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Soft Homogeneous Components and Soft Products

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ABSTRACT

Firstly, for the topological spaces that contain a minimal open set, we obtain various inclusions between minimal open sets and homogeneity components. For a given soft topological space (X, τ, A), we define soft homogeneous components. We show that soft homogeneous components of (X, τ, A) form a soft partition of the absolute soft set. Also, we show that (X, τ, A) is soft homogeneous if and only if it has only one soft homogeneous component. Moreover, we study the relationships between the soft homogeneous components of (X, τ, A) and the homogeneous component. For the soft topological spaces that contain a minimal soft open set, we obtain various inclusions between minimal soft open sets and soft homogeneity components. In addition, we show that soft homeomorphisms stabilize soft homogeneous components. Additionally, we introduce two soft product theorems concerning soft homogeneity and soft minimality, respectively.

KEYWORDS

soft homogeneity; soft minimality; homogeneous components; soft product

1 Introduction and Preliminary

Molodtsov[1] introduced the concept of soft sets as a novel mathematical technique to deal with uncertainties that traditional mathematical tools cannot solve. He has demonstrated numerous applications of this theory in economics, engineering, social science, medical science, and other domains. Papers on soft set theory and its applications in a variety of fields have grown in popularity in recent years. Following that, Refs. [2, 3] employed soft sets to address a decision-making problem and developed a variety of soft set operators, including intersection, union, and subset. The concept of a bijective soft set was introduced and explored using a decision-making issue[4]. Aktaş and Çağman[5] investigated the two types of sets and concluded that any rough and fuzzy set is a soft set. Ali et al.[6] improved on the results of Ref. [3] by changing the essential operators. It should be highlighted that the huge potential for soft set theory applications in a variety of domains encourages rapid research growth (see Refs. [7–9]).

Shabir and Naz[10] proposed soft topological spaces as being defined over an initial universe with a specific set of parameters. Soft interior, soft closure, soft separation, soft open sets, soft closed sets, and soft interior were defined by Shabir and Naz[10]. Many traditional topological notions have been examined and extended in soft set settings (see Refs. [11–26]), but important additions are still possible. As a result, topological

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experts are becoming increasingly interested in studying soft topology.

Al Ghour and Bin-Saadon[13] introduced and investigated soft homogeneity in soft topological spaces, and they also gave some results about soft minimality. The author in Ref. [20] introduced and investigated soft prehomogeneity. The first goal of this paper is to introduce some results regarding homogenous components of topological spaces that have minimal open sets. The second goal of this paper is to continue the study of soft homogeneity and soft minimality, in particular, we give two soft product theorems.

One of the primary motivations for writing this paper comes from the fact that geometric structures in nature are often characterized by some kind of homogeneity.

Assume that X is a non-empty set and A is a set of parameters. A soft set over X relative to A is a function $G : A \longrightarrow \mathcal{P}(X)$, where $\mathcal{P}(X)$ is X's powerset. $\mathrm{SS}(X, A)$ denotes the family of all soft sets over X relative to A. The null soft set and the absolute soft set are denoted by 0_A and 1_A, respectively. If $G \in \mathrm{SS}(X, A)$ such that $G(a) = \{x\}$ and $G(b) = \varnothing$ for all $b \in A - \{a\}$, then G is called a soft point over X relative to A and denoted by a_x. $\mathrm{SP}(X, A)$ denotes the collection of all soft points over X relative to A. If $M \in \mathrm{SS}(X, A)$ and $a_x \in \mathrm{SP}(X, A)$, then a_x is said to belong to M (notation: $a_x \widetilde{\in} M$) if $x \in M(a)$.

Shabir and Naz[10] initiated the structure of soft topological spaces as follows: The triplet (X, τ, A), where $\tau \subseteq \mathrm{SS}(X, A)$, is known as a soft topological space if τ contains the null and the absolute soft sets and τ is closed under finite soft intersection arbitrary soft union. The members of τ are called soft open sets and their soft complements are called soft closed sets.

We will use concepts and terminology from Refs. [13, 14] throughout this work for clarification.

Topological space and soft topological space will now be abbreviated as TS and STS, respectively.

The following definitions will be used in the sequel:

Definition 1.1[27] Let $(X, \Im)$ be a TS. Then the relation $\backsim$ on X is defined as follows: For all $x, y \in X$, $x \backsim y$ if and only if there exists a homeomorphism $p : (X, \Im) \longrightarrow (X, \Im)$ such that $p(x) = y$.

Definition 1.2[27] Let $(X, \Im)$ be a TS and let $x \in X$. The set $C_x = \{y \in X : x \backsim y\}$ is called a homogeneous component of X determined by the point x.

Definition 1.3[28] Let $F \in \mathrm{SS}(X, A)$ and $G \in \mathrm{SS}(Y, B)$. Then the soft Cartesian product of F and G is a soft set denoted by $F \times G \in \mathrm{SS}(X \times Y, A \times B)$ and defined by $(F \times G)(a, b) = F(a) \times G(b)$ for each $(a, b) \in A \times B$.

Definition 1.4[29] Let (X, τ, A) and (Y, σ, B) be two STSs and let $\mathcal{B} = \{F \times G : F \in \tau \text{ and } G \in \sigma\}$. Then the soft topology over $X \times Y$ relative to $A \times B$ having $\mathcal{B}$ as a soft base is called the product soft topology and is denoted by $\tau \times \sigma$.

2 Homogeneous Component

In this section, we obtain several results and examples concerning homogeneous components in TSs, particularly those with minimal open sets.

Proposition 2.1 Let $(X, \Im)$ be a TS. If $\{x\} \in \min(X, \Im)$, then $\{\{y\} : y \in C_x\} \subseteq \min(X, \Im)$.

Proof Let $y \in C_x$. Then there is a homeomorphism $f: (X, \Im) \longrightarrow (X, \Im)$ such that $f(x) = y$ and so $f(\{x\}) = \{y\}$. It follows that $\{y\} \in \min(X, \Im)$.

The following example shows that the inclusion in Proposition 2.1 cannot replaced by equality in general:

Example 2.1 Let $X = \{1, 2, 3\}$ and $\Im = \{\varnothing, X, \{1\}, \{2, 3\}\}$. Then $\{\{y\} : y \in C_1\} = \{\{1\}\}$, but $\min(X, \Im) = \{\{1\}, \{2, 3\}\}$.

Proposition 2.2 Let $(X, \Im)$ be a TS. If $G \in \min(X, \Im)$, then $G \subseteq \bigcap_{x \in G} C_x$.

Proof Let $y \in G$, we will show that $y \in C_x$ for all $x \in G$. Let $x \in G$, define $f: (X, \Im) \longrightarrow (X, \Im)$ by $f(x) = y$, $f(y) = x$, and $f(z) = z$ for all $z \in X - \{x, y\}$. Then f is a bijection with $f(x) = y$.

f is open: Let $U \in \Im - \{\varnothing\}$.

Case 1 $U \cap G = \varnothing$. Then $U \subseteq X - \{x, y\}$ and so $f(U) = U$.

Case 2 $U \cap G \neq \varnothing$. Since $G \in \min(X, \Im)$, then $G \subseteq U$ and so $f(U) = U$.

f^{-1} is open: The proof follows because $f^{-1} = f$.

The following example shows that the inclusion in Proposition 2.2 cannot replaced by equality in general:

Example 2.2 Let $X = \{1, 2, 3, 4\}$ and $\Im = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$. Then $\{1, 2\} \in \min(X, \Im)$, but $C_1 \cap C_2 = X \cap X = X$.

Proposition 2.3 Let $(X, \Im)$ be a TS. If $G \in \min(X, \Im)$ such that $|G| \neq |H|$ for all $H \in \min(X, \Im) - \{G\}$, then for all $x \in G$, $G = C_x$.

Proof By Proposition 2.2, we need only to show that $C_x \subseteq G$ for all $x \in G$. Let $x \in X$ and $y \in C_x$. Choose a homeomorphism $f: (X, \Im) \longrightarrow (X, \Im)$ such that $f(x) = y$. So, $y \in f(G)$. Note that $f(G) \in \min(X, \Im)$, so by assumption $f(G) = G$. Hence $y \in G$.

Corollary 2.1 Let $(X, \Im)$ be a TS such that $\min(X, \Im) = \{G\}$. Then for all $x \in G$, $G = C_x$.

In Examples 2.3 and 2.4, we provide direct applications on Corollary 2.1.

Example 2.3 Let $X = \{1, 2, 3, 4, 5\}$ and $\Im = \{\varnothing, X, \{1, 2\}, \{3, 4, 5\}\}$. Then $\min(X, \Im) = \{\{1, 2\}, \{3, 4, 5\}\}$, and by Proposition 2.3, $C_1 = \{1, 2\}$ and $C_3 = \{3, 4, 5\}$.

Example 2.4 Let $X = \mathbb{R}$ and $\Im = \{\varnothing\} \cup \{U \subseteq \mathbb{R} : \mathbb{N} \subseteq U\}$. Then $\min(X, \Im) = \{\mathbb{N}\}$, and by Corollary 2.1, $C_1 = \mathbb{N}$. On the other hand, it is not difficult to see that $C_{-1} = \mathbb{R} - \mathbb{N}$.

Proposition 2.4 Let $(X, \Im)$ be a TS. If $G \in \min(X, \Im)$, then for all $x \in G$, $\bigcup\{H \in \min(X, \Im) : |G| = |H|\} = C_x$.

Proof Let $x \in G$. By Proposition 2.2, $G \subseteq C_x$. Let $H \in \min(X, \Im)$ with $|G| = |H|$ and $G \neq H$, i.e., $G \cap H = \varnothing$. We are going to show that $H \subseteq C_x$. Let $y \in H$. Choose a bijection $f: G \longrightarrow H$. Choose a bijection $g: H \longrightarrow H$ with $g(f(x)) = y$. Define $h, w: (X, \Im) \longrightarrow (X, \Im)$ by

$$h(z) = \begin{cases} f(z), & \text{if } z \in G; \\ f^{-1}(z), & \text{if } z \in H; \\ z, & \text{if } z \in X - (G \cup H); \end{cases}$$

$$w(z) = \begin{cases} g(z), & \text{if } z \in H; \\ z, & \text{if } z \in X - H. \end{cases}$$

Then $w \circ h: (X, \Im) \longrightarrow (X, \Im)$ is a homeomorphism with $(w \circ h)(x) = y$. Therefore, $y \in C_x$. This shows that $\bigcup\{H \in \min(X,\Im) : |G| = |H|\} \subseteq C_x$.

To show that $C_x \subseteq \bigcup\{H \in \min(X,\Im) : |G| = |H|\}$, let $y \in C_x$. Choose a homeomorphism $f : (X,\Im) \longrightarrow (X,\Im)$ such that $f(x) = y$. So, $y \in f(G)$. Note that $f(G) \in \min(X,\Im)$, so $f(G) \in \{H \in \min(X,\Im) : |G| = |H|\}$. Hence, $y \in \bigcup\{H \in \min(X,\Im) : |G| = |H|\}$.

3 Soft Homogeneous Component

In this section, we present various results and examples addressing soft homogeneous components of STSs, particularly those with minimal soft open sets.

For an STS (X,τ,A), the group of all soft homeomorphisms from (X,τ,A) onto itself will be denoted by $\mathrm{SH}(X,\tau,A)$.

Definition 3.1 Let (X,τ,A) be an STS and let $a_x, b_y \in \mathrm{SP}(X,A)$. Then a_x is said to relate to b_y (notation: $a_x * b_y$) if there exists $f_{pu} \in \mathrm{SH}(X,\tau,A)$ such that $f_{pu}(a_x) = b_y$, or equivalently, $a_x * b_y$ if there exists $f_{pu} \in \mathrm{SH}(X,\tau,A)$ such that $p(x) = y$ and $u(a) = b$.

Theorem 3.1 The relation $*$ in Definition 3.1 is an equivalence relation.

Proof To see that the relation $*$ is reflexive, let $a_x \in \mathrm{SP}(X,A)$. Then $f_{pu} : (X,\tau,A) \longrightarrow (X,\tau,A)$ where $p : X \longrightarrow X$ and $u : A \longrightarrow A$ are the identities, is a soft homeomorphism with $f_{pu}(a_x) = a_x$. Thus, $a_x * a_x$. To see that the relation $*$ is symmetric, let $a_x, b_y \in \mathrm{SP}(X,A)$ such that $a_x * b_y$. Then there exists $f_{pu} \in \mathrm{SH}(X,\tau,A)$ such that $f_{pu}(a_x) = b_y$. It is not difficult to check that $f_{p^{-1}u^{-1}} \in \mathrm{SH}(X,\tau,A)$ with $f_{p^{-1}u^{-1}}(b_y) = a_x$, and hence $b_y * a_x$. To see that the relation $*$ is transitive, let $a_x, b_y, c_z \in \mathrm{SP}(X,A)$ such that $a_x * b_y$ and $b_y * c_z$. Then there exists $f_{pu}, f_{qv} \in \mathrm{SH}(X,\tau,A)$ such that $f_{pu}(a_x) = b_y$ and $f_{qv}(b_y) = c_z$. It is not difficult to check that $f_{(q\circ p)(v\circ u)} \in \mathrm{SH}(X,\tau,A)$ with $f_{(q\circ p)(v\circ u)}(a_x) = c_z$, and hence $a_x * c_z$. This ends the proof that the relation $*$ is an equivalence relation.

Definition 3.2 Let (X,τ,A) be an STS and $a_x \in \mathrm{SP}(X,A)$. The set $\mathrm{SC}_{a_x} = \{b_y \in \mathrm{SP}(X,A) : a_x * b_y\}$ is called a soft homogeneous component of X relative to A determined by the soft point a_x.

Proposition 3.1 An STS (X,τ,A) is soft homogeneous if and only if it has only one soft homogeneous component.

Proposition 3.2 Let (X,τ,A) be an STS and let $a_x, b_y \in \mathrm{SP}(X,A)$. If $a_x * b_y$, then $(X,\tau_a) \cong (X,\tau_b)$.

Proof If $a_x * b_y$, then there exists $f_{pu} \in \mathrm{SH}(X,\tau,A)$ such that $f_{pu}(a_x) = b_y$. By Corollary 5.13 of Ref. [13], $p : (X,\tau_a) \longrightarrow (X,\tau_{u(a)=b})$ is a homeomorphism. Therefore, $(X,\tau_a) \cong (X,\tau_b)$.

Proposition 3.3 Let (X,τ,A) be an STS and let $a_x, a_y \in \mathrm{SP}(X,A)$. If $a_x * a_y$ in (X,τ,A), then $x \sim y$ in (X,τ_a).

Proof If $a_x * b_y$, then there exists $f_{pu} \in \mathrm{SH}(X,\tau,A)$ such that $f_{pu}(a_x) = a_y$. By Corollary 5.13 of Ref. [13], $p : (X,\tau_a) \longrightarrow (X,\tau_{u(a)=a})$ is a homeomorphism with $p(x) = y$. It follows that $x \sim y$.

Example 3.1 Let $X = \{1,2,3,4\}$ and $A = \{a,b\}$. Let

$$F = \{(a, \{1\}), (b, \{1, 2\})\},$$
$$G = \{(a, \{2\}), (b, \{1, 2\})\},$$
$$H = \{(a, \{3\}), (b, \{3, 4\})\},$$
$$K = \{(a, \{4\}), (b, \{3, 4\})\},$$
$$L = \{(a, \varnothing), (b, \{1, 2\})\},$$
$$M = \{(a, \varnothing), (b, \{3, 4\})\},$$
$$\mathcal{B} = \{F, G, H, K, L, M\},$$

and let (X, τ, A) be the STS that has $\mathcal{B}$ as a soft base. Then $\mathrm{SC}_{a_1} = \{a_1, a_2, a_3, a_4\}$ and $\mathrm{SC}_{b_1} = \{b_1, b_2, b_3, b_4\}$.

Proof **Case 1** To show that $a_1 * a_2$, $a_3 * a_4$, $b_1 * b_2$, and $b_3 * b_4$, take $u = \{(a, a), (b, b)\}$ and $p = \{(1, 2), (2, 1), (3, 4), (4, 3)\}$. Then f_{pu} is a bijection with $f_{pu}(a_1) = a_2$, $f_{pu}(a_3) = a_4$, $f_{pu}(b_1) = b_2$, and $f_{pu}(b_3) = b_4$.

f_{pu}: $(X, \tau, A) \longrightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$, $f_{pu}(G) = F$, $f_{pu}(H) = K$, $f_{pu}(K) = H$, $f_{pu}(L) = L$, and $f_{pu}(M) = M$.

f_{pu}: $(X, \tau, A) \longrightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Therefore, $a_1 * a_2$, $a_3 * a_4$, $b_1 * b_2$, and $b_3 * b_4$.

Case 2 To show that $a_1 * a_3$ and $b_1 * b_3$, take $u = \{(a, a), (b, b)\}$ and $p = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(a_1) = a_3$ and $f_{pu}(b_1) = b_3$.

f_{pu}: $(X, \tau, A) \longrightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = H$, $f_{pu}(G) = K$, $f_{pu}(H) = F$, $f_{pu}(K) = G$, $f_{pu}(L) = M$, and $f_{pu}(M) = L$.

$f_{pu}: (X, \tau, A) \longrightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Therefore, $a_1 * a_3$ and $b_1 * b_3$.

Case 3 $a_1 * b_1$ is not true.

If $a_1 * b_1$, then by Proposition 3.2, $(X, \tau_a) \cong (X, \tau_b)$. But τ_a is the discrete topology while τ_b is not the discrete topology.

Example 3.2 Let $X = \{1, 2\}$ and $A = \{a, b\}$. Let

$$F = \{(a, \{1\}), (b, \{1, 2\})\},$$
$$G = \{(a, \{2\}), (b, \{1\})\},$$
$$H = \{(a, \varnothing), (b, \{1\})\},$$

and $\tau = \{0_A, 1_A, F, G, H\}$. Then $\mathrm{SC}_{e_x} = \{e_x\}$ for all $e_x \in \mathrm{SP}(X, A)$.

Proof It is sufficient to see that the only soft homeomorphism from (X, τ, A) onto (X, τ, A) is the identity. Let $f_{pu} \in \mathrm{SH}(X, \tau, A)$. We are going to show that $u(a) = a$ and $p(1) = 1$ which ends the proof.

$u(a) = a$: Since $\min(X, \tau, A) = \{H\}$, then $f_{pu}(H) = H$. So, $(f_{pu}(H))(a) = p(H(u^{-1}(a))) = \varnothing$. Hence, $H(u^{-1}(a)) = \varnothing$ and thus, $u^{-1}(a) = a$. Therefore, $u(a) = a$.

$p(1) = 1$: Since $u(a) = a$, then $u(b) = b$. Since $f_{pu}(H) = H$, then $\left(f_{pu}(H)\right)(b) = p(H(u^{-1}(b))) = p(H(b)) = p(\{1\}) = \{1\}$. Thus, $p(1) = 1$.

Proposition 3.4 Let (X, τ, A) be an STS. If $a_x \in \min(X, \tau, A)$, then $\mathrm{SC}_{a_x} \subseteq \min(X, \tau, A)$.

Proof Let $b_y \in \mathrm{SC}_{a_x}$. Then there is $f_{pu} \in \mathrm{SH}(X, \tau, A)$ such that $f_{pu}(a_x) = b_y$. It follows that $b_y \in \min(X, \tau, A)$. Therefore, $\mathrm{SC}_{a_x} \subseteq \min(X, \tau, A)$.

The following example shows that the inclusion in Proposition 3.4 cannot replaced by equality in general:

Example 3.3 Let $X = \{1, 2, 3\}$ and $A = \{a, b\}$. Let $H = \{(a, \{2\}), (b, \{3\})\}$ and let (X, τ, A) be the STS that has $\mathcal{B} = \{a_1, b_1, H, 1_A\}$ as a soft base. Then $\min(X, \tau, A) = \{a_1, b_1, H\}$. By Proposition 3.3, $\mathrm{SC}_{a_1}=\{a_1, b_1\}$ or $\mathrm{SC}_{a_1}=\{a_1\}$. To see that $a_1 * b_1$, let $u=\{(a, b), (b, a)\}$ and $p = \{(1, 1), (2, 3), (3, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(a_1) = b_1$.

f_{pu}: $(X, \tau, A) \longrightarrow (X, \tau, A)$ is soft open: $f_{pu}(a_1) = b_1$, $f_{pu}(b_1) = a_1$, $f_{pu}(H) = H$, and $f_{pu}(1_A) = 1_A$.

f_{pu}: $(X, \tau, A) \longrightarrow (X, \tau, A)$ is soft continuous: $f^{-1}_{pu} = f_{p^{-1}u^{-1}} = f_{pu}$.

Therefore, $a_1 * b_1$.

The following question about the exact version of Proposition 2.2 is natural:

Let (X, τ, A) be an STS. Suppose $F \in \min(X, \tau, A)$ and let $a_x \in F$. Is it true that $\{b_y : b_y \in F\} \subseteq \mathrm{SC}_{a_x}$?

The following example gives a negative answer to the above question:

Example 3.4 Let $X = \{1, 2, 3\}$, $A = \{a, b\}$, $H = \{(a, \{1\}), (b, \{2, 3\})\}$, and $\tau = \{0_A, 1_A, H\}$. Consider the STS (X, τ, A). Then $\min(X, \tau, A) = \{H\}$ and $a_1, b_2 \in H$. If $a_1 * b_2$, then there exists $f_{pu} \in \mathrm{SH}(X, \tau, A)$ such that $f_{pu}(a_1) = b_2$. So, $u(a) = b$ and $p(1) = 2$. It is not difficult to see that $f_{pu}(H) = H$ and so $(f_{pu}(H))(a) = H(a)$. But $(f_{pu}(H))(a) = p(H(b)) = p(\{2, 3\})$ while $H(a) = \{1\}$.

By Proposition 3.2, $\mathrm{SC}_{a_1} = \{a_1, b_1\}$ or $\mathrm{SC}_{a_1} = \{a_1\}$. To see that $a_1 * b_1$, let $u = \{(a, b), (b, a)\}$ and $p = \{(1, 1), (2, 3), (3, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(a_1) = b_1$.

f_{pu}: $(X, \tau, A) \longrightarrow (X, \tau, A)$ is soft open: $f_{pu}(a_1) = b_1$, $f_{pu}(b_1) = a_1$, and $f_{pu}(H) = H$.

f_{pu}: $(X, \tau, A) \longrightarrow (X, \tau, A)$ is soft continuous: $f^{-1}_{pu} = f_{p^{-1}u^{-1}} = f_{pu}$.

Therefore, $a_1 * b_1$.

In the following result, soft homeomorphisms are shown to stabilize soft homogeneous components.

Theorem 3.2 Let (X, τ, A) be an STS, $f_{pu} \in \mathrm{SH}(X, \tau, A)$, and $a_x \in \mathrm{SP}(X, A)$. If $H = \widetilde{\cup}_{b_y \in \mathrm{SC}_{a_x}} b_y$, $f_{pu}(H) = H$.

Proof Let $f_{pu} \in \mathrm{SH}(X, \tau, A)$ and let $a_x \in \mathrm{SP}(X, A)$. To see that $f_{pu}(H) \widetilde{\subseteq} H$, let $b_y \widetilde{\in} f_{pu}(H)$, then there exists $c_z \in \mathrm{SC}_{a_x}$ such that $f_{pu}(c_z) = b_y$. So, we have $a_x * c_z$ and $c_z * b_y$, hence $a_x * b_y$. Thus, $b_y \in \mathrm{SC}_{a_x}$ and $b_y \widetilde{\in} H$. This ends the proof that $f_{pu}(H) \widetilde{\subseteq} H$. To see that $H \widetilde{\subseteq} f_{pu}(H)$, let $b_y \widetilde{\in} H$, then $b_y * a_x$. Since f_{pu} is surjective, then there exists $c_z \in \mathrm{SP}(X, A)$ such that $f_{pu}(c_z) = b_y$. So, we have $c_z * b_y$ and $b_y * a_x$, hence, $c_z * a_x$. Thus, $c_z \in \mathrm{SC}_{a_x}$. It follows that $b_y \widetilde{\in} f_{pu}(H)$. This ends the proof that $H \widetilde{\subseteq} f_{pu}(H)$. This shows that $f_{pu}(H) = H$.

Theorem 3.3 Let (X, τ, A) be an STS, $a_x \in \mathrm{SP}(X, A)$, and let $H = \widetilde{\cup}\{b_y : b_y \in \mathrm{SC}_{a_x}\}$. If there is $F \in \tau - \{0_A\}$ such that $F \widetilde{\subseteq} H$, then $H \in \tau$.

Proof Let $b_y \widetilde{\in} H$, then $b_y \in \mathrm{SC}_{a_x}$ and so there exists $f_{pu} \in \mathrm{SH}(X, \tau, A)$ such that $f_{pu}(a_x) = b_y$.

We have $F \in \tau - \{0_A\}$. If $b_y \widetilde{\in} F \widetilde{\subseteq} H$, then we are done. So we may assume that $b_y \widetilde{\notin} F$. Choose $c_z \widetilde{\in} F$. Then $c_z \widetilde{\in} H$ and so $c_z \in \mathrm{SC}_{a_x}$. Thus, there exists $f_{qv} \in \mathrm{SH}(X, \tau, A)$ such that $f_{qv}(c_z) = a_x$. Consider $f_{(q \circ p)(v \circ u)} : (X, \tau, A) \longrightarrow (X, \tau, A)$. Then $f_{(q \circ p)(v \circ u)} \in \mathrm{SH}(X, \tau, A)$ and

$$f_{(p \circ q)(u \circ v)}(c_z) = ((u \circ v)(c))_{(p \circ q)(z)} = (u(v(c)))_{p(q(z))} = (u(a))_{p(x)} = b_y.$$

Therefore, $b_y \widetilde{\in} H$. Since $f_{(p \circ q)(u \circ v)}$ is soft open, then $f_{(p \circ q)(u \circ v)}(F) \in \tau$. Since $F \widetilde{\subseteq} H$, then $f_{(q \circ p)(v \circ u)}(F) \widetilde{\subseteq} f_{(q \circ p)(v \circ u)}(H)$. On the other hand, it is not difficult to check that $f_{(q \circ p)(v \circ u)}(H) = H$. Thus, $b_y \widetilde{\in} f_{(q \circ p)(v \circ u)}(F) \widetilde{\subseteq} H$. This shows that $H \in \tau$.

4 Soft Product

In this section, we present two soft product theorems, one for soft homogeneity and one for soft minimality.

Definition 4.1 Let $f_{pu} : \mathrm{SS}(X, A) \longrightarrow SS(Y, B)$ and $f_{qv} : \mathrm{SS}(Z, C) \longrightarrow \mathrm{SS}(W, D)$ be two soft mappings. If $p \times q : X \times Z \longrightarrow Y \times W$ and $u \times v : A \times C \longrightarrow B \times D$ are the product functions defined by $(p \times q)(x, z) = (p(x), q(z))$ and $(u \times v)(a, b) = (u(a), v(b))$, then $f_{(p \times q)(u \times v)} : \mathrm{SS}(X \times Z, A \times C) \longrightarrow \mathrm{SS}(Y \times W, B \times D)$ is called the product soft mapping of f_{pu} and f_{qv}.

Lemma 4.1 Let $f_{pu} : \mathrm{SS}(X, A) \longrightarrow \mathrm{SS}(Y, B)$ and $f_{qv} : \mathrm{SS}(Z, C) \longrightarrow \mathrm{SS}(W, D)$ be two soft mappings and consider the product soft mapping $f_{(p \times q)(u \times v)} : \mathrm{SS}(X \times Z, A \times C) \longrightarrow \mathrm{SS}(Y \times W, B \times D)$. Then $f^{-1}_{(p \times q)(u \times v)}(F \times G) = f^{-1}_{pu}(F) \times f^{-1}_{qv}(G)$ for every $F \in \mathrm{SS}(Y, B)$ and $G \in \mathrm{SS}(W, D)$.

Proof Let $F \in \mathrm{SS}(Y, B)$ and $G \in \mathrm{SS}(W, D)$. Let $a \in A$ and $c \in C$. Then

$$\begin{aligned}
&\left(f^{-1}_{(p \times q)(u \times v)}(F \times G)\right)(a, c) = \\
&(p \times q)((F \times G)((u \times v)^{-1}(a, c))) = \\
&(p \times q)((F \times G)(u^{-1}(a), v^{-1}(c))) = \\
&(p \times q)(F(u^{-1}(a)) \times G(v^{-1}(c))) = \\
&p(F(u^{-1}(a))) \times q(G(v^{-1}(c))) = \\
&((f^{-1}_{pu}(F))(a)) \times ((f^{-1}_{qv}(G))(c)) = \\
&(f^{-1}_{pu}(F) \times f^{-1}_{qv}(G))(a, c).
\end{aligned}$$

It follows that $f^{-1}_{(p \times q)(u \times v)}(F \times G) = f^{-1}_{pu}(F) \times f^{-1}_{qv}(G)$.

Proposition 4.1 If $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ and $f_{qv} : (Z, \delta, C) \longrightarrow (W, \rho, D)$ are soft continuous, then $f_{(p \times q)(u \times v)} : (X \times Z, \tau \times \delta, A \times C) \longrightarrow (Y \times W, \sigma \times \rho, B \times D)$ is soft continuous.

Proof Suppose that f_{pu} and f_{qv} are soft continuous. Let $F \times G$ where $F \in \sigma$ and $G \in \rho$ be a soft basic element for $\sigma \times \rho$. Since f_{pu} and f_{qv} are soft continuous, then $f^{-1}_{pu}(F) \in \tau$ and $f^{-1}_{qv}(G) \in \delta$. By Lemma 4.2, $f^{-1}_{(p \times q)(u \times v)}(F \times G) = f^{-1}_{pu}(F) \times f^{-1}_{qv}(G)$, and so $f^{-1}_{(p \times q)(u \times v)}(F \times G) \in \tau \times \delta$. It follows that $f_{(p \times q)(u \times v)}$ is soft continuous.

Theorem 4.1 If $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ and $f_{qv} : (Z, \delta, C) \longrightarrow (W, \rho, D)$ are soft homeomorphisms, then $f_{(p \times q)(u \times v)} : (X \times Z, \tau \times \delta, A \times C) \longrightarrow (Y \times W, \sigma \times \rho, B \times D)$ is soft homeomorphism.

Proof Suppose that f_{pu} and f_{qv} are soft homeomorphisms. Since f_{pu} and f_{qv} are bijections, then p, u, q, and v are bijections and so $p \times q$ and $u \times v$ are bijections. Therefore, $f_{(p\times q)(u\times v)}$ is a bijection. Since f_{pu} and f_{qv} are soft continuous, then by Proposition 4.1, $f_{(p\times q)(u\times v)}$ is soft continuous. Since $f_{p^{-1}u^{-1}} : (Y, \sigma, B) \longrightarrow (X, \tau, A)$ and $f_{q^{-1}v^{-1}} : (W, \rho, D) \longrightarrow (Z, \delta, C)$ are soft continuous, then again by Proposition 4.1, $f_{\left((p\times q)^{-1}\right)\left((u\times v)^{-1}\right)} = f_{(p^{-1}\times q^{-1})(u^{-1}\times v^{-1})} : (Y \times W, \sigma \times \rho, B \times D) \longrightarrow (X \times Z, \tau \times \delta, A \times C)$ is soft continuous. It follows that $f_{(p\times q)(u\times v)}$ is soft homeomorphism.

Theorem 4.2 If (X, τ, A) and (Y, σ, B) are soft homogeneous STSs, then $(X \times Y, \tau \times \sigma, A \times B)$ is soft homogeneous.

Proof Let $(a, b)_{(x,y)}, (c, d)_{(z,w)} \in \mathrm{SP}(X \times Y, A \times B)$. Since (X, τ, A) is soft homogeneous and $a_x, c_z \in \mathrm{SP}(X, A)$, then there exists $f_{pu} \in \mathrm{SH}(X, \tau, A)$ such that $u(a) = c$ and $p(x) = z$. Also, since (Y, σ, B) is soft homogeneous and $b_y, d_w \in \mathrm{SP}(Y, B)$, then there exists $f_{qv} \in \mathrm{SH}(Y, \sigma, B)$ such that $v(b) = d$ and $q(y) = w$. By Theorem 4.2, $f_{(p\times q)(u\times v)}$ is soft homeomorphism. Note that $(u \times v)(a, b) = (u(a), v(b)) = (c, d)$ and $(p \times q)(x, y) = (p(x), q(y)) = (z, w)$. It follows that $(X \times Y, \tau \times \sigma, A \times B)$ is soft homogeneous.

Theorem 4.3 Let (X, τ, A) and (Y, σ, B) be two STSs. If $c_z \in \mathrm{SC}_{a_x}$ and $d_w \in \mathrm{SC}_{b_y}$, then $(c, d)_{(z,w)} \in \mathrm{SC}_{(a,b)_{(x,y)}}$.

Proof Suppose that $c_z \in \mathrm{SC}_{a_x}$ and $d_w \in \mathrm{SC}_{b_y}$. Then there exists $f_{pu} \in \mathrm{SH}(X, \tau, A)$ and $f_{qv} \in \mathrm{SH}(Y, \sigma, B)$ such that $p(x) = z$, $u(a) = c$, $q(y) = w$, and $v(b) = d$. By Theorem 4.3, $f_{(p\times q)(u\times v)}$ is soft homeomorphism. Note that $(u \times v)(a, b) = (u(a), v(b)) = (c, d)$ and $(p \times q)(x, y) = (p(x), q(y)) = (z, w)$. This shows that $(c, d)_{(z,w)} \in \mathrm{SC}_{(a,b)_{(x,y)}}$.

Theorem 4.4 Let (X, τ, A) and (Y, σ, B) be STSs. If $F \in \min(X, \tau, A)$ and $G \in \min(Y, \sigma, B)$, then $F \times G \in \min(X \times Y, \tau \times \sigma, A \times B)$.

Proof Since $F \in \min(X, \tau, A)$ and $G \in \min(Y, \sigma, B)$, then there exists $a_x \widetilde{\in} F$ and $b_y \widetilde{\in} G$. So, $(a, b)_{(x,y)} \widetilde{\in} F \times G$ and thus, $F \times G \neq 0_{A\times B}$. Suppose that $M \in (\tau \times \sigma) - \{0_{A\times B}\}$ with $M \widetilde{\subseteq} F \times G$. Choose $(c, d)_{(z,w)} \widetilde{\in} M$. Since $M \in \tau \times \sigma$, then there exists $H \in \tau$ and $K \in \sigma$ such that $(c, d)_{(z,w)} \widetilde{\in} H \times K \widetilde{\subseteq} M \widetilde{\subseteq} F \times G$. So, $(c, d)_{(z,w)} \widetilde{\in} (H \times K) \widetilde{\cap} (F \times G)$ and hence, $(c, d)_{(z,w)} \widetilde{\in} \left(F \widetilde{\cap} H\right) \times \left(G \widetilde{\cap} K\right)$. Thus, $c_z \widetilde{\in} F \widetilde{\cap} H$ and $d_w \widetilde{\in} G \widetilde{\cap} K$. Since $F \in \min(X, \tau, A)$ and $a_x \widetilde{\in} F \widetilde{\cap} H$, then by Proposition 3.2 of Ref. [13], $F \widetilde{\subseteq} H$. Since $G \in \min(Y, \sigma, B)$ and $b_y \widetilde{\in} G \widetilde{\cap} K$, then again by Proposition 3.2 of Ref. [13], $G \widetilde{\subseteq} K$. Therefore, $F \times G \widetilde{\subseteq} H \times K$. Hence, $F \times G = H \times K \widetilde{\subseteq} M$. Thus, $M = F \times G$. This shows that $F \times G \in \min(X \times Y, \tau \times \sigma, A \times B)$.

5 Conclusion

Since the creation of the STS structure, efforts have been undertaken to apply traditional topological ideas to this soft structure. According to the published literature, STS offers a fertile setting for the extension of topological concepts; for example, soft homogeneity was presented in Ref. [13], and the author in Ref. [20] introduced and investigated soft prehomogeneity. In this paper, soft homogeneous components, which form a soft partition of the soft points in a given STS, are introduced and analyzed. Various results and examples related to soft homogeneous components of STSs are introduced, particularly those contain a minimal soft open set. Additionally, two soft product theorems concerning soft homogeneity and soft minimality, respectively, are given. Some results related to homogeneity concepts in general topology are also obtained.

The following subjects may be explored in our future research: (1) To define the soft prehomogeneous components; (2) To define soft semihomogeneity.

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