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Incomplete Market Models of Carbon Emissions Markets

Walid Mnif
The University of Western Ontario

Supervisor
Dr. Matt Davison
The University of Western Ontario

Graduate Program in Applied Mathematics
A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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(Thesis format: Integrated Article)

by

Walid Mnif

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A thesis submitted in partial fulfillment
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The School of Graduate and Postdoctoral Studies
The University of Western Ontario
London, Ontario, Canada

THE UNIVERSITY OF WESTERN ONTARIO
School of Graduate and Postdoctoral Studies

CERTIFICATE OF EXAMINATION

Supervisor:

. . . .
Dr. Matt Davison

Supervisory Committee:

. . . .
Dr. Mark Reesor

. . . .
Dr. Adam Metzler

Examiners:

. . . .
Dr. Bruno Rémillard

. . . .
Dr. Reg Kulperger

. . . .
Dr. Rogemar Mamon

. . . .
Dr. Mark Reesor

The thesis by

Walid Mnif

entitled:

Incomplete Market Models of Carbon Emissions Markets

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Ph.D.

. . . .
Date

. . . .
Chair of the Thesis Examination Board

Co-Authorship Statement

This thesis is an integrated article manuscript assembled from research papers co-authored with Dr. Matt Davison. Dr. Davison is a Professor in the Departments of Applied Mathematics and Statistical & Actuarial Sciences. He is also the supervisor of Walid Mnif.

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Abstract

New regulatory frameworks have been developed with the aim of decreasing global greenhouse gas emissions over both short and long time periods. Incentives must be established to encourage the transition to a clean energy economy. Emissions taxes represent a "price" incentive for this transition, but economists agree this approach is suboptimal. Instead, the "quantity" instrument provided by cap-and-trade markets are superior from an economic point of view. This thesis focuses on the cap-and-trade instrument. Carbon emissions markets have recently been implemented in different countries. We summarize the state of world cap-and-trade schemes. We also provide a literature review of existing research that offer pricing and hedging tools.

Based on the European Union Emissions Trading Scheme, we study the impact of the market design on the observed spread between futures contracts with different maturities. Moreover we investigate the relationship between their returns. First we study the spread using a discrete-time model. We propose a pricing procedure arising from quadratic risk minimization hedging strategies. We suggest recommendations for both traders and the regulator in order to efficiently encourage market participation.

We also present a continuous-time model that investigates the way in which the market structure affects the impact of an unexpected release of information on futures returns. We propose a pricing solution based on the Föllmer-Schweizer decomposition. The optimal hedging strategy depends on all traded futures and minimizes the mean conditional square error of the cumulative cost process. Both discrete and continuous time model parameters are estimated to fit real data, and economic conclusions are drawn.

Keywords: Carbon Emissions Market, Incomplete Market, Quadratic hedging, Föllmer-Schweizer decomposition, Indifference price, Econometrics.

Contents

List of Figures

List of Tables

Chapter 1

Introduction and Thesis Road Map

1.1 Introduction

Carbon markets were recently introduced as a policy instrument to spur the transition to a clean energy economy. The literature that covers this kind of market is still relatively sparse. The purpose of this thesis is to help understand detailed specific features of carbon market mechanisms. The work is strongly rooted in empirical evidence, but also uses modern theories of incomplete market pricing which enter into this discussion because of a strong impact of hidden information in these markets. We develop models and fit their parameters to data collected from the European market. We also present pricing and hedging strategies. The thesis ends with conclusions and some suggestions for future work. This thesis is written in manuscript style and it is assembled from:

1. Mnif, W. and M. Davison, 2011a, "Carbon Emissions Markets," (D. D. Wu, ed), 95-108, *Quantitative Financial Risk Management*, Springer-Verlag Berlin, Heidelberg. [**Chap 1**]

2. Mnif, W. and M. Davison, 2011b, "What Can We Learn from the EU ETS Experience? Recommendations for Effective Trading and Market Design," *Working paper. (Submitted to Journal of Derivatives on 27-10-2011)* [**Chap 3, 4, 5, 6**]

3. Mnif, W. and M. Davison, 2012, “EU ETS Futures Spread Analysis and Pricing Contingent Claims under Different Market Schemes”, *Working paper. (Submitted to Management Science on 11-10-2012)* [**Chap 7, 8, 9, 10**]

The second chapter presents carbon markets and related literature review (Mnif and Davison, 2011a). The remaining thesis is divided into two main parts. Because of the manuscript format, the different papers overlap to a small degree, particularly in their introductions. As a corresponding benefit, each paper may be read on its own.

During our investigation, we believe that discrete-time models were necessary in order to deeply understand market specifications (Mnif and Davison, 2011b). The discrete-time framework is very rich and sufficiently flexible to model well documented stylized features, but it is very computationally expensive. In Mnif and Davison (2012), we propose a continuous-time stochastic model that is coherent with the discrete-time model results.

Carbon markets have a specific feature that distinguishes them from all other commodity markets. In fact futures with different maturities are linked through the banking and borrowing possibilities inherent in carbon markets. This banking/borrowing is wholly under the regulatory control, unlike any other commodity markets. The resulting impact of regulatory decisions makes the market very sensitive to the information possessed by the regulator. This structure provides an additional uncertainty factor, and hence the market is incomplete. In this thesis, we suggest mathematical models to show how this can be addressed. We believe a great deal of hidden information related to the market design is embedded in the spread. Our discrete-time model focuses on the excess return that quantifies the return arising from the market incompleteness property. This quantity is a function of the expected market position at a future date. We find that the excess returns dynamics of the contract with shortest maturity T_1 can explain most of the excess returns of the other contracts. Nevertheless the excess returns of other contracts quantify the expectation and the intrinsic risk related to post-T_1 trading periods. We show how this feature could be used to reduce market position risk. Different econometric techniques are used to estimate the model parameters, including the EM algorithm and instrumental variables.

We also study the relationship between the market design and the impact of any unexpected release of information. We propose that futures dynamics are governed by a geometric Brownian motion augmented by a Poisson process which represents the unpredictable component. We estimate the model parameters through the maximum likelihood approach followed by a generalized EM algorithm. Our empirical results show that most of the market uncertainty is driven by the continuous random component of the contract with shortest maturity T_1. Furthermore we find that the impact of an unexpected release of information depends on the market design. This impact alters as the market structure changes. This provides evidence that carbon futures markets are mature and efficient enough to be comparable to many other futures markets.

The Appendices to this thesis are integral to it and contain the proof of the main results. Reading them is not optional to the serious mathematically orientated reader, although readers seeking economic intuition and willing to take mathematical results on faith are welcome to omit them.

The following symbols and notations are used through the thesis:

CERs	Certified emission reductions
CDM	Clean development mechanism
EAT	Emission allowances trading
ERU	Emission reduction units
EM	Expectation-maximization
EU ETS	European Union Emissions Trading Scheme
GEM	Generalized expectation-maximization
IET	International emissions trading
JI	Joint implementation
tCO_2e	Ton of carbon dioxide equivalent
$(\Omega, \mathbb{F}, \mathcal{P})$	Probability space

$\mathcal{L}^2(\mathcal{P})$	Space of $\mathcal{P}$ square integrable 1-dimensional random variables
$\mathcal{L}^2_d(\mathcal{P})$	Space of $\mathcal{P}$ square integrable d-dimensional random variables
$\mathcal{P}^e$	Set of equivalent local martingale measures
$\widetilde{P}$	Variance-optimal measure
$\widehat{P}$	Minimum martingale measure
$E[.]$	Expectation under the historical measure $\mathcal{P}$
$E_{\widetilde{P}}[.]$	Expectation under $\widetilde{P}$
$E_{\widehat{P}}[.]$	Expectation under $\widehat{P}$
$a.s.$	Almost surely
T_1	Compliance date
T_2	Compliance date such that $T_2 > T_1$
S_t	Discrete-time d-dimensional vector of discounted futures allowance price processes such that $S^i_t, t \leq T_i$, is used for compliance purpose at time T_i
ξ^i_t	Discrete-time returns of S^i_t
F_t	Continuous-time vector of discounted futures allowance price process such that the i^{th} element is $F(t, T_i)$ and is used for compliance purpose at time T_i
X_{it}	Continuous-time return of $F(t, T_i)$
W_{it}	$\mathcal{P}$-Standard Brownian motion
N_{it}	$\mathcal{P}$-Poisson process
$\widetilde{N}_{it}$	Compensated Poisson process of N_{it}
$W^{\widehat{P}}_{it}$	$\widehat{P}$-Standard Brownian motion
$N^{\widehat{P}}_{it}$	$\widehat{P}$-Poisson process
H	Contingent claim to be priced
V_0	Initial price
P_{max}	Ceiling price
$\mathcal{N}(.)$	Standard normal cumulative funciton
$\psi(\mu, \sigma)$	Normal probability density function with mean μ and standard deviation σ

ζ. Hedging strategy

The market is said to be short if the number of total permits is less than the total greenhouse gas emissions. Consequently the prices are traded at a high level. In the opposite case the market registers low prices and is said to be long.

This thesis makes many novel contributions arranged here by category:

- Mathematical aspect: We use probability techniques to price contingent claims under various incomplete market models. We propose the solution of a discrete-time multidimensional quadratic hedging problem. Rémillard and Rubenthaler (2009) also report the same result. However we prove it in a different way, similar to Schweizer (1996). Moreover we provide a sufficient condition that permits the reduction of unhedgeable risk by including positions on an additional risky asset.

- Financial aspect: We show that a multiperiod pricing framework is mandatory to obtain more significant contingent claim price signals.

- Economic aspect: We draw economic conclusions based on empirical evidences. Furthermore we provide some recommendations to the regulator to efficiently participate in current market design.

1.2 Thesis Road Map

Empirical evidence from the European Union Emissions Trading Scheme (EU ETS) experience drove our motivation. In fact, as it will be shown later, a spread is observed between futures discounted to an equal time value baseline. Permits are traded on a daily basis. However the settlement is set to be once per year at which time emitters must cover their total emissions. Consequently no “convenience yield ” or “cost of storage ” is included in the annual scale spread. Our objective is to study this spread and understands its principal origins. Based on our analysis, we suggest recommendations for both participants and regulator. Furthermore, we provide pricing tools for different contingent claims that incorporate our empirical findings. We proceed by analyzing a two-period market model. Generalization to the multi-period case is straightforward. Two parts are reported in this thesis, depending on whether a discrete or continuous time framework was used.

The first part of the thesis presents an analysis based on a discrete-time model. We describe futures allowance dynamics by a binomial tree with returns partially driven by the implied expected market position at subsequent compliance dates. By construction, this market is incomplete. We propose a pricing procedure and associated quadratic risk minimization hedging strategies. We show that the best hedging strategy must include positions in futures maturing at subsequent compliance dates. We recommend that the regulator introduce a new additional tradable primary asset to prompt non-emitters to participate in the market in order to increase its liquidity. We present a possible pricing framework based on the indifference pricing technique.

The second part proposes a continuous-time model that depicts the relationship assimilated into the spread. We assume that two futures that mature at subsequent dates are traded. Their dynamic is driven by Brownian motions augmented by two jump processes. The discontinuous component reflects the impact of any unexpected release of information. We estimate the model parameters and draw economic conclusions. Furthermore we present a pricing solution based on the Föllmer-Schweizer decomposition. The optimal hedging strategy depends on all traded futures and minimizes the mean conditional square error of the cumulative cost process. We show how the *fair* price of any contingent claim can theoretically be computed in this context. Pricing examples under different market schemes are investigated.

Principle results of the thesis are summarized in Chapter 11, together with a few suggestions for future work.

Chapter 2

Review of Carbon Markets

In 1997, an international agreement known as the Kyoto Protocol was adopted by over 184 states with the aim of reducing global greenhouse gas emissions. Greenhouse gases (GHGs), as defined by the World Bank, are the gases released by human activity that are responsible for climate change and global warming. The six gases listed in the Kyoto Protocol are carbon dioxide (CO_2), methane (CH_4), and nitrous oxide (N_2O), as well as hydrofluorocarbons (HFCs), perfluorocarbons (PFCs), and sulfur hexafluoride (SF_6). For each gas a Global Warming Potential (GWP) indicator is defined to measure the impact of a particular GHG on the additional heat/energy retained in the earth's ecosystem through an addition of an unit of the gas given to the atmosphere. The unit of measure is ton of CO_2 equivalent (tCO_2e). Table 2.1 summarizes the GWP for each GHG.

GHG	tCO_2e
Carbon Dioxide (CO_2)	1
Methane (CH_4)	21
Nitrous Oxide (N_2O)	310
Perflurocarbons (PFC)	6500
Hydroflurocarbons (HFC)	11700
Sulfur Fluoride (SF_6)	23900

Table 2.1: Global warming potential indicator for greenhouse gas emissions. (Source: World Bank, Sustainable Development Department)

The Kyoto Protocol defines emission caps for industrialized and transition countries with the goal of decreasing GHG emissions by 5.2% relative to 1990 levels during the commitment period 2008-2012. The tools it provides for meeting this goal are the Clean Development Mechanism (CDM), Joint Implementation (JI) and International Emissions Trading (IET). The latter allows for Emission Allowances Trading (EAT) between governments. The CDM is a mechanism designed to assist developing countries in achieving sustainable development by permitting industrialized countries to finance projects for reducing greenhouse gas emission in developing countries and to receive credit for doing so. The JI is a mechanism whereby an industrialized nation as specified by Kyoto's Annex I [1] may acquire Emission Reduction Units (ERU) when it helps to finance projects that reduce net emissions in another industrialized country (including countries with economies in transition). For emission reductions resulting from JI projects, countries are granted Certified Emission Reductions (CERs). Both CER and JI projects have a number of conditions attached to them. Each project, together with the protocol used for measuring its emission reductions, must be approved by the executive board. Countries may use EATs, ERUs and CERs to comply with their emission caps.

The Kyoto commitment was introduced for the period 2008-2012. The role of the post 2012 portion is to stabilize atmospheric concentrations by 40% to 45% by 2050, compared to 1990 levels. As it takes time to achieve the target of the new regulations and to put incentives in place, companies must be confident that the system will endure in order to make decisions that require a long time line. To create this confidence, the World Bank is already buying credit for the post Kyoto commitment, while European policy makers are confident that 2012 will be followed by another compliance period[2].

We focus on allowances markets rather than project-based transactions and secondary Kyoto mechanisms, which suffer from inefficiency and unstable complex regulation. The effect of this inefficiency can be seen in the project-based transactions where the traded volume plum-

[1] Industrialized countries: Australia, Austria, Belarus, Belgium, Bulgaria, Canada, Croatia, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Japan, Latvia, Liechtenstein, Lithuania, Luxembourg, Monaco, Netherlands, New Zealand, Norway, Poland, Portugal, Romania, Russian Federation, Slovakia, Slovenia, Spain, Sweden, Switzerland, Turkey, Ukraine, United Kingdom, United States of America.

[2] This was written in 2010. The post-Kyoto phase has since been set to be from 2013 to 2020.

meted from 636 $MtCO_2e$ in 2007 to 283 $MtCO_2e$ in 2009. Little academic literature is available on this topic.

The chapter is organized as follows. Section 2.1 summarizes the current state of world cap-and-trade schemes. The recent literature devoted to financial quantitative modeling for these markets is presented in Section 2.2.

2.1 Carbon Markets

The new Kyoto regulatory framework forces countries to transition into a clean energy economy. A policy instrument that could be used is a carbon emissions tax. Such a tax imposes a price that an emitter has to pay per unit of GHG emission. Companies will have to choose between paying the emission tax or reducing their pollution, encouraging emissions reductions if the marginal costs of abatement is less than the imposed tax. As a consequence the optimal tax for each company must be equal to the marginal cost of abating. This marginal abatement cost varies across emitters and information about it is often unavailable to the regulator. As a result, the tax instrument is suboptimal. Furthermore it will be difficult to comply with the reduction commitment as the regulator does not directly control the emitted amount. Goers, Wagner and Wegmayr (2010) provide more details about the inefficiency of emission taxes.

Inspired by the U.S. Acid Rain Program (1990) that was designed to control sulfur dioxide (SO_2) and nitrogen oxides (NO_x) from fossil fuel-burning power plants, some regulators decided to implement the cap-and-trade mechanism as the most cost-efficient instrument to comply with emission reduction target.

A cap-and-trade system is a market-based mechanism that uses market principles to achieve emissions reduction. The government running the cap-and-trade program sets an absolute limit, or cap, on the amount of GHG, and issues a limited number of tradable allowances which sum to the cap and represent the right to emit a specific amount. The market is designed to provide price signals describing the true cost of the emission of a tonne of carbon. This is a crucial input for planning the transition to a clean energy economy, while protecting sensitive sectors from undue disruption and keeping local industry internationally competitive.

Higher emissions prices would induce companies with lower abatement costs to profit from the price difference by abating more CO_2 than they would need to comply with regulations, and then to sell the spare certificates for the higher certificate price. Each company faces a basic choice between buying or selling allowances, and reducing emissions through the use of alternative technologies. Three general classes of techniques for the physical reduction of emissions are available. Firstly, emissions can be reduced by lowering the output scale. Secondly, the production process or the inputs used may be modified, for example fuels can be switched (Gas/Coal). Finally, tail end cleaning equipment can be installed to remove pollutants from effluent streams before they are released into the environment.

2.1.1 European Union Emissions Trading Scheme

The European Union Emissions Trading Scheme (EU ETS) market is a cap-and-trade system limited to European industrial installations. It is the largest carbon emission market in the world with 6.3 billion tCO_2e trading volume and US$118.5 billion exchanged value in 2009. It was established in 2005[3], three years before the beginning of the first Kyoto commitment phase. It comprises combustion installations exceeding 20 MW, refineries and coke ovens as well as the metal, pulp and paper, glass, and ceramic industries. In total more than 12,000 installations among 30 countries (27 European Union States plus Iceland, Liechtenstein and Norway). Companies covered by the ETS receive at the end of every February a certain number of EU allowances (EUAs). The initial allocation assigned to each company depends on the National Allocation Plan[4]. Each allowance gives the right to emit one tCO_2 in the current calendar year. On April 30^{th} of the following year, companies must submit EUAs to the national

[3]The first trading started in 2004 in anticipation of the formal initiation of the scheme in January 2005. The traded volume was about 8.5 $MtCO_2$.

[4]An important component of each plan is a quantity deserved aside for new installations and new companies, known as New Entrants' Reserve.

surveillance authorities. If companies do not provide EUAs that cover their total emission, they must pay a penalty[5] and deliver the missing EUAs in the following year. EUAs were initially allocated to the market participants for free with limited information during the first trading period[6]. Some companies, as a consequence, made gains described as windfall profits.

In addition to using carbon trading, only CDM were considered within the phase I (2005-2007). The JI was added during phase II (2008-2012). The contribution of CDM and JI are limited in order to ensure local emission reduction targets.

As the EU ETS market started in 2005, there are differences between the first trading period (2005-2007) and the first Kyoto commitment period (2008-2012). In fact, in most European countries, the EUAs issued in the first trading period were only valid during this trading period (although France and Poland allowed limited banking between 2007 and 2008). In France and Poland, companies could bank at most the difference between the initially allocated allowances and their accumulated emissions. Furthermore, companies can bank CERs from the first period, but we highlight the fact that the use of CERs was limited during this phase.

The EU ETS allows borrowing from a future year within the same trading period. As the compliance date is at the end of April, the company can use the received EUAs at the end of February to comply with the preceding year. The recent global economic crisis decreased the demand side of the market in 2009, with emissions falling by 11.2%. As a result some companies, such as steel and cement, raised cash by taking advantage of the overlap between the issuance of the 2009 allowances and the 2008 deadline for compliance. In fact they sold their 2008 EUAs and borrowed the 2009 allocations to comply with their 2008 emissions. The EUA prices dropped sharply from the € 31 reached in July 2008 to € 8 in February 2009. This is a strong illustration of the importance of banking and borrowing rules in driving spot prices and their volatilities. Table 2.2 summarizes some features of Phase I and II. Carbon futures markets seem to be more liquid than the corresponding spot markets. In fact, an EUA

[5] The amount decided for this penalty is € 40 per metric ton of carbon equivalent above the cap in 2005-2007 period and €100 for the phase 2008-2012.

[6] Very limited number of EUAs were auctioned during the first phase. Referring to Article 10 of the European Directives, auctioning will be up to 10% of total emissions in phase II (2008-2012).

Phase I	Phase II
Excessive free allocation No banking for next phase Abated 3% of total emissions	Number of Allowances ↘ 1.74% annually Intra/inter-phase Banking is allowed Intra-phase borrowing is allowed Borrowing from Phase III is not allowed

Table 2.2: Some features of Phase I and II.

spot transaction is considered as a good so it is subjected to Value-Added Tax (VAT) while a futures and options contracts are VAT exempt because they are treated as financial transactions within the European Union. The largest and most liquid spot market for EUAs is the NYSE Euronext while the key futures market is the European Climate Exchange (ECX). Not only are companies regulated, but private or institutional investors are allowed to buy or sell allowances. The EU ETS allows non-emitting firms or individual investors to trade to increase liquidity and for speculation and diversification purposes. They need only establish an account in the emission registry of an European member state. U.S. funds are responsible for 10 – 15% of traded volume on ECX during the phase II.

Despite of the competition from NYSE Euronext, ECX does not have a spot market. They use the EUA Futures as underlying asset to write an option. For the first period only, futures with monthly expiries were traded in ECX. In 2008, quarterly futures contracts were introduced. These contracts are listed on an expiry cycle of: March, June, September and December contract months and they are listed up to June 2013. December annual contracts are also traded from December 2013 to December 2020. In October 2006, European style put and call options on EUA Futures started to be traded on ECX. In March 2009, ECX introduce EUA Daily futures contracts which are exchange-traded cash contracts. Daily futures Contracts will be physically delivered by the transfer of EUAs from the seller to the buyer.

Several empirical studies were completed to understand the market behavior during phase I. They show that the EU ETS is characterized by a very high historical volatility. Referring to Daskalakis, Psychoyios and Markellos (2009), EUA spot prices in Powernext Carbon[7] and

[7]NYSE Euronext acquired Powernext Carbon in December 2007.

Nord Pool[8] moved closely with the average mean absolute difference being around 7 cents (fixed transaction costs are on the order of 3 cents per EUA). Moreover, the correlation coefficient of weekly spot returns between the Powernext and Nord Pool EUA markets is very strong, reaching almost 90%.

Daskalakis, Psychoyios and Markellos (2009) found that there is no correlation between price returns for CO_2 and power. This result conforms to that reported by Svendsen and Vesterdal (2003) who concluded that the largest CO_2 emitters do not have enough market share and thus all market participants are assumed to be pure price takers. At conventional significance levels, they also show that logarithmic spot process are non stationary. Since EUAs are considered to be commodities for consumption, this result contradicts the common findings of mean reverting behavior observed in commodities and energy markets.

Daskalakis and Markellos (2008) examined the efficiency of EU ETS, concentrating on the weak-form of market efficiency according to which all the information contained in historical prices should be reflected in today's price. They conclude that the historical prices cannot be used to form superior forecasts or to accomplish trading profits above the level justified by the risk assumed.

Paolella and Taschini (2008) undertook an econometric analysis of emission allowance spot market returns and found that the unconditional tails can be well represented by a Pareto distribution while the conditional dynamics can be approximated by GARCH-type innovation structure.

Franke (2005) shows that if companies tacitly collude to manipulate the market, then CO_2 returns should have positive autocorrelations. A brief analysis of these autocorrelations in Seifert, Uhrig-Homburg and Wagner (2008) reveals no strong empirical evidence in favor of this conjecture.

Ben and Trück (2009) analyze the behavior of CO_2 spot prices' log-returns over the period starting from January 3, 2005 until December 29, 2006. They compared results from a sim-

[8] Nord Pool was sold entirely to NASDAQ OMX to create the NASDAQ OMX Commodities unit.

ple normal distribution, AR(1)[9], GARCH(1,1), and a Markov model switching between two regimes, namely a base regime and a spike regime. They concluded that the GARCH(1,1) and Markov switching models outperform both the normal and AR(1) models, and are quite similar.

The European regulator set up the third compliance phase during 2013-2020. The emission target is to reach, in 2020, a level of emissions 21% less than 2005. The detailed regulatory framework remains uncertain, but two major baselines consider carbon leakage and auctioning policy. "Carbon leakage" describes the transfer of a company to another country or state with less stringent constraints on carbon emissions in order to survive international competition. An auctioning policy will spur the carbon leakage as it will likely increase the production cost[10]. Economists agreed that auctioning will offset the downside effect of grandfathering and allow a more significant carbon price signal. Starting from 2013, the European regulator was engaged to set auctioning as an alternative for allowance allocation. To fight carbon leakage, a company in a given sector need pay for only a fraction of their allowances with companies in sector deemed exceptionally (leaky) receiving allowances free. The assistance will decrease annually such that in 2027 full auctioning will be applied in all sectors.

Analysts believe that the EU ETS options market is mature enough to be comparable to other many options markets. Furthermore they expect the market to be short post 2012 which explains the active trading of the December 2013 EUA contracts.

2.1.2 Other Emissions Trading Markets

In 2009, New Zealand (NZ) opted for a carbon trading scheme to comply with its Kyoto protocol commitment. The scheme started in July 2010. It regulates stationary energy, industrial process and liquid fossil fuels for transport. It will progressively include some other sectors

[9] They studied the models AR(p), $p \geq 1$ and found only AR(1) is significant.

[10] European policy makers are studying the possibility of imposing carbon taxes on goods imported from foreign countries which do not penalize emissions. Companies not exposed to foreign competition (e.g. in the electricity sector) will presumably pass the additional marginal cost to the final consumer.

(i.e. synthetic gas and waste on January 2013) until fully implemented by 2015. 2010-2012 is the transition period in which one NZ allowance is used to surrender two tCO_2e. Within this period, the market is a combination between a cap-and-trade and a tax system, known as hybrid market. In fact initially the allowances are distributed for free with a possibility to purchase more from the regulator at a predefined price of NZ$25. In case of non-compliance, the company will have to cancel the allowances they failed to deliver with a penalty of NZ$30 per unit. Borrowing from post 2012 is prohibited while unlimited banking is permitted.

In the U.S., the American Clean Energy and Security Act of 2009, known as the Waxman-Markey Bill, was passed by the House of Representatives in June 2009. It consists of a cap-and-trade scheme to reduce emissions by 17% from 2005 levels by 2020. The Bill still need to be considered by the Senate, probably during the next legislative term[11]. Despite the federal carbon regulation, the Regional Greenhouse Gas Initiative (RGGI) was set up in 2008 among the states of Connecticut, Delaware, Maine, Maryland, Massachusetts, New Hampshire, New Jersey, New York, Rhode Island, and Vermont. It is a mandatory cap-and-trade market covering only the power sector. It aims to reduce 10% of its emission by 2018. In 2009, 805 $MtCO_2$ was traded for an equivalent value of US$ 2.2 billion.

Four Canadian provinces (British Columbia, Manitoba, Ontario, Quebec) have developed the Western Climate Initiative (WCI) program together with seven U.S. states (Arizona, California, Montana, New Mexico, Oregon, Utah, Washington) to jointly implement a cap-and-trade scheme starting in January 2012. The initiative targets 15% emissions reduction below 2005 levels by 2020.

Some voluntary markets are implemented as a domestic initiative to spur transition into clean energy (i.e. China, Japan). Brazil intends to establish a voluntary market-based instrument to reduce emissions up to 38.9% by 2020.

Several questions may arise: Is it possible to set up an international linkage between different emissions trading schemes? If yes, is it the most cost-effective method for abatement?

[11]This was written in 2010. Shortly thereafter it was decided that the Bill will be considered by the Senate during the next legislative term. The current term expires on January 3, 2013.

2.2 Modeling and Pricing in Emission Markets

Cap-and-trade is a policy instrument to combat the climate change impact. This mechanism allows avoidance of climate risk at the corporate level even though it adds some other operational risks (see Labatt and White, 2007). As a consequence companies need a financial modeling framework to price emission allowances and their derivatives for risk management purpose.

Several approaches to this problem were developed during the past decade. Most existing work can be divided into those involving equilibrium models and those using quantitative finance style stochastic modeling. We now review this literature. We notice that some of the models that deal with allowances pricing under one compliance period are not flexible enough to take into consideration the impact of banking and borrowing possibilities under a multi-period trading scheme on allowance price dynamics. However they allow for a good understanding of the market mechanism.

2.2.1 Equilibrium Models

Dales (1969) was the first economist to introduce a market idea for trading the right to pollute. Three years later, Montgomery (1972) provided a theoretical foundation of a market in licenses and developed a decentralized system based on achieving environmental goals at a number of different locations. These two seminal papers are the origins of the development of more recent contributions.

Carmona, Fehr and Hinz (2009) explore the relation between the price evolution of emission allowances and the way in which a multi-agent electricity producer decides when to switch from a hard coal power plant to a cleaner Combined Cycle Gas Turbine (CCGT). A one period discrete time mathematical model is developed to determine the optimal switching policy that minimizes the overall cost under zero net supply conditions. The resulting equilibrium carbon price is equal to the marginal price of an extra allowance to lower the expected penalty payment amount.

Seifert, Uhrig-Homburg and Wagner (2008) assume that emission rate dynamics are given by a stochastic process, where the uncertainty is driven by a standard Brownian process. The existence of this term in the model is explained by a potential emission variation due to some external randomness (e.g. weather changes and economic growth). Under the assumption of risk-neutral market participants, the central planner choose an optimal abatement policy as function of time and total expected accumulated emissions over the entire compliance period. The latter variable is a controlled stochastic process with dynamics derived from the emission rate's stochastic differential equation with a drift controlled by the abatement policy undertaken. The marginal abatement costs is assumed to be linearly increasing with respect to the emissions abatement strategy. It is also defined as the spot price, and has a martingale property under the objective probability measure. Its motion is not correlated to the specification of the emission process rate. A logarithmic utility function was introduced to study the impact of risk aversion on allowances price.

Chesney and Taschini (2008) deal with pricing spot allowances for a one period market scheme and assume that the emitter releases GHG exogenously and continuously under a geometric Brownian motion. A company may trade only at an initial time in order to minimize final costs comprising the sum of the initial trading cost[12] and the expected penalty payment applying to any future allowance shortages. The spot price obtained is equal to the discounted[13] penalty price times the probability weight of non-compliance scenarios. Chesney and Taschini (2008) extend the basic model to the case of an economy where two companies can trade at multiple discrete times. They suppose that the allowance price is equal to the penalty level at the compliance date when at least one company faces an allowance shortage. Companies also trade using only information about their own pollution and the accumulated emissions volume of their counterparty at the previous trading possibility. The equilibrium price process for each trading time is defined as function of the traded quantity. The latter is obtained

[12]It can take positive values and be considered as gain when the company decides to sell allowances at the initial time, as it can be negative and seen as cost if it buys allowances.

[13]The discount rate is the weighted average cost of capital.

by solving a system of two equations, incorporating the market clearing condition. Using the method of moments to approximate the sum of more than one geometric Brownian motion by another geometric Brownian motion, an extension of the model to a multi company framework is possible.

Carmona et al. (2010) propose a competitive equilibrium model under one compliance period. The output goods are assumed to be exogenous and inelastic. A producer in the economy has the choice between several technologies for each good characterized by different marginal cost production, emission factor and production capacity. The stochastic properties of the demand and costs are known for all firms from the beginning. The overall demand is considered always to be satisfied. As such, overall demand must be less than the total production capacity. To avoid paying penalties, the planner switches its production to a cleaner available technologies or has recourse to the ETS to buy allowances. The authors show the existence of market equilibrium such that the zero net supply condition is fulfilled, the demand is covered, and the strategies maximize the expected terminal wealth. To avoid issues with discounting, Carmona et al. (2010) work with forward prices applying at the compliance time T. The forward allowances prices in time T currency is a bounded martingale under the objective probability measure with value less then the penalty level. Also time T spot goods prices and optimal production strategy are merit-type equilibrium with defined adjusted costs. These properties are necessary conditions to the existence of the equilibrium. It is shown that the market equilibrium is equivalent to a representative agent problem where the emission is reduced in a cost-effective way. A generalized cap-and-trade scheme is introduced by including taxes and subsidies in the original formulated problem. Furthermore it allows the regulator to distribute allowances dynamically and linearly in the production quantity. By assigning adequate values to these variables, a comparative analysis is made between the standard cap-and-trade market (a), cap-and-trade market with auctioning of allowances (b), tax scheme (c), and a cap-and-trade scheme with relative allowance allocation (d) vis-*à*-vis emissions reduction, incentives to invest in a cleaner energy, windfall profits, social cost, and end-consumer cost. Table 2.2

reports the results.

	Reduction Target	Incentives	Windfall	Social Cost	Consumer Cost
a	+	+	-	+	-
b	+	-	-	+	-
c	-	-	+	-	-
d	+	+	+	+	+

Table 2.3: Comparison of different market schemes from Carmona et al. (2009).

Hinz and Novikov (2010) solve the central planner problem treated in Seifert, Uhrig-Homburg and Wagner (2008), Carmona, Fehr and Hinz (2009), and Carmona et al. (2010) by including additional assumptions in the equilibrium mathematical model. Under a no-arbitrage condition, they assume the existence of an equivalent risk neutral probability Q such that the equilibrium price is a $Q-$martingale. Also the agent opts immediately to abate when allowance prices exceed its abatement cost. At the compliance date, the spot price is zero if the market is long and equals the penalty level otherwise. As consequence the spot price under Q will depend only on the cumulative abatement volume and the overall allowance shortage. The model is developed under a discrete time framework. As an illustrative example, they focus on the martingale case with independent increments for the cumulative emissions and deterministic abatement functions combined with the least-square Monte-Carlo method of Longstaff and Schwartz (2001). An algorithm is formulated to price allowances and European calls written on the spot allowances price.

Borovkov, Decrouez and Hinz (2010) study the continuous time version of the solution obtained by Hinz and Novikov (2010). They show the existence of the allowance price when the conditional expectation of the total cumulative emissions is a Q-martingale diffusion process with a deterministic volatility. The allowance price is derived by solving a nonlinear partial differential equation (PDE), while a European call option is priced by solving a linear PDE. An extension to a jump diffusion setting is developed and the spot price is obtained by solving a partial integro-differential equation. Borovkov, Decrouez and Hinz (2010) prove uniqueness of the allowance price and use a numerical finite difference scheme to compute it.

Kijima, Maeda and Nishidie (2010) extend the work of Maeda (2004). They suppose the existence of a competitive market within a single-period economy, where the regulated emitters must comply with emission reduction target set up by the regulatory authority at the future time T. Two markets are available: the spot market, and the derivatives market written on the T allowances price and assumed to be complete. Financial traders are considered in the model and trade only in contingent claims market to hedge the risk in their exogenous income. The authors assume that each economic agent has a negative exponential utility with an appropriate risk-aversion coefficient. The key assumptions for their model to obtain closed-form formulas are the following: they suppose the cost abatement function to be continuously differentiable, increasing and strictly convex with a derivative that starts at zero when there is no abatement, and goes to infinity asymptotically. Infinite penalties are imposed, so that the emitter must abate emission or buy allowances at time T to comply with the regulatory emission target. The state price density is provided for each of the cases in which banking and borrowing are allowed or not, giving a pricing solution for any contingent claim. Moreover, the market clearing condition when banking and borrowing are forbidden must be satisfied, otherwise being replaced by the equality between the aggregate abatement target and the whole emission reduction over the entire compliance period. Under a piecewise linear quadratic abatement cost function, price spikes may occur more frequently in the forward than in the spot price (in contrast to intuition deriving from the usual Samuelson effect for commodities). The relationship between the spot and forward prices are analyzed. They show that when there are many financial traders the forward price is smaller than expected future spot price. This forward curve phenomenon is known as normal backwardation.

2.2.2 Stochastic modeling

We introduce the papers that use applied probability techniques in order to provide a pricing and hedging solution to the market participants. These approaches offer general flexible tools for pricing complex contingent claims.

Çetin and Verschuere (2009) present a probabilistic pricing and hedging framework. They assume that the market contains only two forward contracts P_t and S_t with subsequent maturities T_1 and T_2, $T_1 < T_2$, respectively. S_t dynamics are modeled by a Markov process with a drift expressed as an affine function of a right continuous with left limits Markov chain taking values depending on the market position. Under the assumption of no banking, P_t is zero if the market is long; otherwise taking the value of the penalty level plus S_{T_1}. If the market is short the investor must pay the penalty and deliver the missing allowances at later time T_2. The model framework is incomplete because there are two sources of uncertainty in the stochastic differential equation for S_t, and one of them is not tradable. As a result, contingent claims have, in addition to the hedgeable risk, a relative intrinsic risk (Föllmer and Sondermann, 1986) which cannot be covered. Çetin and Verschuere (2009) uses the Föllmer-Schweizer decomposition to price P_t as an expectation under an equivalent probability measure called the minimal martingale measure. The associated hedging strategy is a locally-risk minimizing strategy as defined by Föllmer and Schweizer (1991). A filtration projection technique is used to price the allowance and a digital option, which pays an unit amount of money if the market is short at time T_1, under incomplete information. The effect of intermediate announcements is also studied.

Carmona and Hinz (2009) assume the existence of an equivalent martingale measure Q such that the price process of a future contract A_t is a martingale. Within a single T compliance period model, A_T is equal to the penalty level Π when the emitted quantity is greater than the number of allowances. Carmona and Hinz (2009) define N as a set of allowance shortage events. N is described as the set where some positive-valued random variable Γ is located above the boundary 1. The total normalized emission can be seen as a special choice. Carmona and Hinz (2009) identify a class of parameterized positive Q-martingales with values less than the penalty level. These Q-martingales satisfy the following condition: under the objective probability measure, the probability of the events such that the limit of A_t equals to Π is the same as one minus the probability of the events such that the limit of A_t equals to 0. For ease of calibration to historical data, they provide a formulation of the likelihood density under the assumption that the market price of risk is constant over time. The model is extended to a two-period market model without borrowing, with unlimited banking and withdrawal. The prices of European call options written on futures contracts and maturing before the first compliance date are derived for both models.

Grüll and Kiesel (2009) assume that the emission rate follows a geometric Brownian motion, similar to the assumption of Chesney and Taschini (2008). They use the result of Carmona et al. (2010) and assume that the price of the futures contract maturing at the compliance date T may be computed from the penalty price times the probability of the set of events where the total cumulative emissions at time T exceeds the cap predetermined by the regulator. The spot price is approximated using three different approaches which depend on the approximation method used to compute the total cumulative emissions at time T. In the first, linear approach cumulative time T emissions are estimated to be T times the emissions rate at time T. The second and third approaches are a bit more sophisticated, relying on moment matching techniques for the cumulative emissions estimate. They differ only because the second approach uses a log-normal distribution in the matching while the third uses a reciprocal gamma distribution.

Under a risk-neutral assumption, Huang (2010) models emission rate dynamics as a stochastic process. Instead of a geometric Brownian motion dynamics as in Chesney and Taschini (2008) and Grüll and Kiesel (2009), he assumes that the process can follow either an arithmetic Brownian motion or a mean reversion process. At the compliance date, the spot price is zero if the aggregate emissions exceed the allocated emissions limit and equals the penalty level otherwise. Formulas are provided for spot prices, European options prices (call and put) as well as their Greeks. Futures prices can be derived from the spot price when the convenience yield is neglected.

Part I

Discrete-Time Model

Chapter 3

Discrete-Time Model Introduction

A major environmental challenge facing humanity is the problem of climate change, which has been linked to industrial emissions of carbon dioxide and other greenhouse gases (GHGs). As a consequence, governments around the world have joined to establish a new regulatory framework in order to stabilize and decrease emissions. The Kyoto protocol to the United Nations Framework Convention on Climate Change is a concrete example. Ratified by more than 184 states, the protocol defines emission caps for industrialized and transition countries to decrease GHGs emission by 5.2% below 1990 levels during the commitment period 2008-2012 and to stabilize atmospheric concentrations by 40% to 45% by 2050. It includes three flexible compliance mechanisms: the Clean Development Mechanism (CDM), Joint Implementation (JI) and International Emissions Trading (IET). Kyoto's IET allows for emission allowances trading between governments. The CDM is a mechanism for assisting developing countries to achieve sustainable development by permitting industrialized countries to finance projects for reducing GHG emission in developing countries and to receive credit for doing so. The JI is a mechanism whereby an industrialized country may acquire emission reduction units when it helps to finance projects that reduce net emissions in another industrialized country, including countries with economies in transition. Given these instruments, each country must comply with the emission reduction target by imposing national regulations on its economic agents.

The new regulatory framework is designed to force economies to transition to a “clean energy economy”. The regulator must protect consumers from predatory price increases and spur energy efficiency through the development and deployment of new energy technology. Three practical policy instruments can be used to meet these constraints: emission taxes, cap-and-trade markets, and a hybrid of the two. An emission tax is a price that an emitter must pay per unit of GHG emission. Under an emission tax companies must choose between paying the emission tax or reducing their pollution. As long as the marginal costs of abatement is less than the imposed tax, they will reduce emissions. Therefore the tax rate plays a key role in defining corporate strategies to face the regulatory environment. A cap-and-trade market is a quantitative instrument that uses market principles to achieve emissions reduction. The regulatory agency sets an absolute limit, or cap, on the amount of emissions, and issues a limited number of tradable allowances which sum to the cap and represent the right to emit a specific amount. A hybrid safety valve system combines both the emission tax and the cap-and-trade market. Companies may buy allowances from the national authority at a high, but fixed, rate rather than from the market. Jacoby and Ellerman (2004) describe the origins of the safety valve concept and trace its evolution in the climate policy context.

The European Union Emission Trading Scheme (EU ETS) established in 2005 is the largest multi-country, multi-sector GHG cap-and-trade system. Designed to comply with Kyoto targets, it comprises combustion installations exceeding 20 MW, refineries, coke, metal, pulp and paper, glass, and ceramic industries. The largest and most liquid spot and futures exchanges for European emissions credits are NYSE Euronext and the European Climate Exchange. Not only regulated companies, but also private or institutional investors are allowed to trade allowances. Studies show that the EU ETS market is still illiquid and risky with a high historical volatility. We refer the reader to Benz and Trück (2009), Daskalakis and Markellos (2008), Daskalakis, Psychoyios and Markellos (2009), and Uhrig-Homburg and Wagner (2009) for empirical analyses of the EU ETS market under the first commitment period 2005-2008.

Chesney and Taschini (2009), Seifert, Uhrig-Homburg and Wagner (2008), Carmona, Fehr and Hinz (2009), Carmona et al. (2010), Hinz and Novikov (2010), and Borovkov, Decrouez and Hinz (2010) present emissions pricing frameworks based on equilibrium models. These interesting papers do not offer a flexible pricing framework for complex derivatives. However, the EU ETS experience suggests that allowance prices have a high historical volatility. Therefore options, including exotic options, are attractive for market participants who use them to hedge their net positions. Grüll and Kiesel (2009) and Huang (2010) present a pricing solution by assuming that a) the emission rate follows a stochastic differential equation b) the spot price is the penalty level if the aggregate emission exceed the allocated emissions limit and equals zero otherwise. All of the above introduced models are difficult to extend to a multiperiod compliance setting. Therefore, each compliance period is considered independent from the other, in contrast to the EU ETS. In fact if companies do not deliver allowances to cover their total emissions, they must pay a penalty and withdraw the missing allowances on the following compliance date. Furthermore, borrowing and banking are possible in certain circumstances and drive the spot prices and their volatilities to a great extent before compliance dates.

Kijima, Maeda and Nishidie (2010) suppose the existence of a competitive market within a single-period economy with infinite penalties imposed for any future allowance shortages. Under mild assumptions on the cost abatement function, they provide the state price density for the cases in which banking and borrowing are allowed or forbidden. Carmona and Hinz (2009) model allowance shortage events by a set, $\mathcal{N}$, associated with a positive-valued random variable when it takes values above the boundary 1. A pricing framework is provided by assuming that the spot price at a compliance date is equal to the penalty level for all scenarios in $\mathcal{N}$, and to zero otherwise. The model is extended to a two-period market model without borrowing, with unlimited banking and withdrawal. Hitzemann and Uhrig-Homburg (2011) propose a multiperiod stochastic equilibrium model with banking and abatement possibilities. Under their framework, the emission permit can be seen as a strip of binary options written on net cumulative emissions.

Çetin and Verschuere (2009) use incomplete market results to propose a model for trading in a cap-and-trade market under the assumption of no banking between two compliance periods. The results which underpin their work come from a much more general incomplete markets setting introduced by Föllmer and Sondermann (1986) and generalized by Föllmer and Schweizer (1991). A filtration projection technique is used to price both allowances and a digital option under incomplete information. Mnif and Davison (2011a) present a review of the recent literature devoted to quantitative pricing and hedging tools for emissions markets as well as the state of the world cap-and-trade schemes.

The emission market mechanism is different from all other commodity markets and is, in some ways, more complex. On the one hand, schemes may or may not allow either banking (carrying allowances over to subsequent commitment periods) or to borrow allowances from a later period to comply with the current emissions cap. As consequence, any release of information about the expected market position within a compliance period can dramatically affect prices. Assume that the market allows banking and prices are currently low. If the market is expected at the next subsequent date to be short, emitters behave so as to minimize their future compliance cost. Thus, as a profitable strategy, they buy current cheap allowances and bank them to the next commitment period. It follows that the demand for cheap allowances exceeds their supply and therefore their prices increase. Many profitable strategies could be devised for different combinations between current and expected market states. Hence the market expectation conditioning on the current market state deeply affects the borrowing and banking strategies that drive price fluctuations. On the other hand, emissions trading markets are often dysfunctional, registering only very high prices (in which case the market is said to be *short*) or very low prices (in which case the market is *long*). In either extreme case, liquidity disappears from the market. In order to repair this market failure, the regulator may intervene to adjust the market parameters (i.e. initial endowment, penalty level) at the beginning of a new compliance period to balance the desire for a functioning market with the reduction policy commitment. The result of this manoeuvre is to make such a carbon market intrinsically incomplete. As a consequence not only are claim payoffs affected, the price dynamics also show a jump behavior and a change in their structural parameters. Therefore the financial tools usually applied in the markets (i.e. Black and Scholes, 1973, and all its many extensions) fail to give a *fair* price.

The relationship between information release and stock prices is among the questions that signaling models, in particular those in which information is linked to dividend levels (Battacharya, 1979, John and Williams, 1985, Miller and Rock, 1985, and Litzenberger and Ramaswamy, 1982), are designed to answer. Under the asymmetric information assumption that the firm's manager has more information than the shareholders about the firm's future earnings an unexpected increase of dividends induces a positive share price movement and vice versa. Under the carbon market framework we can match the signal of the change in unanticipated dividends rate to the release of information about the expected market position. The allowance prices increase if the market is expected to be short at compliance date and decrease otherwise. As a consequence modeling the expected market position requires a model of allowance dynamics. In this paper we proceed by an empirical investigation where the allowances returns are driven by the projection on the current data's information structure of the market expectation at a subsequent compliance date. Furthermore we assume that the regulator has more information than the market participants about overall market position and can identify the market state from macro/micro parameters observable to him. We show how the regulator can use this additional information to minimize risk associated with the traded contracts in the market.

We present a framework to illustrate the issues described above and to show how it may be addressed with stochastic tools. We assume that the allowance future prices follow a binomial tree model. We model the market expectation at a subsequent compliance date by random variables that drive the prices processes and affects their return through time. The (untraded) market expectation is unobservable by an investor. By including it in the model, we construct an incomplete market in which market participants trade under incomplete information. We propose a pricing framework that provides a *fair* price as well as its associated strategy for constructing a replicable portfolio that matures close to the contract written on emission permits. Furthermore we study the relationship between the one period model and a multiperiod one. We show that to obtain less risky hedging strategies, a multiperiod pricing framework is essential. Traders are subjected to uncertainty in the long run regulatory environment. So to decrease dependency between compliance periods, we suggest that the regulator should intervene in the market by offering a new tradeable financial asset. However, the regulator is not dynamically hedging this asset so its pricing procedure will be different from that of other market participants and has different goals. We propose a pricing approach based on utility maximization, such that the regulator evaluates the additional financial asset without affecting the overall social wealth. This technique is known in the literature as indifference pricing. We study the impact of such additional instruments on systematic risk.

The reminder of this part is organized as follows. Chapter 4 suggests a mathematical model which describes the futures dynamics and calibrates it to real data. Chapter 5 presents discussions and recommendations for effective trading and market design based on the pricing framework that we propose. Chapter 6 summarizes our discrete time results.

Chapter 4

Modeling Futures Allowance Dynamics

4.1 Futures Allowance Dynamics

Given a fixed probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$[1], we consider a market model where S_t is a d-dimensional vector of discounted futures allowance price processes such that the $S^i_t,\ \ t \leq T_i$, are used for compliance purpose at time T_i. We choose to work with futures contracts, because carbon futures markets are more liquid than the corresponding spot markets. An important reason for this difference in liquidity is that an EUA spot transaction is considered as a "good" and as such is subjected to Value-Added Tax (VAT). In contrast, futures contracts are VAT exempt and they are treated as financial transactions within the European Union. If a financial market is complete, the simplest discrete time model is the binomial in which:

$$S^i_t = \xi^i_t S^i_{t-1}, \forall t \geq 1, \tag{4.1}$$

where ξ^i_t is a $\mathcal{F}_t$-measurable process that takes two possible outcomes $\{\xi^{iu}_t, \xi^{id}_t\},\ \ \xi^{iu}_t > 1 > \xi^{id}_t$.

Define a non-observable process Y_t, the fluctuations of which are generated by the implied investors market expectation position at time t for the subsequent compliance dates. Y_t is nei-

[1] $(\mathcal{F}_t)_{t\geq 0}$ is assumed to satisfy the usual conditions. In other words, $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and every negligible set is measurable.

ther tradeable nor observable by an investor while he is trading at time t. Y_t can be considered as a hidden risky factor that exogenously affects the market behavior. As such, Y_t depends on market dynamics as well as the overall accumulated emissions up to time t, the economic business cycle, and any previous release of information. The collection of information from the dynamics of Y_t generates the missing information needed to have a pricing framework under the larger information set. The market can be long at time t (in which case prices are low), even while the market is expected to be short at the end of the compliance period. The variable Y_t takes values on the set of events where the market ends up short and its distribution is conditioned on the current market state, i.e. low prices. The distribution of Y_t is conditioned on the set of information available as well as that released at time $t-1$ about the projected market state. Moreover the set of Y_t values ($Y_t > 0$ if the market is expected to be short and $Y_t < 0$ otherwise) could be known by market participants when the regulator releases information about the overall expected market position (for instance at compliance dates T_i). In fact the regulator has access to more information about microeconomic[2] as well as macroeconomic variables, which allows him to predict market position given enough data. We assume that the allowance returns are affected by the non-observable process as follows:

$$S^i_t = f^i_{t-1}(\xi^i_t, Y^i_{t-1})S^i_{t-1}, \forall t \geq 1, \tag{4.2}$$

where f^i_{t-1} is an $\mathcal{F}_{t-1}$-measurable return function that defines the effective return of S^i_t given the realized values of ξ^i_t and Y^i_{t-1}. The condition $f^i_{t-1}(\xi^i_t, Y^i_{t-1}) \geq \xi^i_t$ holds if, at time $t-1$, the market is expected to be short, otherwise $f^i_{t-1}(\xi^i_t, Y^i_{t-1}) \leq \xi^i_t$. The no-arbitrage condition imposes that $0 < f^i_{t-1}(\xi^{id}_t, Y^i_{t-1}) < 1 < f^i_{t-1}(\xi^{iu}_t, Y^i_{t-1})$. Note that within a single period, any S-contingent claim can be replicated and we define the market to be *locally complete*; this is no longer the case when we turn to a multiperiod model. Intuitively this property means that any release of information at t stimulates a movement on the prices with a one time step lag, i.e. at time $t+1$.

We now present a simple example with traded asset S under two-period market framework

[2]He is able to require emitters to periodically report their emissions during the compliance period.

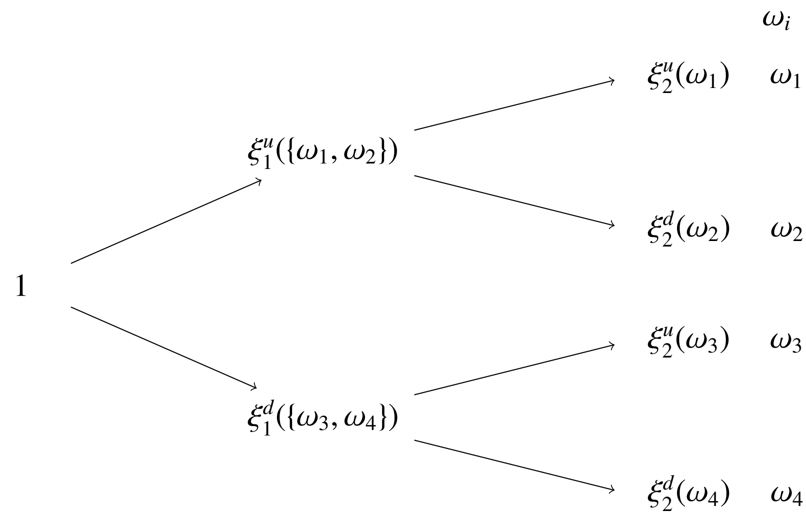

Figure 4.1: Returns per time step of the traded asset in absence of Y_t.

in order to clarify our model. Assume a complete market with at the original time $S = S_0$. At the end of each period, S can take two possible values: $S_t = \xi_t S_{t-1}, t = 1, 2$, where $\xi_t = \xi_t^u$ or $\xi_t^d, \xi_t^u > 1 > \xi_t^d$ (see Figure 4.1). We therefore distinguish 4 states of nature at $t = 2$ and any contingent claim written on S_t is totally hedged with no residual remaining risk. Its price is then equal to the cost of the strategy that allows to replicate its payoff $\mathcal{P} - a.s$[3]. However a non-observable risky factor introduces incompleteness to the model by enlarging the filtration corresponding with the more complete information set without supplying new tradeable assets to span this uncertainty. In fact, as shown in Figure 4.2, we end up with 16 potential states of nature. Consider an agent "representative" in the sense that his market position expectation is typical of market participants. Its market projection (short or long) affects the dynamics of the process Y_t. The latter has an impact on the time $t + 1$ allowance returns by the $\mathcal{F}_t$-measurable function f_t that relates its outcome at time t with ξ_{t+1}. Figure 4.2 shows how the return per period evolves. For example, to obtain the final allowance price, $S_2(\omega_1) = f_0(\xi_1^u(B_2), Y_0, \mathcal{F}_0) f_1(\xi_2^u(\{\omega_1, \omega_2\}), Y_1(A_1), A_1) S_0$. Thereby we provide the flexibility to model the allowance dynamics under a discrete time framework.

[3]An event is $\mathcal{P}$-a.s. if its probability under $\mathcal{P}$ measure is 1.

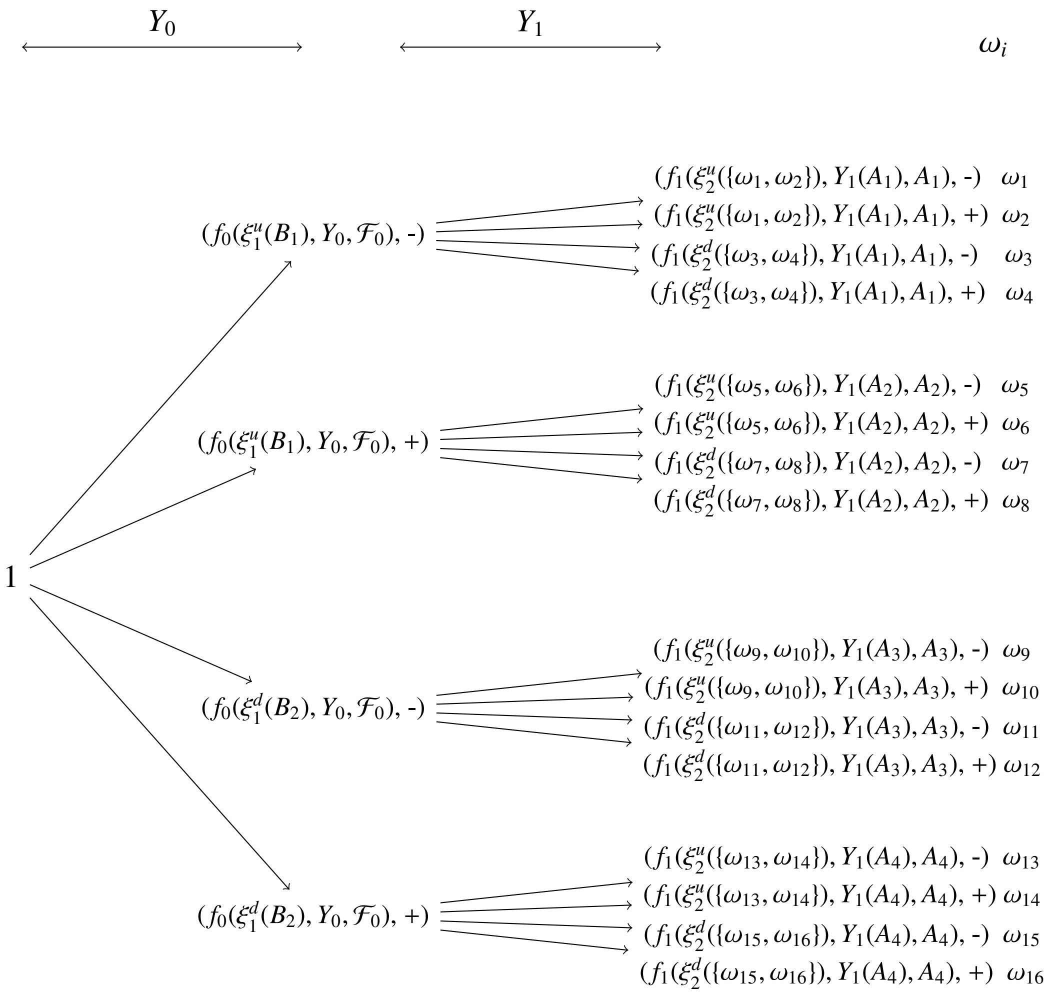

Figure 4.2: Returns per time step of the traded asset in presence of Y_t , where $A_j = \{\omega_{(j-1)*4+i}, i = 1, 2, 3, 4\}, j = 1, 2, 3, 4$, $B_1 = \{\omega_i, i = 1, ..., 8\}$, and $B_2 = \{\omega_i, i = 9, ..., 16\}$. The "-" sign indicates that the market is expected to be short, and the "+" sign denotes that the market is anticipated to be long.

4.2 Data Analysis and Parameter Estimation

4.2.1 Data Analysis

From the EU ETS experience, the emission market displays volatile price behavior due to the lack of information available to the investor as well as market intrinsic features. Figures 4.3 and 4.4 show the spread between two subsequent futures contracts of Phase II discounted to December 2009 money value using the EURIBOR futures term structure. Both backwardation and contango forward curve behaviours are observed depending on the overall environment.

At the beginning of 2008, EURIBOR futures were high (See Figure 4.5). Moreover analysts expected the market to be long; therefore it seemed more likely that emitters would gain windfall profits given that the allowances were initially allocated at no cost. As a result, emissions prices were then relatively high, so emitters started to sell 2008 allowances to raise cash and take advantage of the high rates, keeping in mind that 2009 allowances could be used for 2008 compliance in case of shortage. In fact, during the Kyoto period, the EU ETS allowed borrowing from a future year within the same phase. Moreover the 2009 allowances were distributed in February 2009 while the 2008 surrender date was April 30, 2009. Consequently many emitters purchased futures to hedge their possible 2009 shortfall position. This explains the reason why the Dec-2009 contract is more valuable than other contracts, and Figure 4.3 shows the resulting negative spread (backwardation).

In late 2008, futures prices plummeted to about the €15 level after trading above €25 for June and most of July. This drop was in response to the economic crisis that hurt big emitters reducing their demand and hence the aggregate demand for emissions allowances below the supply of allowances. Besides an expected economic stimulus beginning in 2009, emitters were attracted to buy cheap futures for use in forthcoming compliance periods, including the post 2012 phase III[4]. Unlimited banking is allowed between Phase II and III. Furthermore auctioning will increase from 4-5% on average during Phase II to at least 50% of traded credits

[4]The European regulator set up 2012-2020 as the third compliance phase.

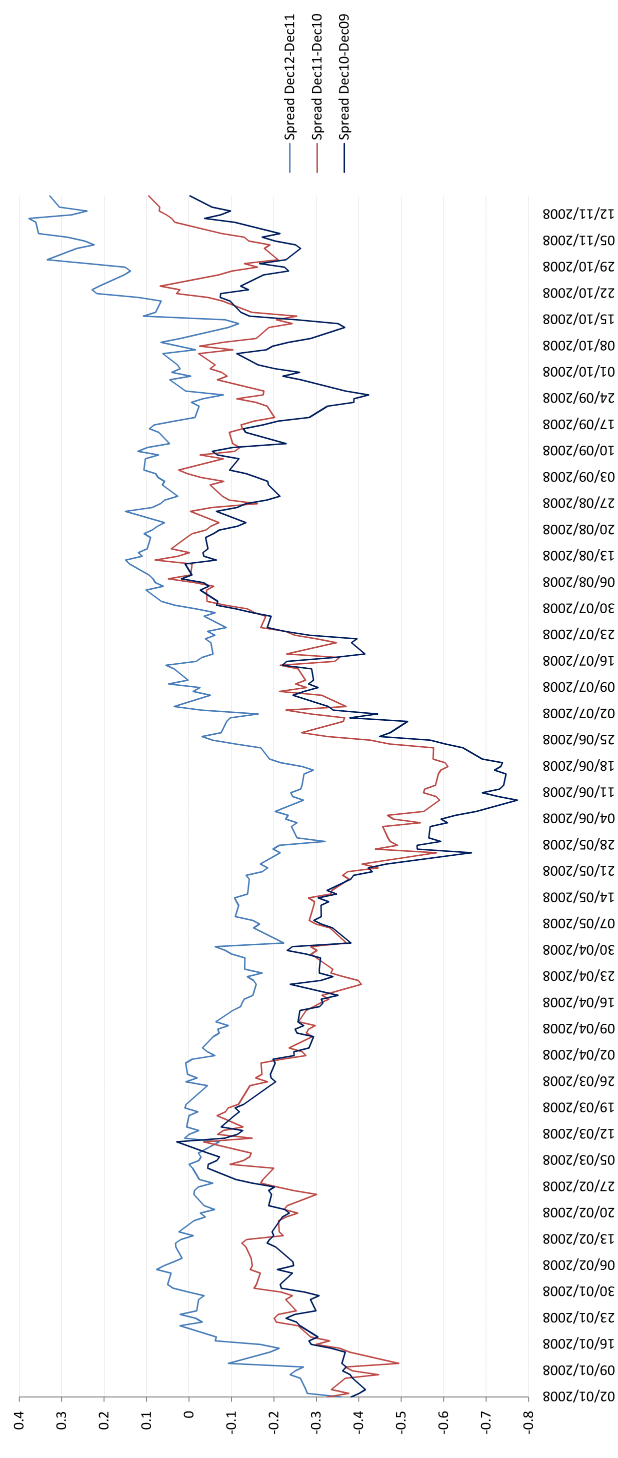

Figure 4.3: Spread between two subsequent futures contracts of Phase II discounted to December 2009 money value using EURIBOR futures during 2008. Prices were quoted in the European market from 02/01/2008 to 17/11/2008. (Source: Bloomberg)

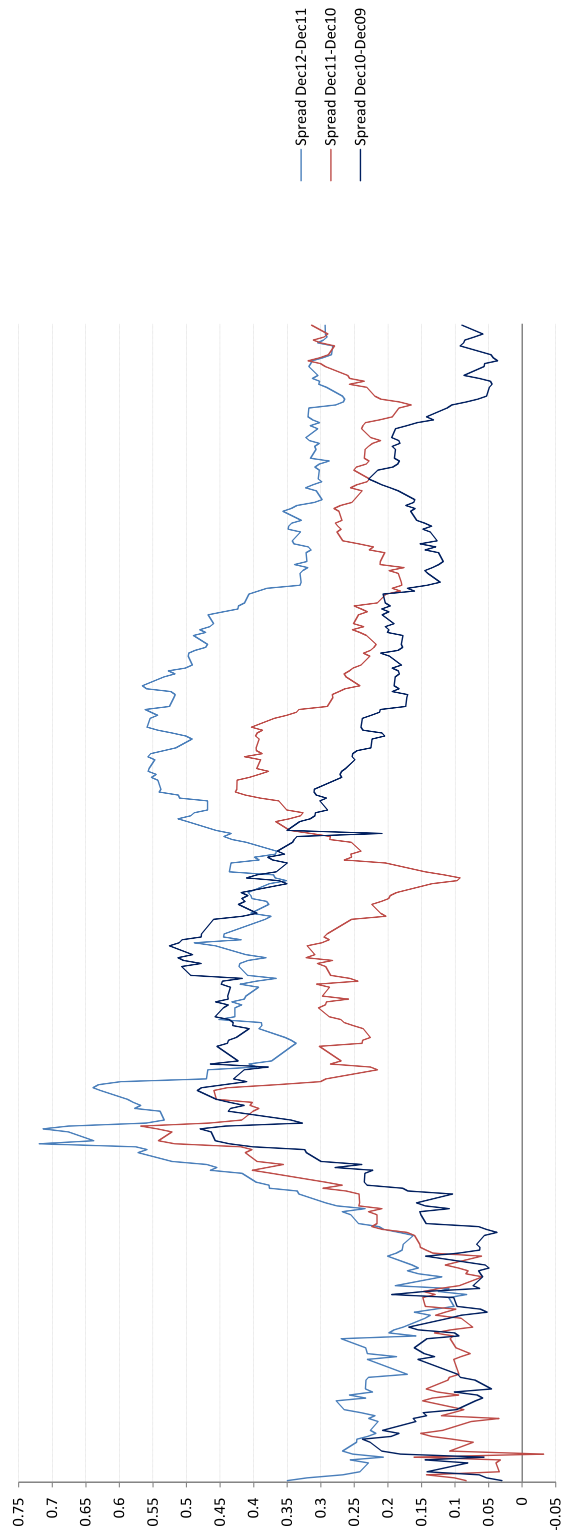

Figure 4.4: Spread between two subsequent futures contracts of Phase II discounted to December 2009 money value using EURIBOR futures during 2009. Prices were quoted in the European market from 18/11/2008 to 14/12/2009. (Source: Bloomberg)

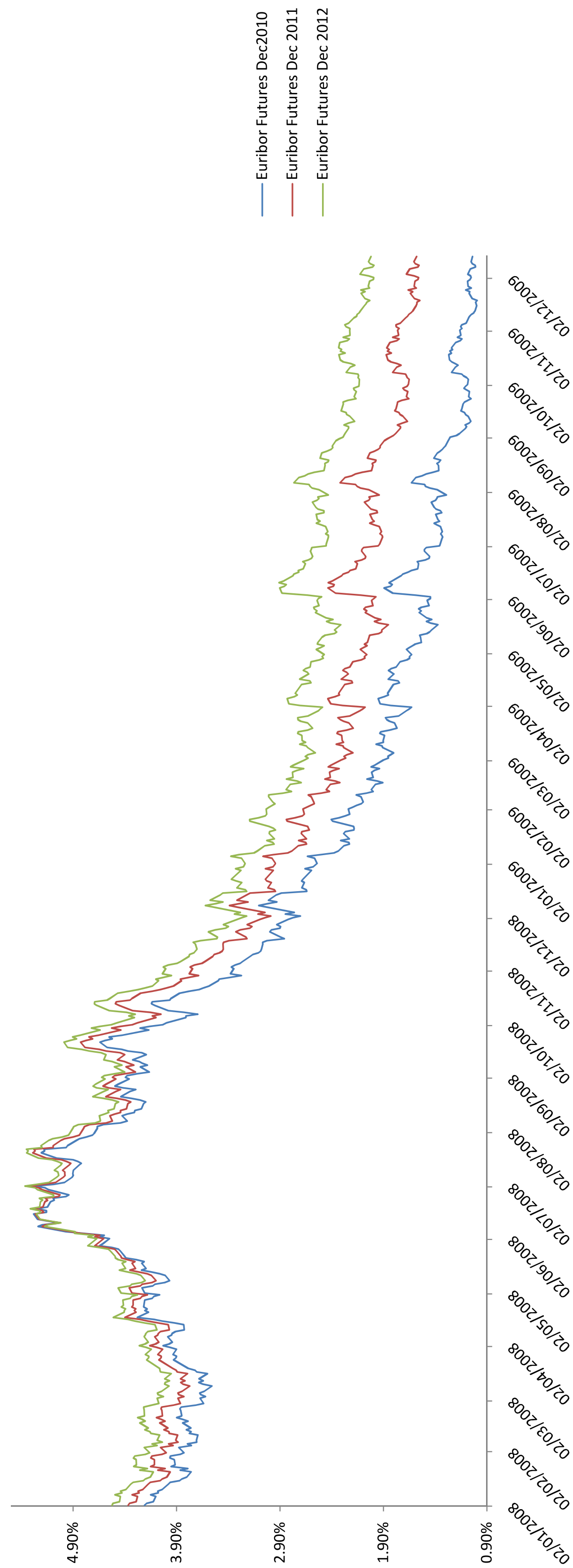

Figure 4.5: EURIBOR futures rates per annum between December 2009 and December 2010-2012. We use daily rates from 02/01/2008 to 14/12/2009. (Source: Bloomberg)

starting with post Kyoto. Emissions credits for electricity producers will be fully auctioned by 2013 and analysts expect the market to be short post 2012. Thus futures contracts with a late delivery date are more valuable and have an intrinsic value that prices the expected market position and the future change of regulatory framework, and the positive contango spread is dominant in Figure 4.4.

Banking and borrowing have an opposite impact on the discounted futures term structure. Banking dominates if the market is currently long and expected to be short later, while borrowing prevails if the market is presently short but projected to be long later. The spread between discounted futures contains information about future market adjustment and its expected position. This leads us to believe in the existence of a relationship that can be useful in pricing contingent claims.

4.2.2 Parameter Estimation

Our analysis focused on Dec-2009 (S_t^1) and Dec-2010 (S_t^2) contracts during the trading period from January 2008 until December 2009 (500 observations). Figure 4.6 reports price level histograms for both contracts. The market has two dominant states: beginning with low prices (i.e. the market was long); prices are high for the second state (i.e. the market was short). For the Dec-2009 contract, the prices for the last trading days varied near the bottom of the €12 to €14.5 overall trading range, which means that the 2009 compliance period ended "long".

We suppose that ξ_t^i is constant over time. We estimate ξ^{iu} by taking the average of the upward movement returns for S_t^i and estimate ξ^{id} by considering only the downward movement returns. The result is:

$$\widehat{\xi^{1u}} = 1.0205 \quad \widehat{\xi^{1d}} = 0.9800 \tag{4.3}$$

$$\widehat{\xi^{2u}} = 1.0380 \quad \widehat{\xi^{2d}} = 0.9827. \tag{4.4}$$

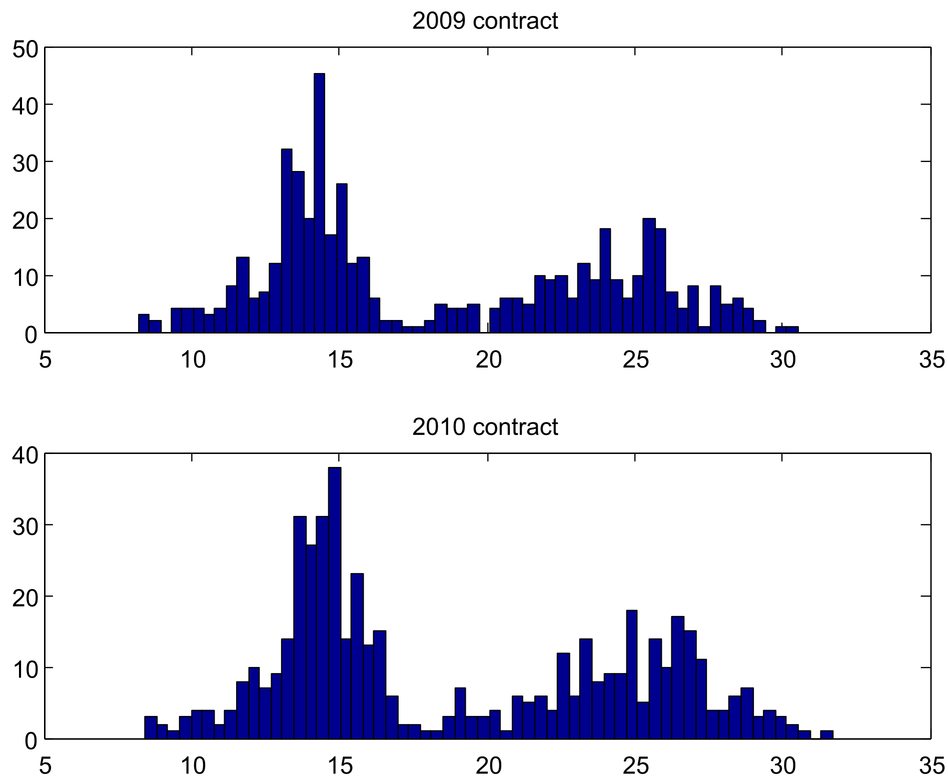

Figure 4.6: Price distribution for Dec-2009 and Dec-2010 contracts. The prices were quoted in the European market from 02/01/2008 to 14/12/2009.

θ	0.92
μ_1	1.34e-04
σ_1	14.06e-03
μ_2	-1.77e-03
σ_2	41.70e-03

Table 4.1: Gaussian mixture parameters estimated using an Expectation Maximization algorithm.

Moreover, we assume that:

$$S_t^1 = f_{t-1}^1(\xi^1, Y_{t-1}^1)S_{t-1}^1, \forall t \geq 1, \tag{4.5}$$

where $f_{t-1}^1(\xi^1, Y_{t-1}^1) = \xi^1 + Y_{t-1}^1$, and $(Y_t^1)_{t\geq 0}$ are independent and identically distributed (i.i.d) random variables and play the role of excess returns. Y_{t-1}^1 takes positive values if at time $t-1$ the market is expected to be short and negative values otherwise. This assumption implies that the expected market position at the end of 2009 is fully described by the Dec-2009 future contract dynamic. We assume Y^1 has a continuous distribution. As shown later, the mixture of two univariate Gaussian distributions can be a decent fit to the Y_t^1 distribution. Define φ as the probability density function (pdf) of Y_t^1:

$$\varphi(x) = \theta\psi(\mu_1, \sigma_1) + (1-\theta)\psi(\mu_2, \sigma_2), \tag{4.6}$$

where $\psi(\mu, \sigma)$ is the normal pdf with mean μ and standard deviation σ. The Gaussian mixture distribution offers a flexibility in modeling the excess return with two possible distribution outcomes depending on the state of nature. With probability θ, the excess return has a normal distribution with a pdf $\psi(\mu_1, \sigma_1)$ and it is normally distributed following the pdf $\psi(\mu_2, \sigma_2)$ with a chance of $1-\theta$.

We estimate the Gaussian mixture parameters using an Expectation Maximization algorithm (Mclachlan and Peel, 2000). Table 4.1 reports the obtained parameters. Figure 4.7 compares the empirical and the fitted Gaussian mixture cumulative distribution function (CDF).

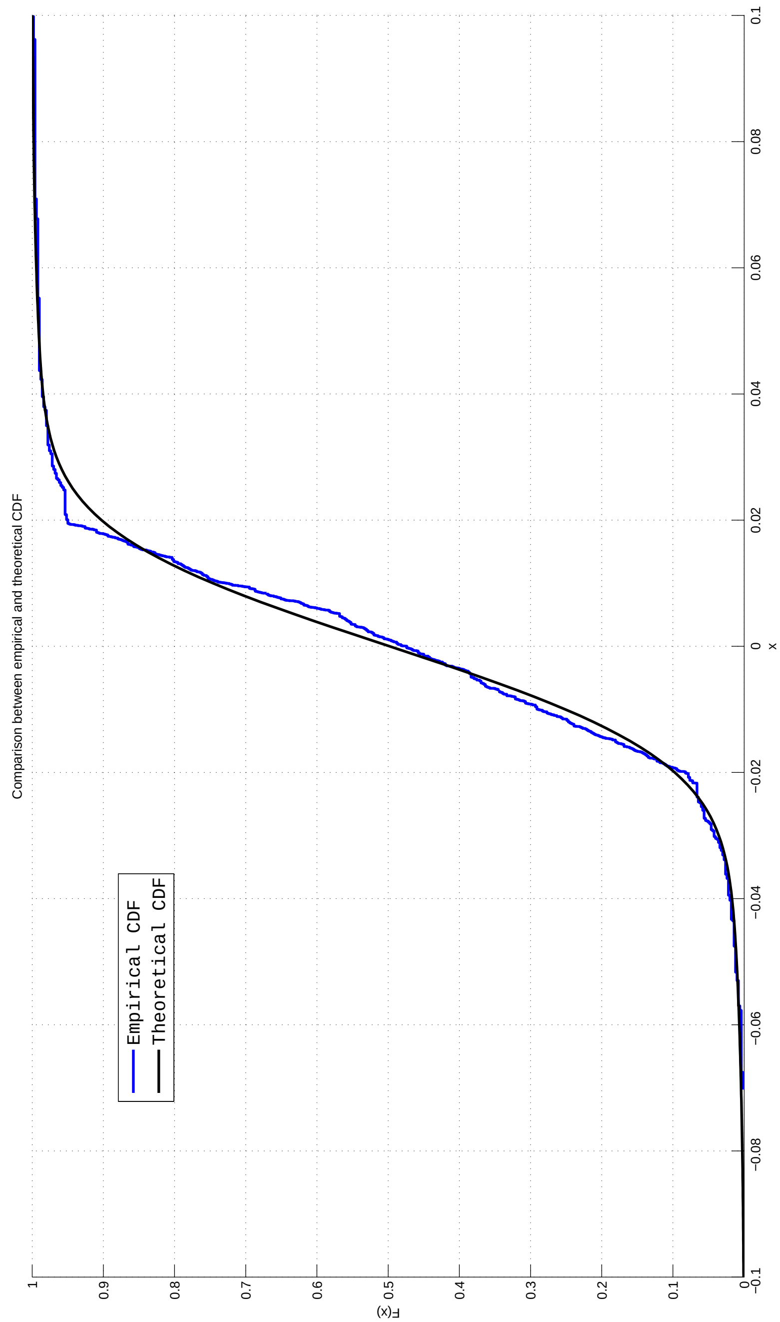

Figure 4.7: This figure shows that the mixture of two univariate Gaussian distributions is a decent fit to the expected market position at the end of 2009. It reports a comparison between the empirical CDF and the theoretical Gaussian mixture CDF. Parameters as given in Table 4.1.

With a probability of 0.92, the market expectation has the same chance either to be long or short. However with a probability of 0.08, the market is expected to be long 52% of the time. At first this seems fairly uninteresting to be a minor and distinction, but recall that the other parameters also vary between states.

After analyzing the 2009 excess return dynamics in depth, we investigate whether we can explain the 2010 excess returns by means of the 2009 excess returns. Figure 4.8 shows that the market reaction depends on the contract. We assume that

$$S_t^2 = f_{t-1}^2(\xi^2, Y_{t-1}^2) S_{t-1}^2, \forall t \geq 1, \tag{4.7}$$

where $f_{t-1}^2(\xi^2, Y_{t-1}^2) = \xi^2 + Y_{t-1}^2$, and $(Y_t^2)_{t\geq 0}$ are i.i.d. We suppose that Y_t^2 has the following structural equation:

$$Y_t^2 = g(Y_t^1) + I_t + u_t, \tag{4.8}$$

where u_t are i.i.d such that $E[u_t \mid (Y_t^1, I_t)] = 0$. The explanatory variable $g(Y_t^1)$ represents the causality of the Y_t^1 outcome on Y_t^2, while I_t represents the component of impact of the expected market position solely affecting Y_t^2. Without lost of generality, we assume $E[I_t] = 0$ because any constant offset can be absorbed by g. However I_t is an unobserved heterogeneity and we have a correlated variable which is the market state MS_t that takes the value +1 if the market is expected to be short ($Y_t^1 > 0$), -1 otherwise. Here MS_t can be a proxy for I_t. We assume that MS_t is exogenous and satisfies $E[Y_t^2 \mid (Y_t^1, I_t, MS_t)] = E[Y_t^2 \mid (Y_t^1, I_t)]$[5]. In other words conditional on (Y_t^1, I_t), the proxy variable MS_t does not provide any additional information and is irrelevant for explaining the conditional mean of Y_t^2. Furthermore we suppose that:

$$I_t = h_0 + h_1 MS_t + \upsilon_t, \tag{4.9}$$

[5]This condition is always valid as MS_t is the sign of Y_t^1.

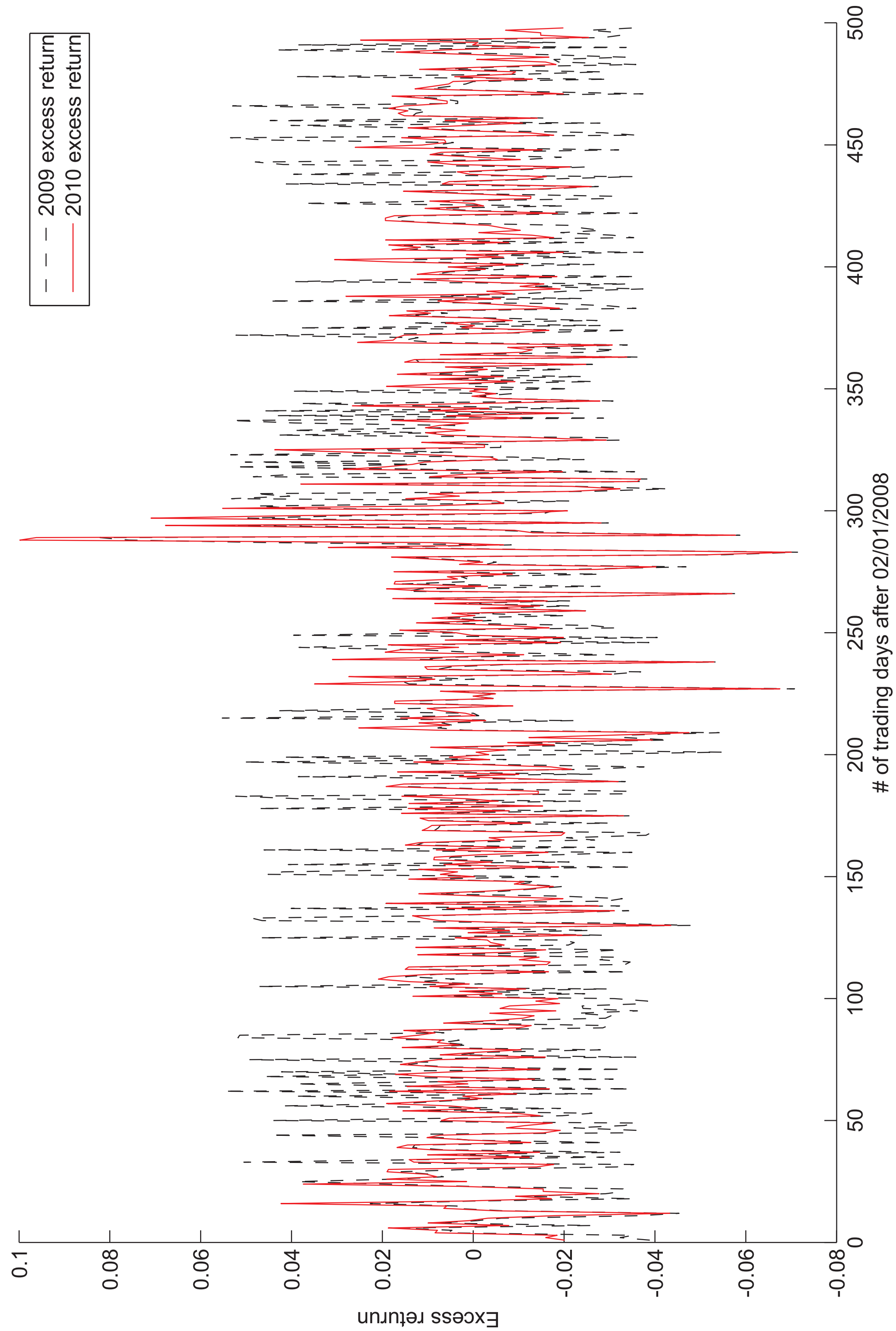

Figure 4.8: This figure shows that the market reaction depends on the contract. It reports the excess returns for Dec-2009 and Dec-2010 contracts.

where $(\upsilon_t)_{t\geq 0}$ are i.i.d such that $E[\upsilon_t] = 0$, $E[\upsilon_t MS_t] = 0$ and $E[\upsilon_t g(Y_t^1)] = 0$. We obtain the structural equation:

$$Y_t^2 = g(Y_t^1) + h_0 + h_1 MS_t + \varepsilon_t, \tag{4.10}$$

where $\varepsilon_t = \upsilon_t + u_t$ is the composite error, and is uncorrelated with $g(Y_t^1)$. See Wooldridge (2002) for a detailed development on the proxy variable technique and its properties. On the other hand we suppose that g is a polynomial of order p so that the estimation of the partial effect of Y_t^1 on Y_t^2 is tractable. As a consequence, we end up with the following linear equation with high order terms:

$$Y_t^2 = (h_0 + a_0) + \sum_{k=1}^{p} a_k (Y_t^1)^k + h_1 MS_t + \varepsilon_t. \tag{4.11}$$

We estimate the structural equation (4.11) using the ordinary least square method, which is, under our assumptions, consistent with $(h_0 + a_0, (a_k)_{k\geq 1}, h_1)$ and is the best linear unbiased estimator (Wooldridge, 2002). Table 4.2 reports some estimated parameters and the R^2 as function of the polynomial order p. Table 4.2 also displays some estimated parameters of the structural equation (4.8) if we omit I_t. The proxy variable MS_t helps to better explain the residual, and therefore it is suitable to include I_t in the structural equation (4.8). The obtained results show that about 75% of the 2010 expiry returns can be explained by the information associated with the compliance period relative to the Dec-2009 contract. This dependency is expected because of the continuity between compliance periods that arises from the ability to borrow. Nevertheless about 25% of the 2010 excess return dynamics depend on the market expectation and the intrinsic risk related to the 2010 and post-2010 trading periods. Moreover the intercept parameter $(h_0 + a_0)$ is adjusted by $\pm h_1$ depending on the market expectation. The coefficient a_1 is always positive, which leads us to think that Y_t^1 and Y_t^2 vary such that their outcomes have the same sign. We focus on this point later after estimating the historical probability.

	With the proxy variable				I_t omitted		
	R^2	$h_0 + a_0$	a_1	h_1	R^2	a_0	a_1
p=0	68.21%	-3.60e-03	-	2.25e-02	-	-	-
p=1	73.86%	-3.20e-03	5.41e-01	1.52e-02	60.35%	-2.40e-03	1.17
p=2	73.98%	-2.81e-03	5.60e-01	1.49e-02	61.39%	-1.29e-03	1.19
p=3	74.98%	-2.28e-03	2.78e-01	1.77e-02	62.67%	-2.05e-03	1.34
p=4	75.22%	-1.56e-03	3.51e-01	1.70e-02	64.73%	-5.18e-05	1.43
p=5	75.48%	-1.05e-03	2.07e-01	1.81e-02	65.34%	-9.16e-04	1.54
p=6	75.92%	-1.09e-04	3.85e-01	1.68e-02	68.08%	1.27e-03	1.73
p=7	75.95%	9.99e-05	3.38e-01	1.71e-02	68.45%	4.34e-04	1.82
p=8	76.10%	3.62e-04	4.93e-01	1.62e-02	70.17%	1.20e-03	2.07
p=9	76.34%	1.27e-03	4.67e-01	1.63e-02	70.34%	1.98e-03	2.05
p=10	76.34%	1.27e-03	4.67e-01	1.63e-02	70.34%	1.98e-03	2.05

Table 4.2: Parameters resulting from the OLS estimator as function of the polynomial degree p in both cases where I_t is omitted or approximated by a proxy variable.

Since we are working within a model for two traded allowances, each node of the tree generates 8 (=2 contracts x 2 tree branches x 2 market states) possible states at the next time step (See Figure 4.9). $p_i, i = 1, ..., 8$, represents the probability, under the historical measure, that the event i occurs. We assume that it is constant over time and we estimate it with the unbiased estimator:

$$p_i = \frac{\text{Number of times that the event } i \text{ occurs}}{\text{Sample size}}. \tag{4.12}$$

Table 4.3 reports the probability weights of the 8 states. We remark that the probability that the returns of S_t^1 and S_t^2 evolve in an opposite direction is almost zero. This is consistent with the previously obtained positive sign of a_1. Thus the returns have the same dynamic pattern for both Dec-2009 and Dec-2010 contracts. However there is a spread between the prices that evaluates the future uncertainty of the market position as well as the market adjustment at the compliance period. As a possible consequence, a more appropriate hedging strategy in order to replicate a derivative written on S_t^1 should incorporate a position on S_t^2. The next section develops this statement further.

We conclude that Dec-2009 and Dec-2010 futures cannot be interpreted as two indepen-

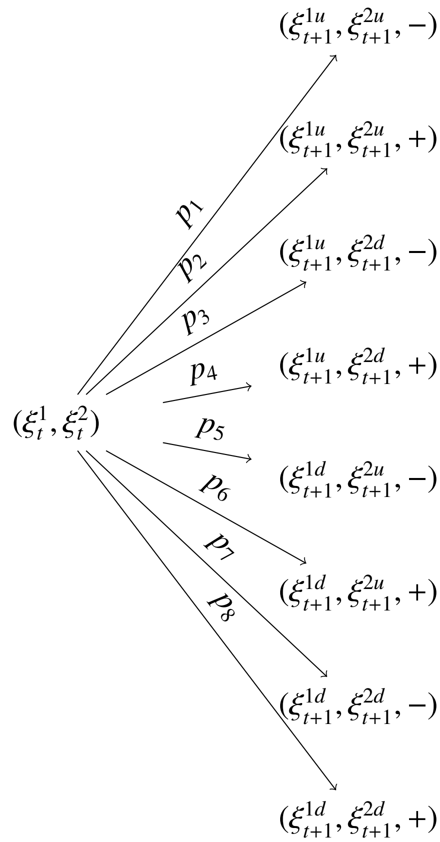

Figure 4.9: States of nature generated at time $t + 1$ by a node at time t.

Probability	Weight
p_1	0.261
p_2	0.205
p_3	0.006
p_4	0.004
p_5	0
p_6	0.002
p_7	0.255
p_8	0.267

Table 4.3: Estimated probability weights of the states of nature that are described in figure 4.9.

dent commodity contracts. Furthermore an effective market design requires a long term stable regulatory framework as the compliance periods are mutually correlated.

Chapter 5

Recommendations for Effective Trading and Market Design

5.1 How effective is hedging in a one period model?

The market design shows long term uncertainty such that participants cannot have a clear information signal to evaluate derivatives written on allowances. Moreover, to ensure liquidity, private or institutional investors are encouraged to trade (which cannot be buying contracts for fundamental reasons). For the EU ETS, U.S. funds have been responsible for 10 – 15% of traded volume starting from 2008 on the European Climate Exchange[1]. On the other hand, the spread between S_t^1 and S_t^2 contains information about the expected market position at the compliance date as well as the expected market adjustment by the regulator for the new compliance period. The implied volatility is strongly affected by this kind of information and consequently the derivatives market written on allowances is also influenced. This suggests that a pricing model should consider the existence of tradable permits that mature at subsequent dates. In the EU ETS, contracts are traded up to Dec-2020, including the post Kyoto phase. We present a pricing framework that considers all these features and is based on some intuitive criterion.

[1] According to the State and Trend of the Carbon Market 2010 World Bank report.

Introduced by Schäl (1994) and developed for more general one dimensional processes by Schweizer (1996), the quadratic criteria consists of solving an optimization problem which defines a *fair* price and its associated attainable portfolio that minimizes the quadratic risk[2]. More precisely, for a payoff given by a particular random variable $H \in \mathcal{L}^2(\mathcal{P})$, the initial capital V_0 which allows the best approximation of H by the cumulative trading gains $G_T(\zeta)$ associated with the self financing portfolio determined by ζ solves:

$$(V_0, \zeta) = \underset{(c,\vartheta)\in\mathbb{R}\times\Theta}{\arg\min}\ E_{\mathcal{P}}[(H - c - G_T(\vartheta))^2], \tag{5.1}$$

where

$$\Theta := \{\text{predictable processes } \vartheta | \vartheta'_k \Delta S_k \in \mathcal{L}^2(\mathcal{P})\}, \tag{5.2}$$

$$G_T(\vartheta) := \sum_{j=1}^{T} \vartheta'_j \Delta S_j. \tag{5.3}$$

Following a similar procedure to that used by Schweizer (1996) to prove the existence and uniqueness of the similar one dimensional problem, we generalize Rémillard and Rubenthaler's work (Rémillard and Rubenthaler, 2009) in order to present the solution of (5.1) under very mild technical assumptions.

Proposition 5.1.1 *Assume a probability space* $(\Omega, \mathbb{F}, \mathcal{P})$ *and stochastic process* $(S_t)_{t\in\mathcal{T}} \in \mathcal{L}^2_d(\mathcal{P})$ *adapted to the filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ *such that* $E[\Delta S_k^2 | \mathcal{F}_{k-1}]$ *is invertible a.s. and satisfies the non-degeneracy condition* [3]. *Therefore, there exists a unique solution* (V_0, ζ) *solving* (5.1),

[2]We assume frictionless trading.
[3]See Appendix A for notations and definitions

where:

$$\zeta_k = \varrho_k - \beta_k(V_0 + G_{k-1}(\zeta)), \tag{5.4}$$

$$V_0 = E_{\widetilde{P}}[H], \tag{5.5}$$

$$\varrho_k = \left(E\left[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)^2|\mathcal{F}_{k-1}\right]\right)^{-1} E\left[H\Delta S_k \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}\right], \tag{5.6}$$

$$\beta_k = \left(E\left[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)^2|\mathcal{F}_{k-1}\right]\right)^{-1} E\left[\Delta S_k \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}\right], \tag{5.7}$$

$$\frac{d\widetilde{P}}{dP} = \frac{\widetilde{Z}_0}{E[\widetilde{Z}_0]}, \tag{5.8}$$

$$\widetilde{Z}_0 = \prod_{j=1}^{T}(1-\beta_j'\Delta S_j). \tag{5.9}$$

Furthermore, the unhedgeable risk defined by (5.1) is:

$$V_0^2 E\left[\widetilde{Z}_0\right] - 2V_0 E\left[H\widetilde{Z}_0\right] + E\left[(H - \sum_{j=1}^{T}\varrho_j'\Delta S_j \prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l))^2\right] \tag{5.10}$$

Proof See Appendix A

With Proposition 5.1.1 in hand, we have a solution for (5.1) under very mild assumptions that cover a wide family of discrete processes. From the perspective of market participants, a better hedging strategy is the one that provides less risk as measured by (5.10).

To sell a contract written on S_t^1 and maturing at the compliance date [4] T_1, an investor needs to hedge its associated risk and so avoid tail loss scenarios. In our case, a two-period market design provides a choice between: strategy A where only S_t^1 is traded, or a strategy B that includes positions in both S_t^1 and S_t^2. With strategy A, the investor considers only the states of nature that are observed by S_t^1 fluctuations, i.e. the S_t^1-filtration. The information provided by S_t^2 is neglected, and so is the market expectation at time T_1^- and during the subsequent compliance period that ends at time[5] T_2. In fact our empirical investigation suggests that about 25%

[4]Note that T_1 can be any date during the earlier compliance period.
[5]We take T_1 to be December 2009 and T_2 to be December 2010.

of S_t^2 is driven by uncertainty originating from the following compliance periods. This implied factor is very important especially when the derivative contract nears maturity at the end of the current compliance date. The market adjustment by the regulator as well as the banking possibility generates a puzzling behavior of the allowance dynamics and thus its implied volatility. Strategy B offers more flexibility to the contract issuer and allows him to partially hedge the regulatory uncertainty at maturity.

Our numerical examples support the statement. Assume 5 business days remain to the last trading day of S_t^1. This last trading day is also the maturity date of the option to be priced. We consider different possible scenarios depending on the current and expected market positions. If the market is currently short (i.e. are high prices) we use initial prices as quoted on 4/4/2008:

$$S_0^1 = €23.96 \text{ and } S_0^2 = €24.61. \tag{5.11}$$

else, i.e. the market is long (low prices), we take the quoted prices on 26/3/2009:

$$S_0^1 = €10.95 \text{ and } S_0^2 = €11.6. \tag{5.12}$$

Using the estimated parameters from the previous section[6], we compare strategies A and B. A technical point about the filtration set size arises. More states result from generating the (S_t^1, S_t^2) tree than from generating the S_t^1 tree. To overcome this obstacle, we compare the average of prices and unhedged risks over 1000 different scenarios. Table 5.1 reports the average prices for call and put options for different moneyness. Define the associated risk gain factors (GF) as the ratio of the unhedgeable risk using strategy A over the unhedgeable risk resulting from strategy B. Table 5.2 shows the minimum, median and mean GF of the 1000 different scenarios repeated 25 times. For both puts and calls, the unhedgeable risk does not depend on the current market state, i.e. if allowance prices are initially low (i.e. market is currently long) or high (i.e. market is currently short). In fact for the same generated uncertainty Y_t^1

[6] We approximate Y_t^2 by $E[Y_t^2|Y_t^1] = (h_0 + a_0) + \sum_{k=1}^{9} a_k (Y_t^1)^k + h_1 MS_t$.

and Y_t^2, the initial price does not impact the risk as the payoff is an affine function of S^1 and we report results in terms of moneyness. Moreover, the unhedgeable risk for the call is equal to the unhedgeable risk for the put with the same parameters (which follows from put-call parity). Neither the call nor the put are totally replicable. Furthermore the results strengthen our statement that the market participant is able to construct a less risky strategy by holding positions in both S_t^1 and S_t^2 ($GF > 1$). Strategy B outperforms strategy A for different maturities with a maximum attained for at the money options, whose prices are strongly correlated to the implied allowance volatility fluctuations at the end of the compliance period. Appendix D presents a sufficient condition that permits the reduction of unhedgeable risk by including positions on an additional risky asset. It can be summarized as follows: S_t^2 must be correlated close to maturity to the non-replicable payoff by hedging with S_t^1. Table 5.1 suggests that the cost of both strategies are almost the same. A possible explanation can be provided by writing equation (5.10) as follows:

$$-V_0^2 + \frac{E\left[(H - \sum_{j=1}^{T} \varrho_j' \Delta S_j \prod_{l=j+1}^{T}(1 - \beta_l' \Delta S_l))^2\right]}{E\left[\widetilde{Z}_0\right]} \tag{5.13}$$

Given constant V_0, trading in both S_t^1 and S_t^2 provides a strategy for which (5.13) has a smaller second term than the equivalent expression formed by trading only S_t^1. Intuitively the former strategy allows the market participant to consider a larger set of information and hedging with correlated primary assets that distinguish more possible scenarios.

To conclude, hedging using a multiperiod pricing framework is more efficient than hedging in a one period model. This is a special feature of the carbon market where the compliance periods are mutually correlated by banking and borrowing possibilities, and thus futures with different maturities cannot be treated as completely dependent or independent contracts.

Call									
Market state	Strategy	Moneyness	.925	.95	.975	1	1.025	1.05	1.075
SL	A	Price	1.8019 (0.12%)	1.2276 (0.53%)	0.7157 (1.36%)	0.3285 (2.03%)	0.1051 (1.86%)	0.0213 (0.70%)	0.0028 (0.14%)
	B	Price	1.8018 (0.13%)	1.2270 (0.53%)	0.7141 (1.34%)	0.3252 (1.97%)	0.1016 (1.81%)	0.0195 (0.65%)	0.0025 (0.12%)
SS	A	Price	1.7989 (0.07%)	1.2150 (0.34%)	0.6869 (1.04%)	0.2927 (1.61%)	0.0834 (1.28%)	0.0170 (0.42%)	0.0026 (0.09%)
	B	Price	1.7989 (0.07%)	1.2147 (0.34%)	0.6848 (1.03%)	0.2883 (1.68%)	0.0795 (1.31%)	0.0158 (0.40%)	0.0024 (0.08%)
LL	A	Price	0.8235 (0.06%)	0.5610 (0.24%)	0.3271 (0.62%)	0.1501 (0.93%)	0.0480 (0.85%)	0.0097 (0.32%)	0.0013 (0.06%)
	B	Price	0.8235 (0.06%)	0.5607 (0.24%)	0.3264 (0.61%)	0.1486 (0.90%)	0.0464 (0.83%)	0.0089 (0.30%)	0.0011 (0.05%)
LS	A	Price	0.8221 (0.03%)	0.5552 (0.16%)	0.3139 (0.48%)	0.1337 (0.74%)	0.0381 (0.59%)	0.0078 (0.19%)	0.0012 (0.04%)
	B	Price	0.8221 (0.03%)	0.5551 (0.16%)	0.3129 (0.47%)	0.1318 (0.77%)	0.0363 (0.60%)	0.0072 (0.18%)	0.0011 (0.04%)
Put									
Market state	Strategy	Moneyness	.925	.95	.975	1	1.025	1.05	1.075
SL	A	Price	0.0049 (0.12%)	0.0295 (0.53%)	0.1167 (1.36%)	0.3285 (2.03%)	0.7040 (1.86%)	1.2193 (0.70%)	1.7998 (0.14%)
	B	Price	0.0048 (0.13%)	0.0290 (0.53%)	0.1151 (1.34%)	0.3253 (1.97%)	0.7006 (1.81%)	1.2175 (0.65%)	1.7995 (0.12%)
SS	A	Price	0.0019 (0.07%)	0.0170 (0.34%)	0.0879 (1.04%)	0.2927 (1.61%)	0.6824 (1.28%)	1.2150 (0.42%)	1.7996 (0.09%)
	B	Price	0.0019 (0.07%)	0.0167 (0.34%)	0.0858 (1.03%)	0.2883 (1.68%)	0.6785 (1.31%)	1.2138 (0.40%)	1.7994 (0.08%)
LL	A	Price	0.0022 (0.06%)	0.0135 (0.24%)	0.0534 (0.62%)	0.1501 (0.93%)	0.3218 (0.85%)	0.5572 (0.32%)	0.8225 (0.06%)
	B	Price	0.0022 (0.06%)	0.0132 (0.24%)	0.0526 (0.61%)	0.1486 (0.90%)	0.3202 (0.83%)	0.5564 (0.30%)	0.8224 (0.05%)
LS	A	Price	0.0008 (0.03%)	0.0078 (0.16%)	0.0402 (0.48%)	0.1338 (0.74%)	0.3119 (0.59%)	0.5553 (0.19%)	0.8224 (0.04%)
	B	Price	0.0009 (0.03%)	0.0077 (0.16%)	0.0392 (0.47%)	0.1318 (0.77%)	0.3101 (0.60%)	0.5547 (0.18%)	0.8223 (0.04%)

Table 5.1: Prices of strategy B with comparison to strategy A for call and put options written on S_t^1 with different strike prices. All possible initial market states are considered. The first letter (S: short and L: long) stands for the current market position and the second letter denotes the expected market position at the subsequent compliance date. For example, SL: Market is short (i.e. high prices) and expected to be long (i.e. low prices).

Call & Put								
Market state	Moneyness	.925	.95	.975	1	1.025	1.05	1.075
SL & LL	Min	1.19	1.21	1.22	1.22	1.20	1.17	1.16
	Median	1.21	1.24	1.24	1.24	1.23	1.20	1.19
	Mean	1.21	1.24	1.24	1.24	1.23	1.20	1.19
SS & LS	Min	1.11	1.18	1.22	1.22	1.23	1.21	1.18
	Median	1.17	1.20	1.23	1.24	1.24	1.23	1.20
	Mean	1.17	1.20	1.23	1.24	1.24	1.23	1.20

Table 5.2: Gain factor (GF) of strategy B with comparison to strategy A in order to price call and put options written on S_t^1 for different strike prices. All possible initial market states are considered. The first letter (S: short and L: long) stands for the current market position and the second letter denotes the expected market position at the subsequent compliance date. For example, SL: Market is short (i.e. high prices) and expected to be long.

5.2 Recommendation for Effective Market Design

The existence of an inter-dependency between compliance periods can be used to reduce the risk for any market position. This special feature distinguishes the emission market from other options markets. However individual investors as well as non-emitting firms, who fear long term regulatory ambiguity, must be enticed to trade in order to increase liquidity for speculation and diversification purposes. The market will attract more participants if the regulator intervenes and reduces the dependency between compliance periods using the fact that it is the most informed market player.

Auctioning offsets the downside effect of free allowance allocation. Economists agreed that this policy minimizes undesirable windfall profits and allows a more relevant carbon signal price. The European regulator is engaged in progressively establishing an auctioning policy. This new policy is to begin in 2013 and will be extended to fully cover all sectors by 2027. The initial funding obtained from such an allocation system can be used to finance green projects. An additional strategy could be an intervention of the regulator in the market by offering *new additional* primary trading assets. The aim of the latter solution is to offer a larger range of available traded financial contracts that can be used for risk management objectives.

This new additional traded asset should allow some of the intrinsic market risk to be priced. Thus the set of non-redundant financial assets will increase to span a larger set of attainable payoffs as well as to reduce the position risk for a general contract written on allowances. An interesting contract for this purpose is the digital option which pays a certain amount if a predefined event happens within a future time interval. In our model, the time t announcement by the regulator about the expected market position makes the set of Y_t^1 outcomes observable to market participants, and hence the current state of the market is identified. In this case the regulator can offer, as a primary financial asset, the option to pay 1 currency unit if the allowance price has just increased and he announces that the market at time T_i is expected to be short, where T_i is the nearest compliance date. For the example presented in Figure 4.2, the regulator will pay the amount if at the end of the second period the state of nature occurs in $\{\omega_1, \omega_5, \omega_9, \omega_{13}\}$. This contract type reduces the carbon leakage challenge that faces any regulatory framework which penalizes emissions. Carbon leakage describes the phenomenon in which high allowance prices spur the transfer of companies to another regulatory framework with less stringent constraints on carbon emissions. Thereby resulting in economic pain for no environment gain. Given this traded financial asset, with almost no additional risk, the investor could minimize the risk related to the market position dynamics.

The new additional traded asset is exogenous to the market participants. The regulator periodically updates its prices and quotes them in the market. Its pricing must consider the social wealth of the market rules initially established to define the market parameters Γ (e.g. endowments, compliance period length and penalty level) rather than dynamic hedging. Moreover the method should use the information provided by allowance dynamics to provide a price that is consistent with the arbitrage free theory. We propose an indifference pricing methodology to price this new additional traded asset. This approach quantifies risk using a nonlinear transformation and derives a coherent price by solving an optimization problem based on the concept of the expected utility of wealth.

To present the approach, we consider a single market model in which only S_t^1 is traded. Define G as the derivative depending only on Y_t^1 that will be offered by the regulator. Consider U as the utility function of the representative agent (Duffie, 2001). Given Γ, the regulator is maximizing:

$$V_0(x, \Gamma) = \sup_{\alpha} E_P \left[U(X^{x,\alpha}, \Gamma) \right], \tag{5.14}$$

where $X^{x,\alpha} = x + \alpha(S_{T_1} - S_0)$ is the final wealth, and x represents the initial wealth. We could interpret S_0 as the auction price and α as the initial allowance allocation that the regulator will distribute across the economy. By introducing G, the new optimization problem replacing (5.14) is:

$$V_G(x, \Gamma) = \sup_{\alpha} E_P \left[U(X^{x,\alpha} - G, \Gamma) \right]. \tag{5.15}$$

The idea behind the indifference price methodology is to define a price $\nu(G_{T_1})$ such that (5.14) and (5.15) have the same supremum:

$$V_0(x, \Gamma) = V_G(x + \nu(G_{T_1}), \Gamma) \tag{5.16}$$

As an instructive example, we present the solution for the exponential utility

$$U(x) = -e^{-\gamma x}, \quad \forall x \in \mathbb{R} \text{ and } \gamma > 0. \tag{5.17}$$

Here γ is a parameter which is selected by the regulator. It looks like the risk aversion parameter in a utility function but this may not be the best way to think of it. More discussion of γ's meaning is given near the end of this section. For a general utility function and setting, we refer the reader to Elliot and Van der Hoek (2009). Let $\mathcal{F}^{S^1} = \sigma\{S^1_t, t = 0, T_1\}$. Define F_1, F_2 as the atoms of $\mathcal{F}^{S^1}$. Also define for each atom F_s, $1 \leq s \leq 2$, $A_s = \{i \in \{1, ..., 4\} | \omega_i \in F_s\}$. We denote by q_1 the price at time 0 of 1 currency unit that is paid at time T_1 if the futures price rises and $q_2 = 1 - q_1$ the price at time 0 of 1 unit of money time T_1 if the futures price goes down. These are Arrow-Debreu securities. It is well known that q_1 and q_2 can be interpreted as the unique equivalent risk neutral probability measure Q associated with $\mathcal{F}^{S^1}$:

$$q_1 = \frac{1-\xi^d}{\xi^u - \xi^d} \tag{5.18}$$

As described in Elliot and Van der Hoek (2009), solving (5.15) is equivalent to solving:

$$V_G(x,\Gamma) = \sup_{\substack{x_1,x_2 \\ x_1q_1+x_2q_2=x}} \sum_{s=1}^{2}\left(\sum_{l\in A_s} U(x_s - g_l)p_l\right) \tag{5.19}$$

The Lagrange function associated to this maximization problem is defined as:

$$F(x_1,x_2) = \sum_{s=1}^{2}\left(\sum_{l\in A_s} U(x_s - g_l)p_l\right) + \lambda\left[x - \sum_{s=1}^{2} x_s q_s\right], \tag{5.20}$$

where $G_{T_1}(\omega_l) = g_l$. The first-order conditions give:

$$\begin{aligned} x_s &= \frac{1}{\gamma}\log\left[\frac{\gamma}{q_s}\sum_{l\in A_s} p_l e^{\gamma g_l}\right] - \frac{1}{\gamma}\log\lambda \\ &= x + \frac{1}{\gamma}\log\left[\frac{\gamma}{q_s}\sum_{l\in A_s} p_l e^{\gamma g_l}\right] \\ &\quad -\frac{1}{\gamma}\sum_{r=1}^{2}\left(q_r\log\left[\frac{\gamma}{q_r}\sum_{l\in A_r} p_l e^{\gamma g_l}\right]\right), \end{aligned} \tag{5.21}$$

$$\lambda = \left[\gamma e^{-\gamma x}\prod_{s=1}^{2}\left(\frac{\sum_{l\in A_s} p_l e^{\gamma g_l}}{q_s}\right)^{q_s}\right]^{\frac{\gamma}{1+\gamma}}. \tag{5.22}$$

So,

$$V_G(x) = -\frac{1}{\gamma}e^{-\gamma x}\exp\left(\sum_{r=1}^{2} q_r\log\left(\frac{\gamma}{q_r}\sum_{l\in A_r} p_l e^{\gamma g_l}\right)\right). \tag{5.23}$$

Therefore,

$$v(G_{T_1}) = \frac{1}{\gamma}E_Q(\log(E_P(e^{\gamma G_{T_1}}|S^1_{T_1}))). \tag{5.24}$$

Because of the exponential utility assumption, this indifference price [7] does not depend on the wealth level. It is worth mentioning that (5.24) evaluates the nonhedgeable part with respect to the risk parameter γ as well as the utility function. The adjusted payoff becomes attainable and then the dynamic hedging principle is used to price it. This methodology combines the utility maximization concept with the usual linear pricing approach for complete markets.

After exploring a one period pricing model, we present the approach under a multiperiod setting. As in Musiela and Zariphopoulou (2004), we define the nonlinear operator:

$$\mathcal{E}_Q^{(s,s+1)}(L_{s+1}) = E_Q\left(\frac{1}{\gamma_s}\log\left(E_P(e^{\gamma_s L_{s+1}}|\mathcal{F}_s \vee \mathcal{F}_{s+1}^{S^1})\right)|\mathcal{F}_s\right), \tag{5.25}$$

where

$$\mathcal{E}_Q^{s,s}(L_s) = L_s, \tag{5.26}$$

and γ_s and L_s are $\mathcal{F}_s$-adapted process. Define $\nu_t(G_{T_1})$ as the indifference price, which is $\mathcal{F}_t$-adapted process. Starting with the final condition

$$\nu_{T_1}(G_{T_1}) = G_{T_1}, \tag{5.27}$$

we compute $\nu_t(G_{T_1})$ by moving backwards and using the following relationship:

$$\nu_t(G_{T_1}) = \mathcal{E}_Q^{(t,t+1)}(\nu_{t+1}(G_{T_1})). \tag{5.28}$$

For a multi-traded assets framework, Lim (2006) presents a numerical algorithm to compute $\nu_t(G_{T_1})$ based on the duality between the exponential utility function and the minimum relative entropy.

[7] In general $\nu(\beta G_{T_1}) \neq \beta\nu(G_{T_1})$ for $\beta \neq 1$. However we ignore this property. The suboptimality can simply be recovered by using the money obtained from the initial auction or the regulator can price the βG_{T_1} contracts that he has already decided to offer and present a scheme to sell them to market traders. Many other solutions for this suboptimality could be proposed.

The additional traded asset υ_t provided from the regulator uses only information with respect to the market position for the current compliance period. We proceed by a numerical investigation where the previously described digital option is considered and γ_s is constant equal to γ. Its injection to the market will not affect the total social cost. However, it provides the alternative of hedging against scenarios in which prices spike at the end of the compliance period. Define strategy C as the strategy that allows to hold a position on υ_t and S_t^1. To generate the υ_t tree, we use the filtration generated by S_t^1. Tables 5.4 and 5.3 compare, over 1000 scenarios repeated 25 times, the minimum, median, and mean GF as described before between A and C, and the average gain factor (AGF), which is defined as the average of the ratio of the unhedgeable risk using strategy A over the unhedgeable risk by following strategy C for each scenario. Table 5.5 shows that strategy C permits the construction of a portfolio that is less risky than either strategy A or B.

Moreover, option prices are at most slightly sensitive to the regulator risk aversion parameter. However γ does impact the degree to which market risk may be mitigated through hedging. The regulator defines his stringency constraint to reduce emissions through γ. If the market is expected to be short at the compliance period, the regulator sets γ to a high level. Market participants can reduce the hedging error by taking positions on υ_t and S_t^1 while ignoring the ambiguity of the long term regulatory framework because the digital option hedges this gross market state. Therefore the interdependency between periods will be counterbalanced by trading the additional traded asset associated with each compliance period. Unregulated participants will in this way be encouraged to participate in the market, as the market will now behave more like other options markets but provide some risk reduction to options trader as it will have some idiosyncratic features related to other markets. We also believe that under this framework the market provides more meaningful price signals.

Put									
Market state	Strategy	Moneyness	.925	.95	.975	1	1.025	1.05	1.075
SS	A	Price	0.0019 (0.07%)	0.0170 (0.34%)	0.0879 (1.04%)	0.2927 (1.61%)	0.6824 (1.28%)	1.2150 (0.42%)	1.7996 (0.09%)
	γ = .25	Price	0.0014 (0.06%)	0.0145 (0.30%)	0.0805 (0.98%)	0.2821 (1.67%)	0.6766 (1.20%)	1.2132 (0.44%)	1.7993 (0.10%)
	γ = 4	Price	0.0022 (0.09%)	0.0173 (0.34%)	0.0854 (1.02%)	0.2866 (1.74%)	0.6804 (1.26%)	1.2156 (0.49%)	1.7996 (0.14%)
	γ = 16	Price	0.0030 (0.19%)	0.0193 (0.55%)	0.0873 (1.36%)	0.2861 (2.28%)	0.6805 (1.51%)	1.2160 (0.87%)	1.7997 (0.30%)
SL	A	Price	0.0049 (0.12%)	0.0296 (0.53%)	0.1167 (1.36%)	0.3285 (2.03%)	0.7041 (1.86%)	1.2193 (0.70%)	1.7998 (0.14%)
	γ = .25	Price	0.0041 (0.12%)	0.0265 (0.50%)	0.1099 (1.38%)	0.3200 (2.11%)	0.6970 (1.84%)	1.2169 (0.66%)	1.7994 (0.13%)
	γ = 4	Price	0.0051 (0.13%)	0.0292 (0.53%)	0.1142 (1.39%)	0.3245 (2.13%)	0.7007 (1.89%)	1.2194 (0.67%)	1.8000 (0.18%)
	γ = 16	Price	0.0059 (0.25%)	0.0305 (0.73%)	0.1151 (1.62%)	0.3250 (2.43%)	0.7008 (2.11%)	1.2203 (0.85%)	1.8001 (0.33%)
LS	A	Price	0.0009 (0.03%)	0.0078 (0.16%)	0.0402 (0.48%)	0.1338 (0.74%)	0.3119 (0.59%)	0.5553 (0.19%)	0.8224 (0.04%)
	γ = .25	Price	0.0006 (0.03%)	0.0066 (0.14%)	0.0368 (0.45%)	0.1289 (0.76%)	0.3092 (0.55%)	0.5545 (0.20%)	0.8223 (0.05%)
	γ = 4	Price	0.0010 (0.04%)	0.0079 (0.15%)	0.0390 (0.47%)	0.1310 (0.79%)	0.3109 (0.58%)	0.5555 (0.22%)	0.8225 (0.06%)
	γ = 16	Price	0.0014 (0.09%)	0.0088 (0.25%)	0.0399 (0.62%)	0.1307 (1.04%)	0.3110 (0.69%)	0.5557 (0.40%)	0.8225 (0.14%)
LL	A	Price	0.0022 (0.06%)	0.0135 (0.24%)	0.0534 (0.62%)	0.1501 (0.93%)	0.3218 (0.85%)	0.5572 (0.32%)	0.8225 (0.06%)
	γ = .25	Price	0.0019 (0.06%)	0.0122 (0.30%)	0.0505 (0.82%)	0.1470 (1.47%)	0.3197 (1.88%)	0.5577 (2.08%)	0.8243 (2.28%)
	γ = 4	Price	0.0027 (0.22%)	0.0148 (0.80%)	0.0559 (1.98%)	0.1552 (3.52%)	0.3299 (4.79%)	0.5684 (5.34%)	0.8343 (5.53%)
	γ = 16	Price	0.0033 (0.29%)	0.0161 (0.97%)	0.0581 (2.36%)	0.1585 (4.19%)	0.3342 (5.69%)	0.5736 (6.31%)	0.8393 (6.52%)

Table 5.3: Comparison between strategies A and C to price put options written on S_t^1 for different strike prices. All possible initial market states are considered. The first letter (S: short and L: long) stands for the current market position and the second letter denotes the expected market position at the subsequent compliance date. For example SL: Market is short (i.e. high prices) and expected to be long (i.e. low prices).

Call									
Market state	Strategy	Moneyness	.925	.95	.975	1	1.025	1.05	1.075
SS	A	Price	1.7989 (0.07%)	1.2150 (0.34%)	0.6869 (1.04%)	0.2927 (1.61%)	0.0834 (1.28%)	0.0170 (0.42%)	0.0026 (0.09%)
	γ = .25	Price	1.7984 (0.06%)	1.2125 (0.30%)	0.6795 (0.98%)	0.2821 (1.67%)	0.0776 (1.20%)	0.0152 (0.44%)	0.0023 (0.10%)
	γ = 4	Price	1.7992 (0.09%)	1.2153 (0.34%)	0.6844 (1.02%)	0.2866 (1.74%)	0.0814 (1.26%)	0.0176 (0.49%)	0.0026 (0.14%)
	γ = 16	Price	1.8000 (0.19%)	1.2173 (0.55%)	0.6863 (1.36%)	0.2861 (2.28%)	0.0815 (1.51%)	0.0180 (0.87%)	0.0027 (0.30%)
SL	A	Price	1.8019 (0.12%)	1.2276 (0.53%)	0.7157 (1.36%)	0.3285 (2.03%)	0.1051 (1.86%)	0.0213 (0.70%)	0.0028 (0.14%)
	γ = .25	Price	1.8011 (0.12%)	1.2245 (0.50%)	0.7089 (1.38%)	0.3200 (2.11%)	0.0980 (1.84%)	0.0189 (0.66%)	0.0024 (0.13%)
	γ = 4	Price	1.8021 (0.13%)	1.2272 (0.53%)	0.7132 (1.39%)	0.3245 (2.13%)	0.1017 (1.89%)	0.0214 (0.67%)	0.0030 (0.18%)
	γ = 16	Price	1.8029 (0.25%)	1.2285 (0.73%)	0.7141 (1.62%)	0.3250 (2.43%)	0.1018 (2.11%)	0.0223 (0.85%)	0.0031 (0.33%)
LS	A	Price	0.8221 (0.03%)	0.5553 (0.16%)	0.3139 (0.48%)	0.1338 (0.74%)	0.0381 (0.59%)	0.0078 (0.19%)	0.0012 (0.04%)
	γ = .25	Price	0.8219 (0.03%)	0.5541 (0.14%)	0.3106 (0.45%)	0.1289 (0.76%)	0.0355 (0.55%)	0.0070 (0.20%)	0.0011 (0.05%)
	γ = 4	Price	0.8223 (0.04%)	0.5554 (0.15%)	0.3128 (0.47%)	0.1310 (0.79%)	0.0372 (0.58%)	0.0080 (0.22%)	0.0012 (0.06%)
	γ = 16	Price	0.8226 (0.09%)	0.5563 (0.25%)	0.3136 (0.62%)	0.1307 (1.04%)	0.0373 (0.69%)	0.0082 (0.40%)	0.0012 (0.14%)
LL	A	Price	0.8235 (0.06%)	0.5610 (0.24%)	0.3271 (0.62%)	0.1501 (0.93%)	0.0480 (0.85%)	0.0097 (0.32%)	0.0013 (0.06%)
	γ = .25	Price	0.8231 (0.05%)	0.5596 (0.23%)	0.3240 (0.63%)	0.1463 (0.97%)	0.0448 (0.84%)	0.0086 (0.30%)	0.0011 (0.06%)
	γ = 4	Price	0.8236 (0.06%)	0.5608 (0.24%)	0.3259 (0.63%)	0.1483 (0.97%)	0.0465 (0.86%)	0.0098 (0.31%)	0.0014 (0.08%)
	γ = 16	Price	0.8239 (0.11%)	0.5614 (0.33%)	0.3264 (0.74%)	0.1485 (1.11%)	0.0465 (0.97%)	0.0102 (0.39%)	0.0001 (0.15%)

Table 5.4: Comparison between strategies A and C to price call options written on S_t^1 for different strike prices. All possible initial market states are considered. The first letter (S: short and L: long) stands for the current market position and the second letter denotes the expected market position at the subsequent compliance date. For example, SL: Market is short (i.e. high prices) and expected to be long (i.e. low prices).

The introduction of the new tradable asset discussed in this section will have little direct impact on regulatory emitters. Rather, its benefit will be felt by speculators which will be more likely to join the market and hence improve liquidity for all participants.

Call & Put										
Market state	γ		Moneyness	.925	.95	.975	1	1.025	1.05	1.075
SL & LL	.25	*GF*	Min	1.95	1.97	1.92	1.91	1.90	1.84	1.78
			Median	2.02	2.01	1.96	1.94	1.94	1.87	1.83
			Mean	2.02	2.01	1.96	1.94	1.94	1.87	1.83
		AGF	Min	2.28	2.08	2.01	1.98	2.04	1.95	1.95
			Median	2.37	2.12	2.04	2.02	2.09	1.98	2.05
			Mean	2.37	2.12	2.05	2.02	2.09	1.98	2.04
	4	*GF*	Min	2.18	2.25	2.28	2.25	2.18	2.11	1.98
			Median	2.27	2.31	2.31	2.27	2.22	2.16	2.05
			Mean	2.27	2.31	2.32	2.27	2.22	2.16	2.04
		AGF	Min	2.53	2.38	2.38	2.32	2.33	2.22	2.19
			Median	2.65	2.44	2.42	2.35	2.39	2.27	2.27
			Mean	2.65	2.44	2.42	2.35	2.38	2.27	2.27
	16	*GF*	Min	2.48	2.65	2.73	2.67	2.54	2.40	2.19
			Median	2.56	2.69	2.78	2.70	2.58	2.46	2.23
			Mean	2.57	2.69	2.78	2.70	2.58	2.46	2.24
		AGF	Min	2.88	2.80	2.83	2.76	2.69	2.51	2.43
			Median	2.98	2.85	2.90	2.79	2.76	2.59	2.49
			Mean	3.00	2.84	2.89	2.79	2.76	2.58	2.49
SS & LS	.25	*GF*	Min	1.93	1.96	1.97	1.94	1.92	1.92	1.85
			Median	2.02	2.01	2.01	1.95	1.95	1.94	1.87
			Mean	2.02	2.02	2.01	1.96	1.95	1.94	1.88
		AGF	Min	3.40	2.19	2.07	2.00	1.99	1.99	1.91
			Median	3.80	2.26	2.12	2.02	2.02	2.02	1.94
			Mean	3.94	2.26	2.11	2.02	2.02	2.02	1.94
	4	*GF*	Min	2.13	2.20	2.26	2.30	2.23	2.19	2.15
			Median	2.23	2.25	2.30	2.32	2.27	2.22	2.18
			Mean	2.23	2.25	2.30	2.32	2.27	2.22	2.18
		AGF	Min	3.91	2.47	2.37	2.37	2.31	2.26	2.22
			Median	4.16	2.53	2.41	2.39	2.35	2.30	2.25
			Mean	4.20	2.53	2.41	2.39	2.35	2.30	2.25
	16	*GF*	Min	2.31	2.48	2.64	2.75	2.68	2.56	2.47
			Median	2.42	2.54	2.68	2.79	2.70	2.59	2.50
			Mean	2.41	2.54	2.68	2.79	2.70	2.59	2.50
		AGF	Min	4.07	2.77	2.76	2.83	2.76	2.63	2.55
			Median	4.65	2.87	2.81	2.87	2.80	2.67	2.58
			Mean	4.60	2.87	2.81	2.88	2.80	2.67	2.58

Table 5.5: Gain factor (GF) of strategy C with comparison to strategy A in order to price call and put options written on S_t^1 for different strike prices. All possible initial market states are considered. The first letter (S: short and L: long) stands for the current market position and the second letter denotes the expected market position at the subsequent compliance date. For example SL: Market is short (i.e. high prices) and expected to be long.

Chapter 6

Discrete-time Model Summary

Functioning cap-and-trade mechanisms require careful attention to market design. Based on the EU ETS experience, Section 4.2.1 provides evidence that market participants adjust their trading strategies to profit from the expected market position. This manoeuvre is a result of the banking and borrowing possibilities that the EU ETS scheme allowed during the Kyoto commitment period. We describe futures allowance dynamics by a binomial tree where the returns are partially driven by the implied expected market position at subsequent compliance dates. Parameters have been estimated for the Dec-2009 and Dec-2010 contracts. Two major results are interpreted: a) about 75% of Dec-2010 dynamic returns could be explained by the Dec-2009 dynamic returns; b) Dec-2009 and Dec-2010 dynamic returns have the same dynamic pattern, and with probability almost zero they evolve in an opposite directions. As a consequence, a multiperiod pricing framework is necessary to consider this feature and offer appropriate quantitative tools to price and hedge derivatives written on allowances. We present a pricing procedure as well as its associated risk minimizing hedging strategies, as defined by a quadratic criteria. From the perspective of market participants, a better hedging strategy is the one that provides less unhedgeable risk. Our numerical investigation suggests that the best hedging strategy must include positions in futures that mature at compliance dates. However this is not good news for non-regulated market participants, who fear long term regulatory ambiguity. We recommend that the regulator intervene by offering a new additional tradable asset. A possible tradable asset is a digital option written on the regulator's release of information about expected market position at subsequent compliance dates. A pricing procedure based on indifference pricing was presented to evaluate such a contract. A novel traded asset of this type will allow the dependency between compliance periods to be reduced and encourage non-emitters to participate in the market hence increasing its liquidity. The efficiency of the new additional asset in reducing risk, even beyond the risk reduction which may be attributed to trading in S^1 and S^2, has been demonstrated through numerical experiments. We believe that market prices for the new traded asset will be efficient signals of longer term market state and volatility.

Part II

Continuous-Time Model

Chapter 7

Continuous-Time Model Introduction

Via the Kyoto protocol and other similar treaties, countries have committed to reduce their emissions of carbon dioxide equivalent (CO_2e) in order to stabilize pollution responsible for climate change and global warming. To that end, different policy instruments have been implemented, most of which are either cap-and-trade market schemes, emission taxes or hybrids of these policies. An emission tax is a price that an emitter must pay per unit of emitted CO_2e. Companies will have to choose between paying the emission tax or reducing their pollution. As long as the marginal costs of abating is less than the imposed tax, they will reduce emissions. On the other hand, a cap-and-trade market is a quantitative instrument that uses market principles to achieve emissions reduction. The regulatory agency sets an absolute limit, or cap, on the amount of CO_2e, and issues a limited number of tradable allowances which sum to the cap and represent the right to emit a specific amount. Those who find it expensive to abate can buy emissions rights from those who can abate more cheaply. A hybrid or "safety valve" system combines features of both emission taxes and cap-and-trade markets. In such hybrid schemes, companies may purchase allowances from the national authority rather than from the market when the market price is high enough to create serious economic inefficiencies. A hybrid scheme may also have a floor price in addition to the ceiling price in order to avoid the crash and collapse of the carbon market.

According to Baumol and Oates (1988), marketable emission permits and emission taxes are equivalent only if complete information is available to all participants. Moreover, Weitzman (1974) shows the existence of an asymmetry between taxes and emission permits when the regulator has incomplete information about the marginal benefits and producer costs curves. As consequence, there are no strict dominance between these instruments. In the case that marginal costs are steeper than marginal benefits, the market price equilibrium may be extremely high, affecting the competitiveness of the local industry. Taxes are then an appropriate policy instrument since the emission costs is predefined to the companies so they can avoid the deadweight losses. On the other side, a tradable system is preferred if an excess of emissions can cause an environmental disaster. Furthermore, Montero (2002) shows that the cap-and-trade instrument is more appropriate provided both the marginal benefits and costs have the same sensitivity with respect to the level of emission. A more detailed discussion about the efficiency of the emission tax vis-*à*-vis to the cap-and-trade system is provided by Goers, Wagner and Wegmayr (2010) and Taschini (2010). Economists propose the safety valve system as a solution to the impairment of the taxes and trading instruments. Referring to Roberts and Spence (1976), the combination of both instruments is more efficient than implementing only one instrument. The efficiency of a hybrid system to reduce emissions was also argued by Jacoby and Ellerman (2004), Pizer (2002), and McKibbin and Wilcoxen (2002). Maeda (2011) develops an analytical model that shows the existence of a specific combination of the valve price and emissions target allowing the control of emissions reduction under business-as-usual emissions uncertainty. The current practical consensus seems to be that hybrid markets are the way to go. For example, Australia has recently adopted such a structure as described below.

Australia's nationwide market based carbon price mechanism (CPM) will be established by 2015 to unconditionally reduce carbon emissions 5% below 2000 levels. The mid-term horizon reduction objective depends on international community action to reach a 25% reduction target by 2020. Starting in July 2012, the CPM allows participants to trade Australian allowances at a fixed level[1] without emissions cap in order to ensure a smooth transition into a trading scheme by July 2015. The allowances will be considered as personal property that acts equivalent to a tax mechanism. Consequently they are regulated as financial products. The period from July 2015 to July 2018 represents the first phase of the market mechanism with ceiling and floor prices. The floor will be equal to A$15 and increases 4% per annum in real terms. The ceiling will be introduced in May 2014 and it will depend on the international carbon permit price as it is set to exceed this permit price by A$20 for 2015-2016, with an annual increase of 5% in real terms until 2018. In the same month an emission cap will be introduced for the first five years. A five year window emission cap will remain available as the fifth year cap is released after each elapsed year. The Australian Climate Change Authority will review the role of the ceiling and floor prices in 2016. Banking and borrowing are only allowed during the flexible price phase (i.e. from July 2015). The carbon price excludes carbon emissions used for agricultural purposes and closely assist the transition to exposed sectors, including industries whose international competitors are based in countries without carbon reduction regulation. At the time of writing, the Australian model was still very young and it is premature to draw any quantitative conclusions from it. For this reason we focus on the much more mature EU ETS market described below.

The European Union Emission Trading Scheme (EU ETS) is the leading cap-and-trade system and covers around 41% of EU's CO_2e emissions. The EU ETS is designed into different phases. The first commitment period ran between 2005 and 2007. This pilot phase acted to introduce the new regulatory framework to participants within the 27 EU member states. Low prices as well as high volatility were observed as this period suffered from excessive free allocation. Benz and Truck (2009), Daskalakis and Markellos (2008), Uhrig-Homburg and Wagner (2009) performed empirical analyses of the EU ETS market under phase I. Despite all these deficiencies, phase I succeeded by abating 3% of total verified emissions. Phase II started at 2008 and ends in 2012. Phase I allowances could not be carried over, or banked, to phase II[2]

[1]The price is set equal to: A$23/tonne in 2012, A$24.15/tonne in 2013 and A$25.40/tonne in 2014.

[2]France and Poland allowed limited banking.

so expired worthless at the end of 2007 trading period. Phase II was characterized both by a geographical extension to include Iceland, Liechtenstein and Norway, and the new possibility of banking credits over to the phase III period. The latter phase has the longest compliance period, from 2013 to 2020. Its target is to reach, by 2020, an emissions level of 21% less than 2005 level. The number of allowances will decrease 1.74% annually until 2020[3]. The trading scheme will be progressively introduced until it is fully applied to all sectors by 2027. Trading will take place on a common EU-wide platform for the majority of the members. The EU ETS allows the intra-period borrowing and banking practice from or to a future year. However inter-period borrowing is forbidden.

Research to understand the functioning of an ETS can be classified into two different groups. The first group adopts a pricing framework based on equilibrium models. This includes the work of Hitzemann and Uhrig-Homburg (2011), Borovkov et al. (2011), Hinz and Novikov (2010), Kijima et al. (2010), Carmona et al. (2009), Chesney and Taschini (2009), Seifert, Fehr and Henz (2009), and Maeda (2004). The second group adopts applied probability techniques. For instance we cite Carmona and Hinz (2011), Mnif and Davison (2011b), Çetin and Verschuere (2009) and Grüll and Kiesel (2009). A part from where specifically needed here, we refer to Mnif and Davison (2011a) for a review of most previously mentioned papers.

This part aims to study the spread observed in the EU ETS between futures that mature at subsequent dates. We focus on futures rather than spot market for the following reasons. First, EU allowances (EUA) are not considered as financial instruments. Their transaction is not protected by the EU financial regulation against observed market abuse and transparency requirements. EUA spot transaction is subjected to Value-Added Tax, which causes a decrease in liquidity. Second, a number of carbon trades involving stolen carbon units from the national registries occurred and caused a temporary suspension on EUA spot trading in couple of exchanges (e.g. ICE, Bluenext, Green Exchange). Consequently the spot market value decreased from US$ 7.5 billion in 2008 to US$2.8 billion in 2011, while at the same time futures

[3]This rate will be applied to the fourth phase between 2021 and 2028, with a possible adjustment by 2025.

transactions have grown steadily to reach a market value of US$130.8 billion in 2011.

We assume that random futures price dynamics are driven by both a continuous part generated by Brownian motions and a discontinuous component modeled by two pure Poisson processes. The size and sign of these Poisson jumps can model the impact of different kind of unexpected information release. Afterwards we fit the model to quoted prices during phase II. The market model framework has more sources of randomness than the number of traded assets. Therefore a wide range of derivatives written on futures carries an intrinsic risk that cannot be hedged by traded assets. We present a flexible pricing framework based on the Föllmer-Schweizer decomposition. The *fair* price as well as the optimal hedging strategy depend on all traded futures. A generalization to a setting with more than two periods is straightforward.

This part is organized as follows. Chapter 8 introduces the underlying stochastic differential equation that models the market system. Also it reports the estimated parameters and the econometric procedure employed to fit the data. Chapter 9 presents the pricing methodology as well as the price of a contingent claim under different market schemes. Chapter 10 summarizes the results of the continuous time work.

Chapter 8

Futures dynamics and Parameter Estimation

8.1 Futures Dynamics

Figure 8.1 (resp. 8.2) shows the quoted prices of December 2011 (resp. 2010) and December 2012 (resp. 2011) futures discounted to a December 2011 (resp. 2010) baseline. We observe a positive spread starting from 2009, where a longer maturity contract is more valuable than the short one. This spread reaches 5% between 2011 and 2012 contracts.

We consider a two-period market model in which two discounted futures contracts $F(t, T_1)$ and $F(t, T_2)$ that respectively mature at subsequent compliance dates T_1 and T_2 are traded. We investigate the impact of information release at any time before T_1 in both contracts prices, and therefore on the spread. We model the uncertainty by two Brownian motions $W_{it},\ \ i = 1, 2$. W_{1t} represents the continuous random part that drives $F(t, T_1)$. W_{2t} is independent of W_{1t}, and it characterizes the risk specific to the post-T_1 phase. We assume that any unexpected release of information generates a jump in the prices and that such jumps can be classified into two types, each modeled by its own Poisson process.

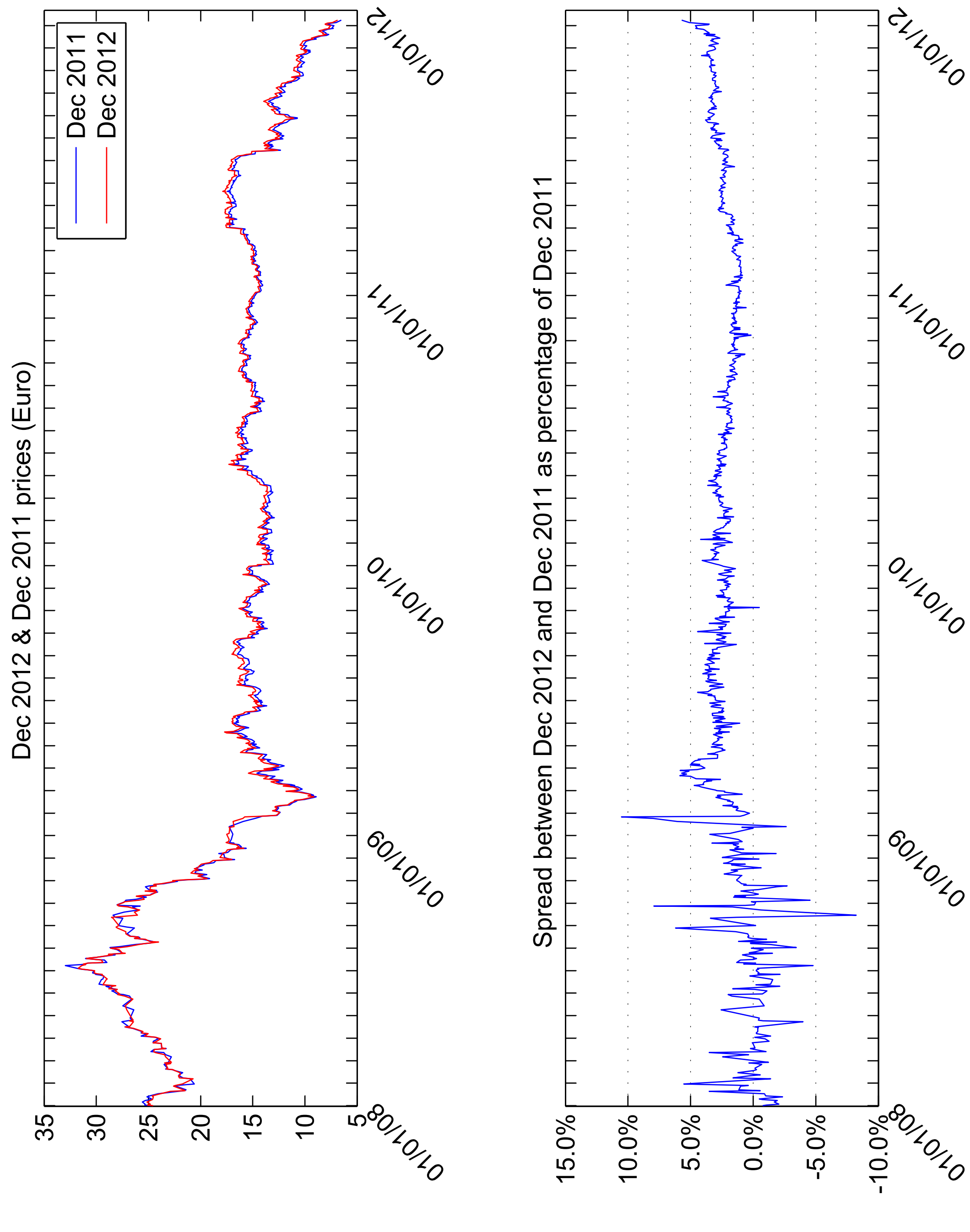

Figure 8.1: Spread between Dec-2012 and Dec-2011 contracts discounted to December 2011 money value using EURIBOR futures. The futures price was quoted in the European market from 02/01/2008 to 19/12/2011. (Source: Bloomberg)

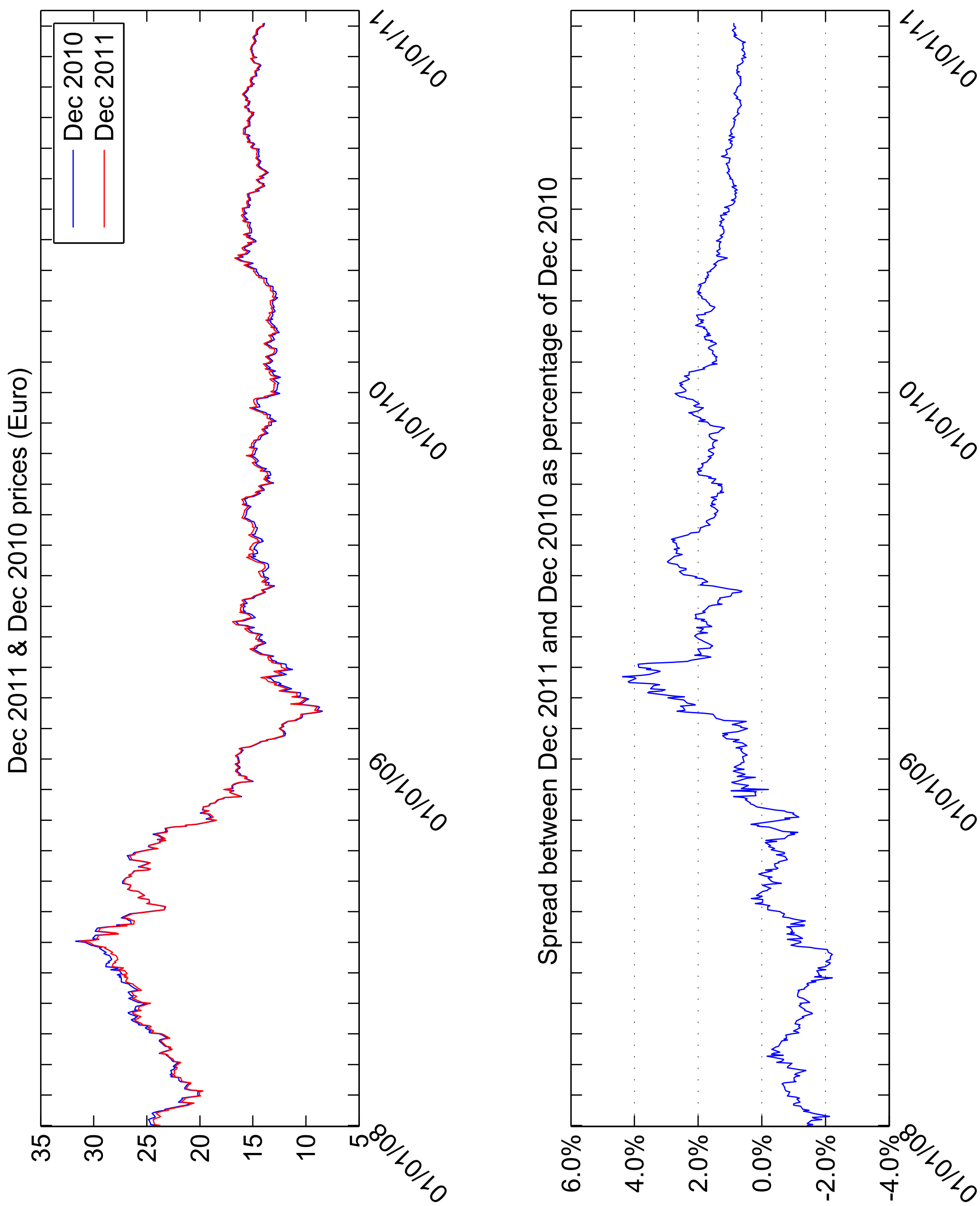

Figure 8.2: Spread between Dec-2011 and Dec-2010 contracts discounted to December 2010 money value using EURIBOR futures. The futures price was quoted in the European market from 02/01/2008 to 20/12/2010. (Source: Bloomberg)

Let $(\Omega, \mathbb{F}, \mathcal{P})$ be a complete[1] probability space with the right continuous filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$. We assume that at the initial time we know all impossible scenarios and scenarios with probability 1. To be more technically accurate, $\mathcal{F}_0$ is trivial and contains all null sets of $\mathbb{F}$. The dynamic of futures is:

$$\frac{dF(t,T_1)}{F(t^-,T_1)} = \mu_1 dt + \sigma_{11} dW_{1t} + \varphi_{11} dN_{1t} + \varphi_{12} dN_{2t}, \quad F(0,T_1) > 0, \tag{8.1}$$

$$\frac{dF(t,T_2)}{F(t^-,T_2)} = \mu_2 dt + \sigma_{21} dW_{1t} + \sigma_{22} dW_{2t} + \varphi_{21} dN_{1t} + \varphi_{22} dN_{2t}, \quad F(0,T_2) > 0, \tag{8.2}$$

where $N_t = (N_{1t}, N_{2t})'$ is a bivariate Poisson process with constant intensity $\lambda = (\lambda_1, \lambda_2)'$, and $\varphi_{ij} > -1, \forall i, j = 1, 2$. We do not impose any restrictions on $(\varphi_{i1}, \varphi_{i2}), i = 1, 2$ instead allowing our econometric investigation to define their signs. σ_{22} determines the impact of W_{2t}'s fluctuation on $F(t, T_2)$. We are interested in studying the effect of:

- W_{1t} on both $F(t, T_1)$ and $F(t, T_2)$. Thus we compare σ_{21} with σ_{11}.

- W_{2t} on $F(t, T_2)$, in particular the relative size of σ_{22} with respect to σ_{21}.

- $N_{it}, i = 1, 2$, on both $F(t, T_1)$ and $F(t, T_2)$. We investigate the sign and relative size of: φ_{11} and φ_{12}; φ_{11} and φ_{21}; φ_{12} and φ_{22}.

Although the parameters are assumed to be deterministic, the underlying dynamics given by (8.1)-(8.2) allow us to acquire deep insights about the emissions market with multiperiod compliance dates. Moreover the model proposed here includes more uncertainty factors than the number of traded assets. Therefore the market model is incomplete and contingent claims may add non-redundant risk to the market.

[1] Every negligible set is measurable.

8.2 Parameter Estimation

We estimate the model for discounted prices[2] of December 2011-12 as well as December 2010-11 futures. The estimation procedure consists of two stages. We begin by numerically maximizing the log-likelihood function. This maximization is over a large space and the outcome of this procedure is an approximate to the maximum likelihood. Afterwards, we plug this approximate solution as an initial guess for the Generalized Expectation-Maximization (GEM) algorithm. The GEM method is a powerful tool that requires to find at least one set of parameters that increases a function Q at each iteration in order to monotonically increase the log-likelihood function. The GEM converges slowly and its efficiency strongly depends on the initial guess. For this reason we begin by using a global maximization approach. This estimation procedure allows us to converge to the best approximation, relatively close to the maximum likelihood. We estimate the asymptotic standard error by two methods: the inverse of the information matrix and the robust variance matrix.

8.2.1 Log-likelihood Function

Let us define $X_t = (X_{1t}, X_{2t})'$, where:

$$X_{it} = \ln(F(t, T_i)), \quad i = 1, 2.$$

Using Itô's lemma, it follows that the stochastic differential equation governing X_{it} is:

$$dX_{1t} = (\mu_1 - \frac{1}{2}\sigma_{11}^2)dt + \sigma_{11}dW_{1t} + \sigma_{12}dW_{2t} + \phi_{11}dN_{1t} + \phi_{12}dN_{2t}, \tag{8.3}$$

$$dX_{2t} = (\mu_2 - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2))dt + \sigma_{21}dW_{1t} + \sigma_{22}dW_{2t} + \phi_{21}dN_{1t} + \phi_{22}dN_{2t}, \tag{8.4}$$

[2] We use Euribor futures rates to discount the prices. We are interested in studying the spread between bivariate process components originating from the market design. So we use the same time reference for both processes in order to neglect the time value effect.

where

$$\phi_{ij} = \ln(1 + \varphi_{ij}), \quad i, j = 1, 2.$$

Therefore, for a time step $\triangle t$, the probability distribution of $\triangle X_t = (\triangle X_{1t}, \triangle X_{2t})'$ is:

$$f(\triangle X_t) = \sum_{i=0}^{M} \sum_{j=0}^{M} \frac{e^{-(\lambda_1+\lambda_2)\triangle t}(\lambda_1 \triangle t)^i (\lambda_2 \triangle t)^j}{i! \;\; j!} \Phi_{ij}, \tag{8.5}$$

where $M = +\infty$ and Φ_{ij} is the bivariate normal distribution with mean θ_{ij} and covariance matrix $\Sigma\triangle t$:

$$\theta_{ij} = \begin{pmatrix} (\mu_1 - \frac{1}{2}\sigma_{11}^2)\triangle t + i\phi_{11} + j\phi_{12} \\ (\mu_2 - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2))\triangle t + i\phi_{21} + j\phi_{22} \end{pmatrix} \tag{8.6}$$

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{11}\sigma_{21} \\ \sigma_{11}\sigma_{21} & \sigma_{22}^2 + \sigma_{21}^2 \end{pmatrix} \tag{8.7}$$

The joint distribution of the return increment is state independent with probability law given by an infinite Gaussian mixture. The probability weight for each regime depends on the number of jumps as well as the Poisson processes intensity, and decreases as the number of jumps increase. However the mean θ_{ij} may change sign as i and j vary. We impose restrictions neither on the jump size nor on its sign. The impact of unexpected information release will be fully determined by the econometric investigation. To numerically compute the joint distribution, we truncate (8.5) to 30 possible jumps for each Poisson process (M=30)[3]. Our aim is to estimate the vector of unknown parameters Θ defined as:

$$\Theta = \begin{pmatrix} \mu_1, & \mu_2, & \sigma_{11}, & \sigma_{21}, & \sigma_{22}, & \delta_1, & \phi_{11}, & \phi_{21}, & \delta_2, & \phi_{12}, & \phi_{22} \end{pmatrix}, \tag{8.8}$$

[3]We use daily returns to estimate the parameters. M=30 means that it is possible to observe a jump in less than one hour. Using the estimated parameters, we find that $\sum_{i=0}^{30} \sum_{j=0}^{30} \frac{e^{-(\lambda_1+\lambda_2)\triangle t}(\lambda_1 \triangle t)^i (\lambda_2 \triangle t)^j}{i! \;\; j!} = 1$ at least to 8 figures accuracy.

where $\delta_1 = \ln(\lambda_1)$ and $\delta_2 = \ln(\lambda_2)$, that maximizes the log-likelhood function:

$$L(\mathcal{X}, \Theta) = \prod_{i=1}^{n} f(\triangle X_i; \Theta). \tag{8.9}$$

n is the number of observed return increments.

8.2.2 The Generalized Expectation-Maximization Algorithm

The EM algorithm is an iterative procedure that was firstly used by Newcomb (1886). In their seminal work, Dempster, Laird, and Rubin (1977) presented its fundamental properties together with some applications. The EM algorithm assumes that the observed data is incomplete to fully describe the joint distribution. Consequently we estimate the maximum log-likelihood in presence of incomplete data.

We suppose that the observed returns $\mathcal{X}$ constitute the incomplete data set. We now construct an artificial problem which requires additional data to complete the missing information. Our hypothetical experiment assumes that the sum of the total number of jumps from both Poisson processes that occurs at each time step $\triangle t$ is constant and equal to $\mathcal{J}$. $\mathcal{J}$ is not observable, and represents the missing information. Thus the complete data $\mathcal{Y}$ is $\mathcal{X}$ augmented with $\mathcal{J}$. An i^{th} complete observation would make the ordered pair $Y_i = (\triangle X_i, \mathcal{J})$ available, and has the probability density function (pdf):

$$f_c(\triangle X_i, \mathcal{J}) = \sum_{k=0}^{\mathcal{J}} \frac{e^{-(\lambda_1+\lambda_2)\triangle t}(\lambda_1 \triangle t)^k (\lambda_2 \triangle t)^{(\mathcal{J}-k)}}{k!\ (\mathcal{J}-k)!} \Phi_{k(\mathcal{J}-k)}, \tag{8.10}$$

The complete log-likelihood function is then:

$$\ln(L_c(\mathcal{Y}; \Theta)) = \ln\left(\prod_{i=1}^{n} f_c\left(\triangle X_i, \mathcal{J}; \Theta\right)\right), \tag{8.11}$$

The iterative EM algorithm consists of successively applying the E-step and the M-step. On the (k+1)st iteration, the E-step computes the Q-function:

$$Q\left(\Theta, \Theta^{(k)}\right) = E_{\Theta^{(k)}}\left[\ln\left(L_c\left(\mathcal{Y}; \Theta\right)\right) \mid \mathcal{X}\right]. \tag{8.12}$$

Appendix E provides details on the computation of the Q-function. The M-step, which follows the E-step, finds the maximum over any Θ of $Q(\Theta, \Theta^{(k)})$ and equates this to $\Theta^{(k+1)}$. Consequently,

$$Q(\Theta^{(k+1)}, \Theta^{(k)}) \geq Q(\Theta, \Theta^{(k)}), \quad \forall\Theta. \tag{8.13}$$

The GEM is an extension of the EM procedure, where $\Theta^{(k+1)}$ is chosen such that:

$$Q(\Theta^{(k+1)}, \Theta^{(k)}) \geq Q(\Theta^{(k)}, \Theta^{(k)}). \tag{8.14}$$

Condition (8.14) guarantees the monotonicity of (8.9) for each iteration[4]:

$$L(\mathcal{X}, \Theta^{(k+1)}) \geq L(\mathcal{X}, \Theta^{(k)}). \tag{8.15}$$

We consider that the GEM algorithm to converge when $\| \ \Theta^{(k+1)} - \Theta^{(k)} \ \| \leq \ 10^{-6}$. We refer to McLachlan and Krishnan (2008) for an extensive development of both the EM and GEM algorithms and for a description of their properties.

8.2.3 Asymptotic Standard Errors

Assume that $\widehat{\Theta}$ is the estimate of the true maximum likelihood estimator Θ^*. We report two asymptotic standard errors. We can approximate the asymptotic distribution of the estimate $\widehat{\Theta}$ as:

$$\sqrt{n}(\widehat{\Theta} - \Theta^*) \sim N(0, (\mathcal{I}(\Theta^*))^{-1}), \tag{8.16}$$

[4]We use numerical algorithm to find such $\Theta^{(k+1)}$.

where the information matrix $\mathcal{I}$ is defined as:

$$\mathcal{I}(\Theta^*) = E\left[\frac{\partial \ln(f(\triangle X_t); \Theta^*)}{\partial \Theta} \frac{\partial \ln(f(\triangle X_t); \Theta^*)'}{\partial \Theta}\right]. \quad (8.17)$$

Here $(\mathcal{I}(\Theta^*))^{-1}$ is the Cramer-Rao lower bound. The outer-product estimate:

$$\mathcal{I}_n(\widehat{\Theta}) = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \ln(f(\triangle X_i); \widehat{\Theta})}{\partial \Theta} \frac{\partial \ln(f(\triangle X_i); \widehat{\Theta})}{\partial \Theta}' \quad (8.18)$$

is an efficient estimate of $\mathcal{I}$ (See Hamilton (1994)).

The second asymptotic standard error is based on the robust covariance matrix, also known as the sandwich matrix. By Theorem 3.2 of White (1982), we have:

$$\sqrt{n}(\widehat{\Theta} - \Theta^*) \sim N(0, J(\Theta^*)^{-1} I(\Theta^*) J(\Theta^*)^{-1}), \quad (8.19)$$

where

$$J(\Theta^*) = E\left[\frac{\partial^2 \ln(f(\triangle X_t); \Theta^*)}{\partial \Theta \partial \Theta'}\right] \quad (8.20)$$

Define:

$$J_n(\Theta) = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 \ln(f(\triangle X_i); \Theta)}{\partial \Theta \partial \Theta'} \quad (8.21)$$

Then,

$$J_n(\widehat{\Theta})^{-1} I_n(\widehat{\Theta}) J_n(\widehat{\Theta})^{-1} \xrightarrow{a.s.} J(\Theta^*)^{-1} I(\Theta^*) J(\Theta^*)^{-1}. \quad (8.22)$$

We use Slutsky's theorem to compute the standard error of the estimated parameters taken from $\widehat{\Theta}$.

8.2.4 Estimated Parameters

We apply the above described estimation procedure to December 2010-11 (D_1) and December 2011-12 (D_2) contracts. We use daily returns from January 2, 2008 to the last trading day of December 2010 for D_1 and December 2011 for D_2. Table 8.1 reports the estimated parameters.

We observe that most of the uncertainty for both D_1 and D_2 is driven by the same Brownian motion. Moreover σ_{11} and σ_{21} have the same sign, which means that this factor causes the same fluctuation pattern. The impact of σ_{22} increases as we get closer to the end of phase II and it remains relatively small with respect to σ_{21}. Two kind of information affect the return dynamics. The first adds value to the future price, and represents the impact of any unexpected release of information that projects the market to be short. The second has the opposite effect and drives the returns down. We emphasize that we did not impose any sign restrictions on $\varphi_{ij}, \;\; i, j = 1, 2$ during the estimation procedure.

We observe that $\mid \varphi_{1j} \mid < \mid \varphi_{2j} \mid, \;\; j = 1, 2$. This implies that any release of information has more impact on returns with longer maturity. Futures on D_1 vary on a similar way as φ_{1j} and φ_{2j} have the same sign, $\forall j = 1, 2$. However this pattern no longer holds for D_2. A jump, when it occurs, has an opposite effect on returns. To explain this, we highlight that each contract in both data set may be banked either within the intra-phase (for December 2010 and 2011) or the inter-phase (December 2012) compliance periods. However the same cannot be said for phase II borrowing practices as participants in 2012 are forbidden to use phase III allowances for compliance purposes. We conclude that this observed feature between D_1 and D_2 is generated by the borrowing condition. Moreover return adjustments via φ_{ij} with respect to this structural change suggests that the futures market of the EU ETS is efficient and mature enough to be considered in the same league as usual futures markets.

		Dec 2010 - Dec 2011	Dec 2011 - Dec 2012
μ_1	$\widehat{\mu}_1$	-42.37%	-21.45 %
	CI_I	[-100.78% ; 16.03%]	[-81.63% ; 38.73%]
	CI_R	[-106.63% ; 21.89%]	[-80.03% ; 37.13%]
μ_2	$\widehat{\mu}_2$	-39.56%	-25.32%
	CI_I	[-96.02% ; 16.89%]	[-84.46% ;33.82%]
	CI_R	[-102.42% ; 23.29%]	[-83.64% ;33.00%]
σ_{11}	$\widehat{\sigma}_{11}$	43.95%	47.24%
	CI_I	[42.46% ; 45.43%]	[44.27% ; 50.20%]
	CI_R	[40.63% ; 47.26%]	[44.27% ; 50.20%]
σ_{21}	$\widehat{\sigma}_{21}$	42.37%	45.95%
	CI_I	[40.89% ; 43.85%]	[44.48% ; 47.42%]
	CI_R	[39.12% ; 45.63%]	[42.93% ; 48.96%]
σ_{22}	$\widehat{\sigma}_{22}$	2.17%	7.86%
	CI_I	[2.05% ; 2.30%]	[7.47% ; 8.25%]
	CI_R	[1.98% ; 2.37%]	[7.32% ; 8.40%]
λ_1	$\widehat{\lambda}_1$	12.79	19.78
	CI_I	[7.04 ; 18.53]	[14.82 ; 24.74]
	CI_R	[7.38 ; 18.19]	[12.68 ; 26.88]
φ_{11}	$\widehat{\varphi}_{11}$	1.53%	0.72%
	CI_I	[0.46% ;2.61%]	[0.31% ; 1.12%]
	CI_R	[0.13% ;2.93%]	[-0.03% ; 1.46%]
φ_{21}	$\widehat{\varphi}_{21}$	1.95%	-1.32%
	CI_I	[0.88% ; 3.03%]	[-1.71% ; -0.93%]
	CI_R	[0.64% ; 3.26%]	[-2.07% ; -0.57%]
λ_2	$\widehat{\lambda}_2$	20.32	27.26
	CI_I	[12.40 ; 28.25]	[20.62 ; 33.91]
	CI_R	[7.09 ; 33.56]	[17.22 ; 37.31]
φ_{12}	$\widehat{\varphi}_{12}$	-0.87%	-0.59%
	CI_I	[-1.81% ; 0.06%]	[-1.20% ; 0.02%]
	CI_R	[-1.98% ; 0.24%]	[-1.07% ; -0.11%]
φ_{22}	$\widehat{\varphi}_{22}$	-1.22%	0.92%
	CI_I	[-2.20% ; -0.25%]	[0.28% ; 1.55%]
	CI_R	[-2.22% ; -0.22%]	[0.47% ; 1.37%]

Table 8.1: The table reports the estimated parameters of (8.1)-(8.2) for Dec 2010 - Dec 2011 and Dec 2011 - Dec 2012 futures. We use daily returns quoted in the European market from: a) 02/01/2008 to 20/12/2010 for Dec 2010 - Dec 2011 futures; b) 02/01/2008 to 19/12/2011 for Dec 2011 - Dec 2012 futures. CI_I: 90% confidence interval computed from the Information matrix. CI_R: 90% confidence interval computed from the Robust Variance matrix.

Chapter 9

Pricing Contingent Claims under Different Market Schemes

9.1 Pricing Criterion

When the market is complete, all risks are generated by the primary assets and any contingent claim may be completely hedged by means of a self-financing strategy. Recall that a self-financing strategy requires no further injection of funds during the life of the claim. Therefore the discounted cumulative cost of the option is constant, and equal to the initial price of the hedging strategy. In the incomplete market framework, we distinguish two sets of contingent claims. The first set contains all claims that can be fully replicated, while the second one comprises claims that add risk to the market. The former set is not attainable and possess an intrinsic risk. The discounted cost process is no longer constant and many strategies could be considered to hedge the claim. Therefore a strategy performance criteria must be introduced in order to choose the most efficient hedging rule. A claim's *fair* price is then defined with respect to this criteria; because of this arbitrary choice this price is not unique.

Föllmer and Sondermann (1986) introduce the risk-minimizing strategy where the primary asset is a martingale under the historical probability measure $\mathcal{P}$. It measures the risk by the mean conditional square error of the cumulative cost process. To illustrate this criteria, we present Föllmer and Sondermann's ideas for a one-period discrete time model, in particular to the pricing of a contingent claim H written on the underlying asset F_t. Consider ζ_0 as the number of stocks to hold at time 0 and $\eta_i, \quad i = 0, 1$ the amount invested in the bank account[1]. The cost C_0 of the claim H is equal to the initial value of the portfolio V_0:

$$C_0 = V_0 = \zeta_0 F_0 + \eta_0. \tag{9.1}$$

At time 1, we must deliver the payoff H. Therefore we must choose η_1 such that $V_1 = H$. Thus

$$\eta_1 = H - \zeta_0 F_1 \tag{9.2}$$

The change in the cost process is:

$$C_1 - C_0 = \eta_1 - \eta_0 \tag{9.3}$$

$$= H - V_0 - \zeta_0(F_1 - F_0). \tag{9.4}$$

$F_1 - F_0$ represents the fluctuation of the underlying asset's price which we denote by ΔF. The performance measure is defined as the expected quadratic cost:

$$R_0(\zeta_0) = E\left[(C_1 - C_0)^2\right] \tag{9.5}$$

$$= E\left[(H - V_0 - \zeta_0 \Delta F)^2\right]. \tag{9.6}$$

The optimal strategy (V_0, ζ_0) is the one which minimizes the risk defined by (9.6). In continuous time, the value of the portfolio is:

$$V_t = \zeta_t F_t + \eta_t \tag{9.7}$$

[1]Recall that we work with discounted prices. So η_1 represents the bank account in time 0 money value.

and the cumulative cost is:

$$C_t = V_t - \int_0^t \zeta_t dF_t. \tag{9.8}$$

Define the trading strategy $\Phi = (V, \zeta)$. The remaining risk at time t of Φ is:

$$R_t(\Phi) = E\left[(C_T - C_t)^2 \mid \mathcal{F}_t\right]. \tag{9.9}$$

The optimal strategy minimizes (9.9) over all possible strategies. Schweizer (1990) extends the work of Föllmer and Sondermann (1986) to the case where the primary asset is a semi-martingale under $\mathcal{P}$ and defines the locally risk-minimizing concept. A strategy is locally risk-minimizing if it continues to minimize (9.9) under all infinitesimal perturbations (Schweizer, 1990). The solution of (9.9) gives us the opportunity to price any contingent claim by taking the expectation under an equivalent measure $\widehat{P}$, called the minimal martingale measure.

Define:

$$\Gamma = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \tag{9.10}$$

$$\Lambda = \begin{pmatrix} \sigma_{11}^2 & \sigma_{21}\sigma_{11} \\ \sigma_{21}\sigma_{11} & \sigma_{22}^2 + \sigma_{21}^2 \end{pmatrix} \tag{9.11}$$

$$\Xi = \begin{pmatrix} \varphi_{11}^2\lambda_1 + \varphi_{12}^2\lambda_2 & \varphi_{11}\varphi_{21}\lambda_1 + \varphi_{12}\varphi_{22}\lambda_2 \\ \varphi_{21}\varphi_{11}\lambda_1 + \varphi_{22}\varphi_{12}\lambda_2 & \varphi_{21}^2\lambda_1 + \varphi_{22}^2\lambda_2 \end{pmatrix} \tag{9.12}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (\Lambda + \Xi)^{-1}(\mu + \Gamma\lambda) \tag{9.13}$$

where $\mu = (\mu_1, \mu_2)'$. The dynamics of futures under $\widehat{P}$ is:

$$\frac{dF(t,T_1)}{F(t,T_1)} = \mu_1^{\widehat{P}} dt + \sigma_{11} dW_{1t}^{\widehat{P}} + \varphi_{11} dN_{1t}^{\widehat{P}} + \varphi_{12} dN_{2t}^{\widehat{P}}, \quad F(0,T_1) > 0, \tag{9.14}$$

$$\frac{dF(t,T_2)}{F(t,T_2)} = \mu_2^{\widehat{P}} dt + \sigma_{21} dW_{1t}^{\widehat{P}} + \sigma_{22} dW_{2t}^{\widehat{P}} + \varphi_{21} dN_{1t}^{\widehat{P}} + \varphi_{22} dN_{2t}^{\widehat{P}}, \quad F(0,T_2) > 0, \tag{9.15}$$

where

$$\mu_1^{\widehat{P}} = \mu_1 - \sigma_{11}(\alpha_1\sigma_{11} + \alpha_2\sigma_{21}), \tag{9.16}$$

$$\mu_2^{\widehat{P}} = \mu_2 - \left(\sigma_{21}(\alpha_1\sigma_{11} + \alpha_2\sigma_{21}) + \sigma_{22}(\alpha_1\sigma_{12} + \alpha_2\sigma_{22})\right), \tag{9.17}$$

$(W_{1t}^{\widehat{P}}, W_{2t}^{\widehat{P}})'$ is a $\widehat{P}$-standard Brownian motion and $(N_{1t}^{\widehat{P}}, N_{2t}^{\widehat{P}})'$ is Poisson process under $\widehat{P}$ with intensity:

$$\lambda^{\widehat{P}} = \begin{pmatrix} \lambda_1(1-\alpha_1\varphi_{11})(1-\alpha_2\varphi_{21}) \\ \lambda_2(1-\alpha_1\varphi_{12})(1-\alpha_2\varphi_{22}) \end{pmatrix} \tag{9.18}$$

The *fair* price of H at time t is computed as a conditional expectation:

$$V_t = E_{\widehat{P}}[H \mid \mathcal{F}_t], \tag{9.19}$$

where $E_{\widehat{P}}[.]$ is the expectation taken under the probability measure $\widehat{P}$. Thus the initial cost is:

$$V_0 = E_{\widehat{P}}[H], \tag{9.20}$$

The details of this derivation are given in Appendix F.

The dynamics of $F(t,T_1)$ under $\widehat{P}$ depends on the parameters of $F(t,T_2)$ under $\mathcal{P}$. Therefore the price of any contingent claim written on $F(t,T_1)$ depends on the parameters of both $F(t,T_1)$ and $F(t,T_2)$ under $\mathcal{P}$. This is consistent with the work reported by Mnif and Davison (2011b), which shows that the fair price of a T_1-claim should depend on both $F(t,T_1)$ and $F(t,T_2)$.

Consequently in a more general multiperiod setting, the optimal hedging strategy depends on all futures with post-T_1 maturity.

9.2 Claim Price Under Cap-and-Trade Market

The pricing equation in (9.20) allows any T_1-contingent claim to be priced as a function of post-T_1 futures parameters. The price is easily computed through an expectation under the minimal martingale measure $\widehat{P}$. Within a cap-and-trade market framework, any derivative that has a closed form solution under the Black and Scholes (1973) setting has a closed form formula under $\widehat{P}$ by conditioning on $N^{\widehat{P}}_{1T_1}$ and $N^{\widehat{P}}_{2T_1}$. The price of a T_1-contingent claim H is given by:

$$\begin{aligned} V_0 &= E_{\widehat{P}}[H] && (9.21)\\ &= E_{\widehat{P}}\Big[E_{\widehat{P}}\big[H \mid (N^{\widehat{P}}_{1T_1} = i, N^{\widehat{P}}_{2T_1} = j)\big]\Big] && (9.22)\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e^{-(\lambda^{\widehat{P}}_1+\lambda^{\widehat{P}}_2)T_1}\frac{(\lambda^{\widehat{P}}_1 T_1)^i(\lambda^{\widehat{P}}_2 T_1)^j}{i!j!} BS(F(0,T_1)(1+\varphi_{11})^i(1+\varphi_{12})^j, \mu^{\widehat{P}}_1, T_1, \sigma_{11}), && (9.23) \end{aligned}$$

where $\lambda^{\widehat{P}} = (\lambda^{\widehat{P}}_1, \lambda^{\widehat{P}}_2)'$. $BS(F(0,T_1)(1+\varphi_{11})^i(1+\varphi_{12})^j, \mu^{\widehat{P}}_1, T_1, \sigma_{11})$ is the Black-Scholes price of H with time to maturity T_1, an initial underlying asset value $F(0,T_1)(1+\varphi_{11})^i(1+\varphi_{12})^j$, a volatility σ_{11} and a drift $\mu^{\widehat{P}}_1$ both in annual units. For a call price with a strike K, we have:

$$\begin{aligned} BS(F(0,T_1)(1+\varphi_{11})^i(1+\varphi_{12})^j, \mu^{\widehat{P}}_1, T_1, \sigma_{11}) &= S^{ij}_0 \mathcal{N}(d^{ij}_1) - Ke^{-\mu^{\widehat{P}}_1 T_1}\mathcal{N}(d^{ij}_2) && (9.24)\\ S^{ij}_0 &= F(0,T_1)(1+\varphi_{11})^i(1+\varphi_{12})^j && (9.25)\\ d^{ij}_1 &= \frac{\ln(\frac{S^{ij}_0}{K}) + (\mu^{\widehat{P}}_1 - \frac{\sigma^2_{11}}{2})T_1}{\sigma_{11}\sqrt{T_1}} && (9.26)\\ d^{ij}_2 &= d^{ij}_1 - \sigma_{11}\sqrt{T_1}, && (9.27) \end{aligned}$$

and $\mathcal{N}(.)$ is the standard normal cumulative distribution function.

Conditioning on $N^{\widehat{P}}_{1T_1}$ and $N^{\widehat{P}}_{2T_1}$, the Black-Scholes price uses $\mu^{\widehat{P}}_1$ as the discount rate, instead of the risk-free rate, which is equal to zero in our framework as we work with discounted prices. As $\mu^{\widehat{P}}_1$ depends on the market model, the discount factor takes into consideration all traded primary assets and is applicable to all market participants.

9.3 Claim Price Under Hybrid Market

A hybrid market offers a guarantee against high prices to market participants in order to avoid carbon leakage. Carbon leakage is defined as the competition motivated transfer of a company to another country or state with less stringent constraints on carbon emissions. The regulator sets a price ceiling P_{max} at the beginning of the compliance period. We assume that P_{max} is constant over the compliance period. Furthermore we suppose that P_{max} is an absorbing state, meaning that the market will become dysfunctional when the price reaches P_{max}, and prices will be traded at P_{max} level during the remaining compliance period. The payoff of any T_1-contingent claim H written on $F(t, T_1)$ with payoff $h(F(T_1, T_1))$ is:

$$\mathbb{I}_{\tau>T_1} h(F(T_1, T_1)) + \mathbb{I}_{\tau\leq T_1} h(P_{max}), \tag{9.28}$$

where τ is the first overshooting[2] time and is defined as:

$$\tau = \min\{t | F(t, T_1) \geq P_{max}\} \tag{9.29}$$

The price of H at an initial time is therefore:

$$\begin{aligned} V_0 &= E_{\widehat{P}}\left[\mathbb{I}_{\tau>T_1} h(F(T_1, T_1))\right] + h(P_{max}) E_{\widehat{P}}\left[\mathbb{I}_{\tau\leq T_1}\right] && (9.30) \\ &= E_{\widehat{P}}\left[\mathbb{I}_{\tau>T_1} h(F(T_1, T_1))\right] + h(P_{max}) \widehat{P}(\tau \leq T_1). && (9.31) \end{aligned}$$

[2]The underlying process is discontinuous, the reason why it is defined as an overshooting time rather than as a hitting time.

$\widehat{P}(\tau \leq T_1)$ represents the probability of the market to be dysfunctional under $\widehat{P}$. It reflects the impact of the market parameters on its functionality and the involvement degree of the regulator to protect his economy from prices spike. This quantity is the same for all contingent claims. Note that in the absence of jump components, closed form solution for digital, call and put options are available.

Monte-Carlo simulation can be used in order to provide an efficient estimate of (9.31). Figure 9.1 shows the box plot of put and call option prices estimated over 10000 paths repeated 1000 times for different degrees of moneyness[3]. Monte-Carlo techniques are easy to implement and, as shown in Figure 9.1, they are efficient and show low variance.

A cap-and-trade scheme is a special case of an hybrid scheme when we let P_{max} approaches $+\infty$. Figures 9.2 and 9.3 show the convergence of both call and put option prices to the closed form solution under a cap-and-trade scheme as P_{max} increases. The reported prices are computed as the average of the 1000 runs. The cap-and-trade prices for put and call options are an upper bound for all different hybrid schemes. Therefore trading these options within an hybrid market is less expensive for market participants.

[3]The results are obtained using the estimated parameters from December 2011-12 futures.

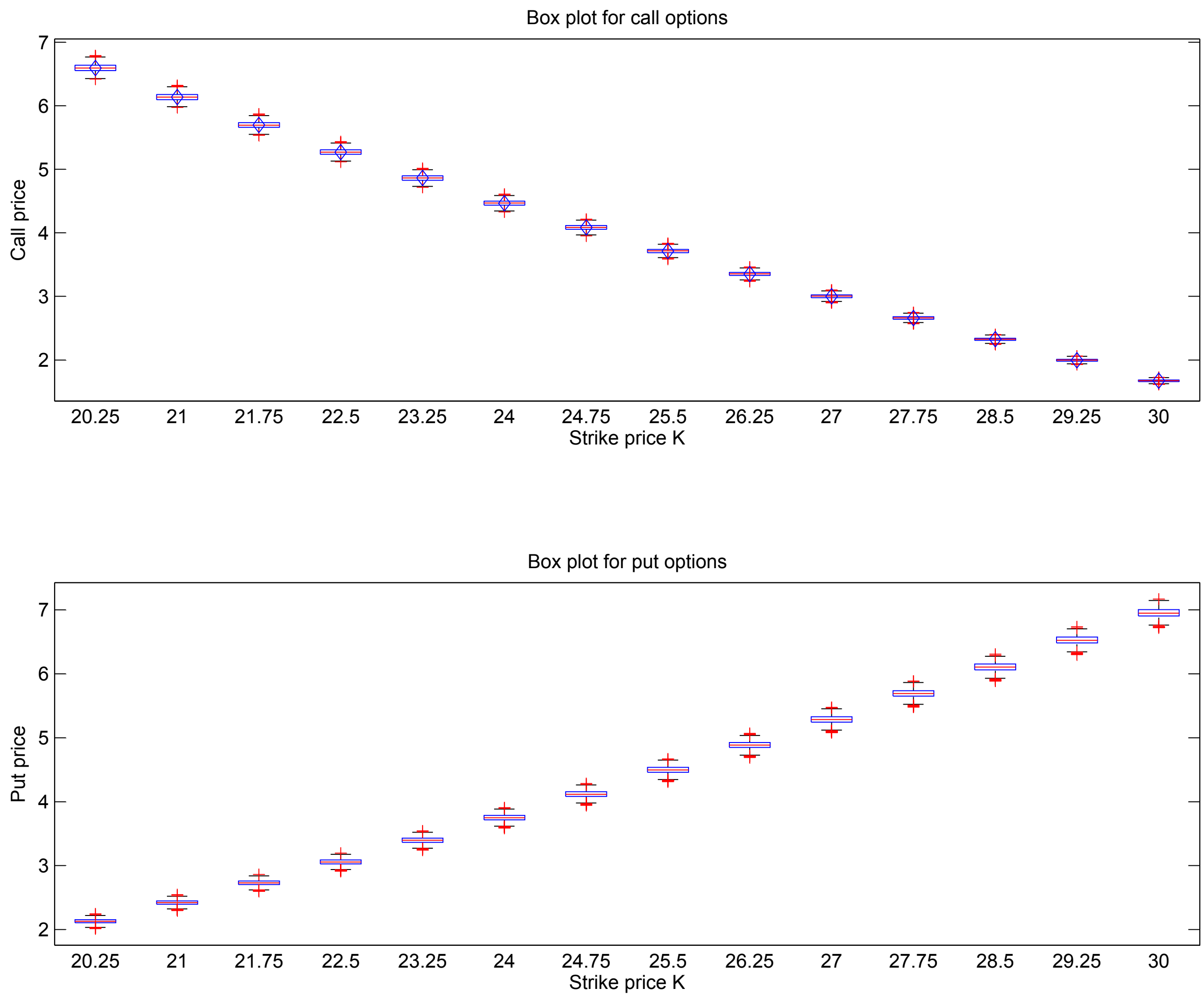

Figure 9.1: Box plot for call and put options under hybrid market scheme with $P_{max} = 35$ for different strike prices. The options mature in one year with initial future price $F(0, T_1) = 24.75$. The results are obtained using the estimated parameters from Dec 2011 - Dec 2012 futures (see table 8.1).

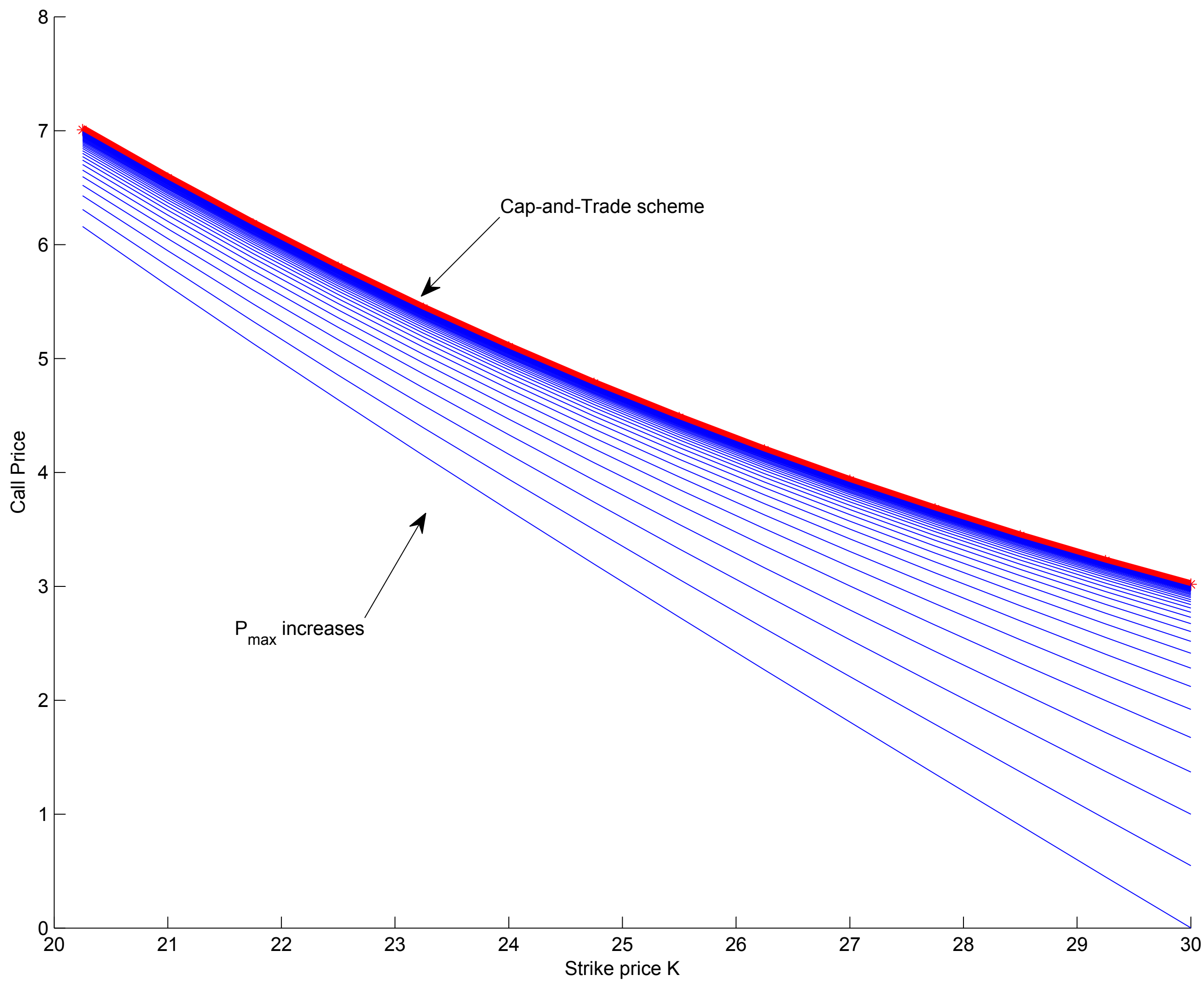

Figure 9.2: Call prices under a hybrid scheme converges to cap-and-trade scheme as P_{max} varies from 30 to 150 with an unit increment. The options mature in one year with initial future price $F(0, T_1) = 24.75$. The results are obtained using the estimated parameters from Dec 2011 - Dec 2012 futures (see table 8.1).

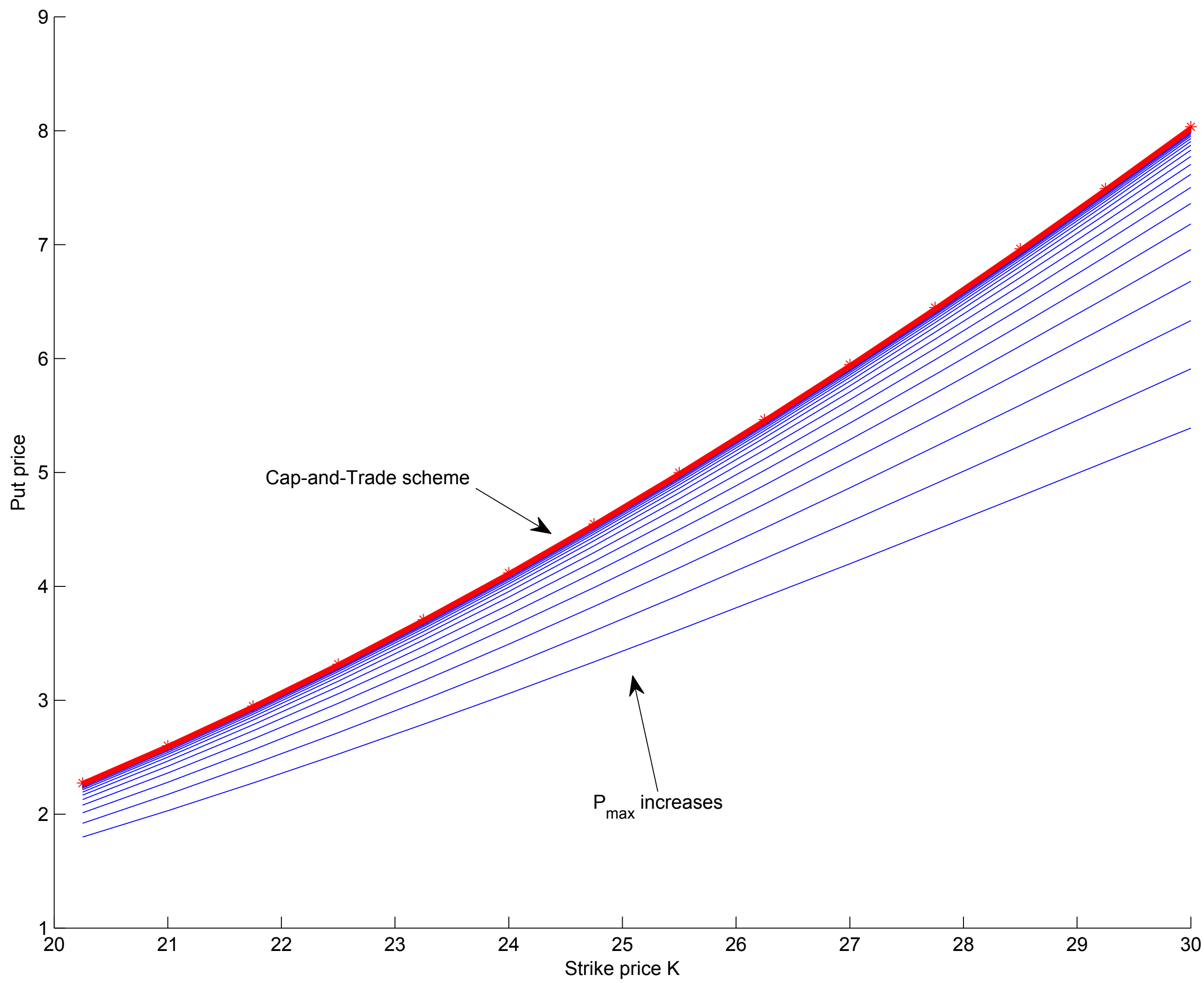

Figure 9.3: Put prices under a hybrid scheme converges to cap-and-trade scheme as P_{max} varies from 30 to 150 with an unit increment. The options mature in one year with initial future price $F(0, T_1) = 24.75$. The results are obtained using the estimated parameters from Dec 2011 - Dec 12 futures (see table 8.1).

Chapter 10

Continuous-Time Model Summary

We propose a stochastic model to study the spread between two futures that mature at subsequent dates. We assume that futures dynamics are mainly driven by Brownian motions. We add two jump components in order to model the impact of an unexpected release of information. We fit the model to the data collected from the EU ETS during phase II. Our estimation procedure consists of two steps. We start by numerically maximizing the maximum likelihood function. Afterwards we use its output as an initial guess for a generalized EM algorithm. We find that the fluctuation of $F(t,T_1)$ explains the major part of $F(t,T_2)$ variation. The impact of any release of information is amplified in contracts with longer maturity. Furthermore jump sizes have the same sign in different contracts when both borrowing and banking are allowed. The way in which the market response (via the φ_{ij}) to a structural change supports the stance that the market is mature and efficient enough to be comparable to many other derivatives markets. We present a pricing procedure under which the cost process is a martingale under the historical probability measure. This approach is based on minimizing the mean conditional square error of the cumulative cost process. The optimal hedging strategy is invariant after any small perturbation as defined in Schweizer (1990). Any contingent claim that has closed formula under Black and Scholes (1973) has a closed formula within a cap-and-trade framework, up to some parameter adjustments. We compare call and put options prices under different market schemes. We observe that options cost less under a hybrid scheme. This result seems to support the Australian government choice of a hybrid market structure.

Chapter 11

Conclusions

The purpose of carbon markets is to implement the least cost policy instrument to achieve a given emissions reduction target. It will provide a time varying price signal equal to the lowest internal abatement cost in the economy. However many obstacles face a regulatory agency as it attempts to establish an appropriate market design, one which yields the decrease of global greenhouse gas emissions over both short and long time periods. On the one hand, regulators are concerned about local economy competitiveness and the related carbon leakage phenomenon. In fact some countries have yet to commit to reduce greenhouse gas emissions through international agreements and compliance processes. On the other hand, the long term regulatory environment remains very complex and dependent on international community action (e.g. Australia's market based carbon price mechanism). Low and high prices were observed during phase I and II of the EU ETS experience. Moreover the EU ETS is characterized by a very high historical volatility. Therefore carbon markets are different from most traditional markets, and the quantitative financial tools that have been built to study other financial markets must be carefully employed within this new context.

This thesis contributes to the carbon markets literature in three different respects. Recall that a market is incomplete if the number of sources of randomness in the market exceeds the number of traded primary financial contracts. We propose mathematical models to describe the futures dynamics under an incompleteness assumption. We fit their parameters to the quoted EU ETS prices. Our empirical study shows that most of the market uncertainty can be explained by one factor that can be either the returns of the contract with the closest maturity or by a Brownian motion. Contracts with further maturities evaluate risk related to the expected market position at future dates. This is a result of carbon market banking and borrowing possibilities. We provide evidence that market participants adjust their trading strategies to profit from the expected market position.

The second aspect is related to options pricing and hedging strategies. We show that a strategy involving all traded assets is more efficient than a strategy that includes only positions on the underlying futures contract. The trading compliance periods are therefore correlated and the one-period pricing framework is less effective than a multi-period model. We present pricing tools under two criteria. We price contingent claims under discrete-time models by quadratic hedging criterion. This criterion consists of solving an optimization problem which defines a *fair* price and its associated attainable portfolio that minimizes the quadratic risk. In other words, the price of any contingent claim H is approximated by the price of an attainable payoff that matures as close as possible to H with respect to the Euclidean norm. This notion was first introduced by Schäl (1994) and generalized by Schweizer (1996) in the one dimensional case. We also provide pricing solution for continuous-time model, in which the hedging strategy minimizes the mean conditional square error of the cumulative cost process and remains the minimizing strategy under all infinitesimal perturbations. This strategy is called the locally risk-minimizing strategy. This concept was introduced by Schweizer (1990) for the case where the primary asset is a semimartingale under the historical probability. We compare call and put options prices under different market designs, and we conclude that options cost less under a hybrid scheme.

The third aspect of this thesis is linked to regulatory policy and market design. We show how the regulator can intervene in order to provide market participants with a tool to hedge against extreme scenarios. We recommend that the regulator introduce a new additional tradable primary asset, exogenous to all market participants. This new financial instrument would play the role of an expected market position indicator. Its pricing solution considers the social wealth of the market rules initially established to define the market parameters. We present a possible pricing framework based on the indifference pricing technique.

Carbon markets are still at an early stage in terms of their real world implementation. As such, related empirically grounded academic research is also by necessity sparse. It is important that uncertainty related to market rules be resolved, since the result of this ambiguity is that companies do not yet have clear information signals for making clean energy investment decisions. Should they make these investments now, or wait for new regulations to be introduced? Quantitative finance techniques are ill suited to address such questions of regulatory risk.

An extension of the thesis is to empirically study the observed spread between certified emissions reductions (CERs) and emissions allowances. Such a study faces many challenges, as CERs are less liquid than allowances and have associated conditions that must be satisfied before they may be issued. Another challenging area for future work would be to provide pricing and hedging tools in the case where different emissions trading schemes are linked. In this setting foreign exchange rates play a key role in defining the optimal trading strategy.

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Part III

Appendices

Appendix A

Proof of Proposition 5.1.1

As first step and before presenting the proof, we assume the following conventions and definitions:

- $\mathcal{L}^2(\mathcal{P})$: the space of square integrable 1-dimensional random variables;
- $\mathcal{L}^2_d(\mathcal{P})$: the space of square integrable d-dimensional random variables;
- $\mathcal{T} = [0, 1, 2, ..., T]$;
- If $X \in \mathbb{R}^d$, then $X^2 = XX'$; which is a $d \times d$ matrix;
- Sum over an empty set is 0;
- Product over an empty set is 1;
- $\frac{0}{0} = 0$;
- if Σ is an $d \times d$ singular matrix, $\Sigma^{-1} = 0_{d \times d}$[1];
- $'$ is the transpose operator;
- if $X, Y \in \mathbb{R}^d$, then $X'Y$ is the Euclidean product defined as: $\sum_{i=1}^{d} X_i Y_i = Y'X$.

[1]Note that we use this assumption to keep a solid proof. A comment in Appendix C states that the nonsingularity condition that we need afterwards is satisfied by the market model under the no-arbitrage condition.

Definition Signed Θ-martingale measure: Q is a signed measure on $(\Omega, \mathbb{F})$ if it satisfies:

- $Q[\Omega] = 1$;
- $Q \ll \mathcal{P}$ such that $\frac{dQ}{d\mathcal{P}} \in \mathcal{L}^2(\mathcal{P})$;
- $\forall \vartheta \in \Theta$,

$$E\left[\frac{dQ}{d\mathcal{P}} G_T(\vartheta)\right] = 0. \tag{A.1}$$

Define $\mathbb{P}_s(\Theta)$ as the set of signed Θ-martingale measures. This set depends on the space of all admissible strategies, as follows from the characterization (A.1). Moreover, as we are working within a finite discrete time framework, every signed measure Q is a martingale measure for S (Schweizer, 1996). In other words, $\forall s, t \in \mathcal{T}$, $s \leq t$,

$$E\left[\frac{dQ}{d\mathcal{P}} S_t | \mathcal{F}_s\right] = S_s, \;\; \mathcal{P} - a.s. \tag{A.2}$$

Q is also called a signed $\mathcal{L}^2$-martingale measure for S. By assuming that the market is free of arbitrage, we have

$$\mathbb{P}_s(\Theta) \neq \emptyset. \tag{A.3}$$

Definition Non-Degeneracy Condition (ND): Suppose $\delta \in (0, 1)$. The process $(S_t)_{t \in \mathcal{T}} \in \mathcal{L}^2_d(\mathcal{P})$ meets the non-degeneracy condition, if $\forall k = 1, ..., T$, the random matrix

$$\delta E[\Delta S_k^2 | \mathcal{F}_{k-1}] - (E[\Delta S_k | \mathcal{F}_{k-1}])^2 \tag{A.4}$$

is positive-semidefinite $\mathcal{P}$-a.s.

To give more intuition, (ND) is equivalent under a one dimensional framework to:

$$Var(\Delta S_k) > 0. \tag{A.5}$$

It means that we are not interested in deterministic processes: process dynamics must depend on the information that arises during the compliance period.

Proof β_k and ϱ_k are well defined. Appendix B contains the proofs of the following properties used later to ensure integrability conditions:

$$\prod_{j=k}^{T}(1-\beta_j'\Delta S_j) \in \mathcal{L}^2(\mathcal{P}), \tag{A.6}$$

$$\beta_k'\Delta S_k \prod_{j=k}^{T}(1-\beta_j'\Delta S_j) \in \mathcal{L}^2(\mathcal{P}), \tag{A.7}$$

$$E\left[\prod_{j=k}^{T}(1-\beta_j'\Delta S_j)^2|\mathcal{F}_{k-1}\right] = E\left[\prod_{j=k}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}\right] \leq 1 \ \ \mathcal{P}-a.s., \tag{A.8}$$

$$E\left[H\beta_k'\Delta S_k \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}\right] = E\left[\varrho_k'\Delta S_k \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}\right] \ \ \mathcal{P}-a.s., \tag{A.9}$$

$\forall k \in \mathcal{T}\backslash\{0\}$.

Consider the following optimization problem: For a fixed $c \in \mathbb{R}$ and $H \in \mathcal{L}^2(\mathcal{P})$, we determine $\zeta = (\zeta_k)_{k=1}^T$ such that

$$\zeta = \underset{\vartheta\in\Theta}{\arg\min} \ \ E_{\mathcal{P}}[(H-c-G_T(\vartheta))^2]. \tag{A.10}$$

Motoczyński (2000) shows that the space $G(\vartheta)$ is closed under the (ND) condition. Therefore, by the projection theorem, there is a unique strategy $\zeta^c \in \vartheta$ solving (A.10). Furthermore a necessary and sufficient condition that $\zeta^c \in \vartheta$ be the unique minimizing strategy is:

$$E\left[(H-c-G_T(\zeta^c))G_T(\vartheta)\right] = 0, \forall\vartheta \in \Theta. \tag{A.11}$$

The condition (A.11) is equivalently written as follows:

$$E\left[(H-c-G_T(\zeta^c))\Delta S_k|\mathcal{F}_{k-1}\right]=0,\ \ \mathcal{P}-a.s.\ \ \forall k\in\mathcal{T}\backslash\{0\}. \tag{A.12}$$

We show by backward induction that:

(i)

$$H-c-G_T(\zeta^c)=H-\sum_{j=k}^{T}\varrho_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l)-(c+G_{k-1}(\zeta^c))\prod_{l=k}^{T}(1-\beta_l'\Delta S_l) \tag{A.13}$$

(ii)

$$\zeta_k^c=\varrho_k-\beta_k(c+G_{k-1}(\zeta^c)) \tag{A.14}$$

For $k=T$, we have from (A.12)

$$\begin{aligned}
0 &= E\left[(H-c-G_T(\zeta^c))\Delta S_T|\mathcal{F}_{T-1}\right] && \text{(A.15)}\\
&= E\left[H\Delta S_T-\Delta S_T^2\zeta_T^c-(c+G_{T-1}(\zeta^c))\Delta S_T|\mathcal{F}_{T-1}\right] && \text{(A.16)}\\
&= E\left[H\Delta S_T|\mathcal{F}_{T-1}\right]-E\left[\Delta S_T^2|\mathcal{F}_{T-1}\right]\zeta_T^c-(c+G_{T-1}(\zeta^c))E\left[\Delta S_T|\mathcal{F}_{T-1}\right] && \text{(A.17)}
\end{aligned}$$

as $G_{T-1}(\zeta^c)$ and ζ_T^c are both $\mathcal{F}_{T-1}$-measurable. So,

$$\begin{aligned}
\zeta_T^c &= \left(E\left[\Delta S_T^2|\mathcal{F}_{T-1}\right]\right)^{-1}E\left[H\Delta S_T|\mathcal{F}_{T-1}\right]\\
&\quad -\left(E\left[\Delta S_T^2|\mathcal{F}_{T-1}\right]\right)^{-1}E\left[\Delta S_T|\mathcal{F}_{T-1}\right](c+G_{T-1}(\zeta^c)) && \text{(A.18)}\\
&= \varrho_T-\beta_T(c+G_{T-1}(\zeta^c)). && \text{(A.19)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned} H - c - G_T(\zeta^c) &= H - \zeta_T^{c\prime}\Delta S_T - (c + G_{T-1}(\zeta^c)) && \text{(A.20)} \\ &= H - \varrho_T'\Delta S_T - (c + G_{T-1}(\zeta^c))(1 - \beta_T'\Delta S_T) && \text{(A.21)} \end{aligned}$$

Assume that (A.13) and (A.14) hold for $j = k+1, ..., T$. Then (A.12) implies that:

$$\begin{aligned} 0 &= E\left[\Delta S_k(H - c - G_T(\zeta^c))|\mathcal{F}_{k-1}\right] && \text{(A.22)} \\ &= E\left[\Delta S_k\left(H - \sum_{j=k+1}^{T}\varrho_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l)\right)|\mathcal{F}_{k-1}\right] && \text{(A.23)} \\ &\quad -(c + G_{k-1}(\zeta^c))E\left[\Delta S_k\prod_{l=k+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right] && \text{(A.24)} \\ &\quad -E\left[\Delta S_k^2\prod_{l=k+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right]\zeta_k^c && \text{(A.25)} \end{aligned}$$

The last equality is obtained due to the $\mathcal{F}_{k-1}$-measurability of ζ_k^c and $G_{k-1}(\zeta^c)$. When $\forall j > k$, (A.9) implies:

$$E\left[H\beta_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{j-1}\right] = E\left[\varrho_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{j-1}\right] \quad \text{(A.26)}$$

By conditioning on $\mathcal{F}_{j-1}$, $\forall j > k$, and armed with (A.26), we get:

$$\begin{aligned} 0 &= E\left[\Delta S_k H|\mathcal{F}_{k-1}\right] - E\left[\Delta S_k H\sum_{j=k+1}^{T}\beta_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right] && \text{(A.27)} \\ &- (c + G_{k-1}(\zeta^c))E\left[\Delta S_k\prod_{l=k+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right] && \text{(A.28)} \\ &- E\left[\Delta S_k^2\prod_{l=k+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right]. && \text{(A.29)} \end{aligned}$$

Or,

$$E\left[\Delta S_k H|\mathcal{F}_{k-1}\right] - E\left[\Delta S_k H \sum_{j=k+1}^{T} \beta_j' \Delta S_j \prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right] =$$
$$E\left[\Delta S_k H \prod_{l=k+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right] \quad \text{(A.30)}$$

Whence,

$$\begin{aligned}\zeta_k^c &= \left(E\left[\Delta S_k^2 \prod_{l=k+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right]\right)^{-1} \\ &\quad \left(E\left[\Delta S_k H|\mathcal{F}_{k-1}\right] - E\left[\Delta S_k \prod_{l=k+1}^{T}(1-\beta_l'\Delta S_l)|\mathcal{F}_{k-1}\right](c+G_{k-1}(\zeta^c))\right) \\ &= \varrho_k - \beta_k(c+G_{k-1}(\zeta^c)). \end{aligned} \quad \text{(A.31)}$$

We can then write:

$$c + G_k(\zeta^c) = c + G_{k-1}(\zeta^c) + \zeta_k^{c\prime}\Delta S_k \quad \text{(A.32)}$$
$$= \varrho_k'\Delta S_k + (c+G_{k-1}(\zeta^c))(1-\beta_k'\Delta S_k) \quad \text{(A.33)}$$

Furthermore, the induction assumption allows us to apply (A.13) for $k+1$ and, plugging in (A.33), we prove that (A.13) is true for k.

On the other hand, from (A.13) we have:

$$E\left[(H-c-G_T(\zeta^c))\right] = E\left[H - \sum_{j=1}^{T}\varrho_j'\Delta S_j \prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l) - c\prod_{l=1}^{T}(1-\beta_l'\Delta X_l)\right] \quad \text{(A.34)}$$
$$= E\left[H - \sum_{j=1}^{T}H\beta_j'\Delta S_j \prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l) - c\prod_{l=1}^{T}(1-\beta_l'\Delta X_l)\right] \text{ (by (A.9))} \quad \text{(A.35)}$$
$$= E\left[(H-c)\prod_{l=1}^{T}(1-\beta_l'\Delta S_l)\right] \quad \text{(A.36)}$$
$$= E[H\widetilde{Z}_0] - cE[\widetilde{Z}_0], \quad \text{(A.37)}$$

where

$$\widetilde{Z}_0 = \prod_{l=1}^{T}(1-\beta_l'\Delta S_l). \tag{A.38}$$

Define

$$\frac{d\widetilde{P}}{dP} = \frac{\widetilde{Z}_0}{E[\widetilde{Z}_0]} \tag{A.39}$$

(Appendix C presents the properties of $\widetilde{P}$.)

Moreover,

$$\begin{aligned}
E\left[(H-c-G_T(\zeta^c))^2\right] &= c^2E\left[\prod_{j=1}^{T}(1-\beta_j'\Delta S_j)^2\right] \\
&\quad +E\left[(H-\sum_{j=1}^{T}\varrho_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l))^2\right] \\
&\quad -2cE[(H-\sum_{j=1}^{T}\varrho_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l))\prod_{k=1}^{T}(1-\beta_k'\Delta S_k)] \qquad \text{(A.40)} \\
&= c^2E\left[\widetilde{Z}_0\right]-2cE\left[H\widetilde{Z}_0\right]+E\left[(H-\sum_{j=1}^{T}\varrho_j'\Delta S_j\prod_{l=j+1}^{T}(1-\beta_l'\Delta S_l))^2\right] \qquad \text{(A.41)}
\end{aligned}$$

In fact,

$$
\begin{aligned}
&E\left[\varrho_j' \Delta S_j \prod_{l=j+1}^{T}(1-\beta_l' \Delta S_l) \prod_{k=1}^{T}(1-\beta_k' \Delta S_k) | \mathcal{F}_{j-1}\right] = \\
&\varrho_j' E\left[\Delta S_j \prod_{l=j+1}^{T}(1-\beta_l' \Delta S_l)^2 (1-\beta_j' \Delta S_j) | \mathcal{F}_{j-1}\right] \prod_{k=1}^{j-1}(1-\beta_k' \Delta S_k) = \\
&\varrho_j' E\left[\Delta S_j E\left[\prod_{l=j+1}^{T}(1-\beta_l' \Delta S_l)^2 | \mathcal{F}_j\right] (1-\beta_j' \Delta S_j) | \mathcal{F}_{j-1}\right] \prod_{k=1}^{j-1}(1-\beta_k' \Delta S_k) = \text{ (by (A.8))} \\
&\varrho_j' E\left[\Delta S_j \prod_{l=1}^{T}(1-\beta_l' \Delta S_l)\right] = \text{ (by martingale property)} \\
&0 \qquad \text{(A.42)}
\end{aligned}
$$

The price V_0 therefore minimizes (A.41). Then,

$$V_0 E\left[\widetilde{Z}_0\right] - E\left[H\widetilde{Z}_0\right] = 0. \qquad \text{(A.43)}$$

This implies:

$$V_0 = \frac{E\left[H\widetilde{Z}_0\right]}{E\left[\widetilde{Z}_0\right]} = E_{\widetilde{P}}\left[H\right]. \qquad \text{(A.44)}$$

To complete the proof, we obtain (5.10) by replacing c with V_0 in (A.41).

Appendix B

Proof of (A.6)-(A.9)

Proof of (A.6)-(A.8)

To prove (A.6)-(A.7), it is sufficient to show that $(\beta'_k \Delta S_k)^2 \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j)^2 \in \mathcal{L}^1(\mathcal{P})$. Define:

$$\begin{aligned} R_n \quad := \quad & E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j)|\mathcal{F}_{k-1}](E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j)^2|\mathcal{F}_{k-1}])^{-1}\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j))^2 \\ & (E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j)^2|\mathcal{F}_{k-1}])^{-1} E[\Delta S_k \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j)|\mathcal{F}_{k-1}] \\ & \mathbf{1}_{\{|\det(E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j)^2|\mathcal{F}_{k-1}])|\geq \frac{1}{n}\}}, \end{aligned} \tag{B.1}$$

where $\det(A)$ is the determinant of the matrix A and $\mathbf{1}_{\{B\}}$ is the indicator function of the subset B. We have R_n are positive and increasing a.s. to $(\beta'_k \Delta S_k \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j))^2$. By the monotone convergence theorem (see Jacod and Protter, 2003), we obtain:

$$E[\beta'_k \Delta S_k \prod_{j=k+1}^{T}(1-\beta'_j \Delta S_j))^2|\mathcal{F}_{k-1}] \stackrel{P-a.s.}{=} \lim_{n\to\infty} E[R_n|\mathcal{F}_{k-1}] \tag{B.2}$$

$$\begin{aligned}
= \quad & E[E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}](E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}])^{-1}\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2 \\
& (E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}])^{-1}E[\Delta S_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}] \\
& \mathbf{1}_{\{|det(E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}])|\geq \frac{1}{n}\}} \qquad \text{(B.3)} \\
& = E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}](E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}])^{-1} \\
& E[\Delta S_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}]\mathbf{1}_{\{|det(E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}])|\geq \frac{1}{n}\}} \leq 1
\end{aligned}$$

The last inequality is obtained from the positive semi-definite property of the variance covariance matrix. We have:

$$Var(\Delta S_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}) \geq 0 \tag{B.4}$$

$$\Rightarrow E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}] - (E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}])^2 \geq 0 \tag{B.5}$$

We are interested in the set of events where $E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}]$ is invertible, then

$$\begin{aligned}
& E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}]E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}] \\
& \Bigg(1-(E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}])(E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}])^{-1} \\
& E[\Delta S'_k \prod_{j=k+1}^{T}(1-\beta_j\Delta S_j)|\mathcal{F}_{k-1}]\Bigg) \geq 0,
\end{aligned} \tag{B.6}$$

proving the inequality. Note that the inequality still holds if $E[\Delta S_k \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}] = 0_{d\times 1}$. As a consequence,

$$E[(\beta_k'\Delta S_k)^2 \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)^2] \leq 1, \;\; \mathcal{P}-a.s.$$
$$\Rightarrow (\beta_k'\Delta S_k)^2 \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)^2 \in \mathcal{L}^1(\mathcal{P}). \tag{B.7}$$

We prove (A.8) by backward induction. For $k = T$, note that from the definition of β_T we have

$$E[(\beta_T'\Delta S_T)^2|\mathcal{F}_{T-1}] = \beta_T' E[\Delta S_T|\mathcal{F}_{T-1}] \tag{B.8}$$
$$= E[\beta_T'\Delta S_T|\mathcal{F}_{T-1}]. \tag{B.9}$$

Therefore, we conclude that

$$E[(1-\beta_T'\Delta S_T)^2|\mathcal{F}_{T-1}] = E[1-\beta_T'\Delta S_T|\mathcal{F}_{T-1}] \tag{B.10}$$
$$= 1 - E[\Delta S_T'|\mathcal{F}_{T-1}]E[\Delta S_T^2|\mathcal{F}_{T-1}]E[\Delta S_T|\mathcal{F}_{T-1}] \leq 1. \tag{B.11}$$

Suppose that (A.8) holds for $k+1$, then

$$E[\prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)^2|\mathcal{F}_k] = E[\prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_k] \leq 1 \;\; \mathcal{P}-a.s. \tag{B.12}$$

We have

$$E[\prod_{j=k}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}] = E[\prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}] - E[\beta_k'\Delta S_k \prod_{j=k+1}^{T}(1-\beta_j'\Delta S_j)|\mathcal{F}_{k-1}] \tag{B.13}$$

and

$$E[\beta'_k \Delta S_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] = \beta'_k E[\Delta S_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] \tag{B.14}$$

$$= E[\Delta S'_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] (E[\Delta S_k^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}])^{-1}$$

$$(E[\Delta S_k^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}])$$

$$(E[\Delta S_k^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}])^{-1} E[\Delta S_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}]$$

$$= E\,[\, E[\Delta S'_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] (E[\Delta S_k^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}])^{-1}$$

$$\Delta S_k^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 (E[\Delta S_k^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}])^{-1} E[\Delta S_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] | \mathcal{F}_{k-1}\,] \tag{B.15}$$

$$= E[(\beta'_k \Delta S_k)^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}]. \tag{B.16}$$

It follows that:

$$E[\prod_{j=k}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] = E[E[\prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_k] | \mathcal{F}_{k-1}] - E[(\beta'_k \Delta S_k)^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}] \tag{B.17}$$

$$= E[\prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}] - E[(\beta'_k \Delta S_k)^2 \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}] \tag{B.18}$$

$$= E[\prod_{j=k}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}]. \tag{B.19}$$

Note that from (B.18), we obtain

$$E[\prod_{j=k}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] \leq 1 \;\; \mathcal{P} - a.s. \tag{B.20}$$

which completes the proof.

Proof of (A.9)

We have

$$E[H\beta'_k\Delta S_k\prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}] = E[\Delta S'_k\prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}]$$

$$(E[\Delta S_k^2\prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)^2|\mathcal{F}_{k-1}])^{-1}E[H\Delta S_k\prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}] \qquad \text{(B.21)}$$

$$= E[\varrho'_k\Delta S_k\prod_{j=k+1}^{T}(1-\beta'_j\Delta S_j)|\mathcal{F}_{k-1}]. \qquad \text{(B.22)}$$

Appendix C

Properties of $\widetilde{P}$

$\widetilde{P}$ is a signed $\mathcal{L}^2$-martingale measure. In fact, we have from (A.8):

$$0 \leq E[\widetilde{Z}_0] = E[(\widetilde{Z}_0)^2] \leq 1 \tag{C.1}$$

$$\Rightarrow \widetilde{Z}_0 \in \mathcal{L}^2(\mathcal{P})$$

and $\forall k = 1, ..., T$,

$$E[\widetilde{Z}_0 \Delta S'_k | \mathcal{F}_{k-1}] = E[\prod_{j=1}^{T} (1 - \beta'_j \Delta S_j) \Delta S'_k | \mathcal{F}_{k-1}] \tag{C.2}$$

$$= E[\prod_{j=k+1}^{T} (1 - \beta'_j \Delta S_j)(1 - \beta'_k \Delta S_k) \Delta S'_k | \mathcal{F}_{k-1}] \prod_{j=1}^{k-1} (1 - \beta'_j \Delta S_j) \tag{C.3}$$

$$= \left(E[\prod_{j=k+1}^{T} (1 - \beta'_j \Delta S_j) \Delta S'_k | \mathcal{F}_{k-1}] - E[\prod_{j=k+1}^{T} (1 - \beta'_j \Delta S_j) \beta'_k \Delta S_k \Delta S'_k | \mathcal{F}_{k-1}] \right) \prod_{j=1}^{k-1} (1 - \beta'_j \Delta S_j) \tag{C.4}$$

$$= \left(E[\Delta S'_k \prod_{j=k+1}^{T} (1 - \beta'_j \Delta S_j) | \mathcal{F}_{k-1}] - \beta'_k E[\Delta S_k \Delta S'_k E[\prod_{j=k+1}^{T} (1 - \beta'_j \Delta S_j) | \mathcal{F}_k] | \mathcal{F}_{k-1}] \right) \prod_{j=1}^{k-1} (1 - \beta'_j \Delta S_j) \tag{C.5}$$

$$= \left(E[\Delta S'_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] - \beta'_k E[\Delta S_k \Delta S'_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j)^2 | \mathcal{F}_{k-1}] \right) \prod_{j=1}^{k-1} (1-\beta'_j \Delta S_j) \tag{C.6}$$

$$= \left(E[\Delta S'_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] - E[\Delta S'_k \prod_{j=k+1}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] \right) \prod_{j=1}^{k-1} (1-\beta'_j \Delta S_j) \tag{C.7}$$

$$= 0,$$

from which the martingale property of $\widetilde{Z}_0$ follows.

Schweizer (1996) shows that a signed martingale measure $\widetilde{P}$ is a variance-optimal measure if and only if

$$E\left[\frac{dQ}{d\mathcal{P}} \frac{d\widetilde{P}}{d\mathcal{P}}\right] \text{ is constant over all } Q \in \mathbb{P}_s(\Theta). \tag{C.8}$$

Assume that $Q \in \mathbb{P}_s(\Theta)$ and define

$$Z_k := E\left[\frac{dQ}{d\mathcal{P}} | \mathcal{F}_{k-1}\right] \in \mathcal{L}^2(\mathcal{P}). \tag{C.9}$$

By backward induction we show that

$$E[Z_T \prod_{j=k}^{T} (1-\beta'_j \Delta S_j) | \mathcal{F}_{k-1}] = Z_{k-1}, \mathcal{P}\text{-a.s. for } k = 1, ..., T. \tag{C.10}$$

For k=T, (C.10) is satisfied due to:

$$E[Z_k \Delta S_k | \mathcal{F}_{k-1}] = 0 \;\; \mathcal{P} - a.s. \text{ for } k = 1, ..., T. \tag{C.11}$$

Supposing that (C.10) is true for $k+1$, then

$$E[Z_T \prod_{j=k}^{T}(1-\beta_j' \Delta S_j)|\mathcal{F}_{k-1}] = E[Z_T \prod_{j=k+1}^{T}(1-\beta_j' \Delta S_j)(1-\beta_k' \Delta S_k)|\mathcal{F}_{k-1}] \tag{C.12}$$

$$= E[E[Z_T \prod_{j=k+1}^{T}(1-\beta_j' \Delta S_j)|\mathcal{F}_k](1-\beta_k' \Delta S_k)|\mathcal{F}_{k-1}] \tag{C.13}$$

$$= E[Z_k(1-\beta_k' \Delta S_k)|\mathcal{F}_{k-1}] \tag{C.14}$$

$$= E[Z_k|\mathcal{F}_{k-1}] - \beta_k' \underbrace{E[Z_k \Delta S_k|\mathcal{F}_{k-1}]}_{\text{0 by martingale property}} \tag{C.15}$$

$$= Z_{k-1}. \tag{C.16}$$

We note that integrability is ensured by (A.7). This completes the proof of (C.10).

Armed with (C.10), we obtain by inspection that

$$E[Z_T \widetilde{Z}_0] = E[Z_0] = 1. \tag{C.17}$$

We conclude that $\widetilde{P}$ is a variance-optimal measure.

Furthermore, under the no-arbitrage condition, $\widetilde{P}$ exists and is unique (see Schweizer, 1996). This implies that the invertibility of $(E[\Delta S_k^2 \prod_{j=k+1}^{T}(1-\beta_j' \Delta S_j)^2|\mathcal{F}_{k-1}], \forall k = 1, ..., T, \mathcal{P}-$ $a.s.$ is guaranteed by the assumption (A.3) and the $\mathcal{P}-a.s.$ non-singularity condition of $E[\Delta S_k^2|\mathcal{F}_{k-1}]$.

Appendix D

Sufficient Condition to Reduce the Unhedged Risk

Assume $(S'_t, \Xi_t)_{t\in\mathcal{T}} \in \mathcal{L}^2_{d+1}(\mathcal{P})$ and $H \in \mathcal{L}^2(\mathcal{P})$. Define

$$\Theta^1 := \{\text{predictable processes } \vartheta | \vartheta'_k(\Delta S'_k, \Delta\Xi_k)' \in \mathcal{L}^2(\mathcal{P})\}, \tag{D.1}$$

$$G_T(\vartheta, S) := \sum_{j=1}^{T} \vartheta'_j \Delta S_j. \tag{D.2}$$

The unhedged risk of given strategy $(\vartheta, \eta) \in \Theta^1$ and initial capital V_0 is:

$$R(V_0, \vartheta, \eta) = E_{\mathcal{P}}[(H - V_0 - G_T(\vartheta, S) - G_T(\eta, \Xi))^2]. \tag{D.3}$$

Define $(O)_{t\in\mathcal{T}} \in \mathcal{L}^2(\mathcal{P})$, such as $P(O_t = 0, \forall t \leq T) = 1$. $R(V_0, \vartheta, O)$ is the unhedged risk by following strategy ϑ on S with an initial capital V_0.

Proposition D.1 *Assume a probability space* $(\Omega, \mathbb{F}, \mathcal{P})$*,* $H \in \mathcal{L}^2(\mathcal{P})$*, and stochastic process* $(S'_t, \Xi)'_{t\in\mathcal{T}} \in \mathcal{L}^2_{d+1}(\mathcal{P})$ *adapted to the filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ *such that* $E[\Delta S^2_k|\mathcal{F}_{k-1}]$ *and* $E[(\Delta S'_k, \Delta\Xi_k)'^2|\mathcal{F}_{k-1}]$ *are invertible* $\mathcal{P}$*-a.s. and satisfy the non-degeneracy condition. Define:*

$$(V_0, \zeta) = \underset{(c,\vartheta)\in\mathbb{R}\times\Theta}{\operatorname{argmin}} R(c, \vartheta, O). \tag{D.4}$$

If $P(E[(H - V_0 - G_T(\zeta, S))\Delta\Xi_T|\mathcal{F}_{T-1}] \neq 0) > 0$*, then*

$$R(V_0, \zeta, O) > \min_{(c,(\vartheta,\eta))\mathbb{R}\times\Theta^1} R(c, \vartheta, \eta). \tag{D.5}$$

Therefore, hedging with $(S'_t, \Xi_t)_{t\in\mathcal{T}}$ *is more efficient than hedging with* $(S_t)_{t\in\mathcal{T}}$*.*

Proof The existence and uniqueness of a solution to the optimization problems (either with or without $(\Xi_t)_{t\in\mathcal{T}}$) are guaranteed by proposition 5.1.1. To clarify the steps, we start with presenting the proof for one time period model, $T = 1$. We firstly prove that it exists at least a η_0 such as:

$$R(V_0, \zeta, O) > R(V_0, \zeta, \eta_0). \tag{D.6}$$

As $R(V_0, \zeta, \eta_0) \geq \min_{(c,\vartheta,\eta)} R(c, \vartheta, \eta)$, then (D.5) is satisfied.

We have:

$$\begin{aligned} R(V_0, \zeta, \eta) &= E[(H - V_0 - \zeta'\Delta S_1 - \eta\Delta\Xi_1)^2] && \text{(D.7)} \\ &= E[(H - V_0 - \zeta'_0\Delta S_1)^2] + \eta^2 E[(\Delta\Xi_1)^2] - 2\eta E[(H - V_0 - \zeta'\Delta S_1)\Delta\Xi_1] && \text{(D.8)} \\ &= R(V_0, \zeta, 0) + \eta^2 E[(\Delta\Xi_1)^2] - 2\eta E[(H - V_0 - \zeta'\Delta S_1)\Delta\Xi_1]. && \text{(D.9)} \end{aligned}$$

As $P(E[(H - V_0 - \zeta'_0\Delta S_1)\Delta\Xi_1] \neq 0) > 0$, then $E[(H - V_0 - \zeta'_0\Delta S_1)\Delta\Xi_1] \neq 0$. We can always find η_0 such as:

$$\eta_0^2 E[(\Delta\Xi_1)^2] - 2\eta_0 E[(H - V_0 - \zeta_0' \Delta S_1)\Delta\Xi_1] < 0. \tag{D.10}$$

Thus (D.5) is satisfied.

For a multiperiod model, consider the following strategy (ζ, η) such that $(\eta_t)_{t \le T-1}$ is indistinguishable from $\mathcal{O}$ under the historical probability measure $\mathcal{P}$. In probability terms,

$$P(\eta_t = 0, \forall t \le T-1) = 1. \tag{D.11}$$

Therefore,

$$\begin{aligned} R(V_0, \zeta, \eta) &= E[(H - V_0 - G_T(\zeta, S) - \eta_T \Delta\Xi_T)^2] && \text{(D.12)} \\ &= E[(H - V_0 - G_T(\zeta, S))^2] + E\left[\eta_T^2 E[(\Delta\Xi_T)^2 | \mathcal{F}_{T-1}] - \right. \\ &\quad \left. 2\eta_T E[(H - V_0 - G_T(\zeta, S))\Delta\Xi_T | \mathcal{F}_{T-1}]\right] && \text{(D.13)} \\ &= R(V_0, \zeta, 0) + E\left[\eta_T^2 E[(\Delta\Xi_T)^2 | \mathcal{F}_{T-1}] - 2\eta_T E[(H - V_0 - G_T(\zeta, S))\Delta\Xi_T | \mathcal{F}_{T-1}]\right]. && \text{(D.14)} \end{aligned}$$

Define

$$A = \{\omega \in \Omega | E[(H - V_0 - G_T(\zeta, S))\Delta\Xi_T | \mathcal{F}_{T-1}](\omega) \neq 0\} \tag{D.15}$$

Now A is $\mathcal{F}_{T-1}-$ measurable and $P(A) > 0$. It is always possible to construct a strategy η_T such that there exists $M < 0$:

$$\begin{cases} M < \eta_T^2(\omega) E[(\Delta\Xi_T)^2 | \mathcal{F}_{T-1}](\omega) - 2\eta_T(\omega) E[(H - V_0 - G_T(\zeta, S))\Delta\Xi_T | \mathcal{F}_{T-1}](\omega) < 0 & \text{if } \omega \in A, \\ 0 & \text{otherwise.} \end{cases}$$

As A is $\mathcal{F}_{T-1}-$measurable, then η_T is $\mathcal{F}_{T-1}-$measurable too. Also $(\zeta, \eta) \in \Theta^1$. Furthermore,

$$E\left[\eta_T^2 E[(\Delta\Xi_T)^2|\mathcal{F}_{T-1}] - 2\eta_T E[(H - V_0 - G_T(\zeta, S))\Delta\Xi_T|\mathcal{F}_{T-1}]\right] < 0. \tag{D.16}$$

Thus,

$$R(V_0, \zeta, \eta) < R(V_0, \zeta, \mathcal{O}). \tag{D.17}$$

Consequently, (D.5) follows.

Appendix E

E-step

$\mathcal{J}$ has the Poisson distribution with parameters $(\lambda_1 + \lambda_2)\triangle t$, as it is the sum of two independent Poisson processes dN_{1t} and dN_{2t} with parameter $\lambda_1 \triangle t$ and $\lambda_2 \triangle t$, respectively.

To compute the E-step, we need to determine the conditional probability of $\mathcal{J}$ given the observed returns:

$$\begin{aligned} P(\mathcal{J} = J \mid \mathcal{X}; \Theta) &= \frac{P(\mathcal{J} = J, \mathcal{X})}{P(\mathcal{X})} && \text{(E.1)} \\ &= \frac{P(\mathcal{J} = J) P(\mathcal{X} \mid \mathcal{J} = J)}{P(\mathcal{X})} && \text{(E.2)} \\ &= \frac{e^{-(\lambda_1+\lambda_2)\triangle t} \left((\lambda_1 + \lambda_2)\triangle t\right)^J}{J!} \frac{\prod_{i=1}^{n} f_c\left(\triangle X_i, J; \Theta\right)}{\prod_{i=1}^{n} f\left(\triangle X_i; \Theta\right)} && \text{(E.3)} \end{aligned}$$

Given $P(\mathcal{J} = J \mid \mathcal{X}; \Theta)$, the Q-function on the $(k+1)^{st}$ iteration is:

$$\begin{aligned} Q\left(\Theta, \Theta^{(k)}\right) &= E_{\Theta^{(k)}}\left[\ln\left(L_c\left(\mathcal{Y}; \Theta\right)\right) \mid \mathcal{X}\right] && \text{(E.4)} \\ &= \sum_{J=0}^{M} P(\mathcal{J} = J \mid \mathcal{X}; \Theta^{(k)}) \ln\left(L_c\left((\mathcal{X}, J); \Theta\right)\right), && \text{(E.5)} \end{aligned}$$

where $M = +\infty$. We choose $M = 60$ to compute (E.5)[1].

[1]It is discussed in page 77 that 30 is a decent choice to truncate the number of jumps for each discontinuous component. As $\mathcal{J}$ represents the sum of the total number of jumps from both Poisson processes, we choose 60 as a truncation level.

Appendix F

Föllmer-Schweizer Decomposition

Consider a more general case where:

$$\frac{dF(t,T_i)}{F(t^-,T_i)} = \mu_i dt + \sigma_{i1} dW_{1t} + \sigma_{i2} dW_{2t} + \varphi_{i1} dN_{1t} + \varphi_{i2} dN_{2t}, \quad F(0,T_i) > 0, i = 1,2. \tag{F.1}$$

We set $\sigma_{12} = 0$ in order to reduce this to our framework. The process $F(t,T_i)$, $i = 1,2$, is a semimartingale and it has the following Doob-Meyer decomposition:

$$F(t,T_i) = F(0,T_i) + M_{it} + A_{it}, \tag{F.2}$$

where

$$M_{it} = \int_0^t F(t^-,T_i)(\sigma_{i1} dW_{1s} + \sigma_{i2} dW_{2s} + \varphi_{i1} d\widetilde{N}_1 s + \varphi_{i2} d\widetilde{N}_{2s}),$$
$$A_{it} = \int_0^t F(t^-,T_i)(\mu_i + \varphi_{i1}\lambda_1 + \varphi_{i2}\lambda_2) ds,$$

with $M_{i0} = A_{i0} = 0$. $\widetilde{N}_{it}$ is the compensated Poisson process of N_{it}. Since $\widetilde{N}_{it}$ is a $\mathcal{P}$-martingale, therefore M_{it} is a $\mathcal{P}$-martingale. Furthermore A_{it} is a predictable process with finite variation. Consequently $F(t,T_i)$ is a special semimartingale. Given $F(0,T_i)$, this decomposition is unique.

Denote $\mathcal{L}^2(\mathcal{P})$ as the set of square integrable random variables with respect to $\mathcal{P}$:

$$\mathcal{L}^2(\mathcal{P}) = \left\{X; E[|X|^2] < \infty\right\} \tag{F.3}$$

(Unless explicitly stated otherwise, all expectations are taken under the $\mathcal{P}$ measure).

We assume that the market is free of arbitrage and therefore the set of equivalent local martingale measures:

$$\mathcal{P}^e = \left\{Q \sim \mathcal{P} : \frac{dQ}{d\mathcal{P}} \in \mathcal{L}^2(\mathcal{P}), F(t, T_i), i = 1, 2, \text{ is a } Q - \text{local martingale}\right\} \tag{F.4}$$

is non-empty. As the market is incomplete and $\mathcal{P}^e \neq \emptyset$, $\mathcal{P}^e$ contains infinitely many elements (see Delbaen and Schachermayer, 2006).

A portfolio strategy Φ includes both the value of the portfolio process V as well as the trading strategy ζ. V is an adapted process with $V_T \in \mathcal{L}^2(\mathcal{P})$. ζ is a $\mathbb{R}^2$-predictable process that lives in:

$$\Psi = \left\{(\zeta)_t : \mathbb{R}^2 - \text{predictable process}; \left(E\left[\int_0^{T_1} \zeta_t' d < M >_t \zeta_t\right]\right)^{\frac{1}{2}} < \infty \text{ and } E\left[\left(\int_0^{T_1} |\zeta_t' dA_t|\right)^2\right] < \infty\right\},$$

where $M_t = (M_{1t}, M_{2t})'$, $A_t = (A_{1t}, A_{2t})'$, and $< M >$ is the sharp bracket process[1] of M. Consequently, we have

$$\int \zeta dF \text{ is a martingale under } Q, \quad \forall Q \in \mathcal{P}^e. \tag{F.5}$$

where $F_t = (F(t, T_1), F(t, T_2))'$(see Pham, 2000).

Given a T_1-claim $H \in \mathcal{L}^2(\mathcal{P})$, an H-admissible strategy $\Phi^H = (V^H, \zeta^H)$ is a portfolio strategy that pays H at maturity T_1, $V_{T_1}^H = H$, $\mathcal{P}-$ almost surely. Schweizer (1990) shows that an H-admissible strategy Φ is locally risk minimizing if the associated cost process is a square-

[1]The sharp bracket process of M is the compensator of its quadratic variation process. We refer to Klebaner (2005) for a detailed definition of the sharp bracket process.

integrable martingale orthogonal to M_i, $i = 1,2$, under $\mathcal{P}$. Recall that two square integrable martingales M^1 and M^2 are orthogonal if their product M^1M^2 is a martingale with initial value $M^1M^2 = 0$. This is equivalent to:

$$< M^1, M^2 >:= \frac{1}{2}\left(< M^1 + M^2 > - < M^1 > - < M^2 >\right) = 0. \tag{F.6}$$

Föllmer and Schweizer (1991) show that a T_1-claim H has an H-admissible strategy $\Phi^H = (V^H, \zeta^H)$ if and only if H:

$$H = H_0 + \int_0^{T_1} \zeta_t^H dF_t + L_T^H,\ \ \mathcal{P} - \text{almost surely}, \tag{F.7}$$

where $H_0 \in \mathbb{R}$ and is $\mathcal{F}_0$-measurable, $\zeta^H \in \Psi$ and L^H is a square-integrable $\mathcal{P}-$martingale orthogonal to $M_i, i = 1,2$, such that $L_0^H = 0$. This decomposition is called the Föllmer-Schweizer decomposition of H under $\mathcal{P}$. The associated value of the portfolio is:

$$V_t^H = H_0 + \int_0^t \zeta_s^H dF_s + L_t^H,\ \ \mathcal{P} - \text{almost surely},\ \ 0 \le t \le T_1, \tag{F.8}$$

and the associated cost process is:

$$C_t(\Phi^H) = H_0 + L_t^H,\ \ 0 \le t \le T_1. \tag{F.9}$$

Set $\widehat{\lambda}_{it} := \frac{\alpha_i}{F(t^-,T_i)}, i = 1,2, \forall 0 \le t \le T_1$. The mean-variance trade-off process $\widehat{K}$ is the predictable process:

$$\widehat{K}_t \;\; = \;\; \int_0^t \widehat{\lambda}'_{is} d < M >_s \widehat{\lambda}_{is},\ \ 0 \le t \le T_1. \tag{F.10}$$

In our framework, the mean-variance trade-off process is:

$$\widehat{K}_t = [\alpha_1[\alpha_1\beta + \alpha_2\chi] + \alpha_2[\alpha_1\chi + \alpha_2\gamma]]t, \tag{F.11}$$

where

$$\beta = [\sigma_{11}^2 + \sigma_{12}^2 + \varphi_{11}^2\lambda_1 + \varphi_{12}^2\lambda_2 + 2\varphi_{12}\varphi_{11}(\lambda_1 + \lambda_2)]$$
$$\gamma = [\sigma_{21}^2 + \sigma_{22}^2 + \varphi_{22}\lambda_2 + \varphi_{21}^2\lambda_1 + 2\varphi_{21}\varphi_{22}(\lambda_1 + \lambda_2)]$$
$$\chi = [\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} + \varphi_{11}\varphi_{21}\lambda_1 + \varphi_{11}\varphi_{22}(\lambda_1 + \lambda_2) + \varphi_{12}\varphi_{21}(\lambda_1 + \lambda_2) + \varphi_{22}\varphi_{12}\lambda_2].$$

$\widehat{K}$ is uniformly bounded. Thus, every $\mathcal{P}$ square-integrable contingent claim has an unique Föllmer-Schweizer decomposition (Monat and Stricker, 1995).

Assume now that there exists $\widehat{P} \in \mathcal{P}^e$ such that both $\int_0^{\cdot} \zeta_t^H dF_t$ and L^H are $\widehat{P}$-martingales. Thus,

$$E_{\widehat{P}}[H|\mathcal{F}_t] = E_{\widehat{P}}\left[H_0 + \int_0^{T_1} \zeta_s^H dF_s + L_T^H|\mathcal{F}_t\right] \quad (F.12)$$
$$= H_0 + \int_0^t \zeta_s^H dF_s + L_t^H \quad (F.13)$$
$$= V_t^H. \quad (F.14)$$

Therefore, the *fair* price of H at time t is given by:

$$V_t^H = E_{\widehat{P}}[H|\mathcal{F}_t]. \quad (F.15)$$

$\widehat{P}$ is called the minimum martingale measure. Referring to Arai (2004), such a measure exists and its density process $\widehat{Z}$ is:

$$\begin{aligned}\widehat{Z}_t &= \exp(-(\alpha_1\sigma_{11}+\alpha_2\sigma_{21})W_{1t}-(\sigma_{12}\alpha_1+\sigma_{22}\alpha_2)W_{2t}\\ &-\frac{1}{2}(\alpha_1^2\sigma_{11}^2+\alpha_1^2\sigma_{12}^2+\alpha_1\alpha_2\sigma_{11}\sigma_{21}+\alpha_1\alpha_2\sigma_{12}\sigma_{22}\\ &\alpha_2\alpha_1\sigma_{21}\sigma_{11}+\alpha_2\alpha_1\sigma_{22}\sigma_{12}+\alpha_2^2\sigma_{21}^2+\alpha_2^2\sigma_{22}^2)t\\ &+\ln(1-\alpha_1\varphi_{11})N_{1t}+\alpha_1\varphi_{11}\lambda_1 t\\ &+\ln(1-\alpha_2\varphi_{21})N_{1t}+\alpha_2\varphi_{21}\lambda_1 t\\ &+\ln(1-\alpha_1\varphi_{12})N_{2t}+\alpha_1\varphi_{12}\lambda_2 t\\ &+\ln(1-\alpha_2\varphi_{22})N_{2t}+\alpha_2\varphi_{22}\lambda_2 t)\end{aligned} \tag{F.16}$$

$$\begin{aligned}\widehat{Z}_t &= \exp(-(\alpha_1\sigma_{11}+\alpha_2\sigma_{21})W_{1t}-(\sigma_{12}\alpha_1+\sigma_{22}\alpha_2)W_{2t}\\ &-\frac{1}{2}((\alpha_1\sigma_{11}+\alpha_2\sigma_{21})^2+(\sigma_{12}\alpha_1+\sigma_{22}\alpha_2)^2)t\\ &+\ln(1-\alpha_1\varphi_{11})N_{1t}+\alpha_1\varphi_{11}\lambda_1 t\\ &+\ln(1-\alpha_2\varphi_{21})N_{1t}+\alpha_2\varphi_{21}\lambda_1 t\\ &+\ln(1-\alpha_1\varphi_{12})N_{2t}+\alpha_1\varphi_{12}\lambda_2 t\\ &+\ln(1-\alpha_2\varphi_{22})N_{2t}+\alpha_2\varphi_{22}\lambda_2 t)\end{aligned} \tag{F.17}$$

$$\begin{aligned}\widehat{Z}_t &= \exp(-(\alpha_1\sigma_{11}+\alpha_2\sigma_{21})W_{1t}-\frac{1}{2}(\alpha_1\sigma_{11}+\alpha_2\sigma_{21})^2 t\\ &-(\sigma_{12}\alpha_1+\sigma_{22}\alpha_2)W_{2t}-\frac{1}{2}(\sigma_{12}\alpha_1+\sigma_{22}\alpha_2)^2 t\\ &+\ln(1-\alpha_1\varphi_{11})N_1(t)+\alpha_1\varphi_{11}\lambda_1 t\\ &+\ln(1-\alpha_2\varphi_{21})N_{1t}+\alpha_2\varphi_{21}\lambda_1 t\\ &+\ln(1-\alpha_1\varphi_{12})N_{2t}+\alpha_1\varphi_{12}\lambda_2 t\\ &+\ln(1-\alpha_2\varphi_{22})N_{2t}+\alpha_2\varphi_{22}\lambda_2 t).\end{aligned} \tag{F.18}$$

$\widehat{Z}$ is strictly positive if there exists a constant $c > 0$, such that

$$(\alpha'\Gamma)_i \leq 1 - c, \quad i = 1, 2. \tag{F.19}$$

$\widehat{K}$ is uniformly bounded, therefore $\widehat{Z}$ is a square-integrable local martingale under $\mathcal{P}$[2]. Define $W_{1t}^{\widehat{P}} = W_{1t} + (\alpha_1\sigma_{11} + \alpha_2\sigma_{21})t$ and $W_{1t}^{\widehat{P}} = W_{1t} + (\sigma_{12}\alpha_1 + \sigma_{22}\alpha_2)t$. Therefore, the dynamics of $F(t, T_i)$ under $\widehat{P}$ is:

$$\begin{aligned} \frac{dF(t,T_i)}{F(t^-,T_i)} = \quad & ((\mu_i - \sigma_{i1}(\alpha_1\sigma_{11} + \alpha_2\sigma_{21}) - \sigma_{i2}(\sigma_{12}\alpha_1 + \sigma_{22}\alpha_2))dt \\ & +\sigma_{i1}dW_{1t}^{\widehat{P}} + \sigma_{i2}dW_{2t}^{\widehat{P}} + \varphi_{i1}dN_{1t}^{\widehat{P}} + \varphi_{i2}dN_{2t}^{\widehat{P}}), \quad F(0,T_i) > 0 \end{aligned} \tag{F.20}$$

where $W_{1t}^{\widehat{P}}$ and $W_{2t}^{\widehat{P}}$ are $\widehat{P}$-standard Brownian motions, and $N_t^{\widehat{P}} = (N_{1t}^{\widehat{P}}, N_{2t}^{\widehat{P}})'$ is a bivariate Poisson process under $\widehat{P}$ with intensity

$$\lambda^{\widehat{P}} = (\lambda_1(1 - \alpha_1\varphi_{11})(1 - \alpha_2\varphi_{21}), \lambda_2(1 - \alpha_1\varphi_{12})(1 - \alpha_2\varphi_{22}))' \tag{F.21}$$

and the initial *fair* price is:

$$V_0 = E_{\widehat{P}}[H]. \tag{F.22}$$

[2]In our case, it is easy to show that $\widehat{Z}$ is a $\mathcal{P}$-martingale. To prove the martingale property for the Poisson component, you might use the following result $E[\exp(\ln(1-\beta)N_{it} + \beta\lambda_i t)] = 1, \forall 0 < \beta < 1$ and $i = 1, 2$.

Curriculum Vitae

Academic background

• PhD Candidate, Financial Mathematics, University of Western Ontario, 2009-2012.
• High Performance Computing Summer School, Sharcnet, June 2012.
• Mathematical Methods for Finance, Euro-Mediterranean Research Center for Mathematics and its Applications, May 2011.
• Thematic Program on Quantitative Finance: Foundations and Applications, Fields Institute-University of Toronto, January-June 2010.
• M.Sc. Financial Engineering, HEC Montréal, 2007-2008.
• B.Sc. Engineering in Economics & Scientific Management, Tunisia Polytechnic School, 2003-2006.
• Preparatory School for Engineering Studies of Sfax-Tunisia, Maths-Physics, 2001-2003.

Publication & Working papers

• "What Can We Learn from the EU ETS Experience? Recommendations for Effective Trading and Market Design ", with M. Davison (University of Western Ontario). *(Submitted)*
• "Carbon Emission Markets", with M. Davison (University of Western Ontario), Book Chapter in "*Quantitative Financial Risk Management*", D.D. Wu (ed.), Springer-Verlag Berlin Heidelberg, 2011.
• "EU ETS Futures Spread Analysis and Pricing Contingent Claims under Different Market Schemes", with M. Davison (University of Western Ontario). *(Submitted)*
• "Pricing Interest Rate Derivatives with Multilinear Interpolations and Transitions Densities", with H. Ben-Ameur (HEC Montréal) and L. Karoui (Goldman, Sachs & Co.). *(Submitted)*

Professional Experience

• Instructor, University of Western Ontario, 2012.
• Teaching Assistant, University of Western Ontario, 2009-2011.

Oral Presentations

• 12^{th} Symposium on Finance, Banking, and Insurance,"Emissions Markets: Recommendations for Effective Design to Facilitate Efficient Trading ", December-2011, Karlsruhe, Germany.
• QMF 2011 Conference,"Emissions Markets: Recommendations for Effective Design to Facilitate Efficient Trading ", December-2011, Sydney, Australia.
• 5^{th} Annual Risk Management Conference "Pricing Interest Rate Derivatives with Multilinear Interpolations and Transitions Densities", July-2011, National University of Singapore, Singapore.
• The International AMMCS Conference, "Introducing New Tradeable Instruments for Pricing and Hedging in Incomplete Emissions Markets", July-2011, Waterloo, Canada.
• ASMDA International Society, "Emissions Markets: Recommendations for Effective Design to Facilitate Efficient Trading ", June-2011, Rome, Italy.
• 21^{st} Annual Derivatives Securities and Risk Management Conference, "Pricing Interest Rate Derivatives with Multilinear Interpolations and Transitions Densities", March-2011, FDIC-Arlington, United States.
• Bachelier Finance Society, "Pricing and Hedging Strategies for Contingent Claims in an Incomplete Hybrid Emissions Market ", June-2010, Toronto, Canada.
• Industrial-Academic Forum on Commodities, Energy Markets, and Emissions Trading, "Pricing and Hedging Strategies for Contingent Claims in an Incomplete Hybrid Emissions Market", April-2010, Toronto, Canada.