Furthermore, let the claim sizes, say X_1, X_2, ..., be i.i.d. random variables, with each X_i having the same distribution as X. Each claim size X_i is also independent of time. The insurer then purchases a reinsurance contract that ensures reinsurance coverage for all insurance policies the insurer had issued. For each insurance policy included in the reinsurance contract, the reinsurer covers the claim amount exceeding R when a claim payment needs to be made. Then the insurer's share of all claims is given by

$$X_I = \sum_{i=1}^{N}(X_i \wedge R),$$

and the reinsurer's share of all claims is given by

$$X_R = \sum_{i=1}^{N}(X_i - R)_+.$$

The rest of this section is organized as follows:

- In Subsection 3.4.1, we present an expression for $\mathbf{Cov}[X_I, X_R]$.
- In Subsection 3.4.2, we compare the critical points of $\mathbf{Cov}[X_I, X_R]$ and $\mathbf{Cov}[(X \wedge R), (X - R)_+]$. Note that the latter covariance arises when only one insurance policy requires a claim payment during the given time period.
- In Subsection 3.4.3, we use four illustrative examples to show how the optimal retention R^* is affected by parameter values of the distribution of N.

3.4.1 Covariance between the insurer's and the reinsurer's shares of the claims

Here we derive an expression for $\mathbf{Cov}[X_I, X_R]$.

Theorem 3.4.1 *Suppose that N follows a discrete distribution. Furthermore, assume that the random variables X_1, X_2, $\ldots$ are i.i.d. Finally, under treaty reinsurance in the excess of loss form, let the retention be $R > 0$. Then the covariance between X_I and X_R is given by*

$$\begin{aligned}\mathbf{Cov}[X_I, X_R] = \mathbf{E}[N]R\mathbf{E}[(X - R)_+] + \mathbf{Var}[N]\mathbf{E}[X \wedge R]\mathbf{E}[(X - R)_+] \\ - \mathbf{E}[N]\mathbf{E}[X \wedge R]\mathbf{E}[(X - R)_+]. \qquad (3.16)\end{aligned}$$

Proof The covariance between X_I and X_R can be written as follows:

$$\mathbf{Cov}[X_I, X_R] = \mathbf{E}[X_I X_R] - \mathbf{E}[X_I]\mathbf{E}[X_R]. \qquad (3.17)$$

We next calculate the three expectations on the right-hand side of equation (3.17). We begin with the expectation of the product $X_I X_R$. To calculate the expectation, we consider the law of iterated expectations.

Definition 3.4.2 (cf., e.g., Weiss et al., 2006) The law of iterated expectations states that if X is an integrable random variable and Y is any random variable, not necessarily integrable, on the same probability space, then $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]]$.

Using the law of iterated expectations, we have

$$\begin{aligned}
\mathbf{E}[X_I X_R] &= \mathbf{E}\Big[\mathbf{E}\Big[\sum_{i=j}(X_i \mathbf{1}\{X_i \le R\} + R\mathbf{1}\{X_i > R\})(X_j - R)\mathbf{1}\{X_j > R\} \\
&\qquad + \sum_{i \ne j}(X_i \mathbf{1}\{X_i \le R\} + R\mathbf{1}\{X_i > R\})(X_j - R)\mathbf{1}\{X_j > R\} \mid N\Big]\Big] \\
&= \mathbf{E}[\mathbf{E}[NR(X - R)\mathbf{1}\{X > R\}]] + \mathbf{E}[N(N-1)\mathbf{E}[X\mathbf{1}\{X \le R\} + R\mathbf{1}\{X > R\}] \\
&\qquad \times \mathbf{E}[(X - R)\mathbf{1}\{X > R\}]] \\
&= \mathbf{E}[N]R\mathbf{E}[(X - R)_+] + \mathbf{E}[N^2 - N]\mathbf{E}[X \wedge R]\mathbf{E}[(X - R)_+]. \qquad (3.18)
\end{aligned}$$

Similarly, the other two expectations on the right-hand side of equation (3.17) become

$$\begin{aligned}
\mathbf{E}[X_I] &= \mathbf{E}\Big[\mathbf{E}\Big[\sum_{i=1}^{N}(X_i \mathbf{1}\{X_i \le R\} + R\mathbf{1}\{X_i > R\}) \mid N\Big]\Big] \\
&= \mathbf{E}[N]\Big(\int_0^R x \mathrm{d}F_X(x) + R(1 - F_X(R))\Big) \\
&= \mathbf{E}[N]\mathbf{E}[X \wedge R] \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}[X_R] &= \mathbf{E}\Big[\mathbf{E}\Big[\sum_{i=1}^{N}(X_i - R)\mathbf{1}\{X > R\} \mid N\Big]\Big] \\
&= \mathbf{E}[N]\Big(\int_R^\infty x \mathrm{d}F_X(x) - R\int_R^\infty \mathrm{d}F_X(x)\Big) \\
&= \mathbf{E}[N]\mathbf{E}[(X - R)_+]. \qquad (3.20)
\end{aligned}$$

Having thus calculated the three expectations with formulas given in equations (3.18), (3.19), and (3.20), we obtain an expression for the covariance, which is given in equation (3.16). This completes the proof of Theorem 3.4.1. □

3.4.2 Comparison of critical points

We now compare the critical points of $\mathbf{Cov}[X_I, X_R]$ and $\mathbf{Cov}[(X \wedge R), (X - R)_+]$. Since the critical point(s) must include the value of the optimal retention, this may show how the optimal retention is affected by N. Let us first find the critical point(s) of $\mathbf{Cov}[X_I, X_R]$.

Theorem 3.4.3 *Assume that conditions of Theorem 3.4.1 are satisfied. Furthermore, assume that the distribution function of X is twice differentiable. Then the value $\tilde{R}_1$ that satisfies the equation*

$$\begin{aligned}\mathbf{E}[N]\mathbf{E}[X\mathbf{1}\{X > \tilde{R}_1\}] = 2\tilde{R}_1\mathbf{E}[N](1 - F_X(\tilde{R}_1)) - (\mathbf{Var}[N] - \mathbf{E}[N])((1 - F_X(\tilde{R}_1)))\\ \times (\mathbf{E}[(X - \tilde{R}_1)_+] - \mathbf{E}[X \wedge \tilde{R}_1])\end{aligned} \tag{3.21}$$

is a critical point of $\mathbf{Cov}[X_I, X_R]$. *Moreover, the critical point $\tilde{R}_1$ maximizes* $\mathbf{Cov}[X_I, X_R]$ *if the following condition is satisfied*

$$\begin{aligned}\mathbf{E}[N](\tilde{R}_1 f_X(\tilde{R}_1) - 2(1 - F_X(\tilde{R}_1))) + (\mathbf{Var}[N] - \mathbf{E}[N])(f_X(\tilde{R}_1)\\ \times (\mathbf{E}[X \wedge \tilde{R}_1] - \mathbf{E}[(X - \tilde{R}_1)_+]) - 2(1 - F_X(\tilde{R}_1))^2) < 0.\end{aligned} \tag{3.22}$$

Proof To find the critical point(s), we differentiate $\mathbf{Cov}[X_I, X_R]$ with respect to R and get:

$$\begin{aligned}\frac{\partial}{\partial R}\mathbf{Cov}[X_I, X_R] &= \mathbf{E}[N]\int_R^\infty x f_X(x)\mathrm{d}x - 2R\mathbf{E}[N](1 - F_X(R))\\ &\quad + (\mathbf{Var}[N] - \mathbf{E}[N])(1 - F_X(R))\\ &\quad \times (\mathbf{E}[(X - R)_+] - \mathbf{E}[X \wedge R])\\ &= \mathbf{E}[N]\mathbf{E}[X\mathbf{1}\{X > R\}] - 2R\mathbf{E}[N](1 - F_X(R))\\ &\quad + (\mathbf{Var}[N] - \mathbf{E}[N])(1 - F_X(R))(\mathbf{E}[(X - R)_+] - \mathbf{E}[X \wedge R]).\end{aligned} \tag{3.23}$$

Next, we set the right-hand side of equation (3.23) to 0 and solve for R. The solution, which we denote by $\tilde{R}_1$, satisfies equation (3.21). The critical point $\tilde{R}_1$ maximizes $\mathbf{Cov}[X_I, X_R]$ if

$$\left(\frac{\partial}{\partial R}\right)^2 \mathbf{Cov}[X_I, X_R]\bigg|_{R=\tilde{R}_1} < 0.$$

The second derivative can be written as follows:

$$\left(\frac{\partial}{\partial R}\right)^2 \mathbf{Cov}[X_I, X_R] = \mathbf{E}[N](Rf_X(R) - 2(1 - F_X(R)))$$
$$+ (f_X(R)(\mathbf{E}[X \wedge R] - \mathbf{E}[(X - R)_+]) - 2(1 - F_X(R))^2)$$
$$\times (\mathbf{Var}[N] - \mathbf{E}[N]).$$

The critical point $\tilde{R}_1$ that maximizes $\mathbf{Cov}[X_I, X_R]$ should satisfy the property

$$\mathbf{E}[N](\tilde{R}_1 f_X(\tilde{R}_1) - 2(1 - F_X(\tilde{R}_1))) + (\mathbf{Var}[N] - \mathbf{E}[N])(f_X(\tilde{R}_1)$$
$$\times (\mathbf{E}[X \wedge \tilde{R}_1] - \mathbf{E}[(X - \tilde{R}_1)_+]) - 2(1 - F_X(\tilde{R}_1))^2) < 0,$$

which is condition (3.22). This completes the proof of Theorem 3.4.3. □

Next, we obtain the critical point(s) of $\mathbf{Cov}[(X \wedge R), (X - R)_+]$.

Theorem 3.4.4 *Suppose that under treaty reinsurance, a claim payment is required for only one insurance policy during the given time period. Furthermore, let the distribution function of X be differentiable. Finally, let the reinsurer cover the amount exceeding $R > 0$. Then the value $\tilde{R}_2$ is a critical point of* $\mathbf{Cov}[(X \wedge R), (X - R)_+]$ *if*

$$\int_{\tilde{R}_2}^{\infty} x f_X(x)\mathrm{d}x - 2\tilde{R}_2(1 - F_X(\tilde{R}_2)) - (1 - F_X(\tilde{R}_2))(\mathbf{E}[(X - \tilde{R}_2)_+] - \mathbf{E}[X \wedge \tilde{R}_2]) = 0. \quad (3.24)$$

Proof We can write the covariance as follows:

$$\mathbf{Cov}[(X \wedge R), (X - R)_+] = \mathbf{E}[(X \wedge R)(X - R)_+] - \mathbf{E}[X \wedge R]\mathbf{E}[(X - R)_+]$$
$$= R\mathbf{E}[(X - R)_+] - \mathbf{E}[X \wedge R]\mathbf{E}[(X - R)_+]. \quad (3.25)$$

To obtain the critical point(s) of $\mathbf{Cov}[(X \wedge R), (X - R)_+]$, we differentiate $\mathbf{Cov}[(X \wedge R), (X - R)_+]$ with respect to R and get

$$\frac{\partial}{\partial R}\mathbf{Cov}[(X \wedge R), (X - R)_+] = \int_R^{\infty} x f_X(x)\mathrm{d}x - 2R(1 - F_X(R)) - (1 - F_X(R))$$
$$\times (\mathbf{E}[(X - R)_+] - \mathbf{E}[X \wedge R]). \quad (3.26)$$

We then set

$$\frac{\partial}{\partial R}\mathbf{Cov}[(X \wedge R), (X - R)_+] = 0,$$

and solve for R. The solution, which we denote by $\tilde{R}_2$, must satisfy the condition for the critical point(s) given by equation (3.24). This completes the proof of Theorem 3.4.4. $\square$

Comparing results from Theorems 3.4.3 and 3.4.4, we note that when $\mathbf{Var}[N] = 0$, then the critical points of $\mathbf{Cov}[\sum_{i=1}^{N}(X_i \wedge R), \sum_{i=1}^{N}(X_i - R)_+]$ and $\mathbf{Cov}[(X \wedge R), (X - R)_+]$ are the same, that is, equation (3.21) becomes equation (3.24). Next, we provide examples that show how the optimal retention is affected by $\mathbf{E}[N]$ and $\mathbf{Var}[N]$.

3.4.3 Illustrative examples

We use four examples to demonstrate how the optimal retention R^* is affected by the parameter values of the distribution of N. In each example, we first obtain optimal retentions when $\mathbf{E}[N]$ is fixed and $\mathbf{Var}[N]$ changes. Next, we find optimal retentions when $\mathbf{Var}[N]$ is fixed and $\mathbf{E}[N]$ changes. The distributions of the random variables X_i and N used in each example are as follows:

- In Example 3.4.5, the random variable N follows a binomial distribution and each claim size X_i follows an exponential distribution.
- In Example 3.4.6, the random variable N follows a negative binomial distribution and each claim size X_i follows an exponential distribution.
- In Example 3.4.7, the random variable N follows a binomial distribution and each claim size X_i follows a two-parameter Pareto distribution.
- In Example 3.4.8, the random variable N follows a negative binomial distribution and each claim size X_i follows a two-parameter Pareto distribution.

Recall that when N follows the binomial distribution with parameters $n = 1, 2, \ldots$ and $0 \leq p \leq 1$, then the PDF of N is given by

$$\mathbf{P}[N = k] = \binom{n}{k} p^k (1 - p)^{n-k}$$

for $k = 0, 1, \ldots, n$. When N follows the negative binomial distribution with parameters $r > 0$ and $\beta > 0$, then the PDF of N is given by

$$\mathbf{P}[N = k] = \binom{k + r - 1}{k} \left(\frac{1}{1 + \beta}\right)^k \left(\frac{\beta}{1 + \beta}\right)^r$$

for $k = 0, 1, \ldots$. When X follows the exponential distribution with parameter $\theta > 0$, then the PDF of X is given by

$$f_X(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0.$$

We let N follow either the binomial distribution or the negative binomial distribution, because both distributions are commonly used to model the occurrence of events during the considered period of time. We do not consider the Poisson distribution here, because the mean and the variance of it are the same.

Example 3.4.5 Let N follow the binomial distribution with parameters $n > 0$ and $0 \leq p \leq 1$. Furthermore, let each claim size X_i follow the exponential distribution with parameter $\theta = 10$. Under these conditions, we find the optimal retention R^* that maximizes $\mathbf{Cov}[X_I, X_R]$ for each pair (n, p). Results are given in Tables 3.2 and 3.3. In detail, given the specified distributions of N and X_i, we have the formula

$$\mathbf{Cov}[X_I, X_R] = npR\theta e^{-R/\theta} + np(1-p)\theta^2(1 - e^{-R/\theta})e^{-R/\theta} - np\theta^2(1 - e^{-R/\theta})e^{-R/\theta}.$$

From Table 3.2, we note that, for each pair (n, p), the mean $\mathbf{E}[N]$ is fixed, that is,

n	p
2	1/2
3	1/3
4	1/4
5	1/5
6	1/6
7	1/7
8	1/8

Table 3.2: Values of n and p such that $\mathbf{E}[N]$ is fixed and $\mathbf{Var}[N]$ varies.

n	p
2	$\frac{1}{2}$
3	$\frac{1}{2} - \frac{1}{6}\sqrt{3}$
4	$\frac{1}{2} - \frac{1}{4}\sqrt{2}$
5	$\frac{1}{2} - \frac{1}{10}\sqrt{15}$
6	$\frac{1}{2} - \frac{1}{6}\sqrt{6}$
7	$\frac{1}{2} - \frac{1}{14}\sqrt{35}$
8	$\frac{1}{2} - \frac{1}{4}\sqrt{3}$

Table 3.3: Values of n and p such that $\mathbf{Var}[N]$ is fixed and $\mathbf{E}[N]$ varies.

$$\mathbf{E}[N] = 1,$$

but the variance $\mathbf{Var}[N]$ varies. An optimal retention R^* that satisfies the two conditions of Theorem 3.4.3, which are given by (3.21) and (3.22), is obtained for each value of $\mathbf{Var}[N]$. Results are provided in Table 3.4. To show graphically how R^* is affected by $\mathbf{Var}[N]$, in Figure 3.8, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when $\mathbf{Var}[N] = 1/2, 2/3$, and $3/4$, and $\mathbf{E}[N] = 1$. Note that

$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
1	1/2	11.983
1	2/3	11.146
1	3/4	10.802
1	4/5	10.616
1	5/6	10.500
1	6/7	10.421
1	7/8	10.363

Table 3.4: Optimal retention R^* when $\mathbf{E}[N]$ is fixed, N is binomial, and X is exponential.

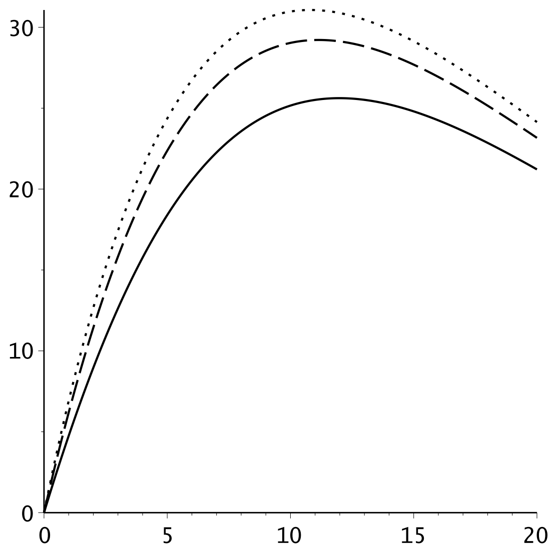

Figure 3.8: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{Var}[N] = 1/2$ (solid curve), $2/3$ (dashed curve), and $3/4$ (dotted curve).

for fixed $\mathbf{E}[N]$, the optimal retention R^* decreases when the variance $\mathbf{Var}[N]$ increases.

Next, from Table 3.3, we see that, for each pair (n, p),

$$\mathbf{Var}[N] = \frac{1}{2},$$

but $\mathbf{E}[N]$ varies. An optimal retention R^* that satisfies the two conditions in Theorem 3.4.3 is obtained for each value of $\mathbf{E}[N]$. Results are provided in Table 3.5. To show graphically how R^* is affected by $\mathbf{E}[N]$, in Figure 3.9, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when the mean $\mathbf{E}[N] = 1, (3/2) - (1/2)\sqrt{3}$, and $2 - \sqrt{2}$, and the variance $\mathbf{Var}[N] = 1/2$. Note that for fixed $\mathbf{Var}[N]$,

$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
1	$1/2$	11.983
$\frac{3}{2} - \frac{1}{2}\sqrt{3}$	$1/2$	10.657
$2 - \sqrt{2}$	$1/2$	10.432
$\frac{5}{2} - \frac{1}{2}\sqrt{15}$	$1/2$	10.324
$3 - \sqrt{6}$	$1/2$	10.260
$\frac{7}{2} - \frac{1}{2}\sqrt{35}$	$1/2$	10.217
$4 - 2\sqrt{3}$	$1/2$	10.186

Table 3.5: Optimal retention R^* when $\mathbf{Var}[N]$ is fixed, N is binomial, and X is exponential.

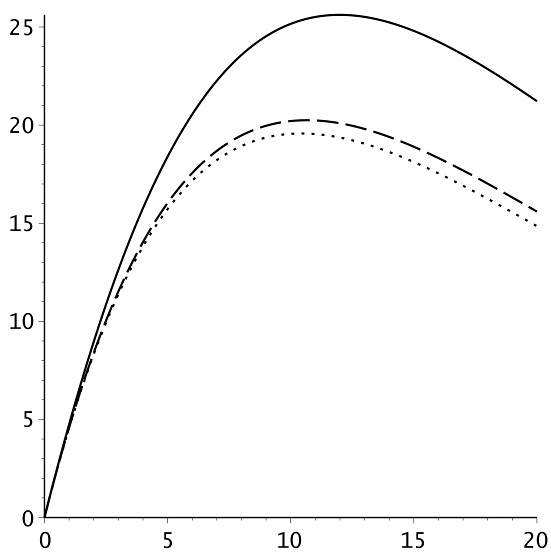

Figure 3.9: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{E}[N] = 1$ (solid curve), $(3/2)-(1/2)\sqrt{3}$ (dashed curve), and $2 - \sqrt{2}$ (dotted curve).

the optimal retention R^* decreases when $\mathbf{E}[N]$ decreases. This concludes Example 3.4.5.

Example 3.4.6 Let N follow the negative binomial distribution with parameters $r > 0$ and $\beta > 0$. Furthermore, let each claim size X_i follow the exponential distribution with parameter $\theta = 10$. Then we find the optimal retention R^* that maximizes the covariance $\mathbf{Cov}[X_I, X_R]$ for each pair (r, β). The results are given in Tables 3.6 and 3.7. In detail, given the specified distributions of N and X_i, we have the formula

$$\mathbf{Cov}[X_I, X_R] = r\beta R\theta e^{-R/\theta} + r\beta(1+\beta)\theta^2(1-e^{-R/\theta})e^{-R/\theta} - r\beta\theta^2(1-e^{-R/\theta})e^{-R/\theta}.$$

From Table 3.6, note that for each pair (r, β), we have

r	β
1	2
2	1
3	2/3
4	1/2
5	2/5

Table 3.6: Values of r and β such that $\mathbf{E}[N]$ is fixed and $\mathbf{Var}[N]$ varies.

r	β
1	2
2	$-\frac{1}{2}+\frac{1}{2}\sqrt{13}$
3	1
4	$-\frac{1}{2}+\frac{1}{2}\sqrt{7}$
5	$-\frac{1}{2}+\frac{1}{10}\sqrt{145}$

Table 3.7: Values of r and β such that $\mathbf{Var}[N]$ is fixed and $\mathbf{E}[N]$ varies.

$$\mathbf{E}[N] = 2,$$

but $\mathbf{Var}[N]$ varies. An optimal retention R^* that satisfies the two conditions of Theorem 3.4.3 is obtained for each value of $\mathbf{Var}[N]$. Results are provided in Table 3.8. To show graphically how R^* is affected by $\mathbf{Var}[N]$, in Figure 3.10, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when the variance $\mathbf{Var}[N] = 6, 4$, and $10/3$, and the mean $\mathbf{E}[N] = 2$. We note the same pattern as in Example

$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
2	6	7.990
2	4	8.526
2	10/3	8.841
2	3	9.047
2	14/5	9.191

Table 3.8: Optimal retention R^* when $\mathbf{E}[N]$ is fixed, N is negative binomial, and X is exponential.

3.4.5, that is, for fixed $\mathbf{E}[N]$, the optimal retention R^* decreases when $\mathbf{Var}[N]$ increases.

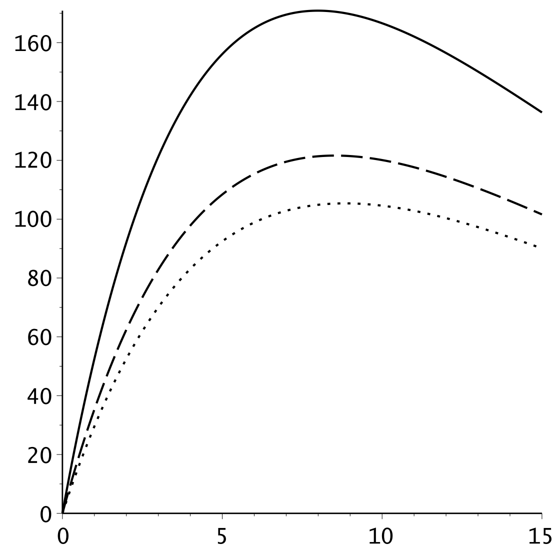

Figure 3.10: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{Var}[N] = 6$ (solid curve), 4 (dashed curve), and 10/3 (dotted curve).

Next, from Table 3.7, we see that, for each pair (r, β),

$$\mathbf{Var}[N] = 6,$$

but $\mathbf{E}[N]$ varies. An optimal retention R^* that satisfies the two conditions of Theorem 3.4.3 is obtained for each value of $\mathbf{E}[N]$. Results are provided in Table 3.9. To show graphically how R^* is affected by $\mathbf{E}[N]$, in Figure 3.11, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when the mean $\mathbf{E}[N] = 2, -1 + \sqrt{13}$, and 3, and the variance $\mathbf{Var}[N] = 6$. We note that for fixed $\mathbf{Var}[N]$, the optimal retention R^* increases when $\mathbf{E}[N]$ increases. This concludes Example 3.4.6.

$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
2	6	7.990
$-1 + \sqrt{13}$	6	8.316
3	6	8.526
$-2 + 2\sqrt{7}$	6	8.680
$-\frac{5}{2} + \frac{1}{2}\sqrt{145}$	6	8.800

Table 3.9: Optimal retention R^* when $\mathbf{Var}[N]$ is fixed, N is negative binomial, and X is exponential.

Example 3.4.7 Let N follow the binomial distribution with parameters $n > 0$ and $0 \le p \le 1$. Furthermore, let each claim size X_i follow the two-parameter Pareto distribution with parameters $\alpha = 5$ and $\theta = 10$. Then we obtain the optimal retention R^* that maximizes $\mathbf{Cov}[X_I, X_R]$

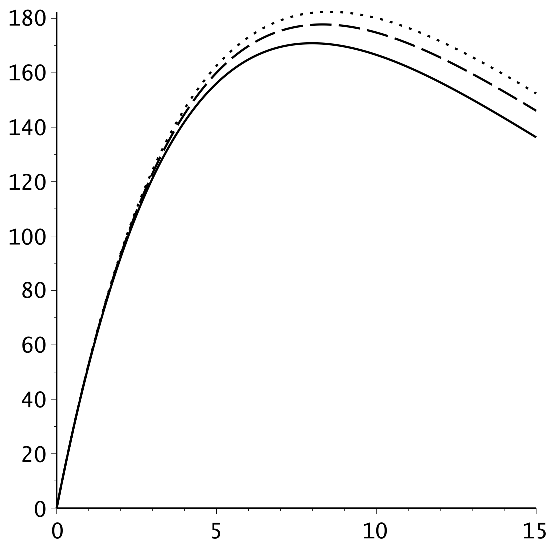

Figure 3.11: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{E}[N] = 2$ (solid curve), $-1 + \sqrt{13}$ (dashed curve), and 3 (dotted curve).

for each pair (n, p), with results given in Tables 3.10 and 3.11. In detail, given the specified distributions of N and X_i, we have the formula

$$\begin{aligned}\mathbf{Cov}[X_I, X_R] = {} & npR\frac{\theta}{\alpha-1}\left(\frac{\theta}{R+\theta}\right)^{\alpha-1} + np(1-p)\frac{\theta}{\alpha-1} \\ & \times\left(1-\left(\frac{\theta}{R+\theta}\right)^{\alpha-1}\right)\frac{\theta}{\alpha-1}\left(\frac{\theta}{R+\theta}\right)^{\alpha-1} - np\frac{\theta}{\alpha-1} \\ & \times\left(1-\left(\frac{\theta}{R+\theta}\right)^{\alpha-1}\right)\frac{\theta}{\alpha-1}\left(\frac{\theta}{R+\theta}\right)^{\alpha-1}.\end{aligned}$$

From Table 3.10, we see that for each pair (n, p), we have the fixed mean

n	p
2	1/2
3	1/3
4	1/4
5	1/5
6	1/6
7	1/7
8	1/8

Table 3.10: Values of n and p such that $\mathbf{E}[N]$ is fixed and $\mathbf{Var}[N]$ varies.

$$\mathbf{E}[N] = 1,$$

n	p
2	$\frac{1}{2}$
3	$\frac{1}{2} - \frac{1}{6}\sqrt{3}$
4	$\frac{1}{2} - \frac{1}{4}\sqrt{2}$
5	$\frac{1}{2} - \frac{1}{10}\sqrt{15}$
6	$\frac{1}{2} - \frac{1}{6}\sqrt{6}$
7	$\frac{1}{2} - \frac{1}{14}\sqrt{35}$
8	$\frac{1}{2} - \frac{1}{4}\sqrt{3}$

Table 3.11: Values of n and p such that $\mathbf{Var}[N]$ is fixed and $\mathbf{E}[N]$ varies.

but the variance $\mathbf{Var}[N]$ varies. An optimal retention R^* that satisfies the two conditions of Theorem 3.4.3 is obtained for each value of $\mathbf{Var}[N]$. Results are provided in Table 3.12. To show graphically how R^* is affected by $\mathbf{Var}[N]$, in Figure 3.12, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when the variance $\mathbf{Var}[N] = 1/2, 2/3$, and $3/4$, and the mean $\mathbf{E}[N] = 1$.

α	θ	$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
5	10	1	1/2	7.187
5	10	1	2/3	6.610
5	10	1	3/4	6.327
5	10	1	4/5	6.160
5	10	1	5/6	6.050
5	10	1	6/7	5.972
5	10	1	7/8	5.914

Table 3.12: Optimal retention R^* when $\mathbf{E}[N]$ is fixed, N is binomial, and X is two-parameter Pareto.

Next, from Table 3.11, we see that, for each pair (n, p), the variance is

$$\mathbf{Var}[N] = \frac{1}{2},$$

but the mean $\mathbf{E}[N]$ changes. An optimal retention R^* that satisfies the two conditions of Theorem 3.4.3 is obtained for each value of $\mathbf{E}[N]$. Results are provided in Table 3.13. To show graphically how R^* is affected by $\mathbf{E}[N]$, in Figure 3.13, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when the mean $\mathbf{E}[N] = 1, (3/2) - \sqrt{3}/2$, and $2 - \sqrt{2}$, and the variance $\mathbf{Var}[N] = 1/2$. This concludes Example 3.4.7.

Example 3.4.8 Let N follow the negative binomial distribution with parameters $r > 0$ and $\beta > 0$. Furthermore, let each claim size X_i follow the two-parameter Pareto distribution with the parameters $\alpha = 3$ and $\theta = 10$. Then we find the optimal retention R^* that maximizes

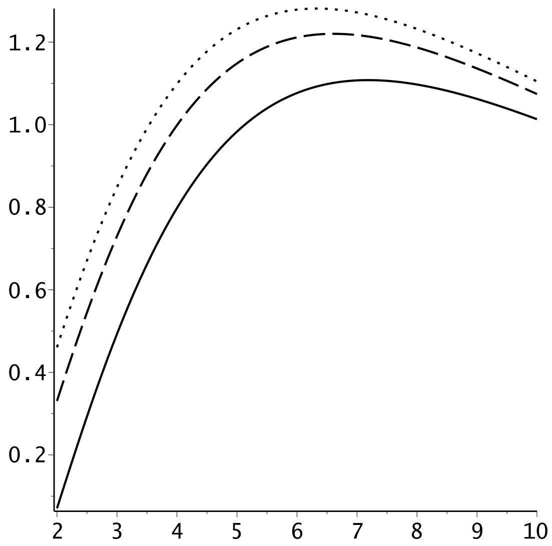

Figure 3.12: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{Var}[N] = 1/2$ (solid curve), $2/3$ (dashed curve), and $3/4$ (dotted curve).

α	θ	$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
5	10	1	1/2	7.187
5	10	$\frac{3}{2} - \frac{1}{2}\sqrt{3}$	1/2	6.198
5	10	$2 - \sqrt{2}$	1/2	5.984
5	10	$\frac{5}{2} - \frac{1}{2}\sqrt{15}$	1/2	5.874
5	10	$3 - \sqrt{6}$	1/2	5.806
5	10	$\frac{7}{2} - \frac{1}{2}\sqrt{35}$	1/2	5.761
5	10	$4 - 2\sqrt{3}$	1/2	5.727

Table 3.13: Optimal retention R^* when $\mathbf{Var}[N]$ is fixed, N is binomial, and X is two-parameter Pareto.

$\mathbf{Cov}[X_I, X_R]$ for each pair (r, β), with results given in Tables 3.14 and 3.15. In detail, given the specified distributions of N and X_i, we have the formula

$$\mathbf{Cov}[X_I, X_R] = r\beta R \frac{\theta}{\alpha - 1}\left(\frac{\theta}{R+\theta}\right)^{\alpha-1} + r\beta^2 \frac{\theta}{\alpha-1}\left(1 - \left(\frac{\theta}{R+\theta}\right)^{\alpha-1}\right)\frac{\theta}{\alpha-1}\left(\frac{\theta}{R+\theta}\right)^{\alpha-1}.$$

From Table 3.14, we see that, for each pair (r, β), the mean is

$$\mathbf{E}[N] = 2,$$

but the variance $\mathbf{Var}[N]$ changes. An optimal retention R^* that satisfies the two conditions of Theorem 3.4.3 is obtained for each value of $\mathbf{Var}[N]$. Results are provided in Table 3.16. To

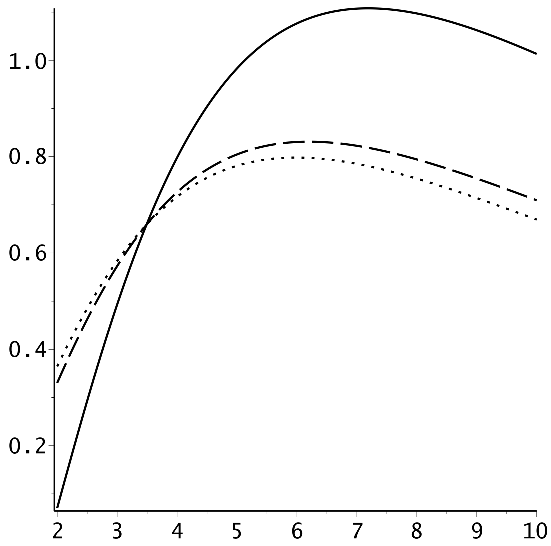

Figure 3.13: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{E}[N] = 1$ (solid curve), $(3/2) - \sqrt{3}/2$ (dashed curve), and $2 - \sqrt{2}$ (dotted curve).

r	β
1	2
2	1
3	2/3
4	1/2
5	2/5

Table 3.14: Values of r and β such that $\mathbf{E}[N]$ is fixed and $\mathbf{Var}[N]$ varies.

r	β
1	2
2	$-\frac{1}{2} + \frac{1}{2}\sqrt{13}$
3	1
4	$-\frac{1}{2} + \frac{1}{2}\sqrt{7}$
5	$-\frac{1}{2} + \frac{1}{10}\sqrt{145}$

Table 3.15: Values of r and β such that $\mathbf{Var}[N]$ is fixed and $\mathbf{E}[N]$ varies.

show graphically how R^* is affected by $\mathbf{Var}[N]$, in Figure 3.14, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when the variance $\mathbf{Var}[N] = 6, 4$, and $10/3$, and the mean $\mathbf{E}[N] = 2$.

Next, from Table 3.15, we see that, for each pair (r, β), the variance is

$$\mathbf{Var}[N] = 6,$$

but the mean $\mathbf{E}[N]$ varies. An optimal retention R^* that satisfies the two conditions of Theorem

α	θ	$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
3	10	2	6	5.874
3	10	2	4	6.956
3	10	2	10/3	7.625
3	10	2	3	8.064
3	10	2	14/5	8.370

Table 3.16: Optimal retention R^* when $\mathbf{E}[N]$ is fixed, N is negative binomial, and X is two-parameter Pareto.

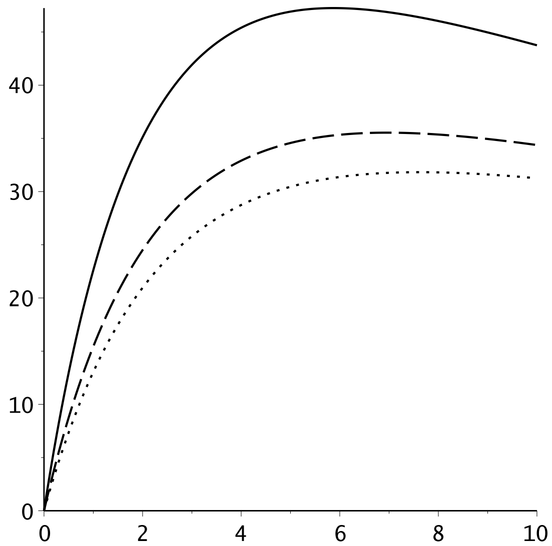

Figure 3.14: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{Var}[N] = 6$ (solid curve), 4 (dashed curve), and 10/3 (dotted curve).

3.4.3 is obtained for each value of $\mathbf{E}[N]$. Results are provided in Table 3.17. To show graphically how R^* is affected by $\mathbf{E}[N]$, in Figure 3.15, we plot R versus $\mathbf{Cov}[X_I, X_R]$ when the mean $\mathbf{E}[N] = 2, -1 + \sqrt{13}$, and 3, and the variance $\mathbf{Var}[N] = 6$. We observe the same pattern as in Examples 3.4.5, 3.4.6, and 3.4.7. This concludes Example 3.4.8.

α	θ	$\mathbf{E}[N]$	$\mathbf{Var}[N]$	R^*
3	10	2	6	5.874
3	10	$-1 + \sqrt{13}$	6	6.520
3	3	10	6	6.956
3	10	$-2 + 2\sqrt{7}$	6	7.282
3	10	$-\frac{5}{2} + \frac{1}{2}\sqrt{145}$	6	7.537

Table 3.17: Optimal retention R^* when $\mathbf{E}[N]$ is fixed, N is negative binomial, and X is two-parameter Pareto.

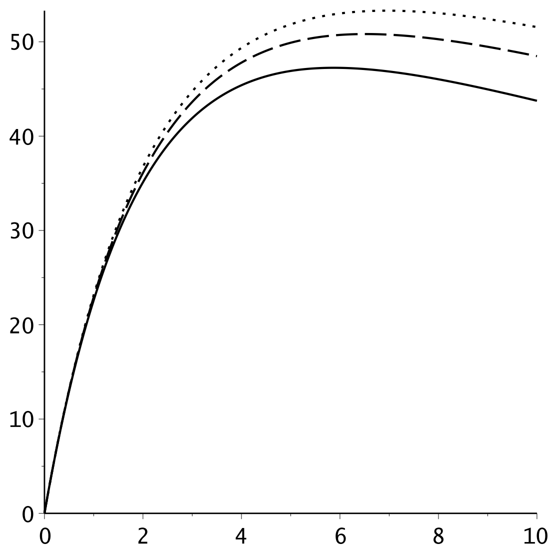

Figure 3.15: Retention R versus $\mathbf{Cov}[X_I, X_R]$ for $\mathbf{E}[N] = 2$ (solid curve), $-1 + \sqrt{13}$ (dashed curve), and 3 (dotted curve).

In summary, we have shown how changing the parameter values of the distribution of N affects the optimal retention. To obtain the optimal retention, simple assumptions are made, but they do not always hold true in the real world. Therefore, the patterns that we have observed in these examples will not always hold true either. Nevertheless, they are instructive.

Chapter 4

Claim sizes depend on time

In this chapter, we obtain an optimal reinsurance policy when claim sizes are dependent on preceding inter-claim times.

During the time interval $(0, t]$, let $N(t)$ be the number of insurance policies that require a claim payment from the insurer up to and including the time t. Furthermore, let $T_1, T_2, \ldots$ be the claim arrival times, and let $X_1, X_2, \ldots$ be the corresponding claim sizes. Then the insurer's aggregate claim size without the reinsurance coverage up to and including the time t is $\sum_{i=1}^{N(t)} X_i$, where $0 < T_1 < T_2 < \cdots < T_{N(t)} \leq t$, and $\sum_{i=1}^{N(t)} X_i = 0$ when $N(t) = 0$. When the policies are reinsured, the insurer's share of the claims is given by

$$X_I = \sum_{i=1}^{N(t)} (X_i \wedge R),$$

and the reinsurer's share of the claims is given by

$$X_R = \sum_{i=1}^{N(t)} (X_i - R)_+.$$

Next, let V_i be the i^{th} inter-claim time, which is given by

$$V_i = T_i - T_{i-1} \quad \text{for} \quad i \geq 1, \quad \text{and} \quad T_0 = 0.$$

We assume that each claim size X_i depends on the inter-claim time V_i. The rest of this chapter is organized as follows:

- In Section 4.1, we define the order statistic point process and review studies in the literature related to modelling claim sizes that depend on time.

- In Section 4.2, we present an expression for $\mathbf{Cov}[X_I, X_R]$.
- In Section 4.3, we present the steps for deriving $\mathbf{Cov}[X_I, X_R]$.
- In Section 4.4, we illustrate how to find the optimal retention R^* that maximizes the covariance $\mathbf{Cov}[X_I, X_R]$.

4.1 Order statistic point process

Since each claim size X_i depends on the inter-claim time V_i, we must understand the claim arrival process, which is a general point process. An order statistic point process is useful for this purpose.

Definition 4.1.1 (cf., e.g., Debrabant, 2008) A point process $N(t)$ is an order statistic point process if, provided $\mathbf{P}[N(t) = n] > 0$ with $t > 0$ and $n \geq 0$, the arrival times of the claims $T_1, T_2, \ldots, T_n$ conditioned upon $N(t) = n$, are distributed like the order statistics of n i.i.d. random variables with a common distribution function $F_t(x)$, for $x > 0$, such that $F_t(t) = 1$.

For more on the properties of order statistic point processes, we refer to Neuts and Resnick (1971), Crump (1975), Berg (1981), Puri (1982), and Huang and Shoung (1994). A simple example of order statistics point processes is the homogeneous Poisson process with rate parameter $\lambda > 0$, whose definition we recall next.

Definition 4.1.2 (cf., e.g., Focardi and Fabozzi, 2004) A homogeneous Poisson process is defined as a process $N(t)$ that starts at zero and has independent stationary increments. In addition, the random variable $N(t)$ is distributed as a Poisson variable with parameter λt, where $N(t)$ is a time dependent discrete variable that can assume nonnegative integer values. In this case, the distribution function F_t that we mentioned in Definition 4.1.1 is given by

$$F_t(x) = \frac{x}{t} \quad \text{for} \quad 0 \leq x \leq t,$$

which is the uniform distribution on $[0, t]$.

We will use the homogeneous Poisson process as the point process for our model. In more complicated cases, a non-homogeneous Poisson process with rate parameter $\lambda(t)$ is assumed, when during the time interval $(0, t]$, the mean is $\int_0^t \lambda(t)\mathrm{d}t$. Some examples of past studies on the topic include the estimation of the intensity function of a cyclic Poisson process (cf., e.g., Helmers and Zitikis, 1999; Helmers et al., 2003, 2005; Bebbington and Zitikis, 2004) and the non-parametric estimation of the doubly periodic Poisson intensity function (cf., e.g., Helmers et al., 2007). Other examples of point processes include the Cox point process and the determinantal point process. For more information, we refer to Cox and Isham (2000).

Various studies on modelling claim sizes that depend on time have been reported in the literature. For example, by assuming that the subsequent claim size depends on the previous inter-claim time, Albrecher and Teugels (2006) provide exponential estimates for infinite and finite time ruin probabilities, and Boudreault et al. (2006) derive an explicit expression for the Laplace transform of the time to ruin. Sendova and Zitikis (2012) model aggregate insurance claims when claim sizes depend on the claim arrival times and/or the inter-claim times. For more examples on modelling claim sizes that depend on the inter-claim times, we refer to Albrecher and Boxma (2004), Cossette et al. (2008), Asimit and Badescu (2010), and Cheung et al. (2010).

4.2 Expression for the covariance

To obtain the optimal retention R^* that maximizes $\mathbf{Cov}[X_I, X_R]$, we first derive an expression for the covariance.

Theorem 4.2.1 *Suppose $N(t)$ follows the homogeneous Poisson process with rate parameter $\lambda > 0$. Furthermore, let each claim size X_i depend on the inter-claim time V_i. Finally, let the conditional variables $X_i \mid V_i = v$ be i.i.d. Then*

$$\begin{aligned}
&\mathbf{Cov}[X_I, X_R] \\
&= \int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1)\mathbf{E}[(X_1 \wedge R)(X_1 - R)_+ \mid V_1 = v]\mathrm{d}v + \int_0^t \int_0^{t-y} \lambda^2 e^{-\lambda(y+v)}((\lambda(t-y-v)+2)^2 - 2) \\
&\times \mathbf{E}[(X_1 \wedge R) \mid V_1 = v]\mathbf{E}[(X_1 - R)_+ \mid V_1 = v]\mathrm{d}v\mathrm{d}y - \int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1)\mathbf{E}[(X_1 \wedge R) \mid V_1 = v]\mathrm{d}v \\
&\qquad \times \int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1)\mathbf{E}[(X_1 - R)_+ \mid V_1 = v]\mathrm{d}v. \qquad (4.1)
\end{aligned}$$

Corollary 4.2.2 *Suppose the conditions of Theorem 4.2.1 hold true. Furthermore, let F_l be the*

CDF of the larger claims, and let F_s be the CDF of the smaller claims. Finally, let the CDF of the conditional variables $X_i \mid V_i = v$ be given by (cf., e.g., Boudreault et al., 2006*)*

$$\mathbf{P}[X_i \le x \mid V_i = v] = (1 - e^{-\beta v})F_l(x) + e^{-\beta v}F_s(x) \tag{4.2}$$

for every $x \ge 0$, where $\beta \ge 0$ and $0 \le v \le t$. Then

$$\begin{aligned}
\mathbf{Cov}[X_I, X_R] = &\int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1)R\int_R^\infty (x-R)((1-e^{-\beta v})f_l(x) + e^{-\beta v}f_s(x))\mathrm{d}x\mathrm{d}v \\
&+ \int_0^t \int_0^{t-y} \lambda^2 e^{-\lambda(y+v)}(((\lambda(t-y-v)+2)^2)-2) \\
&\times \left\{ \int_0^R x((1-e^{-\beta y})f_l(x) + e^{-\beta y}f_s(x))\mathrm{d}x + \int_R^\infty R((1-e^{-\beta y})f_l(x) + e^{-\beta y}f_s(x))\mathrm{d}x \right\} \\
&\times \int_R^\infty (x-R)((1-e^{-\beta v})f_l(x) + e^{-\beta v}f_s(x))\mathrm{d}x\mathrm{d}v\mathrm{d}y \\
&- \int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1)\left\{ \int_0^R x((1-e^{-\beta v})f_l(x) + e^{-\beta v}f_s(x))\mathrm{d}x \right. \\
&\left. + \int_R^\infty R((1-e^{-\beta y})f_l(x) + e^{-\beta y}f_s(x))\mathrm{d}x \right\} \int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1) \\
&\times \int_R^\infty (x-R)((1-e^{-\beta v})f_l(x) + e^{-\beta v}f_s(x))\mathrm{d}x\mathrm{d}v,
\end{aligned} \tag{4.3}$$

where f_l is the PDF of larger claims and f_s is the PDF of smaller claims.

Proof Given the CDF of conditional variables $X_i \mid V_i = v$, we write

$$\mathbf{E}[(X_1 \wedge R)(X_1 - R)_+ | V_1 = v] = \int_R^\infty R(x-R)((1-e^{-\beta v})f_l(x) + e^{-\beta v}f_s(x))\mathrm{d}x. \tag{4.4}$$

Similarly, we obtain

$$\begin{aligned}
\mathbf{E}[(X_1 \wedge R) \mid V_1 = v] = &\int_0^R x((1-e^{-\beta y})f_l(x) + e^{-\beta y}f_s(x))\mathrm{d}x \\
&+ \int_R^\infty R((1-e^{-\beta y})f_l(x) + e^{-\beta y}f_s(x))\mathrm{d}x
\end{aligned} \tag{4.5}$$

and

$$\mathbf{E}[(X_1 - R)_+ | V_i = v] = \int_R^\infty (x-R)((1-e^{-\beta v})f_l(x) + e^{-\beta v}f_s(x))\mathrm{d}x. \tag{4.6}$$

Recall from Theorem 4.2.1 that an expression for the covariance $\mathbf{Cov}[X_I, X_R]$ is given by equation (4.1). By substituting equations (4.4), (4.5), and (4.6) into the right-hand side of equation

(4.1), we obtain a new expression for the covariance, which is given by equation (4.3). This completes the proof of Corollary 4.2.2. □

4.3 Proof of Theorem 4.2.1

Here we adopt the approach in Sendova and Zitikis (2012). The covariance of X_I and X_R can be written as follows:

$$\mathbf{Cov}[X_I, X_R] = \mathbf{E}[X_I X_R] - \mathbf{E}[X_I]\mathbf{E}[X_R]. \tag{4.7}$$

Next, we calculate the expectations of X_I and X_R (Subsection 4.3.1), and the expectation of $X_I X_R$ (Subsection 4.3.2).

4.3.1 Calculating expectations of X_I and X_R

We begin with the expectation of X_I. To calculate the expectation, we shall take three steps:

- In the first step, we assume that each claim X_i depends on claim arrival times T_{i-1} and T_i.
- In the second step, we assume that each claim X_i depends on the previous claim arrival time T_{i-1} and V_i, the inter-claim time between the $(i-1)^{th}$ claim and the i^{th} claim.
- In the third step, we assume that each claim X_i depends only on V_i, which is the condition in our theorem.

Each claim X_i depends on claim arrival times T_{i-1} and T_i

Using repeated conditioning, we have

$$\begin{aligned}\mathbf{E}[X_I] &= \sum_{n=1}^{\infty} \mathbf{P}[N(t)=n] \Big\{ \sum_{i=1}^{n} \mathbf{E}[X_i \wedge R \mid N(t)=n] \Big\} \\ &= \sum_{n=1}^{\infty} \mathbf{P}[N(t)=n] \Big\{ \int_0^t \mathbf{E}[X_1 \wedge R \mid T_1 = y, N(t)=n] \mathrm{d}F_{1|t,n}(y) \\ &\quad + \sum_{i=2}^{n} \int\int_{0 \le x \le y \le t} \mathbf{E}[X_i \wedge R \mid T_{i-1} = x, T_i = y, N(t)=n] \mathrm{d}F_{i-1,i|t,n}(x,y) \Big\},\end{aligned} \tag{4.8}$$

where, for $i \geq 1$,

$$F_{i|t,n}(y) = \mathbf{P}[T_i \leq y \mid N(t)=n] \quad \text{for} \quad 0 \leq y \leq t,$$

and, for $1 \leq i \leq j$,

$$F_{i,j|t,n}(x, y) = \mathbf{P}[T_i \leq x, T_j \leq y \mid N(t) = n] \quad \text{for} \quad 0 \leq x \leq y \leq t.$$

Using formulas (2.1.6) and (2.2.1) provided by David and Nagaraja (2003), the density corresponding to $F_{1|t,n}(y)$ is given by

$$f_{1|t,n}(y) = n f_t(y)(1 - F_t(y))^{n-1}, \tag{4.9}$$

and the density corresponding to $F_{i-1,i|t,n}(x, y)$ is given by

$$f_{i-1,i|t,n}(x, y) = \frac{n!}{(i-2)!(n-i)!} f_t(x) f_t(y) F_t^{i-2}(x)(1 - F_t(y))^{n-i}. \tag{4.10}$$

Since $N(t)$ follows the homogeneous Poisson process with rate parameter λ, we have $f_t(x) = 1/t$ and $F_t(x) = x/t$, where $x \in [0, t]$. Consequently, equation (4.9) becomes

$$f_{1|t,n}(y) = \frac{n(t-y)^{n-1}}{t^n}, \tag{4.11}$$

and equation (4.10) becomes

$$f_{i-1,i|t,n}(x, y) = \frac{n!}{(i-2)!(n-i)!} \frac{x^{i-2}(t-y)^{n-i}}{t^n}. \tag{4.12}$$

By substituting equations (4.11) and (4.12) into equation (4.8), we have

$$\begin{aligned} \mathbf{E}[X_I] = \sum_{n=1}^{\infty} \mathbf{P}[N(t) = n] \bigg\{ & \int_0^t \mathbf{E}[X_1 \wedge R \mid T_1 = y] \frac{n(t-y)^{n-1}}{t^n} \mathrm{d}y + \sum_{i=2}^{n} \frac{n!}{(i-2)!(n-i)!} \\ & \times \int_0^t \int_x^t \mathbf{E}[X_i \wedge R \mid T_{i-1} = x, T_i = y] \frac{x^{i-2}(t-y)^{n-i}}{t^n} \mathrm{d}y \mathrm{d}x \bigg\}. \end{aligned} \tag{4.13}$$

Each claim X_i depends on the previous claim arrival time T_{i-1} and V_i

By using the change of variables technique, equation (4.13) becomes

$$\mathbf{E}[X_I] = \sum_{n=1}^{\infty} \mathbf{P}[N(t) = n]\bigg\{ \int_0^t \mathbf{E}[X_1 \wedge R \mid T_1 = y] \frac{n(t-y)^{n-1}}{t^n} \mathrm{d}y$$
$$+ \sum_{i=2}^{n} \frac{n!}{(i-2)!(n-i)!} \int_0^t \int_0^{t-x} \mathbf{E}[X_i \wedge R \mid T_{i-1} = x, V_i = v]$$
$$\times \frac{x^{i-2}(t-x-v)^{n-i}}{t^n} \mathrm{d}v\mathrm{d}x \bigg\}. \tag{4.14}$$

Each claim X_i depends only on V_i

Since $T_0 = 0$ and $V_1 = T_1 - T_0$, we have

$$\mathbf{E}[X_1 \wedge R \mid T_1 = y] = \mathbf{E}[X_1 \wedge R \mid V_1 = y]$$

and

$$\mathbf{E}[X_i \wedge R \mid T_{i-1} = x, V_i = v] = \mathbf{E}[X_i \wedge R \mid V_i = v].$$

Then equation (4.14) becomes

$$\mathbf{E}[X_I] = \sum_{n=1}^{\infty} \mathbf{P}[N(t) = n]\bigg\{ \int_0^t \mathbf{E}[X_1 \wedge R \mid V_1 = y] \frac{n(t-y)^{n-1}}{t^n} \mathrm{d}y$$
$$+ \sum_{i=2}^{n} \int_0^t \int_0^{t-x} \mathbf{E}[X_i \wedge R \mid V_i = v] \frac{n!}{(i-2)!(n-i)!} \frac{x^{i-2}(t-x-v)^{n-i}}{t^n} \mathrm{d}v\mathrm{d}x \bigg\}. \tag{4.15}$$

Remaining steps for finding the expectation of X_I

By changing the order of integration on the right-hand side of equation (4.15), we obtain

$$\mathbf{E}[X_I] = \sum_{n=1}^{\infty} \mathbf{P}[N(t) = n]\bigg\{ \int_0^t \mathbf{E}[X_1 \wedge R \mid V_1 = y] \frac{n(t-y)^{n-1}}{t^n} \mathrm{d}y$$
$$+ \sum_{i=2}^{n} \int_0^t \mathbf{E}[X_i \wedge R \mid V_i = v] \int_0^{t-v} \frac{n!}{(i-2)!(n-i)!} \frac{x^{i-2}(t-x-v)^{n-i}}{t^n} \mathrm{d}x\mathrm{d}v \bigg\}, \tag{4.16}$$

where

$$\int_0^{t-v} \frac{n!}{(i-2)!(n-i)!} \frac{x^{i-2}(t-x-v)^{n-i}}{t^n} \mathrm{d}x = \frac{n!}{(i-2)!(n-i)!} \frac{1}{t^n} \int_0^{t-v} x^{i-2}\left(1 - \frac{x}{t-v}\right)^{n-i} (t-v)^{n-i} \mathrm{d}x. \tag{4.17}$$

Let $x = (t-v)y$ and $\mathrm{d}x = (t-v)\mathrm{d}y$. Then equation (4.17) becomes

$$\begin{aligned}\int_0^{t-v} \frac{n!}{(i-2)!(n-i)!}\frac{x^{i-2}(t-x-v)^{n-i}}{t^n}\mathrm{d}x &= \frac{n!}{(i-2)!(n-i)!}\frac{1}{t^n}\int_0^1 (t-v)^{i-2}y^{i-2}(1-y)^{n-i} \\ &\quad \times (t-v)^{n-i}(t-v)\mathrm{d}y \\ &= \frac{n!}{(i-2)!(n-i)!}\frac{(t-v)^{n-1}}{t^n}\int_0^1 y^{i-2}(1-y)^{n-i}\mathrm{d}y.\end{aligned} \tag{4.18}$$

Recall the complete beta function, which is defined as follows:

$$B(z,w) = \int_0^1 x^{z-1}(1-x)^{w-1}\mathrm{d}x,$$

where $z, w > 0$. Also recall the well-known formula for the complete beta function:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{4.19}$$

Using equation (4.19), we rewrite equation (4.18) as follows:

$$\begin{aligned}\int_0^{t-v} \frac{n!}{(i-2)!(n-i)!}\frac{x^{i-2}(t-x-v)^{n-i}}{t^n}\mathrm{d}x &= \frac{n!}{(i-2)!(n-i)!}\frac{(t-v)^{n-1}}{t^n}\frac{(i-2)!(n-i)!}{(i-1+n-i+1-1)!} \\ &= \frac{n!}{(i-2)!(n-i)!}\frac{(t-v)^{n-1}}{t^n}\frac{(i-2)!(n-i)!}{(n-1)!} \\ &= \frac{n(t-v)^{n-1}}{t^n}.\end{aligned} \tag{4.20}$$

By substituting equation (4.20) into equation (4.16), we obtain

$$\begin{aligned}\mathbf{E}[X_I] &= \sum_{n=1}^{\infty}\mathbf{P}[N(t)=n]\Bigg\{\int_0^t \mathbf{E}[X_1 \wedge R \mid V_1 = y]\frac{n(t-y)^{n-1}}{t^n}\mathrm{d}y \\ &\quad + \sum_{i=2}^{n}\int_0^t \mathbf{E}[X_i \wedge R \mid V_i = v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v\Bigg\} \\ &= \sum_{n=1}^{\infty}\mathbf{P}[N(t)=n]\Bigg\{\sum_{i=1}^{n}\int_0^t \mathbf{E}[X_i \wedge R \mid V_i = v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v\Bigg\}.\end{aligned} \tag{4.21}$$

Since

$$\mathbf{P}[N(t)=n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!},$$

equation (4.21) becomes

$$\mathbf{E}[X_I] = \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\Big\{\sum_{i=1}^{n}\int_0^t \mathbf{E}[X_i \wedge R \mid V_i = v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v\Big\}. \tag{4.22}$$

By rearranging terms on the right-hand side of equation (4.22), we have

$$\begin{aligned}\mathbf{E}[X_I] &= \int_0^t \lambda e^{-\lambda v}\sum_{n=1}^{\infty} e^{-\lambda(t-v)}\frac{(\lambda(t-v))^{n-1}}{(n-1)!}\sum_{i=1}^{n}\mathbf{E}[X_i \wedge R \mid V_i = v]\mathrm{d}v \\ &= \int_0^t \lambda e^{-\lambda v}\sum_{n=0}^{\infty} e^{-\lambda(t-v)}\frac{\lambda^n(t-v)^n}{n!}\sum_{i=1}^{n+1}\mathbf{E}[X_i \wedge R \mid V_i = v]\mathrm{d}v. \end{aligned} \tag{4.23}$$

Since the conditional variables $X_i \mid V_i$ are i.i.d., we have

$$\mathbf{E}[X_i \wedge R \mid V_i = v] = \mathbf{E}[X_1 \wedge R \mid V_1 = v] \quad \text{for} \quad i \geq 1.$$

Therefore, equation (4.23) becomes

$$\mathbf{E}[X_I] = \int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1)\mathbf{E}[X_1 \wedge R \mid V_1 = v]\mathrm{d}v. \tag{4.24}$$

Result for the expectation of X_R

Similar to the way $\mathbf{E}[X_I]$ was obtained, we obtain the equation

$$\mathbf{E}[X_R] = \int_0^t \lambda e^{-\lambda v}(\lambda(t-v)+1)\mathbf{E}[(X_1 - R)_+ \mid V_1 = v]\mathrm{d}v. \tag{4.25}$$

4.3.2 Finding the expectation of $X_I X_R$

We start by decomposing the expectation of $X_I X_R$ as follows:

$$\begin{aligned}\mathbf{E}[X_I X_R] = \sum_{n=1}^{\infty}\mathbf{P}[N(t) = n]\Big\{&\mathbf{E}\Big[\sum_{1\leq i=j\leq n}(X_i \wedge R)(X_j - R)_+ | N(t) = n\Big] \\ &+ \mathbf{E}\Big[\sum_{1\leq i\neq j\leq n}(X_i \wedge R)(X_j - R)_+ | N(t) = n\Big]\Big\}. \end{aligned} \tag{4.26}$$

We first obtain expressions for the two expectations in the braces on the right-hand side of equation (4.26). Then we calculate

$$\sum_{n=1}^{\infty}\mathbf{P}[N(t) = n]\mathbf{E}\Big[\sum_{1\leq i=j\leq n}(X_i \wedge R)(X_j - R)_+ \mid N(t) = n\Big]$$

and

$$\sum_{n=1}^{\infty}\mathbf{P}[N(t)=n]\mathbf{E}\Big[\sum_{1\le i\neq j\le n}(X_i\wedge R)(X_j-R)_+ \mid N(t)=n\Big].$$

Finding the expectation of $\sum_{1\le i=j\le n}(X_i\wedge R)(X_j-R)_+ \mid N(t)=n$

To simplify the presentation, denote $\mathbf{E}[\sum_{1\le i=j\le n}(X_i\wedge R)(X_j-R)_+|N(t)=n]$ by b_t. Using repeated conditioning, for each claim X_i that depends on T_{i-1} and T_i, we have

$$\begin{aligned}
b_t &= \int_0^t \mathbf{E}[(X_1\wedge R)(X_1-R)_+|T_1=y,N(t)=n]\mathrm{d}F_{1|t,n}(y)\\
&\quad+\sum_{i=2}^{n}\int_0^t\int_0^{t-x}\mathbf{E}[(X_i\wedge R)(X_i-R)_+\mid T_{i-1}=x,T_i=y,N(t)=n]\mathrm{d}F_{i-1,i|t,n}(x,y)\\
&=\int_0^t \mathbf{E}[(X_1\wedge R)(X_1-R)_+\mid T_1=y]\frac{n(t-y)^{n-1}}{t^n}\mathrm{d}y\\
&\quad+\sum_{i=2}^{n}\int_0^t\int_0^{t-x}\mathbf{E}[(X_i\wedge R)(X_i-R)_+\mid T_{i-1}=x,T_i=y]\frac{n!}{(i-2)!(n-i)!}\frac{x^{i-2}(t-y)^{n-i}}{t^n}\mathrm{d}y\mathrm{d}x.
\end{aligned}\tag{4.27}$$

Next, similar to the way an expression for $\mathbf{E}[X_I]$ was obtained when each claim size X_i depends only on the inter-claim time V_i, equation (4.27) becomes

$$\begin{aligned}
b_t &= \int_0^t \mathbf{E}[(X_1\wedge R)(X_1-R)_+\mid V_1=y]\frac{n(t-y)^{n-1}}{t^n}\mathrm{d}y\\
&\quad+\sum_{i=2}^{n}\int_0^t\mathbf{E}[(X_i\wedge R)(X_i-R)_+\mid V_i=v]\int_0^{t-v}\frac{n!}{(i-2)!(n-i)!}\frac{x^{i-2}(t-v-x)^{n-i}}{t^n}\mathrm{d}x\mathrm{d}v\\
&=\int_0^t \mathbf{E}[(X_1\wedge R)(X_1-R)_+\mid V_1=y]\frac{n(t-y)^{n-1}}{t^n}\mathrm{d}y\\
&\quad+\sum_{i=2}^{n}\int_0^t\mathbf{E}[(X_i\wedge R)(X_i-R)_+\mid V_i=v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v\\
&=\sum_{i=1}^{n}\int_0^t\mathbf{E}[(X_i\wedge R)(X_i-R)_+\mid V_i=v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v.
\end{aligned}\tag{4.28}$$

Finding the expectation of $\sum_{1\le i\neq j\le n}(X_i\wedge R)(X_j-R)_+ \mid N(t)=n$ when $i<j$ and $i>j$

We consider two scenarios: $i<j$ and $i>j$. When $i<j$, we denote $\mathbf{E}[\sum_{1\le i<j\le n}(X_i\wedge R)(X_j-R)_+|N(t)=n]$ by c_t. Then

$$c_t = \mathbf{E}[(X_1 \wedge R)(X_2 - R)_+ \mid N(t) = n] + \sum_{j=3}^{n} \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ \mid N(t) = n]$$
$$+ \sum_{i=2}^{n-1} \mathbf{E}[(X_i \wedge R)(X_{i+1} - R)_+ \mid N(t) = n] + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \mathbf{E}[(X_i \wedge R)(X_j - R)_+ | N(t) = n]. \quad (4.29)$$

Next, we calculate each term on the right-hand side of equation (4.29).

Finding the first term on the right-hand side of equation (4.29)

When X_i depends on T_{i-1} and T_i, the expectation of $(X_1 \wedge R)(X_2 - R)_+ | N(t) = n$ becomes

$$\mathbf{E}[(X_1 \wedge R)(X_2 - R)_+ | N(t) = n] = \int_0^t \int_y^t \mathbf{E}[(X_1 \wedge R)(X_2 - R)_+ \mid T_1 = y, T_2 = x] \times \frac{n!}{(n-2)!} \frac{(t-x)^{n-2}}{t^n} \mathrm{d}x\mathrm{d}y.$$

When X_i depends on T_{i-1} and V_i, the expectation is

$$\mathbf{E}[(X_1 \wedge R)(X_2 - R)_+ | N(t) = n] = \int_0^t \int_0^{t-y} \mathbf{E}[(X_1 \wedge R)(X_2 - R)_+ | T_1 = y, V_2 = v] \times \frac{n!}{(n-2)!} \frac{(t-y-v)^{n-2}}{t^n} \mathrm{d}v\mathrm{d}y.$$

When X_i depends only on V_i, the expectation becomes

$$\mathbf{E}[(X_1 \wedge R)(X_2 - R)_+ | N(t) = n] = n(n-1) \int_0^t \int_0^{t-y} \mathbf{E}[(X_1 \wedge R) | V_1 = y] \mathbf{E}[(X_2 - R)_+ | V_2 = v] \times \frac{(t-y-v)^{n-2}}{t^n} \mathrm{d}v\mathrm{d}y. \quad (4.30)$$

Finding the second term on the right-hand side of equation (4.29)

We write

$$\sum_{j=3}^{n} \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ | N(t) = n]$$
$$= \sum_{j=3}^{n} \int \int \int \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ \mid T_1 = y, T_{j-1} = w, T_j = z, N(t) = n] \mathrm{d}F_{1,j-1,j|t,n}(y, w, z),$$

where, for $3 \leq j \leq n$,

$$F_{1,j-1,j|t,n}(y,w,z) = \mathbf{P}[T_1 \le y, T_{j-1} \le w, T_j \le z \mid N(t) = n]$$

for $0 \le y \le w \le z \le t$. Using formula (2.2.2) provided by David and Nagaraja (2003), the density corresponding to $F_{1,j-1,j|t,n}$ is equal to

$$f_{1,j-1,j|t,n}(y,w,z) = \frac{n!}{(j-3)!(n-j)!} f_t(y) f_t(w) f_t(z) (F_t(w) - F_t(y))^{j-3} (1 - F_t(z))^{n-j}.$$

When X_i depends on T_{i-1} and T_i, we obtain

$$\begin{aligned}
&\sum_{j=3}^{n} \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ | N(t) = n] \\
&= \sum_{j=3}^{n} \int_0^t \int_0^z \int_0^w \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ \mid T_1 = y, T_{j-1} = w, T_j = z] \\
&\quad \times \frac{n!}{(j-3)!(n-j)!} f_t(y) f_t(w) f_t(z) (F_t(w) - F_t(y))^{j-3} \\
&\quad \times (1 - F_t(z))^{n-j} \mathrm{d}y \mathrm{d}w \mathrm{d}z \\
&= \sum_{j=3}^{n} \int_0^t \int_0^z \int_0^w \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ \mid T_1 = y, T_{j-1} = w, T_j = z] \\
&\quad \times \frac{n!}{(j-3)!(n-j)!} \frac{(w-y)^{j-3}(t-z)^{n-j}}{t^n} \mathrm{d}y \mathrm{d}w \mathrm{d}z.
\end{aligned}$$

When X_i depends on T_{i-1} and V_i, we have

$$\begin{aligned}
&\sum_{j=3}^{n} \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ | N(t) = n] \\
&= \sum_{j=3}^{n} \int_0^t \int_y^t \int_0^{t-w} \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ | T_1 = y, T_{j-1} = w, V_j = v] \\
&\quad \times \frac{n!}{(j-3)!(n-j)!} \frac{(w-y)^{j-3}(t-w-v)^{n-j}}{t^n} \mathrm{d}v \mathrm{d}w \mathrm{d}y.
\end{aligned}$$

When X_i depends only on V_i, we obtain

$$\begin{aligned}
\sum_{j=3}^{n} \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ | N(t) = n] &= \sum_{j=3}^{n} \frac{n!}{(j-3)!(n-j)!} \int_0^t \int_y^t \int_0^{t-w} \frac{(w-y)^{j-3}(t-w-v)^{n-j}}{t^n} \\
&\quad \times \mathbf{E}[X_1 \wedge R \mid V_1 = y] \mathbf{E}[(X_j - R)_+ \mid V_j = v] \mathrm{d}v \mathrm{d}w \mathrm{d}y.
\end{aligned} \tag{4.31}$$

Interchanging the order of integration on the right-hand side of equation (4.31), we have

$$\sum_{j=3}^{n}\mathbf{E}[(X_1\wedge R)(X_j-R)_+|N(t)=n]=\sum_{j=3}^{n}\frac{n!}{(j-3)!(n-j)!}\int_0^t\int_0^{t-y}\int_y^{t-v}\mathbf{E}[X_1\wedge R\mid V_1=y]$$
$$\times\,\mathbf{E}[(X_j-R)_+\mid V_j=v]\frac{(w-y)^{j-3}(t-w-v)^{n-j}}{t^n}\mathrm{d}w\mathrm{d}v\mathrm{d}y, \tag{4.32}$$

where

$$\int_y^{t-v}\frac{(w-y)^{j-3}(t-w-v)^{n-j}}{t^n}\mathrm{d}w=\frac{1}{t^n}\int_y^{t-v}(w-y)^{j-3}\Big(\frac{t-w-v}{t-y-v}\Big)^{n-j}(t-y-v)^{n-j}\mathrm{d}w. \tag{4.33}$$

Let $w=z(t-y-v)+y$ and $\mathrm{d}w=(t-y-v)\mathrm{d}z$. Then equation (4.33) becomes

$$\begin{aligned}\int_y^{t-v}\frac{(w-y)^{j-3}(t-w-v)^{n-j}}{t^n}\mathrm{d}w&=\frac{1}{t^n}\int_0^1 z^{j-3}(1-z)^{n-j}(t-y-v)^{j-3}(t-y-v)^{n-j}\\&\quad\times(t-y-v)\mathrm{d}z\\&=\frac{(t-y-v)^{n-2}}{t^n}\int_0^1 z^{j-3}(1-z)^{n-j}\mathrm{d}z.\end{aligned} \tag{4.34}$$

Using the formula for the complete beta function, which is given by equation (4.19), we have

$$\int_0^1 z^{j-3}(1-z)^{n-j}\mathrm{d}z=\frac{(j-3)!(n-j)!}{(n-2)!}.$$

Hence, equation (4.34) becomes

$$\sum_{j=3}^{n}\mathbf{E}[(X_1\wedge R)(X_j-R)_+|N(t)=n]=\frac{(t-y-v)^{n-2}}{t^n}\frac{(j-3)!(n-j)!}{(n-2)!}. \tag{4.35}$$

By substituting equation (4.35) into equation (4.32), we obtain

$$\begin{aligned}&\sum_{j=3}^{n}\mathbf{E}[(X_1\wedge R)(X_j-R)_+|N(t)=n]\\&=\sum_{j=3}^{n}n(n-1)\int_0^t\int_0^{t-y}\mathbf{E}[(X_1\wedge R)|V_1=y]\mathbf{E}[(X_j-R)_+|V_j=v]\frac{(t-y-v)^{n-2}}{t^n}\mathrm{d}v\mathrm{d}y.\end{aligned} \tag{4.36}$$

Finding the third term on the right-hand side of equation (4.29)

Denote $\sum_{i=2}^{n-1}\mathbf{E}[(X_i\wedge R)(X_{i+1}-R)_+|N(t)=n]$ by d_t. When X_i depends on T_{i-1} and T_i, we have

$$d_t = \sum_{i=2}^{n-1} \int\int\int \mathbf{E}[(X_i \wedge R)(X_{i+1} - R)_+ \mid T_{i-1} = y, T_i = w, T_{i+1} = z] \mathrm{d}F_{i-1,i,i+1|t,n}(y, w, z), \quad (4.37)$$

where, for $2 \leq i \leq n$,

$$F_{i-1,i,i+1|t,n}(y, w, z) = \mathbf{P}[T_{i-1} \leq y, T_i \leq w, T_{i+1} \leq z \mid N(t) = n]$$

for $0 \leq y \leq w \leq z \leq t$. Using formula (2.2.2) from David and Nagaraja (2003), the density corresponding to $F_{i-1,i,i+1|t,n}$ is equal to

$$f_{i-1,i,i+1|t,n}(y, w, z) = \frac{n!}{(i-2)!(n-i-1)!} f_t(y) f_t(w) f_t(z) F_t(y)^{i-2} (1 - F_t(z))^{n-i-1}. \quad (4.38)$$

Hence, equation (4.37) becomes

$$\begin{aligned} d_t = \sum_{i=2}^{n-1} \int_0^t \int_0^z \int_0^w & \mathbf{E}[(X_i \wedge R)(X_{i+1} - R)_+ \mid T_{i-1} = y, T_i = w, T_{i+1} = z] \\ & \times \frac{n!}{(i-2)!(n-i-1)!} \frac{y^{i-2}(t-z)^{n-i-1}}{t^n} \mathrm{d}y\mathrm{d}w\mathrm{d}z. \end{aligned} \quad (4.39)$$

When X_i depends on T_{i-1} and V_i, equation (4.39) is

$$\begin{aligned} d_t = \sum_{i=2}^{n-1} \int_0^t \int_0^{t-y} \int_0^{t-y-u} & \mathbf{E}[(X_i \wedge R)(X_{i+1} - R)_+ | T_{i-1} = y, V_i = u, V_{i+1} = v] \\ & \times \frac{n!}{(i-2)!(n-i-1)!} \frac{y^{i-2}(t-y-u-v)^{n-i-1}}{t^n} \mathrm{d}v\mathrm{d}u\mathrm{d}y. \end{aligned} \quad (4.40)$$

When X_i depends only on V_i, similar to the way $\sum_{j=3}^{n} \mathbf{E}[(X_1 \wedge R)(X_j - R)_+ | N(t) = n]$ was obtained, equation (4.40) becomes

$$d_t = \sum_{i=2}^{n-1} n(n-1) \int_0^t \int_0^{t-u} \mathbf{E}[(X_i \wedge R) \mid V_i = u] \mathbf{E}[(X_{i+1} - R)_+ \mid V_{i+1} = v] \frac{(t-u-v)^{n-2}}{t^n} \mathrm{d}v\mathrm{d}u. \quad (4.41)$$

Finding the fourth term on the right-hand side of equation (4.29)

Denote $\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \mathbf{E}[(X_i \wedge R)(X_j - R)_+ | N(t) = n]$ by e_t. When X_i depends on T_{i-1} and T_i, we have

$$e_t = \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \Big\{ \int\int\int\int E_1 \mathrm{d}F_{i-1,i,j-1,j|t,n}(x,y,w,z) \Big\},$$

where

$$E_1 = \mathbf{E}[(X_i \wedge R)(X_j - R)_+ \mid T_{i-1} = x, T_i = y, T_{j-1} = w, T_j = z]$$

and

$$F_{i-1,i,j-1,j|t,n}(x,y,w,z) = \mathbf{P}[T_{i-1} \leq x, T_i \leq y, T_{j-1} \leq w, T_j \leq z \mid N(t) = n]$$

for $2 \leq i < j \leq n$, $j - i \geq 2$, and $0 \leq x \leq y \leq w \leq z \leq t$. Using formula (2.2.2) from David and Nagaraja (2003), the density corresponding to $F_{i-1,i,j-1,j|t,n}$ is equal to

$$\begin{aligned} &f_{i-1,i,j-1,j|t,n}(x,y,w,z) \\ &= \frac{n!}{(i-2)!(j-i-2)!(n-j)!} f_t(x) f_t(y) f_t(w) f_t(z) F_t^{i-2}(x)(F_t(w) - F_t(y))^{j-i-2}(1 - F_t(z))^{n-j}. \end{aligned} \tag{4.42}$$

Consequently,

$$\begin{aligned} e_t = &\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \int_0^t \int_0^z \int_0^w \int_0^y \mathbf{E}[(X_i \wedge R)(X_j - R)_+ \mid T_{i-1} = x, T_i = y, T_{j-1} = w, T_j = z] \\ &\times \frac{n!}{(i-2)!(j-i-2)!(n-j)!} \frac{x^{i-2}(w-y)^{j-i-2}(t-z)^{n-j}}{t^n} \mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}z. \end{aligned} \tag{4.43}$$

When X_i depends on T_{i-1} and V_i, equation (4.43) becomes

$$\begin{aligned} e_t = &\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \frac{n!}{(i-2)!(j-i-2)!(n-j)!} \\ &\times \int_0^t \int_0^{t-x} \int_{x+u}^t \int_0^{t-w} \mathbf{E}[(X_i \wedge R)(X_j - R)_+ | T_{i-1} = x, V_i = u, T_{j-1} = w, V_j = v] \\ &\quad \times \frac{x^{i-2}(w-x-u)^{j-i-2}(t-w-v)^{n-j}}{t^n} \mathrm{d}v\mathrm{d}w\mathrm{d}u\mathrm{d}x. \end{aligned} \tag{4.44}$$

When X_i depends only on V_i, equation (4.44) becomes

$$\begin{aligned} e_t = &\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \frac{n!}{(i-2)!(j-i-2)!(n-j)!} \int_0^t \int_0^{t-x} \int_{x+u}^t \int_0^{t-w} \mathbf{E}[(X_i \wedge R) | V_i = u] \\ &\times \mathbf{E}[(X_j - R)_+ | V_j = v] \frac{x^{i-2}(w-x-u)^{j-i-2}(t-w-v)^{n-j}}{t^n} \mathrm{d}v\mathrm{d}w\mathrm{d}u\mathrm{d}x. \end{aligned} \tag{4.45}$$

Let $w = w - x - u$. Then equation (4.45) becomes

$$e_t = \sum_{i=2}^{n-2}\sum_{j=i+2}^{n}\frac{n!}{(i-2)!(j-i-2)!(n-j)!}\int_0^t\int_0^{t-x}\int_0^{t-x-u}\int_0^{t-x-u-w}\mathbf{E}[(X_i\wedge R)|V_i=u]$$
$$\times\,\mathbf{E}[(X_j-R)_+|V_j=v]\frac{x^{i-2}w^{j-i-2}(t-x-u-v-w)^{n-j}}{t^n}\mathrm{d}x\mathrm{d}w\mathrm{d}v\mathrm{d}u. \tag{4.46}$$

Rearranging terms on the right-hand side of equation (4.46), we have

$$e_t = \sum_{i=2}^{n-2}\sum_{j=i+2}^{n}\frac{n!}{(i-2)!(j-i-2)!(n-j)!}\int_0^t\int_0^{t-u}\mathbf{E}[(X_i\wedge R)|V_i=u]\mathbf{E}[(X_j-R)_+|V_j=v]$$
$$\times\int_0^{t-u-v}\int_0^{t-u-v-w}\frac{x^{i-2}w^{j-i-2}(t-x-u-v-w)^{n-j}}{t^n}\mathrm{d}x\mathrm{d}w\mathrm{d}v\mathrm{d}u, \tag{4.47}$$

where

$$\int_0^{t-u-v-w}\frac{x^{i-2}w^{j-i-2}(t-x-u-v-w)^{n-j}}{t^n}\mathrm{d}x = \frac{w^{j-i-2}}{t^n}\int_0^{t-u-v-w}x^{i-2}\left(\frac{t-u-v-w-x}{t-u-v-w}\right)^{n-j}$$
$$\times\,(t-u-v-w)^{n-j}\mathrm{d}x. \tag{4.48}$$

Let $x = (t-u-v-w)y$ and $\mathrm{d}x = (t-u-v-w)\mathrm{d}y$. Then equation (4.48) becomes

$$\int_0^{t-u-v-w}\frac{x^{i-2}w^{j-i-2}(t-x-u-v-w)^{n-j}}{t^n}\mathrm{d}x = \frac{w^{j-i-2}}{t^n}\int_0^1 y^{i-2}(1-y)^{n-j}(t-u-v-w)^{i-2}$$
$$\times\,(t-u-v-w)^{n-j}(t-u-v-w)\mathrm{d}y.$$

Consequently,

$$\int_0^{t-u-v-w}\frac{x^{i-2}w^{j-i-2}(t-x-u-v-w)^{n-j}}{t^n}\mathrm{d}x = \frac{w^{j-i-2}}{t^n}(t-u-v-w)^{n-j+i-1}$$
$$\times\int_0^1 y^{i-2}(1-y)^{n-j}\mathrm{d}y. \tag{4.49}$$

Using the formula for the complete beta function, which is given in equation (4.19), we have

$$\int_0^1 y^{i-2}(1-y)^{n-j}\mathrm{d}y = \frac{(i-2)!(n-j)!}{(n-j+i-1)!}.$$

Then equation (4.49) becomes

$$\int_0^{t-u-v-w} \frac{x^{i-2}w^{j-i-2}(t-x-u-v-w)^{n-j}}{t^n}\mathrm{d}x = \frac{w^{j-i-2}}{t^n}(t-u-v-w)^{n-j+i-1} \times \frac{(i-2)!(n-j)!}{(n-j+i-1)!}. \tag{4.50}$$

By substituting equation (4.50) into equation (4.47), we have

$$\begin{aligned} e_t &= \sum_{i=2}^{n-2}\sum_{j=i+2}^{n} \frac{n!}{(i-2)!(j-i-2)!(n-j)!}\int_0^t\int_0^{t-u} \mathbf{E}[(X_i\wedge R)|V_i=u]\mathbf{E}[(X_j-R)_+|V_j=v] \\ &\quad \times \int_0^{t-u-v} \frac{w^{j-i-2}(t-u-v-w)^{n-j+i-1}}{t^n}\mathrm{d}w\mathrm{d}v\mathrm{d}u \\ &= \sum_{i=2}^{n-2}\sum_{j=i+2}^{n} n(n-1)\int_0^t\int_0^{t-u} \mathbf{E}[(X_i\wedge R)\mid V_i=u]\mathbf{E}[(X_j-R)_+|V_j=v]\frac{(t-u-v)^{n-2}}{t^n}\mathrm{d}v\mathrm{d}u, \end{aligned} \tag{4.51}$$

where

$$\int_0^{t-u-v} \frac{w^{j-i-2}(t-u-v-w)^{n-j+i-1}}{t^n}\mathrm{d}w = \frac{(t-u-v)^{n-2}}{t^n}\frac{(j-i-2)!(n-j+i-1)!}{(n-2)!}. \tag{4.52}$$

Equation (4.52) is obtained using the formula for the complete beta function, which is similar to the way equation (4.50) was obtained.

Expression for the expectation of $\sum_{1\le i<j\le n}(X_i\wedge R)(X_j-R)_+ \mid N(t)=n$

Having calculated each term on the right-hand side of equation (4.29), the expectation of $\sum_{1\le i<j\le n}(X_i\wedge R)(X_j-R)_+ \mid N(t)=n$, which we denoted by $c(t)$, becomes

$$\begin{aligned} c_t &= n(n-1)\int_0^t\int_0^{t-y} \mathbf{E}[(X_1\wedge R)|V_1=y]\mathbf{E}[(X_2-R)_+|V_2=v]\frac{(t-y-v)^{n-2}}{t^n}\mathrm{d}v\mathrm{d}y \\ &\quad + \sum_{j=3}^{n} n(n-1)\int_0^t\int_0^{t-y} \mathbf{E}[(X_1\wedge R)|V_1=y]\mathbf{E}[(X_j-R)_+|V_j=v]\frac{(t-y-v)^{n-2}}{t^n}\mathrm{d}v\mathrm{d}y \\ &\quad + \sum_{i=2}^{n-1} n(n-1)\int_0^t\int_0^{t-u} \mathbf{E}[(X_i\wedge R)|V_i=u]\mathbf{E}[(X_{i+1}-R)_+|V_{i+1}=v]\frac{(t-u-v)^{n-2}}{t^n}\mathrm{d}v\mathrm{d}u \\ &\quad + \sum_{i=2}^{n-2}\sum_{j=i+2}^{n} n(n-1)\int_0^t\int_0^{t-u} \mathbf{E}[(X_i\wedge R)|V_i=u]\mathbf{E}[(X_j-R)_+|V_j=v]\frac{(t-u-v)^{n-2}}{t^n}\mathrm{d}v\mathrm{d}u \\ &= n(n-1)\int_0^t\int_0^{t-y} \sum_{1\le i<j\le n} \mathbf{E}[(X_i\wedge R)|V_i=y]\mathbf{E}[(X_j-R)_+|V_j=v]\frac{(t-y-v)^{n-2}}{t^n}\mathrm{d}v\mathrm{d}y. \end{aligned} \tag{4.53}$$

Expression for the expectation of $\sum_{1\le j<i\le n}(X_i \wedge R)(X_j - R)_+ \mid N(t) = n$

Similarly to the way c_t was calculated, the expectation of $\sum_{1\le j<i\le n}(X_i \wedge R)(X_j - R)_+ \mid N(t) = n$ becomes

$$\mathbf{E}\Big[\sum_{1\le j<i\le n}(X_i \wedge R)(X_j - R)_+ \mid N(t) = n\Big] = n(n-1)\int_0^t\int_0^{t-v}\sum_{1\le j<i\le n}\mathbf{E}[(X_i \wedge R) \mid V_i = v] \times \mathbf{E}[(X_j - R)_+ \mid V_j = y]\frac{(t-y-v)^{n-2}}{t^n}\mathrm{d}y\mathrm{d}v. \quad (4.54)$$

Remaining steps for finding the expectation of $X_I X_R$

Next, we calculate the sums

$$\sum_{n=1}^{\infty}\mathbf{P}[N(t) = n]\mathbf{E}\Big[\sum_{1\le i=j\le n}(X_i \wedge R)(X_j - R)_+ \mid N(t) = n\Big]$$

and

$$\sum_{n=1}^{\infty}\mathbf{P}[N(t) = n]\mathbf{E}\Big[\sum_{1\le i\ne j\le n}(X_i \wedge R)(X_j - R)_+ \mid N(t) = n\Big],$$

which we denote by g_t and h_t, respectively.

Calculation of the sum g_t

We have calculated the expectation in the sum g_t, which is provided in equation (4.28). The expectation is given by

$$\mathbf{E}\Big[\sum_{1\le i=j\le n}(X_i \wedge R)(X_i - R)_+ | N(t) = n\Big] = \sum_{i=1}^{n}\int_0^t \mathbf{E}[(X_i \wedge R)(X_i - R)_+ \mid V_i = v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v.$$

Then

$$g_t = \sum_{n=1}^{\infty}\mathbf{P}[N(t) = n]\Big\{\sum_{i=1}^{n}\int_0^t \mathbf{E}[(X_i \wedge R)(X_i - R)_+ \mid V_i = v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v\Big\}.$$

Since $N(t)$ follows the homogeneous Poisson process, we have

$$g_t = \sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\Big\{\sum_{i=1}^{n}\int_0^t \mathbf{E}[(X_i \wedge R)(X_i - R)_+ \mid V_i = v]\frac{n(t-v)^{n-1}}{t^n}\mathrm{d}v\Big\}. \quad (4.55)$$

Rearranging terms on the right-hand side of equation (4.55), we have

$$\begin{aligned}
g_t &= \int_0^t \lambda e^{-\lambda v} \sum_{n=1}^{\infty} e^{-\lambda(t-v)} \frac{t^n \lambda^{n-1}}{n!} \sum_{i=1}^{n} \mathbf{E}[(X_i \wedge R)(X_i - R)_+ \mid V_i = v] \frac{n(t-v)^{n-1}}{t^n} \mathrm{d}v \\
&= \int_0^t \lambda e^{-\lambda v} \sum_{n=1}^{\infty} e^{-\lambda(t-v)} \frac{(\lambda(t-v))^{n-1}}{(n-1)!} \sum_{i=1}^{n} \mathbf{E}[(X_i \wedge R)(X_i - R)_+ \mid V_i = v] \mathrm{d}v \\
&= \int_0^t \lambda e^{-\lambda v} \sum_{n=0}^{\infty} e^{-\lambda(t-v)} \frac{(\lambda(t-v))^{n}}{n!} \sum_{i=1}^{n} \mathbf{E}[(X_i \wedge R)(X_i - R)_+ \mid V_i = v] \mathrm{d}v. \qquad (4.56)
\end{aligned}$$

By combining the two sums on the right-hand side of equation (4.56), we obtain

$$\begin{aligned}
g_t &= \int_0^t \lambda e^{-\lambda v} \sum_{i=1}^{N(t-v)+1} \mathbf{E}[(X_i \wedge R)(X_i - R)_+ \mid V_i = v] \mathrm{d}v \\
&= \int_0^t \lambda e^{-\lambda v} \mathbf{E}[N(t-v)+1] \mathbf{E}[(X_1 \wedge R)(X_1 - R)_+ \mid V_1 = v] \mathrm{d}v.
\end{aligned}$$

Since

$$\mathbf{E}[N(t-v)+1] = \lambda(t-v)+1,$$

we have

$$g_t = \int_0^t \lambda e^{-\lambda v} (\lambda(t-v)+1) \mathbf{E}[(X_1 \wedge R)(X_1 - R)_+ \mid V_1 = v] \mathrm{d}v. \qquad (4.57)$$

Calculation of the sum h_t

We need to consider two cases: $i < j$ and $i > j$. Then

$$\begin{aligned}
h_t &= \sum_{n=1}^{\infty} \mathbf{P}[N(t) = n] \Big\{ \mathbf{E}\Big[\sum_{1 \le i < j \le n} (X_i \wedge R)(X_j - R)_+ \mid N(t) = n \Big] \\
&\quad + \mathbf{E}\Big[\sum_{1 \le j < i \le n} (X_i \wedge R)(X_j - R)_+ \mid N(t) = n \Big] \Big\} \\
&= \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \Big\{ \mathbf{E}\Big[\sum_{1 \le i < j \le n} (X_i \wedge R)(X_j - R)_+ \mid N(t) = n \Big] \\
&\quad + \mathbf{E}\Big[\sum_{1 \le j < i \le n} (X_i \wedge R)(X_j - R)_+ \mid N(t) = n \Big] \Big\}. \qquad (4.58)
\end{aligned}$$

We have calculated the two expectations in the braces on the right-hand side of equation (4.58), and the results are provided in equations (4.53) and (4.54). Using these results, equation (4.58) becomes

$$\begin{aligned}
h_t &= \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \Big\{ n(n-1) \int_0^t \int_0^{t-y} \sum_{1 \le i < j \le n} \mathbf{E}[(X_i \wedge R)|V_i = y] \mathbf{E}[(X_j - R)_+|V_j = v] \\
&\quad \times \frac{(t-y-v)^{n-2}}{t^n} \mathrm{d}v\mathrm{d}y + n(n-1) \int_0^t \int_0^{t-v} \sum_{1 \le j < i \le n} \mathbf{E}[(X_i \wedge R) \mid V_i = v] \\
&\quad \times \mathbf{E}[(X_j - R)_+ \mid V_j = y] \frac{(t-y-v)^{n-2}}{t^n} \mathrm{d}y\mathrm{d}v \Big\} \\
&= \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \Big\{ n(n-1) \int_0^t \int_0^{t-y} \sum_{1 \le i \ne j \le n} \mathbf{E}[(X_i \wedge R) \mid V_i = y] \mathbf{E}[(X_j - R)_+ \mid V_j = v] \\
&\quad \times \frac{(t-y-v)^{n-2}}{t^n} \mathrm{d}v\mathrm{d}y \Big\}. \qquad (4.59)
\end{aligned}$$

Rearranging terms on the right-hand side of equation (4.59), we have

$$\begin{aligned}
h_t &= \int_0^t \int_0^{t-y} \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{1 \le i \ne j \le n} \mathbf{E}[(X_i \wedge R) \mid V_i = y] \mathbf{E}[(X_j - R)_+ \mid V_j = v] \\
&\quad \times n(n-1) \frac{(t-y-v)^{n-2}}{t^n} \mathrm{d}v\mathrm{d}y \\
&= \int_0^t \int_0^{t-y} \lambda^2 e^{-\lambda(y+v)} \sum_{n=2}^{\infty} e^{-\lambda(t-y-v)} \frac{(\lambda(t-y-v))^{n-2}}{(n-2)!} \sum_{1 \le i \ne j \le n} \mathbf{E}[(X_i \wedge R)|V_i = y] \\
&\quad \times \mathbf{E}[(X_j - R)_+|V_j = v] \mathrm{d}v\mathrm{d}y \\
&= \int_0^t \int_0^{t-y} \lambda^2 e^{-\lambda(y+v)} \sum_{n=0}^{\infty} e^{-\lambda(t-y-v)} \frac{(\lambda(t-y-v))^{n}}{n!} \sum_{1 \le i \ne j \le n+2} \mathbf{E}[(X_i \wedge R)|V_i = y] \\
&\quad \times \mathbf{E}[(X_j - R)_+|V_j = v] \mathrm{d}v\mathrm{d}y. \qquad (4.60)
\end{aligned}$$

By combining the two sums on the right-hand side of equation (4.60), we obtain

$$\begin{aligned}
h_t &= \int_0^t \int_0^{t-y} \lambda^2 e^{-\lambda(y+v)} \sum_{1 \le i \ne j \le N(t-y-v)+2} \mathbf{E}[(X_i \wedge R)|V_i = y] \mathbf{E}[(X_j - R)_+|V_j = v] \mathrm{d}v\mathrm{d}y \\
&= \int_0^t \int_0^{t-y} \lambda^2 e^{-\lambda(y+v)} \mathbf{E}[(N(t-y-v)+2)(N(t-y-v)+1)] \mathbf{E}[(X_1 \wedge R)|V_1 = y] \\
&\quad \times \mathbf{E}[(X_1 - R)_+|V_1 = v] \mathrm{d}v\mathrm{d}y, \qquad (4.61)
\end{aligned}$$

where

$$\mathbf{E}[(N(t-y-v)+2)(N(t-y-v)+1)] = \mathbf{E}[(N(t-y-v))^2] + 3\mathbf{E}[N(t-y-v)] + 2.$$

Since

$$\mathbf{E}[N(t-y-v)] = \lambda(t-y-v)$$

and

$$\begin{aligned}\mathbf{E}[(N(t-y-v))^2] &= (\mathbf{E}[N(t-y-v)])^2 + \mathbf{Var}[N(t-y-v)] \\ &= \lambda^2(t-y-v)^2 + \lambda(t-y-v),\end{aligned}$$

we obtain

$$\begin{aligned}\mathbf{E}[(N(t-y-v)+2)(N(t-y-v)+1)] &= \lambda^2(t-y-v)^2 + \lambda(t-y-v) + 3\lambda(t-y-v) + 2 \\ &= \lambda^2(t-y-v)^2 + 4\lambda(t-y-v) + 2 \\ &= \lambda^2(t-y-v)^2 + 4\lambda(t-y-v) + 4 - 2 \\ &= (\lambda(t-y-v)+2)^2 - 2. \qquad (4.62)\end{aligned}$$

Substituting equation (4.62) into equation (4.61), we obtain

$$\begin{aligned}h_t = &\int_0^t \int_0^{t-y} \lambda^2 e^{-\lambda(y+v)}((\lambda(t-y-v)+2)^2 - 2)\mathbf{E}[X_1 \wedge R \mid V_1 = y] \\ &\times \mathbf{E}[(X_1 - R)_+ \mid V_1 = v]\mathrm{d}v\mathrm{d}y. \qquad (4.63)\end{aligned}$$

Concluding the proof

Having calculated the two sums g_t and h_t, the expectation $\mathbf{E}[X_I X_R]$ is obtained. Having calculated the three expectations on the right-hand side of equation (4.7), we obtain an expression for the covariance $\mathbf{Cov}[X_I, X_R]$, which is given in equation (4.1). This completes the proof of Theorem 4.2.1.

4.4 Finding an optimal retention

Using two examples, we shall next show how to obtain the optimal retention R^* that maximizes the covariance $\mathbf{Cov}[X_I, X_R]$ up to and including the time t. We shall also illustrate how R^* is affected by t assuming that conditions of Corollary 4.2.2 hold true.

Example 4.4.1 Assume that larger claim sizes follow the exponential distribution with parameter $l_1 > 0$. Furthermore, assume that smaller claim sizes follow the exponential distribution with parameter $s_1 > 0$. Hence, the probability density functions (PDFs) f_l and f_s are given by the equations

$$f_l(x) = \frac{1}{l_1} e^{-x/l_1} \tag{4.64}$$

and

$$f_s(x) = \frac{1}{s_1} e^{-x/s_1}, \tag{4.65}$$

respectively, where $x \geq 0$. Then we find the optimal retention R^* that maximizes the covariance for $t = 1, 2, \ldots, 15$, where $\lambda = 2$, $\beta = 1$, $l_1 = 1000$, and $s_1 = 10$. Results for R^* are shown in Table 4.1. In addition, to show how R^* is affected by t, we have plotted t versus R^* for

t	R^*
1	1062.320
2	1109.076
3	1130.102
4	1140.722
5	1146.992
6	1151.117
7	1154.036
8	1156.211
9	1157.894
10	1159.235
11	1160.328
12	1161.236
13	1162.004
14	1162.660
15	1163.228

Table 4.1: Optimal retention R^* for $t = 1, 2, \ldots, 15$.

$t = 1, 2, \ldots, 50$ in Figure 4.1. This concludes Example 4.4.1.

Example 4.4.2 Assume that larger claim sizes follow the Weibull distribution with parameters $\tau_1 > 0$ and $l_2 > 0$. Furthermore, assume that smaller claim sizes follow the Weibull distribution with parameters $\tau_2 > 0$ and $s_2 > 0$. Hence, the PDFs f_l and f_s are given by the equations

$$f_l(x) = \frac{\tau_1}{l_2}\left(\frac{x}{l_2}\right)^{\tau_1 - 1} e^{-(x/l_2)^{\tau_1}} \tag{4.66}$$

and

$$f_s(x) = \frac{\tau_2}{s_2}\left(\frac{x}{s_2}\right)^{\tau_2 - 1} e^{-(x/s_2)^{\tau_2}}, \tag{4.67}$$

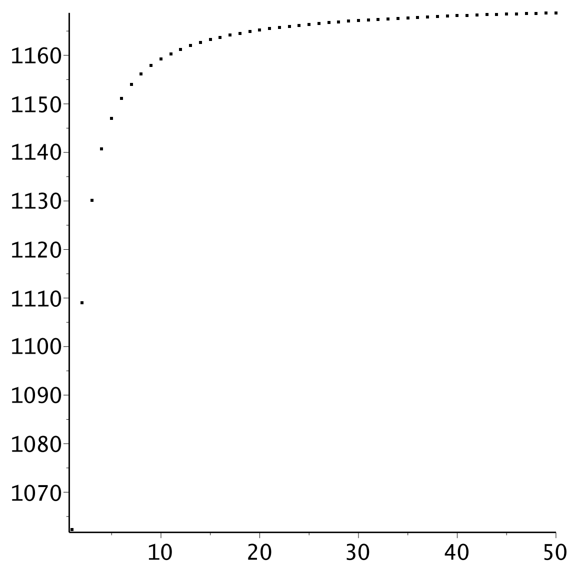

Figure 4.1: Time t versus R^* for exponential f_l and f_s.

respectively, where $x \geq 0$. Then we obtain the optimal retention R^* that maximizes the covariance for $t = 1, 2, \ldots, 10$, where $\beta = 1$, $\tau_1 = \tau_2 = 0.5$, $l_2 = 6$, $s_2 = 1$, and $\lambda = 1$.

The Weibull distribution with parameters $\tau > 0$ and $\theta > 0$ is heavy-tailed when $0 < \tau < 1$. In Figure 4.2, we plot the PDF of the Weibull distribution when $\theta = 1$, $\tau = 0.8, 1.5$, and 5. Note that when $\tau = 0.8$, then the distribution is heavy-tailed. When $\tau = 5$, we have a symmetric distribution. In this example, the Weibull distribution is always heavy-tailed since $\tau_1 = \tau_2 = 0.5$.

Using Maple, we obtain R^* for $t = 1, 2, \ldots, 10$. Results are provided in Table 4.2. In

t	R^*
1	44.190
2	46.364
3	47.327
4	47.827
5	48.119
6	48.307
7	48.438
8	48.534
9	48.608
10	48.666

Table 4.2: Optimal retention R^* for $t = 1, 2, \ldots, 10$.

addition, to show how R^* is affected by t, we have plotted t versus R^* for $t = 1, 2, \ldots, 10$ in

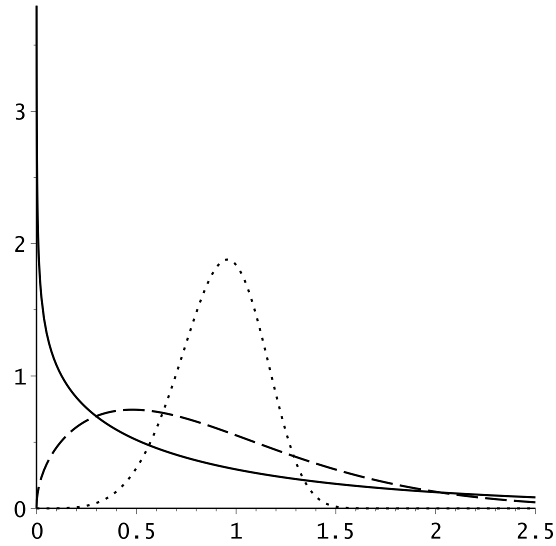

Figure 4.2: The PDF of the Weibull distribution when $\theta = 1$, $\tau = 0.8$ (solid curve), 1.5 (dashed curve), and 5 (dotted curve).

Figure 4.3. This concludes Example 4.4.2.

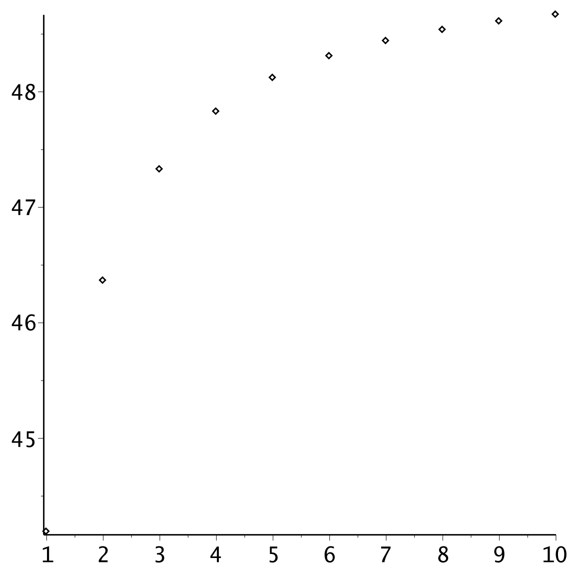

Figure 4.3: Time t versus R^* for Weibull f_l and f_s.

Summarizing this chapter, we have considered finding optimal reinsurance using the variance reduction approach under various scenarios. The assumptions we have made do not always hold true in the real world. Therefore, the patterns we have noted from the illustrative examples will not always hold true either, but they are instructive.

Chapter 5

A CTE-based optimal criterion

5.1 Introduction

In this chapter, we discuss a CTE-based approach for constructing an optimal reinsurance policy. Recall that the CTE of a random variable X at the confidence level $1 - \alpha$, where $0 \leq \alpha \leq 1$, is defined by the equation

$$CTE_X(\alpha) = \mathbf{E}[X \mid X \geq VaR_X(\alpha)].$$

Unlike the optimal criteria considered by Cai and Tan (2007), and Tan et al. (2009), we shall find a common solution for both the insurer and the reinsurer. Specifically, consider the following problem:

Under facultative reinsurance, the insurer and the reinsurer agree on an excess of loss reinsurance contract. The reinsurer needs to cover the claim amount exceeding the retention $R > 0$ on one insurance policy issued by the insurer. Let p_R be the reinsurance policy premium, and let T be the insurer's total cost in the presence of reinsurance. The insurer's total cost includes the insurer's share of the claim and the cost of purchasing the reinsurance policy, that is,

$$T = X_I + p_R, \tag{5.1}$$

where $p_R = (1 + \rho)\mathbf{E}[(X - R)_+]$, and $\rho > 0$ is the premium loading coefficient.

In Section 5.2, we shall provide expressions for $CTEs$ of T and X_R at a given confidence level $1 - \alpha$. Then, we shall show graphically that in this case it is not reasonable to obtain an optimal reinsurance policy that is beneficial to the insurer and the reinsurer by minimizing $CTEs$ of X_I and X_R with respect to R. In Section 5.3, we shall introduce our CTE-based method that connects with the variance reduction approach discussed in Chapter 3. In Section 5.4, we shall use two illustrative examples to show how an optimal retention can be obtained using the criterion of Section 5.3.

5.2 $CTEs$ of T and X_R at a given confidence level $1 - \alpha$

From Propositions 2.1 and 3.1 of Cai and Tan (2007), we have that the VaR and the CTE of T can be expressed by the formulas:

$$VaR_T(\alpha, R) = \begin{cases} R + (1+\rho)\mathbf{E}[(X-R)_+] & \text{if} \quad 0 < R \le S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) + (1+\rho)\mathbf{E}[(X-R)_+] & \text{if} \quad R > S_X^{-1}(\alpha), \end{cases} \tag{5.2}$$

and

$$CTE_T(\alpha, R) = \begin{cases} R + (1+\rho)\mathbf{E}[(X-R)_+] & \text{if} \quad 0 < R \le S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^{R} S_X(x)\mathrm{d}x + (1+\rho)\mathbf{E}[(X-R)_+] & \text{if} \quad R > S_X^{-1}(\alpha), \end{cases} \tag{5.3}$$

where S_X is the survival function of X and α is such that $0 < \alpha < S_X(0)$.

Next, we calculate VaR and CTE of R. To do so, we first derive an expression for the survival function of X_R.

Theorem 5.2.1 *Let the excess of loss reinsurance policy have retention $R > 0$. Then the survival function of X_R is given by*

$$S_{X_R}(x) = \mathbf{P}[X > x + R] \tag{5.4}$$

for all $x \ge 0$.

Proof We start by writing the survival function as follows:

$$\begin{aligned} S_{X_R}(x) &= \mathbf{P}[(X-R)_+ > x] \\ &= \mathbf{E}[\mathbf{1}\{(X-R)_+ > x\}]. \end{aligned} \tag{5.5}$$

The indicator function **1** can be written as follows:

$$\mathbf{1}\{(X-R)_+ > x\} = \begin{cases} 1 & \text{if } X-R > x, X > R, \text{or if } 0 > x, X \leq R, \\ 0 & \text{if } X-R \leq x, X > R, \text{or if } 0 \leq x, X \leq R. \end{cases} \tag{5.6}$$

By substituting equation (5.6) into equation (5.5), we obtain

$$S_{X_R}(x) = \mathbf{P}[X-R > x, X > R] + \mathbf{P}[0 > x, X \leq R]. \tag{5.7}$$

Next, we investigate the two terms on the right-hand side of equation (5.7). Starting with the first term, we have

$$\mathbf{P}[X-R > x, X > R] = \begin{cases} \mathbf{P}[X > x+R] & \text{if} \quad 0 \leq x < R, \\ \mathbf{P}[X > x+R] & \text{if} \quad x \geq R. \end{cases} \tag{5.8}$$

As to the second term, we obtain

$$\mathbf{P}[0 > x, X \leq R] = \begin{cases} 0 & \text{if} \quad 0 \leq x < R, \\ 0 & \text{if} \quad x \geq R. \end{cases} \tag{5.9}$$

By substituting equations (5.8) and (5.9) into equation (5.7), we obtain

$$S_{X_R}(x) = \begin{cases} \mathbf{P}[X > x+R] & \text{if} \quad 0 \leq x < R, \\ \mathbf{P}[X > x+R] & \text{if} \quad x \geq R, \end{cases} \tag{5.10}$$

which matches the survival function of X_R given in equation (5.4). This completes the proof of Theorem 5.2.1. □

Now we are ready to derive expressions for $VaR_{X_R}(\alpha, R)$ and $CTE_{X_R}(\alpha, R)$.

Theorem 5.2.2 *Under conditions of Theorem 5.2.1, we have*

$$VaR_{X_R}(\alpha, R) = \begin{cases} S_X^{-1}(\alpha) - R & \textit{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ 0 & \textit{if} \quad R > S_X^{-1}(\alpha), \end{cases} \tag{5.11}$$

where $0 < \alpha < S_X(0)$.

Proof The VaR of X at the confidence level $1-\alpha$ is given by

$$VaR_X(\alpha) = S_X^{-1}(\alpha), \tag{5.12}$$

and when the retention is $R > 0$, the VaR of X_I at the confidence level $1 - \alpha$ is given by

$$VaR_{X_I}(\alpha, R) = \begin{cases} R & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) & \text{if} \quad R > S_X^{-1}(\alpha). \end{cases} \tag{5.13}$$

If we can show that

$$VaR_X(\alpha) = VaR_{X_I}(\alpha, R) + VaR_{X_R}(\alpha, R), \tag{5.14}$$

then by substituting equations (5.12) and (5.13) into equation (5.14), we shall obtain $VaR_{X_R}(\alpha, R)$. To prove that equation (5.14) holds true, we first define comonotonicity (cf., e.g., Schmeidler, 1986).

Definition 5.2.3 Random variables Y and Z are comonotonic if

$$F_{Y,Z}(y, z) = \min(F_Y(y), F_Z(z)) \quad \text{for} \quad \text{all} \quad y, z \geq 0,$$

where $F_{Y,Z}$ is the joint CDF of Y and Z, F_Y is the CDF of Y, and F_Z is the CDF of Z.

Denuit et al. (2005) have shown that if random variables $Y_1, Y_2, \ldots, Y_n$ are comonotonic and $S_n = \sum_{i=1}^n Y_i$, then

$$VaR_{S_n}(\alpha) = \sum_{i=1}^{n} VaR_{Y_i}(\alpha).$$

In our case, we have

$$X = X_I + X_R.$$

Wang and Dhaene (1998) have proven that X_I and X_R are comonotonic. Therefore, equation (5.14) holds true, and then the formula for $VaR_{X_R}(\alpha, R)$ given by equation (5.11) is obtained. This completes the proof of Theorem 5.2.2. □

Theorem 5.2.4 *Under conditions of Theorem 5.2.2, we have*

$$CTE_{X_R}(\alpha, R) = \begin{cases} S_X^{-1}(\alpha) - R + \frac{1}{\alpha} \int_{S_X^{-1}(\alpha) - R}^{\infty} S_X(x + R) \mathrm{d}x & \textit{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ \frac{1}{\alpha} \int_0^{\infty} S_X(x + R) \mathrm{d}x & \textit{if} \quad R > S_X^{-1}(\alpha). \end{cases} \tag{5.15}$$

Proof We write

$$\begin{aligned}
CTE_{X_R}(\alpha, R) &= \mathbf{E}[X_R \mid X_R \geq VaR_{X_R}(\alpha, R)] \\
&= \mathbf{E}[VaR_{X_R}(\alpha, R) + X_R - VaR_{X_R}(\alpha, R) \mid X_R \geq VaR_{X_R}(\alpha, R)] \\
&= VaR_{X_R}(\alpha, R) + \frac{1}{\alpha}\int_{VaR_{X_R}(\alpha,R)}^{\infty} S_{X_R}(x)\mathrm{d}x. \qquad (5.16)
\end{aligned}$$

We have calculated the survival function of X_R and the VaR of X_R at a given confidence level $1-\alpha$ in equations (5.4) and (5.11). Using these results, we obtain an expression for the CTE of X_R at a given confidence level $1-\alpha$. Then calculations immediately lead to equation (5.15), and thus complete the proof of Theorem 5.2.4. □

To exemplify the above proved formulas, we note that when X follows the exponential distribution with mean $\theta > 0$, we have

$$CTE_T(\alpha, R) = \begin{cases} R + (1+\rho)\theta e^{-R/\theta} & \text{if} \quad 0 < R \leq -\theta \ln \alpha, \\ -\theta \ln \alpha + (1+\rho)\theta e^{-R/\theta} + \frac{1}{\alpha}(\alpha\theta - \theta e^{-R/\theta}) & \text{if} \quad R > -\theta \ln \alpha, \end{cases} \qquad (5.17)$$

and

$$CTE_{X_R}(\alpha, R) = \begin{cases} -\theta \ln \alpha - R + \theta & \text{if} \quad 0 < R \leq -\theta \ln \alpha, \\ \frac{1}{\alpha}\theta e^{-R/\theta} & \text{if} \quad R > -\theta \ln \alpha. \end{cases} \qquad (5.18)$$

Next, in Figure 5.1, we plot $CTE_T(\alpha, R)$ and $CTE_{X_R}(\alpha, R)$ as functions of R when $\alpha = 0.05$ and $\theta = 10$. We see from Figure 5.1 that a common solution cannot be obtained for the insurer and the reinsurer by minimizing $CTE_T(\alpha, R)$ or $CTE_{X_R}(\alpha, R)$ with respect to R. To construct an optimal reinsurance policy that is beneficial to both parties, we therefore consider another CTE-based approach, which is the topic of the next section.

5.3 An optimal criterion and the variance reduction approach

We now discuss an optimal criterion that connects with the variance reduction approach considered in Chapter 3. We write

$$CTE_{X_I}(\alpha, R) = \frac{1}{\alpha}\mathbf{E}[X_I \mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}] \qquad (5.19)$$

and

$$CTE_{X_R}(\alpha, R) = \frac{1}{\alpha}\mathbf{E}[X_R \mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}]. \qquad (5.20)$$

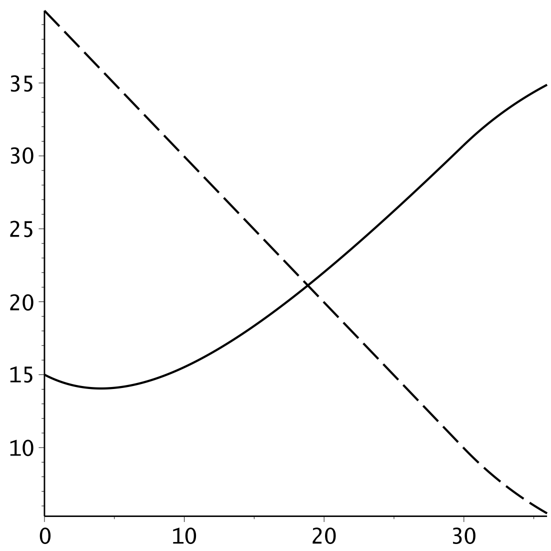

Figure 5.1: Retention R versus $CTE_T(\alpha, R)$ (solid curve), and R versus $CTE_{X_R}(\alpha, R)$ (dashed curve).

Next, we investigate the two expectations on the right-hand side of equations (5.19) and (5.20). Denote $\mathbf{Cov}[X_I \mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}, X_R \mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}]$ by a_1. Then

$$\begin{aligned} a_1 &= \mathbf{E}[X_I \mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\} X_R \mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] \\ &\quad - \mathbf{E}[X_I \mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}] \mathbf{E}[X_R \mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}]. \end{aligned} \tag{5.21}$$

Note that when α approaches 1, then the terms $VaR_{X_I}(\alpha, R)$ and $VaR_{X_R}(\alpha, R)$ in the indicator functions become small. Hence, the first term on the right-hand side of equation (5.21) converges to $\mathbf{E}[X_I X_R]$ and the second term, which is the product of the two expectations on the right-hand side of equation (5.21), converges to $\mathbf{E}[X_I]\mathbf{E}[X_R]$. Then we can consider a_1 to be a special case of the covariance $\mathbf{Cov}[X_I, X_R]$, which we used to obtain an optimal retention in Chapter 3. To obtain an optimal retention that maximizes a_1, we first need to obtain an expression for a_1.

Theorem 5.3.1 *Under conditions of Theorem 5.2.2, we have*

$$a_1 = \begin{cases} b_1 & \textit{when } 0 < R \leq S_X^{-1}(\alpha), \\ b_2 & \textit{when } R > S_X^{-1}(\alpha), \end{cases} \tag{5.22}$$

where

$$b_1 = R\int_{S_X^{-1}(\alpha)}^{\infty}(x-R)\mathrm{d}F_X(x) - \alpha R\Big(\int_{S_X^{-1}(\alpha)-R}^{\infty} S_X(x+R)\mathrm{d}x + \alpha(S_X^{-1}(\alpha)-R)\Big)$$

and

$$b_2 = R\int_{R}^{\infty}(x-R)\mathrm{d}F_X(x) - \Big(\int_{S_X^{-1}(\alpha)}^{R} S_X(x)\mathrm{d}x + \alpha S_X^{-1}(\alpha)\Big)\int_0^{\infty} S_X(x+R)\mathrm{d}x.$$

Proof We investigate the three expectations on the right-hand side of equation (5.21) separately. We begin with the first expectation:

$$\begin{aligned}\mathbf{E}[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha,R)\}]&\\ = \mathbf{E}[R(X-R)\mathbf{1}\{X \geq S_X^{-1}(\alpha)\}\mathbf{1}\{X \geq R\}].& \quad (5.23)\end{aligned}$$

By combining the two indicator functions on the right-hand side of equation (5.23), we obtain

$$\mathbf{E}[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha,R)\}] = \mathbf{E}[R(X-R)\mathbf{1}\{X \geq \max(S_X^{-1}(\alpha),R)\}]. \quad (5.24)$$

The right-hand side of equation (5.24) can be written as follows:

$$\mathbf{E}[R(X-R)\mathbf{1}\{X \geq \max(S_X^{-1}(\alpha),R)\}] = \begin{cases} R\int_{S_X^{-1}(\alpha)}^{\infty}(x-R)\mathrm{d}F_X(x) & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha),\\ R\int_{R}^{\infty}(x-R)\mathrm{d}F_X(x) & \text{if} \quad R > S_X^{-1}(\alpha).\end{cases} \quad (5.25)$$

Next, we calculate the expectation of $X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}$ on the right-hand side of equation (5.21) and have that

$$\begin{aligned}\mathbf{E}[X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}] &= \mathbf{E}[(X_I + VaR_{X_I}(\alpha,R) - VaR_{X_I}(\alpha,R))\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}]\\ &= \mathbf{E}[(X_I - VaR_{X_I}(\alpha,R))\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}]\\ &\quad + VaR_{X_I}(\alpha,R)\mathbf{P}[X_I \geq VaR_{X_I}(\alpha,R)]\\ &= \mathbf{E}[(X_I - VaR_{X_I}(\alpha,R))\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}] + \alpha VaR_{X_I}(\alpha,R).\end{aligned} \quad (5.26)$$

The expectation on the right-hand side of equation (5.26) can be written as follows:

$$\mathbf{E}[(X_I - VaR_{X_I}(\alpha,R))\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha,R)\}] = \int_{VaR_{X_I}(\alpha,R)}^{\infty} S_{X_I}(x)\mathrm{d}x.$$

We now write the above expectation in terms of the survival function of X. We have

$$\mathbf{E}[(X_I - VaR_{X_I}(\alpha, R))\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}] = \int_{VaR_{X_I}(\alpha,R)}^{R} S_X(x)\mathrm{d}x. \tag{5.27}$$

Using the result for $VaR_{X_I}(\alpha, R)$ in equation (5.13), we have that equation (5.27) can be written as follows:

$$\mathbf{E}[(X_I - VaR_{X_I}(\alpha, R))\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}] = \begin{cases} 0 & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ \int_{S_X^{-1}(\alpha)}^{R} S_X(x) & \text{if} \quad R > S_X^{-1}(\alpha). \end{cases} \tag{5.28}$$

By substituting equation (5.28) into equation (5.26), we obtain

$$\mathbf{E}[(X_I\mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}] = \begin{cases} \alpha R & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ \int_{S_X^{-1}(\alpha)}^{R} S_X(x)\mathrm{d}x + \alpha S_X^{-1}(\alpha) & \text{if} \quad R > S_X^{-1}(\alpha). \end{cases} \tag{5.29}$$

Similarly, we can rewrite the expectation of $X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}$ on the right-hand side of equation (5.21). We obtain the equation

$$\mathbf{E}[X_R\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] = \mathbf{E}[(X_R - VaR_{X_R}(\alpha, R))\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] + \alpha VaR_{X_R}(\alpha, R). \tag{5.30}$$

The expectation on the right-hand side of equation (5.30) can be written as follows:

$$\begin{aligned} \mathbf{E}[(X_R - VaR_{X_R}(\alpha, R))\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] &= \int_{VaR_{X_R}(\alpha,R)}^{\infty} S_{X_R}(x)\mathrm{d}x \\ &= \int_{VaR_{X_R}(\alpha,R)}^{\infty} S_X(x + R)\mathrm{d}x. \end{aligned} \tag{5.31}$$

Note that using the result from Theorem 5.2.1, which states that $S_{X_R}(x) = S_X(x + R)$ for $x \geq 0$, we obtain the survival function of X instead of X_R on the right-hand side of equation (5.31). Using the result for $VaR_{X_R}(\alpha, R)$ given in equation (5.11), we have that equation (5.31) can be written as follows:

$$\mathbf{E}[(X_R - VaR_{X_R}(\alpha, R))\mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] = \begin{cases} \int_{S_X^{-1}(\alpha)-R}^{\infty} S_X(x + R)\mathrm{d}x & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ \int_0^{\infty} S_X(x + R)\mathrm{d}x & \text{if} \quad R > S_X^{-1}(\alpha). \end{cases} \tag{5.32}$$

By substituting equation (5.32) into equation (5.30), we obtain

$$\mathbf{E}[X_R \mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}] = \begin{cases} \int_{S_X^{-1}(\alpha)-R}^{\infty} S_X(x+R)\mathrm{d}x + \alpha(S_X^{-1}(\alpha) - R) & \text{if} \quad 0 < R \leq S_X^{-1}(\alpha), \\ \int_0^{\infty} S_X(x+R)\mathrm{d}x & \text{if} \quad R > S_X^{-1}(\alpha). \end{cases} \tag{5.33}$$

Using the results that we have calculated for the three expectations on the right-hand side of equation (5.21), equation (5.22) follows, which completes the proof of Theorem 5.3.1. □

5.4 Finding optimal retention

We now obtain the retention that maximizes

$$a_1 := \mathbf{Cov}[X_I \mathbf{1}\{X_I \geq VaR_{X_I}(\alpha, R)\}, X_R \mathbf{1}\{X_R \geq VaR_{X_R}(\alpha, R)\}]$$

when X follows the exponential distribution (Subsection 5.4.1) and also the two-parameter Pareto distribution (Subsection 5.4.2). We call this maximizing R the optimal retention and denote it by R^* for the rest of this chapter.

5.4.1 Optimal retention when X is exponential

When X follows the exponential distribution with mean $\theta > 0$, then we have

$$a_1 = \begin{cases} k_1 & \text{if } 0 < R \leq -\theta \ln \alpha, \\ k_2 & \text{if } R > -\theta \ln \alpha, \end{cases} \tag{5.34}$$

where

$$k_1 = R \int_{-\theta \ln \alpha}^{\infty} (x - R) \frac{1}{\theta} e^{-x/\theta} \mathrm{d}x - \alpha R \Big(\int_{-\theta \ln \alpha - R}^{\infty} e^{-(x+R)/\theta} \mathrm{d}x + \alpha(-\theta \ln \alpha - R) \Big) \tag{5.35}$$

and

$$k_2 = R \int_R^{\infty} (x - R) \frac{1}{\theta} e^{-x/\theta} \mathrm{d}x - \Big(\int_{-\theta \ln \alpha}^{R} e^{-x/\theta} \mathrm{d}x - \alpha\theta \ln \alpha \Big) \int_0^{\infty} e^{-(x+R)/\theta} \mathrm{d}x. \tag{5.36}$$

Next, we obtain values that maximize k_1 and k_2.

Theorem 5.4.1 *Let X follow the exponential distribution with mean $\theta > 0$. Furthermore, let the excess of loss reinsurance policy have retention $R > 0$. Finally, let the given confidence* *level be* $1-\alpha$ *for some* $0<\alpha<1$. *Then* k_1*, which is given by equation (5.35), is maximized at*

$$R_1 = \frac{\theta\ln\alpha - \theta + \alpha\theta - \alpha\theta\ln\alpha}{2\alpha-2}, \tag{5.37}$$

and k_2*, which is given in equation (5.36), is maximized at*

$$R_2 = -\theta(-LambertW(-2\exp\{-1-\alpha+\alpha\ln\alpha\})), \tag{5.38}$$

where LambertW is the Lambert function. Moreover, the value R_2 *is assumed to satisfy the condition*

$$\frac{R_2}{\theta} - 2 - \alpha + \alpha\ln\alpha + 4e^{-R_2/\theta} < 0. \tag{5.39}$$

Note 5.4.2 The Lambert W function, which is also known as the product algorithm or the omega function, is a set of functions, namely the branches of the inverse relation of the function $z = We^W$, where e^W is the exponential function and W is any complex number (cf., e.g., Disney and Lambrecht, 2008).

Proof We start by finding the critical point(s) of k_1. By differentiating k_1 with respect to R, we have

$$\frac{\partial}{\partial R}k_1 = -\alpha\theta\ln\alpha + \alpha\theta - 2\alpha R - \alpha^2\theta + \alpha^2\theta\ln\alpha + 2\alpha^2 R. \tag{5.40}$$

Next, we set

$$\frac{\partial}{\partial R}k_1 = 0,$$

and solve for R. The solution, which we denote by R_1, is given by equation (5.37). The critical point R_1 maximizes k_1 if

$$\left(\frac{\partial}{\partial R}\right)^2 k_1\Big|_{R=R_1} < 0.$$

The second derivative of k_1 with respect to R is

$$\begin{aligned}\left(\frac{\partial}{\partial R}\right)^2 k_1 &= -2\int_{-\theta\ln\alpha}^{\infty}\frac{1}{\theta}e^{-x/\theta}\mathrm{d}x - \alpha\Big(-e^{-R/\theta}\int_{-\theta\ln\alpha-R}^{\infty}\frac{1}{\theta}e^{-x/\theta}\mathrm{d}x + e^{(\theta\ln\alpha)/\theta} - \alpha\Big)\\ &\quad - \alpha\Big(-e^{-R/\theta}\int_{-\theta\ln\alpha-R}^{\infty}\frac{1}{\theta}e^{-x/\theta}\mathrm{d}x\Big) - \alpha R\Big(\frac{1}{\theta}e^{-R/\theta}\int_{-\theta\ln\alpha-R}^{\infty}\frac{1}{\theta}e^{-x/\theta}\mathrm{d}x - \frac{1}{\theta}e^{(\theta\ln\alpha)/\theta}\Big)\\ &= 2\alpha^2 - 2\alpha < 0.\end{aligned}$$

We conclude that R_1 maximizes k_1.

Similarly, we obtain the critical point(s) of k_2. By differentiating k_2 with respect to R, we obtain

$$\frac{\partial}{\partial R}k_2 = \theta e^{-R/\theta} - Re^{-R/\theta} + \alpha\theta e^{-R/\theta} - 2\theta e^{-2R/\theta} - \alpha\theta e^{-R/\theta}\ln\alpha. \tag{5.41}$$

Then we set

$$\frac{\partial}{\partial R}k_2 = 0,$$

and solve for R using the "solve" feature in Maple. The solution, which we denote by R_2, is given by equation (5.38). The critical point R_2 maximizes k_2 if

$$\left(\frac{\partial}{\partial R}\right)^2 k_2\bigg|_{R=R_2} < 0. \tag{5.42}$$

We differentiate k_2 twice with respect to R and obtain

$$\begin{aligned}\left(\frac{\partial}{\partial R}\right)^2 k_2 &= -\frac{R}{\theta}e^{-R/\theta} - 2\int_R^\infty \frac{1}{\theta}e^{-x/\theta}\mathrm{d}x + 2R\frac{1}{\theta}e^{-R/\theta} + \frac{2}{\theta}e^{-2R/\theta}\int_0^\infty e^{-x/\theta}\mathrm{d}x \\ &\quad + e^{-R/\theta}\frac{1}{\theta}e^{-R/\theta}\int_0^\infty e^{-x/\theta}\mathrm{d}x - \frac{1}{\theta^2}e^{-R/\theta}\int_0^\infty e^{-x/\theta}\mathrm{d}x\left(\int_{-\theta\ln\alpha}^R e^{-x/\theta}\mathrm{d}x - \alpha\theta\ln\alpha\right) \\ &= \left(\frac{R}{\theta} - 2 - \alpha + \alpha\ln\alpha\right)e^{-R/\theta} + 4e^{-2R/\theta}.\end{aligned}$$

Condition (5.42) is satisfied provided that

$$\left(\frac{R_2}{\theta} - 2 - \alpha + \alpha\ln\alpha\right)e^{-R_2/\theta} + 4e^{-2R_2/\theta} < 0. \tag{5.43}$$

When multiplying both sides of (5.43) by $e^{R_2/\theta}$, we obtain condition (5.39). This completes the proof of Theorem 5.4.1. □

To illustrate graphically, we plot R versus a_1 when $\alpha = 0.2$ and $\theta = 10$ (left panel in Figure 5.2), and when $\alpha = 0.6$ and $\theta = 10$ (right panel in Figure 5.2). We note that on the left panel, the covariance a_1 is maximized in the interval $0 < R \leq -\theta\ln\alpha$, and the optimal retention R^* is equal to 13.047, which corresponds to the value of R_1. On the right panel, the covariance a_1 is maximized in the interval $R > -\theta\ln\alpha$, and the optimal retention R^* is equal to 14.259, which corresponds to the value of R_2.

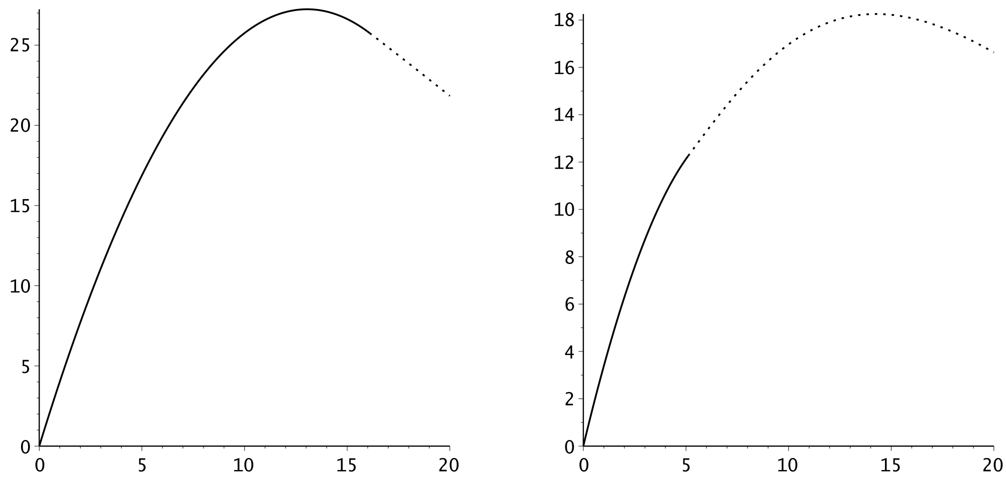

Figure 5.2: Retention R versus a_1 for $0 < R \leq -\theta \ln \alpha$ (solid curve) and $R > -\theta \ln \alpha$ (dotted curve). Left panel: $\alpha = 0.2$ and $\theta = 10$. Right panel: $\alpha = 0.6$ and $\theta = 10$.

5.4.2 Optimal retention when X is the two-parameter Pareto

When X follows the two-parameter Pareto distribution with parameters $a > 1$ and $b > 0$, then we have

$$a_1 = \begin{cases} k_3 & \text{when } 0 < R \leq b/(\alpha^{1/a}) - b, \\ k_4 & \text{when } R > b/(\alpha^{1/a}) - b, \end{cases} \tag{5.44}$$

where

$$k_3 = R\int_{b/(\alpha^{1/a})-b}^{\infty}(x-R)\frac{ab^a}{(x+b)^{a+1}}\mathrm{d}x-\alpha R\Big(\int_{b/(\alpha^{1/a})-b-R}^{\infty}\Big(\frac{b}{x+b+R}\Big)^a\mathrm{d}x+\alpha\Big(\frac{b}{\alpha^{1/a}}-b-R\Big)\Big) \tag{5.45}$$

and

$$k_4 = R\int_R^{\infty}(x-R)\frac{ab^a}{(x+b)^{a+1}}\mathrm{d}x - \Big(\int_{b/(\alpha^{1/a})-b}^{R}\Big(\frac{b}{x+b}\Big)^a\mathrm{d}x + \alpha\Big(\frac{b}{\alpha^{1/a}} - b\Big)\Big)\int_0^{\infty}\Big(\frac{b}{x+b+R}\Big)^a\mathrm{d}x. \tag{5.46}$$

Next, we obtain values that maximize k_3 and k_4.

Theorem 5.4.3 *Let X follow the two-parameter Pareto distribution with parameters $a > 1$ and $b > 0$. Furthermore, let the excess of loss reinsurance policy have retention $R > 0$. Finally, let the confidence level be $1 - \alpha$ for some $0 < \alpha < 1$. Then k_3, which is given in equation (5.45), is maximized at*

$$R_3 = \frac{1}{\alpha-2}\Big(-\frac{b}{a-1}\alpha^{-1/a} - \alpha^{-1/a}b + b + \alpha^{(a-1)/a}\frac{b}{a-1} + \alpha^{(a-1)/a}b - \alpha b\Big), \tag{5.47}$$

and the value R_4 that maximizes k_4, which is given in equation (5.46), must satisfy the following two conditions:

$$\frac{b}{a-1}\Big(\frac{b}{b+R_2}\Big)^{a-1} - R_2\Big(\frac{b}{b+R_2}\Big)^{a} - \frac{2b^{2a}}{(a-1)(b+R_2)^{2a-1}} + ab^{a+1}\alpha^{(a-1)/a} - a\alpha b^{a+1} + \alpha b^{a+1} = 0, \tag{5.48}$$

$$-\Big(\frac{b}{b+R_2}\Big)^{a} + aR_2\frac{b^a}{(b+R_2)^{a+1}} + (2a-1)\frac{2b^{2a}}{(a-1)(b+R_2)^{2a}} - \frac{a^2b^{a+1}\alpha^{(a-1)/a} - a^2\alpha b^{a+1} + a\alpha b^{a+1}}{(a-1)(b+R_2)^{a+1}} < 0. \tag{5.49}$$

Proof We start by finding the critical point(s) of k_3. By differentiating k_3 with respect to R, we obtain

$$\begin{aligned}\frac{\partial}{\partial R}k_3 &= \Big(\frac{b}{a-1}\alpha^{(a-1)/a} + \Big(\frac{b}{\alpha^{1/a}} - b\Big)\alpha - R\alpha\Big) - R\alpha - \alpha\Big(- \frac{b}{(1-a)\alpha^{(1-a)/a}} + \alpha\Big(\frac{b}{\alpha^{1/a}} - b - R\Big)\Big)\\ &\quad + \alpha^2 R\\ &= \frac{b}{a-1}\alpha^{(a-1)/a} + \Big(\frac{b}{\alpha^{1/a}} - b\Big)\alpha - 2\alpha R + \alpha\frac{b}{(1-a)\alpha^{(1-a)/a}} - \alpha^2\frac{b}{\alpha^{1/a}} + \alpha^2 b + \alpha^2 R. \end{aligned} \tag{5.50}$$

Next, we set

$$\frac{\partial}{\partial R}k_3 = 0,$$

and solve for R. The solution, which we denote by R_3, is provided by equation (5.47). The critical point R_3 maximizes k_3 if

$$\Big(\frac{\partial}{\partial R}\Big)^2 k_3\Big|_{R=R_3} < 0.$$

This is equivalent to $-2\alpha + \alpha^2 < 0$, which always holds. Therefore, we conclude that the value R_3 maximizes k_3.

Similarly, we obtain the critical point(s) of k_4. By differentiating k_4 with respect to R, we obtain

$$\begin{aligned}\frac{\partial}{\partial R}k_4 &= \Big(\frac{b}{a-1}\Big(\frac{b}{b+R}\Big)^{a-1} + R\Big(\frac{b}{b+R}\Big)^{a}\Big)\\ &\quad + R\Big(\frac{b^a}{a-1}(1-a)(b+R)^{-a} + \Big(\frac{b}{b+R}\Big)^{a} - aRb^a(b+R)^{-a-1}\Big) - 2R\Big(\frac{b}{b+R}\Big)^{a}\\ &\quad + aR^2b^a(b+R)^{-a-1} - \frac{b^a}{a-1}(1-a)(b+R)^{-a}\\ &\quad \times\Big(\frac{b^a}{1-a}(b+R)^{-a+1} - \frac{b^a}{1-a}\Big(\frac{b}{\alpha^{1/a}}\Big)^{-a+1} + \alpha\Big(\frac{b}{\alpha^{1/a}} - b\Big)\Big)\\ &\quad - \frac{b^a}{a-1}(b+R)^{-a+1}\frac{b^a}{1-a}(1-a)(b+R)^{-a}.\end{aligned}$$

Then

$$\frac{\partial}{\partial R}k_4 = \frac{b}{a-1}\left(\frac{b}{b+R}\right)^{a-1} - R\left(\frac{b}{b+R}\right)^{a} - \frac{2b^{2a}}{(a-1)(b+R)^{2a-1}} + ab^{a+1}\alpha^{(a-1)/a} - a\alpha b^{a+1} + \alpha b^{a+1}. \tag{5.51}$$

We set

$$\frac{\partial}{\partial R}k_4 = 0,$$

and solve for R. The solution, which we denote by R_4, satisfies condition (5.48). The critical point R_4 maximizes k_4 if the second derivative of k_4 with respect to R evaluated at R_4 is negative. The second derivative of k_4 with respect to R is

$$\begin{aligned}
\left(\frac{\partial}{\partial R}\right)^2 k_4 &= \frac{b^a}{a-1}(1-a)(b+R)^{-a} + aRb^a(b+R)^{-a-1} - \frac{2b^{2a}}{a-1}(1-2a)(b+R)^{-2a} \\
&\quad - a(b+R)^{-a-1}\frac{ab^{a+1}\alpha^{(a-1)/a} - a\alpha b^{a+1} + \alpha b^{a+1}}{a-1} \\
&= -\left(\frac{b}{b+R}\right)^{a} + aR\frac{b^a}{(b+R)^{a+1}} + (2a-1)\frac{2b^{2a}}{(a-1)(b+R)^{2a}} \\
&\quad - \frac{a^2b^{a+1}\alpha^{(a-1)/a} - a^2\alpha b^{a+1} + a\alpha b^{a+1}}{(a-1)(b+R)^{a+1}}.
\end{aligned}$$

We note that

$$\left(\frac{\partial}{\partial R}\right)^2 k_4\bigg|_{R=R_4} < 0$$

if condition (5.49) is satisfied. In other words, condition (5.49) must be satisfied for k_4 to be maximized at R_4. This completes the proof of Theorem 5.4.3. □

To illustrate graphically, we plot R versus a_1 when $\alpha = 0.05$, $a = 3$, and $b = 10$ (left panel in Figure 5.3), and when $\alpha = 0.5$, $a = 3$, and $b = 10$ (right panel in Figure 5.3). On the left panel, the covariance a_1 is maximized in the interval $0 < R \leq b/(\alpha^{1/a}) - b$, and the optimal retention R^* is equal to 14.964, which corresponds to the value of R_3. On the right panel, the covariance a_1 is maximized in the interval $R > b/(\alpha^{1/a}) - b$, and the optimal retention R^* is equal to 15.922, which corresponds to the value of R_4.

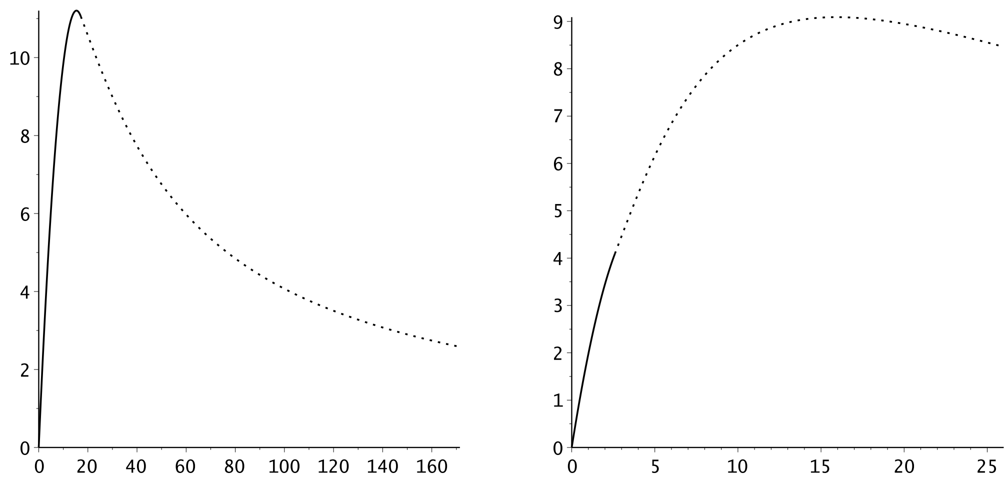

Figure 5.3: Retention R versus a_1 for $0 < R \leq b/(\alpha^{1/a}) - b$ (solid curve) and $R > b/(\alpha^{1/a}) - b$ (dotted curve). Left panel: $\alpha = 0.05$, $a = 3$, and $b = 10$. Right panel: $\alpha = 0.5$, $a = 3$, and $b = 10$.

Chapter 6

Likelihood of purchasing a property

6.1 Introduction

When buying or negotiating selling properties, a number of factors could influence the outcome. Among them are the seller's reservation (i.e., minimal) price, the shape of the negotiated selling price distribution, and the buyer's reservation (i.e., maximal affordable) price. Various studies of these key factors have been reported in the literature.

For example, Rothschild (1974) discusses an optimal search strategy from the buyer's perspective that focuses on minimizing the total cost of buying properties. The buyer's reservation price is determined based on the total expected expense from searching and purchasing. Gastwirth (1976) considers a model for obtaining the expected minimal price after searching through a number of targets. Egozcue et al. (2013) derive an optimal strategy that maximizes the expected real estate selling price when only one of the two remaining buyers has made an offer.

Various distributions of the negotiated selling prices have been discussed. Some examples are the uniform distribution (Stigler, 1962), the normal distribution (Nelson, 1970), and the triangular distribution (Gastwirth, 1976).

Other approaches to modelling negotiated selling prices include the hedonic pricing model discussed in Gundimeda (2006), the repeated negotiated selling method considered by Baroni et al. (2007), and the replication method suggested by Lai et al. (2008).

Our research has been motivated by the following problem: Consider a buyer who wants to purchase a property, and suppose that there are a number of similar properties on the market for sale. The buyer has a reservation price in mind and looks at the offers one at a time. If the negotiated selling price is below the buyer's reservation price, the buyer purchases the property immediately and avoids the risk of losing the property to another potential buyer. If the negotiated selling price is above the buyer's reservation price, then the buyer moves on to the next offer. Unlike in the case considered by Stigler (1962) and Gastwirth (1976), we assume that if the buyer passes on an offer, he/she does not have the option to go back and review it. This scenario is as realistic as those of Stigler (1962) and Gastwirth (1976) because sellers and real estate agents often have multiple buying offers and, therefore, may not wait for one potential buyer's reply. The search ends when the buyer purchases a property with a negotiated selling price lower than her/his reservation price. If all of the properties under consideration are being sold at a price higher than the reservation price, then the buyer does not purchase a property.

The rest of this chapter is organized as follows:

- In Section 6.2, we derive a formula for the probability of purchasing a property during a specified time period under the simplest yet practically possible scenario. We assume that the buyer's reservation price does not change during the search period. Furthermore, we assume that the negotiated selling prices are i.i.d. random variables.

- In Section 6.3, we consider various scenarios under which one or more assumptions made in Section 6.2 no longer hold true and obtain a formula for the probability of buying a property during a given time interval for each case.

- In Sections 6.4 and 6.5, we discuss modelling the dependence among negotiated selling prices. Three methods are considered including direct representation, copula representation, and background risk model.

- In Section 6.6, we review past studies on finding the bounds and direct representations of the tail probability for some discrete distributions. We may need the results from these studies to calculate the probability of buying a property.

- In Section 6.7, we calculate the (unconditional) probability of the buyer purchasing one condominium or one detached property in the London and St. Thomas area under assumptions made in Section 6.2.

6.2 An illustrative case

We are interested in the probability of the buyer purchasing a property during a search time-interval, say $(t_0, t_1]$. Whatever the initial impression we have gotten about the problem, it is far from trivial. To unearth the crux of the matter, we next provide an illustrative solution in a highly simplified, yet reasonable, scenario.

Let H be a binary random variable taking on two values: 1 if a property is purchased during the time period $(t_0, t_1]$ and 0 otherwise. Naturally, the number of properties that interest the buyer and are available on the market during the noted time period is random, and we denote it by N, which can take on any integer value $n = 0, 1, \ldots$ (e.g., N might follow the Poisson distribution). The rule of total probability immediately gives us the equation

$$\mathbf{P}[H = 1] = \sum_{n=1}^{\infty} \mathbf{P}[H = 1 \mid N = n]\mathbf{P}[N = n], \tag{6.1}$$

where the summation starts at $n = 1$ because the probability $\mathbf{P}[H = 1 \mid N = 0]$ is obviously equal to 0: you cannot buy a property if there is not any on the market. The above equation has reduced our problem of calculating $\mathbf{P}[H = 1]$ to that of calculating the conditional probability $\mathbf{P}[H = 1 \mid N = n]$ of purchasing a property for every $n \geq 1$ when there are n properties on the market during the search period $(t_0, t_1]$. To successfully proceed further, we need to make additional assumptions on the model, which we shall do later. An illustrative example follows next.

We assume the simplest yet practically reasonable model: First, during the search period, let the buyer's budget stay the same and, therefore, her/his reservation price is the same for every property to be considered irrespectively on the time of negotiations and/or sale during the search period. Furthermore, we assume that the properties that the buyer is targeting have similar features and that the sellers hold similar negotiating power. Hence, the (abstract) negotiated selling prices, say $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$, follow the same distribution for each property considered. Next, we assume that the sellers of the properties do not communicate with each other on the negotiated selling matters. Therefore, the negotiated selling prices $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$ become independent random variables. In summary, we assume for the rest of this section that $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$ are i.i.d. random variables. Denote the CDF of each of these random variables $Z_{n,i}$ by ζ, that is,

$$\zeta(x) = \mathbf{P}[Z_{n,i} \leq x]$$

for every $x \geq 0$.

Let the buyer's reservation price be u. Then, for every $n \geq 1$, the probability $\mathbf{P}[H = 1 \mid N = n]$ is equal to $1 - (1 - \zeta(u))^n$, where $(1 - \zeta(u))^n$ is the probability that the buyer is unable to purchase a property after the search since the price of every searched property on the market exceeds the buyer's reservation price. Hence, we have the formula

$$\begin{aligned}\mathbf{P}[H = 1] &= 1 - \mathbf{P}[N = 0] - \sum_{n=1}^{\infty}(1 - \zeta(u))^n \mathbf{P}[N = n] \\ &= 1 - G(1 - \zeta(u)), \end{aligned} \tag{6.2}$$

where $G(y) := \sum_{n=0}^{\infty} y^n \mathbf{P}[N = n]$ is the probability generating function of N.

Example 6.2.1 Let N follow the Poisson distribution with some mean, which we denote by μ_{t_0,t_1}, with subindices t_0 and t_1 because it depends on the search time-interval $(t_0, t_1]$. Then the probability generating function G is given by

$$G(y) = \exp\{-(1 - y)\mu_{t_0,t_1}\}.$$

Therefore, equation (6.2) becomes

$$\mathbf{P}[H = 1] = 1 - \exp\{-\zeta(u)\mu_{t_0,t_1}\}. \tag{6.3}$$

Equation (6.3) is useful from the practical point of view because it allows us to estimate the probability $\mathbf{P}[H = 1]$ given values of the parameters u, t_0, and t_1, as well as either historical data or some knowledge-based considerations to get an estimate of the mean $\mu(t_0, t_1]$. Next, we view the mean μ_{t_0,t_1} of the underlying Poisson process governed by an intensity function $\lambda(t)$, which gives the expression $\mu(t_0, t_1] = \int_{t_0}^{t_1} \lambda(t)\mathrm{d}t$. In this case, we have the equation

$$\mathbf{P}[H = 1] = 1 - \exp\left\{-\zeta(u)\int_{t_0}^{t_1} \lambda(t)\mathrm{d}t\right\}. \tag{6.4}$$

This equation is particularly useful to see the dynamics of the probability of purchasing a property when the search period $(t_0, t_1]$ varies, which can be utilized by the buyer to make certain time adjustments to her/his property-hunting strategy.

So far, we have presented formulas for the likelihood of the successful purchasing of a property under simplified assumptions. Yet, the obtained results are illuminating and convey basic features of what we shall see in the following sections under relaxed and thus more practical assumptions.

6.3 The likelihood of purchasing

This section consists of four subsections, where we impose, step by step, additional simplifying assumptions. Table 6.1 overviews the subsections from the viewpoint of assumptions. Naturally, the first subsection is the most general.

	Negotiated sales price	
Subsection	Identically distributed	Independent
6.3.1	-	-
6.3.2	-	✓
6.3.3	✓	-
6.3.4	✓	✓

Table 6.1: An overview of the following subsections.

6.3.1 Most general case

We now drop all the assumptions made in Section 6.2. In other words, the reservation price may or may not be the same for each property available on the market. We shall denote the reservation prices by $u_{n,1}, u_{n,2}, \ldots, u_{n,n}$. Furthermore, the sales prices may or may not be independent. Finally, the sales prices may or may not be identically distributed. Under these circumstances, we next derive an expression for the conditional probability $\mathbf{P}_n[H = 1] := \mathbf{P}[H = 1 \mid N = n]$.

Theorem 6.3.1 *In the most general case, the conditional probability of the buyer purchasing a property is given by*

$$\mathbf{P}_n[H = 1] = \mathbf{P}_n[Z_{n,1} \le u_{n,1}] + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]\mathbf{P}_n[Z_{n,1} > u_{n,1}], \tag{6.5}$$

where, for $n = 0, 1$*, we have*

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = 0,$$

and for $n = 2$, we have

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = \mathbf{P}_n[Z_{n,2} \le u_{n,2} \mid Z_{n,1} > u_{n,1}],$$

and for $n \ge 3$, we have

$$\begin{aligned}\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = \mathbf{P}_n[Z_{n,2} \le u_{n,2} \mid Z_{n,1} > u_{n,1}]&\\ + \sum_{j=3}^{n} \Big(\mathbf{P}_n[Z_{n,j} \le u_{n,j} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}]&\\ \times \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k} \mid Z_{n,k-1} > u_{n,k-1}]\Big).&\end{aligned}$$

Proof When $n = 0, 1$, then $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = 0$, and when $n = 2$, then $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = \mathbf{P}_n[Z_{n,2} \le u_{n,2} \mid Z_{n,1} > u_{n,1}]$. Next, we shall prove that for every $i = 1, 2, \ldots, n-2$ with $n \ge 3$, we have

$$\begin{aligned}&\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i} > u_{n,i}]\\ &\quad = \mathbf{P}_n[Z_{n,i+1} \le u_{n,i+1} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i} > u_{n,i}] + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i+1} > u_{n,i+1}]\\ &\qquad \times \mathbf{P}_n[Z_{n,i+1} > u_{n,i+1} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i} > u_{n,i}], \end{aligned} \tag{6.6}$$

and furthermore, for $i = n-1$ and $n \ge 2$, we have

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = \mathbf{P}_n[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]. \tag{6.7}$$

Note from equations (6.6) and (6.7) that for $i = 1, 2, \ldots, n-1$ with $n \ge 3$, we have

$$\begin{aligned}\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i} > u_{n,i}] = \mathbf{P}_n[Z_{n,i+1} \le u_{n,i+1} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i} > u_{n,i}]&\\ + \sum_{j=i+2}^{n} \Big(\mathbf{P}_n[Z_{n,j} \le u_{n,j} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}]&\\ \times \prod_{k=i+1}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,k-1} > u_{n,k-1}]\Big),&\end{aligned}$$

and for $i = 1$ with $n \ge 3$, we have

$$\begin{aligned}
&\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] \\
&\quad = \mathbf{P}_n[Z_{n,2} \le u_{n,2} \mid Z_{n,1} > u_{n,1}] + \sum_{j=3}^{n} \Big(\mathbf{P}_n[Z_{n,j} \le u_{n,j} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,j-1} > u_{n,j-1}] \\
&\qquad\qquad \times \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k} \mid Z_{n,k-1} > u_{n,k-1}] \Big).
\end{aligned}$$

This would complete the proof of Theorem 6.3.1. Hence, our goal is to establish equations (6.6) and (6.7).

At the time of the first offer, there are two possible outcomes: First, if the selling price is below $u_{n,1}$, then the buyer will purchase the first property and $\mathbf{P}_n[H = 1 \mid Z_{n,1} \le u_{n,1}] = 1$. Second, if the first offer is above $u_{n,1}$, then the buyer will not purchase the property, and instead, will move on to the second property. The conditional probability of the buyer purchasing a property can then be expanded in the following way:

$$\begin{aligned}
\mathbf{P}_n[H = 1] &= \mathbf{P}_n[H = 1, Z_{n,1} \le u_{n,1}] + \mathbf{P}_n[H = 1, Z_{n,1} > u_{n,1}] \\
&= \mathbf{P}_n[H = 1 \mid Z_{n,1} \le u_{n,1}]\mathbf{P}_n[Z_{n,1} \le u_{n,1}] + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]\mathbf{P}_n[Z_{n,1} > u_{n,1}] \\
&= \mathbf{P}_n[Z_{n,1} \le u_{n,1}] + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]\mathbf{P}_n[Z_{n,1} > u_{n,1}].
\end{aligned}$$

When checking the second property, the buyer will once again face two outcomes. The selling price is either below $u_{n,2}$ or above $u_{n,2}$. We would then need to expand $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]$ in the following way:

$$\begin{aligned}
\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] &= \mathbf{P}_n[Z_{n,2} \le u_{n,2} \mid Z_{n,1} > u_{n,1}] \\
&\quad + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, Z_{n,2} > u_{n,2}]\mathbf{P}_n[Z_{n,2} > u_{n,2} \mid Z_{n,1} > u_{n,1}].
\end{aligned}$$

This process will continue until the final offer. Similar to the expansion of $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]$, we have

$$\begin{aligned}
\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, Z_{n,2} > u_{n,2}] &= \mathbf{P}_n[Z_{n,3} \le u_{n,3} \mid Z_{n,1} > u_{n,1}, Z_{n,2} > u_{n,2}] \\
&\quad + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, Z_{n,2} > u_{n,2}, Z_{n,3} > u_{n,3}] \\
&\quad \times \mathbf{P}_n[Z_{n,3} > u_{n,3} \mid Z_{n,1} > u_{n,1}, Z_{n,2} > u_{n,2}].
\end{aligned}$$

Analogously, we have

$$\begin{aligned}\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-2} > u_{n,n-2}] &= \mathbf{P}_n[Z_{n,n-1} \le u_{n,n-1} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-2} > u_{n,n-2}] \\ &\quad + \mathbf{P}_n[Z_{n,n-1} > u_{n,n-1} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-2} > u_{n,n-2}] \\ &\quad \times \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}].\end{aligned}$$

When the buyer checks the final offer, he/she has two choices remaining: If the price is below $u_{n,n}$, then the buyer will purchase the property and $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} \le u_{n,n}] = 1$. If the price is above $u_{n,n}$, then the buyer will not make a purchase and $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}] = 0$. Therefore,

$$\begin{aligned}&\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] \\ &\quad = \mathbf{P}_n[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}] \\ &\quad\quad \times \mathbf{P}_n[Z_{n,n} > u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}].\end{aligned}$$

Then we obtain

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = \mathbf{P}_n[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}].$$

In order to express $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]$ in terms of the conditional distribution function of $Z_{n,i+1} \mid Z_{n,1}, \ldots, Z_{n,i}$, we start with the expression for $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$. Then we work backwards to find $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-2} > u_{n,n-2}]$, and so on. In general, for $i = 1, 2, \ldots, n-2$ with $n \ge 3$, we obtain equation (6.6) and for $i = n-1$, we obtain equation (6.7). This completes the proof of Theorem 6.3.1. □

From equation (6.1) and Theorem 6.3.1, we immediately obtain the formula

$$\mathbf{P}[H = 1] = s_1 + s_2 + s_3, \tag{6.8}$$

where

$$s_1 = \sum_{n=1}^{\infty} \mathbf{P}_n[Z_{n,1} \le u_{n,1}]\mathbf{P}[N = n],$$

$$s_2 = \sum_{n=2}^{\infty} \mathbf{P}_n[Z_{n,2} \le u_{n,2} \mid Z_{n,1} > u_{n,1}]\mathbf{P}_n[Z_{n,1} > u_{n,1}]\mathbf{P}[N = n],$$

and

$$s_3 = \sum_{n=3}^{\infty} \mathbf{P}_n[Z_{n,1} > u_{n,1}] \sum_{j=3}^{n} \mathbf{P}_n[Z_{n,j} \leq u_{n,j} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}]$$
$$\times \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,k-1} > u_{n,k-1}]\mathbf{P}[N = n].$$

When the negotiated selling prices $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,n}$ are independent of the number N of properties on sale, the conditional probability $\mathbf{P}_n$ becomes the unconditional $\mathbf{P}$. We then obtain the formula

$$\begin{aligned}\mathbf{P}[H = 1] = \mathbf{P}[Z_{n,1} \leq u_{n,1}](1 - \mathbf{P}[N = 0])& \\ + \mathbf{P}[Z_{n,2} \leq u_{n,2} \mid Z_{n,1} > u_{n,1}]\mathbf{P}[Z_{n,1} > u_{n,1}](1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1])& \\ + \sum_{j=3}^{\infty} \mathbf{P}[Z_{n,1} > u_{n,1}]\mathbf{P}[Z_{n,j} \leq u_{n,j} \mid Z_{n,i} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}]& \\ \times \prod_{k=2}^{j-1} \mathbf{P}[Z_{n,k} > u_{n,k} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,k-1} > u_{n,k-1}]\mathbf{P}[N \geq j].& \end{aligned} \tag{6.9}$$

Next, if we assume that the number N of properties on sale during the search interval $(t_0, t_1]$ follows the Poisson distribution with mean μ_{t_0,t_1}, then $\mathbf{P}[N = 0]$ in equation (6.9) can be replaced by $\exp\{-\mu_{t_0,t_1}\}$, and $\mathbf{P}[N \geq k]$ by $\sum_{i=k}^{\infty}(\mu_{t_0,t_1}^i / i!) \exp\{-\mu_{t_0,t_1}\}$.

6.3.2 Independent negotiated selling prices

Here we drop the assumption that the reservation price is the same for each property available on the market. We also drop the assumption that the sales prices are identically distributed. Under these circumstances, we next obtain an expression for $\mathbf{P}_n[H = 1]$.

Corollary 6.3.2 *Let the sales prices be independent. Then we obtain the following expression for the conditional probability of buying a property:*

$$\mathbf{P}_n[H = 1] = \mathbf{P}_n[Z_{n,1} \leq u_{n,1}] + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]\mathbf{P}_n[Z_{n,1} > u_{n,1}], \tag{6.10}$$

where

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]$$
$$= \begin{cases} 0 & \textit{when} \quad n = 0, 1, \\ \mathbf{P}_n[Z_{n,2} \le u_{n,2}] & \textit{when} \quad n = 2, \\ \mathbf{P}_n[Z_{n,2} \le u_{n,2}] + \sum_{j=3}^{n} \Big(\mathbf{P}_n[Z_{n,j} \le u_{n,j}] \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k}] \Big) & \textit{when} \quad n \ge 3. \end{cases} \tag{6.11}$$

Proof We shall prove that, for $i = 1, 2, \ldots, n-2$ with $n \ge 3$, the conditional probability $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]$ can be obtained using the following recursive relationship:

$$\begin{aligned} &\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i} > u_{n,i}] \\ &\quad = \mathbf{P}_n[Z_{n,i+1} \le u_{n,i+1}] + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i+1} > u_{n,i+1}] \mathbf{P}_n[Z_{n,i+1} > u_{n,i+1}], \end{aligned} \tag{6.12}$$

and for $i = n-1$ with $n \ge 2$,

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = \mathbf{P}_n[Z_{n,n} \le u_{n,n}]. \tag{6.13}$$

Note that from equations (6.12) and (6.13), for $i = 1, 2, \ldots, n-1$ with $n \ge 3$, we immediately get

$$\begin{aligned} &\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,i} > u_{n,i}] \\ &\qquad = \mathbf{P}_n[Z_{n,i+1} \le u_{n,i+1}] + \sum_{j=i+2}^{n} \Big(\mathbf{P}_n[Z_{n,j} \le u_{n,j}] \prod_{k=i+1}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k}] \Big). \end{aligned}$$

When $i = 1$ with $n \ge 3$, we have

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = \mathbf{P}_n[Z_{n,2} \le u_{n,2}] + \sum_{j=3}^{n} \Big(\mathbf{P}_n[Z_{n,j} \le u_{n,j}] \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k}] \Big).$$

When we set $k = 1$ with $n = 0$ or 1, we have

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = 0,$$

and when $n = 2$, we have

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] = \mathbf{P}_n[Z_{n,2} \le u_{n,2}].$$

This would complete the proof of Corollary 6.3.2. Hence, to complete the proof of the corollary, we need to establish equations (6.12) and (6.13).

Similar to the proof of Theorem 6.3.1, we have

$$\begin{aligned}\mathbf{P}_n[H=1] &= \mathbf{P}_n[H=1, Z_{n,1}\le u_{n,1}] + \mathbf{P}_n[H=1, Z_{n,1}>u_{n,1}]\\ &= \mathbf{P}_n[H=1 \mid Z_{n,1}\le u_{n,1}]\mathbf{P}_n[Z_{n,1}\le u_{n,1}]\\ &\quad + \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}]\mathbf{P}_n[Z_{n,1}>u_{n,1}]\\ &= \mathbf{P}_n[Z_{n,1}\le u_{n,1}] + \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}]\mathbf{P}_n[Z_{n,1}>u_{n,1}].\end{aligned}$$

Then we write

$$\begin{aligned}\mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}] &= \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, Z_{n,2}\le u_{n,2}]\mathbf{P}_n[Z_{n,2}\le u_{n,2} \mid Z_{n,1}>u_{n,1}]\\ &\quad + \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, Z_{n,2}> u_{n,2}]\mathbf{P}_n[Z_{n,2}> u_{n,2} \mid Z_{n,1}>u_{n,1}]\\ &= \mathbf{P}_n[Z_{n,2}\le u_{n,2} \mid Z_{n,1}>u_{n,1}]\\ &\quad + \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, Z_{n,2}> u_{n,2}]\mathbf{P}_n[Z_{n,2}> u_{n,2} \mid Z_{n,1}>u_{n,1}]\\ &= \mathbf{P}_n[Z_{n,2}\le u_{n,2}] + \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, Z_{n,2}> u_{n,2}]\mathbf{P}_n[Z_{n,2}> u_{n,2}].\end{aligned}$$

Since $Z_{n,1}$ and $Z_{n,2}$ are independent of each other, we have the equation $\mathbf{P}_n[Z_{n,2}\le u_{n,2} \mid Z_{n,1} > u_{n,1}] = \mathbf{P}_n[Z_{n,2}\le u_{n,2}]$. Similarly, we have

$$\begin{aligned}\mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, Z_{n,2}>u_{n,2}] &= \mathbf{P}_n[Z_{n,3}\le u_{n,3}]\\ &\quad + \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, Z_{n,2}>u_{n,2}, Z_{n,3}>u_{n,3}]\\ &\quad \times \mathbf{P}_n[Z_{n,3}>u_{n,3}].\end{aligned}$$

Proceeding in a similar fashion, we obtain

$$\begin{aligned}&\mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, \dots, Z_{n,n-2}>u_{n,n-2}]\\ &\quad = \mathbf{P}_n[Z_{n,n-1}\le u_{n,n-1}] + \mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, \dots, Z_{n,n-1}>u_{n,n-1}]\mathbf{P}_n[Z_{n,n-1}>u_{n,n-1}].\end{aligned}$$

Finally, for $\mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, \dots, Z_{n,n-1}>u_{n,n-1}]$, we have

$$\mathbf{P}_n[H=1 \mid Z_{n,1}>u_{n,1}, \dots, Z_{n,n-1}>u_{n,n-1}] = \mathbf{P}_n[Z_{n,n}\le u_{n,n}].$$

In order to express $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]$ in terms of the distribution function of $Z_{n,i}$, we start with $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$, then work backwards to find $\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-2} > u_{n,n-2}]$, and so on. In general, for $i = 1, 2, \ldots, n-2$ with $n \geq 3$, we obtain equation (6.12), and when $i = n-1$, then we obtain equation (6.13). This completes the proof of Corollary 6.3.2. □

From equation (6.1) and Corollary 6.3.2, we have that the (unconditional) probability of the buyer purchasing a property is given by

$$\mathbf{P}[H = 1] = \sum_{n=1}^{\infty} \mathbf{P}_n[Z_{n,1} \leq u_{n,1}]\mathbf{P}[N = n] + \sum_{n=2}^{\infty} \mathbf{P}_n[Z_{n,2} \leq u_{n,2}]\mathbf{P}_n[Z_{n,1} > u_{n,1}]\mathbf{P}[N = n]$$
$$+ \sum_{n=3}^{\infty} \mathbf{P}_n[Z_1 > u_1] \sum_{j=3}^{n} \Big(\mathbf{P}_n[Z_{n,j} \leq u_{n,j}] \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k}]\Big)\mathbf{P}[N = n]. \tag{6.14}$$

When the negotiated selling prices are independent of the number N of properties on sale, the (unconditional) probability of the buyer purchasing a property becomes

$$\begin{aligned}\mathbf{P}[H = 1] = \mathbf{P}[Z_{n,1} \leq u_{n,1}](1 - \mathbf{P}[N = 0])& \\ + \mathbf{P}[Z_{n,2} \leq u_{n,2}]\mathbf{P}[Z_{n,1} > u_{n,1}](1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1])& \\ + \sum_{j=3}^{\infty} \mathbf{P}[Z_{n,1} > u_{n,1}]\mathbf{P}[Z_{n,j} \leq u_{n,j}]\Big(\prod_{k=2}^{j-1} \mathbf{P}[Z_{n,k} > u_{n,k}]\Big)\mathbf{P}[N \geq j].& \end{aligned} \tag{6.15}$$

6.3.3 Identically distributed negotiated selling prices

Here we drop the assumption that the reservation price is the same for each property on sale. We also drop the assumption that the negotiated selling prices are independent. Under these circumstances, we next derive an expression for $\mathbf{P}_n[H = 1]$.

Corollary 6.3.3 *Let the distributions of the sales prices be identical. Then we obtain the following expression for the conditional probability of buying a property:*

$$\mathbf{P}_n[H = 1] = \zeta_n(u_{n,1}) + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}](1 - \zeta_n(u_{n,1})), \tag{6.16}$$

where

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}]$$
$$= \begin{cases} 0 & \textit{when } n = 0, 1, \\ \mathbf{P}_n[Z_{n,2} \leq u_{n,2} \mid Z_{n,1} > u_{n,1}] & \textit{when } n = 2, \\ \mathbf{P}_n[Z_{n,2} \leq u_{n,2} \mid Z_{n,1} > u_{n,1}] & \\ + \sum_{j=3}^{n} \Big(\mathbf{P}_n[Z_{n,j} \leq u_{n,j} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}] & \\ \quad \times \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k} \mid Z_{n,k-1} > u_{n,k-1}]\Big) & \textit{when } n \geq 3. \end{cases} \tag{6.17}$$

Proof The proof is similar to that of Theorem 6.3.1. The main difference is that we now assume that the sales prices are identically distributed, that is, they follow the same distribution as that of a random variable Z. This concludes the proof of Corollary 6.3.3. □

From equation (6.1) and Corollary 6.3.3, we immediately obtain the (unconditional) probability of the buyer purchasing a property:

$$\begin{aligned} \mathbf{P}[H = 1] = & \sum_{n=1}^{\infty} \zeta_n(u_{n,1})\mathbf{P}[N = n] + \sum_{n=2}^{\infty} \mathbf{P}_n[Z_{n,2} \leq u_{n,2} \mid Z_{n,1} > u_{n,1}](1 - \zeta_n(u_{n,1}))\mathbf{P}[N = n] \\ & + \sum_{n=3}^{\infty}(1 - \zeta_n(u_{n,1})) \sum_{j=3}^{n} \mathbf{P}_n[Z_{n,j} \leq u_{n,j} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}] \\ & \times \prod_{k=2}^{j-1} \mathbf{P}_n[Z_{n,k} > u_{n,k} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,k-1} > u_{n,k-1}]\mathbf{P}[N = n], \end{aligned} \tag{6.18}$$

where

$$\zeta_n(u) = \mathbf{P}_n[Z \leq u].$$

When the negotiated selling prices are independent of the number N of properties on sale, the conditional probability $\mathbf{P}_n$ becomes the unconditional $\mathbf{P}$, and ζ_n becomes ζ, which has been defined in Section 6.2. Therefore, we have

$$\begin{aligned} \mathbf{P}[H = 1] = & \zeta(u_{n,1})(1 - \mathbf{P}[N = 0]) \\ & + \mathbf{P}[Z_{n,2} \leq u_{n,2} \mid Z_{n,1} > u_{n,1}](1 - \zeta(u_1))(1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1]) \\ & + \sum_{j=3}^{\infty}(1 - \zeta(u_{n,1}))\mathbf{P}[Z_{n,j} \leq u_{n,j} \mid Z_{n,i} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}] \\ & \times \prod_{k=2}^{j-1} \mathbf{P}[Z_{n,k} > u_{n,k} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,k-1} > u_{n,k-1}]\mathbf{P}[N \geq j]. \end{aligned} \tag{6.19}$$

6.3.4 IID negotiated selling prices

We now drop the assumption that the buyer's reservation price stays the same and derive an expression for $\mathbf{P}_n[H = 1]$.

Corollary 6.3.4 *Let the sales prices be i.i.d. Then the conditional probability of the buyer purchasing a property is given by*

$$\mathbf{P}_n[H = 1] = \zeta_n(u_{n,1}) + \mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}](1 - \zeta_n(u_{n,1})), \tag{6.20}$$

where

$$\mathbf{P}_n[H = 1 \mid Z_{n,1} > u_{n,1}] \\ = \begin{cases} 0 & \textit{when} \quad n = 0, 1, \\ \zeta_n(u_{n,2}) & \textit{when} \quad n = 2, \\ \zeta_n(u_{n,2}) + \sum_{j=3}^{n} \left(\zeta_n(u_{n,j}) \prod_{k=2}^{j-1} (1 - \zeta_n(u_{n,k})) \right) & \textit{when} \quad n \geq 3. \end{cases} \tag{6.21}$$

Proof The proof of Corollary 6.3.4 is similar to that of Corollary 6.3.2. The main difference is that the random variables of the sales prices are now identically distributed, say, like a random variable Z. This concludes the proof of Corollary 6.3.4. □

From equation (6.1) and Corollary 6.3.4, we have the formula

$$\begin{aligned} \mathbf{P}[H = 1] = &\sum_{n=1}^{\infty} \zeta_n(u_{n,1}) \mathbf{P}[N = n] + \sum_{n=2}^{\infty} \zeta_n(u_{n,2})(1 - \zeta_n(u_{n,1})) \mathbf{P}[N = n] \\ &+ \sum_{n=3}^{\infty} (1 - \zeta_n(u_{n,1})) \sum_{j=3}^{n} \left(\zeta_n(u_{n,j}) \prod_{k=2}^{j-1} (1 - \zeta_n(u_{n,k})) \right) \mathbf{P}[N = n]. \end{aligned} \tag{6.22}$$

Next, when the negotiated selling price Z is independent of the number N of properties on sale, then the conditional probability $\mathbf{P}_n$ becomes the unconditional $\mathbf{P}$, and ζ_n becomes ζ. Consequently, we have

$$\mathbf{P}[H = 1] = m_1 + m_2 + m_3, \tag{6.23}$$

where

$$m_1 = \zeta(u_{n,1})(1 - \mathbf{P}[N = 0]),$$

$$m_2 = \zeta(u_{n,2})(1 - \zeta(u_{n,1}))(1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1]),$$

and

$$m_3 = \sum_{j=3}^{\infty}(1-\zeta(u_{n,1}))\zeta(u_{n,j})\Big(\prod_{k=2}^{j-1}(1-\zeta(u_{n,k})\Big)\mathbf{P}[N \geq j].$$

Note that when the reservation price stays the same, say u, equation (6.23) becomes equation (6.2) given in Section 6.2.

6.4 Modelling price dependence

The dependence has come into our above considerations via the conditional probability $\mathbf{P}[Z_{n,j} \leq u_{n,j} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,j-1} > u_{n,j-1}]$, which can also be written as follows:

$$\mathbf{P}[Z_{n,j} \leq u_{n,j} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,j-1} > u_{n,j-1}] = 1 - \frac{\mathbf{P}[Z_{n,1} > u_{n,1}, \dots, Z_{n,j} > u_{n,j}]}{\mathbf{P}[Z_{n,1} > u_{n,1}, \dots, Z_{n,j-1} > u_{n,j-1}]}. \quad (6.24)$$

Hence, in general, we have to learn how to calculate, or estimate, the joint survival probability of the negotiated selling prices. In what follows, we shall discuss direct representations of the joint survival probability as well as via survival copulas. These are the topics of Subsections 6.4.1 and 6.4.2, respectively.

6.4.1 Direct representation

Here we present three direct representations of the joint survival function. They come from various studies in the literature and are very popular in actuarial literature.

Multivariate Pareto of the second kind

The joint survival function of the multivariate Pareto of the second kind is given by (cf., e.g., Arnold, 1983; Asimit et al., 2010)

$$\mathbf{P}[X_1 > x_1, \dots, X_n > x_n] = \Big(\sum_{i=1}^{n}\frac{x_i - \mu_{n,i}}{\theta_i} + 1\Big)^{-a} \quad (6.25)$$

for $x_i \geq \mu_{n,i}$ with $\mu_{n,i} \in \mathbb{R}$ and $i = 1, 2, \dots, n$, where $\theta_i > 0$ and $a > 0$.

Marshall and Olkin's multivariate exponential distribution

The joint survival function of the Marshall and Olkin's (1967) multivariate exponential distribution is given by

$$\mathbf{P}[X_1 > x_1, \ldots, X_n > x_n] = \exp\left\{ -\sum_{i=1}^{n} \lambda_i x_i - \lambda_{n+1} \max(x_1, x_2, \ldots, x_n) \right\} \tag{6.26}$$

for $x_i > 0$ with $i = 1, 2, \ldots, n$, where $\lambda_i > 0$ and $\lambda_{n+1} \geq 0$.

Multivariate Weibull distribution

The joint survival function of the multivariate Weibull distribution is given by (cf., e.g., Hougaard, 1986)

$$\mathbf{P}[X_1 > x_1, \ldots, X_n > x_n] = \exp\left\{ -\left(\sum_{i=1}^{n} \theta_i x_i^p \right)^l \right\} \tag{6.27}$$

for $x_i \geq 0$ with $i = 1, 2, \ldots, n$, where $l > 0$, $p > 0$, and $\theta_i > 0$.

6.4.2 Copula representation

We now consider using the survival copula to obtain the joint survival function. A copula is defined as follows (cf., e.g., Nelson, 2006):

Definition 6.4.1 A copula is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform.

The copula representation of a joint survival function is given by (cf., e.g., Nelson, 2006)

$$\mathbf{P}[X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n] = \bar{C}(S_{X_1}(x_1), S_{X_2}(x_2), \ldots, S_{X_n}(x_n)) \tag{6.28}$$

for $x_i \in \mathbb{R}$ with $i = 1, 2, \ldots, n$, where $\bar{C}$ is the survival copula and S_i is the survival function of X_i. From equation (6.28), we note that to find the copula representation of a joint survival function, marginal survival functions must be obtained. To illustrate, we next present the survival copula that can be used to obtain the joint survival function of the multivariate Pareto distribution of the second kind. Its marginal survival functions are given by

$$S_{X_i}(x_i) = \left(\frac{x_i - \mu_{n,i}}{\theta_i} + 1 \right)^{-a} \tag{6.29}$$

for $x_i \geq \mu_{n,i}$ with $i = 1, 2, \ldots, n$.

Lemma 6.4.2 *The joint survival function of the multivariate Pareto distribution in equation (6.25) can be obtained using the following survival copula (Clayton copula):*

$$\bar{C}(S_{X_1}(x_1), S_{X_2}(x_2), \ldots, S_{X_n}(x_n)) = \Big(\sum_{i=1}^{n} (S_{X_i}(x_i))^{-1/a} - n + 1 \Big)^{-a}. \tag{6.30}$$

Proof We start with the equation

$$\begin{aligned} \frac{x_i - \mu_{n,i}}{\theta_i} &= \left(\left(\frac{x_i - \mu_{n,i}}{\theta_i} + 1 \right)^{-a} \right)^{-1/a} - 1 \\ &= (S_{X_i}(x_i))^{-1/a} - 1. \end{aligned} \tag{6.31}$$

Now we recall the joint survival function of the multivariate Pareto distribution, which is given in equation (6.25). Next, we replace the term $(x_i - \mu_{n,i})/\theta_i$ by the expression on the right-hand side of equation (6.31). Then we obtain the survival copula given in equation (6.30). This completes the proof of Theorem 6.4.2. □

Note 6.4.3 Schweizer and Wolff (1981) have shown Kendall's tau of two variables X and Y is given by $a/(a + 2)$. The Clayton copula belongs to the family of Archimedean copulas. Some of the other commonly known copulas in this family include the Ali-Mikhail-Haq, Gumbel, Frank, and Joe copulas.

Next, we discuss using a background risk model to describe dependencies among negotiated selling prices.

6.5 Background risk

We begin by introducing the background risk, which can be viewed as follows (cf., e.g., Gollier, 2001): Since decision making under uncertainty often takes place in the presence of multiple risks, choices about endogenous risks must sometimes be made while facing exogenous risks that are independent of the endogenous risks and not under the control of the agent. Such exogenous risks are known as background risks.

Coming now back to our real estate problem, we assume that the negotiated selling prices are affected by either the additive background risk or the multiplicative background risk. Specifically, under the additive background risk, the negotiated selling prices are given by

$$Z_{n,i} = Y_i + Y_0 \tag{6.32}$$

for $i = 1, 2, \ldots, n$, and in the case of the multiplicative background risk, the negotiated selling prices are given by

$$Z_{n,i} = Y_0 Y_i, \tag{6.33}$$

where Y_0 denotes the background price. Random variables Y_i of the negotiated selling prices (stand-alone prices) without the presence of the background risk are assumed to be i.i.d. and independent of Y_0.

The additive and multiplicative background risks have been very popular. Vernic (1997) has modelled the number of claims reported to the insurance company during a given time interval in the presence of an additive background risk. Guiso et al. (1996) have discussed how the demand for risky assets is affected by additive background risks. Alai et al. (2013) have considered the effect an additive background risk has on the survival time of individual policyholders. Doherty and Schlesinger (1983), and Meyer and Meyer (1998) have shown that the outcome of an optimal insurance policy is different when an additive background risk exists. For more on the application of the additive background risk, we refer to Pratt and Zeckhauser (1987), Kimball (1993), Gollier and Pratt (1996), and Vernic (2000). Asimit et al. (2013) have evaluated risk measures, premiums, and capital allocation based on dependent multi-losses, which follow the multivariate Pareto distribution of the second kind. The losses in their model become dependent when a multiplicative background risk is present. For more on the application of the multiplicative background risk, we refer to Nachman (1982) and Pratt (1988).

Various studies in the literature have been reported related to the technique for finding a pair of dependent random variables from three or more random variables that may or may not be independent. This is commonly known as the trivariate reduction technique. The trivariate reduction technique has been used to derive various bivariate distributions, including the Cherian's bivariate gamma distribution (Gupta and Nadarajah, 2006) and the bivariate Marshall and Olkin's exponential distribution (Marshall and Olkin, 1967).

The idea behind using the background risk to model the dependence among negotiated selling prices is to create dependent random variables from more than two initially independent random variables. In our case, random variables $Y_0, Y_1, \ldots, Y_n$ are independent but $Z_{n,1}$, $Z_{n,2}$, $\ldots$, $Z_{n,n}$ are dependent. Next, we obtain expressions for $\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$ and the (unconditional) probability $\mathbf{P}[H = 1]$ of the buyer purchasing a property under additive or multiplicative background risks. Results are presented in the following two subsections.

6.5.1 Additive background price

Under the additive background risk, we first obtain an expression for $\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$.

Theorem 6.5.1 *Assume that the additive background risk is present. Furthermore, assume that the distribution of each Y_i is the same as that of the random variable Y for $i = 1, 2, \ldots, n$. Then we have the formula*

$$\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = 1 - \frac{\mathbf{E}[\prod_{i=1}^{n} \bar{F}_Y(u_{n,i} - Y_0)]}{\mathbf{E}[\prod_{i=1}^{n-1} \bar{F}_Y(u_{n,i} - Y_0)]}. \tag{6.34}$$

Proof Recall the expression for the conditional probability $\mathbf{P}[Z_{n,j} \le u_{n,j} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}]$ given in equation (6.24). We need to calculate the two terms $\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}]$ and $\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$ on the right-hand side of equation (6.24). We start with the first term, which can be written as follows:

$$\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}] = \mathbf{E}[\mathbf{1}\{Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}\}].$$

Since $Z_{n,i} = Y_0 + Y_i$ for $i = 1, 2, \ldots, n$, using the law of iterated expectations, we have

$$\begin{aligned}
\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}] &= \mathbf{E}[\mathbf{E}[\mathbf{1}\{Y_0 + Y_1 > u_{n,1}, \ldots, Y_0 + Y_n > u_{n,n}\} \mid Y_0]] \\
&= \mathbf{E}[\mathbf{E}[\mathbf{1}\{Y_1 > u_{n,1} - Y_0, \ldots, Y_n > u_{n,n} - Y_0\}]] \\
&= \mathbf{E}[\mathbf{E}[\mathbf{1}\{Y_1 > u_{n,1} - Y_0\} \ldots \mathbf{1}\{Y_n > u_{n,n} - Y_0\}]] \\
&= \mathbf{E}[\mathbf{E}[\mathbf{1}\{Y_1 > u_{n,1} - Y_0\}] \ldots \mathbf{E}[\mathbf{1}\{Y_n > u_{n,n} - Y_0\}]] \\
&= \mathbf{E}\Big[\prod_{i=1}^{n} \bar{F}_Y(u_{n,i} - Y_0)\Big].
\end{aligned} \tag{6.35}$$

Similarly, the second term on the right-hand side of equation (6.24) becomes

$$\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = \mathbf{E}\Big[\prod_{i=1}^{n-1} \bar{F}_Y(u_{n,i} - Y_0)\Big]. \tag{6.36}$$

Having thus calculated the two terms on the right-hand side of equation (6.24), we obtain a new expression for the conditional probability $\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$, which is given by equation (6.34). This completes the proof of Theorem 6.5.1. □

Suppose that $u_{n,1} = u_{n,2} = \cdots = u_{n,n} = u$. Then from Theorem 6.5.1, we immediately get

$$\begin{aligned}
\mathbf{P}[H = 1] = {} & \mathbf{E}_{Y_0}[F_Y(u - Y_0)](1 - \mathbf{P}[N = 0]) \\
& + \Big(1 - \frac{\mathbf{E}[(\bar{F}_Y(u - Y_0))^2]}{\mathbf{E}[\bar{F}_Y(u - Y_0)]}\Big)(1 - \mathbf{E}_{Y_0}[F_Y(u - Y_0)])(1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1]) \\
& + \sum_{j=3}^{\infty}(1 - \mathbf{E}_{Y_0}[F_Y(u - Y_0)])\Big(1 - \frac{\mathbf{E}[(\bar{F}_Y(u - Y_0))^j]}{\mathbf{E}[(\bar{F}_Y(u - Y_0))^{j-1}]}\Big) \\
& \times \prod_{k=2}^{j-1} \frac{\mathbf{E}[(\bar{F}_Y(u - Y_0))^k]}{\mathbf{E}[(\bar{F}_Y(u - Y_0))^{k-1}]}\mathbf{P}[N \geq j].
\end{aligned} \tag{6.37}$$

When the number of properties on sale during the time interval $(t_0, t_1]$ follows the Poisson distribution with rate parameter $\lambda > 0$, equation (6.37) becomes

$$\begin{aligned}
\mathbf{P}[H = 1] = {} & \mathbf{E}_{Y_0}[F_Y(u - Y_0)](1 - e^{-\lambda(t_1 - t_0)}) \\
& + \Big(1 - \frac{\mathbf{E}[(\bar{F}_Y(u - Y_0))^2]]}{\mathbf{E}[\bar{F}_Y(u - Y_0)]}\Big)(1 - \mathbf{E}_{Y_0}[F_Y(u - Y_0)]) \\
& \times (1 - e^{-\lambda(t_1 - t_0)} - \lambda(t_1 - t_0)e^{-\lambda(t_1 - t_0)}) \\
& + \sum_{j=3}^{\infty}(1 - \mathbf{E}_{Y_0}[F_Y(u - Y_0)])\Big(1 - \frac{\mathbf{E}[(\bar{F}_Y(u - Y_0))^j]}{\mathbf{E}[(\bar{F}_Y(u - Y_0))^{j-1}]}\Big) \\
& \times \prod_{k=2}^{j-1} \frac{\mathbf{E}[(\bar{F}_Y(u - Y_0))^k]}{\mathbf{E}[(\bar{F}_Y(u - Y_0))^{k-1}]}\mathbf{P}[N \geq j].
\end{aligned}$$

Next, we assume that the stand-alone prices follow the two-parameter exponential distribution with parameters $\lambda_1 > 0$ and $\mu_1 \geq 0$. Furthermore, we assume that the reservation price stays the same, that is, $u_{n,1} = u_{n,2} = \cdots = u_{n,n} = u$. Then the conditional probability given by equation (6.34) in the presence of the additive background risk becomes

$$\begin{aligned}\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}] \\ = 1 - \frac{\mathbf{E}_{Y_0}[\exp\{-n\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}]}{\mathbf{E}_{Y_0}[\exp\{-(n-1)\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}]}. \quad (6.38)\end{aligned}$$

We next obtain expressions for $\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}]$ and $\mathbf{P}[H = 1]$ when Y_0 follows the two-parameter exponential distribution, the uniform distribution, and the log-normal distribution. Throughout the following consideration of the three distributions, we assume that the conditions of Theorem 6.5.1 are satisfied.

Exponential background prices

Recall that the PDF of Y_0 that follows the two-parameter exponential distribution with parameters $\lambda_0 > 0$ and $\mu_0 > 0$ for $y \ge \mu_0$ is given by

$$f_{Y_0}(y) = \lambda_0 e^{-\lambda_0(y-\mu_0)}.$$

Theorem 6.5.2 *Let the stand-alone prices follow the two-parameter exponential distribution with parameters $\lambda_1 > 0$ and $\mu_1 \ge 0$. Furthermore, let Y_0 follow the two-parameter exponential distribution with parameters $\lambda_0 > 0$ and $\mu_0 \ge 0$. Then we have the formula*

$$\mathbf{P}[Z_{n,n} \le u \mid Z_{n,1} > u, \dots, Z_{n,n-1} > u] = 1 - \frac{h_1(n)}{h_1(n-1)}, \quad (6.39)$$

where, for $j = n - 1$ and n,

$$\begin{aligned}h_1(j) = \mathbf{1}\{u - \mu_1 \le \mu_0\} + \mathbf{1}\{u - \mu_1 > \mu_0\}\lambda_0\Big(\frac{1}{\lambda_1 j - \lambda_0}\exp\{-\lambda_0(u - \mu_0 - \mu_1)\} \\ - \frac{1}{\lambda_1 j - \lambda_0}\exp\{-\lambda_1 j(u - \mu_0 - \mu_1)\}\Big). \quad (6.40)\end{aligned}$$

Proof Recall the conditional probability given in equation (6.38), which is

$$\begin{aligned}\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}] \\ = 1 - \frac{\mathbf{E}_{Y_0}[\exp\{-n\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}]}{\mathbf{E}_{Y_0}[\exp\{-(n-1)\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}]}.\end{aligned}$$

Next, we calculate the numerator and the denominator on the right-hand side of equation (6.38). We begin with the numerator. When Y_0 follows the two-parameter exponential distribution, we have

$$\mathbf{E}_{Y_0}[\exp\{-n\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}] = \int_{\mu_0}^{\infty} \exp\{-\lambda_1 n(u - y - \mu_1)\mathbf{1}\{y < u - \mu_1\}\}\lambda_0 e^{-\lambda_0(y-\mu_0)}\mathrm{d}y.$$

By considering the scenarios $u - \mu_1 \le \mu_0$ and $u - \mu_1 > \mu_0$, we obtain

$$\begin{aligned}\mathbf{E}_{Y_0}&[\exp\{-n\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}] \\ &= \mathbf{1}\{u - \mu_1 \le \mu_0\} + \mathbf{1}\{u - \mu_1 > \mu_0\}\lambda_0\Big(\frac{1}{\lambda_1 n - \lambda_0}\exp\{-\lambda_0(u - \mu_0 - \mu_1)\} \\ &\qquad - \frac{1}{\lambda_1 n - \lambda_0}\exp\{-\lambda_1 n(u - \mu_0 - \mu_1)\}\Big). \end{aligned} \tag{6.41}$$

Similarly, the denominator on the right-hand side of equation (6.38) becomes

$$\begin{aligned}\mathbf{E}_{Y_0}&[\exp\{-(n-1)\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}] \\ &= \mathbf{1}\{u - \mu_1 \le \mu_0\} + \mathbf{1}\{u - \mu_1 > \mu_0\}\lambda_0\Big(\frac{1}{\lambda_1(n-1) - \lambda_0}\exp\{-\lambda_0(u - \mu_0 - \mu_1)\} \\ &\qquad - \frac{1}{\lambda_1(n-1) - \lambda_0}\exp\{-\lambda_1(n-1)(u - \mu_0 - \mu_1)\}\Big). \end{aligned} \tag{6.42}$$

Having thus calculated the numerator and the denominator, we obtain the conditional probability $\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$ given by equations (6.39) and (6.40). This completes the proof of Theorem 6.5.2. □

From Theorem 6.5.2 and the formula for the (unconditional) probability of the buyer purchasing a property in the most general case, which is given by equation (6.9), we immediately get

$$\begin{aligned}\mathbf{P}[H = 1] &= w_1(1 - \mathbf{P}[N = 0]) + \Big(1 - \frac{h_1(2)}{h_1(1)}\Big)(1 - w_1)(1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1]) \\ &\quad + \sum_{j=3}^{\infty}(1 - w_1)\Big(1 - \frac{h_1(j)}{h_1(j-1)}\Big)\prod_{k=2}^{j-1}\frac{h_1(k)}{h_1(k-1)}\mathbf{P}[N \ge j],\end{aligned}$$

where

$$w_1 = \mathbf{1}\{u - \mu_1 > \mu_0\}\Big(1 - \lambda_0\Big(\frac{1}{\lambda_1 - \lambda_0}\exp\{-\lambda_0(u - \mu_0 - \mu_1)\} - \frac{1}{\lambda_1 - \lambda_0}\exp\{-\lambda_1(u - \mu_0 - \mu_1)\}\Big)\Big).$$

Uniform background prices

Recall that the PDF of Y_0 that follows the uniform distribution with parameters a and b is given by

$$f_{Y_0}(y) = \frac{1}{b-a}.$$

Theorem 6.5.3 *Let the stand-alone prices follow the two-parameter exponential distribution with parameters $\lambda_1 > 0$ and $\mu_1 \geq 0$. Furthermore, let Y_0 follow the uniform (a, b) distribution. Then we have the formula*

$$\mathbf{P}[Z_{n,n} \leq u \mid Z_{n,1} > u, \ldots, Z_{n,n-1} > u] = 1 - \frac{h_2(n)}{h_2(n-1)}, \tag{6.43}$$

where, for $j = n-1$ and n,

$$\begin{aligned} h_2(j) = I\{u-\mu_1 \leq a\} &+ I\{a < u-\mu_1 \leq b\}\Big(\frac{1}{b-a}\Big(\frac{1}{\lambda_1 j} - \frac{1}{\lambda_1 j}e^{-\lambda_1 j(u-a-\mu_1)}\Big)\Big) \\ &+ I\{u-\mu_1 > b\}\Big(\frac{1}{b-a}\Big(\frac{1}{\lambda_1 j}e^{-\lambda_1 j(u-b-\mu_1)} - \frac{1}{\lambda_1 j}e^{-\lambda_1 j(u-a-\mu_1)}\Big)\Big). \end{aligned} \tag{6.44}$$

Proof Similar to the proof of Theorem 6.5.2, we calculate the numerator and the denominator on the right-hand side of equation (6.38). We begin with the numerator. Since Y_0 follows the uniform distribution, we have

$$\begin{aligned} &\mathbf{E}_{Y_0}[\exp\{-n\lambda_1(u-Y_0-\mu_1)\mathbf{1}\{Y_0 < u-\mu_1\}\}] \\ &\qquad = \int_a^b \exp\{-\lambda_1 n(u-y-\mu_1)\mathbf{1}\{y < u-\mu_1\}\}\frac{1}{b-a}\mathrm{d}y. \end{aligned}$$

By considering the three scenarios $u-\mu_1 \leq a$, $a < u-\mu_1 \leq b$, and $u-\mu_1 > b$, we obtain

$$\begin{aligned} &\mathbf{E}_{Y_0}[\exp\{-n\lambda_1(u-Y_0-\mu_1)\mathbf{1}\{Y_0 < u-\mu_1\}\}] \\ &\quad = \mathbf{1}\{u-\mu_1 \leq a\} + \Big(\frac{1}{b-a}\Big(\frac{1}{\lambda_1 n} - \frac{1}{\lambda_1 n}\exp\{-\lambda_1 n(u-a-\mu_1)\}\Big)\Big)\mathbf{1}\{a < u-\mu_1 \leq b\} \\ &\qquad + \Big(\frac{1}{b-a}\Big(\frac{1}{\lambda_1 n}\exp\{-\lambda_1 n(u-b-\mu_1)\} - \frac{1}{\lambda_1 n}\exp\{-\lambda_1 n(u-a-\mu_1)\}\Big)\Big) \\ &\qquad \times \mathbf{1}\{u-\mu_1 > b\}. \end{aligned} \tag{6.45}$$

Similarly, the denominator on the right-hand side of equation (6.38) is written as follows:

$$
\begin{aligned}
&\mathbf{E}_{Y_0}[\exp\{-(n-1)\lambda_1(u-Y_0-\mu_1)\mathbf{1}\{Y_0<u-\mu_1\}\}]\\
&= \mathbf{1}\{u-\mu_1 \le a\} + \Big(\frac{1}{b-a}\Big(\frac{1}{\lambda_1(n-1)} - \frac{1}{\lambda_1(n-1)}\exp\{-\lambda_1(n-1)(u-a-\mu_1)\}\Big)\Big)\mathbf{1}\{a<u-\mu_1\le b\}\\
&\quad + \Big(\frac{1}{b-a}\Big(\frac{1}{\lambda_1(n-1)}\exp\{-\lambda_1(n-1)(u-b-\mu_1)\} - \frac{1}{\lambda_1(n-1)}\exp\{-\lambda_1(n-1)(u-a-\mu_1)\}\Big)\Big)\\
&\qquad \times \mathbf{1}\{u-\mu_1>b\}. \quad (6.46)
\end{aligned}
$$

Having thus calculated the numerator and the denominator in equations (6.45) and (6.46), we calculate the conditional probability $\mathbf{P}[Z_{n,n} \le u \mid Z_{n,1} > u, \ldots, Z_{n,n-1} > u]$ and arrive at equations (6.43) and (6.44). This completes the proof of Theorem 6.5.3. □

From Theorem 6.5.3 and the formula for the (unconditional) probability of the buyer purchasing a property given by equation (6.9), we immediately get

$$
\begin{aligned}
\mathbf{P}[H=1] &= w_2(1-\mathbf{P}[N=0]) + \Big(1-\frac{h_2(2)}{h_2(1)}\Big)(1-w_2)(1-\mathbf{P}[N=0]-\mathbf{P}[N=1])\\
&\quad + \sum_{j=3}^{\infty}(1-w_2)\Big(1-\frac{h_2(j)}{h_2(j-1)}\Big)\prod_{k=2}^{j-1}\frac{h_2(k)}{h_2(k-1)}\mathbf{P}[N\ge j],
\end{aligned}
$$

where

$$
\begin{aligned}
w_2 &= \mathbf{1}\{a<u-\mu_1\le b\}\Big(1-\frac{1}{b-a}\Big(\frac{1}{\lambda_1}-\frac{1}{\lambda_1}\exp\{-\lambda_1(u-a-\mu_1)\}\Big)\Big)\\
&\quad + \mathbf{1}\{u-\mu_1>b\}\Big(1-\frac{1}{b-a}\Big(\frac{1}{\lambda_1}\exp\{-\lambda_1(u-b-\mu_1)\}-\frac{1}{\lambda_1}\exp\{-\lambda_1(u-a-\mu_1)\}\Big)\Big).
\end{aligned}
$$

Log-normal background prices

Recall that the PDF of Y_0 that follows the three-parameter log-normal distribution with parameters $\mu \in \mathbb{R}$, $\sigma > 0$, and $\gamma \ge 0$ is given by

$$
f_{Y_0}(y) = \frac{1}{(y-\gamma)\sigma\sqrt{2\pi}}\exp\Big\{-\frac{(\ln(y-\gamma)-\mu)^2}{2\sigma^2}\Big\}.
$$

Theorem 6.5.4 *Let the stand-alone prices follow the two-parameter exponential distribution with parameters $\lambda_1 > 0$ and $\mu_1 \ge 0$. Furthermore, let Y_0 follow the three-parameter log-normal distribution with parameters $\mu \in \mathbb{R}$, $\sigma > 0$, and $\gamma \ge 0$. Then we have the formula*

$$
\mathbf{P}[Z_{n,n} \le u \mid Z_{n,1} > u, \ldots, Z_{n,n-1} > u] = 1 - \frac{\mathbf{E}[(h^*(G_{0,1}))^n]}{\mathbf{E}[(h^*(G_{0,1}))^{n-1}]}, \quad (6.47)
$$

where $G_{0,1}$ follows the standard normal distribution and the function $h^*(x)$ is given by

$$h^*(x) = \exp\left\{(-\lambda_1 u + \lambda_1\mu_1 + \lambda_1\gamma + \lambda_1 \exp\{\mu + \sigma x\})\mathbf{1}\left\{x < \frac{\ln(u - \mu_1 - \gamma) - \mu}{\sigma}\right\}\right\}. \tag{6.48}$$

Proof Similar to the proof of Theorem 6.5.2, we calculate the numerator and the denominator on the right-hand side of equation (6.38). We begin with the numerator:

$$\mathbf{E}[\exp\{-n\lambda_1(u-Y_0-\mu_1)\mathbf{1}\{Y_0 < u-\mu_1\}\}] = \mathbf{E}[\exp\{(-\lambda_1 u+\lambda_1\mu_1+\lambda_1 Y_0)n\mathbf{1}\{Y_0 < u-\mu_1\}\}]. \tag{6.49}$$

The background price Y_0 is given by

$$Y_0 = \exp\{\mu + \sigma G_{0,1}\} + \gamma. \tag{6.50}$$

By substituting equation (6.50) into equation (6.49), we obtain

$$\begin{aligned}
&\mathbf{E}[\exp\{-n\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}] \\
&= \mathbf{E}\left[\exp\left\{(-\lambda_1 u + \lambda_1\mu_1 + \lambda_1\gamma + \lambda_1 \exp\{\mu + \sigma G_{0,1}\} + \gamma)n\mathbf{1}\left\{G_{0,1} < \frac{\ln(u - \mu_1 - \gamma) - \mu}{\sigma}\right\}\right\}\right].
\end{aligned} \tag{6.51}$$

Similarly, the denominator on the right-hand side of equation (6.38) is written as follows:

$$\begin{aligned}
&\mathbf{E}[\exp\{-(n - 1)\lambda_1(u - Y_0 - \mu_1)\mathbf{1}\{Y_0 < u - \mu_1\}\}] \\
&= \mathbf{E}\left[\exp\left\{(-\lambda_1 u + \lambda_1\mu_1 + \lambda_1\gamma + \lambda_1 \exp\{\mu + \sigma G_{0,1}\} + \gamma)(n - 1)\mathbf{1}\left\{G_{0,1} < \frac{\ln(u - \mu_1 - \gamma) - \mu}{\sigma}\right\}\right\}\right].
\end{aligned} \tag{6.52}$$

Having thus calculated the numerator and the denominator in equations (6.51) and (6.52), we obtain the conditional probability $\mathbf{P}[Z_{n,n} \le u \mid Z_{n,1} > u, \dots, Z_{n,n-1} > u]$ and establish equations (6.47) and (6.48). This completes the proof of Theorem 6.5.4. □

From Theorem 6.5.4 and the formula for the (unconditional) probability of the buyer purchasing a property given by equation (6.9), we immediately get

$$\mathbf{P}[H = 1] = n_1 + n_2 + n_3,$$

where

$$n_1 = (1 - \mathbf{E}[h^*(G_{0,1})])(1 - \mathbf{P}[N = 0]),$$

$$n_2 = \Big(1 - \frac{\mathbf{E}[(h^*(G_{0,1}))^2]}{\mathbf{E}[h^*(G_{0,1})]}\Big)\mathbf{E}[h^*(G_{0,1})](1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1]),$$

and

$$n_3 = \sum_{j=3}^{\infty} \mathbf{E}[h^*(G_{0,1})]\Big(1 - \frac{\mathbf{E}[(h^*(G_{0,1}))^j]}{\mathbf{E}[(h^*(G_{0,1}))^{j-1}]}\Big)\prod_{k=2}^{j-1} \frac{\mathbf{E}[(h^*(G_{0,1}))^k]}{\mathbf{E}[(h^*(G_{0,1}))^{k-1}]}\mathbf{P}[N \geq j].$$

6.5.2 Multiplicative background price

Under the multiplicative background risk, the negotiated selling price $Z_{n,i}$ is given in equation (6.33). We next obtain an expression for $\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$.

Theorem 6.5.5 *Assume that the multiplicative background risk is present. Furthermore, assume that the distribution of each Y_i is the same as that of the random variable Y. Then we have*

$$\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = 1 - \frac{\mathbf{E}_{Y_0}[\prod_{i=1}^{n} \bar{F}_Y(u_{n,i}/Y_0)]}{\mathbf{E}_{Y_0}[\prod_{i=1}^{n-1} \bar{F}_Y(u_{n,i}/Y_0)]}. \tag{6.53}$$

Proof Recall the expression for the conditional probability $\mathbf{P}[Z_{n,j} \leq u_{n,j} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,j-1} > u_{n,j-1}]$ given in equation (6.24). Hence, we need to calculate the two terms $\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}]$ and $\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$ on the right-hand side of equation (6.24). We start with the first term, which can be written as follows:

$$\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}] = \mathbf{E}[\mathbf{1}\{Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}\}].$$

Since $Z_{n,i} = Y_0 Y_i$ for $i = 1, 2, \ldots, n$, we have

$$\mathbf{P}[Z_{n,1} > u_{n,1}, \ldots, Z_{n,n} > u_{n,n}] = \mathbf{E}[\mathbf{1}\{Y_0 Y_1 > u_{n,1}, \ldots, Y_0 Y_n > u_{n,n}\}].$$

Using the law of iterated expectations, we obtain

$$\begin{aligned}
\mathbf{P}[Z_{n,1} > u_{n,1}, \dots, Z_{n,n} > u_{n,n}] &= \mathbf{E}_{Y_0}[\mathbf{E}_Y[\mathbf{1}\{Y_0Y_1, \dots, Y_0Y_n > u_{n,n}\} \mid Y_0]] \\
&= \mathbf{E}_{Y_0}\Big[\mathbf{E}_Y\Big[\mathbf{1}\Big\{Y_1 > \frac{u_{n,1}}{Y_0}, \dots, Y_n > \frac{u_{n,n}}{Y_0}\Big\}\Big]\Big] \\
&= \mathbf{E}_{Y_0}\Big[\prod_{i=1}^{n} \bar{F}_Y\Big(\frac{u_{n,i}}{Y_0}\Big)\Big]. \qquad (6.54)
\end{aligned}$$

Similarly, the second term on the right-hand side of equation (6.24) can be written as follows:

$$\mathbf{P}[Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}] = \mathbf{E}_{Y_0}\Big[\prod_{i=1}^{n-1} \bar{F}_Y\Big(\frac{u_{n,i}}{Y_0}\Big)\Big]. \qquad (6.55)$$

Using equations (6.54) and (6.55), we calculate the conditional probability $\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}]$ and arrive at equation (6.53). This completes the proof of Theorem 6.5.5. □

Next, we assume that the stand-alone prices Y_i follow the two-parameter exponential distribution with parameters $\lambda_1 > 0$ and $\mu_1 \geq 0$. Furthermore, we assume that the reservation price stays the same, that is, let $u_{n,1} = u_{n,2} = \cdots = u_{n,n} = u$. Then the conditional probability becomes

$$\begin{aligned}
&\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}] \\
&\qquad = 1 - \frac{\mathbf{E}_{Y_0}[\exp\{-\lambda_1 n(u/Y_0 - \mu_1)\mathbf{1}\{u/Y_0 > \mu_1\}\}]}{\mathbf{E}_{Y_0}[\exp\{-\lambda_1 (n-1)(u/Y_0 - \mu_1)\mathbf{1}\{u/Y_0 > \mu_1\}\}]}. \qquad (6.56)
\end{aligned}$$

Using this formula, we obtain expressions for $\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}]$ and $\mathbf{P}[H = 1]$ when Y_0 follows the two-parameter exponential distribution, the uniform distribution, and the log-normal distribution. Throughout the following consideration of the three distributions, we assume that conditions of Theorem 6.5.5 are satisfied.

Exponential background prices

Theorem 6.5.6 *Let the stand-alone prices follow the two-parameter exponential distribution with parameters $\lambda_1 > 0$ and $\mu_1 \geq 0$. Furthermore, let Y_0 follow the two-parameter exponential distribution with parameters $\lambda_0 > 0$ and $\mu_0 \geq 0$. Then we have the formula*

$$\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}] = 1 - \frac{g_1(n)}{g_1(n-1)}, \qquad (6.57)$$

where, for $j = n - 1$ and n,

$$g_1(j) = \mathbf{1}\{u/\mu_1 \leq \mu_0\} + \mathbf{1}\{u/\mu_1 > \mu_0\} \int_{\mu_0}^{u/\mu_1} e^{-\lambda_1 j((u/y)-\mu_1)} \lambda_0 e^{-\lambda_0(y-\mu_0)} \mathrm{d}y. \tag{6.58}$$

Proof Recall the conditional probability given in equation (6.56), which is

$$\begin{aligned}\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]& \\ = 1 - \frac{\mathbf{E}_{Y_0}[\exp\{-\lambda_1 n(u/Y_0 - \mu_1)\mathbf{1}\{u/Y_0 > \mu_1\}\}]}{\mathbf{E}_{Y_0}[\exp\{-\lambda_1 (n-1)(u/Y_0 - \mu_1)\mathbf{1}\{u/Y_0 > \mu_1\}\}]}&.\end{aligned}$$

Next, we calculate the numerator and the denominator on the right-hand side of this equation. We begin with the numerator. When Y_0 follows the two-parameter exponential distribution, we have

$$\begin{aligned}&\mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1 n\Big(\frac{u}{Y_0} - \mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0} > \mu_1\Big\}\Big\}\Big] \\ &\quad = \mathbf{1}\Big\{\frac{u}{\mu_1} \leq \mu_0\Big\} + \mathbf{1}\Big\{\frac{u}{\mu_1} > \mu_0\Big\} \int_{\mu_0}^{u/\mu_1} \exp\Big\{-\lambda_1 n\Big(\frac{u}{y} - \mu_1\Big)\Big\}\lambda_0 \exp\{-\lambda_0(y-\mu_0)\}\mathrm{d}y. \quad (6.59)\end{aligned}$$

The denominator can be written as follows:

$$\begin{aligned}&\mathbf{E}\Big[\exp\Big\{-\lambda_1(n-1)\Big(\frac{u}{Y_0} - \mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0} > \mu_1\Big\}\Big\}\Big] \\ &= \mathbf{1}\Big\{\frac{u}{\mu_1} \leq \mu_0\Big\} + \mathbf{1}\Big\{\frac{u}{\mu_1} > \mu_0\Big\} \int_{\mu_0}^{u/\mu_1} \exp\Big\{-\lambda_1(n-1)\Big(\frac{u}{y} - \mu_1\Big)\Big\}\lambda_0 \exp\{-\lambda_0(y-\mu_0)\}\mathrm{d}y. \quad (6.60)\end{aligned}$$

Having thus calculated the numerator and the denominator in equations (6.59) and (6.60), we calculate the conditional probability $\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$ and obtain equations (6.57) and (6.58). This completes the proof of Theorem 6.5.6. □

From Theorem 6.5.6 and the formula for the (unconditional) probability of the buyer purchasing a property given in equation (6.9), we immediately get

$$\begin{aligned}\mathbf{P}[H = 1] &= (1 - g_1(1))(1 - \mathbf{P}[N = 0]) + \Big(1 - \frac{g_1(2)}{g_1(1)}\Big)g_1(1)(1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1]) \\ &\quad + \sum_{j=3}^{\infty} g_1(1)\Big(1 - \frac{g_1(j)}{g_1(j-1)}\Big)\prod_{k=2}^{j-1} \frac{g_1(k)}{g_1(k-1)}\mathbf{P}[N \geq j].\end{aligned}$$

Uniform background prices

Theorem 6.5.7 *Let the stand-alone prices follow the two-parameter exponential distribution with parameters* $\lambda_1 > 0$ *and* $\mu_1 \geq 0$*. Furthermore, let* Y_0 *follow the uniform distribution* (a, b)*. Then we have the formula*

$$\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = 1 - \frac{g_2(n)}{g_2(n-1)}, \tag{6.61}$$

where, for $j = n$ *and* $n - 1$*,*

$$\begin{aligned} g_2(j) = \mathbf{1}\Big\{\frac{u}{\mu_1} \leq a\Big\} + \mathbf{1}\Big\{a < \frac{u}{\mu_1} \leq b\Big\}\frac{1}{b-a}e^{\lambda_1\mu_1 j}\int_a^{u/\mu_1} e^{-\lambda_1 ju/y}\mathrm{d}y \\ + \mathbf{1}\Big\{\frac{u}{\mu_1} > b\Big\}\frac{1}{b-a}e^{\lambda_1\mu_1 j}\int_a^{b} e^{-\lambda_1 ju/y}\mathrm{d}y. \end{aligned} \tag{6.62}$$

Proof Similar to the proof of Theorem 6.5.6, we calculate the numerator and the denominator on the right-hand side of equation (6.56). We begin with the numerator. When Y_0 follows the uniform distribution, we have

$$\mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1 n\Big(\frac{u}{Y_0} - \mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0} > \mu_1\Big\}\Big\}\Big] = \int_a^b \exp\Big\{-\lambda_1\Big(\frac{u}{y} - \mu_1\Big)\mathbf{1}\Big\{y < \frac{u}{\mu_1}\Big\}\Big\}\frac{1}{b-a}\mathrm{d}y.$$

By considering the scenarios $u/\mu_1 \leq a$, $a < u/\mu_1 \leq b$, and $u/\mu_1 > b$, we obtain

$$\begin{aligned} &\mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1 n\Big(\frac{u}{Y_0} - \mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0} > \mu_1\Big\}\Big\}\Big] \\ &\quad = \mathbf{1}\Big\{\frac{u}{\mu_1} \leq a\Big\} + \mathbf{1}\Big\{a < \frac{u}{\mu_1} \leq b\Big\}\frac{1}{b-a}\exp\{\lambda_1\mu_1 n\}\int_a^{u/\mu_1} \exp\Big\{-\lambda_1\frac{nu}{y}\Big\}\mathrm{d}y \\ &\qquad + \mathbf{1}\Big\{\frac{u}{\mu_1} > b\Big\}\frac{1}{b-a}\exp\{\lambda_1\mu_1 n\}\int_a^{b} \exp\Big\{-\lambda_1\frac{nu}{y}\Big\}\mathrm{d}y. \end{aligned} \tag{6.63}$$

Similarly, the denominator can be written as follows:

$$\begin{aligned} &\mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1(n-1)\Big(\frac{u}{Y_0} - \mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0} > \mu_1\Big\}\Big\}\Big] \\ &\quad = \mathbf{1}\Big\{\frac{u}{\mu_1} \leq a\Big\} + \mathbf{1}\Big\{a < \frac{u}{\mu_1} \leq b\Big\}\frac{1}{b-a}e^{\lambda_1\mu_1(n-1)}\int_a^{u/\mu_1} e^{-\lambda_1(n-1)u/y}\mathrm{d}y \\ &\qquad + \mathbf{1}\Big\{\frac{u}{\mu_1} > b\Big\}\frac{1}{b-a}e^{\lambda_1\mu_1(n-1)}\int_a^{b} e^{-\lambda_1(n-1)u/y}\mathrm{d}y. \end{aligned} \tag{6.64}$$

Using these results for the numerator and the denominator in equations (6.63) and (6.64), we calculate the conditional probability $\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}]$ and obtain equations (6.61) and (6.62). This completes the proof of Theorem 6.5.7. □

From Theorem 6.5.7 and the formula for the (unconditional) probability of the buyer purchasing a property given in equation (6.9), we immediately get

$$\begin{aligned}\mathbf{P}[H = 1] = {} & (1 - g_2(1))(1 - \mathbf{P}[N = 0]) \\ & + \Big(1 - \frac{g_2(2)}{g_2(1)}\Big) g_2(1)(1 - \mathbf{P}[N = 0] - \mathbf{P}[N = 1]) \\ & + \sum_{j=3}^{\infty} g_2(1)\Big(1 - \frac{g_2(j)}{g_2(j-1)}\Big) \prod_{k=2}^{j-1} \frac{g_2(k)}{g_2(k-1)} \mathbf{P}[N \geq j].\end{aligned}$$

Log-normal background prices

Theorem 6.5.8 *Let the stand-alone prices follow the two-parameter exponential distribution with parameters $\lambda_1 > 0$ and $\mu_1 \geq 0$. Furthermore, let Y_0 follow the three-parameter log-normal distribution with parameters $\mu \in \mathbb{R}$, $\sigma > 0$, and $\gamma \geq 0$. Then the conditional probability given in equation (6.56) becomes*

$$\mathbf{P}[Z_{n,n} \leq u_{n,n} \mid Z_{n,1} > u_{n,1}, \ldots, Z_{n,n-1} > u_{n,n-1}] = 1 - \frac{\mathbf{E}[(g^*(G_{0,1}))^n]}{\mathbf{E}[(g^*(G_{0,1}))^{n-1}]}, \tag{6.65}$$

where $G_{0,1}$ follows the standard normal distribution and the function $g^*(x)$ is given by

$$g^*(x) = \exp\Big\{\lambda_1\Big(\frac{u}{\exp\{\mu + \sigma x\} + \gamma} - \mu_1\Big)\mathbf{1}\Big\{x < \frac{\ln(u/\mu_1 - \gamma) - \mu}{\sigma}\Big\}\Big\}. \tag{6.66}$$

Proof Similar to the proof of Theorem 6.5.6, we calculate the numerator and the denominator on the right-hand side of equation (6.56). We begin with the numerator:

$$\mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1 n\Big(\frac{u}{Y_0} - \mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0} > \mu_1\Big\}\Big\}\Big] = \mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1\Big(\frac{u}{Y_0} - \mu_1\Big) n \mathbf{1}\Big\{Y_0 < \frac{u}{\mu_1}\Big\}\Big\}\Big]. \tag{6.67}$$

The background price is given by

$$Y_0 = \exp\{\mu + \sigma G_{0,1}\} + \gamma. \tag{6.68}$$

By substituting equation (6.68) into equation (6.67), we obtain

$$\begin{aligned}&\mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1 n\Big(\frac{u}{Y_0}-\mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0}>\mu_1\Big\}\Big\}\Big]\\&\qquad=\mathbf{E}\Big[\exp\Big\{-\lambda_1\Big(\frac{u}{\exp\{\mu+\sigma G_{0,1}\}+\gamma}-\mu_1\Big)n\mathbf{1}\Big\{G_{0,1}<\frac{\ln(u/\mu_1-\gamma-\mu)}{\sigma}\Big\}\Big\}\Big].\end{aligned}\tag{6.69}$$

Similarly, the denominator can be written as follows:

$$\begin{aligned}&\mathbf{E}_{Y_0}\Big[\exp\Big\{-\lambda_1(n-1)\Big(\frac{u}{Y_0}-\mu_1\Big)\mathbf{1}\Big\{\frac{u}{Y_0}>\mu_1\Big\}\Big\}\Big]\\&\qquad=\mathbf{E}\Big[\exp\Big\{-\lambda_1\Big(\frac{u}{\exp\{\mu+\sigma G_{0,1}\}+\gamma}-\mu_1\Big)(n-1)\mathbf{1}\Big\{G_{0,1}<\frac{\ln(u/\mu_1-\gamma)-\mu}{\sigma}\Big\}\Big\}\Big].\end{aligned}\tag{6.70}$$

Using the above results for the numerator and the denominator, we calculate the conditional probability $\mathbf{P}[Z_{n,n} \le u_{n,n} \mid Z_{n,1} > u_{n,1}, \dots, Z_{n,n-1} > u_{n,n-1}]$ and arrive at equations (6.65) and (6.66). This completes the proof of Theorem 6.5.8. □

From Theorem 6.5.8 and the formula for the (unconditional) probability of the buyer purchasing a property given in equation (6.9), we immediately get

$$\begin{aligned}\mathbf{P}[H=1]&=(1-\mathbf{E}[g^*(G_{0,1})])(1-\mathbf{P}[N=0])\\&\quad+\Big(1-\frac{\mathbf{E}[(g^*(G_{0,1}))^2]}{\mathbf{E}[g^*(G_{0,1})]}\Big)\mathbf{E}[g^*(G_{0,1})](1-\mathbf{P}[N=0]-\mathbf{P}[N=1])\\&\quad+\sum_{j=3}^{\infty}\mathbf{E}[g^*(G_{0,1})]\Big(1-\frac{\mathbf{E}[(g^*(G_{0,1}))^j]}{\mathbf{E}[(g^*(G_{0,1}))^{j-1}]}\Big)\prod_{k=2}^{j-1}\frac{\mathbf{E}[(g^*(G_{0,1}))^k]}{\mathbf{E}[(g^*(G_{0,1}))^{k-1}]}\mathbf{P}[N\ge j].\end{aligned}$$

Note that when the negotiated selling prices are independent of the number of properties on sale, the probability $\mathbf{P}[H = 1]$ of the buyer purchasing one property is written in terms of $\mathbf{P}[N \ge j]$ for $j = 1, 2, \dots$. Hence, we must obtain the tail probability $\mathbf{P}[N \ge j]$ and thus next review literature studies related to finding the bounds and direct representations of the tail probabilities for discrete distributions.

6.6 Literature review related to bounds on tail probabilities for discrete distributions

Various methods for obtaining the bounds and direct representations of the tail probabilities for discrete distributions have been reported in the literature. Approaches based on the Markov inequality include: the Chernoff bound (Chernoff, 1952), the moment bound (Philips and Nelson, 1995), the factorial moment bound (Naveau, 1997), and the fractional moments (Goria and Tagliani, 2003).

In addition to Markov inequality based techniques, Peizer and Pratt (1968), Pratt (1968), Glynn (1987), and Fox and Glynn (1988) have considered the normal approximation method. Gideon and Gurland (1971) have used a weighted sum of exponential distributions to estimate the tail probability. Andrews (1973) has introduced the Andrews approximation. Ross (1998) and Klar (2000) have discussed the importance sampling identity approach.

Next, we present the bounds and direct representations of the tail probabilities for the Poisson, negative binomial, binomial, log-series, and zeta distributions that we have found in the literature.

Tail probability of the Poisson distribution

When a random variable N follows the Poisson distribution with mean $\lambda > 0$, then the probability mass function of N is given by

$$\mathbf{P}[N = n] = e^{-\lambda}\frac{\lambda^n}{n!} \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Gideon and Gurland (1971) have used the chi-squared distribution to obtain

$$\mathbf{P}[N \geq n] \approx 1 - \mathbf{P}[\chi_v^2 > 2\lambda] + \mathbf{P}[N = n],$$

where $v = 2(n+1)$. Recall that a chi-squared distribution with k degrees of freedom, which we denote by χ_k^2, is the distribution of a sum of the squares of k independent standard normal random variables (cf., e.g., Timm, 2002). Furthermore, Glynn (1987) has obtained the following upper bound

$$\mathbf{P}[N \geq n] \leq \left(1 - \left(\frac{\lambda}{n+1}\right)^m\right)^{-1} \sum_{k=n}^{n+m-1} e^{-\lambda}\frac{\lambda^k}{k!},$$

where $n > \lambda - 1$ and $m \geq 1$. For more results on the tail probability of the Poisson distribution, we refer to Gross and Hosmer (1978), and Klar (2000).

Tail probability of the negative binomial distribution

When N follows the negative binomial distribution with parameters $r > 0$ and $\beta > 0$, then the probability mass function of N is given by

$$\mathbf{P}[N = n] = \binom{r+n-1}{n}\left(\frac{1}{1+\beta}\right)^{r}\left(\frac{\beta}{1+\beta}\right)^{n} \quad \text{for} \quad n = 0, 1, \ldots .$$

Best and Gipps (1974) have shown that

$$\mathbf{P}[N > n] \approx 1 - \int_{h}^{n+0.5+r(1-p_1)/(2-p_1)} g\left(y; \frac{4r(1-p_1)}{(2-p_1)^2}, \frac{2-p_1}{2p_1}\right)\mathrm{d}y,$$

where

$$p_1 = \frac{1}{1+\beta},$$

$$h = \max\left(0, \frac{r(1-p_1)}{2-p_1} - 0.5\right),$$

and

$$g(y; a, b) = \frac{1}{b^a\Gamma(a)} y^{a-1} e^{-y/b}$$

for $y > 0$, $a > 0$, and $b > 0$. Klar (2000) has proven that if $r > 1$ and $n \geq r(1-p_1)/p_1$, then

$$\frac{1}{p_1}\mathbf{P}[N = n] < \mathbf{P}[N \geq n] < \left(1 - \frac{n+r}{n+1}(1-p_1)\right)^{-1}\mathbf{P}[N = n],$$

and if $r < 1$ and $n \geq r(1-p_1)/p_1$, then

$$\left(1 - \frac{n+r}{n+1}(1-p_1)\right)^{-1}\mathbf{P}[N = n] < \mathbf{P}[N \geq n] < \frac{1}{p_1}\mathbf{P}[N = n].$$

Tail probability of the binomial distribution

When N follows a binomial distribution with parameters $m = 1, 2, \ldots$ and $0 \leq p \leq 1$, then the probability mass function of N is given by

$$\mathbf{P}[N = n] = \binom{m}{n} p^n (1-p)^{m-n} \quad \text{for} \quad n = 0, 1, \ldots, m.$$

Klar (2000) has shown that for $mp \leq n \leq m$, the lower and upper bounds of $\mathbf{P}[N \geq n]$ are as follows:

$$\binom{m}{n} p^n (1-p)^{m-n} \leq \mathbf{P}[N \geq n] \leq \frac{(n+1)(1-p)}{n+1-(m+1)p} \binom{m}{n} p^n (1-p)^{m-n}.$$

For more on the tail probability of the binomial distribution, we refer to Gross and Hosmer (1978).

Tail probability of the log-series distribution

When N follows the log-series distribution with parameter $0 \leq p < 1$, then the probability mass function of N is given by

$$\mathbf{P}[N = n] = \frac{-1}{\ln(1-p)} \frac{p^n}{n} \quad \text{for} \quad n = 1, 2, \ldots.$$

Klar (2000) has presented the following lower and upper bounds of $\mathbf{P}[N \geq n]$:

$$\left(1 - \frac{np}{n+1}\right)^{-1} \frac{-1}{\ln(1-p)} \frac{p^n}{n} < \mathbf{P}[N \geq n] < (1-p)^{-1} \frac{-1}{\ln(1-p)} \frac{p^n}{n}.$$

Tail probability of the zeta distribution

When N follows the zeta distribution with parameter $s > 1$, then the probability mass function of N is given by

$$\mathbf{P}[N = n] = \frac{n^{-s}}{\zeta_1(s)} \quad \text{for} \quad n = 1, 2, \ldots,$$

where $\zeta_1(s) = \sum_{k=1}^{\infty} 1/k^s$. Klar (2000) has obtained the following lower and upper bounds of $\mathbf{P}[N \geq n]$:

$$\frac{n}{s-1} \frac{n^{-s}}{\zeta_1(s)} < \mathbf{P}[N \geq n] < \left(\frac{n}{n-1}\right)^{s-1} \frac{n}{s-1} \frac{n^{-s}}{\zeta_1(s)}$$

for $n \geq 2$.

6.7 An illustrative example

We now calculate the (unconditional) probability of purchasing a condominium or a detached property in the London and St. Thomas area under assumptions made in Section 6.2. The data, which are reported in the London and St. Thomas Association of Realtors statistical report from the year 2012, includes the average prices of condominiums and detached properties for each month in the year 2012, which we reproduce in Table 6.2. The average number of condominiums and detached properties sold, and their variances for each month from the year 2003 to 2012 are reproduced in Table 6.6.

Note from equation (6.2) that to calculate $\mathbf{P}[H = 1]$, we first need to obtain ζ, which is the CDF of each $Z_{n,i}$, and then $G(1 - \zeta(u))$, which is the probability generating function of N. In other words, we need to find parameter values for the distributions of $Z_{n,i}$ and N.

The rest of this section is organized as follows: In Subsection 6.7.1, we obtain expressions for ζ when the negotiated selling prices follow the exponential, uniform, and log-normal distributions. We then estimate the parameter values of these distributions for condominiums and detached properties sold in May and December of 2012. In Subsection 6.7.2, we find a suitable distribution to model the number of condominiums and detached properties available on the market. In Subsection 6.7.3, we present plots of the (unconditional) probabilities of purchasing condominiums and detached properties in the months of May and December of 2012, for various negotiated selling price distributions.

6.7.1 Parameter estimates of the negotiated selling price distributions

Here we consider three distributions for the negotiated selling prices: the exponential, uniform, and log-normal distributions. When negotiated selling prices follow the two-parameter exponential distribution with parameters $\lambda > 0$ and u_0, then we have

$$\zeta(u) = 1 - \exp\{-\lambda(u - u_0)\} \quad \text{for} \quad u > u_0 \geq 0.$$

When negotiated selling prices follow the uniform distribution (u_0, u_1), for $0 \leq u_0 < u < u_1$, we have

$$\zeta(u) = \frac{u - u_0}{u_1 - u_0}.$$

When negotiated selling prices follow the three-parameter log-normal distribution with parameters $\mu_1 \in \mathbb{R}$, $\sigma > 0$, and $u_0 \geq 0$, then we have

$$\zeta(u) = \Phi\left(\frac{\ln(u - u_0) - \mu_1}{\sigma}\right) \quad \text{for} \quad u > u_0.$$

Next, we estimate parameter values of the three selling price distributions for condominiums and detached properties sold in the months of May and December of 2012. The parameter estimates are based on the average prices of condominiums and detached properties in the London and St. Thomas area for each month in the year 2012, which are presented in Table 6.2. The sample data are taken from London St. Thomas Association of Realtors statistical report from the year 2012.

Month	Condominiums	Detached properties
January	$169,069$	$236,101$
February	$169,897$	$252,523$
March	$170,193$	$251,533$
April	$166,717$	$266,457$
May	$169,358$	$265,756$
June	$180,749$	$252,451$
July	$176,594$	$260,172$
August	$174,475$	$242,042$
September	$160,869$	$246,376$
October	$182,223$	$257,302$
November	$164,593$	$252,259$
December	$168,304$	$254,740$

Table 6.2: Average prices of condominiums and detached properties.

Parameter estimates for condominiums sold in May of 2012

From Table 6.2, we see that the average price for condominiums sold in May is \$169,358. The lowest average price is \$160,869 in September. To estimate parameter values of the negotiated selling price distribution for condominiums sold in May, we equate the mean of the selling price distribution to \$169,360. This value is obtained by rounding up from the average price \$169,358 in May. We also equate the minimum negotiated selling price to \$150,000, which is \$10869 less than the lowest average price of \$160,869. Next, we obtain parameter estimates for the two-parameter exponential, uniform, and three-parameter log-normal distributions.

Assume that $Z_{n,i}$ follows the two-parameter exponential distribution with parameters λ and u_0, where u_0 is the minimum negotiated selling price for condominiums sold in May, and thus

$$u_0 = 150,000. \tag{6.71}$$

Since the mean is \$169,360, we have $u_0 + (1/\lambda) = 169,360$. Hence, we immediately get

$$\lambda = \frac{1}{19360}.$$

Assume that $Z_{n,i}$ follows the uniform distribution (u_0, u_1), where u_0 is the minimum negotiated selling price for condominiums sold in the month of May, and thus

$$u_0 = 150,000. \tag{6.72}$$

Since the mean of the uniform distribution is \$169,360, we have $(u_0+u_1)/2 = 169,360$. Hence, we immediately get

$$u_1 = 188,720.$$

Assume that $Z_{n,i}$ follows the three-parameter log-normal distribution with parameters u_0, μ_1, and σ, where u_0 is the minimum negotiated selling price for condominiums in the month of May, and thus

$$u_0 = 150,000. \tag{6.73}$$

The mean of the three-parameter log-normal distribution is \$169,360. Assume that the standard deviation of the log-normal distribution is the same as that of the average price of condominiums from January to December, which is \$6338. Then $u_0 + e^{\mu_1+\sigma^2/2} = 169,360$ and $e^{2\mu_1+\sigma^2}(e^{\sigma^2} - 1) = 6338^2$. Hence, we immediately get

$$\mu_1 = 9.8 \quad \text{and} \quad \sigma = 0.35.$$

Parameter estimates for condominiums sold in December of 2012

From Table 6.2, we see that the average price of condominiums sold in December is \$168,304. To estimate parameter values of the negotiated selling price distribution for condominiums sold in December, we equate the mean of the selling price distribution to \$168,300 (rounding down from \$168,304). The minimum negotiated selling price remains at \$150,000. Similar to the parameter estimates for condominiums sold in May, the following results are obtained for condominiums sold in December:

- When negotiated selling prices follow the two-parameter exponential distribution, then the parameter estimates are $u_0 = 150,000$ and $\lambda = 1/18300$.

- When negotiated selling prices follow the uniform distribution, then the parameter estimates are $u_0 = 150,000$ and $u_1 = 186,600$.

- When negotiated selling prices follow the three-parameter log-normal distribution, then the parameter estimates are $u_0 = 150,000$, $\mu_1 = 9.7$, and $\sigma = 0.37$.

Parameter estimates for detached properties sold in May of 2012

From Table 6.2, we see that the average price for detached properties sold in May is \$265,756, and the lowest average price of detached properties is \$242,042 in August. To estimate parameter values of the selling price distribution for detached properties sold in May, we equate the mean of the selling price distribution to \$265,800 (rounding up from \$265,756). We also equate the minimum selling price to \$230,000, which is \$12042 less than the minimum average price of \$242,042. Similar to the parameter estimates for condominiums sold in May, the following results are obtained for detached properties sold in May:

- When negotiated selling prices follow the two-parameter exponential distribution, then the parameter estimates are $u_0 = 230,000$ and $\lambda = 1/35800$.

- When negotiated selling prices follow the uniform distribution, then the parameter estimates are $u_0 = 230,000$ and $u_1 = 301,600$.

- When negotiated selling prices follow the three-parameter log-normal distribution, then the parameter estimates are $u_0 = 230,000$, $\mu_1 = 10.5$, and $\sigma = 0.25$.

Parameter estimates for detached properties sold in December of 2012

From Table 6.2, we see that the average price of detached properties sold is \$254,740 in December. To estimate parameter values of the negotiated selling price distribution for detached properties sold in December, we equate the mean of the negotiated selling price distribution to \$254,700 (rounding down from \$254,740). The minimum negotiated selling price remains at \$230,000. Similar to the parameter estimates for condominiums sold in May, the following results are obtained for detached properties sold in December:

- When negotiated selling prices follow the two-parameter exponential distribution, then the parameter estimates are $u_0 = 230,000$ and $\lambda = 1/24700$.

- When negotiated selling prices follow the uniform distribution, then the parameter estimates are $u_0 = 230,000$ and $u_1 = 279,400$.

- When negotiated selling prices follow the three-parameter log-normal distribution, then the parameter estimates are $u_0 = 230,000$, $\mu_1 = 10.1$, and $\sigma = 0.35$.

Results are presented in Tables 6.3, 6.4, and 6.5.

	Condominiums		Detached properties	
Month	u_0	λ	u_0	λ
May	$150,000$	$1/19360$	$230,000$	$1/35800$
December	$150,000$	$1/18300$	$230,000$	$1/24700$

Table 6.3: Parameter values of the two-parameter exponential distribution.

	Condominiums		Detached properties	
Month	u_0	u_1	u_0	u_1
May	$150,000$	$188,720$	$230,000$	$301,600$
December	$150,000$	$186,600$	$230,000$	$279,400$

Table 6.4: Parameter values of the uniform distribution.

	Condominiums			Detached properties		
Month	u_0	μ_1	σ	u_0	μ_1	σ
May	$150,000$	9.8	0.35	$230,000$	10.5	0.25
December	$150,000$	9.7	0.37	$230,000$	10.1	0.35

Table 6.5: Parameter values of the log-normal distribution.

6.7.2 Finding the distribution of N

In Table 6.6, we have recorded the average number of condominiums and detached properties sold, as well as their variances for all months from the year 2003 to 2012. Although the number of properties sold is not the same as the number of properties available on the market, this is the most relevant information we have managed to gather over the 10-year period. Note that there is a significant difference between the mean and the variance of properties sold each month. Hence, it is not appropriate to assume that N follows the Poisson distribution. We shall next find a more suitable discrete distribution for the random variable of the number N of properties available on the market. We consider five discrete distributions: binomial, log-series, zeta, hypergeometric, and negative binomial.

	Condominiums		Detached	
Month	Average	Variance	Average	Variance
January	116	8682	337	9698
February	132	280	490	2765
March	157	645	669	6243
April	170	420	706	3656
May	191	673	762	5592
June	180	401	733	2965
July	169	1047	673	4795
August	155	554	612	3650
September	131	300	549	2152
October	121	357	518	1322
November	106	259	467	4738
December	82	68	300	1253

Table 6.6: The averages and variances of sold properties for each month from the year 2003 to 2012 (numbers are rounded down to the nearest integer).

The binomial distribution

When N follows the binomial distribution with parameters $n = 1, 2, \ldots$ and $0 \leq p \leq 1$, then the expectation and the variance of N are given by

$$\mathbf{E}[N] = np \quad \text{and} \quad \mathbf{Var}[N] = np(1-p).$$

Note that $\mathbf{E}[N] \geq \mathbf{Var}[N]$ always holds true. This contradicts results in Table 6.6, except for condominiums sold in December. We now equate $\mathbf{E}[N]$ and $\mathbf{Var}[N]$ to the mean and the variance of the number of condominiums sold in December. Hence, $np = 82$, and $np(1-p) = 68$. Consequently,

$$n = 480 \quad \text{and} \quad p = 0.171.$$

The log-series distribution

When N follows the log-series distribution with parameter $0 \leq p < 1$, then the expectation and the variance of N are given by

$$\mathbf{E}[N] = \frac{-1}{\ln(1-p)} \frac{p}{1-p}$$

and

$$\mathbf{Var}[N] = -p \frac{p + \ln(1-p)}{(1-p)^2 (\ln(1-p))^2}.$$

If we equate $\mathbf{E}[N]$ and $\mathbf{Var}[N]$ to the mean and the variance of the number of condominiums and detached properties sold in each month, no solution can be obtained. Therefore, we conclude that the log-series distribution is not suitable for N.

The zeta distribution

When N follows the zeta distribution with parameter $s > 1$, then the expectation and the variance of N are given by

$$\mathbf{E}[N] = \frac{\zeta_1(s-1)}{\zeta_1(s)} \quad \text{for} \quad s > 2$$

and

$$\mathbf{Var}[N] = \frac{\zeta_1(s)\zeta_1(s-2) - (\zeta_1(s-1))^2}{(\zeta_1(s))^2} \quad \text{for} \quad s > 3,$$

where

$$\zeta_1(s) = \sum_{n=1}^{\infty} 1/n^s.$$

Standard packages are available to compute $\zeta_1(s)$. For the sake of simplicity, we consider other discrete distributions.

The hypergeometric distribution

When N follows the hypergeometric distribution with parameters n ($n = 0, 1, \ldots, M$), M ($M = 0, 1, \ldots$), and K ($K = 0, 1, \ldots, M$), such that $\max(0, n + K - M) \leq k \leq \min(K, n)$, then the probability mass function, the expectation, and the variance of N are given by

$$\mathbf{P}[N = k] = \frac{\binom{K}{k}\binom{M-K}{n-k}}{\binom{M}{n}},$$

$$\mathbf{E}[N] = n\frac{K}{M},$$

$$\mathbf{Var}[N] = n\frac{K}{M}\frac{M-K}{M}\frac{M-n}{M-1}.$$

We note that $\mathbf{E}[N] \geq \mathbf{Var}[N]$ always holds true. The conclusion is similar to that of the binomial distribution, that is, the hypergeometric distribution is not suitable for modelling the distribution of N, except for the number of condominiums available in December.

The negative binomial distribution

When N follows a negative binomial distribution with parameters $r > 0$ and $\beta > 0$, then the expectation and the variance of N are given by

$$\mathbf{E}[N] = r\beta \quad \text{and} \quad \mathbf{Var}[N] = r\beta(1+\beta).$$

Next, we equate $\mathbf{E}[N]$ and $\mathbf{Var}[N]$ to the mean and the variance of the number of condominiums and detached properties sold in each month. For condominiums sold in January, we have

$$r\beta = 116 \quad \text{and} \quad r\beta(1+\beta) = 8682.$$

Consequently,

$$r = 1.571 \quad \text{and} \quad \beta = 73.845.$$

Similarly, we obtain parameter estimates for r and β for the number of condominiums and detached properties available in each month. Results are presented in Table 6.7. We see that

	Condominiums		Detached	
Month	r	β	r	β
January	1.571	73.845	12.132	27.777
February	117.730	1.121	105.538	4.643
March	50.510	3.108	80.294	8.332
April	115.600	1.470	168.961	4.178
May	75.687	2.524	120.216	6.338
June	146.606	1.228	240.721	3.045
July	32.530	5.195	109.881	6.125
August	60.213	2.574	123.286	4.964
September	101.544	1.290	188.023	2.920
October	62.038	1.950	333.736	1.552
November	73.438	1.443	51.063	9.146
December	−480.286	−0.171	94.439	3.177

Table 6.7: Negative binomial distribution with parameters r and β.

the negative binomial distribution is a reasonable choice for the distribution of N, except for condominiums available in December. Hence, we use the binomial distribution to model the number of condominiums available in December.

6.7.3 Plots of the probabilities of purchasing a property

Here we plot the probabilities of a buyer purchasing a condominium or a detached property in May and December. For the negotiated selling price distributions, we use results of Tables 6.3, 6.4, and 6.5. For the distribution of N, our assumptions are as follows:

- The number of condominiums available on the market in May follows the negative binomial distribution with parameters $r = 75.687$ and $\beta = 2.524$.
- The number of condominiums available on the market in December follows the binomial distribution with parameters $n = 480$ and $p = 0.171$.
- The number of detached properties available on the market in May follows the negative binomial distribution with parameters $r = 120.216$ and $\beta = 6.338$.
- The number of detached properties available on the market in December follows the negative binomial distribution with parameters $r = 94.439$ and $\beta = 3.177$.

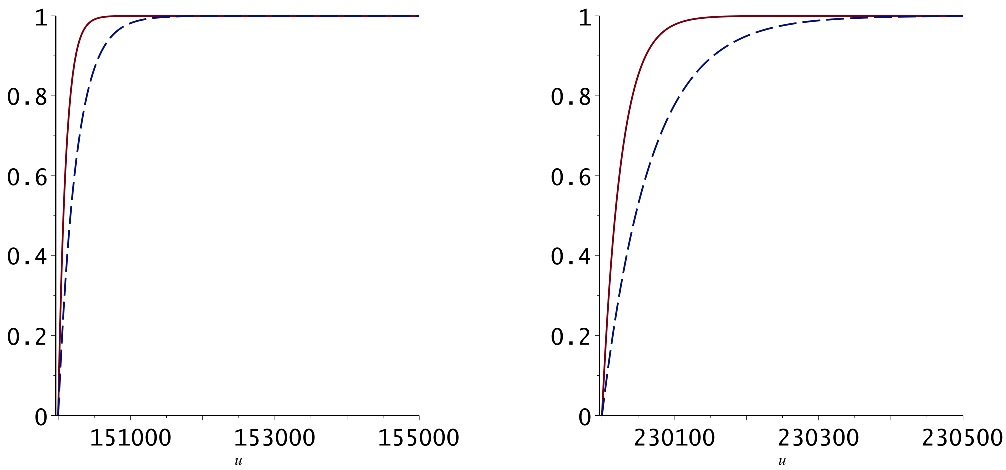

Figure 6.1: The probabilities of purchasing condominiums (left panel) and detached properties (right panel) in May (solid line) and December (dashed line) when negotiated selling prices follow the exponential distribution.

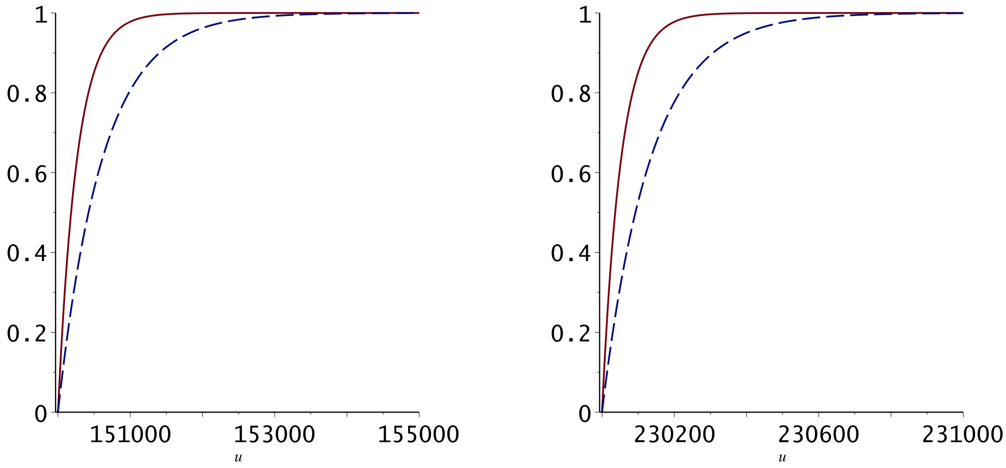

Figure 6.2: The probabilities of purchasing condominiums (left panel) and detached properties (right panel) in May (solid line) and December (dashed line) when negotiated selling prices follow the uniform distribution.

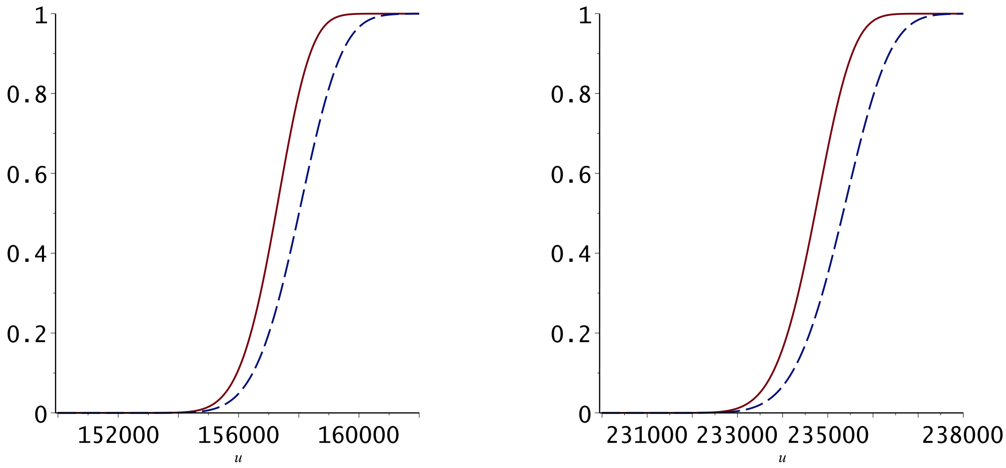

Figure 6.3: The probabilities of purchasing condominiums (left panel) and detached properties (right panel) in May (solid line) and December (dashed line) when negotiated selling prices follow the log-normal distribution.

Chapter 7

Concluding remarks and future work

7.1 Concluding remarks

When multiple parties are involved in the decision making process, the final outcome depends on everybody's decision. To illustrate how decisions might be made in such situations, we have considered two scenarios: one in insurance and another in real estate industries. Of course, numerous other scenarios can be considered, but each of them usually carries some specific features that cannot always be generalized easily or naturally.

Specifically, in the insurance industry, we have discussed two criteria for finding an optimal reinsurance policy that is beneficial to both the insurer and the reinsurer. The variance reduction approach was the first introduced criterion. We have shown that to maximize the variance reduction, the covariance(s) should be maximized. We have also demonstrated through a numerical example that maximizing the correlation coefficient with respect to the retention is not a suitable criterion.

Under facultative reinsurance, we have considered using the variance reduction approach to find optimal reinsurance in three cases. In the first case, conditions for the optimal retention have been obtained when the policy limit is already determined. In the second case, conditions for the optimal policy limit have been derived when the retention is already determined. In the final case, we have proposed a negotiation process when both the retention and the policy limit are still negotiable. Numerical results have been presented for the first thirteen rounds of negotiations.

Under treaty reinsurance, we have used the variance reduction approach to optimize the reinsurance policy in two cases. In the first case, the claim size of each insurance policy is independent of time. We have used four examples to illustrate how the optimal retention is affected by parameter values of the distribution of the number of claims that require a claim payment during a considered time interval. In the second case, claim sizes depend on preceding inter-claim times. By adopting the approach of Sendova and Zitikis (2012), we have obtained an explicit expression for the covariance of the insurer's and the reinsurer's shares of claims. Next, to illustrate how the optimal retention is affected by time, two numerical examples have been presented.

In addition to the variance reduction approach, we considered a method based on the CTE. A connection between these two methods has been established. Explicit expressions for the CTEs of the insurer's and the reinsurer's shares of the claim have also been presented. Furthermore, we have obtained the optimal retention using the CTE-based criterion when the total claim amount follows the exponential distribution and also the two-parameter Pareto distribution.

As to the illustrative scenario in the real estate industry, we have formulated the probability of the buyer purchasing one property under various scenarios. We started by assuming that the reservation price stays the same, and that the negotiated selling prices are i.i.d. One or more of these assumptions were then dropped as our research progressed.

When the assumption of independent negotiated selling prices was dropped, three methods for modelling the dependence structure among selling prices were discussed. Under the method of direct representation, joint survival functions of several multivariate distributions from past studies have been provided. Next, under the method of copula representation, we have used the multivariate Pareto distribution of the second kind as an example to show how the joint survival function can be obtained from the survival copula. Finally, the additive and multiplicative background risk models have been discussed in detail.

In an illustrative example for properties in the London and St. Thomas area, we have obtained parameter estimates for three negotiated selling price distributions. Furthermore, we have successfully determined distributions that are suitable for modelling the number of condominiums and detached properties available on the market. Finally, we have calculated the probability of the buyer purchasing a condominium or a detached property in two months: May and December.

Overall, this thesis proposes methodologies that facilitate decision making when multiple parties are involved in the process. Certain adjustments may be required to make the methodologies more attractive from the practical point of view. Nevertheless, the developed approaches are very general in nature and can be applied to various fields beyond insurance and real estate.

7.2 Future work

In the first illustrative scenario, we have proposed methodologies for optimizing reinsurance policies in the excess of loss form. We plan to apply the proposed methodologies to other forms of reinsurance policies (e.g., quota share reinsurance). Furthermore, when the variance reduction approach was applied under treaty reinsurance, we assumed independent claim sizes. We will extend our model to include dependent claim sizes. Finally, we only considered the CTE-based criterion under facultative reinsurance. We plan to also investigate the case when the method of treaty reinsurance is assumed.

In the second illustrative scenario, we have assumed only the buyer and the seller are involved in buying or negotiating selling properties. We plan to consider the involvement of agents. We can also extend our model to include other factors (e.g., inflation) that influence the negotiated selling prices. Furthermore, to incorporate the feature of seasonal trends, we plan to use the Poisson process with a variety of periodic and other intensity functions to model the number of properties available on the market (cf., e.g., Bebbington and Zitikis, 2004).

Finally, it is important to keep in mind that situations like the one just discussed about buying or selling arise in a variety of contexts, including of course insurance: indeed, a person may have a budget in mind and searches for an insurance policy to purchase. In short, the results of this thesis can be extended to a multitude of scenarios, well beyond the two illustrative ones considered in the thesis.

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