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Optimal Sale Across Venues and Auctions with a Buy-Now Option*

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Abstract: We characterize the optimal selling mechanism for a seller who faces demand demarcated by a high and a low end and who can access an (online) auction site (by paying an access cost) in addition to using his own store that can be used as a posted price selling venue. We first solve for the optimal mechanism of a direct revelation game in which there is no venue-restriction constraint. We find that the direct optimal mechanism must necessarily incorporate a certain kind of pooling. We then show that even with the venue constraint, the seller can use a two stage indirect mechanism that implements the allocation rule from the optimal direct mechanism, and uses the venues in an optimal fashion. The first stage of the indirect mechanism is a posted price at the store. If the object is not sold, we move to stage two, which involves an auction at the auction site. A feature of this auction is a buy-now option which is essential for implementing the pooling feature of the optimal direct mechanism. We also show that the buy-now option in the optimal mechanism is of a "temporary" variety, and that a "permanent" buy-now option, in contrast, cannot implement the optimal mechanism. Auctions with a temporary buy-now option are in widespread use on eBay.

KEYWORDS: Optimal Auction, eBay Auctions, Temporary Buy-Now Option, Permanent Buy-Now Option, Heterogeneous Sales Venues, Posted Price, Price Discrimination

JEL CLASSIFICATION: D44

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1 Introduction

Low transactions cost of trading on the Internet has led to a tremendous growth in the scope of auctions. Moreover, the advent of listing sites such as eBay and Yahoo! have made it very easy to sell an object through an auction and provide access to a large pool of buyers. Initially, most of the goods sold through online auctions were collectables[(2)]. Increasingly, however, goods traditionally sold at posted prices (through bricks-and-mortar stores and web sites) are being offered also through online auctions. A large part of the standard goods sold through auctions are overstock items from wholesalers and standard retailers. Overstock.com, which describes itself as "an Internet leader for name-brands at clearance prices," sells items through both posted prices and auctions. IBM has launched its own eBay auctions to sell discontinued products. Other companies such as Dell, HP, Gateway, and Sears, who have well established posted price sales channels, have opened up eBay stores to sell overstock items and refurbished systems through auctions. [(3)]

Many sellers sell through conventional stores making use of a posted price. Evidently, such stores might increase revenue if they could arrange an auction, but typically holding an auction at such venues is very costly, ruling it out as a practical option. It is of course much cheaper to set up auctions online, and the advent of such auctions in recent years has given sellers access to a sales channel that augments existing posted price channels, allowing the sellers to expand their market base.

This role of auctions as an additional sales channel is the focus in our paper. Specifically, we characterize the optimal sales mechanism in an environment where a seller, facing demand that has a clearly demarcated high and low end, can access a sales venue restricted to posted prices, and, at a small cost, access another venue to use an auction that the seller designs. Our results show that in the optimal selling mechanism, the seller uses different venues to target the different ends of the market: the good is sold to the "high-end" buyers through a posted price and if the good is not sold at the posted price, to " lower-end" buyers through an auction that includes a non-standard feature in the form of a buy-now option. An important insight from our analysis is that since the optimal design involves second degree price discrimination *across* individual selling mechanisms,

[(2)] See Lucking-Reiley (2000).

[(3)] Also see Miller (2005) for a variety of examples.

individual components of the overall optimal mechanism may seem suboptimal if considered in isolation. Our analysis also provides a novel theoretical justification for auctions with buy-now options by showing that they form part of the optimal selling mechanism.[4] Note also that even though our formal model has a single unit, the intuition should carry over to more general frameworks incorporating multiple items as long as the stochastic nature of the demand implies that items are sometimes left unsold at the (posted price) store. Taking a broader perspective, our work connects the literature on optimal auctions to the industrial literature on optimal pricing by considering the optimal auction design exercise in a scenario where an auction is part of a bigger set of sales methods used by a seller.

We model the environment as follows. There is a seller who owns a store (we refer to this as selling venue 1) which uses a posted price to sell the object. A second selling venue is an (online) auction site, which the seller can access at some cost $c > 0$.[5] The seller can choose the posted price and the auction format. To capture the high and low ends of the demand side we model the distribution from which buyers draw valuations as follows: with probability μ a buyer has value v_h for the object, an event that can be interpreted as the " high end" of the market. With probability $(1-\mu)$ the buyer can have a range of possible values all of which are less than v_h. Specifically, with probability $(1-\mu)$ a buyer's value is drawn from the interval $[\underline{v}, \overline{v}]$ where $\overline{v} < v_h$. We make the standard assumption that a buyer's value is private information. Given this environment, we ask what the optimal selling strategy for the seller is and use techniques from the mechanism design literature to answer this question.[6]

Instead of solving for the seller's optimal strategy directly, we follow an indirect approach

[4]The literature typically describes environments where such auctions are superior only to certain standard auctions (usually the English auction). As far as we know, we are the first to show that these auctions can arise as part of the *optimal mechanism* in certain environments. We discuss buy-now options at greater length later in the introduction.

[5]For example, the sellers using eBay have to pay a listing fee as well as a percentage of the final selling price to eBay. Similar charges are also applied by other auction platforms. However, we emphasize that here c is an *additional* cost of accessing the auction site. In order to not carry unnecessary notations, we assume that the cost of accessing the store that the seller owns is zero (i.e. any cost incurred is already sunk) and the cost of accessing the auction site is $c > 0$.

[6]See the comments at the end of this subsection for our choice of the atom to represent the high-end of the market.

that we find useful and instructive. We first consider a direct revelation game ignoring any venue restrictions. In other words, we first analyze the seller's problem *as if* it was a standard mechanism design exercise. We then consider the problem incorporating the venue restrictions.

Let us first describe the direct revelation game. Note that we have a non-regular distribution which cannot be addressed by standard methods. The presence of the gap (zero density over some range) and the atom makes it fall outside the analysis of Myerson (1981). We therefore derive the optimal mechanism directly. However, as we explain later (section 3.5), the environment we consider can be thought of as the limit of a non-regular case of Myerson. Thus, an additional (minor) contribution of our work is the verification that the optimal mechanism has no discontinuity at the limit.

We characterize the optimal direct mechanism and show that this involves a cutoff $\widehat{v^*} < \overline{v}$, such that types above this cutoff are pooled. The intuition is as follows. The payment that can be extracted from type v_h depends on the information rent it can earn which in turn depends on its payoff when reporting some type other than v_h. Extracting the optimal amount of surplus from type v_h while maintaining its incentive constraint requires "downgrading" its chances of winning the object if it reports its type untruthfully. This is what the pooling accomplishes. The allocation rule is efficient over the interval $[v_*, \widehat{v^*})$, where v_* is the reserve type, and takes the *maximal possible* (constant) value over the interval $[\widehat{v^*}, \overline{v}]$.

This pooling feature plays an important role in the next step, which considers the actual problem of the seller incorporating the venue restrictions. We show that despite such restrictions, the seller can implement the same allocation as the optimal direct mechanism through a two-stage indirect mechanism. The first stage involves a posted price offer at the store. If the object is not sold, the second stage uses the auction site. Importantly, the optimal auction format is one that includes a temporary buy-now option which implements the pooling in the optimal direct mechanism. Let us discuss this feature briefly.

A puzzling non-standard feature of many online auctions is the availability of a "buy-now" option. This is a posted price at which a bidder can buy the object immediately, superseding the auction[7]. In some cases (e.g. Yahoo!, Amazon, Overstock) the buy price

[7] This is called "Buy-It-Now" on eBay auctions, "Buy" price on Yahoo! auctions, "Take It" price on Amazon auctions, "UBuy it" on uBid auctions and "Make it Mine" price on Overstock.com auctions.

is available throughout the duration of the auction. However, the predominant site eBay uses a format where the buy-now option is temporary: it vanishes as soon as the auction becomes active (i.e. a bid higher than the reserve price but lower than the buy price is placed). As mentioned, the seller's optimal choice of auction format includes a buy-now feature which essentially implements the pooling of the optimal direct mechanism. Moreover, the buy-now phase precedes the standard auction and the buy-now offer is withdrawn when the auction begins. Thus the buy-now option is *temporary*. Indeed, we show that in general it is not possible to implement the optimal mechanism using a permanent buy-now option. This contrasts with the literature – discussed below – which focuses primarily on a permanent buy-now option and note (implicitly or explicitly) the suboptimality of a temporary buy-now option in their settings.

As noted above, eBay auctions use buy-now options that are temporary in nature. To be clear, the forms of the game considered in this paper and the actual eBay game are formally different (the eBay buy-price vanishes as soon as the first auction bid appears). However, we show (in section 4.5) that they are strategically equivalent and give rise to the same outcome.

Let us now comment on the role of the atom at v_h. As stated before, our objective is to consider a situation where an auction is part of a bigger set of sales methods, and therefore the optimal design has features that might seem puzzling when viewed in isolation. In particular, in our setting an auction is used to reach lower value customers in the presence of posted price selling to buyers with higher values. The presence of the atom allows us to derive a posted price as part of the optimal direct mechanism in the cleanest manner, and we then study the optimal distortion in the auction arising from the need to preserve the incentive of high value buyers to pay the posted price rather than participate in the auction.[(8)]

Many smaller auctions also have such a buy-now feature.

[(8)] Apart from modelling convenience, the atom in the distribution might arise naturally as a " reduced form" version of a more general model. We thank the editor of this symposium volume, George Deltas, for pointing out this line of reasoning to us. For example, suppose others sell similar items at some posted price. Even if the values of high end buyers are distributed over an interval, the presence of this (exogenous) outside option could then imply that the entire mass of the high end buyers have the same valuation. See Deltas (2002) for an example where members of a bidding ring obtain a good at an auction at a price that is uninformative of their valuations, and subsequently "anchors" the auction sub-game.

The above discussion goes to justify the atom in the value distribution which makes a posted price part of the optimal direct mechanism. However, our theory has broader appeal beyond cases in which a posted price is explicitly derived as part of an optimal design. Indeed, a posted price could arise because of a variety of unmodelled reasons. For example, many customers find it helpful to get to know a product by physically inspecting it (or through advice from sales personnel) before deciding their maximum willingness to pay, and a bricks-and-mortar store using a posted price could facilitate such a " browsing" function in addition to being a sales channel. In this case, even though the posted price may not itself be derived as part of the overall optimal selling strategy of the *formal* model, the fact that the seller must sell a certain amount at the posted price implies that he is confronted with the same problem as in our model: to ensure that certain customers have an incentive to buy at the posted price rather than migrate to the auction, the auction will be distorted optimally using a buy-now option.

Finally, note that even if the values of the high end buyers have a non-degenerate distribution, as long as the distribution is over a sufficiently narrow range, there is very little difference between the expected revenues from the posted price and an auction for types in this range. Hence, given the presence of the access cost c for the auction site, it is still optimal for the seller to use a posted price at his store to try to sell to the high end types before accessing the auction site. And again, the optimal auction design involves a buy-now option.

Relating to the literature

Section 3.5 develops the relation with the literature on optimal auctions. Here we discuss the (relatively small) literature on auctions with a buy-now option.

This literature usually takes the auction format as given (often taken to be the English auction) and focuses on cases under which revenue might increase if a buy-now option is added. There is no suggestion that an auction with such an option is an optimal arrangement in such cases.

Our approach, in contrast, is to model the environment, and ask what the optimal selling mechanism is. We derive the optimal mechanism and show that implementing it involves selling through a posted price followed by an auction with a " temporary" buy-now option.

Such auctions are in widespread use on eBay. As far as we are aware, ours is the first attempt to explain an auction with a buy-now option as part of an optimal response to the online pricing environment. As discussed in the concluding section, our approach also generates different testable implications compared to the literature.

An early contribution to the literature is by Budish and Takeyama (2001). They provide a simple example with discrete values to show that adding a maximum price to an English auction increases revenue if the bidders are risk averse. Others have subsequently investigated this idea in more general settings. Reynolds and Wooders (2008) and Hidvégi, Wang, and Whinston (2006) show that this result holds in a setting in which bidders draw values from a continuous distribution. The latter paper as well as Mathews and Katzman (2006) show that the result holds even with risk neutral buyers if the seller is risk averse. However, auctions with a buy-now option are not optimal in the environments they model. With continuous distributions, and risk averse buyers, Maskin and Riley (1984) characterize the optimal auction[(9)]. They also show that even if we limit attention to standard auctions, so long as either the bidders are risk averse or the seller is risk averse, first price auctions are preferred to English auctions by the seller[(10)]. However, none of the papers that study buy-price English auctions under a continuous distribution and risk averse bidders/seller compare such auctions with the optimal auction in their setting, or even the first price auction.[(11)] Therefore even with risk averse bidders/seller, the reason for choosing a buy-price auction remains unclear.

Milgrom (2004) considers a model with sequential entry of bidders who face a cost of learning their own type. A bidder who wants to buy at a (permanent) buy price wins only if no other bidder in a higher queue position exercises this right. Bidders do not know own queue positions, and consider all positions equally likely. In this setting types above a certain cutoff have an incentive to buy at the buy price, and the revenue maximizing auction involves a buy price. Our model, in contrast, is much closer to the standard private values model so that entry is simultaneous, and there is no cost of learning own value. The only departure from the standard model is the presence of an atom at v_h.

(9) See also Matthews (1983). The optimal auction is in fact quite complex, involving payments by some losing bidders, and is very different from standard auctions with a buy-now option.

(10) Holt (1980) previously derived this result for the case of risk averse bidders and a risk neutral seller.

(11) Budish and Takeyama (2001) show that with a discrete value distribution (two possible values), an English auction with a buy-now option can raise more revenue than a first price auction.

A further difference between the literature and our approach is the treatment of a temporary buy-now option. Such an option is seemingly even more puzzling (compared to a permanent buy-now option) in the sense that even if one could find a reason for introducing a buy-now option, taking it away as the auction starts begs a further question. Indeed, the literature has noted that in the standard independent private values setting (with risk-neutral seller and bidders), while an auction involving a suitably chosen permanent buy price is revenue equivalent to an English auction (and therefore an optimal auction), an auction with a temporary buy price seems not to be optimal. Reynolds and Wooders (2008) consider both forms of buy-price and show that with CARA risk averse bidders, an auction with a permanent buy price raises more revenue. The concluding section of Hidvégi, Wang, and Whinston (2006) also contain a remark to this effect. The same conclusion applies to the sequential costly entry model of Milgrom (2004). In contrast, our analysis shows that posted price selling followed by a standard auction with a temporary buy-now option – used widely by eBay – implements the optimal mechanism, while a permanent buy-now option fails to do so.

2 The Model

An object is offered for sale by a risk neutral seller with reservation utility zero. The seller has access to two different selling venues. Venue 1 is a posted price selling mechanism. The seller owns this site and can access it at zero cost (any cost incurred in setting up the site is already sunk). Venue 2 is an (online) auction site which the seller can access by paying some cost $c > 0$. We assume c to be small enough to rule out the uninteresting case where the seller does not want to use the auction site at all. The seller can choose the posted price for venue 1 and can design the auction mechanism for venue 2.[12]

There are $N \geq 2$ potential risk neutral buyers. The buyers are ex ante symmetric and draw values independently from the following distribution. With probability μ a buyer has valuation $v_h > 1$, and with probability $(1-\mu)$ the buyer's valuation is drawn from the distribution $F(v)$ with support $[0, 1]$. We assume $F(v)$ has continuous density $f(v)$, where $f(v) > 0$ for all $v \in [0, 1]$, and that $F(v)$ satisfies the monotone hazard rate condition. In

[12] We follow the standard mechanism design approach and assume that the seller can commit to the mechanism.

other words, other than an atom at v_h, the rest of the model is essentially the standard independent private values model [13].

3 The Optimal Direct Mechanism

As noted in the introduction, the plan of the paper is as follows. Instead of solving for the seller's optimal selling strategy directly, we follow an indirect approach that we find useful and instructive. We first consider (in this section) a direct revelation game without a venue-restriction constraint. In other words, we first analyze the seller's problem *as if* it was a standard mechanism design exercise by ignoring the fact that the two available venues have different properties. Once we derive the optimal direct mechanism, we consider the problem incorporating venue restrictions and show that the seller can nevertheless implement the optimal allocation from the direct mechanism with an indirect mechanism that obeys the venue restrictions and also uses the venues in the most profitable way. This optimal implementing mechanism involves a suitably chosen posted price at venue 1 (the store) and an auction with a temporary buy-now option at venue 2 (the online auction site).

3.1 Formulating the seller's optimization problem

The seller is the mechanism designer. Let $(r_1, \ldots, r_N)$ denote a profile of reported values, where $r_i \in [0,1] \cup \{v_h\}$ for all $i \in \{1, \ldots, N\}$. Let r_{-i} denote the vector of reports of buyers other than i.

For any (r_i, r_{-i}), the direct mechanism specifies a vector $(t_i(r_i, r_{-i}), x_i(r_i, r_{-i}))$ for all $i \in \{1, \ldots, N\}$, where $t_i(\cdot, \cdot)$ is the transfer from buyer i to the seller, and $x_i(\cdot, \cdot)$ is the probability that i wins the object.

[13] We could, for the sake of generality, write the type space as $[\underline{v}, \overline{v}] \cup \{v_h\}$ with $v_h > \overline{v}$. Since we allow the seller to have a non-trivial reserve price, there is no gain in allowing $\underline{v}$ to be less than 0 (the seller's valuation). Further, the only real effect of allowing $\underline{v}$ to be greater than zero is that the optimal auction may not have any reserve price; as will be shown later this is of no importance for the analysis that follows. Therefore, to maintain parsimony of notation, we use the unit interval instead of $[\underline{v}, \overline{v}]$.

Let $X_i(r_i)$ denote the expected probability with which i wins the object conditional on reporting r_i, and $T_i(r_i)$, the expected transfer conditional on reporting r_i. The expectation is with respect to the other buyers' valuation distribution; in other words, buyer i assumes that buyers $-i$ report truthfully and the mechanism is incentive compatible when i reports truthfully as well.

Since the seller is risk neutral and the buyers are ex ante symmetric, we can, without loss of generality, restrict the search for the optimal mechanism by considering symmetric mechanisms only. In what follows, we drop the subscripts in the functions $T_i(\cdot)$ and $X_i(\cdot)$.

The seller's task consists of finding the optimal $X(\cdot)$ and $T(\cdot)$ subject to incentive compatibility and individual rationality constraints. It is well known that this can be transformed into a problem of finding the optimal $X(\cdot)$ only. We now proceed to derive this problem.

A bidder of type v chooses r to maximize $U(v,r) = vX(r) - T(r)$. Satisfying the incentive compatibility constraint implies that the maximum occurs at $r = v$. Following standard arguments it is easy to verify that incentive compatibility requires $X(\cdot)$ to be non-decreasing[(14)].

It is convenient to consider types $v \in [0,1]$ and type v_h separately. From the envelope theorem[(15)] we have,

$$U(v) = U(0) + \int_0^v X(t)\,dt \tag{3.1}$$

Putting $U(0) = 0$, we get,

$$T(v) = vX(v) - \int_0^v X(t)\,dt \tag{3.2}$$

Next, consider type v_h. The highest payoff this type can get by reporting falsely is $\max_{v\in[0,1]} (v_h X(v) - T(v))$.

(14) Consider two possible values v and v' of a bidder. Incentive compatibility requires $vX(v) - T(v) \geqslant vX(v') - T(v')$ and $v'X(v') - T(v') \geqslant v'X(v) - T(v)$ Combining the inequalities we get $(v - v')(X(v) - X(v')) \geqslant 0$. Therefore $X(v) \geqslant X(v')$ whenever $v \geqslant v'$.

(15) See Milgrom and Segal (2002). Even though our model does not fit exactly their framework since our entire type space, $[0,1] \cup v_h$ is not a connected interval, it is clear that since v_h is the highest type, the optimal mechanism involves $x(v_h, r_{-i}) = 1$ when each element of r_{-i} is in $[0,1]$, and $t(v, r_{-i}) = 0$ for all $v \in [0,1]$ whenever at least one element of r_{-i} is v_h. Therefore without loss of generality we can consider $X(v) = (1-\mu)^{N-1}\,\mathbb{E}x(v, v_{-i})$ and $T(v) = (1-\mu)^{N-1}\,\mathbb{E}t(v, v_{-i})$ where in both cases the expectation is taken over v_{-i} assuming each element of v_{-i} is in $[0,1]$. We show later that $X(v)$ is in fact differentiable almost everywhere for $v \in [0,1]$.

For any $v', v'' \in [0,1]$ with $v' > v''$. We have,

$$
\begin{aligned}
& [v_h X(v') - T(v')] - [v_h X(v'') - T(v'')] \\
= \; & v_h [X(v') - X(v'')] - v' X(v') + v'' X(v'') + \int_{v''}^{v'} X(t) dt \\
\geqslant \; & v_h [X(v') - X(v'')] - v' X(v') + v'' X(v'') + (v' - v'') X(v'') \\
= \; & [v_h - v'] [X(v') - X(v'')] \\
\geqslant \; & 0
\end{aligned}
$$

Where the first inequality follows from the fact that $X(v)$ is non-decreasing in v and the last inequality follows since in addition $v_h > v'$. Thus type v_h gets a higher utility by reporting a higher type. Therefore, without loss of generality, we can write,

$$\max_{v \in [0,1]} v_h X(v) - T(v) = v_h X(1) - T(1). \tag{3.3}$$

It is clear that the IC constraint of type v_h must bind at the optimal mechanism. Hence we must have

$$v_h X(v_h) - T(v_h) = \max_{v} v_h X(v) - T(v) \tag{3.4}$$

Using equations (3.2)-(3.3), $T(v_h)$ is given by

$$T(v_h) = v_h X(v_h) - v_h X(1) + X(1) - \int_0^1 X(t)\, dt \tag{3.5}$$

Denote the seller's expected revenue as $\mathbb{E}R$. The seller's *per capita* expected revenue is given by

$$\frac{\mathbb{E}R}{N} = \mu T(v_h) + (1 - \mu) \int_0^1 T(v) f(v) dv$$

Following standard procedure (see Myerson (1981)), we can derive:

$$\int_0^1 T(v) f(v) dv = \int_0^1 \left(v - \frac{1 - F(v)}{f(v)} \right) X(v) f(v) dv$$

Hence,

$$
\begin{aligned}
\frac{\mathbb{E}R}{N} \; = \; & \mu \left(v_h X(v_h) - (v_h - 1) X(1) - \int_0^1 X(v) dv \right) \\
& + (1 - \mu) \int_0^1 \left(v - \frac{1 - F(v)}{f(v)} \right) X(v) f(v) dv
\end{aligned}
$$

This can be further rewritten as:

$$\frac{\mathbb{E}R}{N} = \mu v_h X(v_h) - \mu\,(v_h - 1)\,X(1) + \int_0^1 \psi(v)X(v)f(v)dv \tag{3.6}$$

where

$$\psi(v) \equiv (1-\mu)\left(v - \frac{1-F(v)}{f(v)}\right) - \frac{\mu}{f(v)} \tag{3.7}$$

We call $\psi(v)$ the "modified virtual valuation" of type v.

To make our environment as close as possible to that under which standard auctions are optimal, we assume the following.

Assumption 1 ***(Modified regularity)*** *$\psi(v)$ is increasing in v.*[(16)]

Next, if $\psi(v)$ is always negative for $v \in [0,1]$, the seller's problem is trivial: never allocate to these types. To avoid this uninteresting case, the weakest possible assumption is to ensure $\psi(1) > 0$ We assume

Assumption 2 ***(Non-trivial Optimal Mechanism)*** *$\frac{\mu}{1-\mu} < f(1)$.*

This ensures $\psi(1) > 0$ and hence by continuity that there is a range of values of v such that $\psi(v) > 0$ for all v in that range[(17)].

Since there is no type higher than v_h (i.e., there is no type who can obtain rent by reporting v_h), the seller should sell with probability 1 if at least one bidder announces v_h. It follows that

$$X(v_h) = \sum_{n=0}^{N-1} \binom{N-1}{n} (1-\mu)^{N-n-1}\mu^n \frac{1}{n+1} \tag{3.8}$$

Since this is a constant, we can ignore this term in maximizing expected revenue (given by (3.6)). This proves the following result.

(16) Monotone hazard rate of $F(v)$ ensures that the standard virtual valuation, i.e. $v - \frac{1-F(v)}{f(v)}$ is increasing in v. Here, this is no longer sufficient to ensure that the modified virtual valuation $\psi(v)$ is increasing in v. To get a sufficient condition that is independent of μ, we also need convexity of $F(v)$ (satisfied by distributions such as uniform and triangular). For other distributions this assumption holds if, along with monotone hazard rate, μ is not too high.

(17) However, this assumption, though necessary, is not sufficient for the seller to want to sell to types other than v_h. If μ is high, it is optimal for the seller to not price discriminate, and only sell to type v_h. The precise condition is derived in proposition 2 below.

Proposition 1 *The optimal mechanism is the solution of the following problem*

$$\max_{X(v)} \quad -\mu\,(v_h - 1)\,X(1) + \int_0^1 \psi(v) X(v) f(v)\,dv$$

when $X(v)$ is non-decreasing and where $\psi(v)$ is given by equation (3.7) *and the transfers $T(v)$ and $T(v_h)$ are given by equations (3.2) and (3.5) respectively.*

Intuitively, the first term is the loss from having an auction which allows types below v_h to participate. This generates a loss because a pure posted price mechanism can charge v_h, while the mechanism including an auction must charge a lower posted price to satisfy incentive compatibility. The second term is the gain from including an auction. Note that the form of the second term is similar to the corresponding expression in a standard setting. However, the solution for $X(\cdot)$ does not coincide with the standard solution, even though $\psi(v)$ is strictly increasing in v, because of the first term. This also clarifies the price discriminating role of the auction: $X(\cdot)$ must be set so that we can separate the high type from the rest and extract surplus optimally from both segments of the market.

Note that we have already imposed the condition that $X(v)$ must satisfy $U(0) = 0$. The condition is used following equation (3.1), which also shows that the solution to the above problem satisfies individual rationality. To satisfy incentive compatibility, the solution to the above problem must also satisfy monotonicity of $X(v)$. The following analysis takes this into account explicitly in deriving the optimal mechanism.

3.2 Solving the seller's optimization problem

We now derive a set of results that characterize the optimal mechanism.

First, let v_* be such that

$$\psi(v_*) = 0. \tag{3.9}$$

Note that for v close to zero, $\psi(v) < 0$. From assumptions 1 and 2 it follows that v_* exists and $0 < v_* < 1$.

Since $\psi(v) < 0$ for $v < v_*$, a simple inspection of the seller's objective function shows that the optimal mechanism must be characterized by

$$X(v) = 0 \;\text{ for any } v \in [0, v_*). \tag{3.10}$$

In other words, similar to standard auctions, the seller does not sell to types whose (modified) virtual valuation is negative.

If μ is close to 1, it is clearly optimal to simply sell to type v_h at price v_h and not sell to lower types at all. In other words, in this case a large information rent must be ceded to type v_h if the seller wants to sell to lower types and maintain incentive compatibility. The next result clarifies the precise condition required for the optimal mechanism to involve a positive probability of allocating to lower types.

Proposition 2 *Let* $\lambda(v) \equiv \int_v^1 \psi(t)f(t)dt - \mu(v_h - 1)$. *The optimal mechanism involves* $X(v) \neq 0$ *for some* $v \in [0,1]$ *of positive measure only if* $\lambda(v_*) > 0$, *where* v_* *is given by equation* (3.9) *In other words, if* $\lambda(v_*) \leqslant 0$ *the optimal mechanism is simply a posted price of* v_h.

Proof. Using equation (3.10), the seller's optimization problem can be rewritten as

$$\max_{X(v)} \quad -\mu\left(v_h - 1\right) X(1) + \int_{v_*}^1 \psi(v) X(v) f(v)\, dv.$$

Let us reduce $X(v)$ for all v by ϵ. The maximand increases by $-\epsilon\lambda(v_*)$. Further, $\lambda'(v) = -\psi(v)f(v) < 0$ for $v > v_*$ and therefore $\lambda(\cdot)$ is maximized at $v = v_*$. Therefore if $\lambda(v_*) \leqslant 0$, $\lambda(v) < 0$ for all $v > v_*$. Therefore reducing $X(v)$ increases the maximand by a positive amount, and it is then optimal to set $X(v) = 0$ for all v. ■

If $\lambda(v_*) \leqslant 0$, the information rent type v_h gets if the seller tries to reach the "lower end" of the market as well is too high and it is better for the seller to sell to type v_h only. In what follows, we assume that the optimal design is non-trivial, i.e. $\lambda(v_*) > 0$.

The next result, which is one of our main results, shows that the optimal mechanism must involve pooling for some neighborhood of 1.

Proposition 3 *Let* $X^*(v)$ *denote the optimal mechanism. There exists a* $\widehat{v} < 1$, *such that* $X^*(v)$ *is constant for* $v \in [\widehat{v}, 1]$.

Proof. Let $\widetilde{X}(v)$ be incentive compatible, individually rational, and strictly increasing for any neighborhood of 1 no matter how small. The following argument shows that such a $\widetilde{X}(v)$ cannot be optimal.

Consider $\varepsilon > 0$, small, and define $K(\varepsilon)$ as

$$K(\varepsilon) = \mathbb{E}\left[\widetilde{X}(v)|1-\varepsilon \leq v \leq 1\right] = \int_{1-\varepsilon}^{1} \widetilde{X}(v)\frac{f(v)}{1-F(1-\varepsilon)}dv$$

Note that because $\widetilde{X}(v)$ is strictly increasing, $\widetilde{X}(1-\varepsilon) < K(\varepsilon) < \widetilde{X}(1)$.

Consider the following $X(\cdot)$:

$$X(v) = \begin{cases} \widetilde{X}(v) & \text{for } v \in [0, 1-\varepsilon) \\ K(\varepsilon) & \text{for } v \in [1-\varepsilon, 1] \end{cases}$$

$X(v)$ is non-decreasing since $\widetilde{X}(v)$ is non-decreasing. In particular, $X(v)$ must satisfy the incentive compatibility and individual rationality constraints given that $\widetilde{X}(v)$ does.

Now, let

$$\widetilde{R} = -\mu(v_h - 1)\widetilde{X}(1) + \int_0^1 \widetilde{X}(v)\psi(v)f(v)dv$$

and

$$R = -\mu(v_h - 1)X(1) + \int_0^1 X(v)\psi(v)f(v)dv.$$

We have,

$$\begin{aligned} R - \widetilde{R} &= \left[\widetilde{X}(1) - K(\varepsilon)\right]\mu(v_h - 1) + K(\varepsilon)\int_{1-\varepsilon}^{1}\psi(v)f(v)dv - \int_{1-\varepsilon}^{1}\psi(v)\widetilde{X}(v)f(v)dv \\ &> \left[\widetilde{X}(1) - K(\varepsilon)\right]\mu(v_h - 1) + K(\varepsilon)\int_{1-\varepsilon}^{1}\psi(v)f(v)dv - \widetilde{X}(1)\int_{1-\varepsilon}^{1}\psi(v)f(v)dv \\ &= \left[\widetilde{X}(1) - K(\varepsilon)\right]\left[\mu(v_h - 1) - \int_{1-\varepsilon}^{1}\psi(v)f(v)dv\right] \end{aligned}$$

where the inequality follows since $\widetilde{X}(v)$ is strictly increasing. Now, for ε sufficiently small but strictly positive both terms in the last line above is positive and hence $R > \widetilde{R}$ which shows that $\widetilde{X}(v)$ cannot be optimal. It follows that any optimal mechanism must satisfy the condition in the statement of the proposition. ■

Let $\widehat{v}$ denote the cutoff type such that all types above $\widehat{v}$ are pooled. Let K denote the allocation probability of the pooled types. In other words, $K = X(v)$ for $v \in [\widehat{v}, 1]$. Using this, as well as (3.10), the seller's objective function can be rewritten as

$$\max_{X(v),\, K,\, \widehat{v}} \; K\left[-\mu\left(v_h - 1\right) + \int_{\widehat{v}}^{1}\psi(v)f(v)dv\right] + \int_{v_*}^{\widehat{v}} X(v)\psi(v)f(v)dv \tag{3.11}$$

The remaining task is to characterize K, $\widehat{v}$, and $X(v)$ for $v \in [v_*, \widehat{v}\,)$. Proposition 4 shows that the exercise boils down to a simple exercise. Before stating the formal result, we discuss intuition behind it below.

Note first that the (expected) probability allocation function $X(v)$ can be discontinuous at $\widehat{v}$. Given a $\widehat{v}$, the seller needs to determine how to allocate the probabilities across the types in $[v_*, \widehat{v}\,)$ and in $[\widehat{v}, 1]$. Since there is no gain in "wasting probabilities" (i.e. not selling with positive probability even if at least one bidder's type is greater than v_*) or in having an inefficient allocation rule if types of all bidders lie in the interval $[v_*, \widehat{v}\,)$, the only case we need to consider is the nature of the allocation function when some reported types are in $[v_*, \widehat{v}\,)$ and some others in $[\widehat{v}, 1]$. Let $\beta_n(v)$ denote the transfer of probability of winning from types in $[\widehat{v}, 1]$ to types in $[v_*, \widehat{v}\,)$ when n bidders report types below $\widehat{v}$, with at least one type in $[v_*, \widehat{v}\,)$, and the other $N-n$ bidders report types in $[\widehat{v}, 1]$. Again, since there is no reason to allocate this probability inefficiently among types in the latter interval, $\beta_n(v)$ must optimally be the probability transfer to the highest among the types in the interval $[v_*, \widehat{v}\,)$. The expected allocation function is as follows.

Since types in $[0, 1]$ win only if there is no other bidder of type v_h, all terms in the expression for $X(v)$ contain a common factor $(1-\mu)^{N-1}$. For economy of expression, we keep this common factor on the left hand side.

$$\frac{X(v)}{(1-\mu)^{N-1}} = \begin{cases} 0 & \text{if} \quad v < v_* \\ F^{N-1}(v) + \sum_{n=1}^{N-1} \binom{N-1}{n-1} \left(1 - F(\widehat{v}\,)\right)^{N-n} F^{n-1}(v)\beta_n(v) & \text{if} \quad v \in [v_*, \widehat{v}\,) \\ \sum_{n=1}^{N-1} \binom{N-1}{n} \frac{(1-F(\widehat{v}\,))^{N-1-n}}{N-n} \left[F^n(v_*) + \int_{v_*}^{\widehat{v}} (1 - \beta_n(t))\, dF^n \right] & \\ \qquad + \dfrac{(1-F(\widehat{v}\,))^{N-1}}{N} & \text{if} \quad v \in [\widehat{v}, 1] \end{cases} \tag{3.12}$$

Note that the expected allocation for $v \in [\widehat{v}, 1]$ can be rewritten as

$$G(\widehat{v}\,) - \sum_{n=1}^{N-1} \binom{N-1}{n} \frac{(1-F(\widehat{v}\,))^{N-1-n}}{N-n} \int_{v_*}^{\widehat{v}} \beta_n(t)\, dF^n$$

where

$$G(\widehat{v}\,) = \sum_{n=0}^{N-1} \binom{N-1}{n} \frac{(1-F(\widehat{v}\,))^{N-1-n}\; F^n(\widehat{v}\,)}{N-n} = \frac{1 - F(\widehat{v}\,)^N}{N(1 - F(\widehat{v}\,))} \tag{3.13}$$

The next (and crucial) observation is that the function $\beta_n(\cdot)$ should optimally be such that the allocation function for $v > v_*$ has only a single point of discontinuity, and is flat beyond this point at the maximal height.

Suppose a K has been chosen. Since $\psi(v)$ is strictly increasing, so long as some probability is being transferred from the types above $\widehat{v}$ to types below it is optimal for the seller to transfer probability to the highest possible types below $\widehat{v}$. That is, irrespective of the value of K, the seller "packs the right side" of the interval $[v_*, \widehat{v}\,)$ as much as possible. Preserving incentive compatibility requires that $\frac{X(v)}{(1-\mu)^{N-1}}$ be non-decreasing and therefore $\frac{X(v)}{(1-\mu)^{N-1}} \leqslant K$ for any $v < \widehat{v}$. Because $\psi(v)$ is strictly increasing the constraint is actually satisfied as an equality. Hence, irrespective of the value of K, the form of the optimal expected allocation function remains the same: $\frac{X(v)}{(1-\mu)^{N-1}}$ equals $F(v)$ up to a cutoff point, and is flat at the maximal height beyond the cutoff. This observation helps simplify the analysis tremendously since the exercise boils down to finding a *single* cut-off value such that all types beyond this type are pooled (and have the same, maximum possible, expected probability of winning). We state this formally in the proposition below.

Proposition 4 *The optimal mechanism involves solving the following maximization problem*

$$\max_{y} \quad \int_{v_*}^{y} F(v)\psi(v)f(v)\,dv + G(y)\left[-\mu(v_h - 1) + \int_{y}^{1} \psi(v)f(v)\,dv\right]$$

where $G(y)$ is given by equation (3.13)*, and where the argmax gives the optimal pooling cutoff.*

We denote the optimal pooling cutoff as $\widehat{v}^*$. The next result shows that this exists.

Proposition 5 *The optimal pooling cutoff $\widehat{v}^*$ exists.*

Proof. Define $H(\cdot)$ as follows.

$$\begin{aligned} H(y) &\equiv \left(F(y) - G(y)\right)\psi(y) + \frac{G'(y)}{f(y)}\left(-\mu(v_h - 1) + \int_{y}^{1} \psi(t)f(t)dt\right) \\ &= \left(F(y) - G(y)\right)\psi(y) + \frac{G'(y)}{f(y)}\lambda(y), \end{aligned}$$

where $\lambda(y)$ is defined in proposition 2, and $G(y)$ is given by equation (3.13). Note that $H(y) = 0$ is the first order condition for an interior optimum of the expression in proposition 4 above. Now, using L'Hospital's rule, it can be easily calculated that $\lim_{y\to 1} G(y) = 1$. Further, $\lim_{y\to 1} \frac{G'(y)}{f(y)} = \frac{N-1}{2}$. Using these, and the fact that $\lambda(1) < 0$, it follows that $\lim_{y\to 1} H(y) < 0$.

Next, recall that following proposition 2, we assumed that $\lambda(v_*) > 0$ (to ensure a non-trivial price discrimination problem). Further, after some simplifying,

$$\frac{G'(y)}{f(y)} = \frac{1}{N}\sum_{n=1}^{N-1} nF^{n-1}(y)$$

Thus $\frac{G'(y)}{f(y)} > 0$ for all $y \geqslant v_*$. This, coupled with the fact that $\psi(v_*) = 0$, implies that $H(v_*) = \frac{G'(v_*)}{f(v_*)}\lambda(v_*) > 0$.

Note that $H(\cdot)$ is a continuous function which is strictly positive at $y = v_*$ and strictly negative at $y = 1$. Hence there must exist $\widehat{v}^* \in (v_*, 1)$ such that both $H(\widehat{v}^*) = 0$, and the second order condition is satisfied at $\widehat{v}^*$. ■

From proposition 4, the general form of the mechanism can be rewritten as follows.

$$\frac{X(v)}{(1-\mu)^{N-1}} = \begin{cases} 0 & \text{if } v < v_* \\ F^{N-1}(v) & \text{if } v \in [v_*, \widehat{v}^*) \\ G(\widehat{v}) & \text{if } v \in [\widehat{v}^*, 1] \end{cases} \tag{3.14}$$

3.3 The optimal direct mechanism

Finally, we collect together the results from above and describe the optimal mechanism. This is given by the allocation function $X^*(\cdot)$ and the expected payment function $T^*(\cdot)$ given by equations (3.15) to (3.18) below.

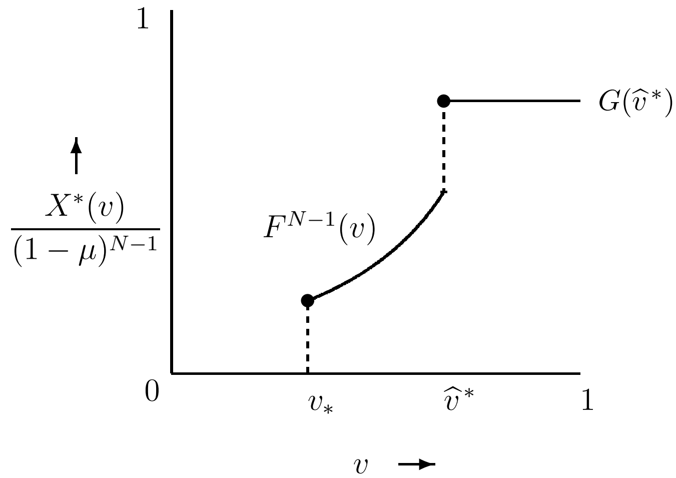

Figure 1: The figure shows the optimal allocation function $X^*(v)/(1-\mu)^{N-1}$. The function jumps up at the reserve cutoff v_* and at the pooling cutoff $\widehat{v}^*$. The extent of the jump at the pooling cutoff is maximal.

$$X^*(v_h) = \sum_{n=0}^{N-1} \binom{N-1}{n} (1-\mu)^{N-n-1} \mu^n \frac{1}{n+1}, \tag{3.15}$$

$$X^*(v) = \begin{cases} 0 & \text{if } v < v_* \\ (1-\mu)^{N-1}\, F^{N-1}(v) & \text{if } v \in [v_*, \widehat{v}^*) \\ (1-\mu)^{N-1}\, G(\widehat{v}^*) & \text{if } v \in [\widehat{v}^*, 1] \end{cases} \tag{3.16}$$

$$T^*(v_h) = v_h X^*(v_h) - (v_h - 1)\, X^*(1) - \int_0^1 X^*(t)\, dt, \tag{3.17}$$

$$T^*(v) = vX^*(v) - \int_0^v X^*(t)\, dt \qquad \text{for } v \in [0,1]. \tag{3.18}$$

where the "reserve type" v_* is given by $\psi(v_*) = 0$, $G(\cdot)$ is given by equation (3.13), and the "pooling cutoff" $\widehat{v}^*$ maximizes the expression in proposition 4.

$X^*(v_h)$ follows from equation (3.8), and $T^*(v_h)$ follows from equation (3.5). The optimal allocation $X^*(v)$ follows from equation (3.14). $T^*(v)$ then follows from equation (3.2). Figure 1 shows the optimal mechanism.

Also note that other than the pooling of types in $[\widehat{v}^*, 1]$ and the presence of the reserve type v_*, the optimal mechanism is efficient. Whenever at least one bidder draws a type above v_*, the object is sold with probability 1. We now present a simple numerical example to elucidate the design of the optimal direct mechanism.

3.4 An Example

Suppose $N = 2$, and suppose the bidders draw values from a uniform distribution over the unit interval.[18] Then $F(v) = v$, and $f(v) = 1$ for all $v \in [0, 1]$. For the optimal mechanism to involve selling to high as well as lower types, we need to satisfy assumption 2, which requires $\mu < \frac{1}{2}$. From proposition 2, we also need $\lambda(v_*) > 0$, which implies $4\mu(1-\mu)v_h < 1$. Solving for μ and using the restriction $\mu < 1/2$, we get

$$\mu < \frac{1}{2}\left(1 - \sqrt{\frac{v_h - 1}{v_h}}\right) \equiv \overline{\mu}(v_h)$$

For any $\mu < \overline{\mu}(v_h)$, the optimal mechanism involves an auction. The reserve type v_* and the "pooling cutoff" $\widehat{v}^*$ are given by $v_* = \frac{1}{2(1-\mu)}$ and $\widehat{v}^* = 1 - \sqrt{\frac{\mu(v_h-1)}{1-\mu}}$. The expected revenue in this case is given by equation (3.6). For $\mu \geqslant \overline{\mu}(v_h)$, the optimal mechanism is a posted price of v_h and the expected revenue is[19] $\mathbb{E}R = 2\mu v_h X^*(v_h)$.

Finally consider a mechanism that combines a posted price with a standard auction. Calculating the expected revenue from this suboptimal mechanism[20] clarifies the gain from the pooling feature in the optimal auction. The expected revenue is given by

$$\mathbb{E}R = 2\left[\mu v_h X^*(v_h) - \mu\left(v_h - 1\right)X(1) + \int_0^1 \psi(v)X(v)f(v)dv\right]$$

where $X^*(v_h)$ is as before, but now $X(\cdot) = (1-\mu)F(v)$ for all $v \in [v_*, 1]$.

As an example, consider $v_h = 1.5$ and $\mu = .1$. Since $\overline{\mu}(1.5) = 0.21$, the optimal mechanism involves a posted price and an auction. The auction is characterized by $v_* = .56$ and $\widehat{v}^* = .76$, i.e. the lowest 56% types are not served and the top 24% types in the unit

[18] We present the example in a concise manner here. See Bose and Daripa (2007) for a detailed exposition.

[19] Alternatively, this can be derived as v_h times the probability that at least one bidder is of type v_h, i.e. $\mathbb{E}R = (1-(1-\mu)^2)v_h$.

[20] Such an auction (with the right reserve price) is optimal in the standard independent private values setting (i.e. without an atom at v_h), but suboptimal in our setting.

interval are pooled. The revenue from the optimal mechanism is 0.47. A simple posted price fetches a revenue of .29. Thus there is a gain from price discrimination.

The following table compares the expected revenue from the optimal mechanism ($\mathbb{E}R$ Opt), posted price plus standard auction ($\mathbb{E}R$ Std), and a pure posted price mechanism ($\mathbb{E}R$ PP) across different values of v_h. The numbers reported are for $\mu = 0.1$. For $v_h \leqslant 2.75$, $\overline{\mu}(v_h) > 0.1$. Therefore over this range of v_h, the optimal mechanism involves non-trivial price discrimination. The table also compares the values of the optimal posted price (PP Opt) with the posted price if a standard auction is used (PP Std). For either mechanism, the posted price is given by $T(v_h)/X(v_h)$.

v_h	$\mathbb{E}R$ Opt	$\mathbb{E}R$ Std	$\mathbb{E}R$ PP
1	0.463	0.463	0.190
1.25	0.468	0.465	0.238
1.75	0.483	0.470	0.333
2.25	0.503	0.475	0.428
2.75	0.526	0.480	0.523

PP Opt	PP Std
0.673	0.673
0.705	0.686
0.815	0.712
0.959	0.738
1.130	0.765

3.5 Relating to the literature on optimal auctions

Finally, let us relate our work to the literature on optimal auctions. In our model the value of a bidder is v_h with probability μ and with probability $(1-\mu)$ the value is drawn from a distribution $F(\cdot)$ (with density $f(\cdot)$) on the domain $[0,1]$. Instead, consider a distribution $\mathcal{F}(\cdot)$ on $[0, v_h+\epsilon]$ where the density is $(1-\mu)f$ on $[0,1]$, ϵ on $(1, v_h)$ and $\frac{1}{\epsilon}(\mu - \epsilon(v_h - 1))$ on $[v_h, v_h+\epsilon]$. This "perturbed" distribution is within the scope of the analysis of Myerson (1981). It is instructive to see that the pooling result we obtain here can be obtained from the standard methods applied to this perturbed distribution, and that the mechanism constructed here can be obtained by a limiting argument.[21]

Since the limiting case is the one we are interested in, and since one cannot directly apply the standard methods in this case, we have derived the optimal mechanism directly. As a minor technical point, our results can be taken as a verification that there is no discontinuity in the optimal mechanism in the limit. The optimal mechanism for distributions

[21]We are grateful to a referee for pointing this out.

away from the limit (derived using standard methods), converge to the one we derive for the limiting case.

To derive the optimal mechanism for the perturbed distribution, we adopt the "marginal-revenue" approach of Bulow and Roberts (1989) who reinterpret the problem of designing an optimal auction as a standard monopolist's problem. Define $q = 1 - F(v)$ as quantity and v as price. The (inverse) "demand curve" is given by:

$$v(q) = \begin{cases} v_h + \varepsilon - \dfrac{\varepsilon q}{\mu - (v_h - 1)\varepsilon} & \text{if} \quad 0 \leqslant q \leqslant \mu - (v_h - 1)\varepsilon \\ v_h - \dfrac{q - \mu + (v_h - 1)\varepsilon}{\varepsilon} & \text{if} \quad \mu - (v_h - 1)\varepsilon < q < \mu \\ F^{-1}\left(1 - \frac{q-\mu}{1-\mu}\right) & \text{if} \quad \mu \leqslant q \leqslant 1 \end{cases}$$

The corresponding marginal revenue (MR) is given by

$$\mathrm{MR}(q) = \begin{cases} \mathrm{MR}_1(q) \equiv v_h + \varepsilon - \dfrac{\varepsilon q}{\mu - (v_h - 1)\varepsilon} & \text{if} \quad 0 \leqslant q \leqslant \mu - (v_h - 1)\varepsilon \\ \mathrm{MR}_2(q) \equiv v_h - \dfrac{q - \mu + (v_h - 1)\varepsilon}{\varepsilon} & \text{if} \quad \mu - (v_h - 1)\varepsilon < q < \mu \\ \mathrm{MR}_3(q) \equiv F^{-1}\left(1 - \frac{q-\mu}{1-\mu}\right) - \frac{q}{(1-\mu) f\left(F^{-1}\left(1 - \frac{q-\mu}{1-\mu}\right)\right)} & \text{if} \quad \mu \leqslant q \leqslant 1 \end{cases}$$

MR_2 and MR_3 are clearly decreasing in q. By rewriting as a function of v, it is easy to check that MR_1 is simply the modified virtual value, and thus assumptions 1 and 2 ensure that this is decreasing in q and present a non-trivial optimization problem.

For ε small, clearly MR_2 is negative for all q in its domain and lies below the other two segments. Figure 2 shows the form of the MR curve for ε small. It also shows the ironing required to construct the optimal mechanism: the MR must be ironed at w such that areas A_1 (trapezoid) and A_2 (triangle) are equal.

The area of A_1 is given by $\varepsilon(v_h - 1)(w - 2v_h + 1 + \mu/\varepsilon) + \frac{1}{2}\varepsilon(v_h - 1)(2v_h - 2)$. As $\varepsilon \to 0$, the area of A_1 goes to $(v_h - 1)\mu$. Let q_w be implicitly defined by $\mathrm{MR}_3(q_w) = w$. The area of A_2 is given by $\frac{1}{2}(q_w - \mu)\left(1 - \frac{\mu}{(1-\mu)f(1)} - w\right)$. Equating the two areas we get the values of w and q_w. Clearly, type $v(q_w)$ is the "pooling cutoff" and types $[v(q_w), 1]$ are pooled.

As an example, if F is the uniform distribution, $v_h = 1.5$ and $\mu = .1$, by the above limiting argument we can obtain $v(q_w) = .76$, which is indeed the same cutoff as in the example in the previous section.

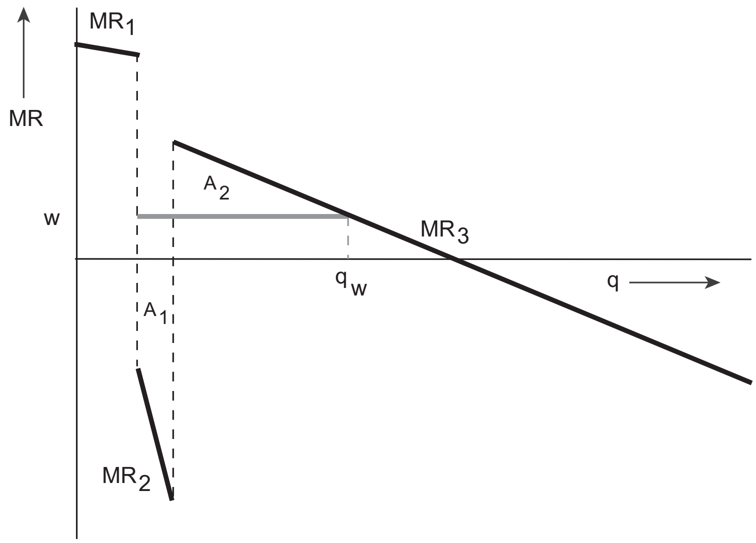

Figure 2: Marginal revenue and ironed marginal revenue.

4 Implementing the Optimal Mechanism

In this section we return to the seller's problem of designing the optimal selling mechanism (an "indirect" mechanism) in the presence of the constraints involving the selling venues. Recall that venue 1 (a regular store) uses posted prices. In particular, it is not practical for the seller to run an auction at venue 1. Venue 2 is an (online) auction site. The seller can access venue 1 at zero cost and venue 2 by incurring a cost of $c > 0$. The seller can optimally choose the posted price at venue 1 and the auction format at venue 2.

We solve the seller's problem by establishing that the seller can in fact implement[(22)] the optimal direct mechanism from the previous section with a two stage mechanism even under the additional venue restrictions.[(23)] Further, the two stage mechanism also uses the two venues optimally. Therefore the two stage mechanism is the optimal mechanism in our model.

The mechanism works as follows. The first stage involves a posted price P at venue 1. If the object is not sold in the first stage then the seller uses venue 2 where the mechanism

[(22)] In the sense that the probability allocation in the indirect game are the same as those in the direct revelation game; the seller's expected profit is, of course, smaller (and depends on the value of c).

[(23)] We remind the reader that the direct revelation game in the previous section addressed the same optimization problem *except* that it ignored restrictions posed by the presence of heterogeneous venues.

used is an auction involving a buy-now option. We also show that the buy-now option is *temporary* rather than *permanent*. We discuss this point further after describing the selling mechanism, and show that a permanent buy price cannot implement the optimal mechanism.

4.1 Description of the selling mechanism

The selling mechanism is implemented in two stages. Stage 1 takes place at venue 1 (store using posted price). Stage 2 is carried out at venue 2 (auction site).

Stage 1. (Posted Price) The item is offered for sale at a posted price P. If any buyer wants to buy at that price, the item is sold and the game is over. If there is a tie, it is resolved by randomly allocating the item to one of the tied buyers. If the item is not sold, we proceed to stage 2.

Stage 2. (Auction) This is an auction augmented by a buy-now option. Let B denote the buy price. Stage 2 has two sub-stages.

First sub-stage. (buy-now option) In the first sub-stage, the auction opens with a buy price B. If a single bidder bids B, the object is awarded to that bidder. If two or more bidders submit B, the object is allocated randomly with the winning bidder paying the price B. If no bidder bids B, the game proceeds to the second sub-stage.

Second sub-stage. (Vickrey auction) The second sub-stage is a standard Vickrey auction with a reserve price. In this stage the object is allocated to the highest bidder at a price which is the maximum of the reserve price and the second highest bid provided the highest bid is above the reserve price. If the highest bid is below the reserve price, the seller keeps the object and the game is over.

We now show that this selling mechanism implements the optimal direct mechanism from the previous section, and is also the optimal mechanism taking into account the venue restrictions.

4.2 Implementation

To economize on notation, in what follows we refer to the optimal pooling cutoff as $\widehat{v}$ (rather than $\widehat{v}^*$).

Proposition 6 *The two stage selling mechanism described above implements the optimal direct mechanism when the reserve price in the Vickrey auction is chosen as v_*, and the posted price P and temporary buy price B are chosen as:*

$$P = v_h - \frac{(1-\mu)^{N-1}}{X^*(v_h)} \left[\Big(v_h - F(\widehat{v}\,) \Big) G(\widehat{v}\,) + \int_{v_*}^{\widehat{v}} F^{N-1}(t)\, dt \right] \tag{4.1}$$

$$B = \widehat{v} - \frac{1}{G(\widehat{v}\,)} \int_{v_*}^{\widehat{v}} F^{N-1}(t)\, dt \tag{4.2}$$

where $X^*(v_h) = \sum_{n=0}^{N-1} \binom{N-1}{n} (1-\mu)^{N-n-1} \mu^n \frac{1}{n+1}$, *$v_*$ is given by $\psi(v_*) = 0$, and $\widehat{v}$ is given by proposition 4.*[24] *Finally, the two stage mechanism makes optimal use of the venues.*

Proof. STEP 1. **(Comparing with the optimal direct mechanism)** Consider the following strategies: Type v_h buys the object in stage 1 by paying the posted price P. Types in the interval $[\widehat{v}, 1]$ submit the buy price B in the first sub-stage of stage 2. Types in $[v_*, \widehat{v}\,)$ submit their true valuations as bids in the second sub-stage. (Types below v_* do not win the object so it is irrelevant whether they participate or not. Without loss of generality, assume they submit bids equal to their valuations also.)

Given the strategies, type v_h is the only type to buy at price P which means that if a buyer of type v_h follows the prescribed strategy, the expected probability of winning the object is the same as $X^*(v_h)$ (given by equation (3.15)).

Using the values of $X^*(v_h)$ and $X^*(v)$ for $v \in [0, 1]$, we get

$$\int_0^1 X^*(t)\, dt = (1-\mu)^{N-1} \left[\int_{v_*}^{\widehat{v}} F^{N-1}(t)\, dt + (1 - F(\widehat{v}\,)) G(\widehat{v}\,) \right]$$

Substituting in equation (3.17),

$$\begin{aligned} T^*(v_h) = v_h & \left(\sum_{n=0}^{N-1} \binom{N-1}{n} (1-\mu)^{N-n-1} \mu^n \frac{1}{n+1} \right) \\ & - (1-\mu)^{N-1} \left[\Big(v_h - F(\widehat{v}\,) \Big) G(\widehat{v}\,) + \int_{v_*}^{\widehat{v}} F^{N-1}(t)\, dt \right] \end{aligned}$$

[24] Reynolds and Wooders (2008) analyze auctions with a temporary buy price. It is worth noting that equation (4.2) coincides with the equation in their proposition 1 determining the cutoff value beyond which buyers accept the buy price. The difference, of course, is that unlike Reynolds-Wooders, here the reserve price and the cutoff $\widehat{v}$ are not arbitrary but derived as parts of an optimal mechanism. Given optimal values of the reserve price and the cutoff, we determine the buy price. Reynolds-Wooders, on the other hand, start from any given reserve price and buy price and determine the cutoff.

Hence, if the posted price P is given by equation (4.1), the expected payment of type v_h is the same in the indirect mechanism as it is in the optimal mechanism. For types in $[v_*, \widehat{v}\,)$, the probability of winning and expected payments are exactly as in the optimal mechanism (given by equation (3.16)) since they are simply taking part in a Vickrey auction. The types in $[\widehat{v}, 1]$ also have the same expected probability of winning as in the optimal mechanism since if they submit the buy price, they are being "pooled" in the indirect mechanism in the same way as in the optimal direct mechanism. Now, the expected payment of these pooled types in the optimal mechanism is given by $\widehat{v}\, X^*(\widehat{v}\,) - \int_0^{\widehat{v}} X^*(t)\, dt$. Since $\int_0^{\widehat{v}} X^*(t)\, dt = (1-\mu)^{N-1} \int_{v_*}^{\widehat{v}} F^{N-1}(t)\, dt$, the expected payment of the pooled types can be written as

$$(1-\mu)^{N-1} \left[\widehat{v}\, G(\widehat{v}\,) - \int_{v_*}^{\widehat{v}} F^{N-1}(t)\, dt \right]$$

Conditional on reaching stage 2, if a type $v \in [\widehat{v}, 1]$ bids B, the expected payment is $G(\widehat{v}\,)B$, and hence this type's overall expected payment from participating in the mechanism is given by $(1-\mu)^{N-1} G(\widehat{v}\,)B$. Thus if the buy price B is given by equation (4.2), the expected payments are the same in the direct and the indirect mechanisms.

STEP 2. **(Checking for equilibrium)** We now show that the strategies described above do constitute an equilibrium. From the way the payoffs are constructed, type v_h is indifferent between buying at the price P at the first stage and waiting to bid B in the second stage and strictly prefers buying at price P to bidding any other price in the second sub-stage auction. Consider now a type $v \in [0, 1]$. It is clear that no such type wants to mimic type v_h and buy the object in the first stage. To derive the optimal bid of a type $v \in [0, 1]$ in the second stage, all we need to do is to compare the type's expected payoff from only two possible bids: bidding the buy price B in the first sub-stage, or bidding the true value of v in the Vickrey auction in the second sub-stage.

Let $D(v)$ denote the difference in expected surplus of type v from the bids B and v. The proof now proceeds through the following lemma, which is proved in the appendix.

Lemma 1 *$D(v)$ is strictly increasing in v and $D(\widehat{v}\,) = 0$.*

This shows that, as required, type $\widehat{v}$ is indeed indifferent between bidding B and $\widehat{v}$. Further, since $D'(v) > 0$, types below $\widehat{v}$ strictly prefer to bid their true value rather than B, and types $v \in (\widehat{v}, 1]$ strictly prefer to bid B.

Since the mechanism implements the same probability allocation as derived in the last section without the venue-restriction constraint, there cannot be a more profitable allocation.

STEP 3. **(Optimal use of venues)** Finally, it is straightforward to verify that the indirect selling mechanism described above makes optimal use of venues. By assumption, the seller does not want to use a posted price mechanism only and so using only venue 1 is not a better alternative. Of course by using only venue 2 and having two sequential posted price offers followed by the standard Vickrey auction would also give rise to the same expected allocation probabilities and expected revenues as in the mechanism described in this section. Thus, it is possible to not use venue 1 at all but still implement the allocation and revenue from the direct mechanism. However since there is the additional cost c of using venue 2, this would reduce overall expected profit by an amount $[1-(1-\mu)^N]c$. It is thus better for the seller to use the store (venue 1) first and to use the auction site (venue 2) only when the object is not sold at the store. ■

The two stage mechanism implements the optimal direct allocation ignoring venue restrictions. It also makes optimal use of venues. Therefore even when we incorporate the venue restrictions, it remains an optimal selling arrangement.

4.3 Properties of the selling mechanism: A temporary buy-now option

We now clarify that the indirect mechanism which implements the optimal direct mechanism includes a *temporary* buy-now option. Later, in section 4.5 we point out the connection between the temporary buy-now option we use and the one used by eBay.

In stage 2 of the selling mechanism described in the previous (sub)section, bidders are offered the chance to buy the item at price B, followed by a Vickrey auction. In other words, the mechanism involves an auction but with a buy-now option that is only temporary - the option is withdrawn when the actual auction takes place. It is interesting to note that under such a buy-now option, the auction price can actually exceed the buy price. To see this, note that types above $\widehat{v}$ plan to buy the item at buy price B and types $[v_*, \widehat{v})$ follow the standard strategy of a Vickrey auction. Hence, at the auction stage, the price range belongs to the interval $[v_*, \widehat{v}]$ and since $B < \widehat{v}$ (see equation (4.2)), once the buy-now option vanishes, the subsequent auction price in the optimal mechanism could be higher than the buy price.

4.4 Suboptimality of a permanent buy-now option

As noted in the introduction, the literature has focused mainly on augmenting standard auctions with a permanent buy price feature and has remarked that this generates more revenue compared to a standard auction augmented by a temporary buy-price auction. We show, on the other hand, that in general it is not possible to implement our optimal mechanism with a permanent buy price.

Papers by Reynolds and Wooders (2008) and Hidvégi, Wang, and Whinston (2006) (henceforth HWW) analyze an English auction augmented by a permanent buy price b.[(25)] We make use of some of their results to show that in our context a permanent buy price is suboptimal.[(26)]

Result (HWW): Consider an independent private values setting with bidders drawing values from a distribution on $[\underline{v}, \overline{v}]$. Consider an English auction with a reserve price r and permanent buy price b. In (the unique) equilibrium, there are cutoffs v_{c} and v_{uc} such that the type space can be partitioned into the following (possibly empty) intervals:

- types $v \in [r, b)$ use a "traditional strategy" : remain active till price rises to v,
- types $v \in [b, v_{\mathrm{c}})$ use a "threshold strategy" : remain active till the auction price reaches a critical point $t(v, b)$ and then jumps to the buy price b.
- types $v \in [v_{\mathrm{c}}, v_{\mathrm{uc}})$ use a "conditional strategy" : bid b if (and only if) the auctions clocks moves from r (implying that there is at least one other bidder with value above r), and
- types $v \in [v_{\mathrm{uc}}, \overline{v}]$ use an "unconditional strategy" : bid b right at the start of the auction.

(25) It should be noted that their basic environment is different from ours. They have a standard private-value auction setting, and the distribution of values does not have a counterpart of the atom at v_h that features in our model. However, this distinction is not important for what follows.

(26) Reynolds and Wooders (2008) describe threshold strategies (a bidder bids as in an usual English auction up to a threshold, then accepts the buy price), and add an assumption of "no-regret" to ensure existence. HWW use a strategy specification that ensures existence under general conditions. In what follows we use the treatment of HWW.

The threshold $t(v, b)$ is decreasing in v, and if $\lim_{v \to \overline{v}} t(v, b) \geq r$, $v_{\text{c}} = v_{\text{uc}} = \overline{v}$, and all types $v \geqslant b$ play the threshold strategy. On the other hand if $\lim_{v \to \overline{v}} t(v, b) < r$, there are types who play the conditional and unconditional strategies.

Translating this to our context, given a permanent buy price B and the reserve type v_*, a posted price followed by an auction can implement the optimal mechanism if types $[\widehat{v}, 1]$ play the unconditional strategy (i.e. bid B immediately) and types $[v_*, \widehat{v}\,)$ play strategies such that their bids are revenue-equivalent to the bids in the temporary buy-price auction.

It is clear that in the temporary buy-price auction (as in the optimal direct mechanism), *only* types $v \in [\widehat{v}, 1]$ are pooled, and there is no pooling in the interval $[v_*, \widehat{v}\,)$. In this interval the allocation probability and expected payment are strictly increasing. Therefore if any indirect mechanism involves pooling (same allocation probability, same expected payment) in the latter region, this would introduce additional inefficiency, and prevent the mechanism from implementing the optimal mechanism.

Consider an auction with a permanent buy price B. Types playing the conditional strategy or unconditional strategy are clearly pooled. Therefore to avoid pooling of types below $\widehat{v}$, it must be that no sub-interval of types in $[v_*, \widehat{v}\,)$ play either the conditional or unconditional strategies. But from the result above, we know that this happens if and only if $\lim_{v \to \widehat{v}} t(v, B) \geq v_*$. We state this below.

Corollary: A necessary condition for implementing the optimal direct mechanism using a posted price combined with an English auction with a permanent buy price B is given by $\lim\limits_{v \to \widehat{v}} t(v, B) \geqslant v_*$.

Can we ensure this condition holds? As the following result shows, the problem is that $\widehat{v}$, B as well as the function $t(\cdot, \cdot)$ are already determined by conditions of implementation of the optimal mechanism. Thus there is no parameter we could vary to ensure this inequality holds. It might hold for some distribution F by lucky coincidence, but in general it is not possible to ensure this. The proof of the following result shows that the inequality is not satisfied for a uniform distribution.

For the following result, we use the usual definition of implementation: for any distribution F (satisfying the basic assumptions) an indirect mechanism implements the optimal mechanism if there exists an equilibrium of the indirect mechanism whose outcome is the same as that of the optimal mechanism. The following result shows that a permanent buy-now option fails this test.

Proposition 7 *It is not possible to implement the optimal direct mechanism using an English auction with a permanent buy price B.*

Proof. Since the indirect mechanism, by definition, is required to implement the optimal mechanism for all allowable parameter values and distributions, it suffices to provide an example where it does not. In particular, we show here that for $N = 2$, under a uniform distribution, $\lim_{t \to \widehat{v}} t(v, B) < v_*$, so that the necessary condition for implementation identified in the corollary above is violated.

We have already shown that an auction with a temporary buy price does implement the optimal mechanism. If an auction with a permanent buy price also implements the optimal mechanism, all types should win with the same probability and get the same expected surplus in the two buy-price auctions.

Let us derive explicitly the threshold strategies of the types in $[B, \widehat{v}\,)$[27] for $N = 2$. Under a temporary buy price, a type $v \in [B, \widehat{v}\,)$ obtains a surplus

$$(v - v_*)\, F(v_*) + \int_{v_*}^{v} (v - y)\, dF$$

Under a permanent buy price, assuming, for the time being, that there is no pooling sub-interval (i.e. the condition in the corollary is satisfied), expected surplus from following the threshold strategy is:

$$(v - v_*)\, F(v_*) + \int_{v_*}^{t(v,B)} (v - y)\, dF + (v - B)\,(F(v) - F(t(v, B)))$$

Suppose now that $F(\cdot)$ is the uniform distribution. Equating the two expressions above, we get $t(v, B) = 2B - v$. Using the value of B under the uniform distribution (from equation (4.2)) for $N = 2$, we have

$$\lim_{v \to \widehat{v}} t(v, B) - v_* \;=\; \widehat{v} - \frac{2}{1 + \widehat{v}} \left(\widehat{v}^2 - v_*^2 \right) - v_* \;=\; (\widehat{v} - v_*)\, \frac{(1 - \widehat{v} - 2v_*)}{1 + \widehat{v}}$$

Under a uniform distribution, v_* (defined as $\psi(v_*) = 0$) is given by $v_* = \dfrac{1}{2(1-\mu)}$. Since $\mu \in (0, 1)$, $1 - 2v_* < 0$, which proves that $\lim_{t \to \widehat{v}} t(v, B) - v_* < 0$. This violates the implementation condition in the corollary above. ■

[27] Types $v \in [v_*, B)$ play the standard Vickrey auction strategy and English auction strategy in the two auctions so their probability of winning and expected surplus are the same in the two auctions.

4.5 The temporary buy-now option used by eBay

In the final part of this section we discuss the relation between our temporary buy-price auction and the one used by eBay.

In our indirect mechanism the seller makes an initial offer to sell the object at a pre-specified price B but this buy price offer is temporary since it is withdrawn by the seller if there are no takers. Therefore the buy now option is absent during the subsequent auction. eBay uses a slightly different form of temporary buy now option. The chief distinction is that in the eBay auction the disappearance of the buy price is endogenous: the buy-now option vanishes whenever a bidder places a bid above the reserve price. Even though the eBay auction differs from ours, we now argue that in the scenario modeled here, the outcome of the two should be the same. To see this, recall the four possible strategies in a buy-price auction from the HWW result stated in section 4.4 above. These are traditional, threshold, conditional and unconditional strategies. Now, the threshold strategy (under which a bidder waits for the price in the auction to rise to a certain level before exercising the buy-now option), and the conditional strategy (under which a bidder waits and bids the buy price only if some other bidder bids above the reserve price) are not available under the rules of eBay auctions, since in both cases the buy-now option would vanish. Therefore, in an eBay buy-price auction, only two – the standard and the unconditional strategies – are feasible. But this is just like our selling mechanism. Hence, under the setting of our model, the optimal eBay auction is to choose the reserve price and the buy price to be v_* and B respectively, and in this case the eBay auction should implement the optimal mechanism.

5 Conclusion

We study the optimal price discriminating mechanism across heterogeneous venues, and relate our results to a widely used but seemingly suboptimal online auction format. The starting point of our analysis is the observation that in addition to traditional posted price selling through bricks-and-mortar stores (as well as own web sites), sellers can now access online auction sites as a sales channel. Such auctions often enable sellers to reach buyers who are typically priced out in traditional markets. Since the sellers use different sales mechanisms to target different groups of buyers, the optimal design of the overall mechanism involves second degree price discrimination *across* selling methods.

We assume that the seller has access to a sales venue such as a store that can only use a posted price, and can also pay a small fee to access an (online) auction site, which allows unrestricted design of the sales mechanism. We characterize the optimal selling mechanism in this environment, and show that it involves a posted price at the store and an auction where there is "pooling at the top" amongst the types who decide to participate in the auction. This feature of the optimal mechanism corresponds exactly to a buy-now option. Thus, the phenomenon of a buy-now option in an auction, something that might appear puzzling when seen in isolation (i.e. in the context of the auction alone) emerges as a *necessary* feature of the overall optimal selling mechanism. Interestingly, we show that posted price selling followed by a standard auction with a *temporary* buy-now option – used by eBay and seemingly even more of a puzzling phenomenon[(28)] – implements the optimal mechanism, but the same is not true of an auction with a *permanent* buy-now option.

Of course, eBay as well as other online auctions are rich in institutional detail. While we do not claim to capture all of these, we do believe that for many sellers online auctions form *part* of an optimal selling strategy across heterogeneous venues, and therefore it is important to understand the overall strategy in order to analyze its constituent parts. While other theories have sought to explain a buy-now option as an improvement over a standard auction in certain cases (e.g. risk averse bidders), this is the first paper to explain such an option as part of an optimal mechanism. However, further empirical tests taking into account overall market data are required to distinguish between competing theories. To this end, we briefly mention how the testable implications of our model differ from other theories.

[(28)]Indeed, as noted in the introduction, the literature often concludes that auctions with a temporary buy-now option are revenue inferior to those with a buy-now option that is permanent.

Testable implications

The risk-aversion-based theory as well as other approaches mentioned in the introduction look at auctions in isolation. An implication is that there is no systematic relation between prices posted elsewhere by the seller for similar goods and the buy price in the auction. A further implication of the risk aversion theory is that if similar items are offered for sale through posted prices elsewhere, this might weaken the incentive to have a buy price, since the most risk averse buyers might prefer to buy at a posted price. Thus the theory implies that if similar items are sold elsewhere through posted prices by a seller, this makes it less likely that the seller would use a buy-now option.

The theory presented here, on the other hand, shows that posted price sales make it more likely that the seller uses a buy price in the auction. Further, our theory predicts a negative relation between the posted price and the buy price. A higher posted price necessitates greater inefficiency in the auction to preserve incentives. Further empirical work is needed to test the relative importance of theories in explaining the emergence of a buy-now option in standard auctions.

6 Appendix: Proofs

A.1 Proof of Proposition 4

From (3.11), the objective is to maximize

$$\int_{v_*}^{\widehat{v}} \psi(v)X(v)f(v)\,dv + \left[-\mu(v_h - 1) + \int_{\widehat{v}}^{1} \psi(v)f(v)\,dv\right] X(\widehat{v}\,)$$

We know from equation (3.12) that all terms in the expression for $X(v)$ contain a common factor $(1-\mu)^{N-1}$. Define $Y(v) \equiv \dfrac{X(v)}{(1-\mu)^{N-1}}$. Without loss of generality, we can optimize in terms of $Y(v)$ rather than $X(v)$.

Equation (3.12) specifies $Y(v)$. Let us find the optimal $\beta_n(\cdot)$ for any given value of $n \in \{1, \ldots, N-1\}$. Let $Y(v,n)$ denote the relevant term for this n in the summed expression $Y(v)$.

Let $\rho(n) \equiv \binom{N-1}{n}\left(1 - F(\widehat{v}\,)\right)^{N-1-n}$.

Suppose $\beta_n(\cdot)$ is not always zero, and let

$$Y(v,n) = (1-\theta)G(\widehat{v}\,) \qquad \text{for } v \in [\widehat{v}, 1] \tag{A.1}$$

where $\theta \in (0,1)$ and fixed exogenously. From the expression for $Y(v)$, this implies we must obey the constraint

$$\frac{\rho(n)}{N-n}\int_{v_*}^{\widehat{v}} \beta_n(t)\,dF^n = \theta G(\widehat{v}\,) \tag{A.2}$$

Further, incentive compatibility requires

$$Y(\widehat{v}_-) \equiv \lim_{v\uparrow\widehat{v}} Y(v) \leqslant (1-\theta)G(\widehat{v}\,) \tag{A.3}$$

For any given $(\widehat{v}, \theta)$, the optimization problem can be written (using equations (3.12) and (A.1)) as

$$\max_{\beta_n(v)} \int_{v_*}^{\widehat{v}} \rho(n-1)\beta_n(v)F^{n-1}(v)\psi(v)f(v)\,dv + \int_{v_*}^{\widehat{v}} F^{N-1}(v)\psi(v)f(v)\,dv$$
$$+ \left[-\mu(v_h - 1) + \int_{\widehat{v}}^{1} \psi(v)f(v)\,dv\right](1-\theta)G(\widehat{v}\,)$$

subject to (A.2) and (A.3).

Given $(\widehat{v}, \theta)$, the second and third terms are constant terms. Thus the optimization problem is simply given by the first term, which can be rewritten so that the maximization problem is:

$$\max_{\beta_n(v)} \frac{\rho(n-1)}{n} \int_{v_*}^{\widehat{v}} \beta_n(v)\psi(v)\, dF^n$$

subject to (A.3) and (A.2).

Since $\frac{\rho(n-1)}{n}$ is a constant term, we can ignore this, and simply maximize $\int_{v_*}^{\widehat{v}} \beta_n(v)\psi(v)\, dF^n$.

Let us approximate the $\beta_n(v)$ function by a step function with s steps at $v_0 < v_1 < \ldots < v_s$. Let $v_0 = v_*$ and $v_s = \widehat{v}$. Let the value of the function be $\beta_n^{(k)}$ over the k-th step.

Let $\widehat{R}$ denote the maximand. The optimization problem is

$$\max_{\beta_n^{(1)},\ldots,\beta_n^{(s)}} \widehat{R}$$

where

$$\widehat{R} \equiv \int_{v_*}^{v_1} \beta_n^{(1)} F(v)\psi(v)\, dF^n + \int_{v_1}^{v_2} \beta_n^{(2)} F(v)\psi(v)\, dF^n + \ldots + \int_{v_{s-1}}^{\widehat{v}} \beta_n^{(s)} F(v)\psi(v)\, dF^n$$

Subject to (A.3) and (A.2), which becomes

$$\theta G(\widehat{v}\,) = \frac{\rho(n)}{N-n} \left[\beta_n^{(1)}(F^n(v_1) - F^n(v_*)) + \ldots + \beta_n^{(s)}\left(F^n(\widehat{v}\,) - F^n(v_{s-1})\right)\right] \tag{A.4}$$

Let us increase $\beta_n^{(k)}$ and reduce $\beta_n^{(j)}$ in a way such that the constraint is preserved. From the constraint,

$$\frac{\partial \beta_n^{(j)}}{\partial \beta_n^{(k)}} = -\left(\frac{F^n(v_k) - F^n(v_{k-1})}{F^n(v_j) - F^n(v_{j-1})}\right)$$

Using this, $\dfrac{d\widehat{R}}{d\beta_n^{(k)}} = (F^n(v_k) - F^n(v_{k-1}))\ Z$, where

$$Z = \int_{v_{k-1}}^{v_k} \psi(v) \frac{dF^n}{(F^n(v_k) - F^n(v_{k-1}))} - \int_{v_{j-1}}^{v_j} \psi(v) \frac{dF^n}{(F^n(v_j) - F^n(v_{j-1}))}$$

Clearly, $Z \gtreqless 0$ as $k \gtreqless j$.

Therefore, if constraint (A.3) is slack to start with, it is optimal to increase $\beta_n^{(s)}$ until it binds, so that over the interval $[v_{s-1}, \widehat{v}\,)$, $Y(v)$ is raised to $(1-\theta)G(\widehat{v}\,)$. Therefore the optimal $\beta_n^{(s)}$, denoted by $\beta_n^{(s)*}$, is given by

$$\rho(n)\beta_n^{(s)*}F^{\,n}(\widehat{v}\,) + F^{\,N-1}(\widehat{v}\,) = (1-\theta)G(\widehat{v}\,)$$

At this point the incentive compatibility constraint (A.3) binds, so that $\beta_n^{(s)}$ cannot be increased any further.

Next, in the same way, increase $\beta_n^{(s-1)}$ so that $\rho(n)\beta_n^{(s-1)*}\,F^{\,n}(v_{s-1}) + F^{\,N-1}(v_{s-1}) = (1-\theta)G(\widehat{v}\,)$.

Continuing in this manner, we get $\ell \in 1,\ldots,s-1$ such that

$$\Phi(v_\ell) \geqslant 0 \quad \text{and} \quad \Phi(v_{\ell-1}) < 0,$$

where

$$\Phi(v_\ell) = \frac{\rho(n)}{N-n}\sum_{k=\ell+1}^{s}\int_{v_{k-1}}^{v_k} \beta_n^{(k)*}\,dF^{\,n} - \theta\,G(\widehat{v}\,)$$

Note that the expression on the right hand side simply reflects the constraint (A.4), with $\beta_n^{(k)}$ replaced by its optimal value for $k \in \{\ell,\ldots,s\}$. As we consider a finer and finer grid (i.e. as $(v_k - v_{k-1}) \to 0$ for all k), in the limit v_ℓ is such that

$$\frac{\rho(n)}{N-n}\int_{v_\ell}^{\widehat{v}} \beta_n^*(v)\,dF^{\,n} = \theta\,G(\widehat{v}\,)$$

Therefore v_ℓ is such that types below receive no transfer (and therefore any type $v < v_\ell$ wins with probability $F^{\,N-1}(v)$ only), and types above v_ℓ are pooled.

If originally we had put $\beta_n(\cdot) = 0$ always, types above $\widehat{v}$ would be pooled with expected probability of winning $G(\widehat{v}\,)$.

Now, types above v_ℓ are pooled, and types below receive no transfer. Thus, apart from $\widehat{v}$ changing to v_ℓ, the situation is exactly the same as the case in which $\beta_n(\cdot) = 0$. Therefore v_ℓ must be such that $G(v_\ell) = \theta G(\widehat{v}\,)$. Note that the solution for v_ℓ does not depend on n. Therefore we can rewrite the optimization problem as the following unconstrained maximization problem:

$$\max_{v_\ell}\int_{v_*}^{v_\ell} F(v)\psi(v)f(v)\,dv + G(v_\ell)\left[-\mu(v_h - 1) + \int_{v_\ell}^{1}\psi(v)f(v)\,dv\right]$$

Thus the general form of the maximization problem is as stated. This completes the proof. ❋

A.2 Proof of Lemma 1

Case 1: $v \leqslant \widehat{v}$.

In stage 2, in equilibrium, the expected payment from bidding v is $v_* F^{N-1}(v_*)+\int_{v_*}^{v} t\, dF^{N-1}$. The second term can be rewritten as $vF^{N-1}(v) - v_* F^{N-1}(v_*) - \int_{v_*}^{v} F^{N-1}(t)dt$. Therefore the expected payment from bidding v can be written as

$$vF^{N-1}(v) - \int_{v_*}^{v} F^{N-1}(t)dt$$

The expected value is $vF^{N-1}(v)$. Therefore the expected net payoff from bidding v is $\int_{v_*}^{v} F^{N-1}(t)dt$. The expected payoff from bidding B is $(v - B)G(\widehat{v}\,)$.

$D(v)$ denotes the difference between the expected payoff from bidding B in the first stage or v in the second stage. From the above, this is given by $D(v) = (v - B)\, G(\widehat{v}\,) - \int_{v_*}^{v} F^{N-1}(t)dt$. Using the value of B from equation (4.2), this can be rewritten as

$$D(v) = (v - \widehat{v}\,)G(\widehat{v}\,) + \int_{v}^{\widehat{v}} F^{N-1}(t)dt$$

Clearly, $D(\widehat{v}\,) = 0$. Next, $D'(v) = G(\widehat{v}\,) - F^{N-1}(v)$.

From equation (3.13),

$$G(\widehat{v}\,) = F^{N-1}(\widehat{v}\,) + \sum_{n=0}^{N-2} \binom{N-1}{n} \frac{\left(1 - F(\widehat{v}\,)\right)^{N-1-n}\ F^{n}(\widehat{v}\,)}{N-n} > F^{N-1}(\widehat{v}\,) \tag{A.5}$$

Since $v \leqslant \widehat{v}$, $G(\widehat{v}\,) > F^{N-1}(v)$. Thus $D'(v) > 0$.

Case 2: $v > \widehat{v}$.

The expected payment from bidding v is $v_* F^{N-1}(v_*) + \int_{v_*}^{\widehat{v}} t\, dF^{N-1}$. This simplifies to $\widehat{v}F^{N-1}(\widehat{v}\,) - \int_{v_*}^{\widehat{v}} F^{N-1}(t)dt$. The expected value is $vF^{N-1}(\widehat{v}\,)$. Therefore the expected net payoff is $(v - \widehat{v})\, F^{N-1}(\widehat{v}\,) + \int_{v_*}^{\widehat{v}} F^{N-1}(t)dt$. Hence for types $v > \widehat{v}$, $D(v) = (v - B)\, G(\widehat{v}\,) - \Big((v - \widehat{v})\, F^{N-1}(\widehat{v}\,) + \int_{v_*}^{\widehat{v}} F^{N-1}(t)dt\Big)$. Using the value of B from equation (4.2), this can be rewritten as $D(v) = (v - \widehat{v}\,)\,\big(G(\widehat{v}\,) - F^{N-1}(\widehat{v}\,)\big)$. Clearly, $D(\widehat{v}\,) = 0$, and $D'(v) = G(\widehat{v}\,) - F^{N-1}(\widehat{v}\,) > 0$, where the last inequality follows from (A.5). ❋

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