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Noncommutative complex geometry of quantum projective spaces

Ali Moatadelro
The University of Western Ontario

Supervisor
Professor Masoud Khalkhali
The University of Western Ontario

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Ali Moatadelro

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Chief Adviser:

Professor Masoud Khalkhali

Examining Board:

Professor Tatyana Foth

Professor Jan Minac

Professor John Bell

Professor Giovanni Landi

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Abstract

In this thesis, we study complex structures of quantum projective spaces that was initiated in [19] for the quantum projective line, $\mathbb{C}P^1_q$. In Chapters 2 and 3, which are the main parts of this thesis, we generalize the the results of [19] to the spaces $\mathbb{C}P^2_q$ and $\mathbb{C}P^\ell_q$. We consider a natural holomorphic structure on the quantum projective space already presented in [11, 9], and define holomorphic structures on its canonical quantum line bundles. The space of holomorphic sections of these line bundles then will determine the quantum homogeneous coordinate ring of the quantum projective space as the space of twisted polynomials.

We also introduce a twisted positive Hochschild 2ℓ-cocycle on $\mathbb{C}P^\ell_q$, by using the complex structure of $\mathbb{C}P^\ell_q$, and show that it is cohomologous to its fundamental class which is represented by a twisted cyclic cocycle. This fits with the point of view of holomorphic structures in noncommutative geometry advocated in [4, 5], that holomorphic structures in noncommutative geometry are represented by (extremal) positive Hochschild cocycles within the fundamental class.

In Chapter 4, we directly verify that the main statements of Riemann-Roch formula and Serre duality theorem hold true for $\mathbb{C}P^1_q$ and $\mathbb{C}P^2_q$.

In Chapter 5, a quantum version of the Borel-Weil theorem for $SU_q(3)$ is proved and is generalized to the case of $SU_q(n)$.

Finally, in the last chapter the noncommutative complex structure of finite spaces is investigated. The space of holomorphic functions are determined and it is also proved that there is no holomorphic structure on the nontrivial vector bundle $\mathcal{E}_a \oplus \mathcal{E}_b$ over the space of two points $X = \{a, b\}$, where dim $\mathcal{E}_a = 2$ and dim $\mathcal{E}_b = 1$.

Keywords: Noncommutative geometry, noncommutative complex geometry, positive Hochschild cocycle.

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To Yalda

and

My Parents

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λ_N	twisted isomorphism	59
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$\mathbb{C}P^n$	complex projective space	3
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L_λ	homogeneous line bundle	82
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$\lvert\underline{m}\rangle, \lvert n_1, n_2, i_1, i_2, m\rangle$	Gelfand-Tsetlin basis	18, 44
$t^n_{\underline{m}', \underline{m}}, t(n_1, n_2)^{\underline{i}}_{\underline{j}}$	matrix coefficients of irreducible representations	45, 20
$U_q(\mathfrak{g})$	the quantum universal enveloping algebra of $\mathfrak{g}$	10
Z^2_+	the space of positive Hochschild 2-cocycles	3

Preface

The correspondence between geometry and algebra is not a new idea in mathematics. Classically, it amounts to a correspondence or duality between commutative algebras and classical spaces. The classical space appears as the spectrum of the commutative algebra. For example, the celebrated theorem of Gelfand and Naimark states that the category of locally compact Hausdorff spaces is equivalent to the dual of the category of commutative C^*-algebras. Hence one can regard not necessarily commutative C^*-algebras as representing noncommutative spaces. In general one seeks an algebraic formulation of geometric notions based on which one can then try to find their analogues in the noncommutative world.

Tools of (differential) topology such as K-theory, de Rham cohomology and Chern-Weil theory of characteristic classes have been extended to noncommutative algebras. A major discovery of Alain Connes, namely cyclic cohomology, can be regarded as the noncommutative analogue of de Rham homology [6].

During the past thirty years, different aspects of noncommutative differential and Riemannian geometry have been developed. For instance now it is a well known fact that the metric information of a Riemannian (spin) manifold can be encoded by a triple of algebra of smooth functions on the manifold, the Hilbert space of spinors and the associated Dirac operator. More precisely a spectral triple over a noncommutative unital $*$-algebra $\mathcal{A}$ is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is represented by bounded operators in a Hilbert space $\mathcal{H}$ and D is a self-adjoint unbounded operator on $\mathcal{H}$ with the following properties. First of all, the commutator $[D, a]$ must be bounded for any element $a \in \mathcal{A}$. Moreover, the resolvent $(D - \lambda)^{-1}$ is a compact operator for any $\lambda \notin \mathbb{R}$. To any Riemannian spin manifold M, a spectral triple is associated canonically with $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(M, S)$ the space of L^2-sections of the spin bundle over M and D is the associated Dirac operator on $\mathcal{H}$. The geodesic distance can be recovered by a formula of Connes

$$d(p, q) = \sup\{|f(p) - f(q)|; ||\,[D, f]\,|| \leq 1,\ f \in \mathcal{A}\}.$$

There also exists an analogue of volume form and even Yang-Mills action in this set up. A theorem of Connes assures that every Riemannian spin^c manifold can be reconstructed in this way from a commutative spectral triple satisfying some natural axioms [7].

While there has been much progress in noncommutative geometry, the progress in noncommutative complex geometry has been very slow and much remains to be done in this area. A beginning step in this direction was made by Alain Connes, who pointed out that positive Hochschild 2-cocycles on the algebra $\mathcal{A} = C^\infty(M)$, on the two dimensional closed oriented surface M, can encode the information needed to define a conformal (or equivalently, complex) structure on the surface M. More precisely, let $\varphi : \mathcal{A}^{\otimes 3} \to \mathbb{C}$, defined by

$$\varphi(f^0, f^1, f^2) := -\frac{1}{2\pi i} \int_M f^0 \, \mathrm{d}f^1 \wedge \mathrm{d}f^2,$$

be the cyclic 2-cocycle representing the fundamental class of the 2-dimensional manifold M. If M carries a conformal structure g, one can define a functional $\varphi_g : \mathcal{A}^{\otimes 3} \to \mathbb{C}$ by

$$\varphi_g(f^0, f^1, f^2) := \frac{i}{\pi} \int_M f^0 \partial f^1 \wedge \overline{\partial} f^2.$$

The 2-cocycle φ is a cyclic 2-cocycle while φ_g is just a Hochschild 2-cocycle in the same Hochschild cohomology class of φ. The cocycle φ_g has another property which is called positivity. To be more explicit, the following gives a positive sequilinear form on $\mathcal{A}^{\otimes 2}$.

$$\langle a_0 \otimes a_1, b_0 \otimes b_1 \rangle := \varphi(b_0^* a_0, a_1, b_1^*).$$

The 2-cocycle φ_g is the unique point in the convex cone $Z_+^2 \cap [\varphi]$ with an extremal property. Here Z_+^2 is the space of positive Hochschild 2-cocycles and extremality is with respect to the following functional. First let

$$G = \sum_1^d g_{\mu\nu} \, \mathrm{d}x^\mu (\, \mathrm{d}x^\nu)^* \in \Omega^2(\mathcal{A}),$$

now define

$$\langle G, \phi \rangle = \sum \phi(g_{\mu\nu}, x^\mu, (x^\nu)^*).$$

This functional takes its minimum in a unique point in $Z_+^2 \cap [\varphi]$ which is φ_g. Conversely the complex structure can also be recovered by φ_g. This strategy has been applied by Connes himself to the case of noncommutative two torus $\mathbb{T}_\theta^2$ and the positive Hochschild 2-cocycle representing the noncommutative complex structure is given explicitly[4, VI, lemma 9]. On the other hand Polishchuk and Schwarz have considered the holomorphic vector bundles on $\mathbb{T}_\theta^2$ and gave a classification of noncommutative holomorphic vector bundles [29, 30]. In [19] M. Khalkhali, G. Landi and W. van Suijlekom proposed a definition for a holomorphic vector bundle on an involutive algebra as a finitely generated projective module that admits a flat $\overline{\partial}$-connection. They applied the appropriate techniques to the case of quantum projective line and its canonical quantum line bundles to derive quantum version of some well known classical results.

To deal with complex structure of some noncommutative spaces, such as quantum projective spaces, the correspondence of complex structure and positivity must be extended to twisted positivity of Hochschild cocyles and twisted cyclic cocycles. Here twist is defined via the modular automorphism of the Haar state on the quantum group $SU_q(n)$. So authors in [19] also defined the notion of twisted positivity and gave an example of twisted positive Hochschild 2-cocycle in the same cohomology class of fundamental class of quantum projective line. The paper [19] by itself has given rise to several questions. Among them, two that took our attention are as following. First, what can we say about higher dimensional quantum projective spaces? The second question is: could we give a classification of all holomorphic vector bundles on $\mathbb{C}P_q^1$? In particular, how to formulate and prove a Grothendieck type theorem for holomorphic vector bundles on $\mathbb{C}P^1$ in our noncommutative setting.

The complex projective spaces $\mathbb{C}P^n$ are among the most important complex manifolds so it is totally natural to work with the noncommutative version of these spaces if we are willing to investigate the noncommutative complex geometry. The quantum group version of the fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, gives the $\mathcal{A}(\mathbb{C}P_q^n)$ as the invariant elements of $\mathcal{A}(S_q^{2n+1})$ under the action of S^1. Here $\mathcal{A}(\mathbb{C}P_q^n)$ denotes the algebra of functions on the quantum space $\mathbb{C}P_q^n$. The canonical quantum line bundles L_N then is defined by characters of $U(1)$. Each L_N is a $\mathcal{A}(\mathbb{C}P_q^n)$-bimodule. A classical result in complex geometry states that the space of holomorphic sections of canonical line bundles is isomorphic to homogeneous polynomial space as a vector space. The homogeneous coordinate ring of these line bundles which is defined by $\mathcal{R} := \bigoplus_{m\geq 0} H^0(\mathbb{C}P^n, \mathcal{O}(m))$ is isomorphic to the ring of polynomials in $n+1$ variables.

To have a noncommutative complex geometry on $A := \mathcal{A}(\mathbb{C}P_q^n)$ in the sense of [19], we need a bigraded differential algebra $\Omega^{(\bullet,\bullet)}(A)$ together with two differentials ∂ and $\overline{\partial}$ with some appropriate properties. The space of forms and differential maps for quantum projective spaces have been worked out in [9, 12] and in more general case of quantum flag manifolds in [17].

This thesis is structured as follows. In Chapter 1, we review basic concepts of quantum groups that will be needed in forthcoming chapters. The notion of noncommutative complex geometry in the sense of [19] is recalled and the main problem that chapters 2 and 3 are devoted to is stated.

In Chapter 2, the holomorphic structures on canonical quantum line bundles on the quantum projective plane are investigated. It is shown that these line bundles admit a flat $\overline{\partial}$-connection and the compatibility of this with bimodule structure of line bundles is also established. This compatibility together with the determination of the space of holomorphic sections, led us to derive the structure of the quantum homogeneous coordinate ring of $\mathbb{C}P_q^2$ as a twisted polynomial algebra in three variables. We also extended our results from polynomial holomorphic sections to continuous and L^2-sections. In addition we prove the existence of a positive twisted Hochschild 4-cocycle that represents the fundamental class of the quantum projective plane. The question of the relation between positivity and complex structure in complex dimension ≥ 2, even in the classical case, is still open.

Chapter 3 is an extension of our results to higher dimensional quantum projective spaces where the quantum homogeneous coordinate ring of a projective space is determined as a space of twisted polynomials. This result has a perfect analogue in the classical case as q approaches 1.

In Chapter 4, we determine the Dolbeault cohomology of $\mathbb{C}P_q^1$ and $\mathbb{C}P_q^2$ as the first step of seeking an analogue of the Riemann-Roch theorem for quantum projective spaces.

In Chapter 5, we prove a quantum version of the Borel-Weil theorem for $SU_q(3)$ and generalize it to the case of $SU_q(n)$. Classically the Borel-Weil theorem gives a concrete geometric realization of irreducible representations of a compact Lie group as the space of holomorphic sections of the line bundles on the associated flag manifold.

In the last chapter, we investigate the noncommutative complex structure of finite spaces and determine the space holomorphic functions of these spaces. We also prove that there is no nontrivial holomorphic vector bundle on the space of two points.

Chapter 1

A review on quantum groups and noncommutative complex geometry

1.1 Quantum groups

In this section we review some basic notions of quantum groups following [23].

1.1.1 Hopf algebras

In this thesis, by an algebra we mean an associative algebra over $\mathbb{C}$ with unit. More precisely, an algebra is a vector space $\mathcal{A}$ over $\mathbb{C}$ with two linear maps $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, called the *product* and $\eta : \mathbb{C} \to \mathcal{A}$ called the *unit* such that

$$m \circ (m \otimes id) = m \circ (id \otimes m), \quad m \circ (\eta \otimes id) = id = m \circ (id \otimes \eta). \tag{1.1}$$

On the elements of $\mathcal{A}$, equations (1.1) can be written as:

$$a(bc) = (ab)c, \quad a1 = 1a = a, \quad \forall\, a, b, c \in \mathcal{A}.$$

A *coalgebra* is a vector space over $\mathbb{C}$ with two maps $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and $\varepsilon : \mathcal{A} \to \mathbb{C}$ such that

$$\begin{aligned}(\Delta \otimes id) \circ \Delta &= (id \otimes \Delta) \circ \Delta,\\ (\varepsilon \otimes id) \circ \Delta &= id = (id \otimes \varepsilon) \circ \Delta.\end{aligned} \tag{1.2}$$

The maps Δ and ε are called *coproduct* and *counit* respectively. A *bialgebra* is a tuple $(\mathcal{A}, m, \eta, \Delta, \varepsilon)$ such that $\mathcal{A}$ is simultaneously an algebra and a coalgebra and also ε and Δ are morphisms of algebras.

Definition 1.1.1. *A Hopf algebra is a bialgebra* $(\mathcal{A}, m, \eta, \Delta, \varepsilon)$ *with a linear map* $S : \mathcal{A} \to \mathcal{A}$*, called the* antipode*, such that*

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta. \tag{1.3}$$

In Sweedler's notation (i.e. $\Delta a = \sum a_{(1)} \otimes a_{(2)}$), formula (1.3) can be written as

$$\sum S(a_{(1)})a_{(2)} = \varepsilon(a)1 = \sum a_{(1)}S(a_{(2)}).$$

The map S has the following properties:

$$S(1) = 1, \quad S(ab) = S(b)S(a), \quad \forall a, b \in \mathcal{A},$$
$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta, \quad \varepsilon \circ S = \varepsilon.$$

Here $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the flip $\tau(a \otimes b) = b \otimes a$.

1.1.2 Dual pairing of Hopf algebras

Definition 1.1.2. *A dual pairing of Hopf algebras* $\mathcal{U}$ *and* $\mathcal{A}$ *is a bilinear pairing* $\langle\, , \rangle : \mathcal{U} \otimes \mathcal{A} \to \mathbb{C}$ *such that*

$$\langle \Delta(X), a \otimes b \rangle = \langle X, ab \rangle, \quad \langle XY, a \rangle = \langle X \otimes Y, \Delta(a) \rangle \tag{1.4}$$
$$\langle X, 1 \rangle = \varepsilon(X), \quad \langle 1, a \rangle = \varepsilon(a) \tag{1.5}$$

for all $X, Y \in \mathcal{U}$ *and for all* $a, b \in \mathcal{A}$*. The pairing is called nondegenerate if* $\langle X, a \rangle = 0$ *for all* $a \in \mathcal{A}$ *implies* $X = 0$ *and if* $\langle X, a \rangle = 0$ *for all* $X \in \mathcal{U}$ *implies* $a = 0$*.*

For a pairing of Hopf algebras $\langle\, , \rangle$, we have

$$\langle S(X), a \rangle = \langle X, S(a) \rangle \quad \forall X \in \mathcal{U}, \forall a \in \mathcal{A}.$$

Definition 1.1.3. *A Hopf* $*$*-algebra is a Hopf algebra* $\mathcal{A}$ *over* $\mathbb{C}$ *with an involution* $*$ *on* $\mathcal{A}$ *such that* $(ab)^* = b^* a^*$ *and* $\Delta(a^*) = \Delta(a)^*$*.*

Note that in any Hopf $*$-algebra we have $1^* = 1$, $\varepsilon(a^*) = \overline{\varepsilon(a)}$ and $S^{-1} = * \circ S \circ *$.

Definition 1.1.4. *A dual pairing between Hopf $*$-algebras $\mathcal{U}$ and $\mathcal{A}$ is a dual Hopf pairing such that*

$$\langle X^*, a\rangle = \overline{\langle X, S(a^*)\rangle}, \quad \langle X, a^*\rangle = \overline{\langle S(X^*), a\rangle}.$$

1.1.2.1 Examples of Hopf algebras

- **The universal enveloping algebra $U(\mathfrak{g})$.**

  For a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the universal enveloping algebra $U(\mathfrak{g})$ is defined to be the quotient of the tensor algebra $T(\mathfrak{g})$ by the two sided ideal I generated by elements $x \otimes y - y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$. It has the following universal property:

  Given a linear map $\varphi : \mathfrak{g} \to A$ to an algebra A satisfying

  $$\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x), \quad \forall x, y \in \mathfrak{g},$$

  there is a unique algebra homomorphism $\Phi : U(\mathfrak{g}) \to A$ such that $\Phi(x) = \varphi(x)$, $x \in \mathfrak{g}$.

  There exists a unique Hopf algebra structure on $U(\mathfrak{g})$ satisfying

  $$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -x, \quad \forall x \in \mathfrak{g}.$$

  In some cases, for example when $\mathcal{A}(G)$ is the Hopf algebra of a matrix Lie group G with Lie algebra $\mathfrak{g}$, the elements of $U(\mathfrak{g})$ act as left invariant differential operators on G. For $X_i \in \mathfrak{g}$, $i = 1, 2, \cdots n$, the element $X = X_1 X_2 \cdots X_n$ acts on $f \in C^\infty(G)$ as

  $$X f(g) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_i=0} f(g\, e^{t_1 X_1} \cdots e^{t_n X_n}).$$

  The pairing $\langle\, ,\rangle$ between $U(\mathfrak{g})$ and $\mathcal{A}(G)$ given by $\langle X, f\rangle := X f(e)$ is a nondegenerate dual pairing of Hopf algebras. Here e is the identity element of G.

- **The group algebra $\mathbb{C}G$.**

  Let G be a discrete group. The group algebra $\mathbb{C}G$ as a vector space has a basis given by G. The product of G extends linearly to this space and the unit element is the unit of G. There is a unique Hopf algebra structure on $\mathbb{C}G$ such that

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad \forall g \in G.$$

- **The Drinfield-Jimbo algebras.**

In this example $\mathfrak{g}$ is a complex semisimple Lie algebra. The following theorem by Serre, characterizes $U(\mathfrak{g})$ in terms of generators and relations.

Theorem 1.1.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan matrix $A = (a_{ij})$ and simple roots $\alpha_1, \alpha_2, \cdots, \alpha_l$ with $l =$rank $\mathfrak{g}$, then $E_i = E_{\alpha_i}$, $F_i = E_{-\alpha_i}$ and $H_i = [E_i, F_i]$, $i = 1, 2, \cdots, l$, can be chosen in such a way that the universal enveloping algebra $U(\mathfrak{g})$ is generated by E_i, F_i, H_i, subject to the relations*

$$[H_i, H_j] = 0, \quad [E_i, F_i] = H_i, \quad [E_i, F_j] = 0, \quad i \neq j,$$

$$[H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j,$$
$$(ad\, E_i)^{1-a_{ij}} E_j = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \quad i \neq j,$$
$$(ad\, F_i)^{1-a_{ij}} F_j = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^{1-a_{ij}-k} F_j F_i^k = 0, \quad i \neq j.$$

We recall that for a complex semisimple Lie algebra, the Cartan matrix $A = (a_{ij})$ is defined by $a_{ij} := 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle$. Here α_i's are simple roots and $\langle , \rangle$ is the Killing form. This matrix which is a square integer matrix has the following properties:

- $a_{ij} \in \{-3, -2, -1, 0, 2\}$
- $a_{ii} = 2$.
- $a_{ij} \leq 0$ if $i \neq j$
- $a_{ij} = 0$ iff $a_{ji} = 0$
- There exists a diagonal matrix D such that DAD^{-1} gives a symmetric and positive definite quadratic form.

Note. The elements E_i, F_i, H_i produce a PBW basis for $U(\mathfrak{g})$.

Let q be a nonzero complex number and let $q_i = q^{d_i}$ where $d_i := (\alpha_i, \alpha_i)/2$ such that $q_i^2 \neq 1$ for $i = 1, 2, \cdots, l$. Let $U_q(\mathfrak{g})$ be the associative unital algebra with generators E_i, F_i, K_i, K_i^{-1} for $1 \leq i \leq l$ subject to the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i,$$

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \quad K_iF_j = q_i^{-a_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} E_i^{1-a_{ij}-k}E_jE_i^k = 0, \quad i \neq j,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} F_i^{1-a_{ij}-k}F_jF_i^k = 0, \quad i \neq j.$$

where,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Proposition 1.1.1. *There is a unique Hopf algebra structure on the algebra $U_q(\mathfrak{g})$ with product Δ, counit ε and antipode S such that*

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i.$$

Definition 1.1.5. *The Hopf algebra of the proposition (1.1.1) is called the Drinfeld-Jimbo algebra corresponding to the Lie algebra $\mathfrak{g}$ and the complex number q.*

There exists another Hopf algebra associated to $\mathfrak{g}$, denoted by $\check{U}_q(\mathfrak{g})$ which we will use in forthcoming chapters. The algebra $\check{U}_q(\mathfrak{g})$ is the algebra generated by E_i, F_i, K_i, K_i^{-1}

for $1 \le i \le l$ subject to the relations

$$
\begin{aligned}
&K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i,\\
&K_iE_j = q_i^{a_{ij}/2}E_jK_i, \quad K_iF_j = q_i^{-a_{ij}/2}F_jK_i,\\
&E_iF_j - F_jE_i = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}},\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} E_i^{1-a_{ij}-k}E_jE_i^k = 0, \quad i \neq j,\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} F_i^{1-a_{ij}-k}F_jF_i^k = 0, \quad i \neq j.
\end{aligned}
$$

The Hopf algebra structure on the algebra $\check{U}_q(\mathfrak{g})$ is given by

$$
\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},\\
&\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i,\\
&\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,\\
&S(K_i) = K_i^{-1}, \quad S(E_i) = -q_iE_i, \quad S(F_i) = -q_i^{-1}F_i.
\end{aligned}
$$

One advantage of this Hopf algebra is the fact that comultiplications of generators E_i and F_i are given by the same formula. There is a Hopf algebra homomorphism $\varphi : U_q(\mathfrak{g}) \to \check{U}_q(\mathfrak{g})$ given on generators by

$$
\varphi(E_i) = E_iK_i, \qquad \varphi(F_i) = K_i^{-1}F_i, \qquad \varphi(K_i) = K_i^2.
$$

The map φ is injective so $U_q(\mathfrak{g})$ can be considered as a Hopf subalgebra of $\check{U}_q(\mathfrak{g})$. Since these two Hopf algebra have different number of one dimensional representations, they are not isomorphic.

In this thesis, we are interested in $U_q(\mathfrak{sl}_n)$, its compact real form $U_q(\mathfrak{su}_n)$ and its irreducible representations. The generators and relations are given in forthcoming chapters.

1.1.3 The Haar functional

A linear functional h on $\mathcal{A}$ is called *invariant* if

$$(id \otimes h) \circ \Delta = h = (h \otimes id) \circ \Delta. \tag{1.6}$$

Definition 1.1.6. *A linear functional h on $\mathcal{A}$ which is invariant in the sense of (1.6) is called a Haar functional of $\mathcal{A}$.*

We will see in our case of interest i.e. $SU_q(n)$ such a map h exists and is unique with properties $h(a^*a) \geq 0$ and $h(1) = 1$. Because of these properties h is also called the *Haar state* of $SU_q(n)$.

Definition 1.1.7. *[35] A Hopf $*$-algebra $\mathcal{A}$ is called a compact quantum group (CQG), if there exists a linear functional h on $\mathcal{A}$ such that*

$$(id \otimes h) \circ \Delta(a) = h(a)1, \quad \forall a \in \mathcal{A}.$$

1.2 Noncommutative complex geometry

In this section we review the general setup of a noncommutative complex structure on a given $*$-algebra as introduced in [19].

Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$. A *differential $*$-calculus* for $\mathcal{A}$ is a pair $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$, where $\Omega^\bullet(\mathcal{A}) = \bigoplus_{n\geq 0} \Omega^n(\mathcal{A})$ is a graded differential $*$-algebra with $\Omega^0(\mathcal{A}) = \mathcal{A}$. The differential map $\mathrm{d} : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet+1}(\mathcal{A})$ satisfies the graded Leibniz rule,

$$\mathrm{d}(\omega_1\omega_2) = (\mathrm{d}\omega_1)\omega_2 + (-1)^{deg(\omega_1)}\omega_1(\mathrm{d}\omega_2)$$

and $\mathrm{d}^2 = 0$. The differential also commutes with the $*$-structure: $\mathrm{d}(a^*) = (\mathrm{d}a)^*$.

Definition 1.2.1. *A complex structure on an algebra $\mathcal{A}$, equipped with a differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$, is a bigraded differential $*$-algebra $\Omega^{(\bullet,\bullet)}(\mathcal{A})$ and two differential maps*

$\partial : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p+1,q)}(\mathcal{A})$ and $\overline{\partial} : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p,q+1)}(\mathcal{A})$ such that:

$$\Omega^n(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{(p,q)}(\mathcal{A})\,, \quad \partial a^* = (\overline{\partial} a)^*\,, \quad \mathrm{d} = \partial + \overline{\partial}. \tag{1.7}$$

Also, the involution $*$ maps $\Omega^{(p,q)}(\mathcal{A})$ to $\Omega^{(q,p)}(\mathcal{A})$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^{(0,2)} \arrow[u] \arrow[r] & \Omega^{(1,2)} \arrow[u] \arrow[r] & \Omega^{(2,2)} \arrow[u] \arrow[r] & \cdots \\
\Omega^{(0,1)} \arrow[u] \arrow[r] & \Omega^{(1,1)} \arrow[u] \arrow[r] & \Omega^{(1,2)} \arrow[u] \arrow[r] & \cdots \\
\Omega^{(0,0)} \arrow[u, "\overline{\partial}"'] \arrow[r, "\partial"] & \Omega^{(1,0)} \arrow[u] \arrow[r] \arrow[ul, dotted, "*"'] & \Omega^{(2,0)} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

We will use the simple notation $(\mathcal{A}, \overline{\partial})$ for a complex structure on $\mathcal{A}$.

Definition 1.2.2. *Let $(\mathcal{A}, \overline{\partial})$ be an algebra with a complex structure. The space of holomorphic elements of $\mathcal{A}$ is defined as*

$$\mathcal{O}(\mathcal{A}) := Ker\{\overline{\partial} : \mathcal{A} \to \Omega^{(0,1)}(\mathcal{A})\}.$$

By the Leibniz rule one can see that $\mathcal{O}(\mathcal{A})$ is an algebra over $\mathbb{C}$.

1.2.1 Holomorphic connections

Suppose we are given a differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$. We recall that a connection on a left $\mathcal{A}$-module $\mathcal{E}$ for the differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$ is a linear map $\nabla : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ with left Leibniz property:

$$\nabla(a\xi) = a\nabla\xi + \mathrm{d}a \otimes_{\mathcal{A}} \xi, \quad \forall a \in \mathcal{A}, \forall \xi \in \mathcal{E}. \tag{1.8}$$

By the graded Leibniz rule, i.e.

$$\nabla(\omega\xi) = (-1)^n \omega\nabla\xi + \mathrm{d}\omega \otimes_{\mathcal{A}} \xi, \quad \forall\omega \in \Omega^n(\mathcal{A}), \forall\xi \in \Omega(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}, \tag{1.9}$$

this connection can be uniquely extended to a map, which will be denoted again by ∇, $\nabla : \Omega^\bullet(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^{\bullet+1}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$.

The curvature of such a connection is defined by $F_\nabla = \nabla \circ \nabla$. One can show that, F_∇ is an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E})$.

Definition 1.2.3. *Suppose $(\mathcal{A}, \overline{\partial})$ is an algebra with a complex structure. A holomorphic structure on a left $\mathcal{A}$-module $\mathcal{E}$ with respect to this complex structure is given by a linear map $\nabla^{\overline{\partial}} : \mathcal{E} \to \Omega^{(0,1)} \otimes_{\mathcal{A}} \mathcal{E}$ such that*

$$\nabla^{\overline{\partial}}(a\xi) = a\nabla^{\overline{\partial}}\xi + \overline{\partial}a \otimes_{\mathcal{A}} \xi, \quad \forall a \in \mathcal{A}, \forall\xi \in \mathcal{E}, \tag{1.10}$$

and such that $F_{\nabla^{\overline{\partial}}} = (\nabla^{\overline{\partial}})^2 = 0$.

Such a connection will be called a flat $\overline{\partial}$-connection. In the case which $\mathcal{E}$ is a finitely generated $\mathcal{A}$-module, $(\mathcal{E}, \nabla^{\overline{\partial}})$ will be called a holomorphic vector bundle.

The motivation for this definition comes from the classical case.

Theorem 1.2.1. *[16]. Let E be a complex vector bundle on a complex manifold X. A holomorphic structure on E is uniquely determined by a $\mathbb{C}$-linear operator $\overline{\partial}_E : \mathcal{A}^0(E) \to \mathcal{A}^{(0,1)}(E)$ satisfying the Leibniz rule and the integrability condition $\overline{\partial}_E^2 = 0$.*

In fact there is a one to one correspondence between holomorphic structures on a complex vector bundle E and flat $\overline{\partial}$-connections on E up the gauge equivalence. Two connections ∇_1 and ∇_2 are said to be gauge equivalent if there exists an invertible element $g \in End_{\mathcal{A}}(E)$ such that $\nabla_1 = g^{-1}\nabla_2 g$.

Associated to a flat $\overline{\partial}$-connection, there exists a complex of vector spaces

$$0 \to \mathcal{E} \to \Omega^{(0,1)} \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^{(0,2)} \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \tag{1.11}$$

Here $\nabla^{\overline{\partial}}$ is extended to $\Omega^{(0,q)} \otimes_{\mathcal{A}} \mathcal{E}$ by the graded Leibniz rule. The zeroth cohomology group of this complex is called the space of holomorphic sections of $\mathcal{E}$ and will be denoted by $H^0(\mathcal{E}, \nabla^{\overline{\partial}})$.

1.2.2 Holomorphic structures on bimodules

Definition 1.2.4. *Let $\mathcal{A}$ be an algebra with a differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$. A bimodule connection on an $\mathcal{A}$-bimodule $\mathcal{E}$ is given by a connection ∇ which satisfies a left Leibniz rule as in formula (1.8) and a right σ-twisted Leibniz property with respect to a bimodule isomorphism $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$. i.e.*

$$\nabla(\xi a) = (\nabla\xi)a + \sigma(\xi \otimes \mathrm{d}a)\,, \quad \forall \xi \in \mathcal{E}, \forall a \in \mathcal{A}. \tag{1.12}$$

The tensor product connection of two bimodule connections ∇_1 and ∇_2 on two $\mathcal{A}$-bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$ with respect to the bimodule isomorphisms σ_1 and σ_2 is a map $\nabla : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ defined by

$$\nabla := \nabla_1 \otimes 1 + (\sigma_1 \otimes 1)(1 \otimes \nabla_2).$$

It can be checked that, ∇ has the right σ-twisted property with $\sigma : \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \mathcal{E}_1 \otimes \mathcal{E}_2$ given by $\sigma = (\sigma_1 \otimes 1) \circ (1 \otimes \sigma_2)$.

Definition 1.2.5. *A holomorphic structure on a $\mathcal{A}$-bimodule $\mathcal{E}$ is a given by a flat $\overline{\partial}$-bimodule connection.*

It is worth mentioning that the tensor product of two flat connection is not a flat connection in general, even in the case of finite dimensional vector spaces [19]. Therefore, in general we cannot expect that the tensor product of two holomorphic structures $(\mathcal{E}_1, \nabla_1^{\overline{\partial}})$ and $(\mathcal{E}_2, \nabla_2^{\overline{\partial}})$ gives a holomorphic structure on $\mathcal{E}_1 \otimes \mathcal{E}_2$.

Let us recall the results of [19] that in chapters 2 and 3 of this thesis we generalize them. In [19], beside the general setup of noncommutative complex (NCC) structure on an algebra and its holomorphic vector bundles, the authors considered the case of $\mathbb{C}P^1_q$ as the quotient space of $S^3_q = SU_q(2)$ by the action of $U(1)$. The canonical quantum line bundles L_N are defined by the characters of $U(1)$. For the standard complex structure on $\mathbb{C}P_q^1$ which is induced by the left invariant first order differential calculus (in the sense of Woronowicz) on $SU_q(2)$, they showed that the space of holomorphic sections of line bundles L_N is described as follows. (cf. [19] theorems 4.4, 4.5 and the proposition 5.2)

Theorem 1.2.2. *[19] Let N be a positive integer. Then*

- $H^0(L_N, \overline{\nabla}) = 0$,
- $H^0(L_{-N}, \overline{\nabla}) \simeq \mathbb{C}^{N+1}$.

These results continue to hold when considering continuous sections $\Gamma(L_N)$ as modules over C^*-algebra $C(\mathbb{C}P_q^1)$.

Theorem 1.2.3. *[19] The space $R = \bigoplus_{N\geq 0} H^0(L_{-N}, \overline{\nabla})$ carries a ring structure and is isomorphic to the quantum plane:*

$$R \simeq \mathbb{C}\langle a, c\rangle/(ac - qca).$$

The following proposition shows the existence of a twisted positive Hochschild cocycle which is cohomologous to the fundamental twisted cyclic cocycle defined via smooth structure of the space $\mathbb{C}P_q^1$.

Proposition 1.2.1. *[19] The cochain $\varphi \in C^2(\mathcal{A}(\mathbb{C}P_q^1))$ defined by*

$$\varphi(a_0, a_1, a_2) = \int_h a_0 \partial a_1 \overline{\partial} a_2$$

is a twisted Hochschild 2-cocycle on $\mathcal{A}(\mathbb{C}P_q^1)$, that is to say $b_\sigma\varphi = 0$ and $\lambda_\sigma^3\varphi = \varphi$; it is also positive, with positivity expressed as:

$$\int_h a_0 \partial a_1 (a_0 \partial a_1)^* \geq 0$$

for all $a_0, a_1 \in \mathcal{A}(\mathbb{C}P_q^1)$.

Chapter 2

Noncommutative complex structure of $\mathbb{C}P_q^2$

2.1 The quantum projective plane $\mathbb{C}P_q^2$

In this section, we recall the definition of the quantum enveloping algebra $U_q(\mathfrak{su}(3))$, the quantum group $\mathcal{A}(SU_q(3))$ and the pairing between them. We also recall the definition of the quantum projective plane $\mathbb{C}P_q^2$ and its canonical quantum line bundles [12].

2.1.1 The quantum enveloping algebra $U_q(\mathfrak{su}(3))$

Let $0 < q < 1$. We use the following notation

$$[a,b]_q = ab - q^{-1}ba,\ [z] = \frac{q^z - q^{-z}}{q - q^{-1}},\ \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}\,,$$
$$[j,k,l]! = q^{-(jk+kl+lj)}\frac{[j+k+l]!}{[j]![k]![l]!}\,.$$

The Hopf $*$-algebra $U_q(\mathfrak{su}(3))$ as a $*$-algebra is generated by K_i, K_i^{-1}, E_i, F_i, $i = 1, 2$ with $K_i^* = K_i$, $E_i^* = F_i$ subject to the relations

$$\begin{aligned}&[K_i, K_j] = 0, \quad K_iE_i = qE_iK_i, \quad [E_i, F_i] = (q - q^{-1})^{-1}(K_i^2 - K_i^{-2}),\\ &K_iE_j = q^{-1/2}E_jK_i, \quad [E_i, F_j] = 0, \quad i \neq j\,,\end{aligned}$$

and

$$E_i^2E_j + E_jE_i^2 = (q + q^{-1})E_iE_jE_i \quad i \neq j\,.$$

Its coproduct, counit and antipode are defined on generators as

$$\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i,$$
$$\Delta(K_i) = K_i \otimes K_i, \quad \epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0,$$
$$S(K_i) = K_i^{-1}, \quad S(E_i) = -qE_i, \quad S(F_i) = -q^{-1}F_i.$$

Let $V(n_1, n_2)$ be the irreducible finite dimensional $*$-representation of $U_q(\mathfrak{su}(3))$ [23] with the orthonormal basis $|n_1, n_2, j_1, j_2, m\rangle$, where indices are restricted by

$$j_i = 0, 1, 2, ..., n_i, \quad \frac{1}{2}(j_1 + j_2) - |m| \in \mathbb{N}. \tag{2.1}$$

The generators of $U_q(\mathfrak{su}(3))$ act on this basis as

$$\begin{aligned}
K_1|n_1, n_2, j_1, j_2, m\rangle &= q^m|n_1, n_2, j_1, j_2, m\rangle, \\
K_2|n_1, n_2, j_1, j_2, m\rangle &= q^{\frac{3}{4}(j_1-j_2)+\frac{1}{2}(n_2-n_1-m)}|n_1, n_2, j_1, j_2, m\rangle, \\
E_1|n_1, n_2, j_1, j_2, m\rangle &= \sqrt{[\frac{1}{2}(j_1 + j_2) - m][\frac{1}{2}(j_1 + j_2) + m + 1]} \\
&\quad |n_1, n_2, j_1, j_2, m + 1\rangle, \\
E_2|n_1, n_2, j_1, j_2, m\rangle &= \sqrt{[\frac{1}{2}(j_1 + j_2) - m + 1]}A_{j_1,j_2}|n_1, n_2, j_1 + 1, j_2, m - \frac{1}{2}\rangle \\
&+\sqrt{[\frac{1}{2}(j_1 + j_2) + m]}B_{j_1,j_2}|n_1, n_2, j_1, j_2 - 1, m - \frac{1}{2}\rangle,
\end{aligned} \tag{2.2}$$

where

$$A_{j_1,j_2} := \sqrt{\frac{[n_1 - j_1][n_2 + j_1 + 2][j_1 + 1]}{[j_1 + j_2 + 1][j_1 + j_2 + 2]}}, \tag{2.3}$$

$$B_{j_1,j_2} := \begin{cases} \sqrt{\frac{[n_1+j_2+1][n_2-j_2+1][j_2]}{[j_1+j_2][j_1+j_2+1]}} & \text{if } j_1 + j_2 \neq 0, \\ 1 & \text{if } j_1 + j_2 = 0. \end{cases} \tag{2.4}$$

2.1.2 The quantum group $\mathcal{A}(SU_q(3))$

As a $*$-algebra, $\mathcal{A}(SU_q(3))$ is generated by u^i_j, $i,j = 1,2,3$, satisfying the following commutation relations

$$\begin{aligned} &u^i_k u^j_k = q u^j_k u^i_k, && u^k_i u^k_j = q u^k_j u^k_i \quad \forall\, i<j,\\ &[u^i_l, u^j_k] = 0, && [u^i_k, u^j_l] = (q-q^{-1}) u^i_l u^j_k \quad \forall\, i<j,\, k<l, \end{aligned}$$

and a cubic relation

$$\sum_{\sigma\in S_3} (-q)^{l(\sigma)} u^1_{\sigma(1)} u^2_{\sigma(2)} u^3_{\sigma(3)} = 1.$$

In the last equation, sum is taken over all permutations σ on three letters and $l(\sigma)$ is the length of σ. The involution $*$ is defined as

$$(u^i_j)^* := (-q)^{j-i} (u^{k_1}_{l_1} u^{k_2}_{l_2} - q u^{k_1}_{l_2} u^{k_2}_{l_1}), \tag{2.5}$$

where as an ordered set, $\{k_1, k_2\} = \{1,2,3\} \setminus \{i\}$ and $\{l_1, l_2\} = \{1,2,3\} \setminus \{j\}$. The Hopf algebra structure is given by

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j, \quad \epsilon(u^i_j) = \delta^i_j, \quad S(u^i_j) = (u^j_i)^*.$$

There exists a non-degenerate pairing between Hopf algebras $\mathcal{A}(SU_q(3))$ and $U_q(\mathfrak{su}(3))$, which allows us to define a left and a right action of $U_q(\mathfrak{su}(3))$ on $\mathcal{A}(SU_q(3))$. These actions make $\mathcal{A}(SU_q(3))$ an $U_q(\mathfrak{su}(3))$-bimodule $*$-algebra.
The actions are defined as

$$h \triangleright a = a_{(1)} \langle h, a_{(2)} \rangle, \quad a \triangleleft h = \langle h, a_{(1)} \rangle a_{(2)}.$$

Here we used Sweedler's notation. Left and right actions on generators are given by (see

[9])

$$K_i \triangleright u^j_k = q^{\frac{1}{2}(\delta_{i+1,k}-\delta_{i,k})} u^j_k, \quad E_i \triangleright u^j_k = \delta_{i,k} u^j_{i+1}, \quad F_i \triangleright u^j_k = \delta_{i+1,k} u^j_i,$$
$$u^j_k \triangleleft K_i = q^{\frac{1}{2}(\delta_{i+1,j}-\delta_{i,j})} u^j_k, \quad u^j_k \triangleleft E_i = \delta_{i+1,j} u^i_k, \quad u^j_k \triangleleft F_i = \delta_{i,j} u^{i+1}_k. \tag{2.6}$$

A linear basis of $\mathcal{A}(SU_q(3))$ corresponding to the Peter-Weyl decomposition is given by (see [9, 12])

$$t(n_1, n_2)^{l_1,l_2,k}_{j_1,j_2,m} := X^{n_1,n_2}_{j_1,j_2,m} \triangleright \{(u^1_1)^*\}^{n_1} (u^3_3)^{n_2} \triangleleft (X^{n_1,n_2}_{l_1,l_2,k})^*. \tag{2.7}$$

where $X^{n_1,n_2}_{j_1,j_2,m}$ is defined as

$$X^{n_1,n_2}_{j_1,j_2,m} := N^{n_1,n_2}_{j_1,j_2,m}$$
$$\sum_{k=0}^{n_1-j_1} \frac{q^{-k(j_1+j_2+k+1)}}{[j_1+j_2+k+1]!} \begin{bmatrix} n_1 - j_1 \\ k \end{bmatrix} F_1^{1/2(j_1+j_2)-m+k} [F_2, F_1]_q^{n_1-j_1-k} F_2^{j_2+k}.$$

The coefficients $N^{n_1,n_2}_{j_1,j_2,m}$ are defined by

$$N^{n_1,n_2}_{j_1,j_2,m} = \sqrt{[j_1+j_2+1]} \sqrt{\frac{[\frac{j_1+j_2}{2}+m]![n_2-j_2]![j_1]![n_1+j_2+1]![n_2+j_1+1]!}{[\frac{j_1+j_2}{2}-m]![n_1-j_1]![j_2]![n_1]![n_2]![n_1+n_2+1]!}}.$$

The Peter-Weyl isomorphism $Q : \mathcal{A}(SU_q(3)) \to \bigoplus_{(n_1,n_2)} V(n_1,n_2) \otimes V(n_1,n_2)$ has the following property for all $h \in U_q(\mathfrak{su}(3))$:

$$Q(h \triangleright t(n_1,n_2)^{l_1,l_2,k}_{j_1,j_2,m}) = h|n_1,n_2,j_1,j_2,m\rangle \otimes |n_1,n_2,l_1,l_2,k\rangle,$$
$$Q(t(n_1,n_2)^{l_1,l_2,k}_{j_1,j_2,m} \triangleleft h) = |n_1,n_2,j_1,j_2,m\rangle \otimes \theta(h)|n_1,n_2,l_1,l_2,k\rangle, \tag{2.8}$$

where $\theta : U_q(\mathfrak{su}(3)) \to U_q(\mathfrak{su}(3))^{op}$ is the Hopf $*$-algebra isomorphism which is defined on generators as

$$\theta(K_i) = K_i,\ \theta(E_i) = F_i,\ \theta(F_i) = E_i,$$

and satisfying $\theta^2 = id$.

We define the quantum projective plane $\mathbb{C}P^2_q$ as a quotient of the 5-dimensional quantum sphere ([12]). By definition

$$\mathcal{A}(S_q^5) := \{a \in \mathcal{A}(SU_q(3)) |\, a \triangleleft h = \epsilon(h)a \,,\, \forall h \in U_q(\mathfrak{su}(2))\}.$$

As a $*$-algebra, $\mathcal{A}(S_q^5)$ is generated by elements $z_j = u_j^3$, $j = 1, 2, 3$ of $\mathcal{A}(SU_q(3))$. Abstractly, this algebra is the algebra with generators z_i, z_i^* $i = 1, 2, 3$ and subject to the following relations

$$\begin{aligned} &z_i z_j = q z_j z_i \quad \forall\, i < j, && z_i^* z_j = q z_j z_i^*, \quad \forall\, i \neq j, \\ &[z_1^*, z_1] = 0, && [z_2^*, z_2] = (1 - q^2) z_1 z_1^*, \\ &[z_3, z_3] = (1 - q^2)(z_1 z_1^* + z_2 z_2^*), && z_1 z_1^* + z_2 z_2^* + z_3 z_3^* = 1. \end{aligned}$$

Now we define the algebra $\mathcal{A}(\mathbb{C}P_q^2)$ of the quantum projective plane as a $*$-subalgebra of $\mathcal{A}(S_q^5)$.

$$\mathcal{A}(\mathbb{C}P_q^2) := \{a \in \mathcal{A}(S_q^5) |\, a \triangleleft K_1 K_2^2 = a\}.$$

One can show that [12], $\mathcal{A}(S_q^5) \simeq \bigoplus_{(n_1,n_2)\in\mathbb{N}^2} V(n_1, n_2)$ with the basis $t(n_1, n_2)^{0}_{\underline{j}}$, where n_1 and n_2 are non-negative integers. Also $\mathcal{A}(\mathbb{C}P_q^2) \simeq \bigoplus_{n\in\mathbb{N}} V(n, n)$ with the basis $t(n, n)^{0}_{\underline{j}}$. Here we have used the multi index notation $\underline{j} = j_1, j_2, m$ and indices j_1, j_2, m are restricted by (2.1).

For any integer N, we define the space of the canonical quantum line bundle L_N on $\mathbb{C}P_q^2$ by

$$L_N := \{a \in \mathcal{A}(S_q^5) : a \triangleleft K_1 K_2^2 = q^N a\}.$$

These spaces are $\mathcal{A}(\mathbb{C}P_q^2)$-bimodules. One can see that [12],

$$L_N = \bigoplus_{n\in\mathbb{N}} V(n, n+N) \quad \text{if } N \geq 0, \text{ and } L_N = \bigoplus_{n\in\mathbb{N}} V(n-N, n) \quad \text{if } N < 0.$$

The basis elements are given by $t(n, n+N)^{0}_{\underline{j}}$ for $N \geq 0$ and $t(n-N, n)^{0}_{\underline{j}}$ for $N < 0$.

2.2 The complex structure of $\mathbb{C}P_q^2$

There is a complex structure on $\mathbb{C}P_q^2$ defined in [9, 12]. For future use, we give an explicit description of the spaces $\Omega^{(0,0)}$, $\Omega^{(0,1)}$ and $\Omega^{(0,2)}$:

$$\Omega^{(0,0)} = L_0 = \mathcal{A}(\mathbb{C}P_q^2), \quad \Omega^{(0,2)} = L_3,$$

and as a subspace of $\mathcal{A}(SU_q(3))^2$, $\Omega^{(0,1)}$ contains all pairs (v_+, v_-) such that the following conditions hold

$$\begin{aligned} &(v_+, v_-) \triangleleft K_1 K_2^2 = q^{\frac{3}{2}}(v_+, v_-), && (v_+, v_-) \triangleleft K_1 = (q^{\frac{1}{2}} v_+, q^{-\frac{1}{2}} v_-), \\ &(v_+, v_-) \triangleleft F_1 = (0, v_+), && (v_+, v_-) \triangleleft E_1 = (v_-, 0). \end{aligned} \tag{2.9}$$

The complex structure on $\mathbb{C}P_q^2$ is given by the maps $\partial : \mathcal{A}(\mathbb{C}P_q^2) \to \Omega^{(1,0)}(\mathbb{C}P_q^2)$ and $\overline{\partial} : \mathcal{A}(\mathbb{C}P_q^2) \to \Omega^{(0,1)}(\mathbb{C}P_q^2)$, which (up to multiplicative constants) are $\partial a = (a \triangleleft E_2, a \triangleleft E_2 E_1)^t$, $\overline{\partial} a = (a \triangleleft F_2 F_1, a \triangleleft F_2)^t$.

In this section we identify the space of holomorphic functions on $\mathbb{C}P_q^2$ and holomorphic sections of L_N.

2.2.1 Holomorphic functions

Proposition 2.2.1. *There are no non-trivial holomorphic polynomials on $\mathbb{C}P_q^2$.*

Proof. Let $a = \sum_{n,\underline{j}} \lambda_{n,\underline{j}} t(n,n)^{0}_{\underline{j}}$. Then $\overline{\partial} a = 0$ implies that $a \triangleleft F_2 = 0$ and $a \triangleleft F_2 F_1 = 0$. A simple computation shows that $a \triangleleft F_2 = \sum \lambda_{n,\underline{j}} \gamma_n t(n,n)^{1,0,-\frac{1}{2}}_{\underline{j}}$, where $\gamma_n = A_{0,0} = \sqrt{\frac{[n][n+2]}{[2]}}$. This can be obtained by (2.8), (2.2) and (2.3) because
$E_2|n,n,0,0,0\rangle = A_{0,0}|n,n,1,0,-\frac{1}{2}\rangle$.
Since $\gamma_n = 0$ iff $n = 0$, all coefficients need to be zero except $c_{0,0}$. Note that the action of F_1 does not put more restrictions on the coefficients. This demonstrates that

$$Ker\{\overline{\partial} : \mathcal{A}(\mathbb{C}P_q^2) \to \Omega^{(0,1)}(\mathbb{C}P_q^2)\} = \langle t(0,0)^{0}_{\underline{0}} \rangle = \mathbb{C}.$$

$\square$

This preposition, has already been proved in [12] as a result of a Hodge decomposition.

2.2.2 Canonical line bundles

Like [9], we define the connection ∇_N on L_N by $\nabla_N := q^{-N}\Psi_N^\dagger \, \mathrm{d}\Psi_N$, where Ψ_N is the column vector with components $\psi^N_{i,j,k}$ given by

$$(\psi^N_{j,k,l})^* = \sqrt{[j,k,l]!}\, z_1^j z_2^k z_3^l, \qquad \text{if } N \geq 0 \quad \text{and with } j+k+l = N,$$
$$(\psi^N_{j,k,l})^* = \sqrt{[j,k,l]!}\, (z_1^j z_2^k z_3^l)^*, \qquad \text{if } N \leq 0 \quad \text{and with } i+j+k = -N.$$

Notice that we put an extra coefficient q^{-N}. This is needed for compatibility with the twist map in section (2.2.3).

The anti holomorphic part of this connection will be $\nabla_N^{\overline{\partial}} = q^{-N}\Psi_N^\dagger \overline{\partial}\Psi_N$. The curvature of $\nabla_N^{\overline{\partial}}$ can be computed as follows

$$(\nabla_N^{\overline{\partial}})^2 = q^{-2N}\Psi_N^\dagger(\overline{\partial}P_N\overline{\partial}P_N)\Psi_N,$$

where $P_N := \Psi_N\Psi_N^\dagger$ is a projection map due to the fact that $\Psi_N^\dagger\Psi_N = 1$.

Proposition 2.2.2. *The connection $\nabla_N^{\overline{\partial}}$ is flat.*

Proof. We will prove this for $N \geq 0$ and a similar discussion will cover the case $N < 0$.

It suffices to show that

$$\Psi_N^\dagger\overline{\partial}P_N = \Psi_N^\dagger(P_N \triangleleft F_2F_1, P_N \triangleleft F_2)^t = 0.$$

The second component

$$\begin{aligned}
\Psi_N^\dagger(P_N \triangleleft F_2) &= \Psi_N^\dagger((\Psi_N\Psi_N^\dagger) \triangleleft F_2) \\
&= \Psi_N^\dagger\{(\Psi_N \triangleleft F_2)(\Psi_N^\dagger \triangleleft K_2) + (\Psi_N \triangleleft K_2^{-1})(\Psi_N^\dagger \triangleleft F_2)\} \\
&= 0\,.
\end{aligned}$$

and this last equality is obtained by (see [9], section 6)

$$\Psi_N^\dagger \triangleleft F_2 = 0\,, \quad \Psi_N^\dagger(\Psi_N \triangleleft F_2) = 0\,. \tag{2.10}$$

Similar computation shows that $\Psi_N^\dagger(P_N \triangleleft F_2F_1)$ also vanishes. For this the following identity is needed.

$$\Psi_N^\dagger(\Psi_N \triangleleft F_2F_1) = 0. \tag{2.11}$$

Hence $(\nabla_N^{\overline{\partial}})^2 = 0$. □

Alternatively, as it was kindly pointed out to us by Francesco D'Andrea, using Lemma 6.1 in [9], the full connection (holomorphic + antiholomorphic part) has curvature of type $(1,1)$. This implies that the square of the holomorphic and antiholomorphic part is zero.

Proposition (2.2.2) verifies that the operator $\nabla_N^{\overline{\partial}}$ satisfies the condition of holomorphic structure as given in the definition (1.2.3).

Flatness of $\nabla_N^{\overline{\partial}}$ gives the following complex of vector spaces

$$0 \to L_N \to \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} L_N \to \Omega^{(0,2)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} L_N \to 0\,.$$

The zeroth cohomology group $H^0(L_N, \nabla_N^{\overline{\partial}})$ of this complex is called the *space of holomorphic sections of* L_N. The structure of this space is best described by the following theorem.

Theorem 2.2.1. *Let N be a positive integer. Then*

$$(1) \qquad H^0(L_N, \nabla_N^{\overline{\partial}}) \simeq \mathbb{C}^{\frac{(N+1)(N+2)}{2}}$$

$$(2) \qquad H^0(L_{-N}, \nabla_{-N}^{\overline{\partial}}) = 0.$$

Proof. First we recall that

$$\nabla_N^{\overline{\partial}}\xi = q^{-N}\Psi_N^\dagger\overline{\partial}\Psi_N\xi = q^{-N}\Psi_N^\dagger((\Psi_N\xi) \triangleleft F_2F_1, (\Psi_N\xi) \triangleleft F_2)^t.$$

Using (2.10), (2.11) and the following identities

$$\Psi_N \triangleleft F_1 = 0, \quad \Psi_N \triangleleft K_1 = \Psi_N, \quad \Psi_N \triangleleft K_2 = q^{-N/2}\Psi_N, \tag{2.12}$$

we prove that $\nabla_N^{\overline{\partial}}\xi = 0$ is equivalent to the equations $\xi \triangleleft F_2 = 0$ and $\xi \triangleleft F_2F_1 = 0$.

First we compute the second component of $\nabla_N^{\overline{\partial}}\xi$.

$$\begin{aligned} q^{-N}\Psi_N^\dagger((\Psi_N\xi)\triangleleft F_2) &= q^{-N}\Psi_N^\dagger\{(\Psi_N \triangleleft F_2)(\xi \triangleleft K_2) + (\Psi_N \triangleleft K_2^{-1})(\xi \triangleleft F_2)\} \\ &= q^{-N/2}\xi \triangleleft F_2. \end{aligned}$$

In addition to (2.10) and (2.12), here we have used $\Psi_N^\dagger\Psi_N = 1$. In a similar manner, one can show that the first component is

$$\begin{aligned} &q^{-N}\Psi_N^\dagger((\Psi_N\xi)\triangleleft F_2F_1) \\ &\qquad = q^{-N}\Psi_N^\dagger\{(\Psi_N \triangleleft F_2)(\xi \triangleleft K_2) + (\Psi_N \triangleleft K_2^{-1})(\xi \triangleleft F_2)\} \triangleleft F_1 \\ &\qquad = q^{-N}\Psi_N^\dagger\{q^{N/2}(\Psi_N \triangleleft F_2)\xi + q^{N/2}\Psi_N(\xi \triangleleft F_2)\} \triangleleft F_1 \\ &\qquad = q^{-N/2}\Psi_N^\dagger\{(\Psi_N \triangleleft F_2F_1)(\xi \triangleleft K_1) + (\Psi_N \triangleleft F_2K_1^{-1})(\xi \triangleleft F_1) \\ &\qquad + (\Psi_N \triangleleft F_1)(\xi \triangleleft F_2K_1) + (\Psi_N \triangleleft K_1^{-1})(\xi \triangleleft F_2F_1)\} \\ &\qquad = q^{-N/2}\xi \triangleleft F_2F_1. \end{aligned}$$

Let $N \geq 0$. In this case, a basis element of L_N is of the form $t(n, n+N)^{0}_{\underline{j}}$. Similar computation to the proof of proposition 2.2.1, using (2.8), (2.2) and (2.3), shows that $t(n, n+N)^{0}_{\underline{j}} \triangleleft F_2 = \gamma_n t(n, n+N)^{1,0,-\frac{1}{2}}_{\underline{j}}$, where $\gamma_n = A_{0,0} = (\frac{[n][n+N+2]}{[2]})^{1/2}$. If $\xi \in L_N$, then ξ can be written as $\sum_{n,\underline{j}} \lambda_{n,\underline{j}} t(n, n+N)^{0}_{\underline{j}}$. So $\xi \triangleleft F_2 = \sum \lambda_{n,\underline{j}}\gamma_n t(n, n+N)^{1,0,-1/2}_{\underline{j}}$. Since $\gamma_n = 0$ iff $n = 0$, $\xi \triangleleft F_2 = 0$ implies that the set $\{t(0, N)^{0}_{\underline{j}}\}$ will form a basis for the space of $\mathrm{Ker}\nabla_N^{\overline{\partial}}$. Remembering that by (2.1), the indices are restricted by $j_1 = 0, j_2 = 0, ..., N$, and $j_2/2 - |m| \in \mathbb{N}$, we will find that dim Ker $\nabla_N^{\overline{\partial}} = \frac{(N+1)(N+2)}{2}$.

When N is a negative integer, γ_n will be $(\frac{[n-N][n+2]}{[2]})^{1/2}$ which is nonzero. So dim Ker $\nabla_N^{\overline{\partial}} = 0$.

$\square$

2.2.3 Bimodule connections

There exists a $\mathcal{A}(\mathbb{C}P_q^2)$-bimodules isomorphism $\sigma : \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} L_N \to L_N \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \Omega^{(0,1)}$ which acts as

$$\sigma(\omega \otimes \xi) = q^{-N} \xi' \otimes \omega',$$

such that both elements $\omega \otimes \xi$ and $\xi' \otimes \omega'$ in $\mathcal{A}(SU_q(3))^2$, after multiplication are the same. We try to illustrate this in the case of $N = 1$. More precisely let us define the maps ϕ_1 and ϕ_2 as follows:

$$\phi_1 : \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} L_1 \to \mathcal{A}(SU_q(3))^2,$$
$$\phi_1((v_+, v_-)^t \otimes \xi) = q^{\frac{1}{2}} (v_+\xi, v_-\xi)^t,$$

and

$$\phi_2 : L_1 \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \Omega^{(0,1)} \to \mathcal{A}(SU_q(3))^2,$$
$$\phi_2(\xi \otimes (v_+, v_-)^t) = q^{-\frac{1}{2}} (\xi v_+, \xi v_-)^t.$$

We will prove that Im ϕ_1=Im ϕ_2. Therefore $\sigma = \phi_1^{-1}\phi_2$ gives an isomorphism from $L_1 \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \Omega^{(0,1)}$ to $\Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} L_1$ which is coming from the multiplication map. Let us first recall that as a $*$-algebra $\mathcal{A}(\mathbb{C}P_q^2)$ is generated by elements $p_{jk} = z_j^* z_k = (u_j^3)^* u_k^3$.

Lemma 2.2.1. *With above notation $Im\, \phi_1 = Im\, \phi_2$.*

Proof. case1. $\alpha \in$ Im ϕ_2 is a basis element.

$$\alpha = \phi_2(t(n, n+1)^{\underline{0}}_{\underline{i}} \otimes p_{rs}\overline{\partial} p_{jk}) = q^{-1/2} t(n, n+1)^{\underline{0}}_{\underline{i}} p_{rs} \begin{pmatrix} -q^{-3/2}(u_j^1)^* \\ q^{-1/2}(u_j^2)^* \end{pmatrix} u_k^3$$
$$= q^{-1/2} \begin{pmatrix} -q^{-3/2} t(n, n+1)^{\underline{0}}_{\underline{i}} p_{rs}(u_j^1)^* \\ q^{-1/2} t(n, n+1)^{\underline{0}}_{\underline{i}} p_{rs}(u_j^2)^* \end{pmatrix} u_k^3 = q^{-1} \phi_1(T_{\underline{i}rsj} \otimes u_k^3),$$

where

$$T_{\underline{i}rsj} = (-q^{-3/2} t(n, n+1)^{\underline{0}}_{\underline{i}} p_{rs}(u_j^1)^*, q^{-1/2} p_{rs} t(n, n+1)^{\underline{0}}_{\underline{i}} (u_j^2)^*)^t.$$

Since $u_k^3 \in L_1$, it is enough to prove that $T_{\underline{i}rsj} \in \Omega^{(0,1)}$. In order to do so, we need to show that the pair (v_+, v_-) defined as below, satisfies the properties given in (2.9).

$$(v_+, v_-)^t = (-q^{-3/2} t(n, n+1)^{0}_{\underline{i}} p_{rs} (u_j^1)^*,\ q^{-1/2} t(n, n+1)^{0}_{\underline{i}} p_{rs} (u_j^2)^*)^t.$$

We will check $(v_+, v_-) \triangleleft E_1 = (v_-, 0)$.

$$\begin{aligned}
v_+ \triangleleft E_1 &= -q^{-3/2} t(n, n+1)^{0}_{\underline{i}} p_{rs} (u_j^1)^* \triangleleft E_1 \\
&= -q^{-3/2} \{(t(n, n+1)^{0}_{\underline{i}} \triangleleft E_1)((p_{rs}(u_j^1)^*) \triangleleft K_1) \\
&+ (t(n, n+1)^{0}_{\underline{i}} \triangleleft K_1^{-1})((p_{rs}(u_j^1)^*) \triangleleft E_1)\} \\
&= -q^{-3/2} t(n, n+1)^{0}_{\underline{i}} \{(p_{rs} \triangleleft E_1)((u_j^1)^* \triangleleft K_1) \\
&+ (p_{rs} \triangleleft K_1^{-1})((u_j^1)^* \triangleleft E_1)\} \\
&= -q^{-3/2} t(n, n+1)^{0}_{\underline{i}} p_{rs} (-q)(u_j^2)^* \\
&= q^{-1/2} t(n, n+1)^{0}_{\underline{i}} p_{rs} (u_j^2)^* \\
&= v_-.
\end{aligned}$$

Here we have used the following identities which are obtained from (2.2), (2.6) and (2.8).

$$\begin{aligned}
&t(n, n+1)^{0}_{\underline{i}} \triangleleft K_1 = t(n, n+1)^{0}_{\underline{i}}, && t(n, n+1)^{0}_{\underline{i}} \triangleleft E_1 = 0 \\
&p_{ij} \triangleleft E_1 = 0, && (u_j^1)^* \triangleleft K_1 = q^{1/2}(u_j^1)^*, \\
&p_{ij} \triangleleft K_1 = p_{ij}, && (u_j^1)^* \triangleleft E_1 = (-q)(u_j^2)^*.
\end{aligned}$$

Similarly

$$
\begin{aligned}
v_- \triangleleft E_1 &= q^{-1/2} t(n,n+1)^{0}_{\underline{i}} p_{rs} (u^2_j)^* \triangleleft E_1 \\
&= q^{-1/2} \{ (t(n,n+1)^{0}_{\underline{i}} \triangleleft E_1)((p_{rs}(u^2_j)^*) \triangleleft K_1) \\
&+ (t(n,n+1)^{0}_{\underline{i}} \triangleleft K_1^{-1})((p_{rs}(u^2_j)^*) \triangleleft E_1) \} \\
&= q^{-1/2} t(n,n+1)^{0}_{\underline{i}} \{ (p_{rs} \triangleleft E_1)((u^2_j)^* \triangleleft K_1) \\
&+ (p_{rs} \triangleleft K_1^{-1})((u^2_j)^* \triangleleft E_1) \} \\
&= 0.
\end{aligned}
$$

Two more identities which have been used above, are

$$
(u^2_j)^* \triangleleft K_1 = q^{-1/2} (u^1_j)^*, \qquad (u^2_j)^* \triangleleft E_1 = 0.
$$

The case $(v_+, v_-) \triangleleft F_1 = (0, v_+)$ is similar and the other two cases $(v_+, v_-) \triangleleft K_1 = (q^{1/2} v_+, q^{-1/2} v_-)$ and $(v_+, v_-) \triangleleft K_1 K_2^2 = q^{3/2} (v_+, v_-)$ are straightforward, but the following relations are needed.

$$
\begin{aligned}
& t(n,n+1)^{0}_{\underline{i}} \triangleleft K_2 = q^{1/2} t(n,n+1)^{0}_{\underline{i}}, && t(n,n+1)^{0}_{\underline{i}} \triangleleft F_1 = 0, \\
& (u^1_j)^* \triangleleft K_2 = (u^1_j)^*, && (u^2_j)^* \triangleleft K_2 = q^{1/2} (u^2_j)^*, \\
& (u^1_j)^* \triangleleft F_1 = 0, && (u^2_j)^* \triangleleft F_1 = (-q)^{-1} (u^1_j)^*, \\
& p_{ij} \triangleleft K_2 = p_{ij}, && p_{ij} \triangleleft F_1 = 0.
\end{aligned}
$$

Case2. $\alpha \in \mathrm{Im}\ \phi_2$ is a general element.

$$\begin{aligned}
\alpha &= \phi_2(\sum_{n,\underline{i}} c_{n\underline{i}} t(n,n+1)^{\underline{0}}_{\underline{i}} \otimes \sum_{r,s,j,k} d_{rsjk} p_{rs} \overline{\partial} p_{jk}) \\
&= q^{-1/2} \sum_{n,\underline{i},r,s,j,k} c_{n\underline{i}} t(n,n+1)^{\underline{0}}_{\underline{i}} d_{rsjk} p_{rs} \begin{pmatrix} -q^{-3/2}(u^1_j)^* \\ q^{-1/2}(u^2_j)^* \end{pmatrix} u^3_k \\
&= q^{-1} \phi_1(\sum_k \{ \sum_{\underline{i},r,s,j} c_{\underline{i}} d_{rsjk} t(n,n+1)^{\underline{0}}_{\underline{i}} p_{rs} \begin{pmatrix} -q^{-3/2}(u^1_j)^* \\ q^{-1/2}(u^2_j)^* \end{pmatrix} \} \otimes u^3_k) \\
&= q^{-1} \phi_1(\sum_k A_k \otimes u^3_k),
\end{aligned}$$

where

$$A_k = \sum_{n,\underline{i},r,s,j} c_{n\underline{i}} d_{rsjk} t(n,n+1)^{\underline{0}}_{\underline{i}} p_{rs} \begin{pmatrix} q^{-3/2}(u^1_j)^* \\ q^{-1/2}(u^2_j)^* \end{pmatrix} \in \Omega^{(0,1)}.$$

The proof for $Im\ \phi_2 \subset Im\ \phi_1$ is similar.

$\square$

In general the maps ϕ_1 and ϕ_2 will be defined as

$$\begin{aligned}
&\phi_1 : \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} L_N \to \mathcal{A}(SU_q(3))^2, \\
&\phi_1((v_+, v_-)^t \otimes \xi) = q^{\frac{N}{2}} (v_+\xi, v_-\xi)^t,
\end{aligned}$$

and

$$\begin{aligned}
&\phi_2 : L_N \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \Omega^{(0,1)} \to \mathcal{A}(SU_q(3))^2, \\
&\phi_2(\xi \otimes (v_+, v_-)) = q^{-\frac{N}{2}} (\xi v_+, \xi v_-).
\end{aligned}$$

Now, we prove that $\nabla^{\overline{\partial}}_N$ has the right σ-twisted Leibniz property with respect to the map $\sigma = \phi_1^{-1}\phi_2$.

Proposition 2.2.3. *Taking σ as above, the following holds*

$$\nabla^{\overline{\partial}}_N(\xi a) = (\nabla^{\overline{\partial}}_N \xi) a + \sigma(\xi \otimes \overline{\partial} a), \quad \forall a \in \mathcal{A}(\mathbb{C}P^2_q), \forall \xi \in L_N. \tag{2.13}$$

Proof. By (2.10), (2.12) and the fact that $\xi \triangleleft K_2 = q^{N/2}\xi$, we compute the second component of the left hand side as follows

$$
\begin{aligned}
& q^{-N}\Psi_N^\dagger((\Psi_N\xi a) \triangleleft F_2) \\
&\qquad = q^{-N}\Psi_N^\dagger\{(\Psi_N \triangleleft F_2)((\xi a) \triangleleft K_2) + (\Psi_N \triangleleft K_2^{-1})((\xi a) \triangleleft F_2)\} \\
&\qquad = q^{-N/2}(\xi \triangleleft F_2)a + q^{-N}\xi(a \triangleleft F_2).
\end{aligned}
$$

(Note that this actually is $\phi_1\nabla_N^{\overline{\partial}}$.) For the second component of the right hand side we will get

$$
q^{-N/2}(\xi \triangleleft F_2)a + \sigma(\xi \otimes a \triangleleft F_2).
$$

The previous lemma says that q^{-N} will appear after acting σ on the second term. It can be seen that ϕ_1 of both sides coincides. Computation for the second component will be similar. □

Now we will come up to the analogue of proposition 3.8 of ([19]).

Proposition 2.2.4. *The tensor product connection $\nabla_N^{\overline{\partial}} \otimes 1 + (\sigma \otimes 1)(1 \otimes \nabla_M^{\overline{\partial}})$ coincides with the holomorphic structure on $L_N \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} L_M$ when identified with L_{N+M}.*

Proof.

$$
\begin{aligned}
& \nabla_{N+M}^{\overline{\partial}}(\xi_1\xi_2) \\
&= q^{-(N+M)}\Psi_{N+M}^\dagger\overline{\partial}\Psi_{N+M}(\xi_1\xi_2) \\
&= q^{-(N+M)}\Psi_{N+M}^\dagger\begin{pmatrix}(\Psi_{N+M}\xi_1\xi_2) \triangleleft F_2F_1 \\ (\Psi_{N+M}\xi_1\xi_2) \triangleleft F_2\end{pmatrix} \\
&= q^{-(N+M)}\Psi_{N+M}^\dagger\begin{pmatrix}\{(\Psi_{N+M} \triangleleft F_2)((\xi_1\xi_2) \triangleleft K_2)\} \triangleleft F_1 \\ (\Psi_{N+M} \triangleleft F_2)((\xi_1\xi_2) \triangleleft K_2)\end{pmatrix} \\
&+ q^{-(N+M)}\Psi_{N+M}^\dagger\begin{pmatrix}\{(\Psi_{N+M} \triangleleft K_2^{-1})((\xi_1\xi_2) \triangleleft F_2)\} \triangleleft F_1 \\ (\Psi_{N+M} \triangleleft K_2^{-1})((\xi_1\xi_2) \triangleleft F_2)\end{pmatrix} \\
&= q^{-\frac{N+M}{2}}\begin{pmatrix}(\xi_1\xi_2) \triangleleft F_2F_1 \\ (\xi_1\xi_2) \triangleleft F_2\end{pmatrix} \\
&= q^{-\frac{N}{2}}\begin{pmatrix}\{(\xi_1 \triangleleft F_2)\xi_2 + (q^{-N-M/2}\xi_1(\xi_2 \triangleleft F_2)\} \triangleleft F_1 \\ (\xi_1 \triangleleft F_2)\xi_2 + q^{-N-M/2}\xi_1(\xi_2 \triangleleft F_2)\end{pmatrix}.
\end{aligned}
$$

Besides (2.10) and (2.11), we also applied the identities $\xi_i \triangleleft K_1 = 0$, $\xi_i \triangleleft F_1 = 0$.

On the other hand

$$((\nabla_N^{\overline{\partial}} \otimes 1) + (\sigma \otimes 1)(1 \otimes \nabla_M^{\overline{\partial}}))(\xi_1 \otimes \xi_2) =$$
$$q^{-N/2}\begin{pmatrix} \xi_1 \triangleleft F_2F_1 \\ \xi_1 \triangleleft F_2 \end{pmatrix} \otimes \xi_2 + (\sigma \otimes 1)(\xi_1 \otimes q^{-M/2}\begin{pmatrix} \xi_2 \triangleleft F_2F_1 \\ \xi_2 \triangleleft F_2 \end{pmatrix}).$$

Interpreting this expression as an element of $\Omega^{(0,1)} \otimes L_{N+M}$, after applying the map σ, which gives us q^{-N} on the second summand, we will get the same result.

□

Thanks to proposition (2.2.4), the space $R := \bigoplus H^0(L_N, \nabla_N^{\overline{\partial}})$ has a ring structure under the natural tensor product of bimodules. In the following, we identify the quantum homogeneous coordinate ring R with a twisted polynomial algebra in three variables

Theorem 2.2.2. *We have the algebra isomorphism*

$$R := \bigoplus_{N \geq 0} H^0(L_N, \nabla_N^{\overline{\partial}}) \simeq \frac{\mathbb{C}\langle z_1, z_2, z_3 \rangle}{\langle\, z_i z_j - q z_j z_i : 1 \leq i < j \leq 3 \,\rangle}$$

Proof. The ring structure on R is coming from the tensor product $L_{N_1} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} L_{N_2} \simeq L_{N_1+N_2}$. The following discussion shows that $H^0(L_1, \nabla_1^{\overline{\partial}}) = \mathbb{C}z_1 \oplus \mathbb{C}z_2 \oplus \mathbb{C}z_3$. The explicit formula for the basis elements of $H^0(L_N, \nabla_N^{\overline{\partial}})$, i.e. $t(0,N)^{\underline{0}}_{\underline{j}}$ is given by Proposition 3.3 and Proposition 3.4 of [9] as following

$$t(0,N)^{\underline{0}}_{\underline{j}} = \frac{1}{[j_2]!}\sqrt{\frac{[\frac{j_2}{2}+m]![N-j_2]!}{[\frac{j_2}{2}-m]![N]!}}\, F_1^{1/2j_2-m} F_2^{j_2} \triangleright z_3^N. \tag{2.14}$$

We just mention that up to a multiplicative constant this equals to $z_1^{1/2j_2-m} z_2^{1/2j_2+m} z_3^{N-j_2}$. In the case $N=1$, $t(0,1)^{\underline{0}}_{0,1,-\frac{1}{2}} = z_1$, $t(0,1)^{\underline{0}}_{0,1,\frac{1}{2}} = z_2$ and $t(0,1)^{\underline{0}}_{\underline{0}} = z_3$. Now the isomorphism follows from the identities $z_i \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} z_j - q z_j \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} z_i = 0$ in L_2, which can easily be seen.

□

2.3 The C^*-algebras $C(SU_q(3))$ and $C(\mathbb{C}P_q^2)$

In this section we extend the results of Proposition (2.2.1) and Theorem (2.2.1) which are stated for polynomial functions and polynomial sections to L^2-functions and sections, respectively.

Let $C(SU_q(3))$ denotes the C^* completion of $\mathcal{A}(SU_q(3))$, i.e. the universal C^*-algebra generated by the elements u^i_j subject to the relations given in section 2.1.2. This is a compact quantum group in the sense of Woronowicz [23]. There exists a unique left invariant normalized Haar state on this compact quantum group denoted by h. The functional h is faithful and it also has a twisted tracial property which will be considered in the next section. We denote the C^*-norm completion of $\mathcal{A}(\mathbb{C}P_q^2))$ inside $C(SU_q(3))$ by $C(\mathbb{C}P_q^2))$ and regard it as the space of continuous functions on the quantum projective plane. Similarly, we denote the C^*-norm completion of L_N inside $C(SU_q(3))$ by $\Gamma(L_N)$.

We denote the Hilbert space completion of $\mathcal{A}(SU_q(3))$ with respect to the inner product $\langle a, b\rangle := h(a^*b)$ by $L^2(SU_q(3))$. Since the Haar state on the C^*-algebra $C(SU_q(3))$ is faithful [28], the GNS map $\eta : C(SU_q(3)) \to L^2(SU_q(3))$ will be injective. An orthogonal basis of $L^2(SU_q(3))$ is given by $\eta(t(n_1, n_2)^{\underline{l}}_{\underline{j}})$. Similarly, we denote the L^2-completion of $\Gamma(L_N)$ inside $L^2(SU_q(3))$ by $L^2(L_N)$, and we have

$$L^2(L_N) = \mathrm{Span}\{t(n, n+N)^{\underline{0}}_{\underline{j}} |\ n \in \mathbb{N},\ \ \underline{j}\,\text{satisfies}\,(2.1)\,\}^{closure}.$$

Note that the last equality is for $N \geq 0$. For $N < 0$, basis elements are of the form $t(n-N, n)^{\underline{0}}_{\underline{j}}$.

The operator $Z = \triangleleft(F_2F_1, F_2)$, in its original definition, is a densely defined unbounded operator on $L^2(SU_q(3))$. There is however a natural extension of this operator to a larger domain that we specify now. First note that the action of Z on basis elements is given by:

$$t(n_1, n_2)^{\underline{i}}_{\underline{j}} \triangleleft Z = \alpha^{\underline{i}}_{n_1,n_2} \binom{\theta^{I_1}_{n_1,n_2} t(n_1, n_2)^{I_1'}_{\underline{j}}}{t(n_1, n_2)^{I_1}_{\underline{j}}} + \beta^{\underline{i}}_{n_1,n_2} \binom{\theta^{I_2}_{n_1,n_2} t(n_1, n_2)^{I_2'}_{\underline{j}}}{t(n_1, n_2)^{I_2}_{\underline{j}}}$$

where

$$I_1 = (i_1+1, i_2, m-1/2), \quad I_2 = (i_1, i_2-1, m-1/2),$$
$$I_1' = (i_1+1, i_2, m+1/2), \quad I_2' = (i_1, i_2-1, m+1/2),$$
$$\alpha^{\underline{i}}_{n_1,n_2} = [1/2(i_1+i_2) - m + 1]^{1/2} A_{i_1,i_2},$$
$$\beta^{\underline{i}}_{n_1,n_2} = [1/2(i_1+i_2) + m]^{1/2} B_{i_1,i_2},$$
$$\theta^{\underline{i}}_{n_1,n_2} = [1/2(i_1+i_2) - m]^{1/2} [1/2(i_1+i_2) + m + 1]^{1/2}.$$

The coefficients A_{i_1,i_2} and B_{i_1,i_2} are given by (2.3). Suppose that $(a^{\underline{i},\underline{j}}_{n_1,n_2})$ denote the coefficients of $a \in L^2(SU_q(3))$ in the given basis, i.e., $a = \sum a^{\underline{i},\underline{j}}_{n_1,n_2} t(n_1,n_2)^{\underline{i}}_{\underline{j}}$. Now the second component of $a \triangleleft Z$ is

$$\sum_{n_1,n_2} \sum_{\underline{i},\underline{j}} a^{\underline{i},\underline{j}}_{n_1,n_2} \alpha^{\underline{i}}_{n_1,n_2} t(n_1,n_2)^{I_1}_{\underline{j}} + \sum_{n_1,n_2} \sum_{\underline{i},\underline{j}} a^{\underline{i},\underline{j}}_{n_1,n_2} \beta^{\underline{i}}_{n_1,n_2} t(n_1,n_2)^{I_2}_{\underline{j}} =$$
$$\sum_{n_1,n_2} \sum_{\underline{i},\underline{j}} \{a^{\underline{i},\underline{j}}_{n_1,n_2} \alpha^{\underline{i}}_{n_1,n_2} + a^{\underline{i}',\underline{j}}_{n_1,n_2} \beta^{\underline{i}'}_{n_1,n_2}\} t(n_1,n_2)^{I_1}_{\underline{j}} +$$
$$\sum_{n_1,n_2} \sum_{\underline{i},\underline{j}} a^{\underline{i},\underline{j}}_{n_1,n_2} \beta^{\underline{i}}_{n_1,n_2} t(n_1,n_2)^{I_2}_{\underline{j}}$$

where

$$i' = (i_1+1, i_2+1, m).$$

Note that all sums are subject to admissibility of I_1 and I_2. Moreover, the last sum is taken over all indices such that $I_1 \neq I_2$. With a similar computation for the first component, we can now define $a \in \mathrm{Dom}(Z)$ if

$$\sum_{n_1,n_2} \sum_{\underline{i},\underline{j}} |a^{\underline{i},\underline{j}}_{n_1,n_2} \alpha^{\underline{i}}_{n_1,n_2} + a^{\underline{i}',\underline{j}}_{n_1,n_2} \beta^{\underline{i}'}_{n_1,n_2}|^2 + \sum_{n_1,n_2} \sum_{\underline{i},\underline{j}} |a^{\underline{i},\underline{j}}_{n_1,n_2} \beta^{\underline{i}}_{n_1,n_2}|^2 < \infty$$

and

$$\sum_{n_1,n_2}\sum_{\underline{i},\underline{j}} |a^{\underline{i},\underline{j}}_{n_1,n_2}\alpha^{\underline{i}}_{n_1,n_2}\theta^{I_1}_{n_1,n_2} + a^{\underline{i}',\underline{j}}_{n_1,n_2}\beta^{\underline{i}'}_{n_1,n_2}\theta^{I_1}_{n_1,n_2}|^2 +$$
$$\sum_{n_1,n_2}\sum_{\underline{i},\underline{j}} |a^{\underline{i},\underline{j}}_{n_1,n_2}\beta^{\underline{i}}_{n_1,n_2}\theta^{I_2}_{n_1,n_2}|^2 < \infty.$$

Here the last summation is over the set of indices such that $I_1 \neq I_2$. This can be denoted, with some abuse of notation perhaps, by

$$\mathrm{Dom}(Z) := \{a \in L^2(SU_q(3)) | \, (a \triangleleft F_2F_1, a \triangleleft F_2) \in L^2(SU_q(3)^2)\}.$$

Now Proposition 2.2.1 can easily be generalized to the following proposition.

Proposition 2.3.1. *The Kernel of the map Z restricted to $L^2(\mathbb{C}P^2_q)$ is $\mathbb{C}$.*

Proof. Since any element of $L^2(\mathbb{C}P^2_q)$ is a L^2-linear combination of the elements $t(n,n)^{\underline{0}}_{\underline{j}}$, proof is exactly like Proposition 2.2.1. □

Corollary 2.3.1. *There is no non-constant holomorphic function in $C(\mathbb{C}P^2_q)$.*

With a similar discussion, the analogue of 2.2.1 continues to hold if we work with L^2-sections of L_N. We give the statement of the theorem and leave its similar proof to the reader.

Theorem 2.3.1. *Let N be a positive integer. Then*

$$(1) \qquad H^0(L^2(L_N), \nabla^{\overline{\partial}}_N) \simeq \mathbb{C}^{\frac{(N+1)(N+2)}{2}},$$
$$(2) \qquad H^0(L^2(L_{-N}), \nabla^{\overline{\partial}}_{-N}) = 0.$$

We note that our approach here as well as in [19], is somehow the opposite of the approach adopted in [1, 2] to noncommutative projective spaces. We started with a C^*-algebra defined as the quantum homogeneous space of the quantum group $SU_q(3)$ and its natural line bundles, and endowed them with holomorphic structures. The quantum homogeneous coordinate ring is then defined as the algebra of holomorphic sections of these line bundles. This ring coincides with the twisted homogeneous ring associated in [1, 2] to the line bundle $\mathcal{O}(1)$ under a suitable twist.

2.4 Existence of a twisted positive Hochschild 4-cocycle on $\mathbb{C}P_q^2$

In [4], Section VI.2, Connes shows that extremal positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a holomorphic structure on the surface. There is a similar result for holomorphic structures on the noncommutative two torus (cf. *Loc cit.*). In particular the positive Hochschild cocycle is defined via the holomorphic structure and represents the fundamental cyclic cocycle. In [19] a notion of twisted positive Hochschild cocycle is introduced and a similar result is proved for the holomorphic structure of $\mathbb{C}P_q^1$. Although the corresponding problem of characterizing holomorphic structures on higher dimensional (commutative or noncommutative) manifolds via positive Hochschild cocycles is still open, nevertheless these results suggest regarding (twisted) positive Hochschild cocycles as a possible framework for holomorphic noncommutative structures. In this section we prove an analogous result for $\mathbb{C}P_q^2$.

First we recall the notion of twisted Hochschild and cyclic cohomologies. Let $\mathcal{A}$ be an algebra and σ an automorphism of $\mathcal{A}$. For each $n \geq 0$, $C^n(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}^{\otimes(n+1)}, \mathbb{C})$ is the space of *n-cochains* on $\mathcal{A}$. Define the space of *twisted Hochschild n-cochains* as $C^n_\sigma(\mathcal{A}) :=\mathrm{Ker}\{(1-\lambda_\sigma^{n+1}) : C^n(\mathcal{A}) \to C^n(\mathcal{A})\}$, where the *twisted cyclic* map $\lambda_\sigma : C^n(\mathcal{A}) \to C^n(\mathcal{A})$ is defined as

$$(\lambda_\sigma\phi)(a_0, a_1, ..., a_n) = (-1)^n \phi(\sigma(a_n), a_0, a_1, ..., a_{n-1}).$$

The *twisted Hochschild coboundary* map $b_\sigma : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ is given by

$$\begin{aligned} b_\sigma\phi(a_0, a_1, ..., a_{n+1}) = &\sum_{i=0}^{n} (-1)^i \phi(a_0, ..., a_i a_{i+1}, ..., a_{n+1}) \\ &+ (-1)^{n+1} \phi(\sigma(a_{n+1})a_0, ..., a_n). \end{aligned}$$

The cohomology of the complex $(C^*_\sigma(\mathcal{A}), b_\sigma)$ is called the *twisted Hochschild cohomology* of $\mathcal{A}$. We also need the notion of *twisted cyclic cohomology* of $\mathcal{A}$. It is by definition the cohomology of the complex $(C^*_{\sigma,\lambda}(\mathcal{A}), b_\sigma)$, where

$$C^n_{\sigma,\lambda} := Ker\{(1-\lambda) : C^n_\sigma(\mathcal{A}) \to C^{n+1}_\sigma(\mathcal{A})\}.$$

Now we come back to the case of our interest, that is $\mathbb{C}P^2_q$. Let τ be the fundamental class on $\mathbb{C}P^2_q$ defined as in [9] by

$$\tau(a_0, a_1, a_2, a_3, a_4) := -\int_h a_0 \, \mathrm{d}a_1 \, \mathrm{d}a_2 \, \mathrm{d}a_3 \, \mathrm{d}a_4 \,, \quad \forall a_0, a_1, ..., a_4 \in \mathcal{A}(\mathbb{C}P^2_q). \tag{2.15}$$

Here h stands for the Haar state functional of the quantum group $\mathcal{A}(SU_q(3))$ which has a twisted tracial property $h(xy) = h(\sigma(y)x)$. Here the algebra automorphism σ is defined by

$$\sigma : \mathcal{A}(SU_q(3)) \to \mathcal{A}(SU_q(3)), \quad \sigma(x) = K \triangleright x \triangleleft K.$$

where $K = (K_1K_2)^{-4}$. The map σ, restricted to the algebra $\mathcal{A}(\mathbb{C}P^2_q)$ is given by $\sigma(x) = K \triangleright x$. Non-triviality of τ has been shown in [9]. Now we recall the definition of a twisted positive Hochschild cocycle as given in [19].

Definition 2.4.1. *A twisted Hochschild 2n-cocycle ϕ on a $*$-algebra $\mathcal{A}$ is said to be twisted positive if the following map defines a positive sesquilinear form on the vector space $\mathcal{A}^{\otimes(n+1)}$:*

$$\langle a_0 \otimes a_1 \otimes ... \otimes a_n, b_0 \otimes b_1 \otimes ... \otimes b_n \rangle = \phi(\sigma(b_n^*)a_0, a_1, ..., a_n, b_n^*, ..., b_1^*).$$

We would like to define a twisted Hochschild cocycle φ which is cohomologous to τ and it is positive. For simplicity, we introduce first the maps φ_i, for $i = 1, 2$ as follows

$$\begin{aligned} \varphi_1(a_0, a_1, a_2, a_3, a_4) &= -3 \int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4, \\ \varphi_2(a_0, a_1, a_2, a_3, a_4) &= -3 \int_h a_0 \overline{\partial} a_1 \overline{\partial} a_2 \partial a_3 \partial a_4. \end{aligned} \tag{2.16}$$

Now we define $\varphi \in C^4(\mathcal{A}(\mathbb{C}P_q^2))$ by

$$\varphi := \varphi_1 + \varphi_2 \,. \tag{2.17}$$

We will need the following simple lemma for future computations.

Lemma 2.4.1. *For any $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}(\mathbb{C}P_q^2)$ the following identities hold:*

$$\int_h a_0(\partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4) a_5 = \int_h \sigma(a_5) a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4,$$
$$\int_h a_0(\overline{\partial} a_1 \overline{\partial} a_2 \partial a_3 \partial a_4) a_5 = \int_h \sigma(a_5) a_0 \overline{\partial} a_1 \overline{\partial} a_2 \partial a_3 \partial a_4.$$

Proof. We give the proof of the first one. The proof for the second equality will be similar. The space of $\Omega^{(2,2)}$ is a rank one free $\mathcal{A}(\mathbb{C}P_q^2)$-module. Let ω be the central basis element for the space of $\Omega^{(2,2)}$ and let $\partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 = x\omega$. Then

$$\begin{aligned}
\int_h a_0(\partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4) a_5 - \int_h \sigma(a_5) a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 &= \int_h (a_0 x \omega a_5 - \sigma(a_5) a_0 x \omega) \\
&= \int_h (a_0 x a_5 \omega - \sigma(a_5) a_0 x \omega) \\
&= h(a_0 x a_5 - \sigma(a_5) a_0 x) = 0.
\end{aligned}$$

The last equality comes from the twisted property of the Haar state. □

Proposition 2.4.1. *The functional φ defined by formula (3.17), is a twisted positive Hochschild 4-cocycle.*

Proof. We first verify the twisted cocycle property. In order to do so, we consider this property for each φ_i. We will prove the statement for φ_1. The proof for φ_2 is similar.

$$\begin{aligned}
&\varphi_1(\sigma(a_0),\sigma(a_1),\sigma(a_2),\sigma(a_3),\sigma(a_4))\\
&= -3\int_h \sigma(a_0)\partial\sigma(a_1)\partial\sigma(a_2)\overline{\partial}\sigma(a_3)\overline{\partial}\sigma(a_4)\\
&= -3\int_h (K\triangleright a_0)(K\triangleright \partial a_1)(K\triangleright \partial a_2)(K\triangleright \overline{\partial} a_3)(K\triangleright \overline{\partial} a_4)\\
&= -3\int_h K\triangleright (a_0\partial a_1\partial a_2\overline{\partial}a_3\overline{\partial}a_4) = -3\,\epsilon(K)\int_h a_0\partial a_1\partial a_2\overline{\partial}a_3\overline{\partial}a_4\\
&= \varphi_1(a_0,a_1,a_2,a_3,a_4).
\end{aligned}$$

Now let us prove that $b_\sigma\varphi = 0$. Again we just prove for φ_1 and leave the similar proof of the other one.

$$\begin{aligned}
b_\sigma\varphi_1(a_0,a_1,a_2,a_3,a_4,a_5) &= \varphi_1(a_0a_1,a_2,a_3,a_4,a_5) - \varphi_1(a_0,a_1a_2,a_3,a_4,a_5)\\
&+ \varphi_1(a_0,a_1,a_2a_3,a_4,a_5) - \varphi_1(a_0,a_1,a_2,a_3a_4,a_5)\\
&+ \varphi_1(a_0,a_1,a_2,a_3,a_4a_5) - \varphi_1(\sigma(a_5)a_0,a_1,a_2,a_3,a_4)
\end{aligned}$$

Using (2.16), this equals to

$$\begin{aligned}
&-3\int_h a_0a_1\partial a_2\partial a_3\overline{\partial}a_4\overline{\partial}a_5 + 3\int_h a_0\partial(a_1a_2)\partial a_3\overline{\partial}a_4\overline{\partial}a_5\\
&-3\int_h a_0\partial a_1\partial(a_2a_3)\overline{\partial}a_4\overline{\partial}a_5 + 3\int_h a_0\partial a_1\partial a_2\overline{\partial}(a_3a_4)\overline{\partial}a_5\\
&-3\int_h a_0\partial a_1\partial a_2\overline{\partial}a_3\overline{\partial}(a_4a_5) + 3\int_h \sigma(a_5)a_0\partial a_1\partial a_2\overline{\partial}a_3\overline{\partial}a_4.
\end{aligned}$$

Using the Leibniz property we get

$$b_\sigma\varphi_1(a_0,a_1,a_2,a_3,a_4,a_5) = -3\int_h (a_0a_1\partial a_2\partial a_3\overline{\partial}a_4\overline{\partial}a_5 - \sigma(a_5)a_0\partial a_1\partial a_2\overline{\partial}a_3\overline{\partial}a_4),$$

which is zero by the previous lemma.

Now we will show that all φ_1 and φ_2 are positive.

Positivity of φ_1:

$$\begin{aligned}\varphi_1(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) &= -3\int_h \sigma(a_0^*)a_0\partial a_1\partial a_2\overline{\partial}a_2^*\overline{\partial}a_1^* \\ &= -3\int_h a_0\partial a_1\partial a_2\overline{\partial}a_2^*\overline{\partial}a_1^*a_0^* \\ &= 3\int_h (a_0\partial a_1\partial a_2)(a_0\partial a_1\partial a_2)^*.\end{aligned}$$

One can take $\partial a_1 = (v_1, v_2)$ and $\partial a_2 = (w_1, w_2)$, then using the multiplication rule of type (1,0) forms (c.f. [9] Proposition A.1), we find that $(a_0\partial a_1\partial a_2)(a_0\partial a_1\partial a_2)^* = c_4^2[2]^{-1}\mu\mu^*$, where $\mu = q^{1/2}a_0v_1w_2 - q^{-1/2}a_0v_2w_1$. Hence

$$\varphi_1(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) = h(3c_4^2[2]^{-1}\mu\mu^*) \geq 0.$$

Positivity of φ_2:

$$\begin{aligned}\varphi_2(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) &= -3\int_h \sigma(a_0^*)a_0\overline{\partial}a_1\overline{\partial}a_2\partial a_2^*\partial a_1^* \\ &= -3\int_h a_0\overline{\partial}a_1\overline{\partial}a_2\partial a_2^*\partial a_1^*a_0^* \\ &= 3\int_h (a_0\overline{\partial}a_1\overline{\partial}a_2)(a_0\overline{\partial}a_1\overline{\partial}a_2)^*.\end{aligned}$$

Similar to the above discussion, one can take $\overline{\partial}a_1 = (v_1, v_2)$ and $\overline{\partial}a_2 = (w_1, w_2)$ and use the multiplication of type (0,1) forms to find that

$$\varphi_2(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) = h(3c_0^2[2]^{-1}\nu\nu^*) \geq 0,$$

where $\nu = q^{1/2}a_0v_1w_2 - q^{-1/2}a_0v_2w_1$. Here c_0 and c_4 are two real constants. This concludes the positivity of φ.

□

Now we want to show that the twisted Hochschild cocycle φ as defined by formula (3.17) and the twisted cyclic cocycle τ as in formula (3.15) are cohomologous. To this end, we need an appropriate twisted Hochschild cocycle ψ such that $\tau - \varphi = b_\sigma \psi$. Let ψ_i for , i=1,2,3,4 be defined by

$$\begin{aligned}
\psi_1(a_0, a_1, a_2, a_3) &= -\int_h a_0 \partial a_1 \overline{\partial} a_2 \partial \overline{\partial} a_3, \\
\psi_2(a_0, a_1, a_2, a_3) &= 2\int_h a_0 \partial a_1 \partial \overline{\partial} a_2 \overline{\partial} a_3, \\
\psi_3(a_0, a_1, a_2, a_3) &= 2\int_h a_0 \overline{\partial} a_1 \overline{\partial} \partial a_2 \partial a_3, \\
\psi_4(a_0, a_1, a_2, a_3) &= -\int_h a_0 \overline{\partial} a_1 \partial a_2 \overline{\partial} \partial a_3.
\end{aligned}$$

and let $\psi = \sum_{i=1}^4 \psi_i$. Then we will have the following result.

Proposition 2.4.2. *The twisted Hochschild cocycles τ and φ are cohomologous.*

Proof.

$$\begin{aligned}
b_\sigma \psi_1(a_0, a_1, a_2, a_3, a_4) &= \psi_1(a_0 a_1, a_2, a_3, a_4) - \psi_1(a_0, a_1 a_2, a_3, a_4) \\
&\quad + \psi_1(a_0, a_1, a_2 a_3, a_4) - \psi_1(a_0, a_1, a_2, a_3 a_4) \\
&\quad + \psi_1(\sigma(a_4) a_0, a_1, a_2, a_3)
\end{aligned}$$

which equals to

$$\begin{aligned}
-\int_h \{ & a_0 a_1 \partial a_2 \overline{\partial} a_3 \partial \overline{\partial} a_4 - a_0 \partial(a_1 a_2) \overline{\partial} a_3 \partial \overline{\partial} a_4 + a_0 \partial a_1 \overline{\partial}(a_2 a_3) \partial \overline{\partial} a_4 \\
& - a_0 \partial a_1 \overline{\partial} a_2 \partial \overline{\partial}(a_3 a_4) + \sigma(a_4) a_0 \partial a_1 \overline{\partial} a_2 \partial \overline{\partial} a_3 \}.
\end{aligned}$$

Applying the Leibniz rule, one can see that in the expanded form, all but two terms will cancel. That is

$$b_\sigma\psi_1 = \int\limits_h a_0(\partial a_1\overline{\partial} a_2\partial a_3\overline{\partial} a_4 - \partial a_1\overline{\partial} a_2\overline{\partial} a_3\partial a_4).$$

Similar computation for ψ_i, $i = 2, 3$ and 4 shows that

$$\begin{aligned}
b_\sigma\psi_2 &= 2\int\limits_h a_0(\partial a_1\partial a_2\overline{\partial} a_3\overline{\partial} a_4 - \partial a_1\overline{\partial} a_2\partial a_3\overline{\partial} a_4),\\
b_\sigma\psi_3 &= 2\int\limits_h a_0(\overline{\partial} a_1\overline{\partial} a_2\partial a_3\partial a_4 - \overline{\partial} a_1\partial a_2\overline{\partial} a_3\partial a_4),\\
b_\sigma\psi_4 &= \int\limits_h a_0(\overline{\partial} a_1\partial a_2\overline{\partial} a_3\partial a_4 - \overline{\partial} a_1\partial a_2\partial a_3\overline{\partial} a_4).
\end{aligned}$$

Therefore

$$\begin{aligned}
b_\sigma\psi = 2&\int\limits_h a_0(\partial a_1\partial a_2\overline{\partial} a_3\overline{\partial} a_4 + \overline{\partial} a_1\overline{\partial} a_2\partial a_3\partial a_4)\\
&-\int\limits_h a_0(\overline{\partial} a_1\partial a_2\partial a_3\overline{\partial} a_4 + \partial a_1\overline{\partial} a_2\overline{\partial} a_3\partial a_4)\\
&-\int\limits_h a_0(\overline{\partial} a_1\partial a_2\overline{\partial} a_3\partial a_4 + \partial a_1\overline{\partial} a_2\partial a_3\overline{\partial} a_4).
\end{aligned} \tag{2.18}$$

Now from (3.15), (3.17) and (2.18), we can easily find that $\tau - \varphi = b_\sigma\psi$. $\square$

Chapter 3

Noncommutative complex structure of $\mathbb{C}P_q^\ell$

In this chapter we continue the study of complex structures on quantum projective spaces.

In Section 3.1, we review the preliminaries on irreducible representations of quantum groups $U_q(\mathfrak{su}(\ell+1))$ and the Gelfand-Tsetlin basis for these representations. We refer to Chapter 1 for the the definition of a complex structure, holomorphic line bundles and bimodule connections. In Section 3.3 we recall the definition of the quantum projective space $\mathbb{C}P_q^\ell$, and endow its canonical line bundles with holomorphic connections. We also identify the space of holomorphic sections of these line bundles. In Section 3.4, we define bimodule connections on canonical line bundles. This enables us to define the quantum homogeneous coordinate ring of $\mathbb{C}P_q^\ell$ and identify this ring with the ring of twisted polynomials. In Section 3.5 we introduce a twisted positive Hochschild cocycle 2ℓ-cocycle on $\mathbb{C}P_q^\ell$, by using the complex structure of $\mathbb{C}P_q^\ell$, and show that it is cohomologous to its fundamental class which is represented by a twisted cyclic cocycle. This certainly provides further evidence for the belief, advocated by Alain Connes [4, 5], that holomorphic structures in noncommutative geometry should be represented by (extremal) positive Hochschild cocycles within the fundamental class.

3.1 Preliminaries on $U_q(\mathfrak{su}(\ell+1))$ and $\mathcal{A}(SU_q(\ell+1))$

3.1.1 The quantum enveloping algebra $U_q(\mathfrak{su}(\ell+1))$

Let $0<q<1$. We use the following notation

$$[a,b]_q = ab - q^{-1}ba, \quad [z] = \frac{q^z - q^{-z}}{q-q^{-1}}, \quad [n]! = [n][n-1]\cdots[1],$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}, \quad [j_1, j_2, \cdots, j_k]! = q^{-\sum_{r<s} j_r j_s} \frac{[j_1 + j_2 + \cdots + j_k]!}{[j_1]![j_2]!\cdots[j_k]!}\,.$$

The quantum enveloping algebra $U_q(\mathfrak{su}(\ell+1))$, as a $*$-algebra, is generated by elements K_i, K_i^{-1}, E_i, F_i, $i = 1, 2, \cdots, \ell$, with $K_i^* = K_i$ and $E_i^* = F_i$, subject to the following relations for $0 \leq i, j \leq \ell$ [23],

$$\begin{aligned}
&K_iK_j = K_jK_i \qquad E_iK_i = q^{-1}K_iE_i \\
&E_iK_j = q^{1/2}K_jE_i \qquad \text{if} \quad |i-j| = 1 \\
&E_iK_j = K_jE_i \qquad \text{if} \quad |i-j| > 1 \\
&E_iF_j - F_jE_i = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q - q^{-1}} \\
&E_iE_j = E_jE_i \qquad \text{if} \quad |i-j| > 1,
\end{aligned} \tag{3.1}$$

and the Serre relation

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \qquad \text{if} \quad |i-j| = 1.$$

The coproduct, counit and antipode of this Hopf algebra is given by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i,$$
$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0, \quad S(K_i) = K_i^{-1}, \quad S(E_i) = -qE_i.$$

3.1.2 The quantum group $\mathcal{A}(SU_q(\ell+1))$

As a $*$-algebra, $\mathcal{A}(SU_q(\ell+1))$ is generated by $(\ell+1)^2$ elements u^i_j, where $i, j = 1, 2, ..., \ell+1$ subject to the following commutation relations

$$\begin{aligned}
&u^i_ku^j_k = qu^j_ku^i_k, \qquad && u^k_iu^k_j = qu^k_ju^k_i \quad \forall\, i < j, \\
&[u^i_l, u^j_k] = 0, && [u^i_k, u^j_l] = (q - q^{-1})u^i_lu^j_k \quad \forall\, i < j,\ k < l,
\end{aligned}$$

and

$$\sum_{\pi \in S_{\ell+1}} (-q)^{||\pi||} u^1_{\pi(1)}u^2_{\pi(2)} \cdots u^{\ell+1}_{\pi(\ell+1)} = 1,$$

where the sum is taken over all permutations of the $\ell+1$ elements and $||\pi||$ is the number of simple inversions of the permutation π. The involution is given by

$$(u^i_j)^* = (-q)^{j-i} \sum_{\pi \in S_\ell} (-q)^{||\pi||} u^{k_1}_{\pi(n_1)} u^{k_2}_{\pi(n_2)} \cdots u^{k_\ell}_{\pi(n_\ell)}$$

with $\{k_1, \cdots, k_\ell\} = \{1, 2, \cdots, \ell+1\} \setminus \{i\}$ and $\{n_1, \cdots, n_\ell\} = \{1, 2, \cdots, \ell+1\} \setminus \{j\}$ as ordered sets, and the sum is over all permutations π of the set $\{n_1, \cdots, n_\ell\}$. The Hopf algebra structure is given by

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j, \quad \epsilon(u^i_j) = \delta^i_j, \quad S(u^i_j) = (u^j_i)^*.$$

3.1.3 Irreducible representations of $U_q(\mathfrak{su}(\ell+1))$ and the related Gelfand-Tsetlin tableaux

The finite dimensional irreducible $*$-representations of $U_q(\mathfrak{su}(\ell+1))$ are indexed by $\ell-$tuples of non-negative integers $n := (n_1, n_2, ..., n_\ell)$. We denote this representation by V_n. A basis for V_n is given by Gelfand-Tsetlin (GT) tableaux that we denote it here by

$$|\underline{m}\rangle := \begin{bmatrix} m_{1,\ell+1} & m_{2,\ell+1} & \dots & m_{\ell,\ell+1} & m_{\ell+1,\ell+1} \\ m_{1,\ell} & m_{2,\ell} & \dots & m_{\ell,\ell} & \\ \vdots & \vdots & & & \\ m_{1,2} & m_{2,2} & & & \\ m_{1,1} & & & & \end{bmatrix}$$

where $n_i = m_{i,\ell+1} - m_{i+1,\ell+1}$ and $m_{i+1,j+1} \le m_{ij} \le m_{i,j+1}$ for $i = 1, 2, ..., \ell$. Fixing n_i fixes $m_{i,\ell+1}$ up to an additive constant. It is also known that two tableaux $|\underline{m}\rangle$ and $|\underline{m}'\rangle$ correspond to the same basis vector if there is a constant c (independent of i and j) such that $m_{ij} - m'_{ij} = c$. The action of generators on this basis is given by (see [23]),

$$K_k |\underline{m}\rangle = q^{\frac{a_k}{2}} |\underline{m}\rangle,$$

where

$$a_k = \sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1} - \sum_{i=1}^{k+1} m_{i,k+1} + \sum_{i=1}^{k} m_{i,k} \tag{3.2}$$
$$= 2\sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1} - \sum_{i=1}^{k+1} m_{i,k+1},$$

and the action of E_k is given by

$$E_k|\underline{m}\rangle = \sum_{j=1}^{k} A_k^j |\underline{m}_k^j\rangle, \tag{3.3}$$

where $|\underline{m}_k^j\rangle$ is obtained from $|\underline{m}\rangle$ when $m_{j,k}$ is replaced by $m_{j,k}+1$ and

$$A_k^j = \Big(- \frac{\Pi_{i=1}^{k+1}[l_{i,k+1} - l_{j,k}]\Pi_{i=1}^{k-1}[l_{i,k-1} - l_{j,k} - 1]}{\Pi_{i\neq j}[l_{i,k} - l_{j,k}][l_{i,k} - l_{j,k} - 1]} \Big)^{1/2}. \tag{3.4}$$

Here $l_{i,j} = m_{i,j} - i$, and the positive square root is taken. For the inner product $\langle \underline{m}'|\underline{m}\rangle := \delta_{\underline{m}',\underline{m}}$ this will be a $*$-representation and the matrix coefficients of $\rho^n : U_q(\mathfrak{su}(\ell+1)) \to End(V_n)$ will be $\rho^n_{\underline{m}',\underline{m}}(h) = \langle \underline{m}'|h|\underline{m}\rangle$. Note that the basic representation of $U_q(\mathfrak{su}(\ell+1))$ is given by $\sigma : U_q(\mathfrak{su}(\ell+1)) \to M_{\ell+1}(\mathbb{C})$ where

$$\sigma^i_j(K_r) = \delta^i_j q^{\frac{1}{2}(\delta_{r+1,i} - \delta_{r,i})}, \quad \sigma^i_j(E_r) = \delta^i_{r+1}\delta^r_j,$$

and the Hopf pairing $\langle\, , \rangle : U_q(\mathfrak{su}(\ell+1)) \times \mathcal{A}(SU_q(\ell+1)) \to \mathbb{C}$ is defined by $\langle h, u^i_j\rangle := \sigma^i_j(h)$. Therefore

$$\langle K_r, u^i_j\rangle = \sigma^i_j(K_r) = \delta^i_j q^{\frac{1}{2}(\delta_{r+1,i} - \delta_{r,i})},$$
$$\langle E_r, u^i_j\rangle = \sigma^i_j(E_r) = \delta^i_{r+1}\delta^r_j. \tag{3.5}$$

Using the Peter-Weyl decomposition theorem, we have $\mathcal{A}(SU_q(\ell+1)) \simeq \bigoplus_n V_n \otimes V_n$, where the sum is over all irreducible representations of $U_q(\mathfrak{su}(\ell+1))$. For any basis elements $|\underline{m}'\rangle$ and $|\underline{m}\rangle$ of V_n, $|\underline{m}'\rangle \otimes |\underline{m}\rangle$ corresponds to a basis element $t^n_{\underline{m}',\underline{m}}$. The action of $U_q(\mathfrak{su}(\ell+1))$

on $t^n_{\underline{m}',\underline{m}}$ under the Peter-Weyl isomorphism is given by

$$h \triangleright t^n_{\underline{m}',\underline{m}} = h|\underline{m}'\rangle \otimes |\underline{m}\rangle, \qquad t^n_{\underline{m}',\underline{m}} \triangleleft h = |\underline{m}'\rangle \otimes \theta(h)|\underline{m}\rangle,$$

where $\theta : U_q(\mathfrak{su}(\ell+1)) \to U_q(\mathfrak{su}(\ell+1))^{op}$ is the Hopf $*$-algebra isomorphism which is defined on generators as

$$\theta(K_i) = K_i, \quad \theta(E_i) = F_i, \quad \theta(F_i) = E_i,$$

and satisfying $\theta^2 = id$. The basis $\{t^n_{\underline{m}',\underline{m}}\}$ for $\mathcal{A}(SU_q(\ell+1))$ is implicitly given by $\langle h, t^n_{\underline{m}',\underline{m}}\rangle = \rho^n_{\underline{m}',\underline{m}}(h)$. For later use it is worth mentioning here that for $n = (0,0,...,0,1)$ these basis $t^n_{\underline{m}',\underline{m}}$ are just generators u^i_j. In order to show this, it is enough to compute $\rho^n_{\underline{m}',\underline{m}}(h)$ for generators of $U_q(\mathfrak{su}(\ell+1))$. Indeed for $n = (0,0,...,0,1)$ a basis element $|\underline{m}\rangle$ takes the following form

$$|\underline{m}\rangle := \begin{bmatrix} m & m & \dots & m & m & m-1 \\ m & m & \dots & m & m_l & \\ \vdots & \vdots & & & & \\ m & m_2 & & & & \\ m_1 & & & & & \end{bmatrix}$$

where each of the m_i's is either m or $m-1$ such that $m_1 \geq m_2 \geq ... \geq m_l$. So $|\underline{m}\rangle$ can be parametrized just by one integer i. Let us denote $|\underline{m}\rangle$ by $|i\rangle$ when $m_j = m$ for $j \leq i-1$ and $m_j = m-1$ for $j \geq i$.

$$\rho^n_{i,j}(K_r) = \langle i|K_r|j\rangle = q^{\frac{a_r}{2}}\langle i|j\rangle = q^{\frac{a_r}{2}}\delta_{i,j}.$$

where

$$a_r = 2\sum_{k=1}^{r} m_{k,r} - \sum_{k=1}^{r-1} m_{k,r-1} - \sum_{k=1}^{r+1} m_{k,r+1}.$$

So for our case we will end up with

$$\rho^n_{i,j}(K_r) = \langle i|K_r|j\rangle = q^{\alpha/2}\langle i|j\rangle = q^{\alpha/2}\delta_{i,j},$$

where

$$\alpha = \begin{cases} 0 \quad \text{if} \quad r \geq j \quad \text{or} \quad r \leq j-2 \\ 1 \quad \text{if} \quad r = j-1 \\ -1 \quad \text{if} \quad r = j. \end{cases}$$

One can easily see that $\alpha = \delta_{r+1,j} - \delta_{r,j}$ and we get the same answer as (3.5). Also we have

$$\rho^n_{i,j}(E_r) = \langle i|E_r|j\rangle = \delta^r_j \langle i|r+1\rangle = \delta_{i,r+1}\delta^r_j = \langle E_r, u^i_j\rangle,$$

which can be obtained from (3.3) and (3.4) since

$$E_r|r\rangle = A^r_r|r+1\rangle$$

and

$$A^r_r = \Big(- \frac{\Pi_{i=1}^{r+1}[l_{i,r+1} - l_{r,r}]\Pi_{i=1}^{r-1}[l_{i,r-1} - l_{r,r} - 1]}{\Pi_{i\neq j}[l_{i,r} - l_{r,r}][l_{i,r} - l_{r,r} - 1]} \Big)^{1/2}.$$

The fact that only A^r_r contributes in the summation (3.3) is simply because of the form of $|r\rangle$. We also have $E_r|j\rangle = 0$ if $j \neq r$. The value of this fraction is one since both the numerator and the denominator are equal to $[r]![r-1]!$. In particular we have $t^n_{\ell+1,j} = u^{\ell+1}_j = z_j$, the generators of the quantum sphere $\mathcal{A}(S^{2\ell+1}_q)$ to be defined in the next section.

3.2 The complex structure of $\mathbb{C}P^\ell_q$

In this section we shall define a complex structure on $\mathbb{C}P^\ell_q$ and its canonical line bundles following closely [11]. For a review on general setup of noncommutative complex geometry, we refer the reader to the first chapter.

3.3 Noncommutative complex geometry of $\mathbb{C}P_q^\ell$

In this section we recall the definition of the quantum projective space $\mathbb{C}P_q^\ell$ as the quantum homogeneous space of the quantum group $SU_q(\ell+1)$ and its quantum subgroup $U_q(\ell)$ from [11].

3.3.1 $\mathbb{C}P_q^\ell$ and the associated quantum line bundles

Let $\hat{K} := (K_1 K_2^2 \cdots K_\ell^\ell)^{2/\ell+1}$ and $\mathcal{L}_h a := a \triangleleft S^{-1}(h)$. Then we define the quantum $2\ell+1$ sphere as

$$\mathcal{A}(S_q^{2\ell+1}) := \{a \in \mathcal{A}(SU_q(\ell+1)) |\, \mathcal{L}_h(a) = \epsilon(h)a, \quad \forall h \in U_q(\mathfrak{su}(\ell))\}.$$

The invariant elements of this space under the action of $\hat{K}$ will provide the coordinate functions of the quantum projective space

$$\mathcal{A}(\mathbb{C}P_q^\ell) := \{a \in \mathcal{A}(S_q^{2\ell+1}) | \mathcal{L}_{\hat{K}} a = a\}.$$

The space of sections of the canonical line bundles L_N, $N \in \mathbb{Z}$, are defined by

$$L_N := \{a \in \mathcal{A}(S_q^{2\ell+1}) | \mathcal{L}_{\hat{K}} a = q^{\frac{N\ell}{\ell+1}} a\}. \tag{3.6}$$

Let

$$M_{jk} := [E_j, [E_{j+1}, ..., [E_{k-1}, E_k]_q ...]_q]_q \quad \text{for } 1 \leq j < k \leq \ell,$$

and

$$N_{jk} := (K_j K_{j+1} ... K_\ell).(K_{k+1} K_{k+2} ... K_\ell).\hat{K}^{-1} \quad \text{for } 1 \leq j < k \leq \ell.$$

Let $X_i := N_{i\ell} M_{i\ell}^*$ for $i = 1, ..., \ell$. We will also use a right black action defined by $h \blacktriangleright a := a \triangleleft \theta(h)$.

For any r-dimensional $*$-representation of $U_q(\mathfrak{u}(\ell))$ like σ, we define the $\mathcal{A}(\mathbb{C}P_q^\ell)$-bimodule $\mathfrak{M}(\sigma) := \{v \in \mathcal{A}(SU_q(\ell+1))^r \,|\, v \triangleleft h = \sigma(h)v, \forall h \in U_q(\mathfrak{u}(\ell))\}$ [11, 12]. Suppose that σ_1^N is obtained from the basic representation $\sigma_1 : U_q(\mathfrak{su}(\ell)) \to \mathrm{End}(\mathbb{C}^\ell)$ lifted to a representation of $U_q(\mathfrak{u}(\ell))$ by $\sigma_1^N(\hat{K}) = q^{1-\frac{\ell N}{\ell+1}} Id_{\mathbb{C}^\ell}$. Then the space of anti-holomorphic 1-forms is given by $\Omega^{(0,1)} := \mathfrak{M}(\sigma_1^0)$. Hence, any anti-holomorphic 1-form is a ℓ-tuple $v := (v_1, ..., v_\ell)$ such that $v \triangleleft h = \sigma_1^0(h)v$. The complex structure of $\mathbb{C}P_q^\ell$ is given by

$$\overline{\partial} := \sum \mathcal{L}_{\hat{K}X_i} \otimes \mathfrak{e}^L_{e^i}.$$

Here e_i's are elements of the standard basis and $\mathfrak{e}^L_{e^i}$ is the left exterior product by e_i. We show that on $\mathcal{A}(\mathbb{C}P_q^\ell)$ we have

$$\overline{\partial}a = -\Big(a \triangleleft F_\ell F_{\ell-1}...F_1, a \triangleleft F_\ell F_{\ell-1}...F_2, ..., a \triangleleft F_\ell F_{\ell-1}, a \triangleleft F_\ell\Big). \tag{3.7}$$

In fact,

$$\begin{aligned}
\mathcal{L}_{X_i} a &= a \triangleleft S^{-1}\Big(\hat{K}K_\ell K_{\ell-1}\cdots K_i \hat{K}^{-1}[...[[F_\ell, F_{\ell-1}]_q, F_{\ell-2}]_q, ..., F_i]_q\Big) \\
&= (-q^{-1})^{\ell-i}(-q)^{\ell-i+1} a \triangleleft F_\ell F_{\ell-1}\cdots F_i \hat{K} K_i^{-1} K_{i+1}^{-1} ... K_\ell^{-1} \hat{K}^{-1} \\
&= (-1)^{2(\ell-i)+1} a \triangleleft \hat{K} K_i^{-1} K_{i+1}^{-1} ... K_\ell^{-1} \hat{K}^{-1} F_\ell F_{\ell-1} ... F_i \\
&= -a \triangleleft F_\ell F_{\ell-1} ... F_i
\end{aligned}$$

Here we used the commutation relations (3.1). The only order of F_j's in the commutators that takes part in computation is $F_\ell F_{\ell-1}...F_i$ and others vanish because $a \triangleleft F_j = 0$ for $j < \ell$, $a \in \mathcal{A}(\mathbb{C}P_q^\ell)$. Note that, all elements of $\mathcal{A}(\mathbb{C}P_q^\ell)$ are fixed under the of action of all K_i's.

We would like to find a basis for the space of sections of the canonical quantum line bundles L_N. Note that $L_0 = \mathcal{A}(\mathbb{C}P_q^\ell)$. By (3.6), the conditions that must hold are as follows

$$\begin{aligned}
&K_i \blacktriangleright a = a, \quad E_i \blacktriangleright a = F_i \blacktriangleright a = 0, \quad i = 1, 2, ..., \ell - 1, \\
&K_1 K_2^2 ... K_\ell^\ell \blacktriangleright a = q^{-N\ell/2} a.
\end{aligned} \tag{3.8}$$

Proposition 3.3.1. *Let N be an integer. The set equations (3.8) forces the tableaux $|\underline{m}\rangle$, corresponding to the element $a = t^n_{\underline{m}',\underline{m}}$, be of the form of*

$$\begin{bmatrix} m_{1,\ell+1} & m & \dots & m & 2m - m_{1,\ell+1} + N \\ m & m & \dots & m & \\ \vdots & \vdots & & & \\ m & m & & & \\ m & & & & \end{bmatrix} \quad or \quad \begin{bmatrix} 2k+N & k & \dots & k & 0 \\ k & k & \dots & k & \\ \vdots & \vdots & & & \\ k & k & & & \\ k & & & & \end{bmatrix}.$$

Proof. $K_1 \blacktriangleright a = a$ and $E_1 \blacktriangleright a = 0$ give the equality for $m_{11} = m_{12} = m_{22}$. We know that $K_k \blacktriangleright a = q^{\frac{a_k}{2}} a$, where a_k is given by (3.2). For instance $a_1 = 2m_{11} - m_{12} - m_{22}$ and $a_2 = 2(m_{12} + m_{22}) - m_{11} - (m_{13} + m_{23} + m_{33})$ and so on. By (3.3) and (3.4) we have

$$\begin{aligned} E_1|\underline{m}\rangle &= \Big(- [m_{11} - m_{12}][m_{11} - m_{22} + 1]\Big)^{1/2} |\underline{m}_1^1\rangle \\ E_2|\underline{m}\rangle &= \Big(\frac{[m_{13} - m_{12}][m_{23} - m_{12} - 1][m_{33} - m_{12} - 2][m_{12} - m_{11} + 1]}{[m_{12} - m_{22} + 1][m_{12} - m_{22} + 2]}\Big)^{\frac{1}{2}} |\underline{m}_2^1\rangle \\ &+ \Big(\frac{[m_{13} - m_{22} + 1][m_{23} - m_{22}][m_{33} - m_{22} - 1][m_{11} - m_{22}]}{[m_{12} - m_{22} + 1][m_{12} - m_{22}]}\Big)^{\frac{1}{2}} |\underline{m}_2^2\rangle. \end{aligned}$$

and

$$\begin{aligned} F_2|\underline{m}\rangle &= \Big(\frac{[m_{13} - m_{12} + 1][m_{23} - m_{12}][m_{33} - m_{12} - 1][m_{11} - m_{12} - 1]}{[m_{12} - m_{22}][m_{12} - m_{22} + 1]}\Big)^{\frac{1}{2}} |\underline{m}_2^{-1}\rangle \\ &+ \Big(\frac{[m_{13} - m_{22} + 2][m_{23} - m_{22} + 1][m_{33} - m_{22}][m_{11} - m_{22} - 2]}{[m_{12} - m_{22} + 1][m_{12} - m_{22} + 2]}\Big)^{\frac{1}{2}} |\underline{m}_2^{-2}\rangle. \end{aligned}$$

Now it is not difficult to see that $K_1|\underline{m}\rangle = |\underline{m}\rangle$ and $E_1|\underline{m}\rangle = 0$, imposing

$$2m_{11} - m_{12} - m_{22} = 0,$$

$$m_{11} - m_{12} = 0.$$

So $m_{11} = m_{12} = m_{22}$. In the same manner $K_2|\underline{m}\rangle = |\underline{m}\rangle$, $E_2|\underline{m}\rangle = 0$ and $F_2|\underline{m}\rangle = 0$ give

$$\begin{aligned} 2m_{12} + 2m_{22} - m_{11} - m_{13} - m_{23} - m_{33} &= 0, \\ m_{12} - m_{13} &= 0, \\ m_{22} - m_{33} &= 0. \end{aligned}$$

So we have $m_{11} = m_{12} = m_{22} = m_{13} = m_{23} = m_{33}$. Suppose that rows 1 to k with $k+1 < \ell+1$, have been found equal to m. Let us prove that $E_k|\underline{m}\rangle = 0$ and $F_k|\underline{m}\rangle = 0$ will make the equality of all elements up to and including row $k+1$. First note that in row $k+1$, we have $m_{2,k+1} = ... = m_{k,k+1} = m$. Let us look at A_k^1.

$$\begin{aligned} A_k^1 &= \Big(- \frac{\Pi_{i=1}^{k+1}[l_{i,k+1} - l_{1,k}]\Pi_{i=1}^{k-1}[l_{i,k-1} - l_{1,k} - 1]}{\Pi_{i\neq j}[l_{i,k} - l_{1,k}][l_{i,k} - l_{1,k} - 1]} \Big)^{1/2} \\ &= \Big(- \frac{[l_{1,k+1} - l_{1,k}]...[l_{k+1,k+1} - l_{1,k}][l_{1,k-1} - l_{1,k} - 1]...[l_{k-1,k-1} - l_{1,k} - 1]}{\Pi_{i\neq 1}[l_{i,k} - l_{1,k}][l_{i,k} - l_{1,k} - 1]} \Big)^{1/2}. \end{aligned}$$

It is not hard to see that $A_k^1 = 0$ if $[l_{1,k+1} - l_{1,k}] = [m_{1,k+1} - m_{1,k}] = 0$. So $m_{1,k+1} = m_{1,k} = m$ and by a similar observation the action of F_k gives the equality $m_{k,k+1} = m_{kk} = m$. But to get to the very top row we need to use the action of K_ℓ. We have

$$\begin{aligned} a_\ell &= 2\sum_{i=1}^{\ell} m_{i,\ell} - \sum_{i=1}^{\ell-1} m_{i,\ell-1} - \sum_{i=1}^{\ell+1} m_{i,\ell+1} \\ &= 2\ell m - (\ell-1)m - m_{1,\ell+1} - (\ell-1)m - m_{\ell+1,\ell+1} \\ &= 2m - m_{1,\ell+1} - m_{\ell+1,\ell+1}. \end{aligned}$$

Since $\ell a_\ell/2 = -N\ell/2$, we see that $m_{1,\ell+1} = 2m - m_{\ell+1,\ell+1} + N$.

So we will find a Peter-Weyl basis for line bundles L_N as $\langle t^n_{\underline{0},\underline{m}}\rangle$, where

$$n = (n_1 + N, 0, ..., 0, n_1)$$

and

$$|\underline{0}\rangle = \begin{bmatrix} 2m - m_{\ell+1,\ell+1} + N & m & \dots & m & m_{\ell+1,\ell+1} \\ m & m & \dots & m & \\ \vdots & \vdots & & & \\ m & m & & & \\ m & & & & \end{bmatrix}. \tag{3.9}$$

Assuming $k = m - m_{\ell+1,\ell+1}$, this tableaux is equivalent to the following tableaux already presented in [8].

$$\begin{bmatrix} 2k+N & k & \dots & k & 0 \\ k & k & \dots & k & \\ \vdots & \vdots & & & \\ k & k & & & \\ k & & & & \end{bmatrix}. \tag{3.10}$$

□

Therefore we have

$$L_N \simeq \bigoplus_{k\geq 0} V_{(k+N,0,\dots,0,k)};\ N > 0, \qquad L_N \simeq \bigoplus_{k\geq -N} V_{(k+N,0,\dots,0,k)};\ N \leq 0.$$

Theorem 3.3.1. *Let N be an integer. Then*

$$dimKerE_\ell\Big|_{L_N} = \binom{|N|+\ell}{\ell} \qquad\qquad if \quad N \leq 0,$$
$$dimKerE_\ell\Big|_{L_N} = 0 \qquad\qquad if \quad N > 0.$$

Proof. First, one can see that $E_\ell \blacktriangleright t^n_{\underline{m}',\underline{m}} = \gamma^n_{\underline{m}',\underline{m}} t^n_{\underline{m}',\underline{m}''}$, where,

$$\gamma^n_{\underline{m}',\underline{m}} = \sqrt{\frac{[k+N][k+\ell]}{[\ell]}}$$

Indeed,

$$E_\ell|\underline{m}\rangle = \sum A_\ell^j|\underline{m}_\ell^j\rangle = A_\ell^1|\underline{m}_\ell^1\rangle$$

Other A_ℓ^j vanish because of the existence of the factor $\Pi_{i=1}^{\ell+1}[l_{i,\ell+1}-l_{j,\ell}]$. For each $j = 2, \cdots, \ell$, one of the brackets would be zero. For the coefficient A_ℓ^1, we have

$$\begin{aligned}
A_\ell^1 &= \Big(- \frac{\Pi_{i=1}^{\ell+1}[l_{i,\ell+1} - l_{1,\ell}]\Pi_{i=1}^{\ell-1}[l_{i,\ell-1} - l_{1,\ell} - 1]}{\Pi_{i\neq j}[l_{i,\ell} - l_{1,\ell}][l_{i,\ell} - l_{1,\ell} - 1]}\Big)^{1/2} \\
&= \Big(- \frac{[l_{1,\ell+1} - l_{1,\ell}]...[l_{\ell+1,\ell+1} - l_{1,\ell}][l_{1,\ell-1} - l_{1,\ell} - 1]...[l_{\ell-1,\ell-1} - l_{1,\ell} - 1]}{\Pi_{i\neq 1}[l_{i,\ell} - l_{1,\ell}][l_{i,\ell} - l_{1,\ell} - 1]}\Big)^{1/2} \\
&= \sqrt{\frac{[k+N][l-1]!^2[k+\ell]}{[l-1]![l]!}} = \sqrt{\frac{[k+N][k+\ell]}{[\ell]}}
\end{aligned}$$

Now, let $\xi = \sum c^n_{\underline{m}',\underline{m}} t^n_{\underline{m}',\underline{m}} \in L_N$,

$$E\xi = E\sum c^n_{\underline{m}',\underline{m}} t^n_{\underline{m}',\underline{m}} = \sum c^n_{\underline{m}',\underline{m}} \gamma^n_{\underline{m}',\underline{m}} t^n_{\underline{m}',\underline{m}''}.$$

Here, $|\underline{m}'\rangle = |\underline{0}\rangle$, as given by (3.9) or (3.10). For $N > 0$, $\gamma^n_{\underline{m}',\underline{m}}$ is never zero, but for $N \leq 0$, $\gamma^n_{\underline{m}',\underline{m}} = 0$ iff $k = -N$. This implies that

$$|\underline{m}'\rangle = \begin{bmatrix} -N & -N & \dots & -N & 0 \\ x_{1,\ell} & x_{2,\ell} & \dots & x_{\ell,\ell} & \\ \vdots & \vdots & & & \\ x_{1,2} & x_{2,2} & & & \\ x_{1,1} & & & & \end{bmatrix} = \begin{bmatrix} -N & -N & \dots & -N & 0 \\ -N & -N & \dots & x_\ell & \\ \vdots & \vdots & & & \\ -N & x_2 & & & \\ x_1 & & & & \end{bmatrix}$$

with $x_i = x_{i,i}$. The question turns into a simple combinatorial problem of counting the number of non-decreasing sequences $-N \geq x_1 \geq x_2 \geq ... \geq x_\ell \geq 0$, which is $\binom{|N|+l}{\ell}$. $\square$

Corollary 3.3.1. *There are no non-constant holomorphic polynomials in* $\mathcal{A}(\mathbb{C}P_q^\ell)$.

Proof. By (3.7) it is obvious that $\overline{\partial}a = 0$ iff $E_\ell \blacktriangleright a = 0$. Now the previous lemma for $N = 0$ gives the result. $\square$

3.3.2 Holomorphic line bundles

An anti-holomorphic connection on the line bundle L_N is given by

$$\nabla_N^{\overline{\partial}} : L_N \to \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_N$$
$$\nabla_N^{\overline{\partial}}(\xi) := \Psi_N^\dagger \overline{\partial} \Psi_N,$$

where Ψ_N is a column vector [11], given by $\Psi_N := (\psi^N_{j_1,\dots,j_{\ell+1}})$ with

$$\psi^N_{j_1,\dots,j_{\ell+1}} := [j_1,...,j_{\ell+1}]!^{1/2}(z_1^{j_1}...z_{\ell+1}^{j_{\ell+1}})^*, \quad \forall\, j_1+...+j_{\ell+1}=N, \quad \text{for} \quad N \geq 0,$$

which are a generating family of L_N for $N \geq 0$ as one-sided and as bimodule [10]. For $N \leq 0$ a generating family of L_N is given by

$$\psi^N_{j_1,\dots,j_{\ell+1}} := [j_1,...,j_{\ell+1}]!^{1/2} q^{\sum_{r=1}^{\ell+1} rj_r} z_1^{j_1}...z_{\ell+1}^{j_{\ell+1}}, \quad \forall\, j_1+...+j_{\ell+1}=-N, \quad \text{for} \quad N \leq 0.$$

This is a flat connection as can be verified directly with a computation as previous chapter. This gives us the following Dolbeault complex

$$0 \to L_N \to \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_N \to \cdots \to \Omega^{(0,\ell)} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_N \to 0\,.$$

The structure of the zeroth cohomology group $H^0(L_N, \nabla_N^{\overline{\partial}})$ of this complex which is called the *space of holomorphic sections of* L_N, is best described by the following theorem.

Corollary 3.3.2. *For any integer N, the space of holomorphic sections of the canonical line bundles of $\mathbb{C}P_q^\ell$ is*

$$\begin{cases} H^0(L_N, \nabla_N^{\overline{\partial}}) \simeq \mathbb{C}^{\binom{|N|+\ell}{\ell}}, & \textit{if} \quad N \leq 0 \\ H^0(L_N, \nabla_N^{\overline{\partial}}) = 0, & \textit{if} \quad N > 0 \end{cases}$$

Proof. It is not difficult to see that the kernel of $\nabla_N^{\overline{\partial}}$ coincides with the kernel of $E_\ell \blacktriangleright (.)$. Now the result is an obvious consequence of theorem (3.3.1). $\square$

Alternative proof of the Corollary 3.3.2 without Theorem 3.3.1.

By Lemma 6.1. [11], we know that $\nabla_N^{\overline{\partial}} = \overline{\partial}|_{\Omega_N^0}$, where $\Omega_N^0 = L_N$. From [12] Prop. 6.4 and $\overline{\partial}^{\dagger}|_{\Omega_N^0} = 0$, we have:

$$H^0(L_N, \nabla_N^{\overline{\partial}}) = ker\, \overline{\partial}^{\dagger}\overline{\partial}|_{\Omega_N^0}.$$

Lemma 6.3 and 6.5 of [11] gives:

$$q^{\ell+1}\overline{\partial}^{\dagger}|_{\Omega_N^0} = q^{\frac{2N}{\ell+1}}\mathcal{C}_q + \frac{q^{N-\ell}[N] - q^{\frac{N}{\ell+1}}[\frac{N}{\ell+1}][\ell+1]}{q^{-1}-q},$$

where $\mathcal{C}_q$ is the Casimir. Prop. 5.5 of [11] gives the following decomposition for Ω_N^0.

$$\Omega_N^0 \simeq \bigoplus_{m\geq 0} V_{(m+N,0,\ldots,0,m)};\ N > 0, \qquad \Omega_N^0 \simeq \bigoplus_{m\geq -N} V_{(m+N,0,\ldots,0,m)};\ N \leq 0.$$

The operator $\overline{\partial}^{\dagger}\overline{\partial}$ is constant on each subspace $V_{(m+N,0,\ldots,0,m)}$ and its value can be obtained from Prop. 3.3 [11] or Lemma 3.4 (for $k = 1$, $n_1 = m + N$ and $n_\ell = m$ in the formula (3.17)). For example, if $q = 1$ we have

$$\overline{\partial}^{\dagger}\overline{\partial}|_{V_{(m+N,0,\ldots,0,m)}} = (m+\ell)(N+m),$$

This vanishes if and only if $m = -N$, which holds ony if $N \leq 0$. If $q \neq 1$ the formula is more complicated but the same result holds. Therefore

$$\dim H^0(L_N, \nabla_N^{\overline{\partial}}) = \begin{cases} 0 \qquad \text{if} \quad N > 0, \\ \dim V_{(0,0,\ldots,0,-N)} = \binom{-N+\ell}{\ell} \qquad \text{if} \quad N \leq 0. \end{cases}$$

Here we would like to establish the fact that for any integers N and M we have a bimodule isomorphism $L_N \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_M \simeq L_{N+M}$. The multiplication map from left to right is an injective $\mathcal{A}(\mathbb{C}P_q^\ell)$-bilinear map. Indeed, it is enough to show that $L_N \otimes_{L_0} L_1 \simeq L_{N+1}$ for all $N \in \mathbb{Z}$ and $L_1 \otimes_{L_0} L_{-1} \simeq L_0$. Let

$$\phi : L_{N+1} \to, L_N \otimes L_1, \quad \phi(\eta) = \sum_k \eta z_k \otimes z_k^*,$$

$$\chi : L_0 \to, L_1 \otimes L_{-1}, \quad \chi(a) = \sum_k q^{6-2k} a z_k^* \otimes z_k$$

Now one can see that,

$$m \circ \phi(\eta) = m(\sum_k \eta z_k \otimes z_k^*) = \eta \sum_k z_k z_k^* = \eta,$$

$$m \circ \chi(a) = m(\sum_k q^{6-2k} a z_k^* \otimes z_k) = a \sum_k q^{6-2k} z_k^* z_k = a.$$

Elements z_i (resp. z_i^*) are generating family of L_1 (resp. L_{-1}), so any element $\eta \in L_N \otimes L_1$ can be written as $\sum \eta_i \otimes z_i^*$ with $\eta_i \in L_N$, and any element $\xi \in L_1 \otimes_{L_0} L_{-1}$ can be written as $\xi = \sum \xi_i \otimes z_i$, with $\xi_i \in L_1$.

$$\phi \circ m(\eta) = \phi(\sum \eta_i z_i^*) = \sum_{i,k} \eta_i z_i^* z_k \otimes z_k^* = \sum_{i,k} \eta_i \otimes z_i^* z_k z_k^* = \sum_i \eta_i \otimes z_i^* = \eta,$$

$$\chi \circ m(\xi) = \chi(\sum \xi_i z_i) = \sum_{i,k} q^{6-2k} \xi_i z_i z_k^* \otimes z_k = \sum_{i,k} q^{6-2k} \xi_i \otimes z_i z_k^* z_k = \sum_i \xi_i \otimes z_i = \xi.$$

Here we used the fact that $z_i z_k^*$ and $z_i^* z_k$ belong to L_0 and also

$$\sum_k z_k z_k^* = 1, \qquad \sum_k q^{6-2k} z_k^* z_k = 1.$$

Alternatively, using generating elements $\psi_{\underline{j}}$ and $\psi_{\underline{j}}^*$ one can define

$$\phi : L_{N+M} \to L_N \otimes L_M$$

$$\phi(\eta) = \sum_k \eta (\psi_{\underline{k}}^M)^* \otimes \psi_{\underline{k}}^M, \quad M \geq 0,\, k_1 + ... + k_{\ell+1} = M.$$

Here, one uses $\sum_{\underline{k}} (\psi_{\underline{k}}^M)^* \psi_{\underline{k}}^M = 1$. The same idea works for $M \leq 0$ and $k_1 + ... + k_{\ell+1} = -M$.

To see that the map m is a surjection, we use a PBW-basis for $\mathcal{A}(S_q^{2\ell+1})$ generated by

$$\{z_1^{s_1}z_2^{s_2}\cdots z_\ell^{s_\ell}(z_1^*)^{t_1}(z_2^*)^{t_2}\cdots(z_{\ell-1}^*)^{t_{\ell-1}},\, z_1^{s_1}z_2^{s_2}\cdots z_{\ell-1}^{s_{\ell-1}}(z_1^*)^{t_1}(z_2^*)^{t_2}\cdots(z_\ell^*)^{t_\ell}\},$$

for non-negative integers s_i and t_i. Since

$$K_j \blacktriangleright z_i = z_i, \quad K_j \blacktriangleright z_i^* = z_i^*,\ j<\ell$$

and

$$K_\ell \blacktriangleright z_i = q^{1/2}z_i, \quad K_\ell \blacktriangleright z_i^* = q^{-1/2}z_i^*,$$

we have

$$K_1K_2^2\cdots K_\ell^\ell \blacktriangleright Z = q^{\ell/2\{\sum s_i-\sum t_i\}}Z,$$

where

$$Z = z_1^{s_1}z_2^{s_2}\cdots z_\ell^{s_\ell}(z_1^*)^{t_1}(z_2^*)^{t_2}\cdots(z_{\ell-1}^*)^{t_{\ell-1}}$$

or

$$Z = z_1^{s_1}z_2^{s_2}\cdots z_{\ell-1}^{s_{\ell-1}}(z_1^*)^{t_1}(z_2^*)^{t_2}\cdots(z_\ell^*)^{t_\ell}.$$

It is obvious that $Z\in L_N$ iff $\sum s_i - \sum t_i = -N$.

Now suppose that $Z = z_1^{s_1}z_2^{s_2}\cdots z_\ell^{s_\ell}(z_1^*)^{t_1}(z_2^*)^{t_2}\cdots(z_{\ell-1}^*)^{t_{\ell-1}} \in L_{N+M}$ and suppose k is the first positive integer such that $\sum_{i=1}^k s_i > N$. Then take a partition of N as $\sum_{i=1}^k r_i = N$, such that $s_i - r_i \geq 0$. Now the following is a preimage of Z.

$$q^R\big(z_1^{r_1}z_2^{r_2}\cdots z_k^{r_k}\otimes z_1^{s_1-r_1}z_2^{s_2-r_2}\cdots z_k^{s_k-r_k}z_{k+1}^{r_{k+1}}\cdots z_\ell^{s_\ell}(z_1^*)^{t_1}(z_2^*)^{t_2}\cdots(z_{\ell-1}^*)^{t_{\ell-1}}\big),$$

where

$$R = r_k\{(s_{k-1}-r_{k-1})+\cdots+(s_1-r_1)\}+r_{k-1}\{(s_{k-2}-r_{k-2})+\cdots+(s_1-r_1)\}+\cdots+r_2(s_1-r_1).$$

By the above discussion it is obvious that $Z_1 := z_1^{r_1}z_2^{r_2}\cdots z_k^{r_k}\in L_N$ and

$$Z_2 := z_1^{s_1-r_1}z_2^{s_2-r_2}\cdots z_k^{s_k-r_k}z_{k+1}^{r_{k+1}}\cdots z_\ell^{s_\ell}(z_1^*)^{t_1}(z_2^*)^{t_2}\cdots(z_{\ell-1}^*)^{t_{\ell-1}}\in L_M.$$

The result is obtained by noting that the product $Z_1 Z_2 = q^{-R} Z$.

For later use we would like to mention here that $\Omega^{(0,\ell)} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_N \simeq L_{-\ell-1} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_N \simeq L_{N-\ell-1}$. In order to see this we recall the definition of $\Omega^{(0,\ell)} := \mathfrak{M}(\sigma_\ell^0)$, where σ_k^0 is obtained from the representation $\sigma_k : U_q(\mathfrak{su}(\ell)) \to \mathrm{End}(W_k)$ lifted to a representation of $U_q(\mathfrak{u}(l))$ by $\sigma_k^0(\hat{K}) = q^k Id_{W_k}$ [11]. We define the $\mathcal{A}(\mathbb{C}P_q^\ell)$-bimodule $\mathfrak{M}(\sigma) := \{v \in \mathcal{A}(SU_q(\ell+1))^r \,|\, v \triangleleft h = \sigma(h)v, \forall h \in U_q(\mathfrak{u}(\ell))\}$, where σ is an r-dimensional $*$-representation of $U_q(\mathfrak{u}(\ell))$. So in our case σ_ℓ^0 will be a 1-dimensional $*$-representation of $U_q(\mathfrak{su}(\ell))$. Hence, any anti-holomorphic ℓ-form is an element like v such that $v \triangleleft h = \sigma_\ell^0(h)v$. The conditions that must hold are:

$$
\begin{aligned}
&K_i \blacktriangleright a = a, \quad E_i \blacktriangleright a = F_i \blacktriangleright a = 0, \quad i = 1, 2, ..., \ell - 1. \\
&K_1 K_2^2 ... K_\ell^\ell \blacktriangleright a = q^{\ell(\ell+1)/2} a.
\end{aligned}
$$

This gives us $\Omega^{(0,\ell)} \simeq L_{-\ell-1}$.

3.4 Bimodule connections

In this section we would like to show that line bundles L_N accept a bimodule connection in the sense of [19]. This means that there exists an isomorphism $\lambda_N : L_N \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} \Omega^{(0,1)} \to \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_N$ such that

$$
\nabla_N^{\overline{\partial}}(\xi a) := (\nabla_N^{\overline{\partial}} \xi) a + \lambda_N(\xi \otimes a).
$$

Let us define

$$
\Omega_N^{(0,1)} := \{\omega = (\omega_1, \cdots, \omega_\ell)\} \,|\quad \omega \triangleleft h = \sigma_1^N(h)\omega,\ \forall h \in U_q(\mathfrak{u}(\ell))\},
$$

where $U_q(\mathfrak{u}(\ell))$ and σ_1^N are introduced in Section 3.3. One can prove that the multiplication map $m_N : \Omega^{(0,1)} \otimes_{L_0} L_N \to \Omega_N^{(0,1)}$, where $m_N((v_1, \cdots, v_\ell) \otimes \xi) = (v_1\xi, \cdots, v_\ell\xi)$, gives an isomorphism of $\mathcal{A}(\mathbb{C}P_q^\ell)$-bimodules. The same is true for $m'_N : L_N \otimes_{L_0} \Omega^{(0,1)} \to \Omega_N^{(0,1)}$ given by $m'_N(\xi \otimes (v_1, \cdots, v_\ell)) = (\xi v_1, \cdots, \xi v_\ell)$. We just give the proof of the former case for

$N \geq 0$. One can easily check that the map

$$\phi_N : \Omega_N^{(0,1)} \to \Omega^{(0,1)} \otimes_{L_0} L_N,$$

defined as

$$\phi_N(\omega_1, \cdots, \omega_\ell) = (\sum_{\underline{k}} \omega_1 (\psi_{\underline{k}}^N)^* \otimes \psi_{\underline{k}}^N, \cdots, \sum_{\underline{k}} \omega_\ell (\psi_{\underline{k}}^N)^* \otimes \psi_{\underline{k}}^N)$$

is the inverse of m_N. Now we define the map $\lambda_N : L_N \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} \Omega^{(0,1)} \to \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_N$ as follows:

$$\lambda_N := q^N \phi'_N m_N,$$

where, ϕ'_N is the inverse of multiplication map m'_N. In fact

$$\lambda_N(\xi \otimes v) = q^N \phi'_N(\xi v) = q^N \sum_{\underline{k}} (\xi v)(\psi_{\underline{k}}^N)^* \otimes \psi_{\underline{k}}^N.$$

Let us mention that why we put the factor q^N in the definition of λ_N. A simple computation shows that

$$\begin{aligned} \nabla_N^{\overline{\partial}} \xi &= -\Psi_N^\dagger \Big((\Psi_N \xi) \triangleleft F_\ell \cdots F_1, \cdots, (\Psi_N \xi) \triangleleft F_\ell \Big) \\ &= -q^{N/2} \Big(\xi \triangleleft F_\ell \cdots F_1, \cdots, \xi \triangleleft F_\ell \Big). \end{aligned}$$

Hence,

$$\begin{aligned} \nabla_N^{\overline{\partial}}(a\xi) &= -q^{N/2} \Big\{ a \Big(\xi \triangleleft F_\ell \cdots F_1, \cdots, \xi \triangleleft F_\ell \Big) + q^{-N/2} \Big(a \triangleleft F_\ell \cdots F_1, \cdots, a \triangleleft F_\ell \Big) \xi \Big\} \\ &= -\Big\{ q^{N/2} a \Big(\xi \triangleleft F_\ell \cdots F_1, \cdots, \xi \triangleleft F_\ell \Big) + \Big(a \triangleleft F_\ell \cdots F_1, \cdots, a \triangleleft F_\ell \Big) \xi \Big\} \\ &= a \nabla_N^{\overline{\partial}}(\xi) + \overline{\partial} a \xi, \end{aligned}$$

which is the left Leibniz property. On the other hand

$$\begin{aligned}
\nabla_N^{\overline{\partial}}(\xi a) &= -q^{N/2}\Big\{\Big(\xi \triangleleft F_\ell \cdots F_1, \cdots, \xi \triangleleft F_\ell\Big)a + q^{N/2}\xi\Big(a \triangleleft F_\ell \cdots F_1, \cdots, a \triangleleft F_\ell\Big)\Big\} \\
&= -\Big\{q^{N/2}\Big(\xi \triangleleft F_\ell \cdots F_1, \cdots, \xi \triangleleft F_\ell\Big)a + q^N\Big(a \triangleleft F_\ell \cdots F_1, \cdots, a \triangleleft F_\ell\Big)\xi\Big\} \\
&= \nabla_N^{\overline{\partial}}(\xi)a + q^N\overline{\partial}a\xi, \qquad (3.11)
\end{aligned}$$

Indeed for the ℓ th component we have

$$\begin{aligned}
\Psi_N^\dagger((\Psi_N\xi a) \triangleleft F_\ell) &= \Psi_N^\dagger\Big\{(\Psi_N \triangleleft F_\ell)((\xi a) \triangleleft K_\ell) + (\Psi_N \triangleleft K_\ell^{-1})((\xi a) \triangleleft F_\ell)\Big\} \\
&= q^{N/2}(\xi a) \triangleleft F_\ell \qquad (3.12) \\
&= q^{N/2}\Big\{(\xi \triangleleft F_\ell)a + q^{N/2}\xi(a \triangleleft F_\ell)\Big\}.
\end{aligned}$$

Here we used $\xi \triangleleft K_\ell = q^{-N/2}\xi$ and $\Psi_N \triangleleft K_\ell = q^{-N/2}\Psi_N$. Other components can be obtained as follows:

$$\begin{aligned}
&\Psi_N^\dagger((\Psi_N\xi a) \triangleleft F_\ell F_{\ell-1} \cdots F_i) \\
&\quad = \Psi_N^\dagger\Big\{(\Psi_N \triangleleft F_\ell)((\xi a) \triangleleft K_\ell) + (\Psi_N \triangleleft K_\ell^{-1})((\xi a) \triangleleft F_\ell)\Big\}F_{\ell-1} \cdots F_i \\
&\quad = q^{N/2}\Big\{(\xi \triangleleft F_\ell)a + q^{N/2}\xi(a \triangleleft F_\ell)\Big\}F_{\ell-1} \cdots F_i \\
&\quad = q^{N/2}(\xi \triangleleft F_\ell \cdots F_i)a + q^N\xi(a \triangleleft F_\ell \cdots F_i).
\end{aligned}$$

In fact (3.11) says that $\nabla^{\overline{\partial}}$ does not satisfy a right Leibniz rule, instead it enjoys a λ_N-twisted right Leibniz property.

Proposition 3.4.1. *Taking λ_N as above, the following holds*

$$\nabla_N^{\overline{\partial}}(\xi a) = (\nabla_N^{\overline{\partial}}\xi)a + \lambda_N(\xi \otimes \overline{\partial}a), \quad \forall a \in \mathcal{A}(\mathbb{C}P_q^\ell), \quad \forall \xi \in L_N,$$

i.e. $\nabla_N^{\overline{\partial}}$ is a bimodule connection on L_N.

Proof. By the above discussion, the proof is obvious. □

Now we can prove that the two holomorphic structures on $L_N \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_M$ and L_{N+M}

are identical after the canonical isomorphism of these two spaces.

Proposition 3.4.2. *The tensor product connection* $\nabla_N^{\overline{\partial}} \otimes 1 + (\lambda_N \otimes 1)(1 \otimes \nabla_M^{\overline{\partial}})$ *coincides with the holomorphic structure on* $L_N \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_M$ *when identified with* L_{N+M}.

Proof. We will look at the last component first.

$$\begin{aligned}\left\{\nabla_{N+M}^{\overline{\partial}}(\xi_1\xi_2)\right\}_\ell &= q^{\frac{N+M}{2}}(\xi_1\xi_2) \triangleleft F_\ell \\ &= q^{\frac{N}{2}}(\xi_1 \triangleleft F_\ell)\xi_2 + q^{N+M/2}\xi_1(\xi_2 \triangleleft F_\ell). \end{aligned} \tag{3.13}$$

On the other hand

$$\begin{aligned}&\left\{((\nabla_N^{\overline{\partial}} \otimes 1) + (\lambda_N \otimes 1)(1 \otimes \nabla_M^{\overline{\partial}}))(\xi_1 \otimes \xi_2)\right\}_\ell \\ &\qquad = q^{N/2}\xi_1 \triangleleft F_\ell \otimes \xi_2 + (\lambda_N \otimes 1)(\xi_1 \otimes q^{M/2}\xi_2 \triangleleft F_\ell) \\ &\qquad = q^{N/2}\xi_1 \triangleleft F_\ell \otimes \xi_2 + q^{N+M/2}\sum_{\underline{k}} \xi_1(\xi_2 \triangleleft F_\ell)(\psi_{\underline{k}}^N)^* \otimes \psi_{\underline{k}}^N. \end{aligned} \tag{3.14}$$

Interpreting the expression (3.14) as an element of $\Omega_{N+M}^{(0,1)}$, one can see that (3.14) coincides with left hand side. The same argument as previous proposition gives the result for other components. □

Now the quantum homogeneous coordinate ring $R := \bigoplus_{N \leq 0} H^0(L_N, \nabla_N^{\overline{\partial}})$ of the quantum projective space can be described as follows. This result was first obtained for $\ell = 1, 2$ in [19] and the previous chapter where its relation with the work in [1, 2] is also explained.

Theorem 3.4.1. *We have the algebra isomorphism*

$$\bigoplus_{N \leq 0} H^0(L_N, \nabla_N^{\overline{\partial}}) \simeq \frac{\mathbb{C}\langle z_1, z_2, ..., z_{\ell+1}\rangle}{\langle\, z_i z_j - q z_j z_i : 1 \leq i < j \leq \ell + 1\,\rangle}$$

Proof. The ring structure on R is coming from the tensor product $L_{N_1} \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} L_{N_2} \simeq L_{N_1+N_2}$. Since the basis elements $t_{0,j}^{(0,...,0,1)}$ of $H^0(L_1, \nabla_1^{\overline{\partial}})$, as shown in section 2 are z_j for $j = 1, 2, ..., \ell + 1$, one can easily see that $H^0(L_1, \nabla_1^{\overline{\partial}}) = \mathbb{C}z_1 \oplus \mathbb{C}z_2 \oplus \cdots \oplus \mathbb{C}z_{\ell+1}$. Now the isomorphism follows from the identities $z_i \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} z_j - q z_j \otimes_{\mathcal{A}(\mathbb{C}P_q^\ell)} z_i = 0$ in L_2, which is obvious.

□

3.5 Existence of a twisted positive Hochschild cocycle for $\mathbb{C}P_q^\ell$

We refer to section for a review on twisted Hochschild and cyclic cohomology. In [4], Section VI.2, Connes shows that extremal positive Hochschild cocycles in the sense of [5] on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a holomorphic structure on the surface. In [19] a notion of twisted positive Hochschild cocycle is introduced and a similar result is proved for the holomorphic structure of $\mathbb{C}P_q^1$ and $\mathbb{C}P_q^2$ in [19] and chapter 2. Although the corresponding problem of characterizing holomorphic structures on higher dimensional (commutative or noncommutative) manifolds via positive Hochschild cocycles is still open, nevertheless these results suggest regarding (twisted) positive Hochschild cocycles as a possible framework for holomorphic noncommutative structures. In this section we prove an analogous result for $\mathbb{C}P_q^\ell$ for all ℓ.

Now we come back to the case of our interest, that is $\mathbb{C}P_q^\ell$. Let τ be the fundamental class on $\mathbb{C}P_q^\ell$ defined as in [9] by a twisted cyclic cocycle

$$\tau(a_0, a_1, a_2, \cdots a_{2\ell}) := \int_h a_0 \, \mathrm{d}a_1 \, \mathrm{d}a_2 \cdots \mathrm{d}a_{2\ell}\,, \quad \forall a_i \in \mathcal{A}(\mathbb{C}P_q^\ell). \tag{3.15}$$

Here h stands for the Haar state functional of the quantum group $\mathcal{A}(SU_q(\ell+1))$ which has a twisted tracial property $h(xy) = h(y\sigma(x))$. Here the algebra automorphism σ is defined by

$$\sigma : \mathcal{A}(SU_q(\ell+1)) \to \mathcal{A}(SU_q(\ell+1)), \quad \sigma(x) = K^{-1} \triangleright x \triangleleft K^{-1}.$$

where $K = (K_1^\ell K_2^{2(\ell-1)} \cdots K_j^{j(\ell-j+1)} \cdots K_\ell^\ell)^2$, see[11]. The map σ, restricted to the algebra $\mathcal{A}(\mathbb{C}P_q^\ell)$ is given by $\sigma(x) = K^{-1} \triangleright x$. Non-triviality of τ has been shown in [9]. Now we recall the definition of a twisted positive Hochschild cocycle as given in [19].

Definition 3.5.1. *A twisted Hochschild 2n-cocycle ϕ on a $*$-algebra $\mathcal{A}$ is said to be twisted positive if the following map defines a positive sesquilinear form on the vector space $\mathcal{A}^{\otimes(n+1)}$:*

$$\langle a_0 \otimes a_1 \otimes ... \otimes a_n, b_0 \otimes b_1 \otimes ... \otimes b_n\rangle = \phi(\sigma(b_n^*)a_0, a_1, ..., a_n, b_n^*, ..., b_1^*).$$

3.5.1 A twisted positive Hochschild cocycle on $\mathbb{C}P_q^\ell$.

We recall that the set of (ℓ, ℓ)-shuffles (denoted by $S_{\ell,\ell}$) is set of all permutations $\pi \in S_{2\ell}$ such that $\pi(1) < \pi(2) < \cdots < \pi(\ell)$ and $\pi(\ell+1) < \pi(\ell+2) < \cdots < \pi(2\ell)$. Here we would like to look at a shuffle π as an increasing function from $\{\ell+1, \cdots, 2\ell\}$ to $\{1, 2, \cdots 2\ell\}$. Let us define $\theta^\pi : \{1, 2, \cdots, 2\ell\} \to \{\pm\}$ by $\theta^\pi|_{Im\,\pi} = -$ and $\theta^\pi|_{(Im\,\pi)^c} = +$. For any $\pi \in S_{\ell,\ell}$ define

$$\varphi_\pi(a_0, a_1, \cdots a_{2\ell}) := \int_h a_0(\partial^{\theta_1^\pi} a_1)(\partial^{\theta_2^\pi} a_2) \cdots (\partial^{\theta_{2\ell}^\pi} a_{2\ell}). \tag{3.16}$$

Here $\partial^+ = \partial$, $\partial^- = \overline{\partial}$ and $\theta_i^\pi = \theta^\pi(i)$. Now suppose that π and π' are two shuffles that are just different in their values on a single value i such that $|\pi'(i) - \pi(i)| = 1$. We define a cochain $\psi_{\pi,\pi'}$ by

$$\psi_{\pi,\pi'}(a_0, a_1, a_2, \cdots, a_{2\ell-1}) := \int_h a_0(\partial^{\theta_1^\pi} a_1)(\partial^{\theta_2^\pi} a_2) \cdots (\partial^{\theta_j^\pi} \partial^{\theta_j^{\pi'}} a_j)(\partial^{\theta_{j+2}^\pi} a_{j+1}) \cdots (\partial^{\theta_{2\ell}^\pi} a_{2\ell-1}).$$

Here $j = min\{\pi(i), \pi'(i)\}$. It is then easy to prove that $b_\sigma \psi_\pi = \pm(\varphi_\pi - \varphi_{\pi'})$. The proof is based on the following easy observation.

$$\partial\overline{\partial}(ab) = a\partial\overline{\partial}b + \partial a\overline{\partial}b - \overline{\partial}a\partial b + (\partial\overline{\partial}a)b.$$

The term $\partial^{\theta_j^\pi}\partial^{\theta_j^{\pi'}}$ is either $\partial\overline{\partial}$ or $\overline{\partial}\partial$ simply because of our choice of π and π'.

Now we recall an easy combinatorial fact. The number of permutations of 2ℓ letters including ℓ letter A and ℓ letter B is $\binom{2\ell}{\ell} = \frac{(2\ell)!}{\ell!\ell!}$. All permutations can be grouped in two groups and in each group there exists an order on permutations $\{\pi_1, ..., \pi_r\}$ and $\{\pi'_1, ..., \pi'_r\}$ with $r = \frac{1}{2}\binom{2\ell}{\ell}$, such that π_{i+1} (respectively π'_{i+1}), can be obtained from π_i (resp. π'_i) just with replacing the two letters in the spots j and $j+1$ where $1 \leq j \leq r-1$. In addition we can always choose $\pi_1 = AA\cdots ABB\cdots B$ and $\pi'_1 = BB\cdots BAA\cdots A$. The permutation π_r has the above mentioned property with respect to one of π'_i's.

Now we come back to the case $\mathbb{C}P^{\ell}_q$. We consider a complex structure $(\Omega^{(\bullet,\bullet)}(\mathcal{A}), \partial, \overline{\partial})$ on the $*$-algebra $\mathcal{A}(\mathbb{C}P^{\ell}_q)$ with $* : \Omega^{(p,q)} \to \Omega^{(q,p)}$ such that $\overline{\partial}a^* = (\partial a)^*$. We have seen that $\Omega^{(0,1)} = \mathfrak{M}(\sigma^{0,1})$, where $\sigma^{0,1}$ restricted to $\mathcal{U}_q(\mathfrak{su}(\ell))$ is the fundamental representation of $\mathcal{U}_q(\mathfrak{su}(\ell))$ in $\mathbb{C}^{\ell}$ and $\sigma^{0,1}(K_1K_2^2\cdots K_{\ell}^{\ell}) = q^{\frac{\ell+1}{2}}I$. The representation $\sigma^{1,0}$ can be obtained from $\sigma^{0,1}$ by conjugation. Define

$$\partial a := \triangleleft(E_{\ell}, E_{\ell}E_{\ell-1}, \cdots, E_{\ell}\cdots E_2E_1), \quad \overline{\partial}a := \triangleleft(F_{\ell}\cdots F_2F_1, \cdots F_{\ell}F_{\ell-1}, F_{\ell}).$$

For an anti-holomorphic 1-form $\omega = (\omega_1, \omega_2, \cdots, \omega_{\ell})$ we define

$$\omega^* := (-q\omega^*_{\ell}, q^2\omega^*_{\ell-1}, \cdots, (-q)^{\ell-1}\omega^*_2, (-q)^{\ell}\omega^*_1).$$

The property $\overline{\partial}a^* = (\partial a)^*$ holds simply because

$$(a^* \triangleleft F_{\ell}F_{\ell-1}\cdots F_i)^* = a \triangleleft S(F_{\ell}F_{\ell-1}\cdots F_i)^* = (-q)^{-(\ell-i+1)}a \triangleleft E_{\ell}E_{\ell-1}\cdots E_i.$$

One can define $*$ on anti-holomorphic forms such that $(\omega \wedge_q \omega')^* = (-1)^{deg(\omega)deg(\omega')}\omega'^* \wedge_q \omega^*$, then extend it to all holomorphic and anti-holomorphic forms with $\overline{\partial}a^* = (\partial a)^*$. Note that we can extend $\wedge_q$ to holomorphic forms as [11]. One can see that

$$\begin{aligned}\partial a_1\partial a_2\cdots\partial a_{\ell}\overline{\partial}a^*_{\ell}\cdots\overline{\partial}a^*_2\overline{\partial}a^*_1 &= \partial a_1\partial a_2\cdots\partial a_{\ell}(\overline{\partial}a_{\ell})^*\cdots(\overline{\partial}a_2)^*(\overline{\partial}a_1)^* \\ &= -\partial a_1\partial a_2\cdots\partial a_{\ell}(\partial a_1\partial a_2\cdots\partial a_{\ell})^*.\end{aligned}$$

We will need the following simple lemma for future computations.

Lemma 3.5.1. *For any $a_0, a_1, a_2, \cdots, a_{2\ell+1} \in \mathcal{A}(\mathbb{C}P^{\ell}_q)$ the following identities hold:*

$$\int_h a_0(\partial a_1\cdots\partial a_{\ell}\overline{\partial}a_{\ell+1}\cdots\overline{\partial}a_{2\ell})a_{2\ell+1} = \int_h \sigma(a_{2\ell+1})a_0\partial a_1\cdots\partial a_{\ell}\overline{\partial}a_{\ell+1}\cdots\overline{\partial}a_{2\ell}.$$

Proof. The space of $\Omega^{(\ell,\ell)}$ is a rank one free $\mathcal{A}(\mathbb{C}P^{\ell}_q)$-module. Let ω be the central basis element for the space of $\Omega^{(\ell,\ell)}$ and let $\partial a_1 \cdots \partial a_\ell \overline{\partial} a_{\ell+1} \cdots \overline{\partial} a_{2\ell} = x\omega$. Then

$$\begin{aligned}
&\int_h \left\{ a_0(\partial a_1 \cdots \partial a_\ell \overline{\partial} a_{\ell+1} \cdots \overline{\partial} a_{2\ell}) a_{2\ell+1} - \sigma(a_{2\ell+1}) a_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_{\ell+1} \cdots \overline{\partial} a_{2\ell} \right\} \\
&= \int_h (a_0 x\omega a_{2\ell+1} - \sigma(a_{2\ell+1}) a_0 x\omega) \\
&= \int_h (a_0 x a_{2\ell+1}\omega - \sigma(a_{2\ell+1}) a_0 x\omega) \\
&= h(a_0 x a_{2\ell+1} - \sigma(a_{2\ell+1}) a_0 x) = 0.
\end{aligned}$$

The last equality comes from the twisted property of the Haar state. □

Using $\mathrm{d} = \partial + \overline{\partial}$, we have

$$\tau = \sum_{\pi \in S_{\ell,\ell}} \varphi_\pi,$$

where φ_π is given by (3.16). Let $\pi_1 = id$, i.e. π_1 is the shuffle that keeps every letter at the same spot. Define the Hochschild cocycle

$$\varphi := -2r\varphi_{\pi_1}, \tag{3.17}$$

where $r = \frac{1}{2}\binom{2\ell}{\ell}$.

Theorem 3.5.1. *The 2ℓ-cocycle φ defined by (3.17), is a twisted positive Hochschild cocycle and it is cohomologous to the fundamental twisted cyclic cocycle τ.*

Proof. We first verify the twisted cocycle property.

$$\begin{aligned}
&\varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2), \cdots, \sigma(a_{2\ell})) \\
&\qquad = 2r \int_h \sigma(a_0) \partial\sigma(a_1) \cdots \partial\sigma(a_\ell) \overline{\partial}\sigma(a_{\ell+1}) \cdots \overline{\partial}\sigma(a_{2\ell}) \\
&\qquad = 2r \int_h K \triangleright (a_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_{\ell+1} \cdots \overline{\partial} a_{2\ell}) \\
&\qquad = 2r\, \epsilon(K) \int_h a_0 \partial a_1 \cdots \partial a_\ell \overline{\partial} a_{\ell+1} \cdots \overline{\partial} a_{2\ell} \\
&\qquad = \varphi(a_0, a_1, a_2, \cdots, a_{2\ell}).
\end{aligned}$$

For positivity one can see that

$$\begin{aligned}\varphi(\sigma(a_0^*)a_0, a_1, a_2, \cdots, a_\ell, a_\ell^*, \cdots, a_2^*, a_1^*) &= -2r \int\limits_h \sigma(a_0^*)a_0\partial a_1\partial a_2\cdots\partial a_\ell\overline{\partial}a_\ell^*\cdots\overline{\partial}a_2^*\overline{\partial}a_1^* \\ &= -2r \int\limits_h a_0\partial a_1\partial a_2\cdots\partial a_\ell\overline{\partial}a_\ell^*\cdots\overline{\partial}a_2^*\overline{\partial}a_1^*a_0^* \\ &= 2r \int\limits_h (a_0\partial a_1\partial a_2\cdots\partial a_\ell)(a_0\partial a_1\partial a_2\cdots\partial a_\ell)^*.\end{aligned}$$

One can take $\partial a_i = (v_1^i, v_2^i, \cdots, v_\ell^i)$, then using the multiplication rule of type (1,0) forms (for (0,1) forms c.f. [11]), we find that $(a_0\partial a_1\partial a_2\cdots\partial a_3)(a_0\partial a_1\partial a_2\cdots\partial a_3)^* = \mu\mu^*$, where

$$\mu = a_0 \sum_{\pi\in S_\ell} (-q^{-1})^{||\pi||} v^1_{\pi(1)} v^2_{\pi(2)} \cdots v^\ell_{\pi(\ell)}.$$

Hence

$$\varphi(\sigma(a_0^*)a_0, a_1, a_2, \cdots, a_\ell, a_\ell^*, \cdots, a_2^*, a_1^*) = 2r\, h(\mu\mu^*) \geq 0.$$

Here we used the positivity of the Haar functional h.

Now we would like to find the coefficients m, k such that $m\tau - k\varphi_{\pi_1} = b_\sigma\psi$ for a suitable $(2\ell-1)$-cocycle ψ. Here we order all φ_π's as explained at the beginning of the section, i.e. we use the order for permutations of ∂ and $\overline{\partial}$ to make two sets $\{\varphi_{\pi_1}, \varphi_{\pi_2}, ..., \varphi_{\pi_r}\}$ and $\{\varphi_{\pi'_1}, \varphi_{\pi'_2}, ..., \varphi_{\pi'_r}\}$, where $r = \frac{1}{2}\binom{2\ell}{\ell}$. For instance we give the formula for one choice of φ_{π_2}.

$$\varphi_{\pi_2}(a_0, a_1, ..., a_{2\ell}) := \int\limits_h a_0\overline{\partial}a_1\overline{\partial}a_2\cdots\overline{\partial}a_{\ell-1}\partial a_\ell\overline{\partial}a_{\ell+1}\partial a_{\ell+2}\cdots\partial a_{2\ell}.$$

One can show that there exist $2r-1$ twisted cochains $\psi_{\pi,\pi'}$ such that

$$\begin{aligned}
b_\sigma\psi_{\pi_1,\pi_2} &= \varphi_{\pi_1} - \varphi_{\pi_2}, \\
b_\sigma\psi_{\pi_2,\pi_3} &= \varphi_{\pi_2} - \varphi_{\pi_3}, \\
&\vdots \\
b_\sigma\psi_{\pi_{r-1},\pi_r} &= \varphi_{\pi_{r-1}} - \varphi_{\pi_r}, \\
b_\sigma\psi_{\pi_r,\pi'_k} &= \varphi_{\pi_r} - \varphi_{\pi'_k}, \\
b_\sigma\psi_{\pi'_1,\pi'_2} &= \varphi_{\pi'_1} - \varphi_{\pi'_2}, \\
b_\sigma\psi_{\pi'_2,\pi'_3} &= \varphi_{\pi'_2} - \varphi_{\pi'_3}, \\
&\vdots \\
b_\sigma\psi_{\pi'_{r-1}\pi'_r} &= \varphi_{\pi'_{r-1}} - \varphi_{\pi'_r}
\end{aligned} \tag{3.18}$$

For instance ψ_{π_1,π_2} (up to a $\pm$ sign) is defined by

$$\psi_{\pi_1,\pi_2}(a_0, a_1, ..., a_{2\ell-1}) := \int_h a_0\partial a_1...\partial a_{\ell-1}(\partial\overline{\partial}a_\ell)\overline{\partial}a_{\ell+1}...\overline{\partial}a_{2\ell-1}.$$

Define

$$\psi := \sum_{i=1}^{r-1} x_i\psi_{\pi_i,\pi_{i+1}} + x_r\psi_{\pi_i,\pi'_k} + \sum_{i=1}^{r-1} x_{r+i}\psi_{\pi'_i,\pi'_{i+1}},$$

with constants x_i's $i = 1, 2, \cdots, 2r-1$ have to be determined. We find the following linear system of equations for $m\tau - k\varphi_{\pi_1} = b_\sigma\psi$.

$$\begin{cases} m - k - x_1 = 0 \\ m + x_1 - x_2 \\ \vdots \\ m + x_{r-1} - x_r = 0 \\ m + x_{r+1} = 0 \\ m + x_{r+1} - x_{r+2} = 0 \\ \vdots \\ m + x_{r+k-1} - x_{r+k} = 0 \\ m + x_r - x_{r+k-1} - x_{r+k} = 0 \\ m + x_{r+k} - x_{r+k+1} = 0 \\ \vdots \\ m + x_{2r-2} - x_{2r-1} = 0 \\ m + x_{2r-1} = 0 \end{cases}$$

This system has the one parameter family of solutions given by

$$x_i = -(2r-i)m \quad \text{for} \quad i \in \{1, 2, \cdots, 2r-1\} - \{r+1\}, \quad x_{r+1} = -m, \quad k = 2rm.$$

For $m = 1$, we have $\tau - 2r\varphi_1 = b_\sigma\psi$. Note that ψ_i's are defined up to sign. □

3.5.2 A positive cocycle φ on $\mathbb{C}P_q^3$

In this subsection we would like to delve into the case of $\mathbb{C}P_q^3$ in details. We consider the complex structure on $(\Omega^{\bullet,\bullet}(\mathcal{A}), \partial, \overline{\partial})$ on the $*$-algebra $\mathcal{A}$. There exists $* : \Omega^{(p,q)} \to \Omega^{(q,p)}$ such that $\overline{\partial} a^* = -(\partial a)^*$.

We have seen that $\Omega^{(0,1)} = \mathfrak{M}(\sigma^{(0,1)})$, where the representation $\sigma^{(0,1)}$ on $\mathcal{U}_q(\mathfrak{su}(3))$ is the fundamental representation of $\mathcal{U}_q(\mathfrak{su}(3))$ in $\mathbb{C}^3$ and on the generator of $U(1)$ is given by

$\sigma^{(0,1)}(K_1K_2^2K_3^3) = q^2I$. Here the representation on the basis is given by

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} q^{1/2} & 0 & 0 \\ 0 & q^{-1/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q^{1/2} & 0 \\ 0 & 0 & q^{-1/2} \end{bmatrix}.$$

The representation $\sigma^{(1,0)}$ can be obtained from $\sigma^{(0,1)}$ by conjugation. Define

$$\partial a := \triangleleft(E_3,\, E_3E_2,\, E_3E_2E_1), \quad \overline{\partial} a := \triangleleft(F_3F_2F_1,\, F_3F_2,\, F_3).$$

For $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega^{(0,1)}$, let define

$$\omega^* := q^2(q\omega_3^*, -\omega^*, q^{-1}\omega_1^*).$$

One can see that

$$\begin{aligned}
&(a^* \triangleleft F_3F_2F_1, a^* \triangleleft F_3F_2,\, a^* \triangleleft F_3)^* \\
&\qquad = q^2(q^{-1}a \triangleleft S(F_3F_2F_1)^*,\, -a \triangleleft S(F_3F_2)^*,\, qa \triangleleft S(F_3)^*) \\
&\qquad = q^2(-q^{1-3}a \triangleleft E_3E_2E_1,\, -q^{-2}a \triangleleft E_3E_2,\, -q^{-1-1}a \triangleleft E_3) \\
&\qquad = -(a \triangleleft E_3E_2E_1,\, a \triangleleft E_3E_2,\, a \triangleleft E_3).
\end{aligned}$$

Hence

$$\overline{\partial} a^* = -(\partial a)^*.$$

For $\eta = (\eta_1, \eta_2, \eta_3) \in \Omega^{(0,2)}$ we define

$$\eta^* := q^4(q\eta_3^*, -\eta_2, q^{-1}\eta_1^*).$$

One can prove that for anti-holomorphic 1-forms we have

$$(\omega \wedge_q \omega')^* = (-1)^{deg(\omega)deg(\omega')}\omega'^* \wedge_q \omega^*,$$

Then one can extend $*$ to all holomorphic and anti-holomorphic forms with $\overline{\partial} a^* = -(\partial a)^*$. Note that we can extend $\wedge_q$ to holomorphic forms as [11]. One can see that

$$\partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_3^* \overline{\partial} a_2^* \overline{\partial} a_1^* = -\partial a_1 \partial a_2 \partial a_3 (\overline{\partial} a_3)^* (\overline{\partial} a_2)^* (\overline{\partial} a_1)^* = \partial a_1 \partial a_2 \partial a_3 (\partial a_1 \partial a_2 \partial a_3)^*.$$

We will need the following simple lemma for future computations.

Lemma 3.5.2. *For any $a_0, a_1, a_2, \cdots, a_7 \in \mathcal{A}(\mathbb{C}P_q^3)$ the following identity hold:*

$$\int_h a_0(\partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \overline{\partial} a_6) a_7 = \int_h \sigma(a_7) a_0 \partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \overline{\partial} a_6,$$

Proof. The space of $\Omega^{(3,3)}$ is a rank one free $\mathcal{A}(\mathbb{C}P_q^3)$-module. Let ω be the central basis element for the space of $\Omega^{(3,3)}$ and let $\partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \overline{\partial} a_6 = x\omega$. Then

$$\begin{aligned}
&\int_h a_0(\partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \overline{\partial} a_6) a_7 - \int_h \sigma(a_7) a_0 \partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \overline{\partial} a_6 \qquad (3.19)\\
&= \int_h (a_0 x \omega a_7 - \sigma(a_7) a_0 x \omega)\\
&= \int_h (a_0 x a_7 \omega - \sigma(a_7) a_0 x \omega)\\
&= h(a_0 x a_7 - \sigma(a_7) a_0 x) = 0.
\end{aligned}$$

The last equality comes from the twisted property of the Haar state. □

Theorem 3.5.2. *The 6-cocycle φ defined by*

$$\varphi(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := 20 \int_h a_0 \partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \overline{\partial} a_6$$

is a twisted positive Hochschild cocycle on $\mathcal{A}(\mathbb{C}P_q^3)$ and is cohomologous to τ.

Proof. We will give φ's and ψ's explicitly for this case, i.e. $\mathbb{C}P_q^3$. We first introduce cocycles φ_i, $i = 1, \cdots, 20$

$$\varphi_1(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \overline{\partial} a_6,$$

$$\varphi_2(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \partial a_4 \overline{\partial} a_5 \overline{\partial} a_6,$$

$$\varphi_3(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 \partial a_5 \overline{\partial} a_6,$$

$$\varphi_4(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 \overline{\partial} a_5 \partial a_6,$$

$$\varphi_5(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \overline{\partial} a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \partial a_6,$$

$$\varphi_6(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \overline{\partial} a_2 \partial a_3 \overline{\partial} a_4 \partial a_5 \overline{\partial} a_6,$$

$$\varphi_7(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \overline{\partial} a_2 \overline{\partial} a_3 \partial a_4 \partial a_5 \overline{\partial} a_6,$$

$$\varphi_8(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \overline{\partial} a_2 \overline{\partial} a_3 \partial a_4 \overline{\partial} a_5 \partial a_6,$$

$$\varphi_9(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \partial a_1 \overline{\partial} a_2 \overline{\partial} a_3 \overline{\partial} a_4 \partial a_5 \partial a_6,$$

$$\varphi_{10}(a_0, a_1, a_2, a_3, a_4, a_5, a_6) := \int_h a_0 \overline{\partial} a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 \partial a_5 \partial a_6,$$

$$\varphi_{11}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\overline{\partial}a_2\overline{\partial}a_3\partial a_4\partial a_5\partial a_6,$$

$$\varphi_{12}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\overline{\partial}a_2\partial a_3\overline{\partial}a_4\partial a_5\partial a_6,$$

$$\varphi_{13}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\overline{\partial}a_2\partial a_3\partial a_4\overline{\partial}a_5\partial a_6,$$

$$\varphi_{14}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\overline{\partial}a_2\partial a_3\partial a_4\partial a_5\overline{\partial}a_6,$$

$$\varphi_{15}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\partial a_2\overline{\partial}a_3\partial a_4\partial a_5\overline{\partial}a_6,$$

$$\varphi_{16}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\partial a_2\overline{\partial}a_3\partial a_4\overline{\partial}a_5\partial a_6,$$

$$\varphi_{17}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\partial a_2\partial a_3\overline{\partial}a_4\overline{\partial}a_5\partial a_6,$$

$$\varphi_{18}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\partial a_2\partial a_3\overline{\partial}a_4\partial a_5\overline{\partial}a_6,$$

$$\varphi_{19}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\overline{\partial}a_1\partial a_2\partial a_3\partial a_4\overline{\partial}a_5\overline{\partial}a_6,$$

$$\varphi_{20}(a_0,a_1,a_2,a_3,a_4,a_5,a_6) := \int_h a_0\partial a_1\overline{\partial}a_2\partial a_3\partial a_4\overline{\partial}a_5\overline{\partial}a_6.$$

We also define twisted cochains ψ_i's as follows

$$\psi_1(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\partial a_1\partial a_2\partial\bar\partial a_3\bar\partial a_4\bar\partial a_5,$$
$$\psi_2(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\partial a_1\partial a_2\bar\partial a_3\partial\bar\partial a_4\bar\partial a_5,$$
$$\psi_3(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\partial a_1\partial a_2\bar\partial a_3\bar\partial a_4\partial\bar\partial a_5,$$
$$\psi_4(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\partial a_1\partial\bar\partial a_2\bar\partial a_3\bar\partial a_4\partial a_5,$$
$$\psi_5(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\partial a_1\bar\partial a_2\partial a_3\bar\partial a_4\bar\partial\partial a_5,$$
$$\psi_6(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\partial a_1\bar\partial a_2\bar\partial\partial a_3\partial a_4\bar\partial a_5,$$
$$\psi_7(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\partial a_1\bar\partial a_2\bar\partial a_3\partial a_4\partial\bar\partial a_5,$$
$$\psi_8(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\partial a_1\bar\partial a_2\bar\partial a_3\partial\bar\partial a_4\partial a_5,$$
$$\psi_9(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\partial\bar\partial a_1\bar\partial a_2\bar\partial a_3\partial a_4\partial a_5,$$
$$\psi_{10}(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\bar\partial a_1\partial\bar\partial a_2\bar\partial a_3\partial a_4\partial a_5,$$
$$\psi_{11}(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\bar\partial a_1\bar\partial a_2\partial\bar\partial a_3\partial a_4\partial a_5,$$
$$\psi_{12}(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\bar\partial a_1\bar\partial a_2\partial a_3\bar\partial\partial a_4\partial a_5,$$
$$\psi_{13}(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\bar\partial a_1\bar\partial a_2\partial a_3\partial a_4\bar\partial\partial a_5,$$
$$\psi_{14}(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\bar\partial a_1\partial\bar\partial a_2\partial a_3\partial a_4\bar\partial a_5,$$
$$\psi_{15}(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\bar\partial a_1\partial a_2\bar\partial a_3\partial a_4\partial\bar\partial a_5,$$
$$\psi_{16}(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\bar\partial a_1\partial a_2\partial\bar\partial a_3\bar\partial a_4\partial a_5,$$
$$\psi_{17}(a_0,a_1,a_2,a_3,a_4,a_5) := -\int_h a_0\bar\partial a_1\partial a_2\partial a_3\bar\partial a_4\bar\partial\partial a_5,$$
$$\psi_{18}(a_0,a_1,a_2,a_3,a_4,a_5) := \int_h a_0\bar\partial a_1\partial a_2\partial\bar\partial a_3\partial a_4\bar\partial a_5,$$

Now let us define the map $\psi = \sum_{i=1}^{19} x_i\psi_i$ and $\varphi := -k\varphi_1$. One can check that for all $i = 1, 2\cdots, 19$ except $i = 10, 11$, we have

$$b_\sigma\psi_i = \varphi_i - \varphi_{i+1},$$

and

$$b_\sigma\psi_{10} = \varphi_{10} - \varphi_{12}, \quad b_\sigma\psi_{11} = \varphi_{12} - \varphi_{11}.$$

We only show the computation for $b_\sigma\psi_1 = \varphi_1 - \varphi_2$ and the rest can be proven in a similar way.

$$\begin{aligned}
b_\sigma\psi_1(a_0, a_1, a_2, a_3, a_4, a_5, a_6) &= \psi_1(a_0a_1, a_2, a_3, a_4, a_5, a_6) \\
&- \psi_1(a_0, a_1a_2, a_3, a_4, a_5, a_6) + \psi_1(a_0, a_1, a_2a_3, a_4, a_5, a_6) \\
&- \psi_1(a_0, a_1, a_2, a_3a_4, a_5, a_6) + \psi_1(a_0, a_1, a_2, a_3, a_4a_5, a_6) \\
&- \psi_1(a_0, a_1, a_2, a_3, a_4, a_5a_6) + \psi_1(\sigma(a_6)a_0, a_1, a_2, a_3, a_4, a_5)
\end{aligned}$$

Hence

$$\begin{aligned}
&b_\sigma\psi_1(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = \\
&\quad -\int_h a_0a_1\partial a_2\partial a_3\partial\overline{\partial} a_4\overline{\partial} a_5\overline{\partial} a_6 + \int_h a_0\partial(a_1a_2)\partial a_3\partial\overline{\partial} a_4\overline{\partial} a_5\overline{\partial} a_6 \\
&\quad -\int_h a_0\partial a_1\partial(a_2a_3)\partial\overline{\partial} a_4\overline{\partial} a_5\overline{\partial} a_6 + \int_h a_0\partial a_1\partial a_2\partial\overline{\partial}(a_3a_4)\overline{\partial} a_5\overline{\partial} a_6 \\
&\quad -\int_h a_0\partial a_1\partial a_2\partial\overline{\partial} a_3\overline{\partial}(a_4a_5)\overline{\partial} a_6 + \int_h a_0\partial a_1\partial a_2\partial\overline{\partial} a_3\overline{\partial} a_4\overline{\partial}(a_5a_6) \\
&\quad -\int_h \sigma(a_6)a_0\partial a_1\partial a_2\partial\overline{\partial} a_3\overline{\partial} a_4\overline{\partial} a_5 \\
&\quad = \int_h a_0\partial a_1\partial a_2\partial a_3\overline{\partial} a_4\overline{\partial} a_5\overline{\partial} a_6 - \int_h a_0\partial a_1\partial a_2\overline{\partial} a_3\partial a_4\overline{\partial} a_5\overline{\partial} a_6, \\
&\quad = \varphi_1(a_0, a_1, a_2, a_3, a_4, a_5, a_6) - \varphi_2(a_0, a_1, a_2, a_3, a_4, a_5, a_6).
\end{aligned}$$

Here we applied Leibniz rule several times. Solving the equation $\tau - \varphi = b_\sigma\psi$ with respect to coefficients is equivalent to the following system.

$$\begin{cases} m+k-x_1=0 \\ m+x_1-x_2=0 \\ m+x_2-x_3=0 \\ \vdots \\ m+x_8-x_9=0 \\ m+x_9-x_{10}=0 \\ m+x_{11}=0 \\ m+x_{10}-x_{11}-x_{12}=0 \\ m+x_{12}-x_{13}=0 \\ \vdots \\ m+x_{18}-x_{19}=0 \\ m+x_{19}=0. \end{cases}$$

This system has the following solution

$$x_i=-(20-i)m \quad \text{for} \quad i\in\{1,2,\cdots,19\}-\{11\}, \quad x_{11}=-m, \quad k=-20m.$$

For $m=1$, we have $\tau-20\varphi_1=b_\sigma\psi$. For positivity, one can see that

$$\begin{aligned}\varphi(\sigma(a_0^*)a_0,a_1,a_2,a_3,a_3^*,a_2^*,a_1^*) &= 20\int_h \sigma(a_0^*)a_0\partial a_1\partial a_2\partial a_3\overline{\partial}a_3^*\overline{\partial}a_2^*\overline{\partial}a_1^* \\ &= 20\int_h a_0\partial a_1\partial a_2\partial a_3\overline{\partial}a_3^*\overline{\partial}a_2^*\overline{\partial}a_1^*a_0^* \\ &= 20\int_h (a_0\partial a_1\partial a_2\partial a_3)(a_0\partial a_1\partial a_2\partial a_3)^*.\end{aligned}$$

One can take $\partial a_i=(v_1^i,v_2^i,v_3^i)$, then using the multiplication rule of type (1,0) forms (for

(0,1) forms c.f. [11]), we find that $(a_0 \partial a_1 \partial a_2 \partial a_3)(a_0 \partial a_1 \partial a_2 \partial a_3)^* = \mu\mu^*$, where

$$\mu = a_0(v_1^1 v_2^2 v_3^3 - q^{-1} v_2^1 v_1^2 v_3^3 - q^{-1} v_1^1 v_3^2 v_2^3 + q^{-2} v_2^1 v_3^2 v_1^3 + q^{-2} v_3^1 v_1^2 v_2^3 - q^{-3} v_3^1 v_2^2 v_1^3.$$

Hence

$$\varphi(\sigma(a_0^*)a_0, a_1, a_2, a_3, a_3^*, a_2^*, a_1^*) = 20\, h(\mu\mu^*) \geq 0.$$

Here we used the positivity of the Haar functional h. $\square$

Chapter 4

The Riemann-Roch theorem for $\mathbb{C}P_q^\ell$, $\ell = 1, 2$

First recall that, for classical projective space $\mathbb{C}P^n$, its sheaf (or equivalently Dolbeault) cohomology with coefficients in the sheaf of holomorphic sections of line bundles $\mathcal{O}(m)$ are given by

$$H^i(\mathbb{C}P^n, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, z_1, ..., z_n]_m & \text{if} \quad i = 0,\ m \geq 0, \\ 0 & \text{if} \begin{cases} i = 0,\ m < 0 \\ 0 < i < n \\ i = n,\ m > -n-1 \end{cases} \\ H^0(\mathbb{C}P^n, \mathcal{O}(-m-n-1))^* & \text{if} \quad i = n,\ m \leq -n-1. \end{cases}$$

Therefore for the holomorphic Euler characteristic of $\mathcal{O}(m)$, we get

$$\chi(\mathbb{C}P^1, \mathcal{O}(m)) := \dim H^0(\mathbb{C}P^1, \mathcal{O}(m)) - \dim H^1(\mathbb{C}P^1, \mathcal{O}(m)) = m + 1.$$

4.0.3 The case of $\mathbb{C}P_q^1$

This last formula has an analogue in the case of $\mathbb{C}P_q^1$. The zeroth cohomology has been computed in [19], but for completeness we recall it here again. First let us recall that finite dimensional irreducible representations of $U_q(\mathfrak{su}(2))$ are given by vector spaces V_l, where $2l \in \mathbb{N}$ with basis $|l, m\rangle$, $m \in \{-l, ..., l\}$. The action on generators are given by

$$K|l, m\rangle = q^m |l, m\rangle,$$
$$E|l, m\rangle = \sqrt{[l-m+1][l+m]}\,|l, m-1\rangle,$$
$$F|l, m\rangle = \sqrt{[l+m+1][l-m]}\,|l, m+1\rangle.$$

We will have the isomorphism $A(SU_q(2)) = \bigoplus V_l \otimes V_l^*$ and under this isomorphism the space of canonical quantum line bundle $L_N := \{a \in A(SU_q(2))| \quad h \triangleright a = q^{N/2}a\}$ corresponds to $\{|l, N/2\rangle \otimes |l, m\rangle| \quad l \geq |N/2|, m = -2l, ..., 2l\}$. From now on we will use the notation $|l, n, m\rangle = |l, n\rangle \otimes |l, m\rangle$.

The anti-holomorphic part of the connection on L_N is given by $\nabla^{\overline{\partial}}|l, \frac{N}{2}, n\rangle := E|l, \frac{N}{2}, n\rangle$. Consider the Dolbeault complex of $\mathbb{C}P_q^1$

$$0 \to L_N \to \Omega^{(0,1)} \otimes L_N \to 0,$$

or equivalently

$$0 \to L_N \to L_{N-2} \to 0.$$

One can easily see that $\nabla^{\overline{\partial}}\xi = E|l, \frac{N}{2}, m\rangle = \sqrt{[l - \frac{N}{2} + 1][l + \frac{N}{2}]}|l, \frac{N}{2} - 1, m\rangle$. To find the holomorphic Euler characteristic $\chi(\mathbb{C}P_q^1, L_N)$, we will consider the following three cases.

- $N \geq 2$.

In this case, the kernel of $\nabla^{\overline{\partial}}$ is zero, simply because $l + \frac{N}{2}$ cannot be zero and $l - \frac{N}{2} + 1$ is zero only if $l = \frac{N}{2} - 1$, which is impossible in this case, since by assumption $l \geq \frac{N}{2}$. The Image of $\nabla^{\overline{\partial}}$ will be generated by the basis elements $|l, \frac{N}{2} - 1, m\rangle$ with $l \geq \frac{N}{2}$. But it differs from basis of L_{N-2} by elements $|\frac{N}{2} - 1, \frac{N}{2} - 1, m\rangle$ which can be counted as $N-1$ elements.

- $N = 1$.

Here we have $\nabla^{\overline{\partial}}\xi = \sqrt{[l - \frac{1}{2} + 1][l + \frac{1}{2}]}|l, \frac{1}{2} - 1, m\rangle$. So $E|l, \frac{1}{2}, m\rangle = [l + \frac{1}{2}]|l, -\frac{1}{2}, m\rangle$ and it is not hard to see that $\mathrm{Im}\nabla^{\overline{\partial}} = L_{N-2}$. The same argument as case $N \geq 2$ shows that $\mathrm{Ker}\nabla^{\overline{\partial}} = 0$. Hence $\chi(\mathbb{C}P_q^1, L_N) = 0$.

- $N \leq 0$.

If $N \leq 0$, $l + \frac{N}{2} = 0$ when $l = -\frac{N}{2}$ and this gives the set $\{|-\frac{N}{2}, \frac{N}{2}, m\rangle| \, m = \frac{N}{2}, \frac{N}{2} + \frac{1}{2}, ..., -\frac{N}{2}\}$

as a basis for the space of holomorphic sections of L_N. So dim Ker $\nabla^{\overline{\partial}} = |N| + 1$. In a similar manner to case $N = 1$ one can show that the map $\nabla^{\overline{\partial}}$ is surjective. Therefore we will come to the following result

$$\chi(\mathbb{C}P^1_q, L_N) = -N + 1.$$

Note that there is a switch between N and $-N$ with respect to the classical case.

4.0.4 Serre duality for $\mathbb{C}P^2_q$

There exists a non-degenerate pairing $\langle\, ,\rangle : L_N \times L_{-N} \to \mathbb{C}$, given by

$$\langle \xi, \eta\rangle := h(\xi\eta), \quad \forall \xi \in L_N, \quad \forall \eta \in L_{-N}. \tag{4.1}$$

Here h is the Haar state of the quantum group $\mathcal{A}(SU_q(3))$. The map is obviously bilinear and the nondegeneracy comes from the facts that $L_N^* \subset L_{-N}$ and h is faithful. Now consider the $(0, q)$-Dolbeault complex of $\mathbb{C}P^2_q$

$$0 \to L_N \to \Omega^{(0,1)} \otimes L_N \to \Omega^{(0,2)} \otimes L_N \to 0. \tag{4.2}$$

We would like to state an analogue of Serre duality theorem for this complex as

Proposition 4.0.1. *There exists a non-degenerate pairing defined by*

$$\langle\, ,\, \rangle : H^2(\nabla, L_N) \times H^0(\nabla, L_{-N-3}) \to \mathbb{C}$$

$$\langle [\xi], [\eta]\rangle := h(\xi\eta), \quad \forall \xi \in L_{N+3}, \quad \forall \eta \in L_{-N-3}.$$

Proof. First note that $H^2(\nabla, L_N)$ is a quotient of L_{N+3} and $H^0(\nabla, L_{-N-3})$ is a subspace of L_{-N-3}. We show that this map is well defined. For this, suppose that ξ and ξ' are in the same cohomology class. Hence $h(\xi\eta) - h(\xi'\eta) = h((\xi - \xi')\eta) = h(\overline{\partial}\alpha\eta) = h(\overline{\partial}(\alpha\eta) - \alpha\overline{\partial}\eta) = 0$, by noting that $\eta \in \mathrm{Ker}\overline{\partial}$ and h has invariance property with respect to the map $\overline{\partial}$. Now non-degeneracy is obvious by the above discussion.

□

The above result easily can be lifted to the general case of $\mathbb{C}P_q^\ell$ in the following way. The pairing

$$\langle \xi, \eta \rangle := h(\xi\eta), \quad \forall \xi \in L_N, \forall \eta \in L_{-N}. \tag{4.3}$$

is a nondegenerate pairing and hold true passing to the cohomology

$$\langle \, , \, \rangle : H^\ell(\nabla, L_N) \times H^0(\nabla, L_{-N-\ell-1}) \to \mathbb{C}$$
$$\langle [\xi], [\eta] \rangle := h(\xi\eta), \quad \forall \xi \in L_{N+\ell+1}, \quad \forall \eta \in L_{-N-\ell-1}.$$

In the following we will compute the $(0,q)$-Dolbeault cohomology of $\mathbb{C}P_q^2$. The result is analogue of the classical case. i.e.

Theorem 4.0.3. *With the above notations*

$$H^i(\nabla^{\overline{\partial}}, L_N) = \begin{cases} \mathbb{C}\langle z_1, z_2, z_3 \rangle_N & \text{if} \quad i = 0,\ N \geq 0, \\ 0 & \text{if} \begin{cases} i = 0,\ N < 0 \\ i = 1,\ N = 0 \\ i = 2,\ N > -3 \end{cases} \\ \mathbb{C}\langle z_1, z_2, z_3 \rangle^*_{-N-3} & \text{if} \quad i = 2,\ N \leq -3. \end{cases}$$

Proof. The zeroth-cohomology has been computed in Chapter 2 and the second cohomology comes from the Serre duality. So we just have to prove that the triviality of the first cohomology. In order to do so, we will calculate the Im $\overline{\partial}_1$ and the Ker $\overline{\partial}_2$ and show the equality.

$$\begin{aligned} \overline{\partial}_1(t(n, n+N)^0_{\underline{j}}) &= (E_1E_2 \blacktriangleright t(n, n+N)^0_{\underline{j}}, E_2 \blacktriangleright t(n, n+N)^0_{\underline{j}}) \\ &= (t(n, n+N)^{1,0,1/2}_{\underline{j}}, t(n, n+N)^{1,0,-1/2}_{\underline{j}}) \end{aligned}$$

For Ker $\overline{\partial}_2$ we will use the $\overline{\partial}_2(v_+, v_-) = -E_2 \blacktriangleright v_+ - E_2E_1 + 2[2]^{-1}E_1E_2 \blacktriangleright v_-$. Applying

$v_+ = t(n, n+3)_{\underline{j}}^{1,0,1/2}$ and $v_- = t(n, n+3)_{\underline{j}}^{1,0,-1/2}$ we will have

$$\begin{aligned}
&- E_2 \blacktriangleright t(n, n+3)_{\underline{j}}^{1,0,1/2} \\
&= -\sqrt{\frac{[n][n+3+2]}{[2][3]}} t(n, n+3)_{\underline{j}}^{1,1,0} - \sqrt{\frac{[n+2][n+3]}{[2]}} t(n, n+3)_{\underline{j}}^{\underline{0}}, \\
&- E_2 E_1 \blacktriangleright t(n, n+3)_{\underline{j}}^{1,0,-1/2} = -E_2 \blacktriangleright t(n, n+3)_{\underline{j}}^{1,0,1/2} \\
&= -\sqrt{\frac{[n][n+3+2]}{[2][3]}} t(n, n+3)_{\underline{j}}^{1,1,0} - \sqrt{\frac{[n+2][n+3]}{[2]}} t(n, n+3)_{\underline{j}}^{\underline{0}},
\end{aligned}$$

and

$$\begin{aligned}
&2[2]^{-1} E_1 E_2 \blacktriangleright t(n, n+3)_{\underline{j}}^{1,0,-1/2} = \\
&2[2]^{-1} E_1 \blacktriangleright (\sqrt{[2]} \sqrt{\frac{[n][n+3+2]}{[2][3]}} t(n, n+3)_{\underline{j}}^{1,1,-1}) = 2\sqrt{\frac{[n][n+5]}{[2][3]}} t(n, n+3)_{\underline{j}}^{1,1,0}
\end{aligned}$$

Hence

$$\overline{\partial}_2 (t(n, n+3)_{\underline{j}}^{1,0,1/2}, t(n, n+3)_{\underline{j}}^{1,0,-1/2}) = -2\sqrt{\frac{[n+2][n+3]}{[2]}} t(n, n+3)_{\underline{j}}^{\underline{0}}$$

This shows that $H^1 = \frac{Ker\,\overline{\partial}_2}{Im\,\overline{\partial}_1} = 0$ in the case of $N = 0$. $\square$

By a similar but lengthier calculation, one can prove that $H^0(\nabla^{\overline{\partial}}, L_N) = 0$ for all $N \neq 0$.

Chapter 5

A q-analogue of the Borel-Weil theorem

5.1 The Borel-Weil theorem

Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is the complexification of $\mathfrak{g}$ and $G_{\mathbb{C}}$ is the corresponding Lie group. The Cartan decomposition is

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^{+} \oplus \mathfrak{n}^{-},$$

where

$$\mathfrak{n}^{+} = \bigoplus_{\alpha \in R^{+}} \mathfrak{g}_{\alpha}, \quad \mathfrak{n}^{-} = \bigoplus_{\alpha \in R^{-}} \mathfrak{g}_{\alpha}.$$

Here R^{+} and R^{-} are the space of positive and negative roots. The Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is defined by $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^{+}$. Let B be its associated Lie group, which is called the Borel subgroup. Then one can see that $G/T = G_{\mathbb{C}}/B$. The right one is a complex manifold. It is known that associated to a weight λ on G (i.e. an irreducible representation of a maximal torus T inside G), there exists a line bundle L_{λ} on G/T defined by

$$L_{\lambda} := \{(g, c) \in G \times \mathbb{C} : (g, c) \sim (gh, h^{-1}c)\}, \tag{5.1}$$

and the space of sections of this line bundle is given by

$$\Gamma(L_{\lambda}) = \{f : G \to \mathbb{C} |\, f(gh) = \lambda(h^{-1}) f(g), \quad \forall h \in T, \forall g \in G\}. \tag{5.2}$$

The holomorphic sections are defined by

$$\begin{aligned} \Gamma_{hol}(L_{\lambda}) &= \{f : G \to \mathbb{C} |\, f(gh) = \lambda(h^{-1}) f(g), \quad \forall h \in B, \forall g \in G\} \\ &= \{f : G \to \mathbb{C} | X \blacktriangleright f = 0, \quad \forall X \in \mathfrak{n}^{-}\}. \end{aligned}$$

The last equality is because for any $X \in \mathfrak{n}^-$, [34]

$$X \blacktriangleright f = \frac{d}{dt} f(ge^{-tX})\Big|_{t=0}.$$

The group G acts on $\Gamma(L_\lambda)$ by $(g, f)(g') = f(g^{-1}g')$, for all g and g' in G.

The classical Borel-Weil theorem gives a geometric characterization of all irreducible representations of G.

Theorem 5.1.1. *For a dominant weight λ of G, the space $\Gamma_{hol}(L_\lambda)$ is a non trivial irreducible representation of G with the highest weight λ and all irreducible representations can be obtained in this way.*

In the case of $G = SU(2)$, the maximal torus is $U(1)$ and all weights are indexed by integers $n \in \mathbb{Z}$. The space of sections of line bundle L_n on the projective line $SU(2)/U(1) = \mathbb{C}P^1$ can be also given by

$$\Gamma(L_n) = \{f : SU(2) \to \mathbb{C} | \quad H \blacktriangleright f = nf\}. \tag{5.3}$$

The holomorphic sections are

$$\begin{aligned}\Gamma_{hol}(L_n) &= \{f : SU(2) \to \mathbb{C} | \quad f(gh) = \lambda(h^{-1})f(g), \quad \forall h \in B, \forall g \in G\} \\ &= \{f : SU(2) \to \mathbb{C} | \quad H \blacktriangleright f = nf, \quad E \blacktriangleright f = 0\}.\end{aligned}$$

Let us look at the case $SU(3)$. In this case weights are indexed by a pair of integers $\lambda = (m_1, m_2) \in \mathbb{Z}^2$ and the associated line bundles on the flag manifold $Fl(3) := SU(3)/S(U(1) \times U(1))$ are given by

$$\Gamma(L_\lambda) = \{f : SU(3) \to \mathbb{C} | \quad H_i \blacktriangleright f = m_i f, \, i = 1, 2\}. \tag{5.4}$$

and the holomorphic sections are

$$\Gamma_{hol}(L_\lambda) := \{f : SU(3) \to \mathbb{C} | \quad H_i \blacktriangleright f = m_i f, \; E_i \blacktriangleright f = 0, \; i = 1, 2\}. \tag{5.5}$$

In the quantum case this definition changes to

$$\Gamma_{hol}(L_\lambda) := \{f \in \mathcal{A}(SU_q(3)) \to \mathbb{C} | K_i \blacktriangleright f = q^{m_i/2} f,\ E_i \blacktriangleright f = 0,\ i = 1,2\}. \tag{5.6}$$

Now suppose that λ is a dominant weight for $SU_q(3)$, that is a pair of non-negative integers, then we want to show that the space of holomorphic sections $\Gamma_{hol}(L_{(m_1,m_2)})$ is an irreducible representation of $SU_q(3)$ of dimension $\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$.

Lemma 5.1.1. *With the above notation* $dim\Gamma_{hol}(L_{(m_1,m_2)}) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$.

Proof. Taking a basis element $|m'\rangle \otimes |m\rangle$ where $|m\rangle$ is a GT-basis element of $U_q(su(3))$. These set of four conditions gives the following restrictions on

$$|m\rangle = \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} & \\ m_{11} & & \end{bmatrix}$$

$$m_{11} = m_{12} = m_{13} := m, \quad m_{22} = m - m_1, \quad m_{23} + m_{33} = 2m - 2m_1 - m_2.$$

A combinatorial argument then will complete the proof. We must have $m - m_1 - m_2/2 \leq m_{23} \leq m$, but among these values just $m_{23} = m - m_1$ and $m_{33} = m - m_1 - m_2$ are acceptable. For the obvious reason of restriction on weights. Now we find the possibilities for the following matrix

$$\begin{bmatrix} m & m-m_1 & m-m_1-m_2 \\ m_{12} & m_{22} & \\ m_{11} & & \end{bmatrix}.$$

The number of possible values for the m_{11}, m_{12} and m_{22} is

$$\sum_{j=0}^{m_1} \sum_{i=0}^{m_2+1} (i+j) = \frac{1}{2}(m_1+1)(m_2+2)(m_1+m_2+2).$$

Case $m_{12} = m$: In this case we will have the total $(m_1+1) + (m_1+2) \cdots + (m_1+m_2+1)$ solutions. If $m_{12} = m-1 : m_1 + (m_1+1) \cdots + (m_1+m_2)$ and so on until $m_{12} = m - m_1 :$

$0 + 1 + 2 + \cdots + (m_2 + 1)$.

$\square$

Let us show the case $(0, 1)$ and $(1, 0)$. In case $(1, 0)$ the only possibility for m_{23} is $m - 1$ and then we will have the following options

$$|m'\rangle \otimes |m\rangle = \begin{bmatrix} m & m-1 & m-1 \\ m_{12} & m_{22} & \\ m_{11} & & \end{bmatrix} \otimes \begin{bmatrix} m & m-1 & m-1 \\ m & m-1 & \\ m & & \end{bmatrix}$$

which there exists exactly 3 solutions for this case. The same argument shows options for the case $(0, 1)$ are of the form of

$$|m'\rangle \otimes |m\rangle = \begin{bmatrix} m & m & m-1 \\ m & m_{22} & \\ m_{11} & & \end{bmatrix} \otimes \begin{bmatrix} m & m & m-1 \\ m & m & \\ m & & \end{bmatrix}$$

which again just gives us dim$= 3$.

Theorem 5.1.2. *(The q-analogue of the Borel-Weil theorem) If $\lambda = (m_1, m_2)$ is a dominant weight for $SU_q(3)$, the space of holomorphic sections $\Gamma_{hol}(L_\lambda)$ of the associated line bundle over the quantum flag manifold $Fl_q(3)$ is an irreducible representation of $SU_q(3)$ of the highest weight λ. If λ is not dominant $\Gamma_{hol}(L_\lambda) = 0$. All the irreducible representations of $SU_q(3)$ will be obtained in this way.*

Proof. It is shown that for a dominant λ, the space $\Gamma_{hol}(L_\lambda)$ is finite dimensional. It is easy to see that $SU_q(3)$ coacts on $\Gamma_{hol}(L_\lambda)$ since $U_q(\mathfrak{su}(3))$ acts on $\Gamma_{hol}(L_\lambda)$. The fact that this is an irreducible representation can be seen by the existence of the highest weight vector v given by

$$v = \begin{bmatrix} m & m-m_1 & m-m_1-m_2 \\ m & m-m_1 & \\ m & & \end{bmatrix} \otimes \begin{bmatrix} m & m-m_1 & m-m_1-m_2 \\ m-m_1 & m-m_1-m_2 & \\ m-m_1-m_2 & & \end{bmatrix}$$

It is not hard to see that $E_i \blacktriangleright v = 0$ and $K_i \blacktriangleright v = q^{m_i/2}v$. Using the highest weight theorem one can see this is the only irreducible representation of $SU_q(3)$. $\square$

This theorem can be generalized to the following case.

Theorem 5.1.3. *If $\lambda = (m_1, m_2, \cdots, m_l)$ is a dominant weight for $SU_q(l+1)$, the space of holomorphic sections $\Gamma_{hol}(L_\lambda)$ of the associated line bundle over the quantum flag manifold $Fl_q(l+1)$ is an irreducible representation of $SU_q(l+1)$ of the highest weight λ. If λ is not dominant $\Gamma_{hol}(L_\lambda) = 0$. All the irreducible representations of $SU_q(l+1)$ will be obtained in this way.*

Chapter 6

Noncommutative complex structures of finite spaces

6.1 Complex structures on $\bigoplus_{i=1}^{k} M_{n_i}$

Let $(\mathcal{A}, \mathcal{H}, D)$ be the spectral triple associated to $X = \{a, b\}$ by $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ acting diagonaly on $\mathcal{H}_a \oplus \mathcal{H}_b$ i.e. $f \to \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix}$. The Dirac operator is given by the matrix $D = \begin{bmatrix} 0 & m \\ \overline{m} & 0 \end{bmatrix}$. So $\mathrm{d}f := [D, f] = (f(b) - f(a)) \begin{bmatrix} 0 & m \\ -\overline{m} & 0 \end{bmatrix}$. We would like to define a complex structure on $\mathcal{A}$ by $\partial f := [D^{(1,0)}, f]$ and $\overline{\partial} f := [D^{(0,1)}, f]$ where $D^{(1,0)} := \begin{bmatrix} 0 & 0 \\ \overline{m} & 0 \end{bmatrix}$ and $D^{(0,1)} := \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$. It is easy to see that $\mathrm{d} = \partial + \overline{\partial}$, $\partial f^* = (\overline{\partial} f)^*$ and $\overline{\partial} f = 0$ iff f is constant.

The next case that we consider is $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C}$. In this case

$$D^{(1,0)} = \begin{bmatrix} 0 & 0 \\ \overline{m} & 0 \end{bmatrix}, \quad D^{(0,1)} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$$

where m is a column vector. Define ∂ and $\overline{\partial}$ as above. An element $f \in \mathcal{A}$ has the form $f = \begin{bmatrix} A & 0 \\ 0 & \lambda \end{bmatrix}$, where A is a 2×2 matrix and λ is an scalar. Let $\chi_A(x) = A - xI$.

Proposition 6.1.1. *With the above notation f is holomorphic iff rows of $\chi_A(\lambda)$ are orthogonal to the vector m. In particular if f is holomorphic then λ is an eigenvalue of A.*

Proof. A direct calculation shows that $\overline{\partial} f = \begin{bmatrix} 0 & m_1(\lambda - a_{11}) - m_2 a_{12} \\ 0 & m_2(\lambda - a_{22}) - m_1 a_{21} \end{bmatrix}$. So f is holomorphic if and only if

$$\begin{cases} m_1(\lambda - a_{11}) - m_2 a_{12} = 0 \\ -m_1 a_{21} + m_2(\lambda - a_{22}) = 0 \end{cases}$$

Hence two vectors $(\lambda - a_{11}, -a_{12})$ and $(-a_{21}, \lambda - a_{22})$ are orthogonal to the vector m, since they must be colinear and det $\begin{bmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{bmatrix} = 0$. □

This result could be generalized easily to the following.

Proposition 6.1.2. *If $\mathcal{A} = M_n(\mathbb{C}) \oplus \mathbb{C}$, f=$\begin{bmatrix} A & 0 \\ 0 & \lambda \end{bmatrix}$ is holomorphic iff rows of $\chi_A(\lambda)$ are orthogonal to the vector m. In particular, if f is holomorphic then λ is an eigenvalue of A.*

Here A is a $n \times n$ matrix and m is column vector in $\mathbb{C}^n$.

Proof. With the same argument we end up with the following system

$$\begin{cases} m_1(\lambda - a_{11}) - m_2 a_{12} - \ldots - m_n a_{1n} & = 0 \\ -m_1 a_{21} + m_2(\lambda - a_{22}) - \ldots - m_n a_{2n} & = 0 \\ \vdots & \\ -m_1 a_{n1} - m_2 a_{n2} - \ldots + m_n(\lambda - a_{nn}) & = 0. \end{cases}$$

The result is obvious now. □

If $\mathcal{A} = M_n(\mathbb{C}) \oplus M_k(\mathbb{C})$, we can formulate the following.

Proposition 6.1.3. *If f=*$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ *is holomorphic then*

$$det \begin{bmatrix} \chi_B(a_{11}) & a_{12}I & \cdots & a_{1n}I \\ a_{21}I & \chi_B(a_{22}) & \cdots & a_{2n}I \\ \vdots & & & \\ a_{n1}I & \cdots & a_{n-1,1}I & \chi_B(a_{nn}) \end{bmatrix} = 0. \tag{6.1}$$

Proof. Easy. □

Notation. We say $det(\chi_B(A)) = 0$ if (6.1) holds.

On the space of k points if we take the $\mathcal{A} = \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$ and we define

$$D^{(0,1)} = \begin{bmatrix} 0 & M_{1,2} & M_{1,3} & \cdots & M_{1,k} \\ 0 & 0 & M_{2,3} & \cdots & M_{2,k} \\ \vdots & & & & \\ 0 & 0 & & \cdots & M_{k-1,k-1} \\ 0 & 0 & & \cdots & 0 \end{bmatrix},$$

where $M_{i,j}$ is of order $n_i \times n_j$.

Proposition 6.1.4. *With the above notation if an element f=diag*$(A_1, A_2, \cdots, A_k) \in \mathcal{A}$ *is holomorphic, then the conditions det*$(\chi_{A_i}(A_j)) = 0$ *for* $1 \leq j < i \leq k$ *must hold.*

Proof. Easy. □

6.2 Holomorphic vector bundles on the space of two points

Let $\mathcal{E} = \mathcal{E}_a \oplus \mathcal{E}_b$ be a nontrivial vector bundle on the space $X = \{a, b\}$ with dimension 2 and 1 respectively as in [4]. We have $\mathcal{E} = f\mathcal{A}^2$, where $f = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$. The Grassmannian connection is given by

$$\nabla_0 \sigma = f\,\mathrm{d}\sigma = f(\partial\sigma + \overline{\partial}\sigma),$$

Any other connection will be $\nabla = \nabla_0 + \rho$ where $\rho \in M_2(\Omega^1_D(\mathcal{A}))$ [4]. To have a holomorphic structure on $\mathcal{E}$, we must have $(\nabla^{(0,1)})^2 = 0$.

$$\begin{aligned} \nabla^2 = (f\,\mathrm{d} + \rho)^2 &= f\,\mathrm{d}f\,\mathrm{d} + f\,\mathrm{d}\rho + \rho f\,\mathrm{d} + \rho^2 \\ &= f\,\mathrm{d}f\,\mathrm{d}f + f\,\mathrm{d}\rho + \rho^2. \end{aligned} \tag{6.2}$$

In fact, since $\mathrm{d}\xi = \mathrm{d}f\xi = (\mathrm{d}f)\xi + f\,\mathrm{d}\xi$, then

$$f\,\mathrm{d}(f\,\mathrm{d}\xi) = f\,\mathrm{d}(f(\mathrm{d}f)\xi + f\,\mathrm{d}\xi) = f\,\mathrm{d}f\,\mathrm{d}f\xi - f\,\mathrm{d}f\,\mathrm{d}\xi + f\,\mathrm{d}f\,\mathrm{d}\xi,$$

which gives the first term in 6.2. We also have $f\,\mathrm{d}(\rho\xi) + \rho f\,\mathrm{d}\xi = f\,\mathrm{d}(\rho)\xi - f\rho\,\mathrm{d}\xi + f\rho\,\mathrm{d}\xi = f\,\mathrm{d}\rho\xi$, which gives the second term.

We recall from ([4], chapter 6) that $\rho^* = \rho$ and $f\rho = \rho f = \rho$, which implies:

$$\rho_{11} = -\overline{\Phi}_1 e\,\mathrm{d}e + \Phi_1(1-e)\,\mathrm{d}e, \quad \rho_{21} = \overline{\Phi}_2 e\,\mathrm{d}e \quad \rho_{12} = \rho_{21}^*, \quad \rho_{22} = 0.$$

Suppose that $\rho = (\rho_{ij})$, then the curvature F is given by

$$\begin{bmatrix} 0 & 0 \\ 0 & e\,\mathrm{d}e\,\mathrm{d}e \end{bmatrix} + \begin{bmatrix} \mathrm{d}\rho_{11} & (\mathrm{d}\rho_{12})e \\ e(\mathrm{d}\rho_{21}) & 0 \end{bmatrix} + \begin{bmatrix} \rho_{11}^2 + \rho_{12}\rho_{21} & \rho_{11}\rho_{12} \\ \rho_{21}\rho_{11} & \rho_{21}\rho_{12} \end{bmatrix}.$$

Now it is not hard to find the $(0,2)$ part of the curvature. Let $F^{(0,2)} = (a_{ij})$, then

$$\begin{aligned} a_{11} &= (-\Phi_1 - \overline{\Phi}_1 - |\Phi_1|^2)\overline{\partial}e\overline{\partial}e - |\Phi_2|^2(1-e)\overline{\partial}e\overline{\partial}e, \\ a_{12} &= \Phi_2(1 + \overline{\Phi}_1)e\overline{\partial}e\overline{\partial}e, \\ a_{21} &= \overline{\Phi}_2(1 + \Phi_1)e\overline{\partial}e\overline{\partial}e, \\ a_{22} &= (1 - |\Phi_2|^2)e\overline{\partial}e\overline{\partial}e \end{aligned}$$

One can easily see that there are no complex numbers Φ_1 and Φ_2 such that the entries a_{ij} vanish. Therefore,

Theorem 6.2.1. *With the above notation there is no holomorphic structure on $\mathcal{E}_a \oplus \mathcal{E}_b$.*

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Curriculum Vitae

Ali Moatadelro

Education

- Ph.D. Mathematics, University of Western Ontario, Canada (2007-2011)
  - *Supervisor:* Prof. Masoud Khalkhali
  - *Thesis:* Noncommutative complex geometry of the quantum projective space
- M.Sc. Mathematics, University of Tehran, Iran (1999-2002)
  - *Supervisor:* Prof. Ahmad Shafiei Deh Abad
  - *Thesis:* Quantum Cyclotomic Orders of 3-manifolds
- B.Sc. Mathematics, University of Tehran, Iran (1995-1999)

Research Publications

- M. Khalkhali, **A. Moatadelro**, The quantum homogeneous coordinate ring of projective plane, *J. Geom. Phys. Volume 61, Issue 1, January 2011, 276-289*
- M. Khalkhali, **A. Moatadelro**, Noncommutative complex geometry of the quantum projective spaces, to appear in *J. Geom. Phys.*, arXiv:1105.0456
- M. Khalkhali, **A. Moatadelro**, The Borel-Weil-Bott theorem and quantum flag manifolds, Preprint

Selected Presentations

- Noncommutative complex geometry of the quantum projective space, Canadian Operator Symposium (COSY 2011), University of Victoria, (May 2011)
- Noncommutative complex geometry of the quantum projective space, University of Western Ontario (April 2011)
- The quantum homogeneous coordinate ring of $\mathbb{C}P_q^2$, University of New Brunswick (August 2010)
- Nonommutative variations on Laplace's equation, Noncommutative geometry Seminar, University of Western Ontario(November 2009)
- CKM invariants in Noncommutative geometry, Noncommutative geometry Seminar, University of Western Ontario (November 2008)