11-28-2012 12:00 AM

On the Distribution of Quadratic Expressions in Various Types of Random Vectors

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A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

ON THE DISTRIBUTION OF QUADRATIC EXPRESSIONS IN VARIOUS TYPES OF RANDOM VECTORS

(Spine title: QUADRATIC EXPRESSIONS IN RANDOM VECTORS)
(Thesis format: Monograph)

by

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of the requirements for the degree of
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies
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London, Ontario, Canada

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Doctor of Philosophy

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Date

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Abstract

Several approximations to the distribution of indefinite quadratic expressions in possibly singular Gaussian random vectors and ratios thereof are obtained in this dissertation. It is established that such quadratic expressions can be represented in their most general form as the difference of two positive definite quadratic forms plus a linear combination of Gaussian random variables. New advances on the distribution of quadratic expressions in elliptically contoured vectors, which are expressed as scalar mixtures of Gaussian vectors, are proposed as well. Certain distributional aspects of Hermitian quadratic expressions in complex Gaussian vectors are also investigated. Additionally, approximations to the distributions of quadratic forms in uniform, beta, exponential and gamma random variables as well as order statistics thereof are determined from their exact moments, for which explicit representations are derived. Closed form representations of the approximations to the density functions of the various types of quadratic expressions being considered herein are obtained by adjusting the base density functions associated with the quadratic forms appearing in the decompositions of the expressions by means of polynomials whose coefficients are determined from the moments of the target distributions. Quadratic forms being ubiquitous in Statistics, the proposed distributional results should prove eminently useful.

Keywords: Real quadratic expressions, Hermitian quadratic forms, density approximation, cumulant generating function, moments, singular Gaussian vectors, order statistics, generalized gamma distribution, uniform random variables, beta random variables, exponential random variables, elliptically contoured random vectors.

Acknowledgements

First of all, I would like to express my sincere appreciation to my supervisor, Professor Serge B. Provost, for his valuable guidance and financial support. This thesis would not have been possible without his help and patience. I am very appreciative of his generosity with his time and for numerous stimulating discussions.

I would like to express my sincere thanks to the members of my supervisory committee and thesis examiners, Drs Sudhir Paul, Rićardas Zitikis, GuangYong Zou, David Stanford and Jiandong Ren. I am also grateful to Emeritus Professor A. M. Mathai for helpful suggestions.

I am indebted to the Department of Statistical and Actuarial Sciences and the Faculty of Graduate Studies for their financial support. I would also like to thank to Professor W. John Braun (Graduate Chair), Jennifer Dungavell (Administrative Officer) and Lisa Hunt (Academic Coordinator) for their assistance.

I wish to extend my deepest gratitude to my family: my parents, Morteza Mohsenipour and Tayebeh Majlesi, my wife, Maryam Majlesi, my children, Shahrokh and Hasti, my brothers, Dr. Ali Asghar Mohsenipour, Mohammad Ali Mohsenipour and Gholam Reza Mohsenipour, as well as my sisters, my uncle Mohammad Bagher Majlesi and his wife, Forough Tootonchi, for their unwavering love and encouragement. Thank you for your affection, support and patience during the past four years.

Special thanks also go to my friends Mir Hashem Moosavi and Taha Kowsari for their support and encouragement.

Finally, I wish to acknowledge the fond memory of Abbas Mohsenipour (1967-2007), my kind and unforgettable brother.

Dedicated To:

My Parents
My Wife
And My Children

With All My Love

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Chapter 1

Introduction

1.1 Introduction

Numerous distributional results are already available in connection with quadratic forms in normal random variables and ratios thereof. Various representations of the density and distribution functions of a quadratic form have been derived, and several procedures have been proposed for computing percentage points and preparing tables. Box (1954b) considered a linear combination of chi-square variables having even degrees of freedom. Gurland (1953), Pachares (1955), Ruben (1960, 1962), Shah and Khatri (1961), and Kotz *et al.* (1967a,b) among others, have obtained expressions involving MacLaurin series and the density function of chi-square variables. Gurland (1956) and Shah (1963) respectively considered central and noncentral indefinite quadratic forms, but as pointed by Shah (1963), the expansions obtained are not practical. Imhof (1961), Davis (1973) and Rice (1980) determined the exact density and distribution functions of indefinite quadratic forms in normal vectors. As pointed out in Mathai and Provost (1992), which contains a wealth of related results, a wide array of statistics can be expressed in terms of quadratic forms in normal random vectors.

An accessible approach is proposed in this thesis for approximating the density of positive definite and indefinite quadratic forms and expressions in normal random variables in terms of gamma, generalized gamma and Pearson-type densities. The case of quadratic forms and quadratic expressions in possibly singular normal vectors and their ratios had yet to be fully developed. So far, when dealing with quadratic forms in singular normal vectors, it has been implicitly assumed in the literature that the rank of the matrix associated with the quadratic form is greater than or equal to that of the covariance matrix of the singular normal vector. This is the case, for instance, within Representation 3.1a.5 in Mathai and Provost (1992) and Equation (1) in Tong *et al.* (2010), neither of which involves a linear term. Such a term is indeed present in the general representation given in Equation (2.4). It should also be noted that, as pointed out in Provost (1996), bilinear expressions can be expressed in terms of quadratic expressions. Thus, all the results presented in this thesis can also be utilized to approximate the distributions of bilinear forms and bilinear expressions in random vectors.

Chapter 2 provides a methodology that yields very accurate approximations to the density and distribution functions of *any* quadratic form or expression in singular normal vectors. Such quadratic forms are involved for instance in singular linear models as pointed out in Rao (1978), in least-squares estimators as discussed in Hsuan *et al.* (1985) and in genetic studies in connection with genome scans and the determination of haplotype frequencies, as explained in Tong *et al.* (2010). It should be noted that the computational routines that are currently available for determining the distribution of quadratic forms do not adequately address the singular case.

It is shown in Chapter 3 that the results derived in Chapter 2 can be utilized to determine the approximate distribution of certain ratios of quadratic forms. Such ratios arise for example in regression theory, linear models, analysis of variance and time series. For instance, the sample serial correlation coefficient as defined in Anderson (1990) and discussed in Provost and Rudiuk (1995), as well as the sample innovation cross-correlation function for an ARMA time series whose asymptotic distribution was derived by McLeod (1979), have such a structure. Koerts and Abrahamse (1969) investigated the distribution of ratios of quadratic forms in the context of the general linear model. Shenton and Johnson (1965) derived the first few terms of the series expansions of the first two moments of this sample circular serial correlation coefficient. Inder (1986) developed an approximation to the null distribution of the Durbin-Watson statistic to test for autoregressive disturbances in a linear regression model with a lagged dependent variable and obtained its critical values. This statistic can be expressed as a ratio of quadratic forms wherein the matrix of the quadratic form appearing in the denominator is idempotent. One may also consider the lagged regression residuals developed by De Gooijer and MacNeill (1999) and discussed in Provost *et al.* (2005), or certain change point test statistics obtained by MacNeill (1978). In fact, one of the first papers that extended the study of quadratic forms to the study of their ratios is due to Robbins and Pitman (1949). Other statistics that can be expressed as ratios of quadratic forms include the ratio of the mean square successive differences to the variance is studied in von Neumann *et al.* (1941); a statistic involved in a two-stage test is considered in Toyoda and Ohtani (1986); test statistics having this structure are derived in connection with a two-way analysis of variance for stationary periodic time series in Sutradhar and Bartlett (1989); certain ratios used in time series analysis were investigated in Geisser (1957) and Meng (2005); and test statistics related to some general linear models are considered in Koerts and Abrahamse (1969).

Ratios of quadratic forms that are connected to certain analysis of variance problems such as the determination of the effects of inequality of variance and of correlation between errors in the two-way classification, are considered in Box (1954b). Another example involves the sample circular serial correlation coefficient associated with a first order Gaussian auto-regressive process, X_t, which, in White (1957), was taken to be an estimator of the parameter ρ in the stochastic difference equation, $X_t = \rho X_{t-1} + U_t$, where the U_t's are independent standard normal variables. The first few terms in the series expansions of the first and second moments of this serial correlation coefficient are derived in Shenton and Johnson (1965). An approximation to the distribution of the ratio of two quadratic forms in connection with time series valued designs is discussed in Sutradhar and Bartlett (1989). A statistic whose structure is a ratio of two sums of gamma variables for the problem of testing the equality of two gamma populations with common shape parameter is derived in Shiue and Bain (1983).

The notion of mixture distributions was utilized to obtain convergent series expansions for the distribution of positive definite quadratic forms as well as that of certain ratios thereof; for instance, a mixture representation is utilized in Baldessari (1965) to derive the moments of the ratios. Inequalities applying to ratios of quadratic forms in independent normal random variables were obtained by Kadiyala (1968).

Ratios of independent quadratic forms involving chi-squares having even degrees of freedom are considered in Box (1954a). An inversion formula for the distribution of ratios of linear combinations of chi-square random variables is derived in Gurland (1948). An expressions for the moments of the ratios of certain quadratic forms as well as conditions for their existence is provided in Magnus (1990). Other results on the moments of ratios of quadratic forms may be found in Magnus (1986), Jones (1987), Smith (1989) and Roberts (1995).

The moments of the quantity Q_1/Q_2 with $Q_1 = \sum a_i X_i + \sum c_i Z_i$ and $Q_2 = \sum b_i Y_i + \sum d_i Z_i$ where X_i, Y_i, Z_i are independently distributed chi-square random variables, are derived in Chaubey and Nur Enayet Talukder; a representation of the moments about the origin of the ratio Q_1/Q_2 was obtained in closed form by Morin-Wahhab (1985). Representations of the distribution function of ratios of sums of gamma random variables were derived in Provost (1989a). Gurland (1948) derived an inversion formula for the distribution of ratios of the form $R = (c_1 Y_1 + \cdots + c_n Y_n)/(d_1 Y_1 + \cdots + d_n Y_n)$, where the Y_i's are independently distributed chi-square random variables. On expressing quadratic forms as sums of gamma random variables, a representation of the distribution function of ratios thereof was obtained by Provost (1989b).

The distribution of Hermitian quadratic forms and quadratic expressions in complex normal vectors is discussed in Chapter 4. Such quadratic forms and expressions frequently arise in binary hypothesis testing problems, especially in the performance analysis of systems whose inputs are affected by random noise such as radars, sonars, communications receivers and signal acquisition devices. This is explained, for instance, in Kac and Sieger (1947), Divsalar *et al.* (1990), and Kailath (1960). As pointed out by Biyari and Lindsey (1993), the decision variables in many systems can also be characterized by means of Hermitian quadratic forms in complex Gaussian vectors. Moreover, as explained in Provost and Rudiuk (1995), Section 2.16, several statistics used for testing hypotheses on the parameters of complex random vectors involve Hermitian quadratic forms. As well, Hermitian quadratic forms were employed as cost functions by Kwon *et al.* (1994) and as characteristic functions in correlated Rician fading environments by Annamalai *et al.* (2005).

Some distributional properties of Hermitian quadratic forms in complex Gaussian random vectors have been studied by Bello and Nelin (1962), Khatri (1970), Goodman (1963), Fang *et al.* (1990), Sultan (1999) and Mathai (1997), Provost and Cheong (2002), among others. Kac and Sieger (1947), Turin (1958, 1959), Kailath (1960), Bello and Nelin (1962), Simon and Divsalar (1988), Divsalar *et al.* (1990), Cavers and Ho (1992) and Biyari and Lindsey (1993) make use of such results in the computation of pairwise error probabilities of system output decision variables. Shah and Li (2005) pointed out an application involving bit error rate calculation in a certain wireless relay network. While considering a full-duplex decode-and-forward relay system in a Rician fading environment, Zhu *et al.* (2008) expressed the highest achievable information rate of the system as a Hermitian quadratic form.

As pointed out by Kay (1989) and Monzigo and Miller (1980), complex random vectors are utilized in many areas of signal processing such as spectral analysis and array processing. Picinbono (1996) provides an informative account of the uses of complex normal vectors and discusses related distributional results.

A general form of the moment generating function of a scalar random variable, which covers many cases including that of a Hermitian quadratic forms in complex normal variables, is presented in Sultan (1999). A representation of the characteristic function of Hermitian quadratic forms in complex normal variables was derived by Turin (1960). Shah and Li (2005) obtained an alternative representation of the moment generating function by contour integration. Soong (1984) provides the expected values of certain Hermitian quadratic forms in closed form. It should be pointed out that, up to now, no general representation of Hermitian quadratic forms in singular Gaussian vectors was available.

Chapter 5 addresses the case of quadratic expressions in elliptically contoured vectors. Several fields of applications involve elliptically contoured distributions, including for instance, anomalous change detection in hyperspectral imagery: Theiler *et al.* (2010); option pricing: Hamada and Valdez (2008); filtering and stochastic control: Chu (1973); random input signal: McGraw and Wagner (1968); financial analysis: Zellner (1976) and the references therein; the analysis of stock market data: Mandelbrot (1963) and Fama (1965); and Bayesian Kalman filtering: Girón and Rojano (1994). Additionally, studies on the robustness of statistical procedures when the probability model departs from the multivariate normal distribution to the broader class of elliptically contoured distributions were carried out by King (1980) and Osiewalski and Steel (1993). Several multivariate applications are also discussed in Devlin *et al.* (1976). Results related to regression analysis can be found for example in Fraser and Ng (1980). Heavy-tailed time series models were discussed in Resnick (1997). A new family of life distributions, that are generated from an elliptically contoured distribution, is discussed by Díaz-García and Leiva-Sánchez (2005). Recently Ipa *et al.* (2007) derived some results applicable to Bayesian inference for a general multivariate linear regression model with matrix variate elliptically distributed errors. In fact, the class of elliptically contoured distributions, which contains the multivariate normal distribution, enjoys several of its properties while allowing for more flexibility in modeling various random processes.

Quadratic forms in uniform and beta random variables are discussed in Chapter 6. As explained in Guttorp and Lockhart (1988), many tests of the hypothesis that a distribution is uniform over $(0,1)$ are based on statistics of the form, $T = M_{ij}(U_i - i/(n+i))(U_j - j/(n+1))$, where $U_1 < \cdots < U_n$ are order statistics from a uniform distribution over the interval $(0,1)$ and the matrix M is such that $n\,M_{ij}$ is a function of i/n and j/n. The Cramér-von Mises statistic, Watson's U2 statistic, Greenwood's statistic and Cressie's overlapping spacings statistics are all of this type. For instance, Greenwood's (1946) statistic is $\sum_{i=0}^{n}(U_{i+1} - U_i)^2$ where $U_0 = 0$ and $U_{n+1} = 1$. Cressie (1976, 1979) studied the overlapping m-spacings generalizations, $C_m = \sum_{i=0}^{n}(U_{m+1} - U_i)^2$ where $U_{n+1+k} = 1 + U_k$, and $C^*_m = \sum_{i=0}^{n+1-m}(U_{i+m} - U_i)^2$, whereas del Pino (1979) restricted the sum to a subset of i such that the m-spacings do not overlap. The large-sample distribution of such statistics has been studied by several authors. For approaches based on empirical processes and U-statistics, the reader is referred to Durbin (1973) and Gregory (1977), respectively. Hartley and Pfaffenberger (1972) pointed out the connection to some goodness-of-fit criteria, determined the exact small-sample distribution in a certain instance and showed that the family of criteria presents certain asymptotic optimal power properties.

Chapter 6 also provides distributional results on quadratic forms in exponential and gamma random variables. Let $Y_1 < \cdots < Y_n$ be order statistics from an exponential distribution with mean θ; several tests of fit with respect to the exponential distribution are based on certain quadratic forms in the Y_i's divided by an estimate of the scaling parameter. Hartley and Pfaffenberger (1972), Lockhart (1985) and McLaren and Lockhart (1987) considered tests based on correlations involving the Y_i's. Some distributional limit theorems such as those that are discussed in del Barrio *et al.* (2005) in connection with a certain empirical quantile process, involve quadratic forms in exponential random variables. Moreover, Donald and Paarsch (2002) described three test statistics that can be expressed as quadratic forms in exponential random variables.

This thesis provides functional representations of the approximate densities associated with quadratic forms and expressions in various types of random vectors. The distributional results are often compared with simulated distributions when the exact densities are not tractable. The Monte Carlo and analytical approaches have their own merits and shortcomings. Monte Carlo simulations where artificial data are generated, wherefrom sampling distributions and moments are estimated, can be implemented with relative ease on an extensive range of models and error probability distributions. There are, however, some limitations on the range of applicability of this approach as the results may be subject to sampling variations and simulation inadequacies, and may depend on the assumed parameter values. Recent efforts to cope with these issues are reported for example in Hendry (1979), Hendry and Harrison (1974), Hendry and Mizon (1980) and Dempster *et al.* (1977). The analytical approach, on the other hand, derives results that hold over the whole parameter space but may find limitations in terms of simplifications on the model, which have to be imposed to make the problem tractable. When exact theoretical results can be obtained, the resulting expressions can then be fairly complicated.

The thesis is organized as follows. The distribution of quadratic forms and quadratic expressions in nonsingular and singular Gaussian vectors is discussed in Chapter 2. Distributional results on moment generating functions, moments, cumulant generating functions and cumulants are also provided in this chapter. Approximations based on a Pearson-type density function or generalized gamma-type densities, which are polynomially adjusted for increased precision, are also proposed. Ratios of quadratic forms and quadratic expressions are investigated in Chapter 3. More specifically, ratios whose distribution can be determined from that of the difference of positive definite quadratic forms and ratios involving idempotent or positive definite matrices in their denominators are being considered. It is shown in Chapter 4 that Hermitian quadratic forms or quadratic expressions in singular Gaussian vectors can be expressed in terms of real positive definite quadratic forms and an independently distributed normal random variable; representations of their moment generating functions and cumulants—wherefrom the moments can be determined—are also provided. A methodology for approximating the distribution of Hermitian quadratic forms and quadratic expressions is also introduced. Chapter 5 includes distributional results in connection with quadratic expressions in elliptically contoured random vectors: A decomposition of quadratic expressions in elliptically contoured vectors is derived and the distribution of such quadratic expressions is obtained by expressing the elliptically contoured vectors as scale mixtures of Gaussian vectors. Representations of the moments of quadratic forms in uniform and gamma random variables are derived in Chapter 6. Closed form expressions are also obtained for the moments of quadratic forms in *order statistics* from uniform and exponential populations. Quadratic forms in beta and gamma random variables are considered as well. Some concluding remarks and suggestions for future work are included in the last chapter.

Each chapter is meant to be essentially self-contained. As a result, certain preliminary results, definitions and derivations will appear more than once in this dissertation.

Chapter 2

The Distribution of Real Quadratic Expressions in Normal Vectors

2.1 Introduction

Some basic results related to the decomposition of matrices and the definiteness of the associated quadratic forms are presented in Section 2.2. Several distributional results on quadratic forms in nonsingular normal vectors are included in Section 2.3. This section contains a definition of quadratic forms in random variables and a representation of nonsingular normal vectors in terms of standard normal vectors. Indefinite quadratic expressions in nonsingular normal vectors are discussed in Section 2.4 which also includes results on their moments, cumulants, moment generating functions and cumulant generating functions. Representations of singular quadratic forms and quadratic expressions are respectively given in Sections 2.5 and 2.6. Approximate distributions based on Pearson's density and generalized gamma-type densities, as well as their polynomially adjusted counterparts, are proposed in Section 2.7. This section also includes a closed form representation of the exact density of a quadratic form whose associated matrix has eigenvalues occurring in pairs, as well as closed form density functions for the general case. In addition, a step-by-step algorithm for implementing the proposed density approximation methodology is provided and several numerical examples are presented for various cases. The last section is specifically devoted to the evaluation of approximate distributions for quadratic expressions in singular normal vectors.

2.2 Preliminary Results

Several relevant concepts and definitions as well as some preliminary results are included in this section.

Definition 2.2.1. *Characteristic roots and vectors* If A is an $n \times n$ matrix, then a non-null vector $\mathbf{x}$ in $\Re^n$ is called a *characteristic vector* or *eigenvector* of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$, i.e.,

$$A\mathbf{x} = \lambda\mathbf{x} \quad \text{or} \quad (A - \lambda I)\mathbf{x} = \mathbf{0} \tag{2.1}$$

for some scalar λ. A necessary and sufficient condition for the existence of a non-null vector $\mathbf{x}$ satisfying this equation is that λ be a root of the determinantal equation

$$|A - \lambda I| = 0 \ . \tag{2.2}$$

This equation is called the characteristic equation of A. As a polynomial in λ, the right-hand side possesses n roots, distinct or not, which are called the *characteristic roots* or *eigenvalues* of A and denoted $ch(A)$.

Theorem 2.2.1. If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then the following identities hold:

(i) $tr(A^k) = \sum_{i=1}^{n} \lambda_i^k \ , \ k = 1, 2, \ldots$

(ii) $|A| = \prod_{i=1}^{n} \lambda_i$

(iii) $|I_n \pm A| = \prod_{i=1}^{n} (1 \pm \lambda_i)$.

Theorem 2.2.2. *Spectral decomposition theorem* Let A be a real $n \times n$ symmetric matrix. Then there exists an orthogonal matrix $P = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ such that $P'AP$ is a diagonal matrix whose diagonal elements $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the characteristic roots of A, that is,

$$P'AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} \equiv L,$$

with $\mathbf{p}_i = \boldsymbol{\nu}_i / (\boldsymbol{\nu}_i' \boldsymbol{\nu}_i)^{\frac{1}{2}}$, $\boldsymbol{\nu}_i$ being a characteristic vector corresponding to $\lambda_i, i = 1, \ldots, n$. It follows that $A = PLP'$ or, equivalently, that

$$A = \sum_{i=1}^{n} \lambda_i \mathbf{p}_i \mathbf{p}_i' \ .$$

Theorem 2.2.3. Let A be a real $n \times n$ symmetric matrix. Then the characteristic roots of A are all real.

Theorem 2.2.4. Let A be a real $n \times n$ symmetric matrix. If the rank of A, $\rho(A) = r < n$, then zero will be a characteristic root of multiplicity $(n - r)$.

Theorem 2.2.5. If A is an idempotent matrix, then its characteristic roots are either zero or one. If all are unities then $A = I_n$.

Definition 2.2.2. The moment-generating function of an n-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is

$$M_{\mathbf{X}}(\mathbf{t}) = E(e^{\mathbf{t}'\mathbf{X}}), \qquad \mathbf{t} \in \Re^n \tag{2.3}$$

whenever this expectation exists. $M_{\mathbf{X}}(\mathbf{0})$ always exists and is equal to 1.

A key problem with moment-generating functions is that moments and the moment-generating function may not exist, as the integrals need not converge absolutely. By contrast, the characteristic function always exists (because it is the integral of a bounded function on a space of finite measure), and thus may be used instead.

Definition 2.2.3. The characteristic function of an n-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is

$$\varphi_{\mathbf{X}}(\mathbf{t}) = E(e^{i\,\mathbf{t}'\mathbf{X}}), \qquad \mathbf{t} \in \Re^n\,. \tag{2.4}$$

Definition 2.2.4. *Quadratic form* Let $\mathbf{X} = (X_1, \ldots, X_n)'$ denote a random vector with mean $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$ and covariance matrix Σ. The quadratic form in the random variables $X_1, \ldots, X_n$ with associated $n \times n$ symmetric matrix $A = (a_{ij})$ is defined as

$$Q(\mathbf{X}) = Q(X_1, \ldots, X_n) = \mathbf{X}'A\mathbf{X} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}X_iX_j.$$

We note that if A is not symmetric, it suffices to replace this matrix by $(A + A')/2$ in any quadratic form where A' denotes the transpose of A. Accordingly, it will be assumed without any loss of generality that the matrices of the quadratic forms being considered are symmetric. (Vectors are denoted by bold letters in this thesis.)

Definition 2.2.5. A *central quadratic form* is a quadratic form in random variables whose means are all equal to zero (that is, in central random variables); otherwise the quadratic form is said to be noncentral. Thus, when $E(\mathbf{X}) \equiv \boldsymbol{\mu}$ is a null vector, $\mathbf{X}'A\mathbf{X}$ is a central quadratic form in $\mathbf{X}$; when $\boldsymbol{\mu}$ is a non-null vector, $\mathbf{X}'A\mathbf{X}$ is said to be a *noncentral quadratic* form.

Definition 2.2.6. *Positive definite quadratic form* A real quadratic form $\boldsymbol{X}'A\boldsymbol{X}$ is said to be positive definite if $\boldsymbol{X}'A\boldsymbol{X} > 0$ for all $\boldsymbol{X} \neq \boldsymbol{0}$. A matrix A is said to be positive definite, denoted by $A > 0$, if the quadratic form $\boldsymbol{X}'A\boldsymbol{X}$ is positive definite. A symmetric matrix A is said to be negative definite if $-A$ is positive definite.

Theorem 2.2.6. Let A be a symmetric $n \times n$ positive definite matrix; then

(i) A is nonsingular

(ii) the eigenvalues of A are all positive

(iii) A can be written as $R'R$ where R is nonsingular; the converse also holds true

(iv) if B is a $p \times n$ matrix of rank p where $p \leq n$, BAB' will also be positive definite

(v) A^{-1} is also positive definite

(vi) there exists a symmetric positive definite matrix denoted, $A^{\frac{1}{2}}$, called the *symmetric square root* of A, which is such that

$$A = A^{\frac{1}{2}} A^{\frac{1}{2}}$$

with

$$A^{\frac{1}{2}} = P' L^{\frac{1}{2}} P$$

where P and L are as defined in Theorem 2.2.2, $L^{\frac{1}{2}}$ being equal to $\mathcal{D}iag(\lambda_1^{\frac{1}{2}}, \ldots, \lambda_n^{\frac{1}{2}})$.

Definition 2.2.7. *Positive semidefinite quadratic form* A real quadratic form $\boldsymbol{X}'A\boldsymbol{X}$ and its matrix A are said to be positive semidefinite if $\boldsymbol{X}'A\boldsymbol{X} \geq 0$ for all $\boldsymbol{X}$ and we shall use the notation $A \geq 0$. The term nonnegative definite is used to indicate that the quadratic form is either positive definite or positive semidefinite. In that case, all the eigenvalues are nonnegative.

Definition 2.2.8. *Negative semidefinite quadratic form* A quadratic form and its matrix A are said to be negative semidefinite if $-A$ is positive semidefinite.

Theorem 2.2.7. Let A be a symmetric $n \times n$ positive semidefinite matrix, then

(i) its eigenvalues are nonnegative and so is its trace

(ii) if A has rank ρ, it can be written as $S'S$ where S is a square matrix of rank ρ. The converse also holds true

(iii) $B'AB \geq 0$ for any $n \times m$ matrix B

(iv) there exists a symmetric positive semidefinite matrix denoted by $A^{\frac{1}{2}}$ and called the symmetric square root of A such that

$$A = A^{\frac{1}{2}} A^{\frac{1}{2}}$$

(v) if $\rho(A) = r \leq n$, exactly r eigenvalues of A will be positive while the remaining $(n - r)$ eigenvalues of A will be equal to zero.

Definition 2.2.9. *Indefinite quadratic form* A quadratic form and its matrix are said to be indefinite if they do not belong to any of the categories, positive definite, positive semidefinite, negative definite or negative semidefinite. An indefinite matrix has both positive and negative eigenvalues.

Theorem 2.2.8. *Cholesky's Decomposition* Let A be a symmetric $n \times n$ positive definite matrix, then A has a unique factorization of the form $A = TT'$ where T is a lower triangular matrix whose diagonal elements are all positive. One can then write $\boldsymbol{X}'A\boldsymbol{X}$ as $(T\boldsymbol{X})'(T\boldsymbol{X})$.

The elements of the matrix T can easily be found by multiplying out TT' and equating the resulting expressions to the elements of A. Other methods such as Doolittle's method and Crout's method for factoring matrices into a product of triangular matrices are discussed for instance in Burden and Faires (1988).

2.3 Quadratic Forms in Nonsingular Normal Vectors

Let $\mathbf{X}$ be a $p \times 1$ normal random vector with mean $\boldsymbol{\mu}$ and positive definite covariance matrix Σ, that is, $E(\mathbf{X}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\mathbf{X}) = E[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))'] = \Sigma > 0$. Then, letting $\mathbf{Y} = \Sigma^{-\frac{1}{2}}\mathbf{X}$, one has

$$E(\mathbf{Y}) = \Sigma^{-\frac{1}{2}}\, \boldsymbol{\mu},\ \mathrm{Cov}(\mathbf{Y}) = \Sigma^{-\frac{1}{2}}\, \mathrm{Cov}(\mathbf{X})\Sigma^{-\frac{1}{2}} = \Sigma^{-\frac{1}{2}}\Sigma\Sigma^{-\frac{1}{2}} = I \text{ and } \mathbf{Y} \sim \mathcal{N}_p(\Sigma^{-\frac{1}{2}}\, \boldsymbol{\mu}, I)\ .$$

Thus, letting $\mathbf{Z} = \Sigma^{-\frac{1}{2}}(\mathbf{X} - \boldsymbol{\mu})$,

$$\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I_p)\,,$$

and one can express the quadratic form $Q(\mathbf{X})$ as follows:

$$Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X} = \mathbf{Y}'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}\mathbf{Y} = (\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})\ .$$

Note that one can use any decomposition of the form $\Sigma = BB'$ where B is $p\times p$ and $|B| \neq 0$ instead of the symmetric square root $\Sigma^{\frac{1}{2}}$. Then, the standardizing transformation will be of the form $\mathbf{Z} = B^{-1}(\mathbf{X}-\boldsymbol{\mu})$. For notational convenience, we shall use the symmetric square root $\Sigma^{\frac{1}{2}}$ throughout this thesis.

Let P be a $p\times p$ orthogonal matrix which diagonalizes $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$. That is,

$$P'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}P = \mathcal{D}iag(\lambda_1,\ldots,\lambda_p), \qquad P'P = PP' = I,$$

where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ or equivalently those of ΣA. Note that all orthogonal matrices are assumed to be orthonormal in this thesis. Letting $\mathbf{U} = P'\mathbf{Z}$, one has that

$$\mathbf{Z} = P\mathbf{U} \ \text{ where } \ \mathbf{U} \sim \mathcal{N}_p(\mathbf{0}, I_p)\ .$$

Then,

$$\begin{aligned} Q(\mathbf{X}) &= (\mathbf{Z}+\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}(\mathbf{Z}+\Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) \\ &= (\mathbf{U}+\mathbf{b})'P'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}P(\mathbf{U}+\mathbf{b}) \\ &= (\mathbf{U}+\mathbf{b})'\mathcal{D}iag(\lambda_1,\ldots,\lambda_p)(\mathbf{U}+\mathbf{b}), \end{aligned} \tag{2.5}$$

where $\mathbf{U}' = (U_1,\ldots,U_p)$, $\mathbf{U}\sim\mathcal{N}_p(\mathbf{0}, I)$ and $\mathbf{b}' = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})' = (b_1,\ldots,b_p)$. Accordingly, one has

Representation 2.3.1. Let $\mathbf{X}\sim\mathcal{N}_p(\boldsymbol{\mu},\Sigma)$, $\Sigma > 0$ and $A = A'$. Then

$$\begin{aligned} Q(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} = \sum_{j=1}^{p}\lambda_j(U_j+b_j)^2 \\ &= \sum_{j=1}^{p}\lambda_j U_j^2, \quad \text{whenever } \boldsymbol{\mu} = \mathbf{0} \end{aligned} \tag{2.6}$$

where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$, the U_i's are independently distributed standard normal variables, $(b_1,\ldots,b_p)\equiv\mathbf{b}' = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'$, P being an orthogonal matrix such that $P'\Sigma^{1/2}A\Sigma^{1/2}P = \mathcal{D}iag(\lambda_1,\ldots,\lambda_p)$. Thus, $Q(\mathbf{X})$ is distributed as a linear combination of independent noncentral (central) chi-square variables when $\boldsymbol{\mu}\neq\mathbf{0}$ ($\boldsymbol{\mu}=\mathbf{0}$).

2.4 Indefinite Quadratic Expressions: The Nonsingular Case

A decomposition of noncentral indefinite quadratic expressions in nonsingular normal vectors is given in terms of the difference of two positive definite quadratic forms whose moments are determined from a certain recursive relationship involving their cumulants. An integral representation of the density function of an indefinite quadratic form is also provided.

We first show that an indefinite quadratic expression in a nonsingular normal random vector can be expressed in terms of standard normal variables. Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$, that is, $\mathbf{X}$ is distributed as a p-variate normal random vector with mean $\boldsymbol{\mu}$ and positive definite covariance matrix Σ. On letting $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$, where I is a $p \times p$ identity matrix, one has $\mathbf{X} = \Sigma^{\frac{1}{2}}\mathbf{Z}+\boldsymbol{\mu}$ where $\Sigma^{\frac{1}{2}}$ denotes the symmetric square root of Σ. Then, in light of the spectral decomposition theorem, the quadratic expression $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ where A is a $p \times p$ real symmetric matrix, $\mathbf{a}$ is a p-dimensional constant vector and d is a scalar constant can be expressed as

$$\begin{aligned} Q^*(\mathbf{X}) &= (\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) + \mathbf{a}'\Sigma^{\frac{1}{2}}(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) + d \\ &= (\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'PP'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}PP'(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) \\ &\quad +\mathbf{a}'\Sigma^{\frac{1}{2}}PP'(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) + d \end{aligned} \tag{2.7}$$

where P is an orthogonal matrix that diagonalizes $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$, that is, $P'\Sigma^{\frac{1}{2}}A\ \Sigma^{\frac{1}{2}}P = \mathcal{D}iag(\lambda_1, \ldots, \lambda_p)$, $\lambda_1, \ldots, \lambda_p$ being the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ in *decreasing* order with $\lambda_1, \ldots, \lambda_r$ positive, $\lambda_{r+1} = \cdots = \lambda_{r+\theta} = 0$ and $\lambda_{r+1+\theta}, \ldots, \lambda_p$ negative. Let $\mathbf{v}_i$ denote the *normalized* eigenvector of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ *corresponding* to λ_i, $i = 1, \ldots, p$, (such that $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ and $\mathbf{v}_i{}'\mathbf{v}_i = 1$) and $P = (\mathbf{v}_1, \ldots, \mathbf{v}_p)$. Letting $\mathbf{U} = P'\mathbf{Z}$ where $\mathbf{U} = (U_1, \ldots, U_p)' \sim \mathcal{N}_p(\mathbf{0}, I)$, $\mathbf{b} = P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu}$ with $\mathbf{b} = (b_1, \ldots, b_p)'$, $\mathbf{g}' = (g_1, \ldots, g_p) = \mathbf{a}'\Sigma^{\frac{1}{2}}P$ and $c = \mathbf{b}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p)\mathbf{b} + \mathbf{g}'\mathbf{b} + d$, one has

$$\begin{aligned} Q^*(\mathbf{X}) &= (\mathbf{U} + \mathbf{b})'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p)(\mathbf{U} + \mathbf{b}) + \mathbf{a}'\Sigma^{\frac{1}{2}}P(\mathbf{U} + \mathbf{b}) + d \\ &= \mathbf{U}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p)\mathbf{U} + (2\mathbf{b}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p) + \mathbf{g}')\mathbf{U} + c \\ &= \sum_{j=1}^{p} \lambda_j U_j^2 + \sum_{j=1}^{p} k_j U_j + c \\ &= \sum_{j=1}^{r} \lambda_j U_j^2 + \sum_{j=1}^{r} k_j U_j - \sum_{j=r+\theta+1}^{p} |\lambda_j| U_j^2 + \sum_{j=r+\theta+1}^{p} k_j U_j \\ &\qquad + \sum_{j=r+1}^{r+\theta} k_j U_j + c \\ &= \sum_{j=1}^{r} \lambda_j \Big(U_j + \frac{k_j}{2\lambda_j}\Big)^2 - \sum_{j=r+\theta+1}^{p} |\lambda_j| \Big(U_j + \frac{k_j}{2\lambda_j}\Big)^2 \end{aligned}$$

$$
\begin{aligned}
& + \sum_{j=r+1}^{r+\theta} k_j U_j + \Big(c - \sum_{j=1}^{r} \frac{k_j^2}{4\lambda_j} - \sum_{j=r+\theta+1}^{p} \frac{k_j^2}{4\lambda_j}\Big) \\
\equiv\ & Q_1(\mathbf{V}^+) - Q_2(\mathbf{V}^-) + \sum_{j=r+1}^{r+\theta} k_j U_j + \kappa \\
\equiv\ & Q_1(\mathbf{V}^+) - Q_2(\mathbf{V}^-) + T, \qquad (2.8)
\end{aligned}
$$

where $\mathbf{k}' = (k_1, \ldots, k_p) = 2\mathbf{b}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_p) + \mathbf{g}'$, $\kappa = \Big(c - \sum_{j=1}^{r} k_j^2/(4\lambda_j) - \sum_{j=r+\theta+1}^{p} k_j^2/(4\lambda_j)\Big)$, $T = (\sum_{j=r+1}^{r+\theta} g_j U_j + \kappa) \sim \mathcal{N}(\kappa\,, \sum_{j=r+1}^{r+\theta} g_j^2)$, $Q_1(\mathbf{V}^+)$ and $Q_2(\mathbf{V}^-)$ are positive definite quadratic forms with $\mathbf{V}^+ = (U_1 + k_1/(2\lambda_1),\ \ldots, U_r + k_r/(2\lambda_r))' \sim \mathcal{N}_r(\mathbf{m}_1, I)$, $\mathbf{V}^- = (U_{r+\theta+1} + k_{r+\theta+1}/(2\lambda_{r+\theta+1}), \ldots, U_p + k_p/(2\lambda_p))' \sim \mathcal{N}_{p-r-\theta}(\mathbf{m}_2, I)$, where $\mathbf{m}_1 = (k_1/(2\lambda_1), \ldots, k_r/(2\lambda_r))'$ and $\mathbf{m}_2 = (k_{r+\theta+1}/(2\lambda_{r+\theta+1}), \ldots, k_p/(2\lambda_p))'$, θ being number of zero eigenvalues of $A\,\Sigma$. It should be emphasized that the three terms in Representation (2.8) are independently distributed, which facilitates the determination of the distribution of $Q^*(\mathbf{X})$.

In particular, when $\mathbf{a} = \mathbf{0}$ and $d = 0$, one has

$$
\begin{aligned}
Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X} &= \sum_{j=1}^{p} \lambda_j (U_j + b_j)^2 \\
&= \sum_{j=1}^{r} \lambda_j (U_j + b_j)^2 - \sum_{j=r+\theta+1}^{p} |\lambda_j| (U_j + b_j)^2 \\
&\equiv Q_1(\mathbf{Y}^+) - Q_2(\mathbf{Y}^-)\,, \qquad (2.9)
\end{aligned}
$$

where $\mathbf{Y}^+ = (U_1 + b_1, \ldots, U_r + b_r)' \sim \mathcal{N}_r(\mathbf{m}_1, I)$, $\mathbf{Y}^- = (U_{r+\theta+1} + b_{r+\theta+1}, \ldots, U_p + b_p)' \sim \mathcal{N}_{p-r-\theta}(\mathbf{m}_2, I)$ with $\mathbf{m}_1 = (b_1, \ldots, b_r)'$, $\mathbf{m}_2 = (b_{r+\theta+1}, \ldots, b_p)'$ and $\mathbf{b} = (b_1, \ldots, b_p)' = P'\Sigma^{-1/2}\boldsymbol{\mu}$. Thus, a noncentral indefinite quadratic expression, $Q^*(\mathbf{X})$, can be expressed as a difference of independently distributed linear combinations of independent noncentral chi-square random variables having one degree of freedom each plus linear combination of normal random variables, or equivalently, as the difference of two positive definite quadratic forms plus linear combination of normal random variables. It is seen from (2.7) that, in the nonsingular case, a noncentral indefinite *quadratic form* can be represented as the difference of two positive definite quadratic forms. It should be noted that the chi-square random variables are central whenever $\boldsymbol{\mu} = \mathbf{0}$. When the matrix A is positive semidefinite, so is the quadratic form $Q(\mathbf{X})$, and then, $Q(\mathbf{X}) \sim Q_1(\mathbf{Y}^+)$, as defined in Equation (2.9).

The cumulants and moments of quadratic forms and quadratic expressions, which are useful for determining the parameters of the distributions involved in the density approximations, are discussed in the next section.

2.4.1 Moments and Cumulants of Quadratic Expressions

Representations of the moment generating functions and the moments of quadratic expressions in nonsingular normal vectors are included in this section. As shown in Mathai and Provost (1992), if A be a real symmetric $p \times p$ matrix, $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$, $\mathbf{a}'$ be a p dimensional constant vector and d be a scalar constant, then the moment generating function of $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ is

$$\begin{aligned} M_{Q^*}(t) &= |I - 2t\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}|^{-\frac{1}{2}} \exp\{t(d + \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu}) \\ &\quad +(t^2/2)(\Sigma^{\frac{1}{2}}\mathbf{a} + 2\Sigma^{\frac{1}{2}}A\boldsymbol{\mu})'(I - 2t\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}})^{-1}(\Sigma^{\frac{1}{2}}\mathbf{a} + 2\Sigma^{\frac{1}{2}}A\boldsymbol{\mu})\} \end{aligned} \tag{2.10}$$

and that of $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ is

$$M_Q(t) = |I - 2tA\Sigma|^{-\frac{1}{2}} \exp\Big\{-\frac{1}{2}\boldsymbol{\mu}'[I - (I - 2tA\Sigma)^{-1}]\Sigma^{-1}\boldsymbol{\mu}\Big\}. \tag{2.11}$$

In terms of the eigenvalues of $A\Sigma$, the moment generating functions of $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ and $Q(\mathbf{X})^* = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ can respectively be expressed as

$$\begin{aligned} M_Q(t) &= \exp\Big\{-\frac{1}{2}\sum_{j=1}^{p} b_j^2\Big\} \exp\Big\{\frac{1}{2}\sum_{j=1}^{p} b_j^2(1 - 2t\lambda_j)^{-1}\Big\} \prod_{j=1}^{p}(1 - 2t\lambda_j)^{-\frac{1}{2}} \\ &= \exp\Big\{t\sum_{j=1}^{p} b_j^2\lambda_j(1 - 2t\lambda_j)^{-1}\Big\} \prod_{j=1}^{p}(1 - 2t\lambda_j)^{-\frac{1}{2}}\,, \qquad \text{for } \ \boldsymbol{\mu} \neq \mathbf{0} \\ &= \prod_{j=1}^{p}(1 - 2t\lambda_j)^{-\frac{1}{2}} \quad \text{for } \ \boldsymbol{\mu} = \mathbf{0} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} M_{Q^*}(t) &= \exp\big\{t(d + \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu}) \\ &\quad + \frac{t^2}{2}\sum_{j=1}^{p} b_j^{*2}(1 - 2t\lambda_j)^{-1}\big\} \prod_{j=1}^{p}(1 - 2t\lambda_j)^{-\frac{1}{2}}\,, \end{aligned} \tag{2.13}$$

where $P'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}P = \mathcal{D}iag(\lambda_1, \ldots, \lambda_p)$, $PP' = P'P = I$, $P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu} = \mathbf{b} = (b_1, \ldots, b_p)'$, $P'(\Sigma^{\frac{1}{2}}\mathbf{a} + 2\Sigma^{\frac{1}{2}}A\boldsymbol{\mu}) = \mathbf{b}^* = (b_1^*, \ldots, b_p^*)'$.

We now provide explicit expressions for the cumulants of a quadratic expression and discuss some special cases of interest.

Definition 2.4.1. Let $M(t)$ be the moment generating function of a random variable X and let $M(t_1,\ldots,t_k)$ denote the joint moment generating function of k random variables $X_1,\ldots,X_k$. Then the logarithms $\ln M(t)$ and $\ln M(t_1,\ldots,t_k)$ are defined as the *cumulant generating function* of X and the *joint cumulant generating function of* $X_1,\ldots,X_k$, respectively.

Definition 2.4.2. If $\ln M(t)$ of Definition 2.3.1 admits a power series expansion, then the coefficient of $t^s/s!$ in the power series of $\ln M(t)$ is defined as the s^{th} cumulant of X, which is denoted by $k(s)$. That is,

$$\ln M(t) = \sum_{s=1}^{\infty} k(s)\frac{t^s}{s!}\ .$$

If $\ln M(t)$ is differentiable, then

$$k(s) = \frac{d^s}{dt^s}[\ln M(t)]|_{t=0}\ .$$

The s^{th} cumulant, $k(s)$, of $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ is specified in the following result.

Result 2.4.1. Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$, $A = A'$, $\mathbf{a}'$ be a p dimensional constant vector, d be a scaler constant, $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ and $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$; then s^{th} cumulants of $Q^*(\mathbf{X})$ and $Q(\mathbf{X})$ are, respectively,

$$\begin{aligned} k^*(s) &= 2^{s-1}s!\Big\{\frac{\text{tr}(A\Sigma)^s}{s} + \frac{1}{4}\mathbf{a}'(\Sigma A)^{s-2}\Sigma\mathbf{a} + \boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu} \\ &\quad +\mathbf{a}'(\Sigma A)^{s-1}A\boldsymbol{\mu}\Big\}, \qquad s \geq 2 \\ &= \text{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d, \qquad s = 1\,; \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} k(s) &= 2^{s-1}s!\Big\{\frac{\text{tr}(A\Sigma)^s}{s} + \boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu}\Big\}, \quad s \geq 2 \\ &= \text{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu}\,, \qquad s = 1. \end{aligned} \tag{2.15}$$

For any random variable Y, $k(1) = E(Y)$ and $k(2) = \text{ Var}(Y)$. We observe that for the quadratic form, $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$, one has

$$\begin{aligned} k(s) &= 2^{s-1}s!\Big(\mathrm{tr}(A\,\Sigma)^s/s \;+\; \boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu}\Big) \\ &= 2^{s-1}s!\sum_{j=1}^{p}\lambda_j^s(b_j^2+1/s) \\ &= 2^{s-1}(s-1)!\sum_{j=1}^{p}\lambda_j^s(sb_j^2+1) \\ &= 2^{s-1}(s-1)!\;\theta_s \end{aligned} \tag{2.16}$$

where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$, $\mathbf{b}' = (b_1,\ldots,b_p) = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'$, $\mathrm{tr}(\cdot)$ denotes the trace of $(\cdot)$ and $\theta_s = \sum_{j=1}^{p}\lambda_j^s(sb_j^2+1)$, $s = 1,2,\ldots$. Note that $\mathrm{tr}(A\Sigma)^s = \sum_{j=1}^{p}\lambda_j^s$.

As explained in Smith (1995), the moments of a random variable can be obtained from its cumulants by means of the recursive relationship that is specified by Equation (2.17). Accordingly, the h^{th} moment of $Q^*(\mathbf{X})$ is given by

$$\mu_h^* \;=\; \sum_{i=0}^{h-1}\frac{(h-1)!}{(h-1-i)!\,i!}\,k^*(h-i)\,\mu_i^*\,, \tag{2.17}$$

where $k^*(s)$ is as given in Equation (2.14) and μ_h^* denotes the h^{th} moment about the origin.

2.5 Quadratic Forms in Singular Normal Vectors

Singular covariance matrices occur in many contexts. For example, consider a standard linear regression model $\mathbf{y} = X\boldsymbol{\beta}+\boldsymbol{\varepsilon}$ where $\mathbf{y}\in\mathbb{R}^n$, X is a non stochastic $n\times k$ matrix of full column rank and $\boldsymbol{\varepsilon}\sim\mathcal{N}_n(\mathbf{0},\sigma^2 I_n)$, I_n denoting identity matrix order n. The distribution of the residuals, $\boldsymbol{e} = \mathbf{y} - X\hat{\boldsymbol{\beta}} = (I_n - X(X'X)^{-1}X')\mathbf{y}$, where $\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\mathbf{y}$, is

$$\boldsymbol{e}\sim\mathcal{N}_n\Big(\mathbf{0}\,,\,\sigma^2(I_n - X(X'X)^{-1}X')\Big)$$

where the covariance matrix, $\sigma^2(I_n - X(X'X)^{-1}X')$, is of rank $n-k$.

Another example of application of singular covariance matrices pertains to economic data, which may be subject to constraints such as the requirement for a company's profits equal its turnover expenses. If, for example, the data vector $\mathbf{X} = (X_1,\ldots,X_k)'$ must satisfy the restriction $X_1+\cdots+X_{k-1} = X_k$, then Σ, the covariance matrix of X, will be singular.

When $\Sigma_{p\times p}$ is a singular matrix of rank $r < p$, we make use of the spectral decomposition theorem to express Σ as UWU' where W is a diagonal matrix whose first r diagonal elements are positive, the remaining diagonal elements being equal to zero. Next, we let $B^*_{p\times p} = UW^{1/2}$ and remove the $p-r$ last columns of B^*, which are null vectors, to obtain the matrix $B_{p\times r}$. Then, it can be verified that $\Sigma = BB'$.

Let X be a $p\times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}$ and $\text{Cov}(\mathbf{X}) = \Sigma$ of rank $r \le p$. Since Σ is positive semidefinite and symmetric, as previously explained, one can write $\Sigma = BB'$ where B is a $p\times r$ matrix of rank r. Now, consider the linear transformation

$$\mathbf{X} = \boldsymbol{\mu} + B\,\mathbf{Z}_1 \ \text{ where } \ \mathbf{Z}_1 \sim \mathcal{N}_r(\mathbf{0}, I)\,;$$

then, one has the following decomposition of the quadratic form $Q(\mathbf{X})$:

$$\begin{aligned} Q(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} = (\boldsymbol{\mu} + B\mathbf{Z}_1)'A(\boldsymbol{\mu} + B\mathbf{Z}_1) \\ &= \boldsymbol{\mu}'A\boldsymbol{\mu} + 2\,\mathbf{Z}_1'B'A\boldsymbol{\mu} + \mathbf{Z}_1'B'AB\mathbf{Z}_1, \quad \text{whenever } A = A'. \end{aligned}$$

Let P be an orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1,\ldots,\lambda_r)$, $\lambda_1,\ldots,\lambda_r$ being the eigenvalues of $B'AB$ in decreasing order, with $\lambda_{r_1+1},\ldots,\lambda_{r_1+\theta}$ denoting null eigenvalues, if any. Note that when $B'AB = O$, the null matrix, $Q(\mathbf{X})$ reduces to a linear form. Then, assuming that $B'AB \ne O$, one has $\mathbf{Z} \equiv P'\mathbf{Z}_1 \sim \mathcal{N}_r(\mathbf{0}, I)$, and

$$Q(\mathbf{X}) \ = \boldsymbol{\mu}'A\,\boldsymbol{\mu} + 2\mathbf{Z}'P'B'A\,\boldsymbol{\mu} + \mathbf{Z}'\mathcal{D}iag(\lambda_1,\ldots,\lambda_r)\mathbf{Z}\,.$$

Thus, the quadratic form $Q(\mathbf{X}) = \mathbf{X}'\,A\,\mathbf{X}$ has the following representation.

Representation 2.5.1. Letting $A = A'$, $\mathbf{X}$ be a $p\times 1$ normal vector with $E(\mathbf{X}) = \boldsymbol{\mu}$, $\text{Cov}(\mathbf{X}) = \Sigma \ge 0$, $\text{rank}(\Sigma) = r \le p$, $\Sigma = BB'$ where B is a $p\times r$ matrix and assuming that $B'AB \ne O$, one has

$$\begin{aligned} Q(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} = \sum_{j=1}^{r}\lambda_j Z_j^2 + 2\sum_{j=1}^{r} b_j^* Z_j + c^* \\ &= \sum_{j=1}^{r_1}\lambda_j Z_j^2 + 2\sum_{j=1}^{r_1} b_j^* Z_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| Z_j^2 + 2\sum_{j=r_1+\theta+1}^{r} b_j^* Z_j \\ &\qquad + 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j + c^* \\ &= \sum_{j=1}^{r_1}\lambda_j\Big(Z_j + \frac{b_j^*}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r}|\lambda_j|\Big(Z_j + \frac{b_j^*}{\lambda_j}\Big)^2 + 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j \\ &\qquad + \Big(c^* - \sum_{j=1}^{r_1}\frac{b_j^{*2}}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r}\frac{b_j^{*2}}{\lambda_j}\Big) \end{aligned}$$

$$\begin{aligned}&\equiv\ Q_1(\mathbf{W}_1)-Q_2(\mathbf{W}_2)+2\sum_{j=r_1+1}^{r_1+\theta}b_j^*Z_j+\kappa^*\\&\equiv\ Q_1(\mathbf{W}_1)-Q_2(\mathbf{W}_2)+T^*,\end{aligned}\tag{2.18}$$

where $Q_1(\mathbf{W}_1)$ and $Q_2(\mathbf{W}_2)$ are positive definite quadratic forms with $\mathbf{W}_1=(W_1,\ldots,W_{r_1})'$, $\mathbf{W}_2=(W_{r_1+\theta+1},\ldots,W_r)'$, $W_j=Z_j+b_j^*/\lambda_j$, $j=1,\ldots,r_1,r_1+\theta+1,\ldots,r$, $\mathbf{b}^{*'}=(b_1^*,\ldots,b_r^*)=\boldsymbol{\mu}'A'BP$, $\mathbf{Z}=(Z_1,\ldots,Z_r)'\sim\mathcal{N}_r(\mathbf{0},I)$, $P'B'ABP=\mathcal{D}iag(\lambda_1,\ldots,\lambda_r)$, $PP'=P'P=I$, $c^*=\boldsymbol{\mu}'A\boldsymbol{\mu}$, $\kappa^*=\Big(c^*-\sum_{j=1}^{r_1}b_j^{*2}/\lambda_j-\sum_{j=r_1+\theta+1}^{r}b_j^{*2}/\lambda_j\Big)$, $\lambda_j>0$, $j=1,\ldots,r_1$; $\lambda_j=0$, $j=r_1+1,\ldots,r_1+\theta$; $\lambda_j<0$, $j=r_1+\theta+1,\ldots,r$, and $T^*=2\sum_{j=r_1+1}^{r_1+\theta}b_j^*Z_j+\kappa^*\sim\mathcal{N}(\kappa^*,\,4\sum_{j=r_1+1}^{r_1+\theta}b_j^{*2})$.

2.6 Quadratic Expressions in Singular Normal Vectors

Let the $p\times 1$ random vector $\mathbf{X}$ be a singular p-variate normal random variables with $E(\mathbf{X})=\boldsymbol{\mu}$ and $\text{Cov}(\mathbf{X})=\Sigma=BB'$ where B is $p\times r$ of rank $r\le p$. Consider the quadratic expression

$$Q^*(\mathbf{X})=\mathbf{X}'A\mathbf{X}+\mathbf{a}'\mathbf{X}+d\tag{2.19}$$

where $A=A'$, $\mathbf{a}$ is a p-dimensional vector and d is a constant.

Representation of $Q^*(\mathbf{X})$ and its cumulants are provided in next two subsections.

2.6.1 A Decomposition of $Q^*(\mathbf{X})$

Letting $\mathbf{X}=\boldsymbol{\mu}+B\,\mathbf{Z}$ where $\mathbf{Z}\sim\mathcal{N}_r(\mathbf{0},I)$, one can write

$$\begin{aligned}Q^*(\mathbf{X})\equiv Q^*(\mathbf{Z})&=(\boldsymbol{\mu}+B\mathbf{Z})'A(\boldsymbol{\mu}+B\mathbf{Z})+\mathbf{a}'(\boldsymbol{\mu}+B\mathbf{Z})+d\\&=\boldsymbol{\mu}'A\boldsymbol{\mu}+2\boldsymbol{\mu}'A'B\mathbf{Z}+\mathbf{Z}'B'AB\mathbf{Z}+\mathbf{a}'B\mathbf{Z}+\mathbf{a}'\boldsymbol{\mu}+d\,.\end{aligned}$$

Let P be an orthogonal matrix such that $P'B'ABP=\mathcal{D}iag(\lambda_1,\ldots,\lambda_r)$, with $\lambda_1,\ldots,\lambda_r$ being the eigenvalues of $B'AB$, $PP'=P'P=I$, $\mathbf{m}'=\mathbf{a}'BP$, $\mathbf{b}^{*'}=\boldsymbol{\mu}'ABP$ and $c_1=\boldsymbol{\mu}'A\boldsymbol{\mu}+\mathbf{a}'\boldsymbol{\mu}+d$ and $\mathbf{W}=P'\mathbf{Z}\sim\mathcal{N}_r(\mathbf{0},I)$. Then, assuming that $B'AB\neq O$, one has

$$\begin{aligned}Q^*(\mathbf{X})\equiv Q^*(\mathbf{W})&=\mathbf{W}'P'B'ABP\mathbf{W}+2\boldsymbol{\mu}'A'BP\mathbf{W}+\mathbf{a}'BP\mathbf{W}+\boldsymbol{\mu}'A\boldsymbol{\mu}+\mathbf{a}'\boldsymbol{\mu}+d\\&=\mathbf{W}'\mathcal{D}iag(\lambda_1,\ldots,\lambda_r)\mathbf{W}+(2\mathbf{b}^{*'}+\mathbf{m}')\mathbf{W}+c_1\,.\end{aligned}$$

which yields the decomposition that follows.

Representation 2.6.1. Let $A = A'$, $\mathbf{X}$ be a p-dimensional normal vector with $E(\mathbf{X}) = \boldsymbol{\mu}$, $\text{Cov}(\mathbf{X}) = \Sigma \geq 0$, $\text{rank}(\Sigma) = r \leq p$, $\Sigma = BB'$ where B is a $p \times r$ matrix, $\mathbf{a}$ is a p-dimensional vector, $P'B'ABP = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$ with $PP' = P'P = I$, $\lambda_1, \ldots, \lambda_{r_1}$ be the positive eigenvalues $B'AB$, $\lambda_{r_1+1} = \cdots = \lambda_{r_1+\theta} = 0$, $\lambda_{r_1+\theta+1}, \ldots, \lambda_r$ be the negative eigenvalues of $B'AB$, $\mathbf{m}' = (m_1, \ldots, m_r) = \mathbf{a}'BP$, $\mathbf{b}^{*'} = (b_1^*, \ldots, b_r^*) = \boldsymbol{\mu}'A'BP$, and d is a real constant, and assume that $B'AB \neq O$, then

$$\begin{aligned}
Q^*(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d \\
&\equiv Q^*(\mathbf{W}) = \sum_{j=1}^{r} \lambda_j W_j^2 + 2\sum_{j=1}^{r}\Big(\frac{1}{2}m_j + b_j^*\Big)W_j + c_1 \\
&= \sum_{j=1}^{r_1} \lambda_j W_j^2 + 2\sum_{j=1}^{r_1} n_j W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| W_j^2 + 2\sum_{j=r_1+\theta+1}^{r} n_j W_j \\
&\qquad + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + c_1 \\
&= \sum_{j=1}^{r_1} \lambda_j\Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j|\Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j \\
&\qquad + \Big(c_1 - \sum_{j=1}^{r_1}\frac{n_j^2}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r}\frac{n_j^2}{\lambda_j}\Big) \\
&\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1 \\
&\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1, \qquad (2.20)
\end{aligned}$$

where $\mathbf{W}' = (W_1, \ldots, W_r) \sim \mathcal{N}_r(\mathbf{0}, I)$, $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are positive definite quadratic forms with $\mathbf{W}^+ = (W_1 + n_1/\lambda_1, \ldots, W_{r_1} + n_{r_1}/\lambda_{r_1})' \sim \mathcal{N}_{r_1}(\boldsymbol{\nu}_1, I)$, $\mathbf{W}^- = (W_{r_1+\theta+1} + n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, W_r + n_r/\lambda_r)' \sim \mathcal{N}_{r-r_1-\theta}\ (\boldsymbol{\nu}_2, I)$ with $\boldsymbol{\nu}_1 = (n_1/\lambda_1, \ldots, n_{r_1}/\lambda_{r_1})'$ and $\boldsymbol{\nu}_2 = (n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, n_r/\lambda_r)'$, θ being number of null eigenvalues of $B'AB$, $n_j = \frac{1}{2}m_j + b_j^*$, $c_1 = \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d$, $\kappa_1 = \Big(c_1 - \sum_{j=1}^{r_1} n_j^2/\lambda_j - \sum_{j=r_1+\theta+1}^{r} n_j^2/\lambda_j\Big)$ and $T_1 = (2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1) \sim \mathcal{N}(\kappa_1\,,\, 4\sum_{j=r_1+1}^{r_1+\theta} n_j^2)$.

When $\boldsymbol{\mu} = \mathbf{0}$, one has

$$Q^*(\mathbf{X}) \equiv Q^*(\mathbf{W}) = \sum_{j=1}^{r} \lambda_j W_j^2 + \sum_{j=1}^{r} m_j W_j + d$$

$$\begin{aligned}
&= \sum_{j=1}^{r_1}\lambda_j W_j^2 + \sum_{j=1}^{r_1} m_j W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| W_j^2 + \sum_{j=r_1+\theta+1}^{r} m_j W_j \\
&\qquad + \sum_{j=r_1+1}^{r_1+\theta} m_j W_j + d \\
&= \sum_{j=1}^{r_1}\lambda_j \Big(W_j + \frac{m_j}{2\lambda_j}\Big)^2 - \sum_{j=r+\theta+1}^{r} |\lambda_j| \Big(W_j + \frac{m_j}{2\lambda_j}\Big)^2 \\
&\quad + \sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \Big(d - \sum_{j=1}^{r_1}\frac{m_j^2}{4\lambda_j} - \sum_{j=r_1+\theta+1}^{r}\frac{m_j^2}{4\lambda_j}\Big) \\
&\equiv Q_1(\mathbf{W}_1^+) - Q_2(\mathbf{W}_1^-) + \sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \kappa_1^* \\
&\equiv Q_1(\mathbf{W}_1^+) - Q_2(\mathbf{W}_1^-) + T_1^*, \qquad (2.21)
\end{aligned}$$

where $Q_1(\mathbf{W}_1^+)$ and $Q_2(\mathbf{W}_1^-)$ are positive definite quadratic forms with $\mathbf{W}_1^+ = (W_1 + m_1/(2\lambda_1), \ldots, W_{r_1} + m_{r_1}/(2\lambda_{r_1}))' \sim \mathcal{N}_{r_1}(\boldsymbol{\mu}_1, I)$, $\boldsymbol{\mu}_1 = (m_1/(2\lambda_1), \ldots,\ m_{r_1}/(2\lambda_{r_1}))'$, $\mathbf{W}_1^- = (W_{r_1+\theta+1} + m_{r_1+\theta+1}/(2\lambda_{r_1+\theta+1}), \ldots, W_r + m_r/(2\lambda_r))' \sim \mathcal{N}_{r-r_1-\theta}(\boldsymbol{\mu}_2, I)$, $\boldsymbol{\mu}_2 = (m_{r_1+\theta+1}/(2\lambda_{r_1+\theta+1}), \ldots, m_r/(2\lambda_r))'$, $\kappa_1^* = \Big(d - \sum_{j=1}^{r_1} m_j^2/(4\lambda_j) - \sum_{j=r_1+\theta+1}^{r} m_j^2/(4\lambda_j)\Big)$ and $T_1^* = (\sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \kappa_1^*) \sim \mathcal{N}(\kappa_1^*,\ \sum_{j=r_1+1}^{r_1+\theta} m_j^2)$.

2.6.2 Cumulants and Moments of Quadratic Expressions in Singular Normal Vectors

The cumulant generating functions of $Q^* = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ and $Q = \mathbf{X}'A\mathbf{X}$ where $A = A'$, $\mathbf{X}$ has a singular p-variate normal distribution with $E(\mathbf{X}) = \boldsymbol{\mu}$, $\mathrm{Cov}(\mathbf{X}) = \Sigma = BB'$, with $B_{p\times r}$ of rank r, $\mathbf{a}$ is a p–dimensional constant vector and d is a scalar constant, are respectively

$$\begin{aligned}
\ln(M_{Q^*}(t)) &= t(d + \mathbf{a}'\boldsymbol{\mu} + \boldsymbol{\mu}'A\boldsymbol{\mu}) + \frac{1}{2}\sum_{j=1}^{\infty}\frac{(2t)^j}{j}\mathrm{tr}(A\Sigma)^j \\
&\quad + \sum_{j=0}^{\infty}(2t)^{j+2}\Big\{\frac{1}{8}\,\mathbf{a}'(\Sigma A)^j\Sigma\,\mathbf{a} + \frac{1}{2}\,\boldsymbol{\mu}'\,(A\Sigma)^{j+1}A\,\boldsymbol{\mu} \\
&\qquad + \frac{1}{2}\,\mathbf{a}'(\Sigma A)^{j+1}\boldsymbol{\mu}\Big\} \qquad (2.22)
\end{aligned}$$

and

$$\ln(M_Q(t)) = -\frac{1}{2}\sum_{j=1}^{r}\ln(1-2\,t\lambda_j) + c^*t + 2\,t^2\sum_{j=1}^{r}\frac{b_j^{*2}}{(1-2\,t\lambda_j)}$$

where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of $B'AB,\ B'AB \neq O$, $c^* = \boldsymbol{\mu}'A\boldsymbol{\mu}$, $\mathbf{b}^* = P'B'A\boldsymbol{\mu}$, and P is an orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1,\ldots,\lambda_r)$.

It is also shown in Mathai and Provost (1992) that s^{th} cumulant of Q^* is

$$\begin{aligned}
k^*(s) &= 2^{s-1}s!\left\{(1/s)\,\mathrm{tr}(B'AB)^s + (1/4)\,\mathbf{a}'B\,(B'AB)^{s-2}B'\mathbf{a}\right.\\
&\qquad \left. + \,\boldsymbol{\mu}'AB(B'AB)^{s-2}B'A\boldsymbol{\mu} + \mathbf{a}'B(B'AB)^{s-2}B'A\,\boldsymbol{\mu}\right\}\\
&= 2^{s-1}s!\left\{(1/s)\mathrm{tr}(A\Sigma)^s + (1/4)\mathbf{a}'(\Sigma A)^{s-2}\Sigma\,\mathbf{a}\right.\\
&\qquad \left. +\boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu} + \mathbf{a}'(\Sigma A)^{s-1}\boldsymbol{\mu}\right\},\ \text{ for } s \geq 2.\\
&= \mathrm{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d,\ \text{ for } s = 1\,.
\end{aligned} \tag{2.23}$$

The moments of $Q^*(\mathbf{X})$ can then be readily determined via the recursive relationship given in Equation (2.17).

2.7 Approximating the Distribution of Quadratic Forms

Since the representations of indefinite quadratic expressions involve Q_1-Q_2 where Q_1 and Q_2 are independently distributed positive definite quadratic forms, some approximations to the density function of Q_1-Q_2 are provided in Sections 2.7.1 and 2.7.2. An algorithm describing proposed methodology is provided in Section 2.7.5.

Letting $Q(\mathbf{X}) = Q_1(\mathbf{X}_1) - Q_2(\mathbf{X}_2)$ and $h_Q(q)\,\mathcal{I}_{\Re}(q)$, $f_{Q_1}(q_1)\,\mathcal{I}_{(\tau_1,\infty)}(q_1)$ and $f_{Q_2}(q_2)$ $\mathcal{I}_{(\tau_2,\infty)}(q_2)$ respectively denote the approximate densities of $Q(\mathbf{X}), Q_1(\mathbf{X}_1) > 0$ and $Q_2(\mathbf{X}_2) > 0$, where $\mathbf{X}' = (\mathbf{X}_1', \mathbf{X}_2')$ and $\mathbf{X}_1'$ and $\mathbf{X}_2'$ are independently distributed, $\mathcal{I}_A(.)$ being the indicator function with respect to the set A, an approximation to density function of the indefinite quadratic form $Q(\mathbf{X})$ can be obtained as follows via the transformation variables technique:

$$h_Q(q) = \begin{cases} h_p(q) & \text{for } q \geq \tau_1 - \tau_2 \\ h_n(q) & \text{for } q < \tau_1 - \tau_2, \end{cases} \tag{2.24}$$

where

$$h_p(q) = \int_{q+\tau_2}^{\infty} f_{Q_1}(y) f_{Q_2}(y-q)\mathrm{d}y \tag{2.25}$$

and

$$h_n(q) = \int_{\tau_1}^{\infty} f_{Q_1}(y) f_{Q_2}(y-q)\mathrm{d}y\,. \tag{2.26}$$

These integral representations hold whether τ_1 and τ_2 are positive or negative and whether $\tau_1 > \tau_2$ or $\tau_1 \le \tau_2$.

Note that in the case of gamma-type density functions without location parameters, τ_1 and τ_2 are equal to zero in Equations (2.24), (2.25) and (2.26).

2.7.1 Approximation via Pearson's Approach

Let σ_Q denote the standard deviation of the positive definite quadratic form $Q(\mathbf{X})$. According to Pearson (1959), one has $Q(\mathbf{X}) \approx U$ with

$$U \sim \left(\frac{\chi^2_\nu - \nu}{\sqrt{2\nu}}\right) \sigma_Q + E(Q(\mathbf{X})) \tag{2.27}$$

where the symbol $\approx$ means "is approximately distributed as" and ν is such that both $Q(\mathbf{X})$ and U have equal third cumulants. Since $E(\chi^2_\nu) = \nu$ and $\mathrm{Var}(\chi^2_\nu) = 2\nu$, $E(U) = E(Q(\mathbf{X}))$ and $\mathrm{Var}(U) = \sigma_Q^2$. Letting θ_i be as defined in Equation (2.16), the third cumulant of U is $8\nu\,\sigma_Q^3/(2\nu)^{3/2} = 2^{3/2}k(2)^{3/2}/\sqrt{\nu} = 8\theta_2^{3/2}/\sqrt{\nu}$, while the first and second cumulants of U coincide with those of $Q(\mathbf{X})$. On equating the third cumulants of U and $Q(\mathbf{X})$, which according to (2.16) is $8\,\theta_3$, one has

$$\nu = \frac{\theta_2^3}{\theta_3^2}\,. \tag{2.28}$$

Thus,

$$Q(\mathbf{X}) \approx \frac{\theta_3}{\theta_2}\,\chi^2_\nu - \frac{\theta_2^3}{\theta_3^2} + \theta_1\,, \tag{2.29}$$

or equivalently,

$$Q(\mathbf{X}) \approx c\,\chi^2_\nu + \tau, \tag{2.30}$$

where $c = \frac{\theta_3}{\theta_2}$ and $\tau = -\frac{\theta_2^3}{\theta_3^2} + \theta_1$. That is, Pearson's approximant to the exact density of $Q(\mathbf{X})$ is given by

$$f_Q(q) \;=\; \frac{(q-\tau)^{\nu/2-1}\, e^{-(q-\tau)/(2c)}}{\Gamma(\frac{\nu}{2})(2c)^{\nu/2}} \mathcal{I}_{(\tau,\Re)}(q)\,. \tag{2.31}$$

Accordingly, the density function of the indefinite quadratic form $Q(\mathbf{X}) = Q_1(\mathbf{X}) - Q_2(\mathbf{X})$, where $Q_1(\mathbf{X})$ and $Q_2(\mathbf{X})$ are positive definite quadratic forms, can be approximated by making use of Equation (2.24) where $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$ respectively denote the Pearson-type density approximants of $Q_1(\mathbf{X})$ and $Q_2(\mathbf{X})$ with parameters τ_i, c_i and $\nu_i/2$, $i = 1, 2$, which are available from Equation (2.31). Explicit representations of $h_p(q)$ and $h_n(q)$ as specified by Equations (2.25) and (2.26), respectively, can be obtained as follows:

$$\begin{aligned} h_n(q) &= \int_{\tau_1}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y\,, \qquad\qquad q<0 \\ &= \int_{\tau_1}^{\infty} \frac{(y-\tau_1)^{\nu_1/2-1}\,(y-q-\tau_2)^{\nu_2/2-1}\, e^{-(y-\tau_1)/(2c_1)}\, e^{-(y-q-\tau_2)/(2c_2)}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,(2c_1)^{\nu_1/2}\,(2c_2)^{\nu_2/2}}\,\mathrm{d}y \end{aligned}$$

where $\tau_1 - \tau_2 > q$, $\nu_1 > 0$, $\nu_2 > 0$, $c_1 > 0$, $c_2 > 0$; and

$$\begin{aligned} h_p(q) &= \int_{q+\tau_2}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y\,, \qquad\qquad q>0 \\ &= \int_{q+\tau_2}^{\infty} \frac{(y-\tau_1)^{\nu_1/2-1}\,(y-q-\tau_2)^{\nu_2/2-1}\, e^{-(y-\tau_1)/(2c_1)}\, e^{-(y-q-\tau_2)/(2c_2)}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,(2c_1)^{\nu_1/2}\,(2c_2)^{\nu_2/2}}\,\mathrm{d}y \end{aligned} \tag{2.32}$$

where $\tau_1 - \tau_2 < q$, $\nu_1 > 0$, $\nu_2 > 0$, $c_1 > 0$, $c_2 > 0$. One can express $h_n(q)$ and $h_p(q)$ in terms of the Whittaker function, which has the following representation, see Section 9.220 in Gradshteyn and Ryzhik (1980):

$$W_{\lambda,\mu}(z) \;=\; \frac{z^{\lambda}\, e^{-z/2}}{\Gamma(\mu-\lambda+\frac{1}{2})} \int_0^{\infty} t^{\mu-\lambda-\frac{1}{2}}\, e^{-t}\,(1+\frac{t}{z})^{\mu+\lambda-\frac{1}{2}}\,\mathrm{d}t\,, \tag{2.33}$$

which is real for all positive real-valued z and $Re(\mu - \lambda) > -\frac{1}{2}$. The value at zero is easily obtained by evaluating $W_{\lambda,\mu}(z)$ at $\epsilon > 0$ and letting ϵ tend to zero.

Letting $y - q - \tau_2 = x$ in Equation (2.32) and then replacing $(c_1 + c_2)/(2c_1\, c_2)$ by ϑ and ω by $q + \tau_2 - \tau_1$, one has

$$\begin{aligned} h_p(q) &= \int_0^\infty \frac{(x+\omega)^{\frac{\nu_1}{2}-1}\, x^{\frac{\nu_2}{2}-1}\, e^{-(x+\omega)/(2c_1)}\, e^{-x/(2c_2)}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,(2c_1)^{\frac{\nu_1}{2}}\,(2c_2)^{\frac{\nu_2}{2}}}\,\mathrm{d}x \\ &= \int_0^\infty \frac{(1+\frac{x}{\omega})^{\frac{\nu_1}{2}-1}\omega^{\frac{\nu_1}{2}-1}\, x^{\frac{\nu_2}{2}-1}\, e^{-x(\frac{1}{2c1}+\frac{1}{2c2})}\, e^{-\omega/(2c_1)}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,(2c_1)^{\frac{\nu_1}{2}}\,(2c_2)^{\frac{\nu_2}{2}}}\,\mathrm{d}x \\ &= \frac{\omega^{\frac{\nu_1}{2}-1}\, e^{-\omega/(2c_1)}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,(2c_1)^{\frac{\nu_1}{2}}\,(2c_2)^{\frac{\nu_2}{2}}}\int_0^\infty (1+\frac{x}{\omega})^{\frac{\nu_1}{2}-1}\, x^{\frac{\nu_2}{2}-1}\, e^{-x\vartheta}\,\mathrm{d}x\,. \end{aligned}$$

Now, letting $x\,\vartheta = t$, $\frac{\nu_1}{2}-1 = \mu-\lambda-\frac{1}{2}$ and $\frac{\nu_2}{2}-1 = \mu+\lambda-\frac{1}{2}$, which implies that $\lambda = (\nu_1-\nu_2)/4$ and $\mu = (\nu_2+\nu_1-2)/4$, one has

$$\begin{aligned} h_p(q) &= \frac{\omega^{\frac{\nu_1}{2}-1}\, e^{-\omega/(2c_1)}\, \vartheta^{-\frac{\nu_2}{2}}\,(\vartheta\,\omega)^{-\lambda}\, e^{\vartheta\,\omega/2}}{\Gamma(\frac{\nu_1}{2})\,(2c_1)^{\frac{\nu_1}{2}}\,(2c_2)^{\frac{\nu_2}{2}}} \\ &\qquad\times\frac{(\vartheta\,\omega)^{\lambda}\, e^{-\vartheta\,\omega/2}}{\Gamma(\mu-\lambda+\frac{1}{2})}\int_0^\infty t^{\mu-\lambda-\frac{1}{2}}\,(1+\frac{t}{\vartheta\,\omega})^{\mu+\lambda-\frac{1}{2}}\, e^{-t}\,\mathrm{d}t \\ &= \frac{\omega^{\frac{\nu_1}{2}-1}\, e^{-\omega/(2c_1)}\, \vartheta^{-\frac{\nu_2}{2}}\,(\vartheta\,\omega)^{-\lambda}\, e^{\vartheta\,\omega/2}}{\Gamma(\frac{\nu_1}{2})\,(2c_1)^{\frac{\nu_1}{2}}\,(2c_2)^{\frac{\nu_2}{2}}}\, W_{(\nu_1-\nu_2)/4,(\nu_2+\nu_1-2)/4}(\omega\,\vartheta) \\ &= \frac{\omega^{(\nu_1+\nu_2-4)/4}\, e^{\omega/(\vartheta/2-1/(2c_1))}\, \vartheta^{-(\nu_1+\nu_2)/4}}{\Gamma(\frac{\nu_1}{2})\,(2c_1)^{\frac{\nu_1}{2}}\,(2c_2)^{\frac{\nu_2}{2}}}\, W_{(\nu_1-\nu_2)/4,(\nu_2+\nu_1-2)/4}(\omega\,\vartheta) \end{aligned}$$

$$\begin{aligned} &= \frac{\vartheta^{-(\nu_1+\nu_2)/4}}{\Gamma(\frac{\nu_1}{2})\,(2c_1)^{\frac{\nu_1}{2}}\,(2c_2)^{\frac{\nu_2}{2}}}\,(q+\tau_2-\tau_1)^{(\nu_1+\nu_2-4)/4}\, e^{(q+\tau_2-\tau_1)/(\vartheta/2-1/(2c_1))} \\ &\qquad\times W_{(\nu_1-\nu_2)/4,(\nu_2+\nu_1-2)/4}((q+\tau_2-\tau_1)\,\vartheta)\,. \end{aligned} \tag{2.34}$$

Since $h_n(q;\frac{\nu_1}{2},c_1,\frac{\nu_2}{2},c_2) = h_p(-q;\frac{\nu_2}{2},c_2,\frac{\nu_1}{2},c_1)$, one has

$$\begin{aligned} h_n(q) &= \frac{\vartheta^{-(\nu_2+\nu_1)/4}}{\Gamma(\frac{\nu_2}{2})\,(2c_2)^{\frac{\nu_2}{2}}\,(2c_1)^{\frac{\nu_1}{2}}}\,(-q+\tau_1-\tau_2)^{(\nu_2+\nu_1-4)/4}\, e^{(-q+\tau_1-\tau_2)/(\vartheta/2-1/(2c_2))} \\ &\qquad\times W_{(\nu_2-\nu_1)/4,(\nu_1+\nu_2-2)/4}((-q+\tau_1-\tau_2)\,\vartheta)\,. \end{aligned} \tag{2.35}$$

Note that the condition $Re(\mu-\lambda) > -\frac{1}{2}$ in (2.33) is not restrictive since $\mu-\lambda+\frac{1}{2} = \nu_2/2$ in Equation (2.34), $\mu-\lambda+\frac{1}{2} = \nu_1/2$ in Equation (2.35), and ν_1 and ν_2 are positive parameters. Thus, the density function of $Q_1(\mathbf{X}) - Q_2(\mathbf{X})$ is

$$h_Q(q) \; = h_n(q)\,\mathcal{I}_{(-\infty,\,\tau_1)}(q) + h_p(q)\,\mathcal{I}_{(\tau_2,\,\infty)}(q)\,.$$

The corresponding cumulative distribution function is obtained by numerical integration. When ν_1 and ν_2 are equal to two, a limiting procedure has to be applied to determine the cumulative distribution function.

2.7.2 Approximations via Generalized Gamma Distributions

Positive definite quadratic forms are approximated by gamma-type distributions in this section. First, let us consider the gamma distribution whose density function is given by

$$\psi(x) \;\; = \;\; \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}\,\mathcal{I}_{(0\,,\,\infty)}(x) \tag{2.36}$$

where $\alpha > 0$ and $\beta > 0$ can be specified as follows on the basis of μ_1 and μ_2, the first two integer moments of the distribution being approximated:

$$\alpha = \mu_1^2/(\mu_2 - \mu_1^2) \;\text{ and }\; \beta = \mu_2/\mu_1 - \mu_1\,.$$

The generalized gamma density function that we are considering has the following parameterization:

$$\psi(x) = \frac{\gamma}{\beta^{\alpha\,\gamma}\Gamma(\alpha)}\,x^{\alpha\,\gamma-1}e^{-(x/\beta)^{\gamma}}\;\mathcal{I}_{(0\,,\,\infty)}(x) \tag{2.37}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Denoting its integer moments by m_j, $j = 0, 1, \ldots,$ one has

$$m_j = \frac{\beta^j\,\Gamma(\alpha\,+j/\gamma)}{\Gamma(\alpha)}\,. \tag{2.38}$$

Its three parameters can readily be determined by solving numerically the equations,

$$\mu_i \;\; = \;\; m_i\,, \qquad \text{for } i = 1, 2, 3, \tag{2.39}$$

where μ_i denotes the i^{th} moment of a certain positive definite quadratic form Q.

A four-parameter gamma, referred to as a shifted generalized gamma density function, is given by

$$\psi(x) = \frac{\gamma}{\beta^{\alpha\,\gamma}\Gamma(\alpha)}\,(x-\tau)^{\alpha\,\gamma-1}e^{-(\frac{x-\tau}{\beta})^{\gamma}}\;\mathcal{I}_{(\tau,\infty)}(x) \tag{2.40}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. One can determine the moments of the shifted generalized gamma distribution by applying the binomial expansion to the moments of the generalized gamma.

Let $Q_1(\mathbf{Y}^+)$ and $Q_2(\mathbf{Y}^-)$ be two independently distributed positive definite quadratic forms such as those defined in Equation (2.9). Then, an approximate density function for $Q_1(\mathbf{Y}^+) - Q_2(\mathbf{Y}^-)$ can be obtained from Equation (2.24). Consider the non-shifted gamma distribution whose density function is given in Equation (2.36). Let α_i and β_i be determined from the first two moments of $Q_i(\mathbf{X})$, $i = 1, 2$. In this case, the negative part of the density function of $Q(\mathbf{X})$ is

$$\begin{aligned} h_n(q) &= \int_{-q}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y\,, \qquad q < 0 \\ &= \int_{0}^{\infty} \frac{y^{\alpha_1-1}\,(y-q)^{\alpha_2-1}\,e^{-y/\beta_1}\,e^{-(y-q)/\beta_2}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\beta_1^{\alpha_1}\,\beta_2^{\alpha_2}}\,\mathrm{d}y \end{aligned}$$

the positive part of the density being

$$\begin{aligned} h_p(q) &= \int_{q}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y\,, \qquad q > 0 \\ &= \int_{q}^{\infty} \frac{y^{\alpha_1-1}\,(y-q)^{\alpha_2-1}\,e^{-y/\beta_1}\,e^{-(y-q)/\beta_2}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\beta_1^{\alpha_1}\,\beta_2^{\alpha_2}}\,\mathrm{d}y \end{aligned}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 > 0$ and $\beta_2 > 0$. One can express $h_p(q)$ and $h_n(q)$ in terms of the Whittaker function by letting $\tau_1 = 0, \tau_2 = 0, 2c_1 = \beta_1, \nu_1/2 = \alpha_1, 2c_2 = \beta_2$ and $\nu_2/2 = \alpha_2$ in (2.34) and (2.35), respectively, as follows:

$$\begin{aligned} h_p(q) &= \frac{\vartheta_1^{-(\alpha_1+\alpha_2)/2}}{\Gamma(\alpha_1)\,\beta_1^{\alpha_1}\,\beta_2^{\alpha_2}}\, q^{(\alpha_1+\alpha_2-2)/2}\, e^{q(\vartheta_1/2-1/\beta_1)} \\ &\qquad \times W_{(\alpha_1-\alpha_2)/2,(\alpha_1+\alpha_2-1)/2}(\vartheta_1 q) \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} h_n(q) &= \frac{\vartheta_1^{-(\alpha_1+\alpha_2)/2}}{\Gamma(\alpha_2)\,\beta_1^{\alpha_1}\,\beta_2^{\alpha_2}}\, (-q)^{(\alpha_1+\alpha_2-2)/2}\, e^{-q(\vartheta_1/2-1/\beta_2)} \\ &\qquad \times W_{(\alpha_2-\alpha_1)/2,(\alpha_1+\alpha_2-1)/2}(-\vartheta_1 q) \end{aligned} \tag{2.42}$$

where $\vartheta_1 = \frac{\beta_1+\beta_2}{\beta_1\beta_2}$, $\vartheta_1 q \neq 0$. A limiting procedure yields the density function at the point zero. Once again, it should be pointed out that the Whittaker function as specified by

(2.33) is defined for $Re(\mu-\lambda) > -\frac{1}{2}$ which merely requires that α_1 and α_2 be positive in (2.41) and (2.42).

Thus, the density function of $Q_1(\mathbf{Y}^+) - Q_2(\mathbf{Y}^-)$ is

$$h_Q(q) = h_n(q)\,\mathcal{I}_{(-\infty,0)}(q) + h_p(q)\,\mathcal{I}_{(0,\infty)}(q)\,. \tag{2.43}$$

The corresponding cumulative distribution function of $Q(\mathbf{X})$ is obtained by numerical integration. When $\alpha_1 = 1$ or $\alpha_2 = 1$, the cumulative distribution function is determined by letting $\alpha_i = 1 \pm \epsilon$ and ϵ tend 0 for $i = 1, 2$.

2.7.3 Polynomially Adjusted Density Functions

In this section, the density approximations are adjusted with polynomials whose coefficients are such that the first n moments of the approximation coincide with the first moments of a given quadratic form. The larger n is, the more accurate the approximation. Accordingly, the value of n can be increased until a satisfactory level of accuracy is attained.

In order to approximate the density function of a noncentral quadratic form $Q(\mathbf{X})$, one should first approximate the density functions of the two positive definite quadratic forms, $Q_1(\mathbf{X})$ and $Q_2(\mathbf{X})$ as defined in (2.9). According to Equation (2.17), the moments of the positive definite quadratic form $Q_1(\mathbf{X})$ denoted by $\mu_{Q_1}(\cdot)$ can be obtained recursively from the cumulants. Then, on the basis of the first n moments of $Q_1(\mathbf{X})$, a density approximation of the following form is assumed for $Q_1(\mathbf{X})$:

$$f_n(x) = \varphi(x)\sum_{j=0}^{n} \xi_j x^j \tag{2.44}$$

where $\varphi(x)$ is an initial density approximant referred to as base density function, which could be a gamma, generalized gamma, generalized shifted gamma or Pearson-type density function.

In order to determine the polynomial coefficients, ξ_j, we equate the h^{th} moment of $Q_1(\mathbf{X})$ to the h^{th} moment of the approximate distribution specified by $f_n(x)$. That is,

$$\begin{aligned}
\mu_{Q_1}(h) &= \int_{\tau_1}^{\infty} x^h \varphi(x) \sum_{j=0}^{n} \xi_j x^j \mathrm{d}x \\
&= \sum_{j=0}^{n} \xi_j \int_{\tau_1}^{\infty} x^{h+j}\varphi(x)\mathrm{d}x \\
&= \sum_{j=0}^{n} \xi_j\, m_{h+j}, \qquad h = 0, 1, \ldots, n,
\end{aligned} \tag{2.45}$$

where m_{h+j} is the $(h+j)^{\text{th}}$ moment associated with $\varphi(x)$. For the generalized gamma, m_j is given by (2.38), and for the Pearson-type distribution,

$$m_j = \begin{cases} \frac{2^{-\nu/2}\, c^h\, e^{\tau/(2\,c)}}{\Gamma(1-\frac{\nu}{2})\,\Gamma(h+1+\frac{\nu}{2})\,\Gamma(\frac{\nu}{2})}\Big(\Gamma(h+1)\Big(\Gamma(-h-\frac{\nu}{2})\,\Gamma(h+\frac{\nu}{2}+1)\,(-\frac{\tau}{c})^h + (\frac{\tau}{c})^h\,\Gamma(1-\frac{\nu}{2}) \\ \quad\times\Gamma(\frac{\nu}{2})\Big)\,{}_1F_1\Big(h+1; h+\frac{\nu}{2}+1; -\frac{\tau}{2c}\Big)\,(-\frac{\tau}{c})^{\nu/2} + 2^{h+\nu/2}\,\Gamma(1-\frac{\nu}{2})\,\Gamma(h+\frac{\nu}{2}) \\ \quad\times\Gamma(h+\frac{\nu}{2}+1)\,{}_1F_1(1-\frac{\nu}{2}; \frac{1}{2}(-2\,h-\nu+2); -\frac{\tau}{2c})\Big) & \text{for} \quad \tau\le 0 \\ 2^h\,c^h\,U(-h, 1-h-\frac{\nu}{2}, \frac{\tau}{2c}) & \text{for} \quad \tau > 0. \end{cases}$$

where $U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-z\,t}\,t^{a-1}\,(1+t)^{b-a-1}\mathrm{d}t$ is the confluent hypergeometric function. This leads to a linear system of $(n+1)$ equations in $(n+1)$ unknowns whose solution is

$$\begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & \cdots & m_n & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & \cdots & m_{2n-1} & m_{2n} \end{bmatrix}^{-1} \begin{bmatrix} \mu_{Q_1}(0) \\ \mu_{Q_1}(1) \\ \vdots \\ \mu_{Q_1}(n) \end{bmatrix}. \tag{2.46}$$

The resulting representation of the density function of $Q_1(\mathbf{X})$ will be referred to as a polynomially adjusted density approximant, which can be readily evaluated. As long as higher moments are available, more accurate approximations can always be obtained by making use of additional moments.

The density function for $Q_2(\mathbf{X})$ can similarly be approximated using the same procedure. The density approximant to the noncentral indefinite quadratic form $Q(\mathbf{X}){=}Q_1(\mathbf{X}) - Q_2(\mathbf{X})$ is obtained from Equation (2.24), with τ_1 and τ_2 equal to zero.

2.7.4 Polynomially Adjusted Gamma Density Approximations

This section provides an alternative representation of the polynomial adjustment when the base density is a gamma density function.

As explained in Provost (2005), the density functions of numerous statistics distributed on the positive half-line can be approximated from their exact moments by making use of gamma-type density functions that are adjusted by means of linear combinations of Laguerre polynomials. For conditions ensuring that a distribution be uniquely defined by its moments, the reader is referred to Rao (1965).

Consider a random variable Y defined on the interval $[0,\infty)$, whose j^{th} moment is denoted by $\mu_j, j = 0,1,2,\ldots,$ and let $c = (\mu_2-\mu_1^2)/\mu_1, v = (\mu_1/c)-1$ and $X = Y/c$. Denoting the j^{th} moment of X by $\mu_j^* = E\left[(Y/c)^j\right]$, the density function of the random variable X, also defined on the interval $[0,\infty)$, can be expressed as

$$f(x) = x^{\nu} e^{-x} \sum_{j=0}^{\infty} \delta_j L_j(\nu, x), \tag{2.47}$$

where

$$L_j(v, x) = \sum_{k=0}^{j} (-1)^k \frac{\Gamma(v + j + 1)\, x^{j-k}}{k!\,(j-k)!\,\Gamma(v + j - k + 1)} \tag{2.48}$$

is a Laguerre polynomial of order j in x with parameter v and

$$\delta_j = \sum_{k=0}^{j} (-1)^k \frac{j\,!}{k!\,(j-k)!\,\Gamma(v + j - k + 1)}\, \mu^*_{j-k}\,, \tag{2.49}$$

see for instance Szegö (1959) or Devroye (1989). Then, on truncating the series appearing in Equation (2.47) and making the change of variable $Y = cX$, one obtains the following density approximant for Y:

$$f_n(y) = \frac{y^v e^{-y/c}}{c^{v+1}} \sum_{j=0}^{n} \delta_j L_j\left(v, y/c\right)\ . \tag{2.50}$$

Remark 2.7.1. Note that $f_0(y)$ is a gamma density function with parameters $\alpha \equiv v + 1 = \mu_1^2/\left(\mu_2 - \mu_1^2\right)$ and $\beta \equiv c = (\mu_2 - {\mu_1}^2)/\mu_1$ whose mean, $\alpha\,\beta = \mu_1$, and variance, $\alpha\,\beta^2 = \mu_2 - {\mu_1}^2$, match the mean and variance of Y and that, in light of Equation (2.50), we can express $f_n(y)$ as the product of an initial gamma density approximation specified by $f_0(y)$ times a polynomial adjustment, that is,

$$f_n(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)} \sum_{j=0}^{n} \omega_j L_j \left(\alpha - 1, \frac{y}{\beta}\right) \tag{2.51}$$

where $\omega_j = \Gamma(\alpha)\,\delta_j$.

2.7.5 Algorithm for Approximating the Distribution of $Q(\mathbf{X})$

The following algorithm can be utilized to approximate the density function of the quadratic form $Q = \mathbf{X}' A \mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu},\, \Sigma)$, $\Sigma > 0$ and A is a symmetric indefinite real matrix.

1. The eigenvalues of $A\Sigma$ denoted by $\lambda_1 \geq \cdots \geq \lambda_r > 0 > \lambda_{r+\theta+1} \geq \cdots \geq \lambda_p$, and the *corresponding normalized* eigenvectors, $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p$, are determined.

2. Letting $P = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p)$, $\gamma_1, \ldots, \gamma_p$ be the eigenvalues of Σ, $\mathbf{t}_1, \ldots, \mathbf{t}_p$ be the *normalized* eigenvectors corresponding to $\gamma_1, \ldots, \gamma_p$, $T = (\mathbf{t}_1, \ldots, \mathbf{t}_p)$, $\Sigma^{-1/2} = T\,\mathcal{D}iag(\gamma_1^{-1/2}, \ldots, \gamma_p^{-1/2})\;T'$, $\boldsymbol{b} = (b_1, \cdots, b_p)' = P'\,\Sigma^{-1/2}\,\boldsymbol{\mu}$ and the U_j's be independently distributed standard normal variables, one has the decomposition $Q = \sum_{j=1}^{r} \lambda_j (U_j + b_j)^2 - \sum_{j=r+\theta+1}^{p} |\lambda_j| (U_j + b_j)^2 \equiv Q_1 - Q_2$, where $Q_1 \equiv \mathbf{W}_1' A_1 \mathbf{W}_1$, $\mathbf{W}_1 \sim \mathcal{N}_r(\boldsymbol{b_1}, I)$, $\boldsymbol{b_1} = (b_1, \ldots, b_r)'$, $A_1 = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, and $Q_2 \equiv \mathbf{W}_2' A_2 \mathbf{W}_2$, $\mathbf{W}_2 \sim \mathcal{N}_{p-r-\theta}(\boldsymbol{b_2}, I)$, $\boldsymbol{b_2} = (b_{r+\theta+1}, \ldots, b_p)'$, $A_2 = \mathcal{D}iag(|\lambda_{r+\theta+1}|, \ldots, |\lambda_p|)$. Clearly, $\mathbf{b} = \mathbf{0}$ whenever $\boldsymbol{\mu} = \mathbf{0}$ and, in that case, there is no need to determine the matrices P or T.

3. The cumulants and the moments of Q_1 and Q_2 are obtained from Equations (2.15) and (2.17), respectively.

4. Density approximants are determined for each of the positive definite quadratic forms Q_1 and Q_2 on the basis of their respective moments and denoted by $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$, $f_{Q_i}(\cdot)$ being given by Equation (2.31) for a Pearson-type density function.

5. Given $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$, the approximate density of $Q(\mathbf{X})$ is obtained from Equation (2.24) where $h_p(\cdot)$ and $h_n(\cdot)$ are respectively specified by Equation (2.26) and (2.25). When making use of Pearson's approach, $h_n(\cdot)$ and $h_p(\cdot)$ are explicitly given by (2.35) and (2.34) while (2.42) and (2.41) are to be used in the case of gamma approximations. Otherwise, numerical integration can be used.

6. A polynomial adjustment of degree d can be made as explained in Section 2.7.3, the resulting density approximation being

$$f_d(z) = \varphi(z) \sum_{j=0}^{d} \xi_j z^j \,.$$

   Additional accuracy can be attained by increasing d.

Remark 2.7.2. For a nonnegative definite quadratic form, in which case $Q(\mathbf{X}) = \mathbf{X}' A \mathbf{X}$ where $A = A'$ and $A \geq 0$, all the eigenvalues of A are nonnegative, and only the distribution of $Q_1(\mathbf{X})$ needs be approximated. This remark, of course, applies to positive definite quadratic forms.

2.7.6 Exact Density of Central Quadratic Forms When the Eigenvalues Occur in Pairs

The following result is useful for comparison purposes. Consider the following general linear combination of independently distributed central chi-square random variables;

$$Q(\mathbf{X}) = Q_1(\mathbf{X}) - Q_2(\mathbf{X}) = \sum_{i=1}^{r} \lambda_i Y_i - \sum_{j=r+\theta+1}^{p} |\lambda_j| Y_j \,, \tag{2.52}$$

where the Y_j's, $j = 1, \ldots, p$, are independently distributed central chi-square random variables, each having one degree of freedom. Suppose that the eigenvalues occur in pairs in the right-hand side of Equation (2.52). Then, $Q(\mathbf{X})$ can be expressed as

$$Q(\mathbf{X}) = \sum_{i=1}^{s} \lambda_i' \, T_i - \sum_{j=s+1}^{t} |\lambda_j'| \, T_j \,, \tag{2.53}$$

where $s = r/2$, $t = p/2$, $\lambda_k' = \lambda_{k/2}$, $k = 1, \ldots, t$, and the T_i's and T_j's are independently distributed chi-square random variables, each having *two* degrees of freedom. Imhof (1961) derived the following representation of the exact density function of $Q(\mathbf{X})$:

$$\psi(q) = \begin{cases} \sum_{j=1}^{s} \dfrac{\lambda_j'^{\,t-2}\, e^{-q/(2\lambda_j')}}{2\left(\prod_{k=1,k\neq j}^{s}(\lambda_j'-\lambda_k')\right)\left(\prod_{k=s+1}^{t}(|\lambda_j'|+|\lambda_k'|)\right)} \,, & q \geq 0 \\[2ex] \sum_{j=s+1}^{t} \dfrac{|\lambda_j'|^{t-2}\, e^{q/(2|\lambda_j'|)}}{2\left(\prod_{k=s+1,k\neq j}^{t}(|\lambda_j'|-|\lambda_k'|)\right)\left(\prod_{k=1}^{s}(\lambda_j'+\lambda_k')\right)} \,, & q < 0. \end{cases} \tag{2.54}$$

2.7.7 Numerical Examples

Four numerical examples are presented in this section. The first example involves a positive definite central quadratic form whose exact density is compared to various approximations. Secondly, we consider the case of a central indefinite quadratic form. The third example involves a noncentral indefinite quadratic form and the last one, which is the most general, involves a noncentral singular quadratic form.

Example 2.7.1. We first consider the case of a positive definite central quadratic form in independently distributed standard normal variables, which, according to Representation 2.3.1, can be expressed as

$$Q_1(\mathbf{X}) = \mathbf{X}'A\mathbf{X} = \sum_{j=1}^{r} \lambda_j Y_j \,, \tag{2.55}$$

Table 2.1: Four approximations to the distribution function of $Q_1(\mathbf{X})$ evaluated at certain exact percentage points (Exact %).

CDF	Exact %	Gamma	Ge.G.	Ge.S.G.	Pear.
0.0001	1.2626	0.556672	1.20013	2.364263	4.990738
0.0010	2.3608	1.358368	2.25234	3.134820	5.267700
0.01	4.6406	3.42151	4.49223	4.99449	6.29310
0.05	7.9534	6.85298	7.82310	8.01495	8.52796
0.10	10.388	9.50466	10.2952	10.3562	10.5203
0.50	24.421	24.8204	24.5012	24.4035	24.1541
0.90	51.182	51.6342	51.1048	51.2234	51.5235
0.95	61.874	61.6360	61.7067	61.8650	62.2407
0.99	86.268	83.4670	86.1000	86.1370	86.1563
0.9990	120.88	112.890	121.560	120.850	119.120
0.9999	155.40	141.202	158.201	156.100	151.301

where $A > 0$, $\mathbf{X} \sim \mathcal{N}_p(\mathbf{0}, I)$, λ_j, $j = 1, \ldots, r$, are the positive eigenvalues of A, the Y_j's, $j = 1, \ldots, r$ are independently distributed central chi-square random variables, each having one degree of freedom.

Let $r = 8$ and $\lambda_1 = \lambda_2 = 1.2$, $\lambda_3 = \lambda_4 = 1.45$, $\lambda_5 = \lambda_6 = 4$, and $\lambda_7 = \lambda_8 = 7.5$. Since the eigenvalues occur in pairs, the exact density function can be determined from the positive part of Equation (2.54) wherein $\lambda'_k = \lambda_{k/2}$, $s = t = r/2$, $\rho = 0$ and an empty product is interpreted as 1. In Table 2.1, we compare certain quantiles determined from the exact distribution of $Q_1(\mathbf{X})$ with those obtained from various approximate distributions, namely, the gamma, generalized gamma (Ge.G), generalized shifted gamma (Ge.S.G.) and Pearson-type (Pear.) as defined in (2.36), (2.37), (2.40), and (2.31), respectively. In this case, no polynomial adjustments were made. The most accurate approximation is highlighted for each value of the cdf being considered.

As can be seen from Table 2.1, the approximations obtained by means of the generalized shifted gamma distribution are generally more accurate. (A shaded background designates the most accurate approximation in a given row of the table.) Certain extreme tail quantiles determined from the exact distribution function of $Q_1(\mathbf{X})$ and the approximated distributions are presented in Table 2.1 as well. In this case, the generalized gamma is more accurate for extreme lower quantiles while, for higher quantiles, the generalized shifted gamma provides more accurate quantiles.

We now refine our approximations with polynomial adjustments of degree 10. The results are presented in Table 2.2 for many quantiles of interest. Table 2.2 indicates that, in this case, the generalized gamma distribution is more accurate than the other

Table 2.2: Four polynomially-adjusted approximations to the distribution function of $Q_1(\mathbf{X})$ evaluated at certain exact percentage points (Exact %).

CDF	Exact %	G.P.	Ge.G.P.	Ge.S.G.P	Pear.P.
0.0001	1.2626	1.214325	1.245530	2.335322	4.960818
0.0010	2.3608	2.295100	2.328056	3.076903	5.162420
0.01	4.6406	4.59770	4.60548	4.89275	5.94111
0.05	7.9534	7.95705	7.94169	7.90754	7.84305
0.10	10.388	10.4089	10.3928	10.2809	9.81260
0.50	24.421	24.3937	24.4129	24.4999	24.8680
0.90	51.182	51.1884	51.2178	51.0612	50.7508
0.95	61.874	61.7905	61.9075	61.8875	62.6298
0.99	86.268	86.4170	86.1600	86.5220	86.0780
0.9990	120.88	120.480	121.140	119.810	119.370
0.9999	155.40	156.002	155.702	158.101	162.500

distributions under consideration, even for extreme lower and higher percentage points.

Figures 2.1 and 2.2 clearly show that the gamma, generalized gamma and generalized shifted gamma densities provide close approximations throughout the range of distribution. Figure 2.2 suggests that Pearson's approximation is not as accurate for $0 \le x < 30$. The corresponding cumulative distribution functions are plotted in Figures 2.3 and 2.4.

Example 2.7.2. Consider the following general linear combination of independently distributed central chi-square random variables;

$$Q_2(\mathbf{X}) = \sum_{i=1}^{s} \lambda_i' T_i - \sum_{j=s+1}^{t} |\lambda_j'| \, T_j \, ,$$

where $s = 6$, $t = 10$, $\lambda_k' = \lambda_{k/2}$, $k = 1, \ldots, 10$, the T_i's and T_j's are independently distributed chi-square random variables, each having *two* degrees of freedom and $\lambda_1 = \lambda_2 = 23.1$, $\lambda_3 = \lambda_4 = 4.5$, $\lambda_5 = \lambda_6 = 6.8$, $\lambda_7 = \lambda_8 = 8.13$, $\lambda_9 = \lambda_{10} = 10.3$, $\lambda_{11} = \lambda_{12} = 20.1$, $\lambda_{13} = \lambda_{14} = -3.4$, $\lambda_{15} = \lambda_{16} = -12.4$, $\lambda_{17} = \lambda_{18} = -2$ and $\lambda_{19} = \lambda_{20} = -1.3$.

Since the eigenvalues occur in pairs; the exact density of $Q_2(\mathbf{X})$ can be determined from Equation (2.54). In this example, we compare the exact density and distribution functions of $Q_2(\mathbf{X})$ with various approximations. Exact and approximate percentiles are listed in Tables 2.3 and 2.4, polynomial adjustments of degree 10 being used in the latter.

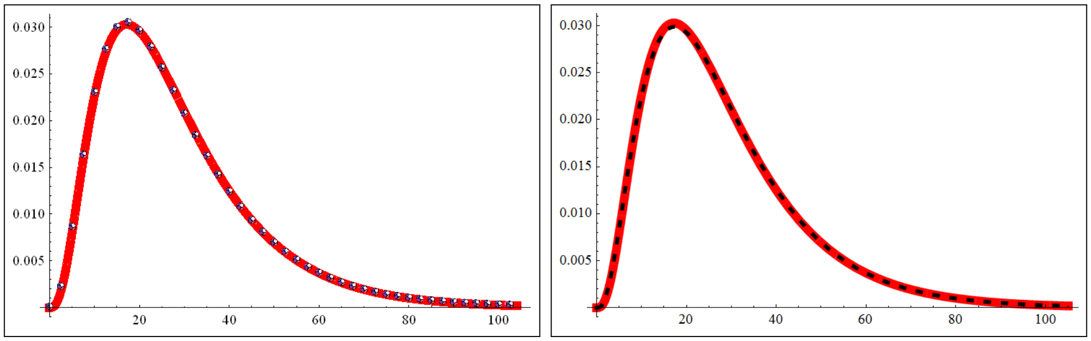


Figure 2.1: *Exact density (light solid line), gamma pdf approximation (left) and generalized gamma pdf approximation (right)*

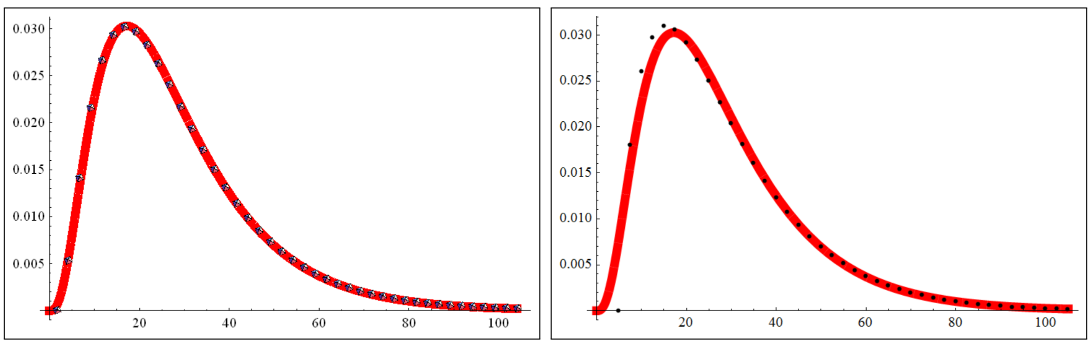


Figure 2.2: *Exact density (light solid line), generalized shifted gamma pdf approximation (left) and Pearson's pdf approximation (right)*

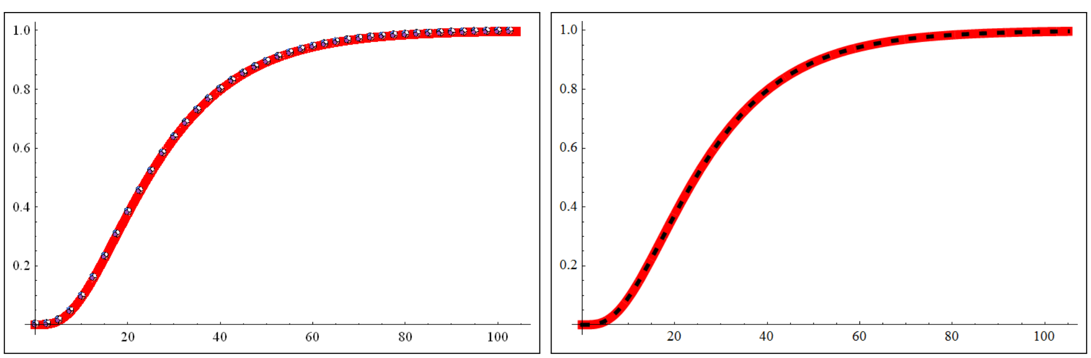


Figure 2.3: *Exact cdf (light solid line), gamma cdf approximation (left) and generalized gamma cdf approximation (right)*

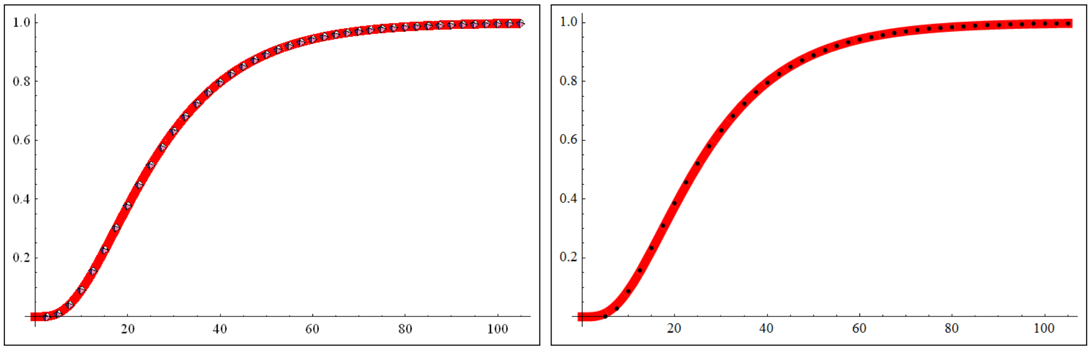


Figure 2.4: *Exact cdf (light solid line), generalized shifted gamma cdf approximation (left) and Pearson's cdf approximation (right)*

Table 2.3: Four approximations to the distribution function of $Q_2(\mathbf{X})$ evaluated at certain exact percentage points (Exact %).

CDF	Exact %	Gamma	Ge.G.	Ge.S.G.	Pear.
0.0001	−147.47	0.000040090	0.00012745	0.00010296	0.000080655
0.0010	−90.366	0.000689575	0.00104074	0.00098547	0.000895522
0.01	−33.257	0.010198	0.00981096	0.009887	0.0095731
0.05	7.0176	0.055784	0.0499524	0.049864	0.0482842
0.10	25.734	0.108681	0.100281	0.100013	0.0981895
0.50	98.008	0.494008	0.499698	0.500128	0.503077
0.90	203.27	0.898124	0.900115	0.899893	0.898534
0.95	241.73	0.950857	0.950186	0.950052	0.949254
0.99	325.86	0.991558	0.990045	0.990057	0.990126
0.9990	440.25	0.999399	0.998977	0.998997	0.999104
0.9999	551.20	0.999961	0.999889	0.999895	0.999922

Table 2.4: Four polynomially-adjusted approximations to the distribution function of $Q_2(\mathbf{X})$ evaluated at certain exact percentage points (Exact %).

CDF	Exact %	G.P.	Ge.G.P.	Ge.S.G.P	Pear.P.
0.0001	−147.47	0.00009764	0.00011283	0.00009848	0.00013783
0.0010	−90.366	0.00100590	0.00097822	0.00097477	0.00090439
0.01	−33.257	0.009993	0.010023	0.010031	0.010036
0.05	7.0176	0.050010	0.050009	0.049983	0.050146
0.10	25.734	0.099996	0.100029	0.100153	0.101126
0.50	98.008	0.500075	0.499928	0.499725	0.498507
0.90	203.27	0.899967	0.900005	0.899984	0.899103
0.95	241.73	0.950023	0.949979	0.949858	0.948979
0.99	325.86	0.989989	0.990005	0.990030	0.990473
0.9990	440.25	0.998996	0.999003	0.999004	0.998887
0.9999	551.20	0.999901	0.999899	0.999894	0.999907

The results included in Table 2.3 indicate that the approximations obtained from the generalized shifted gamma distribution are more accurate when enhanced with polynomial adjustments. The results presented in Table 2.4 show that after making a polynomial adjustment, the generalized gamma distribution is more accurate, even for extreme higher percentage points.

Figures 2.5 and 2.6 indicate that all the density approximations closely follow the exact density. In Figures 2.7 and 2.8, the cumulative distribution functions of the various approximations are superimposed on the exact distribution function. Again, close agreement is observed. The tables prove more informative as to which approximation is more accurate.

Example 2.7.3. Consider the noncentral indefinite quadratic form, $Q_3(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$, where $\mathbf{X} \sim \mathcal{N}_4(\boldsymbol{\mu}, \Sigma)$,

$$A = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 2 & 8 & 0 & 4 \\ 2 & 0 & -1/4 & 1 \\ 5 & 4 & 1 & -2 \end{pmatrix},$$

$\boldsymbol{\mu} = (1, 2, 3, 4)'$ and

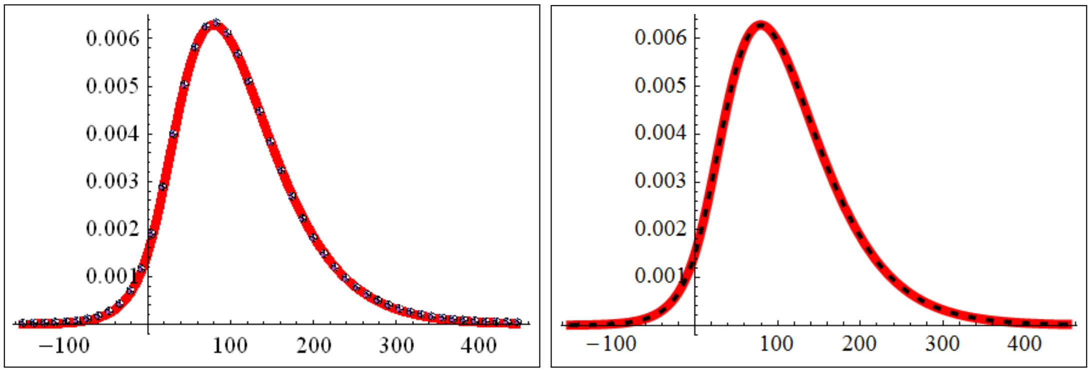


Figure 2.5: *Exact density (light solid line), gamma pdf approximation (left) and generalized gamma pdf approximation (right)*

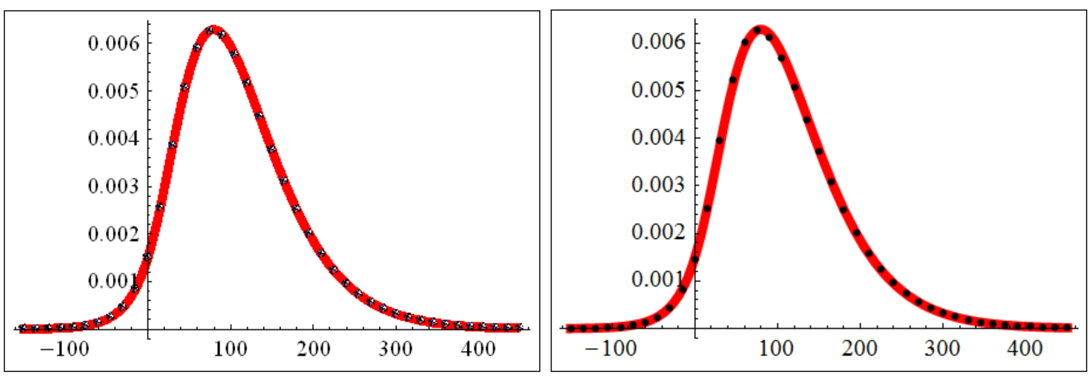


Figure 2.6: *Exact density (light solid line), generalized shifted gamma pdf approximation (left) and Pearson's pdf approximation (right)*

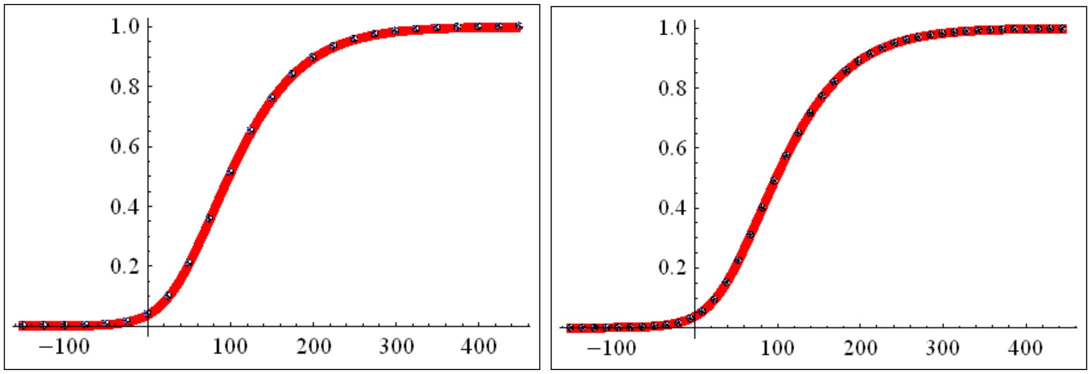


Figure 2.7: *Exact cdf (light solid line), gamma cdf approximation (left) and generalized gamma cdf approximation (right)*

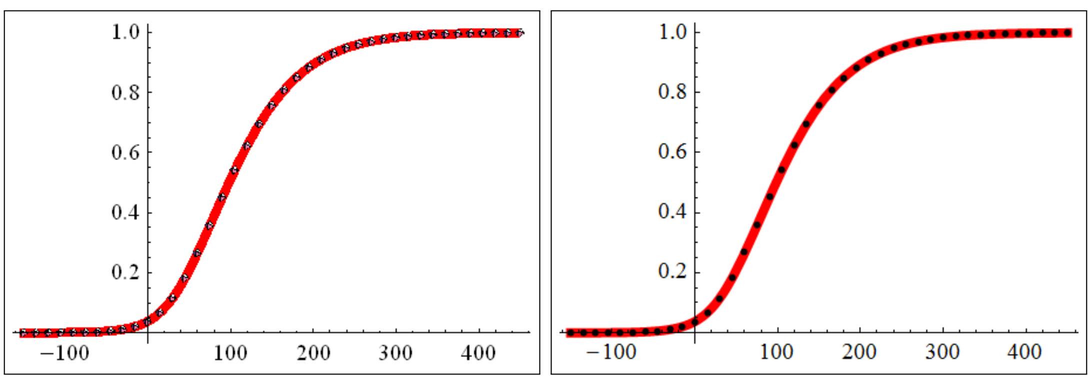


Figure 2.8: *Exact cdf (light solid line), generalized shifted gamma cdf approximation (left) and Pearson's cdf approximation (right)*

$$\Sigma = \begin{pmatrix} 1 & -1/2 & 2/5 & 1/2 \\ -1/2 & 1 & 1/4 & -3/8 \\ 2/5 & 1/4 & 1 & 1/3 \\ 1/2 & -3/8 & 1/3 & 1 \end{pmatrix}.$$

In light of Equation (2.9), $Q_3(\mathbf{X})$ can be re-expressed as

$$Q_3(\mathbf{X}) = Q^I(\mathbf{X}) - Q^{II}(\mathbf{X}) = \sum_{i=1}^{2} \lambda_i (U_i + b_i)^2 - \sum_{j=3}^{4} |\lambda_j| (U_j + b_j)^2 \tag{2.56}$$

where the U_i's, $i = 1, 2, 3, 4$, are standard normal random variables, $\lambda_1 = 8.29749$, $\lambda_2 = 4.61802$, $\lambda_3 = -3.25405$, $\lambda_4 = -0.644806$, $b_1 = 2.13221$, $b_2 = 0.519464$, $b_3 = -1.67346$, and $b_4 = -2.52353$. In this case, the matrices $\Sigma^{1/2}$ and P are respectively

$$\Sigma^{1/2} = \begin{pmatrix} 0.90931 & -0.27212 & 0.22259 & 0.22264 \\ -0.27212 & 0.92651 & 0.18280 & -0.18472 \\ 0.22259 & 0.18280 & 0.94269 & 0.16846 \\ 0.22264 & -0.18472 & 0.16846 & 0.94230 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0.59391 & 0.35170 & 0.53923 & 0.48251 \\ -0.39961 & 0.90875 & -0.11103 & -0.04643 \\ 0.47283 & 0.17968 & 0.12569 & -0.85343 \\ 0.51382 & 0.13490 & -0.82529 & 0.19153 \end{pmatrix}.$$

The approximate density functions of $Q^I(\mathbf{X})$ and $Q^{II}(\mathbf{X})$ were obtained by making use of Pearson's approximation, as well as the gamma and generalized gamma approximations. The resulting approximations to the density of $Q_3(\mathbf{X})$, as evaluated from steps 4 and 5 (Section 2.7.5) of the proposed algorithm, are plotted in Figure 2.10 (left panel). The cumulative distribution functions were determined by making use of the last step of the algorithm described in Section 2.7.5. They are respectively plotted in Figures 2.9 and 2.10 (right panel) where they are superimposed on the simulated distribution function which was determined on the basis of 1,000,000 replications.

Example 2.7.4. Consider the quadratic form, $Q_4(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$, in the singular normal vector $\mathbf{X} \sim \mathcal{N}_5(\boldsymbol{\mu}, \Sigma)$ where

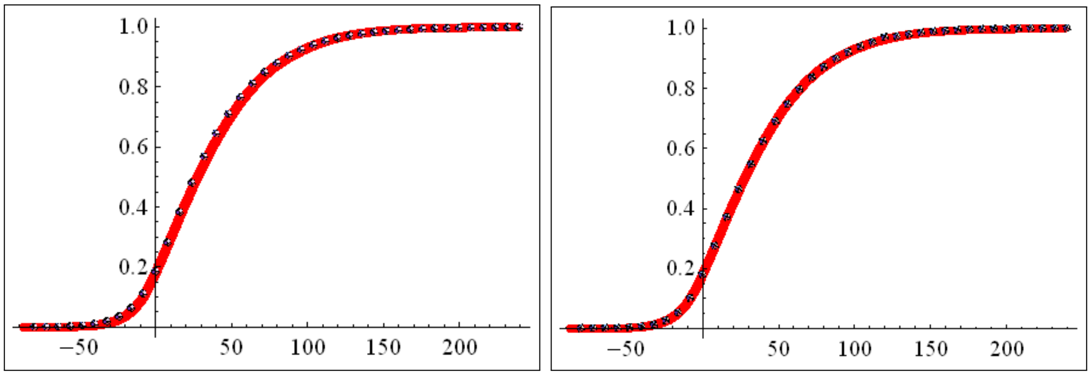


Figure 2.9: *Simulated cdf (light solid line), gamma cdf approximation (left) and generalized gamma cdf approximation (right)*

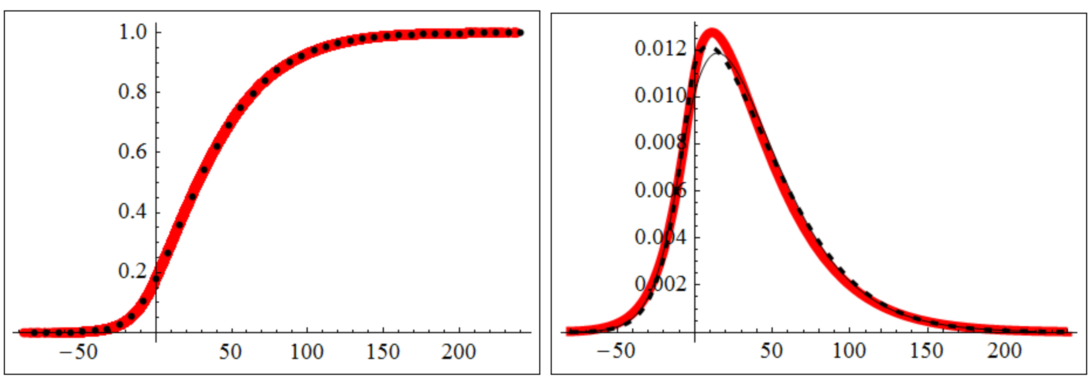


Figure 2.10: *Simulated cdf (light solid line) and Pearson's cdf approximation (right). Three Density Approximants: Gamma (light solid line), Generalized Gamma (dashed line) and Pearson's (dark solid line) (left)*

$$A = \begin{pmatrix} 1 & 4 & 3 & 1 & 3 \\ 4 & 4 & 1 & 2 & 1 \\ 3 & 1 & 3 & 3 & 2 \\ 1 & 2 & 3 & 1 & 5 \\ 3 & 1 & 2 & 5 & 2 \end{pmatrix},$$

$\boldsymbol{\mu} = (1, 0, 1, -1)'$ and

$$\Sigma = \begin{pmatrix} 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 5 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

On making use of the representation given in Section 2.5, it was determined that B and P are respectively

$$B = \begin{pmatrix} 1.66591 & 0.39015 & 0 & -0.26930 \\ 1.66591 & 0.39015 & 0 & -0.26930 \\ 2.03287 & -0.92672 & 0 & 0.09291 \\ 1.18171 & 0.49418 & 0 & 0.59945 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} -0.98651 & 0.08692 & -0.07525 & 0.11651 \\ -0.04021 & 0.49908 & -0.20028 & -0.84213 \\ -0.15866 & -0.67263 & 0.50984 & -0.51231 \\ -0.00168 & 0.53938 & 0.83324 & 0.12157 \end{pmatrix},$$

that the eigenvalues of $B'AB$ are $\lambda_1 = 106.028$, $\lambda_2 = -3.45476$, $\lambda_3 = 2.13033$, $\lambda_4 = 1.29687$, and $b_1 = -61.1512$, $b_2 = -3.99144$, $b_3 = 2.57186$ and $b_4 = 3.31448$.

The approximate density functions of $Q^I(\mathbf{X})$ and $Q^{II}(\mathbf{X})$ were obtained from the gamma, generalized gamma approximations and the generalized shifted gamma approximations. For comparison purposes, the distribution was determined on the basis of 1,000,000 replications. The resulting approximated cdf's of $Q_4(\mathbf{X})$, as evaluated from Step 4 and 5 of the algorithm described in Section 2.7.5 are presented in Tables of 2.5 to 2.7.

In Table 2.5, the approximations are determined without polynomial adjustments. The results show that for the extreme lower percentage points, the generalized gamma

Table 2.5: Four approximations to the distribution of $Q_4(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation.

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.0001	−56.232	0.00020343	0.00010682	0.00003158
0.0010	−36.489	0.00159707	0.00121260	0.00035982
0.01	−15.281	0.015662	0.014260	0.004442
0.05	0.0292	0.104459	0.090615	0.025754
0.10	7.0079	0.168562	0.152575	0.057528
0.50	73.634	0.478706	0.474164	0.494155
0.90	388.74	0.898058	0.900882	0.899273
0.95	540.29	0.950187	0.951229	0.949404
0.99	904.36	0.990558	0.990163	0.989841
0.9990	1433.5	0.999097	0.998875	0.998910
0.9999	1930.8	0.999760	0.999844	0.999942

Table 2.6: Four approximations to the distribution of $Q_4(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation.

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P
0.0001	−56.232	0.00000890	0.00009990	0.00002528
0.0010	−36.489	0.00113033	0.00106195	0.00038488
0.01	−15.281	0.012767	0.012474	0.004358
0.05	0.0292	0.081759	0.079405	0.026191
0.10	7.0079	0.135374	0.135192	0.057579
0.50	73.634	0.463551	0.465501	0.494400
0.90	388.74	0.898585	0.901356	0.899710
0.95	540.29	0.944821	0.946199	0.949768
0.99	904.36	0.991395	0.990308	0.989689
0.9990	1433.5	0.998585	0.999015	0.998977
0.9999	1930.8	0.999733	0.999909	0.999822

Table 2.7: Two approximations with and without polynomial adjustment ($d = 10$) to the distribution of $Q_4(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation.

CDF	Simul. %	Ge.G	Ge.G.P.	Ge.S.G.	Ge.S.G.P.
0.0001	−56.232	0.00010682	0.00009990	0.00003158	0.00002528
0.0010	−36.489	0.00121260	0.00106195	0.00036000	0.00038488
0.01	−15.281	0.014260	0.012474	0.004442	0.00435772
0.05	0.0292	0.090615	0.079405	0.025754	0.026191
0.10	7.0079	0.152575	0.135192	0.057528	0.0575786
0.50	73.634	0.474164	0.465501	0.494155	0.494400
0.90	388.74	0.900882	0.901356	0.899273	0.899710
0.95	540.29	0.951229	0.946199	0.949404	0.949768
0.99	904.36	0.990163	0.990308	0.989841	0.989689
0.9990	1433.5	0.998875	0.999015	0.99891	0.998977
0.9999	1930.8	0.999844	0.999733	0.999942	0.999822

provides accurate approximations but that for cdf's exceeding 0.1, the generalized shifted gamma is clearly more accurate in the majority of the cases.

The approximations that are adjusted with polynomials of degree 10 are presented in Table 2.6. The results indicate that for the extreme lower and higher points, the generalized gamma approximation is more accurate than the other approximations. Moreover, the generalized gamma approximations are more accurate than the other approximations at certain percentage points exceeding 0.1. Table 2.7 includes approximate percentiles obtained from the generalized gamma and generalized shifted gamma distributions with and without polynomial adjustments. The results show that the polynomially-adjusted generalized shifted gamma and that generalized shifted gamma are more accurate in a majority of cases. The polynomially-adjusted generalized gamma approximation is more accurate for extreme lower percentage points.

Figures 2.6 and 2.6 show plots of the gamma, generalized gamma and the generalized shifted gamma approximations superimposed on the simulated distribution function, which was determined on the basis of 1,000,000 replications.

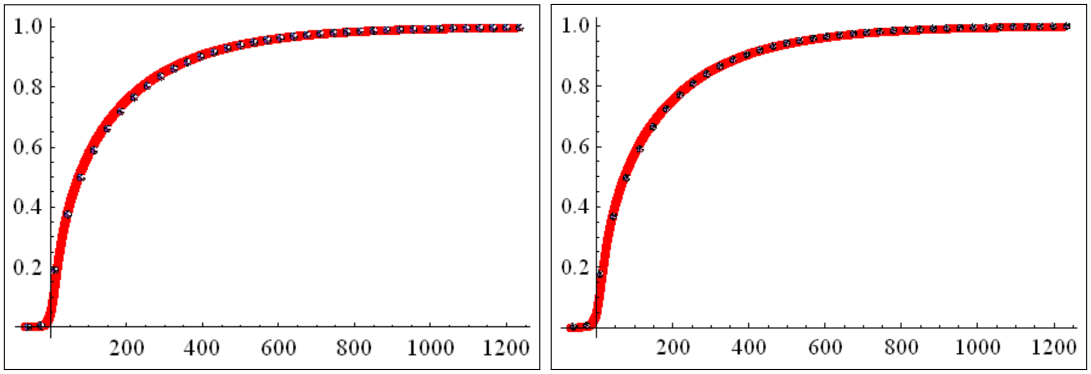


Figure 2.11: *Simulated cdf (light solid line), gamma cdf approximation (left) and generalized gamma (right)*

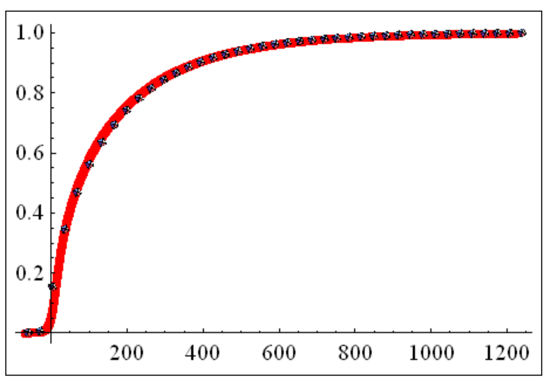


Figure 2.12: *Simulated cdf (light solid line) and generalized shifted gamma cdf approximation*

2.8 Approximating the Distribution of Quadratic Expressions

Quadratic expressions are represented as the difference of two positive definite quadratic forms plus a linear combination of normal random variables in Equations (2.8), (2.9), (2.20) or (2.21).

Consider the case of a singular quadratic expression $Q^*(\mathbf{X})$, which is decomposed as in Equation (2.20) into $Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)+T_1$, where the approximate density function of $Q=Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)$ is as given in Equation (2.43) and $T_1=(2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j+\kappa_1)\sim\mathcal{N}(\kappa_1\,,\,4\sum_{j=r_1+1}^{r_1+\theta} n_j^2)$ with $\kappa_1=\Big(c_1-\sum_{j=1}^{r_1} n_j^2/\lambda_j\,-\sum_{j=r_1+\theta+1}^{r} n_j^2/\lambda_j\Big)$, T_1 being distributed independently of $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$. In this case, the density function of T_1 is $\eta(t)=\big(1\big/\big(\sqrt{2\pi}\sigma\big)\big)\;e^{-(t-\kappa_1)^2/\left(2\sigma^2\right)}$ where $\sigma^2=4\sum_{j=r_1+1}^{r_1+\theta} n_j^2$. Then, the approximate density function of $V=Q+T_1$ is

$$\begin{aligned}
g(v) &= \int_{-\infty}^{\infty} g_{V,U}(v\,,\,u)\,\mathrm{d}u\\
&= \int_{-\infty}^{\infty} h_Q(v-u)\eta(u)\,\mathrm{d}u\\
&= \int_{-\infty}^{\infty}\Big(h_N(v-u)\,\mathcal{I}_{(-\infty,0)}(v-u)\eta(u)+\\
&\qquad\qquad h_P(v-u)\,\mathcal{I}_{(0,\infty)}(v-u)\eta(u)\Big)\,\mathrm{d}u\\
&= \int_{-\infty}^{v} h_N(v-u)\,\eta(u)\,\mathrm{d}u+\int_{v}^{\infty} h_P(v-u)\,\eta(u)\,\mathrm{d}u\\
&\equiv g_n(v)\;+g_p(v)
\end{aligned}\tag{2.57}$$

where

$$\begin{aligned}
g_n(v) &= \int_{-\infty}^{v} h_N(v-u)\,\eta(u)\,\mathrm{d}u\\
&= \int_{-\infty}^{0}\sum_{k=0}^{\infty}\left(\exp\left\{-\frac{(u-\kappa_1)^2}{2\sigma^2}-\frac{u}{\beta_2}+\frac{v}{\beta_2}\right\}\beta_1^{\alpha_2-2}\beta_2^{\alpha_1-2}b^{-a+1}\left(\zeta(u-v)\right)^{k-1}\right.\\
&\quad\times\Big(\beta_1\beta_2\Gamma(\alpha_1)^2\Gamma(k-\alpha_2+1)\Gamma(-a+1)\Gamma(-a+2)\Gamma(k+a)
\end{aligned}$$

$$
\begin{aligned}
&\times (\zeta\,(u-v))^{a} + (u-v)\,b\,\Gamma(k+\alpha_1)\,\Gamma(1-\alpha_2)^2\,\Gamma(k-a+2) \\
&\times\Gamma(a-1)\Gamma(a)\Big) \Big/ \Big((\sqrt{2\pi}\sigma k!\Gamma(\alpha_1)^2\Gamma(1-\alpha_2)^2\Gamma(k-a+2) \\
&\qquad\qquad\qquad\qquad \times\Gamma(\alpha_2)\Gamma(k+a))\Big)\Bigg)\mathrm{d}u \\
= &\sum_{k=0}^{\infty}\Bigg(\frac{1}{\sqrt{\pi}k!\Gamma(\alpha_2)}2^{\frac{k}{2}-2}e^{-(v-\kappa_1)^2/(2\sigma^2)}\beta_1^{\alpha_2}\zeta^k\beta_2^{\alpha_1-2}b^{-a}\,\sigma^{k-2} \\
&\times\bigg(\frac{1}{\Gamma(1-\alpha_2)^2\Gamma(k-a+2)}2^{\frac{a}{2}}\beta_2\Gamma(k-\alpha_2+1)\Gamma(-a+1) \\
&\times\Gamma(-a+2)\bigg(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{1}{2}(k+a)\right)\,{}_1F_1\left(\frac{1}{2}(k+a);\frac{1}{2};\gamma\right) \\
&-2\left(\sigma^2+v\beta_2-\beta_2\,\kappa_1\right)\Gamma\left(\frac{1}{2}(k+a+1)\right)\,{}_1F_1\Big(\frac{1}{2}(k+a+1); \\
&\times\frac{3}{2};\gamma\Big)\bigg)(\zeta\sigma)^a + \frac{1}{\beta_1\Gamma(\alpha_1)^2\Gamma(k+a)}\sqrt{2}\beta_2\sigma\Gamma(k+\alpha_1)\Gamma(a-1) \\
&\times\Gamma(a)\left(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\,{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\gamma\right)\right. \\
&\left.-2\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right)\,{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\gamma\right)\right) \\
&+\frac{1}{\Gamma(\alpha_1)^2\Gamma(k+a)}2\sigma\Gamma(k+\alpha_1)\Gamma(a-1)\Gamma(a) \\
&\times\left(\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\,{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\gamma\right)\right. \\
&\left.\left.-\sqrt{2}\left(\sigma^2+v\beta_2-\beta_2\,\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right)\,{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\gamma\right)\right)\bigg)\Bigg). \qquad (2.58)
\end{aligned}
$$

and

$$
\begin{aligned}
g_p(v) &= \int_v^{\infty} h_P(v-u)\,\eta(t)\,\mathrm{d}u \\
&= \int_v^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2\pi}\sigma k!\Gamma(\alpha_1)}\exp\left\{\frac{v-u}{\beta_2}-\frac{(u-\kappa_1)^2}{2\sigma^2}\right\}(v-u)^{\alpha_2-1}\beta_1^{-\alpha_1}\beta_2^{-\alpha_2}\right. \\
&\times(\zeta(u-v))^k\left(\frac{\Gamma(k+\alpha_1)\Gamma(-a+1)\Gamma(a)(v-u)^{\alpha_1}}{\Gamma(1-\alpha_1)\Gamma(\alpha_1)\Gamma(k+a)}\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\left(\frac{\left(\frac{1}{\zeta}\right)^{a-1}\Gamma(k-\alpha_2+1)\Gamma(-a+2)\Gamma(a-1)(v-u)^{1-\alpha_2}}{\Gamma(1-\alpha2)\Gamma(k-a+2)\Gamma(\alpha2)}\right)\Bigg)\mathrm{d}u \\
= & \sum_{k=0}^{\infty}\left\{\frac{\frac{1}{\sqrt{\pi}\sigma^2 k!\Gamma(\alpha_1)^2}2^{k/2-2}e^{-(v-\kappa_1)^2/(2\sigma^2)}\beta_1^{-\alpha_1}\beta_2^{-\alpha_2-2}}{\Gamma(1-\alpha_2)\Gamma(k-a+2)\Gamma(\alpha_2)}\right. \\
& \times\left[\frac{1}{\Gamma(1-\alpha_1)\Gamma(k+a)}2^{\frac{a}{2}}\beta_2\sigma^{k+a}\Gamma(k+\alpha_1)\Gamma(-a+1)\Gamma(a)\right. \\
& \times\left(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{1}{2}(k+a)\right)\,{}_1F_1\left(\frac{1}{2}(k+a);\frac{1}{2};\gamma\right)\right. \\
& +2\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{1}{2}(k+a+1)\right) \\
& \times{}_1F_1\left(\frac{1}{2}(k+a+1);\frac{3}{2};\gamma\right)(-\zeta)^k+\left(2\beta_2\left(\frac{1}{\zeta}\right)^{a-1}\sigma\left(\zeta\sigma\right)^k\right. \\
& \times\Gamma(\alpha_1)\Gamma(k-\alpha_2+1)\Gamma(-a+2)\Gamma(a-1)\left(\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\right. \\
& \times{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\gamma\right)+\sqrt{2}\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right) \\
& \left.\left.\left.\left.\left.\times{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\gamma\right)\right)\right)\right)\right]\right\}
\end{aligned}
\tag{2.59}
$$

where $a=\alpha_1+\alpha_2$, $b=\beta_1+\beta_2$, $\zeta=(\beta_1+\beta_2)/(\beta_1\beta_2)$, $\gamma=(\sigma^2+v\beta_2-\beta_2\kappa_1)^2/(2\beta^2\sigma^2)$ and $1-\alpha_1$ and $1-\alpha_2$ are not zero or negative integer and $a\neq 3,4,\ldots$.

2.8.1 Algorithm for Approximating the Distribution of $Q^*(\mathbf{X})$

The following algorithm can be utilized to approximate the density function of the quadratic expression $Q^*(\mathbf{X})=\mathbf{X}'A\mathbf{X}+\mathbf{a}'\mathbf{X}+d$ where $\mathbf{X}\sim\mathcal{N}_p(\boldsymbol{\mu},\,\Sigma)$, $\Sigma\geq 0$, A is an indefinite symmetric real matrix, $\mathbf{a}$ is a p-dimensional constant vector and d is a scalar constant. When Σ is a singular matrix, the symmetric square root does not exist. In this case, we make use of the spectral decomposition theorem to express Σ as UWU' where W is a diagonal matrix whose first r diagonal elements are positive, the remaining diagonal elements being equal to zero. Next, we let $B^*_{p\times p}=UW^{1/2}$ and remove the $p-r$ last columns of B^*, which are null vectors, to obtain the matrix $B_{p\times r}$, and it follows that $\Sigma=BB'$.

1. The eigenvalues of $B'AB$ denoted by $\lambda_1\geq\cdots\geq\lambda_r>\lambda_{r+1}=\cdots=\lambda_{r+\theta}=0>\lambda_{r+\theta+1}\geq\cdots\geq\lambda_p$, and the *corresponding normalized* eigenvectors, $\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_p$, are

determined; then, we let $P = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p)$.

2. In the singular case, one can decompose $Q^*(\mathbf{X})$ as $Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1$ where $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are positive definite quadratic forms with $\mathbf{W}^+ = (W_1 + n_1/\lambda_1, \ldots, W_{r_1} + n_{r_1}/\lambda_{r_1})' \sim \mathcal{N}_{r_1}(\boldsymbol{\nu}_1, I)$, $\boldsymbol{\nu}_1 = (n_1/\lambda_1, \ldots, n_{r_1}/\ \lambda_{r_1})'$, $\mathbf{W}^- = (W_{r_1+\theta+1} + n_{r_1+\theta+1}/(\lambda_{r_1+\theta+1}), \ldots, W_r + n_r/(\lambda_r))' \sim \mathcal{N}_{r-r_1-\theta}(\boldsymbol{\nu}_2\ , I)$, $\boldsymbol{\nu}_2 = (n_{r_1+\theta+1}/(\lambda_{r_1+\theta+1}), \ldots,\ n_r/(\lambda_r))'$, θ being number of null eigenvalues, $\mathbf{b}^{*'} = (b_1^*, \ldots, b_r^*) = \boldsymbol{\mu}' ABP,\ n_j = \frac{1}{2}m_j + b_j^*,\ c_1 = \boldsymbol{\mu}' A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d$ and $\mathbf{W}' = (W_1, \ldots, W_r)$. Letting $\kappa_1 = \left(c_1 - \sum_{j=1}^{r_1} n_j^2/\lambda_j - \sum_{j=r_1+\theta+1}^{r} n_j^2/\lambda_j\right)$, $T_1 = (2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1) \sim \mathcal{N}(\kappa_1\ ,\ 4\sum_{j=r_1+1}^{r_1+\theta} n_j^2)$. Clearly, $\mathbf{b}^* = \mathbf{0}$ whenever $\boldsymbol{\mu} = \mathbf{0}$ and in that case, there is no need to determine the matrix P.

3. The cumulants and the moments of Q_1 and Q_2 are obtained from Equations (2.15) and (2.17), respectively.

4. Density approximants are determined for each of the positive definite quadratic forms Q_1 and Q_2 on the basis of their respective moments and denoted by $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$.

5. Given $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$, we first approximate density of $Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-)$ by using Equation (2.24) and then, determine the density function of $Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1$ by making use of Equation (2.57) .

6. A polynomial adjustment, which improves the accuracy of the approximations, can also be applied to the density approximations determined for $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ as explained in Section 2.7.3. Then, an approximate density function for $Q^*(\mathbf{X})$ is obtained as explained in Step 5 .

Example 2.8.1. Consider the singular quadratic expression $Q^*(\mathbf{X}) = \mathbf{X}' A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ where $\mathbf{X} \sim \mathcal{N}_5(\boldsymbol{\mu},\ \Sigma)$,

$$A = \begin{pmatrix} 1 & -.9 & -1 & 0 & -5 \\ -.9 & 1 & 1 & 2 & 1 \\ -1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 3 & -1 & 0 \\ -5 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$\boldsymbol{\mu} = \mathbf{0}'$, $\mathbf{a}' = (-1, 2, 3, 1, 1)$, $d = 6$ and

$$\Sigma = \begin{pmatrix} 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 5 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrices B and P were found to be

$$B = \begin{pmatrix} 1.66591 & 0.39015 & 0 & -0.26930 \\ 1.66591 & 0.39015 & 0 & -0.26930 \\ 2.03287 & -0.92672 & 0 & 0.09291 \\ 1.18171 & 0.49418 & 0 & 0.59945 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} -0.97731 & 0.00042 & -0.14936 & -0.15022 \\ 0.05695 & -0.58347 & -0.72923 & 0.35290 \\ 0.13922 & 0.69384 & -0.66277 & -0.24484 \\ -0.14916 & 0.42208 & 0.08157 & 0.89048 \end{pmatrix},$$

respectively. The eigenvalues of $B'AB$ are $\lambda_1 = 31.2355$, $\lambda_2 = 3.80066$, $\lambda_3 = -2.92434$, $\lambda_4 = -2.51178$ and the n_i's as defined in Step 2 are $n_1 = 4.47312$, $n_2 = -0.94791$, $n_3 = 0.5$, and $n_4 = 0.304443$. Moreover, $\boldsymbol{\mu}_1 = (0.143206, -0.249407)'$, $\boldsymbol{\mu}_2 = (-0.170979, -0.121206)'$ and $c_1 = 6$.

The approximate density functions of $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are obtained by making use of the gamma, generalized gamma and the generalized shifted gamma approximations. The resulting distribution functions are evaluated at certain simulated percentiles obtained on the basis of 1,000,000 replications. The results are presented in Tables 2.8 and 2.9.

The approximations are determined without polynomial adjustments in Table 2.8. The results indicate that for cdf's lower than .05, the generalized gamma provides accurate approximations but that for cdf's higher than .05, the generalized shifted gamma is more accurate than the others approximations of the cdf in a majority of cases.

The approximations, once adjusted with polynomials of degree 10, are presented in Table 2.9. The cdf values included in Table 2.9 show that, for the extreme lower points, the generalized gamma approximation is more accurate, whereas the generalized shifted gamma approximation produces the best results for the extreme higher points. Approximations obtained from the generalized gamma are more accurate than the other

Table 2.8: Four approximations to the distribution of $Q^*(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation [$\boldsymbol{\mu} = \mathbf{0}$].

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.0001	−38.250	0.00011045	0.0000988	0.00008133
0.0010	−24.571	0.00120036	0.0010863	0.00089474
0.01	−11.369	0.011995	0.010931	0.009010
0.05	−2.1220	0.06010	0.054855	0.045251
0.10	1.8869	0.120792	0.110206	0.090960
0.50	19.792	0.501819	0.495766	0.508848
0.90	90.668	0.896101	0.900322	0.899398
0.95	126.56	0.949267	0.950921	0.949723
0.99	214.63	0.990651	0.990124	0.989887
0.9990	347.62	0.999210	0.998922	0.998983
0.9999	482.47	0.999932	0.999871	0.999895

Table 2.9: Three approximations with and without polynomial adjustment ($d = 10$) to the distribution of $Q^*(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation [$\boldsymbol{\mu} = \mathbf{0}$].

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P.
0.0001	−38.250	0.00009491	0.00009580	0.00008227
0.0010	−24.571	0.00105407	0.00104208	0.00090530
0.01	−11.369	0.010613	0.010490	0.009116
0.05	−2.1220	0.053264	0.052643	0.045791
0.10	1.8869	0.106997	0.105759	0.092028
0.50	19.792	0.486872	0.490738	0.509861
0.90	90.668	0.900936	0.902296	0.899949
0.95	126.56	0.947142	0.948977	0.951033
0.99	214.63	0.990651	0.989648	0.989506
0.9990	347.62	0.998747	0.999094	0.999043
0.9999	482.47	0.999924	0.999827	0.999906

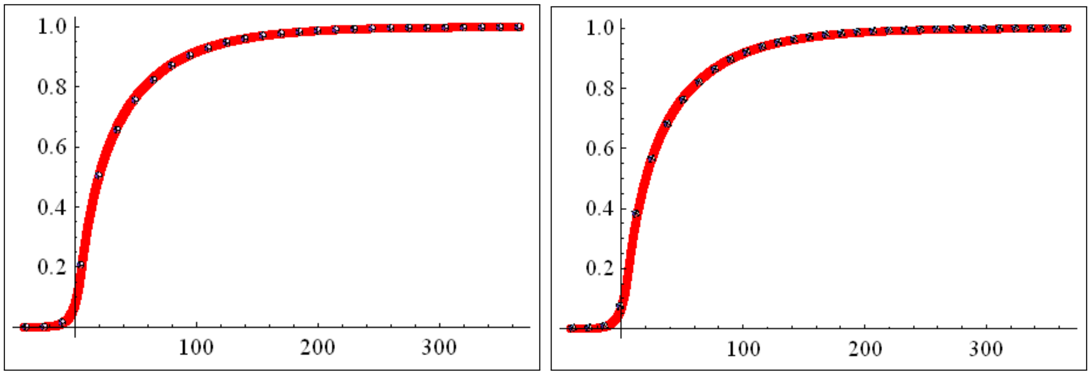


Figure 2.13: *Simulated cdf (light solid lines), Gamma cdf approximation (left) and generalized gamma cdf approximation (right) for* $Q^*(\boldsymbol{X})$ $[\boldsymbol{\mu} = \boldsymbol{0}]$.

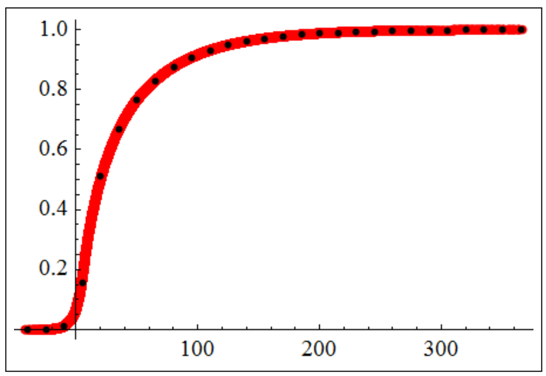


Figure 2.14: *Simulated cdf (light solid lines) and generalized shifted gamma cdf approximation for* $Q^*(\boldsymbol{X})$ $[\boldsymbol{\mu} = \boldsymbol{0}]$.

approximations for cdf's between .01 and .99. Figures 2.13 and 2.14 show that all of these three densities provide accurate approximations.

For comparison purposes, we consider the generalized gamma and the generalized shifted gamma with and without polynomial adjustments in Table 2.10. This table shows that for cdf's larger than .95, accurate results are obtained from the generalized shifted gamma whereas, for extreme lower points, the polynomially-adjusted generalized gamma provides more precision. This table also indicates that polynomial adjustments do not improve the approximations when used in conjunction with the generalized shifted gamma as base density.

In this next example, assume that $\mathbf{X}$ is a noncentral normal vector with mean $\boldsymbol{\mu} = (100, 0, -50, 150, 5)'$ in the quadratic expression $Q^*(\mathbf{X})$ as defined in Example 2.8.1

Table 2.10: Two approximations with and without polynomial adjustment ($d = 10$) to the distribution of $Q^*(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation [$\boldsymbol{\mu} = \mathbf{0}$].

CDF	Simul. %	Ge.G	Ge.G.P.	Ge.S.G.	Ge.S.G.P.
0.0001	−38.250	0.00009877	0.00009580	0.000008133	0.00008227
0.0010	−24.571	0.00108627	0.00104208	0.00089474	0.00090530
0.01	−11.369	0.010931	0.010490	0.009010	0.009116
0.05	−2.1220	0.054855	0.052643	0.045251	0.045791
0.10	1.8869	0.110206	0.105759	0.0909599	0.0920281
0.50	19.792	0.495766	0.490738	0.508848	0.509861
0.90	90.668	0.900322	0.902296	0.899398	0.899949
0.95	126.56	0.950921	0.948977	0.949723	0.951033
0.99	214.63	0.990124	0.989648	0.989887	0.989506
0.9990	347.62	0.998922	0.999094	0.998983	0.999043
0.9999	482.47	0.999871	0.999827	0.999895	0.999906

and denote the resulting quadratic expression by $Q_1^*(\mathbf{X})$. Figures 2.15 and 2.16 show that the gamma, generalized gamma and the generalized shifted gamma all provide accurate approximations. Table 2.11 include various approximate cdf values, which were determined with and without polynomial adjustments. The results in this table indicate that the generalized shifted gamma provides accurate approximations for most points. The approximations adjusted with polynomials of degree 5, which are presented in Table 2.12, are compared with non polynomially-adjusted approximations. The results indicate that polynomially-adjusted generalized gamma approximations are more accurate than the other approximations. Polynomial adjustments do not improve the approximations when the generalized shifted gamma is being utilized as base density.

Example 2.8.2. Consider the singular quadratic expression $Q_2^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ where $\mathbf{X} \sim \mathcal{N}_5(\boldsymbol{\mu}, \Sigma)$,

$$A = \begin{pmatrix} 4 & 4 & 1 & 2 & 1 \\ 4 & 4 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 5 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

Table 2.11: Three approximations to the distribution of $Q_1^*(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation [$\boldsymbol{\mu} = (100, 0, -50,\ 150, 5)'$].

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.0001	−54663.6	0.00012171	0.00012171	0.00012167
0.0010	−53591.0	0.00109316	0.00109315	0.00109291
0.01	−52256.0	0.010282	0.010282	0.010281
0.05	−51039.4	0.050496	0.050496	0.050493
0.10	−50389.2	0.100088	0.100088	0.100083
0.50	−48053.1	0.498438	0.498439	0.498434
0.90	−45661.4	0.900235	0.900235	0.900231
0.95	−44971.4	0.950553	0.950553	0.950549
0.99	−43679.4	0.990187	0.990187	0.990184
0.9990	−42211.5	0.999040	0.999040	0.999036
0.9999	−40911.8	0.999920	0.999920	0.999918

Table 2.12: Three approximations with and without polynomial adjustment ($d = 5$) to the distribution of $Q_1^*(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation [$\boldsymbol{\mu} = (100, 0, -50, 150, 5)'$].

CDF	Simul. %	Gamma	G.P.	Ge.G.	Ge.G.P.	Ge.S.G.
0.0001	−54663.6	0.00012171	0.00010440	0.00012171	0.00010402	0.000121669
0.0010	−53591.0	0.00109316	0.00100100	0.00109315	0.00099966	0.00109291
0.01	−52256.0	0.010282	0.009906	0.010282	0.009907	0.0102805
0.05	−51039.4	0.050496	0.049888	0.050496	0.049920	0.0504927
0.10	−50389.2	0.100088	0.099617	0.100088	0.099693	0.100083
0.50	−48053.1	0.498438	0.499307	0.498439	0.499719	0.498434
0.90	−45661.4	0.900235	0.899218	0.900235	0.899959	0.900231
0.95	−44971.4	0.950553	0.949276	0.950553	0.950064	0.950549
0.99	−43679.4	0.990187	0.98904	0.990187	0.989871	0.990184
0.9990	−42211.5	0.999040	0.998159	0.999040	0.999001	0.999036
0.9999	−40911.8	0.999920	0.999115	0.999920	0.999958	0.999918

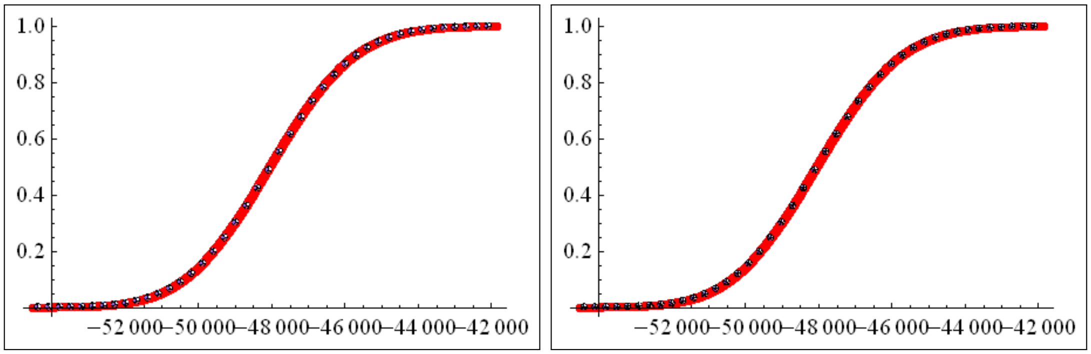


Figure 2.15: *Simulated cdf (light solid lines), Gamma cdf approximation (left) and generalized gamma cdf approximation (right) for* $Q_1^*(\boldsymbol{X})$ $[\boldsymbol{\mu} = (100, 0, -50,\ 150, 5)']$.

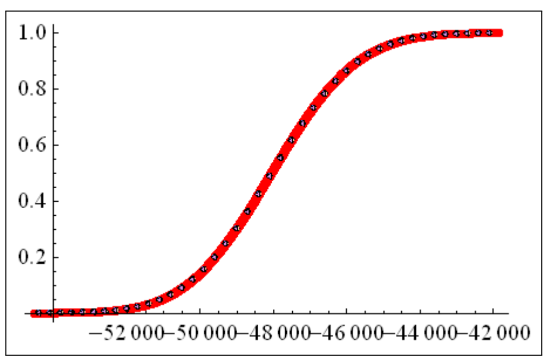


Figure 2.16: *Simulated cdf (light solid lines)and generalized shifted gamma cdf approximation for* $Q_1^*(\boldsymbol{X})$ $[\boldsymbol{\mu} = (100, 0, -50, 150, 5)']$.

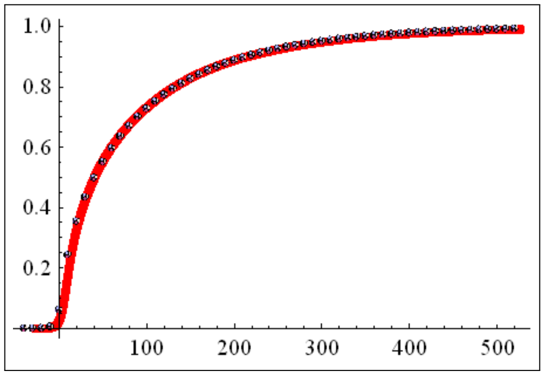


Figure 2.17: *Simulated cdf (light solid lines) and gamma cdf approximation for* $Q_2^*(\boldsymbol{X})$.

$\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{a}' = (1, 2, 3, 4, 5)$ and $d = 6$.

In this case, the matrices B and P were found to be

$$B = \begin{pmatrix} 1.66591 & 0.39015 & 0 & -0.26930 \\ 1.66591 & 0.39015 & 0 & -0.26930 \\ 2.03287 & -0.92672 & 0 & 0.09291 \\ 1.18171 & 0.49418 & 0 & 0.59945 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} -0.97731 & 0.00042 & -0.14936 & -0.15022 \\ 0.05695 & -0.58347 & -0.72923 & 0.352901 \\ 0.13922 & 0.69384 & -0.66277 & -0.24484 \\ -0.14916 & 0.42208 & 0.08157 & 0.89048 \end{pmatrix}.$$

The eigenvalues of $B'AB$ are $\lambda_1 = 76.8865$, $\lambda_2 = 0.9121$, $\lambda_3 = -0.79856$, $\lambda_4 = 0$. Figure 2.17 indicates that the gamma approximation agrees closely with the simulated cdf.

Chapter 3

The Distribution of Ratios of Quadratic Expressions in Normal Vectors

3.1 Introduction

Ratios of quadratic forms and quadratic expressions are discussed in this chapter. More specifically, ratios whose distribution can be determined from that of the difference of positive definite quadratic forms and ratios involving idempotent or positive definite matrices in their denominators are being considered. Suitable approaches are proposed for approximating their distributions. Several illustrative examples are provided, including applications to the Durbin-Watson statistic and Burg's estimator. The last section focuses on the case of ratios of quadratic expressions in singular normal vectors.

3.2 The Distribution of Ratios of Quadratic Forms

Three type of the ratios of quadratic forms are considered in this section: ratios of indefinite quadratic forms (Section 3.2.1) and ratios involving idempotent or positive definite matrices in their denominators (Sections 3.2.2 and 3.2.3, respectively).

3.2.1 The Distribution of Ratios of Indefinite Quadratic Forms

Let $R = Q_1(\mathbf{X})/Q_2(\mathbf{X}) = \mathbf{X}'A\mathbf{X}/\mathbf{X}'B\mathbf{X}$ where the matrices of A and B can be indefinite, the rank of B being at least one and let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$; then, one has

$$\begin{aligned} \Pr(R \le t_0) &= \Pr\left(\frac{\mathbf{X}'A\mathbf{X}}{\mathbf{X}'B\mathbf{X}} \le t_0\right) \\ &= \Pr\left(\mathbf{X}'A\mathbf{X} \le t_0\mathbf{X}'B\mathbf{X}\right) \\ &= \Pr\left(\mathbf{X}'(A - t_0B)\mathbf{X} \le 0\right). \end{aligned} \tag{3.1}$$

On letting $U = \mathbf{X}'(A - t_0B)\mathbf{X}$, U can be re-expressed as a difference of two positive quadratic forms as explained in Section 2.4 and the distribution function of R can be evaluated at each point t_0. This approach is illustrated by the next example which involves the Durbin-Watson statistic.

Example 3.2.1. The statistic proposed by Durbin and Watson (1950), which in fact assesses whether the disturbances in the linear regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ are uncorrelated, can be expressed as

$$D = \frac{\hat{\boldsymbol{\epsilon}}'A^*\hat{\boldsymbol{\epsilon}}}{\hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}}$$

where

$$\hat{\boldsymbol{\epsilon}} = \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}$$

is the vector of residuals, $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ being the ordinary least-squares estimator of $\boldsymbol{\beta}$, and $A^* = (a^*_{ij})$ is a symmetric tridiagonal matrix with $a^*_{11} = a^*_{pp} = 1$; $a^*_{ii} = 2$, for $i = 2, \ldots, p-1$; $a^*_{ij} = -1$ if $|i-j| = 1$; and $a^*_{ij} = 0$ if $|i-j| \ge 2$. Assuming that the error vector is normally distributed, one has $\boldsymbol{\epsilon} \sim \mathcal{N}_p(\mathbf{0}, I)$ under the null hypothesis. Then, on writing $\hat{\boldsymbol{\epsilon}}$ as $M\mathbf{Y}$ where $M_{p\times p} = I - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = M'$ is an idempotent matrix of rank $p - k$, the test statistic can be expressed as the following ratio of quadratic forms:

$$D = \frac{\mathbf{Z}'MA^*M\mathbf{Z}}{\mathbf{Z}'M\mathbf{Z}}, \tag{3.2}$$

where $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$; this can be seen from the fact that $M\mathbf{Y}$ and $M\mathbf{Z}$ are identically distributed singular normal vectors with mean vector $\mathbf{0}$ and covariance matrix MM'.

The cumulative distribution function of D at t_0 is

$$\Pr(D < t_0) = \Pr\left(\mathbf{Z}'M(A^*M - t_0I)\mathbf{Z} < 0\right) \tag{3.3}$$

where $U_1 = \mathbf{Z}'M(A^*M - t_0I)\mathbf{Z}$ is an indefinite quadratic from with $A = M(A^*M - t_0I)$, $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = I$. One can obtain the moments and the various approximations of the density functions of U_1 from Equations (2.17) and (2.24).

We make use of a data set that is provided in Hildreth and Lu (1960). In this case, there are $k = 5$ independent variables, $p = 18$, the observed value of D is 0.96, and the

Table 3.1: Three polynomially-adjusted approximations ($d = 10$) to the distribution function of D evaluated at certain percentage points (Simul. %) obtained by simulation.

CDF	Simul. %	G.P.	Ge.G.P.	Pear.P.
0.01	1.3607	0.010435	0.010420	0.010197
0.05	1.6479	0.050280	0.050277	0.050286
0.10	1.8098	0.099761	0.099770	0.100059
0.25	2.0854	0.247875	0.247909	0.248167
0.50	2.3901	0.495934	0.495953	0.496051
0.75	2.6861	0.748343	0.748288	0.749567
0.90	2.9374	0.902156	0.902100	0.901533
0.95	3.0768	0.952783	0.952788	0.952592
0.99	3.3101	0.991466	0.991457	0.991665

13 non-zero eigenvalues of $M(A^*M - t_0I)$ are those of MA^*M minus t_0. The non-zero eigenvalues of MA^*M are $3.92807, 3.82025$, 3.68089, 3.38335, 3.22043, 2.9572, 2.35303, 2.25696, 1.79483, 1.48804, 0.948635, 0.742294 and 0.378736. For instance, when $t_0 = 1.80977$, which corresponds to the 10$^{\text{th}}$ percentile of the simulated cumulative distribution functions resulting from 1,000,000 replications, the eigenvalues of the positive definite quadratic form $Q_1(\mathbf{X})$ are 2.11817, 2.01035, 1.87099, 1.57345, 1.41053, 1.14734, 0.54-313 and 0.44706, while those of $Q_2(\mathbf{X})$ are 0.01507, 0.3218, 0.86126, 1.06761 and 1.43116.

Polynomially adjusted density functions were obtained for D with gamma and generalized gamma base density functions. The corresponding cumulative distribution functions were evaluated at certain percentiles of the distribution obtained by simulation on the basis of 1,000,000 replications. The results reported in Table 3.1 suggest that the polynomially adjusted generalized gamma approximation is slightly more accurate.

3.2.2 Ratios whose Denominator Involves an Idempotent Matrix

Let $R = \mathbf{X}'A\mathbf{X}\ /\mathbf{X}'B\mathbf{X}$ where A is indefinite, B is idempotent and $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$. Then, as stated in Hannan (1970), the h^{th} moment of the ratio of such quadratic forms is equal to the ratio of their h^{th} moments. Thus, $E(R^h) = E[(\mathbf{X}'A\mathbf{X})^h]\ /E[(\mathbf{X}'B\mathbf{X})^h]$. The following example involves such a ratio.

Example 3.2.2. In Example 3.2.1, M, the matrix of the quadratic form appearing in the denominator of D as defined in (3.2), happens to be idempotent. Thus, the h^{th} moment of D can be obtained as $E(\mathbf{Z}'MA^*M\mathbf{Z})^h/E(\mathbf{Z}'M\mathbf{Z})^h$ and polynomially adjusted generalized gamma density approximants as defined in Section 2.7.2 can be directly determined

Table 3.2: Generalized gamma approximations to the distribution function of D evaluated at certain percentage points (Simul. %) obtained with (Ge.G.P) and without (Ge.G.) polynomial adjustment.

CDF	Simul. %	Ge.G.	Ge.G.P.
0.01	1.3607	0.011744	0.010365
0.05	1.6479	0.050061	0.050308
0.10	1.8098	0.097460	0.099875
0.25	2.0854	0.243139	0.247947
0.50	2.3901	0.495703	0.495807
0.75	2.6861	0.754125	0.748325
0.90	2.9374	0.905234	0.902239
0.95	3.0768	0.952770	0.952814
0.99	3.3101	0.989273	0.991458

from the exact moments of D. A polynomial adjustment of degree $d = 10$ was used. The approximate cumulative distribution function for the generalized gamma and the polynomially-adjusted generalized gamma were evaluated at certain percentiles obtained from the empirical distribution, which was generated from 1,000,000 replications. The results reported in Table 3.3 indicate that the proposed approximations are indeed very accurate.

3.2.3 Ratios whose Denominator Consists of a Positive Definite Quadratic Form

In this section, the denominators are assumed to be positive definite quadratic forms. Accordingly, letting $R = \mathbf{X}'A\mathbf{X}\ /\mathbf{X}'B\mathbf{X} \equiv Q_1/Q_2$ where A is indefinite and B is positive definite, we have the following representation of the h^{th} moments of R whenever it exists:

$$
\begin{aligned}
E(R)^h &= E\left[(\mathbf{X}'A\mathbf{X})^h(\mathbf{X}'B\mathbf{X})^{-h}\right] \\
&= E\left(Q_1^h \ \frac{1}{\Gamma(h)}\int_0^\infty y^{h-1}e^{-Q_2 y}\,\mathrm{d}y\right) \\
&= \frac{1}{\Gamma(h)}\int_0^\infty y^{h-1}E\left(Q_1^h \ \ e^{-Q_2 y}\right)\ \mathrm{d}y \\
&= \frac{1}{\Gamma(h)}\int_0^\infty y^{h-1}\frac{d^h}{ds^h}\ \ M_{Q_1,Q_2}(s,-y)\ |_{s=0}\,\mathrm{d}y \\
&= \frac{1}{\Gamma(h)}\int_0^\infty y^{h-1}\Big(\frac{d^h}{ds^h}\mid I-2sA\Sigma+2yB\Sigma\mid^{-1/2}|_{s=0}\Big)\mathrm{d}y
\end{aligned}
$$

$$= \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} \mid \Sigma^{-1} \mid^{1/2} \left(\frac{d^h}{ds^h} \mid \Sigma^{-1} - 2sA + 2yB \mid^{-1/2}|_{s=0} \right) \mathrm{d}y \qquad (3.4)$$

where $M_{Q_1,Q_2}(s, y)$ is the joint moment generating function of $Q_1(\mathbf{X})$ and $Q_2(\mathbf{X})$.

In the next example, we determine the moments of Burg's estimator and approximate its distribution.

Example 3.2.3. Burg's estimator, $\bar{\alpha}$, of the parameter α in an AR(1) process is defined as

$$\bar{\alpha} = \frac{2\sum_{t=2}^{n} x_t x_{t-1}}{\sum_{t=2}^{n} \left(x_t^2 + x_{t-1}^2\right)} \, ,$$

which can be expressed as follows in matrix form:

$$\bar{\alpha} = \frac{\mathbf{X}' B_1 \mathbf{X}}{\mathbf{X}' B_0 \mathbf{X}} \qquad (3.5)$$

where

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \quad , \quad B_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

and $\mathbf{X} \sim \mathcal{N}_n(\mathbf{0}, \Sigma)$, the inverse of the covariance matrix of an AR(1) process being

$$\Sigma^{-1} = \begin{pmatrix} 1 & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1+\alpha^2 & -\alpha & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -\alpha & 1+\alpha^2 & -\alpha \\ 0 & \cdots & 0 & -\alpha & 1 \end{pmatrix} . \qquad (3.6)$$

In light of Equation (3.4), the h^{th} moment of Burg's estimator is given by

$$E(\bar{\alpha})^h = E\left[(\mathbf{X}' B_1 \mathbf{X})^h (\mathbf{X}' B_0 \mathbf{X})^{-h}\right] ,$$

and letting $Q_0 = \mathbf{X}' B_0 \mathbf{X}$ and $Q_1 = \mathbf{X}' B_1 \mathbf{X}$,

$$\begin{aligned} E(\bar{\alpha})^h &= E\left(Q_1^h \, \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} e^{-Q_0 y} \, dy\right) \\ &= \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} \mid \Sigma^{-1} \mid^{1/2} \quad \Big(\frac{d^h}{ds^h} \mid \Sigma^{-1} - 2sB_1 \\ &\qquad\qquad + 2yB_0 \mid^{-1/2}\mid_{s=0}\Big) dy. \end{aligned} \tag{3.7}$$

In this case, the expression

$$\mid \Sigma^{-1} - 2sB_1 + 2yB_0 \mid^{-1/2}\mid_{s=0} \tag{3.8}$$

is tridiagonal which make it easier to evaluate.

Since the support of the distribution is finite, we approximate the distribution of the ratio from its moments by making use of a beta distribution as base density function. The proposed methodology comprises the following steps:

1. The moments of $\bar{\alpha}$ are determined from Equation (3.7) for $n = 50$ and $\alpha = 0.5$.

2. A beta density function is utilized as base density:

   $$\phi(x) = \frac{1}{\mathrm{B}(a,b)} \, x^{a-1}(1-x)^{b-1} \, \mathcal{I}_{(0,1)}(x), \;\; a > 0, \;\; b > 0,$$

   where $\mathrm{B}(a,b) = \Gamma(a)\,\Gamma(b)/\Gamma(a+b)$.

3. The support (q,r) of the ratio denoted by y is mapped onto the interval (0,1) by means of the affine transformation, $x = (y-q)/(r-q)$, which implies that $y = x(r-q)+q$.

4. The h^{th} moment of x is determined from the binomial expansion of $\Big[(y-q)/(r-q)\Big]^h$.

5. The parameters of the beta density are evaluated as follows:

   $$a = -\mu_1 + \frac{(1-\mu_1)\,\mu_1^2}{\mu_2 - \mu_1^2}\,, \quad b = -1 - a + \frac{(1-\mu_1)\,\mu_1}{\mu_2 - \mu_1^2}.$$

6. Approximate densities are obtained with and without polynomial adjustments using the procedure described in Section 2.7.3.

Table 3.3: Approximate cdf's of $Q(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation based on the moments of $\bar{\alpha}$ ($n = 50$ and $\alpha = .5$).

CDF	Simul. %	Beta	Beta Poly
0.0001	−0.06975	0.00003878	0.00011285
0.0010	0.03410	0.00059530	0.00117713
0.01	0.15748	0.008215	0.010906
0.05	0.26337	0.048012	0.051062
0.10	0.31726	0.100023	0.100270
0.25	0.40260	0.256156	0.248771
0.50	0.49001	0.508722	0.500073
0.75	0.56864	0.750512	0.750520
0.90	0.66737	0.944097	0.949802
0.95	0.66737	0.944097	0.949802
0.99	0.72743	0.986898	0.989970
0.999	0.78539	0.998254	0.999007
0.9999	0.82370	0.999721	0.999890

A polynomial adjustment of degree $d = 3$ was used. The approximate cumulative distribution functions corresponding to the beta and the polynomially-adjusted beta density functions were evaluated at certain percentiles obtained from the empirical distribution, which was generated from 1,000,000 replications. The results reported in Table 3.3 corroborate that the proposed approximations are very accurate.

We now resort to a different approach involving the relationship (3.1) to approximate distribution of $\bar{\alpha}$ using several base densities and various values for n and α.

Example 3.2.4. Let $\bar{\alpha}$ be the Burg estimator of the parameter α in an AR(1) process as defined in (3.5). Then, it follows from the relationship (3.1) that the distribution function of $\bar{\alpha}$ at the point t_0 is

$$\Pr(\bar{\alpha} \leq t_0) \;=\; \Pr\Big(\mathbf{X}'(B_1 - t_0 B_0)\mathbf{X} \leq 0\Big)\,. \tag{3.9}$$

On letting $U = \mathbf{X}'(B_1 - t_0 B_0)\mathbf{X}$, U can be re-expressed as a difference of two positive quadratic forms by applying Steps 1 and 2 of the algorithm provided in Section 2.7.5, with $A = (B_1 - t_0 B_0)$, $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = I$. Polynomially adjusted density functions were obtained via the indefinite quadratic form approach with gamma, generalized gamma and Pearson-type base density functions. The corresponding cumulatve distribution functions were evaluated at certain percentiles of the distribution obtained by simulation. The

Table 3.4: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) without polynomial adjustments ($n = 50$ and $\alpha = 0.25$).

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.0001	−0.29474	0.00016120	0.000110418	0.00010394
0.0010	−0.20229	0.00138816	0.001114110	0.00108598
0.01	−0.09178	0.011440	0.010336	0.0102635
0.05	0.00890	0.052744	0.050636	0.0505776
0.10	0.06238	0.102949	0.100756	0.100751
0.25	0.15008	0.250864	0.249903	0.249990
0.50	0.24513	0.498778	0.499997	0.500061
0.75	0.33566	0.748343	0.750052	0.750028
0.90	0.41276	0.899569	0.900444	0.900416
0.95	0.45684	0.950268	0.950668	0.950660
0.99	0.53496	0.990473	0.990487	0.990524
0.9990	0.61468	0.998968	0.998923	0.999138
0.9999	0.66910	0.999268	0.999175	0.999906

approximate cdf's are presented in Tables 3.4 to 3.11 for $\alpha = 0.25, -0.25, 0.5$ and 0.95 and for $n = 10$ and 50.

3.3 Ratios of Quadratic Expressions in Singular Normal Vectors

Let $A_1 = A_1'$ and $A_2 = A_2'$ be indefinite matrices, $\mathbf{X}$ be a $p \times 1$ normal vector such that $E(\mathbf{X}) = \boldsymbol{\mu}$, $\text{Cov}(\mathbf{X}) = \Sigma \geq 0$, $\rho(\Sigma) = r \leq p$ so that, $\Sigma = BB'$, B being a $p \times r$ matrix, and let $\mathbf{a}_1'$ and $\mathbf{a}_2'$ be $p-$dimensional constant vectors, and d_1 and d_2 be scalar constants. Then, letting $Q_1^*(\mathbf{X}) = \mathbf{X}'A_1\mathbf{X} + \mathbf{a}_1'\mathbf{X} + d_1$ and $Q_2^*(\mathbf{X}) = \mathbf{X}'A_2\mathbf{X} + \mathbf{a}_2'\mathbf{X} + d_2$, the distribution of the ratio of quadratic expressions,

$$R = \frac{\mathbf{X}'A_1\mathbf{X} + \mathbf{a}_1'\mathbf{X} + d_1}{\mathbf{X}'A_2\mathbf{X} + \mathbf{a}_2'\mathbf{X} + d_2}, \tag{3.10}$$

can be determined as follows:

Table 3.5: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) with polynomial adjustments ($d = 10, n = 50$ and $\alpha = 0.25$).

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P
0.0001	−0.29474	0.00011431	0.00010411	0.00010391
0.0010	−0.20229	0.00111173	0.00109919	0.00108635
0.01	−0.09178	0.010282	0.010288	0.010266
0.05	0.00890	0.050576	0.050571	0.050585
0.10	0.06238	0.100735	0.100733	0.100737
0.25	0.15008	0.249963	0.249969	0.250134
0.50	0.24513	0.500062	0.500057	0.500065
0.75	0.33566	0.750043	0.750041	0.750050
0.90	0.41276	0.900420	0.900457	0.900402
0.95	0.45684	0.950504	0.950656	0.950665
0.99	0.53496	0.990489	0.990496	0.990462
0.9990	0.61468	0.998928	0.999244	0.998931
0.9999	0.66910	0.999180	0.999182	0.999903

Table 3.6: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) without polynomial adjustments ($n = 50$ and $\alpha = -0.25$).

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.	Pearson
0.0001	−0.66882	0.00009439	0.00009657	0.00009529	0.00000005
0.0010	−0.61317	0.00090534	0.00091529	0.00091011	0.00000556
0.01	−0.53413	0.009704	0.009682	0.0096720	0.000525
0.05	−0.45660	0.049943	0.0495397	0.049550	0.010598
0.10	−0.41245	0.100882	0.100003	0.100032	0.036816
0.25	−0.33548	0.252116	0.250405	0.250429	0.174675
0.50	−0.24477	0.502276	0.501063	0.500999	0.507978
0.75	−0.14998	0.749359	0.750322	0.750235	0.838852
0.90	−0.06180	0.897741	0.899936	0.899942	0.970393
0.95	0.00845	0.947574	0.949679	0.949737	0.992540
0.99	0.09273	0.988744	0.989838	0.989910	0.999767
0.9990	0.20434	0.998671	0.998937	0.998964	0.999999
0.9999	0.29299	0.999832	0.999884	0.999891	1.000000

Table 3.7: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) with polynomial adjustments ($d = 10, n = 50$ and $\alpha = -0.25$).

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P.	Pear.P.
0.0001	−0.66882	0.00009547	0.00009542	0.00009544	0.00003362
0.0010	−0.61317	0.00091092	0.00091099	0.00091035	0.00135811
0.01	−0.53413	0.009675	0.009675	0.009675	0.004625
0.05	−0.45660	0.049554	0.049551	0.049552	0.074497
0.10	−0.41245	0.100027	0.100026	0.100029	0.056944
0.25	−0.33547	0.250413	0.250412	0.250412	0.280581
0.50	−0.24477	0.500998	0.501001	0.500995	0.443270
0.75	−0.14998	0.750261	0.750259	0.749936	0.737736
0.90	−0.06180	0.899958	0.899960	0.899957	0.915464
0.95	0.00845	0.949738	0.949741	0.949730	0.955555
0.99	0.09273	0.989892	0.989886	0.989906	0.993913
0.9990	0.20434	0.998941	0.998951	0.998964	0.999423
0.9999	0.29299	0.999880	0.999890	0.999891	0.999784

Table 3.8: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) without polynomial adjustments ($n = 50$ and $\alpha = 0.5$).

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.0001	−0.06975	0.00026626	0.00021473	0.00010360
0.0010	0.03410	0.00179833	0.00114713	0.00105810
0.01	0.15748	0.013073	0.010633	0.010400
0.05	0.26337	0.055730	0.051370	0.051171
0.10	0.31726	0.105820	0.101477	0.101431
0.25	0.40260	0.251935	0.250435	0.250672
0.50	0.49001	0.496060	0.498871	0.499085
0.75	0.56864	0.744974	0.748063	0.748038
0.90	0.63233	0.898617	0.899578	0.899501
0.95	0.66737	0.950450	0.950487	0.950443
0.99	0.72743	0.991228	0.990902	0.990909
0.9990	0.78539	0.999318	0.999231	0.999267
0.9999	0.82370	0.999847	0.999820	0.999936

Table 3.9: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) with polynomial adjustments ($d = 10, n = 50$ and $\alpha = 0.5$).

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P.
0.0001	−0.06975	0.00011574	0.00010902	0.00010845
0.0010	0.03410	0.00109257	0.00109750	0.00108890
0.01	0.15748	0.010478	0.010468	0.010483
0.05	0.26337	0.051147	0.0511622	0.051202
0.10	0.31726	0.101386	0.101239	0.101585
0.25	0.40260	0.250651	0.250552	0.250556
0.50	0.49001	0.499041	0.499088	0.499083
0.75	0.56864	0.748100	0.748075	0.748101
0.90	0.63233	0.899523	0.899583	0.899564
0.95	0.66737	0.950499	0.950483	0.950454
0.99	0.72743	0.990899	0.991144	0.990816
0.9990	0.78539	0.999237	0.999328	0.999234
0.9999	0.82370	1.000620	0.999831	0.999917

Table 3.10: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) without polynomial adjustments ($n = 10$ and $\alpha = 0.95$).

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.0001	−0.51901	0.01712640	0.00885508	0.000000000
0.0010	−0.26028	0.03047380	0.01823720	0.000000000
0.01	0.12204	0.0608326	0.043297	0.000001
0.05	0.45421	0.113882	0.093368	0.008579
0.10	0.60456	0.160510	0.140418	0.062949
0.25	0.79469	0.282287	0.268802	0.255151
0.50	0.91359	0.499850	0.497266	0.506946
0.75	0.96542	0.749942	0.750062	0.746778
0.90	0.98341	0.899988	0.900104	0.901998
0.95	0.98888	0.950188	0.950275	0.953367
0.99	0.99446	0.990025	0.990064	0.992074
0.9990	0.99737	0.998970	0.998979	0.999247
0.9999	0.99868	0.999918	0.999919	0.999745

Table 3.11: Approximate cdf's of $\bar{\alpha}$ evaluated at certain percentage points (Simul. %) with polynomial adjustments ($d = 10, n = 10$ and $\alpha = 0.95$).

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P.
0.0001	−0.51901	0.01268780	0.00792530	0.000000000
0.0010	−0.26028	0.02285400	0.01649720	0.000000000
0.01	0.12204	0.047212	0.036901	0.000016
0.05	0.45421	0.093623	0.086329	0.010791
0.10	0.60456	0.136677	0.131447	0.063953
0.25	0.79469	0.259564	0.259198	0.256471
0.50	0.91359	0.493526	0.494551	0.503528
0.75	0.96542	0.750099	0.750100	0.750143
0.90	0.98341	0.900102	0.900103	0.899839
0.95	0.98888	0.950271	0.950271	0.950274
0.99	0.99446	0.990060	0.990061	0.990112
0.9990	0.99737	0.998977	0.998978	0.999067
0.9999	0.99868	0.999919	0.999917	0.999601

$$
\begin{aligned}
F_R(t_0) &= Pr(R \le t_0) = Pr(Q_1^*(\mathbf{X}) - t_0\, Q_2^*(\mathbf{X}) \le 0) \\
&= Pr((\mathbf{X}' A_1 \mathbf{X} + \mathbf{a}_1' \mathbf{X} + d_1) - t_0(\mathbf{X}' A_2 \mathbf{X} + \mathbf{a}_2' \mathbf{X} + d_2) \le 0) \\
&= Pr(\mathbf{X}'(A_1 - t_0 A_2)\mathbf{X} + (\mathbf{a}_1' - t_0\, \mathbf{a}_2')\mathbf{X} + (d_1 - t_0\, d_2) \le 0) \\
&= Pr(\mathbf{X}' A \mathbf{X} + \mathbf{a}' \mathbf{X} + d \le 0) \qquad (3.11)
\end{aligned}
$$

where $A = A_1 - t_0\, A_2,\ \mathbf{a}' = \mathbf{a}_1' - t_0\, \mathbf{a}_2'$ and $d = d_1 - t_0\, d_2$.

On letting $Q^*(\mathbf{X}) = \mathbf{X}' A \mathbf{X} + \mathbf{a}' \mathbf{X} + d$, $Q^*(\mathbf{X})$ can be re-expressed as a difference of two positive quadratic forms plus a constant by making use of Representation 2.6.1. Then, it suffices to evaluate the cdf of $Q^*(\mathbf{X})$ at the point 0 to determine $F_R(t_0)$.

Remark 3.3.1. Note that the numerator and denominator may involve different vectors. For example, consider the ratio

$$
\frac{(\mathbf{W}'\,,\,\mathbf{Y}')\, B_1 \begin{pmatrix} \mathbf{W} \\ \mathbf{Y} \end{pmatrix} + \mathbf{b}_1' \mathbf{Y} + d_1}{(\mathbf{Y}'\,,\,\mathbf{Z}')\, B_2 \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} + \mathbf{b}_2' \begin{pmatrix} \mathbf{W} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} + d_2}
$$

which can be re-expressed as

$$\frac{\mathbf{X}'\begin{pmatrix} B_1 & O \\ O & O \end{pmatrix}\mathbf{X} + \mathbf{a}_1'\mathbf{X} + d_1}{\mathbf{X}'\begin{pmatrix} O & O \\ O & B_2 \end{pmatrix}\mathbf{X} + \mathbf{a}_2'\mathbf{X} + d_2}$$

where $\mathbf{X}' = (\mathbf{W}', \mathbf{Y}', \mathbf{Z}')$, $\mathbf{a}_1' = (\mathbf{0}', \mathbf{b}_1', \mathbf{0}')$ and $\mathbf{a}_2' = \mathbf{b}_2'$.

Example 3.3.1. Let $\mathbf{X} \sim \mathcal{N}_5(\boldsymbol{\mu}, \Sigma)$ where $\boldsymbol{\mu} = (1, 2, 2, 1, 4)'$ and

$$\Sigma = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Consider the following ratio of quadratic expressions:

$$R = \frac{\mathbf{X}'A_1\mathbf{X} + \mathbf{a}_1'\mathbf{X} + 3}{\mathbf{X}'A_2\mathbf{X} + \mathbf{a}_2'\mathbf{X} + 1} \tag{3.12}$$

where $\mathbf{a}_1' = (1, 2, 1, 3, 3)$, $\mathbf{a}_2' = (1, 1, 4, 2, 1)$,

$$A_1 = \begin{pmatrix} -4 & 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 0 & -2 \\ 2 & -2 & 0 & -2 & 2 \\ 2 & 0 & -2 & 0 & 2 \\ 0 & -2 & 2 & 2 & 4 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The matrices B and P were found to be

$$B = \begin{pmatrix} 1.10133 & 0. & -0.76987 & 0.44088 \\ 0.87818 & 0. & -0.26911 & -0.39545 \\ 0.87818 & 0. & -0.26911 & -0.39545 \\ 1.47651 & 0. & 0.89436 & 0.14154 \\ 0. & 1.41421 & 0. & 0. \end{pmatrix}$$

Table 3.12: Three approximations to the distribution of R evaluated at certain percentage points (Simul. %) obtained by simulation.

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.01	−0.13586	0.004943	0.008627	0.011654
0.05	0.27988	0.045237	0.050729	0.051379
0.10	0.46437	0.090279	0.098642	0.100893
0.25	0.76761	0.240652	0.249059	0.249943
0.50	1.13672	0.498319	0.500332	0.499710
0.75	1.57132	0.753442	0.750528	0.750055
0.90	2.05093	0.903139	0.900769	0.901018
0.95	2.40063	0.952203	0.950949	0.951489
0.99	3.30089	0.990830	0.990636	0.991129

and

$$P = \begin{pmatrix} 0.95435 & 0.21694 & -0.00179 & 0.20533 \\ -0.04486 & -0.35524 & -0.73373 & 0.57744 \\ 0.22494 & -0.24681 & -0.51537 & -0.78923 \\ -0.19135 & 0.87512 & -0.44275 & -0.03909 \end{pmatrix}.$$

where P is an orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1,\ldots,\lambda_r)$, $\lambda_1,\ldots,\lambda_r$ being the eigenvalues of $B'(A_1 - t_0\,A_2)B$. In light of Equation (3.11), we proceed as in Example 2.8.1 to determine the cdf of R for various values of t_0. Approximations to the distribution function of R were obtained by making use of the gamma, generalized gamma and the generalized shifted gamma densities. The simulated distribution function was determined on the basis of 5,000,000 replications. Since the simulated values are not as reliable for cdf's less than 0.1, we use the following relationship to obtain cdf values that are less than 0.1 from a given approximation.

Let $T = 1/R = Q_2(\mathbf{X})/Q_1(\mathbf{X})$; noting that

$$\begin{aligned} p = Pr(T \le t'_p) &= Pr\Big(\frac{1}{R} \le t'_p\Big) = Pr(R \ge \frac{1}{t'_p}) \\ &= 1 - Pr(R \le \frac{1}{t'_p}) \end{aligned} \tag{3.13}$$

implies that $Pr(R \le 1/t'_p) = 1 - p$, one has that $1 - 1/t'_p \equiv t_{1-p}$ is the $(1-p)100^{th}$ percentile of R. Thus, one can obtain the $(1-p)100^{th}$ percentile of R by determining the percentile t'_p from the generated values of $Q_2(\mathbf{X})/Q_1(\mathbf{X})$.

Table 3.13: Three polynomially-adjusted ($d = 10$) approximations to the distribution of R evaluated at certain percentage points (Simul. %) obtained by simulation.

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P.
0.01	0.12129	0.011033	0.011337	0.011493
0.05	0.27988	0.050263	0.050293	0.050292
0.10	0.46437	0.100035	0.100215	0.100132
0.50	1.13672	0.500356	0.500352	0.500144
0.90	2.05093	0.900152	0.900048	0.900096
0.95	2.40063	0.950389	0.950250	0.950411
0.99	3.30089	0.990395	0.990332	0.990640

Table 3.14: Two approximations with and without polynomial adjustments ($d = 10$) to the distribution of R evaluated at certain percentage points (Simul. %) obtained by simulation.

CDF	Simul. %	Ge.G.	Ge.G.P.	Ge.Sh.G	Ge.Sh.G.P.
0.01	−0.13586	0.008627	0.011033	0.011654	0.011493
0.05	0.05073	0.050730	0.050293	0.051379	0.050292
0.10	0.46437	0.098642	0.100215	0.100893	0.100132
0.50	1.13672	0.500332	0.500352	0.499710	0.500144
0.90	2.05093	0.900769	0.900048	0.901018	0.900096
0.95	2.40063	0.950949	0.950250	0.951489	0.950411
0.99	3.30089	0.990636	0.990332	0.991129	0.990640

Tables 3.12 to 3.14 include various approximate cdf values that are determined with and without polynomial adjustments. The results presented in Table 3.12 indicate that the generalized gamma distribution provides the most accurate approximations for a majority of the points. The approximations adjusted with polynomials of degree 10, which are presented in Table 3.13, suggest that the generalized shifted gamma approximation is more accurate for cdf's less than 0.75. However, for cdf's higher than 0.75, the generalized gamma approximation produces the best results. In addition, Table 3.14 indicates that, in this case, the polynomial adjustments improve the accuracy of the approximations.

Chapter 4

Hermitian Quadratic Forms in Normal Vectors

4.1 Introduction

It is shown in Section 4.2 that Hermitian quadratic forms or quadratic expressions in singular normal vectors can be expressed in terms of real positive definite quadratic forms and an independently distributed normal random variable; representations of their moment generating functions and cumulants—wherefrom the moments can be determined—are provided in Section 4.4. Several particular cases of interest are mentioned. It should be noted that, when dealing with quadratic forms in singular normal vectors, whether real or Hermitian, the results that are available in the statistical literature such as Equation (1) of Tong *et al.* (2010) and Representation 3.1a.5 in Mathai and Provost (1992) may not hold if the rank of the matrix of the quadratic form is less than that of the covariance matrix of the singular normal vector. Section 4.3 proposes a methodology for approximating the distribution of Hermitian quadratic forms and quadratic expressions. Four numerical examples illustrate the application of the proposed distributional results in Section 4.5.

4.2 Hermitian Quadratic Forms Expressed in Terms of Real Quadratic Forms

A complex random vector $\mathbf{W}$ in $\mathcal{C}^n$ can be written as $\mathbf{W} = \mathbf{U} + i\mathbf{V}$ where $\mathbf{U}$ and $\mathbf{V}$ are real random vectors in $\Re^n$. Accordingly, problems involving a complex random vector $\mathbf{W}$, can be re-expressed in terms of the real random vector $(\mathbf{U}', \mathbf{V}')'$ in $\Re^{2n}$ where, for instance, $\mathbf{U}'$ denotes the transpose of $\mathbf{U}$. When $\mathbf{U}$ and $\mathbf{V}$ are correlated n-dimensional real normal vectors with means $\boldsymbol{\mu}_{\mathbf{U}}$ and $\boldsymbol{\mu}_{\mathbf{V}}$, respectively, the random vector $\mathbf{W} = \mathbf{U} + i\mathbf{V}$

has the complex normal distribution $\mathcal{CN}_n(\boldsymbol{\mu}_{\mathbf{W}}, \Gamma, C)$ where $\boldsymbol{\mu}_{\mathbf{W}} = \boldsymbol{\mu}_{\mathbf{U}} + i\boldsymbol{\mu}_{\mathbf{V}} = E(\mathbf{W})$,

$$\Gamma = E[(\mathbf{W} - \boldsymbol{\mu}_{\mathbf{W}})(\overline{\mathbf{W}} - \bar{\boldsymbol{\mu}}_{\mathbf{W}})'] \text{ and } C = E[(\mathbf{W} - \boldsymbol{\mu}_{\mathbf{W}})(\mathbf{W} - \boldsymbol{\mu}_{\mathbf{W}})'], \tag{4.1}$$

$\overline{\mathbf{W}}$ denoting the complex conjugate of $\mathbf{W}$. The covariance matrix Γ is Hermitian and non-negative definite and the relation matrix C is symmetric and non-negative definite. Moreover, as pointed out in Picinbono (1996), the matrices Γ and C must be such that the matrix $\bar{\Gamma} - \bar{C}'\Gamma^{-1/2}C$ is also non-negative definite, which will be assumed throughout, $\Gamma^{-1/2}$ denoting the inverse of the symmetric square root of Γ. We note that in most practical applications, C is taken to be the null matrix. For instance, Mathai (1997) made that assumption when defining the multivariate normal density in the complex case.

It follows from (4.1) that the matrices Γ and C are related to the covariance matrices associated with $\mathbf{U}$ and $\mathbf{V}$ as follows:

$$\text{Cov}(\mathbf{U}) = E[(\mathbf{U} - \boldsymbol{\mu}_{\mathbf{U}})(\mathbf{U} - \boldsymbol{\mu}_{\mathbf{U}})'] = \frac{1}{2}\,\text{Re}[\Gamma + C],$$

$$\text{Cov}(\mathbf{U}, \mathbf{V}) = E[(\mathbf{U} - \boldsymbol{\mu}_{\mathbf{U}})(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})'] = \frac{1}{2}\,\text{Im}[-\Gamma + C],$$

$$\text{Cov}(\mathbf{U}, \mathbf{V})' = E[(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})(\mathbf{U} - \boldsymbol{\mu}_{\mathbf{U}})'] = \frac{1}{2}\,\text{Im}[\Gamma + C],$$

and

$$\text{Cov}(\mathbf{V}) = E[(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})'] = \frac{1}{2}\,\text{Re}[\Gamma - C],$$

where Re$[\cdot]$ and Im$[\cdot]$ respectively denote the real and imaginary parts of $[\cdot]$.

Accordingly, the real random vector $(\mathbf{U}', \mathbf{V}')'$ corresponding to the complex normal random vector $(\mathbf{U}' + i\mathbf{V}') \sim \mathcal{CN}_n(\boldsymbol{\mu}_{\mathbf{U}} + i\boldsymbol{\mu}_{\mathbf{V}}, \Gamma, C)$ has the following distribution:

$$\begin{pmatrix} \mathbf{U} \\ \mathbf{V} \end{pmatrix} \sim \mathcal{N}_{2n}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{U}} \\ \boldsymbol{\mu}_{\mathbf{V}} \end{pmatrix}, \Sigma \right) \tag{4.2}$$

where

$$\Sigma_{2n\times 2n} = \frac{1}{2}\begin{pmatrix} Re[\Gamma + C] & Im[-\Gamma + C] \\ Im[\Gamma + C] & Re[\Gamma - C] \end{pmatrix} \tag{4.3}$$

and $\mathcal{N}_\nu(\boldsymbol{\mu}, \Sigma)$ denotes a real ν-dimensional normal vector whose mean and covariance matrices are $\boldsymbol{\mu}$ and Σ, respectively. Assuming that the rank of Σ is $r \leq 2n$, one has the following representation of the normal vector $(\mathbf{U}', \mathbf{V}')'$:

$$\begin{pmatrix} \mathbf{U} \\ \mathbf{V} \end{pmatrix} = B\,\mathbf{Z} + \boldsymbol{\mu} \tag{4.4}$$

where $\mathbf{Z} \sim \mathcal{N}_r(\mathbf{0}, I)$, $\boldsymbol{\mu} = (\boldsymbol{\mu}'_{\mathbf{U}}, \boldsymbol{\mu}'_{\mathbf{V}})'$ and $B_{2n\times r}$ is such that $BB' = \Sigma$. In order to determine the matrix $B_{2n\times r}$ when $\Sigma_{2n\times 2n}$ is a possibly singular symmetric real matrix of rank $r \le 2n$, we make use of the spectral decomposition theorem to express Σ as $\Theta\Lambda\Theta'$ where Λ is a diagonal matrix whose first r diagonal elements are the positive eigenvalues of Σ, the remaining diagonal elements being equal to zero, and Θ is an orthogonal matrix whose j^{th} column contains the normalized eigenvector of Σ corresponding to the j^{th} diagonal element of $\Lambda = \mathcal{D}iag(\delta_1, \ldots, \delta_{2r})$, the δ_i's denoting the eigenvalues of Σ in decreasing order. Next, we let $B^*_{2n\times 2n} = \Theta\Lambda^{1/2}$ and remove the last $2n - r$ columns of B^*, which are null vectors, to obtain the matrix $B_{2n\times r}$. Then, it can be verified that $\Sigma = BB'$. When Σ is nonsingular, $B = \Sigma^{1/2}$ is the $2n \times 2n$ symmetric square root of Σ.

A representation of a Hermitian quadratic form in a complex normal vector is now given in terms of real quantities under very general assumptions.

Result 4.2.1. Let $Q(\mathbf{W}) = \overline{\mathbf{W}}' H\mathbf{W}$ be a Hermitian quadratic form where $\mathbf{W} \sim \mathcal{CN}_n(\boldsymbol{\mu}_{\mathbf{W}}\ , \Gamma, C)$, $\boldsymbol{\mu}_{\mathbf{W}} = \boldsymbol{\mu}_{\mathbf{U}} + i\boldsymbol{\mu}_{\mathbf{V}}$ with $\boldsymbol{\mu}_{\mathbf{U}} \in \Re^n$ and $\boldsymbol{\mu}_{\mathbf{V}} \in \Re^n$, C is symmetric and non-negative definite, and H and Γ are Hermitian, Γ being non-negative definite. Then, $Q(\mathbf{W})$ admits the decomposition given in (4.11).

Proof

$$\begin{aligned}
Q(\mathbf{W}) &= \overline{\mathbf{W}}' H\mathbf{W} = (\mathbf{U}' - i\,\mathbf{V}')\,H\,(\mathbf{U} + i\,\mathbf{V})\\
&= \mathbf{U}'\left(\frac{H+H'}{2}\right)\mathbf{U} + \mathbf{V}'\left(\frac{H+H'}{2}\right)\mathbf{V} - i\,\mathbf{U}'\,H'\,\mathbf{V} + i\,\mathbf{U}'\,H\,\mathbf{V}\\
&= \mathbf{U}'\left(\frac{H+H'}{2}\right)\mathbf{U} - i\,(\mathbf{U}'\,,\,\mathbf{V}')\begin{pmatrix} O & H'/2\\ H/2 & O\end{pmatrix}\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix}\\
&\quad + i\,(\mathbf{U}'\,,\,\mathbf{V}')\begin{pmatrix} O & H/2\\ H'/2 & O\end{pmatrix}\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix} + \mathbf{V}'\left(\frac{H+H'}{2}\right)\mathbf{V}\\
&= (\mathbf{U}'\,,\,\mathbf{V}')\begin{pmatrix} (H+H')/2 & i\,(H-H')/2\\ i\,(H'-H)/2 & (H+H')/2\end{pmatrix}\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix}\\
&\equiv (\mathbf{U}'\,,\,\mathbf{V}')\,H_1\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix} && (4.5)\\
&\overset{(4.4)}{=} (B\,\mathbf{Z} + \boldsymbol{\mu})' H_1 (B\,\mathbf{Z} + \boldsymbol{\mu}) && (4.6)\\
&= \mathbf{Z}' B' H_1 B\,\mathbf{Z} + 2\,\boldsymbol{\mu}' H_1\,B\,\mathbf{Z} + \boldsymbol{\mu}' H_1\,\boldsymbol{\mu} && (4.7)
\end{aligned}$$

where

$$H_1 = \begin{pmatrix} (H+H')/2 & i\,(H-H')/2\\ i\,(H'-H)/2 & (H+H')/2\end{pmatrix} \qquad (4.8)$$

is a $2n \times 2n$ symmetric *real* matrix and $\boldsymbol{\mu}$, $\mathbf{Z}$ and $B_{2n\times r}$ are as defined in (4.4), with $BB' = \Sigma$ as specified by (4.3).

Now, let P be an $r\times r$ orthogonal matrix such that $P'B'H_1BP = \mathcal{D}iag(\lambda_1,\ldots,\lambda_r)$, where $\lambda_1,\ldots,\lambda_{r_1}$ are the positive eigenvalues of $B'H_1B$ (or equivalently those of ΣH_1), $\lambda_{r_1+1}=\cdots=\lambda_{r_1+\theta}=0$ and $\lambda_{r_1+\theta+1},\ldots,\lambda_r$ are the negative eigenvalues of $B'H_1B$, $\mathbf{b}'=(b_1,\ldots,b_r)=\boldsymbol{\mu}'H_1BP$, $B'H_1B\neq O$ and $c_1=\boldsymbol{\mu}'H_1\boldsymbol{\mu}$. Then, on letting $\mathbf{X}=P'\mathbf{Z}$ and noting that $\mathbf{X}=(X_1,\cdots,X_r)'\sim\mathcal{N}_r(\mathbf{0},I)$, one has

$$
\begin{aligned}
Q(\mathbf{W}) &= \mathbf{X}'\,\mathcal{D}iag(\lambda_1,\ldots,\lambda_r)\,\mathbf{X}+2\,\mathbf{b}'\mathbf{X}+c_1 && (4.9)\\
&= \sum_{j=1}^{r}\lambda_jX_j^2+2\sum_{j=1}^{r}b_jX_j+c_1\\
&= \sum_{j=1}^{r_1}\lambda_jX_j^2+2\sum_{j=1}^{r_1}b_jX_j-\sum_{j=r_1+\theta+1}^{r}|\lambda_j|X_j^2+2\sum_{j=r_1+\theta+1}^{r}b_jX_j\\
&\qquad +2\sum_{j=r_1+1}^{r_1+\theta}b_jX_j+c_1\\
&= \sum_{j=1}^{r_1}\lambda_j\Big(X_j+\frac{b_j}{\lambda_j}\Big)^2-\sum_{j=r_1+\theta+1}^{r}|\lambda_j|\Big(X_j+\frac{b_j}{\lambda_j}\Big)^2+2\sum_{j=r_1+1}^{r_1+\theta}b_jX_j\\
&\qquad +\Big(c_1-\sum_{j=1}^{r_1}\frac{b_j^2}{\lambda_j}-\sum_{j=r_1+\theta+1}^{r}\frac{b_j^2}{\lambda_j}\Big) && (4.10)\\
&\equiv Q_1(\mathbf{X}^+)-Q_2(\mathbf{X}^-)+2\sum_{j=r_1+1}^{r_1+\theta}b_jX_j+\kappa_1\\
&\equiv Q_1(\mathbf{X}^+)-Q_2(\mathbf{X}^-)+T, && (4.11)
\end{aligned}
$$

where $Q_1(\mathbf{X}^+)$ and $Q_2(\mathbf{X}^-)$ are positive definite quadratic forms, $\mathbf{X}^+=(X_1+b_1/\lambda_1,\ldots,X_{r_1}+b_{r_1}/\lambda_{r_1})'\sim\mathcal{N}_{r_1}(\boldsymbol{\nu}_1,I)$ with $\boldsymbol{\nu}_1=(b_1/\lambda_1,\ldots,b_{r_1}/\lambda_{r_1})'$, $\mathbf{X}^-=(X_{r_1+\theta+1}+b_{r_1+\theta+1}/\lambda_{r_1+\theta+1},\ldots,X_r+b_r/\lambda_r)'\sim\mathcal{N}_{r-r_1-\theta}\,(\boldsymbol{\nu}_2,I)$ with $\boldsymbol{\nu}_2=(b_{r_1+\theta+1}/\lambda_{r_1+\theta+1},\ldots,b_r/\lambda_r)'$, $\kappa_1=\Big(c_1-\sum_{j=1}^{r_1}b_j^2/\lambda_j-\sum_{j=r_1+\theta+1}^{r}b_j^2/\lambda_j\Big)$, θ being number of null eigenvalues of ΣH_1 and $T=(2\sum_{j=r_1+1}^{r_1+\theta}b_jX_j+\kappa_1)\sim\mathcal{N}(\kappa_1\,,\,4\sum_{j=r_1+1}^{r_1+\theta}b_j^2)$. Thus, when $\rho(H_1)<\rho(\Sigma)$ and at least one b_i, $i=r_1+1,\ldots,r_1+\theta$, is non-null, $\rho(\cdot)$ denoting the rank of $(\cdot)$, noncentral Hermitian quadratic forms in possibly singular complex normal vectors can be expressed as the difference of two positive definite real quadratic forms and an independently distributed normal random variable.

It should be pointed out that the representation of the quadratic form (4.5), which is given in (4.11), is more general than any representation currently available in the statistical literature. A special case is discussed in the next result.

Result 4.2.2. Consider $Q(\mathbf{W})$ as defined in Result 4.2.1. Let the rank of $H_1\Sigma$ be equal to the rank of Σ, in which case $\lambda_j \neq 0$, $j = 1, 2, \ldots, r$; then a noncentral Hermitian quadratic form in a possibly singular complex normal vectors can be represented as the difference of two positive definite quadratic forms plus a scalar constant since the linear term in (4.10) is now absent, θ being equal to zero. More specifically,

$$\begin{aligned}
Q(\mathbf{W}) &= \sum_{j=1}^{r} \lambda_j X_j^2 + 2\sum_{j=1}^{r} b_j X_j + c_1 \\
&= \sum_{j=1}^{r_1} \lambda_j X_j^2 + 2\sum_{j=1}^{r_1} b_j X_j - \sum_{j=r_1+1}^{r} |\lambda_j| X_j^2 + 2\sum_{j=r_1+1}^{r} b_j X_j + c_1 \\
&= \sum_{j=1}^{r_1} \lambda_j \Big(X_j + \frac{b_j}{\lambda_j}\Big)^2 - \sum_{j=r_1+1}^{r} |\lambda_j| \Big(X_j + \frac{b_j}{\lambda_j}\Big)^2 \\
&\qquad\qquad + \Big(c_1 - \sum_{j=1}^{r_1} \frac{b_j^2}{\lambda_j} - \sum_{j=r_1+1}^{r} \frac{b_j^2}{\lambda_j}\Big) \\
&\equiv Q_1(\mathbf{X}^+) - Q_2(\mathbf{X}^-) + \kappa_1 \qquad (4.12)
\end{aligned}$$

where $Q_1(\mathbf{X}^+)$, $Q_2(\mathbf{X}^-)$, κ_1, the λ_j's and the b_j's are as specified in Result 4.2.1 wherein it is assumed that $\theta = 0$.

When Σ has full rank, the following result holds.

Result 4.2.3. When a Hermitian quadratic form in a complex normal vector whose associated real covariance Σ as specified by (4.3) is nonsingular, the symmetric square root of Σ denoted by $\Sigma^{1/2}$ exists, and, as an alternative to representation (4.11), one can make use of Equation (4.6) with $B = \Sigma^{1/2}$ to obtain the decomposition of $Q(\mathbf{W})$ given in (4.13).

Proof Let P be a $2n \times 2n$ orthogonal matrix that diagonalizes $\Sigma^{1/2} H_1 \Sigma^{1/2}$, that is, P is such that

$$P'\Sigma^{1/2} H_1\, \Sigma^{1/2} P = \mathcal{D}iag(\lambda_1, \ldots, \lambda_{2n}), \text{ with } P'P = I, \;\; PP' = I,$$

where $\lambda_1, \ldots, \lambda_{2n}$ are the eigenvalues of $\Sigma^{1/2} H_1 \Sigma^{1/2}$ (or equivalently those of ΣH_1) in decreasing order. Then, it follows from (4.6) that

$$\begin{aligned}
Q(\mathbf{W}) &= (\mathbf{Z} + \Sigma^{-1/2}\boldsymbol{\mu})'\Sigma^{1/2} H_1 \Sigma^{1/2}(\mathbf{Z} + \Sigma^{-1/2}\boldsymbol{\mu}) \\
&= (\mathbf{Y} + \mathbf{b}^*)' P'\Sigma^{1/2} H_1 \Sigma^{1/2} P(\mathbf{Y} + \mathbf{b}^*) \\
&= (\mathbf{Y} + \mathbf{b}^*)' \mathcal{D}iag(\lambda_1, \ldots, \lambda_{2n})(\mathbf{Y} + \mathbf{b}^*), \\
&= \sum_{j=1}^{2n} \lambda_j (Y_j + b_j^*)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{r_1} \lambda_j (Y_j + b_j^*)^2 - \sum_{j=r_1+\theta+1}^{2n} |\lambda_j| (Y_j + b_j^*)^2 \\
&\equiv Q_1(\mathbf{Y}^+) - Q_2(\mathbf{Y}^-)\,, \qquad (4.13)
\end{aligned}
$$

where $\mathbf{Y} = (Y_1, \ldots, Y_{2n})' = P'\mathbf{Z}$ with $\mathbf{Y} \sim \mathcal{N}_{2n}(\mathbf{0}, I)$, $\mathbf{Y}^+ = (Y_1 + b_1^*, \ldots, Y_{r_1} + b_{r_1}^*)' \sim \mathcal{N}_{r_1}(\mathbf{m}_1, I)$ with $\mathbf{m}_1 = (b_1^*, \ldots, b_{r_1}^*)'$, $\mathbf{Y}^- = (Y_{r_1+\theta+1} + b_{r_1+\theta+1}^*, \ldots, Y_{2n} + b_{2n}^*)' \sim \mathcal{N}_{2n-r_1-\theta}(\mathbf{m}_2, I)$ with $\mathbf{m}_2 = (b_{r_1+\theta+1}^*, \ldots, b_{2n}^*)'$, $\mathbf{b}^* = (b_1^*, \ldots, b_{2n}^*)' = P'\Sigma^{-1/2}\boldsymbol{\mu}$, $\lambda_1, \ldots, \lambda_{r_1}$ are the positive eigenvalues of $\Sigma^{1/2} H_1 \Sigma^{1/2}$, $\lambda_{r_1+1} = \cdots = \lambda_{r_1+\theta} = 0$ and $\lambda_{r_1+\theta+1}, \ldots, \lambda_{2n}$ are the negative eigenvalues of $\Sigma^{1/2} H_1\, \Sigma^{1/2}$.

The central case is addressed in the next two results.

Result 4.2.4. A central Hermitian quadratic form in the complex normal vector $\mathbf{W} \sim \mathcal{CN}_n(\mathbf{0}, \Gamma, C)$ has the representation given in (4.14).

Proof Letting $\boldsymbol{\mu} = \mathbf{0}$ in Results 4.2.1 and 4.2.2, so that $c_1 = 0$ and $b_j = 0$, $j = 1, \ldots, r$, one has

$$
\begin{aligned}
Q(\mathbf{W}) &= \sum_{j=1}^{r} \lambda_j Y_j^2 = \sum_{j=1}^{r_1} \lambda_j Y_j^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| Y_j^2 \\
&\equiv Q_1(\mathbf{Y}_1^+) - Q_2(\mathbf{Y}_1^-), \qquad (4.14)
\end{aligned}
$$

where $Q_1(\mathbf{Y}_1^+)$ and $Q_2(\mathbf{Y}_1^-)$ are positive definite quadratic forms with $\mathbf{Y}_1^+ = (Y_1, \ldots, Y_{r_1})' \sim \mathcal{N}_{r_1}(\mathbf{0}, I_{r_1})$ and $\mathbf{Y}_1^- = (Y_{r_1+\theta+1}, \ldots, Y_r)' \sim \mathcal{N}_{r-r_1-\theta}(\mathbf{0}, I_{r-r_1-\theta})$, and $\{\lambda_1, \ldots, \lambda_{r_1}\}$ and $\{\lambda_{r_1+\theta+1}, \ldots, \lambda_r\}$ are the sets of positive and negative eigenvalues of ΣH_1, respectively.

Result 4.2.5. When C is a null matrix, $\boldsymbol{\mu} = \mathbf{0}$ and the covariance matrix Γ is Hermitian and non-negative definite (possibly singular), it follows from (4.10) that

$$
Q(\mathbf{W}) = \sum_{j=1}^{r_1} \lambda_j X_j^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| X_j^2 .
$$

Since the eigenvalues of $B' H_1 B$ happen to occur in pairs in this representation, the exact density function of $Q(\mathbf{W})$ can be determined by making use of Equation (2.54).

Result 4.2.6. When the matrix H_1 is *positive semidefinite*, so is $Q(\mathbf{W})$, and it follows from Results 4.2.1 and 4.2.2 that $Q(\mathbf{W}) \sim Q_1(\mathbf{W}^+) + T$ when $\rho(H_1\Sigma) < \rho(\Sigma)$ and $Q(\mathbf{W}) \sim Q_1(\mathbf{W}^+) + \kappa_1$ when $\rho(H_1\Sigma) = \rho(\Sigma)$, where $\kappa_1 = \left(c_1 - \sum_{j=1}^{r_1} b_j^2/\lambda_j\right)$.

4.3 Hermitian Quadratic Expressions

Let $Q^*(\mathbf{W}) = \overline{\mathbf{W}}'H\mathbf{W} + \frac{1}{2}\overline{\mathbf{W}}'\boldsymbol{\alpha} + \frac{1}{2}\bar{\boldsymbol{\alpha}}'\mathbf{W} + \delta$ be a Hermitian quadratic expression in a possibly singular complex normal vector $\mathbf{W}$, where $\boldsymbol{\alpha} = (\mathbf{a}_1' + i\mathbf{a}_2')'$ and δ is real scalar constant, H and $\mathbf{W}$ being as defined in Result 4.2.1. Note $Q^*(\mathbf{W})$ is the counterpart of (2.19) for the complex case. First, we note that

$$\begin{aligned}\frac{1}{2}\overline{\mathbf{W}}'\boldsymbol{\alpha} + \frac{1}{2}\bar{\boldsymbol{\alpha}}'\mathbf{W} &= \frac{1}{2}(\mathbf{U}' - i\mathbf{V}')(\mathbf{a}_1 + i\mathbf{a}_2) + \frac{1}{2}(\mathbf{a}_1' - i\mathbf{a}_2')(\mathbf{U} + i\mathbf{V})\\ &= \mathbf{a}_1'\mathbf{U} + \mathbf{a}_2'\mathbf{V} = (\mathbf{a}_1', \mathbf{a}_2')\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix}.\end{aligned}$$

Thus, in light of (4.5), one has

$$Q^*(\mathbf{W}) = (\mathbf{U}', \mathbf{V}')H_1\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix} + \mathbf{a}'\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix} + \delta \tag{4.15}$$

where $\mathbf{a}' = (\mathbf{a}_1', \mathbf{a}_2')$ and $(\mathbf{U}', \mathbf{V}')'$ is a real normal vector whose distribution is specified in (4.2). Then, letting $(\mathbf{U}', \mathbf{V}')' = B\mathbf{Z} + \boldsymbol{\mu}$ where $\mathbf{Z} \sim \mathcal{N}_r(\mathbf{0}, I)$, as was done in Equation (4.4), the following decomposition of $Q^*(\mathbf{W})$ can be obtained from Equations (4.7) and (4.15):

$$\begin{aligned}Q^*(\mathbf{W}) &= \mathbf{Z}'B'H_1B\mathbf{Z} + 2\boldsymbol{\mu}'H_1\,B\mathbf{Z} + \boldsymbol{\mu}'H_1\boldsymbol{\mu} + \mathbf{a}'(B\mathbf{Z} + \boldsymbol{\mu}) + \delta\\ &= \mathbf{Z}'B'H_1B\mathbf{Z} + 2(\boldsymbol{\mu}'H_1 + \frac{1}{2}\mathbf{a}')B\mathbf{Z} + \boldsymbol{\mu}'H_1\boldsymbol{\mu} + \mathbf{a}'\,\boldsymbol{\mu} + \delta\\ &\equiv \mathbf{Z}'B'H_1B\mathbf{Z} + 2\boldsymbol{\beta}'\mathbf{Z} + c_2 \end{aligned}\tag{4.16}$$

where $\boldsymbol{\beta}' = (\boldsymbol{\mu}'H_1 + \frac{1}{2}\mathbf{a}')B$ and $c_2 = \boldsymbol{\mu}'H_1\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + \delta$. Then letting $A_1 = B'H_1B$ and $2\boldsymbol{\beta}'\mathbf{Z} + c_2 \equiv T_2$ with $T_2 \sim \mathcal{N}(c_2, 4\boldsymbol{\beta}'\boldsymbol{\beta})$, one can represent $Q^*(\mathbf{W})$ as $\mathbf{Z}'A_1\mathbf{Z} + T_2$ that is, an indefinite quadratic form (or the difference of two positive definite quadratic forms) and a normal random variable. Note that, as was shown for instance in Provost (1996), the independence of $\mathbf{Z}'A_1\mathbf{Z}$ and T_2 can be verified with the condition $\boldsymbol{\beta}'A_1 = \mathbf{0}$.

Alternatively, on proceding as in Result 4.2.1, with $\mathbf{b}$ and c_1 in (4.9) respectively replaced by $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)'$ and c_2 as defined in (4.16), one has the following representation, which is analogous to (4.11):

$$Q^*(\mathbf{W}) = Q_1(\mathbf{X}^+) - Q_2(\mathbf{X}^-) + T \tag{4.17}$$

where $Q_1(\mathbf{X}^+)$ and $Q_2(\mathbf{X}^-)$ are positive definite quadratic forms, $\mathbf{X}^+ = (X_1 + \beta_1/\lambda_1, \ldots, X_{r_1} + \beta_{r_1}/\lambda_{r_1})' \sim \mathcal{N}_{r_1}(\boldsymbol{\nu}_1, I)$ with $\boldsymbol{\nu}_1 = (\beta_1/\lambda_1, \ldots, \beta_{r_1}/\lambda_{r_1})'$, $\mathbf{X}^- = (X_{r_1+\theta+1} +$

$\beta_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, X_r + \beta_r/\lambda_r)' \sim \mathcal{N}_{r-r_1-\theta}\ (\boldsymbol{\nu}_2, I)$ with $\boldsymbol{\nu}_2 = (\beta_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, \beta_r/\lambda_r)'$, $\kappa_2 = \left(c_2 - \sum_{j=1}^{r_1} \beta_j^2/\lambda_j - \sum_{j=r_1+\theta+1}^{r} \beta_j^2/\lambda_j\right)$, θ being number of null eigenvalues of ΣH_1 and $T = (2\sum_{j=r_1+1}^{r_1+\theta} \beta_j X_j + \kappa_2) \sim \mathcal{N}(\kappa_2\,,\, 4\sum_{j=r_1+1}^{r_1+\theta} \beta_j^2)$.

4.4 Cumulants, Moments and Generating Functions

Expressions for the characteristic function and the cumulant generating function of a quadratic expression in a central normal vector are, for instance, available in Good (1963a). This section provides representations of the moment and cumulant generating functions of quadratic expressions in possibly singular normal vectors, as well as expressions for their cumulants from which the moments can be determined.

Consider the Hermitian quadratic expression $Q^*(\mathbf{W}) = \overline{\mathbf{W}}'H\mathbf{W} + \frac{1}{2}\overline{\mathbf{W}}'\boldsymbol{\alpha} + \frac{1}{2}\bar{\boldsymbol{\alpha}}'\mathbf{W} + \delta$ and the Hermitian quadratic form $Q(\mathbf{W}) = \overline{\mathbf{W}}'H\mathbf{W}$ where $\mathbf{W} \sim \mathcal{CN}_n(\boldsymbol{\mu}_{\mathbf{W}}\ , \Gamma, C)$, $\boldsymbol{\mu}_{\mathbf{W}} = \boldsymbol{\mu}_{\mathbf{U}} + i\boldsymbol{\mu}_{\mathbf{V}}$, C is symmetric and non-negative definite, H and Γ are Hermitian, Γ being non-negative definite, $\boldsymbol{\alpha}' = (\mathbf{a}_1' + i\,\mathbf{a}_2')$ and d is real scalar constant. On expressing $Q^*(\mathbf{W})$ and $Q(\mathbf{W})$ in terms of real quantities as was done in Equations (4.16) and (4.7), and making use of the representations of the moment generating functions of quadratic expressions which were derived in Mathai and Provost (1992) in Theorems 3.2a.3 and Corollary 3.2a.2, one has

Result 4.4.1.

$$
\begin{aligned}
M_{Q^*}(t) &= |I_r - 2tB'H_1B|^{-\frac{1}{2}} \exp\{t(\boldsymbol{\mu}'H_1\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + \delta) \\
&\quad + \frac{t^2}{2}(B'\mathbf{a} + 2B'H_1\boldsymbol{\mu})'(I - 2tB'H_1B)^{-1}(B'\mathbf{a} + 2B'H_1\boldsymbol{\mu})\} \qquad (4.18)
\end{aligned}
$$

and

$$
\begin{aligned}
M_Q(t) &= |I_r - 2tB'H_1B|^{-1/2} \exp\{t\boldsymbol{\mu}'H_1\boldsymbol{\mu} + 2t^2\boldsymbol{\mu}'H_1B \\
&\qquad \times (I - 2tB'H_1B)^{-1}B'H_1\boldsymbol{\mu}\} \qquad (4.19)
\end{aligned}
$$

where H_1 is the real symmetric $2n \times 2n$ matrix specified by (4.8), $\mathbf{a}' = (\mathbf{a}_1', \mathbf{a}_2')$, $\rho(\Sigma) = r \le 2n$, $\Sigma = BB'$ with $\rho(B_{2n\times r}) = r$ and $B'H_1B \ne O$. Alternatively, in terms of $\lambda_1, \ldots, \lambda_r$, the eigenvalues of $B'H_1B$, one has

Result 4.4.2.

$$\begin{aligned} M_{Q^*}(t) &= \Big\{\prod_{j=1}^{r}(1-2t\lambda_j)^{-\frac{1}{2}}\Big\}\exp\Big\{c_1^*t+\frac{t^2}{2}\sum_{j=1}^{r}(b_j^*)^2(1-2t\lambda_j)^{-1}\Big\},\ \boldsymbol{\mu}\neq\mathbf{0} \\ &= \Big\{\prod_{j=1}^{r}(1-2t\lambda_j)^{-\frac{1}{2}}\Big\}\exp\Big\{d\,t+\frac{t^2}{2}\sum_{j=1}^{r}\beta_j^2(1-2t\lambda_j)^{-1}\Big\},\ \ \boldsymbol{\mu}=\mathbf{0} \end{aligned} \tag{4.20}$$

where $(b_1^*,\ldots,b_r^*)'=P'(2B'H_1\boldsymbol{\mu}+B'\mathbf{a})$, $c_1^*=\boldsymbol{\mu}'H_1\boldsymbol{\mu}+\mathbf{a}'\boldsymbol{\mu}+\delta$ and $(\beta_1,\ldots,\beta_r)'=B'P'\mathbf{a}$, and

$$\begin{aligned} M_Q(t) &= \Big\{\prod_{j=1}^{r}(1-2t\lambda_j)^{-\frac{1}{2}}\Big\}\exp\Big\{c_1t+2t^2\sum_{j=1}^{r}b_j^2(1-2t\lambda_j)^{-1}\Big\},\ \boldsymbol{\mu}\neq\mathbf{0} \\ &= \prod_{j=1}^{r}(1-2t\lambda_j)^{-\frac{1}{2}},\quad \boldsymbol{\mu}=\mathbf{0} \end{aligned} \tag{4.21}$$

where $c_1=\boldsymbol{\mu}'H_1\boldsymbol{\mu}$ and $(b_1,\cdots,b_r)'=P'B'H_1\boldsymbol{\mu}$, with P such that $P'B'H_1BP=\mathcal{D}iag(\lambda_1,\ldots,\lambda_{r_1},0,\ldots,0,\lambda_{r_1+\theta+1},\ldots,\lambda_r)$ and $PP'=P'P=I$.

Result 4.4.3. When $r=2n$ in Equations (4.18) and (4.19), Σ has full rank, and then

$$\begin{aligned} M_{Q^*}(t) &= |I_{2n}-2tH_1\Sigma|^{-\frac{1}{2}}\exp\{t(\boldsymbol{\mu}'H_1\boldsymbol{\mu}+\mathbf{a}'\boldsymbol{\mu}+\delta) \\ &\qquad +\frac{t^2}{2}(\mathbf{a}+2H_1\boldsymbol{\mu})'(I_{2n}-2tH_1\Sigma)^{-1}\,\Sigma\,(\mathbf{a}+2H_1\boldsymbol{\mu})\} \end{aligned}$$

and

$$\begin{aligned} M_Q(t) &= |I_{2n}-2tH_1\Sigma|^{-1/2}\exp\{t\boldsymbol{\mu}'H_1\boldsymbol{\mu}+2t^2\boldsymbol{\mu}'H_1 \\ &\qquad \times(I_{2n}-2tH_1\Sigma)^{-1}\Sigma\,H_1\boldsymbol{\mu}\}. \end{aligned}$$

Result 4.4.4. The *cumulant generating functions* (cgf) of $Q^*(\mathbf{W})$ and $Q(\mathbf{W})$ resulting from Equations (4.18) and (4.19) are respectively

$$\begin{aligned} \ln M_{Q^*}(t) &= -\frac{1}{2}\ln|I_r-2tB'H_1B|+t(\delta+\mathbf{a}'\boldsymbol{\mu}+\boldsymbol{\mu}'H_1\boldsymbol{\mu}) \\ &\qquad +\frac{t^2}{2}(B'\mathbf{a}+2B'H_1\boldsymbol{\mu})'(I_r-2tB'H_1B)^{-1}(B'\mathbf{a}+2B'H_1\boldsymbol{\mu}) \end{aligned} \tag{4.22}$$

and

$$\ln M_Q(t) = -\frac{1}{2}\ln|I_r - 2tB'H_1B| + \{t\boldsymbol{\mu}'H_1\boldsymbol{\mu} + 2t^2\boldsymbol{\mu}'H_1B \times(I - 2tB'H_1B)^{-1}B'H_1\boldsymbol{\mu}\}\,. \tag{4.23}$$

Result 4.4.5. Referring to Result 4.4.2, the cumulant generating functions of $Q^*(\mathbf{W})$ and $Q(\mathbf{W})$ can also be respectively expressed as follows:

$$\begin{aligned} \ln M_{Q^*}(t) &= -\frac{1}{2}\sum_{j=1}^{r}\ln(1-2t\lambda_j) + c_1^*\,t + \frac{t^2}{2}\sum_{j=1}^{r}\frac{(b_j^*)^2}{(1-2t\lambda_j)}\ ,\ \boldsymbol{\mu}\neq\mathbf{0}\,,\\ &= -\frac{1}{2}\sum_{j=1}^{r}\ln(1-2t\lambda_j) + d\,t + \frac{t^2}{2}\sum_{j=1}^{r}\frac{\beta_j^2}{(1-2t\lambda_j)}\,,\ \boldsymbol{\mu}=\mathbf{0}\,, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} \ln M_Q(t) &= -\frac{1}{2}\sum_{j=1}^{r}\ln(1-2t\lambda_j) + c_1\,t + 2\,t^2\sum_{j=1}^{r}\frac{b_j^2}{(1-2t\lambda_j)}\,,\ \boldsymbol{\mu}\neq\mathbf{0}\,,\\ &= -\frac{1}{2}\sum_{j=1}^{r}\ln(1-2t\lambda_j), \qquad \boldsymbol{\mu}=\mathbf{0}\,. \end{aligned} \tag{4.25}$$

An alternative representation of the cumulant generating functions of $Q(\mathbf{W})$ is proposed in the next result.

Result 4.4.6. Referring to (4.11) and applying Result 4.2.3 with $\Sigma = I$, one can determine the cgf of $Q(\mathbf{W}) = Q_1(\mathbf{X}^+) - Q_2(\mathbf{X}^-) + T$ as follows. Let

$$Q^\dagger = \mathbf{X}^{+'}A_1\mathbf{X}^+ - \mathbf{X}^{-'}A_2\mathbf{X}^- = \mathbf{X}'A\mathbf{X}$$

where $A_{r\times r} = \mathcal{D}iag(\lambda_1,\ldots,\lambda_{r_1},0,\ldots,0,\lambda_{r_1+\theta+1},\ldots,\lambda_r)$ and $\mathbf{X}\sim\mathcal{N}_r(\boldsymbol{\nu},I)$, with $\boldsymbol{\nu} = (\boldsymbol{\nu}_1',\mathbf{0}',\boldsymbol{\nu}_2')'$, $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ being as defined in Equation (4.11). On making use of (4.25), one has the following representation of the cumulant generating function of $Q^\dagger$:

$$\ln M_{Q^\dagger}(t) = -\frac{1}{2}\sum_{j=1}^{r}\ln(1-2t\lambda_j) + c_3t + 2t^2\sum_{j=1}^{r}\frac{\delta_j^2}{(1-2t\lambda_j)} \tag{4.26}$$

where $\boldsymbol{\delta}' = (\delta_1,\ldots,\delta_r) = \boldsymbol{\nu}'A$ and $c_3 = \boldsymbol{\nu}'A\boldsymbol{\nu}$.

The cgf of $T \sim \mathcal{N}(\kappa_1\,,\,4\sum_{j=r_1+1}^{r_1+\theta} b_j^2)$ whose parameters are defined in Equation (4.11), is $\kappa_1 t + \sigma^2 t^2/2$ where $\sigma^2 = 4\sum_{j=r_1+1}^{r_1+\theta} b_j^2$. Since $Q^\dagger$ and T are independently distributed,

$$\begin{aligned}\ln M_Q(t) &= \ln M_{Q^\dagger+T}(t) = \ln M_{Q^\dagger}(t) + \ln M_T(t)\\ &= -\frac{1}{2}\sum_{j=1}^{r}\ln(1-2t\lambda_j) + c_3 t + 2t^2\sum_{j=1}^{r}\frac{\delta_j^2}{(1-2t\lambda_j)} + \kappa_1 t + \sigma^2 t^2/2\,.\end{aligned} \tag{4.27}$$

Remark 4.4.1. An expression analogous to (4.27) can be similarly obtained from (4.16) for the cgf of the quadratic expression $Q^*(\mathbf{W})$.

If $\ln M_Q(t)$ admits a power series expansion then the coefficient of $t^s/s!$ in the power series of $\ln M_Q(t)$ is defined to be the s^{th} cumulant of $Q(\mathbf{W})$, which is denoted by $k(s)$. Thus, $\ln M_Q(t) = \sum_{s=1}^{\infty} k(s)\, t^s/s!$ and whenever $\ln M_Q(t)$ is differentiable,

$$k(s) = \frac{\mathrm{d}^s}{\mathrm{d}t^s}[\ln M_Q(t)]|_{t=0}.$$

Then, as explained in Mathai and Provost (1992), the following result can be derived from Equations (4.22) and (4.23):

Result 4.4.7. The s^{th} cumulants of Q^* and Q are

$$\begin{aligned}k^*(s) &= 2^{s-1}s!\left\{\frac{1}{s}\,\mathrm{tr}(B'H_1B)^s + \mathbf{a}'B(B'H_1B)^{s-2}B'\mathbf{a}/4\right.\\ &\quad \left.+\,\boldsymbol{\mu}'H_1B(B'H_1B)^{s-2}B'H_1\boldsymbol{\mu} + \mathbf{a}'B(B'H_1B)^{s-2}B'H_1\boldsymbol{\mu}\right\},\, s\geq 2\,,\\ &= \mathrm{tr}(B'H_1B) + \boldsymbol{\mu}'H_1\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d, \qquad s=1,\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}k(s) &= 2^{s-1}s!\left\{\frac{1}{s}\,\mathrm{tr}(B'H_1B)^s + \boldsymbol{\mu}'H_1B(B'H_1B)^{s-2}B'H_1\boldsymbol{\mu}\right\},\quad s\geq 2\,,\\ &= \mathrm{tr}(B'H_1B) + \boldsymbol{\mu}'H_1\boldsymbol{\mu},\ \ s=1,\end{aligned} \tag{4.29}$$

respectively.

For the special case where Σ is nonsingular, one has

$$\begin{aligned}k^*(s) &= 2^{s-1}s!\Big\{\frac{1}{s}\,\mathrm{tr}(H_1\Sigma)^s + \frac{1}{4}\mathbf{a}'(\Sigma H_1)^{s-2}\Sigma\mathbf{a} + \boldsymbol{\mu}'(H_1\Sigma)^{s-1}H_1\boldsymbol{\mu}\\ &\qquad +\mathbf{a}'(\Sigma H_1)^{s-1}H_1\boldsymbol{\mu}\Big\},\quad s\geq 2\,,\\ &= \mathrm{tr}(H_1\Sigma) + \boldsymbol{\mu}'H_1\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d,\quad s=1\,,\end{aligned} \tag{4.30}$$

and

$$k(s) = 2^{s-1}s!\Big\{\frac{1}{s}\,\mathrm{tr}(H_1\Sigma)^s + \boldsymbol{\mu}'(H_1\Sigma)^{s-1}H_1\boldsymbol{\mu}\Big\}, \quad s \geq 2\,,$$
$$= \mathrm{tr}(H_1\Sigma) + \boldsymbol{\mu}'H_1\boldsymbol{\mu}, \qquad s = 1. \tag{4.31}$$

Result 4.4.8. In light of (4.25), the s^{th} cumulant of $Q(\mathbf{W}) = \overline{\mathbf{W}}'H_1\mathbf{W}$ can also be expressed as

$$k(s) = 2^{s-1}s!\sum_{j=1}^{r}\lambda_j^s(b_j^2 + 1/s) \tag{4.32}$$

where $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $\Sigma^{\frac{1}{2}}H_1\Sigma^{\frac{1}{2}}$ and $\mathbf{b}' = (b_1, \ldots, b_r) = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'$. Note that $\sum_{j=1}^{r}\lambda_j^s = \mathrm{tr}(H_1\Sigma)^s$.

Alternatively, one can make use of Equation (4.27) to obtain the following representations of $k(s)$:

$$k(s) = 2^{s-1}s!\sum_{j=1}^{r}\lambda_j^s\,(\delta_j^2 + 1/s) + \kappa_1 + \sigma^2 t, \quad s = 1\,,$$
$$= 2^{s-1}s!\sum_{j=1}^{r}\lambda_j^s\,(\delta_j^2 + 1/s) + \sigma^2, \quad s = 2\,,$$
$$= 2^{s-1}s!\sum_{j=1}^{r}\lambda_j^s\,(\delta_j^2 + 1/s), \quad s \geq 3\,, \tag{4.33}$$

where σ, κ_1, λ_j and δ_j are as defined in Result 4.4.6.

Result 4.4.9. The moments of a random variable can be obtained from its cumulants by means of a recursive relationship given in Smith (1995), which can also be deduced for instance from Theorem 3.2b.2 of Mathai and Provost (1992). For example, the s^{th} moment of $Q^*(\mathbf{W})$ can be determined as follows:

$$\mu_s = \sum_{i=0}^{s-1}\frac{(s-1)!}{(s-1-i)!\,i!}\,k(s-i)\,\mu_i\,, \tag{4.34}$$

where $k(s)$ is as given in (4.29) or (4.33).

4.5 Numerical Examples

Four numerical examples involving Hermitian quadratic forms and quadratic expressions in singular or nonsingular complex normal vectors are presented in this section.

Example 4.5.1. Let $Q_1(\mathbf{W}) = \overline{\mathbf{W}}' H \mathbf{W}$ where $\mathbf{W} = \mathbf{X}_1 + i\mathbf{Y}_1 \sim \mathcal{CN}_n(\mathbf{0}, \Gamma, O)$,

$$\Gamma = \begin{pmatrix} 1 & 3i/2 \\ -3i/2 & 4 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 2 & 1-i \\ 1+i & -6 \end{pmatrix}.$$

In light of Equation (4.5), one can represent $Q_1(\mathbf{W})$ as the real quadratic form,

$$Q_1(\mathbf{W}) = (\mathbf{X}_1', \mathbf{Y}_1')\, H_1 \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix}, \tag{4.35}$$

where

$$H_1 = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & -6 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & -6 \end{pmatrix}$$

and $\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix} \sim \mathcal{N}_{2n}\Big(\boldsymbol{\mu}_{\mathbf{W}}, \Sigma\Big)$ with $\boldsymbol{\mu}_{\mathbf{W}}' = (\mathbf{0}', \mathbf{0}')$ and

$$\Sigma = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 4 & 3/2 & 0 \\ 0 & 3/2 & 1 & 0 \\ -3/2 & 0 & 0 & 4 \end{pmatrix}.$$

The eigenvalues of $G_1' H_1 G_1$ where G_1 is the symmetric square root of Σ, are $(-12.9722,\ -12.9722, 0.472165, 0.472165)$. Since the eigenvalues occur in pairs, one can make use of the representation of the exact density of $Q_1(\mathbf{W})$ given in Equation (2.54) to determine the exact distribution function of $Q_1(\mathbf{W})$. Certain exact percentiles of this distribution are presented in Table 4.1. The corresponding cdf approximations obtained from a gamma and generalized gamma distribution are tabulated. The results presented in this table as well as the plots included in Figures 4.1 and 4.2 indicate that this approximation is, for all intents and purposes, exact.

Example 4.5.2. Let $Q_2(\mathbf{W}) = \overline{\mathbf{W}}' H \mathbf{W}$ where $\mathbf{W} \sim \mathcal{CN}_n(\boldsymbol{\mu}_{\mathbf{W}}, \Gamma, C)$, $\boldsymbol{\mu}_{\mathbf{W}} = (1+2\,i, 3-3\,i)$,

$$\Gamma = \begin{pmatrix} 5 & 1+\frac{i}{5} \\ 1-\frac{i}{5} & 3 \end{pmatrix}, \; H = \begin{pmatrix} 3 & 1-2\,i \\ 1+2i & -1 \end{pmatrix} \text{and } C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Table 4.1: Gamma and Generalized Gamma approximations to the distribution function of $Q_1(\mathbf{W})$ evaluated at certain exact quantiles (Exact %).

CDF	Exact %	Gamma	Ge.G
0.0001	−238.03	0.0001	0.0001
0.0010	−178.29	0.0010	0.0010
0.01	−118.55	0.01	0.01
0.05	−118.55	0.01	0.01
0.10	−58.812	0.10	0.10
0.25	−35.039	0.25	0.25
0.50	−17.056	0.50	0.50
0.75	−6.5362	0.75	0.75
0.90	−1.8060	0.90	0.90
0.95	−0.4032	0.95	0.95
0.99	1.1863	0.99	0.99
0.9990	3.3607	0.9990	0.9990
0.9999	5.5351	0.9999	0.9999

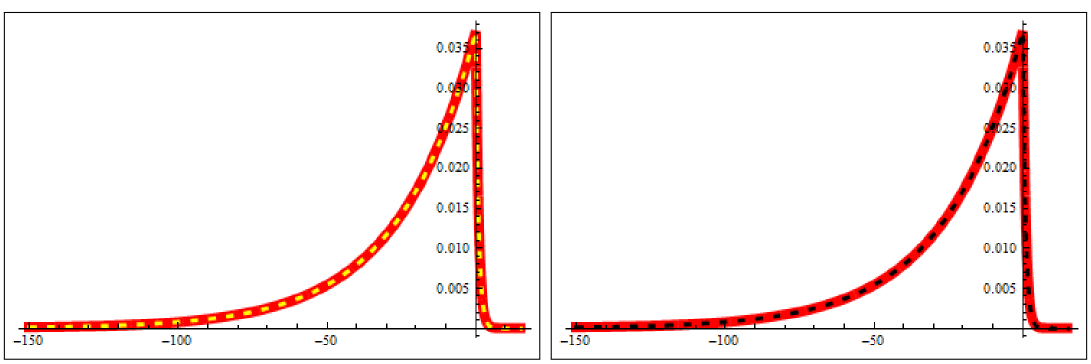


Figure 4.1: *Exact density (light solid line), gamma pdf approximation (left) and generalized gamma pdf approximation (right)*

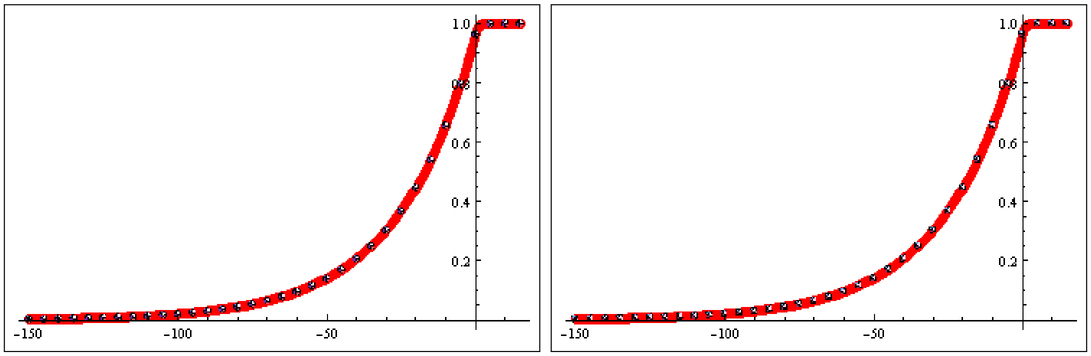


Figure 4.2: *Exact cdf (light solid line), gamma cdf approximation (left) and generalized gamma cdf approximation (right)*

By making use of Equation (4.5), one can represent $Q(\mathbf{W})$ as follows:

$$Q_2(\mathbf{W}) = (\mathbf{X}_1' \,,\, \mathbf{Y}_1')\, H_1 \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix} \tag{4.36}$$

where

$$H_1 = \begin{pmatrix} 3 & 1 & 0 & 2 \\ 1 & -1 & -2 & 0 \\ 0 & -2 & 3 & 1 \\ 2 & 0 & 1 & -1 \end{pmatrix},$$

and $\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix} \sim \mathcal{N}_{2n}\Big(\boldsymbol{\mu}, \Sigma\Big)$ with $\boldsymbol{\mu}' = (1, 3, 2, -3)$ and

$$\Sigma = \frac{1}{2} \begin{pmatrix} 6 & 2 & 0 & -0.2 \\ 2 & 5 & 0.2 & 0 \\ 0 & 0.2 & 4 & 0 \\ -0.2 & 0 & 0 & 1 \end{pmatrix}.$$

The approximate percentiles obtained from gamma, generalized gamma and generalized shifted gamma distributions, with and without Laguerre polynomial adjustments ($d = 7$), are tabulated in Tables 4.2 and 4.3. The results indicate that these approximations are very accurate. The cdf's are also plotted in Figures 4.3 and 4.4.

Table 4.2: Approximate cdf's of $Q_2(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation without polynomial adjustments.

CDF	Simul. %	Gamma	Ge.G.	Ge.S.G.
0.0001	−170.62	0.00009737	0.00010142	0.00010197
0.0010	−140.63	0.00095056	0.00095491	0.00096858
0.01	−107.15	0.010210	0.010177	0.010143
0.05	−81.400	0.050101	0.050128	0.050116
0.10	−69.002	0.099793	0.099865	0.099959
0.25	−50.250	0.249798	0.249798	0.249835
0.50	−31.770	0.500277	0.500312	0.500169
0.75	−14.344	0.749578	0.749595	0.749684
0.90	4.1940	0.900242	0.899939	0.899936
0.95	18.380	0.949920	0.949877	0.949875
0.99	52.890	0.990120	0.990120	0.989987
0.9990	104.45	0.999054	0.998999	0.998999
0.9999	157.92	0.999894	0.999904	0.999904

Table 4.3: Approximate cdf's of $Q_2(\mathbf{X})$ evaluated at certain percentage points (Simul. %) obtained by simulation with polynomial adjustments.

CDF	Simul. %	G.P.	Ge.G.P.	Ge.S.G.P.
0.0001	−170.62	0.00011764	0.00010240	0.00010677
0.0010	−140.63	0.00105284	0.00097506	0.00099144
0.01	−107.15	0.010344	0.010048	0.010064
0.05	−81.400	0.050533	0.050122	0.050027
0.10	−69.002	0.100530	0.100243	0.100076
0.25	−50.250	0.250053	0.250143	0.250076
0.50	−31.770	0.499460	0.499647	0.499949
0.75	−14.344	0.748967	0.749737	0.749780
0.90	4.1940	0.898747	0.900082	0.900021
0.95	18.380	0.949290	0.949838	0.949810
0.99	52.890	0.990438	0.990072	0.990077
0.9990	104.45	0.999197	0.999025	0.999025
0.9999	157.92	0.999938	0.999905	0.999904

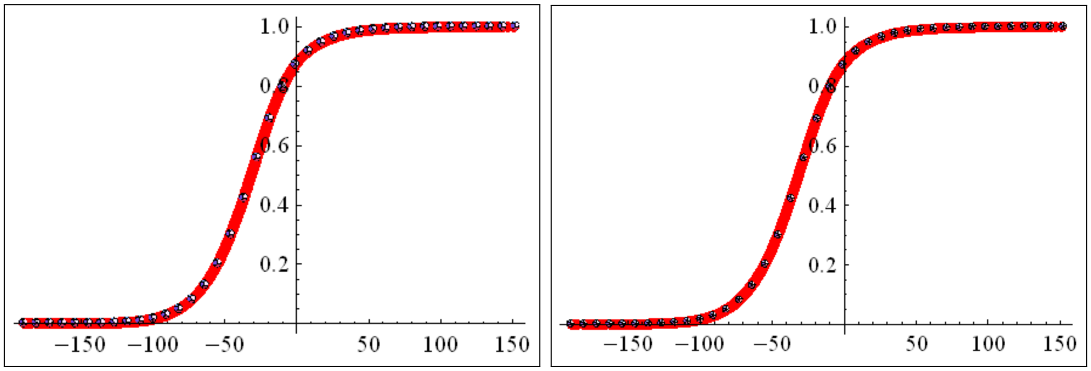


Figure 4.3: *Simulated cdf (solid line), gamma cdf approximation (left) and generalized gamma cdf approximation (right)*

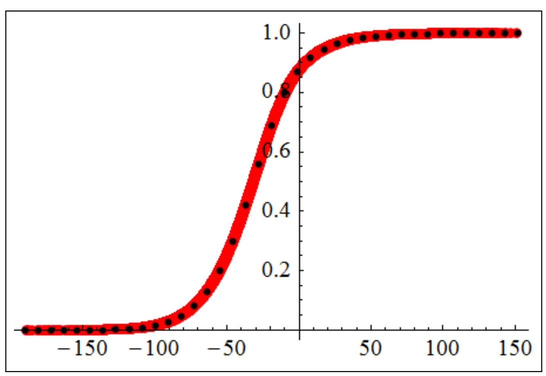


Figure 4.4: *Simulated cdf (solid line) and generalized shifted gamma cdf approximation*

Example 4.5.3. Let

$$Q^*(\mathbf{W}) = \overline{\mathbf{W}}' H \mathbf{W} + \frac{1}{2}\,\overline{\mathbf{W}}' \boldsymbol{\alpha} + \frac{1}{2}\bar{\boldsymbol{\alpha}}'\,\mathbf{W} + \delta$$

where $\mathbf{W} \sim \mathcal{CN}_n(\boldsymbol{\mu}_{\mathbf{W}}, \Gamma, C)$, $\boldsymbol{\mu}_{\mathbf{W}} = (1+2\,i, 3+4\,i, 2.1+3\,i, -3-1.4\,i)'$, $\boldsymbol{\alpha} = (1-2\,i, 2+1.2\,i, -1+3\,i, -5-4\,i)', \delta = 4$,

$$\Gamma = \begin{pmatrix} 10 & 1+i & i & 2-2\,i \\ 1-i & 18 & 1+i & 1+3\,i \\ -i & 1-i & 13 & -i \\ 2+2\,i & 1-3\,i & i & 14 \end{pmatrix},$$

$$H = \begin{pmatrix} 2 & 2 & i & 2 \\ 2 & 2 & i & 2 \\ -i & -i & 4 & 1+2.5\,i \\ 2 & 2 & 1-2.5\,i & -10 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0.3 & 1 & 1 \\ 0.3 & 2.3 & 1.7 & 1 \\ 1 & 1.7 & 2.3 & 2 \\ 1 & 1 & 2 & 2.3 \end{pmatrix}.$$

On making use of Equation (4.16), one can represent $Q^*(\mathbf{W})$ as follows:

$$Q^*(\mathbf{W}) = (\mathbf{X}_1'\,,\,\mathbf{Y}_1')\,H_1 \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix} + \mathbf{a}' \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix} + \delta$$

where

$$H_1 = \begin{pmatrix} 2 & 2 & 0 & 2 & 0 & 0 & -1 & 0 \\ 2 & 2 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 1 & 1 & 1 & 0 & -2.5 \\ 2 & 2 & 1 & -10 & 0 & 0 & 2.5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 \\ -1 & -1 & 0 & 2.5 & 0 & 0 & 4 & 1 \\ 0 & 0 & -2.5 & 0 & 2 & 2 & 1 & -10 \end{pmatrix},$$

$\mathbf{a}' = (1, 2, -1, -5, -2, 1.2, 3, -4)$ and

$$\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix} \sim \mathcal{N}_{2n}\Big(\boldsymbol{\mu}, \Sigma\Big)$$

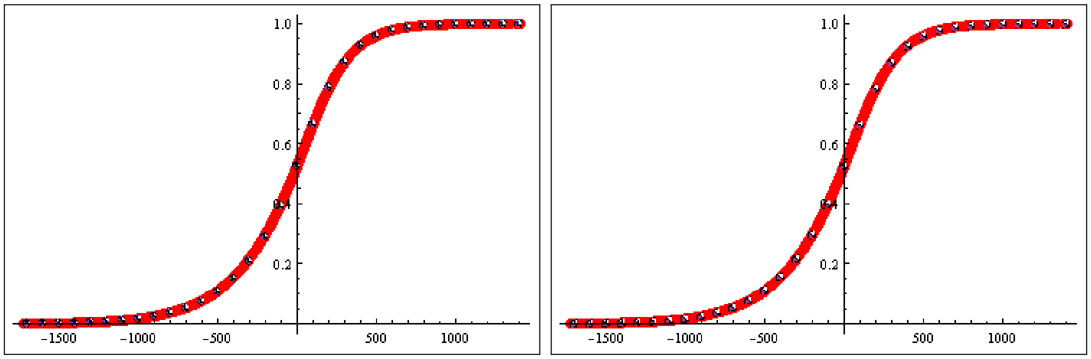


Figure 4.5: *Simulated cdf (solid line), gamma cdf approximation (left) and generalized gamma cdf approximation (right)*

with $\boldsymbol{\mu}' = (1, 3, 2.1, -3, 2, 4, 3, -1.4)$ and

$$\Sigma = \begin{pmatrix} 11 & 1.3 & 1 & 3 & 0 & -1 & -1 & 2 \\ 1.3 & 20.3 & 2.7 & 2 & 1 & 0 & -1 & -3 \\ 1 & 2.7 & 15.3 & 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 2 & 16.3 & -2 & 3 & -1 & 0 \\ 0 & 1 & 1 & -2 & 9 & 0.7 & -1 & 1 \\ -1 & 0 & 1 & 3 & 0.7 & 15.7 & -0.7 & 0 \\ -1 & -1 & 0 & -1 & -1 & -0.7 & 10.7 & -2 \\ 2 & -3 & 1 & 0 & 1 & 0 & -2 & 11.7 \end{pmatrix}.$$

The eigenvalues of $\Sigma^{1/2}H_1\Sigma^{1/2}$ where $\Sigma^{1/2}$ is the symmetric square root of Σ, are $-81.732, -65.954,\ 50.969,\ 37.379, 24.819, 17.519, 0$ and 0. The approximate cdf, which is obtained from a gamma distribution that is adjusted by means of a linear combination of Laguerre polynomials of degrees 1 to 10 by making use of the density approximation methodology described in Provost (2005), was evaluated at certain percentiles of the distribution. These quantiles were determined by simulation on the basis of 1,000,000 replications. The corresponding approximate cdf's based on a simple gamma approximation are also included in Table 4.4 for comparison purposes. The results presented in this table as well as the plots shown in Figure 4.5 suggest that the proposed approximations are very accurate.

Table 4.4: Gamma and Generalized Gamma approximations with and without polynomial adjustments ($d = 10$) to the distribution function of $Q^*(\mathbf{W})$ evaluated at certain percentage points obtained by simulation.

CDF	Simul. %	Gamma	G.P.	Ge.G.	Ge.G.P.
0.0001	−2203.5	0.00006571	0.00002965	0.00003055	0.00003849
0.0010	−1668.6	0.00143960	0.00075039	0.00094030	0.00102410
0.01	−1133.3	0.011215	0.010191	0.009923	0.009990
0.05	−720.23	0.049779	0.049609	0.050367	0.050001
0.10	−530.17	0.096668	0.098333	0.100284	0.099548
0.25	−257.36	0.241247	0.250592	0.249623	0.249623
0.50	−19.165	0.499152	0.500205	0.499175	0.500464
0.75	171.57	0.742547	0.748921	0.750210	0.750012
0.90	346.60	0.903458	0.899686	0.900035	0.899766
0.95	465.09	0.951788	0.950183	0.950032	0.949986
0.99	717.75	0.990153	0.989654	0.989951	0.989795
0.9990	1047.2	0.998770	0.998787	0.998793	0.998994
0.9999	1354.8	0.999681	0.999649	0.999680	0.999693

Example 4.5.4. Let $Q_3(\mathbf{W}) = \overline{\mathbf{W}}'H\mathbf{W}$ be a singular Hermitian quadratic form where $\mathbf{W} = \mathbf{X}_1 + i\mathbf{Y}_1 \sim \mathcal{CN}_n(\mathbf{0}, \Gamma, O)$,

$$\Gamma = \begin{pmatrix} 2 & 2 & i & 2 \\ 2 & 2 & i & 2 \\ -i & -i & 4 & 1+2.5\,i \\ 2 & 2 & 1-2.5\,i & 10 \end{pmatrix}$$

By making use of Equation (4.5) and applying the method described in Section 4.2 to express Γ whose rank is 3 as BB' where B is a matrix of dimension 4×3, one can represent $Q_3(\mathbf{W})$ as the real quadratic form,

$$Q_3(\mathbf{W}) = (\mathbf{X}_1'\,,\,\mathbf{Y}_1')\,H_1\begin{pmatrix}\mathbf{X}_1\\ \mathbf{Y}_1\end{pmatrix}, \tag{4.37}$$

where

$$B = \begin{pmatrix} -0.7079 & -1.1386 & -0.4501 \\ -0.7079 & -1.1386 & -0.4501 \\ -0.4845-0.7993\,i & 0.3360+1.6197\,i & -0.0880-0.6185\,i \\ -3.1251+0.2422\,i & 0.0365-0.1680\,i & 0.3791+0.0440\,i \end{pmatrix},$$

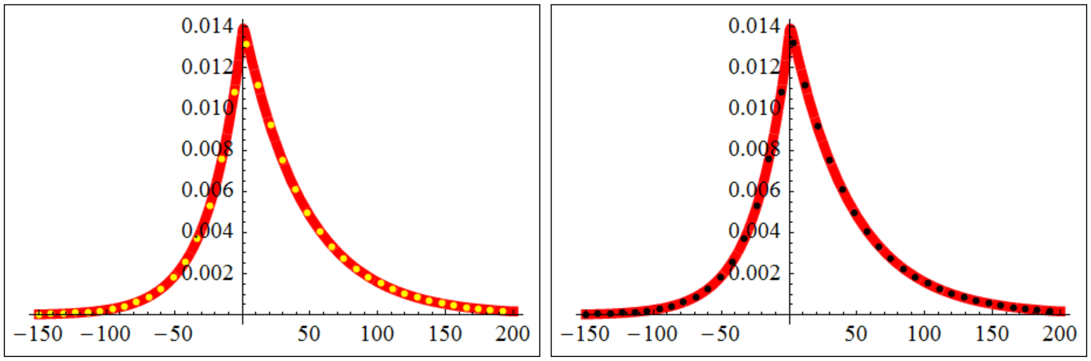


Figure 4.6: *Exact density (light solid line), gamma pdf approximation (left) and generalized gamma pdf approximation (right)*

$$H_1 = \begin{pmatrix} 2 & 1 & 0 & 2 & 0 & -1 & -1 & 2 \\ 1 & 3 & 1 & 1 & 1 & 0 & -1 & -3 \\ 0 & 1 & -6 & 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 2 & -2 & 3 & -1 & 0 \\ 0 & 1 & 1 & -2 & 2 & 1 & 0 & 2 \\ -1 & 0 & 1 & 3 & 1 & 3 & 1 & 1 \\ -1 & -1 & 0 & -1 & 0 & 1 & -6 & 0 \\ 2 & -3 & 1 & 0 & 2 & 1 & 0 & 2 \end{pmatrix}$$

and $\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{Y}_1 \end{pmatrix} \sim \mathcal{N}_{2n}\Big(\boldsymbol{\mu}_{\mathbf{W}}, \Sigma\Big)$ with $\boldsymbol{\mu}'_{\mathbf{W}} = (\mathbf{0}', \mathbf{0}')$ and

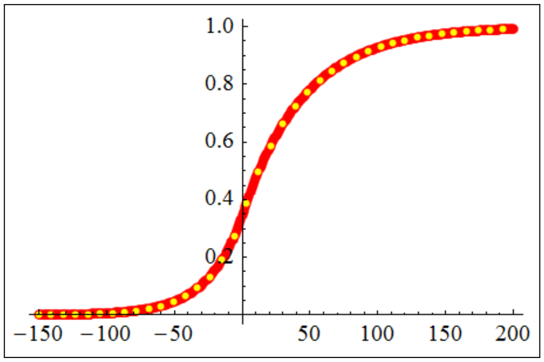
$$\Sigma = \frac{1}{2} \begin{pmatrix} 2 & 2 & 0 & 2 & 0 & 0 & -1 & 0 \\ 2 & 2 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 1 & 1 & 1 & 0 & -2.5 \\ 2 & 2 & 1 & 10 & 0 & 0 & 2.5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 \\ -1 & -1 & 0 & 2.5 & 0 & 0 & 4 & 1 \\ 0 & 0 & -2.5 & 0 & 2 & 2 & 1 & 10 \end{pmatrix}.$$

and

$$H = \begin{pmatrix} 2 & 1+i & i & 2-2\,i \\ 1-i & 3 & 1+i & 1+3\,i \\ -i & 1-i & -6 & -i \\ 2+2\,i & 1-3\,i & i & 2 \end{pmatrix}.$$

Table 4.5: Gamma and Generalized Gamma approximations with and without polynomially adjusted gamma ($d = 10$) to the distribution function of $Q_3(\mathbf{W})$ evaluated at certain exact quantiles.

CDF	Exact %	Gamma	G.P.	Ge.G.	Ge.G.P.
0.0001	−205.68	0.00010063	0.00010003	0.00010025	0.00010002
0.0010	−147.57	0.00100631	0.00100024	0.00100252	0.00100022
0.01	−89.452	0.010063	0.010002	0.010025	0.010003
0.05	−48.832	0.050317	0.050013	0.050127	0.050012
0.10	−31.338	0.100633	0.100026	0.100254	0.100023
0.25	−8.2126	0.251583	0.250065	0.250634	0.250059
0.50	12.229	0.499137	0.498759	0.498679	0.499949
0.75	43.726	0.748555	0.751150	0.749572	0.750943
0.90	85.363	0.899685	0.899392	0.900274	0.899542
0.95	116.86	0.950075	0.949638	0.950238	0.949680
0.99	190.00	0.990165	0.990213	0.990024	0.990016
0.999	294.63	0.999042	0.998933	0.998982	0.998986
0.9999	399.26	0.999907	0.999916	0.999894	0.999910



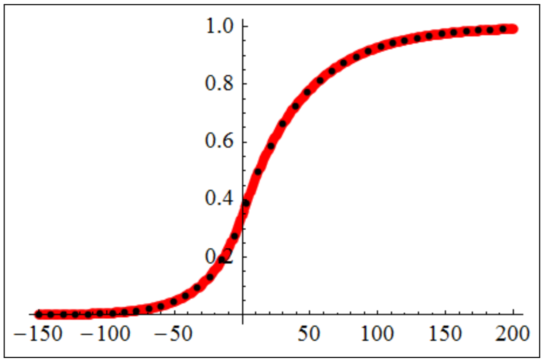


Figure 4.7: *Exact cdf (light solid line), gamma cdf approximation (left) and generalized gamma cdf approximation (right)*

The eigenvalues of $B'H_1B$ with B such that $\Sigma = BB'$, are (22.7205, 22.7205, 0.3987, 0.3987, −12.6192, −12.6192). Since these eigenvalues occur in pairs, one can utilize a representation of the exact density, which is available from Equation (2.54), to determine the exact distribution function of $Q_3(\mathbf{W})$. Certain exact percentiles are included in Table 4.5. The corresponding cdf approximations obtained from a gamma and a generalized gamma distribution, with and without adjustments, by means of a linear combination of Laguerre polynomials of degrees 1 to 10 are also tabulated. The approximation is seen to be nearly exact over the entire range of the distribution.

Chapter 5

Quadratic Expressions in Elliptically Contoured Vectors

5.1 Introduction

A p-dimensional vector $\mathbf{X}$ has an *elliptically contoured* or *elliptical* distribution with mean vector $\boldsymbol{\mu}$ and scale parameter matrix Σ if its characteristic function $\phi(\mathbf{t})$ can be written as

$$\phi(\mathbf{t}) = e^{i\,\mathbf{t}'\boldsymbol{\mu}}\xi(\mathbf{t}'\Sigma\,\mathbf{t})$$

where $\boldsymbol{\mu}$ is a p-dimensional real vector, Σ is a $p \times p$ nonnegative definite matrix and $\xi(\cdot)$ is a nonnegative function, see, for instance, Cambanis *et al.* (1981); this will be denoted

$$\mathbf{X} \sim \mathcal{C}_p(\boldsymbol{\mu}, \Sigma; \xi).$$

Moreover, the densities associated with p-dimensional elliptically contoured vectors $\mathbf{X}$ are of the form $h((\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}))$ where $h(\cdot)$ is a density defined on $(0,\infty)$ whose $(p/2-1)^{\text{th}}$ moment exists, see for example Fang *et al.* (1990), Section 2.2.3. In particular, when $\boldsymbol{\mu}$ is the null vector and Σ is the identity matrix of order p, $\mathbf{X}$ is said to have a spherically symmetric or spherical distribution; this will be denoted

$$\mathbf{X} \sim \mathcal{S}_p(\xi)\ .$$

In fact, whenever $\mathbf{Y} \sim \mathcal{C}_p(\boldsymbol{\mu}, \Sigma; \xi)$ and Σ is a positive definite matrix, $\Sigma^{-\frac{1}{2}}(\mathbf{Y}-\boldsymbol{\mu}) \sim \mathcal{S}_p(\xi)$, where $\Sigma^{-1/2}$ denotes the inverse of the symmetric square root of Σ. Furthermore, spherical distributions are invariant under orthogonal transformations, that is, for any orthogonal matrix P, $\mathbf{X} \sim \mathcal{S}_p(\xi)$ and $P\,\mathbf{X}$ are identically distributed. Other characterizations and properties are available from Kelker (1970), Chmielewski (1981), Fang and Anderson (1990) and Mathai *et al.* (1995), among others.

A decomposition of quadratic expressions in possibly singular elliptically contoured vectors is introduced in Section 5.2 and representations of functions of elliptically contoured vectors such as the moments of a quadratic form, are obtained in Section 5.3. A density approximation methodology that combines these results is described and illustrated by several numerical examples in Section 5.4.

The distributional results derived in this chapter for quadratic forms in elliptically contoured random vectors not only extend, but also make use of, their Gaussian counterparts. Given that elliptically contoured distributions are utilized as models in a host of applications, and quadratic forms are ubiquitous in statistics, the result presented herein should prove useful in a variety of contexts and lead to the development of improved statistical inference techniques.

5.2 A Decomposition of Quadratic Expressions in Elliptically Contoured Vectors

Consider the real quadratic expression $Q^*(\mathbf{X}) = (\mathbf{X}-\boldsymbol{\alpha})'A(\mathbf{X}-\boldsymbol{\alpha}) + \mathbf{a}'(\mathbf{X}-\boldsymbol{\alpha}) + d$ where $\mathbf{X} \sim \mathcal{C}_p(\boldsymbol{\mu}, \Sigma; \xi)$, $rank(\Sigma) = r \le p$, $\boldsymbol{\alpha}$ is a p-dimensional real vector and A is a real symmetric matrix. Letting $\mathbf{X} = \boldsymbol{\mu} + B\,\mathbf{S}$, where $B_{p\times r}$ is such that $BB' = \Sigma$ (cf. Example 5.4.4) and $\mathbf{S} \sim \mathcal{S}_r(\xi)$, one can write

$$\begin{aligned} Q^*(\mathbf{X}) \equiv Q^*(\mathbf{S}) &= (\boldsymbol{\mu} + B\,\mathbf{S} - \boldsymbol{\alpha})'A(\boldsymbol{\mu} + B\,\mathbf{S} - \boldsymbol{\alpha}) + \mathbf{a}'(\boldsymbol{\mu} + B\,\mathbf{S} - \boldsymbol{\alpha}) + d \\ &= [(\boldsymbol{\mu} - \boldsymbol{\alpha}) + B\,\mathbf{S}]'A[(\boldsymbol{\mu} - \boldsymbol{\alpha}) + B\,\mathbf{S}] + \mathbf{a}'[(\boldsymbol{\mu} - \boldsymbol{\alpha}) + B\,\mathbf{S}] + d \\ &= \boldsymbol{\mu}_1' A\,\boldsymbol{\mu}_1 + 2\boldsymbol{\mu}_1' A' B\,\mathbf{S} + \mathbf{S}'B'AB\,\mathbf{S} + \mathbf{a}'B\,\mathbf{S} + \mathbf{a}'\boldsymbol{\mu}_1 + d \end{aligned}$$

where $\boldsymbol{\mu}_1 = \boldsymbol{\mu} - \boldsymbol{\alpha}$. Let P be an orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, $\lambda_1, \ldots, \lambda_r$ denoting the eigenvalues of $B'AB$, with $\lambda_1, \ldots, \lambda_{r_1}$ positive, $\lambda_{r_1+1} = \cdots = \lambda_{r_1+\theta} = 0$ and $\lambda_{r_1+\theta+1}, \ldots, \lambda_r$ negative, $\mathbf{m}' = (m_1, \ldots, m_r) = \mathbf{a}'BP$, $\mathbf{b}^{*'} = (b_1^*, \ldots, b_r^*) = \boldsymbol{\mu}_1' ABP$, $c_1 = \boldsymbol{\mu}_1' A\,\boldsymbol{\mu}_1 + \mathbf{a}'\boldsymbol{\mu}_1 + d$. Then, letting $\mathbf{W} = (W_1, \ldots, W_{r_1}, \ldots, W_{r_1+\theta+1}, \ldots, W_r)' = P'\mathbf{S} \sim \mathcal{S}_r(\xi)$ and assuming that $B'AB \neq O$, one has

$$\begin{aligned} Q^*(\mathbf{X}) \equiv Q^*(\mathbf{W}) &= \mathbf{W}'P'B'ABP\mathbf{W} + 2\boldsymbol{\mu}_1' ABP\mathbf{W} + \mathbf{a}'BP\mathbf{W} + \boldsymbol{\mu}_1' A\boldsymbol{\mu}_1 + \mathbf{a}'\boldsymbol{\mu}_1 + d \\ &= \mathbf{W}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_r)\mathbf{W} + (2\mathbf{b}^{*'} + \mathbf{m}')\mathbf{W} + c_1 \\ &= \sum_{j=1}^{r_1} \lambda_j W_j^2 + 2\sum_{j=1}^{r_1} n_j W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j|\, W_j^2 + 2\sum_{j=r_1+\theta+1}^{r} n_j W_j \\ &\qquad + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + c_1 \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{r_1} \lambda_j \Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| \Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 + 2 \sum_{j=r_1+1}^{r_1+\theta} n_j W_j \\
&\qquad + \Big(c_1 - \sum_{j=1}^{r_1} \frac{n_j^2}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r} \frac{n_j^2}{\lambda_j}\Big) \\
&\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + 2 \sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1 \\
&\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1, \qquad (5.1)
\end{aligned}$$

where $Q_1(\mathbf{W}^+) = \mathbf{W}^{+'}\mathcal{D}iag(\lambda_1, \ldots, \lambda_{r_1})\mathbf{W}^+$ and $Q_2(\mathbf{W}^-) = \mathbf{W}^{-'}\mathcal{D}iag(\lambda_{r_1+\theta+1}, \ldots, \lambda_r)$ $\mathbf{W}^-$ are positive definite quadratic forms with $\mathbf{W}^+ = (W_1 + n_1/\lambda_1, \ldots, W_{r_1} + n_{r_1}/\lambda_{r_1})' \sim \mathcal{C}_{r_1}(\boldsymbol{\nu}_1, I; \xi), \boldsymbol{\nu}_1 = (n_1/\lambda_1, \ldots, n_{r_1}/\lambda_{r_1})'$, $\mathbf{W}^- = (W_{r_1+\theta+1} + n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, W_r + n_r/\lambda_r)' \sim \mathcal{C}_{r-r_1-\theta}\ (\boldsymbol{\nu}_2, I; \xi)$, $\boldsymbol{\nu}_2 = (n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, n_r/\ \lambda_r)'$, θ being number of null eigenvalues of $B'AB$, $n_j = \frac{1}{2} m_j + b_j^*$, $c_1 = \boldsymbol{\mu}_1' A \boldsymbol{\mu}_1 + \mathbf{a}'\boldsymbol{\mu}_1 + d$, $\kappa_1 = \Big(c_1 - \sum_{j=1}^{r_1} n_j^2/\lambda_j - \sum_{j=r_1+\theta+1}^{r} n_j^2/\lambda_j\Big)$ and $T_1 = (2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1) \sim \mathcal{C}_1(\kappa_1\,, 4\sum_{j=r_1+1}^{r_1+\theta} n_j^2; \xi)\,.$ If $rank(A\Sigma) = rank(\Sigma) = r$, $T_1 = \kappa_1$. Note that when $\boldsymbol{\alpha} = \mathbf{0}$ and $\boldsymbol{\mu} = \mathbf{0}$ (the central case), $\boldsymbol{\mu}_1 = \mathbf{0}$ and $\mathbf{b}^* = \mathbf{0}$.

As a particular case, when $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{a} = \mathbf{0}'$ and $d = 0$, one has the following decomposition for the *quadratic form* $\mathbf{X}'A\mathbf{X}$ in the possibly singular elliptically contoured vector $\mathbf{X} \sim \mathcal{C}_p(\boldsymbol{\mu}, \Sigma; \xi), \Sigma$ being of rank $r \le p$:

$$\begin{aligned}
Q(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} = \sum_{j=1}^{r} \lambda_j W_j^2 + 2\sum_{j=1}^{r} b_j^* W_j + c \\
&= \sum_{j=1}^{r_1} \lambda_j W_j^2 + 2\sum_{j=1}^{r_1} b_j^* W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j|\, W_j^2 + 2 \sum_{j=r_1+\theta+1}^{r} b_j^* W_j \\
&\qquad + 2 \sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j + c \\
&= \sum_{j=1}^{r_1} \lambda_j \Big(W_j + \frac{b_j^*}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| \Big(W_j + \frac{b_j^*}{\lambda_j}\Big)^2 + 2 \sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j \\
&\qquad + \Big(c - \sum_{j=1}^{r_1} \frac{b_j^{*2}}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r} \frac{b_j^{*2}}{\lambda_j}\Big)
\end{aligned}$$

$$\begin{aligned} &\equiv Q_1(\mathbf{W}_1) - Q_2(\mathbf{W}_2) + 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j + \kappa \\ &\equiv Q_1(\mathbf{W}_1) - Q_2(\mathbf{W}_2) + T, \end{aligned} \tag{5.2}$$

where $\mathbf{W}' = (W_1, \ldots, W_r) \sim \mathcal{S}_r(\xi)$, $Q_1(\mathbf{W}_1) = \mathbf{W}_1' \mathcal{D}iag(\lambda_1, \ldots, \lambda_{r_1})\mathbf{W}_1$ and $Q_2(\mathbf{W}_2) = \mathbf{W}_2' \mathcal{D}iag(\lambda_{r_1+\theta+1}, \ldots, \lambda_r)\mathbf{W}_2$ are positive definite quadratic forms with $\mathbf{W}_1 = (W_1 + b_1^*/\lambda_1, \ldots, W_{r_1} + b_{r_1}^*/\lambda_{r_1})' \sim \mathcal{C}_{r_1}(\boldsymbol{\nu}_1, I; \xi)$, $\boldsymbol{\nu}_1 = (b_1^*/\lambda_1, \ldots, b_{r_1}^*/\lambda_{r_1})'$, $\mathbf{W}_2 = (W_{r_1+\theta+1} + b_{r_1+\theta+1}^*/\lambda_{r_1+\theta+1}, \ldots, W_r + b_r^*/\lambda_r)' \sim \mathcal{C}_{r-r_1-\theta}(\boldsymbol{\nu}_2, I; \xi)$, $\boldsymbol{\nu}_2 = (b_{r_1+\theta+1}^*/\lambda_{r_1+\theta+1}, \ldots, b_r^*/\lambda_r)'$, θ is the number of null eigenvalues of $A\Sigma$, the λ_j's and b_j^*'s being as previously defined, $c = \boldsymbol{\mu}' A \boldsymbol{\mu}$, $\kappa = \left(c - \sum_{j=1}^{r_1} b_j^{*2}/\lambda_j - \sum_{j=r_1+\theta+1}^{r} b_j^{*2}/\lambda_j\right)$ and $T = 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j + \kappa \sim \mathcal{C}_1(\kappa\,,\, 4\sum_{j=r_1+1}^{r_1+\theta} b_j^{*2})$, whenever $rank(A\Sigma) = r - \theta$, $\theta = 1, \ldots, r-1$. When $rank(\Sigma) = rank(A\Sigma) = r$, $T = \kappa$.

5.3 Elliptically Contoured Distributions as Scale Mixtures of Gaussian Vectors

Elliptically contoured distributions have the stochastic representation $\boldsymbol{\mu} + \Sigma^{1/2} L\,\mathbf{Z}$, where $\boldsymbol{\mu}$ is the mean of the distribution, $\Sigma^{1/2}$ is such that $\Sigma^{1/2}(\Sigma^{1/2})' = \Sigma$, the positive definite scale parameter matrix of the distribution, $\mathbf{Z}$ is a standard Gaussian random vector, and L is a positive random variable that is distributed independently of $\mathbf{Z}$. The density function of $\mathbf{Y} \sim \mathcal{C}_p(\boldsymbol{\mu}, \Sigma; \xi)$ can be expressed in terms of a scale mixture of normal densities as follows:

$$g(\mathbf{y}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \int_0^\infty r^{-p/2} \exp\left\{ -\frac{(\mathbf{y}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{y}-\boldsymbol{\mu})}{2r} \right\} \mathrm{d}U(r) \tag{5.3}$$

where $U(\cdot)$, the distribution function of L^2, is such that $U(0) = 0$. This representation can be found, for example, in Muirhead (1982). We now extend a result due to Chu (1973) to non-central elliptically contoured distributions. This next theorem enables one to express various distributional results involving elliptically contoured vectors in terms of their Gaussian counterparts.

Theorem 5.3.1. *Let $\mathbf{Y} \sim \mathcal{C}_p(\boldsymbol{\mu}, \Sigma; \xi)$ with $\Sigma > 0$, $h(\mathbf{y})$ denotes the density of $\mathbf{Y}$ and $f(s)$ be $h(\mathbf{y})$ wherein $(\mathbf{y}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{y}-\boldsymbol{\mu})/2$ is replaced by s. Then, when the inverse Laplace transform of $f(s)$ exists, the density of $\mathbf{Y}$ denoted by $h(\mathbf{y})$ has the following integral representation:*

$$h(\mathbf{y}) = \int_0^\infty w(t)\, \eta_{\mathbf{Y}}(\boldsymbol{\mu}, t^{-1}\Sigma)\ \mathrm{d}t \tag{5.4}$$

where $\eta_{\mathbf{Y}}(\boldsymbol{\mu}, t^{-1}\Sigma)$ denotes the density function of a p-dimensional Gaussian random vector with mean $\boldsymbol{\mu}$ and covariance matrix $t^{-1}\Sigma$, and the weighting function $w(t)$ is obtained as follows:

$$w(t) = (2\pi)^{p/2}\,|\Sigma|^{1/2}\,t^{-p/2}\,\mathcal{L}^{-1}\,(f(s)),$$

$\mathcal{L}^{-1}(f(s))$ representing the inverse Laplace transform of $f(s)$.

In fact, $\mathcal{L}^{-1}(f(s))$ exists whenever $f(s)$ is an analytic function and $f(s)$ is $O(s^{-k})$ as $s \to \infty$ for $k > 1$; for additional properties of the Laplace transform and its inverse, one may refer to Gradshteyn and Ryzhik (1980), Chapter 17. It follows from Theorem 5.3.1 that an elliptical distribution is completely specified by its mean $\boldsymbol{\mu}$, scale parameter matrix Σ and weighting function $w(t)$, whenever the latter exists. Letting $t = 1/r$ and defining $w(t)$ to be the density function of $1/L^2$, it is seen that (5.3) and (5.4) are equivalent. On integrating $h(\mathbf{y})$ as defined in Theorem 5.3.1 over $\mathcal{R}^p$ and interchanging the order of integration, one can easily establish that $w(t)$ integrates to 1. Thus, $w(t)$ can be regarded as a weighting function. Explicit representations of $w(t)$ are given in Table 5.1 for several p-dimensional elliptically contoured distributions.

Theorem 5.3.1 enables one to determine the distribution of functions of elliptically contoured vectors in terms of their Gaussian counterparts. For instance, let $\mathbf{Y} \sim \mathcal{C}_p(\boldsymbol{\mu}, \Sigma; \xi)$ and its associated weighting function be $w(t)$. Then, the *moment-generating function* of the non-central quadratic form $\mathbf{Y}'A\mathbf{Y}$ can be obtained as follows:

$$\begin{aligned} M_{\mathbf{Y}'A\mathbf{Y}}(\theta) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{\theta\,\mathbf{y}'A\mathbf{y}}\,w(t)\,\eta_{\mathbf{Y}}(\boldsymbol{\mu}, t^{-1}\Sigma)\,\mathrm{d}t\,\mathrm{d}\mathbf{y} \\ &= \int_{0}^{\infty} w(t)\,M^{*}_{Q(\mathbf{W})}(\theta)\,\mathrm{d}t \end{aligned} \tag{5.5}$$

where

$$M^{*}_{Q(\mathbf{W})}(\theta) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\theta\,\mathbf{y}'A\mathbf{y}}\,\eta_{\mathbf{Y}}(\boldsymbol{\mu}, t^{-1}\Sigma)\,\mathrm{d}\mathbf{y}$$

is the moment-generating function of the quadratic form $Q(\mathbf{W}) = \mathbf{W}'A\mathbf{W}$ wherein $\mathbf{W} \sim \mathcal{N}_p(\boldsymbol{\mu}, t^{-1}\,\Sigma)$, which is

$$|I - 2\theta t^{-1}\,A\Sigma|^{-1/2}\,e^{-\frac{1}{2}\boldsymbol{\mu}[I-(I-2\theta t^{-1}\,A\Sigma)^{-1}]\Sigma^{-1}\,\boldsymbol{\mu}},$$

according to Equation (3:2a.1) in Mathai and Provost (1992).

Similarly, the *moments* of $\mathbf{Y}'A\mathbf{Y}$ can be evaluated as follows:

$$E(\mathbf{Y}'A\mathbf{Y})^h \equiv \int_{0}^{\infty} w(t)\,E\big[(\mathbf{W}'A\mathbf{W})^h\big]\,\mathrm{d}t, \tag{5.6}$$

Table 5.1: Some elliptically contoured distributions and their weighting functions.

Distribution	Density function	Weighting function
Gaussian	$e^{-s}/((2\pi)^{p/2}\,\lvert\Sigma\rvert^{1/2})$ $s=\mathbf{x}'\Sigma^{-1}\mathbf{x}/2$ throughout	$\delta(t-1)$ The Dirac delta function
Contaminated Normal	$\{\phi\lambda^{p/2}e^{-\lambda s}+(1-\phi)e^{-s}\}/$ $\{(2\pi)^{p/2}\,\lvert\Sigma\rvert^{1/2}\}$	$\phi\;\delta(t-\lambda)+(1-\phi)\;\delta(t-1)$
$t-$distribution with ν d.f.	$\Big\{\nu^{\nu/2}\Gamma((\nu+p)/2)\lvert\Sigma\rvert^{-1/2}$ $\times(\nu+2s)^{-(\nu+p)/2}\Big\}/\{\pi^{p/2}\Gamma(\nu/2)\}$	$\Big\{\nu(\nu t/2)^{(\nu/2)-1}e^{-\nu t/2}\Big\}/$ $\{2\Gamma(\nu/2)\}$
Multivariate Analog of the Bilateral Exponential Density	$\Big\{\Gamma(p/2)\,e^{-\sqrt{2s}}\Big\}/$ $\Big\{2^{(p+1)/2}\pi^{p/2}\Gamma(p)\lvert\Sigma\rvert^{1/2}\Big\}$	$\Big\{\Gamma(p/2)\,e^{-1/2t}\Big\}$ $\Big\{\Gamma(p)\,2\,\sqrt{\pi}\,t^{(p+3)/2}\Big\}^{-1}$
The Generalized Slash Distribution	$\nu s^{-p/2-v}\lvert\Sigma\rvert^{-1/2}\,\Big\{\Gamma(p/2+v)$ $-\Gamma(p/2+v,s)\Big\}/(2\pi)^{p/2}$	$\begin{cases}\nu\;t^{\nu-1}\,, & 0<\nu<1\\ 0\;, & \nu\geq 1\end{cases}$

where $\mathbf{W}\sim\mathcal{N}_p(\boldsymbol{\mu},t^{-1}\,\Sigma)$ and $E\big[(\mathbf{W}'A\mathbf{W})^h\big]$ can be determined from (5.7).

In general, the moments of a random variable can be obtained from its cumulants by means of a recursive relationship derived in Smith (1995), which can also be deduced for instance from Theorem 3.2b.2 in Mathai and Provost (1992). For example, the h^{th} moment of $Q(\mathbf{W})=\mathbf{W}'A\mathbf{W}$ is given by

$$E(\mathbf{W}'A\mathbf{W})^h=\mu_h=\sum_{i=0}^{h-1}\frac{(h-1)!}{(h-1-i)!\,i!}\,k(h-i)\mu_i \tag{5.7}$$

where $k(h)$, the h^{th} cumulant of $Q(\mathbf{W})$, is given by

$$k(h)=\begin{cases}2^{h-1}h!\left(\operatorname{tr}(\,t^{-1}A\Sigma)^h/h+\boldsymbol{\mu}'(\,t^{-1}A\Sigma)^{h-1}A\,\boldsymbol{\mu}\right), & h\geq 2\,,\\[2ex] \operatorname{tr}(\,t^{-1}A\Sigma)+\boldsymbol{\mu}'A\boldsymbol{\mu}\,, \quad h=1. & \end{cases}$$

5.4 Illustrative Examples

Four numerical examples involving quadratic forms and quadratic expressions in various types of elliptically contoured vectors are presented in this section. The steps to be followed for determining their distributions are described in the first example.

Example 5.4.1. Consider the quadratic form $Q^{\mathrm{I}}(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X}$ has a noncentral $t-$distribution with 10 degrees of freedom whose density function is as given in Table 5.1 with $s = (\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})/2$, $\boldsymbol{\mu} = (0,1,3,2)'$,

$$\Sigma = \begin{pmatrix} 1 & 1/2 & 2/5 & 1/2 \\ 1/2 & 1 & 1/4 & 3/8 \\ 2/5 & 1/4 & 1 & 1/3 \\ 1/2 & 3/8 & 1/3 & 1 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 1 & -6 & 2 & 1 \\ -6 & 7 & 0 & 4 \\ 2 & 0 & -4 & 1 \\ 1 & 4 & 1 & 2 \end{pmatrix}.$$

The proposed methodology comprises the following steps:

1. $Q^{\mathrm{I}}(\mathbf{X})$ is expressed as $Q_1^{\mathrm{I}}(\mathbf{W}_1) - Q_2^{\mathrm{I}}(\mathbf{W}_2) + \kappa$ in accordance with Equation (5.2).

2. The moments of $Q_i^{\mathrm{I}}(\mathbf{W}_i)$, $i = 1,2$ are determined from Equations (5.6) and (5.7).

3. A generalized gamma density function,

$$\psi(z) = \frac{\gamma}{\beta^{\alpha\gamma}\Gamma(\alpha)}\, z^{\alpha\gamma-1} e^{-(z/\beta)^{\gamma}}\, \mathcal{I}_{(0,\infty)}(z)\,, \quad \alpha>0,\ \beta>0,\ \gamma>0, \tag{5.8}$$

   is taken as base density for $Q_i^{\mathrm{I}}(\mathbf{W}_i)$, $i = 1,2$.

4. The parameters α, β and γ are determined by simultaneously solving the following nonlinear equations

$$\mu_j = m_j \qquad \text{for } j = 1,2,3,$$

   where

$$m_j = \frac{\beta^j\,\Gamma(\alpha + j/\gamma)}{\Gamma(\alpha)}\,,\ j = 0,1,\ldots$$

   are the moments associated with the generalized gamma density function and μ_j can be determined from the recursive formula (5.7).

Table 5.2: Approximate cdf of $Q^{\mathrm{I}}(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul %).

CDF	Simul. %	G.Gamma
0.01	−93.013	0.009429
0.05	−56.312	0.052101
0.10	−40.539	0.101342
0.25	−17.485	0.246064
0.50	7.9373	0.509682
0.75	39.771	0.755445
0.90	77.615	0.893073
0.95	106.51	0.942475
0.99	179.76	0.988139

5. A polynomial adjustment of degree d can be made as explained in Section 2.7.3, the resulting density approximation being

$$f_d(z) = \psi(z) \sum_{j=0}^{d} \xi_j \, z^j \, ;$$

in this case, we set $d = 7$.

6. Given the density approximations determined for $Q_1^{\mathrm{I}}(\mathbf{W}_1)$ and $Q_2^{\mathrm{I}}(\mathbf{W}_2)$, the approximate density of the difference is obtained by applying the transformation of variables technique. Shifting this density by κ then yields the desired approximation.

Certain values of the resulting approximate distribution function of $Q^{\mathrm{I}}(\mathbf{X})$ are included in Table 5.2. The percentiles were obtained by simulation on the basis of 1,000,000 replications. The plot shown in Figure 5.1 confirms that the proposed approach yields a very accurate approximation to the distribution of $Q^{\mathrm{I}}(\mathbf{X})$.

Example 5.4.2. Consider the quadratic form $Q^{\mathrm{II}}(\mathbf{X}) = \mathbf{X}' A \mathbf{X}$ where $\mathbf{X}$ is a contaminated normal random vector as specified in Table 5.1, for which $\phi = 0.4$, $\boldsymbol{\mu} = (1, 2, 3)'$,

$$\Sigma = \begin{pmatrix} 1 & 0.2 & 0.7 \\ 0.2 & 1 & 0.2 \\ 0.7 & 0.2 & 1 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 5 & 3 & 2 \\ 3 & -5 & 5 \\ 2 & 5 & -2 \end{pmatrix}.$$

It this case, a gamma distribution (as defined by (5.8) with $\gamma = 1$) was utilized as base density to obtain an approximate distribution for each quadratic form in decomposition of $Q^{\mathrm{II}}(\mathbf{X})$. Letting the integer moments of a non-negative definite quadratic form

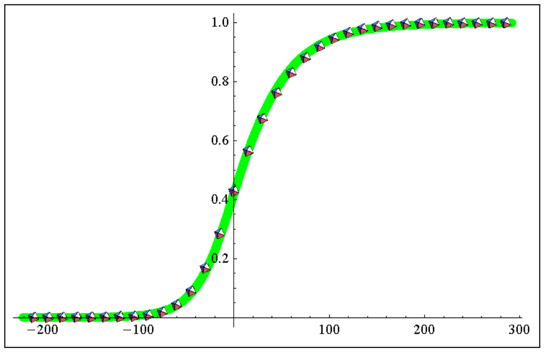


Figure 5.1: *Simulated cdf of* $Q^{\mathrm{I}}(\boldsymbol{X})$ *and cdf approximation (dots).*

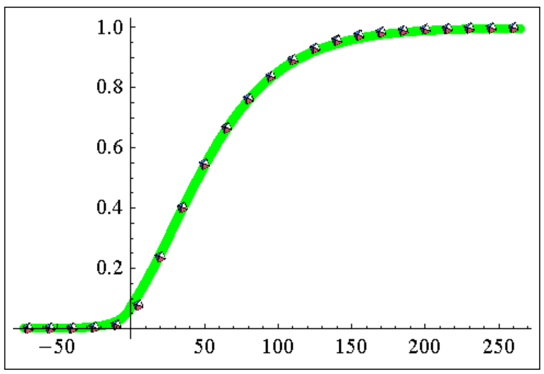


Figure 5.2: *Simulated cdf of* $Q^{\mathrm{II}}(\mathbf{X})$ *and cdf approximation (dots).*

be denoted by μ_j, $j = 1, 2, \ldots$, a gamma approximation can be specified by equating its first two moments to μ_1 and μ_2, respectively, and solving for α and β, that is, $\alpha\beta = \mu_1$ and $\alpha(\alpha+1)\beta^2 = \mu_2$, which yields

$$\alpha = \frac{\mu_1^2}{\mu_2 - \mu_1^2} \quad \text{and} \quad \beta = \frac{\mu_2}{\mu_1} - \mu_1. \tag{5.9}$$

The methodology described in Example 5.4.1 was applied in conjunction with polynomial adjustments of degree six to determine the approximate distribution of $Q^{\mathrm{II}}(\mathbf{X})$. The plot shown in Figure 5.2 indicates that the resulting approximation is very accurate.

Example 5.4.3. Consider the quadratic form $Q^{\mathrm{III}}(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X}$ follows a generalized slash distribution whose density function is as defined in Table 5.1 with $\boldsymbol{\mu} = (0, 1, 2)'$,

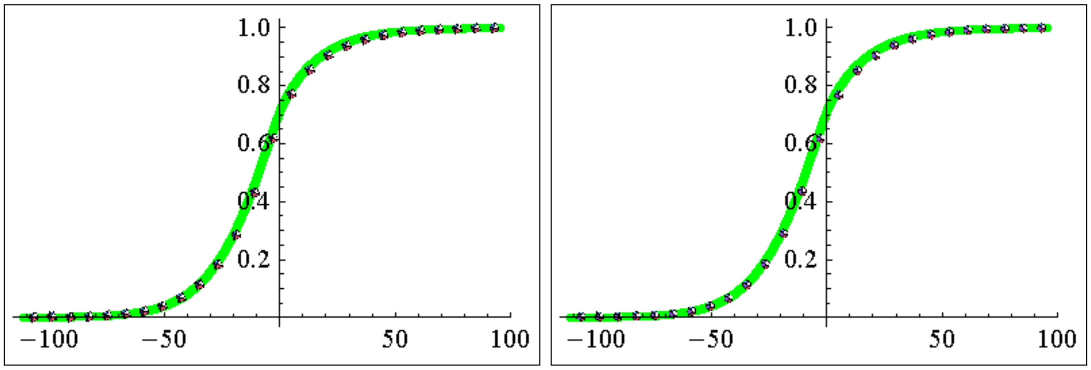


Figure 5.3: *Simulated cdf of $Q^{\text{III}}(\boldsymbol{X})$ and approximations based on polynomially adjusted gamma (left panel) and generalized gamma (right panel) distributions (dots).*

$$\Sigma = \begin{pmatrix} 1 & 1/2 & 2/5 \\ 1/2 & 1 & 1/4 \\ 2/5 & 1/4 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & -6 & 2 \\ -6 & 7 & 0 \\ 2 & 0 & -4 \end{pmatrix}.$$

By making use of the weighting function associated with the generalized slash distribution in order to determine the moments (Equation (5.7)) of the quadratic forms occurring in its decomposition and implementing the steps described in Example 5.4.1 in conjunction with a gamma distribution or a generalized gamma distribution whose associated densities are taken as base densities, one can determine an approximate distribution for $Q^{\text{III}}(\mathbf{X})$.

The left and right panels of Figure 5.3 respectively show the distribution functions resulting from gamma and generalized gamma approximations, which are superimposed on the simulated distribution function determined on the basis of 1,000,000 replications.

Example 5.4.4. Let $Q_1^*(\mathbf{X}) = (\mathbf{X}-\boldsymbol{\alpha})'A(\mathbf{X}-\boldsymbol{\alpha}) + \mathbf{a}'(\mathbf{X}-\boldsymbol{\alpha}) + d$ be a quadratic expression in a singular $t-$vector with 10 degrees of freedom where $\mathbf{X} \sim \mathcal{C}_5(\boldsymbol{\mu}, \Sigma; \xi)$, $\boldsymbol{\mu} = (4, 1, -1, 3, 2)'$,

$$\Sigma = \begin{pmatrix} 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 5 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is singular, $\boldsymbol{\alpha} = (1, 1, 0, 1, 1)'$, A is the following indefinite matrix

Table 5.3: Approximate cdf of $Q_1^*(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul %).

CDF	Simul. %	Gamma
0.01	−364.29	0.011478
0.05	−188.33	0.058634
0.10	−123.21	0.110176
0.25	−44.256	0.248199
0.50	6.1069	0.506286
0.75	39.771	0.755445
0.90	233.11	0.893207
0.95	360.83	0.943116
0.99	722.56	0.988926

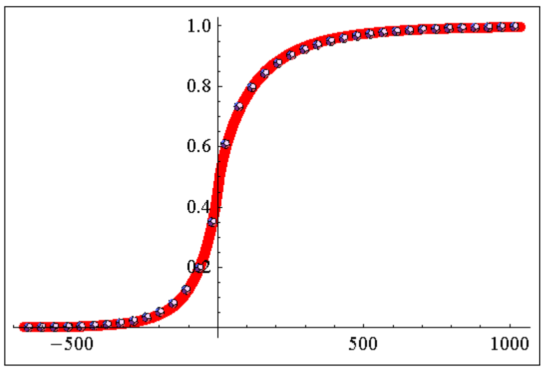


Figure 5.4: *Simulated cdf of* $Q_1^*(\mathbf{X})$ *and cdf approximation (dots).*

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & -5 \\ 1 & 1 & 2 & 3 & -5 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ -5 & -5 & 0 & 0 & -26 \end{pmatrix},$$

$\mathbf{a} = (1, 2, 3, 4, 5)'$ and $d = 6$.

When $\Sigma_{p\times p}$ is a singular matrix of rank $r < p$, we make use of the spectral decomposition theorem to express Σ as UWU' where W is a diagonal matrix whose first r diagonal elements (the non-null eigenvalues of Σ) are positive, the remaining diagonal elements being equal to zero. Next, we let $B^*_{p\times p} = UW^{1/2}$ and remove the last $p - r$

columns of B^*, which are null vectors, to obtain the matrix $B_{p\times r}$. Then, it follows that $\Sigma = BB'$. In this case, the matrices B and P were found to be

$$B = \begin{pmatrix} 1.66591 & 0.39015 & 0 & -0.26930 \\ 1.66591 & 0.39015 & 0 & -0.26930 \\ 2.03287 & -0.92672 & 0 & 0.09291 \\ 1.18171 & 0.49418 & 0 & 0.59945 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} -0.97731 & 0.00042 & -0.14936 & -0.15022 \\ 0.05695 & -0.58347 & -0.72923 & 0.35290 \\ 0.13922 & 0.69384 & -0.66277 & -0.24484 \\ -0.14916 & 0.42208 & 0.08157 & 0.89048 \end{pmatrix},$$

respectively. One can utilize the decomposition of $Q^*(\mathbf{X})$, which is provided in Equation (5.1), to determine an approximation to the distribution function of $Q_1^*(\mathbf{X})$. The approximate density functions of $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are obtained by making use of a gamma approximation, as explained in Example 5.4.2. We first approximated density of $Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)$ and then, determined the density function of $Q_1(\mathbf{W}^+)-Q_2(\mathbf{W}^-)+T_1$ by applying the transformation of variables technique.

Referring again to the decomposition (5.1), the eigenvalues of $B'AB$ were found to be $\lambda_1 = 65.8197$, $\lambda_2 = -29.5759$, $\lambda_3 = -2.24383$, $\lambda_4 = 0$, and it was determined that $n_1 = -43.6247$, $n_2 = 31.6913$, $n_3 = 2.87613$, and $n_4 = -0.154303$, and that $\mu_1 = -0.662791$, $\boldsymbol{\mu}_2 = (-1.07153, -1.2818)'$ and $c_1 = -4$. The resulting distribution function was evaluated at certain simulated percentiles obtained on the basis of 500,000 replications. The results are presented in Table 5.3 and the cdf is plotted in Figure 5.4.

Chapter 6

Quadratic Forms in Uniform, Beta and Gamma Random Variables

6.1 Introduction

A representation of the moments of quadratic forms in uniform random vectors is derived in Section 6.2. A closed form expression is obtained for the moments of quadratic forms in order statistics from a uniform population in Section 6.3. Quadratic forms in beta random variables and their order statistics are respectively considered in Sections 6.4 and 6.5. A representation of quadratic forms in gamma random variables as well as a derivation of their moments are provided in Section 6.6. A closed form representation of the moments of quadratic forms in order statistics from an exponential population is determined in Section 6.7. Several numerical examples illustrate the distributional results.

6.2 Quadratic Forms in Uniform Random Variables

Let $\mathbf{X} = (X_1, \ldots, X_n)'$ denote a random vector of independently distributed random variables whose support is the interval (a, b). Consider the quadratic form,

$$Q(\mathbf{X}) = Q(X_1, \ldots, X_n) = \mathbf{X}'A\mathbf{X} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}X_iX_j\,,$$

where $A = (a_{ij})$ is an $n \times n$ symmetric matrix and $\mathbf{X}'$ denotes the transpose of the vector $\mathbf{X}$. We note that if A is not symmetric, we can replace it without any loss of generality by $(A + A')/2$. Letting $\prod_{i,j}^{n}$ denote the double product $\prod_{i=1}^{n}\prod_{j=1}^{n}$, it follows from the multinomial expansion that

$$Q(\mathbf{X})^m = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}X_iX_j\right)^m = \sum_{(m)}\left[m!\left(\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right)\prod_{\ell=1}^{n}X_\ell^{\delta_\ell}\right],\ m=1,2,\ldots,\quad (6.1)$$

where $\sum_{(m)}$ denotes the sum over all the partitions of m into n^2 terms such that $m_{11}+m_{12}+\cdots+m_{nn}=m$ with $0\le m_{ij}\le m$, the m_{ij}'s being nonnegative integers, for $i=1,\ldots,n$ and $j=1,\ldots,n$, and $\delta_\ell=\sum_{j=1}^{n}(m_{\ell j}+m_{j\ell})$. The following identity is useful for computing sums over partitions:

$$\sum_{p_1+\cdots+p_r=p}\varphi(p_1,\ldots,p_r)=\sum_{p_1=0}^{p}\sum_{p_2=0}^{p-p_1}\cdots\sum_{p_{r-1}=0}^{p-p_1-\cdots-p_{r-2}}\varphi\Big(p_1,p_2,\ldots,p_{r-1},p-\sum_{i=1}^{r-1}p_i\Big),$$

where $p_i=0,1,\ldots,p;\ i=1,2,\ldots,r$.

Alternatively, symbolic computational software packages such as *Mathematica* can readily generate the required partitions and express $Q(\mathbf{X})^m$ as a sum of products of powers of X_ℓ's. Then, assuming that the X_i's are independently distributed, with respective density functions $f_{X_i}(x_i)$, one can determine the m^{th} moment of $Q(\mathbf{X})$ as follows:

$$\begin{aligned}
E(Q(\mathbf{X})^m) &= \int_a^b\int_a^b\cdots\int_a^b Q(\mathbf{X})^m f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&= \int_a^b\int_a^b\cdots\int_a^b\left(\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j\right)^m\prod_{\ell=1}^{n}f_{X_\ell}(x_\ell)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&= \sum_{(m)}m!\left[\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\int_a^b\int_a^b\cdots\int_a^b\left(\prod_{\ell=1}^{n}x_\ell^{\delta_\ell}\,f_{X_\ell}(x_\ell)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&= \sum_{(m)}m!\left[\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\prod_{\ell=1}^{n}\left(\int_a^b x_\ell^{\delta_\ell}\,f_{X_\ell}(x_\ell)\mathrm{d}x_\ell\right). \qquad (6.2)
\end{aligned}$$

Thus, when the X_i's are independently and uniformly distributed on the interval (a,b), one has

$$E(Q(\mathbf{X})^m) = \frac{m!}{(b-a)^n}\sum_{(m)}\left[\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\prod_{\ell=1}^{n}\Big(\frac{b^{\delta_\ell+1}-a^{\delta_\ell+1}}{1+\delta_\ell}\Big). \qquad (6.3)$$

Based on the moments of $Q(\mathbf{X})$, approximations to its distribution can be obtained by making use of an initial beta approximation. The accuracy of the approximations can be improved upon by making use of a polynomial adjustment whose coefficients can be determined from Equation (2.44) of Section 2.7.3. The methodology advocated herein is described in detail in the following example.

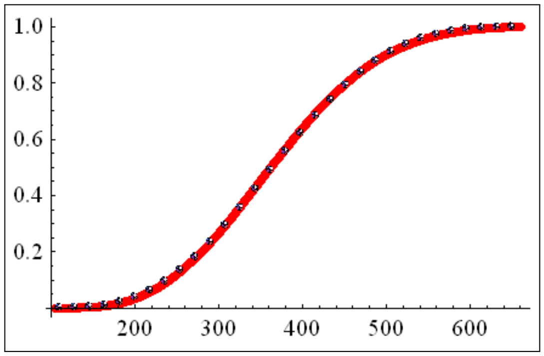


Figure 6.1: *Simulated cdf of $Q_1(\mathbf{X})$ and 7^{th} degree polynomially adjusted beta cdf approximation (dots).*

Example 6.2.1. Consider the quadratic form, $Q_1(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$, where $\mathbf{X}' = (X_1, \dots, X_5)$, the X_i's being independently and uniformly distributed on the interval $(3, 6)$, and

$$A = \begin{pmatrix} 2 & 1 & 1 & 5 & 0 \\ 1 & 0 & -2 & 0 & -1 \\ 1 & -2 & 0 & -2 & 2 \\ 5 & 0 & -2 & 0 & 3 \\ 0 & -1 & 2 & 3 & 2 \end{pmatrix}.$$

We approximate the distribution of $Q_1(\mathbf{X})$ from its moments by making use of a beta density function as base density. The proposed technique comprises the following steps:

1. The moments of $Q_1(\mathbf{X})$ are determined from the representation given in Equation (6.3), with $a = 3$, $b = 6$ and $n = 5$.

2. The base density is taken to be
$$\phi(z) = \frac{1}{\mathrm{B}(\alpha, \beta)}\, z^{\alpha-1}(1-z)^{\beta-1}\, \mathcal{I}_{(0\,,1)}(z), \quad \alpha > 0, \;\; \beta > 0\,, \tag{6.4}$$
where $\mathrm{B}(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha, \beta)$ and $\mathcal{I}_{(0\,,1)}(\cdot)$ denotes the indicator function on the interval $(0, 1)$.

3. The support (q, r) of $Q_1(\mathbf{X})$ is mapped onto the interval $(0, 1)$, the support of the beta distribution, with the affine transformation $z = (y - q)/(r - q)$, the inverse transformation being $y = z(r - q) + q$.

4. The m^{th} moment of the transformed distribution on $(0, 1)$ is given by
$$\mu_m = \frac{1}{(r-q)^m} \sum_{j=1}^{m} \binom{m}{j} E(Q_1(\mathbf{X})^j)(-q)^{m-j}\,.$$

Table 6.1: Approximate cdf of $Q_1(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Beta	Beta Poly
0.0001	111.504	0.00007692	0.00008210
0.0010	132.949	0.00131205	0.00103533
0.01	169.779	0.012625	0.009744
0.05	213.237	0.055130	0.049497
0.10	241.233	0.104888	0.100011
0.25	294.696	0.249791	0.249500
0.50	362.655	0.491610	0.499227
0.75	434.365	0.741042	0.749180
0.90	500.585	0.901307	0.899389
0.95	538.740	0.954795	0.949922
0.99	602.644	0.993672	0.989981
0.9990	654.815	0.999691	0.998937
0.9999	680.786	0.999989	0.999931

where $E(Q_1(\mathbf{X})^j)$ is obtained from (6.3).

5. The parameters of the beta density are taken to be

$$\alpha = -\mu_1 + \frac{(1-\mu_1)\,\mu_1^2}{\mu_2 - \mu_1^2} \quad\text{and}\quad \beta = -1 - \alpha + \frac{(1-\mu_1)\,\mu_1}{\mu_2 - \mu_1^2}\,.$$

6. A polynomial adjustment of degree d can be made as explained in Section 2.7.3, the resulting density approximation being

$$f_d(z) = \varphi(z)\sum_{j=0}^{d} \xi_j z^j\,;$$

in this case, we observed that $d = 7$ provides sufficient accuracy.

7. The approximate density of $Q_1(\mathbf{X})$, as obtained by applying the inverse transformation, is then given by

$$g(y) = \frac{1}{r-q}\, f_d\!\left(\frac{y-q}{r-q}\right) \mathcal{I}_{(q\,,\,r)}(y)\,.$$

The values of the approximate distribution function displayed in Table 6.1 and the plots shown in Figure 6.1 indicate that the polynomially adjusted beta distribution provides a very accurate approximation to the distribution of $Q_1(\mathbf{X})$. The simulated distribution function of $Q_1(\mathbf{X})$ was generated from 1,000,000 replications.

Result 6.2.1. The m^{th} moment of the quadratic expression $Q^*(\mathbf{X}) = \mathbf{X}^{'}A\mathbf{X} + \mathbf{b}'\mathbf{X} + \delta$, where $\mathbf{X}^{'} = (X_1, \ldots, X_n)$ is a vector of independently and uniformly distributed random variables on the interval (a, b), A is symmetric matrix, $\mathbf{b}$ is an $n \times 1$ constant vector and δ is a scalar constant can be obtained in closed form as follows:

$$\begin{aligned}
E(Q_1^*(\mathbf{X})^m) &= \int_a^b \int_a^b \cdots \int_a^b \int_a^b \Big(\frac{1}{b-a}\Big)^n \\
&\quad \times \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j + \sum_{k=1}^n b_k x_k + \delta \right)^m \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1} \mathrm{d}x_n \\
&= \Big(\frac{1}{b-a}\Big)^n \int_a^b \int_a^b \cdots \int_a^b \int_a^b \sum_{s=0}^m \binom{m}{s} \Big[\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \Big]^s \\
&\qquad \Big[\sum_{k=1}^n b_k x_k + \delta \Big]^{m-s} \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1} \mathrm{d}x_n \\
&= \Big(\frac{1}{b-a}\Big)^n \int_a^b \int_a^b \cdots \int_a^b \int_a^b \sum_{s=0}^m \binom{m}{s} \Big[\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \Big]^s \\
&\quad \times \Big[\sum_{f=0}^{m-s} \binom{m-s}{f} \Big(\sum_{k=1}^n b_k x_k \Big)^f \delta^{m-s-f} \Big] \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1} \mathrm{d}x_n \\
&= \Big(\frac{1}{b-a}\Big)^n \int_a^b \int_a^b \cdots \int_a^b \int_a^b \sum_{s=0}^m \binom{m}{s} \Big[\sum_{(s)} s! \Big(\prod_{i,j}^n \frac{a_{ij}^{s_{ij}}}{s_{ij}!} \Big) \prod_{\ell=1}^n x_\ell^{\delta_\ell} \Big] \\
&\quad \times \Big[\sum_{f=0}^{m-s} \binom{m-s}{f} \sum_{k_1, \cdots, k_n} \binom{f}{k_1, \ldots, k_n} \prod_{\ell=1}^n x_\ell^{k_\ell} \, \delta^{m-s-f} \Big] \\
&\qquad \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1} \mathrm{d}x_n \\
&= \Big(\frac{1}{b-a}\Big)^n \sum_{s=0}^m \binom{m}{s} s! \Big[\sum_{(s)} \Big(\prod_{i,j}^n \frac{a_{ij}^{s_{ij}}}{s_{ij}!} \Big) \Big] \Big[\sum_{f=0}^{m-s} \binom{m-s}{f} \\
&\qquad \times \sum_{k_1, \cdots, k_n} \binom{f}{k_1, \ldots, k_n} \delta^{m-s-f} \Big] \int_a^b \int_a^b \cdots \int_a^b \int_a^b \prod_{\ell=1}^n x_\ell^{\delta_\ell + k_\ell} \\
&\qquad \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1} \mathrm{d}x_n
\end{aligned}$$

Table 6.2: Approximate cdf of $Q_1^*(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Beta	Beta Poly
0.0001	34.703	0.00041980	0.00001544
0.0010	42.203	0.00413030	0.00097898
0.01	55.627	0.024080	0.011425
0.05	70.927	0.070212	0.047505
0.10	81.647	0.116757	0.096854
0.25	106.55	0.260830	0.248639
0.50	142.94	0.516285	0.499375
0.75	181.75	0.766097	0.748432
0.90	215.68	0.914014	0.894654
0.95	233.52	0.959897	0.948470
0.99	258.54	0.992134	0.989677
0.9990	654.82	0.999127	0.998911
0.9999	286.70	0.999928	0.999916

$$= \left(\frac{1}{b-a}\right)^n \sum_{s=0}^{m} \binom{m}{s} s! \left[\sum_{(s)} \left(\prod_{i,j}^{n} \frac{a_{ij}^{s_{ij}}}{s_{ij}!}\right)\right] \left[\sum_{f=0}^{m-s} \binom{m-s}{f} \right.$$
$$\left. \times \sum_{k_1,\cdots,k_n} \binom{f}{k_1,\cdots,k_n} \delta^{m-s-f}\right] \prod_{\ell=1}^{n} \frac{b^{k_\ell+\delta_\ell+1} - a^{k_\ell+\delta_\ell+1}}{k_\ell+\delta_\ell+1} \tag{6.5}$$

where $\sum_{i=1}^n k_i = f$.

Example 6.2.2. Consider the quadratic expression, $Q_1^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{b}'\mathbf{X} + \delta$, where $\mathbf{X}' = (X_1, \ldots, X_4)$, the X_i's being uniformly and independently distributed in the interval $(2, 5)$, $\mathbf{b}' = (1, 2, 3, 4)$, $\delta = 3$ and

$$A = \begin{pmatrix} -3 & 1 & 4 & 5 \\ 1 & 0 & -2 & 0 \\ 4 & -2 & 0 & -2 \\ 5 & 0 & -2 & 0 \end{pmatrix}.$$

The beta approximation to the distribution function of $Q_1^*(\mathbf{X})$, as evaluated from Steps 1 to 7 of the proposed approach, is plotted in Figure 6.2 where it is superimposed on the simulated distribution function determined on the basis of 1,000,000 replications.

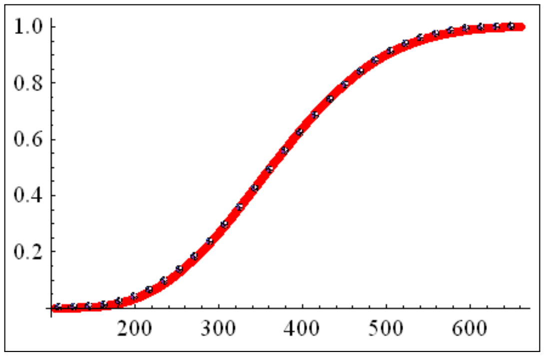


Figure 6.2: *Simulated cdf of* $Q_1^*(\boldsymbol{X})$ *and* 7^{th} *degree polynomially adjusted beta cdf approximation (dots).*

The values of the approximate distribution function presented in Table 6.2 suggest that, following a polynomial adjustment of degree 7, the beta distribution provides a reasonably accurate approximation to the distribution of $Q_1^*(\mathbf{X})$.

6.3 Quadratic Forms in Order Statistics From a Uniform Population

Consider the order statistics $U_1 \le \cdots \le U_k$ obtained from a simple random sample of size n coming from a continuous uniform population on the interval $(0,1)$ and denote the joint density and distribution functions of $U_1, \ldots, U_k$ by $f(\cdot)$ and $F(\cdot)$, respectively. Letting $U_1 = X_{r_1:n}$ be the r_1^{th} order statistic, $U_2 = X_{r_1+r_2:n}$ be the $(r_1+r_2)^{\text{th}}$ order statistic and so on, U_k being the $(r_1+\cdots+r_k)^{\text{th}}$ order statistic, the joint density of $U_1, \ldots, U_k$ is given by

$$
\begin{aligned}
f(u_1, \ldots, u_k) \;=\; & \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)}[F(u_1)]^{r_1-1}\left\{\prod_{i=2}^{k}[F(u_i)-F(u_{i-1})]^{r_i-1}\right\} \\
& \times[1-F(u_k)]^{r_{k+1}-1}\prod_{\ell=1}^{k} f(u_\ell)\,, \qquad (6.6)
\end{aligned}
$$

whenever $0 \le u_1 \le u_2 \le \cdots \le u_k \le 1$, with $r_{k+1}-1 = n - \sum_{i=1}^{k} r_i$.

Letting $\mathbf{U}' = (U_1, \ldots, U_k)$, $\mathbf{c}' = (c_1, \ldots, c_k)$ be a constant vector and making use of the expansion given in Equation (6.1) with $\delta_\ell = \sum_{j=1}^{n}(m_{\ell j} + m_{j\ell})$, $\ell = 1, \ldots, n$, the m^{th} moment of $Q(\mathbf{U}) = (\mathbf{U}-\mathbf{c})'A(\mathbf{U}-\mathbf{c})$ can be determined as follows:

$$
\begin{aligned}
E(Q(\mathbf{U})^m) &= \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)}\sum_{(m)} m!\left[\prod_{i,j}^{k}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\int \underset{0\le u_1\le\cdots\le u_k\le 1}{\int}\cdots\int\left(\prod_{\ell=1}^{k}(u_\ell-c_\ell)^{\delta_\ell}\right)\\
&\quad\times(u_1-c_1)^{r_1-1}[(u_2-u_1)-(c_2-c_1)]^{r_2-1}\cdots[(u_k-u_{k-1})\\
&\quad-(c_k-c_{k-1})]^{r_k-1}(1-(u_k-c_k))^{r_{k+1}-1}\mathrm{d}u_1\ldots\mathrm{d}u_k\\
&= \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)}\sum_{(m)} m!\left[\prod_{i,j}^{k}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\int_{u_k=0}^{1}\int_{u_{k-1}=0}^{u_k}\cdots\int_{u_1=0}^{u_2}\\
&\quad\times\left(\prod_{\ell=1}^{k}\sum_{\alpha_\ell=0}^{\delta_\ell}\binom{\delta_\ell}{\alpha_\ell}u_\ell^{\delta_\ell}(-c_\ell)^{\delta_\ell-\alpha_\ell}\right)\left(\sum_{j_1=0}^{r_1-1}\binom{r_1-1}{j_1}u_1^{j_1}(-c_1)^{r_1-j_1-1}\right)\\
&\quad\times\left(\sum_{j_2=0}^{r_2-1}\binom{r_2-1}{j_2}(u_2-u_1)^{j_2}(c_2-c_1)^{r_2-j_2-1}\right)\cdots\\
&\quad\times\left(\sum_{j_k=0}^{r_k-1}\binom{r_k-1}{j_k}(u_k-u_{k-1})^{j_k}(c_k-c_{k-1})^{r_k-j_2-1}\right)\\
&\quad\times\left(\sum_{j_{k+1}=0}^{r_{k+1}-1}\binom{r_{k+1}-1}{j_{k+1}}(1-u_k)^{j_{k+1}}c_k^{r_{k+1}-j_{k+1}-1}\right)\mathrm{d}u_1\ldots\mathrm{d}u_k\\
&= \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)}\sum_{(m)} m!\left[\prod_{i,j}^{k}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\sum_{j_1=0}^{r_1-1}\sum_{j_2=0}^{r_2-1}\cdots\sum_{j_k=0}^{r_k-1}\sum_{j_{k+1}=0}^{r_{k+1}-1}\binom{r_1-1}{j_1}\\
&\quad\times\binom{r_2-1}{j_2}\binom{r_k-1}{j_k}\cdots\binom{r_{k+1}-1}{j_{k+1}}(-c_1)^{r_1-j_1-1}(c_2-c_1)^{r_2-j_2-1}\\
&\quad\quad\cdots(c_k-c_{k-1})^{r_k-j_2-1}c_k^{r_{k+1}-j_{k+1}-1}\\
&\quad\times\int_{u_k=0}^{1}\int_{u_{k-1}=0}^{u_k}\cdots\int_{u_1=0}^{u_2}\left(\prod_{\ell=1}^{k}\sum_{\alpha_\ell=0}^{\delta_\ell}\binom{\delta_\ell}{\alpha_\ell}u_\ell^{\delta_\ell}(-c_\ell)^{\delta_\ell-\alpha_\ell}\right)\\
&\quad\times u_1^{j_1}(u_2-u_1)^{j_2}\ldots(u_k-u_{k-1})^{j_k}(1-u_k)^{j_{k-1}}\mathrm{d}u_1\ldots\mathrm{d}u_k\,.
\end{aligned}
$$

On integrating the terms involving u and letting $\upsilon=\frac{u_1}{u_2}$, one has

$$
\int_{u_1=0}^{u_2} u_1^{\delta_1+j_1}(u_2-u_1)^{j_2}\mathrm{d}u_1 = \int_{\upsilon=0}^{1} u_2^{j_1+j_2+\delta_1+1}\upsilon^{\delta_1+j_1}(1-\upsilon)^{j_2}\mathrm{d}\upsilon
$$

$$= \; u_2^{j_1+j_2+\delta_1+1}\frac{\Gamma(\delta_1+j_1+1)\Gamma(j_2+1)}{\Gamma(j_1+j_2+\delta_1+2)}\,;$$

similarly,

$$\int_{u_2=0}^{u_3} u_2^{j_1+j_2+\delta_1+\delta_2+1}(u_3-u_2)^{j_3}\mathrm{d}u_2 \;=\; u_3^{j_1+j_2+j_3+\delta_1+\delta_2+2} \times\frac{\Gamma(j_1+j_2+\delta_1+\delta_2+2)\Gamma(j_3+1)}{\Gamma(j_1+j_2+j_3+\delta_1+\delta_2+3)}\,,\ldots,$$

$$\int_{u_{k-1}=0}^{u_k} u_{k-1}^{j_1+\cdots+j_{k-1}+\delta_1+\cdots+\delta_{k-1}+k-2}(u_k-u_{k-1})^{j_k}\mathrm{d}u_{k-1} = u_k^{j_1+j_2+\cdots+j_k+\delta_1+\cdots+\delta_{k-1}+k-1} \times\frac{\Gamma(j_1+j_2+\cdots+j_{k-1}+\delta_1+\cdots+\delta_{k-1}+k-1)\Gamma(j_k+1)}{\Gamma(j_1+\cdots+j_k+\delta_1+\cdots+\delta_{k-1}+k)}\,,$$

and finally,

$$\int_{u_k=0}^{1} u_k^{j_1+j_2+\cdots+j_k+\delta_1+\cdots+\delta_{k-1}+\delta_k+k-1}(1-u_k)^{j_{k+1}}\mathrm{d}u_k$$

$$= \;\frac{\Gamma(j_1+j_2+\cdots+j_k+\delta_1+\cdots+\delta_k+k)\Gamma(j_{k+1}+1)}{\Gamma(j_1+\cdots+j_{k+1}+\delta_1+\cdots+\delta_k+k+1)}\,.$$

Thus,

$$\begin{aligned} E(Q(\mathbf{U})^m) \;=\; & \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)}\sum_{(m)} m!\left[\prod_{i,j}^{k}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\sum_{j_1=0}^{r_1-1}\sum_{j_2=0}^{r_2-1}\cdots\sum_{j_k=0}^{r_k-1}\sum_{j_{k+1}=0}^{r_{k+1}-1}\Bigg(\prod_{\ell=1}^{k}\sum_{\alpha_\ell=0}^{\delta_\ell} \\ & \times\binom{\delta_\ell}{\alpha_\ell}\binom{r_1-1}{j_1}\binom{r_2-1}{j_2}\cdots\binom{r_k-1}{j_k}\binom{r_{k+1}-1}{j_{k+1}} \\ & \times(-c_\ell)^{\delta_\ell-\alpha_\ell}(-c_1)^{r_1-j_1-1}(c_2-c_1)^{r_2-j_2-1}\cdots \\ & \times(c_k-c_{k-1})^{r_k-j_k-1}\,c_k^{r_{k+1}-j_{k+1}-1}\Bigg) \\ & \times\left[\prod_{i=1}^{k}\frac{\Gamma(j_1+j_2+\cdots+j_i+\delta_1+\cdots+\delta_i+i)\Gamma(j_{i+1}+1)}{\Gamma(j_1+\cdots+j_{i+1}+\delta_1+\cdots+\delta_i+i+1)}\right]. \end{aligned} \tag{6.7}$$

Since the r_j's and the δ_j's are non-negative integers, all the gamma functions exist and no further conditions are required.

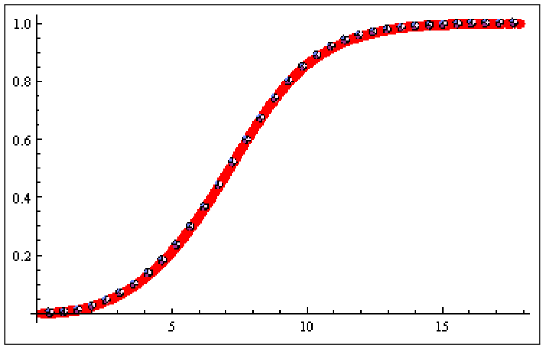


Figure 6.3: *Simulated cdf of $Q_1(\boldsymbol{U})$ and beta cdf approximation (dots).*

Remark 6.3.1. It follows that the m^{th} moment of $Q^*(\mathbf{U}) = \mathbf{U}'A\mathbf{U}$ is

$$E(Q^*(\mathbf{U})^m) = \frac{\Gamma(n+1)}{r_1} \sum_{(m)} m! \left[\prod_{i,j}^{k} \frac{a_{ij}^{m_{ij}}}{m_{ij}!} \right] \left[\prod_{j=1}^{k} \frac{\Gamma(r_1 + \ldots + r_j + \delta_1 + \ldots + \delta_j)}{\Gamma(r_1 + \ldots + r_{j+1} + \delta_1 + \ldots + \delta_j)} \right]. \quad (6.8)$$

Example 6.3.1. Let the order statistics $U_1 \leq \cdots \leq U_5$ originate from a random sample of uniform random variables on $(0,1)$ and $Q_1(\mathbf{U}) = \mathbf{U}'A\mathbf{U}$ be a quadratic form where $\mathbf{U}' = (U_1, \ldots, U_5)$ and

$$A = \begin{pmatrix} -3 & 1 & 4 & 5 & 0 \\ 1 & 0 & -2 & 0 & -1 \\ 4 & -2 & 0 & -2 & 2 \\ 5 & 0 & -2 & 0 & 3 \\ 0 & -1 & 2 & 3 & 4 \end{pmatrix}.$$

One can approximate the distribution function of $Q_1(\mathbf{U})$ by means of a beta distribution by following the seven steps described in Example 6.2.1. This density approximation is plotted in Figure 6.3 where it is superimposed on the simulated distribution function, which was obtained on the basis of 1,000,000 replications. The values of the approximate distribution functions included in Table 6.3 suggest that, following a polynomial adjustment of degree 8, the beta distribution provides a very accurate approximation to the distribution of $Q_1(\mathbf{U})$.

Remark 6.3.2. More generally, suppose that $U_1 \leq \cdots \leq U_n$ are order statistics from a $Uniform(a, b)$ population. In this case, the m^{th} moment of the quadratic form $Q_2(\mathbf{U}) =$

Table 6.3: Approximate cdf of $Q_1(\mathbf{U})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Beta	Beta Poly
0.0001	0.2636	0.00002977	0.000081678
0.0010	0.6019	0.00213030	0.000938185
0.01	1.4747	0.003627	0.010812
0.05	2.8063	0.038119	0.049095
0.10	3.7001	0.094182	0.097786
0.25	5.3459	0.269229	0.252157
0.50	7.1402	0.516618	0.498152
0.75	8.9050	0.738569	0.750771
0.90	10.637	0.887226	0.901552
0.95	11.814	0.946025	0.949137
0.99	14.129	0.992684	0.990303
0.9990	16.524	0.999763	0.998806
0.9999	18.131	0.999997	0.999911

$\mathbf{U}'A\mathbf{U}$, can be obtained numerically from the following expressions:

$$\begin{aligned} E(Q_2(\mathbf{U})^m) &= \int_a^b \int_a^{u_{n-1}} \cdots \int_a^{u_3} \int_a^{u_2} n! \Big(\frac{1}{b-a}\Big)^n \\ &\quad \times \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij} u_i u_j \right)^m \mathrm{d}u_1 \mathrm{d}u_2 \cdots \mathrm{d}u_{n-1} \mathrm{d}u_n \\ &= n! \left(\frac{1}{b-a}\right)^n \sum_{(m)} m! \left[\prod_{i,j}^n \frac{a_{ij}^{m_{ij}}}{m_{ij}!} \right] \int_a^b \int_a^{u_{n-1}} \cdots \int_a^{u_3} \int_a^{u_2} \\ &\qquad \times \left(\prod_{\ell=1}^n u_\ell^{\delta_\ell} \right) \mathrm{d}u_1 \mathrm{d}u_2 \cdots \mathrm{d}u_{n-1} \mathrm{d}u_n \,. \end{aligned} \tag{6.9}$$

Example 6.3.2. Replacing $U_1, \ldots, U_n$ in Example 6.3.1, by the order statistics $U_1 \leq \cdots \leq U_5$ obtained from a random sample generated from a $Uniform(2,5)$ population, denoting the resulting quadratic form by $Q_2(\mathbf{U})$ and following the steps described in Example 6.2.1, one can approximate the density function of $Q_2(\mathbf{U})$ from its moments from the representation given in (6.9). The approximate distribution function is tabulated for certain percentiles in Table 6.4 and superimposed on the simulated distribution in Figure 6.4. Once again, close agreement is observed with the simulated distribution (based on

Table 6.4: Approximate cdf of $Q_2(\mathbf{U})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Beta	Beta Poly
0.0001	112.81	0.00006770	0.00008916
0.0010	131.13	0.00085479	0.00088193
0.01	164.11	0.007038	0.010312
0.05	201.50	0.040145	0.052437
0.10	223.32	0.096718	0.098555
0.25	259.73	0.269143	0.245106
0.50	296.99	0.518558	0.504306
0.75	330.53	0.738310	0.749871
0.90	361.29	0.884750	0.898980
0.95	381.16	0.943191	0.949932
0.99	418.96	0.991702	0.989496
0.9990	457.15	0.999725	0.999255
0.9999	483.37	0.999998	0.999988

1,000,000 replications).

Example 6.3.3. Let $U_1 \leq \cdots \leq U_k$ be order statistics obtained from a simple random sample of size n generated from a continuous uniform population on the interval $(0,1)$. Consider the quadratic form, $S^2 = (\mathbf{U} - \boldsymbol{\mu})' V^{-1} (\mathbf{U} - \boldsymbol{\mu})$ as defined in Equation (4) of Hartley and Pfaffenberger (1972) where $\mathbf{U}' = (U_1, \ldots, U_k)$, $\mu_j = E(U_j) = j/(n+1)$ and the elements v_{ij} of the covariance matrix V associated with the random vector $\mathbf{U}$ are given by

$$v_{ij} = \frac{i\,(n-j+1)}{(n+1)^2(n+2)}\,, \qquad i \leq j\,.$$

Hartley and Pfaffenberger (1972) obtained the exact upper 5^{th} percentage point of the distribution of S^2 by making use of numerical integration recurrence formulas and proposed a Type V Pearson curve approximation. We determined the fifth percentile with the proposed methodology and then by making use of Monte Carlo simulations on the basis of 1,000,000 replications. The results presented in Table 6.5 indicate that the proposed approximation is more accurate than that utilized by Hartley and Pfaffenberger.

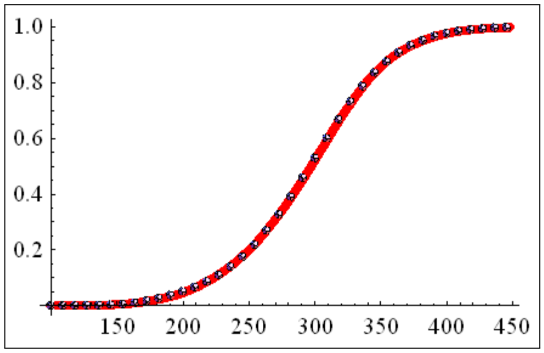


Figure 6.4: *Simulated cdf of $Q_2(\boldsymbol{U})$ and beta cdf approximation (dots).*

Table 6.5: Upper 5^{th} percentage points of S^2 for various values of n.

n	Pearson Type V	Exact	Proposed Method	Monte Carlo
3	6.980	7.390	7.272	7.3850
4	8.980	9.270	9.220	9.2790
5	10.89	11.14	11.11	11.147
6	12.74	12.96	12.94	13.006
7	14.52	14.71	14.71	14.721
8	16.26	16.44	16.43	16.443
9	17.95	18.11	18.04	18.106
10	19.61	19.75	19.68	19.737
11	21.23	21.35	21.28	21.342
12	22.83	22.94	22.85	22.937

6.4 Quadratic Forms in Beta Random Variables

Noting that the uniform distribution is a particular case of the beta distribution, we now extend the results to quadratic form in beta random variables.

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ denote a random vector of independently distributed beta random variables with parameters α and β and $Q(\mathbf{Y}) = Q(Y_1, \ldots, Y_n) = \mathbf{Y}'A\mathbf{Y}$ where $A = (a_{ij})$ is a $n \times n$ symmetric matrix. In light of Equation (6.1) and making use of the same notation, one can determine the m^{th} moment of $Q(\mathbf{Y})$ as follows:

$$\begin{aligned}
E(Q(\mathbf{Y})^m) &= \sum_{(m)} m! \left[\prod_{i,j}^{n} \frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right] \int_0^1 \int_0^1 \cdots \int_0^1 \left(\prod_{l=1}^{n} y_l^{\delta_l}\right) \\
&\quad \times \left(\frac{1}{B(\alpha,\beta)}\right)^n \prod_{j=1}^{n} \left(y_j^{\alpha-1}(1-y_j)^{\beta-1}\right) \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&= m! \sum_{(m)} \left[\prod_{i,j}^{n} \frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right] \int_0^1 \int_0^1 \cdots \int_0^1 \left(\frac{1}{B(\alpha,\beta)}\right)^n \\
&\quad \times \prod_{j=1}^{n} \left(y_j^{\delta_j+\alpha-1}(1-y_j)^{\beta-1}\right) \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right)^n m! \sum_{(m)} \left[\prod_{i,j}^{n} \frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right] \prod_{k=1}^{n} \left(\frac{\Gamma(\alpha+\delta_k)}{\Gamma(\alpha+\beta+\delta_k)}\right). \qquad (6.10)
\end{aligned}$$

Example 6.4.1. Consider the quadratic form, $Q_1(\mathbf{Y}) = \mathbf{Y}'A\mathbf{Y}$, where $\mathbf{Y} = (Y_1, \ldots, Y_4)$ has a beta distribution with parameters $\alpha = 3$ and $\beta = 5$ and

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & -1 & 0 \\ 2 & -1 & 4 & 3 \\ 3 & 0 & 3 & 1 \end{pmatrix}.$$

The steps described in Example 6.2.1 in conjuction with the moment representation provided in Equation (6.10) yield an approximate beta density function for $Q_1(\mathbf{Y})$. The results included in Table 6.6 and Figure 6.5 indicate that the approximate distribution is in close agreement with the simulated distribution (based on 1,000,000 replications).

Table 6.6: Approximate cdf of $Q_1(\mathbf{Y})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Beta	Beta Poly
0.0001	0.1752	0.00020628	0.000037176
0.0010	0.3373	0.00212657	0.000705537
0.01	0.6578	0.015853	0.009003
0.05	1.1054	0.062765	0.049370
0.10	1.4168	0.113877	0.100487
0.25	2.0694	0.256351	0.252974
0.50	3.0047	0.490315	0.500682
0.75	4.1836	0.739495	0.748943
0.90	5.4126	0.896960	0.900290
0.95	6.2319	0.951701	0.951536
0.99	7.9164	0.993377	0.989582
0.9990	9.9633	0.999816	0.998907
0.9999	11.709	0.999999	0.999981

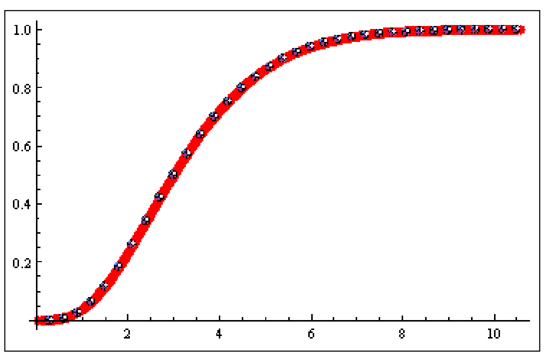


Figure 6.5: *Simulated cdf of $Q_1(\mathbf{Y})$ and beta cdf approximation (dots)*

6.5 Quadratic Forms in Order Statistics From a Beta Population

In this section, we determine the moments of a quadratic form $Q(\mathbf{W}) = \mathbf{W}'A\mathbf{W}$ for the case where $\mathbf{W}$ is a vector of order statistics $W_1 \leq \cdots \leq W_n$ obtained from a random sample of a beta distributed population with parameters α and β, whose density function is as specified in Equation (6.4).

In this case, the m^{th} moment of the quadratic form $Q(\mathbf{W})$, denoted by $\mu_m^{\dagger}$ can be obtained as follows:

$$\begin{aligned}
\mu_m^{\dagger} &= \int_0^1 \int_0^{w_{n-1}} \cdots \int_0^{w_3} \int_0^{w_2} n! \left(\frac{1}{\mathrm{B}(\alpha,\beta)} \right)^n \left(\prod_{k=1}^{n} w_k^{\alpha-1} (1-w_k)^{\beta-1} \right) \\
&\qquad \times \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} w_i w_j \right)^m \mathrm{d}w_1 \mathrm{d}w_2 \cdots \mathrm{d}w_{n-1} \mathrm{d}w_n \\
&= n! \left(\frac{1}{\mathrm{B}(\alpha,\beta)} \right)^n \sum_{(m)} m! \left[\prod_{i,j}^{n} \frac{a_{ij}^{m_{ij}}}{m_{ij}!} \right] \int_0^1 \int_0^{w_{n-1}} \cdots \int_0^{w_3} \int_0^{w_2} \left(\prod_{\ell=1}^{n} w_\ell^{\delta_\ell} \right) \\
&\qquad \times \left(\prod_{k=1}^{n} w_k^{\alpha-1} (1-w_k)^{\beta-1} \right) \mathrm{d}w_1 \mathrm{d}w_2 \cdots \mathrm{d}w_{n-1} \mathrm{d}w_n \\
&= n! \left(\frac{1}{\mathrm{B}(\alpha,\beta)} \right)^n \sum_{(m)} m! \left[\prod_{i,j}^{n} \frac{a_{ij}^{m_{ij}}}{m_{ij}!} \right] \int_0^1 \int_0^{w_{n-1}} \cdots \int_0^{w_3} \int_0^{w_2} \\
&\qquad \left(\prod_{k=1}^{n} w_k^{\alpha-1+\delta_k} (1-w_k)^{\beta-1} \right) \mathrm{d}w_1 \mathrm{d}w_2 \cdots \mathrm{d}w_{n-1} \mathrm{d}w_n .
\end{aligned}$$

On integrating the terms involving w, one has

$$\int_{w_1=0}^{w_2} w_1^{\alpha+\delta_1-1} (1-w_1)^{\beta-1} \mathrm{d}w_1 = \frac{w_2^{\alpha+\delta_1-1}(1-w_2)^{\beta-1}}{\mathrm{B}(\alpha+\delta_1,\beta)} ;$$

similarly,

$$\int_{w_2=0}^{w_3} w_2^{2\,\alpha+\delta_1+\delta_2-2} (1-w_2)^{2\,\beta-2} \mathrm{d}w_2 = \frac{w_3^{2\,\alpha+\delta_1+\delta_2-1}(1-w_3)^{2\,\beta-2}}{\mathrm{B}(2\,\alpha+\delta_1+\delta_2-1, 2\,\beta-1)} , \ldots ,$$

$$\int_{w_{n-1}=0}^{w_n} w_{n-1}^{(n-1)\,\alpha+\delta_1+\delta_2+\cdots+\delta_{n-1}-n+1} (1-w_{n-1})^{(n-1)\,\beta-(n-1)} \mathrm{d}w_{n-1}$$

$$= \frac{w_n^{(n-1)\,\alpha+\delta_1+\delta_2+\cdots+\delta_{n-1}-(n-1)}(1-w_n)^{(n-1)\,\beta-(n-1)}}{\mathrm{B}\Big((n-1)\,\alpha+\delta_1+\delta_2+\cdots+\delta_{n-1}-(n-2),(n-1)\,\beta-(n-2)\Big)}\,,$$

and finally,

$$\int_{w_n=0}^{1} w_n^{n\,\alpha+\delta_1+\delta_2+\cdots+\delta_n-n}(1-w_n)^{n\,\beta-n}\mathrm{d}w_n$$

$$= \frac{1}{\mathrm{B}\Big(n\,\alpha+\delta_1+\delta_2+\cdots+\delta_n-(n-1),n\,\beta-(n-1)\Big)}\,.$$

Thus,

$$\begin{aligned}\mu_m^{\dagger} \;&=\; n!\left(\frac{1}{\mathrm{B}(\alpha,\beta)}\right)^{n}\sum_{(m)} m!\left[\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\\ &\quad\times\prod_{k=1}^{n}\frac{1}{\mathrm{B}\Big(k\,\alpha+\delta_1+\delta_2+\cdots+\delta_k-(k-1),k\,\beta-(k-1)\Big)}\,. \qquad (6.11)\end{aligned}$$

Example 6.5.1. Consider order statistics $W_1 \leq \cdots \leq W_5$ obtained from a random sample generated from a beta distribution with parameters $\alpha = 2$ and $\beta = 3$. Let $Q_1(\mathbf{W}) = \mathbf{W}'A\mathbf{W}$ be a quadratic form where $\mathbf{W}' = (W_1,\ldots,W_5)$ and

$$A = \begin{pmatrix} 3 & 1 & 4 & 2 & 0\\ 1 & 0 & -1 & 0 & -1\\ 4 & -1 & 1 & 3 & 2\\ 2 & 0 & 3 & 0 & 0\\ 0 & -1 & 2 & 0 & 1 \end{pmatrix}.$$

On following the steps outlined in Example 6.2.1 in conjunction with the moments obtained from Equation (6.11), one can approximate distribution function of $Q_1(\mathbf{W})$ at various percentiles by making use of a polynomially adjusted beta density. The values of the approximate distribution function presented in Table 6.7 suggest that, following a polynomial adjustment of degree 8, the adjusted beta distribution function provides a very accurate approximation to the distribution of $Q_1(\mathbf{W})$. This approximation is plotted in Figure 6.6 where it is superimposed on the simulated distribution function determined on the basis of 1,000,000 replications.

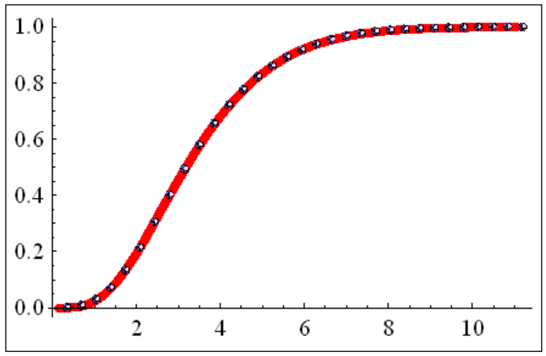


Figure 6.6: *Simulated cdf of $Q_1(\mathbf{W})$ and beta cdf approximation (dots)*

Table 6.7: Approximate cdf of $Q_1(\mathbf{W})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Beta	Beta Poly
0.0001	0.2609	0.00020628	0.000035493
0.0010	0.4417	0.00212657	0.000711698
0.01	0.7919	0.015853	0.009417
0.05	1.2553	0.062765	0.050034
0.10	1.5731	0.113877	0.100828
0.25	2.2342	0.256351	0.252130
0.50	3.1831	0.490315	0.500107
0.75	4.3808	0.739495	0.747402
0.90	5.6653	0.896960	0.899517
0.95	6.5306	0.951701	0.951151
0.99	8.3057	0.993377	0.989656
0.9990	10.507	0.999816	0.998765
0.9999	12.356	0.999999	0.999947

6.6 Quadratic Forms in Gamma Random Variables

Let $\mathbf{X} = (X_1, \ldots, X_n)'$ denote a random vector whose components are independently distributed gamma random variables with parameters α and β whose density function is given by

$$\psi(x) \;=\; \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\,\beta^{\alpha}}\mathcal{I}_{\mathcal{R}^+}(x)\ \ ,\ \alpha>0\,,\ \beta>0\,, \tag{6.12}$$

where $\mathcal{I}_{\mathcal{R}^+}(x)$ denotes the indicator function on the set of positive real numbers. Then, in light of Equation (6.2), one can determine the m^{th} moment of $Q(\mathbf{X})$ as follows:

$$\begin{aligned} E(Q(\mathbf{X})^m) &= m!\sum_{(m)}\left[\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\prod_{\ell=1}^{n}\Big(\int_0^\infty \frac{x_\ell^{\delta_\ell+\alpha-1}e^{-x_\ell/\beta}}{\Gamma(\alpha)\,\beta^\alpha}\Big)\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &= m!\,\Gamma(\alpha)^{-n}\sum_{(m)}\left[\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\beta^{\sum_{\ell=1}^n\delta_\ell}\prod_{\ell=1}^{n}\Gamma(\alpha+\delta_\ell) \\ &\equiv \mu_m\,. \end{aligned} \tag{6.13}$$

Accordingly, when the components of the random vector $\mathbf{X}$ are exponentially distributed with parameter β, their density function is

$$f(x) = \frac{1}{\beta}e^{-x/\beta}\,\mathcal{I}_{\mathcal{R}^+}(x)\ \ ,\ \beta>0\,, \tag{6.14}$$

and the m^{th} moment of $Q(\mathbf{X})$ is

$$E(Q(\mathbf{X})^m) \;=\; m!\sum_{(m)}\left[\prod_{i,j}^{n}\frac{a_{ij}^{m_{ij}}}{m_{ij}!}\right]\beta^{\sum_{\ell=1}^n\delta_\ell}\prod_{\ell=1}^{n}\Gamma(1+\delta_\ell)\,. \tag{6.15}$$

Given the moments of such quadratic forms, approximations to their distribution can be obtained by making use of the techniques advocated in Section 2.7.

Example 6.6.1. Consider the quadratic form $Q_1(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X} = (X_1, \ldots, X_5)$, and

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0\\ 3 & 0 & 2 & 0 & 1\\ 2 & 2 & 0 & 3 & 2\\ 1 & 0 & 3 & 1 & 0\\ 0 & 1 & 2 & 0 & 6 \end{pmatrix},$$

Table 6.8: Approximate cdf's of $Q_1(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Ge.G.Poly
0.01	0.2562	0.008006
0.05	0.6255	0.045688
0.10	0.9717	0.095557
0.25	1.9191	0.247761
0.50	3.8274	0.501751
0.75	8.3808	0.749402
0.90	12.108	0.900084
0.95	16.261	0.949537
0.99	27.589	0.989890

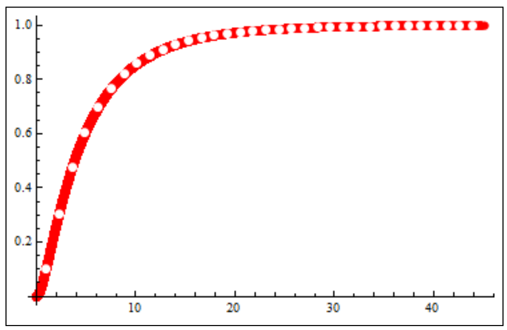


Figure 6.7: *Simulated cdf of* $Q_1(\mathbf{X})$ *and* 7^{th} *degree polynomially adjusted generalized gamma cdf approximation (dots).*

the X_i's being independently and exponentially distributed with parameter $\beta = 3$.

Since the exponential distribution has a semi-infinite support and all the elements of A are nonnegative, a generalized gamma distribution can be used as base density to determine an approximate distribution for $Q_1(\mathbf{X})$. The proposed methodology comprises the steps described in Example 5.4.1. The moments of $Q_1(\mathbf{X})$ are determined from Equation (6.13) wherein $n = 5$ and $\beta = 3$.

Certain values of the resulting approximate distribution function of $Q_1(\mathbf{X})$ are displayed in Table 6.8 where Ge. G. Poly denotes the cdf obtained from the polynomially adjusted generalized gamma density function. The percentiles were determined by simulation on the basis of 1,000,000 replications. The plot shown in Figure 6.7 confirms that the polynomially adjusted generalized gamma distribution provides a very accurate approximation to the distribution of $Q_1(\mathbf{X})$.

Remark 6.6.1. Referring to Equation (6.12), when $\alpha_i = \nu_i/2$, $i = 1, 2, \ldots, n$ and $\beta = 2$, the i^{th} component of the random vector $\mathbf{X} = (X_1, \ldots, X_n)'$ has a chi-square distribution with ν_i degrees of freedom and the representation of the m^{th} moment of $Q(\mathbf{X})$ given in Equation (6.13) applies.

Remark 6.6.2. When the matrix A in the quadratic form $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ contains negative elements, one can utilize the density function of the difference of two gamma random variables as base density in order to determine an approximation to the distribution of $Q(\mathbf{X})$. Such a density function can be determined as follows.

Let Y_1 and Y_2 be independently distributed gamma random variables with parameters $\alpha_1\,, \beta_1$ and $\alpha_2\,, \beta_2$, respectively. By making use of binomial expansion of $(Y_1 - Y_2)^h$, $h = 1, 2, 3, 4$, and simplifying, one can determine the first four moments of $Y_1 - Y_2$, which are

$$\begin{aligned}
E(Y_1 - Y_2) &= \alpha_1\,\beta_1 - \alpha_2\,\beta_2 \\
E(Y_1 - Y_2)^2 &= \alpha_1\,(1+\alpha_1)\,\beta_1^2 - 2\alpha_1\,\alpha_2\,\beta_1\,\beta_2 + \alpha_2\,(1+\alpha_2)\,\beta_2^2 \\
E(Y_1 - Y_2)^3 &= \alpha_1\,(1+\alpha_1)\,(2+\alpha_1)\,\beta_1^3 - \alpha_2\,\beta_2\,(3\,\alpha_1\,(1+\alpha_1)\,\beta_1^2 - 3\,\alpha_1\,(1+\alpha_2)\,\beta_1\,\beta_2 \\
&\quad +(1+\alpha_2)\,(2+\alpha_2)\,\beta_2^2) \\
E(Y_1 - Y_2)^4 &= \alpha_1\,(1+\alpha_1)\,(2+\alpha_1)\,(3+\alpha_1)\,\beta_1^4 + \alpha_2\,(1+\alpha_2)\,(2+\alpha_2)\,(3+\alpha_2)\,\beta_2^4 \\
&\quad -2\,\alpha_1\,\alpha_2\,\beta_1\,\beta_2\,(2\,(1+\alpha_1)\,(2+\alpha_1)\,\beta_1^2 - 3\,(1+\alpha_1)\,(1+\alpha_2)\,\beta_1\,\beta_2 \\
&\quad +2\,(1+\alpha_2)\,(2+\alpha_2)\,\beta_2^2).
\end{aligned} \tag{6.16}$$

Now, on equating these moments to those obtained from (6.13), one can solve the resulting system of equations for $\alpha_1\,, \beta_1, \alpha_2$ and β_2, which can be achieved by utilizing of symbolic computational packages such as Maple and *Mathematica*.

It follows from the results derived in Section 2.7 that the density function of $Q = Y_1 - Y_2$ where Y_1 and Y_2 are independently distributed gamma random variables with parameters $\alpha_1\,, \beta_1$ and $\alpha_2\,, \beta_2$, respectively, can be expressed as

$$h_n(q)\,\mathcal{I}_{(-\infty\,,0)}(q) + h_p(q)\,\mathcal{I}_{[0\,,\infty)}(q) \tag{6.17}$$

where $h_n(q)$ and $h_p(q)$ are specified in (2.42) and (2.41).

Example 6.6.2. Consider the quadratic form $Q_2(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X} = (X_1, X_2, X_3)'$ is a vector of independently distributed chi-square random variables having 4, 3 and 5 degrees of freedom, respectively, and

$$A = \begin{pmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

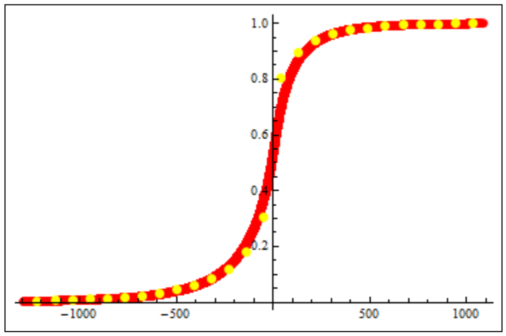


Figure 6.8: *Simulated cdf of $Q_2(\mathbf{X})$ and cdf approximation obtained from the difference of two gamma random variables (dots).*

In light of Remark 6.6.2, one can determine an approximation to the distribution function of $Q_2(\mathbf{X})$ by following the steps described in Example 5.4.1, the base density being given by (6.17) in this instance. This approximation is superimposed in Figure 6.8 on the simulated distribution function which was determined from 1,000,000 replications.

6.7 Quadratic Forms in Order Statistics From an Exponential Population

In this section, we derive the moments of the quadratic form $Q(\mathbf{X}) = \mathbf{X}' A \mathbf{X}$ where $\mathbf{X}$ is a vector of order statistics $X_1 = Y_{r_1:n}$, $X_2 = Y_{r_1+r_2:n}$ and $X_k = Y_{r_1+\cdots+r_k:n}$ obtained from a simple random sample of n observations generated from a standard exponential distribution (with density $g(y) = e^{-y}\mathcal{I}_{\mathcal{R}^+}(x)$).

In this case, the joint density of $X_1, \ldots, X_k$ is

$$
\begin{aligned}
f(x_1, \ldots, x_k) \;&=\; \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)} \Big(\prod_{j=1}^{k} e^{-x_j}\Big)\big(1 - e^{-x_1}\big)^{r_1-1} \\
&\quad \times \prod_{i=2}^{k}\Big(e^{-x_{i-1}} - e^{-x_i}\Big)^{r_i-1}\big(e^{-x_k}\big)^{r_{k+1}-1} \qquad (6.18)
\end{aligned}
$$

whenever $0 < x_1 < \ldots < x_k < \infty$ with $r_{k+1} = n + 1 - \sum_{j=1}^{k} r_j$, and 0, otherwise.

Consider the transformation $z_1 = 1 - e^{-x_1}$ and $z_j = e^{-x_{j-1}} - e^{-x_j}$ for $j = 2, \ldots, k$. The inverse transformation is then

$$x_j = -\ln(1 - z_1 - \cdots - z_j)$$

for $j = 1, \ldots, k$, and its Jacobian is

$$\prod_{j=1}^{k}(1 - z_1 - \cdots - z_j)^{-1} = \prod_{j=1}^{k} e^{x_j} > 0\,. \tag{6.19}$$

Noting that $e^{-x_k} = 1 - z_1 - \cdots - z_k$, the joint density of $Z_1, \ldots, Z_k$ is seen to be

$$h(z_1, \ldots, z_k) = \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)}\Big(\prod_{j=1}^{k} z_j^{r_j - 1}\Big)(1 - z_1 - \cdots - z_k)^{r_{k+1}-1} \tag{6.20}$$

whenever $0 < z_j < 1$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} z_i \leq 1$, and 0 otherwise. Thus, the random vector $\mathbf{Z} = (Z_1, \ldots, Z_k)'$ has a type-one Dirichlet distribution with parameters $r_1, r_2, \ldots, r_{k+1}$.

In view of (6.19), the joint moment-generating function of $\mathbf{U} = (-X_1, \ldots, -X_k)'$ evaluated at the point $\mathbf{t} = (t_1, \ldots, t_k)$ can be expressed as

$$\begin{aligned} M_{\mathbf{U}}(\mathbf{t}) &= E\Big(e^{t_1 \ln(1-Z_1) + \cdots + t_k \ln(1 - Z_1 - \cdots - Z_k)}\Big) = E\Big((1-Z_1)^{t_1} \cdots (1 - Z_1 - \cdots - Z_k)^{t_k}\Big) \\ &= \frac{\Gamma(n+1)}{\prod_{j=1}^{k+1}\Gamma(r_j)} \int \cdots \int (1-z_1)^{t_1}(1-z_1-z_2)^{t_2} \cdots (1 - z_1 - \cdots - z_k)^{t_k} \\ &\qquad \times z_1^{r_1-1} z_2^{r_2-1} \cdots z_k^{r_k-1}(1 - z_1 - \cdots - z_k)^{r_{k+1}-1} \mathrm{d}z_k \cdots \mathrm{d}z_2\, \mathrm{d}z_1 \end{aligned} \tag{6.21}$$

where the domain of integration is $0 < z_i < 1$, $i = 1, \ldots, k$, with $\sum_{i=1}^{k} z_i \leq 1$. Integrating out z_k and making the change of variables $w = z_k/(1 - z_1 - \cdots - z_{k-1})$ yields

$$\int_0^{1 - z_1 - \cdots - z_{k-1}} z_k^{r_k - 1}(1 - z_1 - \cdots - z_k)^{r_{k+1}+t_k-1} \mathrm{d}z_k$$

$$\begin{aligned} &= (1 - z_1 - \cdots - z_{k-1})^{r_k + r_{k+1} + t_k - 1} \int_0^1 w^{r_k - 1}(1-w)^{r_{k+1}+t_k-1}\mathrm{d}w \\ &= (1 - z_1 - \cdots - z_{k-1})^{r_k + r_{k+1} + t_k - 1} \frac{\Gamma(r_k)\Gamma(r_{k+1}+t_k)}{\Gamma(r_k + r_{k+1} + t_k)}\,. \end{aligned}$$

Then, integrating the terms involving z_{k-1} from 0 to $1 - z_1 - \cdots - z_{k-2}$, one has

$$(1 - z_1 - \cdots - z_{k-2})^{r_{k+1}+r_k+r_{k-1}+t_k+t_{k-1}-1} \frac{\Gamma(r_{k-1})\Gamma(r_{k+1}+r_k+t_k+t_{k-1})}{\Gamma(r_{k+1}+r_k+r_{k-1}+t_k+t_{k-1})}$$

and integrating successively the terms involving $z_{k-2}, \ldots, z_2$ and z_1, one obtains

$$M_{\mathbf{U}}(\mathbf{t}) = \frac{\Gamma(n+1)}{\Gamma(r_{k+1})} \prod_{j=1}^{k} \frac{\Gamma(r_{k+1} + \cdots + r_{j+1} + t_k + \cdots + t_j)}{\Gamma(r_{k+1} + \cdots + r_j + t_k + \cdots + t_j)}\,. \tag{6.22}$$

Accordingly,

$$E(X_1^{\delta_1} X_2^{\delta_2} \cdots X_k^{\delta_k}) = (-1)^{\delta_1+\delta_2+\cdots+\delta_k} \frac{\partial^{\delta_1+\delta_2+\cdots+\delta_k} M_{\mathbf{U}}(\mathbf{t})}{\partial^{\delta_1} t_1 \partial^{\delta_2} t_2 \ldots \partial^{\delta_k} t_k}\Big|_{\mathbf{t}=\mathbf{0}}, \tag{6.23}$$

and in light of Equations (6.1), (6.22) and (6.23), the m^{th} moment of the quadratic form $Q(\mathbf{X})$ can be evaluated as follows:

$$E(Q(\mathbf{X})^m) = \sum_{(m)} m! \Big[\prod_{i,j}^{k} \frac{a_{ij}^{m_{ij}}}{m_{ij}!} \Big] (-1)^{\delta_1+\delta_2+\cdots+\delta_k} \frac{\partial^{\delta_1+\delta_2+\cdots+\delta_k} M_{\mathbf{U}}(\mathbf{t})}{\partial^{\delta_1} t_1 \partial^{\delta_2} t_2 \ldots \partial^{\delta_k} t_k}\Big|_{\mathbf{t}=\mathbf{0}} \equiv \mu_m^* . \tag{6.24}$$

For computational purposes, it is simpler to make use of the joint cumulant generating function of $\mathbf{U} = (-X_1, \ldots, -X_k)'$, which is

$$\begin{aligned} C_{\mathbf{U}}^*(\mathbf{t}) \;=\; & \ln[\Gamma(n+1)] - \ln[\Gamma(r_{k+1})] + \sum_{i=1}^{k} \{\ln \Gamma(r_{k+1} + \cdots + r_{j+1} + t_k \\ & + \cdots + t_j) - \ln \Gamma(r_{k+1} + \cdots + r_j + t_k + \cdots + t_j)\}, \end{aligned} \tag{6.25}$$

in order to determine the joint moments needed to evaluate (6.24). The joint cumulants of $-X_1, \ldots, -X_k$ of orders $\xi_1, \ldots, \xi_k$ are then given by

$$\begin{aligned} \kappa_{\mathbf{U}}^*(\xi_1, \ldots, \xi_k) &= \frac{\partial^{\xi_1+\cdots+\xi_k}}{\partial^{\xi_1} t_1 \cdots \partial^{\xi_k} t_k} C_{\mathbf{U}}^*(\mathbf{t}) \,|_{\mathbf{t}=\mathbf{0}} \\ &= \Big(\big(\sum_{j=1}^{k} \xi_j\big) - 1 \Big)! \sum_{\ell=0}^{\nu-1} \Big(-1/(n+1-\nu+\ell) \Big)^{\sum_{j=1}^{k} \xi_j} \end{aligned} \tag{6.26}$$

where $\nu = \sum_{j=1}^{\lambda} r_j$, λ being the position of the first non null component in $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k)'$. On making use of a recursive relationship given in Smith (1995), one can determine the joint moments of $\mathbf{U} = (-X_1, \ldots, -X_k)$ in terms of the joint cumulants as follows:

$$\begin{aligned} \mu_{\mathbf{U}}^*(\delta_1, \ldots, \delta_k) \;=\; & \sum_{i_1=0}^{\delta_1} \cdots \sum_{i_{k-1}=0}^{\delta_{k-1}} \sum_{i_k=0}^{\delta_k - 1} \binom{\delta_1}{i_1} \cdots \binom{\delta_{k-1}}{i_{k-1}} \binom{\delta_k - 1}{i_k} \\ & \times \kappa_{\mathbf{U}}^*(\delta_1 - i_1, \delta_2 - i_2, \ldots, \delta_k - i_k)\, \mu_{\mathbf{U}}^*(i_1, i_2, \ldots, i_k) \end{aligned} \tag{6.27}$$

where $\kappa_{\mathbf{U}}^*(\delta_1 - i_1, \delta_2 - i_2, \ldots, \delta_k - i_k)$ is as specified by (6.26).

Table 6.9: Approximate cdf of $Q_3(\mathbf{X})$ obtained from a generalized gamma (G. Gamma) density function evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Ge.G.Poly
0.0001	0.2257	0.00015054
0.001	0.7141	0.00071070
0.01	2.1704	0.009480
0.05	5.1663	0.050193
0.10	7.8542	0.100647
0.25	14.985	0.250374
0.50	28.884	0.500967
0.75	54.656	0.747110
0.90	84.180	0.897862
0.95	111.50	0.949800
0.99	179.43	0.989827
0.999	293.47	0.998986
0.9999	439.51	0.999919

Example 6.7.1. Let the order statistics $X_1 \leq \cdots \leq X_5$ result from a random sample of size 5 from an exponential distribution with parameter 1. Consider the quadratic form $Q_3(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X} = (X_1, \ldots, X_5)'$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 4 & 2 \\ 3 & 0 & 4 & 1 & 0 \\ 0 & 1 & 2 & 0 & 2 \end{pmatrix}. \tag{6.28}$$

In this example, we approximate the distribution of $Q_3(\mathbf{X})$ whose support is non-negative by making use of a generalized gamma distribution. The moments of $Q_3(\mathbf{X})$ can be determined from Equation (6.24) in terms of the joint moments of $(-X_1, \ldots, -X_k)$ given in (6.27). The steps described in Example 5.4.1 were followed. The results included in Table 6.9 indicate that the generalized gamma density function provides an accurate approximation to the distribution of $Q(\mathbf{X})$. The generalized gamma was adjusted with a seventh degree polynomial and the resulting cdf is plotted in Figure 6.9.

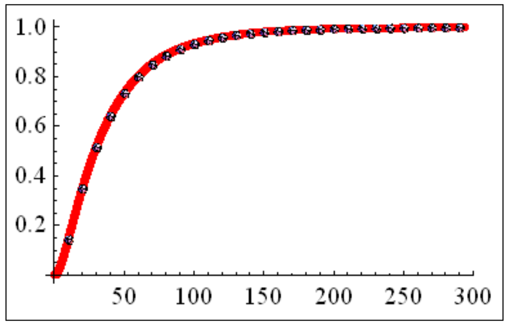


Figure 6.9: *Simulated cdf of $Q_3(\boldsymbol{X})$ and 7^{th} degree polynomially adjusted generalized gamma cdf approximation (dots).*

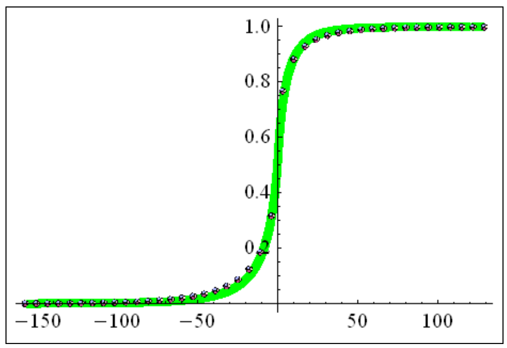


Figure 6.10: *Simulated cdf of $Q_4(\boldsymbol{X})$ and cdf approximation (dots)*

Example 6.7.2. Referring to Example 6.7.1, suppose that A is the matrix

$$\begin{pmatrix} -5 & 1 & 1 & 3 & 0 \\ 1 & 0 & -2 & 0 & -4 \\ 1 & -2 & 0 & 4 & 2 \\ 3 & 0 & 4 & 1 & 0 \\ 0 & -4 & 2 & 0 & -2 \end{pmatrix}.$$

In this case, the base density given in (6.17) is appropriate. Then, on following the steps described in Example 5.4.1, one can determine an approximate distribution for $Q(\mathbf{X})$. Figure 6.10 indicates that the approximated cdf (dots) closely agrees with the simulated cdf.

Remark 6.7.1. More generally, when the order statistics $X_1 \leq \cdots \leq X_n$ are generated from an *Exponential*(β) random variable whose density function is as specified by Equation (6.14), one can represent $Q(\mathbf{X}) = \mathbf{X}' A \mathbf{X}$ as $Q(\mathbf{Y}) = \beta^2\,(Y_1, \ldots, Y_n)\,A\,(Y_1\ , \ldots, Y_n)'$ where the Y_i's are order statistics from an *Exponential*(1) random variable. Once an approximate density is obtained for $(Y_1, \ldots, Y_n)\,A\,(Y_1\ , \ldots, Y_n)'$, a simple change of variables will yield the density function of $Q(\mathbf{X})$. The moments of the quadratic form $Q(\mathbf{X})$, can be also obtained numerically from the following integral representation:

$$
\begin{aligned}
E(Q(\mathbf{X})^m) &= \int\limits_{0\leq x_1\leq\cdots\leq x_k\leq\infty}\int \cdots \int Q(\mathbf{X})^m f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\
&= \int_0^\infty \int_0^{x_{n-1}} \cdots \int_0^{x_3}\int_0^{x_2} \Big(\sum_{i=1}^n\sum_{j=1}^n a_{ij}x_ix_j\Big)^m \prod_{\ell=1}^n n!\, f_{X_\ell}(x_\ell)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\
&= \sum_{(m)} m!\Big[\prod_{i,j}^n \frac{a_{ij}^{m_{ij}}}{m_{ij}!}\Big] \int_0^\infty \int_0^{x_{n-1}} \cdots \int_0^{x_3}\int_0^{x_2} \\
&\qquad\qquad \Big(\prod_{\ell=1}^n n!\, x_\ell^{\delta_\ell}\,\beta^{-1}e^{-x_\ell/\beta})\Big)\mathrm{d}x_1\ldots\mathrm{d}x_n \\
&= n!\,\beta^{-n}\sum_{(m)} m!\Big[\prod_{i,j}^n \frac{a_{ij}^{m_{ij}}}{m_{ij}!}\Big] \int_0^\infty \int_0^{x_{n-1}} \cdots \int_0^{x_3}\int_0^{x_2} \\
&\qquad\qquad \times\Big(\prod_{\ell=1}^n x_\ell^{\delta_\ell}\,e^{-x_\ell/\beta}\Big)\mathrm{d}x_1\ldots\mathrm{d}x_n\,. \qquad (6.29)
\end{aligned}
$$

Example 6.7.3. Suppose that the order statistics $X_1 \leq \cdots \leq X_5$ are generated from a random sample of size 5 from an exponential distribution with parameter 4 and let Q_5 denote the quadratic form $\mathbf{X}'A\mathbf{X}$ with A as given in (6.28). Then, proceeding as in Example 6.7.1 and reexpressing the quadratic form in terms of *Exponential*(1) random variables as explained in Remark 6.7.1, one can approximate the density function of $Q_5(\mathbf{X})$ by making use of a polynomially-adjusted generalized gamma distribution. The results presented in Table 6.10 and Figure 6.11 indicate that approximate distribution agrees with the simulated distribution which was determined on the basis of 1,000,000 replications.

Table 6.10: Approximate cdf of $Q_5(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul. %).

CDF	Simul. %	Ge.G.Poly
0.0001	0.0172	0.00007942
0.001	0.0448	0.00088117
0.01	0.1337	0.009585
0.05	0.3183	0.049363
0.10	0.4866	0.099550
0.25	0.9357	0.250078
0.50	1.8023	0.500105
0.75	2.6956	0.749100
0.90	6.9692	0.899557
0.95	6.9692	0.949828
0.99	11.218	0.989847
0.999	18.358	0.998990
0.9999	26.249	0.999888

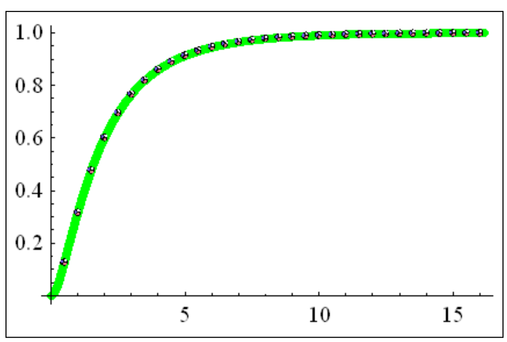


Figure 6.11: *Simulated cdf of $Q_5(\mathbf{X})$ and polynomially-adjusted generalized gamma cdf approximation (dots).*

Chapter 7

Concluding Remarks and Future Work

7.1 Concluding Remarks

The main objective of this dissertation consists in obtaining accurate moment-based approximate distributions for various types of quadratic forms and quadratic expressions. Excluding Chapter 6, the proposed methodology involves the decomposition of quadratic forms and quadratic expressions as the difference of two positive definite real quadratic forms plus possibly a linear combination normal random variables. We would like to reiterate that this last term is not mentioned in the statistical literature. In this general decomposition, the rank of A could be less than the rank of $A\Sigma$. In all cases, the moment generating functions, cumulant generating functions as well as the moments and cumulants are determined. Approximating the distributions by means of polynomially adjusted generalized gamma and generalized shifted gamma as base density, is another novel contribution of this dissertation. Ratios of various types quadratic forms and quadratic expressions were considered in more general settings, including the singular cases. We reexpressed Hermitian quadratic forms and quadratic expressions as well as quadratic forms and quadratic expressions in elliptically contoured vectors in terms of real quadratic forms and quadratic expressions in Gaussian vectors and then, proposed decompositions involving the difference of two real positive definite quadratic forms and a linear combination of normal random variables, which is another innovation of this thesis.

Most of the results derived in the Chapter 6 are original contributions. In this chapter quadratic forms and quadratic expressions in uniform, exponential, gamma and beta variables as well as their order statistics are considered. We have determined the moments of all such types of quadratic forms and quadratic expressions with special techniques. In the case of quadratic forms and quadratic expressions in beta random variables or their order statistics, we are making use of beta density functions as base densities to approximate the distributions. The proposed methodology for approximating the distribution of quadratic forms and quadratic expressions has applications in various fields of scientific investigation. For instance, in finance, the stochastic process for modeling a price Y_t can be described by the stochastic differential equation,

$$\frac{\mathrm{d}Y_t}{Y_t} = \alpha_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \,,$$

where the parameters α_t, σ_t are often considered constant over time, see Šindelář (2010). An estimation of the parameter α can be carried out from the model,

$$Y_{t+1} = \alpha Y_t + e_{t+1} \,,$$

where the innovations could be taken to have normal or Laplace distributions. The Laplace distribution can be viewed as particular case of the bilateral exponential density which was discussed in Chapter 5. The maximum likelihood estimate is of the form

$$\widehat{\alpha}_{GML} = \frac{\sum_{t=2}^{T} y_t \, y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2}$$

which can be expressed as a ratio of quadratic forms.

Another application involves portfolio value-at-risk as pointed out by Glasserman *et al.* (2002) where a quadratic expression in elliptically contoured random vectors is considered in Equation (3.10).

7.2 Future Work

First, I am planning to extend the density approximation methodology advocated in Provost (2005) and Ha and Provost (2007) to random vectors and matrices. This will entail making use of multivariate base densities, which would be adjusted by linear combinations of multivariate orthogonal polynomials on the basis of the joint moments of the variables involved. This semi-parametric approach would allow for much more flexibility than that associated with purely parametric density functions when modeling multivariate or matrix-variate distributions. I shall then consider extensions to the context of density estimation on the basis of sample moments, including stopping rules for the determination of the degree of the polynomial adjustment, which were addressed in Jiang and Provost (2011) for the univariate case.

This would enable me to tackle the problem of determining the distribution of (possibly indefinite) generalized quadratic forms (expressible as X A X' where X is a random matrix), which have applications for instance in multiple time series. A host of test statistics and estimators in this area can be expressed in terms of generalized quadratic forms. Thus, having a methodology for approximating their distributions accurately (without having to resort to zonal polynomials expansions, as discussed for instance in Mathai *et al.* (1995)), should prove eminently useful. I also propose to identify instances where such generalized quadratic forms can be reduced to quadratic forms involving vectors. The matrix X is usually assumed to be normally distributed in the literature. However, such an assumption may not be realistic. Accordingly, I will consider the case of elliptically contoured matrices (whose densities are constant on hyper-ellipsoids). In this case, the quadratic forms could presumably be expressed in terms of their Gaussian counterparts via a certain weight function. The case where A is a Hermitian matrix will also be addressed; it is anticipated that my current results can be extended to the matrix-variate setting. The singular case where the covariance matrices associated with the random matrices may not have full rank will also be studied.

I would also like to investigate the distribution of generalized quadratic forms in random matrices whose elements are distributed as uniform, beta or exponential variables. This would presumably have applications similar to those pointed out in my current work. I shall address the case of generalized quadratic expressions that also involve a linear term of the form B X' and generalized bilinear forms of the type Y B X' where Y and X are random matrices, and develop criteria for their independence along the lines of the results derived in Provost (1996).

Additionally, I have an interest in the saddlepoint density approximation technique [see, for instance, Butler (2007)], as it has been utilized by Kuonen (1999) to approximate the distribution of quadratic forms. It is well-known that in this case the resulting density approximations may be inaccurate in a neighborhood of the mean of a distribution, especially if it happens to be bimodal or irregular. Accordingly, improvements obtained by applying a polynomial adjustment to a base density derived from an appropriately normalized initial saddlepoint-type approximation shall be considered. I also wish to investigate possible generalizations of the saddlepoint approximation in multivariate settings and possibly apply these results to the distribution of generalized quadratic forms. This would involve making use of the joint cumulant-generating functions of the distributions being approximated and generalizing some of the results derived by Barndorff-Nielsen and Kluppelberg (1999).

These results would complement those included in Mathai and Provost (1992) and Mathai *et al.* (1995) as well as those already available in the statistical literature. They could also be included in a monograph on the evaluation of the distribution of quadratic forms, which is currently in preparation.

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  Azad University of Estahban
  Estahban, Iran, 2000–2007

- **Teaching Assistant and Statistical Consultant**
  The University of Western Ontario
  London, Canada, 2008–2012

  - Course Evaluations Supervisor in the Department

  - TA duties, including work in the Statistical Help Center
  - Instructor (substitute): Experimental Design and Statistical Inference (graduate level)
  - Statistical Consultant in the Social Science Network Lab and Data Services (SSNDS)

Special Honours

- **Graduate Thesis Research Award**
  The University of Western Ontario
  London, Canada, 2012

- **Queen Elizabeth II Graduate Scholarship in Science and Technology (QEIIGSST)**
  Ontario, Canada, 2011–2012

- **PhD Program Scholarship Award**
  Department of Statistics and Actuarial Sciences
  The University of Western Ontario
  London, Canada, 2008–2012

- **Master of Science Program Scholarship Award**
  Azad University of Estahban
  Estahban, Iran, 1994–1995

Training

- **Statistics Canada Workshop on Health Data**
  The Department of Epidemiology and Biostatistics
  The University of Western Ontario
  London, Canada, 2012

- **Spatial Statistics for Non-Gaussian Data**
  The Summer Workshop
  The University of Western Ontario
  London, Canada, 2011

- **Communication in Canadian Classroom**
  The University of Western Ontario
  London, Canada, 2009

- **Methodology of Research**
  Azad University of Estahban
  Estahban, Iran, 2004

- **Programming and Sampling of Course Design**
  Azad University of Estahban
  Estahban, Iran, 2004

- **SPSS Software**
  Azad University of Estahban
  Estahban, Iran, 2004

- **Methods of Teaching**
  Azad University of Estahban
  Estahban, Iran, 1999

Publications

- **Mohsenipour, Aliakbar and Provost, Serge**
  On Approximating the Distributions of Ratios and Differences of Noncentral Quadratic Forms in Normal Vectors
  Journal of Statistical Research, Vol. 44, No. 2, pp. 315–334, (2010).

- **Mohsenipour, Aliakbar and Provost, Serge**
  On Approximating the Distribution of Indefinite Quadratic Expressions in Singular Normal Vectors
  Acta et Commentationes Universitatis Tartuensis de Mathematica, Vol 25, No. 1, pp. 61–86, (2011).

- **Provost, Serge and Mohsenipour, Aliakbar**
  The Distribution of Quadratic Expressions in Elliptically Contoured Vectors
  International Journal of Statistics and Probability, Vol. 1, No. 2, pp. 103–112, (2012).

- **Mohsenipour, Aliakbar and Provost, Serge**
  Approximating the Distributions of Singular Quadratic Expressions and their Ratios
  The Journal of Iranian Statistical Society, Vol 11, No.2, pp. 147-171, (2012).

- **Mohsenipour, Aliakbar and Provost, Serge**
  An Approximation to the Distribution of Quadratic Form in Gamma Order Statistics
  Submitted to the *Journal of Statistical Theory and Applications.*

- **Provost, Serge and Mohsenipour, Aliakbar**
  On Approximating the Distribution of Quadratic Forms in Uniform Order Statistics
  Submitted to *Metron.*

- **Provost, Serge and Mohsenipour, Aliakbar**
  A Representation of Hermitian Quadratic Forms in Singular Normal Vectors and Related Distributional Results
  Submitted to the *Journal of Probability and Statistical Science.*

- **Provost, Serge and Mohsenipour, Aliakbar**
  On Evaluating the Distributions of Real and Hermitian Quadratic Forms in Random Variables (working title)
  Monograph in preparation to be submitted to *Springer-Verlag.*