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Revenue Management in Multi-Firm, Multi-Product Price Competition

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Revenue Management in Multi-Firm, Multi-Product Price Competition

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ABSTRACT

Dynamic pricing models in revenue management lack the ability to have multiple firms selling multiple product classes. In this thesis, a framework is created that allows for the construction of revenue management models with multiple firms, each selling multiple product types and where the firms have the ability to alter their prices instantly based on market conditions. The framework is a finite repeated game, where the optimal price for each state can be calculated through backwards induction. Conditions for existence of pure strategy Nash Equilibria are proven and conditions for unique pure strategy Nash Equilibria are discussed. We illustrate the pricing dynamics in a 2x1 and a 2x3 model. We recreate the well-known Netessine and Shumsky airline duopoly model but allow the firms to use dynamic pricing rather than booking limits. We find that in all cases the revenues from a dynamic pricing approach exceed those from booking limits. Through the use of three examples we show that our model provides vastly increased revenues over traditional models as it considers cross-price elasticities and how firms should alter their prices in response to the quantity levels of all products in the market.

CO-AUTHORSHIP

I hereby state that all work presented in this thesis was solely my own and was under the supervision of my advisor Dr. Peter Bell.

DEDICATION

To my parents, Pat and Barb Moffatt, daughter Marguerite Moffatt and my wife Hannah Rasmussen with love.

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Contents

List of Figures

Chapter 1

Introduction - Background and Purpose

The focus of this research is to develop an oligopolistic model that can be used in traditional revenue management scenarios, such as the pricing of rental cars so as to maximize a firm's expected revenue given a set of conditions, such as fleet size. In particular, we wish to develop a model for online pricing scenarios where customers have instant access to each firm's prices (the customer faces no search cost in choosing between different providers of similar goods) and firms can costlessly change prices instantaneously.

An example of such a car rental problem is renting a car from the Indianapolis, IN aiport. On June 14th, 2009 a visitor to *Expedia.com* was presented the following options for a one-day July 25, 2009 rental, in U.S. dollars:

One-Day Car Rental Prices

	Econ.	Comp.	Mid	Stand.	Full	Prem.	SUV	Luxury
Enterprise	$42.90	$43.89	$51.75	$55.49	$55.70	$58.64	$73.64	$97.00
Budget	$41.99	$42.99	$54.99	$59.99	$59.99	$64.99	$78.99	$89.99
Thrifty	$46.09	$47.09	$52.09	$56.09	$56.09	$59.09	$74.09	$89.09
Dollar	$45.90	$46.90	$51.90	$55.90	$55.90	$59.90	$74.90	$87.90
Hertz	$67.49	$67.49	$75.49	$80.49	$80.49	$97.49	$80.49	$101.49

Figure 1.1: Expedia One-Day Car Rental Prices as of June 14th, 2009 For a One-Day July 25, 2009 Rental. All Prices in U.S. Dollars.

There are three observations we can draw from the data above:

- The prices charged by the firms are not identical, and in some cases there are large differences in the prices firms charge for a comparable product (see *Enterprise* vs. *Hertz*). This suggests (but certainly does not prove) that the products are not commodities - a level of brand differentiation exists. See Geraghty and Johnson [45] for a discussion of brand differentiation in the rental car market.

- Firms offer a range of similar, but non-identical products. When Hertz lowers the price of mid-size rentals, this should increase the quantity demanded of these cars, but it may also reduce the demand for compact and standard cars.

- Firms can engage in *diagonal competition*. That is, Hertz may be the preferred brand of a customer, but if Enterprise or Budget price low enough, customers may be willing to choose a higher-end car from these companies, for what they would have paid Hertz for a lower-end car. Alternatively, a consumer may prefer a lower-end car because it is more fuel efficient.

1.1 Literature Review

There are two streams of research that have considered the problem of firms pricing in an environment where there are multiple products. These include both the revenue management literature and the oligopoly literature from microeconomics. Canonical models in the existing revenue management literature do not model all of the three observations mentioned earlier. For this reason, we look to the oligopolistic geometric demand framework developed in microeconomics. This framework balances the need for mathematical tractability with the need to create models that reflect real-world revenue management challenges. As a consequence, a literature review of both areas of study follows.

1.2 Introduction to Revenue Management Literature

One class of models, including Anderson and Wilson [5], Belobaba [10] and Wilson, Anderson, and Kim [56] allows for multiple product classes. However they only allow for a single firm and allow those firms to only alter booking limits, not prices. Others such as Bell and Zhang [9] and Federgruen and Heching [38] allow for dynamic pricing, but in a one-firm, one-product market. To accurately depict markets such as the rental car industry a model that allows for multiple firms each selling multiple products is required. This model should also include dynamic pricing as real-world firms can currently alter their prices in real time.

A number of models of competition with dynamic pricing do exist, Perakis and Sood [82] being a well-known example, but in these models each firm is only able to offer a single-product class. Talluri [97] examines the equilibrium properties of a two-firm/one-product per firm model where product differentiation exists between the firms. A number of extensions to this model exist, including Levin, McGill and Nediak [108], which allow consumers to exhibit strategic behaviour by waiting for prices to fall. Cachon and Feldman [17], Jerath, Netessine and Veeraraghavan [58], Liu and Zhang [67] and Parlakturk and Kabul [81] all allow customers to wait for prices to fall. Marcotte, Savard and Zhu [80] consider the oligopoly problem in the context of airline networks. Xu and Hopp [106] allow for a single inventory replenishment and Mantin, Granot and Granot [7] create a duopoly model that uses the one-arrival-per-period assumption.

Netessine and Shumsky's [76] airline model allows for two firms and two product classes, but only two of each and is a model of booking limits, not of dynamic pricing. Song, Yuan and Mao [54] have an extension that allows for incomplete information.

A number of literature reviews exist in this area, including Talluri et. al. (2009) [37], Phillips (2005) [83] and Talluri and van Ryzin (2004) [98]. Elmaghraby and Keskinocak (2003) [35] provide a detailed literature review of revenue management models. As a starting point, we will consider their literature review, before moving on to articles which have appeared since their article was published.

When examining the dynamic pricing literature in revenue management, Elmaghraby and Keskinocak "postulate that there are three main characteristics of a market environment that influence the type of dynamic pricing problem a retailer faces." Those three characteristics, or dimensions, as identified by Elmaghraby and Keskinocak, are as follows:

1. Replenishment vs. No Replenishment of Inventory (R/NR): Does the retailer have the ability to re-order inventory during the sales period?

2. Dependent vs. Independent Demand over Time (I/D): Do past sales have an impact on future sales?

3. Myopic vs. Strategic Customers (M/S): Do customers always purchase the good immediately, or do they consider waiting for prices to drop?

(Note: Titles and abbreviations borrowed from Elmaghraby and Keskinocak).

Elmaghraby and Keskinocak find that the “bulk” of the dynamic pricing literature falls into two camps: NR-I (both NR-I-M and NR-I-S) and R-I-M (see earlier definitions for ‘R’, ‘NR’, ‘I’, ‘M’ and ‘S’). Thus the major division in the literature, according to Elmaghraby and Keskinocak, is between models where firms can re-order inventory and ones where they cannot. First, we will examine models where no inventory replenishment is possible.

1.2.1 No Inventory Replenishment

Elmaghraby and Keskinocak identify a number of models which study how pricing decisions should be made in these markets. These include: Bitran, Caldentey, and Mondschein (1998) [12]; Bitran and Mondschein (1997) [13]; Feng and Gallego (1995) [39]; Gallego and van Ryzin (1994) [44]; Lazear (1986) [64]; Smith and Achabal (1998) [94] and Zhao and Zheng (2000) [111]. One of the common demoninators is that firms operate in a market of "imperfect competition (e.g. a monopolist)" [35]. There is no competitive interaction between firms in these markets. We wish to develop a model where competitive interaction is possible.

As well, Elmaghraby and Keskinocak indicate that all the papers they reviewed on dynamic pricing in the N-I market consider the case where the firm only sells a single product. Papers which consider simultaneously pricing multiple products do so in a static setting, such as Reibstein and Gatignon (1984) [87].

Elmaghraby and Keskinocak identify a number of features which should be added to NR-I models, in order to "bridge the gap" between theory and practice.

- Multiple products per firm: This is important when considering both substitute products a firm might sell (two different sized bottles of ketchup) and complementary products (a top and the matching pair of pants).

- Multiple stores or multiple channels: Firms often sell the same product, from the same inventory set, over multiple channels. An airport location of a car rental firm sells its cars through its own website, through a third-party website such as *Expedia.com*, to travel agents, and to walk-up-traffic.

- Initial inventory: The initial inventory a firm has is also often treated as given, but in many situations it is a variable that the firm can control.

- Strategic customers: Customers may often "time" their purchases in order to get the best possible price.

- Competitors' Pricing Decisions: Elmaghraby and Keskinocak give an example of how firms take this into account: "In a competitive business environment, consumers' purchasing decisions take into account prices offered by competing firms. IT allows companies to automatically track competitors' prices and incorporate that information into their pricing decisions. For example, Buy.com Inc. developed technology using software "bots" to monitor prices on competing sites such as Amazon.com and Best Buy (Heun 2001) [49]. Competitors' prices, along with other information, are then fed into the dynamic pricing software from KhiMetrics, which suggests price changes on Buy.com Inc."

Keeping in mind the features identified by Elmaghraby and Keskinocak, it is our intention to make a major contribution to bridging this gap between theory and practice by developing a model that allows for dynamic pricing within the context of a multiple-firm, multiple-product-per-firm environment.

1.2.2 Inventory Replenishment

Elmaghraby and Keskinocak find that the literature for models with inventory replenishment fall into three broad categories, based on their modelling assumptions:

1. Markets where the firm's demand is uncertain, faces convex production, holding and ordering costs exist, and production quantity is unlimited. Papers in this

category include Federgruen and Heching (1999) [38], Thowsen (1975) [101] and Zabel (1970) [110].

2. Markets as above with the addition of fixed ordering costs, as in Thomas (1970) [100] and Chen and Simchi-Levi (2002) [18]. Chan, Simchi-Levi and Swann (2002) [62] also incorporate limited production capacity.

3. Markets such as in number 1, expect the firm's demand is deterministic, such as Biller et. al. (2002) [89] and Rajan, Rakesh and Steinberg (1992) [1].

As with NR-I models, these models do not allow for retailers that sell multiple products. Elmaghraby and Keskinocak see this as an area researchers need to consider:

"One can argue that all products' prices are somewhat interdependent and pricing decisions should simultaneously consider all the products offered by a firm and its competitors... One reasonable approach... is to identify families or categories of products whose demands are significantly dependent on each other and simultaneously consider pricing decisions for products in the same family... Advances in IT provide the retailer with the ability and the required data to optimize prices across multiple products and, therefore, we see this as a research direction deserving immediate attention."

Diagonal competition is an important component of real-world industries of interest to revenue managers.

1.2.3 Literature Post-Elmaghraby and Keskinocak

The pricing in revenue management literature has expanded since the publication of Elmaghraby and Keskinocak.

The models in Elmaghraby and Keskinocak fall into two camps: NR-I (both NR-I-M and NR-I-S) and R-I-M. These two camps have the independence assumption in common. A number of recent models have been developed which allow for dependent demand between periods. The models of Anderson and Wilson (2003) [5], Su (2007) [96] and Wilson, Anderson and Kim (2006) [56] consider markets with strategic consumers who may wait for the firm to offer a lower price. In Su, consumers can purchase the good immediately or pay a fee to stay in the market and purchase at a later date.

Another form of dependency between periods is caused by reference effects. Popescu and Wu (2007) [84] examine how reference effects impact the choices made by consumers:

"As customers revisit the firm, they develop price expectations, or reference prices, which become the benchmark against which current prices are compared. Prices above the reference price appear to be "high", whereas prices below the reference price are perceived as "low"... Adaptation level theory (Helson (1964) [48]) predicts that customers respond to the current price of a product by comparing it to an internal standard formed based on past price exposures called the *reference price*... The impact of the reference price on demand [is] called [the] *reference effect*..."

Popescu and Wu find that if reference prices are not taken into account, firms will "systematically price too low and lose revenue."

There are many variations in the structures of the models presented in the literature. In Bell and Zhang (2006) [9], firms have the ability to both alter the price they charge and the quantity for sale. The market structure of Deng and Yano (2006) [29] is that of a monopolist that has the abilities to set production levels, re-order and to alter prices.

Dasci (2003) [24] considers a market with two firms where consumers, if not able to purchase at their most-preferred firm (because they have run out of stock) attempt to purchase from the other firm. Perakis and Sood (2006) [82] consider the pricing problem in a market of multiple firms, each selling similar, but not identical goods. Netessine and Shumsky (2006) [76] model a competitive airline market where each firm allocates its seats between two booking classes. Each of these models considers similar market structures to the one we wish to study and provide an excellent structure off of which to build.

Levin, McGill and Nediak [107] consider a scenario where a monopolist can choose a price to charge but also issue consumers compensation if the price charged in the future goes below the price the consumer paid today. Marcus and Anderson (2006) [71] examine such pricing guarantees using a real options approach and find that in practice firms likely do not benefit by issuing such guarantees.

Netessine, Savin and Xiao (2006) [90] model the scenario where firms 'cross-sell' products, that is they sell goods both individually and as part of a package. They consider both what products should be bundled together and what price point to charge for the bundles.

1.2.4 Lin and Sidbari

Lin and Sidbari (2008) [66] develop a model which allows for dynamic 'real-time' pricing of goods. There are N firms in the model, each of which sells one type of good. Firms hold a limited quantity of each good and cannot re-order when sold out. Following in the footsteps of Lautenbacher and Stidham (1999) [63], Subramanian et. al. (1999) [55] and You (1999) [109], they assume that a single consumer arrives each period with probability λ. We will adopt this assumption in our model as well.

The consumers see the products/brands as being different from one another and make a purchase as to maximize their utility function (the utility function is in multinomial logit form). Using a theorem in Vives [104], they show the game has a pure-strategy Nash Equilibrium. They are unable, however, to prove that the equilibrium is necessarily unique.

Interestingly, Lin and Sidbari find that for a given inventory level, prices are not necessarily increasing over time, due to what they call 'competition effects'.

1.2.5 Martinez de Albeniz and Talluri

Martinez-de-Albeniz and Talluri (2011) [28] create a dynamic pricing model with multiple firms that sell a fixed number of goods over a finite number of periods. In the base model there are two firms, each selling one class of product. Consumers see the products as being identical and will always choose the lower priced good (if prices are identical, consumers choose randomly). As with Lin and Sidbari (2008) [66] each period a single consumer arrives with probability λ. They show for this model a unique subgame-perfect equilibrium in pure strategies. The equilibrium is simply the well-known Bertrand (1883) [11] result.

Martinez-de-Albeniz and Talluri [28] consider a number of extensions to the model, including the case of $N > 2$ firms. They show a unique subgame-perfect equilibrium for this model as well. The results, however, require firms selling only one type of product, and no product/brand differentiation between firms.

A model of differentiated products is briefly considered and it is shown for some functional forms, such as the logit choice function, it is often possible to obtain a unique subgame-perfect Nash Equilibrium.

1.2.6 Lu

Lu (2009) [68] examines a market of N firms, each of which sells a limited inventory of a single type of good. If firms run out of the good they leave the market for the duration of the sales period. Although the model is described as a 'price and inventory' game, re-orders are not possible.

As with previous papers, at most a single customer arrives each period with probability λ. Consumers do not act strategically or wait for higher prices.

Consumers do not see the products from firms as being identical. A lower price (weakly) increases the quantity demanded of that good and (weakly) decreases demand for the other goods.

Lu finds the existence of "a strategy that pushes customer to retailer 2, and hence retailer 2 stocks out, after which retailer 1 becomes a monopolist." In our model, we will refer to such a strategy as a knockout strategy (Lu does not give it a name).

Lu shows that for the $N = 2$ case, a Nash Equilibrium in pure strategies need not exist - it depends on the assumptions placed on demand. The $N > 2$ case is not considered. Lu does also not consider firms that sell multiple products, but indicates it as an area of future research, though it is not obvious from the thesis how the model will be able to incorporate this.

1.2.7 Hu

A doctoral thesis by Hu (2009) [52] examines the N firm, 1 good type per firm dynamic pricing model but does so in continuous time, not discrete time, which is a multi-firm extension of the Gallego and van Ryzin (1994) [44] model. Firms have a finite period in which to sell their goods and assume the goods have no salvage value and all other costs are fixed. They show that a subgame perfect equilibrium can be constructed so long as the pricing options available to each firm are finite and discrete.

1.2.8 Isler and Imhof

Isler and Imhof (2008) [53] develop a two-firm/one-product-per-firm dynamic pricing model in the context of airline ticket pricing. A parameter α is introduced which models product differentiation. They find that when quantities are limited or some product differentiation exists in the market, firms will price above marginal cost.

However, when both firms have a substantially large quantity and product differentiation is assumed to be zero, Isler and Imhof obtain the Bertrand result of zero economic profit. They do not consider the case of multiple seat classes per firm.

In order to formulate the best possible model to reflect real-world revenue management challenges, we next look to microeconomic models for guidance.

1.3 Introduction to Classic Microeconomics Models

Oligopoly theory has a long history in the study of microeconomics, particularly in the field now known as Industrial Organization. Cournot's [23] 1838 treatment remains to this day standard undergraduate textbook fare. Many refinements have been made over the years, from the price-competition model of Bertrand [11] to the spatial competition model of Hotelling [51]. No one model has emerged as the benchmark treatment of oligopoly in Industrial Organization - rather a toolbox full of approaches which vary by the nature of competition (price, quantity, or other factor), assumption used, and the amount of complexity in the model.

1.3.1 Bertrand Competition

Bertrand competition, as developed by Bertrand [11], is a 2-firm model of firm competition where the firms face identical marginal costs and sell their good over one period. Firms do not face capacity constraints and consumers purchase a fixed quantity of goods from the lowest price firm, so long as the lowest price is below some maximum-willingness-to-pay, leading to firms receiving zero economic profits. A logical extension is to increase the number of periods under consideration, but playing a Bertrand game with a finite number of periods yields the same zero economic profit result, as shown by Friedman (1971) [41] and Friedman (1977) [42].

After the publication of Bertrand's model in 1883, a number of extensions were attempted in order to rid the model of the result that both firms price at marginal cost. These extensions include, but are not limited to Fisher (1898) [40], Moore (1906) [73], Schumpeter (1928) [91] and Stigler (1940) [95]. The extensions, and proofs of the results, are available in many microeconomics texts. The results that follow are available in Nichols [77] (2004) and Fudenberg and Tirole (1991) [43].

Fudenberg and Tirole point out a number of problems with the Bertrand model. Firstly, the assumptions of the products being seen by consumers as identical, unlimited quantity available, and wholly symmetric firms lead to the result that each firm prices at marginal cost. Unfortunately, creating the assumption of firms facing differing marginal costs yields the situation that only one firm serves the market. Secondly, problems can be created if the assumption is made that marginal costs are not constant. Allowing for increasing-returns to scale, that is that the marginal cost of production declines as the level of production increases, involves further complications. In the case of homogenous goods, once again the result is that in equilibrium the firms price at marginal cost. Since increasing returns to scale are allowed the marginal cost of producing an additional unit of good is less than average cost - thus this cannot be a Nash Equilibrium. Firms lose money by producing, thus are made worse off by even participating in the game.

In order to get a result where firms do not price at marginal cost, it appears necessary to introduce decreasing returns to scale into the model. Mathematically, the most extreme version of decreasing returns to scale is one where the firms are quantity constrained and cannot produce above a certain level (face an infinite cost of production above a certain point). The quantity constrained version of Bertrand is known as Bertrand-Edgeworth competition.

1.3.2 Bertrand-Edgeworth Competition

In order to create a scenario where firms price at higher than marginal cost, Edgeworth [34] (1888) adds capacity constraints into the finitely repeated Bertrand game. Unfortunately, pure-strategy equilibria do not exist under this scenario. The extension came to be known as Bertrand-Edgeworth competition (see Roll (1940) [88]). Consider the following game:

Two firms sell identical products and face identical constant marginal costs. The firms are capacity constrained, such that no single firm can meet the quantity demanded when price is set to marginal cost. The combined quantities of both firms, however, exceed the quantity demanded at that price. As in the non-capacity constrained model, no Nash Equilibrium can exist where both firms price higher than marginal cost. A firm can always improve its profitability by slightly undercutting the other one. However, no Nash Equilibrium can exist where both firms price at marginal cost. A firm could earn a strictly positive profit by pricing above marginal cost, as it would capture the excess demand not satisfied by the lower priced firm. Thus there is no Nash Equilibrium, as firms have an incentive to raise their prices from marginal cost, yet they face an opposing incentive to lower their prices when above marginal cost. Edgeworth believed that since an equilibrium would not exist, prices would cycle from low to high to low again (known as an *Edgeworth Cycle*; see Maskin and Tirole [72]).

A similar, but not identical, outcome to this model is that the firms will use mixed strategies. Fortunately mixed-strategy Nash Equilibria exist in this game under very weak conditions as shown by Dasgupta and Maskin (1988) [25]. However, the mixed-strategies can be quite intricate and cumbersome.

Since some customers are prevented from purchasing from the lower priced firm, a rationing rule must be used in this model to determine which customers will be served by which firm (or at all). Fortunately, the *existence* of such an equilibrium is invariant to the choice of rule. Unfortunately, the equilibrium (either in pure or mixed strategies) can depend heavily on which rule is used to decide which customers will receive goods when there is excess demand. As discussed in Tirole (1988) [102], the two most common rules, *the proportional rule* and *the surplus maximizing rule* can give very different results and there is generally no *a priori* reason to choose one over the other (see also Davidson and Deneckere (1986) [26], Madden (1998) [69], Osborne and Pitchik (1986) [79] and Vives (1993) [103]).

The Bertrand-Edgeworth model is widely used in the modelling of oligopoly problems (Allen and Helwig (1986) [3], Dixon (1990) [30] and Kruse et. al. (1994) [36] are just three of many examples) particularly in cases where firms compete on price, capacity constraints exist, and where the outcome of pure-strategy equilibrium is not required.

1.3.3 Extensions of Bertrand-Edgeworth

There have been a number of extensions to the Bertrand-Edgeworth framework in order to ensure the existence of Nash Equilibrium in pure strategies. Three such examples are: Dudey (1992) [32], Levitan and Shubik (1972) [65] and Tasnadi (1999) [99]. These models share a number of features. All three have two firms each selling a single identical product. Firms in the models have a finite number of goods to sell (no reorders are possible) and a finite selling horizon. In Tasnadi's model each firm simultaneously chooses the rationing rule it will use if it runs out of stock (this is step 1 of the Tasnadi game). Firms choose price in the second stage and the overall quantity demanded is responsive to price.

In contrast, Dudey's model eliminates the rationing rule by assuming that only one indivisible good can be sold per period; thus bypassing the excess demand problem. Unlike Tasnadi's model, the market demand is constant no matter the price charged by firms, so long as the prices stay within a specified range.

Levitan and Shubik consider a duopoly where the firms sell identical goods and face identical capacity constraints. The firms simultaneously choose what price to charge. Consumers purchase from the lowest price firm first and any excess demand filters to the higher priced firm. They assume that the total demand is the maximum of the high-price demand and the total available amount at the low price (since it is all exhausted). Depending on the overall level of capacity relative to demand, Levitan and Shubik show the final result can be anything between the Bertrand [11] and the Cournot [23] solutions.

1.3.4 Spatial Competition

While Hotelling's [51] (1929) model may at first glance seem significantly different from a typical model of oligopolistic competition, it shares a number of features.

The most basic version of Hotelling's model considers two ice cream vendors who have set-up shop on a linear beach. Hotelling assumes that demand for ice cream is distributed uniformly along the beach and each customer buys a single ice cream cone. It is assumed that the ice cream is costless to produce. In the simple version of the model we ignore price by assuming that each firm charges identical prices and thus has the goal of maximizing market share. Since the products and prices are identical, consumers purchase from the firm closest to their position on the beach. In the two firm model, there is only one Nash Equilibrium in pure strategies - each firm places its stand at the median point of the beach. For more than two firms a pure-strategy Nash Equilibrium in the game cannot be assured. None exist in the three firm case, for example.

The more robust version of Hotelling's model has the firms choose both their locations on the continuum and the price they will charge. Consumers pay a cost proportional to the distance they travel, thus they purchase from the firm with the lowest total cost.

Instead of a one-period game, this model is conducted in two stages: first the choice of location, as before, and in the second stage the firms simultaneously choose the price they will charge. Hotelling believed that the firms would invariably locate close together, but unfortunately d'Aspremont, Gabszewicz, and Thisse (1997) [16] showed that this could not lead to a Nash Equilibrium in pure strategies. A Nash

Equilibrium in pure strategies for this has yet to be found, but it also has not been proven that one does not exist.

Despite the difficulties in obtaining pure strategy Nash Equilibria, Hotelling's model of spatial competition remains an important benchmark model in the study of oligopoly.

1.3.5 Cournot

The Cournot [23] approach differs from many others in Industrial Organization, such as ones based on Bertrand [11], as it considers competition over quantity, instead of over price. Pure-strategy Nash Equilibria exist under fairly general conditions in Cournot models (a sufficient, but not necessary condition is that the profit function for each firm be quasi-concave with respect to own output). For the multi-firm Cournot model, the pure-strategy Nash Equilibrium provides for more output and a lower price than under a monopoly, but for lower output and a higher price than under perfect competition.

There have been a large number models of quantity competition since Cournot. A particularly useful one for revenue management modelers is Manas [70], which is cited frequently in the revenue management literature.

1.3.6 Manas

The model constructed by Manas [70] considers quantity, not price, competition in a market with N firms. Specifically, the model has six basic features:

1. The goods produced by each firm are identical - there is no product differentiation.
2. At the beginning of the game each firm decides what level of quantity to produce - output is the only decision variable for each firm.
3. The equilibrium price for each good is a function of total market output - specifically the equilibrium price is a linear function of overall output.
4. No collusion or cooperation is permitted/available to the firms.
5. All firms are rational profit maximizers.
6. Each firm faces a capacity constraint.

It can be shown that a pure-strategy Nash Equilibrium exists in the game with the aid of a theorem by Nikaido and Isoda [78]. Nikaido and Isoda, through the development of a fixed-point theorem, show that a game has at least one pure-strategy Nash Equilibrium if all strategy spaces in the game are compact (the strategy space is both closed and bounded) and convex (that is any convex combination of two allowable strategies is also allowable), each payoff function is concave with respect to that firm's decision variable, and all payoff variables are concave. That holds in this case, since the strategy spaces are both compact and convex (the real line spanning 0 to k_i), the payoff functions are concave (linear), and all payoff variables are concave (again, linear). Manas goes on to show that in this particular game, the Nash Equilibrium is unique and gives an algorithm on how the Nash Equilibrium can be located.

While the cost and demand functions are relatively simplistic, the model development by Manas provides a useful benchmark for models of quantity competition with differentiated firms (but not products) and capacity constraints.

1.4 If Hertz Charges Twice What Enterprise Does, Are They Competitors at All?

An important question we need to ask, given the data posted in the previous section, is “Are low-price firms such as Dollar and Enterprise in the same market as high-price firms such as Alamo and Hertz?” Dollar’s price for an economy rental is only $5.15 above Enterprise’s, whereas the price differentials are $18.98, $43.22 and $61.65 for Budget, Alamo and Hertz vis-a-vis Enterprise respectively. Renting a car from Alamo could be significantly different than renting a car from Hertz, because of the existence of fences, because they serve two completely separate demographic markets or because of high levels of perceived firm differentiation among customers. If this were the case, then the cross-elasticity between most, if not all brands would be near zero. If cross-elasticity between most brands can be ignored, then we can simply treat Enterprise’s optimization problem as a monopoly problem, or at worst a duopoly problem between Enterprise and Dollar. Dollar need not concern itself with how Budget prices its rental cars.

Without a published study or data on the cross-price elasticity between car rental firms, we cannot say for certain if the elasticity between firms is non-zero. However, there are markets, that share at least somewhat similar features to the car rental market, that have been studied. Consider the market for spray glass cleaners. A 22 ounce bottle of *Windex* may sell for $2.49 whereas a 22 oz bottle of a private label brand with a name such as *Kwik-E-Mart Glass Cleaner* with near identical formulation may sell for $1.09, less than half the price of *Windex*. The *name brand* vs. *generic* problem has been studied extensively in the marketing science literature, and can provide insights on the dynamics of our market.

1.4.1 Sethuraman and Srinivasan

Sethuraman and Srinivasan [92] survey the marketing science literature and note several studies which find that price reductions by high-price brands affect the demand for low-price brands. Similarly price reductions by low-price brands affect the demand for high-price brands. They indicate that the majority of these studies (including Allenby and Rossi [4], Hardie, Johnson and Fader [6], Kamakura and Russell [59] and Sivakumar and Raj [93]) find an *asymmetric price effect* - that is the demand for low-price brand is more sensitive (as measured in terms of cross-price elasticity of demand) to changes in high-price brand price than vice-versa. However, they note that Bronnenberg and Wathieu [15] find that, in some instances, the effect can act in reverse. A later paper by Dawes [27] also casts doubt on the asymmetric price effect by concluding "once the size of the brands are controlled for, lower priced (store) brands induce just as much 'switching' or purchase substitution from higher priced (manufacturer) brands as do the higher priced brands induce switching from the lower priced brands." In the majority of cases examined the high price brands have larger market shares, hence the need to control for the size of the brand.

Through both theory and empirics, Sethuraman and Srinivasan find that when measured in terms of market share rather than price elasticity, the effect runs in reverse - that is price changes in the low-price brand have more impact on the market share of the high-price brand than the converse. This discrepancy is due to the fact that in many markets the high-price brands have larger market-shares than do low-price brands.

In a related study Sethuraman, Srinivasan and Kim [85] find that when a brand discounts its price, not all competing brands are affected equally. Specifically, that the largest negative effect on the demand for a brand is a price reduction in the next-higher priced brand. The second largest negative effect on the demand for a brand is a price reduction in the next-*lowest* priced brand.

1.4.2 Batra and Sinha

Similar to Sethuraman and Srinivasan [92], Batra and Sinha [8] examine the question of competition between *national* brands and *private label* or *store* brands. They find that consumers are more likely to purchase a store brand, ceteris paribus, when:

- Consumers perceive smaller consequences to making a 'mistake' in their brand choice.

- Purchase decisions are more based on *search* characteristics rather than *experience* characteristics. They borrow this distinction from Nelson [75], where *search* characteristics of the products are those that can be verified *ex-ante* whereas *experience* characteristics are ones that can only be verified after purchase.

The second of the two points may be particularly important when it comes to online purchases of rental cars. If online purchases are disproportionately made by younger, more tech-savvy consumers, then online purchases may be associated with less experienced rental car purchasers. Though as time goes on, the demographics of online purchasers are converging to that of the general population.

1.4.3 Van Heerde, Gupta and Wittink

While buying an item of food at a grocery store is not identical to making an online car rental reservation, the markets do share some characteristics. Most notably, both markets involve large oligopolistic producers selling differentiated products. Van Heerde, Gupta and Wittink [50] survey the marketing literature and find that cross-elasticity accounts for the majority of total elasticity:

> "Chiang (1991) [20], Chintagunta (1993) [21], and Bucklin, Gupta and Siddarth (1998) [86] extend Gupta's (1988) [47] approach, which Bell, Chiang and Padmanabhan (1999) [31] then generalize to many categories and brands. Across these decomposition studies, we find that, on average, secondary demand effects (brand switching) account for the vast majority (approximately 74 %) of total elasticity, which leaves 26% for primary demand effects (purchase acceleration and quantity increases)... the percentage of secondary demand effects is never less than 40% (yogurt) and is as high as 94% (margarine)."

1.4.4 Chevalier and Goolsbee

Chevalier and Goolsbee [19] consider price competition between online booksellers, specifically *Amazon.com* and *BarnesandNoble.com*. In order to estimate sales data for Amazon.com, the authors create a formula to translate sales rank data into sales. They find high-levels of cross-price elasticity in the market, though the effect is rather asymmetric:

> "...[A] 1 % increase in the price at Amazon.com reduces quantity by about 0.5 % at Amazon.com but raises quantity at BN.com by 3.5 %. Given that Amazon.com sells somewhere between three and 10 times as many books as BN.com, this is very close to the same number of books, implying that every customer lost by Amazon.com instead buys the book at BN.com... The reverse is not true, however. Raising prices by 1 % at BN.com reduces sales about 4 % but increases sales at Amazon.com by only about 0.2 %. Many of the lost customers from BN.com evidently do not just go buy the book from Amazon.com. [19]

While rental cars and books are two different markets, the Chevalier and Goolsbee study does reveal the existence of cross-price elasticity in the online sphere.

1.4.5 Cross-Price Elasticity in the Rental Car Market

The argument that own-price and cross-price demand elasticities in the rental car market are non-zero and therefore cannot be ignored is not unique to this paper. The Perakis and Sood [82] model exhibits a cross-price demand elasticity for firm i (relative to the price of the other firm, $-i$) as:

$$\frac{\alpha p_{-i}}{D - \beta p_i + \alpha p_{-i}} \tag{1.1}$$

where D, β, α are strictly positive parameter values.

The existence of own price elasticity should not be in doubt, though the level of own price-elasticity is not necessary constant between firms and brand classes. During a 2008 interview, car rental consultant Neil Abrams commented that "[t]here is a lot of elasticity in pricing [on the leisure side]; that is not true on the corporate side." [2].

A larger issue, however, is when firm X raises the price of one of its goods, how much of the quantity demand loss goes to other goods (cross-elasticity) and how much goes to consumers deciding to reduce their overall quantity purchased? While we are unaware of any publicly-available study calculating these elasticities in the rental car industry, there is a significant body of evidence suggesting that cross-elasticities are significantly different than zero. The evidence would suggest that not all consumers simply choose the non-purchase option, rather they choose to rent a car in another class from the same firm or another car from a different firm.

1.4.6 References in the Literature to Cross-Price Elasticity in the Rental Car Market

Although there have been no publicly released studies on cross-price elasticity in the rental car market, so we do not know their magnitude. However, a number of recent papers make references to the existence of this phenomenon. First consider Anderson, Davison and Rasmussen [22]:

> "The car rental industry is not as price sensitive as the airline industry. Price changes do generate subtle changes in demand, but what is more

> important is one car rental firm's price against its competition's." [22]

Geraghty and Johnson [45] make a similar comment in their paper on revenue management at National:

> "The rental car market is extremely competitive. A price move that makes the company more expensive than its competitors can damage utilization levels." [45]

Geraghty and Johnson add:

> "Our initial analysis of National's rate behavior indicated that competitive positioning was the determining factor in its pricing decisions prior to the revenue management program. At times of low demand, sensitivity to competitor behavior is crucial. Utilization levels can suffer drastically from poor rate positioning in the marketplace." [45]

Additional evidence can be found in studies of consumer behaviour when purchasing automobiles.

1.4.7 Cross-Price Elasticity in the Car Market

There have been at least two studies in peer-reviewed journals on cross-price elasticity in the automobile market, both of which found strictly positive levels of cross-price elasticity. Goldberg [46] finds that for automobile purchases:

> "Consistent with utility maximization, and since consumers buy only one car model, all cross price elasticities are positive. Furthermore, their magnitude depends on the degree of similarity between products; automobiles that belong to the same class and origin - and are therefore similar in characteristics - exhibit on average higher cross price semi-elasticities than products of different classes and origins." [46]

Bordley [14] also finds that the level of cross-elasticity is dependent on how closely related the two products are:

> "The loyalty index is higher for higher priced products (as would be expected since higher income buyers are less price-sensitive).. economy car buyers do not view non-economy cars as close substitutes, luxury car buyers do not view non-luxury cars as closed substitutes... But note that luxury car buyers are not, contrary to popular belief substantially less price-sensitive than other buyers. This reflects how competitive the luxury car market has become." [14]

This second study suggests that although an industry (luxury cars) as a whole may have low price-elasticity of demand, the price-elasticity of demand and cross-price elasticity of demand for individual products may be significant.

While we do not have estimates of the exact level of cross-price elasticity in the rental car market, the combined evidence of statements by practitioners on the importance of cross-price elasticity in the rental car market, estimates of cross-price elasticity in the car market and estimates of cross-price elasticity from scanner data suggests that cross-price elasticities in the rental car market are likely to be significantly different than zero.

1.5 Putting it Together

While more direct evidence of cross-price elasticity in the online car rental market is needed, scanner based evidence from the marketing science literature shows that:

- Similar products can have sustainable long-term differences in prices through perceived product differentiation.

- High-priced and low-priced brands are not 'islands' and significant cross-price elasticity can exist between the two.

- The exact level of cross-price elasticity can vary depending on a number of factors, and may (though not necessarily) be higher for the low-priced brand.

1.6 Our Contribution

In this paper we create a dynamic pricing model that allows for $N \times M$ product-firm combinations using a repeated game framework. We share Vives' [104] frustration in the lack of progress made using repeated games to model oligopoly problems. Existing dynamic pricing models in revenue management are lacking as they do not address the question of multiple firms selling multiple product classes, despite the fact that most markets of interest to revenue managers have this feature. We do believe, however, that such a benchmark model can be established with a repeated game model, as the choice of allocation rule for the "excess demand" problem which plagues Bertrand-Edgeworth style capacity constrained models can be eliminated with a set of assumptions which are plausible for many markets. The first key assumption is that customers purchase at most one good at a time from at most one firm, which is reasonable for many consumer markets such as the markets for durable goods and rental cars. The second key assumption is that firms can change their prices instantaneously, which is currently the case in online markets such as Expedia and Orbitz. The impact of the ability to dynamically change online pricing has been discussed for some time now - Ariely (2000) [57], Kannan and Kopalle (2001) [60] and Wurman (2001) [105] are three such examples. It will also likely be possible in the future at retail stores thanks to the ongoing development of digital price displays. The use of RFID (Radio Frequency IDentification) in dynamic digital pricing displays has been heavily discussed - see Eckfeldt (1999) [33], Kourouthanassis and Roussos (2003) [61] and Raza, Bradshaw and Hague (1999) [74] for examples. By introducing these assumptions, we are able to guarantee the existence of Nash Equilibria, a feature which is lacking in many models.

The final goal of this model is to provide a framework which describes price competition in oligopolistic markets. We believe the game-theoretic model, using a geometric representation of the potential market, can provide the benchmark model that oligopoly theory in revenue management is lacking. We hope these ideas will strike a chord with practioners, who will incorporate them into models used by industry, in order to improve the bottom lines of their companies.

Considering cross-price elasticity and the strategic implications of the quantity levels of all products in the market, firms can substantially increase their revenues. If real world car rental firms are using existing pricing models from the revenue management literature, they are leaving a significant amount of revenue on the table. As shown by Lu [68] there may be occasions where firms can increase their profits by using a knockout strategy to eliminate a competitor product from the market.

Chapter 2

Creating and Solving NxM Pricing Demand Models

2.1 Description of the Issue

In this chapter, a framework is created using geometry and the concept of 'maximum-willingness-to-pay' to construct multivariate demand models. The result is the ability to create mathematically tractable models with an unlimited number of product types.

The N firms with M_N products pricing model allows for an accurate representation of the types of markets commonly seen in real-world revenue management situations. Despite 'multiple firm-multiple products per firm' being included in the 'future research' of many revenue management pricing papers, there is very little literature in this area due to the complexity of the problem. However, we show that it is possible to create models that allow for unlimited firms and unlimited products. The results hinge on a number of assumptions, the most important of which is the 'single-customer per period buys at most a single-product' assumption which is relatively common in the literature. Any pricing model also requires a demand framework. A major contribution of this paper is it creates a geometric framework that allows for the required mathematical tractability while still retaining a number of useful properties. We show that we can be assured of the existence of pure strategy subgame perfect Nash Equilibria. It is possible, in some instances, to show that the Nash Equilibria are necessarily unique.

We then solve the NxM pricing model recursively, first for period T then for periods $t < T$. We then prove the existence of subgame-perfect Nash Equilibria for all subgames. The result is the first ever solvable NxM revenue management model with guaranteed existence of a solution (equilibria).

2.2 Assumptions of the Model

There are N competing firms in a market, each selling one or more differentiated products. Specifically, firm n sells m_n different products, for a total of M_N brand-product combinations in the market ($M_N \geq N$). There is no requirement that each firm have the same number of products. The goods are sold over a finite selling period. At the beginning of each period the firms simultaneously set their prices. Firms have limited capacity and are unable to produce more goods. If a firm runs out of a product that product is removed from consideration for the rest of the game. Customers have a preferred brand, but will purchase one of their less preferred brand if the price differential is high enough, or from no firm if all prices are too high.

One difficulty with capacity constrained models is the problem caused by excess demand (the so called 'excess demand' problem). If the demand for a particular product is higher in a given period than remaining inventory, we have to allocate that excess demand somewhere else. This often causes the models to lack Nash Equilibria. The lack of Nash Equilibria provides difficulties to the practitioner. A model that cannot provide guidance in some scenarios is of little use to someone who wishes to use it in practice.

To avoid the excess demand problem, assume that during any period only a single customer arrives (with some probability λ) and she only demands one unit of one of the goods from one of the firms. This eliminates the excess demand problem, because any brand-product combinations remaining in the market will have at least one unit available for sale. Any brand-product combinations that have run out of goods are assumed to have dropped out of the market. This is similar to Lu (2009) [8], which refers to such models as 'price and inventory' games. Thus, there can never be excess demand in this model.

We follow the lead of Dudey (1992) [3], Lin and Sidbari (2008) [7], Martinez-de-Albeniz and Talluri (2011) [2] and Lu (2009) [8] by using the at most one arrival/at most one purchase assumption; an incredibly useful one that we believe is underutilized in the literature. We deviate from Dudey and the subsequent papers by allowing for more than 2 firms, product differentiation and market-level price elasticity of demand. Further extensions are possible as well, such as multiple products per firm as we see in the revenue management literature.

2.3 Introduction to the Oligopoly Pricing Game

Each firm begins the game with an initial inventory $q^1_{n,m}$, where q stands for quantity, $n \in [1, N]$ for the firm, $m \in [1, M]$ for the product. The superscript 1 denotes that it is the quantity at the start of the first period, where $t \in [1, T]$. As with many models in revenue management we will simplify the model, by treating all costs in the model as sunk, thus they play no role in the analysis.

The timing of the game works as follows:

1. At the beginning of each period $t \in [1, T]$, every firm $n \in [1, N]$ with remaining inventory of good $m \in [1, M]$ simultaneously sets a price $p^t_{n,m}$.

2. A customer arrives with probability λ. If a customer arrives, her preferences are drawn randomly from a distribution Δ. Given this distribution, a customer purchases good m from firm n with probability $\pi^t_{n,m}(p^t_{1,1}, ..., p^t_{N,M})$, where the probability is a function of the prices currently offered by each remaining firm. Thus the expected demand for product m from firm n in period t is given by $\lambda\pi^t_{n,m}(p^t_{1,1}, ..., p^t_{N,M})$.

   There is also a chance that the consumer does not purchase any good, which is denoted as $\pi^t_{NS}(p^t_{1,1}, ..., p^t_{N,M})$ (where NS stands for "no sale"). In further analysis, we will drop the $(p^t_{1,1}, ..., p^t_{N,M})$ notation as it should be understood that purchase probabilities are a function of price.

Assumption 1. *We require that inside the range of prices* $[L^t_{n,m}, U^t_{n,m}]$ *our demand space has the following properties:*

(a) The sum of all purchase probabilities, including the probability of no sale, given a customer's arrival, to equal one.

(b) The probability function for a product is twice differentiable (in the interior of the strategy space) with respect to own price.

(c) The probability functions are (weakly) decreasing in own price.

(d) The probability functions are (weakly) increasing in other product prices.

(e) The no-purchase probability is (weakly) increasing in the price of any product.

(f) All probabilities are non-negative.

When a product is no longer for sale in the game, we will note it as having a price $U^t_{n,m}$, which is the price at which there is no demand for that product. By doing so we do not need to alter the probability functions whenever a product is no longer available; 'pricing' the product at $U^t_{n,m}$ allows it to fall away naturally.

If a customer chooses to purchase a particular good, she pays the price the firm selling that good has offered. There is no negotiation. Thus the expected revenue of firm n with M different products in period t is given by $\lambda \sum_{m=1}^{M} \pi^t_{n,m} p^t_{n,m}$.

3. At the end of the game, firms can sell their remaining inventory of each good at the good-specific salvage value $s^T_{n,m}$. Other than $0 \leq s^T_{n,m} < U^t_{n,m}$ we put no restriction on the salvage value. It can be that $s^T_{n,m} > L^t_{n,m}$ and firms could price below the salvage value, if they wished.

2.4 Demand Model

Consumers have a maximum-willingness-to-pay for each of the M_N products. The maximum-willingness-to-pay for each good m from firm n has a lower bound of $L^t_{n,m}$; any prices lower than $L^t_{n,m}$ will not increase quantity demanded. Similarly, the maximum-willingness-to-pay for each good m from firm n has an upper bound of $U^t_{n,m}$. At a price of $U^t_{n,m}$ or higher quantity demanded is zero. Thus our demand space is a closed, compact M_N-dimensional orthotope (hyperrectangle). For mathematical tractability, we will assume that customers are identical with maximum-willingnesses-to-pay uniformly distributed within the orthotope. The demand model in this section will meet the necessary differentiability assumptions. Having a uniform distribution is a sufficient condition, but it is not necessarily necessary; there may be other distributions which meet the necessary conditions.

If only one product m from firm n existed in the market, the probability of sale given one arrival and price $p^t_{n,m}$ would be:

$$\pi^t_{n,m} = \frac{(U^t_{n,m} - p^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} \tag{2.1}$$

And the probability of no-sale, given arrival, equal to:

$$\pi^t_{NS} = \frac{(p^t_{n,m} - L^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} \tag{2.2}$$

Corollary 1 (Single Product Market Meets Demand Space Assumption)**.** *A single product demand model meets all six demand space assumptions.*

Proof. 1. **All Probabilities Given Arrival Equal One**:

$$\frac{(U^t_{n,m} - p^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} + \frac{(p^t_{n,m} - L^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} = \frac{(U^t_{n,m} - L^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} = 1 \tag{2.3}$$

2. **Demand Probability Twice Differentiable**:

$$\frac{\partial \pi^t_{n,m}}{\partial p^t_{n,m}} = \frac{-1}{(U^t_{n,m} - L^t_{n,m})}, \frac{\partial^2 \pi^t_{n,m}}{\partial p^{t2}_{n,m}} = 0 \tag{2.4}$$

3. **Demand Probability Decreasing in Own Price**: Met since the first derivative of the probability function with respect to price is negative.

$$\frac{\partial \pi^t_{n,m}}{\partial p^t_{n,m}} = \frac{-1}{(U^t_{n,m} - L^t_{n,m})} \tag{2.5}$$

4. **Demand Probability Increasing in Other Product Prices**: Irrelevant - no other products in market.

5. **No Purchase Probability Increasing in Prices**: Met since the first derivative of the no-sale probability function with respect to price is positive.

$$\frac{\partial \pi^t_{NS}}{\partial p^t_{n,m}} = \frac{1}{(U^t_{n,m} - L^t_{n,m})} \tag{2.6}$$

6. **Probabilities Non-Negative**: Since our probability is decreasing in price, it reaches a minimum at $p^t_{n,m} = U^t_{n,m}$. When $p^t_{n,m} = U^t_{n,m}, \pi^t_{n,m} = 0$. Non-purchase probability is increasing in price and thus reaches a minimum when $p^t_{n,m} = L^t_{n,m}$. When $p^t_{n,m} = L^t_{n,m}, \pi^t_{NS} = 0$. The condition is met.

   Therefore all six conditions are met.

□

However, when there is more than 1 product in the market, a consumer may be willing to buy *either* product. By definition, however, she will purchase only one. As such, a consumer who is willing to purchase a good does not necessarily purchase that good, if she is also willing to purchase other goods as well.

Define $g(p^t_{-n,-m}) \in [0,1]$ as the fraction of consumers who are willing to purchase product n from firm m and actually do so. $g(p^t_{-n,-m})$ is a function of the price of all of the M_N goods in the market, except product m from firm n. We require $g(p^t_{-n,-m})$ to be continuously differentiable and strictly increasing in all other prices (that is, all partial derivatives exist and are strictly greater than zero). Furthermore we will require that for all $m \neq -m$, $n \neq -n$ $\frac{\partial^2 g(p^t_{-n,-m})}{\partial p^{t2}_{-n,m}} = 0$, $\frac{\partial^2 g(p^t_{-n,-m})}{\partial p^{t2}_{n,m}} = 0$ and $\frac{\partial^2 g(p^t_{-n,-m})}{\partial p^{t2}_{n,-m}} = 0$. We will make this requirement for reasons of mathematical tractability but note that this restriction will be met in the rest of this chapter.

Thus the probability of purchase, given arrival, for this product is now given by:

$$\pi^t_{n,m} = \frac{(U^t_{n,m} - p^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} g(p^t_{-n,-m}) \tag{2.7}$$

Note that as before $\lim_{p^t_{n,m} \to U^t_{n,m}} \pi^t_{n,m} = 0$, regardless of the value of $g(p^t_{-n,-m})$. However, now $\lim_{p^t_{n,m} \to L^t_{n,m}} \pi^t_{n,m} = g(p^t_{-n,-m})$.

To illustrate the logic of such an approach, we will examine a market with 2 products. The products can both be sold by the same firm, or by 2 different firms.

2.5 Two Product Market Illustration

This could be a market of two competing firms each selling a product, which would have prices $p_{1,1}^t$ and $p_{2,1}^t$. This could also be a market of one firm selling two products, which would have prices $p_{1,1}^t$ and $p_{1,2}^t$. We wish to be agnostic about which type of market since this could apply to either, so for the purposes of this section we shall simply label the products A and B with prices p_A^t and p_B^t.

In our market, each customer has a maximum-willingness-to-pay for each product, which we will denote r_A^t and r_B^t. These reservation prices are bounded between L_A^t and U_A^t for product A and L_B^t and U_B^t for product B.

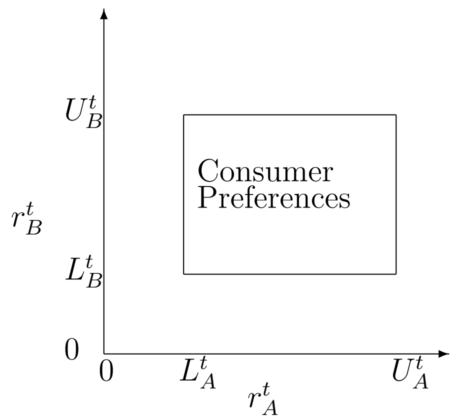

Figure 2.1: Two-Dimensional demand space, with maximum-willingness-to-pay for good A along the X-axis, and maximum-willingness-to-pay for good B along the Y-axis. All consumers' maximum-willingnesses-to-pay exist within the rectangle marked 'Consumer Preferences'; there are no consumers outside the rectangle. Note that L_A^t and L_B^t are drawn as being strictly greater than zero. However, one or both of L_A^t and L_B^t can be equal to zero.

There is no benefit in setting the price of product A (p_A^t) below L_A^t as no consumers have a maximum-willingness-to-pay below this value. Expected quantity demanded would not increase relative to pricing at L_A^t and the firm would receive less revenue per sale. Similarly, there is no incentive to price above U_A^t, as quantity demanded is zero at U_A^t and for all values above this level. As such, we will restrict our analysis to prices that are between (or equal to) L_A^t and U_A^t for firm A and L_B^t and U_B^t for firm B.

Placing prices p_A^t and p_B^t on our figure gives us as follows:

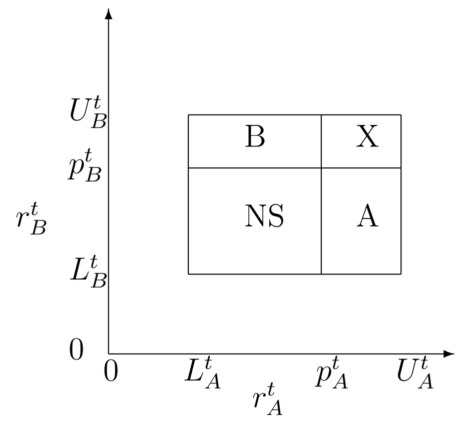

Figure 2.2: Four Segments of a Two-Dimensional Demand Space, with maximum-willingness-to-pay for good A along the X-axis, and maximum-willingness-to-pay for good B along the Y-axis. All consumers' maximum-willingnesses-to-pay exist within the large rectangle; there are no consumers outside the rectangle. Note that L_A^t and L_B^t are drawn as being strictly greater than zero. However, one or both of L_A^t and L_B^t can be equal to zero.

In the NS area, consumers are unwilling to purchase either product, since their maximum-willingness-to-pay for product A is below the price set for A (p_A^t) and the same holds true for product B - their maximum-willingness-to-pay is set below the market price for product B, p_B^t. In area A, customers are willing to buy product A at the market price p_A^t. In area B, customers are willing to buy product B for the price p_B^t. In the section labelled X, customers are willing to purchase either product. Since consumers only purchase one good, we need a mechanism to assign customers who are willing to buy either product to exactly one product.

We need a function that will apportion area X between the two products. One obvious solution (but far from the only one) would be a proportional splitting rule that is simply a straight line from (p_A^t, p_B^t) to U_A^t, U_B^t.

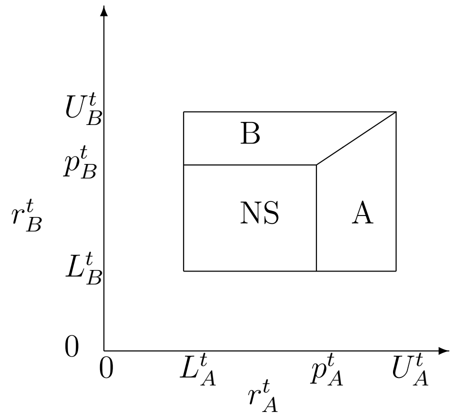

Figure 2.3: Proportional Rule in a Two-Dimensional Demand Space, with maximum-willingness-to-pay for good A along the X-axis, and maximum-willingness-to-pay for good B along the Y-axis. All consumers' maximum-willingnesses-to-pay exist within the large rectangle; there are no consumers outside the rectangle. Note that L_A^t and L_B^t are drawn as being strictly greater than zero. However, one or both of L_A^t and L_B^t can be equal to zero.

Thus, we express the customers who will purchase product A as a simple geometric shape. Note that using the linear splitting rule, if the price of good B, p_B^t is increased, then product A retains its entire area while gaining additional demand from product B.

From the consumer's point of view, this splitting rule is functionally equivalent to a utility function where the consumer chooses the good that has the largest difference between the maximum-willingess-to-pay and the price of that good. However, unlike such a utility function formulation, the geometric model with proportional splitting rule is twice continuously differentiable.

Since customers are distributed uniformly across the rectangle, the probability of a sale for each product, given an arrival, is simply the normalized areas marked A and B respectively. Given prices p_A^t and p_B^t, the demand for product A is given by:

$$\pi_A^t = \frac{(U_A^t - p_A^t)}{(U_A^t - L_A^t)} \frac{(U_B^t + p_B^t)}{2(U_B^t - L_B^t)} \tag{2.8}$$

Demand for product B, given an arrival, given by:

$$\pi_B^t = \frac{(U_B^t - p_B^t)}{(U_B^t - L_B^t)} \frac{(U_A^t + p_A^t)}{2(U_A^t - L_A^t)} \tag{2.9}$$

No sale probability, given an arrival, given by:

$$1 - \pi_A^t - \pi_B^t = \pi_{NS}^t = \frac{(p_A^t - L_A^t)}{(U_A^t - L_A^t)} \frac{(p_B^t - L_B^t)}{(U_B^t - L_B^t)} \tag{2.10}$$

Corollary 2 (Two Product Market Meets Demand Space Assumption)**.** *A two product demand model meets all six demand space assumptions.*

Proof. 1. **All Probabilities Given Arrival Equal One**: Met by construction as $1 - \pi_A^t - \pi_B^t = \pi_{NS}^t$.

2. **Demand Probability Twice Differentiable**:

$$\frac{\partial \pi_A^t}{\partial p_A^t} = \frac{-(U_B^t + p_B^t)}{2(U_A^t - L_A^t)(U_B^t - L_B^t)}, \frac{\partial^2 \pi_A^t}{\partial p_A^{t2}} = 0 \tag{2.11}$$

$$\frac{\partial \pi_B^t}{\partial p_B^t} = \frac{-(U_A^t + p_A^t)}{2(U_B^t - L_B^t)(U_A^t - L_A^t)}, \frac{\partial^2 \pi_B^t}{\partial p_B^{t2}} = 0 \tag{2.12}$$

3. **Demand Probability Decreasing in Own Price**: Met since the first derivative of the probability functions with own price are negative (since $U_A^t \geq L_A^t$,

$U_B^t \geq L_B^t$).

$$\frac{\partial \pi_A^t}{\partial p_A^t} = \frac{-(U_B^t + p_B^t)}{2(U_A^t - L_A^t)(U_B^t - L_B^t)} \tag{2.13}$$

$$\frac{\partial \pi_B^t}{\partial p_B^t} = \frac{-(U_A^t + p_A^t)}{2(U_B^t - L_B^t)(U_A^t - L_A^t)} \tag{2.14}$$

4. **Demand Probability Increasing in Other Product Prices**:

$$\frac{\partial \pi_A^t}{\partial p_B^t} = \frac{(U_A^t - p_A^t)}{2(U_A^t - L_A^t)(U_B^t - L_B^t)} \tag{2.15}$$

$$\frac{\partial \pi_B^t}{\partial p_A^t} = \frac{(U_B^t - p_B^t)}{2(U_A^t - L_A^t)(U_B^t - L_B^t)} \tag{2.16}$$

Holds since $U_A^t \geq L_A^t$, $U_B^t \geq L_B^t$, $U_A^t \geq p_A^t$ and $U_B^t \geq p_B^t$.

5. **No Purchase Probability Increasing in Prices**: Met since the first derivative of the no-sale probability function with respect to prices are positive:

$$\frac{\partial \pi_{NS}^t}{\partial p_A^t} = \frac{(p_B^t - L_B^t)}{(U_A^t - L_A^t)(U_B^t - L_B^t)}, \frac{\partial \pi_{NS}^t}{\partial p_B^t} = \frac{(p_A^t - L_A^t)}{(U_A^t - L_A^t)(U_B^t - L_B^t)} \tag{2.17}$$

Holds since $U_A^t \geq L_A^t$, $U_B^t \geq L_B^t$, $U_A^t \geq p_A^t$ and $U_B^t \geq p_B^t$.

6. **Probabilities Non-Negative**: Since our probability is decreasing in own price but decreasing in competitor price, for A it reaches a minimum at $p_A^t = U_A^t, p_B^t = L_B^t$.

$$\pi_A^t = \frac{(U_A^t - p_A^t)}{(U_A^t - L_A^t)} \frac{(U_B^t + p_B^t)}{2(U_B^t - L_B^t)} \tag{2.18}$$

$$\pi_A^t = \frac{(U_A^t - U_A^t)}{(U_A^t - L_A^t)} \frac{(U_B^t + L_B^t)}{2(U_B^t - L_B^t)} = 0 \tag{2.19}$$

For B:

$$\pi_B^t = \frac{(U_B^t - p_B^t)}{(U_B^t - L_B^t)} \frac{(U_A^t + p_A^t)}{2(U_A^t - L_A^t)} \tag{2.20}$$

$$\pi_B^t = \frac{(U_B^t - U_B^t)}{(U_B^t - L_B^t)} \frac{(U_A^t + L_A^t)}{2(U_A^t - L_A^t)} = 0 \tag{2.21}$$

No sale probability is increasing in all prices, and therefore reaches a minimum at $p_A^t = L_A^t, p_B^t = L_B^t$

$$\pi_{NS}^t = \frac{(p_A^t - L_A^t)}{(U_A^t - L_A^t)} \frac{(p_B^t - L_B^t)}{(U_B^t - L_B^t)} \tag{2.22}$$

$$\pi_{NS}^t = \frac{(L_A^t - L_A^t)}{(U_A^t - L_A^t)} \frac{(L_B^t - L_B^t)}{(U_B^t - L_B^t)} = 0 \tag{2.23}$$

The condition is met.

Therefore all six conditions are met.

□

In a 3 product market, with products A, B and C, the problem becomes more complex. See Appendix A for a discussion of the issue.

For an M-product case, the functional forms become increasingly complicated as the number of products increases, however the $g(p_{-A}^t)$ functions retain the same functional properties. The number of terms π_A^t's probability function is 2^{N-1}, where N is the overall number of products (including A). The number of terms is equal to every possible combination of a set of product prices that are above and below a consumer's maximum-willingness-to-pay.

2.6 Set-Up For Solving an NxM Pricing Problem

For this section, we will analyze the problem from the perspective of a single firm $n \in N$.

In order to compute the optimal price each firm should charge in a given situation, we need to first define our concept of equilibria. First we will assume that all firms in the model are rational profit maximizers. Though, it is easy to adapt this model to scenarios where one or more firms have committed to a different strategy, such as a national advertising campaign that announces “rent a car for just 29.95!”.

Secondly, we will assume that firms do not engage in path-dependent strategies. Thus all strategies in the game are Markovian as they only depend on the state. [1] Here the state is simply the current quantity levels for each firm and the number of periods remaining until the end of the game. We will use the game-theoretic concept of *Markov perfect equilibrium*, if it exists, to determine each firm's optimal pricing strategy for each given state. First we need to define a few terms.

The definition of subgame and subgame perfect Nash Equilibria and Nash Equilibrium used is adapted from Mas-Colell, Whinston, and Green [1]. The definition of Markov perfect equilibrium is adapted from Fudenberg and Tirole [5].

Definition 1 (Subgame)**.** *A subgame begins with an information set containing a single decision node, which contains all the decision nodes that are successors of this node. Furthermore, it contains only those nodes which are successors.*

In the case of the game developed here, every decision node is the initial point of a subgame, including the entire game itself.

From that definition, we can define a subgame perfect Nash Equilibrium in this context.

Definition 2 (Subgame Perfect Nash Equilibrium)**.** *A set of state contingent pricing strategies for the N players in an N-player game is a subgame perfect Nash Equilibrium if it induces a Nash Equilibrium in every subgame of the game.*

Where a Nash Equilibrium is defined as follows:

Definition 3 (Nash Equilibrium)**.** *Given a finite set of players* $1, ..., I$*, and payoff functions* $u = \{u_1()..u_I()\}$*, a strategy profile* $s = (s_1, ..., s_I)$ *constitutes a Nash Equilibrium of game* $\Gamma_N = [I, \{S_i\}, \{u_i()\}]$ *if for every* $i = 1, ..., I$*,* $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ *for all* $s_i' \in S_i$*.*

A Markov perfect equilibrium takes the concept of a Subgame Perfect Nash Equilibrium and applies it to situations where the firms are restricted to using only state-dependent strategies.

Definition 4 (Markov Perfect Equilibrium)**.** *A Markov perfect equilibrium is a profile of Markov strategies that yield a Nash Equilibrium in every subgame. Since the state captures the influence of past play on the strategies and payoffs for each subgame, if all of a player's opponents use Markov strategies, that player has a best response that is Markov as well. A Markov perfect equilibrium thus continues to be a perfect equilibrium when the Markov restriction is dropped.*

We can determine the Markov Perfect Equilibrium, if it exists, for the game through backwards induction of a Markov decision process, where we examine the subgames emanating from every possible combination of quantity levels and time remaining. Each firm's optimal strategy is thus a set of state-contingent strategies for each of these potential combinations.

2.7 Dynamic Programming Formulation

The objective of our Markov decision process is to maximize total profit over the entire sales period, which is equivalent to maximizing our revenue if there are no variable costs in the model. Since there are no variable costs in this model, we will treat the problem as one of maximizing revenue. As discussed earlier, we can break our multi-period planning problem into a series of single-period problems which are solved through backwards induction.

A Markov decision process has four components:

- A set of states S.
- A set of actions $a \in A$.
- $P_a(s, s') = Pr(s^{t+1} = s' | s^t = s, a^t = a)$, which is the probability that an action a in state s at time t will lead to state s' at time $t + 1$.
- $E[R_a(s, s')]$ is the expected revenue received after transition to state s' from state s with probability $P_a(s, s')$.

It is necessary to define a few terms:

- The quantity of goods held by firm n at time t is denoted q_n^t, a vector of length m_n^t. The quantity for a particular good is denoted $q_{n,m}^t$, and the quantity of goods held by other firms is noted q_{-n}^t.

- The arrival rate of customers is given by λ.

- The end-of-game salvage value for a unit of good m held by firm n is $s_{n,m}^T$.

Our four components are then as follows:

- Our state space S at time t is simply the set of quantities of each good held by every firm (not just firm n). Thus $S^t = (q_n^t, q_{-n}^t)$.

- Our set of actions A at time t contains the prices we can set for each good m: $p_{n,m}^t \in [L_{n,m}^t, U_{n,m}^t]$.

- The probability that we make a transition from state s in time t to state s' in time $t+1$ having one less of a given good m is equivalent to the probability of sale of that good (expected demand) and given by $\lambda\pi_{n,m}^t$. The probability that we transition from state t to state $t+1$ with a competitor j having one less good k is equivalent to the probability of sale of that good and is given by $\lambda\pi^t_{j,k}$. The probability that we enter the next state with all quantity levels changed is simply 1 minus the sum of all probabilities of sale.

- The revenue from transitioning to state s' in time $t+1$ from state s at time t is equal to $p^t_{n,m}$ when our firm sells a unit of good m and is equal to zero otherwise.

Our objective is to maximize the sum of expected, discounted revenue over the t periods:

$$\sum_{t=1}^{T} \gamma^t E[R_{a_t}(s_t, s_{t+1})] \tag{2.24}$$

Where γ is the discount rate between periods. Since the time horizon under consideration is small, we will assume $\gamma = 1$ and it will be dropped from the analysis.

The Markovian decision process involves determining an optimal reaction function (also known as a policy function) ρ which contains actions and a value function V. The policy function is a set of state-contingent prices; given the state of the world; what price should the firm set for its goods at time t in order to maximize wealth over the entire game. It is given by:

$$\rho(s) = argmax_a\{\sum_{s'} P_a(s, s')(R_a(s, s') + V(s'))\} \tag{2.25}$$

The value function V for a given state s provides the sum of expected revenue to be earned through the rest of the game if the optimal policies ρ are followed and is given by:

$$V(s) = \sum_{s'} P_{\rho(s)}(s, s')(R_{\rho(s)}(s, s') + V(s')) \tag{2.26}$$

In the next section we will examine the structure of the value function at time T and the optimal reaction functions.

2.7.1 Period T Problem

In the final period T, the only consideration for the firm is that if it sells any good this period, that is one less good that it can sell for salvage. Given the arrival rate λ, the quantity levels of all goods and the salvage value of a good $s^T_{n,m}$, the value function for our firm at time T is given by:

$$V^T_{q^T_n, q^T_{-n}} = \sum_{m=1}^{m_n} \lambda \pi^T_{n,m}(p^T_{n,m} - s^T_{n,m}) + q^T_{n,m} s^T_{n,m} \tag{2.27}$$

Substitute in the probability functions:

$$V^T_{q^T_n, q^T_{-n}} = \sum_{m=1}^{m_n} \lambda \frac{(U^T_{n,m} - p^T_{n,m})}{(U^T_{n,m} - L^T_{n,m})} g(p^T_{-m})(p^T_{n,m} - s^T_{n,m}) + q^T_{n,m} s^T_{n,m} \tag{2.28}$$

To find the optimal reaction functions, we differentiate the value function with respect to each of $n's$ products, set each to zero, giving us a set of m^T_n first order conditions. The first order condition for the mth product is as follows:

$$\frac{\partial V^T_{q^T_n,q^T_{-n}}}{\partial p^T_{n,m}} = \frac{g(p^T_{-m})(U^T_{n,m} + s^T_{n,m} - 2p^T_{n,m})}{U^T_{n,m} - L^T_{n,m}} + \sum_{i=1,i\neq m}^{m_n} \frac{(U^T_{n,i} - p^T_{n,i})}{(U^T_{n,i} - L^T_{n,i})} \frac{\partial g(p^T_{-i})}{\partial p^T_{n,m}} (p^T_{n,i} - s^T_{n,i}) = 0 \quad (2.29)$$

Re-arrange to yield the reaction function for the first price:

$$p^{T*}_{n,m} = 0.5[U^T_{n,m} + s^T_{n,m}] + 0.5[\sum_{i=1,i\neq m}^{m_n} \frac{(U^T_{n,i} - p^T_{n,i})}{(U^T_{n,i} - L^T_{n,i})} \frac{\partial g(p^T_{-i})}{\partial p^T_{n,m}} (p^T_{n,i} - s^T_{n,i})] \frac{(U^T_{n,i} - L^T_{n,i})}{g(P^T_{-i})} \quad (2.30)$$

Note that $[\sum_{i=1,i\neq m}^{m_n} \frac{(U^T_{n,i}-p^T_{n,i})}{(U^T_{n,i}-L^T_{n,i})} \frac{\partial g(p^T_{-i})}{\partial p^T_{n,m}} (p^T_{n,i} - s^T_{n,i})] \frac{(U^T_{n,i}-L^T_{n,i})}{g(P^T_{-i})}$ is necessarily positive since all the terms are positive. Thus $p^{T*}_{n,m} \geq 0.5[U^T_{n,m} + s^T_{n,m}]$. However, we need to recognize that our strategy space is bounded.

2.7.2 Result: Optimal Time T Reaction Functions

Our strategy space for $p^{T*}_{n,m}$ is bounded between $[0, P^T_{n,m}]$. Since $p^{T*}_{n,m} \geq 0.5[U^T_{n,m}+s^T_{n,m}]$ and $s^T_{n,m} \geq 0$, there is no risk of $p^{T*}_{n,m} < 0$. Thus our optimal strategy has the following conditions:

- If $0.5[U^T_{n,m} + s^T_{n,m}] + 0.5[\sum_{i=1,i\neq m}^{m_n} \frac{(U^T_{n,i}-p^T_{n,i})}{(U^T_{n,i}-L^T_{n,i})} \frac{\partial g(p^T_{-i})}{\partial p^T_{n,m}} (p^T_{n,i} - s^T_{n,i})] \frac{(U^T_{n,i}-L^T_{n,i})}{g(P^T_{-i})} > U^T_{n,m}$ then $p^{T*}_{n,m} = U^T_{n,m}$.

- else $p^{T*}_{n,m} = 0.5[U^T_{n,m} + s^T_{n,m}]+$
$0.5[\sum_{i=1,i\neq m}^{m_n} \frac{(U^T_{n,i}-p^T_{n,i})}{(U^T_{n,i}-L^T_{n,i})}\frac{\partial g(p^T_{-i})}{\partial p^T_{n,m}}(p^T_{n,i} - s^T_{n,i})]\frac{(U^T_{n,i}-L^T_{n,i})}{g(P^T_{-i})} > U^T_{n,m}$.

We have a set of M strategy functions and M unknowns. We will prove later in the section that an optimal solution to this system of equations does exist. With the optimal set prices we can calculate the probability of sale and the value functions for each possible state at time T. With this we can then work recursively to determine the optimal price for any period in any state, so long as an optimal price exists.

2.7.3 Period t Problem

Next consider periods $T-1$, $T-2$ and all preceding periods t. Given arrival rate λ and quantity levels q^t_n, q^t_{-n}, of all goods the value function for our firm at time t is as follows:

$$V^t_{q^t_n,q^t_{-n}} = \sum_{m=1}^{m_n} \lambda\pi^t_{n,m}(p^t_{n,m} + V^{t+1}_{q^t_{n,m}-1,q^t_{-n}}) + \sum_{j=1,j\neq n}^{N}\sum_{k=1}^{m_j}\lambda\pi^t_{j,k}(V^{t+1}_{q^t_n,q^t_{j,k}-1})+ \\ (1-\sum_{m=1}^{m_n}\lambda\pi^t_{n,m} - \sum_{j=1,j\neq n}^{N}\sum_{k=1}^{m_j}\lambda\pi^t_{j,k})V^{t+1}_{q^t_n,q^t_{j,k}} \quad (2.31)$$

Here the index j represents all the firms *except* firm n (that is, it is an index of $N-1$ firms). The index k is an index of the m_j products sold by firm j.

There are $M_n + 1$ possible states of the world the firm could find itself in during period $t+1$: One each for a sale in period t of any of the M_n products in the market and one reflecting where there was no sale of any products in period t.

Re-arrange to yield:

$$V^t_{q^t_n, q^t_{-n}} = \sum_{m=1}^{m_n} \lambda \pi^t_{n,m} (p^t_{n,m} + V^{t+1}_{q^t_{n,m}-1, q^t_{-n}} - V^{t+1}_{q^t_{n,m}, q^t_{-n}}) + \\ \sum_{j=1, j \neq n}^{N} \sum_{k=1}^{m_j} \lambda \pi^t_{j,k} (V^{t+1}_{q^t_n, q^t_{j,k}-1} - V^{t+1}_{q^t_n, q^t_{j,k}}) + V^{t+1}_{q^t_n, q^t_{-n}} \quad (2.32)$$

To simplify, we define two functions:

$$c^t_{n,m} = V^{t+1}_{q^t_{n,m}, q^t_{-n}} - V^{t+1}_{q^t_{n,m}-1, q^t_{-n}} \quad (2.33)$$

$c^t_{n,m}$ is the difference in the value functions between having quantity $q^{t+1}_{m,n} = q^t_{m,n} - 1$ and $q^{t+1}_{m,n} = q^t_{m,n}$ (that is, one more) of good $q_{m,n}$ in period $t+1$. Conceptually it can be thought of as the opportunity **cost** of making a sale in period t. The opportunity cost reflects the fact that if we sell a good now, that leaves us with one less good to sell in the future.

The second function we define is as follows:

$$b^t_{j,k} = V^{t+1}_{q^t_n, q^t_{j,k}-1} - V^{t+1}_{q^t_n, q^t_{j,k}} \quad (2.34)$$

$b^t_{j,k}$ is the difference in the value functions between our competitor having $q^{t+1}_{j,k} = q^t_{j,k}$ and $q^{t+1}_{j,k} = q^t_{j,k} - 1$ (that is, one less) of $q^t_{j,k}$ in period $t+1$. Conceptually it can be thought of as the **benefit** of our competitor making a sale in period t, leaving them with one less good in period $t+1$. The benefit reflects the fact that a sale this period increases the chance that our competitor will sell out of this particular good, leaving us with one less competitor product with which to compete.

Substituting in the two functions provides:

$$V^t_{q^t_n, q^t_{-n}} = \sum_{m=1}^{m_n} \lambda \pi^t_{n,m}(p^t_{n,m} - c^t_{n,m}) + \sum_{j=1, j\neq n}^{N} \sum_{k=1}^{m_j} \lambda \pi^t_{j,k}(b^t_{j,k}) + V^{t+1}_{q^t_n, q^t_{-n}} \tag{2.35}$$

Substitute in the probability functions for each:

$$\begin{aligned} V^t_{q^t_n, q^t_{-n}} = \sum_{m=1}^{m_n} \lambda \frac{(U^t_{n,m} - p^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} g(p^t_{-m})(p^t_{n,m} - c^t_{n,m}) + \\ \sum_{j=1, j\neq n}^{N} \sum_{k=1}^{m_j} \lambda \frac{(U^t_{j,k} - p^t_{j,k})}{(U^t_{j,k} - L^t_{j,k})} g(p^t_{j,-k})(b^t_{j,k}) + V^{t+1}_{q^t_n, q^t_{-n}} \end{aligned} \tag{2.36}$$

Note that $g(p^t_{-m})$ is a function of *all* other prices but the price for good m, including both prices of goods sold by other firms and goods sold by our firm.

To find the optimal reaction functions, we differentiate the value function with respect to each of $n's$ products, set each to zero, giving us a set of m_N first order conditions. The first order condition for the mth product is as follows:

$$\frac{\partial V^t_{q^t_n,q^t_{-n}}}{\partial p^t_{n,m}} = \lambda\frac{-1}{(U^t_{n,m}-L^t_{n,m})}g(p^t_{-m})(p^t_{n,m}-c^t_{n,m})+$$
$$\lambda\frac{(U^t_{n,m}-p^t_{n,m})}{(U^t_{n,m}-L^t_{n,m})}g(p^t_{-m})+\sum_{i=1,i\neq m}^{m_n}\lambda\frac{(U^t_{n,i}-p^t_{n,i})}{(U^t_{n,i}-L^t_{n,i})}\frac{\partial g(p^t_{-i})}{\partial p^t_m}(p^t_{n,i}-c^t_{n,i})+$$
$$\sum_{j=1,j\neq n}^{N}\sum_{k=1}^{m_j}\lambda\frac{(U^t_{j,k}-p^t_{j,k})}{(U^t_{j,k}-L^t_{j,k})}\frac{g(p^t_{j,-k})}{p^t_m}(b^t_{j,k})=0 \quad (2.37)$$

Simplify to:

$$\frac{(U^t_{n,m}+c^t_{n,m}-2p^t_{n,m})}{(U^t_{n,m}-L^t_{n,m})}g(p^t_{-m})+$$
$$\sum_{i=1,i\neq m}^{m_n}\frac{(U^t_{n,i}-p^t_{n,i})}{(U^t_{n,i}-L^t_{n,i})}\frac{\partial g(p^t_{-i})}{\partial p^t_m}(p^t_{n,i}-c^t_{n,i})+$$
$$\sum_{j=1,j\neq n}^{N}\sum_{k=1}^{m_j}\frac{(U^t_{j,k}-p^t_{j,k})}{(U^t_{j,k}-L^t_{j,k})}\frac{\partial g(p^t_{j,-k})}{p^t_{n,m}}(b^t_{j,k})=0 \quad (2.38)$$

Note that no term in:

$$\sum_{i=1,i\neq m}^{m_n}\frac{(U^t_{n,i}-p^t_{n,i})}{(U^t_{n,i}-L^t_{n,i})}\frac{\partial g(p^t_{-i})}{\partial p^t_m}(p^t_{n,i}-c^t_{n,i})+$$
$$\sum_{j=1,j\neq n}^{N}\sum_{k=1}^{m_j}\frac{(U^t_{j,k}-p^t_{j,k})}{(U^t_{j,k}-L^t_{j,k})}\frac{\partial g(p^t_{j,-k})}{p^t_{n,m}}(b^t_{j,k}) \quad (2.39)$$

is a function of $p^t_{n,m}$. Define:

$$h(p^t_{-m}) = \sum_{i=1, i\neq m}^{m_n} \frac{(U^t_{n,i} - p^t_{n,i})}{(U^t_{n,i} - L^t_{n,i})} \frac{\partial g(p^t_{-i})}{\partial p^t_m} (p^t_{n,i} - c^t_{n,i}) + \sum_{j=1, j\neq n}^{N} \sum_{k=1}^{m_j} \frac{(U^t_{j,k} - p^t_{j,k})}{(U^t_{j,k} - L^t_{j,k})} \frac{\partial g(p^t_{j,-k})}{p^t_{n,m}} (b^t_{j,k}) \quad (2.40)$$

then:

$$\frac{(U^t_{n,m} + c^t_{n,m} - 2p^t_{n,m})}{(U^t_{n,m} - L^t_{n,m})} g(p^t_{-m}) + h(p^t_{-m}) = 0 \quad (2.41)$$

Re-arrange to yield:

$$p^{t*}_{n,m} = 0.5[U^t_{n,m} + c^t_{n,m} + \frac{h(p^t_{-m})(U^t_{n,m} - L^t_{n,m})}{g(p^t_{-m})}] \quad (2.42)$$

2.7.4 Result: Optimal Time t Reaction Functions

Our strategy space for $p^{t*}_{n,m}$ is bounded between $[0, P^t_{n,m}]$. Thus our optimal strategy has the following conditions:

- If $0.5[U^t_{n,m} + c^t_{n,m} + \frac{h(p^t_{-m})(U^t_{n,m} - L^t_{n,m})}{g(p^t_{-m})}] < 0$, then $p^{t*}_{n,m} = 0$.
- If $0.5[U^t_{n,m} + c^t_{n,m} + \frac{h(p^t_{-m})(U^t_{n,m} - L^t_{n,m})}{g(p^t_{-m})}] > U^t_{n,m}$ then $p^{t*}_{n,m} = U^t_{n,m}$.
- else $p^{t*}_{n,m} = 0.5[U^t_{n,m} + c^t_{n,m} + \frac{h(p^t_{-m})(U^t_{n,m} - L^t_{n,m})}{g(p^t_{-m})}]$.

We have a system of M equations and M unknowns. We can show that this system of equations has a solution.

2.8 Five Factors in Optimal Pricing

Any optimal pricing revenue management model to price a good will consider own-price elasticity, as does this one. It goes beyond, however, by implicitly (and simultaneously) considering five other factors, which we can see as we examine the optimal pricing equation, given an interior solution, $p_{n,m}^{t*} = 0.5[U_{n,m}^t + c_{n,m}^t + \frac{h(p_{-m}^t)(U_{n,m}^t - L_{n,m}^t)}{g(p_{-m}^t)}]$:

1. **Quantity Level of the Product**: The $c_{n,m}^t$ term in our pricing solution reflects that our good is being sold over a limited time period and re-orders are not possible. A sale today means one less good to sell over the sales period (or one less good to sell for salvage), leading to an opportunity cost of making a sale.

2. **Quantity Levels of Firm's Other Products**: In the $h(p_{-m}^t)$ equation we have the term $(p_{n,i}^t - c_{n,i}^t)$, which reflects that our other products have an opportunity cost as well. By changing the price of one good, we change the probability of sale for our *other* goods. As these goods have an opportunity cost, we need to take that into consideration when making our pricing decision.

3. **Quantity Levels of Other Firms' Products**: In the $h(p_{-m}^t)$ equation we have the term $(b_{j,k}^t)$, which reflects the potential benefit to our competitor making a sale. If our competitor makes a sale, this increases the probability that they will run out of that good before the sales period is over. If the competitor

runs out of that product, that is one less competitor to price against.

4. **Cannibalization, Cross-Price Elasticity and the Price of Our Other Products**: The terms $g(p^t_{-m})$, $\frac{\partial g(p^t_{-i})}{\partial p^t_m}$ and $\frac{\partial g(p^t_{j,-k})}{p^t_{n,m}}$ contain the prices of our other products. Thus the optimal price we set must consider the prices set for our other goods and the cross-price elasticity of demand between our products.

5. **Competition, Cross-Price Elasticity and the Price of Competitor Products**: The terms $g(p^t_{-m})$, $\frac{\partial g(p^t_{-i})}{\partial p^t_m}$ and $\frac{\partial g(p^t_{j,-k})}{p^t_{n,m}}$ also contain the prices of our competitor's products. Thus the optimal price we set must consider the prices other firms are setting for their products and that raising our prices will see us losing sales to our competitors.

We will see in the next two chapters how considering each factor increases the revenue of the firm.

2.9 Result: Existence of a Solution

In this section we prove that an optimum solution exists.

Our strategy space is M dimensional, and bounded between $[L^t_{n,m}, U^t_{n,m}]$ in each dimension. Nash [9] proved that the existence of a fixed-point in the strategy space is necessary and sufficient for proving the existence of a Nash Equilibrium. We will follow Nash's lead and use Kakutani's fixed point theorem to prove the existence of a fixed-point (and thus the existence of a Nash Equilibrium).

Definition 5. *Kakutani Fixed Point Theorem Let the strategy space S be a non-empty, compact and convex subset of some Euclidean space R^n. Let ϕ be an upper hemicontinuous set-valued function on S with the property that $\phi(x)$ is non-empty, closed and convex for all $x \in S$. Then ϕ has a fixed point.*

Convexity is guaranteed as our strategy space (the set of prices that can be charged for each product) meets the following conditions.

> **Corollary 3** (Strategy Space of Pricing Game is Convex)**.** *The strategy space, ρ, of our game is convex.*

Proof. Take any two points within the strategy space, each denoted by an N-dimensional vector [call them $P(1)$ and $P(2)$]. Without loss of generality select the i^{th} term of the first vector $p_i(1)$ and the second point $p_i(2)$, where by definition $u_i \geq p_i(1) \geq 0$ and $u_i \geq p_i(2) \geq 0$. Pick any λ such that $\lambda \in [0,1]$. Create a third vector $\lambda P(1) + (1-\lambda)P(2)$. The nth term of the vector is equal to $\lambda p_i(1) + (1-\lambda)p_i(2)$. The nth term is within the bounds 0 and u_i since $u_i \geq max\{p_i(1), p_i(2)\} \geq \lambda p_i(1) + (1-\lambda)p_i(2) \geq min\{p_i(1), p_i(2)\} \geq 0$. Since this holds for all i, then any linear combination of two points inside the strategy space is within the strategy space. Therefore the strategy space is convex. □

Since convexity is proven, we can prove the existence of a Subgame Perfect Nash Equilibrium.

Corollary 4 (Existence of a Subgame Perfect Nash Equilibrium)**.** *Our game has a Subgame Perfect Nash Equilibrium.*

Proof. Our strategy set is non-empty so long as $P^t_{n,m} > 0 \; \forall \; n \in N$, $m \in M$, compact and convex. Our M_n reaction functions are continuous, therefore upper hemicontinuous, non-empty, closed and convex. The conditions of Kakutani's Fixed Point Theorem are met, therefore a Subgame Perfect Nash Equilibrium exists. □

Result: Since our strategy space is convex, the conditions of Kakutani's fixed point theorem are met. A Subgame Perfect Nash Equilibrium exists guaranteeing the existence of (at least) one solution to our NxM pricing model.

2.10 Uniqueness of a Solution

Uniqueness of a solution cannot be shown in general, but a unique solution exists in many specific cases. A unique solution may not necessarily exist since the reaction functions are not-necessarily monotonic, so there may be multiple fixed points. Two methods that are well suited to showing the existence of a unique solution are the functional forms we have chosen are the contraction mapping theorem and Rosen's uniqueness theorem.

Definition 6 (Rosen's Uniqueness Theorem)**.** *If all the best response functions $r_m(s)$, $m \in M$ are twice continuously differentiable and if the Jacobian matrix $J(g)$ of $g(s) \equiv r(s) - s$ is negative quasi-definite, then there is at most one Nash Equilibrium. Rosen (1965) [10]*

The reaction functions by construction are twice continuously differentiable, so that property is necessarily met. The second condition, however, is more cumbersome to determine.

Definition 7 (Contraction Mapping Theorem)**.** *If the best reply function r is a contraction, then there exists a unique equilibrium. [4]. A contraction on a metric space (M, d) is a function f from M to itself, with the property that there is some nonnegative real number $k < 1$ such that for all x and y in M, $d(f(x), f(y)) \leq kd(x, y)$. Lang (1968) [6].*

For this game this may be easier to meet than Rosen's theorem, since it does not require any form of continuous differentiability. Furthermore, in the two-firm/one-product-per-firm case, the best response function is naturally a contraction mapping for two points on the same boundary (since the distance between them is zero). However, we cannot show for all reasonable parameter values that the best-response functions will always hold the contraction mapping property.

It is relatively straight-forward to illustrate the conditions required to have a unique equilibrium in a model with two firms, each selling 1 type of product using the proportional splitting rule. Consider the problem at time t from the perspective of the firm selling product 1:

$$V^t_{q^t_{1,1},q^t_{2,1}} = \lambda\frac{(U^t_{1,1}-p^t_{1,1})(U^t_{2,1}+p^t_{2,1})}{2(U^t_{1,1}-L^t_{1,1})(U^t_{2,1}-L^t_{2,1})}(p^t_{1,1}-c^t_1)+ \\ \lambda\frac{(U^t_{1,1}+p^t_{1,1})(U^t_{2,1}-p^t_{2,1})}{2(U^t_{1,1}-L^t_{1,1})(U^t_{2,1}-L^t_{2,1})}(b^t_{2,1})+V^{t+1}_{q^t_{1,1},q^t_{2,1}} \quad (2.43)$$

Where we can think of c^t_1 as the opportunity cost of firm 1 making a sale this period (the opportunity cost stemming from the fact that if a sale is made that is one less sale that can be made in the future). It is given by (2.33):

$$c^t_{1,1} = V^{t+1}_{q^t_{1,1},q^t_{2,1}} - V^{t+1}_{q^t_{1,1}-1,q^t_{2,1}} \quad (2.44)$$

We can also think of $b^t_{1,1}$ as the benefit of the opponent making a sale (the benefit stemming from that if our opponent makes a sale, that leaves us one step closer to having a monopoly over the market). It is given by (2.34):

$$b^t_{1,1} = V^{t+1}_{q^t_{1,1},q^t_{2,1}-1} - V^{t+1}_{q^t_{1,1},q^t_{2,1}} \quad (2.45)$$

The optimal response function is given by (2.42):

$$p^t_{1,1} = 0.5[c^t_{1,1} + U^t_{1,1} + \frac{b^t_{2,1}(U^t_{2,1}-p^t_{2,1})}{(U^t_{2,1}+p^t_{2,1})}] \quad (2.46)$$

We can determine if we have a unique subgame perfect Nash Equilibrium by using the contraction mapping theorem. By the contraction mapping theorem, we will be assured a unique equilibrium if both $|\frac{\partial p^{t*}_{1,1}}{\partial p^t_{2,1}}| < 1$ and $|\frac{\partial p^{t*}_{2,1}}{\partial p^t_{1,1}}| < 1$. The first term is:

$$|\frac{\partial p_{1,1}^{t*}}{\partial p_{2,1}^{t}}| < 1, |\frac{-2U_{2,1}^{t}b_{2,1}^{t}}{(U_{2,1}^{t} + p_{2,1}^{t})^2}| = 1 \tag{2.47}$$

This will hold for all values of $p_{2,1}^t \geq 0$ so long as $U_{2,1}^t > 2b_{2,1}^t$. Thus as long as both $b_{1,1}^t$ and $b_{2,1}^t$ are sufficiently small, we are guaranteed a unique equilibrium.

2.11 Results

In this section, a geometric demand framework was constructed that allows for mathematically tractable revenue management models with multiple firms selling multiple classes of products. We have proven the existence of pure-strategy Nash Equilibria - the first NxM revenue management model to do so, for any finite value of N and M.

Chapter 3

Two Optimal Dynamic Pricing Examples

In this chapter we create two examples to illustrate the value of our approach. In the first example, a two-firm/one-product-per-firm model illustrates the importance of considering the quantity level of the product under consideration (factor 1 from the previous chapter), the quantity level of the product sold by the other firm (factor 3) and the price of competitor products (factor 5). The second example is a two-firm/three-products-per-firm rental car model. It considers all five factors, but pays particular attention to cannibalization of demand (factor 4).

3.1 Example 1: Behaviour of the Model in a Real World - 2 Firm, 1 Good Per Firm Scenario

This section illustrates through a numerical example the equilibrium properties of the model. Our model does indeed produce the optimal price, making it superior to any other pricing strategy provided that the firm makes an accurate assessment of the state. The model generates a set of prices that shows on average optimal prices are higher when firms have fewer goods remaining. One unusual strategy appears - a knockout strategy where a firm will occasionally price high in order to temporarily cede the market to the other firm. By doing so, it can force the other firm to run out of goods, giving the firm that uses the knockout strategy a monopoly for the rest of the game. The revenue increase from this approach indicates the value of considering factor 3, the quantity level of competitor firms.

3.2 Description of the Problem

A simple two-firm/one-product-per-firm model is created. A numeric simulation is used to illustrate the equilibrium properties of the model and to describe the strategies used by firms.

3.3 Assumptions of the Model

To illustrate the pricing dynamics of the model and the value of optimal pricing strategies, we will use the following numeric example with the following model specifications:

- There are two firms in the game, A and B.
- There are 50 periods in the game. The value of 50 is chosen such that there is a reasonable variance in outcomes including selling out.
- Each firm begins the game with 10 items, which allows for a firm to sell out of a good (but is far from a guarantee).

- A single person arrives each period, for a total of 50 arrivals. This is enough arrivals such that there is a reasonable possibility that a firm will sell out of a good before the end of the game.

- There is no salvage value for either firm for leftover goods.

- The two-dimensional uniform potential demand distribution has endpoints \$0.00 and \$100.00 for Firm A and \$0.00 and \$80.00 for Firm B.

Given these assumptions, there will be on average 40 arrivals. This may seem high, given that there are only 20 goods between the two firms. However, many of these arrivals will not wish to purchase from either firm, as their maximum-willingnesses-to-pay are below the prices charged by each firm, so realized market quantity demanded would be significantly less than 40 even with an unlimited supply of goods. The number of periods was chosen such that there would be a reasonable probability that the firms would sell out before the end of the sales period.

Using these model parameters, we investigate 22 different combinations of pricing strategies. These involve situations where:

- Both firms behave optimally and consider factors 1, 3 and 5 in their pricing.

- One firm uses a fixed price strategy, the other knows this and prices optimally based on that information. The fixed price strategies used can be to always set a low price (the mean of the one-dimension distribution), always set a medium price (75 percent of the one-dimensional distribution), or always set a high price (90 percent of the one-dimension distribution).

- One firm uses a fixed price strategy, the other tries to price optimally but mistakenly assumes that the other firm is also trying to price optimally.

- Both firms commit to fixed price strategies, where the strategies can be fixed prices that are low, medium, or high.

For Firm A, the low, medium, and high prices are 50, 75, and 90 respectively; for Firm B they are 40, 60, and 72 respectively. We will denote the scenario where a firm mistakenly assumes that their competitor is pricing optimally as Op(Mis).

For each scenario we calculated a set of 6050 state-contingent prices for each firm based on each firm's chosen strategy, though in the case where firms follow a fixed price strategy the calculation was trivial. In all four scenarios each firm begins the game with anywhere from 0-10 goods and there are 50 periods in each game. Once again, this gives us a set of 6050 state-contingent prices for each firm in each scenario, where the state is given by the number of periods remaining (1-50), the number of goods Firm A has (0-10) and the number of goods Firm B has (0-10). When a firm runs out of goods we have a monopoly and thus reduce the number of dimensions in our model from 2 to 1. When both firms run out of goods before period T the game is ended prematurely.

We ran 10,000 simulations using Palisade's Risk package for each scenario and recorded the mean revenue for both firms in each scenario, as well as the average number of sales.

3.4 Determining the Optimal Price

In order to run the simulations, we need to know the price each firm should charge in every state. The prices are determined analytically, through backward induction.

3.4.1 Both Firms Remaining

When both firms have quantity remaining, firm A's value function at time t is given by:

$$V^t_{q^t_A,q^t_B} = \lambda\pi^t_A(p^t_A + V^{t+1}_{q^t_A-1,q^t_B}) + \lambda\pi^t_B(V^{t+1}_{q^t_A,q^t_B-1}) + (1 - \lambda\pi^t_A - \lambda\pi^t_B)V^{t+1}_{q^t_A,q^t_B} \tag{3.1}$$

To simplify, we define two functions, first the cost function:

$$c^t_A = V^{t+1}_{q^t_A,q^t_B} - V^{t+1}_{q^t_A-1,q^t_B} \tag{3.2}$$

The second function we define is the benefit function:

$$b^t_A = V^{t+1}_{q^t_A,q^t_B-1} - V^{t+1}_{q^t_A,q^t_B} \tag{3.3}$$

Substituting in the two functions into our value function:

$$V^t_{q^t_A,q^t_B} = \lambda\pi^t_A(p^t_A - c^t_A) + \lambda\pi^t_B(b^t_A) + V^{t+1}_{q^t_A,q^t_B} \tag{3.4}$$

Finally, substitute in our probability functions:

$$V^t_{q^t_A,q^t_B} = \lambda \frac{(U^t_A - p^t_A)(U^t_B + p^t_B)}{2U^t_A U^t_B}(p^t_A - c^t_A) + \lambda \frac{(U^t_A + p^t_A)(U^t_B - p^t_B)}{2U^t_A U^t_B}(b^t_A) + V^{t+1}_{q^t_A,q^t_B} \quad (3.5)$$

If firm A is pricing optimally, the optimal reaction function is obtained by taking the first derivative with respect to p^t_A:

$$p^{t*}_A = 0.5(U^t_A + c^t_A) + \frac{(U^t_B - p^t_B)b^t_A}{2(U^t_B + p^t_B)} \quad (3.6)$$

If B is using a fixed price strategy, we can substitute it into the above equation. If B is pricing optimally, its reaction function is given by:

$$p^{t*}_B = 0.5(U^t_B + c^t_B) + \frac{(U^t_A - p^t_A)b^t_B}{2(U^t_A + p^t_A)} \quad (3.7)$$

If both firms are pricing optimally, we can substitute p^{t*}_B into A's reaction function to obtain the Nash Equilibrium prices. For A:

$$p_A^{t*} = \frac{1}{4(3U_B^t + c_B^t - b_B^t)} *$$
$$(\sqrt{(}(-3U_A^tU_B^t - U_A^tc_B^t - 3U_A^tb_B^t + 3U_B^tc_A^t + U_B^tb_A^t + c_A^tc_B^t - c_A^tb_B^t - c_B^tb_A^t + b_A^tb_B^t)^2 -$$
$$4(-6U_B^t - 2c_B^t + 2b_B^t)(3(U_A^t)^2U_B^t + (U_A^t)^2c_B^t + (U_A^t)^2b_B^t + 3U_A^tU_B^tc_A^t +$$
$$U_A^tU_B^tb_A^t + U_A^tc_A^tc_B^t + U_A^tc_A^tb_B^t - U_A^tc_b^tb_A^t - U_A^tb_A^tb_B^t)) -$$
$$3U_A^tU_B^t - U_A^tc_B^t - 3U_A^tb_B^t + 3U_B^tc_A^t + U_B^tb_A^t + c_A^tc_B^t - c_A^tb_B^t - c_B^tb_A^t + b_A^tb_B^t) \quad (3.8)$$

3.4.2 Only A Remaining

When only A has inventory remaining, its value function at time t is given by:

$$V_{q_A^t,0}^t = \lambda\pi_A^t(p_A^t + V_{q_A^t-1,0}^{t+1}) + (1 - \lambda\pi_A^t)V_{q_A^t,0}^{t+1} \quad (3.9)$$

As above, we substitute in our cost function (there is no benefit function, since our opponent has no inventory) and then substitute in our probability functions:

$$V_{q_A^t,0}^t = \lambda\frac{(U_A^t - p_A^t)}{U_A^t}(p_A^t - c_A^t) + V_{q_A^t,q_B^t}^{t+1} \quad (3.10)$$

If firm A is pricing optimally, the optimal reaction function is obtained by taking the first derivative with respect to p_A^t:

$$p_A^{t*} = 0.5(U_A^t + c_A^t) \quad (3.11)$$

In Appendix B, we give the optimal prices for firm A in periods T and $T-1$ when B is also pricing optimally.

3.5 Simulation Results

Typically we would expect a firm to do better against a firm that is behaving sub-optimally than one that is not. That holds under this experiment, with profits of up to 2 percent higher for Firm A when B uses a fixed price strategy:

Average Revenue and Sales Figures for 13 Different Scenarios

StrategyA	*StrategyB*	*AvgRevA*	*AvgRevB*	*AvgSalesA*	*AvgSalesB*
Optimal	*Optimal*	\$724.95	\$579.18	9.78	9.77
		(70.53)	(57.95)	(0.54)	(0.57)
Optimal	*High*	\$739.98	\$324.78	9.80	4.51
		(68.25)	(143.55)	(0.52)	(1.99)
Optimal	*Medium*	\$726.71	\$559.83	9.76	9.33
		(72.82)	(74.89)	(0.57)	(1.25)
Optimal	*Low*	\$725.93	\$400.00	9.77	10.00
		(71.21)	(0.00)	(0.55)	(0.00)
Op(Mis)	*High*	\$737.78	\$320.74	9.90	4.45
		(59.99)	(140.98)	(0.37)	(1.96)
Op(Mis)	*Medium*	\$726.11	\$559.97	9.79	9.33
		(69.48)	(74.52)	(0.53)	(1.24)
Op(Mis)	*Low*	\$724.82	\$399.99	9.78	10.00
		(69.59)	(0.89)	(0.54)	(0.02)
High	*Optimal*	\$414.05	\$591.98	4.60	9.00
		(184.67)	(70.53)	(2.05)	(0.91)
Medium	*Optimal*	\$720.09	\$581.37	9.60	9.37
		(75.31)	(74.59)	(1.00)	(0.85)
Low	*Optimal*	\$500.00	\$580.05	10.0	9.77
		(0.00)	(58.90)	(0.00)	(0.54)
High	*Op(Mis)*	\$404.58	\$590.44	4.50	9.89
		(178.95)	(47.25)	(1.99)	(0.37)
Medium	*Op(Mis)*	\$700.56	\$580.97	9.34	9.79
		(92.76)	(56.86)	(1.24)	(0.53)
Low	*Op(Mis)*	\$499.99	\$579.26	10.0	9.78
		(1.50)	(56.48)	(0.03)	(0.53)

Figure 3.1: Average Revenue and Sales Figures for 13 Different Scenarios. Standard Deviations in Brackets

Average Revenue and Sales Figures for 9 Different Fixed Price Scenarios

StrategyA	*StrategyB*	*AvgRevA*	*AvgRevB*	*AvgSalesA*	*AvgSalesB*
High	*High*	\$428.94	\$342.17	4.77	4.75
		(185.69)	(147.68)	(2.06)	(2.05)
High	*Medium*	\$406.35	\$572.02	4.52	9.53
		(178.97)	(64.48)	(1.99)	(1.07)
High	*Low*	\$403.79	\$400.00	4.49	10.00
		(178.18)	(0.00)	(1.98)	(0.00)
Medium	*High*	\$715.34	\$321.69	9.54	4.47
		(78.58)	(142.77)	(1.05)	(1.98)
Medium	*Medium*	\$702.55	\$562.30	9.37	9.37
		(91.61)	(72.22)	(1.22)	(1.20)
Medium	*Low*	\$699.50	\$400.00	9.33	10.00
		(94.44)	(0.00)	(1.22)	(0.00)
Low	*High*	\$500.00	\$321.83	10.00	4.47
		(0.00)	(140.99)	(0.00)	(1.96)
Low	*Medium*	\$499.98	\$558.71	10.00	9.31
		(1.66)	(76.24)	(0.03)	(1.27)
Low	*Low*	\$500.00	\$400.00	10.00	10.00
		(0.00)	(0.00)	(0.00)	(0.00)

Figure 3.2: Average Revenue and Sales Figures for 13 Different Fixed Price Scenarios. Standard Deviations in Brackets

While optimal pricing proves to be superior to fixed pricing, mistaking assuming the competition is pricing optimally as well results in lower revenue. This highlights the importance of firms having accurate information about the state.

3.6 Sensitivity of Arrival Rate

The arrival rate for each period was changed from 1 to 0.9 and 0.8 making it less likely that either firm would rent a car. Again, 10,000 simulations were run for each scenario and the average revenue and average number of sales were recorded. As expected, a lower arrival rate results in lower revenue and fewer sales. Otherwise the results were quite similar to the higher arrival rate.

In all cases, the highest revenue was realized when a firm priced optimally against a firm that was using fixed pricing. Having complete information was important as making an incorrect assumption about the opponent's pricing strategy resulted in lower revenue.

Average Revenue and Sales Figures for 13 Different Scenarios - Arrival Rate 0.9

$StrategyA$	$StrategyB$	$AvgRevA$	$AvgRevB$	$AvgSalesA$	$AvgSalesB$
$Optimal$	$Optimal$	\$693.28	\$554.45	9.69	9.69
		(78.07)	(62.45)	(0.66)	(0.67)
$Optimal$	$High$	\$714.41	\$286.60	9.74	3.98
		(73.85)	(136.85)	(0.60)	(1.90)
$Optimal$	$Medium$	\$697.08	\$532.04	9.67	8.87
		(80.48)	(94.81)	(0.68)	(1.58)
$Optimal$	Low	\$693.89	\$399.96	9.70	9.99
		(77.72)	(1.33)	(0.65)	(0.03)
$Op(Mis)$	$High$	\$711.13	\$287.01	9.85	3.99
		(64.12)	(137.05)	(0.46)	(1.90)
$Op(Mis)$	$Medium$	\$695.45	\$531.04	9.71	8.85
		(76.64)	(96.60)	(0.63)	(1.61)
$Op(Mis)$	Low	\$691.42	\$399.94	9.69	10.00
		(78.65)	(2.36)	(0.67)	(0.06)
$High$	$Optimal$	\$369.68	\$571.22	4.11	8.86
		(173.78)	(72.29)	(1.93)	(0.97)
$Medium$	$Optimal$	\$687.46	\$557.66	9.12	9.16
		(106.65)	(77.80)	(1.42)	(0.96)
Low	$Optimal$	\$499.97	\$555.11	10.0	9.70
		(1.59)	(63.11)	(0.03)	(0.63)
$High$	$Op(Mis)$	\$359.59	\$569.19	4.00	9.85
		(171.55)	(51.25)	(1.91)	(0.45)
$Medium$	$Op(Mis)$	\$664.69	\$556.20	8.86	9.73
		(119.04)	(60.49)	(1.59)	(0.63)
Low	$Op(Mis)$	\$499.88	\$553.73	10.0	9.69
		(3.43)	(62.29)	(0.07)	(0.65)

Figure 3.3: Average Revenue and Sales Figures for 13 Different Scenarios. Standard Deviations in Brackets - Arrival Rate 0.9

Average Revenue and Sales Figures for 9 Different Fixed Price Scenarios - Arrival Rate 0.9

StrategyA	*StrategyB*	*AvgRevA*	*AvgRevB*	*AvgSalesA*	*AvgSalesB*
High	*High*	$383.56	$308.82	4.26	4.29
		(175.67)	(142.79)	(1.95)	(1.98)
High	*Medium*	$363.52	$550.32	4.04	9.17
		(170.85)	(84.18)	(1.90)	(1.40)
High	*Low*	$358.27	$400.00	3.98	10.00
		(169.01)	(0.00)	(1.88)	(0.00)
Medium	*High*	$687.55	$289.28	9.61	4.02
		(105.76)	(136.13)	(1.41)	(1.89)
Medium	*Medium*	$666.72	$531.77	8.89	8.86
		(119.31)	(96.11)	(1.59)	(1.60)
Medium	*Low*	$660.58	$399.96	8.81	10.00
		(121.48)	(1.60)	(1.62)	(0.04)
Low	*High*	$500.00	$288.14	10.00	4.00
		(0.50)	(135.62)	(0.01)	(1.88)
Low	*Medium*	$500.00	$529.62	10.00	8.83
		(0.50)	(96.72)	(0.01)	(1.61)
Low	*Low*	$499.97	$399.97	10.00	10.00
		(1.50)	(1.27)	(0.03)	(0.03)

Figure 3.4: Average Revenue and Sales Figures for 13 Different Fixed Price Scenarios. Standard Deviations in Brackets - Arrival Rate 0.9

Average Revenue and Sales Figures for 13 Different Scenarios - Arrival Rate 0.8

StrategyA	*StrategyB*	*AvgRevA*	*AvgRevB*	*AvgSalesA*	*AvgSalesB*
Optimal	*Optimal*	\$654.88	\$524.02	9.53	9.56
		(87.58)	(68.99)	(0.84)	(0.82)
Optimal	*High*	\$682.54	\$252.91	9.66	3.51
		(79.84)	(128.75)	(0.70)	(1.79)
Optimal	*Medium*	\$661.34	\$486.35	9.55	8.11
		(86.11)	(115.27)	(0.81)	(1.92)
Optimal	*Low*	\$652.52	\$399.53	9.53	9.99
		(87.81)	(5.69)	(0.84)	(0.14)
Op(Mis)	*High*	\$677.61	\$249.21	9.78	3.46
		(69.94)	(128.98)	(0.57)	(1.79)
Op(Mis)	*Medium*	\$659.83	\$487.37	9.63	8.12
		(82.41)	(115.55)	(0.75)	(1.93)
Op(Mis)	*Low*	\$650.53	\$399.43	9.56	9.99
		(86.11)	(6.89)	(0.81)	(0.17)
High	*Optimal*	\$324.45	\$546.03	3.61	8.74
		(164.72)	(74.14)	(1.83)	(1.02)
Medium	*Optimal*	\$638.26	\$529.07	8.51	8.88
		(137.01)	(79.57)	(1.83)	(1.08)
Low	*Optimal*	\$499.74	\$522.02	10.0	9.56
		(4.80)	(70.08)	(0.10)	(0.78)
High	*Op(Mis)*	\$312.64	\$543.67	3.47	9.78
		(160.33)	(56.20)	(1.78)	(0.57)
Medium	*Op(Mis)*	\$609.42	\$527.98	8.13	9.61
		(144.06)	(66.53)	(1.92)	(0.77)
Low	*Op(Mis)*	\$499.27	\$520.87	9.99	9.56
		(8.45)	(68.70)	(0.17)	(0.81)

Figure 3.5: Average Revenue and Sales Figures for 13 Different Scenarios. Standard Deviations in Brackets - Arrival Rate 0.8

Average Revenue and Sales Figures for 9 Different Fixed Price Scenarios - Arrival Rate 0.8

StrategyA	*StrategyB*	*AvgRevA*	*AvgRevB*	*AvgSalesA*	*AvgSalesB*
High	*High*	$342.50	$275.65	3.81	3.83
		(167.99)	(135.19)	(1.87)	(1.88)
High	*Medium*	$319.30	$518.00	3.55	8.63
		(163.28)	(103.29)	(1.81)	(1.72)
High	*Low*	$313.75	$399.88	3.49	10.00
		(158.15)	(3.20)	(1.76)	(0.08)
Medium	*High*	$647.45	$255.46	8.63	3.55
		(129.97)	(130.14)	(1.73)	(1.81)
Medium	*Medium*	$616.66	$494.15	8.22	8.24
		(141.54)	(113.01)	(1.89)	(1.88)
Medium	*Low*	$606.20	$399.73	8.08	9.99
		(144.50)	(4.07)	(1.93)	(0.10)
Low	*High*	$499.87	$248.32	10.00	3.45
		(3.31)	(127.73)	(0.07)	(1.78)
Low	*Medium*	$499.61	$483.61	9.99	8.06
		(6.10)	(114.79)	(0.12)	(1.91)
Low	*Low*	$499.20	$399.41	9.99	9.99
		(8.88)	(6.50)	(0.18)	(0.16)

Figure 3.6: Average Revenue and Sales Figures for 13 Different Fixed Price Scenarios. Standard Deviations in Brackets - Arrival Rate 0.8

3.7 Scenario Analysis

Next, to give a feel for the dynamics of the strategies used in the model, we examined four different scenarios. Each firm in this analysis is pricing optimally using the model developed earlier.

Our four scenarios are as follows:

1. Maximum-willingness-to-pay for A's product: Uniform $[0, 100]$

Maximum-willingness-to-pay for B's product: Uniform $[0, 100]$

Probability of customer arrival: .8

2. Maximum-willingness-to-pay for A's product: Uniform $[0, 100]$

   Maximum-willingness-to-pay for B's product: Uniform $[0, 80]$

   Probability of customer arrival: .8

3. Maximum-willingness-to-pay for A's product: Uniform $[0, 80]$

   Maximum-willingness-to-pay for B's product: Uniform $[0, 80]$

   Probability of customer arrival: .8

4. Maximum-willingness-to-pay for A's product: Uniform $[0, 100]$

   Maximum-willingness-to-pay for B's product: Uniform $[0, 100]$

   Probability of customer arrival: .5

3.7.1 Firm A has 1 Good

The following tables illustrate some of the pricing dynamics of the game as well as the impact changing these parameter values has. We investigate the pricing strategies for Firm A and Firm B in each of the four scenarios when B has anywhere from 1-10 goods, Firm A has exactly 1 good and the game is in period 46 (there are 5 periods left including the current period). First, Firm B's pricing strategies in the four scenarios.

B's Prices in Period 46 When A Has 1 Good

B's Quantity	Scenario 1	Scenario 2	Scenario 3	Scenario 4
1	\$70.36	\$56.29	\$56.29	\$64.54
2	\$57.70	\$46.16	\$46.16	\$53.68
3	\$51.77	\$41.42	\$41.42	\$50.50
4	\$50.17	\$40.14	\$40.14	\$50.03
5	\$50.00	\$40.00	\$40.00	\$50.00
5	\$50.00	\$40.00	\$40.00	\$50.00
6	\$50.00	\$40.00	\$40.00	\$50.00
7	\$50.00	\$40.00	\$40.00	\$50.00
8	\$50.00	\$40.00	\$40.00	\$50.00
9	\$50.00	\$40.00	\$40.00	\$50.00
10	\$50.00	\$40.00	\$40.00	\$50.00

Figure 3.7: B's Prices in Period 46 When A Has 1 Good

When B finds itself in this situation, its pricing strategy under scenarios 2 and 3 are identical; changing A's distribution of reservation prices from 0-100 to 0-80 has no impact on B's pricing strategy.

Firm B prices lower when the mean of its potential market demand is lower as in scenarios 2 and 3. In the case where all potential market demands are uniform, then every firm prices at the mean of its distribution if there is zero probability that any firm will run out of goods before the game ends.

Note that Firm B prices higher the less goods it has, since the firm runs a larger risk of running out of goods before the game ends. We will see, however, that this relationship between the number of goods and the number of periods left has one major exception: the knockout strategy.

Firm B prices lower in scenario 4 where the probability of arrival is 0.5, than it does in scenario 1 where the probability of arrival is 0.8. Once again, this is due to the fact that the firm typically (but not always) prices higher when the probability it will run out of goods before the game ends is higher. Thus, our model is very responsive to changes in the state and takes market conditions into account.

3.7.2 Firm A has 10 Goods

Next we will examine the pricing dynamics for Firm A. Note that Firm A cannot possibly run out of goods, given the fact we have assumed that it has 10 goods and there are only 5 periods remaining.

A's Prices in Period 46 When A Has 10 Goods

B's Quantity	Scenario 1	Scenario 2	Scenario 3	Scenario 4
0	\$50.00	\$50.00	\$40.00	\$50.00
1	\$50.98	\$50.98	\$40.78	\$50.92
2	\$50.85	\$50.85	\$40.68	\$50.49
3	\$50.34	\$50.34	\$40.27	\$50.11
4	\$50.05	\$50.05	\$40.04	\$50.01
5	\$50.00	\$50.00	\$40.00	\$50.00
5	\$50.00	\$50.00	\$40.00	\$50.00
6	\$50.00	\$50.00	\$40.00	\$50.00
7	\$50.00	\$50.00	\$40.00	\$50.00
8	\$50.00	\$50.00	\$40.00	\$50.00
9	\$50.00	\$50.00	\$40.00	\$50.00
10	\$50.00	\$50.00	\$40.00	\$50.00

Figure 3.8: A's Prices in Period 46 When A Has 10 Goods

Once again, the prices charged are significantly lower when we lower the mean of the firm's potential market demand. However, note the unusual pricing strategy in this market - the firm actually prices *higher* when it does not have a monopoly than when it does. In all four scenarios, A's prices are higher when B has 1 through 4 goods than when B has none. We will examine this unusual pricing strategy in the next section, where general, non-parameter specific, results are discussed.

Note that the firm prices at the median of their price distribution when there is no risk that they will sell out of the good before the end of the period.

3.8 Results of Example 1

The pricing strategies used by the firms in this model show a number of interesting properties, some quite obvious, others less so. The pricing dynamics that occur in this model are the result of the fact that a firm can run out of goods before the game is over is taken into consideration. This means lost opportunities for sales for the firm who runs out of goods and a change in market structure from duopoly to monopoly for the firm that remains in the game.

All results in this section assume that the salvage value for each firm is set to zero. These results are not altered in any significant way for strictly positive and small salvage values, except for pricing when no firm has the possibility it will run out of goods. Our model generates optimal prices that have the following general characteristics.

1. All else being equal, prices for each firm are (weakly) higher in the amount of time remaining in the game for each pair of quantity levels (q_a, q_b) for the firms. This results from the fact that more periods remaining in the game increase the probability that one or more firms may run out of goods before the game ends.

2. All else being equal, the higher the customer arrival rate, the (weakly) higher each firm prices. Once again, this is because a higher arrival rate increases the probability that one or more firms may run out of goods before the game ends.

3. If there is no possibility that all firms still in the game will run out of goods (i.e. there are fewer periods remaining than goods left for each firm), then each firm prices at the mean of its potential demand distribution, if the dimensions of our overall potential market demand are independent (given our assumptions on demand). This implies that if capacity is not a binding constraint, then the

price set is the same for a monopolist as a duopolist. However, the monopolist has a higher probability of making a sale given a customer arrival, as he does not lose potential sales to another firm.

4. Prices are typically (weakly) higher under a monopoly than under a duopoly. Similarly, prices are typically increasing for both firms as the quantity level is dropped for either firm, as the inventory level of each firm is taken into account. However, there is one interesting exception to this, which we call *the knockout strategy.*

The knockout strategy is employed when a firm, say Firm A, has very few goods, typically one or two, with many periods left to go. In this situation, Firm B will set its price very high, often close to the upper bound of its one-dimensional potential demand distribution. It will do this in order to concede the market to Firm A during this period, so Firm A will make a sale and thus run out of goods (be knocked out of the game), allowing Firm B to have a monopoly for the rest of the game. Under this situation, Firm A will typically price somewhat higher as well, but not so high as to eliminate any chance of making a sale. Firm A knows that the longer it takes to make a sale, the lower the price it will get, since prices, all else being equal, decline as the number of periods remaining is reduced. By considering factor 3, the quantity levels of competitor products, the firm can increase its revenue

Interestingly, Firm B will use the knockout strategy even if it cannot possibly run out of goods, meaning that prices will not be any higher under a monopoly than under a duopoly. A monopoly is still desirable to Firm B, since it raises the probability of a sale under these conditions, so it will be able to sell goods that otherwise would not garner any revenue.

3.9 Example 2: 2 x 3 Model - Rental Car Model

The importance of taking into consideration cross-price elasticities has been covered extensively in the literature and has been demonstrated to be important in markets of interest to revenue managers such as the rental car business. We show that it is possible for a firm to *reduce* its profit using traditional revenue management techniques that consider the pricing of each good in isolation and ignore factor 4. This is caused by firms cannibalizing their high-end market by pricing their lower-end products too low. Using a simultaneous optimization approach, firms achieve higher profits by reducing cannibalization. In our simultaneous optimization model, prices are typically higher for lower end products and lower for higher end products, than in models where pricing decisions are made in isolation. This change in pricing increases profits for the business using it.

3.10 Description of the Problem

The framework developed in this paper allows for the construction of models where there are N firms each selling M similar, but not identical products. In this stylized example we consider the case of 2 rental car brands, each offering 3 classes of cars for rent. The agents in the model have heterogeneous preferences, but *on average* the firms are seen as being identical in quality. The three different classes are described as, in ascending value - compact, mid-sized, and full-sized. Due to the heterogeneity of consumer preferences, some consumers may prefer, all else being equal, compact cars over full-sized (perhaps due to fuel-efficiency reasons), but on average the maximum-willingness-to-pay increases in the size of the car. We consider the performance of four different strategies a firm can use in response to an opposing firm that follows a fixed-priced strategy.

3.11 The Structure of the Market

In our rental car market, we consider 2 firms, which we will denote A and B which each have three types of cars available for rent: compact (C), mid-sized (M), and full-sized (F). Specifically, each firm begins the game with 2 cars of each type available for rent. The number 2 is chosen as the number of state spaces becomes increasingly large as the quantity level for each rises. At each period a single consumer arrives (with probability 1). She chooses either to rent 1 of the available cars in the market or not to rent at all. Firms cannot access any additional inventory. If they rent both of their cars of a particular class, they no long offer that car for rent. There are ten periods in the game; at the end of the 10th period the game ends. A 10 period model is chosen as it allows some chance that a firm will sell out of cars before the end of the game, but without guaranteeing this will happen. Any unrented cars at the end of the game are assumed to have zero value. It is assumed that all the costs in the game are fixed.

3.12 Consumer Preferences

Each consumer that arrives has a maximum-willingness-to-pay for each car from each firm. This set of maximum-willingnesses-to-pay is drawn from a six-dimensional uniform distribution. The dimensions of the distribution are as follows:

Uniform Distribution of Maximum-Willingness-to-Pay

	Compact (C)	Mid-Size (M)	Full-Size (F)
Firm A	[0,50]	[0,80]	[0,100]
Firm B	[0,50]	[0,80]	[0,100]

Figure 3.9: Uniform Distribution of Maximum-Willingness-to-Pay

The upper bound values of 50, 80 and 100 are chosen as they lead to reasonable approximations of real world rental car prices (30, 48 and 60), as shown in Chapter 1. When the consumer arrives, she compares her maximum-willingness-to-pay to the prices being offered in the market (for the products which are still available for rent). If all the prices exceed her maximum-willingness-to-pay (MWP), she will not rent. If a single price is lower than the associated MWP, she will rent that car. If two or more prices are lower, then she will rent exactly one car - the car she will rent will be determined using the proportional splitting rule introduced earlier in this paper, which maximizes consumer surplus while being twice continuously differentiable for the mathematical tractability of the model.

3.13 Firm Strategies

Since in the 'real world' many firms often use fixed-priced or simple pricing rules, we will assume Firm B uses a fixed-priced strategy. A price of 30 dollars for the compact, 48 dollars for the mid-size, and 60 dollars for the full-size are chosen as the fixed price strategies. We believe it is unrealistic to assume that an opposing firm would be using the same model as we are using. For a firm to price optimally, it is important that they have a realistic idea of the strategy their opponent is likely to use.

We consider 4 candidate strategies for Firm A:

1. Firm A follows the same fixed-price strategy as Firm B and ignores all five factors.

2. Firm A follows an isolated optimization approach, where it determines the price it will charge for each good without considering spill-over effects. It does not consider the inventory levels of the other firm. This considers factors 1 and 5. It only partly considers factor 4. It recognizes that the price of the firm's other goods affects the demand for our good. However, it does not consider the reverse. It treats the prices of the firm's other products as a given which cannot be changed. It also completely ignores factors 2 and 3.

3. Firm A follows a global optimization approach, where it determines the price it will charge for all three goods simultaneously, in order to consider spill-over effects. It does not consider the inventory levels of the other firm. This fully considers factor 4 and adds factor 2 to the mix. All factors are now considered other than factor 3.

4. Firm A follows a global optimization approach and considers the inventory level of the other firm. This adds factor 3 and all five factors are now considered.

3.14 Solution Methodology

For each of the four strategy pairs, we compute a state-contingent set of prices through backward induction. The state space is the current period and the inventory levels for each product and firm. However, Firm A does not necessarily use all the information from the state in making its decision. For the four-strategies, the states from A's perspective are as follows:

1. No state; every period identical to the last.
2. The current period and the inventory levels of *only* the product under consideration.
3. The current period and the inventory levels of all Firm A's products.
4. The current period and the inventory levels of all Firm A's *and* Firm B's products.

Once the state-contingent set of prices were obtained for the four different strategies, 10,000 simulations were run for each strategy to obtain the average or expected revenue for each firm. We used the results to investigate the differences in revenues and booking rates between the four strategies.

3.15 Results of Example 2

The average profit for each firm in each strategy was as follows (note revenue and profit in this model are functionally equivalent, as there are no explicit costs).

Average Profit Levels for Each Firm

	Strategy 1	Strategy 2	Strategy 3	Strategy 4
Firm A Profit	\$205.77	\$203.18	\$211.58	\$215.76
(St. Dev)	(48.46)	(46.83)	(48.95)	(50.05)
Gain From Base		-\$2.59	\$5.81	\$9.99
(As a Percent)		(-1.26%)	(2.82%)	(4.85%)
Rental Rate A,C	74.7%	78.7%	71.5%	71.1%
(St. Dev)	(33.74)	(31.63)	(33.77)	(33.35)
Rental Rate A,M	74.7%	79.4%	74.2%	74.2%
(St. Dev)	(34.24)	(34.16)	(33.95)	(33.88)
Rental Rate A,F	74.4%	79.5%	76.1%	76.0%
(St. Dev)	(34.22)	(31.79)	(32.49)	(33.03)
Firm B Profit	\$205.70	\$201.21	\$207.63	\$208.12
(St. Dev)	(48.06)	(47.84)	(47.62)	(47.50)
Rental Rate B,C	74.7%	73.1%	75.2%	75.3%
(St. Dev)	(34.22)	(33.94)	(33.42)	(34.02)
Rental Rate B,M	74.7%	72.7%	75.2%	75.3%
(St. Dev)	(33.92)	(31.62)	(32.31)	(32.82)
Rental Rate B,F	74.4%	73.1%	75.2%	75.2%
(St. Dev)	(33.98)	(34.28)	(33.80)	(33.52)

Figure 3.10: Average Profit Levels for Each Firm

Particularly interesting is the revenue loss that occurs between strategy 1 and strategy 2 for Firm A. In this scenario, the firm actually performs *worse* by trying to behave optimally. This is due to the fact that it is not considering spill-over effects when optimizing each price in isolation. This further illustrates the importance of considering cross-price elasticity of demand when making pricing decisions. It also illustrates the benefit of simultaneously considering the five factors, as a firm can make itself worse off by only considering two or three of them. On the surface, Firm A's strategy seems like a rational one - all else being equal, it prices higher when it has fewer of a good remaining, and prices higher when it has more periods in which to sell the good. Compare the pricing strategy for Firm A's rental of compact cars under the naive approach versus the 'optimal' approach for the first three strategies.

Pricing strategies for Firm A's compact cars when it has only one compact car remaining

Period	Strategy 1	Strategy 2	Strategy 3
1	\$30.00	\$34.70	\$32.49
2	\$30.00	\$34.02	\$32.97
3	\$30.00	\$33.27	\$33.44
4	\$30.00	\$32.45	\$33.89
5	\$30.00	\$31.54	\$34.30
6	\$30.00	\$30.53	\$34.65
7	\$30.00	\$29.40	\$34.90
8	\$30.00	\$28.12	\$35.01
9	\$30.00	\$26.67	\$34.86
10	\$30.00	\$25.00	\$33.86

Figure 3.11: Pricing strategies for Firm A's compact cars when it has only one compact car remaining

These are the prices for the 3 strategies that do *not* depend on B's quantity levels. The optimal price for strategy 4 does depends on B's quantity level of the three car classes. We have omitted it from this table and the ones that follow, as the optimal price for strategy 4 will differ depending on which quantity levels we choose for B (while the other 3 strategies do not).

The optimal isolated pricing strategy (strategy 2) in the above table intuitively seems superior to strategy 1, but it leads to reduced revenue. Strategy 2 for pricing in round 3 of 10 when the firm has 1 or 2 of the goods is intuitive as well - price higher when the firm is more likely to run out of the good before the end of the game.

Firm A's compact car price in round 3

Goods Remaining	Strategy 1	Strategy 2	Strategy 3
1	\$30.00	\$33.27	\$33.44
2	\$30.00	\$27.69	\$31.90

Figure 3.12: Firm A's compact car price in round 3

Firm A (and Firm B, which does not switch strategies) fares worse under this scenario. This will not always be the case - it will depend on the size of the spill-over when the price of one of the goods rises or falls. If the substitutability between the products is *higher* than we allow for in this model, this spill-over effect will increase and this effect will be magnified. However, if the products are highly distinct with little spill-over, then using an individually optimal strategy will yield increased profits and be a good approximation for the global optimization approach. In the case of the rental car market, it would make sense that there is some cross-price elasticity between car classes since needing a vehicle is the overriding factor.

3.15.1 Strategy 2 vs. Strategy 3

Using strategy 2, where each product was 'optimized' without considering spill-over effects, the firm obtained lower profits than by using a fixed-price strategy. This was not the case for strategy 3, where the firm optimizes its prices globally, considering interaction effects between the products. Whereas strategy 2 led to a -1.26% reduction in profit from the base case, strategy 3 yielded a 2.82% *gain* in profit. As the tables illustrate, the gain was due to the firm pricing the lower-end goods higher in order to avoid cannibalization.

Pricing strategies for Firm A's full size cars when it has only one compact car remaining

Period	Strategy 1	Strategy 2	Strategy 3
1	\$60.00	\$69.39	\$66.77
2	\$60.00	\$68.03	\$66.88
3	\$60.00	\$66.54	\$66.88
4	\$60.00	\$64.89	\$66.71
5	\$60.00	\$63.08	\$66.28
6	\$60.00	\$61.05	\$65.50
7	\$60.00	\$58.79	\$64.29
8	\$60.00	\$56.24	\$62.52
9	\$60.00	\$53.33	\$59.98
10	\$60.00	\$50.00	\$56.13

Figure 3.13: Pricing strategies for Firm A's compact cars when it has only one full size car remaining

In period 3, the firm prices strictly higher for the compact car under strategy 3 than for the other 2 strategies. The prices for strategy 3 are for cases when the firm has 2 goods remaining for the two higher-priced goods. This effect is further illustrated when we consider the price set for the compact car when the firm has 1 compact car remaining and 2 of the other cars remaining. By fully adding in factor 4 (cannibalization) and factor 2 (quantity level of the firm's other products) for consideration, the firm can dramatically increase revenue.

Firm A's full size car price in round 3

Goods Remaining	Strategy 1	Strategy 2	Strategy 3
1	\$60.00	\$66.54	\$66.88
2	\$60.00	\$55.37	\$63.39

Figure 3.14: Firm A's full size car price in round 3

3.15.2 Strategy 3 vs. Strategy 4

In strategy 4, Firm A considers the inventory levels of each of the competitor's three products. Thus factor 2 consideration is added and all five factors are simultaneously accounted for. In this strategy Firm A always prices all three types of goods higher, *ceteris paribus*, when the firm has less of any good. In particularly, the prices are raised when the opposing firm runs out of any product class. The difference in prices tend to be small (less than a dollar), but they lead to a 4.85% increase in profit from the base case and a nearly 2% increase over the case where the firm does not consider the opposing firm's inventory levels.

3.16 Sensitivity to Arrival Rate

Our results are maintained if we lower the arrival rate:

Average Profit Levels for Each Firm - 90 Percent Arrival Rate

	Strategy 1	Strategy 2	Strategy 3	Strategy 4
Firm A Profit	$188.88	$188.28	$191.76	$192.33
(St. Dev)	(52.64)	(51.12)	(53.93)	(54.30)
Gain From Base		-$0.60	$2.88	$3.45
(As a Percent)		(-0.32%)	(1.52%)	(1.83%)
Rental Rate A,C	67.7%	71.4%	66.5%	66.9%
(St. Dev)	(36.66)	(34.56)	(35.96)	(36.00)
Rental Rate A,M	68.5%	71.4%	68.9%	69.1%
(St. Dev)	(36.39)	(34.24)	(35.09)	(35.40)
Rental Rate A,F	68.8%	71.4%	70.4%	69.6%
(St. Dev)	(36.14)	(34.68)	(34.53)	(35.08)
Firm B Profit	$187.54	$184.56	$187.59	$188.42
(St. Dev)	(53.17)	(53.17)	(53.20)	(53.32)
Rental Rate B,C	68.0%	66.8%	68.3%	68.4%
(St. Dev)	(36.80)	(36.94)	(36.40)	(36.71)
Rental Rate B,M	67.6%	66.6%	67.7%	68.0%
(St. Dev)	(36.69)	(36.91)	(36.47)	(36.58)
Rental Rate B,F	68.2%	67.1%	68.0%	68.5%
(St. Dev)	(36.77)	(36.76)	(36.27)	(36.43)

Figure 3.15: Average Profit Levels for Each Firm - 90 Percent Arrival Rate

Average Profit Levels for Each Firm - 80 Percent Arrival Rate

	Strategy 1	Strategy 2	Strategy 3	Strategy 4
Firm A Profit	$169.52	$167.69	$171.14	$171.91
(St. Dev)	(56.98)	(54.37)	(56.24)	(57.07)
Gain From Base		-$1.83	$1.62	$2.39
(As a Percent)		(-1.08%)	(0.96%)	(1.41%)
Rental Rate A,C	61.9%	65.4%	59.4%	60.0%
(St. Dev)	(38.12)	(36.46)	(37.67)	(37.38)
Rental Rate A,M	61.9%	64.0%	62.8%	63.0%
(St. Dev)	(38.19)	(36.77)	(36.97)	(36.82)
Rental Rate A,F	60.8%	65.1%	63.3%	63.0%
(St. Dev)	(38.08)	(36.57)	(36.74)	(36.91)
Firm B Profit	$168.33	$164.28	$168.57	$168.73
(St. Dev)	(56.63)	(56.38)	(57.03)	(56.78)
Rental Rate B,C	61.5%	59.8%	61.0%	61.1%
(St. Dev)	(38.17)	(38.30)	(38.08)	(38.12)
Rental Rate B,M	60.6%	59.2%	60.9%	61.6%
(St. Dev)	(38.25)	(38.65)	(38.36)	(38.45)
Rental Rate B,F	61.1%	59.6%	61.3%	60.8%
(St. Dev)	(38.18)	(38.33)	(38.43)	(38.18)

Figure 3.16: Average Profit Levels for Each Firm - 80 Percent Arrival Rate

3.17 Sensitivity to Price Spread

Our results are maintained if we change the price spread. First we narrow the distribution of prices to $[0, 65]$, $[0, 80]$ and $[0, 90]$ which have associated fixed prices of 39, 48 and 54 respectively (60% of each distribution):

Average Profit Levels for Each Firm - Narrow Price Spread

	Strategy 1	Strategy 2	Strategy 3	Strategy 4
Firm A Profit	$210.37	$211.02	$214.87	$215.43
(St. Dev)	(47.34)	(45.66)	(48.29)	(49.23)
Gain From Base		$0.65	$4.50	$5.06
(As a Percent)		(0.31%)	(2.14%)	(2.41%)
Rental Rate A,C	74.5%	76.9%	74.7%	74.8%
(St. Dev)	(34.15)	(32.07)	(32.83)	(32.86)
Rental Rate A,M	74.6%	77.6%	75.3%	75.0%
(St. Dev)	(34.00)	(31.32)	(32.37)	(32.83)
Rental Rate A,F	74.7%	78.0%	74.5%	74.3%
(St. Dev)	(33.99)	(31.45)	(32.96)	(32.77)
Firm B Profit	$210.42	$207.29	$211.21	$210.97
(St. Dev)	(47.18)	(46.97)	(47.00)	(46.95)
Rental Rate B,C	74.6%	73.7%	74.9%	74.4%
(St. Dev)	(33.82)	(34.94)	(33.83)	(34.36)
Rental Rate B,M	74.3%	73.4%	75.1%	75.1%
(St. Dev)	(34.29)	(34.61)	(33.84)	(34.03)
Rental Rate B,F	74.8%	73.4%	74.7%	74.9%
(St. Dev)	(33.87)	(34.52)	(34.24)	(33.75)

Figure 3.17: Average Profit Levels for Each Firm - Narrow Price Spread

Next we widen the distribution of prices to $[0, 20]$, $[0, 80]$ and $[0, 120]$ which have associated fixed prices of 12, 48 and 72 respectively (60% of each distribution):

Average Profit Levels for Each Firm - Wide Price Spread

	Strategy 1	Strategy 2	Strategy 3	Strategy 4
Firm A Profit	\$197.42	\$196.13	\$205.00	\$205.91
(St. Dev)	(53.82)	(52.57)	(53.10)	(55.31)
Gain From Base		-\$1.29	\$7.58	\$8.49
(As a Percent)		(-0.65%)	(3.84%)	(4.30%)
Rental Rate A,C	74.4%	77.1%	61.3%	64.0%
(St. Dev)	(34.03)	(31.88)	(37.58)	(36.9)
Rental Rate A,M	75.3%	77.8%	76.7%	75.9%
(St. Dev)	(33.70)	(31.69)	(32.14)	(32.16)
Rental Rate A,F	74.5%	76.7%	77.6%	76.67%
(St. Dev)	(34.37)	(31.93)	(31.15)	(31.90)
Firm B Profit	\$197.40	\$195.73	\$201.12	\$200.36
(St. Dev)	(53.53)	(53.21)	(52.44)	(52.89)
Rental Rate B,C	74.8%	73.7%	75.7%	75.9%
(St. Dev)	(33.91)	(34.25)	(33.9)	(33.48)
Rental Rate B,M	74.7%	74.0%	76.1%	75.5%
(St. Dev)	(34.12)	(34.43)	(33.23)	(33.70)
Rental Rate B,F	74.8%	74.3%	76.3%	76.2%
(St. Dev)	(33.83)	(34.11)	(33.17)	(33.06)

Figure 3.18: Average Profit Levels for Each Firm - Wide Price Spread

In both cases, a global optimization approach (strategy 4) provided the greatest revenue.

3.18 Example 2 Summary

This example highlights the importance of considering all five factors. If it is impossible to obtain the inventory level of the opposition our model can still provide increased profits. However, the highest profits are made when all five factors are simultaneously accounted for. The two examples outlined in this chapter underline the importance of a revenue management model incorporating as many factors facing business as possible.

Chapter 4

Booking Limits and Optimal Pricing in a 2-Firm, 2 Class Airline Model

Applying our model to the well-known Netessine and Shumsky problem yields significantly increased revenues over the booking limits approach. Substantial gains to revenue are available to firms when they have the ability to alter their prices in response to changing market conditions. There are a number of features of the model that lead to increased revenues. Instead of simply shutting off sales when demand is high, firms can increase their price allowing for higher revenues per seat. When demand is low, firms can lower their prices to assure they are not leaving with a half-empty plane. Firms can also react to the quantity levels of their competitor by employing the knockout strategy. Booking limits consider only quantity limits (factors 1 through 3), while our optimal pricing model considers all five factors and own price elasticity of demand. Optimal pricing provides the firm with more tactical options than the on-off approach of booking limits, which allows the firm to increase their revenues.

4.1 Description of the Problem

In this section, a 2-firm/2-seat class model is created following Netessine and Shumsky [2] where the 2 classes of seat come from a shared inventory. Unlike Netessine and Shumsky, each firm has the ability to alter its prices as well as close booking classes (by setting a price for the class so high that quantity demanded falls to zero).

4.2 Netessine and Shumsky Market Structure

Consider the market structure discussed in Netessine and Shumsky (2005) [2]. They consider the real-life problem of two competing airlines, flying the same route, each with two booking classes.

> "Consider an airline customer looking for an early-morning flight from Rochester, NY, to Chicago in May 2003. The traveller can choose between two airlines, American and United, which offer flights at nearly identical times (6:00 A.M. and 6:10 A.M. respectively) at identical prices (both charge $266 for 14-day and $315 for 7-day advance-purchase round-trip tickets). Now suppose that the customer wishes to purchase American's 14-day advance ticket. If the seats allocated to the 14-day fare class have sold out, it is likely that the customer will attempt to purchase a ticket in the same fare class on the United flight that departs 10 minutes later." [2]

In the Netessine and Shumsky framework, firms choose a booking limit on the maximum number of sales they will allow into the lower class fare. Since this is a competitive model and if the firms "do not collaborate on seat allocations,then the decisions that arise out of the resulting game can differ significantly from the seat allocations that would be optimal for a single decision maker with control over both airlines". Netessine and Shumsky describe the strategy set available to each airline.

> "To maximize expected profits, the airline establishes a booking limit B for low-fare seats. Note that the establishment of a booking limit is an optimal policy for each airline – see Brumelle et al. (1990) [1]. Once the booking limit is reached, the low fare is closed. Sales of high-fare tickets are accepted until either the airplane is full or the flight departs." [2]

We wish to expand the strategy set available to each firm. In the Netessine and Shumsky model prices are taken as given and the firms have a set quantity of seats which they can allocate between the two classes. We also allow seats to be allocated in this manner, but we also allow the firm to change the price of *each* of its fare classes in response to market conditions. This also allows us to have each firm dynamically create booking limits, by raising the price of the lower fare class so high that the probability of sale for that class is zero.

We go on to examine the use of the knockout strategy, where a firm should deliberately increase its price in order to increase the other firm's probability of sale. That is, the optimal price in equilibrium is higher than it normally would be, thanks to this knockout effect. By using the knockout strategy when an opposing firm is close to selling out, a firm can increase the chances that it will have a monopoly for the rest of the period.

4.3 Model Specifications

There are two firms in this model, which we will refer to as Firm A and Firm B. Each firm will offer two qualities of service: a business class denoted by an upper case letter and a coach class which will be denoted by a lower case letter. Consumers see the firms, as well as, the various firm classes being distinct.

To allow the model to be compatible with the Netessine and Shumsky [2] model we use their assumption that coach class consumers purchase tickets before business class customers. Specifically, for the first half of the game, only coach class consumers arrive and for the second half, only business class consumers arrive. This is not necessarily a realistic assumption, but is used to keep our model as close to Netessine and Shumsky's as possible. At each period of the game, each firm can set the price they will charge to the type of consumer they expect to arrive that period (either coach or business class). They can set a price so high that no consumer would purchase a ticket for that class. This is analogous to the Netessine and Shumsky allowance that a firm can close coach class sales so that the firm can have more seats available for sale to business class customers, when they arrive.

The demand space for each class (first-class and coach) is described by a 2-dimensional rectangle. Each dimension represents a range of maximum-willingnesses-to-pay for the product offered in that class by each firm. For product i, this is bounded between 0 and an upper bound u_i, where $u_A > 0$ and $u_B > 0$. The only difference between first class and coach is that the upper-bound for the first class seats is *strictly higher* than the upper-bound for the coach class seats. We assume that customer maximum-willingnesses-to-pay are distributed uniformly within each rectangle.

We employ the proportional splitting rule for areas which are "claimed" by more than one firm.

4.4 Decision variables and strategies employed by each firm

At the beginning of period t, each firm simultaneously sets the price, p_i, for each of its fare classes, taking into account that a sale this period leaves one less available seat next period. The firm, if it wishes, can choose to 'close' the fare class, by setting the fare's price to its upper bound, such that the probability of sale is zero.

We do not allow the airlines to 'bump' coach class seat holders in order to sell additional business class seats.

We restrict the firms to Markovian strategies, so the strategies can depend only on the state, which has three dimensions - the period, the quantity remaining for Firm A, and the quantity remaining for Firm B. Restricting the problem to Markovian strategies allows us to treat the problem as a dynamic program, where the solution is a pure-strategy Markov Perfect Equilibrium.

4.5 Market level elasticity-of-demand, Strategies employed by consumers and the nature of consumer demand

At the beginning of each period, a single customer arrives with probability λ. For the sake of simplicity, for the remainder of the section we will assume $\lambda = 1$. If it is period 1 through $\frac{T}{2}$ then a coach class consumer arrives; if it is period $\frac{T}{2} + 1$ to T then a business class consumer arrives. The consumer's 2-dimensional vector of maximum-willingness-to-pay is drawn randomly from the rectangle. Based on the draw, they purchase one of the available seats or choose not to purchase at all. Since the area of the 'no purchase' option expands as prices rise, the market does show some level of price-elasticity-of-demand.

We assume consumers are myopic in this model. They do not have the option of waiting to see if prices fall. They also do not attempt to cancel a reservation and re-buy, if prices fall.

4.6 Firm A's Optimization Problem

At time t, the firms have quantities q_t^A and q_t^B remaining respectively. We will refer to product A's probability of sale (given an arrival) as π_A, which is a function of both the prices in the market.

Given that the firm can enter three possible states of the world next period (one where A has one less good, one where B has one less good, and one where both firms continue to have the same number of goods), Firm A's value function for period t given quantity levels q_t^A and q_t^B is:

$$\begin{aligned} V_t(q_t^A, q_t^B) = \\ \lambda * \pi_a(p_a - V_{t+1}(q_t^A - 1, q_t^B)) + \\ \lambda * (\pi_B + \pi_b) * V_{t+1}(q_t^A, q_t^B - 1) + \pi_0 * V_{t+1}(q_t^A, q_t^B) \end{aligned} \tag{4.1}$$

Where π_0 is the probability that no firm makes a sale. This is given by:

$$1 - \lambda(\pi_A + \pi_B) \tag{4.2}$$

For Firm A, we place the constraints that $0 \leq p_A^t \leq u_A$ and differentiate with respect to p_A^t to obtain the firm's best response function. Similarly for Firm B, we differentiate with respect to p_B^t, to find the optimal reaction function for Firm B. This gives us a system of two equations and two unknowns. Solving for this gives the optimal price point for the firm at time t given the state of the world at the beginning of the period.

4.7 Results for the 2x2 Airline Problem

In order to better illustrate the dynamics of the game, we created a number of simulations. The parameter values used were as follows:

- The game had 30 rounds. In rounds 1 through 15 a single coach consumer arrived with probability 1. In rounds 16 through 30 a single business class consumer arrived with probability 1.

- Each aircraft has 8 seats.

- Maximum-willingnesses-to-pay for coach class consumers are distributed uniformly over [(0,0),(200,200)]. Maximum-willingnesses-to-pay for business class consumers are distributed uniformly over [(0,0),(400,400)].

These parameter values allow for a reasonable chance that each airline will sell out before the end of the game, with no guarantee that they will do so. The relative upper bounds were chosen to reflect prices one may reasonably see in real world markets.

We allow Firm B two potential strategies: No strategic decisions (prices fixed, no booking limits used) or optimal booking limits (Netessine and Shumsky). For Firm A we allow three potential strategies: No strategic decisions, optimal booking limits, and optimal pricing strategies. For each of the 6 strategy pairs, we consider 3 sets of reference prices which yield 3 sets of average occupancy rates in the absence of booking limits.

In the absence of booking limits occupancy rates vary with fare levels. With low fares (business class \$214, coach \$107) flights run at a 99.1% capacity. With medium (\$260, \$130) and high fares (\$304, \$152) capacity levels drop to 92.9% and 75.9% respectively.

For each of the 18 scenarios (6 strategy pairings by 3 pricing strategies) 10,000 simulations were run in order to calculate the average revenue for each firm. Our results are as follows.

Base Case Pricing Strategies for Airline Model

Fare Prices	Occupancy Rates in the Absence of Booking Limits
Low Fares	99.1% of capacity
Medium Fares	92.9% of capacity
High Fares	75.9% of capacity

Figure 4.1: Occupancy Rates in the Absence of Booking Limits

4.7.1 High-Occupancy Results

In a scenario where, in the absence of booking limits, the firms almost always sell out, it makes sense that an optimal booking limit strategy would yield increased revenue. By limiting coach class sales, a firm has more seats available when higher-margin business consumers arrive. In this scenario, we found that an optimal booking limit strategy increases revenue 22-29% over using fixed prices.

Under a booking limit system, the fare class is closed when realized quantity demanded is too high. Under optimal pricing, the firm can respond to market conditions by altering price at the beginning of each period t. If realized demand is high, the firm will raise its price which will act to both reduce quantity demanded and to increase price-per-sale (and therefore increase revenue).

We see that by using an optimal pricing strategy, the firm can increase its revenue by 36-50% over using the base strategy. Rather than closing coach class seats, the firm should rather increase the price it charges for coach class seats. As with a booking limit, it decreases the number of coach class seats sold, thus leaving more seats available for business class consumers. However, it also *increases* the revenue of the coach class seats that it does sell.

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a High-Occupancy Model (99.1% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	\$1163.72	\$1050.20
(Standard Deviation)	(\$177.68)	(\$168.75)
Optimal Booking Limit	\$1427.33	\$1355.07
(Standard Deviation)	(\$250.79)	(\$232.41)
Gain over Base	\$263.61	\$304.87
(as a Percentage of Base)	(22.65%)	(29.03%)
Optimal Dynamic Pricing	\$1588.88	\$1576.66
(Standard Deviation)	(\$287.41)	(\$285.79)
Gain over Base	\$425.16	\$526.46
(as a Percentage of Base)	(36.53%)	(50.13%)
Pricing Gain over Booking	\$161.55	\$221.59
(as a Percentage of Base)	(13.88%)	(21.10%)
(as a Percentage of Booking Gain)	(61.28%)	(72.68%)

Figure 4.2: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a High-Occupancy Model (99.1% capacity when unrestricted)

4.7.2 Medium-Occupancy Results

In a medium occupancy model, the fixed prices the firm uses are already close to optimal - not so low that the firm sells out and turns away customers, but not so high that the firm is often running planes at half-capacity. Thus it is not surprising that both optimal booking limit strategy and optimal pricing strategy do not lead to great jumps in revenue.

By using an optimal booking limit strategy, a firm can increase its revenue by approximately 5-6%. However, using an optimal pricing strategy, the firm can increase its revenue by approximately 11%. While some increase is provided by both strategies, an optimal pricing approach proves to be superior.

Where optimal pricing strategies shine is in a low occupancy model.

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Medium Occupancy Model (92.9% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	\$1411.15	\$1383.33
(Standard Deviation)	(\$252.09)	(\$243.47)
Optimal Booking Limit	\$1481.77	\$1461.10
(Standard Deviation)	(\$295.02)	(\$307.32)
Gain over Base	\$70.62	\$77.77
(as a Percentage of Base)	(5.00%)	(5.62%)
Optimal Dynamic Pricing	\$1566.55	\$1541.82
(Standard Deviation)	(\$285.59)	(\$284.45)
Gain over Base	\$155.40	\$158.49
(as a Percentage of Base)	(11.01%)	(11.46%)
Pricing Gain over Booking	\$84.78	\$80.72
(as a Percentage of Base)	(6.01%)	(5.84%)
(as a Percentage of Booking Gain)	(120.05%)	(103.79%)

Figure 4.3: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in Medium Occupancy Model (92.9% capacity when unrestricted)

4.7.3 Low-Occupancy Results

In the following scenario, each airline runs at around 76% capacity when both airlines do not use booking limits. Under such a scenario, there is little gained by using booking limits.

If the firm does use booking limits under this scenario; the optimal booking limit in all scenarios is to only allow 7 of the 8 seats to be sold to coach class consumers. This increases revenue by about half of a percent.

An optimal pricing strategy shines in this scenario, since it allows the firm to discount its prices if it appears that the plane will otherwise leave half-empty. By using an optimal pricing strategy, the firm can obtain more than *twenty times* the additional revenue of an optimal booking limit strategy.

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Low Occupancy Model (75.9% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	$1396.38	$1393.89
(Standard Deviation)	($421.19)	($419.34)
Optimal Booking Limit	$1404.47	$1399.73
(Standard Deviation)	($426.30)	($424.27)
Gain over Base	$8.09	$5.84
(as a Percentage of Base)	(0.58%)	(0.42%)
Optimal Dynamic Pricing	$1571.62	$1573.26
(Standard Deviation)	($279.99)	($278.50)
Gain over Base	$175.24	$179.37
(as a Percentage of Base)	(12.55%)	(12.87%)
Pricing Gain over Booking	$167.15	$173.53
(as a Percentage of Base)	(11.97%)	(12.45%)
(as a Percentage of Booking Gain)	(2066.13%)	(2971.40%)

Figure 4.4: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Low Occupancy Model (75.9% capacity when unrestricted)

4.8 Parameter Analysis

Changing the parameters did not change the results. Optimal pricing continued to produce significantly higher revenue than the use of booking limits.

In order to determine the sensitivity of the results from the previous section to our choice of parameters, we adjusted the following three parameters in the medium-results scenario:

- Arrival rate: In the previous section we assumed the arrival rate was 100%. We reduce the rate to 95% and 90%.

- Price ratio: In the medium scenario, we assumed the price ratio between the classes was 2 (business class \$260, \$130). We adjust the ratio by adjusting the price of the business class. We examine a 1.5 ratio (\$195, \$130) and a 2.5 ratio (\$325, \$130).

- Proportion of periods of business class demand: We assumed half of the periods (15 of 30) coach class consumers arrive. We increase this to 60% (18 of 30) and 70% (21 of 30).

We also examined altering several parameters at once, but this tended to either amplify the effects or cancel each other out. One simulation involved both doubling the number of periods and halving the arrival rate. We found in our simulations that the revenue figures from this change were no more than a few dollars different than their original counterparts.

4.8.1 Differences in the Arrival Rate

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Medium Occupancy Model - 95% Arrival Rate (88.2% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	\$1335.93	\$1304.59
(Standard Deviation)	(\$397.94)	(\$390.75)
Optimal Booking Limit	\$1401.83	\$1387.18
(Standard Deviation)	(\$444.81)	(\$430.51)
Gain over Base	\$65.90	\$82.59
(as a Percentage of Base)	(4.93%)	(6.33%)
Optimal Dynamic Pricing	\$1477.69	\$1463.96
(Standard Deviation)	(\$440.55)	(\$421.34)
Gain over Base	\$141.76	\$159.37
(as a Percentage of Base)	(10.61%)	(12.22%)
Pricing Gain over Booking	\$75.68	\$76.78
(as a Percentage of Base)	(5.68%)	(5.89%)
(as a Percentage of Booking Gain)	(114.84%)	(92.97%)

Figure 4.5: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in Medium Occupancy Model - 95% Arrival Rate (88.2% capacity when unrestricted)

In this case, the results are very similar to the base medium-occupancy results. By using an optimal booking limit strategy, a firm can increase its revenue by approximately 5-6%. Whereas, using an optimal pricing strategy increases revenue by 11-12%. This is compared to the 5% and 11% respectively in the 100% arrival model.

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Medium Occupancy Model - 90% Arrival Rate (83.6% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	$1270.81	$1239.65
(Standard Deviation)	($488.91)	($470.64)
Optimal Booking Limit	$1335.02	$1313.05
(Standard Deviation)	($535.82)	($522.94)
Gain over Base	$64.21	$73.40
(as a Percentage of Base)	(5.05%)	(5.92%)
Optimal Dynamic Pricing	$1403.79	$1385.86
(Standard Deviation)	($526.33)	($525.23)
Gain over Base	$132.98	$146.21
(as a Percentage of Base)	(10.46%)	(11.79%)
Pricing Gain over Booking	$68.77	$72.81
(as a Percentage of Base)	(5.41%)	(5.87%)
(as a Percentage of Booking Gain)	(107.10%)	(99.20%)

Figure 4.6: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in Medium Occupancy Model - 90% Arrival Rate (83.6% capacity when unrestricted)

Changing the arrival rate to 90% resulted in a revenue gain under booking limits of 5% and under optimal pricing of 11%. The percentage gains in revenue remained relatively constant under an arrival rate of 100%, 95% and 90% and optimal pricing outperformed booking limits under each scenario.

4.8.2 Differences in the Price Ratio

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Medium Occupancy Model - 1.5x Price Ratio (98.6% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	$1274.51	$1252.55
(Standard Deviation)	($118.24)	($118.19)
Optimal Booking Limit	$1277.61	$1331.21
(Standard Deviation)	($118.19)	($119.37)
Gain over Base	$3.10	$78.66
(as a Percentage of Base)	(0.24%)	(6.28%)
Optimal Dynamic Pricing	$1552.55	$1497.13
(Standard Deviation)	($284.25)	($282.43)
Gain over Base	$278.04	$244.58
(as a Percentage of Base)	(21.82%)	(19.53%)
Pricing Gain over Booking	$274.94	$80.72
(as a Percentage of Base)	(21.58%)	(13.25%)
(as a Percentage of Booking Gain)	(8991%)	(210.99%)

Figure 4.7: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in Medium Occupancy Model - 1.5x Price Ratio (98.6% capacity when unrestricted)

When the price ratio was changed from 2 times to 1.5 times, optimal pricing provided much higher increases in revenue over the base than did the booking limit strategy. Furthermore, optimal pricing performed even better under the 1.5 times scenario than it did in the original 2 times scenario.

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Medium Occupancy Model - 3x Price Ratio (79.8% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	\$1283.50	\$1238.67
(Standard Deviation)	(\$387.61)	(\$366.71)
Optimal Booking Limit	\$1301.77	\$1331.64
(Standard Deviation)	(\$405.63)	(\$402.75)
Gain over Base	\$18.27	\$92.97
(as a Percentage of Base)	(1.42%)	(7.51%)
Optimal Dynamic Pricing	\$1592.08	\$1587.92
(Standard Deviation)	(\$286.82)	(\$283.58)
Gain over Base	\$308.58	\$353.41
(as a Percentage of Base)	(24.04%)	(28.53%)
Pricing Gain over Booking	\$290.31	\$190.61
(as a Percentage of Base)	(22.62%)	(21.02%)
(as a Percentage of Booking Gain)	(1589%)	(205.02%)

Figure 4.8: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in Medium Occupancy Model - 1.5x Price Ratio (79.8% capacity when unrestricted)

Here again using an optimal pricing strategy gave superior results than booking limits and showed an even greater percentage gain in revenue over the 2 times pricing ratio used in the base case.

4.8.3 Differences in Proportion of Business Class Periods

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Medium Occupancy Model - 18/30 Coach Class Periods (93.1% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	\$1234.92	\$1195.72
(Standard Deviation)	(\$226.08)	(\$210.73)
Optimal Booking Limit	\$1320.14	\$1294.28
(Standard Deviation)	(\$234.12)	(\$239.18)
Gain over Base	\$85.22	\$98.56
(as a Percentage of Base)	(6.90%)	(8.24%)
Optimal Dynamic Pricing	\$1388.92	\$1367.17
(Standard Deviation)	(\$257.71)	(\$257.53)
Gain over Base	\$154.00	\$171.45
(as a Percentage of Base)	(12.47%)	(14.34%)
Pricing Gain over Booking	\$68.78	\$72.89
(as a Percentage of Base)	(5.57%)	(6.10%)
(as a Percentage of Booking Gain)	(80.71%)	(73.96%)

Figure 4.9: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in Medium Occupancy Model - 18/30 Coach Class Periods (93.1% capacity when unrestricted)

The percentage gains in revenue were slightly higher for both booking limits and optimal pricing under the 18/30 ratio than the 15/30 base model but again optimal dynamic pricing proved to be superior to a booking limit strategy.

Revenue for Firm A - Booking Limits vs. Dynamic Pricing in a Medium Occupancy Model - 21/30 Coach Class Periods (93.1% capacity when unrestricted)

Firm A's Strategy	Rev. A (B no limit)	Rev. A (B opt. limit)
No Booking Limit - Base Case	\$1140.55	\$1114.82
(Standard Deviation)	(\$195.10)	(\$188.28)
Optimal Booking Limit	\$1209.59	\$1193.61
(Standard Deviation)	(\$234.12)	(\$239.18)
Gain over Base	\$69.04	\$78.79
(as a Percentage of Base)	(6.05%)	(7.07%)
Optimal Dynamic Pricing	\$1273.77	\$1262.84
(Standard Deviation)	(\$235.94)	(\$238.33)
Gain over Base	\$133.22	\$148.02
(as a Percentage of Base)	(11.68%)	(13.28%)
Pricing Gain over Booking	\$64.18	\$69.23
(as a Percentage of Base)	(5.63%)	(6.21%)
(as a Percentage of Booking Gain)	(92.96%)	(87.87%)

Figure 4.10: Revenue for Firm A - Booking Limits vs. Dynamic Pricing in Medium Occupancy Model - 21/30 Coach Class Periods (93.1% capacity when unrestricted)

Here again optimal dynamic pricing provided the largest increase in revenue.

4.8.4 Results from Parameter Analysis

In summary, our results held even though the parameters were altered. Optimal pricing consistently provided increased revenue over the base case and the use of booking limits.

4.9 Knockout Strategy in the Airline Model

Finally, we will consider the knockout strategy in the context of the airline model where a dynamic pricing firm competes against a firm using a fixed-price, no booking limit policy. We choose this comparison to highlight the effect of the knockout strategy. We will examine the increase in our price which is *solely* due to the knockout strategy. The knockout strategy has a number of interesting properties.

4.9.1 Knockout Strategy and Number of Opponent Seats

There is no knockout strategy when the opposing firm has as many or more goods as periods remaining. This makes intuitive sense, as the benefit to knocking out a firm is that our firm gains a monopoly for the rest of the sales period. If the opposing firm has more goods remaining than periods, there is no chance of it selling out, thus there is no value in employing a knockout strategy. Following is the optimal amount a firm with 8 seats remaining should increase its prices when there are 4 periods remaining in the medium-occupancy model:

Increase From Base Price Due to Knockout Strategy With Four Periods Remaining and Our Firm Has 8 Seats Left in Medium-Occupancy Model

Opponent Quantity	Price Increase From Base
1	$10.94
2	$3.61
3	$0.45
4	$0.00
5	$0.00
6	$0.00
7	$0.00
8	$0.00

Figure 4.11: Increase From Base Price Due to Knockout Strategy With Four Periods Remaining and Our Firm Has 8 Seats Left in Medium-Occupancy Model

If there is no possibility of our firm selling out, then the value of the benefit function in our pricing formula is larger the fewer the number of goods our opponent has.

This also makes intuitive sense, as it is more likely we are able to knock them out and have a longer period of being a monopoly.

4.9.2 Knockout Strategy and Number of Our Seats

For a given quantity of our opponents goods, the price increase from the knockout strategy is (weakly) higher as we have more goods. If there is a risk that our firm will sell out before the end of the sales period, then a monopoly is of less value as we will not get to reap the full benefit of monopoly over the entire sales periods. Here is the price effect from the knockout strategy when there are 5 periods remaining and our opponent has only one good remaining:

Increase From Base Price Due to Knockout Strategy With Five Periods Remaining and Opponent Firm Has 1 Seat Left in Medium-Occupancy Model

Our Quantity	Price Increase From Base
1	\$2.40
2	\$6.12
3	\$10.18
4	\$12.98
5	\$13.83
6	\$13.83
7	\$13.83
8	\$13.83

Figure 4.12: Increase From Base Price Due to Knockout Strategy With Four Periods Remaining and Our Firm Has 8 Seats Left in Medium-Occupancy Model

Note that the price increase is constant when there is no risk of our firm selling out and thus we would have a monopoly for the rest of the sales period. This occurs because there is simply no opportunity cost to selling a good if there is no risk in selling out before the end of the period.

4.9.3 Arrival Rates

If there is no risk of selling out, then the price increase is higher when our opponent prices higher than when they price lower. Consider when there are 8 periods remaining and we have 8 goods.

Increase From Base Price Due to Knockout Strategy With Eight Periods Remaining and Our Firm Has 8 Seats Left

Opponent Quantity	Price Increase Opp. Price \$200	Price Increase Opp. Price \$240	Price Increase Opp. Price \$280
1	\$21.54	\$15.63	\$10.21
2	\$19.18	\$12.42	\$6.69
3	\$14.08	\$7.51	\$3.07
4	\$7.83	\$3.26	\$0.97
5	\$3.09	\$0.97	\$0.21
6	\$0.81	\$0.19	\$0.03
7	\$0.13	\$0.02	\$0.00
8	\$0.01	\$0.00	\$0.00

Figure 4.13: Increase From Base Price Due to Knockout Strategy With Eight Periods Remaining and Our Firm Has 8 Seats Left

However, this feature does not necessarily hold in the model when there is a possibility that our firm might sell out. This is due to the fact that when our opponent prices higher, this increases our chance of making a sale (thus in selling out) and as we saw earlier, the pricing increase from the knockout strategy is smaller when the higher the chance of selling out before the end of the sales period.

Increase From Base Price Due to Knockout Strategy With 19 Periods Remaining and Our Firm Has 8 Seats Left

Opponent Quantity	Price Increase Opp. Price \$100	Price Increase Opp. Price \$120	Price Increase Opp. Price \$140
1	\$9.42	\$3.25	\$0.60
2	\$12.31	\$5.46	\$1.42
3	\$14.15	\$7.73	\$2.72
4	\$14.80	\$9.45	\$4.33
5	\$14.15	\$10.15	\$5.77
6	\$11.06	\$9.13	\$6.30
7	\$5.65	\$5.76	\$5.00
8	\$1.31	\$1.76	\$2.03

Figure 4.14: Increase From Base Price Due to Knockout Strategy With 19 Periods Remaining and Our Firm Has 8 Seats Left

Use of the knockout strategy further optimizes revenue by increasing the odds that the firm will be in a monopoly position for the rest of the sales period. The magnitude of the knockout strategy used is highly dependent on the price being charged by the other firm, how many periods there are remaining as well as the quantity levels of the two firms. This further illustrates the dynamic response that the model affords.

4.10 Summary

By examining an extension of Netessine and Shumsky (2005) [2] we find that in all cases dynamic pricing using our model significantly outperforms a booking level approach. In all three cases (high demand, medium demand and low demand), both optimal booking limits and optimal pricing policies outperform the no-booking level policy. In a medium demand scenario (92.9% capacity when unrestricted), optimal booking limits achieve a 5-6% revenue gain over base, whereas optimal pricing yields a

11-12% gain. In high demand situations (99.1% capacity when unrestricted), optimal booking limit policies substantially increase revenue (22-29% revenue gains over base) but are significantly outperformed by optimal pricing policies (36-50% revenue gains over base). Optimal booking limit policies are of marginal value in low demand scenarios (75.9% capacity when unrestricted) providing only a 0.4-0.6% gain in revenue. Optimal pricing policies, however, continue to achieve significant revenue gains of roughly 12-13%. Switching from optimal booking to optimal pricing yields significant revenue increases under any demand scenario including one where our opponent has few goods remaining. This situation allows the firm to further maximize revenue by using the knockout strategy. Optimal pricing is a more nuanced strategy and takes more factors into consideration, which leads to increased revenues for the firm.

Chapter 5

Conclusion and Future Research

5.1 Introduction - Background and Purpose

As discussed in the introduction to this thesis, there are three important pricing properties of revenue management markets of interest:

- The prices charged by the firms are not identical, and in some cases there are large differences in the prices firms charge for a comparable product (see *Enterprise* vs. *Hertz*). This suggests that the products are not commodities - a level of brand differentiation exists.
- Firms offer a range of similar, but non-identical products. When Hertz lowers the price of mid-size rentals, this should increase the quantity demanded of these cars, but it may also reduce the demand for compact and standard cars.
- Firms can engage in *diagonal competition*. That is, Hertz may be the preferred brand of a customer, but if Enterprise or Budget price low enough, customers may be willing to choose a higher-end car from these companies, for what they would have paid Hertz for a lower-end car.

However, typical revenue management models consider at most one of these properties and fails to address all three simultaneously. Revenue management has lacked a model to adequately describe price competition in an oligopoly. The need for such a model is well-known and appears in the "future research" section of many studies, see Lu (2009) [5] as an example. Xavier Vives is quoted as saying:

> "The question is, has oligopoly theory failed? ... Oligopolistic business patterns have lacked a benchmark model of dynamic price formation. A benchmark model could provide a counterpart to well-established static models and some insight toward resolving dynamically the issue of an appropriate game form. The model should be tractable and based on plausible assumptions, and firms should use 'simple' strategies so that the model can deliver relatively robust predictions. Such a result is not possible with the repeated game model, which has probably received too much attention." [11]

While we share Vives' frustration with the lack of progress in modeling oligopoly problems in revenue management, our line of research illustrates that the repeated game model can successfully be used to model the interactions between firms.

We have illustrated the importance of simultaneously considering five factors when pricing a good in a revenue management context:

1. Quantity level of the product.
2. Quantity level of the firm's other products.
3. Quantity level of other firms' products.
4. Cannibalization, cross-price elasticity and the price of our firm's other products.

5. Competition, cross-price elasticity and the price of competitor products.

While there is still much research to be completed in this area, this thesis maps out an ambitious research agenda to create pricing models that adequately resemble real-world markets.

5.2 Creating and Solving NxM Pricing Demand Models

In Chapter 2, we create a twice continuously differentiable geometric demand framework that allows for an unlimited (but finite) number of firms and products. This allows for the creation of mathematically tractable optimal pricing models that have multiple product classes from multiple firms. In Chapter 1 we saw how markets of interest to revenue managers have multiple firms selling multiple classes of goods. However, existing revenue management models allow for at most two of the following three features: multiple firms, multiple goods, optimal pricing. Our framework allows for all three simultaneously. There are two parts of our model that make this possible; a geometric demand model which is unique to this paper and a 'single-customer per period buys at most a single-product' assumption which is relatively common in the literature. We prove the existence of subgame perfect Nash Equilibria and then solve the NxM model recursively. Unlike many models in the Bertrand-Edgeworth competition literature, we can be assured that there is, in fact, a solution to any NxM model. We cannot be assured that a unique solution will exist to all NxM models. In general it is necessarily to solve the model recursively to find the optimal price in each particular state in which it finds itself. We have not found a way to express the optimal price in a particular state as a function of the variables in the model; it is likely only possible to do so for the most trivial of models. We give two methods for proving that a unique solution exists in a particular application of the model. We then give conditions under which a solution will exist in a 2-firm/1-product-per-firm model.

The model created in the chapter considers own price elasticity of demand for a product along with five factors: The quantity level of the product (factor 1), the quantity level of the firm's other products (factor 2), quantity level of the competitor's products (factor 3), cannibalization and cross-price elasticity between products sold by the firm (factor 4) and cross-price elasticity between the firm's product and those of its competitors (factor 5).

5.2.1 Implications For Future Research

While we were able to prove the existence of Nash Equilibria, determining the conditions to guarantee the existence of *unique* Nash Equilibria remains an area for future research. We hope to show that the conditions for unique subgame perfect Nash Equilibria are relatively flexible. While multiple equilibria do not render the model useless, it may be difficult to advise a rental car company to "price their cars at either \$20.00 or \$50.00". Future research can also include studying when a number of classic dynamic properties hold, such as under what conditions can we assure that the optimal price is monotonic in the arrival rate.

5.3 Two Optimal Dynamic Pricing Examples

Chapter 3 contains two examples. A 2-firm/1-product-per-firm car rental model is created. We show that if a firm does not take market structure into account (acts as if it has a monopoly instead of a duopoly), it can do worse by trying to price optimally instead of using a simple fixed-price pricing rule. A firm's revenue will be reduced if both factor 3 (opponent's quantity) and factor 5 (competitive price effects) are not considered. Although the model we develop assumes that firms know the quantity levels of their competitors, we show that the firm can increase its revenues by considering spill-over effects and cross-elasticity, even if it is unaware of the quantity level of the other firm in the market and must ignore factor 3. The highest revenues accrue to the firm when it simultaneously considers all five factors. We show the existence of a knockout strategy, stemming from factor 3, where a firm will price higher, in order to give sales to the other firm. It does so in order to have that firm sell out of its good, so the firm can have a monopoly for the rest of the sales period.

Later in the chapter a 2x3 rental car model is created. We show that a firm can reduce its profit by using traditional revenue management techniques, that ignore factor 4, to price its cars. This is due to existing models not considering spillover effects between their brands. If spillover effects are ignored, firms will price their low end cars too low which will cannibalize the sales of their higher-end/higher-margin offerings. In our framework, prices are typically higher for lower end cars and lower for higher end cars than in traditional revenue management models. Again, all five factors must be considered simultaneously in order to maximize revenue.

5.3.1 Implications For Future Research

There are a great number of potential extensions for future research that would add realism to the model. The first is a realization that not all bookings result in a final sale, thus it is in the interest of the firm to overbook. A customer who books will be a no-show by some probability τ, which can be incorporated into the model. It may be possible to alter the model such that the firm can, in theory, accepted an unlimited number of bookings, but pays a substantial cost for each overbooked customer it is forced to compensate.

An important area for future applied research is determining reasonable estimates for cross-price elasticity in revenue management markets of interest. While Anderson, Davison and Rasmussen [1] and Geraghty and Johnson [4] indicate that substantial cross-price elasticity exists in the rental car market, there are no published cross-price elasticity parameter estimates. It is likely that such research will have to come with the assistance of a rental car company in order to obtain the necessary data. Future research should also consider alternative formulations of cross-price elasticity, as the one used in this model may turn out not to be optimal for real-world applications. However, the common assumption of constant cross-price elasticity may be inappropriate as well.

5.4 Booking Limits and Optimal Pricing in a 2-Firm, 2 Class Airline Model

We consider the well-known 2-firm, 2-seat class Netessine and Shumsky [7] model in Chapter 4. We recreate the model but allow our firm to use an optimal pricing strategy rather than a booking limits strategy, to take advantage of factor 5 and own price elasticity of demand, both of which are lacking in the Netessine and Shumsky model. In a period of high demand (where the occupancy rate is 99.1% of capacity in the absence of booking limits), a booking limit strategy provides a revenue increase of 22-29% over having no strategy at all. However, an optimal pricing strategy provides a 36-50% revenue increase over the no strategy case. We find that rather than closing coach class seats, a firm should increase the price it charges for these seats when demand is high. As with a booking limit, it will decrease the number of seats sold, but do so at a much greater revenue-per-seat. We then test medium demand (92.9% capacity) and low demand (75.9%) situations and find that an optimal pricing strategy significantly outperforms a booking limits strategy here as well.

5.4.1 Implications For Future Research

Elmaghraby and Keskinocak (2003) [3] discuss how a firm will often sell the same inventory through different channels at different prices, such as through its own website, through a third-party website such as *Expedia.com*, to travel agents, and to walk-up-traffic. Future research should consider situations where a firm's inventory is shared through a number of different sales channels.

The 'at most one arrival per period' condition appears necessary to assure the existence of Nash Equilibria - see Tirole (1988) [9], Davidson and Deneckere (1986) [2], Madden (1998) [6], Osborne and Pitchik (1986) [8] and Vives (1993) [10]. It may be possible to loosen the 'an arrival purchases at most one good' assumption to allow for group bookings. One way may be as follows: Treat individual sales and sales of a fixed size (e.g. four) to be two different product classes with a shared inventory.

Firms with enough inventory to cover the group booking set a price for both the group and individual, whereas a firm with not enough inventory to cover a group booking only sets a price for the individual booking. Each period a single customer who is looking for *either* a group booking or an individual booking arrives with probability λ.

Similarly, in a real-world scenario one cannot avoid the issue of bundling of goods. A customer purchases a bundle containing an airline seat to a destination and a return ticket. Online retailers such as Expedia and Orbitz allow the bundling of a car rental, hotel stay and airline ticket for a single price. Length of stay/length of rental can also be thought of as a bundle - a two-day stay in a hotel is a bundle of a room rental for Tuesday night and a room rental for Wednesday night. An area for future research is a dynamic pricing model that allows firms to price and bundle separate goods and services together.

5.5 Final Thoughts

It is our hope that the contribution we have made to the study of modeling multiple-firm/multiple-products per firm problems in revenue management will spark new interest in this area of study and will stimulate the development of revenue management models which are more realistic in this regard.

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Appendix A

A 3 Product Market

In chapter 2, we saw that a 2 product market met the 6 necessary demand space assumptions. In this appendix, we show the same holds for the 3 product model. However, in a 3 product market, with products A, B and C, the problem becomes more complex. From product $A's$ perspective, there is the portion of demand that product A gets to keep. There is a portion split two-ways between A and B, and a portion split two ways between A and C. Finally there is a portion split between all three products. The demand for product A is thus given by:

$$\pi_A^t = \frac{(U_A^t - p_A^t)}{(U_A^t - L_A^t)}\left(\frac{(p_B^t)(p_C)}{(U_B^t - L_B^t)(U_C - L_C)} + \frac{(U_B^t - p_B^t)(p_C)}{2(U_B^t - L_B^t)(U_C - L_C)} + \frac{(U_C - p_C)(p_B^t)}{2(U_B^t - L_B^t)(U_C - L_C)} + \frac{(U_B^t - p_B^t)(U_C - p_C)}{3(U_B^t - L_B^t)(U_C - L_C)}\right) \quad \text{(A.1)}$$

Product B:

$$\pi_B^t = \frac{(U_B^t - p_B^t)}{(U_B^t - L_B^t)}\Big(\frac{(p_A^t)(p_C)}{(U_A^t - L_A^t)(U_C - L_C)} + \frac{(U_A^t - p_A^t)(p_C)}{2(U_A^t - L_A^t)(U_C - L_C)} + \frac{(U_C - p_C)(p_A^t)}{2(U_A^t - L_A^t)(U_C - L_C)} + \frac{(U_A^t - p_A^t)(U_C - p_C)}{3(U_A^t - L_A^t)(U_C - L_C)}\Big) \quad \text{(A.2)}$$

And product C:

$$\pi_C = \frac{(U_C - p_C)}{(U_C - L_C)}\Big(\frac{(p_B^t)(p_A^t)}{(U_B^t - L_B^t)(U_A^t - L_A^t)} + \frac{(U_B^t - p_B^t)(p_A^t)}{2(U_B^t - L_B^t)(U_A^t - L_A^t)} + \frac{(U_A^t - p_A^t)(p_B^t)}{2(U_B^t - L_B^t)(U_A^t - L_A^t)} + \frac{(U_B^t - p_B^t)(U_A^t - p_A^t)}{3(U_B^t - L_B^t)(U_A^t - L_A^t)}\Big) \quad \text{(A.3)}$$

No sale probability, given an arrival, given by:

$$1 - \pi_A^t - \pi_B^t - \pi_C = \pi_{NS}^t = \frac{(p_A^t - L_A^t)}{(U_A^t - L_A^t)}\frac{(p_B^t - L_B^t)}{(U_B^t - L_B^t)}\frac{(p_C - L_C)}{(U_C - L_C)} \quad \text{(A.4)}$$

Once again, we can show this meets all of the necessary assumptions.

Corollary 5 (Three Product Market Meets Demand Space Assumption)**.** *A three product demand model meets all six demand space assumptions.*

Proof.

1. **All Probabilities Given Arrival Equal One**: Met by construction as $1 - \pi_A^t - \pi_B^t - \pi_C = \pi_{NS}^t$.
2. **Demand Probability Twice Differentiable**:

$$\frac{\partial \pi_A^t}{\partial p_A^t} = \frac{(-1)}{(U_A^t - L_A^t)}(\frac{(p_B^t)(p_C)}{(U_B^t - L_B^t)(U_C - L_C)} + \frac{(U_B^t - p_B^t)(p_C)}{2(U_B^t - L_B^t)(U_C - L_C)} + \frac{(U_C - p_C)(p_B^t)}{2(U_B^t - L_B^t)(U_C - L_C)} + \frac{(U_B^t - p_B^t)(U_C - p_C)}{3(U_B^t - L_B^t)(U_C - L_C)}) \quad (A.5)$$

$$\frac{\partial^2 \pi_A^t}{\partial p_A^{t2}} = 0 \quad (A.6)$$

$$\frac{\partial \pi_B^t}{\partial p_B^t} = \frac{(-1)}{(U_B^t - L_B^t)}(\frac{(p_A^t)(p_C)}{(U_A^t - L_A^t)(U_C - L_C)} + \frac{(U_A^t - p_A^t)(p_C)}{2(U_A^t - L_A^t)(U_C - L_C)} + \frac{(U_C - p_C)(p_A^t)}{2(U_A^t - L_A^t)(U_C - L_C)} + \frac{(U_A^t - p_A^t)(U_C - p_C)}{3(U_A^t - L_A^t)(U_C - L_C)}) \quad (A.7)$$

$$\frac{\partial^2 \pi_B^t}{\partial p_B^{t2}} = 0 \quad (A.8)$$

$$\frac{\partial \pi_C}{\partial p_C} = \frac{(-1)}{(U_C - L_C)}(\frac{(p_B^t)(p_A^t)}{(U_B^t - L_B^t)(U_A^t - L_A^t)} + \frac{(U_B^t - p_B^t)(p_A^t)}{2(U_B^t - L_B^t)(U_A^t - L_A^t)} + \frac{(U_A^t - p_A^t)(p_B^t)}{2(U_B^t - L_B^t)(U_A^t - L_A^t)} + \frac{(U_B^t - p_B^t)(U_A^t - p_A^t)}{3(U_B^t - L_B^t)(U_A^t - L_A^t)}) \quad (A.9)$$

3. **Demand Probability Decreasing in Own Price**:

   Met since the first derivatives of the probability functions (see above) with own price are negative (since $U_A^t \geq L_A^t$, $U_B^t \geq L_B^t$, $U_C \geq L_C$).

4. **Demand Probability Increasing in Other Product Prices**:

$$\frac{\partial \pi_A^t}{\partial p_B^t} = \frac{(U_A^t - p_A^t)}{(U_A^t - L_A^t)}\left(\frac{(U_C + 2p_C)}{6(U_B^t - L_B^t)(U_C - L_C)}\right) \tag{A.10}$$

$$\frac{\partial \pi_A^t}{\partial p_C} = \frac{(U_A^t - p_A^t)}{(U_A^t - L_A^t)}\left(\frac{(U_B^t + 2p_B^t)}{6(U_B^t - L_B^t)(U_C - L_C)}\right) \tag{A.11}$$

$$\frac{\partial \pi_B^t}{\partial p_A^t} = \frac{(U_B^t - p_B^t)}{(U_B^t - L_B^t)}\left(\frac{(U_C + 2p_C)}{6(U_A^t - L_A^t)(U_C - L_C)}\right) \tag{A.12}$$

$$\frac{\partial \pi_B^t}{\partial p_C} = \frac{(U_B^t - p_B^t)}{(U_B^t - L_B^t)}\left(\frac{(U_A^t + 2p_A^t)}{6(U_A^t - L_A^t)(U_C - L_C)}\right) \tag{A.13}$$

$$\frac{\partial \pi_C}{\partial p_A^t} = \frac{(U_C - p_C)}{(U_C - L_C)}\left(\frac{(U_B^t + 2p_B^t)}{6(U_A^t - L_A^t)(U_B^t - L_B^t)}\right) \tag{A.14}$$

$$\frac{\partial \pi_C}{\partial p_B^t} = \frac{(U_C - p_C)}{(U_C - L_C)}\left(\frac{(U_A^t + 2p_A^t)}{6(U_B^t - L_B^t)(U_A^t - L_A^t)}\right) \tag{A.15}$$

Holds since $U_A^t \geq L_A^t$, $U_B^t \geq L_B^t$, $U_C \geq L_C$ and $U_A^t \geq p_A^t$,$U_B^t \geq p_B^t$ and $U_C \geq p_C$.

5. **No Purchase Probability Increasing in Prices** Met since the first derivative of the no-sale probability function with respect to prices are positive:

$$\frac{\partial \pi_{NS}^t}{\partial p_A^t} = \frac{1}{(U_A^t - L_A^t)} \frac{(p_B^t - L_B^t)}{(U_B^t - L_B^t)} \frac{(p_C - L_C)}{(U_C - L_C)} \tag{A.16}$$

$$\frac{\partial \pi_{NS}^t}{\partial p_B^t} = \frac{(p_A^t - L_A^t)}{(U_A^t - L_A^t)} \frac{(1)}{(U_B^t - L_B^t)} \frac{(p_C - L_C)}{(U_C - L_C)} \tag{A.17}$$

$$\frac{\partial \pi_{NS}^t}{\partial p_C} = \frac{(p_A^t - L_A^t)}{(U_A^t - L_A^t)} \frac{(p_B^t - L_B^t)}{(U_B^t - L_B^t)} \frac{(1)}{(U_C - L_C)} \tag{A.18}$$

Holds since $U_A^t \geq L_A^t$, $U_B^t \geq L_B^t$, $U_C \geq L_C$ and $U_A^t \geq p_A^t$,$U_B^t \geq p_B^t$ and $U_C \geq p_C$.

6. **Probabilities Non-Negative**: Since our probability is decreasing in own price but decreasing in competitor price, for A it reaches a minimum at $p_A^t = U_A^t, p_B^t = L_B, p_C = L_C$. Substitute $p_A^t = U_A$ and $\pi_A^t = 0$. Similarly substitute $p_B^t = U_B$ and $\pi_B^t = 0$ and $p_C = U_C$ and $\pi_C = 0$.

   No sale probability is increasing in all prices, and therefore reaches a minimum at $p_A^t = L_A^t, p_B^t = L_B, p_C = L_C$

$$\pi_{NS}^t = \frac{(p_A^t - L_A^t)}{(U_A^t - L_A^t)} \frac{(p_B^t - L_B^t)}{(U_B^t - L_B^t)} \frac{(p_C - L_C)}{(U_C - L_C)} \tag{A.19}$$

$$\pi_{NS}^t = \frac{(L_A^t - L_A^t)}{(U_A^t - L_A^t)} \frac{(L_B^t - L_B^t)}{(U_B^t - L_B^t)} \frac{(L_C - L_C)}{(U_C - L_C)} = 0 \tag{A.20}$$

   The condition is met.

   Therefore all six conditions are met.

Since all six conditions are met, we can be assured of the existence of Nash Equilibria in all subgames of the game and therefore in the game itself.

□

Appendix B

Optimal Prices in the 2 Firm, 1 Good Per Firm Scenario

In this appendix, we examine the optimal prices in a 2 firm, 1 good per firm model, to see how the state affects the optimal price for each good in the final 2 periods.

B.1 Time T

To simplify, we assume that U_A and U_B are constants and do not change from period to period. As such, we can lose the time superscript.

B.1.1 Both Firm A and B Have One or More Goods

Optimal price for A:

$$p_A^{T*} = 0.5(U_A) \tag{B.1}$$

Optimal price for B:

$$p_B^{T*} = 0.5(U_B) \tag{B.2}$$

Probability of sale for A (given an arrival):

$$\pi_A^{T*} = \frac{3}{8} \tag{B.3}$$

Probability of sale for B (given an arrival):

$$\pi_B^{T*} = \frac{3}{8} \tag{B.4}$$

Value function for A:

$$V^T_{q_A^T, q_B^T} = \frac{\lambda 3 U_A}{16} \tag{B.5}$$

B.1.2 Firm A Has One or More Good, B Has None

Optimal price for A:

$$p_A^{T*} = 0.5(U_A) \tag{B.6}$$

Probability of sale for A (given an arrival):

$$\pi_A^{T*} = \frac{1}{2} \tag{B.7}$$

Value function for A:

$$V^T_{q_A^T, q_B^T} = \frac{\lambda U_A}{4} \tag{B.8}$$

B.1.3 Firm A Has No Goods, B Has One or More

Optimal price for B:

$$p_B^{T*} = 0.5(U_B) \tag{B.9}$$

Probability of sale for B (given an arrival):

$$\pi_B^{T*} = \frac{1}{2} \tag{B.10}$$

Value function for B:

$$V_{q_A^T, q_B^T}^T = \frac{\lambda U_B}{4} \tag{B.11}$$

B.2 Time T-1

B.2.1 Both Firms Have Two Or More Goods

Optimal price for A:

$$p_A^{T-1*} = 0.5(U_A) \tag{B.12}$$

Optimal price for B:

$$p_B^{T-1*} = 0.5(U_B) \tag{B.13}$$

Probability of sale for A (given an arrival):

$$\pi_A^{T-1*} = \frac{3}{8} \tag{B.14}$$

Probability of sale for B (given an arrival):

$$\pi_B^{T-1*} = \frac{3}{8} \tag{B.15}$$

Value function for A:

$$V^{T-1}_{q_A^{T-1},q_B^{T-1}} = \frac{\lambda 6 U_A}{16} \tag{B.16}$$

Value function for B:

$$V^{T-1}_{q_A^{T-1},q_B^{T-1}} = \frac{\lambda 6 U_B}{16} \tag{B.17}$$

B.2.2 Firm B Has Two Or More Goods, Firm A Has One

For the first time, there is a cost function (for A) and a benefit function (for B). First, the cost function:

$$c_A^{T-1} = V^T_{1,q_B^{T-1}} - V^T_{0,q_B^{T-1}} = \frac{\lambda 3 U_A}{16} \tag{B.18}$$

$$b_B^{T-1} = V^T_{1,q_B^{T-1}} - V^T_{0,q_B^{T-1}} = \frac{\lambda U_A}{16} \tag{B.19}$$

Optimal price for A:

$$p_A^{T-1*} = \frac{U_A}{32}(16 + 3\lambda) \tag{B.20}$$

Optimal price for B:

$$p_B^{T-1*} = \frac{U_B}{32}(16 + \frac{(16+3\lambda)\lambda}{48+3\lambda}) \tag{B.21}$$

Probability of sale for A (given an arrival):

$$\pi_A^{T-1*} = \frac{(16-3\lambda)(48+\frac{16+3\lambda}{48+3\lambda})}{2048} \tag{B.22}$$

Probability of sale for B (given an arrival):

$$\pi_B^{T-1*} = \frac{(48+3\lambda)(16-\frac{16+3\lambda}{48+3\lambda})\lambda}{2048} \tag{B.23}$$

Value for firm A:

$$V_{1,q_B^{T-1}}^{T-1} = \lambda(\frac{(16-3\lambda)(48+\frac{16+3\lambda}{48+3\lambda})}{2048})(\frac{U_A}{32}(16+3\lambda) - \frac{\lambda 3U_A}{16}) + \frac{\lambda 3U_A}{16} \tag{B.24}$$

Value for firm B:

$$V_{1,q_B^{T-1}}^{T-1} = \lambda(\frac{(16-3\lambda)(48+\frac{16+3\lambda}{48+3\lambda})}{2048})(\frac{\lambda U_A}{16})+ \\ \lambda(\frac{U_B}{32}(16+\frac{(16+3\lambda)\lambda}{48+3\lambda})) + \frac{\lambda 3U_A}{16} \tag{B.25}$$

B.2.3 Firm B Has One, Firm A Has None

For the first time, there is a cost function (for B) but no benefit function since A is out of the game:

$$c_B^{T-1} = V_{0,1}^T - V_{0,0}^T = \frac{\lambda U_B}{4} \tag{B.26}$$

Optimal price for B:

$$p_B^{T-1*} = \frac{U_B}{8}(4+\lambda) \tag{B.27}$$

Probability of sale for B (given an arrival):

$$\pi_B^{T-1*} = \frac{4-\lambda}{8} \tag{B.28}$$

Value for firm B:

$$V_{0,1^{T-1}}^{T-1} = \lambda(\frac{4-\lambda}{8})(\frac{U_B}{8}(4+\lambda) - \frac{\lambda U_B}{4}) + \frac{\lambda U_B}{4} \tag{B.29}$$

B.2.4 Firm A Has One, Firm B Has One

Now we have two cost functions and two benefit functions:

$$c_A^{T-1} = V_{1,1}^T - V_{0,1}^T = \frac{\lambda 3 U_A}{16} \tag{B.30}$$

$$c_B^{T-1} = V_{1,1}^T - V_{1,0}^T = \frac{\lambda 3 U_B}{16} \tag{B.31}$$

$$b_A^{T-1} = V_{1,1}^T - V_{1,0}^T = \frac{\lambda U_B}{16} \tag{B.32}$$

$$b_B^{T-1} = V_{1,1}^T - V_{0,1}^T = \frac{\lambda U_A}{16} \tag{B.33}$$

Optimal price for A:

$$p_A^{T-1*} = \frac{1}{64(\lambda U_A + 3\lambda U_B + 48 U_B)}(\mu + \sqrt{\omega} + 48\lambda U_A^2 - 16\lambda U_B^2) \tag{B.34}$$

Optimal price for B:

$$p_B^{T-1*} = \frac{1}{64(3\lambda U_A + \lambda U_B + 48 U_A)}(\mu + \sqrt{\omega} + 48\lambda U_B^2 - 16\lambda U_A^2) \tag{B.35}$$

Where ω is equal to:

$$\begin{aligned}\omega = (&-3\lambda^2U_A^2 - 48\lambda U_A^2 - 10\lambda^2 U_A U_B - 96\lambda U_A U_B + 768 U_A U_B \\ &- 3\lambda^2 U_B^2 + 16\lambda U_B^2)^2 - 4 * (32\lambda U_A + 96\lambda U_B + 1536 U_B) * \\ &(3\lambda^2 U_A^3 + 16\lambda U_A^3 - 8\lambda^2 U_A^2 U_B - 192\lambda U_A^2 U_B - \\ &768 U_A^2 U_B - 3\lambda U_A U_B^2 + 16\lambda U_A U_B^2)\end{aligned} \tag{B.36}$$

And μ is equal to:

$$\mu = 3\lambda^2 U_A^2 + 10\lambda^2 U_A U_B + 96\lambda U_A U_B - 768 U_A U_B + 3\lambda^2 U_B^2 \tag{B.37}$$

It is possible to continue in this vein to periods $T-2$, $T-3$ and so on, however the functional forms get quite complicated. It is relatively straight forward, however, to calculate a matrix of optimal prices for each possible state using a spreadsheet program such as Microsoft Excel.

Curriculum Vitae

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