Uniformly Convergent Finite Element and Finite Difference Methods for Singularly Perturbed Ordinary Differential Equations

A thesis submitted for the degree of Doctor of Philosophy

at the National University of Ireland

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Summary

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This thesis is concerned with uniformly convergent finite element and finite difference methods for numerically solving singularly perturbed two-point boundary value problems.

We examine the following four problems: (i) high order problem of reaction–diffusion type; (ii) high order problem of convection–diffusion type; (iii) second order interior turning point problem; (iv) semilinear reaction–diffusion problem.

Firstly, we consider high order problems of reaction–diffusion type and convection–diffusion type. Under suitable hypotheses, the coercivity of the associated bilinear forms is proved and representation results for the solutions of such problems are given. It is shown that, on an equidistant mesh, polynomial schemes cannot achieve a high order of convergence which is uniform in the perturbation parameter. Piecewise polynomial Galerkin finite element methods are then constructed on a Shishkin mesh. High order convergence results, which are uniform in the perturbation parameter, are obtained in various norms.

Secondly, we investigate linear second order problems with interior turning points. Piecewise linear Galerkin finite element methods are generated on various piecewise equidistant meshes designed for such problems. These methods are shown to be convergent, uniformly in the singular perturbation parameter, in a weighted energy norm and the usual L^2 norm.

Finally, we deal with a semilinear reaction–diffusion problem. Asymptotic properties of solutions to this problem are discussed and analysed. Two simple finite difference schemes on Shishkin meshes are applied to the problem. They are proved to be uniformly convergent of second order and fourth order respectively. Existence and uniqueness of a solution to both schemes are investigated.

Numerical results for the above methods are presented.

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Contents

Chapter 1

Introduction

A singularly perturbed problem is a problem that depends on a parameter (or parameters) in such a way that solutions behave nonuniformly as the parameter tends toward some limiting value of interest. Singularly perturbed differential equations occur in many areas of application and have been widely considered both in asymptotic and numerical analysis. This thesis is concerned with finite element and finite difference methods which are convergent, uniformly in the perturbation parameter, for certain types of singularly perturbed two-point boundary value problems.

1.1 Singularly Perturbed Ordinary Differential Equations

Consider the two-point boundary value problem consisting of the differential equation of order m

$$-\varepsilon u_\varepsilon^{(m)} + P_0(u_\varepsilon, x) = 0, \quad \text{for } x \in (X_1, X_2) \tag{1.1.1a}$$

and the boundary conditions

$$B_k u_\varepsilon = \gamma_k, \quad \text{for } k = 1, \ldots, m. \tag{1.1.1b}$$

Here $\varepsilon \in (0,1]$ is a small parameter, P_0 (which is independent of ε) is a linear or nonlinear ordinary differential operator of order m_0 ($< m$) and the B_k are auxiliary functions.

Suppose that problem (1.1.1) has a solution $u_\varepsilon(x)$. This solution depends not only on the independent variable x but also on the parameter ε. Let us observe the behaviour of $u_\varepsilon(x)$. We start by examining the model second order convection–diffusion problem

$$-\varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) = 0, \quad \text{for } x \in (0,1), \tag{1.1.2a}$$

$$u_\varepsilon(0) = 0, \quad u_\varepsilon(1) = 1. \tag{1.1.2b}$$

The solution of this problem is

$$u_\varepsilon(x) = \frac{\exp(-(1-x)/\varepsilon) - \exp(-1/\varepsilon)}{1 - \exp(-1/\varepsilon)}.$$

For any fixed $x \in [0,1)$, the solution $u_\varepsilon(x)$ converges to 0 as $\varepsilon \to 0$, but $u_\varepsilon(1) = 1$ for all $\varepsilon \in (0,1]$. This indicates that $u_\varepsilon(x)$ changes abruptly in a neighbourhood of $x = 1$ when ε is small. The smaller ε is, the worse $u_\varepsilon(x)$ behaves. Problem (1.1.2) is a singularly perturbed problem.

For the more general problem (1.1.1), the behaviour of the solution $u_\varepsilon(x)$ is more complicated. Here we confine our discussion to singularities of the solution $u_\varepsilon(x)$ itself as $\varepsilon \to 0$. (For certain problems, e.g., Example 1.1.2 below, such singularities occur not in $u_\varepsilon(x)$ but in some of its derivatives.) Under appropriate conditions, $u_\varepsilon(x)$ will converge to a piecewise smooth function $u_0(x)$ for any fixed $x \in [X_1, X_2]$ as $\varepsilon \to 0$, except for a finite set of points x_i ($i = 1, \ldots, l$). We refer to the x_i as critical points. The limiting function $u_0(x)$ is a solution to the reduced problem of (1.1.1). This reduced problem consists of the reduced equation $P_0(u_0, x) = 0$, obtained by setting $\varepsilon = 0$ in (1.1.1a), and m_0 conditions chosen from (1.1.1b). For small ε, $u_\varepsilon(x)$ changes rapidly on small regions adjoining the critical points. The location of these critical points and the nature of the solution $u_\varepsilon(x)$ near them depend on the character of the reduced operator P_0 and the boundary conditions (1.1.1b).

A critical point can be an endpoint of the interval $[X_1, X_2]$. In this case we say that u_ε exhibits a boundary layer in a neighbourhood of the critical point. If the point is in the interior of $[X_1, X_2]$, we say that u_ε has an internal layer at this point.

Many approaches have been developed to derive asymptotic expansions for solutions to problems of type (1.1.1); see O'Malley [29], Smith [38], Wasow [51] and their references. An asymptotic expansion is usually composed of an "outer solution" and an "inner solution". The outer solution approximates the exact solution $u_\varepsilon(x)$ well for values of x away from layers. The inner solution (also known as a boundary or internal layer function) describes the singular behaviour of the solution in layers. The solution of problem (1.1.1) exhibit various singular behaviour in layers. Let us gives some examples which have different layer functions and which are typical of the problems examined in this thesis.

Example 1.1.1 *Convection–diffusion problem*

$$-\varepsilon u_\varepsilon'' + a(x)u_\varepsilon' + b(x)u_\varepsilon = f(x), \quad \textit{for } x \in (0,1),$$

with $u_\varepsilon(0)$ *and* $u_\varepsilon(1)$ *given, where* $a(x) \geq \alpha > 0$ *for* $x \in [0,1]$. *This problem has a boundary layer* $\exp(-\alpha(1-x)/\varepsilon)$ *of exponential type at* $x = 1$.

Example 1.1.2 *Linearized model of the extensible beam problem*

$$-\varepsilon u_\varepsilon^{(4)} + \left(a(x)u_\varepsilon'\right)' + b(x)u_\varepsilon' = f(x), \quad \textit{for } x \in (0,1),$$

where $a(x) \geq \alpha > 0$ *for* $x \in [0,1]$, *with clamped boundary conditions, i.e.,* $u_\varepsilon(0)$, $u'_\varepsilon(0)$, $u_\varepsilon(1)$ *and* $u'_\varepsilon(1)$ *are given. This problem has two boundary layers* $\sqrt{\varepsilon}\exp(-\alpha x/\sqrt{\varepsilon})$ *and* $\sqrt{\varepsilon}\exp(-\alpha(1-x)/\sqrt{\varepsilon})$ *of exponential type.*

Example 1.1.3 *Simple attractive turning point problem*

$$-\varepsilon u''_\varepsilon - x u'_\varepsilon + b(x)u_\varepsilon = f(x), \quad \text{for } x \in (-1,1),$$

with $u_\varepsilon(-1)$ *and* $u_\varepsilon(1)$ *given, where* $b(x) \geq 0$ *and* $b(0) > 0$. *This problem has an internal layer* $\left(|x| + \varepsilon^{1/2}\right)^{\lambda}$ *of cusp type at* $x = 0$, *where* $\lambda = b(0)$.

Example 1.1.4 *Semilinear reaction-diffusion problem*

$$-\varepsilon u''_\varepsilon + b(x, u_\varepsilon) = 0, \quad \text{for } x \in (0,1),$$

with $u_\varepsilon(0)$ *and* $u_\varepsilon(1)$ *given, where* $b_u(x,u) \geq \beta > 0$ *for all* $(x,u) \in [0,1] \times \mathcal{R}^1$. *This problem has two boundary layers* $\exp(-\alpha x/\sqrt{\varepsilon})$ *and* $\exp(-\alpha(1-x)/\sqrt{\varepsilon})$ *of exponential type.*

Singularly perturbed problems of type (1.1.1) arise in many fields of application, notably chemical–reactor theory, optimal–control theory, fluid mechanics, elasticity theory and the physical theory of semiconductors and transistors. The purpose of this thesis is to develop numerical methods for solving certain cases of problem (1.1.1).

1.2 Uniformly Convergent Numerical Methods

It has long been recognized that severe difficulties can arise when "standard" numerical methods are applied to problem (1.1.1) for small ε. These difficulties often result from the instability of the numerical process.

Consider, for example, the model second order convection–diffusion problem (Example 1.1.1) with $b = f = 0$ and a constant. Suppose that we use equidistant meshes with mesh spacing h. On this mesh, the standard central difference scheme suffers from spurious oscillations and is grossly inaccurate, unless $ah/2\varepsilon < 1$. In other words, a stringent restriction on the number of the mesh points is required to preserve stability when ε is small.

To achieve stability without mesh restriction, one can use an upwinding scheme. In fact, such a scheme is convergent as $h \to 0$ for any fixed ε. However, in the typical case $h \geq \varepsilon$, the scheme gives a good approximation of the behaviour of the exact solution only outside the boundary layer. If a node lies inside the boundary layer, the discrete solution at that node may be quite inaccurate when ε is small.

We shall consider methods which are both stable and accurate for all $\varepsilon \in (0, 1]$. Consider a numerical method for (1.1.1). Suppose that this method has a solution u_N. The method is said to be *uniformly convergent*, with respect to a norm $\|\cdot\|$, if $\|u - u_N\| \to 0$ as $N \to \infty$, uniformly in the parameter ε. Here and subsequently, N is the number of subintervals in the mesh used. Furthermore, if $\|u - u_N\| \leq Cg(N)$, where C is a constant independent of N and ε, we say that the method is uniformly convergent with order $g(N)$.

Various approaches have been used to produce uniformly convergent methods. One approach is to work with a uniform mesh and use special difference schemes which take account of the nature of the solution to (1.1.1). Such schemes are called fitted schemes. An alternative is to devise polynomial–based methods on nonequidistant special meshes. The construction of these meshes is based on the boundary and internal layer functions.

Much research has been carried out on the use of uniformly convergent fitted schemes for convection–diffusion problems such as Example 1.1.1. See for example Allen and Southwell [1], Doolan et al. [10], Gartland [16], Stynes and O'Riordan [43] and their references.

Problems like those in Examples 1.1.2–1.1.4 have been less widely studied and it is these types of problem which we shall examine in this thesis. (We give references to earlier work on such problems in the appropriate chapters below.) Since fitted schemes are related to the solution of the differential equation, they can be quite complicated. This disadvantage makes it difficult to use fitted schemes to solve more complex problems. For an example of this complexity in the context of higher–order problems, see Roos and Stynes [35].

Consequently, instead of fitted schemes, we shall use the alternative approach outlined earlier, of using simple schemes on special meshes.

Uniformly convergent polynomial–based methods on nonequidistant special meshes date back to Bakhvalov [2], who introduced a graded mesh constructed by a mesh–generating function to solve a reaction–diffusion problem. Based on this idea several modifications and generalizations of Bakhvalov mesh were constructed. They were used in finite difference schemes for various singularly perturbed second order problems of type (1.1.1); see Vulanović [46, 47, 48] and Herceg [21]. Gartland [17] constructed and analysed a family of compact finite difference schemes on an exponentially graded mesh for higher order problems of the form (1.1.1).

Recently, Shishkin [37] proposed a piecewise equidistant mesh suitable for singularly perturbed problems with layers of exponential type. The mesh of Shishkin type is piecewise uniform and so much simpler than the graded meshes of the above authors. We are not aware of any published work on finite element analysis of uniformly convergent methods on special meshes for problems of type (1.1.1). This leads us to construct and analyse uniformly convergent polynomial–based methods on the meshes of Shishkin type for various singularly perturbed problems of type (1.1.1), using for the most part a finite element framework.

1.3 Outline of Thesis

An outline of this thesis is as follows:

In Chapters 2 and 3 we consider high order problems of reaction–diffusion type and convection–diffusion type respectively. Under suitable hypotheses, the coercivity of the associated bilinear forms is proved and representation results for the solutions of such problems are given. Galerkin finite element methods based on piecewise polynomial test/trial functions and Shishkin meshes are constructed and proved to be uniformly convergent in various norms.

Chapter 4 investigates linear second order problems with interior turning points. Piecewise linear Galerkin finite element methods are generated on various piecewise equidistant meshes designed for such problems. These methods are shown to be uniformly convergent in a weighted energy norm and the usual L^2 norm.

In Chapter 5 we deal with a semilinear reaction–diffusion problem. Asymptotic properties of solutions to this problem are discussed and analysed. Two simple finite difference schemes on Shishkin meshes are applied to the problem. Existence, uniqueness and uniform convergence of a solution to both schemes are investigated.

Finally Chapter 6 draws some conclusions from our work.

Notation: Throughout this thesis we let C, sometimes subscripted, denote a generic positive constant that may take different values in different formulas, but is always independent of N and ε. We shall say that a quantity y is $O(z)$ when we mean that $|y| \leq Cz$ for all sufficiently small z.

Chapter 2

High Order Reaction–Diffusion Problems

2.1 Introduction and Background

Consider the singularly perturbed two–point boundary value problem:

$$L_\varepsilon u \equiv (-1)^m \varepsilon^2 u^{(2m)} + (-1)^{m-1}\left(a_{2(m-1)} u^{(m-1)}\right)^{(m-1)} + L_1 u$$

$$= f(x), \quad \text{for } x \in (0,1), \tag{2.1.1a}$$

$$u^{(j)}(0) = u^{(j)}(1) = 0, \quad \text{for } j = 0, \ldots, m-1, \tag{2.1.1b}$$

where $m \geq 2$ is an integer, $\varepsilon \in (0,1]$ is a perturbation parameter, and

$$L_1 u \equiv \sum_{k=2}^{m} (-1)^{m-k} \left(a_{2(m-k)+1} u^{(m-k+1)} + a_{2(m-k)} u^{(m-k)}\right)^{(m-k)}.$$

The functions a_r(for $r = 0, 1, \ldots, 2(m-1)$) and f are assumed to be sufficiently smooth with

$$a_{2(m-1)}(x) > \alpha > 0 \quad \text{on } [0,1], \tag{2.1.1c}$$

and

$$a_{2(m-k)}(x) - \frac{1}{2} a'_{2(m-k)+1}(x) > \alpha_{m-k}, \quad \text{for } k = 2, \ldots, m, \tag{2.1.1d}$$

for all $x \in [0,1]$ and some constants $\alpha_{m-1} = \alpha$ and α_{m-k} $(k = 2, \ldots, m)$ satisfying

$$\sum_{i=1}^{k} \alpha_{m-i} > 0, \quad \text{for } k = 2, \ldots, m. \tag{2.1.1e}$$

Condition (2.1.1c) excludes turning points, while (2.1.1c) - (2.1.1e) will together guarantee the coercivity of the associated bilinear form, and hence the solvability of the given problem. We consider the homogeneous boundary conditions (2.1.1b), since non-homogeneous conditions $u^{(j)}(0) = A_j$ and $u^{(j)}(1) = B_j$, for $j = 0, 1, \ldots, m-1$, can be homogenized by the transformation $\tilde{u}(x) = u(x) - \sum_{j=0}^{m-1}\{(-1)^j A_j \xi_{m,j}(1-x) + B_j \xi_{m,j}(x)\}$, where the $\xi_{m,j}(\cdot)$ are defined by (2.5.39).

The solution of problem (2.1.1) has, in general, boundary layers at both endpoints of [0,1]. More precisely, $|u^{(m)}(x)|$ is unbounded in the neighbourhoods of $x = 0$ and $x = 1$ as $\varepsilon \to 0$. See (2.2.7) - (2.2.11) below.

If we formally set $m = 1$ in (2.1.1a), we have a model second order reaction–diffusion problem. For this reason we refer to (2.1.1) as being of reaction–diffusion type. In chapter 3 we shall consider $2m$th order singularly perturbed ordinary differential equations which have a nonvanishing $u^{(2m-1)}$ term; we say that such problems are of convection–diffusion type.

When $m = 2$, (2.1.1a) is a variant of the Orr–Sommerfeld equation. This differential equation also governs the deflection of an elastic beam with small flexural rigidity under tension subject to a specified load f, according to the linearized Euler–Bernoulli beam theory; see Semper [36]. The conditions (2.1.1b) correspond to the ends of the beam being clamped.

In this chapter, we consider only "uniformly convergent" (also known as "robust"; see Babuška and Suri [3]) methods; these are methods whose solutions converge to u, uniformly in ε, in some norm.

The second order problem ($m = 1$ and $L_1 \equiv 0$) has been extensively examined. Ways of generating uniformly convergent, exponentially fitted schemes on equidistant meshes are considered for example in Doolan et al. [10], Guo and Lin [19], Hegarty et al. [20], Niijima [28], O'Riordan [30], O'Riordan and Stynes [31], Roos [33] and Sun [44], while a uniformly convergent classical scheme on a special mesh may be found in Herceg [21]. In contrast, there are only a few results on high order problems in the literature (see Roos and Stynes [35] and its references).

Roos and Stynes [35] considered the fourth order problem

$$\varepsilon^2 u^{(4)} - \big(a(x)u'\big)' + b(x)u' + c(x)u = f(x), \quad \text{for } x \in (0,1), \tag{2.1.2a}$$

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{2.1.2b}$$

with

$$a(x) \geq \alpha > 0, \tag{2.1.2c}$$

and

$$c(x) - \frac{1}{2}b'(x) \geq \beta > -\alpha. \tag{2.1.2d}$$

This is the problem (2.1.1) with $m = 2$. In Roos and Stynes [35] an approximate solution is generated by using patched basis functions. The method is uniformly first order convergent in the $H^1[0,1]$ norm. It appears to be the only published scheme which achieves this degree of accuracy for this problem. However, the scheme is quite complicated, since the patched basis functions ϕ are approximate solutions of the problems

$$\varepsilon^2\phi^{(4)} - a\phi'' + b\phi' + c\phi = 1$$

and

$$\varepsilon^2\phi^{(4)} - a\phi'' + b\phi' + c\phi = 0,$$

with some boundary conditions on each mesh interval.

Semper [36] also considered the problem (2.1.2). He examines piecewise polynomial finite element solutions on a quasi–equidistant mesh and gives the error estimate

$$\sqrt{H}\|(u - u_N)'\|_{L^2} + \|u - u_N\|_{L^2} = O(H) + O(\varepsilon/H),$$

when $\varepsilon \ll H$, where u is the solution of problem (2.1.2), u_N is a piecewise cubic finite element approximation and H is the mesh diameter. Numerical results in Semper [36] show that this estimate is sharp. Thus the numerical solution may exhibit a significant order of locking ; see Babuška and Suri [3], i.e., the order of convergence obtained for small ε is significantly inferior to that obtained when $\varepsilon = 1$.

Gartland [17] studied compact finite difference schemes for differential operators of the form

$$L_\varepsilon u \equiv \varepsilon u^{(m)} + \sum_{k=0}^{m-1} a_k u^{(k)}$$

without turning points (i.e., $a_{m-1}(x) \neq 0$ for all x) on a special graded mesh. His results are based on the stability theory of Niederdrenk and Yserentant [27], whose assumption that $a_{m-1} \neq 0$ seems essential, so it is not possible to apply these results directly to our problem (2.1.1).

In this chapter, we shall generate and analyse Galerkin finite element methods for problem (2.1.1). First, in Section 2.2 we prove existence and uniqueness of a solution to (2.1.1) and examine an asymptotic expansion of this solution. In Section 2.3, we consider equidistant meshes. We show (Lemma 2.3.1) that on an equidistant mesh standard polynomial $(2m+1)$–point difference schemes for (2.1.1) cannot be uniformly convergent of order greater than $m-1$ in the discrete maximum norm. Thus the optimal order of convergence is not attained. We also briefly discuss an exponentially fitted scheme which for $m=2$ is uniformly first order convergent in a weighted energy norm and uniformly second order convergent in the usual discrete maximum norm. In Section 2.4, we turn our attention to finite element methods. Using piecewise polynomials as our basis functions on an arbitrary mesh, we obtain a uniform error estimate. Section 2.5 contains an analysis of uniform convergence for the finite element approximation when the mesh is of Shishkin type. This piecewise equidistant mesh is much simpler than the graded meshes of Bakhvalov [2], Gartland [17] and Herceg [21]; in general, it resolves part (but not all) of the boundary layers. Furthermore, the resulting method has polynomial coefficients and is simpler than that of Roos and Stynes [35]. We use a standard finite element analysis to prove that the resulting method is uniformly convergent of order $(N^{-1}\ln N)^m$ with respect to the weighted energy norm $|||\cdot|||$ associated with (2.1.1a). It does not seem possible to use a standard duality argument to deduce a higher order of uniform convergence in the $H^{m-1}[0,1]$ Sobolev norm $\|\cdot\|_{m-1}$. We therefore employ another technique, which is based on that of Stynes and O'Riordan [43], to show that our method is uniformly convergent of order $(N^{-1}\ln N)^{m+1}$ in $\|\cdot\|_{m-1}$. These uniform convergence results are almost optimal (see Remarks 2.5.1 and 2.5.2). Our method is significantly more accurate than those of Roos and Stynes [35] and Semper [36]; see Remark 2.5.3 below. Section 2.6 contains numerical results for the fourth order problem (2.1.2).

2.2 Coercivity, Existence and Uniqueness

We analyze the properties of the continuous solution u to problem (2.1.1). Let us first introduce some notation. We denote by $(\cdot,\cdot)$ the $L^2(0,1)$ inner product and by $H^0 = L^2$ and H^k (for $k = 1,\ldots,m$) the usual Sobolev spaces on $[0,1]$. The norm on H^k will be written as $\|\cdot\|_k$, with the usual associated seminorm $|\cdot|_k$, for $k = 0,\ldots,m$. The essential supremum norm on $L^\infty[0,1]$ is denoted by $\|\cdot\|_\infty$. For $k = 0,1,\ldots,m-1$, the maximum norm on $C^k[0,1]$ is denoted by $\|\cdot\|_{k,\infty}$, i.e., $\|v\|_{k,\infty} = \sum_{j=0}^{k} \|v^{(j)}\|_\infty$, for all $v \in C^k[0,1]$. Set

$$H_0^m = \left\{ v \in H^m : v^{(j)}(0) = v^{(j)}(1) = 0, \quad \text{for } j = 0,\ldots,m-1 \right\}.$$

We define our bilinear form $A_\varepsilon(\cdot,\cdot)$ to be

$$A_\varepsilon(v,w) = \left(\varepsilon^2 v^{(m)}, w^{(m)}\right) + \left(a_{2(m-1)} v^{(m-1)}, w^{(m-1)}\right) + A_1(v,w),$$

where

$$A_1(v,w) = \sum_{k=2}^{m} \left(a_{2(m-k)+1} v^{(m-k+1)} + a_{2(m-k)} v^{(m-k)}, w^{(m-k)}\right),$$

for all $v, w \in H_0^m$. Our weighted energy norm is given by

$$|||v||| = \left\{\varepsilon^2 |v|_m^2 + \|v\|_{m-1}^2\right\}^{1/2}, \quad \forall v \in H_0^m.$$

In what follows we shall make repeated use of the fact that

$$|v|_{s-1} \le |v|_s, \quad \text{for } s \in \{1,\ldots,m\} \quad \text{and all } v \in H_0^m. \tag{2.2.1}$$

We begin the analysis by showing that the bilinear form $A_\varepsilon(\cdot,\cdot)$ is continuous and uniformly coercive over $H_0^m \times H_0^m$.

Lemma 2.2.1 *Assume that (2.1.1c) – (2.1.1e) hold. Then there exist positive constants C_1 and C_2 such that for all $v, w \in H_0^m$,*

$$|A_\varepsilon(v,w)| \le C_1 |||v||| \cdot |||w||| \tag{2.2.2}$$

and

$$C_2 |||v|||^2 \le A_\varepsilon(v,v). \tag{2.2.3}$$

Proof. It is easy to see that (2.2.2) is true, using the Cauchy-Schwarz inequality. For (2.2.3), we have for each $v \in H_0^m$,

$$\begin{aligned} A_\varepsilon(v,v) &= \varepsilon^2 \left(v^{(m)}, v^{(m)}\right) + \left(a_{2(m-1)} v^{(m-1)}, v^{(m-1)}\right) \\ &\quad + \sum_{k=2}^{m} \left(\left(a_{2(m-k)} - \frac{1}{2} a'_{2(m-k)+1}\right) v^{(m-k)}, v^{(m-k)}\right) \\ &\ge \varepsilon^2 |v|_m^2 + \sum_{k=1}^{m} \alpha_{m-k} |v|_{m-k}^2, \end{aligned} \tag{2.2.4}$$

by (2.1.1c) and (2.1.1d).

We now prove, by induction on r, that for $r = 1, \ldots, m$,

$$\sum_{k=1}^{r} \alpha_{r-k} |v|_{r-k}^2 \ge \min_{1 \le j \le r} \sum_{k=1}^{j} \alpha_{r-k} |v|_{r-1}^2, \quad \forall v \in H_0^m. \tag{2.2.5}$$

The case $r = 1$ is trivially true. Fix $s \in \{1, \ldots, m-1\}$. Assume that (2.2.5) holds for $r = s$. Then

$$\begin{aligned} &\sum_{k=1}^{s+1} \alpha_{s+1-k} |v|_{s+1-k}^2 \\ &= \alpha_s |v|_s^2 + \sum_{k=1}^{s} \alpha_{s-k} |v|_{s-k}^2 \\ &\ge \alpha_s |v|_s^2 + \min_{1 \le j \le s} \left\{ \sum_{k=1}^{j} \alpha_{s-k} \right\} |v|_{s-1}^2, \end{aligned}$$

by the inductive hypothesis,

$$\geq \min\left\{\alpha_s, \alpha_s + \min_{1\leq j\leq s}\left\{\sum_{k=1}^{j}\alpha_{s-k}\right\}\right\}|v|_s^2, \quad \text{from (2.2.1)},$$

$$= \min_{0\leq j\leq s}\left\{\sum_{k=0}^{j}\alpha_{s-k}\right\}|v|_s^2$$

$$= \min_{1\leq j\leq s+1}\left\{\sum_{k=1}^{j}\alpha_{s+1-k}\right\}|v|_s^2.$$

This proves the case $r = s+1$. By induction, the proof of (2.2.5) is complete.

Hence, (2.2.4) implies that

$$A_\varepsilon(v,v) \geq \varepsilon^2|v|_m^2 + \min_{1\leq j\leq m}\left\{\sum_{k=1}^{j}\alpha_{m-k}\right\}|v|_{m-1}^2$$

$$\geq \varepsilon^2|v|_m^2 + m^{-1}\min_{1\leq j\leq m}\left\{\sum_{k=1}^{j}\alpha_{m-k}\right\}\|v\|_{m-1}^2,$$

by (2.2.1), which is the desired result with

$$C_2 = \min\left\{1, m^{-1}\min_{1\leq j\leq m}\left\{\sum_{k=1}^{j}\alpha_{m-k}\right\}\right\}. \ \square$$

We can now define our weak formulation of (2.1.1): find $u \in H_0^m$ such that

$$A_\varepsilon(u,v) = (f,v), \quad \forall v \in H_0^m. \tag{2.2.6}$$

Clearly, the mapping $v \longmapsto (f,v)$ is a bounded functional on H_0^m. Combining this with Lemma 2.2.1, the Lax-Milgram Lemma tells us that (2.2.6) has a unique solution $u(x)$ in H_0^m. This weak solution is also the classical solution to (2.1.1), if all the data are smooth.

The reduced problem of (2.1.1) is (see O'Malley [29], p.42)

$$(-1)^{m-1}\left(a_{2(m-1)}z^{(m-1)}\right)^{(m-1)} + L_1 z = f(x), \quad \text{for } x \in (0,1),$$

$$z^{(j)}(0) = z^{(j)}(1) = 0, \quad \text{for } j = 0,\ldots,m-2.$$

From the proof of Lemma 2.2.1, we know that the reduced problem also has a unique solution. It follows from Theorem 2 of O'Malley [29] that the solution $u(x)$ of (2.1.1) has the representation

$$u(x) = G(x) + \varepsilon^{m-1} G_1(x) \exp\left(-\frac{1}{\varepsilon}\int_0^x \sqrt{a_{2(m-1)}(s)\,ds}\right)$$

$$+ \varepsilon^{m-1} G_2(x) \exp\left(-\frac{1}{\varepsilon}\int_x^1 \sqrt{a_{2(m-1)}(s)\,ds}\right), \qquad (2.2.7)$$

where the functions G, G_1, and G_2 have asymptotic power series expansions in ε and are sufficiently differentiable for $x \in [0,1]$. For convenience we will write this in the form

$$u(x) = G(x) + E(x) + F(x), \qquad (2.2.8)$$

where for $x \in [0,1]$ and $j = 0, 1, \ldots$, we have

$$\left|G^{(j)}(x)\right| \le C, \qquad (2.2.9)$$

$$\left|E^{(j)}(x)\right| \le C\varepsilon^{m-1-j} \exp\left(-\alpha x/\varepsilon\right), \qquad (2.2.10)$$

$$\left|F^{(j)}(x)\right| \le C\varepsilon^{m-1-j} \exp\left(-\alpha(1-x)/\varepsilon\right). \qquad (2.2.11)$$

Thus, for $x \in [0,1]$,

$$\left|u^{(j)}(x)\right| \le C, \quad \text{for } j = 0, \ldots, m-1. \qquad (2.2.12)$$

2.3 A Necessary Condition on an Equidistant Mesh

In this section, we consider the numerical solution of (2.1.1) on an equidistant mesh. We show that, if a typical difference scheme is uniformly convergent of sufficiently high order in the discrete maximum norm, then certain coefficients of that scheme must have an exponential nature. This generalizes a result of Doolan, Miller and Schilders [10], who considered the case $m = 1$.

Assume that $G_1(0) \neq 0$ and $G_2(1) \neq 0$, so that $|u^m(x)|$ is not bounded uniformly in ε for $x = 0$ and $x = 1$. (For otherwise the problem is better behaved and is easier to solve numerically.)

Let the mesh be $\{x_i : x_i = ih, \text{ for } i = 0, \ldots, N\}$, where N is a positive integer. Let the difference scheme be

$$(-1)^m \varepsilon^2 \sigma_i(\rho) \frac{\delta^{2m} u_N(x_i)}{h^{2m}} + (-1)^{m-1} \frac{\delta^{m-1}\left(a_{2(m-1)}(x_i)\delta^{m-1} u_N(x_i)\right)}{h^{2m-2}} + L_1^N u_N(x_i) = f(x_i), \quad \text{for } i = m, \ldots, N-m, \tag{2.3.1}$$

with some appropriate discrete boundary conditions, where

$$\delta r(x_i) \equiv r(x_i + h/2) - r(x_i - h/2),$$

and L_1^N is any standard approximation to L_1 satisfying

$$\left|(L_1^N - L_1)u(x_i)\right| \leq Mh, \quad \text{for } i = m, \ldots, N-m, \tag{2.3.2}$$

with

$$M = \max_{0 \leq x \leq 1} \left\{ \left|u^{(j)}(x)\right| : \quad \text{for } j = 0, \ldots, 2m-2 \right\}.$$

We have

Lemma 2.3.1 *Let u be the solution of problem (2.1.1). Recall our assumption that $G_1(0) \neq 0$ and $G_2(1) \neq 0$. Let $\{u_N(x_i) : i = 0, 1, \ldots, N\}$ be a solution of (2.3.1). Suppose that u_N converges to u, uniformly in ε, with order greater than $m-1$ in the* *discrete maximum norm. Then for fixed $\rho = h/\varepsilon$ and each fixed $i \in \{m, \dots, N-m\}$,*

$$\lim_{h\to 0} \sigma_i(\rho) = \left(\frac{\rho_0/2}{\sinh \rho_0/2}\right)^2 \tag{2.3.3}$$

and

$$\lim_{h\to 0} \sigma_{N-i}(\rho) = \left(\frac{\rho_1/2}{\sinh \rho_1/2}\right)^2, \tag{2.3.4}$$

where

$$\rho_0 = \frac{h}{\varepsilon}\sqrt{a_{2(m-1)}(0)} \quad \textit{and} \quad \rho_1 = \frac{h}{\varepsilon}\sqrt{a_{2(m-1)}(1)}.$$

Proof. Fix $\rho > 0$ and $i \in \{m, \dots, N-m\}$. Since u_N converges to u, uniformly in ε, with order greater than $m-1$ in the discrete maximum norm, we have

$$\begin{aligned} L_1^N u_N(x_i) &= L_1^N u(x_i) + O(h^{m-1} h^{-2m+2}) \\ &= L_1 u(x_i) + O(h\varepsilon^{-m+1}) + O(h^{-m+2}), \end{aligned}$$

by (2.3.2), with $M \le C\varepsilon^{-m+1}$ from (2.2.8) - (2.2.11). Hence

$$\begin{aligned} L_1^N u_N(x_i) &= L_1 u(x_i) + O(h^{-m+2}) \\ &= O(h^{-m+2}), \end{aligned}$$

since $|L_1 u(x_i)| \le C\varepsilon^{-m+2}$, from (2.2.8) - (2.2.11).

Multiplying (2.3.1) by $(-1)^m h^{2m-2}$, we obtain

$$\begin{aligned} &\frac{\sigma_i(\rho)}{\rho^2}\delta^{2m} u_N(x_i) - \delta^{m-1}\left(a_{2(m-1)}(x_i)\delta^{m-1} u_N(x_i)\right) \\ &= (-1)^m h^{2m-2}\left(f(x_i) - L_1^N u_N(x_i)\right) \\ &= O(h^m). \end{aligned} \tag{2.3.5}$$

Moreover, our assumption of uniform convergence with order greater than $m-1$ implies that

$$\lim_{h\to 0} \varepsilon^{-m+1}\delta^{2m} u_N(x_i) = \lim_{h\to 0} \varepsilon^{-m+1}\delta^{2m} u(x_i) \tag{2.3.6}$$

and

$$\lim_{h\to 0}\varepsilon^{-m+1}\delta^{m-1}\left(a_{2(m-1)}(x_i)\delta^{m-1}u_N(x_i)\right)$$
$$=\lim_{h\to 0}\varepsilon^{-m+1}\delta^{m-1}\left(a_{2(m-1)}(x_i)\delta^{m-1}u(x_i)\right). \tag{2.3.7}$$

On the other hand, from the decomposition (2.2.7),

$$\lim_{h\to 0}\varepsilon^{-m+1}\delta^{2m}u(x_i)=G_1(0)\delta^{2m}\exp(-\rho_0 i) \tag{2.3.8}$$

and

$$\lim_{h\to 0}\varepsilon^{-m+1}\delta^{m-1}\left(a_{2(m-1)}(x_i)\delta^{m-1}u(x_i)\right)$$
$$=G_1(0)a_{2(m-1)}(0)\delta^{2m-2}\exp(-\rho_0 i). \tag{2.3.9}$$

Combining (2.3.5) - (2.3.9) yields

$$\begin{aligned}\lim_{h\to 0}\sigma_i(\rho)&=\lim_{h\to 0}\left\{\frac{\rho_0^2\delta^{2m-2}\exp(-\rho_0 i)}{\delta^{2m}\exp(-\rho_0 i)}+\frac{\rho^2 O(h^m)}{\varepsilon^{m-1}G_1(0)\delta^{2m}\exp(-\rho_0 i)}\right\}\\&=\frac{\rho_0^2\exp(-\rho_0 i)\left(\exp(\rho_0/2)-\exp(-\rho_0/2)\right)^{2m-2}}{\exp(-\rho_0 i)\left(\exp(\rho_0/2)-\exp(-\rho_0/2)\right)^{2m}}\\&=\left(\frac{\rho_0/2}{\sinh\rho_0/2}\right)^2.\end{aligned}$$

which completes the proof of (2.3.3). Then (2.3.4) can be proven similarly. □

An exponentially fitted scheme can be constructed in the following way on an equidistant mesh. Consider a Galerkin finite element method with a bilinear form based on approximating the coefficients in (2.1.1a) by piecewise linears. The basis functions are simplified $\bar{L}$-splines defined by

$$\varepsilon^2\varphi_k^{(2m)}-\bar{a}_{2(m-1)}\varphi_k^{(2m-2)}=0,\quad \text{for } x\in(x_{k-1},x_k),$$

with some boundary conditions, for $k=1,\ldots,N$, where $\bar{a}_{2(m-1)}$ is a piecewise constant approximation of $a_{2(m-1)}(x)$. When $m=2$, this scheme is much simpler than that of Roos and Stynes [35]. One can prove that it is uniformly first order convergent in a weighted energy norm and uniformly second order convergent in the usual discrete maximum norm, by employing an analysis similar to that of Stynes and O'Riordan [43]. However, the resulting scheme is still complicated because of the exponential fitting factors. In Section 2.5, we shall show that one can obtain uniformly convergent numerical solutions on a certain piecewise equidistant mesh without requiring any exponential factors in the scheme.

2.4 A Galerkin Finite Element Analysis on an Arbitrary Mesh

To construct a Galerkin finite element method for (2.1.1), we first work with a general finite–dimensional approximation space $S^N \subseteq H_0^m$, on an arbitrary mesh

$$X^N: \quad 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1.$$

Set $h_i = x_i - x_{i-1}$, for $i = 1, \ldots, N$, and $H = \max_i h_i$. For each i, denote by I_i the subinterval $[x_{i-1}, x_i]$.

To take into account the effect of quadrature errors, we define a modified bilinear form $A_\varepsilon^N(\cdot,\cdot)$ on $H_0^m \times H_0^m$ by

$$A_\varepsilon^N(v,w) \equiv \varepsilon^2 \left(v^{(m)}, w^{(m)}\right) + \left(a_{2(m-1)}^N v^{(m-1)}, w^{(m-1)}\right) + A_1^N(v,w),$$

where

$$A_1^N(v,w) \equiv \sum_{k=2}^{m} \left(a_{2(m-k)+1}^N v^{(m-k+1)} + a_{2(m-k)}^N v^{(m-k)}, w^{(m-k)}\right)$$

and a_r^N denotes a piecewise polynomial approximation of a_r for $r = 0, 1, \ldots, 2(m-1)$ respectively. For each r, these approximations are assumed to satisfy

$$\left|\left(a_r^N - a_r\right)(x)\right| \le C h_i^l, \quad \text{for } x \in I_i \text{ and } i = 1, \ldots, N, \qquad (2.4.1)$$

where l is a fixed positive integer.

It is easy to see that the modified bilinear form $A_\varepsilon^N(\cdot,\cdot)$ is uniformly bounded, i.e.,

$$|A_\varepsilon^N(v,w)| \le C|||v|||\cdot|||w|||, \quad \forall v,w \in H_0^m. \tag{2.4.2}$$

We show that $A_\varepsilon^N(\cdot,\cdot)$ is also uniformly coercive over $H_0^m \times H_0^m$.

Lemma 2.4.1 *There exists a positive constant h_0 (independent of ε) such that for $H \le h_0$, we have*

$$C|||v|||^2 \le A_\varepsilon^N(v,v), \quad \forall v \in H_0^m.$$

Proof. Let $v \in H_0^m$ be arbitrary but fixed. Now

$$A_\varepsilon^N(v,v) = A_\varepsilon(v,v) + \left(A_\varepsilon^N - A_\varepsilon\right)(v,v). \tag{2.4.3}$$

For the second term of (2.4.3),

$$\begin{aligned}
&\left|\left(A_\varepsilon^N - A_\varepsilon\right)(v,v)\right| \\
&= \left|\left(\left(a_{2(m-1)}^N - a_{2(m-1)}\right)v^{(m-1)}, v^{(m-1)}\right) + \left(A_1^N - A_1\right)(v,v)\right| \\
&\le CH^l\|v\|_{m-1}^2 \\
&\le CH^l|||v|||^2.
\end{aligned}$$

Combining this with (2.4.3) and Lemma 2.2.1 completes the proof. □

We are now in a position to introduce our approximate solution: find $u_N \in S^N$ such that

$$A_\varepsilon^N(u_N, v) = \left(f^N, v\right), \quad \forall v \in S^N, \tag{2.4.4}$$

where f^N is defined analogously to a_r^N in (2.4.1).

It follows from Lemma 2.4.1 that u_N is well defined.

We begin the error analysis with a standard finite element estimate. We give details of the proof here so the reader can see that certain constants are independent of ε.

Theorem 2.4.1 *Let u be the solution of problem (2.1.1) and $u_N \in S^N$ the solution of our method (2.4.4) on an arbitrary mesh X^N. Then*

$$|||u - u_N||| \le C \inf_{v \in S^N} |||u - v||| + CN^{-l}. \tag{2.4.5}$$

Proof. Let $v \in S^N$ be arbitrary. Then by Lemma 2.4.1,

$$\begin{aligned}
&C|||v - u_N|||^2 \\
&\le A_\varepsilon^N(v - u_N, v - u_N) \\
&= A_\varepsilon^N(v - u, v - u_N) + \left(A_\varepsilon^N - A_\varepsilon\right)(u, v - u_N) \\
&\quad + A_\varepsilon(u, v - u_N) - A_\varepsilon^N(u_N, v - u_N) \\
&= A_\varepsilon^N(v - u, v - u_N) + \left(\left(a_{2(m-1)}^N - a_{2(m-1)}\right) u^{(m-1)}, (v - u_N)^{(m-1)}\right) \\
&\quad + \left(A_1^N - A_1\right)(u, v - u_N) + \left(f - f^N, v - u_N\right) \\
&\le C|||v - u||| \cdot |||v - u_N||| + CN^{-l}\|u\|_{m-1}\|v - u_N\|_{m-1} \\
&\quad + CN^{-l}\|v - u_N\|_0 \\
&\le C\left(|||v - u||| + N^{-l}\right)|||v - u_N|||,
\end{aligned}$$

using (2.4.1), (2.4.2) and (2.2.12). Hence

$$|||v - u_N||| \le C|||v - u||| + CN^{-l}.$$

Then

$$\begin{aligned}
|||u - u_N||| &\le |||u - v||| + |||v - u_N||| \\
&\le C|||u - v||| + CN^{-l}.
\end{aligned}$$

Since $v \in S^N$ is arbitrary, (2.4.5) follows. □

2.5 Uniform Convergence Results on a Shishkin Mesh

For arbitrary meshes and general approximation spaces, (2.4.5) will not yield a bound which is uniform in ε. To achieve such uniformity (in various norms), we shall work with piecewise polynomial spaces on a special piecewise equidistant mesh. The construction and analysis of such meshes was initiated by Shishkin [37].

2.5.1 The Mesh

Given a positive integer N which is divisible by 4, the Shishkin mesh X_s^N is constructed by dividing the interval $[0, 1]$ into three subintervals

$$[0, \sigma], \quad [\sigma, 1-\sigma], \quad \text{and} \quad [1-\sigma, 1].$$

Equidistant meshes are then used on each subinterval, with $N/4$ points on each of $[0, \sigma]$ and $[1-\sigma, 1]$, and $N/2$ points on $[\sigma, 1-\sigma]$. The parameter σ is defined by

$$\sigma = \min\left\{1/4, \quad (m+1)\alpha^{-1}\varepsilon \ln N\right\},$$

which depends on ε and N. More explicitly, we have

$$X_s^N : 0 = x_0 < x_1 < \ldots < x_{i_0} < \ldots < x_{N-i_0} < \ldots < x_N = 1,$$

with $i_0 = N/4, x_{i_0} = \sigma,\ x_{N-i_0} = 1 - \sigma$, and

$$h_i = 4\sigma N^{-1}, \quad \text{for } i = 1, \ldots, i_0, N - i_0 + 1, \ldots, N, \tag{2.5.1}$$

$$h_i = 2(1-2\sigma)N^{-1}, \quad \text{for } i = i_0 + 1, \ldots, N - i_0. \tag{2.5.2}$$

This mesh is much simpler than other graded meshes which have been used for singularly perturbed two-point boundary value problems, such as the Bakhvalov mesh; see Bakhvalov [2] and Herceg [21], and the Gartland mesh [17].

We shall assume that

$$\sigma = (m+1)\alpha^{-1}\varepsilon \ln N. \tag{2.5.3}$$

For otherwise $\varepsilon^{-1} \leq 4(m+1)\alpha^{-1}\ln N$, i.e., N^{-1} is small relative to ε, which is unlikely in practice (and in this case the method can be analyzed in the classical way).

From (2.5.1) - (2.5.3), one can easily see that the interval lengths satisfy

$$h_i = (m+1)\alpha^{-1}\varepsilon N^{-1}\ln N, \tag{2.5.4}$$

for $i = 1, \ldots, i_0, N - i_0 + 1, \ldots, N$, and

$$N^{-1} \leq h_i \leq 2N^{-1}, \tag{2.5.5}$$

for $i = i_0 + 1, \ldots, N - i_0$.

2.5.2 Interpolation Error Estimates

We use standard approximation theory to estimate the interpolation errors in the weighted energy norm $|||\cdot|||$ and the Sobolev norm $\|\cdot\|_{m-1}$ on the Shishkin mesh. However as we shall see, the analysis is not entirely straightforward.

Since the solution of the weak formulation (2.2.6) lies in H_0^m, we define our piecewise polynomial approximation space by

$$S^N = \{v(x) \in H_0^m : v|_{I_i} \in P_R(I_i) \text{ for } i = 1, \ldots, N\},$$

where $P_R(I_i)$ is the set of polynomials of degree at most R on I_i and R is some positive integer. In order to guarantee that $S^N \subset C^{m-1} \subset H^m$, we assume that $R \geq 2m-1$.

Consider a finite element $(\Sigma_i, P_R(I_i), I_i)$, where Σ_i is the set of degrees of freedom. It will be assumed that the set Σ_i is P_R–unisolvent, for $i = 1, \ldots, N$. Let s_i be the greatest order of derivatives occuring in the definition of Σ_i. From the P_R–unisolvence of the set Σ_i, we have $R+1 \geq s_i$. Given $v \in C^{s_i}(I_i)$, we denote by $\Pi_i v$ the P_R–interpolant to v on I_i. Set $s = \max\{s_i : i = 1, \ldots, N\}$. Then for $v \in C^s(0,1)$, we will denote by $\Pi^N v$ the piecewise polynomial interpolant from S^N to v. This interpolant satisfies $(\Pi^N v)|_{I_i} = \Pi_i v|_{I_i}$, for $i = 1, \ldots, N$.

Denote by $\|\cdot\|_{j,\infty,I_i}$ the maximum norm on $C^j(I_i)$, with the usual associated seminorm $|\cdot|_{j,\infty,I_i}$. We have

Lemma 2.5.1 *Let k be an integer satisfying $R+1 \geq k \geq s_i$. Let $v \in C^k(I_i)$. Then there exists a constant C_1, which is independent of h_i and v, such that*

$$|v - \Pi_i v|_{j,\infty,I_i} \leq C_1 h_i^{k-j} |v|_{k,\infty,I_i}, \tag{2.5.6}$$

for $j = 0, 1, \ldots, k$.

Proof. From Theorem 3.1.5 of Ciarlet [7], (2.5.6) holds for $R+1 \geq k > s_i$. The case $k = s_i$ can be shown by a similar argument. □

We remark that Lemma 2.5.1 is valid on an arbitrary mesh.

Next, we use Lemma 2.5.1 to estimate $u - \Pi^N u$ on each interval I_i of the Shishkin mesh of subsection 2.5.1.

Recalling (2.2.8) - (2.2.11), we see that

$$\left|u^{(2m)}(x)\right| \leq C\varepsilon^{-m-1}, \quad \text{for } x \in [0,1] \tag{2.5.7}$$

and

$$\left|u^{(2m)}(x)\right| \leq C, \quad \text{for } x \in [\sigma_\varepsilon, 1-\sigma_\varepsilon], \tag{2.5.8}$$

where $\sigma_\varepsilon = \min\left\{1/4, (m+1)\alpha^{-1}\varepsilon\ln(1/\varepsilon)\right\}$.

Consider the realistic situation when $N^{-1} > \varepsilon$. In this case, we see that $[\sigma_\varepsilon, 1-\sigma_\varepsilon] \subseteq [\sigma, 1-\sigma]$. Consequently, as $\varepsilon \to 0$ with N fixed, $|u^{(j)}(x)|$ is unbounded for $j \geq m$ when $x \in J_\varepsilon \equiv ([\sigma, \sigma_\varepsilon) \cup (1-\sigma_\varepsilon, 1-\sigma])$. Recall that the Shishkin mesh X_ε^N is coarse on $[\sigma, 1-\sigma]$. In particular, it is coarse on J_ε. It turns out that a direct application of standard approximation theory to u yields suboptimal results on J_ε. In the proof of Lemma 2.5.2 we shall need an asymptotic decomposition of u in order to achieve the desired optimality. The estimate (2.5.6) with $k = s_i$ then plays a special role in our error analysis.

Set

$$\hat{s} = \max\left\{s_i : i_0 + 1 \leq i \leq i_1 \text{ and } N - i_1 + 1 \leq i \leq N - i_0\right\},$$

where $i_1 = \max\{i : I_i \cap (\sigma, \sigma_\varepsilon) \neq \emptyset\}$. We will assume that $\hat{s} = m-1$, so that (2.5.6) can be used with $k = m-1$ for those coarse subintervals I_i where $|u^{(m)}(x)|$ is unbounded as $\varepsilon \to 0$.

Lemma 2.5.2 *Let u be the solution of problem (2.1.1). Let $i \in \{1, \ldots, N\}$. Then on the Shishkin mesh X_ε^N, we have*

$$\varepsilon\,|u - \Pi_i u|_{m,\infty,I_i} \leq C(N^{-1}\ln N)^m \tag{2.5.9}$$

and

$$|u - \Pi_i u|_{j,\infty,I_i} \le C h_i^{m-j-1} (N^{-1} \ln N)^{m+1}, \tag{2.5.10}$$

for $j = 0, 1, \ldots, m-1$.

Proof. Consider first the fine portions of the mesh, i.e., suppose that $i \in \{1, \ldots, i_0\} \cup \{N - i_0 + 1, \ldots, N\}$. Then it is clear, on taking $k = 2m$ in (2.5.6) and using (2.5.4) and (2.5.7), that (2.5.9) and (2.5.10) hold.

Now suppose that we are on the coarse part of the mesh, i.e., suppose that $i \in \{i_0 + 1, \ldots, N - i_0\}$. We discuss two cases.

Case 1: $N^{-1} \le \varepsilon$. Then $[\sigma, 1-\sigma] \subseteq [\sigma_\varepsilon, 1 - \sigma_\varepsilon]$. Hence

$$|u|_{2m,\infty,I_i} \le C, \quad \text{for } I_i \subset [\sigma, 1-\sigma]. \tag{2.5.11}$$

It is easy to see, on again taking $k = 2m$ in (2.5.6) and using (2.5.5) and (2.5.11), that

$$|u - \Pi_i u|_{j,\infty,I_i} \le C h_i^{m-j-1} N^{-m-1}, \tag{2.5.12}$$

for $j = 0, 1, \ldots, m$, which implies (2.5.9) and (2.5.10).

Case 2: $N^{-1} > \varepsilon$. By the same argument as in Case 1, one can show that (2.5.12) still holds for $i \in \{i_1 + 1, \ldots, N - i_1\}$.

Now suppose that $i \in \{i_0 + 1, \ldots, i_1\} \cup \{N - i_1 + 1, \ldots, N - i)\}$, i.e., we are dealing with the intersection of the coarse mesh and the boundary layers. Here we need the decomposition (2.2.8). Write $\Pi_i u$ in the form

$$\Pi_i u = \Pi_i G + \Pi_i E + \Pi_i F, \tag{2.5.13}$$

where $\Pi_i G$, $\Pi_i E$ and $\Pi_i F$ denote the P_R–interpolants to G, E and F respectively. We shall separately bound $|G - \Pi_i G|_{j,\infty,I_i}$, $|E - \Pi_i E|_{j,\infty,I_i}$ and $|F - \Pi_i F|_{j,\infty,I_i}$ for $j = 0, 1, \ldots, m$.

Firstly, by (2.2.9) and arguments similar to those of Case 1, we have

$$|G - \Pi_i G|_{j,\infty,I_i} \leq C h_i^{m-j-1} N^{-m-1}, \tag{2.5.14}$$

for $j = 0, 1, \ldots, m$.

Secondly, we estimate $|E - \Pi_i E|_{j,\infty,I_i}$. For $j = m$, we obtain, from (2.5.6) with $k = m$,

$$|E - \Pi_i E|_{m,\infty,I_i} \leq C\,|E|_{m,\infty,I_i}\,.$$

Since $I_i \subseteq [\sigma, 1]$, we get by (2.2.10)

$$\begin{aligned} |E - \Pi_i E|_{m,\infty,I_i} &\leq C\varepsilon^{-1}\exp\left(-\alpha\sigma/\varepsilon\right) \\ &= C\varepsilon^{-1} N^{-m-1}, \end{aligned} \tag{2.5.15}$$

by (2.5.3). For $j = 0, 1, \ldots, m-1$, using (2.5.6) with $k = m-1$,

$$\begin{aligned} |E - \Pi_i E|_{j,\infty,I_i} &\leq C h_i^{m-j-1}\,|E|_{m-1,\infty,I_i} \\ &\leq C h_i^{m-j-1}\exp\left(-\alpha\sigma/\varepsilon\right) \\ &= C h_i^{m-j-1} N^{-m-1}. \end{aligned} \tag{2.5.16}$$

Similarly, one may show that

$$|F - \Pi_i F|_{m,\infty,I_i} \leq C\varepsilon^{-1} N^{-m-1} \tag{2.5.17}$$

and

$$|F - \Pi_i F|_{j,\infty,I_i} \leq C h_i^{m-j-1} N^{-m-1}, \tag{2.5.18}$$

for $j = 0, 1, \ldots, m-1$.

Combining (2.2.8) and (2.5.13) – (2.5.18) yields

$$\varepsilon \left|u - \Pi_i u\right|_{m,\infty,I_i} \leq C\left(\varepsilon N^{-m} + N^{-m-1}\right)$$

and

$$\left|u - \Pi_i u\right|_{j,\infty,I_i} \leq C h_i^{m-j-1} N^{-m-1},$$

for $i \in \{i_0+1, \ldots, i_1\} \cup \{N - i_1 + 1, \ldots, N - i_0\}$ and $j \in \{0, 1, \ldots, m-1\}$.

This completes the proof of Case 2. Combining Cases 1 and 2 yields

$$\varepsilon \left|u - \Pi_i u\right|_{m,\infty,I_i} \leq C\left(\varepsilon N^{-m} + N^{-m-1}\right)$$

and

$$\left|u - \Pi_i u\right|_{j,\infty,I_i} \leq C h_i^{m-j-1} N^{-m-1},$$

for all $i \in \{i_0+1, \ldots, N-i_0\}$ and $j \in \{0, 1, \ldots, m-1\}$. Recalling the first paragraph of the proof, we are done. □

The next result follows immediately.

Corollary 2.5.1 *Let u be the solution of problem (2.1.1). Let $\Pi^N u$ be the piecewise polynomial interpolant from S^N to u on the Shishkin mesh X_s^N. Then*

$$|||u - \Pi^N u||| \leq C(N^{-1} \ln N)^m \tag{2.5.19}$$

and

$$\|u - \Pi^N u\|_{m-1} \leq C(N^{-1} \ln N)^{m+1}. \tag{2.5.20}$$

Remark 2.5.1 *Recall that S^N contains piecewise polynomials of degree $2m-1$. Away from the boundary layers the mesh is in practice coarse, with diameter $O(N^{-1})$. Consequently it can be seen a priori that one cannot do better than*

$$\|u - \Pi^N u\|_{m-1} \leq CN^{-m-1}. \tag{2.5.21}$$

Our estimate (2.5.20) thus shows that the Shishkin mesh is at least almost optimal, in the sense that no mesh consisting of $O(N)$ points can improve on (2.5.21).

2.5.3 Convergence Results

We first present a uniform convergence result in the weighted energy norm $|||\cdot|||$.

Theorem 2.5.1 *Let $u_N \in S^N$ be the solution of our method (2.4.4) on the Shishkin mesh X_s^N. Then for N sufficiently large (independently of ε), we have*

$$|||u - u_N||| \leq C\left((N^{-1}\ln N)^m + N^{-l}\right). \tag{2.5.22}$$

Proof. The result follows immediately from Theorem 2.4.1 and Corollary 2.5.1. □

Remark 2.5.2 *Suppose that we use a sufficiently accurate quadrature rule, so that $l \geq m$. Then our polynomially based method on a piecewise equidistant mesh is optimal, in the sense that (2.5.19) and (2.5.22) correspond. In the terminology of Babuška and Suri [3], it is robust (uniformly convergent) with uniform order $(N^{-1}\ln N)^m$ in the weighted energy norm $|||\cdot|||$ and only shows a slight amount of locking, viz., order* $(\ln N)^m$.

Recalling (2.5.20), one may expect that $\|u - u_N\|_{m-1}$ has a higher order of uniform convergence than that implied by (2.5.22). However, for the problem (2.1.1), one cannot in general use an Aubin–Nitsche approach to show that $\|u-u_N\|_{m-1}$ has a higher order of uniform convergence than $|||u-u_N|||$ on an arbitrary mesh. For in Stynes and O'Riordan [42] an example is presented with $m=1$ and piecewise linear functions on an equidistant mesh, where it is shown that $\|u-u_N\|_d$ and $|||u-u_N|||$ are both $O(N^{1/2})$. Here $\|\cdot\|_d$ is a discrete L^2 norm.

We shall see that on a Shishkin mesh one does in fact achieve a higher order of convergence in $\|\cdot\|_{m-1}$ than in $|||\cdot|||$. However it does not seem possible to prove this via an Aubin–Nitsche argument, since to get sharp interpolation error estimates one needs a decomposition similar to (2.2.8). We shall instead show that this higher order convergence occurs by using an analysis similar to that of Stynes and O'Riordan [43].

Since $S^N \subseteq C^{m-1}(0,1)$, it is natural to assume that the set Σ_i includes $p^{(j)}(x_{i-1})$ and $p^{(j)}(x_i)$, for $p \in P_R(I_i)$ and $j=0,1,\ldots,m-1$. Hence for $v \in C^s(0,1)$,

$$\left(\Pi^N v\right)^{(j)}(x_i) = v^{(j)}(x_i), \tag{2.5.23}$$

for $i=0,1,\ldots,N$ and $j=0,1,\ldots,m-1$.

Theorem 2.5.2 *Under the same hypotheses as Theorem 2.5.1,*

$$|||\Pi^N u - u_N||| \le C\left(\left(N^{-1}\ln N\right)^{m+1} + N^{-l}\right), \tag{2.5.24}$$

$$\|u-u_N\|_{m-1} \le C\left(\left(N^{-1}\ln N\right)^{m+1} + N^{-l}\right) \tag{2.5.25}$$

and

$$\|u-u_N\|_{m-2,\infty} \le C\left(\left(N^{-1}\ln N\right)^{m+1} + N^{-l}\right). \tag{2.5.26}$$

Proof. By the coercivity of $A_\varepsilon^N(\cdot,\cdot)$ over $H_0^m \times H_0^m$,

$$\begin{aligned} C|||\Pi^N u - u_N|||^2 &\le A_\varepsilon^N \left(\Pi^N u - u_N, \Pi^N u - u_N\right) \\ &= A_\varepsilon^N \left(\Pi^N u - u, \Pi^N u - u_N\right) \\ &\quad + A_\varepsilon^N \left(u - u_N, \Pi^N u - u_N\right). \end{aligned} \tag{2.5.27}$$

We bound these two terms separately. First

$$\begin{aligned} &A_\varepsilon^N \left(\Pi^N u - u, \Pi^N u - u_N\right) \\ &= \left(\varepsilon^2 \left(\Pi^N u - u\right)^{(m)}, \left(\Pi^N u - u_N\right)^{(m)}\right) \\ &\quad + \left(a_{2(m-1)}^N \left(\Pi^N u - u\right)^{(m-1)}, \left(\Pi^N u - u_N\right)^{(m-1)}\right) \\ &\quad + A_1^N \left(\Pi^N u - u, \Pi^N u - u_N\right) \\ &= \left(a_{2(m-1)}^N \left(\Pi^N u - u\right)^{(m-1)}, \left(\Pi^N u - u_N\right)^{(m-1)}\right) \\ &\quad + A_1^N \left(\Pi^N u - u, \Pi^N u - u_N\right), \end{aligned}$$

by (2.5.23) and using integration by parts, since $\left(\Pi^N u - u_N\right)^{(2m)} \equiv 0$ on each subinterval (x_{i-1}, x_i). Hence,

$$\begin{aligned} A_\varepsilon^N \left(\Pi^N u - u, \Pi^N u - u_N\right) &\le C\|\Pi^N u - u\|_{m-1}\|\Pi^N u - u_N\|_{m-1} \\ &\le C\left(N^{-1}\ln N\right)^{m+1} |||\Pi^N u - u_N|||, \end{aligned} \tag{2.5.28}$$

by Corollary 2.5.1.

Secondly,

$$\begin{aligned} &A_\varepsilon^N \left(u - u_N, \Pi^N u - u_N\right) \\ &= \left(A_\varepsilon^N - A_\varepsilon\right) \left(u, \Pi^N u - u_N\right) \\ &\quad + A_\varepsilon \left(u, \Pi^N u - u_N\right) - A_\varepsilon^N \left(u_N, \Pi^N u - u_N\right) \end{aligned}$$

$$= \left(\left(a_{2(m-1)}^N - a_{2(m-1)}\right) u^{(m-1)}, \left(\Pi^N u - u_N\right)^{(m-1)}\right)$$
$$+ \left(A_1^N - A_1\right)\left(u, \Pi^N u - u_N\right) + \left(f - f^N, \Pi^N u - u_N\right)$$
$$\leq C N^{-l} |||\Pi^N u - u_N|||, \tag{2.5.29}$$

by (2.2.12) and (2.4.1).

From (2.5.27) - (2.5.29), we obtain (2.5.24). Combining (2.5.24) with (2.5.20) yields (2.5.25).

Note that for all $v \in H_0^m$,

$$\|v^{(j)}\|_\infty \leq |v|_{j+1} \leq \|v\|_{j+1}, \quad \text{for } j = 0, \dots, m-2. \tag{2.5.30}$$

We therefore have from (2.5.25)

$$\|(u - u_N)^{(j)}\|_\infty \leq \|u - u_N\|_{j+1}$$
$$\leq C\left(\left(N^{-1}\ln N\right)^{m+1} + N^{-l}\right), \quad \text{for } j = 0, \dots, m-2,$$

This completes the proof of (2.5.26). □

Remark 2.5.3 *Consider the case $m = 2$. If $l \geq 3$, then (2.5.25) shows that we obtain uniform convergence of almost third order in H^1. This is in contrast to the first order uniform convergence in H^1 obtained by Roos and Stynes [35] using a much more complicated scheme. Our uniform convergence rate is also significantly better than the $O(N^{1/2})$ rate obtained by Semper [36].*

When l is odd, we can obtain a stronger convergence result, provided that $a_{2(m-1)}$ is now approximated to a higher order of accuracy than the other a_r in (2.1.1).

Theorem 2.5.3 *Let u be the solution of problem (2.1.1) and $u_N \in S^N$ the solution of (2.4.4) on the Shishkin mesh X_ε^N. Assume also that for $i = 1, \ldots, N$,*

$$\left| \left(a_{2(m-1)}^N - a_{2(m-1)} \right)(x) \right| \leq C h_i^{l+1}, \quad \text{for } x \in I_i, \tag{2.5.31}$$

$$\left| \int_{x_{i-1}}^{x_i} \left(a_r^N - a_r \right)(x)\, dx \right| \leq C h_i^{l+2}, \quad \text{for } r = 0, 1, \ldots, 2m-3, \tag{2.5.32}$$

$$\left| \int_{x_{i-1}}^{x_i} \left(f^N - f \right)(x)\, dx \right| \leq C h_i^{l+2}. \tag{2.5.33}$$

Then for N sufficiently large (independently of ε), we have

$$|||\Pi^N u - u_N||| \leq C \left((N^{-1} \ln N)^{m+1} + N^{-l-1} \right), \tag{2.5.34}$$

$$\|u - u_N\|_{m-1} \leq C \left((N^{-1} \ln N)^{m+1} + N^{-l-1} \right) \tag{2.5.35}$$

and

$$\|u - u_N\|_{m-2,\infty} \leq C \left((N^{-1} \ln N)^{m+1} + N^{-l-1} \right). \tag{2.5.36}$$

Remark 2.5.4 *It is well known in the context of Newton–Cotes integration rules that properties (2.5.32) and (2.5.33) are easily achieved using piecewise polynomials of degree $l-1$ when l is odd.*

Proof of Theorem 2.5.3. Recalling the proof of Theorem 2.5.2, we only need to show that

$$A_\varepsilon^N \left(u - u_N, \Pi^N u - u_N \right) \leq C N^{-l-1} |||\Pi^N u - u_N|||. \tag{2.5.37}$$

Here

$$\begin{aligned} & A_\varepsilon^N \left(u - u_N, \Pi^N u - u_N \right) \\ & = \left(\left(a_{2(m-1)}^N - a_{2(m-1)} \right) u^{(m-1)}, \left(\Pi^N u - u_N \right)^{(m-1)} \right) \\ & \quad + \left(A_1^N - A_1 \right) \left(u, \Pi^N u - u_N \right) + \left(f - f^N, \Pi^N u - u_N \right). \end{aligned}$$

It is easy to see that

$$\left|\left(\left(a^N_{2(m-1)} - a_{2(m-1)}\right) u^{(m-1)}, \left(\Pi^N u - u_N\right)^{(m-1)}\right)\right|$$
$$\leq CN^{-l-1}|\Pi^N u - u_N|_{m-1},$$

by (2.2.12) and (2.5.31).

We now bound $\left|\left(A^N_1 - A_1\right)\left(u, \Pi^N u - u_N\right)\right|$. Fix $i \in \{1, \ldots, N\}$. Then set

$$s_r(x) = \int_{x_{i-1}}^{x} \left(a^N_r - a_r\right)(t)\, dt, \quad \text{for } r = 0, 1, \ldots, 2m-3.$$

We have

$$s_r(x_{i-1}) = 0, \quad |s_r(x_i)| \leq Ch_i^{l+2}$$

and

$$|s_r(x)| \leq Ch_i^{l+1}, \quad \text{for } x \in (x_{i-1}, x_i),$$

by (2.5.32) and (2.4.1). Hence

$$\left|\sum_{k=2}^{m} \int_{x_{i-1}}^{x_i} \left(\left(a^N_{2(m-k)+1} - a_{2(m-k)+1}\right) u^{(m-k+1)} + \left(a^N_{2(m-k)} - a_{2(m-k)}\right) u^{(m-k)}\right) \cdot \left(\Pi^N u - u_N\right)^{(m-k)}(x)\, dx\right|$$
$$= \left|\sum_{k=2}^{m} \left\{ \left(s_{2(m-k)+1} u^{(m-k+1)} + s_{2(m-k)} u^{(m-k)}\right)(x) \left(\Pi^N u - u_N\right)^{(m-k)}(x)\Big|_{x=x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} \left(s_{2(m-k)+1} u^{(m-k+2)} + s_{2(m-k)} u^{(m-k+1)}\right)(x) \left(\Pi^N u - u_N\right)^{(m-k)}(x)\, dx - \int_{x_{i-1}}^{x_i} \left(s_{2(m-k)+1} u^{(m-k+1)} + s_{2(m-k)} u^{(m-k)}\right)(x) \left(\Pi^N u - u_N\right)^{(m-k+1)}(x)\, dx \right\}\right|$$
$$\leq Ch_i^{l+2} \sum_{k=2}^{m} \|u\|_{m-k+1,\infty} \|\Pi^N u - u_N\|_{m-k,\infty}$$
$$+ Ch_i^{l+1} \sum_{k=2}^{m} \left\{ \int_{x_{i-1}}^{x_i} \left(\left|u^{(m-k+2)}(x)\right| + \left|u^{(m-k+1)}(x)\right|\right) dx\, \|\Pi^N u - u_N\|_{m-k,\infty} + \int_{x_{i-1}}^{x_i} \left(\left|u^{(m-k+1)}(x)\right| + \left|u^{(m-k)}(x)\right|\right) \left|\left(\Pi^N u - u_N\right)^{(m-k+1)}(x)\right| dx \right\}.$$

Therefore,

$$
\begin{aligned}
&\left|\left(A_1^N - A_1\right)\left(u, \Pi^N u - u_N\right)\right| \\
&\le C \sum_{i=1}^{N} h_i^{l+2} \sum_{k=2}^{m} \|u\|_{m-k+1,\infty} \|\Pi^N u - u_N\|_{m-k,\infty} \\
&\quad + C \sum_{i=1}^{N} h_i^{l+1} \sum_{k=2}^{m} \left\{ \left(\left\|u^{(m-k+2)}\right\|_{L^1[0,1]} + \left\|u^{(m-k+1)}\right\|_{L^1[0,1]} \right) \|\Pi^N u - u_N\|_{m-k,\infty} \right. \\
&\quad \left. + \left(|u|_{m-k+1} + |u|_{m-k}\right) \left|\Pi^N u - u_N\right|_{m-k+1} \right\} \\
&\le C N^{-l-1} \left\|\Pi^N u - u_N\right\|_{m-1},
\end{aligned}
$$

by (2.2.8) - (2.2.12) and (2.5.30).

The term $\left(f - f^N, \Pi^N u - u_N\right)$ is handled similarly. This completes the proof of (2.5.37).

Combining (2.5.27), (2.5.28) with (2.5.37) yields (2.5.34). Then (2.5.35) and (2.5.36) follow from the arguments similar to those of Theorem 2.5.2. □

Corollary 2.5.2 *Let u and u_N be defined as in Theorem 2.5.3. Then under the same hypotheses as in Theorem 2.5.3,*

$$
|||u - u_N||| \le C\left(\left(N^{-1} \ln N\right)^m + N^{-l-1}\right).
$$

Proof. Combining (2.5.19) and (2.5.34), the result follows. □

2.5.4 A Special Case: $s_i = m - 1$ for each i

In this subsection, we give some results for finite element discretizations based on the following Hermite basis function space:

$$
V^N = V_0 \oplus V_1 \oplus \ldots \oplus V_{m-1},
$$

where V_r = linear span of $\{\varphi_i^r : i = 1, \ldots, N-1\}$, for $r = 0, 1, \ldots, m-1$. The basis functions $\{\varphi_i^r\}_{i=1}^{N-1}$, for $r = 0, \ldots, m-1$, are defined by

$$\varphi_i^r(x) = \begin{cases} h_i^r \xi_{m,r}\left(\frac{x-x_{i-1}}{h_i}\right), & \text{for } x \in (x_{i-1}, x_i), \quad (2.5.38a) \\ (-1)^r h_{i+1}^r \xi_{m,r}\left(\frac{x_{i+1}-x}{h_{i+1}}\right), & \text{for } x \in (x_i, x_{i+1}), \quad (2.5.38b) \\ 0, & \text{elsewhere}, \quad (2.5.38c) \end{cases}$$

where $\xi_{m,r}(s)$ satisfies

$$\xi_{m,r}^{(2m)}(s) = 0, \quad \text{for all } s \in \mathcal{R}^1, \tag{2.5.39a}$$

$$\xi_{m,r}^{(j)}(0) = 0 \quad \text{and} \quad \xi_{m,r}^{(j)}(1) = \delta_{rj}, \quad \text{for } j = 0, 1, \ldots, m-1. \tag{2.5.39b}$$

From (2.1.5.3) of Stoer and Bulirsch [39], one can for example easily compute the following explicit formulae for $\xi_{m,r}(s)$.

m	r=0	r=1	r=2
1	s		
2	$s^2(-2s+3)$	$s^2(s-1)$	
3	$s^3(6s^2-15s+10)$	$s^3(s-1)(-3s+4)$	$s^3(s-1)^2/2$

When $m = 2$ (i.e., we consider problem (2.1.2)) and $l = 1$, we can sharpen Theorem 2.5.1.

We define a discrete H^1-norm by

$$|v|_{d_1} = \left\{ \sum_{i=1}^{N-1} \left(h_i^{-1}(v_i - v_{i-1})^2 + \bar{h}_i w_i^2 \right) \right\}^{1/2},$$

for all $v = \sum_{i=1}^{N-1} \left[v_i \varphi_i^0(x) + w_i \varphi_i^1(x) \right] \in V^N$, where $\bar{h}_i = (h_i + h_{i+1})/2$. By a calculation, one may show that for all $v \in V^N$,

$$\sqrt{2/15}\,|v|_{d_1} \le |v|_1 \le \sqrt{7/5}\,|v|_{d_1}. \tag{2.5.40}$$

That is, on V^N the discrete H^1-norm $|\cdot|_{d_1}$ is equivalent to the usual seminorm $|\cdot|_1$.

Theorem 2.5.4 *Let u be the solution of problem (2.1.2). Let $u_N \in V^N$ be the solution of (2.4.4) on the Shishkin mesh X_ε^N, with $m = 2$. Let*

$$a^N(x) = \frac{a(x_{i-1}) + a(x_i)}{2}, \qquad \text{for } x \in (x_{i-1}, x_i) \text{ and } i = 1, \ldots, N,$$

with similar definitions of $b^N(x)$, $c^N(x)$ and $f^N(x)$, so $l = 1$. Then for N sufficiently large (independently of ε), we have

$$|||u - u_N||| \le CN^{-3/2}$$

Proof. We shall prove that

$$|||\Pi^N u - u_N||| \le CN^{-3/2}. \tag{2.5.41}$$

From (2.5.27) and (2.5.28) with $m = 2$, we need only bound

$$\begin{aligned} &A_\varepsilon^N\left(u - u_N, \Pi^N u - u_N\right) \\ &= \left(f - f^N, \Pi^N u - u_N\right) + \left(\left(a^N - a\right) u', \left(\Pi^N u - u_N\right)'\right) \\ &\quad + \left(\left(b^N - b\right) u' + \left(c^N - c\right) u, \Pi^N u - u_N\right). \end{aligned} \tag{2.5.42}$$

Recalling the proof of Theorem 2.5.3 with $l = 1$, one may see that

$$\begin{aligned} &\left|\left(f - f^N, \Pi^N u - u_N\right) + \left(\left(b^N - b\right) u' + \left(c^N - c\right) u, \Pi^N u - u_N\right)\right| \\ &\le N^{-2}|||\Pi^N u - u_N|||. \end{aligned} \tag{2.5.43}$$

We now examine the term $\left(\left(a^N - a\right) u', \left(\Pi^N u - u_N\right)'\right)$. The term is split into three parts:

$$\begin{aligned} &\left(\left(a^N - a\right) u', \left(\Pi^N u - u_N\right)'\right) \\ &= \int_0^{x_{i_0}} \left(a^N - a\right)(x) u'(x) \left(\Pi^N u - u_N\right)'(x)\, dx \end{aligned}$$

$$+\int_{x_{i_0}}^{1-x_{i_0}} \left(a^N - a\right)(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx$$
$$+\int_{1-x_{i_0}}^{1} \left(a^N - a\right)(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx \qquad (2.5.44)$$

Fix $i \in \{1,2,\ldots,N\}$. In what follows, we shall denote by Z_i any quantity of $O(h_i^2)$. Also set $x_{i-1/2} = (x_{i-1} + x_i)/2$. For $x \in (x_{i-1}, x_i)$, we have

$$\left(a^N - a\right)(x) = (x_{i-1/2} - x)a'(x) + Z_i$$

Set

$$r(x) = \int_{x_{i-1}}^{x} (x_{i-1/2} - t)\,dt, \text{ for } x \in [x_{i-1}, x_i].$$

Then

$$r(x_{i-1}) = r(x_i) = 0 \text{ and } |r(x)| \le Ch_i^2, \text{ for } x \in (x_{i-1}, x_i).$$

We are ready to bound the first term of the right hand side of (2.5.44). For $i \in \{1,\ldots,i_0\}$,

$$\int_{x_{i-1}}^{x_i} \left(a^N - a\right)(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx$$
$$= \int_{x_{i-1}}^{x_i} (x_{i-1/2} - x)a'(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx$$
$$+ \int_{x_{i-1}}^{x_i} Z_i\left(\Pi^N u - u_N\right)'(x)\,dx$$
$$= -\int_{x_{i-1}}^{x_i} r(x)\left(a'(x)u'(x)\right)'\left(\Pi^N u - u_N\right)'(x)\,dx$$
$$- \int_{x_{i-1}}^{x_i} r(x)a'(x)u'(x)\left(\Pi^N u - u_N\right)''(x)\,dx$$
$$+ \int_{x_{i-1}}^{x_i} Z_i\left(\Pi^N u - u_N\right)'(x)\,dx.$$

We have by (2.2.8) - (2.2.12) and (2.5.4)

$$\left|\int_{x_{i-1}}^{x_i} r(x)\left(a'(x)u'(x)\right)'\left(\Pi^N u - u_N\right)'(x)\,dx\right|$$

$$\leq Ch_i^2\varepsilon^{-1}\int_{x_{i-1}}^{x_i}\left|\left(\Pi^N u - u_N\right)'(x)\right|\,dx$$
$$\leq C\left(N^{-1}\ln N\right)^2\varepsilon\int_{x_{i-1}}^{x_i}\left|\left(\Pi^N u - u_N\right)'(x)\right|\,dx$$

and

$$\left|\int_{x_{i-1}}^{x_i} r(x)a'(x)u'(x)\left(\Pi^N u - u_N\right)''(x)\,dx\right|$$
$$\leq Ch_i^2\int_{x_{i-1}}^{x_i}\left|\left(\Pi^N u - u_N\right)''(x)\right|\,dx$$
$$\leq C\left(N^{-1}\ln N\right)^2\varepsilon^2\int_{x_{i-1}}^{x_i}\left|\left(\Pi^N u - u_N\right)''(x)\right|\,dx.$$

Therefore,

$$\left|\int_0^{x_{i_0}}\left(a^N - a\right)(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx\right|$$
$$\leq C\left(N^{-1}\ln N\right)^2\varepsilon\left(\|(\Pi^N u - u_N)'\|_{L_1[0,x_{i_0}]} + \varepsilon\|(\Pi^N u - u_N)''\|_{L_1[0,x_{i_0}]}\right)$$
$$\leq C\left(N^{-1}\ln N\right)^2\varepsilon|||\Pi^N u - u_N|||. \tag{2.5.45}$$

Similarly

$$\left|\int_{1-x_{i_0}}^{1}\left(a^N - a\right)(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx\right|$$
$$\leq C\left(N^{-1}\ln N\right)^2\varepsilon|||\Pi^N u - u_N|||. \tag{2.5.46}$$

In order to estimate the second term of (2.5.44) more carefully, we use the decomposition of (2.2.8). Then

$$\int_{x_{i_0}}^{1-x_{i_0}}\left(a^N - a\right)(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx$$
$$= \int_{x_{i_0}}^{1-x_{i_0}}\left(a^N - a\right)(x)G'(x)\left(\Pi^N u - u_N\right)'(x)\,dx$$
$$+ \int_{x_{i_0}}^{1-x_{i_0}}\left(a^N - a\right)(x)\left(E'(x) + F'(x)\right)\left(\Pi^N u - u_N\right)'(x)\,dx. \tag{2.5.47}$$

We bound these two terms separately. Let $i \in \{i_0+1, \ldots, N-i_0\}$.

Firstly, we have

$$|E'(x) + F'(x)| \le CN^{-3}, \quad \text{for } x \in [x_{i_0}, 1 - x_{i_0}],$$

by (2.2.10), (2.2.11) and (2.5.3). Hence

$$\begin{aligned}
&\left| \int_{x_{i_0}}^{1-x_{i_0}} \left(a^N - a\right)(x) \left(E'(x) + F'(x)\right) \left(\Pi^N u - u_N\right)'(x)\, dx \right| \\
&\le CN^{-4} \left\| \left(\Pi^N u - u_N\right)' \right\|_{L^1[x_{i_0}, 1-x_{i_0}]} \\
&\le CN^{-4} |||\Pi^N u - u_N|||.
\end{aligned} \tag{2.5.48}$$

We now bound the first term of (2.5.47). We introduce some notation. Set $q = \Pi^N u - u_N$. Since $V^N = V_0 \oplus V_1$ (recall $m = 2$), we can write $q = e_0 + e_1$ where $e_0 \in V_0$ and $e_1 \in V_1$. These e_0 and e_1 are uniquely determined by q. Then for each i,

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i} \left(a^N - a\right)(x) G'(x) \left(\Pi^N u - u_N\right)'(x)\, dx \\
&= \int_{x_{i-1}}^{x_i} (x_{i-1/2} - x) a'(x) G'(x) q'(x)\, dx + \int_{x_{i-1}}^{x_i} Z_i q'(x)\, dx \\
&= \int_{x_{i-1}}^{x_i} (x_{i-1/2} - x) a'(x_i) G'(x_i) q'(x)\, dx + \int_{x_{i-1}}^{x_i} Z_i q'(x)\, dx \\
&= a'(x_i) G'(x_i) \int_{x_{i-1}}^{x_i} (x_{i-1/2} - x) \left(e_0'(x) + e_1'(x)\right) dx + \int_{x_{i-1}}^{x_i} Z_i q'(x)\, dx.
\end{aligned} \tag{2.5.49}$$

Now for $x \in (x_{i-1}, x_i)$,

$$e_0(x) = e_0(x_{i-1}) \varphi_{i-1}^0(x) + e_0(x_i) \varphi_i^0(x),$$

with

$$\varphi_{i-1}^0(x) = \xi_{2,0}\left(\frac{x_i - x}{h_i}\right) \quad \text{and} \quad \varphi_i^0(x) = \xi_{2,0}\left(\frac{x - x_{i-1}}{h_i}\right).$$

Since $\frac{d\xi_{2,0}}{ds}(s) = 6s(1-s) = -\frac{d\xi_{2,0}}{ds}(1-s)$,

$$e_0'(x) = h_i^{-1}[e_0(x_i) - e_0(x_{i-1})]6s(1-s),$$

where $s = (x - x_{i-1})/h_i$. But $6s(1-s)$ is symmetric about $s = 1/2$, while $x_{i-1/2} - x$ is antisymmetric about $x = x_{i-1/2}$. Consequently in (2.5.49),

$$\int_{x_{i-1}}^{x_i} (x_{i-1/2} - x)e_0'(x)\,dx = 0. \tag{2.5.50}$$

Next,

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i} (x_{i-1/2} - x)e_1'(x)\,dx \\
&= \int_{x_{i-1}}^{x_i} (x_{i-1/2} - x) \sum_{j=i-1}^{i} e_1'(x_j)\,(\varphi_j^1)'(x)\,dx \\
&= \sum_{j=i-1}^{i} e_1'(x_j) \int_{x_{i-1}}^{x_i} (x_{i-1/2} - x)\,(\varphi_j^1)'(x)\,dx \\
&= \left[e_1'(x_i) - e_1'(x_{i-1})\right] \int_{s=0}^{1} h_i^2 \left(\frac{1}{2} - s\right) \left(\frac{d\xi_{2,1}}{ds}(s)\right) ds \\
&= K h_i^2 \left[e_1'(x_i) - e_1'(x_{i-1})\right],
\end{aligned}$$

where the fixed constant $K = \int_{s=0}^{1} \left(\frac{1}{2} - s\right) \left(\frac{d\xi_{2,1}}{ds}(s)\right) ds$.

Combining this with (2.5.49) and (2.5.50), we get

$$\begin{aligned}
&\int_{x_{i_0}}^{1-x_{i_0}} (a^N - a)(x)G'(x)q'(x)\,dx \\
&= \sum_{i=i_0+1}^{N-i_0} \left\{ a'(x_i)G'(x_i) \int_{x_{i-1}}^{x_i} (x_{1-1/2} - x)e_1'(x)\,dx + \int_{x_{i-1}}^{x_i} Z_i q'(x)\,dx \right\} \\
&= \sum_{i=i_0+1}^{N-i_0} \left\{ K h_i^2 a'(x_i)G'(x_i)\left[e_1'(x_i) - e_1'(x_{i-1})\right] + \int_{x_{i-1}}^{x_i} Z_i q'(x)\,dx \right\} \\
&= K \sum_{i=i_0+1}^{N-i_0-1} e_1'(x_i)\left[h_i^2 a'(x_i)G'(x_i) - h_{i+1}^2 a'(x_{i+1})G'(x_{i+1})\right]
\end{aligned}$$

$$+K\left[h_{N-i_0}^2 a'(x_{N-i_0})G'(x_{N-i_0})e_1'(x_{N-i_0}) - h_{i_0+1}^2 a'(x_{i_0+1})G'(x_{i_0+1})e_1'(x_{i_0})\right]$$
$$+\sum_{i=i_0+1}^{N-i_0}\int_{x_{i-1}}^{x_i} Z_i q'(x)\,dx.$$

But $|a'(x)G'(x)| \le C$ for $x \in [0,1]$, $h_i = O(N^{-1})$ for $i = i_0+1,\ldots,N-i_0$, and $|a'(x_i)G'(x_i) - a'(x_{i+1})G'(x_{i+1})| \le CN^{-1}$, for $i = i_0+1,\ldots,N-i_0-1$. Hence

$$\left|\int_{x_{i_0}}^{1-x_{i_0}} \left(a^N - a\right)(x)G'(x)q'(x)\,dx\right|$$
$$\le CN^{-3}\sum_{i=i_0+1}^{N-i_0-1} |e_1'(x_i)| + CN^{-2}\left[|e_1'(x_{N-i_0})| + |e_1'(x_{i_0})|\right]$$
$$+CN^{-2}\|q'\|_{L^1[0,1]}$$
$$\le CN^{-2}\left(\sum_{i=i_0+1}^{N-i_0-1}\bar{h}_i\right)^{1/2}\left(\sum_{i=i_0+1}^{N-i_0-1}\bar{h}_i e_1^2(x_i)\right)^{1/2}$$
$$+CN^{-2}N^{1/2}\|e_1'\|_{d_1} + CN^{-2}\|q'\|_{L^2[0,1]}$$
$$\le CN^{-3/2}\|e_1'\|_{d_1} + CN^{-2}|||q|||$$
$$\le CN^{-3/2}|||q|||.$$

Substituting this and (2.5.48) into (2.5.47) yields

$$\left|\int_{x_{i_0}}^{1-x_{i_0}} \left(a^N - a\right)(x)u'(x)\left(\Pi^N u - u_N\right)'(x)\,dx\right|$$
$$\le CN^{-3/2}|||\Pi^N u - u_N|||. \tag{2.5.51}$$

Combining (2.5.44) - (2.5.46) and (2.5.51), we get

$$\left|\left(\left(a^N - a\right)u', \left(\Pi^N u - u_N\right)'\right)\right| \le CN^{-3/2}|||\Pi^N u - u_N|||. \tag{2.5.52}$$

Consequently, from (2.5.42), (2.5.43) and (2.5.52), we obtain

$$A_\varepsilon^N\left(u - u_N, \Pi^N u - u_N\right) \le CN^{-3/2}|||\Pi^N u - u_N|||.$$

This completes the proof of (2.5.41). Recalling (2.5.19), we are done. □

Remark 2.5.5 *Numerical results of Section 2.6 show that the result of Theorem 2.5.4 is sharp.*

2.6 Numerical Results

In this section we present numerical results for the method (2.4.4) with Hermite basis functions (see subsection 2.5.4) applied to problem (2.1.2), with

$$a(x) = 1 + x(1-x), \qquad b(x) = c(x) \equiv 0,$$

where $f(x)$ is chosen so that the solution of (2.1.2) is

$$u(x) = \varepsilon \left\{ \frac{\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon)}{1 + \exp(-1/\varepsilon)} - 1 \right\} + \frac{1 - \exp(-1/\varepsilon)}{1 + \exp(-1/\varepsilon)} x(1-x) + x^2(1-x)^2.$$

This $u(x)$ exhibits typical boundary layer behaviour.

Set

$$u_N(x) = \sum_{i=1}^{N-1} \left[u_N(x_i)\varphi_i^1(x) + u_N'(x_i)\varphi_i^2(x) \right].$$

The method (2.4.4), with $m = 2$, may be written in the form

$$\begin{aligned} & r_i^- u_N(x_{i-1}) + r_i^c u_N(x_i) + r_i^+ u_N(x_{i+1}) \\ & \qquad + t_i^- u_N'(x_{i-1}) + t_i^c u_N'(x_i) + t_i^+ u_N'(x_{i+1}) = F_{1i}, \\ & p_i^- u_N(x_{i-1}) + p_i^c u_N(x_i) + p_i^+ u_N(x_{i+1}) \\ & \qquad + q_i^- u_N'(x_{i-1}) + q_i^c u_N'(x_i) + q_i^+ u_N'(x_{i+1}) = F_{2i}, \end{aligned}$$

for $i = 1, \ldots, N-1$. The coefficients r, t, p, q, F_1 and F_2 are some linear combinations of ε^2 and point evaluations of a, b, c, and f, since only polynomials are used as our basis functions. The coefficient matrix of the scheme can be easily permuted to yield a heptadiagonal matrix. The resulting system of $2(N-1)$ equations is solved by Gaussian elimination.

We compute the errors in the following two ways:

(i). The error between the exact solution $u(x)$ and the computed solution $u_N(x)$ in the discrete maximum norm,

$$E_\varepsilon^N = \max_{0\le i\le N} |u(x_i) - u_N(x_i)| .$$

(ii). The error between the interpolant $u_I(x)$ and the computed solution $u_N(x)$ in the discrete H^1-norm,

$$E_\varepsilon^N = |u_I - u_N|_{d_1} .$$

We calculate the convergence rate tables as follows; see Farrell and Hegarty [14]:

(i). Except for the last row, the table entries are given by the classical convergence rate,

$$R_\varepsilon^N = \left(\ln E_\varepsilon^{2N} - \ln E_\varepsilon^N\right) / \ln 2.$$

(ii). The last row of each table is the uniform convergence rate,

$$R^N = \left(\ln E^{2N} - \ln E^N\right) / \ln 2,$$

where $E^N = \max_\varepsilon E_\varepsilon^N$.

ε	N=8	16	32	64	128
2.50000e-01	2.109e-03	5.367e-04	1.348e-04	3.374e-05	8.435e-06
6.25000e-02	6.626e-03	1.709e-03	4.313e-04	1.081e-04	2.704e-05
1.56250e-02	1.872e-02	3.738e-03	7.385e-04	1.489e-04	3.317e-05
3.90625e-03	3.537e-02	8.494e-03	2.020e-03	4.796e-04	1.137e-04
9.76562e-04	4.058e-02	1.013e-02	2.509e-03	6.208e-04	1.535e-04
2.44141e-04	4.195e-02	1.057e-02	2.642e-03	6.597e-04	1.646e-04
6.10352e-05	4.229e-02	1.068e-02	2.675e-03	6.696e-04	1.675e-04
1.52588e-05	4.238e-02	1.071e-02	2.684e-03	6.721e-04	1.682e-04
3.81470e-06	4.240e-02	1.071e-02	2.686e-03	6.727e-04	1.684e-04
9.53674e-07	4.241e-02	1.071e-02	2.686e-03	6.729e-04	1.684e-04

Table 2.6.1: Errors in Maximum Norm $\|\cdot\|_\infty$ for $\bar{A}$ Scheme

ε	N=8	16	32	64
2.50000e-01	1.97	1.99	2.00	2.00
6.25000e-02	1.96	1.99	2.00	2.00
1.56250e-02	2.32	2.34	2.31	2.17
3.90625e-03	2.06	2.07	2.07	2.08
9.76562e-04	2.00	2.01	2.01	2.02
2.44141e-04	1.99	2.00	2.00	2.00
6.10352e-05	1.99	2.00	2.00	2.00
1.52588e-05	1.99	2.00	2.00	2.00
3.81470e-06	1.98	2.00	2.00	2.00
9.53674e-07	1.98	2.00	2.00	2.00
R^N	1.98	2.00	2.00	2.00

Table 2.6.2: Convergence Rates in Maximum Norm $\|\cdot\|_\infty$ for $\bar{A}$ Scheme

Tables 2.6.1 – 2.6.4 present the errors and convergence rates for the $\bar{A}$ scheme; this is the method (2.4.4) with the piecewise constant approximations

$$p^N(x) = \frac{p(x_{i-1}) + p(x_i)}{2}, \quad \text{for } x \in (x_{i-1}, x_i) \text{ and } i = 1, \ldots, N,$$

where p can be a, b, c or f.

ε	N=8	16	32	64	128
2.50000e-01	6.479e-03	1.673e-03	4.215e-04	1.056e-04	2.640e-05
6.25000e-02	2.099e-02	5.471e-03	1.386e-03	3.476e-04	8.697e-05
1.56250e-02	5.678e-02	1.174e-02	2.332e-03	4.800e-04	1.112e-04
3.90625e-03	1.055e-01	2.860e-02	7.432e-03	1.793e-03	4.185e-04
9.76562e-04	1.217e-01	3.629e-02	1.107e-02	3.304e-03	9.132e-04
2.44141e-04	1.260e-01	3.853e-02	1.233e-02	4.022e-03	1.302e-03
6.10352e-05	1.271e-01	3.911e-02	1.267e-02	4.232e-03	1.433e-03
1.52588e-05	1.274e-01	3.925e-02	1.276e-02	4.286e-03	1.468e-03
3.81470e-06	1.275e-01	3.929e-02	1.278e-02	4.300e-03	1.477e-03
9.53674e-07	1.275e-01	3.930e-02	1.278e-02	4.303e-03	1.480e-03

Table 2.6.3 : Errors in Discrete H^1-Norm for $\bar{A}$ Scheme

ε	N=8	16	32	64
2.50000e-01	1.95	1.99	2.00	2.00
6.25000e-02	1.94	1.98	2.00	2.00
1.56250e-02	2.27	2.33	2.28	2.11
3.90625e-03	1.88	1.94	2.05	2.10
9.76562e-04	1.75	1.71	1.74	1.86
2.44141e-04	1.71	1.64	1.62	1.63
6.10352e-05	1.70	1.63	1.58	1.56
1.52588e-05	1.70	1.62	1.57	1.55
3.81470e-06	1.70	1.62	1.57	1.54
9.53674e-07	1.70	1.62	1.57	1.54
R^N	1.70	1.62	1.57	1.54

Table 2.6.4: Convergence Rates in Discrete H^1-Norm for $\bar{A}$ Scheme

We denote by $\hat{A}$ the method (2.4.4) with the piecewise linear approximations

$$p^N(x) = \frac{x_i - x}{h_i} p(x_{i-1}) + \frac{x - x_{i-1}}{h_i} p(x_i),$$

for $x \in (x_{i-1}, x_i)$ and $i = 1, \ldots, N$, where p can be a, b, c and f. Tables 2.6.5 – 2.6.8 give the errors and convergence rates for the $\hat{A}$ scheme.

ε	N=8	16	32	64	128
2.50000e-01	1.065e-03	2.691e-04	6.745e-05	1.687e-05	4.217e-06
6.25000e-02	3.627e-03	8.870e-04	2.180e-04	5.419e-05	1.353e-05
1.56250e-02	1.097e-02	2.321e-03	4.571e-04	8.543e-05	1.699e-05
3.90625e-03	1.648e-02	4.081e-03	9.883e-04	2.380e-04	5.732e-05
9.76562e-04	1.783e-02	4.544e-03	1.134e-03	2.814e-04	6.982e-05
2.44141e-04	1.817e-02	4.662e-03	1.171e-03	2.926e-04	7.305e-05
6.10352e-05	1.826e-02	4.692e-03	1.181e-03	2.955e-04	7.386e-05
1.52588e-05	1.828e-02	4.700e-03	1.183e-03	2.962e-04	7.407e-05
3.81470e-06	1.829e-02	4.701e-03	1.183e-03	2.964e-04	7.412e-05
9.53674e-07	1.829e-02	4.702e-03	1.184e-03	2.964e-04	7.413e-05

Table 2.6.5: Errors in Maximum Norm $\|\cdot\|_\infty$ for $\hat{A}$ Scheme

ε	N=8	16	32	64
2.50000e-01	1.98	2.00	2.00	2.00
6.25000e-02	2.03	2.02	2.01	2.00
1.56250e-02	2.24	2.34	2.42	2.33
3.90625e-03	2.01	2.05	2.05	2.05
9.76562e-04	1.97	2.00	2.01	2.01
2.44141e-04	1.96	1.99	2.00	2.00
6.10352e-05	1.96	1.99	2.00	2.00
1.52588e-05	1.96	1.99	2.00	2.00
3.81470e-06	1.96	1.99	2.00	2.00
9.53674e-07	1.96	1.99	2.00	2.00
R^N	1.96	1.99	2.00	2.00

Table 2.6.6: Convergence Rates in Maximum Norm $\|\cdot\|_\infty$ for $\hat{A}$ Scheme

ε	N=8	16	32	64	128
2.50000e-01	3.294e-03	8.398e-04	2.109e-04	5.279e-05	1.320e-05
6.25000e-02	1.232e-02	2.860e-03	7.008e-04	1.743e-04	4.352e-05
1.56250e-02	4.024e-02	8.283e-03	1.536e-03	2.786e-04	5.655e-05
3.90625e-03	5.753e-02	1.401e-02	3.332e-03	7.851e-04	1.859e-04
9.76562e-04	6.193e-02	1.560e-02	3.875e-03	9.581e-04	2.354e-04
2.44141e-04	6.302e-02	1.600e-02	4.011e-03	1.002e-03	2.498e-04
6.10352e-05	6.330e-02	1.610e-02	4.045e-03	1.012e-03	2.530e-04
1.52588e-05	6.336e-02	1.613e-02	4.053e-03	1.015e-03	2.537e-04
3.81470e-06	6.338e-02	1.613e-02	4.055e-03	1.015e-03	2.539e-04
9.53674e-07	6.339e-02	1.613e-02	4.056e-03	1.015e-03	2.540e-04

Table 2.6.7: Errors in Discrete H^1-Norm for $\hat{A}$ Scheme

ε	N=8	16	32	64
2.50000e-01	1.97	1.99	2.00	2.00
6.25000e-02	2.11	2.03	2.01	2.00
1.56250e-02	2.28	2.43	2.46	2.30
3.90625e-03	2.04	2.07	2.09	2.08
9.76562e-04	1.99	2.01	2.02	2.03
2.44141e-04	1.98	2.00	2.00	2.00
6.10352e-05	1.97	1.99	2.00	2.00
1.52588e-05	1.97	1.99	2.00	2.00
3.81470e-06	1.97	1.99	2.00	2.00
9.53674e-07	1.97	1.99	2.00	2.00
R^N	1.97	1.99	2.00	2.00

Table 2.6.8: Convergence Rates in Discrete H^1-Norm for $\hat{A}$ Scheme

The predicted uniform accuracy is clearly observed. We note that, when errors are measured in $\|\cdot\|_\infty$, the $\bar{A}$ scheme is almost as accurate as the more complicated $\hat{A}$ scheme. We have observed a similar phenomenon with a higher order approximation applied to problems with other data; both piecewise quadratic and piecewise cubic approximations to a, b, c and f yield fourth order accuracy in $\|\cdot\|_\infty$, uniformly in ε. Thus it seems that in cases where coefficients are approximated by piecewise polynomials of even degree and $l < m + 1$, the order of convergence in $\|\cdot\|_\infty$ is one more than that predicted by Theorem 2.5.2.

If instead of $\|\cdot\|_\infty$ one consider the errors measured in $|\cdot|_{d_1}$, then Tables 2.6.4 and 2.6.8 show that the $\hat{A}$ scheme is superior. Nevertheless, the $\bar{A}$ scheme is $O(N^{-3/2})$ convergent, better than the $O(N^{-1})$ predicted by Theorem 2.5.1. Theorem 2.5.4 furnishes a proof of this $O(N^{-3/2})$ result.

Chapter 3

High Order Convection–Diffusion Problems

3.1 Introduction

This chapter is concerned with the numerical approximation by finite element methods of certain singularly perturbed high order two–point boundary value problems with one boundary layer. A model problem of this type is the second order convection–diffusion problem

$$-\varepsilon w'' + a(x)w' + b(x)w = f(x), \quad \text{for } x \in (0,1), \tag{3.1.1a}$$

$$w(0) = w(1) = 0, \tag{3.1.1b}$$

with $a(x) > \alpha > 0$, where ε is a small positive parameter. The problem (3.1.1) has been extensively examined. Ways of using equidistant meshes and locally quasi-equidistant meshes and of generating exponentially fitted schemes which are convergent, uniformly in ε, with respect to various norms are considered for example in Berger et al. [5], El–Mistikawy and Werle [11], Il'in [23], Kellogg and Tsan [24], Stynes [40], and Stynes and O'Riordan [41, 43], while uniformly convergent classical difference schemes on special graded meshes may be found in Gartland [17] and Vulanović [46].

However, there are still some unsolved problems. For example, in Gartland [18] there is numerical evidence that the standard central difference scheme is convergent, uniformly in ε, in the discrete maximum norm, when applied on a special mesh. This is true even though the scheme does not satisfy a discrete maximum principle and admits oscillatory solutions. Gartland [17] has proved that a hybrid scheme, where upwinding is used only in a narrow "transition region" and central differencing is used elsewhere, is uniformly convergent on a special exponentially graded mesh. We shall show that in fact exponential grading of the mesh is unnecessary. Instead, we construct a simpler piecewise equidistant mesh, on which we use polynomial Galerkin methods. Then we analyse their convergence properties in energy and W^k_∞ norms.

In fact, in this chapter, we consider the following more general problem:

$$L_\varepsilon u \equiv (-1)^m \varepsilon u^{(2m)} + (-1)^{m-1}\left(a_{2m-1}(x)u^{(m)}\right)^{(m-1)} + L_1 u$$

$$= f(x), \quad \text{for } x \in (0,1), \tag{3.1.2a}$$

$$u^{(j)}(0) = u^{(j)}(1) = 0, \quad \text{for } j = 0,\dots,m-1, \tag{3.1.2b}$$

where $m \geq 1$ is an integer and $\varepsilon \in (0,1]$ is a perturbation parameter, and

$$L_1 u \equiv \sum_{k=2}^{m} (-1)^{m-k}\left(a_{2(m-k)+1}(x)u^{(m-k+1)}(x)\right)^{(m-k)}$$
$$+ \sum_{k=1}^{m} (-1)^{m-k}\left(a_{2(m-k)}(x)u^{(m-k)}(x)\right)^{(m-k)}.$$

The functions a_r (for $r = 0,1,\dots,2m-1$) and f are assumed to be sufficiently smooth with

$$a_{2m-1}(x) > \alpha > 0 \quad \text{on } [0,1], \tag{3.1.2c}$$

and

$$a_{2(m-k)}(x) - \frac{1}{2}a'_{2(m-k)+1}(x) > \alpha_{m-k}, \quad \text{for } k = 1, \ldots, m, \tag{3.1.2d}$$

for all $x \in [0,1]$ and some constants α and α_{m-k} $(k = 1, \ldots, m)$ satisfying

$$\sum_{i=1}^{k} \alpha_{m-i} > 0, \quad \text{for } k = 1, \ldots, m. \tag{3.1.2e}$$

Under the conditions (3.1.2d) and (3.1.2e), problem (3.1.2) is well posed for $\varepsilon > 0$ and in fact possesses a coercive bilinear form associated with (3.1.2a). The condition (3.1.2c) prohibits the development of turning points or interior layers.

We refer to the problem (3.1.2) as being of convection–diffusion type since it is a generalization of (3.1.1). In Chapter 2 we considered the situation when $a_{2m-1} \equiv 0$; we describe such problems as being of reaction–diffusion type, again using the terminology associated with the second order case. See the discussion of problem (3.1.5) below.

We take the above form of the operator L_ε for convenience. Any linear operator $\tilde{L}_\varepsilon$ of the form

$$\tilde{L}_\varepsilon \equiv (-1)^m \varepsilon u^{(2m)} + \sum_{k=0}^{2m-1} b_k u^{(k)}$$

can be rewritten in the form of L_ε, with each a_r (for $r = 0, \ldots, 2m-1$) equal to a linear combination of b_r, $b_{r+1}, \ldots, b_{2m-1}$ and certain of their derivatives. We consider the homogeneous boundary conditions (3.1.2b), since non-homogeneous conditions $u^{(j)}(0) = A_j$ and $u^{(j)}(1) = B_j$, for $j = 0, 1, \ldots, m-1$, can be homogenized by the transformation $\tilde{u}(x) = u(x) - \sum_{j=0}^{m-1}\{(-1)^j A_j \xi_{m,j}(1-x) + B_j \xi_{m,j}(x)\}$, where the $\xi_{m,j}(\cdot)$ are defined by (3.3.8) below.

In contrast to the second order problem (3.1.1), there are only a few results on higher order problems with one boundary layer; see Gartland [17] and Roos [34]. Gartland [17] studied compact finite difference schemes for a problem of the form

$$\varepsilon v^{(n)} + \sum_{k=0}^{n-1} a_k(x) v^{(k)} = f(x), \quad \text{for } x \in (0,1), \tag{3.1.3}$$

with $a_{n-1}(x) \neq 0$ and appropriate boundary conditions which fulfill certain conditions due to Niederdrenk and Yserentant [27]. His schemes are higher order uniformly convergent on a special graded mesh in the weighted Sobolev norm

$$\|v\|_{N-Y} \equiv \varepsilon \|v^{(n-1)}\|_\infty + \sum_{j=0}^{n-2} \|v^{(j)}\|_\infty, \tag{3.1.4}$$

where $\|\cdot\|_\infty \equiv \|\cdot\|_{L^\infty[0,1]}$. The Niederdrenk and Yserentant conditions guarantee that the differential operator of (3.1.3) is uniformly stable (in the sense of Gartland [17]) with respect to the norm (3.1.4). That is, in the case of homogeneous boundary conditions,

$$\|v\|_{N-Y} \leq C \|f\|_{L^1[0,1]},$$

where the constant C is independent of ε. Now with $n = 2m$, one does not in general have

$$\|u\|_{N-Y} \leq C \|f\|_{L^1[0,1]},$$

where $u(\cdot)$ denotes the solution of (3.1.2). See, e.g., the example in Gartland [17], p.655. That is, (3.1.2) is less stable (and consequently more difficult to solve numerically) than (3.1.3).

Roos [34] applied an iterative approach to problem (3.1.3), assuming the Niederdrenk and Yserentant conditions. This approach is similar to the defect correction method and allows one to generate higher order schemes in a systematic way. However, the method is quite complicated since it is based on obtaining exact solutions of boundary value problems with piecewise constant coefficients.

In Chapter 2 we considered the problem of reaction–diffusion type

$$(-1)^m \varepsilon v^{(2m)} + (-1)^{m-1}\left(b_{2(m-1)}(x)v^{(m-1)}\right)^{(m-1)}$$
$$+\sum_{k=2}^{m}(-1)^{m-k}\left(b_{2(m-k)+1}(x)v^{(m-k+1)} + b_{2(m-k)}(x)v^{(m-k)}\right)^{(m-k)}$$

$$= g(x), \quad \text{for } x \in (0,1), \tag{3.1.5a}$$

$$v^{(j)}(0) = v^{(j)}(1) = 0, \quad \text{for } j = 0,\ldots,m-1, \tag{3.1.5b}$$

with $b_{2(m-1)}(x) > \beta > 0$ and some conditions analogous to (3.1.2d, e) above on b_r (for $r = 0,1,\ldots,2(m-1)$). Problem (3.1.5) is a generalization of the well known second order reaction-diffusion problem. The mth order derivative of its solution has a boundary layer of width $O(\varepsilon^{1/2})$ at each endpoint of [0,1]. Thus (cf. (3.2.6) below) the solution of (3.1.5) is better behaved than that of (3.1.2). In Chapter 2, some finite element methods for problem (3.1.5) were constructed and proved to be convergent, uniformly in ε, in various norms.

Employing a Sturm transformation

$$v(x) = u(x)\varepsilon^{m-1}\exp\left(-\frac{1}{2m\varepsilon}\int_0^x a(s)ds\right), \tag{3.1.6}$$

one may eliminate the $u^{(2m-1)}$ term of (3.1.2a) and reduce problem (3.1.2) to problem (3.1.5) with

$$g(x) = f(x)\varepsilon^{m-1}\exp\left(-\frac{1}{2m\varepsilon}\int_0^x a(s)ds\right). \tag{3.1.7}$$

It is in principle possible to apply the methods in Chapter 2 to solve the problem (3.1.5) with $g(x)$ defined by (3.1.7), then to transform back to (3.1.2). However, from (3.1.6) we can see that, since $u(x) = O(1)$ (cf. (3.2.6) below), $v(x)$ is exponentially small and consequently it will be difficult to accurately compute an approximation to it then transform back to $u(x)$.

In this chapter, we generate and analyse Galerkin finite element methods for problem (3.1.2). We consider only "uniformly convergent" methods; these are methods whose solutions converge to u, uniformly in ε, in some norm. Since the bilinear form associated with (3.1.2a) is not satisfactorily bounded in terms of an associated weighted energy norm (see (3.2.2) below), a classical finite element analysis does not yield uniform convergence results. We therefore use the technique of Stynes and O'Riordan [43], which turns out to be effective for the problem (3.1.2). We obtain convergence results in energy and W^k_∞ norms for $k = 0, \ldots, m-2$. These results show that the accuracy of our method depends both on m and on how well we approximate the coefficients in (3.1.2a). We present numerical experiments to support our claims.

In classical finite element analyses, one expects that by using an Aubin–Nitsche argument one can show enhanced convergence of the computed solution in norms weaker than the energy norm. This is not the case here when $m \geq 2$, as our numerical results show. When $m = 1$ (i.e., the second order convection–diffusion problem (3.1.1)), the situation is different; we shall prove that the order of uniform convergence in the L^2 norm is at least 1/2 higher than that in the energy norm associated with (3.1.1a).

In contrast to the results of Chapter 2 for (3.1.5), we find here that the value of

m poses a more severe restriction on the accuracy of the method; see Remark 3.5.2 for details.

The structure of the chapter is as follows: Section 3.2 contains existence and uniqueness results and an asymptotic decomposition for the solution of (3.1.2). We briefly discuss in Section 3.3 the necessity of using a special scheme to get high order uniform convergence results for the given problem and generate finite element methods using piecewise polynomials as our basis functions on an arbitrary mesh. Section 3.4 gives interpolation error estimates on a piecewise equidistant mesh. This type of mesh, which was recently introduced by Shishkin [37], is much simpler than those of Vulanović [46] and Gartland [17]. In Section 3.5, we prove that (assuming a sufficiently accurate quadrature rule is used) the resulting polynomial methods on the Shishkin mesh are uniformly convergent of order $(N^{-1} \ln N)^m$ in a weighted energy norm associated with (3.1.2a). This implies uniform convergence of the solution and its derivatives of up to the $(m-2)$th order in the maximum norm. In the final section, some numerical results are given for a second order problem and fourth order problems to confirm the theoretical estimates.

3.2 The Continuous Problem

In this section we discuss those properties of (3.1.2) and of its solution u which we shall need later for the analysis of our finite element methods.

Definitions and notation: Let $(\cdot,\cdot)$ denote the usual $L^2(0,1)$ inner product. Let $H^0 = L^2, H^k$ (for $k = 1,\ldots,m$) denote the usual Sobolev spaces on $[0,1]$. Define $\|\cdot\|_k$ to be the norm on H^k and $|\cdot|_k$ to be the usual associated seminorm for $k = 0,\ldots,m$. Let $\|\cdot\|_\infty$ denote the essential supremum norm on $L^\infty[0,1]$. For

$k = 0, 1, \ldots, m-1$, the maximum norm on $C^k[0,1]$ is denoted by $\|\cdot\|_{k,\infty}$, i.e., $\|v\|_{k,\infty} = \sum_{j=0}^{k} \|v^{(j)}\|_\infty$, for all $v \in C^k[0,1]$. Set

$$H_0^m = \{v \in H^m : v^{(j)}(0) = v^{(j)}(1) = 0, \quad \text{for } j = 0, \ldots, m-1\}.$$

Then our bilinear form $A_\epsilon(\cdot,\cdot)$ is defined to be

$$A_\epsilon(v,w) = \left(\varepsilon v^{(m)}, w^{(m)}\right) + \left(a_{2m-1}v^{(m)}, w^{(m-1)}\right) + A_1(v,w), \tag{3.2.1}$$

where

$$A_1(v,w) = \sum_{k=2}^{m} \left(a_{2(m-k)+1}v^{(m-k+1)}, w^{(m-k)}\right) + \sum_{k=1}^{m} \left(a_{2(m-k)}v^{(m-k)}, w^{(m-k)}\right),$$

for all $v, w \in H_0^m$. Our weighted energy norm is given by

$$|||v||| = \{\varepsilon |v|_m^2 + \|v\|_{m-1}^2\}^{1/2}, \quad \forall v \in H_0^m.$$

Our first lemma shows that the bilinear form $A_\epsilon(\cdot,\cdot)$ is uniformly coercive over $H_0^m \times H_0^m$, but is not satisfactorily uniformly bounded in terms of the energy norm.

Lemma 3.2.1 *Assume that (3.1.2d) and (3.1.2e) hold. Then there exist positive constants C_1 and C_2 such that for all $v, w \in H_0^m$,*

$$|A_\epsilon(v,w)| \leq C_1\varepsilon^{-1/2}|||v||| \cdot |||w||| \tag{3.2.2}$$

and

$$C_2|||v|||^2 \leq A_\epsilon(v,v). \tag{3.2.3}$$

Proof. It is easy to see that

$$|A_\epsilon(v,w)| \leq \begin{cases} C_1|||v||| \cdot \|w\|_m \\ C_1\|v\|_m \cdot |||w||| \end{cases} \tag{3.2.4}$$

using the Cauchy-Schwarz inequality. Then (3.2.2) follows immediately, since $0 < \varepsilon \le 1$. For (3.2.3) we have, for each $v \in H_0^m$,

$$
\begin{aligned}
A_\varepsilon(v,v) &= \varepsilon\left(v^{(m)}, v^{(m)}\right) \\
&\quad + \sum_{k=1}^{m}\left(\left(a_{2(m-k)} - \frac{1}{2}a'_{2(m-k)+1}\right) v^{(m-k)}, v^{(m-k)}\right) \\
&\ge \varepsilon |v|_m^2 + \sum_{k=1}^{m} \alpha_{m-k} |v|_{m-k}^2,
\end{aligned}
$$

by (3.1.2d). Using induction on r, one can readily prove (see Chapter 2) that for $r = 1, \ldots, m$,

$$
\sum_{k=1}^{r} \alpha_{r-k} |v|_{r-k}^2 \ge \min_{1\le j\le r} \sum_{k=1}^{j} \alpha_{r-k} |v|_{r-1}^2, \quad \forall v \in H_0^m.
$$

Hence

$$
\begin{aligned}
A_\varepsilon(v,v) &\ge \varepsilon |v|_m^2 + \min_{1\le j\le m} \left\{\sum_{k=1}^{j} \alpha_{m-k}\right\} |v|_{m-1}^2 \\
&\ge \varepsilon |v|_m^2 + m^{-1} \min_{1\le j\le m} \left\{\sum_{k=1}^{j} \alpha_{m-k}\right\} \|v\|_{m-1}^2,
\end{aligned}
$$

which by (3.1.2e) is the desired result with

$$
C_2 = \min\left\{1, m^{-1} \min_{1\le j\le m} \left\{\sum_{k=1}^{j} \alpha_{m-k}\right\}\right\}. \square
$$

We can now define our weak formulation of (3.1.2): find $u \in H_0^m$ such that

$$
A_\varepsilon(u,v) = (f,v), \quad \forall v \in H_0^m. \tag{3.2.5}
$$

For each fixed $\varepsilon \in (0,1]$, Lemma 3.2.1 shows that $A_\varepsilon(\cdot,\cdot)$ is bounded and coercive over $H_0^m \times H_0^m$. Furthermore, the mapping $v \longmapsto (f,v)$ is a bounded linear functional on H_0^m with respect to the norm $\|\cdot\|_m$. Thus the Lax–Milgram Lemma tells us that

(3.2.5) has a unique solution u in H_0^m. Throughout this chapter, u will denote this solution. This weak solution is also a classical solution to (3.1.2), if all the data are smooth.

Using the characterization of the null space of L_ε given in Theorem 3.1.4 of Gartland [17], we can prove a representation result for the solution of problem (3.1.2). This result will be used in the error analysis for the methods derived in Section 3.3.

Lemma 3.2.2 *The solution u of (3.1.2) admits the representation*

$$u(x) = G(x) + \varepsilon^{m-1} G_1(x) \exp\left(-\frac{1}{\varepsilon}\int_x^1 a_{2m-1}(s)\,ds\right), \tag{3.2.6}$$

where G and G_1 and their derivatives up to any prescribed finite order can be bounded independently of ε.

Proof. This is essentially the same result as Theorem 3.1.4 of Gartland [17]. It is proved using Gartland's argument, the only difference being that since we have $|||u||| \le C$, which follows from (3.2.3) and (3.2.5), the layer function component of u must be scaled as in (3.2.6). □

In what follows, we shall denote by $E(x)$ the boundary layer term of (3.2.6). Thus

$$u(x) = G(x) + E(x), \tag{3.2.7}$$

where for $x \in [0,1]$ and $j = 0, 1, \ldots$, we have

$$\left|G^{(j)}(x)\right| \le C, \tag{3.2.8}$$

$$\left|E^{(j)}(x)\right| \le C\varepsilon^{m-1-j} \exp\left(-\alpha(1-x)/\varepsilon\right). \tag{3.2.9}$$

Hence, for $x \in [0,1]$,

$$\left|u^{(j)}(x)\right| \leq C, \quad \text{for } j = 0, \ldots, m-1. \tag{3.2.10}$$

3.3 Galerkin Finite Element Methods

It is well known that classical difference schemes on an equidistant mesh for the second order problem (3.1.1) do not converge uniformly with respect to the discrete $L^\infty(0,1)$ norm. This result can be extended to the more general problem (3.1.2). One may show, by arguments similar to Lemma 2.3.1, that if a typical difference scheme is uniformly convergent of sufficiently high order in the maximum norm, then certain coefficients of that scheme must have an exponential nature.

An exponentially fitted scheme can be constructed in the following way on an equidistant mesh. Consider a Petrov–Galerkin finite element method with a bilinear form based on approximating the coefficients in (3.2.1) by piecewise linear functions. The basis functions for the trial space are simplified $\bar{L}$–splines defined by

$$\varepsilon\varphi_k^{(2m)} - \tilde{a}_{2m-1}\varphi_k^{(2m-1)} = 0 \tag{3.3.1}$$

on the interior of each mesh subinterval, with some boundary conditions, where $\tilde{a}_{2m-1}$ is a piecewise constant approximation of $a_{2m-1}(x)$. The test functions are simplified $\bar{L}^*$–splines satisfying the adjoint equation of (3.3.1). One can expect to prove that this scheme is uniformly convergent in a weighted energy norm, by employing an analysis similar to that of Stynes and O'Riordan [43]. However, the resulting scheme is quite complicated because of the exponential fitting factors.

We therefore consider classical Galerkin finite element methods on special meshes for problem (3.1.2). We first work with an arbitrary mesh

$$X^N: \quad 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1,$$

where $h_i = x_i - x_{i-1}$, for $i = 1, \ldots, N$, and $H = \max_i h_i$.

Since the solution of the weak formulation (3.2.2) lies in H_0^m, we define our piecewise polynomial approximation space by

$$S^N = \{v(x) \in H_0^m : v|_{I_i} \in P_R(I_i) \text{ for } i = 1, \ldots, N\},$$

where $P_R(I_i)$ is the set of polynomials of degree at most R on I_i and R is some positive integer. In order to guarantee that $S^N \subset C^{m-1} \subset H^m$, we assume that $R \geq 2m - 1$.

To generate our computed solution, we define the modified bilinear form $A_\varepsilon^N(\cdot, \cdot)$ on $S^N \times S^N$ to be

$$A_\varepsilon^N(v, w) \equiv \varepsilon(v^{(m)}, w^{(m)}) + \left(a_{2m-1}^N v^{(m)}, w^{(m-1)}\right) + A_1^N(v, w),$$

where

$$A_1^N(v, w) \equiv \sum_{k=2}^{m} \left(a_{2(m-k)+1}^N v^{(m-k+1)}, w^{(m-k)}\right) + \sum_{k=1}^{m} \left(a_{2(m-k)}^N v^{(m-k)}, w^{(m-k)}\right)$$

and a_r^N denotes a piecewise polynomial approximation of a_r for $r = 0, 1, \ldots, 2m - 1$ respectively. For each r, these approximations are assumed to satisfy

$$\left|\left(a_r^N - a_r\right)(x)\right| \leq C h_i^l, \quad \text{for } x \in (x_{i-1}, x_i) \text{ and } i = 1, \ldots, N, \tag{3.3.2}$$

where l is a fixed positive integer. We also require that

$$a_{2m-1}^N \in C[0,1] \bigcap C^1 \left(\bigcup_{i=1}^{N} (x_{i-1}, x_i)\right) \tag{3.3.3}$$

and

$$\left|\left(a_{2m-1}^N - a_{2m-1}\right)'(x)\right| \le C h_i, \tag{3.3.4}$$

for $x \in (x_{i-1}, x_i)$ and $i = 1, \ldots, N$.

The following lemma proves uniform coercivity of the discrete bilinear form $A_\varepsilon^N(\cdot,\cdot)$. This property will be used in the discretization error analyses of Section 3.5.

Lemma 3.3.1 *There exists a positive constant h_0 (independent of ε) such that for $H \le h_0$, we have*

$$C_1 |||v|||^2 \le A_\varepsilon^N(v,v), \quad \forall v \in H_0^m.$$

Proof. Let $v \in H_0^m$ be arbitrary but fixed. Write

$$A_\varepsilon^N(v,v) = A_\varepsilon(v,v) + \left(A_\varepsilon^N - A_\varepsilon\right)(v,v). \tag{3.3.5}$$

For the second term of (3.3.5),

$$\left|\left(A_\varepsilon^N - A_\varepsilon\right)(v,v)\right| = \left|\left(\left(a_{2m-1}^N - a_{2m-1}\right) v^{(m)}, v^{(m-1)}\right) + \left(A_1^N - A_1\right)(v,v)\right|.$$

We have, integrating by parts,

$$\begin{aligned}
&\left|\left(\left(a_{2m-1}^N - a_{2m-1}\right) v^{(m)}, v^{(m-1)}\right)\right| \\
&= \left|\left(-\frac{1}{2}\left(a_{2m-1}^N - a_{2m-1}\right)' v^{(m-1)}, v^{(m-1)}\right)\right| \\
&\le C H |v|_{m-1}^2.
\end{aligned}$$

Also

$$\left|\left(A_1^N - A_1\right)(v,v)\right| \le C H^l \|v\|_{m-1}^2.$$

Hence

$$\left|\left(A_\varepsilon^N - A_\varepsilon\right)(v,v)\right| \le C H |||v|||^2.$$

Combining this with (3.3.5) and Lemma (3.2.1) completes the proof. □

We may then pose the Galerkin discretization of problem (3.1.2): find $u_N \in S^N$ such that

$$A_\epsilon^N(u_N, v) = (f^N, v), \quad \forall v \in S^N, \tag{3.3.6}$$

where f^N approximates f analogously to a_r^N approximating a_r in (3.3.2).

It follows from Lemma 3.3.1 that u_N is well defined.

One choice of the approximation space S^N is the following Hermite basis function space:

$$V^N = V_0 \oplus V_1 \oplus \cdots \oplus V_{m-1},$$

where V_r is the linear span of $\{\varphi_i^r : i = 1, \ldots, N-1\}$, for $r = 0, 1, \ldots, m-1$. The basis functions $\{\varphi_i^r\}_{i=1}^{N-1}$, for $r = 0, \ldots, m-1$, are piecewise Hermite interpolation polynomials of degree $2m-1$. That is,

$$\varphi_i^r(x) = \begin{cases} h_i^r \xi_{m,r}\left(\frac{x - x_{i-1}}{h_i}\right), & \text{for } x \in (x_{i-1}, x_i), & (3.3.7a) \\ (-1)^r h_{i+1}^r \xi_{m,r}\left(\frac{x_{i+1} - x}{h_{i+1}}\right), & \text{for } x \in (x_i, x_{i+1}), & (3.3.7b) \\ 0, & \text{elsewhere}, & (3.3.7c) \end{cases}$$

where $\xi_{m,r}(s)$ satisfies

$$\xi_{m,r}^{(2m)}(s) = 0, \quad \text{for all } s \in \mathcal{R}^1, \tag{3.3.8a}$$

$$\xi_{m,r}^{(j)}(0) = 0 \quad \text{and} \quad \xi_{m,r}^{(j)}(1) = \delta_{rj}, \quad \text{for } j = 0, 1, \ldots, m-1. \tag{3.3.8b}$$

The $\xi_{m,r}(s)$ can be easily computed from (2.1.5.3) of Stoer and Bulirsch [39]. We see that

$$\frac{d^q \varphi_i^r}{dx^q}(x_j) = \delta_{rq}\delta_{ij},$$

for $i = 1, \ldots, N-1$, for $j = 0, 1, \ldots, N$ and for $r, q = 0, \ldots, m-1$.

When $S^N = V^N$, the method (3.3.6) is equivalent to

$$A_\varepsilon^N(u_N, \varphi_i^r) = (f^N, \varphi_i^r), \quad \text{for } i = 1, \ldots, N-1 \text{ and } r = 0, 1, \ldots, m-1.$$

Write

$$u_N(x) = \sum_{i=1}^{N-1} \sum_{r=0}^{m-1} \varphi_i^r(x) u_N^{(r)}(x_i).$$

Set

$$U = \left(u_N(x_1), \ldots, u_N^{(m-1)}(x_1), \ldots, u_N(x_{N-1}), \ldots, u_N^{(m-1)}(x_{N-1})\right)^T.$$

Then the method may be written in the form

$$AU = F, \tag{3.3.9}$$

where A is a $m(N-1) \times m(N-1)$ banded matrix with bandwidth $4m-1$. The non-zero entries of the coefficient matrix A and the right hand side F are certain linear combinations of ε and point evaluations of a_r (for $r = 0, 1, \ldots, 2m-1$) and f, when a_r^N and f^N are Lagrange interpolants to a_r and f respectively. When $m = 1$ (the second order convection-diffusion problem), the resulting scheme is closely related to the classical central difference scheme.

In Section 3.5, we shall show that u_N is uniformly convergent to u in certain norms provided that one takes X^N to be a certain piecewise equidistant mesh.

3.4 Interpolation Error Estimates

In this section, we first introduce a Shishkin mesh, then estimate interpolation errors in energy and Sobolev norms. The results will be used to analyse the uniform convergence of the computed solution.

3.4.1 The Mesh

In the literature, several types of special meshes have been introduced for singularly perturbed two-point boundary value problems. Bakhvalov [2] and Vulanović [46] construct a graded mesh using a special mesh-generating function. Gartland [17] subdivided the interval $[0, 1]$ into three regions: an inner region $[x^*, 1]$, a transition region $[x', x^*]$ and an outer region $[0, x']$, where

$$x^* \approx 1 - K\varepsilon \ln(K/h), \quad x' \approx 1 - K\varepsilon \ln(1/\varepsilon),$$

with K a positive integer, h a prescribed outer mesh spacing and $K\varepsilon < h$. A special mesh was generated by taking an exponentially graded mesh on $[x^*, 1]$, a locally quasi-equidistant mesh on $[x', x^*]$ and an equidistant mesh on $[0, x']$.

In this chapter we shall employ a Shishkin mesh, which is piecewise equidistant and consequently much simpler than the Bakhvalov and Gartland meshes.

Given an even positive integer N, the Shishkin mesh X_s^N is constructed by dividing the interval $[0, 1]$ into two subintervals

$$[0, 1 - \sigma], \quad \text{and} \quad [1 - \sigma, 1].$$

Equidistant meshes with $N/2$ points are then used on each subinterval. The parameter σ is defined by

$$\sigma = \min\{1/2, \quad (m + 1)\alpha^{-1}\varepsilon \ln N\},$$

which depends on ε and N. Set $i_0 = N/2$. Then $x_{i_0} = 1 - \sigma$ is the transition point of the Shishkin mesh

$$X_s^N : 0 = x_0 < x_1 < \cdots < x_{i_0} < \cdots < x_{N-1} < x_N = 1.$$

The mesh spacing on the inner interval $[x_{i_0}, 1]$ is given by

$$h_i = 2\sigma N^{-1}, \quad \text{for } i = i_0 + 1, \ldots, N. \tag{3.4.1}$$

On the outer region $[0, x_{i_0}]$, the mesh spacing is

$$h_i = 2(1-\sigma)N^{-1}, \quad \text{for } i = 0, \ldots, i_0. \tag{3.4.2}$$

Remark 3.4.1 *The distinguished mesh point x_{i_0} is analogous to the point x^* of the Gartland mesh.*

If $\sigma = 1/2$, i.e., $1/2 \le (m+1)\alpha^{-1}\varepsilon \ln N$, then N^{-1} is very small relative to ε. This is unlikely in practice (and in this case the method can be treated in the classical way). We therefore assume that

$$\sigma = (m+1)\alpha^{-1}\varepsilon \ln N. \tag{3.4.3}$$

From this, (3.4.1) and (3.4.2), one may easily obtain that

$$h_i = 2(m+1)\alpha^{-1}\varepsilon N^{-1} \ln N, \tag{3.4.4}$$

for $i = i_0 + 1, \ldots, N$ and

$$N^{-1} \le h_i \le 2N^{-1}, \tag{3.4.5}$$

for $i = 1, \ldots, i_0$.

3.4.2 Interpolation Error Estimates

For the purpose of our interpolation error analysis, we first present some standard approximation error estimates (Lemma 3.4.1) which are valid on an arbitrary mesh.

Consider a finite element $(\Sigma_i, P_R(I_i), I_i)$, where Σ_i is the set of degrees of freedom. It will be assumed that the set Σ_i is P_R-unisolvent, for $i = 1, \ldots, N$. Let s_i be the greatest order of derivatives occuring in the definition of Σ_i. For $v \in C^{s_i}(I_i)$, we denote by $\Pi_i v$ the P_R–interpolant to v on I_i. Set $s = \max\{s_i : i = 1, \ldots, N\}$. Then given $v \in C^s(0,1)$, we will denote by $\Pi^N v$ the piecewise polynomial interpolant from S^N to v. This interpolant satisfies $(\Pi^N v)|_{I_i} = \Pi_i v|_{I_i}$, for $i = 1, \ldots, N$.

Let $\|\cdot\|_{j,\infty,I_i}$ be the maximum norm on $C^j(I_i)$, with the usual associated seminorm $|\cdot|_{j,\infty,I_i}$. Denote by $\|\cdot\|_{j,2,I_i}$ the norm on the Sobolev space $H^j(I_i)$ and by $|\cdot|_{j,2,I_i}$ the associated seminorm. We have

Lemma 3.4.1 *Let k be an integer satisfying $R+1 \geq k \geq s_i$. Let $v \in C^k(I_i)$. Then there exists a constant C_1, which is independent of h_i and v, such that*

$$|v - \Pi_i v|_{j,\infty,I_i} \leq C_1 h_i^{k-j} |v|_{k,\infty,I_i}, \tag{3.4.6}$$

for $j = 0, 1 \ldots, k$. If $R+1 \geq k \geq s_i + 1$, then

$$|v - \Pi_i v|_{j,2,I_i} \leq C_1 h_i^{k-j} |v|_{k,2,I_i}, \tag{3.4.7}$$

for $j = 0, 1 \ldots, k$.

Proof. From Theorem 3.1.5 of Ciarlet [7], (3.4.6) and (3.4.7) hold for $R+1 \geq k \geq s_i + 1$. The case $k = s_i$ of (3.4.6) can be shown by a similar argument. □

We now proceed to the estimation of the interpolation error $u - \Pi^N u$ on the Shishkin mesh of subsection 3.4.1.

Recalling (3.2.7) – (3.2.9), we see that

$$\left|u^{(2m)}(x)\right| \leq C\left(1 + \varepsilon^{-m-1}\exp(-\alpha(1-x)/\varepsilon)\right), \quad \text{for } x \in [0,1] \tag{3.4.8}$$

and

$$\left|u^{(2m)}(x)\right| \leq C, \quad \text{for } x \in [0, 1-\sigma_s], \tag{3.4.9}$$

where $\sigma_\varepsilon = \min\{1/2, (m+1)\alpha^{-1}\varepsilon\ln(1/\varepsilon)\}$.

Consider the local interpolation error $u - \Pi_i u$ on a subinterval I_i. If I_i is in the fine portion of the mesh or lies outside the boundary layer, then the error analysis can be done by standard approximation theory arguments. However, when $\varepsilon < N^{-1}$, the coarse mesh and the boundary layer have nonempty intersection, viz., $J_\varepsilon \equiv (1-\sigma_\varepsilon, 1-\sigma] \neq \emptyset$. In other words, as $\varepsilon \to 0$ with N fixed, $|u^{(j)}(x)|$ is unbounded for $j \geq m$ when $x \in J_\varepsilon$ and the Shishkin mesh is coarse on J_ε. It turns out that a direct application of standard approximation theory will not yield a bound on $|||u - \Pi_i u|||$ which is uniform in ε. In order to obtain such uniformity on those $I_i \subseteq J_\varepsilon$, we need an asymptotic decomposition of u. The estimates (3.4.6) with $k = s_i$ and (3.4.7) with $k = s_i + 1$ then play a special role in our error analysis. To enable us to use (3.4.6) with $k = m-1$ and (3.4.7) with $k = m$, we shall assume that $\hat{s} = m-1$, where $\hat{s} = \max\{s_i : i_1 + 1 \leq i \leq i_0\}$ and $i_1 = \min\{i : I_i \cap J_\varepsilon \neq \emptyset\}$. In particular this assumption is satisfied if we take $S^N = V^N$, the Hermite space defined in Section 3.3.

Lemma 3.4.2 *Let u be the solution of problem (3.1.2). Then on the Shishkin mesh X_ε^N, we have for $j = 0, 1, \ldots, m-1$,*

$$|u - \Pi_i u|_{j,\infty,I_i} \leq C h_i^{m-j-1} N^{-m-1}, \text{ for } i \in \{1, \ldots, i_0\} \tag{3.4.10}$$

and

$$|u - \Pi_i u|_{j,\infty,I_i} \leq C h_i^{m-j-1} (N^{-1}\ln N)^{m+1}, \text{ for } i \in \{i_0 + 1, \ldots, N\}. \tag{3.4.11}$$

Proof. Follows from the argument of Lemma 2.5.2. □

The following interpolation error estimate in the Sobolev norm $\|\cdot\|_{m-1}$ then follows immediately from Lemma 3.4.2.

Corollary 3.4.1 *Let u be the solution of problem (3.1.2). Let $\Pi^N u$ be the piecewise polynomial interpolant from S^N to u on the Shishkin mesh X_s^N. Then*

$$\|u - \Pi^N u\|_{m-1} \le C(N^{-1}\ln N)^{m+1}. \tag{3.4.12}$$

By arguments similar to those of Lemma 2.5.2, one may also show that

$$\varepsilon\, |u - \Pi_i u|_{m,\infty,I_i} \le C(N^{-1}\ln N)^{m}, \tag{3.4.13}$$

for $i = 1, \dots, N$. However, in contrast to the problem of reaction–diffusion type, combining (3.4.13) with (3.4.10) and (3.4.11) here does not yield a bound for $u - \Pi^N u$, which is uniform in ε, in the weighted energy norm $|||\cdot|||$ which was defined in Section 3.2. One needs a more precise analysis to achieve the desired uniform estimate.

Lemma 3.4.3 *Under the same hypotheses as in Corollary 3.4.1, we have*

$$|||u - \Pi^N u||| \le C(N^{-1}\ln N)^{m}. \tag{3.4.14}$$

Proof. We first show that

$$\varepsilon \left\| \left(u - \Pi^N u\right)^{(m)} \right\|^2_{L^2[x_{i_0},\, 1]} \le C\left(N^{-1}\ln N\right)^{2m}. \tag{3.4.15}$$

Let $I_i \subseteq [x_{i_0}, 1]$, i.e., I_i lies in the fine portion of the mesh. Then from (3.4.7) with $j = m$ and $k = 2m$,

$$|u - \Pi_i u|^2_{m,2,I_i} \le C h_i^{2m} |u|^2_{2m,2,I_i} \tag{3.4.16}$$

$$\le C\left(\varepsilon N^{-1}\ln N\right)^{2m} |u|^2_{2m,2,I_i}, \tag{3.4.17}$$

by (3.4.4). Hence

$$\left\| \left(u - \Pi^N u\right)^{(m)} \right\|^2_{L^2[x_{i_0},\, 1]}$$

$$
\begin{aligned}
&\le C\left(\varepsilon N^{-1}\ln N\right)^{2m}\left\|u^{(2m)}\right\|^2_{L^2[x_{i_0},\,1]}\\
&\le C\left(\varepsilon N^{-1}\ln N\right)^{2m}\int_{x_{i_0}}^{1}\left[1+\varepsilon^{-m-1}\exp\left(-\alpha(1-x)/\varepsilon\right)\right]^2 dx\\
&\le C\varepsilon^{-1}\left(N^{-1}\ln N\right)^{2m},
\end{aligned}
$$

by (3.4.8). This completes the proof of (3.4.15).

Secondly, we prove that

$$
\varepsilon\left\|\left(u-\Pi^N u\right)^{(m)}\right\|^2_{L^2[0,\,x_{i_0}]}\le C\left(\varepsilon+N^{-2}\right)N^{-2m}. \tag{3.4.18}
$$

We discuss two cases.

Case 1: $N^{-1}\le\varepsilon$. In this case, we have $[0,x_{i_0}]\subseteq[0,1-\sigma_\varepsilon]$. It is obvious, on again taking $j=m$ and $k=2m$ in (3.4.7) and using (3.4.5) and (3.4.9), that

$$
\begin{aligned}
\left\|\left(u-\Pi^N u\right)^{(m)}\right\|^2_{L^2[0,\,x_{i_0}]}&\le CN^{-2m}\left\|u^{(2m)}\right\|^2_{L^2[0,\,x_{i_0}]}\\
&\le CN^{-2m}.
\end{aligned} \tag{3.4.19}
$$

Case 2: $N^{-1}>\varepsilon$. By the same argument as in Case 1, one can show that

$$
\left\|\left(u-\Pi^N u\right)^{(m)}\right\|^2_{L^2[0,\,x_{i_1}]}\le CN^{-2m}, \tag{3.4.20}
$$

since $[0,x_{i_1}]\subseteq[0,1-\sigma_\varepsilon]$.

We now deal with the intersection of the coarse mesh and the boundary layer. It will be shown that

$$
\left\|\left(u-\Pi^N u\right)^{(m)}\right\|^2_{L^2[x_{i_1},\,x_{i_0}]}\le C\varepsilon^{-1}\left(\varepsilon+N^{-2}\right)N^{-2m}. \tag{3.4.21}
$$

Recall the decomposition (3.2.7). Write $\Pi^N u$ in the form

$$
\Pi^N u=\Pi^N G+\Pi^N E, \tag{3.4.22}
$$

where $\Pi^N G$ and $\Pi^N E$ denote the piecewise polynomial interpolants to G and E respectively. We shall separately bound

$$\left\|(G - \Pi^N G)^{(m)}\right\|^2_{L^2[x_{i_1}, x_{i_0}]} \text{ and } \left\|(E - \Pi^N E)^{(m)}\right\|^2_{L^2[x_{i_1}, x_{i_0}]}. \tag{3.4.23}$$

By (3.2.8) and arguments similar to those of Case 1, we have

$$\left\|(G - \Pi^N G)^{(m)}\right\|^2_{L^2[x_{i_1}, x_{i_0}]} \le CN^{-2m}. \tag{3.4.24}$$

For the second term of (3.4.23), we use (3.4.7) with $j = k = m$. Then

$$\begin{aligned}
\left\|(E - \Pi^N E)^{(m)}\right\|^2_{L^2[x_{i_1}, x_{i_0}]} &\le C \left\|E^{(m)}\right\|^2_{L^2[x_{i_1}, x_{i_0}]} \\
&\le C\varepsilon^{-2} \int_{x_{i_1}}^{x_{i_0}} \exp(-2\alpha(1 - x)/\varepsilon)\, dx \\
&\le C\varepsilon^{-1} \exp(-2\alpha\sigma/\varepsilon) \\
&= C\varepsilon^{-1} N^{-2(m+1)},
\end{aligned}$$

by (3.4.3). Combining this with (3.2.7), (3.4.22) and (3.4.24) yields (3.4.21). This completes the proof of Case 2. Then (3.4.18) follows.

Recalling (3.4.15), we have

$$\varepsilon^{1/2} \left\|(u - \Pi^N u)^{(m)}\right\| \le C\,(N^{-1} \ln N)^m.$$

Combining this with Corollary 3.4.1 yields the desired result. □

3.5 Uniform Convergence Results

In this section, we present uniform convergence results in various norms for the classical finite element method (3.3.6) on the Shishkin mesh.

3.5.1 Analysis of Convergence

Since the bilinear form $A_\varepsilon(\cdot,\cdot)$ is not uniformly bounded in terms of the energy norm $|||\cdot|||$ (see (3.2.4)), a classical finite element approach does not satisfactorily analyse the errors in the computed solution u_N. We shall employ an analysis similar to Stynes and O'Riordan [43] to prove that the method (3.3.6) is uniformly convergent in the energy norm $|||\cdot|||$.

Recall that $S^N \subseteq C^{m-1}(0,1)$. It is natural to assume that for $v \in C^s(0,1)$,

$$\left(\Pi^N v\right)^{(j)}(x_i) = v^{(j)}(x_i), \tag{3.5.1}$$

for $i = 0,1,\ldots,N$ and $j = 0,1,\ldots,m-1$. We shall also assume that $R = 2m-1$, as there seems to be little benefit in using polynomials of higher degree.

Theorem 3.5.1 *Let $u_N \in S^N$ be the solution of (3.3.6) on the Shishkin mesh X_ε^N. Then for N sufficiently large (independently of ε), we have*

$$|||\Pi^N u - u_N||| \le C\left(N^{-m} + (N^{-1}\ln N)^l\right).$$

Proof. By Lemma 3.3.1, we have

$$\begin{aligned}
&C_1|||\Pi^N u - u_N|||^2 \\
&\le A_\varepsilon^N\left(\Pi^N u - u_N, \Pi^N u - u_N\right) \\
&= A_\varepsilon^N\left(\Pi^N u - u, \Pi^N u - u_N\right) + A_\varepsilon^N\left(u - u_N, \Pi^N u - u_N\right).
\end{aligned} \tag{3.5.2}$$

We begin by analyzing the first term:

$$\begin{aligned}
&A_\varepsilon^N\left(\Pi^N u - u, \Pi^N u - u_N\right) \\
&= \left(\varepsilon(\Pi^N u - u)^{(m)}, (\Pi^N u - u_N)^{(m)}\right)
\end{aligned}$$

$$+\left(a_{2m-1}^N(\Pi^N u - u)^{(m)}, (\Pi^N u - u_N)^{(m-1)}\right)$$

$$+A_1^N\left(\Pi^N u - u, \Pi^N u - u_N\right)$$

$$= \left(a_{2m-1}^N(\Pi^N u - u)^{(m)}, (\Pi^N u - u_N)^{(m-1)}\right)$$

$$+A_1^N\left(\Pi^N u - u, \Pi^N u - u_N\right), \tag{3.5.3}$$

by (3.5.1) and using integration by parts, since $(\Pi^N u - u_N)^{(2m)} \equiv 0$ on each element (x_{i-1}, x_i). It is easy to see that

$$\left|A_1^N\left(\Pi^N u - u, \Pi^N u - u_N\right)\right| \le C\|\Pi^N u - u\|_{m-1}\|\Pi^N u - u_N\|_{m-1}$$

$$\le C\left(N^{-1}\ln N\right)^{m+1} |||\Pi^N u - u_N|||, \tag{3.5.4}$$

by Corollary 3.4.1.

We now estimate the first term of (3.5.3). We have

$$\left(a_{2m-1}^N(\Pi^N u - u)^{(m)}, (\Pi^N u - u_N)^{(m-1)}\right)$$

$$= -\left(a_{2m-1}^N(\Pi^N u - u)^{(m-1)}, (\Pi^N u - u_N)^{(m)}\right)$$

$$-\left((a_{2m-1}^N)'(\Pi^N u - u)^{(m-1)}, (\Pi^N u - u_N)^{(m-1)}\right). \tag{3.5.5}$$

Clearly

$$\left|\left((a_{2m-1}^N)'(\Pi^N u - u)^{(m-1)}, (\Pi^N u - u_N)^{(m-1)}\right)\right|$$

$$\le C\left(N^{-1}\ln N\right)^{m+1} |||\Pi^N u - u_N|||. \tag{3.5.6}$$

Next, by (3.4.10),

$$\left|\int_0^{x_{i_0}} a_{2m-1}^N(x)(\Pi^N u - u)^{(m-1)}(x)(\Pi^N u - u_N)^{(m)}(x)\,dx\right|$$

$$\le CN^{-m-1}\|(\Pi^N u - u_N)^{(m)}\|_{L^2[0,\,x_{i_0}]} \tag{3.5.7}$$

$$\le CN^{-m}\|(\Pi^N u - u_N)^{(m-1)}\|_{L^2[0,\,x_{i_0}]}, \tag{3.5.8}$$

by an inverse estimate. Also, by (3.4.11),

$$\left|\int_{s_{i_0}}^{1} a_{2m-1}^{N}(x)(\Pi^N u - u)^{(m-1)}(x)(\Pi^N u - u_N)^{(m)}(x)\,dx\right|$$
$$\leq C\left(N^{-1}\ln N\right)^{m+1}\int_{s_{i_0}}^{1}\left|(\Pi^N u - u_N)^{(m)}(x)\right|\,dx$$
$$\leq C\left(N^{-1}\ln N\right)^{m+1}\sigma^{1/2}\|(\Pi^N u - u_N)^{(m)}\|_{L^2[s_{i_0},\,1]}$$
$$\leq C\left(N^{-1}\ln N\right)^{m+1}\ln^{1/2} N|||\Pi^N u - u_N|||. \tag{3.5.9}$$

Hence, (3.5.8) and (3.5.9) yield

$$\left|\left(a_{2m-1}^{N}(\Pi^N u - u)^{(m-1)}, (\Pi^N u - u_N)^{(m)}\right)\right| \leq CN^{-m}|||\Pi^N u - u_N|||. \tag{3.5.10}$$

Combining (3.5.3) - (3.5.6) and (3.5.10), we have

$$\left|A_\epsilon^N\left(\Pi^N u - u, \Pi^N u - u_N\right)\right| \leq CN^{-m}|||\Pi^N u - u_N|||. \tag{3.5.11}$$

Taking the second term of (3.5.2),

$$A_\epsilon^N\left(u - u_N, \Pi^N u - u_N\right)$$
$$= \left(A_\epsilon^N - A_\epsilon\right)\left(u, \Pi^N u - u_N\right) + A_\epsilon\left(u, \Pi^N u - u_N\right)$$
$$-A_\epsilon^N\left(u_N, \Pi^N u - u_N\right)$$
$$= \left(\left(a_{2m-1}^N - a_{2m-1}\right)u^{(m)}, (\Pi^N u - u_N)^{(m-1)}\right)$$
$$+ \left(A_1^N - A_1\right)\left(u, \Pi^N u - u_N\right) + \left(f - f^N, \Pi^N u - u_N\right). \tag{3.5.12}$$

Since $|u|_k \leq C$ for $k = 0, \ldots, m-1$, clearly

$$\left|\left(A_1^N - A_1\right)\left(u, \Pi^N u - u_N\right) + \left(f - f^N, \Pi^N u - u_N\right)\right| \leq CN^{-l}|||\Pi^N u - u_N|||. \tag{3.5.13}$$

For the first term of (3.5.12), we have by (3.3.2) and (3.2.7) - (3.2.9),

$$\left|\int_{s_{i_0}}^{1} \left(a_{2m-1}^{N} - a_{2m-1}\right)(x)u^{(m)}(x)(\Pi^N u - u_N)^{(m-1)}(x)\,dx\right|$$
$$\leq C\varepsilon^{l-1}\left(N^{-1}\ln N\right)^{l} |||\Pi^N u - u_N|||. \tag{3.5.14}$$

Also

$$\left|\int_{0}^{s_{i_0}} \left(a_{2m-1}^{N} - a_{2m-1}\right)(x)G^{(m)}(x)(\Pi^N u - u_N)^{(m-1)}(x)\,dx\right|$$
$$\leq CN^{-l}\|\Pi^N u - u_N\|_{m-1}. \tag{3.5.15}$$

Since

$$\begin{aligned}\int_{0}^{s_{i_0}} |E^{(m)}(x)|\,dx &\leq C\int_{0}^{s_{i_0}} \varepsilon^{-1}\exp\left(-\alpha(1-x)/\varepsilon\right)dx\\ &= C\alpha^{-1}\left(\exp\left(-\alpha\sigma/\varepsilon\right) - \exp\left(-\alpha/\varepsilon\right)\right)\\ &\leq CN^{-m-1},\end{aligned}$$

we have

$$\left|\int_{0}^{s_{i_0}} \left(a_{2m-1}^{N} - a_{2m-1}\right)(x)E^{(m)}(x)(\Pi^N u - u_N)^{(m-1)}(x)\,dx\right|$$
$$\leq CN^{-l-m-1}\|(\Pi^N u - u_N)^{(m-1)}\|_{L^\infty[0,\,s_{i_0}]}$$
$$\leq CN^{-l-m-1/2}\|(\Pi^N u - u_N)^{(m-1)}\|_{L^2[0,\,s_{i_0}]}, \tag{3.5.16}$$

by an inverse inequality. From (3.5.14) - (3.5.16) and $u = G + E$, we obtain

$$\left|\left(\left(a_{2m-1}^{N} - a_{2m-1}\right)u^{(m)}, (\Pi^N u - u_N)^{(m-1)}\right)\right|$$
$$\leq C\left(N^{-1}\ln N\right)^{l} |||\Pi^N u - u_N|||. \tag{3.5.17}$$

Therefore, from (3.5.12), (3.5.13) and (3.5.17),

$$\left|A_\varepsilon^N\left(u - u_N, \Pi^N u - u_N\right)\right| \leq C\left(N^{-1}\ln N\right)^{l} |||\Pi^N u - u_N|||. \tag{3.5.18}$$

Combining this with (3.5.2) and (3.5.11), we obtain the desired result. □

Remark 3.5.1 *Suppose that $\varepsilon \le N^{-1}$ in Theorem 3.5.1. (This is reasonable in practice.) Then instead of (3.5.14) we have, using a Cauchy-Schwarz inequality,*

$$\begin{aligned}
&\left| \int_{x_{i_0}}^{1} \left(a_{2m-1}^N - a_{2m-1}\right)(x) u^{(m)}(x) (\Pi^N u - u_N)^{(m-1)}(x)\, dx \right| \\
&\le C \left(\varepsilon N^{-1} \ln N\right)^l \int_{x_{i_0}}^{1} \left(1 + \varepsilon^{-1} e^{-\alpha(1-x)/\varepsilon}\right) \left|(\Pi^N u - u_N)^{m-1}(x)\right| dx \\
&\le C \left(\varepsilon N^{-1} \ln N\right)^l \varepsilon^{-1/2} |||\Pi^N u - u_N||| \\
&\le C N^{-l} |||\Pi^N u - u_N|||.
\end{aligned}$$

This inequality clearly allows us to replace the conclusion of Theorem 3.5.1 by

$$|||\Pi^N u - u_N||| \le C N^{-\min\{m,\, l\}}.$$

Corollary 3.5.1 *Let u_N be defined as in Theorem 3.5.1. Then*

$$|||u - u_N||| \le C \left(N^{-1} \ln N\right)^{\min\{m,\, l\}}. \tag{3.5.19}$$

If in addition $m \ge 2$, then for $j \in \{0, \ldots, m-2\}$,

$$\|(u - u_N)^{(j)}\|_\infty \le C \left(N^{-1} \ln N\right)^{\min\{m,\, l\}}. \tag{3.5.20}$$

Proof. First, (3.5.19) follows from Theorem 3.5.1 and Lemma 3.4.3.

Note that for all $v \in H_0^m$,

$$\|v^{(j)}\|_\infty \le |v|_{j+1} \le \|v\|_{j+1}, \quad \text{for } j = 0, \ldots, m-2.$$

We therefore have from (3.5.19) that for $j = 0, 1, \ldots, m-2$,

$$\begin{aligned}
\|(u - u_N)^{(j)}\|_\infty &\le \|u - u_N\|_{j+1} \\
&\le C \left(N^{-1} \ln N\right)^{\min\{m,\, l\}},
\end{aligned}$$

which is the desired result. □

Remark 3.5.2 *Assuming that we use a sufficiently accurate quadrature rule, so that $l \geq m$, Corollary 3.5.1 yields*

$$|||u - u_N||| \leq C\left(N^{-1}\ln N\right)^m . \qquad (3.5.21)$$

In the classical case $\varepsilon = 1$, the Shishkin mesh is equidistant and it is well known that one has

$$|||u - u_N||| = O(N^{-m}) .$$

Consequently we see that (3.5.21) is at least almost optimal.

Remark 3.5.3 *In Chapter 2 we gave the uniform estimate*

$$\|v - v_N\|_{m-1} \leq C\left(\left(N^{-1}\ln N\right)^{m+1} + N^{-l}\right), \qquad (3.5.22)$$

where v is the solution of problem (3.1.5) and v_N is its computed solution, which was obtained by a method similar to that of this chapter. When, e.g., $m = l - 1$, the order of convergence in (3.5.22) is greater than the order implied by (3.5.19). This difference is intrinsic to the two problems under consideration, when $m \geq 2$. Numerical results in Section 3.6 below will show that the exponent m in Corollary 3.5.1 is sharp; hence an inequality such as (3.5.22) does not hold for $\|u - u_N\|_{m-1}$.

Corollary 3.5.2 *Assume that in problem (3.1.2) the functions a_r for $r = 0, 1, \ldots, 2m-1$ and f are constant. Let $u_N \in S^N$ be the solution of (3.3.6) with $a_r^N \equiv a_r$, for $r = 0, \ldots, 2m-1$, and $f^N \equiv f$, on the Shishkin mesh X_s^N. Then for N sufficiently large (independently of ε), we have*

$$|||u - u_N||| \leq C\left(N^{-1}\ln N\right)^m .$$

Proof. Our hypotheses imply that we can take l arbitrarily large in (3.5.19). □

Remark 3.5.4 *Consider now the proof of Theorem 3.5.1 from a classical viewpoint. From (3.5.8), one has*

$$\left|\int_0^{x_{i_0}} a_{2m-1}^N(x)(\Pi^N u - u)^{(m-1)}(x)(\Pi^N u - u_N)^{(m)}(x)\,dx\right|$$
$$\leq C\varepsilon^{-1/2}N^{-m-1}|||\Pi^N u - u_N|||.$$

Combining this with (3.5.3) - (3.5.6) and (3.5.9) yields

$$|A_\varepsilon^N(\Pi^N u - u, \Pi^N u - u_N)|$$
$$\leq C\left(\varepsilon^{-1/2} + \ln^{m+3/2} N\right) N^{-m-1}|||\Pi^N u - u_N|||.$$

Following the proof of Theorem 3.5.1, this leads to

$$|||\Pi^N u - u_N||| \leq C\left(\varepsilon^{-1/2} + \ln^{m+3/2} N\right) N^{-m-1} + \left(N^{-1}\ln N\right)^l.$$

Combining this with Corollary 3.4.1, we have

$$\|u - u_N\|_{m-1} \leq C\left(\varepsilon^{-1/2} + \ln^{m+3/2} N\right) N^{-m-1} + \left(N^{-1}\ln N\right)^l.$$

This shows that in the classical sense (i.e., for ε fixed) the order of convergence in the norm $\|\cdot\|_{m-1}$ is greater than the order of convergence in the energy norm $|||\cdot|||$. On the other hand, when convergence uniformly in ε is considered with $m \geq 2$, this phenomenon does not occur; numerical experiments in Section 3.6 show that one may have the same order of convergence in $\|\cdot\|_{m-1}$ as in $|||\cdot|||$.

When l is odd, some stronger uniform convergence results can be obtained, if a_{2m-1} and a_{2m-2} are approximated to a higher order of accuracy than the other a_r in (3.1.2a).

Theorem 3.5.2 *Let $u_N \in S^N$ be the solution of method (3.3.6) on the Shishkin mesh X_ε^N. Assume also the following: for $i = 1, \dots, N$,*

$$\left|\left(a_r^N - a_r\right)(x)\right| \le C h_i^{l+1} \tag{3.5.23}$$

for $x \in (x_{i-1}, x_i)$ and $r = 2m-2, 2m-1$,

$$\left|\int_{x_{i-1}}^{x_i} \left(a_r^N - a_r\right)(x)\, dx\right| \le C h_i^{l+2}, \quad \text{for } r = 0, 1, \dots, 2m-3 \tag{3.5.24}$$

and

$$\left|\int_{x_{i-1}}^{x_i} (f^N - f)(x)\, dx\right| \le C h_i^{l+2}. \tag{3.5.25}$$

Then for N sufficiently large (independently of ε), we have

$$|||u - u_N||| \le C\left(N^{-1} \ln N\right)^{\min\{m,\, l+1\}}.$$

Proof. We need to prove that

$$\left|A_\varepsilon^N\left(u - u_N, \Pi^N u - u_N\right)\right| \le C\left(N^{-1}\ln N\right)^{l+1} |||\Pi^N u - u_N|||, \tag{3.5.26}$$

where

$$\begin{aligned} &A_\varepsilon^N\left(u - u_N, \Pi^N u - u_N\right) \\ &= \left(\left(a_{2m-1}^N - a_{2m-1}\right) u^{(m)}, \left(\Pi^N u - u_N\right)^{(m-1)}\right) \\ &\quad + \left(A_1^N - A_1\right)\left(u, \Pi^N u - u_N\right) + \left(f - f^N, \Pi^N u - u_N\right). \end{aligned}$$

Inspecting the proof of (3.5.17) and using (3.5.23) with $r = 2m-1$, one can show that

$$\begin{aligned} &\left|\left(\left(a_{2m-1}^N - a_{2m-1}\right) u^{(m)}, \left(\Pi^N u - u_N\right)^{(m-1)}\right)\right| \\ &\le C\left(N^{-1}\ln N\right)^{l+1} |||\Pi^N u - u_N|||. \end{aligned}$$

Imitating the proof of Theorem 2.5.3 of chapter 2 and using (3.5.23) with $r = 2m-2$, (3.5.24) and (3.5.25), one may prove that

$$\left|(A_1^N - A_1)\left(u, \Pi^N u - u_N\right)\right| \le CN^{-l-1}|||\Pi^N u - u_N|||$$

and

$$\left|(f - f^N, \Pi^N u - u_N)\right| \le CN^{-l-1}|||\Pi^N u - u_N|||.$$

This completes the proof of (3.5.26).

Recalling the proof of Theorem 3.5.1, we are done. □

Remark 3.5.5 *It is well known in the context of Newton–Cotes integration rules that properties (3.5.24) and (3.5.25) are easily achieved using piecewise polynomials of degree $l-1$ when l is odd.*

3.5.2 A Special Case

We work with the special case of a second order problem (i.e., $m = 1$). For problem (3.1.1), the conditions (3.1.2d, e) are equivalent to the inequality $b(x) - a'(x)/2 > 0$ and in fact this inequality can be deduced from $a(x) > \alpha > 0$; see Stynes and O'Riordan [43].

Consider the method (3.3.6) with the basis function space V^N (for $m = 1$), i.e., a piecewise linear function space. We take $\sigma = 3\alpha^{-1}\varepsilon \ln N$ in the Shishkin mesh, which is different from our previous value of σ. Let w be the solution of problem (3.1.1). The next result shows that if $l = 2$, then $\|w - u_N\|_0$ has order of uniform convergence at least 1/2 higher than is implied by the bound on $|||w - u_N|||$ given in Corollary 3.5.1.

Theorem 3.5.3 *Let w be the solution of problem (3.1.1) and let $\Pi^N w \in S^N$ interpolate to w at each node x_i, $i = 0, \ldots, N$. Let $u_N \in S^N$ be the solution of the method (3.3.6) on the Shishkin mesh X_s^N, with $\sigma = 3\alpha^{-1}\varepsilon \ln N$. Let*

$$a^N(x) = \frac{x_i - x}{h_i} a(x_{i-1}) + \frac{x - x_{i-1}}{h_i} a(x_i),$$

for $x \in (x_{i-1}, x_i)$ and $i = 1, \ldots, N$, with similar definitions of $b^N(x)$ and $f^N(x)$. Then for N sufficiently large (independently of ε), we have

$$|||\Pi^N w - u_N||| \leq C N^{-3/2} \tag{3.5.27}$$

and consequently

$$\|w - u_N\|_0 \leq C N^{-3/2}. \tag{3.5.28}$$

Proof. Recall the proof of Theorem 3.5.1 and take $m = 1$ and $l = 2$. In order to obtain the desired accuracy, one needs to analyse the term

$$\int_0^{x_{i_0}} a^N(x) \left(\Pi^N w - w\right)(x) \left(\Pi^N w - u_N\right)'(x)\, dx$$

more carefully.

We again use the decomposition (3.2.7). Then

$$\begin{aligned}
&\int_0^{x_{i_0}} a^N(x) \left(\Pi^N w - w\right)(x) \left(\Pi^N w - u_N\right)'(x)\, dx \\
&= \int_0^{x_{i_0}} a^N(x) \left(\Pi^N G - G\right)(x) \left(\Pi^N w - u_N\right)'(x)\, dx \\
&\quad + \int_0^{x_{i_0}} a^N(x) \left(\Pi^N E - E\right)(x) \left(\Pi^N w - u_N\right)'(x)\, dx.
\end{aligned} \tag{3.5.29}$$

Firstly, it is clear, on taking $k = j = 0$ in (3.4.6), that for $i = 1, \ldots, i_0$,

$$\begin{aligned}
|E - \Pi_i E|_{0,\infty,I_i} &\leq C\, |E|_{0,\infty,I_i} \\
&\leq C \exp(-\alpha\sigma/\varepsilon) \\
&= C N^{-3},
\end{aligned}$$

since $\sigma = 3\alpha^{-1}\varepsilon \ln N$. Hence

$$\begin{aligned}
&\left|\int_0^{x_{i_0}} a^N(x)\left(\Pi^N E - E\right)(x)\left(\Pi^N w - u_N\right)'(x)\,dx\right| \\
&\leq CN^{-3}\left\|\left(\Pi^N w - u_N\right)'\right\|_{L^2[0,x_{i_0}]} \\
&\leq CN^{-2}\left\|\Pi^N w - u_N\right\|_0, \qquad (3.5.30)
\end{aligned}$$

by an inverse estimate.

Secondly, we bound the first term of the right hand side in (3.5.29). Let $i \in \{1,\ldots,i_0\}$. For $x \in (x_{i-1}, x_i)$, we have

$$\begin{aligned}
\left(\Pi^N G - G\right)(x) &= \frac{x_i - x}{h_i}G(x_{i-1}) + \frac{x - x_{i-1}}{h_i}G(x_i) - G(x) \\
&= \frac{1}{2}(x - x_{i-1})(x_i - x)G''(x) + O\left(h_i^3\right).
\end{aligned}$$

In the sequel, we shall denote by Z_i any quantity of $O(N^{-2})$. Set

$$e_i = \left(\Pi^N w - u_N\right)(x_i).$$

Also set

$$\begin{aligned}
\Delta_i &= \int_{x_{i-1}}^{x_i} a^N(x)\left(\Pi^N G - G\right)(x)\left(\Pi^N w - u_N\right)'(x)\,dx \\
&= \frac{e_i - e_{i-1}}{2h_i}\int_{x_{i-1}}^{x_i}(x - x_{i-1})(x_i - x)a^N(x)G''(x)\,dx + h_i Z_i(e_i - e_{i-1}) \\
&= \frac{e_i - e_{i-1}}{2h_i}a(x_i)G''(x_i)\int_{x_{i-1}}^{x_i}(x - x_{i-1})(x_i - x)\,dx + h_i Z_i(e_i - e_{i-1}) \\
&= \frac{h_i^2}{12}a(x_i)G''(x_i)(e_i - e_{i-1}) + h_i Z_i(e_i - e_{i-1}).
\end{aligned}$$

Therefore

$$\left|\int_0^{x_{i_0}} a^N(x)\left(\Pi^N G - G\right)(x)\left(\Pi^N w - u_N\right)'(x)\,dx\right| = \left|\sum_{i=1}^{i_0}\Delta_i\right|$$

$$
\begin{aligned}
&= \left| \sum_{i=1}^{i_0} \left[\frac{h_i^2}{12} a(x_i) G''(x_i)(e_i - e_{i-1}) + h_i Z_i (e_i - e_{i-1}) \right] \right| \\
&= \Bigg| - \sum_{i=1}^{i_0 - 1} \frac{1}{12} \left[h_{i+1}^2 a(x_{i+1}) G''(x_{i+1}) - h_i^2 a(x_i) G''(x_i) \right] e_i \\
&\quad + \frac{h_{i_0}}{12} a(x_{i_0}) G''(x_{i_0}) e_{i_0} + \sum_{i=1}^{i_0} h_i Z_i (e_i - e_{i-1}) \Bigg| \\
&= \left| - \sum_{i=1}^{i_0 - 1} h_i Z_i e_i + \frac{h_{i_0}}{12} a(x_{i_0}) G''(x_{i_0}) e_{i_0} + \sum_{i=1}^{i_0} h_i Z_i (e_i - e_{i-1}) \right|, \\
&\qquad\qquad\qquad\qquad \text{since } h_i = h_{i-1} = O(N^{-1}), \\
&\leq C N^{-2} \left(\sum_{i=1}^{i_0} h_i |e_i| + |e_{i_0}| \right) \\
&\leq C N^{-2} \left(\sum_{i=1}^{i_0} h_i \left| \left(\Pi^N w - u_N \right)(x_i) \right| + \left| \left(\Pi^N w - u_N \right)(x_{i_0}) \right| \right) \\
&\leq C N^{-2} \left(\left(\sum_{i=1}^{i_0} h_i \left| \left(\Pi^N w - u_N \right)(x_i) \right|^2 \right)^{1/2} + \left| \left(\Pi^N w - u_N \right)(x_{i_0}) \right| \right) \\
&\leq C N^{-2} \left(1 + \frac{\left| \left(\Pi^N w - u_N \right)(x_{i_0}) \right|}{|||\Pi^N w - u_N|||} \right) |||\Pi^N w - u_N||| \qquad (3.5.31) \\
&\leq C N^{-3/2} |||\Pi^N w - u_N|||,
\end{aligned}
$$

by an inverse inequality. Combining this with (3.5.29) and (3.5.30) yields

$$
\left| \int_0^{x_{i_0}} a^N(x) \left(\Pi^N w - w \right)(x) \left(\Pi^N w - u_N \right)'(x) \, dx \right| \leq C N^{-3/2} |||\Pi^N w - u_N|||.
$$

Recalling (3.5.3) - (3.5.6) and (3.5.9) with $m = 1$, we have

$$
\left| A_\varepsilon^N \left(\Pi^N w - w, \Pi^N w - u_N \right) \right| \leq C N^{-3/2} |||\Pi^N w - u_N|||.
$$

Also,

$$
\left| A_\varepsilon^N \left(w - u_N, \Pi^N w - u_N \right) \right| \leq C \left(N^{-1} \ln N \right)^2 |||\Pi^N w - u_N|||,
$$

by (3.5.18) with $l = 2$.

Returning to (3.5.2), we get

$$C_1|||\Pi^N w - u_N|||^2 \leq CN^{-3/2}|||\Pi^N w - u_N|||.$$

This completes the proof of (3.5.27). Then (3.5.28) follows from (3.5.27) and (3.4.12) with $m = 1$. □

Remark 3.5.6 *Instead of (3.5.27), the proof of Theorem 3.5.3 actually gives (see (3.5.31)) that*

$$|||\Pi^N w - u_N||| \leq \left((N^{-1}\ln N)^2 + N^{-2}\frac{|(\Pi^N w - u_N)(x_{i_0})|}{|||\Pi^N w - u_N|||} \right).$$

Numerical experiments show that if N is sufficiently large one has

$$\frac{|(\Pi^N w - u_N)(x_{i_0})|}{|||\Pi^N w - u_N|||} \leq C,$$

so one may expect to achieve almost second order uniform accuracy of $|||\Pi^N w - u_N|||$.

Remark 3.5.7 *One can get*

$$\max_{1\leq i\leq i_0} |(\Pi^N w - u_N)(x_i)| \leq CN^{-1},$$

on applying an inverse estimate to (3.5.27). Assume that $b(x) \geq 0$ for all $x \in [0,1]$. Now the method defined in Theorem 3.5.3 satisfies a discrete maximum principle, when restricted to $X_s^N \cap [x_{i_0}, 1]$. Hence we have

$$\max_{i_0\leq i\leq N-1} |(w - u_N)(x_i)| \leq CN^{-1}.$$

This shows that the method is uniformly convergent of at least first order in the discrete maximum norm.

3.6 Numerical Results

In this section are reported some numerical experiments to demonstrate the accuracy of the method (3.3.6) applied to fourth order problems and the second order problem (3.1.1). We shall take the trial and test space $S^N = V^N$. Then the scheme has the form of (3.3.9).

We shall examine both the error between the computed solution u_N and the true solution u and the error between the u_N and the interpolant $\Pi^N u$. These errors $u - u_N$ and $\Pi^N u - u_N$ will be measured in various norms $\|\cdot\|$. We calculate the convergence rate tables as follows, where E_ε^N may denote $\|u - u_N\|$ or $\|\Pi^N u - u_N\|$; see Farrell and Hegarty [14]:

(i). Except for the last row, the table entries are given by the classical convergence rate,

$$R_\varepsilon^N = (\ln E_\varepsilon^{2N} - \ln E_\varepsilon^N)/\ln 2.$$

(ii). The last row of each table is the uniform convergence rate,

$$R^N = (\ln E^{2N} - \ln E^N)/\ln 2,$$

where $E^N = \max_\varepsilon E_\varepsilon^N$.

We first consider the fourth order problem (i.e., $m = 2$)

$$\varepsilon^2 u^{(4)} - \left(a(x)u'' + b(x)u'\right)' + c(x)u' + d(x)u = f(x), \quad \text{for } x \in (0,1), \qquad (3.6.1a)$$

$$u(0) = u'(0) = u(1) = u'(1) = 0, \qquad (3.6.1b)$$

with

$$a(x) > \alpha > 0, \qquad (3.6.1c)$$

$$b(x) - \frac{1}{2}a'(x) \geq \alpha_1 > 0 \tag{3.6.1d}$$

and

$$d(x) - \frac{1}{2}c'(x) \geq \alpha_0 > -\alpha_1, \tag{3.6.1e}$$

for $0 \leq x \leq 1$. The matrix of the method (3.3.9), with $m = 2$, is heptadiagonal. The scheme is solved by Gaussian elimination.

We compute the following three errors:

(i). The error between the exact solution $u(x)$ and the computed solution $u_N(x)$ in the discrete maximum norm,

$$E_\varepsilon^N = \max_{0 \leq i \leq N} |u(x_i) - u_N(x_i)|.$$

(ii). The error between the interpolant $\Pi^N u(x)$ and the computed solution $u_N(x)$ in a discrete H^1-norm,

$$E_\varepsilon^N = |\Pi^N u - u_N|_{d_1}.$$

(iii). The error between the interpolant $\Pi^N u(x)$ and the computed solution $u_N(x)$ in a discrete energy norm,

$$E_\varepsilon^N = |||\Pi^N u - u_N|||_d.$$

The discrete H^1-norm and the discrete energy norm are defined respectively by

$$|v|_{d_1} = \left\{ \sum_{i=1}^{N-1} \left(h_i^{-1}(v_i - v_{i-1})^2 + \bar{h}_i w_i^2 \right) \right\}^{1/2}$$

and

$$|||v|||_d = \left\{ \varepsilon \sum_{i=1}^{N} \left(12 h_i^{-1} \left(h_i^{-1} (v_i - v_{i-1}) - (w_i + w_{i-1})/2 \right)^2 + h_i^{-1} (w_i - w_{i-1})^2 \right) + |v|_{d_1} \right\}^{1/2},$$

for all $v = \sum_{i=1}^{N-1} \left[v_i\varphi_i^0(x) + w_i\varphi_i^1(x)\right] \in V^N$, where $\bar{h}_i = (h_i + h_{i+1})/2$. By calculation, one may show that on V^N the discrete H^1-norm $|\cdot|_{d,1}$ is equivalent to the usual seminorm $|\cdot|_1$ and the discrete energy norm is equivalent to the energy norm $|||\cdot|||$.

Example 3.6.1 *(m=2). Consider (3.6.1) with* $a(x) \equiv 10$, $b(x) \equiv 1$, $c(x) \equiv d(x) \equiv 0$ *and*

$$\begin{aligned} u(x) = \varepsilon^2 \exp(-10(1-x)/\varepsilon) - \varepsilon^2(x + (1-x))\exp(-10/\varepsilon) \\ - \left(\varepsilon(10+\varepsilon)\exp(-10/\varepsilon) - \varepsilon^2\right) x(x-1)^2 \\ - \left(\varepsilon(10-\varepsilon) + \varepsilon^2\exp(-10/\varepsilon)\right) x^2(x-1) \\ +0.9x^2(1-x)^2. \end{aligned}$$

The function $f(x)$ *is then chosen to satisfy (3.6.1a); it satisfies*

$$|f^{(j)}(x)| \le C\left(1 + \varepsilon^{1-j}\exp(-10(1-x)/\varepsilon)\right),$$

for $x \in (0,1)$ *and* $j = 0, 1, \ldots$.

Note that, since $\|u''\|_{L^\infty[0,1]} \le C$, the solution is in fact smoother than the typical solution of (3.1.2) given in (3.2.6).

We shall confirm that the exponent m in Theorem 3.5.1 and Corollary 3.5.1 is sharp. To do this, we use the method (3.3.6) with the piecewise quadratic approximations

$$\frac{(x-x_i)(x-x_{i+1})}{2h_i\bar{h}_i}p(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{h_ih_{i+1}}p(x_i)$$
$$+\frac{(x-x_{i-1})(x-x_i)}{2h_{i+1}\bar{h}_i}p(x_{i+1}),$$

for $x \in (x_{i-1}, x_{i+1})$ and $i = 1, 3, \ldots, N-1$, where p can be a, b, c, d or f. For this scheme, which we refer to as the $\tilde{A}$ scheme, we have $l = 3$ in Theorem 3.5.1. We take $\alpha = 9.5$ in (3.4.3).

ε	N=8	16	32	64	128	256
1.00000e+00	1.329e-03	7.760e-05	4.890e-06	3.043e-07	1.903e-08	1.241e-09
2.50000e-01	6.191e-04	1.504e-04	2.310e-05	2.903e-06	3.057e-07	1.902e-08
6.25000e-02	8.812e-04	4.830e-05	2.912e-06	1.795e-07	2.074e-08	2.200e-09
1.56250e-02	2.007e-03	2.108e-04	1.352e-05	8.386e-07	5.195e-08	3.217e-09
3.90625e-03	2.330e-03	4.539e-04	5.261e-05	3.499e-06	2.185e-07	1.363e-08
9.76562e-04	2.394e-03	5.644e-04	1.113e-04	1.317e-05	8.839e-07	5.529e-08
2.44141e-04	2.408e-03	5.965e-04	1.403e-04	2.771e-05	3.296e-06	2.216e-07
6.10352e-05	2.412e-03	6.048e-04	1.490e-04	3.502e-05	6.923e-06	8.244e-07
1.52588e-05	2.412e-03	6.069e-04	1.513e-04	3.725e-05	8.752e-06	1.731e-06
3.81470e-06	2.413e-03	6.074e-04	1.519e-04	3.783e-05	9.311e-06	2.188e-06
9.53674e-07	2.413e-03	6.075e-04	1.520e-04	3.798e-05	9.458e-06	2.328e-06
2.38419e-07	2.413e-03	6.075e-04	1.521e-04	3.802e-05	9.496e-06	2.365e-06
5.96046e-08	2.413e-03	6.075e-04	1.521e-04	3.803e-05	9.505e-06	2.374e-06
1.49012e-08	2.413e-03	6.075e-04	1.521e-04	3.803e-05	9.507e-06	2.376e-06
3.72529e-09	2.413e-03	6.075e-04	1.521e-04	3.803e-05	9.508e-06	2.377e-06
9.31323e-10	2.413e-03	6.075e-04	1.521e-04	3.803e-05	9.508e-06	2.377e-06

Table 3.6.1: $\|u - u_N\|_{\infty,d}$ for $\tilde{A}$ Scheme

ε	N=8	16	32	64	128
1.00000e+00	4.10	3.99	4.01	4.00	3.94
2.50000e-01	2.04	2.70	2.99	3.25	4.01
6.25000e-02	4.19	4.05	4.02	3.11	3.24
1.56250e-02	3.25	3.96	4.01	4.01	4.01
3.90625e-03	2.36	3.11	3.91	4.00	4.00
9.76562e-04	2.08	2.34	3.08	3.90	4.00
2.44141e-04	2.01	2.09	2.34	3.07	3.89
6.10352e-05	2.00	2.02	2.09	2.34	3.07
1.52588e-05	1.99	2.00	2.02	2.09	2.34
3.81470e-06	1.99	2.00	2.01	2.02	2.09
9.53674e-07	1.99	2.00	2.00	2.01	2.02
2.38419e-07	1.99	2.00	2.00	2.00	2.01
5.96046e-08	1.99	2.00	2.00	2.00	2.00
1.49012e-08	1.99	2.00	2.00	2.00	2.00
3.72529e-09	1.99	2.00	2.00	2.00	2.00
9.31323e-10	1.99	2.00	2.00	2.00	2.00
R^N	1.99	2.00	2.00	2.00	2.00

Table 3.6.2: $\|u - u_N\|_{\infty,\bar{\mathcal{A}}}$ Convergence Rates for $\tilde{A}$ Scheme

It is easy to see that

$$\|u - u_N\|_{\infty,\bar{\mathcal{A}}} \leq \|u - u_N\|_1 \leq |||u - u_N|||.$$

Combining this with Corollary 3.5.1, we have

$$\|u - u_N\|_{\infty,\bar{\mathcal{A}}} \leq |||u - u_N||| \leq C\left(N^{-1}\ln N\right)^2.$$

But now Table 3.6.2 implies that the exponent 2 here is best possible, i.e., the exponent m in Corollary 3.5.1 is sharp.

Since $\|u-u_N\|_{\infty,\bar{\mathcal{A}}} = \|\Pi^N u-u_N\|_{\infty,\bar{\mathcal{A}}}$, a similar argument shows that the exponent m in Theorem 3.5.1 is also sharp, since $l = 3$ here.

ε	N=8	16	32	64	128	256
1.56250e-02	7.797e-03	9.620e-04	7.524e-05	4.991e-06	3.102e-07	1.896e-08
3.90625e-03	8.102e-03	1.634e-03	2.160e-04	1.736e-05	1.159e-06	7.344e-08
9.76562e-04	8.112e-03	1.929e-03	3.932e-04	5.274e-05	4.274e-06	2.863e-07
2.44141e-04	8.109e-03	2.015e-03	4.775e-04	9.748e-05	1.312e-05	1.065e-06
6.10352e-05	8.107e-03	2.037e-03	5.028e-04	1.191e-04	2.433e-05	3.276e-06
1.52588e-05	8.107e-03	2.042e-03	5.095e-04	1.257e-04	2.976e-05	6.079e-06
3.81470e-06	8.107e-03	2.044e-03	5.112e-04	1.274e-04	3.141e-05	7.440e-06
9.53674e-07	8.107e-03	2.044e-03	5.116e-04	1.278e-04	3.185e-05	7.853e-06
2.38419e-07	8.107e-03	2.044e-03	5.117e-04	1.279e-04	3.196e-05	7.962e-06
5.96046e-08	8.107e-03	2.044e-03	5.117e-04	1.280e-04	3.198e-05	7.989e-06
1.49012e-08	8.107e-03	2.044e-03	5.117e-04	1.280e-04	3.199e-05	7.996e-06
3.72529e-09	8.107e-03	2.044e-03	5.117e-04	1.280e-04	3.199e-05	7.998e-06
9.31323e-10	8.107e-03	2.044e-03	5.117e-04	1.280e-04	3.199e-05	7.998e-06

Table 3.6.3: $\|\Pi^N u - u_N\|_1$ for $\tilde{A}$ Scheme

ε	N=8	16	32	64	128
1.56250e-02	3.02	3.68	3.91	4.01	4.03
3.90625e-03	2.31	2.92	3.64	3.91	3.98
9.76562e-04	2.07	2.29	2.90	3.63	3.90
2.44141e-04	2.01	2.08	2.29	2.89	3.62
6.10352e-05	1.99	2.02	2.08	2.29	2.89
1.52588e-05	1.99	2.00	2.02	2.08	2.29
3.81470e-06	1.99	2.00	2.00	2.02	2.08
9.53674e-07	1.99	2.00	2.00	2.00	2.02
2.38419e-07	1.99	2.00	2.00	2.00	2.00
5.96046e-08	1.99	2.00	2.00	2.00	2.00
1.49012e-08	1.99	2.00	2.00	2.00	2.00
3.72529e-09	1.99	2.00	2.00	2.00	2.00
9.31323e-10	1.99	2.00	2.00	2.00	2.00
R^N	1.99	2.00	2.00	2.00	2.00

Table 3.6.4: $\|\Pi^N u - u_N\|_1$ Convergence Rates for $\tilde{A}$ Scheme

ε	N=8	16	32	64	128	256
1.56250e-02	1.992e-02	2.852e-03	3.184e-04	3.726e-05	4.216e-06	3.377e-07
3.90625e-03	1.987e-02	4.224e-03	5.328e-04	4.017e-05	2.960e-06	2.765e-07
9.76562e-04	1.977e-02	5.015e-03	1.034e-03	1.273e-04	8.989e-06	5.796e-07
2.44141e-04	1.973e-02	5.251e-03	1.287e-03	2.601e-04	3.175e-05	2.223e-06
6.10352e-05	1.972e-02	5.313e-03	1.364e-03	3.268e-04	6.535e-05	7.950e-06
1.52588e-05	1.971e-02	5.328e-03	1.384e-03	3.472e-04	8.225e-05	1.637e-05
3.81470e-06	1.971e-02	5.332e-03	1.390e-03	3.526e-04	8.741e-05	2.061e-05
9.53674e-07	1.971e-02	5.333e-03	1.391e-03	3.539e-04	8.876e-05	2.191e-05
2.38419e-07	1.971e-02	5.334e-03	1.391e-03	3.542e-04	8.911e-05	2.225e-05
5.96046e-08	1.971e-02	5.334e-03	1.391e-03	3.543e-04	8.919e-05	2.233e-05
1.49012e-08	1.971e-02	5.334e-03	1.391e-03	3.544e-04	8.922e-05	2.235e-05
3.72529e-09	1.971e-02	5.334e-03	1.391e-03	3.544e-04	8.922e-05	2.236e-05
9.31323e-10	1.971e-02	5.334e-03	1.391e-03	3.544e-04	8.922e-05	2.236e-05

Table 3.6.5: $|||\Pi^N u - u_N|||$ for $\tilde{A}$ Scheme

ε	N=8	16	32	64	128
1.56250e-02	2.80	3.16	3.10	3.14	3.64
3.90625e-03	2.23	2.99	3.73	3.76	3.42
9.76562e-04	1.98	2.28	3.02	3.82	3.95
2.44141e-04	1.91	2.03	2.31	3.03	3.84
6.10352e-05	1.89	1.96	2.06	2.32	3.04
1.52588e-05	1.89	1.94	2.00	2.08	2.33
3.81470e-06	1.89	1.94	1.98	2.01	2.08
9.53674e-07	1.89	1.94	1.97	2.00	2.02
2.38419e-07	1.89	1.94	1.97	1.99	2.00
5.96046e-08	1.89	1.94	1.97	1.99	2.00
1.49012e-08	1.89	1.94	1.97	1.99	2.00
3.72529e-09	1.89	1.94	1.97	1.99	2.00
9.31323e-10	1.89	1.94	1.97	1.99	2.00
R^N	1.89	1.94	1.97	1.99	2.00

Table 3.6.6: $|||\Pi^N u - u_N|||$ Convergence Rates for $\tilde{A}$ Scheme

Tables 3.6.4 and 3.6.6 demonstrate the same order uniform convergence of $\|\Pi^N u - u_N\|_1$ and $|||\Pi^N u - u_N|||$.

Example 3.6.2 *(m=2). Consider (3.6.1) with* $a(x) = 2 + \exp(x - 1)$, $b(x) =$

$2\exp(x-1)$ *and* $c(x) \equiv d(x) \equiv 0$, *where* $f(x)$ *is chosen so that the solution of (3.1.2) is*

$$u(x) = y(x) - \left(y'(0) + y(0) - y(1)\right) x(x-1)^2$$
$$- \left(y'(1) + y(0) - y(1)\right) x^2(x-1) - ((1-x)y(0) + xy(1)),$$

with $y(x) = \varepsilon \exp\left(\left(-3 + 2x + e^{x-1}\right)/\varepsilon\right)$.

This $u(x)$ exhibits typical boundary layer behaviour.

We denote by $\hat{A}$ the method (3.3.6) with the piecewise linear approximations

$$p^N(x) = \frac{x_i - x}{h_i} p(x_{i-1}) + \frac{x - x_{i-1}}{h_i} p(x_i),$$

for $x \in (x_{i-1}, x_i)$ and $i = 1, 2, \ldots, N$, where p can be a, b, c, d or f. For this scheme $l = 2$. We choose $\alpha = 2.99$ in (3.4.3).

ε	N=8	16	32	64	128	256
1.00000e+00	9.944e-05	2.067e-05	5.177e-06	1.301e-06	3.257e-07	8.140e-08
2.50000e-01	4.101e-03	2.777e-04	3.362e-05	8.467e-06	2.178e-06	5.491e-07
6.25000e-02	7.773e-03	1.410e-03	2.253e-04	3.041e-05	4.368e-06	8.504e-07
1.56250e-02	2.306e-02	1.900e-03	1.712e-04	2.571e-05	5.852e-06	1.443e-06
3.90625e-03	4.960e-02	5.751e-03	5.452e-04	5.364e-05	7.074e-06	1.622e-06
9.76562e-04	6.434e-02	1.172e-02	1.464e-03	1.489e-04	1.641e-05	2.177e-06
2.44141e-04	6.900e-02	1.510e-02	2.909e-03	3.710e-04	3.882e-05	4.509e-06
6.10352e-05	7.024e-02	1.618e-02	3.734e-03	7.285e-04	9.331e-05	9.886e-06
1.52588e-05	7.055e-02	1.647e-02	3.997e-03	9.330e-04	1.822e-04	2.335e-05
3.81470e-06	7.063e-02	1.654e-02	4.068e-03	9.985e-04	2.331e-04	4.546e-05
9.53674e-07	7.065e-02	1.656e-02	4.085e-03	1.016e-03	2.494e-04	5.814e-05
2.38419e-07	7.065e-02	1.656e-02	4.090e-03	1.020e-03	2.537e-04	6.220e-05
5.96046e-08	7.066e-02	1.657e-02	4.091e-03	1.022e-03	2.548e-04	6.328e-05
1.49012e-08	7.066e-02	1.657e-02	4.091e-03	1.022e-03	2.551e-04	6.356e-05
3.72529e-09	7.066e-02	1.657e-02	4.091e-03	1.022e-03	2.552e-04	6.362e-05
9.31323e-10	7.066e-02	1.657e-02	4.091e-03	1.022e-03	2.552e-04	6.364e-05

Table 3.6.7 : $\|u - u_N\|_{\infty,d}$ for $\hat{A}$ Scheme

ε	N=8	16	32	64	128
1.00000e+00	2.27	2.00	1.99	2.00	2.00
2.50000e-01	3.88	3.05	1.99	1.96	1.99
6.25000e-02	2.46	2.65	2.89	2.80	2.36
1.56250e-02	3.60	3.47	2.74	2.14	2.02
3.90625e-03	3.11	3.40	3.35	2.92	2.12
9.76562e-04	2.46	3.00	3.30	3.18	2.91
2.44141e-04	2.19	2.38	2.97	3.26	3.11
6.10352e-05	2.12	2.12	2.36	2.96	3.24
1.52588e-05	2.10	2.04	2.10	2.36	2.96
3.81470e-06	2.09	2.02	2.03	2.10	2.36
9.53674e-07	2.09	2.02	2.01	2.03	2.10
2.38419e-07	2.09	2.02	2.00	2.01	2.03
5.96046e-08	2.09	2.02	2.00	2.00	2.01
1.49012e-08	2.09	2.02	2.00	2.00	2.01
3.72529e-09	2.09	2.02	2.00	2.00	2.00
9.31323e-10	2.09	2.02	2.00	2.00	2.00
R^N	2.09	2.02	2.00	2.00	2.00

Table 3.6.8: $\|u - u_N\|_{\infty,d}$ Convergence Rates for $\hat{A}$ Scheme

Recalling Remark 3.5.4 and the inequalities

$$\|v\|_{\infty,d} \leq \|v\|_1 \leq |||v||| \quad \forall v \in V^N,$$

one can see from the rates of the first row of Table 3.6.8 that the exponent l in the bound of Theorem 3.5.1 is sharp in general.

However, we observed in all our numerical experiments with $m = 2$ that when piecewise constants are used to approximate the functions a, b, c, d and f (i.e., when $l = 1$), then $|||\Pi^N u - u_N|||$ is second order convergent, uniformly in ε. That is, it appears that when $l = 1$ one can replace l by $l + 1$ in the conclusion of Theorem 3.5.1.

In our last example we consider the second order problem (3.1.1). Many schemes have been proposed for this problem in the literature. We include results for it here in order to demonstrate that one may obtain a higher order of uniform convergence for $|||\Pi^N w - u_N|||$ than is implied by Theorem 3.5.1; see Theorem 3.5.3. We use piecewise linear approximations of a, b and f as described in Theorem 3.5.3. The resulting tridiagonal scheme can be written explicitly as

$$r_i^- u_N(x_{i-1}) + r_i^c u_N(x_i) + r_i^+ u_N(x_{i+1}) = q_i, \quad \text{for } i = 1, \ldots, N-1 \tag{3.6.2a}$$

$$u_N(x_0) = u_N(x_N) = 0, \tag{3.6.2b}$$

where

$$\begin{aligned}
r_i^- &= -\varepsilon h_i^{-1} - (a_{i-1} + 2a_i)/6 + h_i (b_{i-1} + b_i)/12, \\
r_i^+ &= -\varepsilon h_{i+1}^{-1} + (2a_i + a_{i+1})/6 + h_i (b_i + b_{i+1})/12, \\
r_i^c &= -\left(r_i^- + r_i^+\right) + (h_i b_{i-1} + 2(h_i + h_{i+1}) b_i + h_{i+1} b_{i+1})/6, \\
q_i &= (h_i f_{i-1} + 2(h_i + h_{i+1}) f_i + h_{i+1} f_{i+1})/6,
\end{aligned}$$

for $i = 1, \ldots, N-1$.

Example 3.6.3 *(m=1). Consider (3.1.1) with* $a(x) = 5 - \sin(1-x)$, $b(x) = \cos(1-x)$ *and* $f(x)$ *chosen such that*

$$\begin{aligned}
w(x) = \exp\left((-4 + 5x - \cos(1-x))/\varepsilon\right) + (1+x)^4 \\
-17x - \left(\exp\left((-4 - \cos(1))/\varepsilon\right) + 1\right)(1-x).
\end{aligned}$$

In this case we define the discrete energy norm $|||\cdot|||_d$ to be

$$|||v|||_d = \left\{ \varepsilon \sum_{i=1}^{N} h_i^{-1} (v_i - v_{i-1})^2 + \sum_{i=1}^{N-1} \bar{h}_i v_i^2 \right\}^{1/2}$$

for all $v = \sum_{i=1}^{N-1} v_i \varphi_i^0(x) \in V^N$. It can be shown that the discrete energy norm $|||\cdot|||_d$ is equivalent to $|||\cdot|||$ on V^N.

ε	N=64	128	256	512	1024	2048
2.50000e-01	8.896e-03	2.238e-03	5.604e-04	1.413e-04	4.329e-05	8.479e-07
6.25000e-02	2.385e-02	8.388e-03	2.785e-03	8.893e-04	2.771e-04	7.169e-05
1.56250e-02	2.329e-02	8.207e-03	2.730e-03	8.732e-04	2.717e-04	8.184e-05
3.90625e-03	2.318e-02	8.168e-03	2.718e-03	8.696e-04	2.704e-04	8.164e-05
9.76562e-04	2.340e-02	8.161e-03	2.715e-03	8.688e-04	2.702e-04	8.155e-05
2.44141e-04	2.421e-02	8.221e-03	2.715e-03	8.686e-04	2.702e-04	8.208e-05
6.10352e-05	2.468e-02	8.433e-03	2.730e-03	8.686e-04	2.701e-04	8.208e-05
1.52588e-05	2.483e-02	8.556e-03	2.784e-03	8.726e-04	2.701e-04	8.340e-05
3.81470e-06	2.487e-02	8.594e-03	2.816e-03	8.863e-04	2.711e-04	8.173e-05
9.53674e-07	2.488e-02	8.605e-03	2.826e-03	8.943e-04	2.746e-04	8.271e-05
2.38419e-07	2.488e-02	8.607e-03	2.828e-03	8.967e-04	2.766e-04	8.342e-05
5.96046e-08	2.488e-02	8.608e-03	2.829e-03	8.974e-04	2.772e-04	8.352e-05

Table 3.6.9: $|||\Pi^N w - u_N|||_d$ for Scheme (3.6.2)

ε	64	128	256	512	1024
2.50000e-01	1.99	2.00	1.99	1.71	5.67
6.25000e-02	1.51	1.59	1.65	1.68	1.95
1.56250e-02	1.50	1.59	1.64	1.68	1.73
3.90625e-03	1.50	1.59	1.64	1.69	1.73
9.76562e-04	1.52	1.59	1.64	1.69	1.73
2.44141e-04	1.56	1.60	1.64	1.68	1.72
6.10352e-05	1.55	1.63	1.65	1.69	1.72
1.52588e-05	1.54	1.62	1.67	1.69	1.70
3.81470e-06	1.53	1.61	1.67	1.71	1.73
9.53674e-07	1.53	1.61	1.66	1.70	1.73
2.38419e-07	1.53	1.61	1.66	1.70	1.73
5.96046e-08	1.53	1.61	1.66	1.69	1.73
R^N	1.53	1.61	1.66	1.69	1.73

Table 3.6.10: $|||\Pi^N w - u_N|||_d$ Convergence Rates for Scheme (3.6.2)

The experimental rates in Table 3.6.10 verify Theorem 3.5.3 and are consistent with Remark 3.5.6.

Chapter 4

Interior Turning Point Problems

4.1 Introduction

Consider the singularly perturbed two–point boundary value problems

$$L_\varepsilon u \equiv -\varepsilon u'' + x^k b(x)u' + d(x)u = f(x), \quad \text{for } x \in (-1,1), \tag{4.1.1a}$$

$$u(-1) = u(1) = 0, \tag{4.1.1b}$$

with a small parameter $\varepsilon \in (0,1]$ and k a positive integer. These problems arise in modeling the flow of a viscous fluid between two coaxial rotating disks; see Smith [38], Section 8.5.

We assume that b, d and f are sufficiently smooth on $[-1,1]$ and satisfy, for $x \in [-1,1]$,

$$|b(x)| > \beta > 0, \tag{4.1.1c}$$

$$d(x) \geq 0 \quad \text{and} \quad d(0) > 0. \tag{4.1.1d}$$

Condition (4.1.1d) guarantees that the operator L_ε is inverse monotone on $[-1,1]$. From this it follows that (4.1.1) has a unique solution $u(x)$. Since the coefficient of the first derivative vanishes only at $x = 0$ (by (4.1.1c)), problems (4.1.1) have an isolated turning point at $x = 0$. If $k = 1$ the turning point is said to be simple. When $k \geq 2$ it is called a multiple turning point. In the case $b > 0$, the turning point is said to be repulsive and in the case $b < 0$ it is said to be attractive. In what follows we shall denote by $P_k^{\pm}$ the problems (4.1.1), where the superscript is the sign of $b(\cdot)$.

It is well known that the problems $P_k^{\pm}$ may exhibit boundary layers of exponential type or an interior layer of cusp type. The nature of these layers depends on the value of k and the sign of b; see Section 4.2. Special methods must be designed to obtain an accurate numerical solution for $P_k^{\pm}$ without introducing an excessive number of meshpoints. This leads naturally to the consideration of numerical methods which are convergent, uniformly in the parameter ε, in some norm. Finite difference methods for $P_k^{\pm}$ have been extensively considered. Berger, Han and Kellogg [4] applied a modified El–Mistikawy and Werle scheme to the problem $P_1^{\pm}$, with $d(x) > 0$ for $x \in [-1, 1]$. They proved that this scheme is uniformly convergent of order $N^{-\min\{\lambda,1\}}$ in the $L^{\infty}[-1,1]$ norm for P_1^{-}, where $\lambda = d(0)/b(0)$, provided that $\lambda \neq 1$ (when $\lambda = 1$ they obtain order $N^{-1} \ln N$). An improved uniform convergence rate of N^{-1} was obtained by Farrell and Gartland [13], using a scheme involving parabolic cylinder functions. The same problem was considered in Farrell [12], where sufficient conditions for uniform convergence in the discrete L^{∞} norm on an equidistant mesh were investigated. Lin and Sun [25] constructed an exponentially fitted scheme for the problem P_1^{+}, which they proved to be uniformly convergent of order N^{-2}. All of these discretizations use equidistant meshes. The schemes are quite complicated. Clavero and Lisbona [8] consider a family of finite difference schemes, which includes the upwinded scheme, Samarskii scheme and exponentially fitted schemes, for the problem P_1^- with $0 < \lambda < 1$. They showed that on a locally quasi-equidistant mesh the family of schemes is uniformly convergent of order $N^{-\lambda}$ in the discrete maximum norm. Vulanović [47] applied a variation of the Gushchin–Shchennikov scheme, on a special graded mesh, to a simple boundary turning point problem. He proved that the scheme is uniformly convergent of order N^{-2} in the discrete L^∞ norm.

In contrast, there are few results on finite element methods for turning point problems. Stynes and O'Riordan [42] examined problems with arbitrary turning points. In [42], finite element methods, based on an approximate L-spline trial space and an approximate L^*-spline test space, are constructed and proved to be uniformly convergent in a weighted energy norm. Once again, the difference scheme generated is somewhat complicated.

In this chapter, we generate and analyse Galerkin finite element methods for the problems $P_k^\pm$. These methods use piecewise linear functions with special piecewise equidistant discretization meshes. Shishkin meshes [37] are used to handle boundary layers of exponential type. Such layers are the only source of difficulty in the problems $P_k^\pm$, with the exception of P_1^- (see Lemmas 4.2.2 and 4.2.3). The simple attractive turning point problem P_1^- does not have any boundary layers of exponential type but rather an internal layer of cusp type. The interior layer is essentially a Weber parabolic cylinder function. It is not clear how to construct an ordinary Shishkin mesh for the problem P_1^-. We therefore introduce a mesh which is a generalization of Shishkin's. This mesh is equidistant in each of $O(\ln N)$ subintervals. Due to the piecewise equidistance, the meshes used in this chapter are simpler than the Bahkvalov-type mesh used by Vulanović [47].

Our difference schemes are similar to the classical central difference scheme. They do not satisfy a discrete maximum principle. We shall analyse our methods in a framework similar to that of Stynes and O'Riordan [43]. The methods are shown to be uniformly convergent of order $N^{-1}\ln N$ in a weighted energy norm associated with (4.1.1a) and order $(N^{-1}\ln N)^{3/2}$ in the L^2 norm.

In Section 4.2, we present *a priori* estimates for the continuous problems $P_k^{\pm}$. In Section 4.3, Galerkin finite methods are constructed on an arbitrary mesh for the problems $P_k^{\pm}$. Uniform convergence results on the piecewise equidistant Shishkin mesh are given in Section 4.4 for those problems $P_k^{\pm}$ with boundary layers of exponential type. In Section 4.5, we introduce a more general piecewise equidistant mesh. On this mesh, uniform convergence is obtained for the simple attractive turning point problem P_1^-. Section 4.6 gives numerical results.

4.2 The Continuous Problems

In this section we discuss those properties of (4.1.1) and of its solution u which we shall need later for the analysis of our finite element method.

Set $a(x) = x^k b(x)$. First we show that, if we have $d(0) - \frac{1}{2}a'(0) > 0$, then we can deduce an inequality needed later to show that certain bilinear forms associated with the operator L_ϵ are coercive. The proof generalizes an idea of Stynes and O'Riordan [43].

Lemma 4.2.1 *Suppose that $d(0) - \frac{1}{2}a'(0) > 0$. Then without loss of generality, we may assume that there exists $C_1 > 0$ such that for $x \in [-1,1]$ we have*

$$\left(d - \frac{1}{2}a'\right)(x) \geq 2C_1. \qquad (4.2.1)$$

Proof. Set $\gamma = \min_{x\in[-1,1]} \left(d - \frac{1}{2}a'\right)(x)$. If $\gamma > 0$ then we are done, so suppose that $\gamma \leq 0$.

Since $d(0) - \frac{1}{2}a'(0) > 0$, there exist δ, $m > 0$ (both independent of ε) such that

$$\left(d - \frac{1}{2}a'\right)(x) \geq m, \quad \text{for } x \in [-\delta, \delta]. \tag{4.2.2}$$

Let $\beta_1 = \beta \operatorname{sgn} b$, where β is given by (4.1.1d) and $\operatorname{sgn} b = b/|b|$. Without loss of generality, we can assume that ε is so small that

$$\delta^{4k}\beta_1^2 + 4\varepsilon(\gamma - 1) > 0.$$

Set

$$\begin{aligned}\tau &= \frac{\delta^{2k}\beta_1 - (\operatorname{sgn} b)\sqrt{\delta^{4k}\beta_1^2 + 4\varepsilon(\gamma - 1)}}{2\varepsilon(k+1)} \\ &= \frac{2(1-\gamma)}{(k+1)\left(\delta^{2k}\beta_1 + (\operatorname{sgn} b)\sqrt{\delta^{4k}\beta_1^2 + \varepsilon(\gamma - 1)}\right)}.\end{aligned}$$

Then τ satisfies

$$0 < \tau \operatorname{sgn} b \leq C \tag{4.2.3}$$

and

$$-\varepsilon(k+1)^2\tau^2 + (k+1)\delta^{2k}\beta_1\tau + \gamma - 1 = 0. \tag{4.2.4}$$

Consider the differential operator L_τ defined by

$$L_\tau z(x) \equiv -\varepsilon z''(x) + \tilde{a}(x)z'(x) + \tilde{d}(x)z(x).$$

Here

$$\begin{aligned}\tilde{a}(x) &= a(x) - 2\varepsilon(k+1)\tau x^k \\ &= x^k(b(x) - 2\varepsilon(k+1)\tau) \\ &= x^k\tilde{b}(x),\end{aligned}$$

say, and

$$\tilde{d}(x) = d(x) - \varepsilon k(k+1)\tau x^{k-1} - \varepsilon(k+1)^2\tau^2 x^{2k} + (k+1)\tau x^{2k} b(x).$$

It is easy to see that, for $x \in [-1,1]$ and ε sufficiently small,

$$\left|\tilde{b}(x)\right| \geq \beta/2 > 0,$$

$$\tilde{d}(x) \geq 0 \text{ and } \tilde{d}(0) > 0,$$

by (4.2.3). Also for the operator L_τ,

$$\left(\tilde{d} - \frac{1}{2}\tilde{a}'\right)(x) = -\varepsilon(k+1)^2\tau^2 x^{2k} + (k+1)\tau x^{2k} b(x) + \left(d - \frac{1}{2}a'\right)(x).$$

We show that $\left(\tilde{d} - \frac{1}{2}\tilde{a}'\right)(x) \geq 2C_1$ on $[-1,1]$. We discuss two cases.

Case 1: If $x \in [-\delta, \delta]$, then from (4.2.2) and (4.2.3)

$$\begin{aligned}\left(\tilde{d} - \frac{1}{2}\tilde{a}'\right)(x) &\geq -\varepsilon(k+1)^2\tau^2 + m \\ &\geq m/2,\end{aligned}$$

for ε sufficiently small.

Case 2: If $x \in [-1,1] \setminus [-\delta, \delta]$, then using the definition of β_1 and (4.2.3),

$$\begin{aligned}\left(\tilde{d} - \frac{1}{2}\tilde{a}'\right)(x) &\geq -\varepsilon(k+1)^2\tau^2 + (k+1)\delta^{2k}\beta_1\tau + \gamma \\ &= 1,\end{aligned}$$

by (4.2.4).

That is, L_τ satisfies the conditions we would like L_ε to satisfy, with $2C_1 = \min\{m/2, 1\}$. An easy computation shows that

$$L_\tau\left(e^{-\tau x^{k+1}} u(x)\right) = e^{-\tau x^{k+1}} L_\varepsilon u(x) = e^{-\tau x^{k+1}} f(x),$$

so we can now work with the problem $L_\tau(v(x)) = e^{-\tau x^{k+1}} f(x)$, which has the required properties, then transform our results back to (4.1.1) by means of $u(x) = e^{\tau x^{k+1}} v(x)$. By (4.2.3) above, this transformation will at worst scale all quantities by a factor C and so will not alter the orders of convergence in our results. □

Remark 4.2.1 *Note that*

$$d(x) - \frac{1}{2}a'(x) = d(x) - \frac{1}{2}kx^{k-1}b(x) - \frac{1}{2}x^k b'(x).$$

Hence (4.1.1d) implies that the condition $d(0) - \frac{1}{2}a'(0) > 0$ of Lemma 4.2.2 holds for all problems $P_k^\pm$ except possibly the simple repulsive turning point problem P_1^+.

Assumption 4.2.1 *From now on, we shall assume that (4.2.1) is satisfied in addition to (4.1.1a) - (4.1.1d). By Remark 4.2.1 and Lemma 4.2.1, this assumption is restrictive only for the problem P_1^+.*

To construct our special meshes and analyse errors of the finite element schemes, we need *a priori* estimates of the solution $u(x)$ and its derivatives. The boundary or interior layer behaviour of the solutions depends not only on the sign of b, but also on whether k is even or odd.

Lemma 4.2.2 *Fix k and b. Let $u(x)$ be the solution of (4.1.1). Then for $x \in [-1, 1]$ and $j = 0, 1, \ldots,$*

(i) for P_k^+ with k even,

$$\left|u^{(j)}(x)\right| \le C\left(1 + \varepsilon^{-j}\exp(-\beta(1-x)/\varepsilon)\right); \tag{4.2.5}$$

(ii) for P_k^- with k even,

$$\left|u^{(j)}(x)\right| \le C\left(1 + \varepsilon^{-j}\exp(-\beta(1+x)/\varepsilon)\right); \tag{4.2.6}$$

(iii) for P_k^+ with k odd,

$$\left|u^{(j)}(x)\right| \leq C\left(1+\varepsilon^{-j}\exp(-\beta(1-x)/\varepsilon)+\varepsilon^{-j}\exp(-\beta(1+x)/\varepsilon)\right); \qquad (4.2.7)$$

(iv) for P_k^- with odd $k \geq 3$,

$$\left|u^{(j)}(x)\right| \leq C. \qquad (4.2.8)$$

Proof. See Vulanović and Farrell [50]. □

From the above bounds, one can see that in case (iv) no layer is present in the solution and cases (i) – (iii) exhibit one or two boundary layers of exponential type. A numerical method which is suitable for all four cases will be given in Section 4.4.

The solution of the simple attractive turning point problem P_1^- behaves very differently from the cases listed in Lemma 4.2.2. Set $\lambda = d(0)/|b(0)|$. Then $\lambda > 0$. We have

Lemma 4.2.3 *There exists a constant C, which is independent of ε, such that the solution $u(x)$ of problem P_1^- satisfies*

$$\left|u^{(j)}(x)\right| \leq \begin{cases} C\left(1+\left(|x|+\varepsilon^{1/2}\right)^{\lambda-j}\right), & \text{if } \lambda \text{ is not an integer}, \qquad (4.2.9) \\ C\left(1+\left(|x|+\varepsilon^{1/2}\right)^{\lambda-j}\ln\frac{1}{|x|+\varepsilon^{1/2}}\right), & \text{if } \lambda \text{ is an integer}, \qquad (4.2.10) \end{cases}$$

for $x \in (-1,1)$ and $j = 1,2,\ldots$.

Proof. Let $\lambda = m + \Delta$ where m is a non-negative integer and $0 < \Delta \leq 1$. Under the assumption that $d(x) > 0$ for $x \in [-1,1]$, Berger at al. [4] showed that the solution $u(x)$ of problem P_1^- satisfies (4.2.9) and (4.2.10).

For the slightly weaker assumption (4.1.1d), one can obtain the same estimates by applying the result of [4] to the problem

$$-\varepsilon y'' + x^k b(x)y' + d(x)y = f(x), \quad \text{for } x \in (-\delta,\delta),$$

$$y(-\delta) = u(-\delta), \quad y(\delta) = u(\delta),$$

where $\delta \in (0,1)$ is chosen such that $d(x) > 0$, for $x \in [-\delta, \delta]$. □

We remark that a simpler proof of Lemma 4.2.3 for the case $0 < \lambda < 1$ is given by Clavero and Lisbona [8].

From Lemma 4.2.3, one can see that the solution of P_1^- has an internal layer at $x = 0$. If λ becomes smaller, then the solution is more badly behaved. In Section 4.5, we shall discuss a uniformly convergent numerical method for the most difficult case $0 < \lambda < 1$.

4.3 A Galerkin Finite Element Method on an Arbitrary Mesh

In this section, we begin to analyse a Galerkin finite element method for the problems $P_k^{\pm}$. Let us first work with an arbitrary mesh

$$X^N : -1 = x_L < x_{L+1} < \ldots < x_{R-1} < x_R = 1.$$

Set $h_i = x_i - x_{i-1}$ for $i = L+1, \ldots, R$, with $H = \max_i h_i$.

Define the standard piecewise linear basis function φ_i by

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/h_i, & \text{for } x \in (x_{i-1}, x_i), \\ (x_{i+1} - x)/h_{i+1}, & \text{for } x \in (x_i, x_{i+1}), \\ 0, & \text{elsewhere,} \end{cases}$$

for $i = L+1, \ldots, R-1$. Our trial and test spaces S^N are taken to be the linear span of $\{\varphi_i : i = L+1, \ldots, R-1\}$.

Let $(\cdot,\cdot)$ denote the usual $L^2[-1,1]$ inner product. Denote the $L^2[-1,1]$ norm by $\|\cdot\|$. Our weighted energy norm is defined by

$$|||v||| = \{\varepsilon\|v'\|^2 + \|v\|^2\}^{1/2},$$

for all $v \in H_0^1(-1,1)$. Set

$$B_\varepsilon(v,w) = (\varepsilon v', w') + (av', w) + (dv, w),$$

for all $v, w \in H_0^1(-1,1)$. Recall Assumption 4.2.1. A standard argument shows that the bilinear form $B_\varepsilon(\cdot,\cdot)$ is uniformly coercive over $H_0^1(-1,1) \times H_0^1(-1,1)$ in terms of $|||\cdot|||$, i.e., that there exists $C > 0$ such that

$$C|||v|||^2 \leq B_\varepsilon(v,v), \tag{4.3.1}$$

for all $v \in H_0^1(-1,1)$.

We now define our weak formulation of (4.1.1): find $u \in H_0^1(-1,1)$ such that

$$B_\varepsilon(u,v) = (f,v), \quad \text{for all } v \in H_0^1(-1,1). \tag{4.3.2}$$

Clearly (4.3.2) has a unique solution $u(x)$ in $H_0^1(-1,1)$. This weak solution is also the classical solution of (4.1.1) when all the data are smooth.

Let p denote b, d or f. We denote by $\hat{p}$ the piecewise linear interpolant $\hat{p}$ to p on $[-1,1]$, defined by

$$\hat{p}(x) = \frac{x_i - x}{h_i} p(x_{i-1}) + \frac{x - x_{i-1}}{h_i} p(x_i), \tag{4.3.3}$$

for $x \in (x_{i-1}, x_i)$ and $i = L+1, \ldots, R$. Our modified bilinear form is given by

$$\hat{B}(v,w) = (\varepsilon v', w') + (\hat{a}v', w) + (\hat{d}v, w),$$

for all v and $w \in H_0^1(-1,1)$, where $\hat{a}(x) = x^k \hat{b}(x)$.

We begin the analysis by showing that the bilinear form $\hat{B}(\cdot,\cdot)$ is uniformly coercive over $H_0^1(-1,1) \times H_0^1(-1,1)$.

Lemma 4.3.1 *For H sufficiently small, independently of ε, we have*

$$C|||v|||^2 \le \hat{B}_\varepsilon(v,v),$$

for all $v \in H_0^1(-1,1)$.

Proof. For each $v \in H_0^1(-1,1)$,

$$\hat{B}(v,v) = (\varepsilon v', v') + (\hat{a}v', v) + (\hat{d}v, v).$$

Taking the second term of this, we have

$$\begin{aligned}
(\hat{a}v', v) &= \sum_{i=L+1}^{R} \int_{x_{i-1}}^{x_i} \hat{a}v'v\,dx \\
&= \sum_{i=L+1}^{R} \left(\frac{1}{2}\hat{a}v^2 \Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} \frac{1}{2}(\hat{a})'\,v^2\,dx \right) \\
&= \sum_{i=L+1}^{R} \int_{x_{i-1}}^{x_i} \left(-\frac{1}{2}a'(x_i) + O(H) \right) v^2\,dx.
\end{aligned}$$

Also

$$\left(\hat{d}v, v\right) = \sum_{i=L+1}^{R} \int_{x_{i-1}}^{x_i} (d(x_i) + O(H))\,v^2\,dx.$$

Hence

$$\begin{aligned}
\hat{B}(v,v) &\ge \varepsilon\|v'\|^2 + \sum_{i=L+1}^{R} \int_{x_{i-1}}^{x_i} \left(d(x_i) - \frac{1}{2}a'(x_i) + O(H) \right) v^2\,dx \\
&\ge \varepsilon\|v'\|^2 + C_1\|v\|^2,
\end{aligned}$$

for H sufficiently small, by (4.2.1). This implies the result. □

Our discrete solution $u_N \in S^N$ is defined by

$$\hat{B}(u_N, \varphi_i) = (\hat{f}, \varphi_i), \quad \text{for } i = L+1, \ldots, R-1. \tag{4.3.4}$$

It follows from Lemma 4.3.1 that the solution u_N of (4.3.4) is well defined.

When $k = 1$, (4.3.4) is given explicitly by the three-point scheme

$$r_i^- u_N(x_{i-1}) + r_i^c u_N(x_i) + r_i^+ u_N(x_{i+1}) = q_i, \quad \text{for } i = L+1, \ldots, R-1,$$

$$u_N(x_0) = u_N(x_N) = 0,$$

where

$$\begin{aligned}
r_i^- &= -\frac{\varepsilon}{h_i} + \left(\frac{x_i}{6} - \frac{h_i}{12}\right) b_{i-1} + \left(\frac{x_i}{3} - \frac{h_i}{12}\right) b_i + h_i \left(d_{i-1} + d_i\right)/12, \\
r_i^+ &= -\frac{\varepsilon}{h_{i+1}} - \left(\frac{x_i}{3} + \frac{h_{i+1}}{12}\right) b_i - \left(\frac{x_i}{6} - \frac{h_{i+1}}{12}\right) b_{i+1} + h_i \left(d_i + d_{i+1}\right)/12, \\
r_i^c &= -\left(r_i^- + r_i^+\right) + \left(h_i d_{i-1} + 2\left(h_i + h_{i+1}\right) d_i + h_{i+1} d_{i+1}\right)/6, \\
q_i &= \left(h_i f_{i-1} + 2\left(h_i + h_{i+1}\right) f_i + h_{i+1} f_{i+1}\right)/6,
\end{aligned}$$

for $i = L+1, \ldots, R-1$. On general meshes this scheme does not always satisfy a discrete maximum principle. We shall prove that on certain special meshes the scheme (4.3.4) is uniformly convergent in the weighted energy norm $|||\cdot|||$.

4.4 The Shishkin Mesh for Boundary Layers of Exponential Type

In this section, we first introduce a Shishkin mesh. We then present uniform convergence results in the weighted energy norm $|||\cdot|||$ and the L^2 norm for those problems $P_k^{\pm}$ which have boundary layers of exponential type (by Lemma 4.2.2, these are all problems $P_k^{\pm}$ except P_1^-).

Consider the simple repulsive turning point problem P_1^+. The solution has boundary layers of exponential type at both end points $x = -1$ and $x = 1$.

We shall describe the mesh on the subinterval $[0, 1]$, then on $[-1, 0]$ the mesh is constructed by symmetry about $x = 0$. Given an even positive integer N, the

Shishkin mesh X_s^N is constructed by dividing the interval $[0,1]$ into two subintervals

$$[0, 1-\sigma], \quad \text{and} \quad [1-\sigma, 1].$$

Equidistant meshes with $1 + N/2$ points are then used on each of these subintervals. The parameter σ is defined by

$$\sigma = \min\{1/2, \quad 2\beta^{-1}\varepsilon \ln N\},$$

which depends on ε and N. More precisely, we have

$$X_s^N : 0 = x_0 < x_1 < \cdots < x_{i_0} < \cdots < x_{N-1} < x_N = 1,$$

with $i_0 = N/2$ and $x_{i_0} = 1 - \sigma$. The mesh spacing is given by

$$h_i = 2(1-\sigma)N^{-1}, \quad \text{for } i = 0, \ldots, i_0$$

and

$$h_i = 2\sigma N^{-1}, \quad \text{for } i = i_0 + 1, \ldots, N.$$

Theorem 4.4.1 *Let u be the solution of P_1^+. Let $u_N \in S^N$ be the solution of (4.3.4) for the problem P_1^+ on the Shishkin mesh X_s^N. Then for N sufficiently large (independently of ε), we have*

$$|||u - u_N||| \le CN^{-1} \ln N$$

and

$$\|u - u_N\| \le CN^{-3/2}.$$

Proof. Consider the convection–diffusion problems

$$-\varepsilon y_1'' + xb(x)y_1' + d(x)y_1 = f(x), \quad \text{for } x \in (-1, -1/2),$$

$$y_1(-1) = 0, \quad y_1(-1/2) = u(-1/2),$$

and

$$-\varepsilon y_2'' + xb(x)y_2' + d(x)y_2 = f(x), \quad \text{for } x \in (1/2, 1),$$

$$y_2(1/2) = u(1/2), \quad y_2(1) = 0.$$

Recall (4.2.7). We have $|y_1(-1/2)| \le C$ and $|y_2(1/2)| \le C$. Hence, from Gartland [17], Theorem 1.4, $y_1(x)$ and $y_2(x)$ respectively admit the decompositions

$$y_1(x) = G_1(x) + E_1(x), \quad \text{for } x \in (-1, -1/2)$$

and

$$y_2(x) = G_2(x) + E_2(x), \quad \text{for } x \in (1/2, 1).$$

Here

$$\left|G_1^{(j)}(x)\right| \le C, \quad \text{for } x \in (-1, -1/2),$$

$$\left|E_1^{(j)}(x)\right| \le C\varepsilon^{-j}\exp(-\beta(1+x)/\varepsilon), \quad \text{for } x \in (-1, -1/2),$$

$$\left|G_2^{(j)}(x)\right| \le C, \quad \text{for } x \in (1/2, 1),$$

$$\left|E_2^{(j)}(x)\right| \le C\varepsilon^{-j}\exp(-\beta(1-x)/\varepsilon), \quad \text{for } x \in (1/2, 1),$$

for $j = 0, 1, \ldots$. It is easy to see that $u(x) = y_1(x)$ for $x \in [-1, -1/2]$ and $u(x) = y_2(x)$ for $x \in [1/2, 1]$. Therefore,

$$u(x) = \begin{cases} G_1(x) + E_1(x), & \text{for } x \in (-1, -1/2), \\ u(x), & \text{for } x \in [-1/2, 1/2], \\ G_2(x) + E_2(x), & \text{for } x \in (1/2, 1), \end{cases}$$

where

$$\left|u^{(j)}(x)\right| \le C, \quad \text{for } x \in (-1/2, 1/2),$$

for $j = 0, 1, \ldots$, by Lemma 4.2.2.

Using this decomposition with arguments very similar to those of Chapter 3, one can get the desired estimates. □

Analogous results may be obtained for the multiple turning point problems $P_k^\pm$ with $k \geq 2$.

4.5 The Simple Attractive Turning Point Problem P_1^-

We now design a piecewise equidistant mesh on which we apply the method (4.3.4) to the simple attractive turning point problem P_1^-. We shall assume that $0 < \lambda < 1$; as we saw in Sections 4.1 and 4.2, this is the most difficult case. Then (4.2.9) can be written as

$$\left|u^{(j)}(x)\right| \leq C\left(|x| + \varepsilon^{1/2}\right)^{\lambda - j}, \tag{4.5.1}$$

for $x \in (-1, 1)$ and $j = 1, 2, \ldots$. Uniform convergence results are proved in the weighted energy norm $|||\cdot|||$ and the L^2 norm.

4.5.1 The Mesh

The behaviour of the internal layer of problem P_1^- is quite different from that of the boundary layers which occur in the other $P_k^\pm$. The main difference is that the layer of cusp type is "much" wider than $O(\varepsilon)$. In fact as ε varies, $|u'| \leq C$ is not guaranteed on $[-\varepsilon^\theta, \varepsilon^\theta]$ for any fixed positive constant θ. It is not clear how to construct a Shishkin mesh of the usual type for this problem. Instead, we introduce a special mesh which is equidistant on each of $O(\ln N)$ subintervals; this is a generalization of the standard Shishkin approach. We shall again describe the mesh on $[0, 1]$ only, since it is symmetric about $x = 0$.

For any $\varepsilon \in (0, 1]$ and given a positive integer N, set

$$\sigma = \max\left\{\varepsilon^{\frac{1}{2}(1-\frac{\lambda}{2})},\ N^{-3}\right\} \tag{4.5.2}$$

and

$$K = \operatorname{int}\left(1 - \frac{\ln \sigma}{\ln 10}\right), \tag{4.5.3}$$

where $\operatorname{int}(z)$ denotes the largest integer j which satisfies $j \le z$. The interval (0, 1] is divided into $K + 1$ subintervals:

$$(0, 10^{-K}],\quad (10^{-K}, 10^{-K+1}], \ldots, (10^{-1}, 1].$$

The closure of each of these subintervals is then partitioned by an equidistant mesh containing $1 + \operatorname{int}\left(\frac{N}{K+1}\right)$ points. We shall refer to this mesh as X_K^N.

From (4.5.3), it can easily be seen that

$$K + 1 \le 2 + \min\left\{-\frac{1}{2}\left(1 - \frac{\lambda}{2}\right)\frac{\ln \varepsilon}{\ln 10},\ 3\frac{\ln N}{\ln 10}\right\}. \tag{4.5.4}$$

We shall assume that $N \ge 4$; then (4.5.4) implies that $K + 1 \le N$. For convenience, it will also be assumed that $\operatorname{int}\left(\frac{N}{K+1}\right) = \frac{N}{K+1}$. Let $n = \frac{N}{K+1}$. Then the meshpoints on $[0, 10^{-K}]$ are given by

$$x_i = (K + 1)10^{-K}N^{-1}i, \quad \text{for } i = 0, 1, \ldots, n.$$

For $j = 1, \ldots, K$, the meshpoints on $(10^{-j}, 10^{-j+1}]$ are defined by

$$x_i = 10^{-j} + 9(K + 1)10^{-j}N^{-1}(i - (K - j + 1)n),$$

for $i = (K - j + 1)n + 1, \ldots, (K - j + 2)n$. It is obvious that the mesh spacing satisfies

$$h_i = (K + 1)10^{-K}N^{-1}, \quad \text{for } x_i \in (0, 10^{-K}] \tag{4.5.5}$$

and

$$h_i = 9(K+1)10^{-j}N^{-1}, \quad \text{for } x_i \in (10^{-j}, 10^{-j+1}] \text{ and } j = 1, \ldots, K. \tag{4.5.6}$$

Also from (4.5.3), we have

$$10^{-1}\sigma \leq 10^{-K} \leq \sigma. \tag{4.5.7}$$

It is clear from (4.5.4) that

$$K + 1 \leq C \ln N. \tag{4.5.8}$$

This inequality will be used frequently in our analysis.

4.5.2 Analysis of Convergence

Since the bilinear form $B_\varepsilon(\cdot,\cdot)$ is not uniformly bounded in terms of the energy norm $|||\cdot|||$, a classical finite element approach does not satisfactorily analyse the error in the computed solution u_N. We shall employ an analysis similar to that of Stynes and O'Riordan [43] to prove that the method (4.3.4) for P_1^- is uniformly convergent of order $N^{-1}\ln N$ with respect to $|||\cdot|||$ and of order $\left(N^{-1}\ln N\right)^{3/2}$ with respect to $\|\cdot\|$ on the mesh X_K^N. (It does not seem possible to use an Aubin–Nitsche approach to get this higher order in $\|\cdot\|$.)

In what follows, the analysis is performed only on the interval $[0,1]$. The interval $[-1,0]$ can be handled similarly. We shall denote by $u_I \in S^N$ the interpolant to u at each node x_i of an arbitrary mesh X^N. The notation $(\cdot,\cdot)^{\tilde{}}$ denotes that the integration in $(\cdot,\cdot)$ is only over $[-1,1]\setminus X^N$.

We first give some relationships between interpolation errors in different norms on an arbitrary mesh.

Lemma 4.5.1 *Let u be the solution of problem P_1^-. Then on any arbitrary mesh we have*

$$|||u - u_I|||^2 \le C\|u - u_I\| \tag{4.5.9}$$

and

$$\int_{-1}^{1} \left(x(u - u_I)'(x)\right)^2 \, dx \le C\|u - u_I\|. \tag{4.5.10}$$

Proof. We first prove (4.5.9). Integrating by parts, we get, using (4.1.1a) and $u_I'' \equiv 0$ on each subinterval (x_{i-1}, x_i),

$$\begin{aligned} B_\varepsilon(u - u_I, u - u_I) &= \left(-\varepsilon(u - u_I)'' + a(u - u_I)' + d(u - u_I), u - u_I\right) \\ &= \left(f - au_I' + du_I, u - u_I\right) \\ &\le \|f - au_I' + du_I\| \cdot \|u - u_I\|. \end{aligned} \tag{4.5.11}$$

Since $\|u\|_{L^\infty} \le C$ by Lemma 4.2.3,

$$\|f - du_I\| \le C. \tag{4.5.12}$$

To bound $\|au_I'\|$, suppose that $x \in (x_{i-1}, x_i)$, for some fixed $i \in \{1, \ldots, N\}$. From Lemma 4.2.3,

$$|tu'(t)| \le C, \quad \text{for all } t \in [-1, 1]. \tag{4.5.13}$$

Now

$$u(x_i) - u(x_{i-1}) = h_i u'(\theta_i) \quad \text{for some } \theta_i \in (x_{i-1}, x_i).$$

Hence

$$\begin{aligned} |xu_I'(x)| &= x\frac{|u(x_i) - u(x_{i-1})|}{h_i} \\ &= (x - \theta_i)\frac{|u(x_i) - u(x_{i-1})|}{h_i} + \theta_i\,|u'(\theta_i)| \\ &\le C, \end{aligned}$$

using (4.5.13), $|x-\theta_i| \le h_i$ and $\|u\|_{L^\infty} \le C$. It follows that

$$\|au_I'\| \le C.$$

Combining this with (4.5.11), (4.5.12) and (4.3.1) completes the proof of (4.5.9).

We now move to (4.5.10). For each $i \in \{1,\dots,N\}$, using integration by parts twice, we obtain

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i} \left(x(u-u_I)'(x)\right)^2 dx \\
&= \int_{x_{i-1}}^{x_i} \left(x^2(u-u_I)'(x)\right)(u-u_I)'(x)\, dx \\
&= -\int_{x_{i-1}}^{x_i} (u-u_I)(x)\left[2x(u-u_I)'(x) + x^2(u-u_I)''(x)\right] dx \\
&= \int_{x_{i-1}}^{x_i} \left((u-u_I)^2(x) - x^2u''(x)(u-u_I)(x)\right) dx \\
&\le \int_{x_{i-1}}^{x_i} \left((u-u_I)^2(x) + C|(u-u_I)(x)|\right) dx, \qquad (4.5.14)
\end{aligned}$$

using (4.5.1). Hence

$$\begin{aligned}
\int_{-1}^{1} \left(x(u-u_I)'(x)\right)^2 dx &\le \|u-u_I\|^2 + C\|u-u_I\|_{L^1[-1,1]} \\
&\le \|u-u_I\|^2 + C\|u-u_I\| \\
&\le C\|u-u_I\|,
\end{aligned}$$

using $\|u_I\|_{L^\infty} \le \|u\|_{L^\infty} \le C$. □

The next result gives a bound on $|||u_I - u_N|||$ on an arbitrary mesh.

Lemma 4.5.2 *Let u be the solution of problem P_1^- and u_N the solution of (4.3.4) on an arbitrary mesh. Then for H sufficiently small (independently of ε), we have*

$$|||u_I - u_N||| \le C\left(\|u-u_I\|^{1/2} + H^2\right).$$

Proof. By Lemma 4.3.1, we have

$$C|||u_I - u_N|||^2 \le \hat{B}(u_I - u_N, u_I - u_N)$$
$$= \hat{B}(u_I - u, u_I - u_N) + \hat{B}(u - u_N, u_I - u_N). \qquad (4.5.15)$$

We bound these two terms separately. Firstly, integrating by parts, we obtain

$$\begin{aligned}
&\left|\hat{B}(u_I - u, u_I - u_N)\right| \\
&= \left|(\varepsilon(u_I - u), (u_I - u_N)'') + \left(\hat{a}(u_I - u)' + \hat{d}(u_I - u), u_I - u_N\right)\right| \\
&= \left|\left(\hat{a}(u_I - u)' + \hat{d}(u_I - u), u_I - u_N\right)\right| \\
&\le C\left\{\left(\int_{-1}^{1} (x(u - u_I)'(x))^2\, dx\right)^{1/2} + \|u_I - u\|\right\} \|u_I - u_N\| \\
&\le C\|u_I - u\|^{1/2}\|u_I - u_N\|, \qquad (4.5.16)
\end{aligned}$$

by (4.5.10) and $\|u_I\|_{L^\infty} \le \|u\|_{L^\infty} \le C$.

Secondly,

$$\begin{aligned}
&\left|\hat{B}(u - u_N, u_I - u_N)\right| \\
&= \left|\left(\hat{B} - B_\varepsilon\right)(u, u_I - u_N) + B_\varepsilon(u, u_I - u_N) - \hat{B}(u_N, u_I - u_N)\right| \\
&= \left|((\hat{a} - a)u', u_I - u_N) + \left((\hat{d} - d)u, u_I - u_N\right) + \left(f - \hat{f}, u_I - u_N\right)\right| \\
&\le CH^2\|u_I - u_N\|, \qquad (4.5.17)
\end{aligned}$$

since $|xu'(x)| \le C$, for $x \in [-1, 1]$.

Returning to (4.5.15), we have, from (4.5.16) and (4.5.17),

$$|||u_I - u_N|||^2 \le C\left(\|u_I - u\|^{1/2} + H^2\right)\|u_I - u_N\|,$$

which with $\|\cdot\| \le |||\cdot|||$ yields the desired result. □

The following lemma contains some technical bounds on the piecewise equidistant mesh X_K^N.

Lemma 4.5.3 *Consider the piecewise equidistant mesh* X_K^N. *Let* $x_i \in (10^{-K}, 1]$. *Then*

$$h_i^2 \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2} \le CN^{-2}\ln^2 N. \tag{4.5.18}$$

If $\sigma = \varepsilon^{\frac{1}{2}(1-\frac{\lambda}{2})}$, *then*

$$h_i^2 \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2} \le CN^{-2}\ln^2 N, \quad \textit{for } x_i \in (0, 10^{-K}]. \tag{4.5.19}$$

If $\sigma = N^{-2}$, *then*

$$h_i^2 \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2} \le C(i-1)^{-2}, \quad \textit{for } x_i \in (x_1, 10^{-K}]. \tag{4.5.20}$$

Proof. Firstly, let $x_i \in (10^{-j}, 10^{-j+1}]$ for some $j \in \{1, \ldots, K\}$. From (4.5.6), we have

$$\begin{aligned} h_i^2 \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2} &\le 81\left((K+1)10^{-j}N^{-1}\right)^2 10^{(2-\lambda)j} \\ &\le CN^{-2}\ln^2 N, \end{aligned}$$

by (4.5.8).

If $\sigma = \varepsilon^{\frac{1}{2}(1-\frac{\lambda}{2})}$, then for $x_i \in (0, 10^{-K}]$,

$$\begin{aligned} h_i^2 \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2} &\le C\left((K+1)10^{-K}N^{-1}\right)^2 \varepsilon^{-1+\frac{\lambda}{2}} \\ &\le CN^{-2}\ln^2 N, \end{aligned}$$

using (4.5.5) - (4.5.8).

If $\sigma = N^{-2}$, then for $x_i \in (x_1, 10^{-K}]$,

$$\begin{aligned} h_i^2 \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2} &\le Ch_i^2 x_{i-1}^{-2} \\ &= C(i-1)^{-2}. \end{aligned}$$

This completes the proof. □

We now estimate the interpolation error $u - u_I$ on the mesh X_K^N.

Lemma 4.5.4 *Let u be the solution of problem P_1^-. Let $u_I \in S^N$ interpolate to u at each node of the piecewise equidistant mesh X_K^N. Then*

$$\|u - u_I\| \le CN^{-2}\ln^2 N \tag{4.5.21}$$

and

$$|||u - u_I||| \le CN^{-1}\ln N. \tag{4.5.22}$$

Proof. We first prove (4.5.21).

Let $x \in (x_{i-1}, x_i)$ for some i, where $x_i \in (10^{-K}, 1]$. Then for some $\xi_i \in (x_{i-1}, x_i)$,

$$\begin{aligned}|(u - u_I)(x)| &\le Ch_i^2|u''(\xi_i)| \\ &\le Ch_i^2\left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2}, \quad \text{by (4.5.1)}, \\ &\le CN^{-2}\ln^2 N,\end{aligned}$$

by (4.5.18). Consequently

$$\int_{10^{-K}}^{1} (u - u_I)^2(x) \le CN^{-4}\ln^4 N. \tag{4.5.23}$$

Next, let $x \in (x_{i-1}, x_i)$, where $x_i \in (0, 10^{-K}]$. There are two cases to be considered, depending on the value of σ generated by (4.5.2).

If $\sigma = \varepsilon^{\frac{1}{2}(1-\frac{\lambda}{2})}$, then as above

$$|(u - u_I)(x)| \le Ch_i^2\left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-2} \le CN^{-2}\ln^2 N,$$

by (4.5.19). Consequently

$$\int_0^{10^{-K}} (u - u_I)^2(x) \le CN^{-4}\ln^4 N. \tag{4.5.24}$$

If $\sigma = N^{-2}$, we first let $x \in (x_{i-1}, x_i) \subseteq (x_1, 10^{-K}]$. Then

$$
\begin{aligned}
|(u - u_I)(x)| &\le C h_i^2 \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda - 2} \\
&\le C(i-1)^{-2}, \quad \text{for } i = 2, \ldots, \frac{N}{K+1},
\end{aligned}
$$

by (4.5.20). Hence

$$
\begin{aligned}
\int_{x_1}^{10^{-K}} |(u - u_I)(x)|^2 \, dx &= \sum_{i=2}^{\frac{N}{K+1}} \int_{x_{i-1}}^{x_i} |(u - u_I)(x)|^2 \, dx \\
&\le C \sum_{i=2}^{\frac{N}{K+1}} h_i (i-1)^{-4} \\
&\le C(K+1) 10^{-K} N^{-1} \\
&\le C N^{-4} \ln N,
\end{aligned}
$$

where we used (4.5.5) – (4.5.8). Secondly,

$$
\begin{aligned}
\int_0^{x_1} |(u - u_I)(x)|^2 \, dx &\le C x_1 \\
&\le C(K+1) 10^{-K} N^{-1} \\
&\le C N^{-4} \ln N.
\end{aligned}
$$

Thus when $\sigma = N^{-2}$,

$$
\int_0^{10^{-K}} |(u - u_I)(x)|^2 \, dx \le C N^{-4} \ln N. \tag{4.5.25}
$$

Combining (4.5.23) – (4.5.25) and using symmetry on $[-1, 0]$ yields (4.5.21). Recalling (4.5.9), (4.5.22) follows immediately from (4.5.21). □

We now prove uniform convergence results in the energy norm and the L^2 norm.

Theorem 4.5.1 *Let u be the solution of problem P_1^-. Let $u_N \in S^N$ be the solution of* (4.3.4) *on the mesh X_K^N for P_1^-. Then for N sufficiently large, independently of ε, we have*

$$|||u - u_N||| \leq CN^{-1} \ln N \tag{4.5.26}$$

and

$$\|u - u_N\| \leq C \left(N^{-1} \ln N\right)^{3/2} \tag{4.5.27}$$

Proof. The bound (4.5.26) follows from a triangle inequality and Lemmas 4.5.2 and 4.5.4.

We now prove (4.5.27), by sharpening the argument of Lemma 4.5.2. The main step is to use a more careful analysis to show that

$$\left|\left(\hat{a}(u_I - u)', u_I - u_N\right)\right| \leq C \left(N^{-1} \ln N\right)^{3/2} \|u_I - u_N\|. \tag{4.5.28}$$

We discuss two cases, depending on the value of σ in (4.5.2). Set

$$e_i = (u_I - u_N)(x_i), \quad \text{for } i = 0, 1, \ldots, N.$$

Case 1: $\sigma = \varepsilon^{\frac{1}{2}(1-\frac{k}{2})}$.

Integrating by parts, one has

$$\begin{aligned} &\left(\hat{a}(u_I - u)', u_I - u_N\right) \\ &= \left(\hat{a}'(u_I - u), u_I - u_N\right) - \left(\hat{a}(u_I - u), (u_I - u_N)'\right). \end{aligned} \tag{4.5.29}$$

From (4.5.21),

$$\left(\hat{a}'(u_I - u), u_I - u_N\right) \leq CN^{-2} \ln^2 N \|u_I - u_N\|. \tag{4.5.30}$$

Next,

$$
\begin{aligned}
&(\hat{a}(u - u_I), (u_I - u_N)') \\
&= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \hat{a}(x)(u - u_I)(x)(u_I - u_N)'(x)\, dx \\
&= \sum_{i=1}^{N} \frac{e_i - e_{i-1}}{h_i} \int_{x_{i-1}}^{x_i} \hat{a}(x)(u - u_I)(x)\, dx \\
&= \sum_{i=1}^{N-1} e_i \left\{ \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) \hat{a}(x)(u - u_I)(x)\, dx \right\}, \\
&\qquad\qquad\qquad\qquad\qquad \text{since } e_0 = e_N = 0, \\
&= \sum_{i=1}^{N-1} e_i a(x_{i-1}) \left\{ \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) (u - u_I)(x)\, dx \right\} \\
&\quad + \sum_{i=1}^{N-1} e_i \left\{ \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) (\hat{a}(x) - a(x_{i-1}))(u - u_I)(x)\, dx \right\} \\
&= Y_1 + Y_2, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.5.31)
\end{aligned}
$$

say. Set $\bar{h}_i = (h_i + h_{i+1})/2$ for $i = 1, 2, \ldots, N-1$. Inspecting the proof of (4.5.21) and recalling that $\sigma = \varepsilon^{\frac{1}{2}(1-\frac{\lambda}{2})}$, we have

$$\|u - u_I\|_{L^\infty[0,1]} \le CN^{-2} \ln^2 N.$$

Hence

$$
\begin{aligned}
|Y_2| &\le CN^{-2} \ln^2 N \sum_{i=1}^{N-1} \bar{h}_i |e_i| \\
&\le CN^{-2} \ln^2 N \left(\sum_{i=1}^{N-1} \bar{h}_i e_i^2 \right)^{1/2} \\
&\le CN^{-2} \ln^2 N \|u_I - u_N\|, \qquad\qquad (4.5.32)
\end{aligned}
$$

as it is easy to show that the discrete L^2 norm $\left(\sum_{i=1}^{N-1} \bar{h}_i ((\cdot)(x_i))^2\right)^{1/2}$ is equivalent to $\|\cdot\|$ on S^N.

For Y_1, we write

$$Y_1 = \left(\sum_i{}' + \sum_i{}''\right) e_i a(x_{i-1}) \left\{ \left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}\right)(u - u_I)(x)\,dx \right\},$$

where $\sum_i'$ means summation over those i for which $h_i = h_{i+1}$, and $\sum_i''$ sums over the remaining i.

Suppose $h_i = h_{i+1}$; then

$$\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}\right)(u - u_I)(x)\,dx$$
$$= \frac{1}{h_i}\int_{x_{i-1}}^{x_i} [(u - u_I)(x) - (u - u_I)(x + h_i)]\,dx.$$

By the usual interpolation error estimate, for $x \in [x_{i-1}, x_i]$,

$$(u - u_I)(x) = \frac{1}{2}(x - x_{i-1})(x - x_i)u''(\xi_i), \quad \text{where } x_{i-1} < \xi_i < x_i,$$

also

$$(u - u_I)(x + h_i) = \frac{1}{2}(x + h_i - x_i)(x + h_i - x_{i+1})u''(\eta_i), \quad \text{where } x_i < \eta_i < x_{i+1}.$$

Therefore,

$$\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} [(u - u_I)(x) - (u - u_I)(x + h_i)]\,dx\right|$$
$$= \left|\frac{1}{2h_i}\int_{x_{i-1}}^{x_i} (x - x_{i-1})(x - x_i)\left(u''(\xi_i) - u''(\eta_i)\right)\,dx\right|$$
$$\le C h_i \int_{x_{i-1}}^{x_i} |\xi_i - \eta_i| \max_{x_{i-1} \le t \le x_i} |u'''(t)|\,dx$$
$$\le C h_i^2 \bar{h}_i \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda - 3}$$
$$\le C \bar{h}_i \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda - 1} N^{-2} \ln^2 N,$$

by (4.5.18) and (4.5.19). Consequently,

$$\left|\sum_i{}' e_i a(x_{i-1}) \left\{ \left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}\right)(u - u_I)(x)\,dx \right\}\right|$$

$$
\begin{aligned}
&\le C \sum_i{}' |e_i| x_{i-1} \bar{h}_i \left(x_{i-1} + \varepsilon^{1/2}\right)^{\lambda-1} N^{-2} \ln^2 N \\
&\le C N^{-2} \ln^2 N \sum_i{}' \bar{h}_i |e_i| \\
&\le C N^{-2} \ln^2 N \left(\sum_i{}' \bar{h}_i e_i^2 \right)^{1/2} \\
&\le C N^{-2} \ln^2 N \|u_I - u_N\|.
\end{aligned}
\tag{4.5.33}
$$

We now deal with $\sum_i''$.

$$
\begin{aligned}
&\left| \sum_i{}'' e_i a(x_{i-1}) \left\{ \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} - \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \right) (u - u_I)(x)\, dx \right\} \right| \\
&\le C \sum_{j=1}^{K+1} |e_{jn}| x_{jn-1} \left[h_{jn}^2 \left(x_{jn-1} + \varepsilon^{1/2}\right)^{\lambda-2} \right. \\
&\qquad\qquad \left. + h_{jn+1}^2 \left(x_{jn} + \varepsilon^{1/2}\right)^{\lambda-2} \right] \\
&\le C \sum_{j=1}^{K+1} |e_{jn}| \left[h_{jn}^2 \left(x_{jn-1} + \varepsilon^{1/2}\right)^{\lambda-1} + h_{jn+1}^2 \left(x_{jn} + \varepsilon^{1/2}\right)^{\lambda-1} \right] \\
&\le C N^{-1} \ln N \sum_{j=1}^{K+1} \bar{h}_{jn} |e_{jn}| \\
&\le C N^{-1} \ln N \left(\sum_{j=1}^{K+1} \bar{h}_{jn} \right)^{1/2} \left(\sum_{j=1}^{K+1} \bar{h}_{jn} e_{jn}^2 \right)^{1/2} \\
&\le C \left(N^{-1} \ln N\right)^{3/2} \|u_I - u_N\|,
\end{aligned}
\tag{4.5.34}
$$

since $\sum_{j=1}^{K+1} \bar{h}_{jn} = \sum_{j=1}^{K+1} 10^{-K-1+j}(K+1)N^{-1} \le CN^{-1} \ln N$.

Combining (4.5.31) – (4.5.34) yields

$$
\left| (\hat{a}(u_I - u), (u_I - u_N)') \right| \le C \left(N^{-1} \ln N\right)^{3/2} \|u_I - u_N\|.
$$

Hence, recalling (4.5.29) and (4.5.30), we have proven (4.5.28) when $\sigma = \varepsilon^{\frac{1}{2}(1-\frac{\lambda}{2})}$.

Case 2: $\sigma = N^{-2}$.

It is easy to see that the argument of Case 1 yields

$$\left| \int_{10^{-K}}^{1} \hat{a}(x)(u - u_I)'(x)(u_I - u_N)(x)\, dx \right| \le C \left(N^{-1} \ln N\right)^{3/2} \|u_I - u_N\|. \quad (4.5.35)$$

Adding (4.5.14) over $i = 1, 2, \ldots, n$, we obtain

$$\left| \int_0^{10^{-K}} \left(x(u - u_I)'(x)\right)^2 dx \right|$$

$$\le \|u - u_I\|^2_{L^2[0,10^{-K}]} + C\|u - u_I\|_{L^1[0,10^{-K}]}$$

$$\le \|u - u_I\|^2_{L^2[0,10^{-K}]} + C10^{-K/2}\|u - u_I\|_{L^2[0,10^{-K}]},$$

by a Cauchy–Schwarz inequality,

$$\le CN^{-7/2} \ln^{1/2} N,$$

using (4.5.7), (4.5.25) and $\sigma = N^{-3}$. Hence

$$\left| \int_0^{10^{-K}} \hat{a}(x)(u - u_I)'(x)(u_I - u_N)(x)\, dx \right| \le CN^{-7/4} \ln^{1/4} N \|u_I - u_N\|.$$

Combining this with (4.5.35) completes the proof of (4.5.28) for Case 2.

Therefore, using (4.5.28) and (4.5.21) in the proof of Lemma 4.5.2, we obtain

$$|||u_I - u_N||| \le C(N^{-1} \ln N)^{3/2}.$$

Recalling (4.5.21), the proof is completed. □

4.6 Numerical Results and Conclusions

In this section we give some numerical experiments for the method (4.3.4) applied to the simple attractive turning point problem P_1^-.

Our test problem is

$$-\varepsilon u'' - x\left(1 + x^2\right) u' + \lambda \left(1 + x^2\right) u = f, \quad \text{for } x \in (-1, 1),$$

$$u(-1) = u(1) = 0,$$

where $f(x)$ is chosen so that the solution is

$$u(x) = \left(x^2 + \varepsilon\right)^{\lambda/2} + x\left(x^2 + \varepsilon\right)^{(\lambda-1)/2} - (1+\varepsilon)^{\lambda/2}\left[1 + x(1+\varepsilon)^{-1/2}\right].$$

This $u(x)$ exhibits typical internal layer behaviour of cusp type. Similar examples may be found in Berger et al. [4] and Farrell [12]. Since the behaviour of $u(x)$ near the turning point $x = 0$ depends specifically on ε, we shall examine errors and experimental rates of convergence for different values of ε. We take $\lambda = 0.25$ below; numerical experiments with $\lambda = 0.5$ and $\lambda = 0.75$ yielded similar results.

We compute the errors in the following two ways:

(i). The error between the interpolant $u_I(x)$ and the computed solution $u_N(x)$ in a discrete L^2 norm,

$$E_\varepsilon^N = \|u_I - u_N\|_d.$$

(ii). The error between the exact solution $u(x)$ and the computed solution $u_N(x)$ in the discrete maximum norm,

$$E_\varepsilon^N = \|u - u_N\|_{\infty,d} = \max_{L \le i \le R} |u(x_i) - u_N(x_i)|.$$

The discrete L^2 norm is defined by

$$\|v\|_d = \left\{\sum_{i=L+1}^{R-1} \bar{h}_i v_i^2\right\}^{1/2},$$

for all $v = \sum_{i=L+1}^{R-1} v_i\varphi_i(x) \in S^N$. By a calculation, one may easily show that on S^N the discrete L^2 norm $\|\cdot\|_d$ is equivalent to the usual L^2 norm $\|\cdot\|$.

We calculate the convergence rate tables as follows; see Farrell and Hegarty [14]:

(i). Except for the last row, the table entries are given by the classical convergence rate,

$$R_\varepsilon^N = \left(\ln E_\varepsilon^{2N} - \ln E_\varepsilon^N\right) / \ln 2. \qquad (4.6.1)$$

(ii). The last row of each table is the uniform convergence rate,

$$R^N = \left(\ln E^{2N} - \ln E^N\right) / \ln 2,$$

where $E^N = \max_\varepsilon E_\varepsilon^N$.

Recall the definition of the mesh X_K^N. The number of meshpoints on the interval $(0, 1]$ is $(K+1)n$, which is less or equal to N. In order to use the formula (4.6.1) to compute our convergence rates, we need exactly N meshpoints on $(0, 1]$. Hence we adjust the mesh X_K^N as follows:

Let $N_0 = N - (K+1)n$. Then n points are used on each of the subintervals

$$(0, 10^{-K}], \quad \ldots, \quad (10^{-N_0-1}, 10^{-N_0}],$$

and $n+1$ points are taken on each of the remaining subintervals

$$(10^{-N_0}, 10^{-N_0+1}], \quad \ldots, \quad (10^{-1}, 1].$$

The mesh is still uniform on each of the above subintervals.

ε	N=16	32	64	128	256
1.00000e+00	2.057e-04	5.161e-05	1.291e-05	3.229e-06	8.072e-07
2.50000e-01	1.535e-03	3.834e-04	9.582e-05	2.395e-05	5.988e-06
6.25000e-02	4.755e-03	1.184e-03	2.956e-04	7.387e-05	1.847e-05
1.56250e-02	9.929e-03	2.415e-03	5.985e-04	1.493e-04	3.731e-05
3.90625e-03	1.583e-02	4.660e-03	1.159e-03	3.009e-04	7.504e-05
9.76562e-04	1.314e-02	3.675e-03	8.766e-04	2.112e-04	5.126e-05
2.44141e-04	1.324e-02	3.799e-03	1.027e-03	2.202e-04	5.021e-05
6.10352e-05	1.279e-02	3.911e-03	1.197e-03	2.575e-04	5.572e-05
1.52588e-05	2.535e-02	8.118e-03	2.287e-03	5.896e-04	1.341e-04
3.81470e-06	2.531e-02	8.278e-03	2.358e-03	6.090e-04	1.500e-04
9.53674e-07	2.541e-02	8.337e-03	2.428e-03	6.294e-04	1.555e-04

Table 4.6.1: $\|u_I - u_N\|$ Errors

ε	N=16	32	64	128
1.00000e+00	1.99	2.00	2.00	2.00
2.50000e-01	2.00	2.00	2.00	2.00
6.25000e-02	2.01	2.00	2.00	2.00
1.56250e-02	2.04	2.01	2.00	2.00
3.90625e-03	1.76	2.01	1.95	2.00
9.76562e-04	1.84	2.07	2.05	2.04
2.44141e-04	1.80	1.89	2.22	2.13
6.10352e-05	1.71	1.71	2.22	2.21
1.52588e-05	1.64	1.83	1.96	2.14
3.81470e-06	1.61	1.81	1.95	2.02
9.53674e-07	1.61	1.78	1.95	2.02
R^N	1.61	1.78	1.95	2.02

Table 4.6.2: $\|u_I - u_N\|$ Convergence Rates

ε	N=16	32	64	128	256
1.00000e+00	2.970e-04	7.434e-05	1.859e-05	4.648e-06	1.162e-06
2.50000e-01	2.063e-03	5.152e-04	1.291e-04	3.229e-05	8.075e-06
6.25000e-02	7.291e-03	1.823e-03	4.595e-04	1.149e-04	2.874e-05
1.56250e-02	1.824e-02	4.890e-03	1.192e-03	2.964e-04	7.398e-05
3.90625e-03	2.765e-02	1.140e-02	3.281e-03	7.986e-04	1.993e-04
9.76562e-04	2.633e-02	1.060e-02	3.439e-03	8.866e-04	1.856e-04
2.44141e-04	2.599e-02	1.030e-02	3.802e-03	1.097e-03	2.625e-04
6.10352e-05	2.472e-02	1.017e-02	3.990e-03	1.181e-03	3.203e-04
1.52588e-05	4.282e-02	1.860e-02	6.741e-03	2.148e-03	6.074e-04
3.81470e-06	4.292e-02	1.899e-02	6.896e-03	2.167e-03	6.214e-04
9.53674e-07	4.308e-02	1.912e-02	7.058e-03	2.211e-03	6.260e-04

Table 4.6.3: $\|u - u_N\|_{\infty,d}$ Errors

ε	N=16	32	64	128
1.00000e+00	2.00	2.00	2.00	2.00
2.50000e-01	2.00	2.00	2.00	2.00
6.25000e-02	2.00	1.99	2.00	2.00
1.56250e-02	1.90	2.04	2.01	2.00
3.90625e-03	1.28	1.80	2.04	2.00
9.76562e-04	1.31	1.62	1.96	2.26
2.44141e-04	1.33	1.44	1.79	2.06
6.10352e-05	1.28	1.35	1.76	1.88
1.52588e-05	1.20	1.46	1.65	1.82
3.81470e-06	1.18	1.46	1.67	1.80
9.53674e-07	1.17	1.44	1.67	1.82
R^N	1.17	1.44	1.67	1.82

Table 4.6.4: $\|u - u_N\|_{\infty,d}$ Convergence Rates

From Table 4.6.2, one can clearly see that the uniform convergence rate of $\|u_I - u_N\|$ is $O(N^{-2})$. Lemma 4.5.4 shows that $\|u - u_I\| = O(N^{-2}\ln^2 N)$, so we have numerical evidence that $\|u - u_N\|$ is also $O(N^{-2}\ln^2 N)$. This is better than the $O\left(N^{-3/2}\ln^{3/2} N\right)$ result proven in Theorem 4.5.1. We also notice that the uniform convergence rates in the discrete maximum norm reported in Table 4.6.4 are almost second order.

Conclusions: In this work we introduced piecewise linear Galerkin finite element methods on various piecewise equidistant meshes for the singularly perturbed interior turning point problem (4.1.1). The resulting schemes are much simpler than exponentially fitted schemes. The meshes used in this chapter are relatively simple. We proved the convergence, uniformly in ε, of our methods in a weighted energy norm $|||\cdot|||$ and the usual L^2 norm. Numerical experiments verified the convergence in L^2 and showed that the schemes are also uniformly convergent of almost second order in the discrete maximum norm.

Chapter 5

A Semilinear Reaction–Diffusion Problem

5.1 Introduction

Singularly perturbed nonlinear boundary value problems occur frequently in engineering applications such as catalytic reactions or adsorption processes and fluid dynamics.

In this chapter, we consider the semilinear problem

$$F_\varepsilon u(x) \equiv -\varepsilon^2 u''(x) + b(x,u) = 0, \quad \text{for } x \in (0,1), \tag{5.1.1a}$$

$$u(0) = u(1) = 0, \tag{5.1.1b}$$

where ε is a small positive parameter. Set $X = [0,1]$. We shall assume that $b \in C^\infty(X \times \mathcal{R}^1)$ for convenience.

Asymptotic and numerical solutions of problem (5.1.1) have been considered by many authors, under various hypotheses on $b(x,u)$. See for example Chang and Howes [6], D'Annunzio [9], Fife [15], Herceg [21], Herceg and Petrović [22] and Lorenz [26].

One of the conditions occurring frequently in the literature is

$$b_u(x,u) > b_0^2 > 0, \quad \text{for all } (x,u) \in X \times \mathcal{R}^1. \tag{5.1.2}$$

Under this condition, there exists a unique solution $u \in C^\infty(X)$ to the problem (5.1.1), (5.1.2); see Lorenz [26].

The reduced problem of (5.1.1) is defined by

$$b(x,u) = 0, \quad \text{for } x \in X. \tag{5.1.3}$$

Under the condition (5.1.2), this reduced problem has a unique solution $u_0 \in C^\infty(X)$, as can be seen using the implicit function theorem and the compactness of X. Note that in general u_0 does not satisfy either of the boundary conditions in (5.1.1b).

Generally speaking, the reduced problem (5.1.3) may have more than one solution if condition (5.1.2) is not satisfied. Fife [15] and D'Annunzio [9] considered problem (5.1.1) under the assumption that it has a stable reduced solution, i.e., that there exists a solution $u_0 \in C^\infty(X)$ of (5.1.3) which satisfies

$$b_u(x,u_0) > b_0^2 > 0, \quad \text{for all } x \in X, \tag{5.1.4a}$$

$$\int_{u_0(0)}^{\tau} b(0,s)ds > 0, \text{ for } \begin{cases} \tau \in (u_0(0),0], \text{ whenever } 0 > u_0(0), & (5.1.4b) \\ \tau \in [0,u_0(0)), \text{ whenever } u_0(0) > 0, & (5.1.4c) \end{cases}$$

and

$$\int_{u_0(1)}^{\tau} b(1,s)ds > 0, \text{ for } \begin{cases} \tau \in (u_0(1),0], \text{ whenever } 0 > u_0(1), & (5.1.4d) \\ \tau \in [0,u_0(1)), \text{ whenever } u_0(1) > 0. & (5.1.4e) \end{cases}$$

The conditions (5.1.4) are obviously weaker than condition (5.1.2). Problem (5.1.1) under the conditions (5.1.4) may exhibit multiple solutions. D'Annunzio [9] showed existence and local uniqueness of a solution satisfying (5.1.1) and (5.1.4) using degree theory.

In what follows, (5.1.1) under condition (5.1.2) and (5.1.1) under conditions (5.1.4) will be referred to as problem (A) and problem (B) respectively.

In this chapter, we only consider uniform convergence with respect to the discrete L^∞ norm.

A solution $u(x)$ of (5.1.1) usually exhibits sharp boundary layers at the endpoints of the interval X when the parameter ε is near zero. When polynomial-based numerical methods are applied to (5.1.1), one does not obtain accurate results on all of X, even in the linear case. This has lead to the development of uniformly convergent numerical methods. In the linear case both uniformly convergent exponentially fitted schemes on equidistant meshes and uniformly convergent polynomial based schemes on special meshes have been considered; see Doolan et al. [10], O'Riordan and Stynes [31] and Vulanović [49].

Herceg [21] considered problem (A) with additional hypotheses on $b(x,u)$, namely that there exist functions q and $Q \in C^1[X]$ satisfying

$$q(x) \leq b_u(x,u) \leq Q(x), \quad \text{for } (x,u) \in X \times \mathcal{R}^1 \tag{5.1.5a}$$

and

$$\Delta_1 = \min_{x \in X}\{5q(x) - 2Q(x)\} > 0. \tag{5.1.5b}$$

He constructed a scheme by requiring it to be exact on all polynomials of degree at most 4 and proved that this scheme is fourth order uniformly convergent on a

Bakhvalov mesh, which is a graded mesh specially constructed *a priori* to fit the problem.

D'Annunzio [9] examined a solution of problem (B), using a simple central difference scheme on a special locally quasi–equidistant mesh. This mesh contains $O(h^{-1} \ln 1/\varepsilon)$ mesh points when $\varepsilon \leq h$, where h is the maximum mesh spacing over the interval X. She showed existence of a solution to this discrete problem and $O(h)$ uniform convergence of this solution to a solution of problem (B).

In this chapter, we consider both D'Annunzio's scheme and the higher order scheme of Herceg, which we refer to as the D–scheme and H–scheme respectively. We shall use a piecewise equidistant mesh. This type of mesh, which was recently introduced by Shishkin [37], is much simpler than the meshes of Herceg [21] and D'Annunzio [9].

On this mesh, we shall apply both the D–scheme and the H–scheme to problem (A). Existence and uniqueness of a solution to the D–scheme is proved by using Hadamard's Theorem; see Ortega and Rheinboldt [32]. We show that the D–scheme is uniformly convergent of order $\varepsilon^2 N^{-1} + N^{-2} \ln^2 N$. Similar existence and uniqueness results are obtained for the H–scheme under the assumptions (5.1.5). We also discuss existence and local uniqueness of a solution for the H–scheme without the extra conditions (5.1.5), using degree theory. The H–scheme is shown to be uniformly convergent of order $\varepsilon^2 N^{-3} + N^{-4} \ln^4 N$.

For problem (B), we consider only the D–scheme on our Shishkin mesh. We use degree theory to analyse the existence of a solution to the scheme. We construct super and sub solutions which are within order $\varepsilon^2 \ln^2(1/\varepsilon)$ of a solution of problem (B); we also consider their discrete analogues for the D–scheme. This allows us to obtain uniform convergence of order $N^{-2}\ln^2 N$ for the D–scheme under the nonrestrictive assumption $\varepsilon \leq N^{-1}$. This result is a significant improvement on the first order convergence obtained by D'Annunzio [9] for the same scheme on a different mesh.

A summary of this chapter is as follows. Section 5.2 contains results concerning the exact solutions of problem (5.1.1), including an asymptotic expansion of the solution to problem (A) and super and sub solutions of problem (B). In Section 5.3, we bound truncation errors of the D–scheme and the H–scheme on Shishkin meshes for problem (A). In Section 5.4, we analyse existence, uniqueness and uniform convergence of solutions of both the D–scheme and the H–scheme for problem (A). Section 5.5 shows the almost second order uniform accuracy of the D–scheme for problem (B). In Section 5.6, we present numerical computations which confirm our results.

5.2 The Continuous Problems

In this section, we discuss properties of the exact solutions of problem (A) and problem (B). In the sequel, we use J to denote an arbitrary positive constant.

For problem (A), we have

Lemma 5.2.1 *There exists a unique solution $u \in C^{\infty}(X)$ of problem (A). This solution admits the decomposition*

$$u(x) = Y(x) + V(x), \quad \text{for } x \in X, \tag{5.2.1}$$

where

$$\left|Y^{(j)}(x)\right| \leq C \tag{5.2.2}$$

and

$$\left|V^{(j)}(x)\right| \le C\varepsilon^{-j}\left[\exp\left(-b_0 x/\varepsilon\right) + \exp\left(-b_0(1-x)/\varepsilon\right)\right], \qquad (5.2.3)$$

for $x \in X$ *and* $j = 0, 1, \ldots, J$.

Proof. See Vulanović [48]. □

From (5.2.1) – (5.2.3), one may see that, in general, the solution $u(x)$ of problem (A) exhibits boundary layers at the endpoints of the interval X and has no interior layers.

We now move on to problem (B). We shall suppose without loss of generality that $u_0(0) < 0$ and $u_0(1) < 0$, as other cases can be handled similarly. The concepts of super and sub solutions are important for the study of problem (B). Suppose that there exist two functions α and $\beta \in C^2(X)$ with the following properties:

$$F_\varepsilon\alpha(x) \le 0 \le F_\varepsilon\beta(x), \quad \text{for } x \in X,$$

$$\alpha(0) \le 0 \le \beta(0),$$

$$\alpha(1) \le 0 \le \beta(1),$$

$$\alpha(x) \le \beta(x), \quad \text{for } x \in X.$$

Then $\beta(x)$ and $\alpha(x)$ are said to be super and sub solutions respectively of problem (5.1.1).

In order to prove higher order convergence of the D–scheme for problem (B), we shall introduce super and sub solutions which are more accurate than those in D'Annunzio [9]. Let us first give some notation and definitions.

We define the usual cut off function $\sigma(x)$ for asymptotic analysis by

$$\sigma(x) = \begin{cases} 1, & \text{for } 0 \le x \le 1/4, \\ 0, & \text{for } 1/2 \le x \le 1, \end{cases}$$

with $\sigma(x) \geq 0$ infinitely differentiable for $x \in X$.

Let $v \in C^\infty(0, \infty)$. Let b_1 be a positive constant. If for each $\delta \in (0, b_1)$ there exists a positive constant C_δ, depending on δ and J, such that

$$\left|v^{(j)}(\eta)\right| \leq C_\delta \exp(-(b_1 - \delta)\eta),$$

for $\eta > 0$ and $j = 0, 1, \ldots, J$, then the function $v(\eta)$ will be said to belong to the class of $e(b_1, J)$.

The following two lemmas are modifications of Lemmas 2.1 and 2.2 of Fife [15].

Lemma 5.2.2 *Let $\lambda > 0$ be a constant. Let $g \in C^\infty[0, \lambda]$ satisfy $g(0) = 0$, $g'(0) > 0$ and*

$$\int_0^\tau g(s) > 0, \quad \textit{for } \tau \in (0, \lambda].$$

Then for $\eta \geq 0$, there exists a unique monotone solution $v(\eta)$ of

$$v'' - g(v) = 0, \quad \textit{for } \eta > 0, \tag{5.2.4}$$

$$v(0) = \lambda, \quad v(\infty) = 0. \tag{5.2.5}$$

Furthermore, v belongs to the class of $e(b_1, J)$ with $b_1 = \sqrt{g'(0)}$.

Proof. By Lemma 2.1 of Fife [15], the solution v of (5.2.4) and (5.2.5) exists, is monotonic and satisfies

$$C_\delta^{-1} \exp(-(b_1 + \delta)\eta) \leq v^{(j)}(\eta) \leq C_\delta \exp(-(b_1 - \delta)\eta),$$

for $j = 0, 1$ and $\eta > 0$, where $b_1 = \sqrt{g'(0)}$, $\delta \in (0, b_1)$ and $C_\delta > 0$ are constants.

Since $g(0) = 0$ and $g'(s)$ is bounded for $s \in (0, \lambda)$, we have from (5.2.4)

$$\begin{aligned}
|v''(\eta)| &= |g'(v^*)|\, v(\eta), \quad \text{where } v^* \in (0, v) \subseteq (0, \lambda), \\
&\leq C_\delta \exp(-(b_1 - \delta)\eta), \quad \text{for } \eta > 0,
\end{aligned}$$

where we recall that C_δ is a generic constant. The result then follows from differentiating (5.2.4) repeatedly and induction on j, since the derivatives of $g(s)$ up to any prescribed order are bounded for $s \in (0, \lambda)$. □

Lemma 5.2.3 *Let λ and $g(s)$ be as in Lemma 5.2.2. Let $v(\eta)$ be the monotone solution of (5.2.4) and (5.2.5). Let $a(\eta)$ belong to the class of $e(b_1, J)$ with $b_1 = \sqrt{g'(0)}$. Then there exists a unique solution $v_1(\eta)$ of*

$$v_1'' - g'(v(\eta))v_1 = a(\eta), \quad \textit{for } \eta > 0, \tag{5.2.6}$$

$$v_1(0) = \lambda_1, \quad v_1(\infty) = 0. \tag{5.2.7}$$

Moreover, $v_1(\eta)$ belongs to the class of $e(b_1, J)$.

Proof. The result follows easily from an inspection of the proof of Lemma 2.2 in Fife [15]. □

The next lemma is a modification of Lemma 3.1 of D'Annunzio [9].

Lemma 5.2.4 *Let λ and $g(s)$ be as in Lemma 5.2.2. Let p be a constant. Then there is a $p_0 > 0$ such that if $|p| < p_0$, there exists a unique solution $v(\eta, p)$ of*

$$\ddot{v} - g(v) = -pv, \quad \textit{for } \eta > 0, \tag{5.2.8}$$

$$v(0, p) = \lambda, \quad v(\infty, p) = 0. \tag{5.2.9}$$

(Here and in what follows a dot denotes partial differentiation with respect to η.) For each fixed $p \in (-p_0, p_0)$, the solution $v(\eta, p)$ is monotone in η and belongs to the class of $e(b_1, J)$ with $g'(0) > p$ and $b_1 = \sqrt{g'(0) - p}$.

Also the derivative $z(\eta,p) = \frac{\partial v}{\partial p}(\eta,p)$ *exists and satisfies*

$$\ddot{z} - \left(g'(v) - p\right) z = -v, \quad \textit{for } \eta > 0, \tag{5.2.10}$$

$$z(0,p) = z(\infty,p) = 0 \tag{5.2.11}$$

and

$$z(\eta,p) > 0 \quad \textit{for } \eta > 0. \tag{5.2.12}$$

Furthermore, there exist positive constants C_1 *and* C_2*, independent of* p*, such that*

$$|z(\eta,p)| \leq C_1 \eta e^{-C_2\eta}, \quad \textit{for } (\eta,p) \in (0,\infty) \times [-p_0, p_0]. \tag{5.2.13}$$

Proof. The results follow from arguments similar to those of Lemma 3.1 of D'Annunzio [9]. □

We now define the required boundary layer functions. These are more accurate than those of D'Annunzio. They will be used to construct our super and sub solutions. Let

$$w(x,\varepsilon,p) = \begin{cases} \left(v_0^0(x/\varepsilon,p) + \varepsilon v_1^0(x/\varepsilon)\right)\sigma(x), & \text{for } 0 \leq x \leq 1/2, \\ \left(v_0^1((1-x)/\varepsilon,p) + \varepsilon v_1^1((1-x)/\varepsilon)\right)\sigma(1-x), & \\ & \text{for } 1/2 < x \leq 1, \end{cases} \tag{5.2.14}$$

where $v_0^0(\eta,p)$, $v_0^1(\eta,p)$, $v_1^0(\eta)$, and $v_1^1(\eta)$ are respectively defined by

$$\ddot{v}_0^0 - b\left(0, u_0(0) + v_0^0\right) = -pv_0^0, \quad \text{for } \eta > 0, \tag{5.2.15}$$

$$v_0^0(0,p) = -u_0(0), \quad v_0^0(\infty,p) = 0, \tag{5.2.16}$$

$$\ddot{v}_0^1 - b\left(1, u_0(1) + v_0^1\right) = -pv_0^1, \quad \text{for } \eta > 0, \tag{5.2.17}$$

$$v_0^1(0,p) = -u_0(1), \quad v_0^1(\infty,p) = 0, \tag{5.2.18}$$

$$\ddot{v}_1^0 - b_u\left(0, u_0(0) + v_0^0(\eta,0)\right) v_1^0$$
$$= \eta\left[b_x\left(0, u_0(0) + v_0^0(\eta,0)\right) + b_u\left(0, u_0(0) + v_0^0(\eta,0)\right) u_0'(0)\right], \quad \text{for } \eta > 0, \tag{5.2.19}$$

$$v_1^0(0,p) = 0, \quad v_1^0(\infty,p) = 0, \tag{5.2.20}$$

and

$$\ddot{v}_1^1 - b_u\left(1, u_0(1) + v_0^1(\eta,1)\right) v_1^1$$
$$= \eta\left[b_x\left(1, u_0(1) + v_0^1(\eta,1)\right) + b_u\left(1, u_0(1) + v_0^1(\eta,1)\right) u_0'(1)\right], \quad \text{for } \eta > 0, \tag{5.2.21}$$

$$v_1^1(0,p) = 0, \quad v_1^1(\infty,p) = 0. \tag{5.2.22}$$

We remark that D'Annunzio uses only the first terms of our expansions, i.e., $v_1^0 \equiv v_1^1 \equiv 0$ in D'Annunzio [9].

From Lemmas 5.2.2 – 5.2.4, one can see that there is a $p_0 > 0$, independent of ε, such that $w(x,\varepsilon,p)$ is well defined for $|p| < p_0$. Furthermore, we have

$$0 \le \frac{\partial w}{\partial p}(x,\varepsilon,p) \le C \tag{5.2.23}$$

and

$$\left|\frac{\partial^j w}{\partial x^j}(x,\varepsilon,p)\right| \le C\varepsilon^{-j}\left(\exp\left(-(\bar{b}-\delta)x/\varepsilon\right) + \exp\left(-(\bar{b}-\delta)(1-x)/\varepsilon\right)\right), \tag{5.2.24}$$

for $x \in X$ and $j = 0, 1, \ldots, J$, where $b_0^2 > p$, $\bar{b} = \sqrt{b_0^2 - p}$ (b_0 is given by (5.1.3)) and δ is any fixed number in $(0, \bar{b})$. Thus w essentially models boundary layers at $x = 0$ and $x = 1$.

Lemma 5.2.5 *Set $p_\varepsilon = \varepsilon^2 \ln^2(1/\varepsilon)$. Then we can choose positive constants C_1 and C_2, which are independent of ε, such that when ε is sufficiently small, $w(x, \varepsilon, C_1 p_\varepsilon)$ and $w(x, \varepsilon, -C_1 p_\varepsilon)$ are well defined, and*

$$\beta(x, \varepsilon) = u_0(x) + w(x, \varepsilon, C_1 p_\varepsilon) + C_2 p_\varepsilon \tag{5.2.25}$$

and

$$\alpha(x, \varepsilon) = u_0(x) + w(x, \varepsilon, -C_1 p_\varepsilon) - C_2 p_\varepsilon \tag{5.2.26}$$

are super and sub solutions respectively of problem (B).

Proof. Fix $\varepsilon \in (0, 1]$. We shall specify C_1 and C_2 later in the proof. It is easy to see from (5.2.23) that

$$\alpha(x, \varepsilon) < \beta(x, \varepsilon), \quad \text{for } x \in X. \tag{5.2.27}$$

By the construction of $w(x, \varepsilon, p)$, we have

$$\alpha(0, \varepsilon) = -C_2 p_\varepsilon < 0 < C_2 p_\varepsilon = \beta(0, \varepsilon), \tag{5.2.28}$$

$$\alpha(1, \varepsilon) = -C_2 p_\varepsilon < 0 < C_2 p_\varepsilon = \beta(1, \varepsilon). \tag{5.2.29}$$

To be a super solution, β must satisfy $F_\varepsilon \beta \geq 0$, for $x \in X$. Proof of this will be shown only for $x \in [0, 1/2]$ since the result for $x \in [1/2, 1]$ may be obtained similarly. In the rest of this argument, the notation $\zeta = O(M)$ stands for $|\zeta| \leq \tilde{C} M$, where $\tilde{C} > 0$ is any constant independent of C_1, C_2 and ε.

Set

$$x^* = \frac{4\varepsilon}{b_0} \ln(1/\varepsilon). \tag{5.2.30}$$

We have $x^* \in (0, 1/4)$, when ε is sufficiently small.

Consider first $x \in [0, x^*)$. Then $\sigma(x) \equiv 1$ and $\sigma(1-x) \equiv 0$. Hence

$$\beta(x,\varepsilon) = u_0(x) + v_0^0(\eta, C_1 p_\varepsilon) + \varepsilon v_1^0(\eta) + C_2 p_\varepsilon,$$

where $\eta = x/\varepsilon$. Therefore

$$\begin{aligned} F_\varepsilon \beta(x,\varepsilon) &= -\varepsilon^2 \beta''(x,\varepsilon) + b(x,\beta) \\ &= -\varepsilon^2 u_0'' - \ddot{v}_0^0 - \varepsilon \ddot{v}_1^0 + b(x, u_0 + v_0^0 + \varepsilon v_1^0 + C_2 p_\varepsilon) \\ &= -\ddot{v}_0^0 - \varepsilon \ddot{v}_1^0 + b(x, u_0 + v_0^0 + \varepsilon v_1^0) + b_u(x, u_0 + v_0^0 + \varepsilon v_1^0) C_2 p_\varepsilon \\ &\quad + O(\varepsilon^2 + (C_2 p_\varepsilon)^2). \end{aligned} \tag{5.2.31}$$

For the third term of (5.2.31), we have

$$b(x, u_0 + v_0^0 + \varepsilon v_1^0) = b(x, u_0 + v_0^0) + b_u(x, u_0 + v_0^0)\varepsilon v_1^0 + O(\varepsilon^2).$$

A Taylor expansion gives

$$\begin{aligned} & b(x, u_0 + v_0^0) \\ &= b\left(x, u_0(0) + v_0^0 + x u_0'(0) + \frac{x^2}{2} u_0''(\xi)\right), \quad \text{where } \xi \in (0, x^*), \\ &= b\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) + x b_x\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) \\ &\quad + x b_u\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) u_0'(0) + O(x^2) \\ &= b\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) + x b_x\left(0, u_0(0) + v_0^0(\eta, 0)\right) \\ &\quad + x b_u\left(0, u_0(0) + v_0^0(\eta, 0)\right) u_0'(0) \\ &\quad + C_1 x p_\varepsilon\, b_{xu}\left(0, u_0(0) + v_0^0(\eta, p)\right) \frac{\partial v_0^0}{\partial p}(\eta, p)\Big|_{p=p^*} \\ &\quad + C_1 x p_\varepsilon u_0'(0)\, b_{uu}\left(0, u_0(0) + v_0^0(\eta, p)\right) \frac{\partial v_0^0}{\partial p}(\eta, p)\Big|_{p=p^{**}} \\ &\quad + O(x^2), \quad \text{where } p^*, p^{**} \in (0, C_1 p_\varepsilon), \\ &= b\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) + x b_x\left(0, u_0(0) + v_0^0(\eta, 0)\right) \end{aligned}$$

$$+xb_u\left(0, u_0(0) + v_0^0(\eta, 0)\right) u_0'(0) + O\left(x^2 + C_1 x p_\varepsilon\right),$$

using (5.2.13). Also

$$b_u(x, u_0 + v_0^0) = b_u\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) + O(x)$$

$$= b_u\left(0, u_0(0) + v_0^0(\eta, 0)\right) + O(x + C_1 p_\varepsilon).$$

Hence

$$\begin{aligned} &b(x, u_0 + v_0^0 + \varepsilon v_1^0) \\ &= b\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) \\ &\quad + \varepsilon\eta\left(b_x\left(0, u_0(0) + v_0^0(\eta, 0)\right) + b_u\left(0, u_0(0) + v_0^0(\eta, 0)\right) u_0'(0)\right) \\ &\quad + b_u\left(0, u_0(0) + v_0^0(\eta, 0)\right) \varepsilon v_1^0 + O\left(\varepsilon^2 + x^2 + x\varepsilon + C_1 x p_\varepsilon + C_1 \varepsilon p_\varepsilon\right) \end{aligned} \tag{5.2.32}$$

and

$$\begin{aligned} &b_u(x, u_0 + v_0^0 + \varepsilon v_1^0) C_2 p_\varepsilon \\ &= b_u(x, u_0) C_2 p_\varepsilon + b_{uu}(x, u^*) v_0^0 C_2 p_\varepsilon + O\left(C_2 \varepsilon p_\varepsilon\right), \end{aligned} \tag{5.2.33}$$

where $u^* \in (u_0, u_0 + v_0^0)$.

Therefore, by (5.2.31) - (5.2.33),

$$\begin{aligned} F_\varepsilon \beta(x, \varepsilon) = & -\ddot{v}_0^0(\eta, C_1 p_\varepsilon) + b\left(0, u_0(0) + v_0^0(\eta, C_1 p_\varepsilon)\right) \\ & + \varepsilon\left\{-\ddot{v}_1^0(\eta) + b_u\left(0, u_0(0) + v_0^0(\eta, 0)\right) v_1^0(\eta)\right. \\ & \left. + \eta\left(b_x\left(0, u_0(0) + v_0^0(\eta, 0)\right) + b_u\left(0, u_0(0) + v_0^0(\eta, 0)\right) u_0'(0)\right)\right\} \\ & + b_u(x, u_0) C_2 p_\varepsilon + b_{uu}(x, u^*) v_0^0(\eta, C_1 p_\varepsilon) C_2 p_\varepsilon \\ & + O\left(\varepsilon^2 + x^2 + x\varepsilon + C_1 x p_\varepsilon + C_1 \varepsilon p_\varepsilon + C_2 \varepsilon p_\varepsilon\right) \end{aligned}$$

$$= (C_1 + C_2 b_{uu}(x, u^*))\, p_\varepsilon v_0^0(\eta, C_1 p_\varepsilon) + b_u(x, u_0) C_2 p_\varepsilon$$
$$+ O\left(\varepsilon^2 + x^2 + C_1 x p_\varepsilon + x\varepsilon + C_1 \varepsilon p_\varepsilon + C_2 \varepsilon p_\varepsilon\right),$$

by (5.2.15) and (5.2.19). Thus, there exists a constant $\bar{C} > 0$, independent of C_1, C_2 and ε, such that

$$\begin{aligned} F_\varepsilon \beta(x,\varepsilon) &\geq (C_1 - \bar{C} C_2) p_\varepsilon v_0^0(\eta, C_1 p_\varepsilon) \\ &\quad + b_0^2 C_2 p_\varepsilon - \bar{C}\left(\varepsilon^2 + x^2 + x\varepsilon + C_1 x p_\varepsilon + C_1 \varepsilon p_\varepsilon + C_2 \varepsilon p_\varepsilon\right) \\ &\geq (C_1 - C_2 \bar{C}) p_\varepsilon v_0^0(\eta, C_1 p_\varepsilon) \\ &\quad + \left(b_0^2 C_2 - \bar{C}\left(1 + (C_1 + C_2)\varepsilon \ln(1/\varepsilon)\right)\right) p_\varepsilon, \end{aligned}$$

by (5.2.30) and the definition of p_ε. Choosing C_1 and C_2 such that $b_0^2 C_2 > 2\bar{C}$ and $C_1 > \bar{C} C_2$ and taking ε_0 sufficiently small, so that $(C_1 + C_2)\varepsilon_0 \ln(1/\varepsilon_0) < 1$, then for $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} F_\varepsilon \beta(x,\varepsilon) &\geq (C_1 - C_2 \bar{C}) p_\varepsilon v_0^0(\eta, C_1 p_\varepsilon) + \left(b_0^2 C_2 - 2\bar{C}\right) p_\varepsilon \\ &> 0, \quad \text{for } x \in [0, x^*), \end{aligned}$$

since $v_0^0(\eta, C_1 p_\varepsilon) > 0$ by Lemma 5.2.4.

We now deal with the case $x \in [x^*, 1/2]$. Take $p = C_1 p_\varepsilon$ and $\delta = b_0/4$ in (5.2.24). Then, when ε is so small that $C_1 p_\varepsilon < b_0^2/4$, we get

$$\begin{aligned} \left|\frac{\partial^j w}{\partial x^j}(x, \varepsilon, p_\varepsilon)\right| &\leq \bar{C} \varepsilon^{-j}\left(\exp\left(-b_0 x/2\varepsilon\right) + \exp\left(-b_0(1-x)/2\varepsilon\right)\right) \\ &\leq \bar{C} \varepsilon^{-j+2}, \end{aligned}$$

for $x \in [x^*, 1/2]$ and $j = 0, 1, \ldots, J$, from the definition of x^* in (5.2.30). Hence

$$F_\varepsilon \beta(x,\varepsilon) = -\varepsilon^2 \beta''(x,\varepsilon) + b(x, \beta)$$

$$
\begin{aligned}
&= -\varepsilon^2 \left(u_0''(x) + \frac{\partial^2 w}{\partial x^2}(x, \varepsilon, C_1 p_\varepsilon) \right) + b(x, u_0 + w + C_2 p_\varepsilon) \\
&= b(x, u_0) + b_u(x, u_0)(w + C_2 p_\varepsilon) + O\left(\varepsilon^2 + \varepsilon^2 C_2 p_\varepsilon + (C_2 p_\varepsilon)^2\right) \\
&= b_u(x, u_0)(w + C_2 p_\varepsilon) + O\left(\varepsilon^2 + \varepsilon^2 C_2 p_\varepsilon + (C_2 p_\varepsilon)^2\right) \\
&\geq b_0^2 \left(C_2 p_\varepsilon - \bar{C}\varepsilon^2\right) - \bar{C}\left(\varepsilon^2 + \varepsilon^2 C_2 p_\varepsilon + (C_2 p_\varepsilon)^2\right) \\
&> 0,
\end{aligned}
$$

by arguments similar to the case $x \in [0, x^*)$. This completes the proof of

$$F_\varepsilon \beta(x, \varepsilon) > 0, \quad \text{for } x \in X. \tag{5.2.34}$$

Analogously, one may show that

$$F_\varepsilon \alpha(x, \varepsilon) < 0, \quad \text{for } x \in X. \tag{5.2.35}$$

Combining (5.2.34) and (5.2.35) with (5.2.27) – (5.2.29) concludes the proof. □

Theorem 5.2.1 *Under the same hypotheses as in Lemma 5.2.5, problem (B) has a solution $u(x)$, which is the only solution satisfying*

$$\alpha(x, \varepsilon) \leq u(x) \leq \beta(x, \varepsilon), \quad \textit{for } x \in X. \tag{5.2.36}$$

Here $\beta(x, \varepsilon)$ and $\alpha(x, \varepsilon)$ are the super and sub solutions given by (5.2.25) and (5.2.26).

Proof. Corollary 3.1 of D'Annunzio [9] tells us that if problem (B) has a super solution $\beta(x, \varepsilon)$ and a sub solution $\alpha(x, \varepsilon)$, then there exists a solution $u(x)$ of problem (B) such that

$$\alpha(x, \varepsilon) \leq u(x) \leq \beta(x, \varepsilon), \quad \text{for } x \in X.$$

Hence the existence of a solution follows from Lemma 5.2.5 above. The uniqueness of the solution satisfying (5.2.36) can be shown by arguments similar to those of Theorem 3.6 in D'Annunzio [9]. □

From the definition of our super solution $\beta(x,\varepsilon)$ and sub solution $\alpha(x,\varepsilon)$ and recalling (5.2.23), one can see that

$$|\beta(x,\varepsilon)-\alpha(x,\varepsilon)| \leq C\varepsilon^2 \ln^2(1/\varepsilon), \quad \text{for } x \in X.$$

This shows that we have tighter control on the solution $u(x)$ of Theorem 5.2.1 than in Corollary 3.4 in D'Annunzio [9], where the super and sub solutions yield only an $O(\varepsilon)$ estimate of u.

5.3 Discretizations and Truncation Errors on Shishkin Meshes

We analyse the truncation errors of two schemes applied to problem (A) on Shishkin meshes.

For a given positive integer N, we denote by X^N an arbitrary mesh

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1,$$

with $h_i = x_i - x_{i-1}$, for $i = 1,\ldots,N$, and $\bar{h}_i = (h_i + h_{i+1})/2$ for $i = 1,\ldots,N-1$.

We shall denote by $\mathcal{R}^{N+1}$ the real $N+1$ dimensional linear space of all column vectors

$$z = (z_0, z_1, \ldots, z_N)^T.$$

In what follows, for any function $y \in C[X]$, we shall abuse the notation by also writing $y \in \mathcal{R}^{N+1}$ with $y_i = y(x_i)$ for $i = 0,1,\ldots,N$.

The space $\mathcal{R}^{N+1}$ will be assumed to be equipped with the usual l_∞-norm:

$$\|z\|_\infty = \max_{0 \le i \le N} |z_i|.$$

The induced norm of a linear mapping $\bar{A} = (a_{ij}) : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ is given by

$$\|\bar{A}\|_\infty = \max_{0 \le i \le N} \sum_{j=0}^{N} |a_{ij}|.$$

Let A be the $(N+1) \times (N+1)$ tridiagonal matrix defined by

$$A = \begin{pmatrix} 1 & 0 & 0 & & & & \\ r_1^- & r_1^c & r_1^+ & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & r_{N-1}^- & r_{N-1}^c & r_{N-1}^+ \\ & & & & 0 & 0 & 1 \end{pmatrix},$$

where

$$r_i^- = \frac{1}{h_i \bar{h}_i}, \quad r_i^c = -\frac{2}{h_i h_{i+1}}, \quad r_i^+ = \frac{1}{h_{i+1} \bar{h}_i}.$$

Let $B : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ be the mapping:

$$(Bz)_i = \begin{cases} 0, & \text{for } i = 0, \\ s_i^- b(x_{i-1}, z_{i-1}) + s_i^c b(x_i, z_i) + s_i^+ b(x_{i+1}, z_{i+1}), & \\ \quad \text{for } i = 1, \dots, N-1, & \\ 0, & \text{for } i = N, \end{cases} \tag{5.3.1}$$

where s_i^-, s_i^c and s_i^+ are as yet unspecified. Set

$$F = -\varepsilon^2 A + B. \tag{5.3.2}$$

We shall use $\{F, X^N\}$ to denote the three-point scheme

$$Fu_N = 0. \tag{5.3.3}$$

Define $(F_\varepsilon u)(0) = u(0)$ and $(F_\varepsilon u)(1) = u(1)$. The truncation error of F in approximating F_ε is defined by $\|Fu - F_\varepsilon u\|_\infty = \|Fu\|_\infty$, where $u(x)$ is the solution of problem (A). It is clear that $(Fu)_0 = (Fu)_N = 0$. We shall bound $|(Fu)_i|$, for $i = 1, 2, \ldots, N-1$, in the truncation error analysis of this section.

5.3.1 The Mesh

Since $u'(x)$ is in general unbounded in the boundary layers at $x = 0$ and $x = 1$ when $\varepsilon \to 0$, a polynomial based discretization cannot be consistent uniformly in ε, unless it is constructed on a special mesh. In the literature, several types of special meshes have been introduced for singularly perturbed two–point boundary value problems; see Herceg [21], D'Annunzio [9] and Gartland [17]. In this chapter we shall employ a Shishkin mesh [37], which is piecewise equidistant and consequently much simpler than the above meshes.

Given positive integers m and N, where N is divisible by 4, the Shishkin mesh X_m^N is constructed by dividing the interval $[0, 1]$ into the three subintervals

$$[0, \sigma_m], \quad [\sigma_m, 1-\sigma_m], \quad \text{and} \quad [1-\sigma_m, 1].$$

Equidistant meshes are then used on each subinterval, with $1 + N/4$ points in each of $[0, \sigma_m]$ and $[1-\sigma_m, 1]$, and $1 + N/2$ points in $[\sigma_m, 1-\sigma_m]$. The parameter σ_m is defined by

$$\sigma_m = \min\left\{1/4, \quad m b_0^{-1} \varepsilon \ln N\right\}, \tag{5.3.4}$$

which depends on ε, N and m. The basic idea here is to use a fine mesh to resolve part of the boundary layers.

More explicitly, we have

$$X_m^N : 0 = x_0 < x_1 < \ldots < x_{i_0} < \ldots < x_{N-i_0} < \ldots < x_N = 1,$$

with $i_0 = N/4$, $x_{i_0} = \sigma_m$, $x_{N-i_0} = 1 - \sigma_m$, and

$$h_i = 4\sigma_m N^{-1}, \quad \text{for } i = 1, \dots, i_0, N - i_0 + 1, \dots, N, \tag{5.3.5}$$

$$h_i = 2(1 - 2\sigma_m)N^{-1}, \quad \text{for } i = i_0 + 1, \dots, N - i_0. \tag{5.3.6}$$

If $\sigma_m = 1/4$, i.e., $1/4 \le m b_0^{-1} \varepsilon \ln N$, then N^{-1} is very small relative to ε. This is unlikely in practice (and in this case the method can be analysed using standard techniques). We therefore assume that

$$\sigma_m = m b_0^{-1} \varepsilon \ln N. \tag{5.3.7}$$

From (5.3.5) and (5.3.6), it is clear that the interval lengths satisfy

$$h_i = 4 m b_0^{-1} \varepsilon N^{-1} \ln N, \tag{5.3.8}$$

for $i = 1, \dots, i_0, N - i_0 + 1, \dots, N$, and

$$N^{-1} \le h_i \le 2N^{-1}, \tag{5.3.9}$$

for $i = i_0 + 1, \dots, N - i_0$.

5.3.2 The D–scheme

The D–scheme is described by (5.3.3) with

$$s_i^- = 0, \quad s_i^c = 1 \quad \text{and} \quad s_i^+ = 0.$$

We shall denote by $F_D = -\varepsilon^2 A + B_D$ the mapping corresponding to this scheme.

Lemma 5.3.1 *Let u be the solution of problem (A). Then on the Shishkin mesh X_2^N, the truncation error of the D–scheme satisfies*

$$\|F_D u\|_\infty \le C\left(\varepsilon^2 N^{-1} + N^{-2} \ln^2 N\right). \tag{5.3.10}$$

Proof. Let $i \in \{1, 2, \dots, N-1\}$. By a Taylor expansion, there exist $\xi_i \in (x_{i-1}, x_i)$ and $\eta_i \in (x_i, x_{i+1})$ such that the truncation error of the scheme is

$$
\begin{aligned}
(F_D u)_i &= -\frac{\varepsilon^2}{h_i \bar{h}_i} u(x_{i-1}) + \frac{2\varepsilon^2}{h_i h_{i+1}} u(x_i) - \frac{\varepsilon^2}{h_{i+1}\bar{h}_i} u(x_{i+1}) + b(x_i, u(x_i)) \\
&= -\frac{\varepsilon^2}{h_i \bar{h}_i}\left(u(x_i) - h_i u'(x_i) + \frac{h_i^2}{2} u''(x_i) - \frac{h_i^3}{6} u'''(x_i) + \frac{h_i^4}{24} u^{(4)}(\xi_i)\right) \\
&\quad + \frac{2\varepsilon^2}{h_i h_{i+1}} u(x_i) \\
&\quad - \frac{\varepsilon^2}{h_{i+1}\bar{h}_i}\left(u(x_i) + h_{i+1} u'(x_i) + \frac{h_{i+1}^2}{2} u''(x_i) + \frac{h_{i+1}^3}{6} u'''(x_i)\right. \\
&\quad \left. + \frac{h_{i+1}^4}{24} u^{(4)}(\eta_i)\right) + b(x_i, u(x_i)) \\
&= -\varepsilon^2 u''(x_i) + b(x_i, u(x_i)) \\
&\quad + \frac{h_i^2 - h_{i+1}^2}{6\bar{h}_i}\varepsilon^2 u'''(x_i) - \frac{h_i^3}{24\bar{h}_i}\varepsilon^2 u^{(4)}(\xi_i) - \frac{h_{i+1}^3}{24\bar{h}_i}\varepsilon^2 u^{(4)}(\eta_i) \\
&= \frac{h_i^2 - h_{i+1}^2}{6\bar{h}_i}\varepsilon^2 u'''(x_i) - \frac{h_i^3}{24\bar{h}_i}\varepsilon^2 u^{(4)}(\xi_i) - \frac{h_{i+1}^3}{24\bar{h}_i}\varepsilon^2 u^{(4)}(\eta_i),
\end{aligned}
\tag{5.3.11}
$$

by (5.1.1a).

On the other hand, if a Taylor expansion with integral remainder is used, we obtain instead

$$
\begin{aligned}
(F_D u)_i &= -\frac{\varepsilon^2}{h_i \bar{h}_i}\left(u(x_i) - h_i u'(x_i) + \frac{h_i^2}{2} u''(x_i)\right. \\
&\qquad \left. - \frac{1}{2}\int_{x_{i-1}}^{x_i} (s - x_{i-1})^2 u'''(s)\, ds\right) + \frac{2\varepsilon^2}{h_i h_{i+1}} u(x_i) \\
&\quad - \frac{\varepsilon^2}{h_{i+1}\bar{h}_i}\left(u(x_i) + h_{i+1} u'(x_i) + \frac{h_{i+1}^2}{2} u''(x_i)\right. \\
&\qquad \left. + \frac{1}{2}\int_{x_i}^{x_{i+1}} (x_{i+1} - s) u'''(s)\, ds\right) + b(x_i, u(x_i)) \\
&= \frac{\varepsilon^2}{2h_i \bar{h}_i}\int_{x_{i-1}}^{x_i} (s - x_{i-1})^2 u'''(s)\, ds - \frac{\varepsilon^2}{2h_{i+1}\bar{h}_i}\int_{x_i}^{x_{i+1}} (x_{i+1} - s) u'''(s)\, ds.
\end{aligned}
\tag{5.3.12}
$$

Using the decomposition (5.2.1), one may split the truncation error in the form

$$(F_D u)_i = (I_Y)_i + (I_V)_i. \tag{5.3.13}$$

Here for any $y \in C^4(X)$, we define

$$(I_y)_i = \frac{h_i^2 - h_{i+1}^2}{6\bar{h}_i}\varepsilon^2 y'''(x_i) - \frac{h_i^3}{24\bar{h}_i}\varepsilon^2 y^{(4)}(\xi_i) - \frac{h_{i+1}^3}{24\bar{h}_i}\varepsilon^2 y^{(4)}(\eta_i), \tag{5.3.14}$$

where $\xi_i \in (x_{i-1}, x_i)$ and $\eta_i \in (x_i, x_{i+1})$ depend now on the function y, or, equivalently,

$$(I_y)_i = \frac{\varepsilon^2}{2h_i\bar{h}_i}\int_{x_{i-1}}^{x_i}(s - x_{i-1})^2 y'''(s)\,ds - \frac{\varepsilon^2}{2h_{i+1}\bar{h}_i}\int_{x_i}^{x_{i+1}}(x_{i+1} - s)y'''(s)\,ds. \tag{5.3.15}$$

It is easy to see from (5.3.14), (5.2.2), (5.3.8) and (5.3.9) that

$$|(I_Y)_i| \le C\varepsilon^2 N^{-1}. \tag{5.3.16}$$

We now bound $(I_V)_i$. For $i \in \{1, \ldots, i_0 - 1\} \cup \{N - i_0 + 1, \ldots, N - 1\}$, the first term of (5.3.14) is zero, because of the uniformity of the mesh on $[0, \sigma_2]$ and $[1 - \sigma_2, 1]$. Hence, for these values of i,

$$|(I_V)_i| \le CN^{-2}\ln^2 N,$$

by (5.3.14), (5.2.3) and (5.3.8). If $i = i_0$, then from (5.3.15) and (5.2.3) we have

$$\begin{aligned}
|(I_V)_{i_0}| &\le C\varepsilon^{-1}\int_{x_{i_0-1}}^{x_{i_0+1}}(\exp(-b_0 s/\varepsilon) + \exp(-b_0(1 - s)/\varepsilon))\,ds \\
&= \frac{C}{b_0}(\exp(-b_0 x_{i_0-1}/\varepsilon) - \exp(-b_0 x_{i_0+1}/\varepsilon) \\
&\quad + \exp(-b_0(1 - x_{i_0+1})/\varepsilon) - \exp(-b_0(1 - x_{i_0-1})/\varepsilon)) \\
&\le C\exp(-b_0 x_{i_0-1}/\varepsilon) \\
&= CN^{-2}\exp(-b_0 h_{i_0}/\varepsilon), \text{ since } x_{i_0} = \sigma_2 = 2\varepsilon b_0^{-1}\ln N, \\
&\le CN^{-2}.
\end{aligned}$$

Analogously, one may show that

$$|(I_V)_i| \leq CN^{-2}, \quad \text{for } i = i_0 + 1, \dots N - i_0.$$

Thus

$$|(I_V)_i| \leq CN^{-2}\ln^2 N, \quad \text{for } i = 1, \dots N - 1.$$

Combining this with (5.3.16) and (5.3.13) completes the proof. □

Under the reasonable assumption $\varepsilon \leq N^{-1}$, the estimate (5.3.10) becomes $\|F_D u\|_\infty \leq CN^{-2}\ln^2 N$. This is much better than the $O(h)$ result obtained by D'Annunzio [9] for the same scheme with a more complicated mesh, where h is the maximum mesh spacing.

5.3.3 The H–scheme

We now take

$$s_i^- = \frac{h_i^2 - h_{i+1}^2 + h_i h_{i+1}}{12 h_i \bar{h}_i}, \quad s_i^c = \frac{h_i^2 + h_{i+1}^2 + 3h_i h_{i+1}}{6 h_i h_{i+1}}$$

and

$$s_i^+ = \frac{h_{i+1}^2 - h_i^2 + h_i h_{i+1}}{12 h_{i+1} \bar{h}_i} \tag{5.3.17}$$

in (5.3.1). The mapping corresponding to the H–scheme will be referred to as $F_H = -\varepsilon^2 A + B_H$.

This scheme can be found in Herceg [21]. He derives the scheme by using a difference formula of Hermite type to approximate the differential equation (5.1.1a) and requiring this formula to be exact on all polynomials of degree at most 4. The scheme can be also constructed by modifying a finite element scheme, as we now demonstrate.

We use piecewise linear "hat" functions as our trial and test functions, viz., set

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/h_i, & \text{for } x \in (x_{i-1}, x_i), \\ (x_{i+1} - x)/h_{i+1}, & \text{for } x \in (x_i, x_{i+1}), \\ 0, & \text{elsewhere,} \end{cases}$$

for $i = 1, \ldots, N-1$. Then the basis function space S^N is taken to be the linear span of $\{\varphi_i : i = 1, \ldots, N-1\}$. A finite element scheme is defined as follows: find $u_N = \sum_{i=1}^{N-1} u_N(x_i)\varphi_i(x) \in S^N$ such that

$$\varepsilon^2 \left(u_N', \varphi_i'\right) + \left(\hat{b}, \varphi_i\right) = 0, \quad \text{for } i = 1, \ldots, N-1, \tag{5.3.18}$$

where we denote by $\hat{b}$ the piecewise linear interpolant to b, viz.,

$$\hat{b}(x) = \frac{x_i - x}{h_i} b(x_{i-1}, u_N(x_{i-1})) + \frac{x - x_{i-1}}{h_i} b(x_i, u_N(x_i)),$$

for $x \in (x_{i-1}, x_i)$ and $i = 1, \ldots, N$.

Note that $u_N(x_0) = u_N(x_N) = 0$. We write (5.3.18) together with these boundary conditions as $F_B u_N = 0$, where $F_B : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$. A lengthy calculation shows that for $i = 1, \ldots, N-1$, we obtain for its truncation error

$$\begin{aligned} (F_B u)_i = {} & \frac{h_i^3 + h_{i+1}^3}{24\bar{h}_i} \frac{d^2}{dx^2} b(x, u(x))\bigg|_{x=x_i} - \frac{7(h_i^4 - h_{i+1}^4)}{360\bar{h}_i} \varepsilon^2 u^{(5)}(x_i) \\ & + \frac{h_i^5}{720\bar{h}_i} \varepsilon^2 \left(-u^{(6)}(\bar{\xi}_i) + 5u^{(6)}(\xi_i^*)\right) \\ & + \frac{h_{i+1}^5}{720\bar{h}_i} \varepsilon^2 \left(-u^{(6)}(\bar{\eta}_i) + 5u^{(6)}(\eta_i^*)\right), \end{aligned} \tag{5.3.19}$$

where $\bar{\xi}_i$, $\xi_i^* \in (x_{i-1}, x_i)$ and $\bar{\eta}_i$, $\eta_i^* \in (x_i, x_{i+1})$. Now set

$$\begin{aligned} (F_H z)_i = (F_B z)_i - \frac{h_i^3 + h_{i+1}^3}{24\bar{h}_i} \bigg(& \frac{1}{h_i \bar{h}_i} b(x_{i-1}, z_{i-1}) \\ & - \frac{2}{h_i h_{i+1}} b(x_i, z_i) + \frac{1}{h_{i+1}\bar{h}_i} b(x_{i+1}, z_{i+1}) \bigg), \end{aligned}$$

for $i = 1, \ldots, N-1$. This yields a higher order scheme

$$F_H u_N = 0. \tag{5.3.20}$$

By a simple calculation, we see that F_H has the form of (5.3.2) with s_i^-, s_i^c and s_i^+ given by (5.3.17).

Lemma 5.3.2 *Let u be the solution of problem (A). Then on the Shishkin mesh X_4^N, the truncation error of the H-scheme satisfies*

$$\|F_H u\|_\infty \le C\left(\varepsilon^2 N^{-3} + N^{-4}\ln^4 N\right).$$

Proof. Using (5.3.19) and a Taylor expansion, we obtain

$$\begin{aligned}
(F_H u)_i = {} & \frac{h_i^3 + h_{i+1}^3}{24\bar{h}_i}\left(-\frac{h_i^2 - h_{i+1}^2}{6\bar{h}_i}\frac{d^3}{dx^3}b(x,u(x))\bigg|_{x=x_i}\right.\\
& \left. + \frac{h_i^3}{24\bar{h}_i}\frac{d^4}{dx^4}b(x,u(x))\bigg|_{x=\xi_i'} \frac{h_{i+1}^3}{24\bar{h}_i}\frac{d^4}{dx^4}b(x,u(x))\bigg|_{x=\eta_i'}\right)\\
& -\frac{7(h_i^4 - h_{i+1}^4)}{360\bar{h}_i}\varepsilon^2 u^{(5)}(x_i) + \frac{h_i^5}{720\bar{h}_i}\varepsilon^2\left(-u^{(6)}(\bar{\xi}_i) + 5u^{(6)}(\xi_i^*)\right)\\
& +\frac{h_{i+1}^5}{720\bar{h}_i}\varepsilon^2\left(-u^{(6)}(\bar{\eta}_i) + 5u^{(6)}(\eta_i^*)\right),\\
& \qquad \text{where } \xi_i',\ \bar{\xi}_i,\ \xi_i^* \in (x_{i-1}, x_i) \text{ and } \eta_i',\ \bar{\eta}_i,\ \eta_i^* \in (x_i, x_{i+1}),\\
= {} & \left(-\frac{(h_i^3 + h_{i+1}^3)(h_i^2 - h_{i+1}^2)}{144\bar{h}_i^2} + \frac{7(h_i^4 - h_{i+1}^4)}{360\bar{h}_i}\right)\varepsilon^2 u^{(5)}(x_i)\\
& +\frac{h_i^3 + h_{i+1}^3}{576\bar{h}_i^2}\varepsilon^2\left(h_i^3 u^{(6)}(\xi_i') + h_{i+1}^3 u^{(6)}(\eta_i')\right)\\
& +\frac{h_i^5}{720\bar{h}_i}\varepsilon^2\left(-u^{(6)}(\bar{\xi}_i) + 5u^{(6)}(\xi_i^*)\right)\\
& +\frac{h_{i+1}^5}{720\bar{h}_i}\varepsilon^2\left(-u^{(6)}(\bar{\eta}_i) + 5u^{(6)}(\eta_i^*)\right).
\end{aligned}$$

On the other hand, by a Taylor expansion with integral remainder and using

$s_i^- + s_i^c + s_i^+ = 1$, we have

$$(F_H u)_i = -\frac{\varepsilon^2}{h_i \bar{h}_i}\left(u(x_i) - h_i u'(x_i) + \frac{h_i^2}{2}u''(x_i) - \frac{1}{2}\int_{x_{i-1}}^{x_i}(t - x_{i-1})^2 u'''(t)\,dt\right)$$
$$+\frac{2\varepsilon^2}{h_i h_{i+1}}u(x_i) - \frac{\varepsilon^2}{h_{i+1}\bar{h}_i}\left(u(x_i) + h_{i+1}u'(x_i) + \frac{h_{i+1}^2}{2}u''(x_i)\right.$$
$$\left.+\frac{1}{2}\int_{x_i}^{x_{i+1}}(x_{i+1} - t)u'''(t)\,dt\right)$$
$$+s_i^-\left(b(x_i, u(x_i)) - \int_{x_{i-1}}^{x_i}\frac{d}{dt}b(t, u(t))\,dt\right)$$
$$+s_i^c b(x_i, u(x_i))$$
$$+s_i^+\left(b(x_i, u(x_i)) + \int_{x_i}^{x_{i+1}}\frac{d}{dt}b(t, u(t))\,dt\right)$$
$$= \frac{\varepsilon^2}{2h_i\bar{h}_i}\int_{x_{i-1}}^{x_i}(t - x_{i-1})^2 u'''(t)\,dt - s_i^-\varepsilon^2\int_{x_{i-1}}^{x_i}u'''(t)\,dt$$
$$-\frac{\varepsilon^2}{2h_{i+1}\bar{h}_i}\int_{x_i}^{x_{i+1}}(x_{i+1} - t)u'''(t)\,dt + s_i^+\varepsilon^2\int_{x_i}^{x_{i+1}}u'''(t)\,dt,$$

where s_i^- and s_i^+ are given by (5.3.17).

Recalling the choice of σ_4, one may show the desired estimate by arguments similar to those of Lemma 5.3.1. □

5.4 Uniform Convergence of the Schemes for Problem (A)

We investigate the existence, uniqueness and uniform convergence of solutions of the D–scheme and the H–scheme on the Shishkin meshes X_m^N for problem (A). We prove that the D–scheme has a unique solution by employing Hadamard's Theorem. We give an $O(\varepsilon^2 N^{-1} + N^{-2}\ln^2 N)$ error bound for this scheme. This bound is uniform in ε. In a separate argument, we analyse the H–scheme with and without conditions (5.1.5). Under the conditions (5.1.5), a similar uniqueness result is obtained for the

H–scheme and the uniform error bound is proved to be $O(\varepsilon^2 N^{-3} + N^{-4}\ln^4 N)$. In order to dispense with the strong conditions (5.1.5), we use degree theory and a local version of the Hadamard Theorem. Existence, local uniqueness of a solution and uniform accuracy of order $\varepsilon^2 N^{-3} + N^{-4}\ln^4 N$ for the H–scheme are proved without the conditions (5.1.5).

It is easy to see that F, as defined in (5.3.2), is continuously differentiable on $\mathcal{R}^{N+1}$. The Frechet–derivative $F'(z)$ of F at any $z = (z_0, z_1, \ldots, z_N)^T \in \mathcal{R}^{N+1}$ is the tridiagonal matrix

$$F'(z) = \begin{pmatrix} 1 & 0 & 0 & & & \\ f_1^- & f_1^c & f_1^+ & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & f_{N-1}^- & f_{N-1}^c & f_{N-1}^+ \\ & & & 0 & 0 & 1 \end{pmatrix},$$

where for $i = 1, 2, \ldots, N-1$,

$$f_i^- = -\varepsilon^2 r_i^- + s_i^- b_u(x_{i-1}, z_{i-1}), \tag{5.4.1}$$

$$f_i^c = -\varepsilon^2 r_i^c + s_i^c b_u(x_i, z_i), \tag{5.4.2}$$

$$f_i^+ = -\varepsilon^2 r_i^+ + s_i^+ b_u(x_{i+1}, z_{i+1}). \tag{5.4.3}$$

Set

$$\mu = \min_{1 \le i \le N-1} \left\{ |f_i^c| - |f_i^-| - |f_i^+| \right\}.$$

For each scheme, we shall find a constant $\mu^* > 0$, which is independent of N and ε, such that

$$\mu \ge \mu^* > 0, \quad \text{for all } z \in S, \tag{5.4.4}$$

where S is an open ball in $\mathcal{R}^{N+1}$. By Theorem A of Varga [45], (5.4.4) implies that

$F'(z)^{-1}$ exists and

$$\|F'(z)^{-1}\|_\infty \le \frac{1}{\min\{1,\mu^*\}}, \quad \text{for } z \in S. \tag{5.4.5}$$

This inequality plays an important role in proving existence, uniqueness and uniform convergence of a discrete solution of both the D–scheme and the H–scheme for problem (A).

We also use degree theory to analyse the local uniqueness of a solution of the H–scheme.

5.4.1 Uniform Convergence of the D–scheme

We use Hadamard's Theorem to show the uniform convergence of the D–scheme on the Shishkin mesh for problem (A).

Theorem 5.4.1 *Assume that (5.1.1) and (5.1.2) hold. Let u denote the solution of problem (A). For any arbitrary mesh X^N, the D-scheme $\{F_D, X^N\}$ has a unique solution in $\mathcal{R}^{N+1}$. If $u_N \in \mathcal{R}^{N+1}$ is the solution of $\{F_D, X_2^N\}$, then*

$$\|u - u_N\|_\infty \le C\left(\varepsilon^2 N^{-1} + N^{-2}\ln^2 N\right). \tag{5.4.6}$$

Proof. For the D–scheme on an arbitrary mesh X^N, we have in (5.4.1) - (5.4.3)

$$f_i^- = -\frac{\varepsilon^2}{h_i\bar{h}_i}, \quad f_i^c = \frac{2\varepsilon^2}{h_i h_{i+1}} + b_u(x_i, z_i), \quad f_i^+ = -\frac{\varepsilon^2}{h_{i+1}\bar{h}_i}.$$

Hence, for $i = 1, 2, \ldots, N-1$,

$$\begin{aligned} |f_i^c| - |f_i^-| - |f_i^+| &= b_u(x_i, z_i) \\ &\ge b_0^2, \quad \text{for all } z \in \mathcal{R}^{N+1}, \end{aligned}$$

by (5.1.2). Thus Theorem A of Varga [45] implies that $F'_D(z)^{-1}$ exists and

$$\|F'_D(z)^{-1}\|_\infty \le \frac{1}{\min\{1, b_0^2\}}, \quad \text{for all } z \in \mathcal{R}^{N+1}. \tag{5.4.7}$$

It follows from Hadamard's Theorem (see Theorem 5.3.10 of Ortega and Rheinboldt [32]) that F_D is a homeomorphism of $\mathcal{R}^{N+1}$ onto $\mathcal{R}^{N+1}$. This implies that $F_D u_N = 0$ has a unique solution in $\mathcal{R}^{N+1}$. Moreover by the inverse function theorem, the function F_D^{-1} is continuously differentiable on $\mathcal{R}^{N+1}$ and

$$(F_D^{-1})'(F_D z) = F'_D(z)^{-1}, \quad \text{for all } z \in \mathcal{R}^{N+1}. \tag{5.4.8}$$

From Lemma 5.3.1, we have

$$\|F_D u - F_D u_N\|_\infty \le C\left(\varepsilon^2 N^{-1} + N^{-2}\ln^2 N\right), \tag{5.4.9}$$

where u_N is the solution of $\{F_D, X_2^N\}$.

Now by Theorem 3.2.3 of Ortega and Rheinboldt [32], we obtain

$$\begin{aligned}
&\|u - u_N\|_\infty \\
&= \|F_D^{-1}(F_D u) - F_D^{-1}(F_D u_N)\|_\infty \\
&\le \sup_{0 \le t \le 1} \|(F_D^{-1})'(F_D u + t(F_D u - F_D u_N))\|_\infty \cdot \|F_D u - F_D u_N\|_\infty \\
&= \sup_{0 \le t \le 1} \|(F_D^{-1})'(F_D z_t)\|_\infty \cdot \|F_D u - F_D u_N\|_\infty,
\end{aligned}$$

for some $z_t \in \mathcal{R}^{N+1}$, since we know that F_D maps onto $\mathcal{R}^{N+1}$. Now by (5.4.7) - (5.4.9),

$$\begin{aligned}
\|u - u_N\|_\infty &\le \frac{1}{\min\{1, b_0^2\}}\|F_D u - F_D u_N\|_\infty \\
&\le C\left(\varepsilon^2 N^{-1} + N^{-2}\ln^2 N\right),
\end{aligned}$$

which is the desired result. □

5.4.2 Analysis of Uniform Convergence of the H–scheme

In order to apply the Hadamard Theorem to the H–scheme, we need the extra assumptions (5.1.5) as in Herceg [21].

Theorem 5.4.2 *Assume that (5.1.1), (5.1.2) and (5.1.5) hold. Let*

$$\Delta_2 = \max_{0\le x\le 1} \left\{ |q'(x)|, |Q'(x)| \right\}.$$

Then for $N > 8\Delta_2/\Delta_1$ (cf. (5.1.5)), the H-scheme $\{F_H, X_4^N\}$ has a unique solution $u_N \in \mathcal{R}^{N+1}$. Moreover, with u denoting the solution of problem (A),

$$\|u - u_N\|_\infty \le C\left(\varepsilon^2 N^{-3} + N^{-4}\ln^4 N\right). \tag{5.4.10}$$

Proof. For the scheme $\{F_H, X^N\}$, we have in (5.4.1) – (5.4.3)

$$\begin{aligned}
f_i^- &= -\frac{\varepsilon^2}{h_i\bar{h}_i} + \frac{h_i^2 - h_{i+1}^2 + h_ih_{i+1}}{12h_i\bar{h}_i} b_u(x_{i-1}, z_{i-1}),\\
f_i^c &= \frac{2\varepsilon^2}{h_ih_{i+1}} + \frac{h_i^2 + h_{i+1}^2 + 3h_ih_{i+1}}{6h_ih_{i+1}} b_u(x_i, z_i),\\
f_i^+ &= -\frac{\varepsilon^2}{h_{i+1}\bar{h}_i} + \frac{h_{i+1}^2 - h_i^2 + h_ih_{i+1}}{12h_{i+1}\bar{h}_i} b_u(x_{i+1}, z_{i+1}),
\end{aligned}$$

for $i = 1, 2, \ldots, N-1$. It follows immediately from Herceg [21] that

$$|f_i^c| - |f_i^-| - |f_i^+| \ge \min\left\{b_0^2, \Delta_1/6 - \Delta_2 h_{i+1}/3\right\}, \quad \text{for } i = 1, 2, \ldots, N-1.$$

On the Shishkin mesh X_4^N, we have $h_i \le 2N^{-1}$, for $i = 1, 2, \ldots, N$. Hence for $N > 8\Delta_2/\Delta_1$,

$$\begin{aligned}
|f_i^c| - |f_i^-| - |f_i^+| &\ge \min\left\{b_0^2, \Delta_1/6 - 2\Delta_2/3N\right\}\\
&\ge \min\left\{b_0^2, \Delta_1/12\right\}, \quad \text{for } i = 1, 2, \ldots, N-1.
\end{aligned}$$

The theorem then follows from Lemma 5.3.2 and arguments similar to those of Theorem 5.4.1. □

The strong assumptions (5.1.5) are imposed to guarantee the uniqueness of a solution of the H–scheme in the whole space $\mathcal{R}^{N+1}$. Since the continuous problem (A) has a unique solution assuming only the condition (5.1.2), we would naturally prefer to eliminate the assumptions (5.1.5) from the discrete problem. The following theorem gives existence of a solution for the scheme $\{F_H, X_4^N\}$ without the condition (5.1.5).

Given $z^0 \in \mathcal{R}^{N+1}$ and $r > 0$, we shall denote by $S(z^0, r)$ the open ball

$$\{z \in \mathcal{R}^{N+1} : \|z - z^0\|_\infty < r\}$$

in $\mathcal{R}^{N+1}$.

Theorem 5.4.3 *Assume that (5.1.1), (5.1.2) hold. Then there exists a constant $C_0 > 0$, independent of ε, such that the H-scheme $\{F_H, X_4^N\}$ has a solution $u_N \in \mathcal{R}^{N+1}$ which satisfies*

$$\|u - u_N\|_\infty \le C_0 \left(\varepsilon^2 N^{-3} + N^{-4} \ln^4 N\right), \tag{5.4.11}$$

for $N \ge N_0$, where N_0 depends on C_0 but is independent of ε.

Proof. On the Shishkin mesh X_4^N, Lemma 5.3.2 yields

$$\|F_H u\|_\infty \le C_1 \left(\varepsilon^2 N^{-3} + N^{-4} \ln^4 N\right),$$

where C_1 is a fixed positive constant, independent of ε and N. Set

$$C_0 = 2C_1 / \min\{1, b_0^2/6\}.$$

Then

$$\frac{1}{\min\{1, b_0^2/6\}} \|F_H u\|_\infty < C_0 \left(\varepsilon^2 N^{-3} + N^{-4} \ln^4 N\right). \tag{5.4.12}$$

We now prove that there exists a positive integer N_0, depending on C_0 but independent of ε, such that for $N \geq N_0$,

$$\|F_H'(z)^{-1}\|_\infty \leq \frac{1}{\min\{1, b_0^2/6\}}, \quad \text{for all } z \in \left(u, C_0 \left(\varepsilon^2 N^{-3} + N^{-4} \ln^4 N\right)\right). \tag{5.4.13}$$

We have, for $i = 1, 2, \ldots, N-1$, in the notation of (5.4.1) - (5.4.3),

$$\begin{aligned}
&|f_i^c| - |f_i^-| - |f_i^+| \\
&\geq \frac{h_i^2 + h_{i+1}^2 + 3h_i h_{i+1}}{6h_i h_{i+1}} b_u(x_i, z_i) - \frac{h_i^2 + h_{i+1}^2 + h_i h_{i+1}}{12 h_i \bar{h}_i} b_u(x_{i-1}, z_{i-1}) \\
&\quad - \frac{h_i^2 + h_{i+1}^2 + h_i h_{i+1}}{12 h_{i+1} \bar{h}_i} b_u(x_{i+1}, z_{i+1}) \\
&= \frac{1}{3} b_u(x_i, z_i) \\
&\quad - \frac{h_i^2 + h_{i+1}^2 + h_i h_{i+1}}{12 h_i \bar{h}_i} \left(-h_i b_{xu}(\bar{x}_i, \bar{z}_i) - (z_i - z_{i-1}) b_{uu}(\bar{x}_i, \bar{z}_i)\right) \\
&\quad - \frac{h_i^2 + h_{i+1}^2 + h_i h_{i+1}}{12 h_{i+1} \bar{h}_i} \left(h_{i+1} b_{xu}(\tilde{x}_i, \tilde{z}_i) - (z_{i+1} - z_i) b_{uu}(\tilde{x}_i, \tilde{z}_i)\right),
\end{aligned} \tag{5.4.14}$$

where $(\bar{x}_i, \bar{z}_i)$ is between (x_{i-1}, z_{i-1}) and (x_i, z_i), and $(\tilde{x}_i, \tilde{z}_i)$ is between (x_i, z_i) and (x_{i+1}, z_{i+1}).

By Lemma 5.2.1, $\max_{x \in X} |u(x)| \leq C_2$ for some positive constant C_2. Hence

$$|b_{xu}(x, z)| + |b_{uu}(x, z)| \leq C, \quad \text{for } (x, z) \in X \times [-C_2 - 1, C_2 + 1]. \tag{5.4.15}$$

Let $z = (z_0, \ldots, z_N) \in S\left(u, C_0 \left(\varepsilon^2 N^{-3} + N^{-4} \ln^4 N\right)\right)$. We choose N_1 such that for all $N \geq N_1$, $C_0 \left(\varepsilon^2 N^{-3} + N^{-4} \ln^4 N\right) \leq 1$. Hence $|z_i| \leq C_2 + 1$ for $i = 0, 1, \ldots, N$. Consequently for $N \geq N_1$,

$$|\bar{z}_i| \leq C_2 + 1 \quad \text{and} \quad |\tilde{z}_i| \leq C_2 + 1, \quad \text{for } i = 1, \ldots, N-1. \tag{5.4.16}$$

On the other hand, using Lemma 5.2.1, one may easily show that on the Shishkin mesh X_4^N

$$|u_i - u_{i-1}| \le CN^{-1}\ln N, \quad \text{for } i = 1, \dots, N.$$

Therefore for $N \ge N_1$,

$$\begin{aligned} |z_i - z_{i-1}| &\le |z_i - u_i| + |z_{i-1} - u_{i-1}| + |u_i - u_{i-1}| \\ &\le 2C_0\left(\varepsilon^2 N^{-3} + N^{-4}\ln^4 N\right) + CN^{-1}\ln N, \end{aligned} \tag{5.4.17}$$

for $i = 1, \dots, N$.

From (5.4.14) - (5.4.17), we obtain for $N \ge N_1$ that

$$|f_i^c| - |f_i^-| - |f_i^+| \ge \frac{1}{3}b_u(x_i, z_i) - C_3 N^{-1}\ln N, \quad \text{for } i = 1, \dots, N,$$

where C_3 is a positive constant which depends on C_0 but is independent of N and ε. Choose $N_2 > 0$ such that $C_3 N^{-1}\ln N < b_0^2/6$ for $N \ge N_2$. Set $N_0 = \max\{N_1, N_2\}$. Then for $N \ge N_0$,

$$|f_i^c| - |f_i^-| - |f_i^+| \ge b_0^2/6, \quad \text{for } i = 1, \dots, N-1.$$

Thus

$$|f_i^c| - |f_i^-| - |f_i^+| \ge \min\{1, b_0^2/6\}, \quad \text{for } i = 0, 1, \dots, N.$$

This yields (5.4.13), by Theorem A of Varga [45].

Now from (5.4.12) and (5.4.13) we see that the mapping $F_H : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ satisfies the conditions of Theorem 5.3.11 of Ortega and Rheinboldt [32]. Hence $F_H u_N = 0$ has a solution $u_N \in S\left(u_0, C_0\left(\varepsilon^2 N^{-3} + N^{-4}\ln^4 N\right)\right)$, i.e., $\|u - u_N\|_\infty \le C_0\left(\varepsilon^2 N^{-3} + N^{-4}\ln^4 N\right)$. □

To prove local uniqueness of solutions of the scheme $\{F_H, X_4^N\}$ for problem (A), we shall use degree theory. In order to do this, we imbed problem (A) in the following family of problems:

$$\tilde{F}_\varepsilon(\tilde{u}, t) \equiv -\varepsilon^2 \tilde{u}_{xx}(x,t) + \tilde{b}(x, t, \tilde{u}(x,t)) = 0 \quad \text{for } x \in (0,1), \tag{5.4.18}$$

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \tag{5.4.19}$$

where $t \in [0,1]$ is a parameter,

$$\tilde{b}(x, t, \tilde{u}(x,t)) = tb(x, \tilde{u}(x,t)) + (1-t)(\tilde{u}(x,t) - u_0(x)), \tag{5.4.20}$$

for $(x, t, \tilde{u}) \in [0,1] \times [0,1] \times \mathcal{R}^1$, and u_0 is the solution of (5.1.3). Clearly, for each x and t, $\tilde{b}(x, t, u_0(x)) = 0$.

Set $\tilde{b}_0^2 = \min\{b_0^2, 1\}$. Then

$$\begin{aligned} \tilde{b}_u(x,t,u) &= tb_u(x,u) + (1-t) \\ &= tb_0^2 + (1-t) \\ &\geq \tilde{b}_0^2, \end{aligned} \tag{5.4.21}$$

for all $(x, u, t) \in [0,1] \times \mathcal{R}^1 \times [0,1]$. Hence, for each t, problem (5.4.18) – (5.4.19) is of the same type as problem (A).

Define the mapping $\tilde{F}_H(\cdot,\cdot) : \mathcal{R}^{N+1} \times [0,1] \to \mathcal{R}^{N+1}$ by

$$\tilde{F}_H(z,t) = -\varepsilon^2 Az + \tilde{B}_H(z,t),$$

where $\tilde{B}_H(\cdot,\cdot) : \mathcal{R}^{N+1} \times [0,1] \to \mathcal{R}^{N+1}$ is given by

$$(\tilde{B}_H(z,t))_i = \begin{cases} 0, & \text{for } i = 0, \\ s_i^- \tilde{b}(x_{i-1}, t, z_{i-1}) + s_i^c \tilde{b}(x_i, t, z_i) + s_i^+ \tilde{b}(x_{i+1}, t, z_{i+1}), & \\ \qquad \text{for } i = 1, \dots, N-1, & \\ 0, & \text{for } i = N. \end{cases} \tag{5.4.22}$$

Here and in the rest of this section s_i^-, s_i^c and s_i^+ are given by (5.3.17). Then the H–scheme for problem (A) is imbedded in the family of schemes

$$\tilde{F}_H(z,t) = 0. \tag{5.4.23}$$

Let us introduce some more notation and definitions.

For z^1 and $z^2 \in \mathcal{R}^{N+1}$, we denote by $z^1 \leq z^2$ (or $z^1 < z^2$) the natural partial ordering on $\mathcal{R}^{N+1}$, i.e., $z_i^1 \leq z_i^2$ (or $z_i^1 < z_i^2$) for $i = 0, 1, \ldots, N$.

Let $M : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ be a mapping. Let α, $\beta \in \mathcal{R}^{N+1}$. If

$$M\alpha < 0, \tag{5.4.24}$$

$$M\beta > 0 \tag{5.4.25}$$

and

$$\alpha < \beta, \tag{5.4.26}$$

then β and α are said to be super and sub solutions of $Mz = 0$, respectively.

Let α, $\beta \in \mathcal{R}^{N+1}$ satisfy $\alpha < \beta$. Let G be a mapping: $\mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$. Define $G^m : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ by

$$(G^m z)_i = \begin{cases} (G\beta)_i + (z_i - \beta_i), & \text{if } z_i \geq \beta_i, \\ (G\beta)_i, & \text{if } \alpha_i < z_i < \beta_i, \\ (G\beta)_i + (\alpha_i - z_i), & \text{if } z_i \leq \alpha_i, \end{cases} \tag{5.4.27}$$

for $i = 0, 1, \ldots, N$. Then G^m is called a modification of G.

We give a strengthening of Theorem 5.1 of D'Annunzio [9].

Lemma 5.4.1 *Let $D = (d_{ij})$ be an $(N+1) \times (N+1)$ matrix satisfying*

$$d_{ij} \leq 0, \quad \text{for } 0 \leq i, j \leq N \text{ and } i \neq j \tag{5.4.28}$$

and

$$\sum_{j=1}^{N} d_{ij} \geq 0, \quad \textit{for } 0 \leq i \leq N. \tag{5.4.29}$$

Let $G : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ be a mapping. Let α, $\beta \in \mathcal{R}^{N+1}$ satisfy $\alpha < \beta$. Let G^m be as in (5.4.27). Define $M : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ by

$$M = D + G^m. \tag{5.4.30}$$

If

$$Mz = 0, \tag{5.4.31}$$

$$M\alpha < 0 \tag{5.4.32}$$

and

$$M\beta > 0, \tag{5.4.33}$$

then

$$\alpha < z < \beta.$$

Proof. We shall only prove $z < \beta$, since $z > \alpha$ may be proved analogously.

Set $\nu = z - \beta$. We prove that $\nu < 0$. Suppose that $\nu < 0$ is false. Then for some $i \in \{0, 1, \ldots, N\}$, $\nu_i \geq 0$. Let k be an integer such that

$$\nu_k = \max_{0 \leq i \leq N} \{\nu_i\}. \tag{5.4.34}$$

Clearly

$$\nu_k \geq 0. \tag{5.4.35}$$

By (5.4.31),

$$0 = (Dz)_k + (G^m z)_k$$

$$= (Dz)_k + (G\beta)_k + (z_k - \beta_k),$$

from (5.4.27) and (5.4.35). Hence, using (5.4.27) and (5.4.33),

$$
\begin{aligned}
-\nu_k &= (Dz)_k + (G\beta)_k \\
&= (Dz)_k + (G^m\beta)_k \\
&> (Dz)_k - (D\beta)_k \\
&= (D\nu)_k \\
&= \sum_{j=0}^{N} d_{kj}\nu_j \\
&\geq \left(\sum_{j=0}^{N} d_{kj}\right)\nu_k, \quad \text{by (5.4.28) and (5.4.34),} \\
&\geq 0,
\end{aligned}
$$

by (5.4.29) and (5.4.35). That is, $\nu_k < 0$. This contradicts (5.4.35) and the proof of Lemma 5.4.1 is completed. □

D'Annunzio [9] obtained the same result under the extra conditions $d_{ii} > 0$ for $i = 0, 1, \ldots, N$ and assuming that strict inequality holds in (5.4.29) for at least one i.

For each $t \in [0, 1]$, set

$$
\tilde{w}(x,t,\varepsilon,p) = \begin{cases} \tilde{v}_0^0(x/\varepsilon,t,p)\sigma(x), & \text{for } 0 \leq x \leq 1/2, \\ \tilde{v}_0^1((1-x)/\varepsilon,t,p)\sigma(1-x), & \text{for } 1/2 < x \leq 1, \end{cases}
$$

where $\tilde{v}_0^0(\eta,t,p)$ and $\tilde{v}_0^1(\eta,t,p)$ are respectively defined by

$$
\ddot{\tilde{v}}_0^0 - \tilde{b}\left(0,t,u_0(0) + \tilde{v}_0^0\right) = -p\tilde{v}_0^0, \quad \text{for } \eta > 0, \tag{5.4.36}
$$

$$
\tilde{v}_0^0(0,t,p) = -u_0(0), \quad \tilde{v}_0^0(\infty,t,p) = 0 \tag{5.4.37}
$$

and

$$
\ddot{\tilde{v}}_0^1 - \tilde{b}\left(1,t,u_0(1) + \tilde{v}_0^1\right) = -p\tilde{v}_0^1, \quad \text{for } \eta > 0, \tag{5.4.38}
$$

$$\tilde{v}_0^1(0,t,p) = -u_0(1), \quad \tilde{v}_0^1(\infty,t,p) = 0. \tag{5.4.39}$$

Recalling (5.4.21), one may show, by the arguments of Section 5.2, that there is a $\tilde{p}_0 > 0$, independent of ε and t, such that $w(x,t,\varepsilon,p)$ is well defined for $|p| \le \tilde{p}_0$. Furthermore, we have

$$0 \le \frac{\partial \tilde{w}}{\partial p}(x,t,\varepsilon,p) \le C \tag{5.4.40}$$

and

$$\left|\frac{\partial^j \tilde{w}}{\partial x^j}(x,t,\varepsilon,p)\right| \le C\varepsilon^{-j}\left(\exp\left(-(\tilde{b}-\delta)x/\varepsilon\right) + \exp\left(-(\tilde{b}-\delta)(1-x)/\varepsilon\right)\right), \tag{5.4.41}$$

for $(x,t) \in [0,1] \times [0,1]$ and $j = 0,1,\ldots,J$. Here $\tilde{b}_0^2 > p$ and $\tilde{b} = \sqrt{\tilde{b}_0^2 - p}$ with $\tilde{b}_0$ given by (5.4.21) and δ any fixed number in $(0,\tilde{b})$.

Assumption 5.4.1 *In what follows, we shall assume that $\varepsilon \le N^{-1}$, which is non-restrictive in practice.*

Lemma 5.4.2 *Set $\tilde{p}_N = N^{-1}\ln N$. Let $t \in [0,1]$. Then we can choose a constant $\tilde{C}_1 > 0$, which is independent of N, ε and t, and a positive integer N_0, which depends on $\tilde{C}_1$ but is independent of ε and t, such that for each fixed $t \in [0,1]$, when $N \ge N_0$, $\tilde{w}(x,t,\varepsilon,\tilde{C}_1\tilde{p}_N)$ and $\tilde{w}(x,t,\varepsilon,-\tilde{C}_1\tilde{p}_N)$ are well defined, and*

$$\tilde{\beta}^N(x,t) = u_0(x) + \tilde{w}(x,t,\varepsilon,\tilde{C}_1\tilde{p}_N) + \tilde{C}_1\tilde{p}_N \tag{5.4.42}$$

and

$$\tilde{\alpha}^N(x,t) = u_0(x) + \tilde{w}(x,t,\varepsilon,-\tilde{C}_1\tilde{p}_N) - \tilde{C}_1\tilde{p}_N, \tag{5.4.43}$$

are super and sub solutions respectively of (5.4.23) on the Shishkin mesh X_4^N.

Proof. For each $t \in [0,1]$, it is clear that

$$\tilde{\alpha}^N(\cdot,t) < \tilde{\beta}^N(\cdot,t). \tag{5.4.44}$$

We now prove that $\tilde{F}_H(\tilde{\beta}^N,t) > 0$. From the definitions of the terms involved,

$$\left(\tilde{F}_H(\tilde{\beta}^N,t)\right)_0 = \left(\tilde{F}_H(\tilde{\beta}^N,t)\right)_N = \tilde{C}_1\tilde{p}_N > 0. \tag{5.4.45}$$

For $i \in \{1,2,\dots N-1\}$, we have

$$\begin{aligned}\left(\tilde{F}_H(\tilde{\beta}^N,t)\right)_i = &\left(\left(\tilde{F}_H(\tilde{\beta}^N,t)\right)_i - \left(\tilde{F}_\varepsilon(\tilde{\beta}^N,t)\right)(x_i,t,\varepsilon)\right) \\ &+ \left(\tilde{F}_\varepsilon(\tilde{\beta}^N,t)\right)(x_i,t,\varepsilon).\end{aligned} \tag{5.4.46}$$

We separately analyse these two terms. In the following argument, the notation $\zeta = O(M)$ stands for $|\zeta| \le \tilde{C}M$, where $\tilde{C} > 0$ is any constant independent of $\tilde{C}_1$, N, ε and t.

Firstly, take $N_1 > 0$ such that $\tilde{C}_1\tilde{p}_N < \tilde{b}_0^2/4$ for $N > N_1$. Then for $N > N_1$ and $\delta = \tilde{b}_0/4$ in (5.4.41), we have

$$\left|\frac{\partial^j \tilde{w}}{\partial x^j}(x,t,\varepsilon,\tilde{C}_1\tilde{p}_N)\right| \le \tilde{C}\varepsilon^{-j}\left(\exp\left(-\tilde{b}_0 x/2\varepsilon\right) + \exp\left(-\tilde{b}_0(1-x)/2\varepsilon\right)\right), \tag{5.4.47}$$

for $(x,t) \in [0,1]\times[0,1]$ and $j = 0,1,\dots,J$. By arguments similar to those of Lemma 5.3.2, we see that on the Shishkin mesh X_4^N one has, using $\varepsilon \le N^{-1}$,

$$\left|\left(\tilde{F}_H(\tilde{\beta}^N,t)\right)_i - \left(\tilde{F}_\varepsilon(\tilde{\beta}^N,t)\right)(x_i,t,\varepsilon)\right| \le \tilde{C}\tilde{p}_N. \tag{5.4.48}$$

Secondly, we have for $x_i \in (0, x_{i_0}]$

$$\tilde{\beta}^N(x,t) = u_0(x) + \tilde{v}_0^0(\eta,t,\tilde{C}_1\tilde{p}_N) + \tilde{C}_1\tilde{p}_N,$$

where $\eta = x/\varepsilon$. Therefore

$$
\begin{aligned}
&\tilde{F}_\varepsilon(\tilde{\beta}^N, t)(x_i, t, \varepsilon) \\
&= -\varepsilon^2 u_0'' - \ddot{\tilde{v}}_0^0 + \tilde{b}(x_i, t, u_0 + \tilde{v}_0^0 + \tilde{C}_1 \tilde{p}_N) \\
&= -\ddot{\tilde{v}}_0^0 + \tilde{b}(x, t, u_0 + \tilde{v}_0^0) + \tilde{b}_u(x_i, t, u_0 + \tilde{v}_0^0)\tilde{C}_1 \tilde{p}_N \\
&\quad + O\left(\varepsilon^2 + (\tilde{C}_1 \tilde{p}_N)^2\right) \\
&= -\ddot{\tilde{v}}_0^0 + \tilde{b}(0, t, u_0(0) + \tilde{v}_0^0) + \tilde{b}_u(x_i, t, u_0 + \tilde{v}_0^0)\tilde{C}_1 \tilde{p}_N \\
&\quad + O\left(x_i + \varepsilon^2 + (\tilde{C}_1 \tilde{p}_N)^2\right) \\
&= \tilde{C}_1 \tilde{p}_N \tilde{v}_0^0 + \tilde{b}_u(x_i, t, u_0 + \tilde{v}_0^0)\tilde{C}_1 \tilde{p}_N \\
&\quad + O\left(x_i + \varepsilon^2 + (\tilde{C}_1 \tilde{p}_N)^2\right) \\
&\geq \tilde{C}_1 \tilde{p}_N \tilde{v}_0^0 + \left(\tilde{C}_1 \tilde{b}_0^2 - \tilde{C}\right) \tilde{p}_N - \tilde{C}\left(1 + \tilde{C}_1^2\right) \tilde{p}_N^2, \qquad (5.4.49)
\end{aligned}
$$

by (5.4.21), since $0 < x_i \leq x_{i_0} = O(\varepsilon \ln N)$.

For $x_i \in (x_{i_0}, 1/2]$, we have by (5.4.47)

$$
\begin{aligned}
\left|\frac{\partial^j \tilde{w}}{\partial x^j}(x, t, \varepsilon, \tilde{C}_1 \tilde{p}_N)\right| &\leq \tilde{C}\varepsilon^{-j}\left(\exp\left(-\tilde{b}_0 x_{i_0}/2\varepsilon\right) + \exp\left(-\tilde{b}_0(1 - x_{i_0})/2\varepsilon\right)\right) \\
&\leq \tilde{C}\varepsilon^{-j} N^{-2},
\end{aligned}
$$

for $j = 0, 1, \ldots, J$. Hence

$$
\begin{aligned}
&\tilde{F}_\varepsilon(\tilde{\beta}^N, t)(x_i, t, \varepsilon) \\
&= -\varepsilon^2 \left(u_0''(x_i) + \frac{\partial^2 \tilde{w}}{\partial x^2}(x_i, t, \varepsilon, \tilde{C}_1 \tilde{p}_N)\right) \\
&\quad + \tilde{b}(x_i, t, u_0 + \tilde{w} + \tilde{C}_1 \tilde{p}_N) \\
&= \tilde{b}(x_i, t, u_0(x_i)) + \tilde{b}_u(x_i, t, u_0(x_i))\left(\tilde{w}(x_i, t, \varepsilon, \tilde{C}_1 \tilde{p}_N) + \tilde{C}_1 \tilde{p}_N\right) \\
&\quad + O\left(\varepsilon^2 + N^{-2} + (\tilde{C}_1 \tilde{p}_N)^2\right)
\end{aligned}
$$

$$\geq \tilde{b}_0^2 \left(\tilde{C}_1 \tilde{p}_N - \tilde{C} N^{-2}\right) - \tilde{C}\left(\varepsilon^2 + N^{-2} + (\tilde{C}_1 \tilde{p}_N)^2\right)$$
$$\geq \tilde{b}_0^2 \left(\tilde{C}_1 - \tilde{C}\right) \tilde{p}_N - \tilde{C}\left(1 + \tilde{C}_1^2 \tilde{p}_N\right) \tilde{p}_N. \tag{5.4.50}$$

Recalling $\tilde{v}_0^0 > 0$ (by Lemma 5.2.4), (5.4.46) and (5.4.48) - (5.4.50), one may choose $\tilde{C}_1$ (independent of N, ε and t) and N (depending on $\tilde{C}_1$ but independent of ε and t) sufficiently large such that

$$\left(\tilde{F}_H(\tilde{\beta}^N, t)\right)_i > 0, \quad \text{for } x_i \in (0, 1/2]. \tag{5.4.51}$$

Similarly, one may show that

$$\left(\tilde{F}_H(\tilde{\beta}^N, t)\right)_i > 0, \quad \text{for } x_i \in (1/2, 1).$$

Combining this with (5.4.45) and (5.4.51) yields $\tilde{F}_H(\tilde{\beta}^N, t) > 0$.

Analogously, one can prove that $\tilde{F}_H(\tilde{\alpha}^N, t) > 0$. The proof is complete. □

We now introduce a modified problem corresponding to (5.4.23). Consider

$$\tilde{F}_H^m(z, t) = 0, \tag{5.4.52}$$

where the mapping $\tilde{F}_H^m(\cdot,\cdot) : \mathcal{R}^{N+1} \times [0,1] \to \mathcal{R}^{N+1}$ is defined by

$$\tilde{F}_H^m(z, t) = -\varepsilon^2 A z + \tilde{B}_H^m(z, t).$$

Here $\tilde{B}_H^m(\cdot, t)$ is the modification of $\tilde{B}_H(\cdot, t)$ with $\tilde{\beta}^N$ and $\tilde{\alpha}^N$ given by (5.4.42) and (5.4.43) respectively for each t; see (5.4.27).

Define an open and bounded set $D_t \subset \mathcal{R}^{N+1}$ for each $t \in [0,1]$ by

$$D_t = \left\{ z \in \mathcal{R}^{N+1} : \tilde{\alpha}^N(\cdot, t) < z < \tilde{\beta}^N(\cdot, t) \right\}.$$

We shall denote by $\bar{D}_t$ and ∂D_t the closure and the boundary respectively of D_t in $\mathcal{R}^{N+1}$.

Define the mapping $T(\cdot,\cdot) : \bar{D}_1 \times [0,1] \to \mathcal{R}^{N+1}$ by

$$(T(z,t))_i = \left(z_i - \tilde{\alpha}_i^N(\cdot,1)\right) \frac{\tilde{\beta}_i^N(\cdot,t)}{\tilde{\beta}_i^N(\cdot,1) - \tilde{\alpha}_i^N(\cdot,1)} + \left(\tilde{\beta}_i^N(\cdot,1) - z_i\right) \frac{\tilde{\alpha}_i^N(\cdot,t)}{\tilde{\beta}_i^N(\cdot,1) - \tilde{\alpha}_i^N(\cdot,1)},$$

for $i = 0,1,\ldots,N$. It is easy to see that for each $t \in [0,1]$, $T(\cdot,t)$ is a linear transformation from $\bar{D}_1$ onto $\bar{D}_t$.

We finally define a mapping $\tilde{H}(\cdot,\cdot) : \bar{D}_1 \times [0,1] \to \mathcal{R}^{N+1}$ by

$$\tilde{H}(z,t) = \tilde{F}_H^m(T(z,t),t), \quad \text{for } (z,t) \in \bar{D}_1 \times [0,1].$$

This is a continuously differentiable mapping. We shall prove that

$$\left| Deg(\tilde{H}(\cdot,1), D_1, 0) \right| = 1,$$

where Deg denotes topological degree (see, e.g., Ortega and Rheinboldt [32]), by using the Homotopy Invariance Theorem; see Ortega and Rheinboldt [32], Theorem 6.2.2. We first show the following

Lemma 5.4.3

$$\tilde{H}(z,t) \neq 0 \quad \textit{for all } (z,t) \in \partial D_1 \times [0,1].$$

Proof. Suppose that $\tilde{H}(z^*,t^*) = 0$ for some $(z^*,t^*) \in \bar{D}_1 \times [0,1]$. Set $T^* = T(z^*,t^*)$. Then $T^* \in \bar{D}_{t^*}$ satisfies

$$\tilde{F}_H^m(T^*,t^*) = 0. \tag{5.4.53}$$

From the definition of $\tilde{F}_H^m(\cdot,\cdot)$ and Lemma 5.4.2, we have

$$\tilde{F}_H^m(\tilde{\alpha}^N(\cdot,t^*),t^*) = \tilde{F}_H(\tilde{\alpha}^N(\cdot,t^*),t^*) < 0 \tag{5.4.54}$$

$$\tilde{F}_H^m(\tilde{\beta}^N(\cdot,t^*),t^*) = \tilde{F}_H(\tilde{\beta}^N(\cdot,t^*),t^*) > 0. \tag{5.4.55}$$

Setting $D = -\varepsilon^2 A$, the condition (5.4.28) and (5.4.29) are fulfilled. Combining (5.4.53) – (5.4.55) with Lemma 5.4.1 yields

$$\tilde{\alpha}^N(\cdot,t^*) < T^* < \tilde{\beta}^N(\cdot,t^*). \tag{5.4.56}$$

From the definition of $T(\cdot,\cdot)$, we obtain

$$z_i^* = (T_i^* - \tilde{\alpha}_i^N(\cdot,t^*)) \frac{\tilde{\beta}_i^N(\cdot,1)}{\tilde{\beta}_i^N(\cdot,t^*) - \tilde{\alpha}_i^N(\cdot,t^*)} + \left(\tilde{\beta}_i^N(\cdot,t^*) - T_i^*\right) \frac{\tilde{\alpha}_i^N(\cdot,1)}{\tilde{\beta}_i^N(\cdot,t^*) - \tilde{\alpha}_i^N(\cdot,t^*)},$$

for $i = 0, 1, \ldots, N$. Hence

$$\tilde{\alpha}^N(\cdot,1) < z^* < \tilde{\beta}^N(\cdot,1),$$

i.e., $z^* \notin \partial D_1$, which is the desired result. □

Lemma 5.4.4 *If $\tilde{C}_1$ in (5.4.42) and (5.4.43) is chosen sufficiently large, then*

$$\left|Deg(\tilde{H}(\cdot,0), D_1, 0)\right| = 1.$$

Proof. We start with the problem

$$\tilde{F}_H(z,0) = 0, \quad \text{for } z \in \mathcal{R}^{N+1}. \tag{5.4.57}$$

Set

$$S = \begin{pmatrix} 0 & 0 & 0 & & & & \\ s_1^- & s_1^c & s_1^+ & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & s_{N-1}^- & s_{N-1}^c & s_{N-1}^+ \\ & & & & 0 & 0 & 0 \end{pmatrix}.$$

Then (5.4.57) can be written in the form

$$(-\varepsilon^2 A + S)z - S u_0 = 0,$$

from (5.4.20) and (5.4.22). Since a calculation shows that

$$\left|\frac{2\varepsilon^2}{h_i h_{i+1}} + s_i^c\right| - \left|-\frac{\varepsilon^2}{h_i \bar{h}_i} + s_i^-\right| - \left|-\frac{\varepsilon^2}{h_{i+1}\bar{h}_i} + s_i^+\right| \geq \frac{1}{3},$$

for $i = 0, 1, \ldots, N$, we obtain, by Theorem A of Varga [45], that $(-\varepsilon^2 A + S)^{-1}$ exists and

$$\|(-\varepsilon^2 A + S)^{-1}\|_\infty \leq 3. \tag{5.4.58}$$

Consequently (5.4.57) has a unique solution $z^* = (-\varepsilon^2 A + S)^{-1} S u_0 \in \mathcal{R}^{N+1}$.

We wish to prove that $z^* \in D_\theta$. For this purpose, set

$$\bar{z}(x, \varepsilon) = u_0(x) + \tilde{w}(x, 0, \varepsilon, 0), \quad \text{for } x \in [0, 1].$$

Along the lines of Lemma 5.4.2, we can show that on the Shishkin mesh X_4^N

$$\|\tilde{F}_H(\bar{z}, 0)\|_\infty \leq C N^{-1} \ln N.$$

Thus, by (5.4.58),

$$\begin{aligned}
\|\bar{z} - z^*\|_\infty &= \|(-\varepsilon^2 A + S)^{-1}(-\varepsilon^2 A + S)(\bar{z} - z^*)\|_\infty \\
&\leq 3\|(-\varepsilon^2 A + S)(\bar{z} - z^*)\|_\infty \\
&= 3\|(-\varepsilon^2 A + S)\bar{z} - S u_0 - (-\varepsilon^2 A + S)z^* + S u_0\|_\infty \\
&= 3\|\tilde{F}_H(\bar{z}, 0)\|_\infty \\
&\leq C N^{-1} \ln N.
\end{aligned}$$

For $i = 0, 1, \ldots, N$ and $\tilde{C}_1$ chosen as in Lemma 5.4.2, using (5.4.40) we have

$$\begin{aligned}
z_i^* &\leq \bar{z}_i + C N^{-1} \ln N \\
&\leq u_0(x_i) + \tilde{w}\left(x_i, 0, \varepsilon, \tilde{C}_1 \tilde{p}_N\right) + C N^{-1} \ln N \\
&< \tilde{\beta}_i^N(\cdot, 0),
\end{aligned} \tag{5.4.59}$$

provided that $\tilde{C}_1 > C$, where C is the constant of (5.4.59).

Similarly,

$$z_i^* > \tilde{\alpha}_i^N(\cdot,0),$$

for $i = 0, 1, \ldots, N$. That is, $z^* \in D_0$.

We now consider the problem

$$\tilde{H}(z,0) = 0, \quad \text{for } z \in \bar{D}_1. \tag{5.4.60}$$

As $T(z,0) \in \bar{D}_0$, the problem (5.4.60) is equivalent to $\tilde{F}_H(T(z,0),0) = 0$. But from above (5.4.57) has a unique solution $z^* \in D_0$. Consequently we only need look for solutions $z^* \in \bar{D}_1$ of

$$T(z,0) = z^*. \tag{5.4.61}$$

Recalling that $T(\cdot,0)$ is a linear mapping from $\bar{D}_1$ onto $\bar{D}_0$, so $\partial D_0 = T(\partial D_1, 0)$, we conclude that (5.4.61) has a unique solution $\tilde{z} \in D_1$. That is, (5.4.60) has a unique solution, which lies in D_1.

Furthermore, we have for $z \in D_1$,

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial z}(z,0) &= \frac{\partial \tilde{F}_H^m}{\partial T}(T(z,0),0)\frac{\partial T}{\partial z}(z,0)\\ &= \frac{\partial \tilde{F}_H}{\partial T}(T(z,0),0)\frac{\partial T}{\partial z}(z,0)\\ &= (-\varepsilon^2 A + S)\frac{\partial T}{\partial z}(z,0).\end{aligned}$$

From above, we know that

$$det(-\varepsilon^2 A + S) \neq 0.$$

Since $\tilde{\alpha}^N(\cdot,0) < \tilde{\beta}^N(\cdot,0)$, we have

$$det\left(\frac{\partial T}{\partial z}(z,0)\right) \neq 0, \quad \text{for all } z \in \mathcal{R}^{N+1}.$$

Therefore

$$det\left(\frac{\partial\tilde{H}}{\partial z}(z,0)\right) \neq 0, \quad \text{for all } z \in D_1.$$

We have shown that (5.4.60) has a unique solution z^*, which lies in D_1, with $det\left(\frac{\partial\tilde{H}}{\partial z}\Big|_{z=z^*}\right) \neq 0$. This completes the proof. □

Theorem 5.4.4 *There exists a positive integer N_0, independent of ε, such that for $N \geq N_0$, the H-scheme $\{F_H, X_4^N\}$ has a solution $u_N \in D_1$. Moreover, this solution is unique in $\bar{D}_1$.*

Proof. From Lemma 5.4.3,

$$Deg(\tilde{H}(\cdot,t), D_1, 0) \text{ is constant for } t \in [0,1],$$

using the Homotopy Invariance Theorem; see Ortega and Rheinboldt [32], Theorem 6.2.2. Hence

$$\left|Deg(\tilde{H}(\cdot,1), D_1, 0)\right| = \left|Deg(\tilde{H}(\cdot,0), D_1, 0)\right| = 1, \tag{5.4.62}$$

by Lemma 5.4.4. This implies that the equation

$$\tilde{H}(z,1) = 0 \tag{5.4.63}$$

has at least one solution $u_N \in D_1$.

We now prove the uniqueness of this solution in D_1. Since $T(\cdot,1)$ is an identity mapping for $z \in \bar{D}_1$,

$$\begin{aligned} \tilde{H}(z,1) &= \tilde{F}_H^m(z,1) \\ &= \tilde{F}_H(z,1) \\ &= F_H(z). \end{aligned} \tag{5.4.64}$$

By arguments similar to those of Theorem 5.4.3 we may show that there exists a constant $\tilde{C} > 0$, which is independent of N and ε, such that $F'_H(z)$ is nonsingular for all $z \in S\left(u, \tilde{C}N^{-1/2}\right)$. Note that, for N sufficiently large,

$$\tilde{\alpha}^N(\cdot,1) \leq u(\cdot) \leq \tilde{\beta}^N(\cdot,1),$$

when $\tilde{C}_1$ is chosen (independently of N) as in (5.4.42) and (5.4.43). Hence, for any $z \in \bar{D}_1$,

$$\begin{aligned} \|z - u\|_\infty &\leq \|\tilde{\beta}^N(\cdot,1) - \tilde{\alpha}^N(\cdot,1)\|_\infty \\ &\leq \tilde{C}_1 N^{-1} \ln N. \end{aligned}$$

One may choose N_0, depending on $\tilde{C}$ and $\tilde{C}_1$ but independent of ε, such that $\bar{D}_1 \subset S\left(u, \tilde{C}N^{-1/2}\right)$ for $N \geq N_0$. Consequently $det\left(F'_H(z)\right)$ has constant sign on $\bar{D}_1$. Since

$$Deg\left(\tilde{H}(\cdot,1), D_1, 0\right) = \sum_{\{z \in D_1 : H(z,0)=1\}} det\left(\tilde{H}'(z,1)\right),$$

it follows from (5.4.62) and (5.4.64) that our solution is unique in D_1. □

Let u_N be a solution of $Fu_N = 0$ specified in any of the Theorems 5.4.1 – 5.4.4. In each case, the proofs of these theorems show that $det\left(F'(u_N)\right) \neq 0$. Hence u_N is a point of attraction of a Newton iteration. In Section 5.6, we shall give numerical results for both the D–scheme and the H–scheme by using Newton's method with an initial guess obtained from sampling the reduced solution.

5.5 Uniform Convergence of the D–scheme for Problem (B)

We analyse the D–scheme applied to problem (B) on the Shishkin mesh X_4^N. We shall prove that the D–scheme is uniformly convergent of order $N^{-2}\ln^2 N$ on this piecewise equidistant mesh.

The D–scheme is: find $u_N \in \mathcal{R}^{N+1}$ such that

$$F_D u_N = 0, \tag{5.5.1}$$

where F_D is given by (5.3.2) with $s_i^- = s_i^+ = 0$, $s_i^c = 1$ for each i.

Recall the boundary layer function w of (5.2.14). Define

$$\beta^N(x,\varepsilon) = u_0(x) + w(x,\varepsilon,C_3 p_N) + C_4 p_N \tag{5.5.2}$$

and

$$\alpha^N(x,\varepsilon) = u_0(x) + w(x,\varepsilon,-C_3 p_N) - C_4 p_N. \tag{5.5.3}$$

Here $p_N = N^{-2}\ln^2 N$; C_3 and C_4 are positive constants, independent of N and ε.

Lemma 5.5.1 *One can choose positive constants C_3 and C_4, which are independent of N and ε, and a positive integer N_0, which depends on C_3 and C_4 but is independent of ε, such that when $N \geq N_0$, $w(x,\varepsilon,C_3 p_N)$ and $w(x,\varepsilon,-C_3 p_N)$ are well defined, and $\beta^N(x,\varepsilon)$ and $\alpha^N(x,\varepsilon)$ are super and sub solutions respectively of the D-scheme $\{F_D, X_4^N\}$.*

Proof. By inspection of the proof of Lemma 5.2.5, one may show that

$$F_\varepsilon \beta^N(x_i,\varepsilon) \geq (C_3 - C_4\bar{C}) p_N v_0^0(x_i/\varepsilon, C_3 p_N)$$
$$+ \left(b_0^2 C_4 - \bar{C}\left(1 + (C_3 + C_4)N^{-1}\ln N\right)\right) p_N,$$

for $x_i \in (0, x_{i_0}]$, and

$$F_\varepsilon \beta^N(x_i,\varepsilon) \geq b_0^2\left(C_4 p_N - \bar{C}N^{-2}\right) - \bar{C}\left(N^{-2} + (C_4 p_N)^2\right),$$

for $x_i \in (x_{i_0}, 1/2]$. The result then follows from arguments similar to those of Lemma 5.4.2. □

Let $F_D^m : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ be the modification of F_D defined by

$$(F_D^m)_= \begin{cases} -\varepsilon^2(Az)_i + (z_i - \beta_i^N), & \text{if } z_i \geq \beta_i^N, \\ -\varepsilon^2(Az)_i + b(x_i, z_i), & \text{if } \alpha_i^N < z_i < \beta_i^N, \\ -\varepsilon^2(Az)_i + (\alpha_i^N - z_i), & \text{if } z_i \leq \alpha_i^N, \end{cases}$$

for $i = 0, 1, \ldots, N$. This modification is of the same type as (5.4.27).

The following theorem gives almost second order uniform convergence for the D–scheme applied to problem (B).

Theorem 5.5.1 *Let $u(x)$ be the solution of problem (B) guaranteed by Theorem 5.2.1. Assume that $\varepsilon \leq N^{-1}$. For N sufficiently large, independently of ε, the scheme $\{F_D, X_4^N\}$ has a solution u_N such that*

$$\|u - u_N\|_\infty \leq CN^{-2} \ln^2 N.$$

Proof. Let α and β be given by (5.2.25) and (5.2.26). Then

$$\alpha \leq u \leq \beta, \tag{5.5.4}$$

by Theorem 5.2.1.

By arguments analogous to those of subsection 5.4.2, one can show, using degree theory, that $F_D^m z = 0$ has a solution $u_N \in \mathcal{R}^{N+1}$. Also, from Lemma 5.5.1, we have $F_D^m \beta = F_D \beta > 0$ and $F_D^m \alpha = F_D \alpha < 0$. Then Lemma 5.4.1 yields

$$\alpha^N < u^N < \beta^N. \tag{5.5.5}$$

This implies that $F_D u_N = F_D^m u_N = 0$.

Choose C_3 and C_4 in Lemma 5.5.1 sufficiently large such that $C_3 \geq C_1$ and $C_4 \geq C_2$, where C_1 and C_2 are given in Lemma 5.2.5. Then $C_1 p_\varepsilon \leq C_3 p_N$ and $C_2 p_\varepsilon \leq C_4 p_N$, since $\varepsilon \leq N^{-1}$. Hence

$$\alpha^N \leq \alpha < \beta \leq \beta^N, \tag{5.5.6}$$

by (5.2.23).

We have

$$\begin{aligned}\beta_0^N - \alpha_0^N &= \beta_N^N - \alpha_N^N \\ &= 2C_4 N^{-2} \ln^2 N\end{aligned}$$

and, for $i = 1, \ldots, N-1$,

$$\begin{aligned}|\beta_i^N - \alpha_i^N| &= |\beta^N(x_i,\varepsilon) - \alpha^N(x_i,\varepsilon)| \\ &\leq 2C_3 N^{-2} \ln^2 N \left|\frac{\partial w}{\partial p}(x_i,\varepsilon,p^*)\right| + 2C_4 N^{-2} \ln^2 N, \\ &\qquad \text{where } p^* \in (-C_3 p_N, C_3 p_N), \\ &\leq C N^{-2} \ln^2 N,\end{aligned}$$

by (5.2.23).

Therefore, from (5.5.4) – (5.5.6),

$$\begin{aligned}\|u - u_N\|_\infty &\leq \|\beta^N - \alpha^N\|_\infty \\ &\leq C N^{-2} \ln^2 N,\end{aligned}$$

which is the desired result. □

The uniform accuracy of Theorem 5.5.1 is almost one order higher than that of D'Annunzio [9], who used a more complicated locally quasi-equidistant mesh.

5.6 Numerical Examples

In this section we present numerical results to confirm the uniform accuracy of the schemes$\{F_D, X_2^N\}$ and $\{F_H, X_4^N\}$ analysed in Sections 5.4 and 5.5.

When $\varepsilon^2 \le N^{-1}$, the uniform error estimates obtained in Theorems 5.4.1 – 5.4.3 and 5.5.1 have the form of

$$\|u - u_N\|_\infty \le C(N^{-1}\ln N)^r,$$

where r equals 2 for the D-scheme and 4 for the H-scheme.

For both F_D and F_H, the nonlinear system of equations is solved using Newton's method with the initial guess $u_N^0 = (0, u_0(x_1), \dots, u_0(x_{N-1}), 0)^T$. Here, in the case of Problem (A), u_0 is the reduced solution and in the case of Problem (B), u_0 is a stable reduced solution. We iteratively compute u_N^k, for $k = 1, 2, \dots$. The stopping criterion used is

$$\max\left\{\|Fu_N^k\|_\infty, \|u_N^k - u_N^{k-1}\|_\infty < 0.1N^{-r}\right\}.$$

For each N and ε in the tables, it only takes about 5 iterations to satisfy this criterion.

The exact solutions of our test problems are unknown. We use a double mesh method; see Doolan et al. [10], to compute the experimental rates of convergence. In order to do this, we shall in addition to computing u_N also compute another approximate solution $\tilde{u}_N$ which we now describe.

Let $\tilde{u}_N \in \mathcal{R}^{N+1}$ be a solution of $\{F_D, \tilde{X}_2^N\}$ or $\{F_H, \tilde{X}_4^N\}$, where $\tilde{X}_m^N$ is a Shishkin mesh with the mesh parameter σ_m of (5.3.4) altered slightly to

$$\tilde{\sigma}_m = \min\{1/4, mb_0^{-1}\varepsilon\ln(N/2)\}.$$

Then for $i = 0, 1, \ldots, N$, the ith point of the mesh X_m^N coincides with the $(2i)$th point of the mesh $\tilde{X}_m^{2N}$.

By inspecting the arguments of Sections 5.4 and 5.5, one may see that

$$\|u - \tilde{u}_N\|_\infty \le C(N^{-1}\ln N)^r,$$

where C is independent of N and ε. Hence for $i = 0, 1, \ldots, N$,

$$|(u_N)_i - (\tilde{u}_{2N})_{2i}| \le C(N^{-1}\ln N)^r.$$

For each N and ε, we shall report

$$\tilde{E}_\varepsilon^N = \max_{0 \le i \le N} |(u_N)_i - (\tilde{u}_{2N})_{2i}|$$

in the error tables below.

Assuming convergence of order $(N^{-1}\ln N)^r$ for some r, the classical convergence rate r will be computed by

$$\begin{aligned} R_\varepsilon^N &= \frac{\ln \tilde{E}_\varepsilon^{2N} - \ln \tilde{E}_\varepsilon^N}{\ln\left(\frac{2\ln N}{\ln 2N}\right)} \\ &= \frac{\ln \tilde{E}_\varepsilon^{2N} - \ln \tilde{E}_\varepsilon^N}{\ln\left(\frac{2k}{k+1}\right)}, \quad \text{for } N = 2^k \text{ and } k = 5, 6, \ldots, 11. \end{aligned}$$

The last row of each rate table is the *uniform* convergence rate,

$$R^N = \frac{\ln \tilde{E}^{2N} - \ln \tilde{E}^N}{\ln\left(\frac{2k}{k+1}\right)},$$

where $\tilde{E}^N = \max_\varepsilon \tilde{E}_\varepsilon^N$.

Example 5.6.1 *Consider the problem*

$$-\varepsilon^2 u'' + (1+u)\left(1 + (1+u)^2\right) = 0, \quad \textit{for } x \in (0,1),$$

$$u(0) = u(1) = 0.$$

Since $b_u(x,u) = 1 + 3(1+u)^2 \geq 1$, for all $(x,u) \in [0,1] \times \mathcal{R}^1$, this is a problem of type (A). The reduced solution is $u_0 = -1$.

ε	N=64	128	256	512	1024
2.500000e-01	1.3865e-04	3.4760e-05	8.6956e-06	2.1744e-06	5.4361e-07
6.250000e-02	2.0211e-03	5.2618e-04	1.3331e-04	3.3418e-05	8.3622e-06
1.562500e-02	4.2603e-03	1.5289e-03	4.9980e-04	1.5632e-04	4.7415e-05
3.906250e-03	4.2592e-03	1.5288e-03	4.9978e-04	1.5632e-04	4.7415e-05
9.765625e-04	4.2599e-03	1.5288e-03	4.9976e-04	1.5631e-04	4.7415e-05
2.441406e-04	4.2604e-03	1.5289e-03	4.9978e-04	1.5631e-04	4.7414e-05
6.103516e-05	4.2605e-03	1.5289e-03	4.9979e-04	1.5631e-04	4.7415e-05
1.525879e-05	4.2606e-03	1.5289e-03	4.9980e-04	1.5632e-04	4.7415e-05
3.814697e-06	4.2606e-03	1.5289e-03	4.9980e-04	1.5632e-04	4.7415e-05
9.536743e-07	4.2606e-03	1.5289e-03	4.9980e-04	1.5632e-04	4.7415e-05

Table 5.6.1: Example 5.6.1, D–scheme errors

ε	N=64	128	256	512
2.500000e-01	2.57	2.48	2.41	2.36
6.250000e-02	2.50	2.45	2.40	2.36
1.562500e-02	1.90	2.00	2.02	2.03
3.906250e-03	1.90	2.00	2.02	2.03
9.765625e-04	1.90	2.00	2.02	2.03
2.441406e-04	1.90	2.00	2.02	2.03
6.103516e-05	1.90	2.00	2.02	2.03
1.525879e-05	1.90	2.00	2.02	2.03
3.814697e-06	1.90	2.00	2.02	2.03
9.536743e-07	1.90	2.00	2.02	2.03
R^N	1.90	2.00	2.02	2.03

Table 5.6.2: Example 5.6.1, D–scheme convergence rates

ε	N=64	128	256	512	1024
2.500000e-01	4.1617e-07	2.6122e-08	1.6339e-09	1.0208e-10	6.1062e-12
6.250000e-02	9.9833e-05	6.6272e-06	4.1955e-07	2.6356e-08	1.6486e-09
1.562500e-02	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07
3.906250e-03	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07
9.765625e-04	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07
2.441406e-04	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07
6.103516e-05	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07
1.525879e-05	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07
3.814697e-06	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07
9.536743e-07	1.9653e-03	2.4192e-04	2.6229e-05	2.5718e-06	2.3682e-07

Table 5.6.3: Example 5.6.1, H–scheme errors

ε	N=64	128	256	512
2.500000e-01	5.14	4.95	4.82	4.79
6.250000e-02	5.03	4.93	4.81	4.72
1.562500e-02	3.89	3.97	4.04	4.06
3.906250e-03	3.89	3.97	4.04	4.06
9.765625e-04	3.89	3.97	4.04	4.06
2.441406e-04	3.89	3.97	4.04	4.06
6.103516e-05	3.89	3.97	4.04	4.06
1.525879e-05	3.89	3.97	4.04	4.06
3.814697e-06	3.89	3.97	4.04	4.06
9.536743e-07	3.89	3.97	4.04	4.06
R^N	3.89	3.97	4.04	4.06

Table 5.6.4: Example 5.6.1, H–scheme convergence rates

Tables 5.6.2 and 5.6.4 show respectively that the D–scheme is second order accurate but the H–scheme is fourth order accurate, as predicted by our theory.

Example 5.6.2 *Consider the problem (Herceg [21])*

$$-\varepsilon^2 u'' + (u^2 + u - 0.75)(u^2 + u - 3.75) = 0, \quad \textit{for } x \in (0,1), \tag{5.6.1a}$$

$$u(0) = u(1) = 0. \tag{5.6.1b}$$

We have

$$b_u(x,u) = (2u+1)(2u^2+2u-4.5).$$

The reduced problem

$$b(x,u) = 0$$

has four solutions $u_1 = -2.5$, $u_2 = -1.5$, $u_3 = 0.5$ and $u_4 = 1.5$. It is easy to get

$$b_u(x,u_1) = -12, \quad b_u(x,u_2) = 6, \quad b_u(x,u_3) = -6 \text{ and } b_u(x,u_4) = 12.$$

Hence u_1 and u_3 are not stable reduced solutions of (5.6.1). By a calculation, one may show that u_2 and u_4 satisfy the condition (5.1.4). Therefore, (5.6.1) is a problem of type (B) with two stable reduced solutions u_2 and u_4. Each of u_2 and u_4 is "close" (in the sense of Theorem 5.2.1) to a solution of (5.6.1) when ε is sufficiently small. We apply the D–scheme to compute these solutions of (5.6.1).

ε	N=64	128	256	512	1024
2.500000e-01	3.4316e-04	8.6686e-05	2.1679e-05	5.4227e-06	1.3557e-06
6.250000e-02	3.5179e-03	1.1715e-03	3.4384e-04	8.6876e-05	2.1726e-05
1.562500e-02	3.5180e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
3.906250e-03	3.5180e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
9.765625e-04	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
2.441406e-04	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
6.103516e-05	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
1.525879e-05	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
3.814697e-06	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
9.536743e-07	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05

Table 5.6.5: Example 5.6.2, D–scheme errors with solution near u_2

ε	N=64	128	256	512
2.500000e-01	2.55	2.48	2.41	2.36
6.250000e-02	2.04	2.19	2.39	2.36
1.562500e-02	2.04	2.04	2.03	2.03
3.906250e-03	2.04	2.04	2.03	2.03
9.765625e-04	2.04	2.04	2.03	2.03
2.441406e-04	2.04	2.04	2.03	2.03
6.103516e-05	2.04	2.04	2.03	2.03
1.525879e-05	2.04	2.04	2.03	2.03
3.814697e-06	2.04	2.04	2.03	2.03
9.536743e-07	2.04	2.04	2.03	2.03
R^N	2.04	2.04	2.03	2.03

Table 5.6.6: Example 5.6.2, D–scheme convergence rates with solution near u_2

ε	N=64	128	256	512	1024
2.500000e-01	1.1820e-03	2.9456e-04	7.3582e-05	1.8397e-05	4.5991e-06
6.250000e-02	5.5968e-03	1.8000e-03	5.6725e-04	1.7490e-04	5.2884e-05
1.562500e-02	5.6164e-03	1.8021e-03	5.6737e-04	1.7490e-04	5.2884e-05
3.906250e-03	5.6057e-03	1.8022e-03	5.6760e-04	1.7493e-04	5.2885e-05
9.765625e-04	5.5976e-03	1.8008e-03	5.6743e-04	1.7493e-04	5.2888e-05
2.441406e-04	5.5951e-03	1.8002e-03	5.6731e-04	1.7491e-04	5.2886e-05
6.103516e-05	5.5944e-03	1.8000e-03	5.6727e-04	1.7490e-04	5.2884e-05
1.525879e-05	5.5943e-03	1.8000e-03	5.6726e-04	1.7490e-04	5.2884e-05
3.814697e-06	5.5942e-03	1.8000e-03	5.6726e-04	1.7490e-04	5.2884e-05
9.536743e-07	5.5942e-03	1.8000e-03	5.6726e-04	1.7490e-04	5.2884e-05

Table 5.6.7: Example 5.6.2, D–scheme errors with solution near u_4

ε	N=64	128	256	512
2.500000e-01	2.58	2.48	2.41	2.36
6.250000e-02	2.10	2.06	2.04	2.03
1.562500e-02	2.11	2.07	2.05	2.04
3.906250e-03	2.11	2.06	2.05	2.04
9.765625e-04	2.10	2.06	2.05	2.04
2.441406e-04	2.10	2.06	2.04	2.03
6.103516e-05	2.10	2.06	2.04	2.03
1.525879e-05	2.10	2.06	2.04	2.03
3.814697e-06	2.10	2.06	2.04	2.03
9.536743e-07	2.10	2.06	2.04	2.03
R^N	2.10	2.06	2.04	2.03

Table 5.6.8: Example 5.6.2, D–scheme convergence rates with solution near u_4

The numerical results for Example 5.6.2 show that the D–scheme is capable of computing those solutions of the problem (B) which lie close to particular reduced solutions. Furthermore, the scheme achieves second order accuracy for this difficult problem, confirming our theoretical results.

Chapter 6

Conclusions

In this work we considered four singularly perturbed two–point boundary value problems. They were (i) $2m$th order problem of reaction–diffusion type (which has two boundary layers of exponential type in the $(m-1)$th order derivative of the solution), (ii) $2m$th order problem of convection–diffusion type (which exhibits one boundary layer of exponential type in the $(m-1)$th order derivative of the solution), (iii) second order interior turning point problem (which has a boundary layer of exponential type or an internal layer of cusp type), and (iv) semilinear reaction–diffusion problem (which has two boundary layers of exponential type). Classical numerical methods do not in general yield satisfactory numerical solutions for any of these problems. We set out to construct and analyse uniformly convergent methods for these problems; that is, methods whose solutions converge, uniformly in the singular perturbation parameter, to the analytical solution of the problem.

We constructed and analysed polynomial–based finite element and finite difference methods on piecewise equidistant meshes for our four problems. The idea of using such a mesh is due to Shishkin; he considered only schemes which satisfied a discrete maximum principle, but we have extended his approach to more general schemes generated by finite element methods. This mesh is fine only in part of the layer(s) and coarse elsewhere. It works well for those problems with layers of exponential type, such as occur in problems (i), (ii) and (iv) above. For layers of cusp type, which may occur in problem (iii), it does not yield satisfactory results. Consequently for (iii) we devised a mesh which is a generalization of Shishkin's. This mesh is also piecewise equidistant.

Galerkin finite element methods based on piecewise polynomial basis functions and Shishkin meshes were constructed for problems (i) and (ii). Almost optimal uniform convergence results were obtained in the weighted energy norms associated with the original equations for both problems. We achieved a higher order of uniform convergence in the Sobolev norm $\|\cdot\|_{m-1}$ than in the energy norm for the problems of reaction–diffusion type. On the other hand, this phenomenon does not occur, in general, for the higher order problems of convection–diffusion type. This is in contrast to convergence results in standard finite element analysis.

Piecewise linear Galerkin finite element methods were generated on the generalized Shishkin mesh for simple attractive turning point problems, which form a subclass of problem (iii). These methods were shown to be uniformly convergent in a weighted energy norm and the usual L^2 norm.

We also investigated the use of finite difference methods on Shishkin meshes. Two simple difference schemes for problem (iv) were proved to be uniformly convergent of second order and fourth order respectively in the discrete maximum norm.

The methods of this thesis are polynomially based and are uniformly convergent. No exponential fitting factors are used.

We believe that uniformly convergent polynomial–based finite element and finite difference methods on piecewise equidistant meshes can be also devised for other singularly perturbed one–dimensional problems, such as boundary turning point problems, initial value problems and systems of equations. It also seems possible to extend the methods and analyses to problems in more than one dimension, using dimension–splitting arguments.

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