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Novel Classes of Bipolar Soft Generalized Topological Structures: Compactness and Homeomorphisms

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ABSTRACT

The purpose of this paper is to define bipolar soft generalized compact sets and bipolar soft generalized compact spaces. The structures of $\widetilde{\widetilde{g}}$-centralized bipolar soft generalized closed sets collection in a bipolar soft generalized compact space are given. Moreover, some main properties of bipolar soft generalized compactness are discussed and their relationships are studied. The concept of a bipolar soft generalized compactness is introduced and it investigates under what condition a bipolar soft generalized topological space forms a bipolar soft generalized compact space. The relation between bipolar soft generalized compact space and soft generalized compact space is proposed. Furthermore, some further properties of bipolar soft mappings, such as bipolar soft composite mappings, are presented and some of their characteristics are explained. Additionally, novel classes of bipolar soft mapping such as bipolar soft generalized continuous, bipolar soft generalized open, and bipolar soft generalized closed mappings are defined. Finally, some results and counterexamples are obtained.

KEYWORDS

bipolar soft generalized compact space (set); bipolar soft generalized continuous mappings; bipolar soft generalized homeomorphism mappings

1 Introduction

For formal modeling, reasoning, and computing, most traditional tools are characterized by being their crisp, deterministic, and precise. However, many complex problems exist in the domains of economics, engineering, the environment, social science, medical science, and so on. Therefore, traditional methods based on cases may not be suitable for solving or modeling these issues. Based on this, a set of theories has been proposed to tackle these problems. In 1999, Molodtsov[1] introduced the soft set theory, which was designed to solve sophisticated problems. Molodtsov's theory has been implemented in several branches of mathematics, including decision making problems, medical science, social sciences, operation research, etc. Other researchers have improved the theory, such as Maji et al.[2], who defined the operation family of special information systems. Çağman and Enginoğlu[3] redefined the operations of Molodtsov's soft sets and defined products soft sets and uni-int decision making method. Then, Aktaş and

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Çağman[4] compared the soft sets to both fuzzy and rough sets. For the time being, there are two definitions of soft topological spaces. The concept of soft topological spaces on a universe set was first defined by Shabir and Naz[5]. Likewise, the notion of soft topological spaces was also demonstrated using soft sets by Çağman et al.[6] Ma et al.[7] extended of soft set and redefined and simplified operations between soft set. Ameen and Al Ghour[8] presented minimal soft topology and investigated some of their properties. Ameen and Alqahtani[9] introduced some classes of soft functions defined by soft open sets modulo soft sets of the first category. In Ref. [10], they investigated some properties of soft ideals in soft topological spaces.

Császár[11, 12] introduced the notion of a generalized neighborhood system and generalized topological spaces where he defined a family of subsets of Ω to be a generalized topology on Ω if it includes the empty set and arbitrary union of its member and studied some of their basic properties, such as continuous functions, associated interior, and closure operations, also compared his findings to those of the usual topology.

In 2013, Shabir and Naz[13] explained that the bipolar soft set (BSS) structure has clearer and more general results than the soft set structure. Naz and Shabir[14] came up the fuzzy bipolar soft sets and bipolar fuzzy soft sets and established some algebraic structures of these two classes of bipolar soft sets. They presented an application of fuzzy bipolar soft sets in decision making problems. Based on Dubois and Prade[15], decision making is constructed on two sides, namely negative and positive. Bipolarity is significant for characterization between positive and negative information to differentiation between a reasonable and an unreasonable events. Fadel and Dzul-Kifli[16] defined the concept of bipolar soft topological spaces via bipolar soft sets and some properties. In Ref. [17], they introduced bipolar soft functions interms of bipolar soft sets and discussed some of their characterizations. Öztürk[18, 19] presented more properties and operations on bipolar soft sets as well as the bipolar soft points are introduced. Subsequently, a number of definitions, operations, and applications on bipolar soft sets and bipolar soft structures have been investigated. For instance, Dizman and Öztürk[20] introduced fuzzy bipolar soft topological spaces via fuzzy bipolar soft sets. Abdullah et al.[21] proposed a bipolar fuzzy soft set which is a new idea of bipolar soft set and they introduced some basic operations and an application of bipolar fuzzy soft set into decision making problem. They gave an algorithm to solve decision making problems by using bipolar fuzzy soft set. Al-shami[22] defined some ordinary points on bipolar soft sets as well as he presented an application of optimal choices by applying the idea of bipolar soft sets. Karaaslan and Karataş[23] redefined concept of bipolar soft set and bipolar soft set operations and presented decision making method with application. Karaaslana et al.[24] defined normal bipolar soft subgroups. Mahmood[25] defined a novel approach towards bipolar soft sets and discussed an application on decision-making problems. Wang et al.[26] and Rehman and Mahmood[27] combined some generalization of fuzzy sets and bipolar soft sets. They investigated applications in decision-making problems. Hussain[28] defined and discussed binary soft connected spaces in binary soft topological spaces. He proposed an application of a decision making problem by using approach of rough sets. Bera and Pal[29] presented m-polar interval-valued fuzzy graph and application of it.

Yet, studies conducted on the limit point concept were required by mathematicians to bring about more developments in mathematics. Shabir and Bakhtawar[30] and Aras and Bayramov[31] introduced the concept of bipolar soft compact spaces and studied some of their properties in detail. Musa and Asaad[32] introduced the concept of bipolar hypersoft set as a combination of hypersoft set with bipolarity setting and investigated some of its basic operations. They also discussed some topological notions in the frame of bipolar hypersoft setting[33, 34]. In 2022, Saleh et al.[35, 36] studied bipolar soft generalized topological spaces (BSGTSs) and defined it as a set of $\widetilde{\widetilde{g}}$ of bipolar soft collections over the universe Ω. As such, the basic notions of BSGTSs, bipolar soft (BS) $\widetilde{\widetilde{g}}$-open and BS $\widetilde{\widetilde{g}}$-closed sets, BS $\widetilde{\widetilde{g}}$-closure, BS $\widetilde{\widetilde{g}}$-interior, BS $\widetilde{\widetilde{g}}$-limit point, BS $\widetilde{\widetilde{g}}$-boundary, and bipolar soft generalized topological subspaces (BSGTSS) were defined side by side with the investigation of a number of their properties. After that, in Ref. [37], they presented some new ideas on BSGTSs such as BS $\widetilde{\widetilde{g}}$-connected sets, BS $\widetilde{\widetilde{g}}$-connected spaces, and BS $\widetilde{\widetilde{g}}$-disconnected spaces. Then, they devoted towards the idea of BS $\widetilde{\widetilde{g}}$-separated sets, BS $\widetilde{\widetilde{g}}$-separation sets, and BS $\widetilde{\widetilde{g}}$-hereditary property. The concept of BS $\widetilde{\widetilde{g}}$-locally connected spaces and BS $\widetilde{\widetilde{g}}$-components are introduced. In addition, the concept of BS $\widetilde{\widetilde{g}}$-components that the family of all BS $\widetilde{\widetilde{g}}$-components forms a partition for BSGTS is given. Then, some properties of BS $\widetilde{\widetilde{g}}$-components in BSGTSs are proposed.

It endeavors to cast light on the concepts of compactness into the structure of BSGTS. Some substantial concepts forwarded of BS $\widetilde{\widetilde{g}}$-compact sets and BS $\widetilde{\widetilde{g}}$-compact spaces are investigated. The use of bipolar soft sets provides us with a fundamental means to tackle and investigate compactness in a more general way. BS $\widetilde{\widetilde{g}}$-compact proves to be a good extension of bipolar soft compact. A number of findings have been drawn upon, e.g., the finite bipolar soft union of BS $\widetilde{\widetilde{g}}$-compact sets is also BS $\widetilde{\widetilde{g}}$-compact. Later on, the notion of $\widetilde{\widetilde{g}}$-centralized BS collection is presented and if it is BS $\widetilde{\widetilde{g}}$-closed set, then a BSGTS is BS $\widetilde{\widetilde{g}}$-compact and some counterexamples are established to indicate that the converse in general may not be true. Some further properties of bipolar soft mappings, such as bipolar soft composite mappings, investigate and discuss some of its characteristics. Furthermore, BS $\widetilde{\widetilde{g}}$-continuous, BS $\widetilde{\widetilde{g}}$-open, BS $\widetilde{\widetilde{g}}$-closed, and BS $\widetilde{\widetilde{g}}$-homeomorphism are introduced. Some of their characterizations are obtained. we conclude the paper and make recommendations for further research.

2 Preliminary

We commence our paper by elucidating fundamental concepts related to our research prior to engaging in its discussion. Throughout this work, let Ω be an initial universe, $\Upsilon(\Omega)$ be denoted the collection of all subsets of Ω, and ϖ be a set of parameters. Let $\varsigma, \sigma \subseteq \varpi$ and BSS(Ω) be the set of all bipolar soft sets over Ω with parameters ϖ, where $\varsigma = \{\rho_1, \rho_2, \ldots, \rho_n\}$ be a set of parameters.

2.1 Bipolar soft set

This subsection investigates the concepts of bipolar soft sets and some properties and theorems related to this work.

Definition 2.1[2] Let $\varsigma = \{\rho_1, \rho_2, \ldots, \rho_n\}$ be a subset of ϖ. The Not set of ς is denoted by $\neg\varsigma = \{\neg\rho_1, \neg\rho_2, \ldots, \neg\rho_n\}$, where $\neg\rho_i =$ Not ρ_i, for all i.

Definition 2.2[13] A triple $(\Theta, \Lambda, \varsigma)$ is called a bipolar soft set, denoted by BSS, on Ω, where Θ and Λ

are mappings defined by $\Theta : \varsigma \longrightarrow P(\Omega)$ and $\Lambda : \neg\varsigma \longrightarrow P(\Omega)$ such that $\Theta(\rho) \cap \Lambda(\neg\rho) = \varnothing$ for all $\rho \in \varsigma$ and $\neg\rho \in \neg\varsigma$. In other words, a BSS $(\Theta, \Lambda, \varsigma)$ can be written as

$$(\Theta, \Lambda, \varsigma) = \{(\rho, \Theta(\rho), \Lambda(\neg\rho)) : \rho \in \varsigma, \Theta(\rho) \cap \Lambda(\neg\rho) = \varnothing\}.$$

Definition 2.3[13] For any two BSSs $(\Theta_1, \Lambda_1, \varsigma)$ and $(\Theta_2, \Lambda_2, \sigma)$, the BSS $(\Theta_1, \Lambda_1, \varsigma)$ is said to be a bipolar soft subset of $(\Theta_2, \Lambda_2, \sigma)$ if

(1) $\varsigma \subseteq \sigma$ and,

(2) $\Theta_1(\rho) \subseteq \Theta_2(\rho)$ and $\Lambda_2(\neg\rho) \subseteq \Lambda_1(\neg\rho)$, for all $\rho \in \varsigma$ and $\neg\rho \in \neg\varsigma$.

This relationship is denoted by $(\Theta_1, \Lambda_1, \varsigma)\widetilde{\widetilde{\subseteq}}(\Theta_2, \Lambda_2, \sigma)$. Similarly, the BSS $(\Theta_1, \Lambda_1, \varsigma)$ is said to be a bipolar soft superset of $(\Theta_2, \Lambda_2, \sigma)$, denoted by $(\Theta_1, \Lambda_1, \varsigma)\widetilde{\widetilde{\supseteq}}(\Theta_2, \Lambda_2, \sigma)$, if $(\Theta_2, \Lambda_2, \sigma)$ is a bipolar soft subset of $(\Theta_1, \Lambda_1, \varsigma)$.

Definition 2.4[13] Two BSSs $(\Theta_1, \Lambda_1, \varsigma)$ and $(\Theta_2, \Lambda_2, \sigma)$ are said to be BS equal, denoted by $(\Theta_1, \Lambda_1, \varsigma) = (\Theta_2, \Lambda_2, \sigma)$, if $(\Theta_1, \Lambda_1, \varsigma)$ is a bipolar soft subset of $(\Theta_2, \Lambda_2, \sigma)$ and $(\Theta_2, \Lambda_2, \sigma)$ is a bipolar soft subset of $(\Theta_1, \Lambda_1, \varsigma)$.

Definition 2.5[13] The complement of a BSS $(\Theta, \Lambda, \varsigma)$ is denoted by $(\Theta, \Lambda, \varsigma)^c$ and defined by $(\Theta, \Lambda, \varsigma)^c = (\Theta^c, \Lambda^c, \varsigma)$ where Θ^c and Λ^c are mappings given by $\Theta^c(\rho) = \Lambda(\neg\rho)$ and $\Lambda^c(\neg\rho) = \Theta(\rho)$, for all $\rho \in \varsigma$ and $\neg\rho \in \neg\varsigma$.

Definition 2.6[13] A BSS $(\Theta, \Lambda, \varsigma)$ is said to be a relative null BSS (with respect to the parameter set ς), denoted by $(\Phi, \widetilde{\widetilde{\Omega}}, \varsigma)$, if $\Theta(\rho) = \varnothing$ for all $\rho \in \varsigma$ and $\Lambda(\neg\rho) = \Omega$ for all $\neg\rho \in \neg\varsigma$. The relative null BSS (with respect to the universe set parameters ϖ) is called the null BSS on Ω, denoted by $(\Phi, \widetilde{\widetilde{\Omega}}, \varpi)$.

Obviously, a BSS $(\Theta, \Lambda, \varpi)$ is said to be a non-null BSS if $\Theta(\rho) \neq \varnothing$ for some $\rho \in \varpi$.

Definition 2.7[13] A BSS $(\Theta, \Lambda, \varsigma)$ is said to be a relative absolute BSS (with respect to the parameter set ς), denoted by $(\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$, if $\Theta(\rho) = \Omega$ for all $\rho \in \varsigma$ and $\Lambda(\neg\rho) = \varnothing$ for all $\neg\rho \in \neg\varsigma$. The relative absolute BSS (with respect to the universe set of parameter ϖ), denoted by $(\widetilde{\widetilde{\Omega}}, \Phi, \varpi)$, is called the absolute BSS on Ω.

Obviously, a BSS $(\Theta, \Lambda, \varpi)$ is said to be a non-absolute BSS if $\Theta(\rho) \neq \Omega$ for some $\rho \in \varpi$.

Definition 2.8 Let $(\Theta, \Lambda, \varsigma)\widetilde{\widetilde{\in}}\mathrm{BSS}(\Omega)$ and Π be a non-empty subset of Ω. Then, the sub bipolar soft set of $(\Theta, \Lambda, \varsigma)$ over Π denoted by $({}^{\Pi}\Theta, {}^{\Pi}\Lambda, \varsigma)$, is defined as follows:

$${}^{\Pi}\Theta(\rho) = \Pi \cap \Theta(\rho) \text{ and } {}^{\Pi}\Lambda(\neg\rho) = \Pi \cap \Lambda(\neg\rho), \text{ for each } \rho \in \varsigma \text{ and } \neg\rho \in \neg\varsigma.$$

Definition 2.9[38] Let $(\Theta, \Lambda, \varsigma) \widetilde{\widetilde{\in}} \mathrm{BSS}(\Omega)$. The BSS $(\Theta, \Lambda, \varsigma)$ is called a bipolar soft point (BSP) if there exists $\pi, \upsilon \in \Omega$ (it is possible to $\pi = \upsilon$) and $\rho \in \varsigma$ such that

$$\Theta(\gamma) = \begin{cases} \{\pi\}, & \gamma = \rho; \\ \varnothing, & \gamma \neq \rho. \end{cases}$$

$$\Lambda(\gamma') = \begin{cases} \Omega \setminus \{\pi, \upsilon\}, & \gamma' = \rho; \\ \Omega, & \gamma' \neq \rho. \end{cases}$$

We denote BSP $(\Theta, \Lambda, \varsigma)$ by π_υ^ρ, and the family of all BSPs on Ω is denoted by BSP $(\Omega)_{(\varsigma,\neg\varsigma)}$.

Definition 2.10[36] Let π_{υ}^{ρ}, $\pi_{\upsilon'}^{\prime\rho'} \widetilde{\widetilde{\in}} \mathrm{BSP}(\Omega)_{(\varsigma,\neg\varsigma)}$ be two BSPs. Then, π_{υ}^{ρ} and $\pi_{\upsilon'}^{\prime\rho'}$ are said to be not different BSPs if $\pi = \pi'$ and $\rho = \rho'$. Clearly, $\upsilon = \upsilon'$ or $\upsilon \neq \upsilon'$.

Definition 2.11[19] Let π_{υ}^{ρ}, $\pi_{\upsilon'}^{\prime\rho'} \widetilde{\widetilde{\in}} \mathrm{BSP}(\Omega)_{(\varsigma,\neg\varsigma)}$. Then π_{υ}^{ρ} and $\pi_{\upsilon'}^{\prime\rho'}$ are called different BSPs, denoted by $\pi_{\upsilon}^{\rho} \neq \pi_{\upsilon'}^{\prime\rho'}$, if $\pi \neq \pi'$ or $\rho \neq \rho'$.

Definition 2.12[19] Let $(\Theta, \Lambda, \varsigma) \widetilde{\widetilde{\in}} \mathrm{BSS}(\Omega)$ and $\pi_{\upsilon}^{\rho} \widetilde{\widetilde{\in}} \mathrm{BSP}(\Omega)_{(\varsigma,\neg\varsigma)}$. Then, π_{υ}^{ρ} is said to be contained in $(\Theta, \Lambda, \varsigma)$, denoted by $\pi_{\upsilon}^{\rho} \widetilde{\widetilde{\in}} (\Theta, \Lambda, \varsigma)$, if $\pi \in \Theta(\rho)$ and $\upsilon \in \Omega \setminus \Lambda(\neg\rho)$.

Definition 2.13[30] A collection $\Psi = \{(\Theta_\gamma, \Lambda_\gamma, \varsigma) : (\Theta_\gamma, \Lambda_\gamma, \varsigma) \widetilde{\widetilde{\in}} \widetilde{\widetilde{g}}\}_{\gamma \in \Gamma}$ of BSSs is said to be a BS cover of a BSS $(\Theta, \Lambda, \varsigma)$ if

$$(\Theta, \Lambda, \varsigma) \widetilde{\widetilde{\subseteq}} \widetilde{\widetilde{\cup}}_{\gamma \in \Gamma} (\Theta_\gamma, \Lambda_\gamma, \varsigma).$$

Furthermore, it is called the BS open cover of a BSS $(\Theta, \Lambda, \varsigma)$, if each member of Ψ is a BS open set. A bipolar soft subcover of Ψ is a subcollection of $\{(\Theta_\gamma, \Lambda_\gamma, \varsigma)\}_{\gamma \in \Gamma}$ which is also a BS open cover.

Definition 2.14[30] A bipolar soft subset $(\Theta, \Lambda, \varsigma)$ of $(\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$ is called a bipolar soft compact set, denoted by BS compact set, if each BS open cover of $(\Theta, \Lambda, \varsigma)$ has a finite BS subcover. A bipolar soft topological space (BSTS) $(\Omega, \widetilde{\widetilde{\tau}}, \varsigma, \neg\varsigma)$ is said to be BS compact space, if $(\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$ is a BS compact subset of itself.

The readers can be found more properties on bipolar soft sets and bipolar soft topological spaces in Ref. [13, 16, 18, 23, 30].

2.2 Bipolar soft mapping

This subsection investigates the concepts of bipolar soft sets with some properties and theorems related to this work.

Definition 2.15[17] Let $\vartheta : \Omega \longrightarrow \Omega'$ be an injective mapping, $\mu : \varsigma \longrightarrow \varsigma'$ and $\rho : \neg\varsigma \longrightarrow \neg\varsigma'$ be two mappings such as $\rho(\neg\rho) = \neg\mu(\rho)$ for all $\neg\rho \in \neg\varsigma$. Then, a bipolar soft mapping, denoted by BS mapping, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_\varsigma) \longrightarrow \mathrm{BS}(\Omega'_{\varsigma'})$, is defined as

For any BSS $(\Theta, \Lambda, \sigma) \widetilde{\widetilde{\in}} \mathrm{BS}(\Omega_\varsigma)$:

(1) The image of $(\Theta, \Lambda, \sigma) \widetilde{\widetilde{\in}} \mathrm{BS}(\Omega_\varsigma)$ under $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\big((\Theta, \Lambda, \sigma)\big) = \big(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta), \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Lambda), \varsigma'\big)$, is a BSS in $\mathrm{BS}(\Omega'_{\varsigma'})$, given as, for all $\rho' \in \varsigma'$,

$$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta)(\rho') = \begin{cases} \vartheta\big(\bigcup_{\rho \in \mu^{-1}(\rho') \cap \sigma} \Theta(\rho)\big), & \mu^{-1}(\rho') \cap \sigma \neq \varnothing; \\ \varnothing, & \text{otherwise.} \end{cases}$$

$$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Lambda)(\neg\rho') = \begin{cases} \vartheta\big(\bigcap_{\neg\rho \in \rho^{-1}(\neg\rho') \cap \neg\sigma} \Lambda(\neg\rho)\big), & \rho^{-1}(\neg\rho') \cap \neg\sigma \neq \varnothing; \\ \Omega', & \text{otherwise.} \end{cases}$$

(2) The inverse image of $(\chi, \psi, \sigma') \widetilde{\widetilde{\in}} \mathrm{BS}(\Omega'_{\varsigma'})$ under $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$, $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\big((\chi, \psi, \sigma')\big) = \big(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi), \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\psi), \varsigma\big)$, is a BSS in $\mathrm{BS}(\Omega_\varsigma)$, given as, for all $\rho \in \sigma$,

$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi)(\rho) = \begin{cases} \vartheta^{-1}(\chi(\mu(\rho))), & \mu(\rho) \in \sigma'; \\ \varnothing, & \mu(\rho) \notin \sigma'. \end{cases}$$

$$\widetilde{\widetilde{\phi}}^{-1}_{\vartheta\mu\rho}(\psi)(\neg\rho)=\begin{cases}\vartheta^{-1}(\psi(\rho(\neg\rho))), & \rho(\neg\rho)\in\neg\sigma';\\ \Omega, & \rho(\neg\rho)\notin\neg\sigma'.\end{cases}$$

Definition 2.16[17] Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:\mathrm{BS}(\Omega_\varsigma)\longrightarrow\mathrm{BS}(\Omega'_{\varsigma'})$ be a BS mapping and $(\Theta_1,\Lambda_1,\sigma_1)$, $(\Theta_2,\Lambda_2,\sigma_2)\ \widetilde{\widetilde{\in}}\ \mathrm{BS}(\Omega_\varsigma)$. Then

(1) The BS union image of $(\Theta_1,\Lambda_1,\sigma_1)$, $(\Theta_2,\Lambda_2,\sigma_2)\ \widetilde{\widetilde{\in}}\ \mathrm{BSS}(\Omega'_{\varsigma'})$ under $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is defined as: for $\rho'\in\varsigma'$,

$$\left(\left(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1)\right)\widetilde{\widetilde{\cup}}\left(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2)\right)\right)(\rho')=$$
$$(\rho',\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1)(\rho')\cup\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2)(\rho'),\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Lambda_1)(\neg\rho')\cap\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Lambda_2)(\neg\rho')).$$

(2) The BS intersection image of $(\Theta_1,\Lambda_1,\sigma_1)$, $(\Theta_2,\Lambda_2,\sigma_2)\ \widetilde{\widetilde{\in}}\ \mathrm{BSS}(\Omega'_{\varsigma'})$ under $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is defined as: for $\rho'\in\varsigma'$,

$$\left(\left(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1)\right)\widetilde{\widetilde{\cap}}\left(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2)\right)\right)(\rho')=$$
$$(\rho',\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1)(\rho')\cap\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2)(\rho'),\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Lambda_1)(\neg\rho')\cup\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Lambda_2)(\neg\rho')).$$

Definition 2.17[17] Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:\mathrm{BS}(\Omega_\varsigma)\longrightarrow\mathrm{BS}(\Omega'_{\varsigma'})$ be a BS mapping and $(\chi_1,\psi_1,\sigma'_1)$, $(\chi_2,\psi_2,\sigma'_2)\ \widetilde{\widetilde{\in}}\ \mathrm{BS}(\Omega'_{\varsigma'})$. Then

(1) The BS union of inverse image of $(\chi_1,\psi_1,\sigma'_1)$, $(\chi_2,\psi_2,\sigma'_2)\ \widetilde{\widetilde{\in}}\ \mathrm{BS}(\Omega'_{\varsigma'})$ under $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}$ is defined as: for $\rho\in\varsigma$,

$$\left(\left(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\sigma'_1)\right)\widetilde{\widetilde{\cup}}\left(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2,\psi_2,\sigma'_2)\right)\right)(\rho)=$$
$$(\rho,\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1)(\rho)\cup\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2)(\rho),\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\psi_1)(\neg\rho)\cap\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\psi_2)(\neg\rho)).$$

(2) The BS intersection of inverse image of $(\chi_1,\psi_1,\sigma'_1)$, $(\chi_2,\psi_2,\sigma'_2)\ \widetilde{\widetilde{\in}}\ \mathrm{BS}(\Omega'_{\varsigma'})$ under $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}$ is defined as: for $\rho\in\varsigma$,

$$\left(\left(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\sigma_1)\right)\widetilde{\widetilde{\cap}}\left(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2,\psi_2,\sigma_2)\right)\right)(\rho)=$$
$$(\rho,\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1)(\rho)\cap\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2)(\rho),\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\psi_1)(\neg\rho)\cup\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\psi_2)(\neg\rho)).$$

Definition 2.18[17] Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:\mathrm{BS}(\Omega_\varsigma)\longrightarrow\mathrm{BS}(\Omega'_{\varsigma'})$ be a BS mapping, where $\vartheta:\Omega\longrightarrow\Omega'$ is an injective mapping, $\mu:\varsigma\longrightarrow\varsigma'$ and $\rho:\neg\varsigma\longrightarrow\neg\varsigma'$ are two mappings, such as $\rho(\neg\rho)=\neg\mu(\rho)$ for all $\neg\rho\in\neg\varsigma$. Then, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is called a

(1) BS surjective mapping, if ϑ and μ are surjective mappings;

(2) BS injective mapping, if μ is an injective mapping;

(3) BS bijective mapping, if ϑ and μ are bijective mappings.

Proposition 2.1[17] Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:\mathrm{BS}(\Omega_\varsigma)\longrightarrow\mathrm{BS}(\Omega'_{\varsigma'})$ be a BS mapping, where $\vartheta:\Omega\longrightarrow\Omega'$ is an injective mapping, $\mu:\varsigma\longrightarrow\varsigma'$ and $\rho:\neg\varsigma\longrightarrow\neg\varsigma'$ are two mappings such as $\rho(\neg\rho)=\neg\mu(\rho)$ for all $\neg\rho\in\neg\varsigma$. If $(\Theta_1,\Lambda_1,\sigma_1)$, $(\Theta_2,\Lambda_2,\sigma_2)\ \widetilde{\widetilde{\in}}\ \mathrm{BS}(\Omega_\varsigma)$, then

(1) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}((\Phi,\widetilde{\widetilde{\Omega}},\varsigma))\widetilde{\widetilde{\supseteq}}(\Phi,\widetilde{\widetilde{\Omega'}},\varsigma')$. The equality holds when ϑ is surjective mapping.

(2) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}((\widetilde{\widetilde{\Omega}},\Phi,\varsigma))\widetilde{\widetilde{\subseteq}}(\widetilde{\widetilde{\Omega'}},\Phi,\varsigma')$.

(3) If $(\Theta_1,\Lambda_1,\sigma_1)\tilde{\tilde{\subseteq}}(\Theta_2,\Lambda_2,\sigma_2)$, then $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1)\tilde{\tilde{\subseteq}}\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2)$.

(4) $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}((\Theta_1,\Lambda_1,\sigma_1)\tilde{\tilde{\cup}}(\Theta_2,\Lambda_2,\sigma_2)) = \tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1)\tilde{\tilde{\cup}}\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2)$.

(5) $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}((\Theta_1,\Lambda_1,\sigma_1)\tilde{\tilde{\cap}}(\Theta_2,\Lambda_2,\sigma_2)) \tilde{\tilde{\subseteq}} \tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1)\tilde{\tilde{\cap}}\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2)$. The equality holds if $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS injective mapping.

Proposition 2.2[17] Let $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}: \mathrm{BS}(\Omega_\varsigma) \longrightarrow \mathrm{BS}(\Omega'_{\varsigma'})$ be a BS mapping, where $\vartheta:\Omega\longrightarrow\Omega'$ is an injective mapping, $\mu:\varsigma\longrightarrow\varsigma'$ and $\rho:\neg\varsigma\longrightarrow\neg\varsigma'$ are two mappings such as $\rho(\neg\rho)=\neg\mu(\rho)$ for all $\neg\rho\in\neg\varsigma$. If $(\chi_1,\psi_1,\sigma'_1)$, $(\chi_2,\psi_2,\sigma'_2)$ $\tilde{\tilde{\in}}$ $\mathrm{BS}(\Omega'_{\varsigma'})$, then

(1) $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\Phi,\tilde{\tilde{\Omega'}},\varsigma')) = (\Phi,\tilde{\tilde{\Omega}},\varsigma)$.

(2) $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\tilde{\tilde{\Omega'}},\Phi,\varsigma')) = (\tilde{\tilde{\Omega}},\Phi,\varsigma)$.

(3) If $(\chi_1,\psi_1,\sigma'_1)$ $\tilde{\tilde{\subseteq}}$ $(\chi_2,\psi_2,\sigma'_2)$, then $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\sigma'_1)$ $\tilde{\tilde{\subseteq}}$ $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2,\psi_2,\sigma'_2)$.

(4) $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\chi_1,\psi_1,\sigma'_1)\ \tilde{\tilde{\cup}}\ (\chi_2,\psi_2,\sigma'_2)) = \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\sigma'_1)\ \tilde{\tilde{\cup}}\ \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2,\psi_2,\sigma'_2)$.

(5) $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\chi_1,\psi_1,\sigma'_1)\ \tilde{\tilde{\cap}}\ (\chi_2,\psi_2,\sigma'_2)) = \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\sigma'_1)\ \tilde{\tilde{\cap}}\ \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2,\psi_2,\sigma'_2)$.

(6) $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\chi_1,\psi_1,\varsigma')^c) = (\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\varsigma'))^c$.

Proposition 2.3[17] Let $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}: \mathrm{BS}(\Omega_\varsigma) \longrightarrow \mathrm{BS}(\Omega'_{\varsigma'})$ be a BS mapping, where $\vartheta:\Omega\longrightarrow\Omega'$ is an injective mapping, $\mu:\varsigma\longrightarrow\varsigma'$ and $\rho:\neg\varsigma\longrightarrow\neg\varsigma'$ are two mappings such as $\rho(\neg\rho)=\neg\mu(\rho)$ for all $\neg\rho\in\neg\varsigma$. If (Θ,Λ,σ) $\tilde{\tilde{\in}}$ $\mathrm{BS}(\Omega_\varsigma)$ and (χ,ψ,σ') $\tilde{\tilde{\in}}$ $\mathrm{BS}(\Omega'_{\varsigma'})$, then

(1) (Θ,Λ,σ) $\tilde{\tilde{\subseteq}}$ $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\sigma))$. The equality holds if $\sigma=\varsigma$ and $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS injective mapping.

(2) If ϑ is a surjective (bijective) mapping, then $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))$ $\tilde{\tilde{\subseteq}}$ (χ,ψ,ς'). The equality holds if $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective mapping.

2.3 Bipolar soft generalized topology

This subsection presented the concepts of bipolar soft generalized topology, and it investigated some properties and theorems related to this work.

Definition 2.19[35] Let $\tilde{\tilde{g}}$ be a collection of bipolar soft subsets on Ω, then $\tilde{\tilde{g}}$ is said to be a bipolar soft generalized topology (BSGT) on Ω if it satisfies the following conditions:

(1) $(\Phi,\tilde{\tilde{\Omega}},\varsigma)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{g}}$.

(2) If $(\Theta_j,\Lambda_j,\varsigma)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{g}}$ for all $j\in\mathcal{J}$, then $\tilde{\tilde{\bigcup}}_{j\in\mathcal{J}}(\Theta_j,\Lambda_j,\varsigma)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{g}}$.

Then, $(\Omega,\tilde{\tilde{g}},\varsigma,\neg\varsigma)$ is called a BSGTS on Ω. The members of $\tilde{\tilde{g}}$ are said to be bipolar soft $\tilde{\tilde{g}}$-open (BS $\tilde{\tilde{g}}$-open) sets in Ω. The complement of a BS $\tilde{\tilde{g}}$-open set is said to be BS $\tilde{\tilde{g}}$-closed.

We can notice that $(\Phi,\tilde{\tilde{\Omega}},\varsigma)$ is a BS $\tilde{\tilde{g}}$-open, but $(\tilde{\tilde{\Omega}},\Phi,\varsigma)$ needs not be BS $\tilde{\tilde{g}}$-open.

Definition 2.20[35] A BSGT $\tilde{\tilde{g}}$ is said to be strong if $(\tilde{\tilde{\Omega}},\Phi,\varsigma)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{g}}$.

Proposition 2.4[35] Every BSTS on Ω is a BSGTS on Ω.

Theorem 2.1[35] Let $(\Omega,\tilde{\tilde{g}},\varsigma,\neg\varsigma)$ be a BSGTS, then $\tilde{g} = \{(\Theta,\varsigma): (\Theta,\Lambda,\varsigma)\ \tilde{\tilde{\in}}\ \tilde{\tilde{g}}\}$ is soft generalized topology (SGT).

Theorem 2.2[35] Let $(\Omega,\widetilde{g},\varsigma)$ be an SGT. Then the collection $\widetilde{\widetilde{g}}$ consisting of BSSs $(\Theta,\Lambda,\varsigma)$ such that $(\Theta,\varsigma)\ \widetilde{\in}\ \widetilde{g}$ and $\Lambda(\neg\rho)=\Omega\setminus\Theta(\rho)$ for all $\neg\rho\in\neg\varsigma$ define a BSGT on Ω.

Proposition 2.5[35] Let $(\Omega,\widetilde{\widetilde{g}},\rho,\neg\rho)$ be a BSGTS and θ be a non-empty subset of Ω. Then $\widetilde{\widetilde{g}}_\theta=\{({}^\theta\Lambda,{}^\theta\Theta,\rho):(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{g}}\}$ is a BSGTS on Ω.

Theorem 2.3[35] Let $(\Omega,\widetilde{g},\rho)$ be an SGT. Then $\widetilde{\widetilde{g}}$ is the class including BSSs (Λ,Θ,ρ) in which $(\Lambda,\rho)\in\widetilde{g}$ and $\Theta(\neg\rho)=\Omega\setminus\Lambda(\varsigma)$ for all $\varsigma\in\rho$ and $\neg\varsigma\in\neg\rho$ define a BSGT on Ω.

Definition 2.21[36] Let $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ be a BSGTS and $\widetilde{\widetilde{\mathfrak{B}}}\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{g}}$. Then $\widetilde{\widetilde{\mathfrak{B}}}$ is called a bipolar soft generalized basis for the BSGT $\widetilde{\widetilde{g}}$, denoted by BSGB, if every element in $\widetilde{\widetilde{g}}$ can be written as the bipolar soft union of elements of $\widetilde{\widetilde{\mathfrak{B}}}$.

Definition 2.22[36] Let $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ be a BSGTS and $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\in}}\ \mathrm{BSS}(\Omega)$. Then the collection

$$\widetilde{\widetilde{g}}_{(\Theta,\Lambda,\varsigma)}=\{(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}(\Theta_i,\Lambda_i,\varsigma):(\Theta_i,\Lambda_i,\varsigma)\widetilde{\widetilde{\in}}\widetilde{\widetilde{g}},i\in\mathcal{I}\},$$

is said to be a bipolar soft generalized subspace (BSGSS) on $(\Theta,\Lambda,\varsigma)$.

Remark 2.1[36] $\widetilde{\widetilde{g}}_{(\Theta,\Lambda,\varsigma)}$ is a BSGSS on $(\Theta,\Lambda,\varsigma)$.

Theorem 2.4[36] Let $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ be a BSGTS and $\widetilde{\widetilde{\mathfrak{B}}}\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{g}}$. Then

(1) The family $\widetilde{\widetilde{\mathfrak{B}}}$ is a BSGB of $\widetilde{\widetilde{g}}$ if and only if there exists $\widetilde{\widetilde{\mathfrak{B}}}_{\pi_v^\rho}\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}$ such that $\pi_v^\rho\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}_{\pi_v^\rho}\ \widetilde{\widetilde{\subseteq}}\ (\Theta,\Lambda,\varsigma)$ for every $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{g}}$ and every $\pi_v^\rho\ \widetilde{\widetilde{\in}}\ (\Theta,\Lambda,\varsigma)$.

(2) If the family $\widetilde{\widetilde{\mathfrak{B}}}=\{\mathfrak{B}_i\}_{i\in\mathfrak{I}}$ is a BSGB of $\widetilde{\widetilde{g}}$, then there exists $\mathfrak{B}_{i_3}\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}$ such that $\pi_v^\rho\ \widetilde{\widetilde{\in}}\ \mathfrak{B}_{i_3}\ \widetilde{\widetilde{\subseteq}}\ \mathfrak{B}_{i_1}\ \widetilde{\widetilde{\cap}}\ \mathfrak{B}_{i_2}$ for every $\mathfrak{B}_{i_1},\mathfrak{B}_{i_2}\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}$ and every $\pi_v^\rho\ \widetilde{\widetilde{\in}}\ \mathfrak{B}_{i_1}\ \widetilde{\widetilde{\cap}}\ \mathfrak{B}_{i_2}$.

Definition 2.23[36] Let $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ be a BSGTs defined on Ω and $\pi_v^\rho\ \widetilde{\widetilde{\in}}\ \mathrm{BSP}(\Omega)_{(\varsigma,\neg\varsigma)}$. Then, π_v^ρ is called BS $\widetilde{\widetilde{g}}$-limit point of $(\Theta,\Lambda,\varsigma)$ if for every $(\chi,\psi,\varsigma)\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{g}}$ such that $\pi_v^\rho\ \widetilde{\widetilde{\in}}\ (\chi,\psi,\varsigma)$, we have

$$(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\cap}}((\chi,\psi,\varsigma)\setminus(\pi_v^\rho))\neq(\Phi,\Lambda,\varsigma).$$

3 Bipolar Soft Generalized Compact Space

In this section, another significant property of BSGTS called a bipolar soft generalized compactness of BSGTS. Some results of bipolar soft generalized compact spaces and bipolar soft generalized compact sets are derived. Further, we show that the finite bipolar soft union of bipolar soft generalized compact sets is also bipolar soft generalized compact set, and any bipolar soft generalized closed subset of bipolar soft generalized compact set is also bipolar soft generalized compact.

Remark 3.1 Throughout this section, we assume that BSGTSs are strong BSGTSs.

Definition 3.1 A collection $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma):(\Theta_\gamma,\Lambda_\gamma,\varsigma)\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{g}}\}_{\gamma\in\Gamma}$ of BS $\widetilde{\widetilde{g}}$-open sets on Ω is said to be a BS $\widetilde{\widetilde{g}}$-open cover of a BSS $(\Theta,\Lambda,\varsigma)$ if

$$(\Theta,\Lambda,\varsigma)\widetilde{\widetilde{\subseteq}}\widetilde{\widetilde{\cup}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma).$$

Furthermore, a BS subcover is a subcollection of $\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$, which is also a BS $\widetilde{\widetilde{g}}$-open cover.

Definition 3.2 A bipolar soft subset $(\Theta,\Lambda,\varsigma)$ of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ is called a bipolar soft generalized compact set, which is denoted by BS $\widetilde{\widetilde{g}}$-compact set, if each BS $\widetilde{\widetilde{g}}$-open cover of $(\Theta, \Lambda, \varsigma)$ has a finite BS subcover. A BSGTS $(\Omega, \widetilde{\widetilde{g}}, \varsigma, \neg\varsigma)$ is said to be BS $\widetilde{\widetilde{g}}$-compact space if $(\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact subset of itself.

Example 3.1 Let $\Omega = \mathbb{N}$ be the set of natural numbers, $\varsigma = \{\rho_1, \rho_2, \rho_3, \ldots\}$, $\Pi = \{(\Theta_n, \Lambda_n, \varsigma) : n \in \mathbb{N}\}$ be a collection of BS $\widetilde{\widetilde{g}}$-open and let $\widetilde{\widetilde{\mathfrak{B}}} = \{(\rho_1, \{1, n+1\}, \mathbb{N} \setminus \{1, n+1\}), (\rho_2, \{1, 2, n+2\}, \mathbb{N} \setminus \{1, 2, n+2\}), (\rho_3, \{1, 2, 3, n+3\}, \mathbb{N} \setminus \{1, 2, 3, n+3\}), (\rho_4, \{1, 2, 3, 4, n+4\}, \mathbb{N} \setminus \{1, 2, 3, 4, n+4\}), \ldots : n \in \mathbb{N}, n \neq 1\}$ for each $\rho \in \varsigma$. Consider a BSGT $\widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}$ generated on BSGTS $(\Pi, \widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}, \varsigma, \neg\varsigma)$ by the bipolar soft basis $\widetilde{\widetilde{\mathfrak{B}}}$. Then, a BSGTS $(\Pi, \widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}, \varsigma, \neg\varsigma)$ is not BS $\widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}$-compact, we get

$$(\Theta_1, \Lambda_1, \varsigma) = \{(\rho_1, \{1,2\}, \mathbb{N} \setminus \{1,2\}), (\rho_2, \{1,2,3\}, \mathbb{N} \setminus \{1,2,3\}), (\rho_3, \{1,2,3,4\}, \mathbb{N} \setminus \{1,2,3,4\}), \ldots\},$$
$$(\Theta_2, \Lambda_2, \varsigma) = \{(\rho_1, \{1,3\}, \mathbb{N} \setminus \{1,3\}), (\rho_2, \{1,2,4\}, \mathbb{N} \setminus \{1,2,4\}), (\rho_3, \{1,2,3,5\}, \mathbb{N} \setminus \{1,2,3,5\}), \ldots\},$$
$$(\Theta_3, \Lambda_3, \varsigma) = \{(\rho_1, \{1,4\}, \mathbb{N} \setminus \{1,4\}), (\rho_2, \{1,2,5\}, \mathbb{N} \setminus \{1,2,5\}), (\rho_3, \{1,2,3,6\}, \mathbb{N} \setminus \{1,2,3,6\}), \ldots\},$$
$$\vdots$$
$$(\Theta_n, \Lambda_n, \varsigma) = \{(\rho_1, \{1, n+1\}, \mathbb{N} \setminus \{1, n+1\}), (\rho_2, \{1, 2, n+2\}, \mathbb{N} \setminus \{1, 2, n+2\}), (\rho_3, \{1, 2, 3, n+3\}, \mathbb{N} \setminus \{1, 2, 3, n+3\}), \ldots\},$$

form a BS $\widetilde{\widetilde{g}}$-open cover of BSGTS $(\Pi, \widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}, \varsigma, \neg\varsigma)$ with no finite BS subcover.

Remark 3.2 It was pointed out that a BS $\widetilde{\widetilde{g}}$-closed subset of a BS $\widetilde{\widetilde{g}}$-compact space is also a BS $\widetilde{\widetilde{g}}$-compact but a BS $\widetilde{\widetilde{g}}$-closed subset of a BSGTS is not necessary true as shown in the following example:

Example 3.2 Let $\Omega = \mathbb{R}$ be the set of real numbers, $\varsigma = \{\rho_1, \rho_2, \rho_3, \ldots\}$, $\Pi = \{(\Theta_n, \Lambda_n, \varsigma) : n \in \mathbb{N}\}$ and let $\widetilde{\widetilde{\mathfrak{B}}} = \{(\rho, V, \mathbb{R} \setminus V) : V \subseteq \mathbb{R}, V^c \text{ is a countable }\}$ for each $\rho \in \varsigma$. Consider a BSGT $\widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}$ generated on a BSGTS $(\Pi, \widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}, \varsigma, \neg\varsigma)$ by the bipolar soft basis $\widetilde{\widetilde{\mathfrak{B}}}$. Clearly, a BSGTS $(\Pi, \widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathfrak{B}}}}, \varsigma, \neg\varsigma)$ is not a BS $\widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathcal{B}}}}$-compact space, since the only BS $\widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathcal{B}}}}$-open cover is $(\widetilde{\widetilde{\Pi}}, \Phi, \varsigma)$ itself, which has no finite BS subcover.

Now, let $(\Theta_\gamma, \Lambda_\gamma, \varsigma) = \{(\rho_i, \mathbb{N}, \mathbb{R} \setminus \mathbb{N}) : i \in \mathbb{N}\}$ be a BS $\widetilde{\widetilde{g}}$-closed but not a BS $\widetilde{\widetilde{g}}$-open set and let induced BSGT on $(\Theta_\gamma, \Lambda_\gamma, \varsigma)$ be generated by the bipolar soft basis $\widetilde{\widetilde{\mathcal{H}}} = \{(\rho, \{n\}, \mathbb{N} \setminus \{n\}) : n \in \mathbb{N}\}$ for each $\rho \in \varsigma$. Then a BSGTSS $((\Theta_\gamma, \Lambda_\gamma, \varsigma), \widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathcal{H}}}}, \varsigma, \neg\varsigma)$ is not BS $\widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathcal{H}}}}$-compact, since a collection $\{(\rho_i, \mathbb{N}, \mathbb{R} \setminus \mathbb{N}) : i \in \mathbb{N}\}$, where

$$(\Theta_1, \Lambda_1, \varsigma) = \{(\rho_1, \{1\}, \mathbb{N} \setminus \{1\}), (\rho_2, \{1,2\}, \mathbb{N} \setminus \{1,2\}), (\rho_3, \{1,2,3\}, \mathbb{N} \setminus \{1,2,3\}), \ldots\},$$
$$(\Theta_2, \Lambda_2, \varsigma) = \{(\rho_1, \{2\}, \mathbb{N} \setminus \{2\}), (\rho_2, \{1,3\}, \mathbb{N} \setminus \{1,3\}), (\rho_3, \{1,2,4\}, \mathbb{N} \setminus \{1,2,4\}), \ldots\},$$
$$(\Theta_3, \Lambda_3, \varsigma) = \{(\rho_1, \{3\}, \mathbb{N} \setminus \{3\}), (\rho_2, \{1,4\}, \mathbb{N} \setminus \{1,4\}), (\rho_3, \{1,2,5\}, \mathbb{N} \setminus \{1,2,5\}), \ldots\},$$
$$\vdots$$
$$(\Theta_n, \Lambda_n, \varsigma) = \{(\rho_1, \{n\}, \mathbb{N} \setminus \{n\}), (\rho_2, \{1, n+1\}, \mathbb{N} \setminus \{1, n+1\}), (\rho_3, \{1, 2, n+2\}, \mathbb{N} \setminus \{1, 2, n+2\}), \ldots\},$$

is a BS $\widetilde{\widetilde{g}}_{\widetilde{\widetilde{\mathcal{H}}}}$-open cover of $(\Theta_\gamma, \Lambda_\gamma, \varsigma)$ with no finite BS subcover.

Proposition 3.1 If $(\Omega, \widetilde{\widetilde{g}}, \varsigma, \neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact space, then $(\Omega, \widetilde{g}, \varsigma)$ is a $\mathcal{S}$ $\widetilde{g}$-compact space.

Proof Straightforward.

Proposition 3.2 If $(\Omega, \widetilde{g}, \varsigma)$ is a $\mathcal{S}$ $\widetilde{g}$-compact space and $(\Omega, \widetilde{\widetilde{g}}, \varsigma, \neg\varsigma)$ is a BSGTS constructed since Theorem 2.2, then $(\Omega, \widetilde{\widetilde{g}}, \varsigma, \neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact space.

Proof Let $(\Omega,\widetilde{g},\varsigma)$ be a $\mathcal{S}$ $\widetilde{g}$-compact space and $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ be a BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. That is

$$(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)\widetilde{\widetilde{\subseteq}}\widetilde{\widetilde{\cup}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma).$$

Then, $\Omega=\cup\{\Theta_\gamma(\rho)\}_{\gamma\in\Gamma}$ for all $\rho\in\varsigma$. Since $(\Omega,\widetilde{g},\varsigma)$ is a $\mathcal{S}$ $\widetilde{g}$-compact space, thus $\Omega=\cup\{\Theta_{\gamma_i}(\rho):i=1,2,\ldots,n\}_{\gamma_i\in\Gamma}$. Since $\Lambda(\neg\rho)=\Omega\setminus\Theta(\rho)$ for all $\rho\in\varsigma$, so $\Phi=\cap\{\Lambda_{\gamma_i}(\neg\rho):i=1,2,\ldots,n\}_{\gamma_i\in\Gamma}$. Hence, $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\cup}}_{i=1}^{n}\ (\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)$. Therefore, $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ is BS $\widetilde{\widetilde{g}}$-compact.

Theorem 3.1 In BSGTS $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$, the finite bipolar soft unions of BS $\widetilde{\widetilde{g}}$-compact sets is BS $\widetilde{\widetilde{g}}$-compact.

Proof Let $(\Theta_k,\Lambda_k,\varsigma)$, $(\Theta_h,\Lambda_h,\varsigma)$ be any two BS $\widetilde{\widetilde{g}}$-compact subset of $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$. Let Ψ be a BS $\widetilde{\widetilde{g}}$-open cover of $(\Theta_k,\Lambda_k,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\Theta_h,\Lambda_h,\varsigma)$. Then, Ψ will also be a BS $\widetilde{\widetilde{g}}$-open cover of both $(\Theta_k,\Lambda_k,\varsigma)$ and $(\Theta_h,\Lambda_h,\varsigma)$. So, by assumption, there exists a finite subcollection of Ψ of BS $\widetilde{\widetilde{g}}$-open sets, say, $\{(\Theta_{k_i},\Lambda_{k_i},\varsigma)\}_{i=1}^{n}$ and $\{(\Theta_{h_j},\Lambda_{h_j},\varsigma)\}_{j=1}^{m}$ are finite BS subcover for $(\Theta_k,\Lambda_k,\varsigma)$ and $(\Theta_h,\Lambda_h,\varsigma)$, respectively. Then, clearly that $\{(\Theta_{k_i},\Lambda_{k_i},\varsigma)\}_{i=1}^{n}\ \widetilde{\widetilde{\cup}}\ \{(\Theta_{h_j},\Lambda_{h_j},\varsigma)\}_{j=1}^{m}$ is a finite subcollection of Ψ of BS $\widetilde{\widetilde{g}}$-open sets. Thus, it is a finite BS subcover for $(\Theta_k,\Lambda_k,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\Theta_h,\Lambda_h,\varsigma)$. Therefore, $(\Theta_k,\Lambda_k,\varsigma)\ \widetilde{\widetilde{\cup}}\ (\Theta_h,\Lambda_h,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact. By induction, every finite bipolar soft unions of BS $\widetilde{\widetilde{g}}$-compact sets is BS $\widetilde{\widetilde{g}}$-compact.

Remark 3.3 The following example shows that the Theorem 3.1 for infinite bipolar soft union of BS $\widetilde{\widetilde{g}}$-compact sets in general dose not hold.

Example 3.3 Let $\Omega=\mathbb{N}$, $\varsigma=\{\rho_1,\rho_2\}$ and let $\widetilde{\widetilde{g}}$ be a BSGTS over $\mathbb{N}$ consisting of all BSSs, generated by the BSSs.

$$\begin{aligned}
(\Theta_1,\Lambda_1,\varsigma)&=\{(\rho_1,\{1,2\},\mathbb{N}\setminus\{1,2\}),(\rho_2,\{1,2\},\mathbb{N}\setminus\{1,2\})\},\\
(\Theta_2,\Lambda_2,\varsigma)&=\{(\rho_1,\{1,3\},\mathbb{N}\setminus\{1,3\}),(\rho_2,\{1,3\},\mathbb{N}\setminus\{1,3\})\},\\
(\Theta_3,\Lambda_3,\varsigma)&=\{(\rho_1,\{1,4\},\mathbb{N}\setminus\{1,4\}),(\rho_2,\{1,4\},\mathbb{N}\setminus\{1,4\})\},\\
&\vdots\\
(\Theta_n,\Lambda_n,\varsigma)&=\{(\rho_1,\{1,n+1\},\mathbb{N}\setminus\{1,n+1\}),(\rho_2,\{1,n+1\},\mathbb{N}\setminus\{1,n+1\})\}.
\end{aligned}$$

Now, let $\widetilde{\widetilde{\mathfrak{B}}}=\{(\rho_1,\{n,n+1\},\mathbb{N}\setminus\{n,n+1\}),(\rho_2,\{n,n+1\},\mathbb{N}\setminus\{n,n+1\})\}$ be a collection of BS $\widetilde{\widetilde{g}}$-compact sets in BSGTS $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$. But the infinite bipolar soft union of the collection $\widetilde{\widetilde{\mathfrak{B}}}$ is not BS $\widetilde{\widetilde{g}}$-compact.

Theorem 3.2 Let $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ be a BSGTS on Ω and $(\Theta,\Lambda,\varsigma)$ be a BSS. Then, $(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact set if and only if every BS $\widetilde{\widetilde{g}}$-open cover of $(\Theta,\Lambda,\varsigma)$ has a finite BS subcover in $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$.

Proof Let $(\Theta,\Lambda,\varsigma)$ be a BS $\widetilde{\widetilde{g}}$-compact subset of $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ and the family $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ be a BS $\widetilde{\widetilde{g}}$-open cover of $(\Theta,\Lambda,\varsigma)$ in $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$. Then, $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\cup}}_{\gamma\in\Gamma}\ (\Theta_\gamma,\Lambda_\gamma,\varsigma)$. Now the family $\{(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\cap}}\ (\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ is a BS $\widetilde{\widetilde{g}}_{(\Theta,\Lambda,\varsigma)}$-open cover of $\left((\Theta,\Lambda,\varsigma),\widetilde{\widetilde{g}}_{(\Theta,\Lambda,\varsigma)},\varsigma,\neg\varsigma\right)$. Since $(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact, there exists a finite BS subcover $\{(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\cap}}\ (\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)\}_{i=1}^{n}$. Therefore, $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\cup}}_{i=1}^{n}\ (\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)$ is obtained.

Conversely it is clear.

Theorem 3.3 Let $(\Omega,\widetilde{\widetilde{g}}_1,\varsigma,\neg\varsigma)$ and $(\Omega,\widetilde{\widetilde{g}}_2,\varsigma,\neg\varsigma)$ be BSGTSs. Then

(1) If $(\Omega,\widetilde{\widetilde{g}}_2,\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}_2$-compact space on Ω and $\widetilde{\widetilde{g}}_1 \mathrel{\widetilde{\widetilde{\subseteq}}} \widetilde{\widetilde{g}}_2$. Then $(\Omega,\widetilde{\widetilde{g}}_1,\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}_1$-compact space on Ω.

(2) If $(\Omega,\widetilde{\widetilde{g}}_1,\varsigma,\neg\varsigma)$ is not BS $\widetilde{\widetilde{g}}_1$-compact space on Ω and $\widetilde{\widetilde{g}}_1 \mathrel{\widetilde{\widetilde{\subseteq}}} \widetilde{\widetilde{g}}_2$. Then $(\Omega,\widetilde{\widetilde{g}}_2,\varsigma,\neg\varsigma)$ is also not BS $\widetilde{\widetilde{g}}_2$-compact space on Ω.

Proof

(1) Let $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ be a BS $\widetilde{\widetilde{g}}_1$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ in $(\Omega,\widetilde{\widetilde{g}}_1,\varsigma,\neg\varsigma)$. Since $\widetilde{\widetilde{g}}_1 \mathrel{\widetilde{\widetilde{\subseteq}}} \widetilde{\widetilde{g}}_2$, then $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ is the BS $\widetilde{\widetilde{g}}_2$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ by the BS $\widetilde{\widetilde{g}}_2$-open sets of $(\Omega,\widetilde{\widetilde{g}}_2,\varsigma,\neg\varsigma)$. Since $(\Omega,\widetilde{\widetilde{g}}_2,\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}_2$-compact space. Thus,

$$(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)\mathrel{\widetilde{\widetilde{\subseteq}}}\widetilde{\widetilde{\bigcup}}_{\gamma=1}^{n}(\Theta_\gamma,\Lambda_\gamma,\varsigma),\ \text{for some } \gamma_1,\gamma_2,\ldots,\gamma_n\in\Gamma.$$

Therefore, $(\Omega,\widetilde{\widetilde{g}}_1,\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}_1$-compact space.

(2) The proof is similar to part (1).

Theorem 3.4 A BSGTS $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ is BS $\widetilde{\widetilde{g}}$-compact if and only if there exists a bipolar soft basis $\widetilde{\widetilde{\mathfrak{B}}}$ for $\widetilde{\widetilde{g}}$ such that every BS $\widetilde{\widetilde{g}}$-open cover of $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ by members of $\widetilde{\widetilde{\mathfrak{B}}}$ has a finite BS subcover.

Proof Suppose that $\widetilde{\widetilde{\mathfrak{B}}}$ is a bipolar soft basis for $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ with the property that every BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ by members of $\widetilde{\widetilde{\mathfrak{B}}}$ has a finite BS subcover. Now let $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ be a BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$, not necessarily by members of $\widetilde{\widetilde{\mathfrak{B}}}$. For each BSS $(\Theta_\gamma,\Lambda_\gamma,\varsigma)\mathrel{\widetilde{\widetilde{\in}}}\Psi$, there exists a subcollection $\widetilde{\widetilde{\mathfrak{B}}}_{(\Theta_\gamma,\Lambda_\gamma,\varsigma)}$ of $\widetilde{\widetilde{\mathfrak{B}}}$ such that

$$(\Theta_\gamma,\Lambda_\gamma,\varsigma)=\widetilde{\widetilde{\bigcup}}_{(\Theta_\alpha,\Lambda_\alpha,\varsigma)\widetilde{\widetilde{\in}}\widetilde{\widetilde{\mathfrak{B}}}_{(\Theta_\gamma,\Lambda_\gamma,\varsigma)}}(\Theta_\alpha,\Lambda_\alpha,\varsigma).$$

Now, let $\zeta=\widetilde{\widetilde{\bigcup}}_{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\widetilde{\widetilde{\in}}\Psi}\widetilde{\widetilde{\mathfrak{B}}}_{(\Theta_\gamma,\Lambda_\gamma,\varsigma)}$. Clearly, ζ is a BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ by members of $\widetilde{\widetilde{\mathfrak{B}}}$, since Ψ is a BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ and $\zeta\mathrel{\widetilde{\widetilde{\subseteq}}}\Psi$. By hypothesis, then there is a finite subcollection, say, $\{(\Theta_{\beta_1},\Lambda_{\beta_1},\varsigma),(\Theta_{\beta_2},\Lambda_{\beta_2},\varsigma),\ldots,(\Theta_{\beta_n},\Lambda_{\beta_n},\varsigma)\}$ of ζ which a finite BS subcover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Therefore, for each $k=1,2,\ldots,n$, there exists $(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)\mathrel{\widetilde{\widetilde{\in}}}\Psi$ such that $(\Theta_{\beta_k},\Lambda_{\beta_k},\varsigma)\mathrel{\widetilde{\widetilde{\in}}}\widetilde{\widetilde{\mathfrak{B}}}_{(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)}$. So, clearly, $(\Theta_{\beta_k},\Lambda_{\beta_k},\varsigma)\mathrel{\widetilde{\widetilde{\subseteq}}}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)$. Thus $\{(\Theta_{\gamma_1},\Lambda_{\gamma_1},\varsigma),(\Theta_{\gamma_2},\Lambda_{\gamma_2},\varsigma),\ldots,(\Theta_{\gamma_n},\Lambda_{\gamma_n},\varsigma)\}$ is a finite subcollection of Ψ of BS $\widetilde{\widetilde{g}}$-open sets which BS subcovers of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Therefore, $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact. Conversely is obvious.

Theorem 3.5 Let $(\Pi,\widetilde{\widetilde{g}}_\Pi,\varsigma,\neg\varsigma)$ be a BSGTSS of $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$. Then $(\Pi,\widetilde{\widetilde{g}}_\Pi,\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}_\Pi$-compact space if and only if every BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Pi}},\Phi,\varsigma)$ by BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ contains a finite BS subcover.

Proof Let $(\Pi,\widetilde{\widetilde{g}}_\Pi,\varsigma,\neg\varsigma)$ be a BS $\widetilde{\widetilde{g}}_\Pi$-compact space and $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ be a BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Pi}},\Phi,\varsigma)$ by BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$. Now, $\Pi\subseteq\bigcup_{\gamma\in\Gamma}(\Pi\cap\Theta_\gamma(\rho))$ for each $\rho\in\varsigma$ and

$\varnothing \supseteq \bigcap_{\gamma\in\Gamma} (\Pi \cap \Lambda_\gamma(\neg\rho))$ for each $\neg\rho \in \neg\varsigma$. Thus, $\Psi_\Pi = \{({}^\Pi\Theta_\gamma, {}^\Pi\Lambda_\gamma, \varsigma)\}_{\gamma\in\Gamma}$ is a BS $\tilde{\tilde{g}}_\Pi$-open cover of $(\tilde{\tilde{\Pi}}, \Phi, \varsigma)$. Since $(\Pi, \tilde{\tilde{g}}_\Pi, \varsigma, \neg\varsigma)$ is a BS $\tilde{\tilde{g}}_\Pi$-compact space, then there is a finite BS subcover, say, $\{({}^\Pi\Theta_{\gamma_i}, {}^\Pi\Lambda_\gamma, \varsigma)\}_{i=1}^n$ such that,

$$(\tilde{\tilde{\Pi}}, \Phi, \varsigma) \tilde{\tilde{\subseteq}} \tilde{\tilde{\bigcup}}_{i=1}^{n} ({}^\Pi\Theta_{\gamma_i}, {}^\Pi\Lambda_{\gamma_i}, \varsigma), \text{ for some } \gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma.$$

Thus, implies that $\{(\Theta_{\gamma_i}, \Lambda_{\gamma_i}, \varsigma)\}_{i=1}^n$ is a finite BS subcover of $(\tilde{\tilde{\Pi}}, \Phi, \varsigma)$ by BS $\tilde{\tilde{g}}$-open set in $(\Omega, \tilde{\tilde{g}}, \varsigma, \neg\varsigma)$.

Conversely, suppose that $\Psi_\Pi = \{({}^\Pi\Theta_\gamma, {}^\Pi\Lambda_\gamma, \varsigma)\}_{\gamma\in\Gamma}$ which is a BS $\tilde{\tilde{g}}_\Pi$-open cover of $(\tilde{\tilde{\Pi}}, \Phi, \varsigma)$. Then, clearly, $\Psi = \{(\Theta_\gamma, \Lambda_\gamma, \varsigma)\}_{\gamma\in\Gamma}$ is a BS $\tilde{\tilde{g}}$-open cover of $(\tilde{\tilde{\Pi}}, \Phi, \varsigma)$ by BS $\tilde{\tilde{g}}$-open set in $(\Omega, \tilde{\tilde{g}}, \varsigma, \neg\varsigma)$. Thus, by given hypothesis we have, $\{(\Theta_{\gamma_i}, \Lambda_\gamma, \varsigma)\}_{i=1}^n$ is a finite BS subcover of $(\tilde{\tilde{\Pi}}, \Phi, \varsigma)$. Therefore, $(\Pi, \tilde{\tilde{g}}_\Pi, \varsigma, \neg\varsigma)$ is a BS $\tilde{\tilde{g}}_\Pi$-compact space.

Definition 3.3 Let $(\Omega, \tilde{\tilde{g}}, \varsigma, \neg\varsigma)$ be a BSGTS on Ω and $\Psi = \{(\Theta_\gamma, \Lambda_\gamma, \varsigma)\}_{\gamma\in\Gamma}$ be a collection of BSSs. If bipolar soft intersection of every finite subcollection of Ψ is different since null BSS, then Ψ is called a $\tilde{\tilde{g}}$-centralized bipolar soft ($\tilde{\tilde{g}}$-centralized BS) collection.

Theorem 3.6 A BSGTS $(\Omega, \tilde{\tilde{g}}, \varsigma, \neg\varsigma)$ on Ω is BS $\tilde{\tilde{g}}$-compact if and only if every collection of BS $\tilde{\tilde{g}}$-closed sets with null BS intersection in $(\Omega, \tilde{\tilde{g}}, \varsigma, \neg\varsigma)$ has a finite subcollection with null BS intersection.

Proof Let $(\Omega, \tilde{\tilde{g}}, \varsigma, \neg\varsigma)$ be a BS $\tilde{\tilde{g}}$-compact space and the bipolar soft intersection of $\Psi = \{(\Theta_\gamma, \Lambda_\gamma, \varsigma)\}_{\gamma\in\Gamma}$ which is null BSS be a collection of BS $\tilde{\tilde{g}}$-closed sets, such that

$$\tilde{\tilde{\bigcap}}_{\gamma\in\Gamma} (\Theta\gamma, \Lambda_\gamma, \varsigma) = (\Phi, \tilde{\tilde{\Omega}}, \varsigma).$$

Thus,

$$\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma} (\Theta_\gamma, \Lambda_\gamma, \varsigma)^c = (\tilde{\tilde{\Omega}}, \Phi, \varsigma),$$

where $\{(\Theta_\gamma, \Lambda_\gamma, \varsigma)^c\}_{\gamma\in\Gamma}$ is a collection of BS $\tilde{\tilde{g}}$-open cover of $(\tilde{\tilde{\Omega}}, \Phi, \varsigma)$ such that

$$(\tilde{\tilde{\Omega}}, \Phi, \varsigma) = \tilde{\tilde{\bigcup}}_{\gamma\in\Gamma} (\Theta_\gamma, \Lambda_\gamma, \varsigma)^c = (\tilde{\tilde{\bigcap}}_{\gamma\in\Gamma} (\Theta_\gamma, \Lambda_\gamma, \varsigma))^c,$$

is obtained. Since $(\tilde{\tilde{\Omega}}, \Phi, \varsigma)$ is a BS $\tilde{\tilde{g}}$-compact space, then there is a finite subcollection, say, $\{(\Theta_{\gamma_i}, \Lambda_{\gamma_i}, \varsigma)^c\}_{i=1}^n$ is a BS $\tilde{\tilde{g}}$-open cover of $(\tilde{\tilde{\Omega}}, \Phi, \varsigma)$. Therefore, $\tilde{\tilde{\bigcup}}_{i=1}^n (\Theta_{\gamma_i}, \Lambda_{\gamma_i}, \varsigma)^c = (\tilde{\tilde{\Omega}}, \Phi, \varsigma)$. Hence, $\tilde{\tilde{\bigcap}}_{i=1}^n (\Theta_{\gamma_i}, \Lambda_{\gamma_i}, \varsigma) = (\Phi, \tilde{\tilde{\Omega}}, \varsigma)$.

Conversely, let $\Psi = \{(\Theta_{\gamma_i}, \Lambda_{\gamma_i}, \varsigma)^c\}_{\gamma\in\Gamma}$ be a BS $\tilde{\tilde{g}}$-open cover of $(\tilde{\tilde{\Omega}}, \Phi, \varsigma)$. Then, the bipolar soft intersection of BS $\tilde{\tilde{g}}$-closed sets is null BSS such that

$$\tilde{\tilde{\bigcap}}_{\gamma\in\Gamma} (\Theta_{\gamma_i}, \Lambda_{\gamma_i}, \varsigma) = (\Phi, \tilde{\tilde{\Omega}}, \varsigma).$$

Since the condition of theorem

$$\widetilde{\widetilde{\bigcap}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)=(\Phi,\widetilde{\widetilde{\Omega}},\varsigma),$$

is obtained. Thus,

$$\widetilde{\widetilde{\bigcup}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)^c=(\widetilde{\widetilde{\bigcap}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma))^c=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma).$$

Hence, $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact space.

Theorem 3.7 A BSGTS $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ is BS $\widetilde{\widetilde{g}}$-compact space if and only if the bipolar soft intersection of every $\widetilde{\widetilde{g}}$-centralized BS $\widetilde{\widetilde{g}}$-closed sets collection is different since null BSS.

Proof Suppose that $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact and $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ is a $\widetilde{\widetilde{g}}$-centralized BS $\widetilde{\widetilde{g}}$-closed sets collection. Assume, if possible, $\widetilde{\widetilde{\bigcap}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma)=(\Phi,\widetilde{\widetilde{\Omega}},\varsigma)$. Then, $\widetilde{\widetilde{\bigcup}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma)^c=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Thus, $\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)^c\}_{\gamma\in\Gamma}$ is a BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Since $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact. We have $\widetilde{\widetilde{\bigcup}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)^c=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$, by Theorem 3.6, $\widetilde{\widetilde{\bigcap}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma)=(\Phi,\widetilde{\widetilde{\Omega}},\varsigma)$ implies that $\widetilde{\widetilde{\bigcap}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)=(\Phi,\widetilde{\widetilde{\Omega}},\varsigma)$. Therefore, $\widetilde{\widetilde{\bigcap}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)\neq(\Phi,\widetilde{\widetilde{\Omega}},\varsigma)$ implies that $\widetilde{\widetilde{\bigcap}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma)\neq(\Phi,\widetilde{\widetilde{\Omega}},\varsigma)$.

Conversely, suppose that $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ is not a BS $\widetilde{\widetilde{g}}$-compact space. Then, there is no finite BS subcover of $\Psi=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ in $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$. So, $\widetilde{\widetilde{\bigcup}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)\neq(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. We consider that BS $\widetilde{\widetilde{g}}$-closed sets collection $\Psi^c=\{(\Theta_\gamma,\Lambda_\gamma,\varsigma)^c\}_{i=1}^{n}$. Since

$$\widetilde{\widetilde{\bigcap}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma)^c=(\widetilde{\widetilde{\bigcup}}_{i=1}^{n}(\Theta_{\gamma_i},\Lambda_{\gamma_i},\varsigma))^c\neq(\Phi,\widetilde{\widetilde{\Omega}},\varsigma).$$

Ψ^c is a $\widetilde{\widetilde{g}}$-centralized BS $\widetilde{\widetilde{g}}$-closed sets collection. Since the condition of theorem,

$$\widetilde{\widetilde{\bigcap}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma)^c\neq(\Phi,\widetilde{\widetilde{\Omega}},\varsigma).$$

Hence, $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)=\widetilde{\widetilde{\bigcup}}_{\gamma\in\Gamma}(\Theta_\gamma,\Lambda_\gamma,\varsigma)\neq(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. The assumption is wrong. Therefore, $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact.

Theorem 3.8 Every infinite bipolar soft subset $(\Theta,\Lambda,\varsigma)$ of a BS $\widetilde{\widetilde{g}}$-compact space $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ has at least one BS $\widetilde{\widetilde{g}}$-limit point in $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$.

Proof Suppose that $(\Omega,\widetilde{\widetilde{g}},\varsigma,\neg\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact space and $(\Theta,\Lambda,\varsigma)$ is an infinite bipolar soft subset of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Assume that $(\Theta,\Lambda,\varsigma)$ has no BS $\widetilde{\widetilde{g}}$-limit point in $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Then, for each $\pi_v^\rho\ \widetilde{\widetilde{\in}}\ (\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$, there exists a BS $\widetilde{\widetilde{g}}$-open set (χ,ψ,ς) containing π_v^ρ such that $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\sqcap}}\ (\chi,\psi,\varsigma)=(\pi_v^\rho)$ or $(\Phi,\widetilde{\widetilde{\Omega}},\varsigma)$. Then, the collection $\{(\chi_\gamma,\psi_\gamma,\varsigma)\}_{\gamma\in\Gamma}$ is a BS $\widetilde{\widetilde{g}}$-open cover of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Since $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ is BS $\widetilde{\widetilde{g}}$-compact, there exists a BSPs $\pi_{1v_1}^\rho$, $\pi_{2v_2}^\rho,\ldots,\pi_{nv_n}^\rho\ \widetilde{\widetilde{\in}}\ (\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ such that the bipolar soft union $\widetilde{\widetilde{\bigcup}}_{i=1}^{n}(\chi_{\gamma_i},\psi_{\gamma_i},\varsigma)=(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$, then

$$((\chi_{\gamma_1},\psi_{\gamma_1},\varsigma)\widetilde{\widetilde{\sqcap}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\sqcup}}((\chi_{\gamma_2},\psi_{\gamma_2},\varsigma)\widetilde{\widetilde{\sqcap}}(\Theta,\Lambda,\varsigma))\widetilde{\widetilde{\sqcup}}\ldots\widetilde{\widetilde{\sqcup}}((\chi_{\gamma_n},\psi_{\gamma_n},\varsigma)\widetilde{\widetilde{\sqcap}}(\Theta,\Lambda,\varsigma)),$$

is a finite BSS or null BSS $(\Phi, \tilde{\tilde{\Omega}}, \varsigma)$. But

$$\tilde{\tilde{\cup}}_{i=1}^{n}\big((\chi_{\gamma_i}, \psi_{\gamma_i}, \varsigma)\tilde{\tilde{\cap}}(\Theta, \Lambda, \varsigma)\big) = \big(\tilde{\tilde{\cup}}_{i=1}^{n}(\chi_{\gamma_i}, \psi_{\gamma_i}, \varsigma)\big)\tilde{\tilde{\cap}}(\Theta, \Lambda, \varsigma) = (\tilde{\tilde{\Omega}}, \Phi, \varsigma)\tilde{\tilde{\cap}}(\Theta, \Lambda, \varsigma) = (\Theta, \Lambda, \varsigma),$$

is a finite BSS or null BSS $(\Phi, \tilde{\tilde{\Omega}}, \varsigma)$. This contradicts that $(\Theta, \Lambda, \varsigma)$ is an infinite bipolar soft subset of $(\tilde{\tilde{\Omega}}, \Phi, \varsigma)$.

Therefore, $(\Theta, \Lambda, \varsigma)$ has at least one BS $\tilde{\tilde{g}}$-limit point in $(\tilde{\tilde{\Omega}}, \Phi, \varsigma)$.

4 Further Property of Bipolar Soft Mappings

This section presented further properties related to bipolar soft mappings. The following example shows the definition of the BS mappings that was defined in Ref. [16].

Example 4.1[16] Let $\Omega = \{\pi_1, \pi_2, \pi_3\}$ and $\Omega' = \{\pi'_1, \pi'_2, \pi'_3, \pi'_4\}$ be two sets, $\varsigma = \{\rho_1, \rho_2, \rho_3, \rho_4\}$ and $\varsigma' = \{\rho'_1, \rho'_2, \rho'_3, \rho'_4\}$ be two sets of parameters, let $\vartheta : \Omega \longrightarrow \Omega'$ be a mapping defined as $\vartheta(\pi_i) = \pi'_i$, $i = 1, 2, 3$, and $\mu : \varsigma \longrightarrow \varsigma'$ be defined as $\mu(\rho_1) = \mu(\rho_2) = \rho'_1$, $\mu(\rho_3) = \rho'_3$, $\mu(\rho_4) = \rho'_4$, the mapping $\rho : \neg\varsigma \longrightarrow \neg\varsigma'$ be defined as $\rho(\neg\rho_i) = \neg\mu(\rho_i)$, $i = 1, 2, 3, 4$ and $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_\varsigma) \longrightarrow \mathrm{BS}(\Omega'_{\varsigma'})$ be a BS mapping. Let $(\Theta, \Lambda, \sigma) = \{(\rho_1, \{\pi_1\}, \{\pi_2\}), (\rho_2, \{\pi_3\}, \{\pi_1, \pi_2\}), (\rho_3, \{\pi_3\}, \{\pi_1\})\}$. Then the image of $(\Theta, \Lambda, \sigma)$ under $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$, we have $\mu(\sigma) = \mu(\{\rho_1, \rho_2, \rho_3\}) = \{\rho'_1, \rho'_3\}$.

Now, for $\rho'_1 : \mu^{-1}(\rho'_1) \cap \sigma = \{\rho_1, \rho_2\} \cap \{\rho_1, \rho_2, \rho_3\} = \{\rho_1, \rho_2\}$. Thus,

$$\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta)(\rho'_1) = \vartheta\Big(\bigcup_{\rho \in \mu^{-1}(\rho'_1)\cap\sigma} \Theta(\rho)\Big) = \vartheta\big(\Theta(\rho_1) \cup \Theta(\rho_2)\big) = \vartheta\big(\{\pi_1\} \cup \{\pi_3\}\big) = \{\pi'_1, \pi'_3\}.$$

Thus, $\rho(\neg\sigma) = \{\neg\rho'_1, \neg\rho'_3\}$.

For $\neg\rho'_1 : \rho^{-1}(\neg\rho'_1) \cap \neg\sigma = \{\neg\rho_1, \neg\rho_2\}$. We have

$$\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Lambda)(\neg\rho'_1) = \vartheta\Big(\bigcap_{\neg\rho \in \rho^{-1}(\neg\rho'_1)\cap\neg\sigma} \Lambda(\neg\rho)\Big) = \vartheta\big(\Lambda(\neg\rho_1) \cap \Lambda(\neg\rho_2)\big) = \vartheta\big(\{\pi_2\} \cap \{\pi_1, \pi_2\}\big) = \vartheta\big(\{\pi_2\}\big) = \{\pi'_2\}.$$

We can write $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}\big((\Theta, \Lambda, \sigma)\big)(\rho'_1) = (\rho'_1, \{\pi'_1, \pi'_3\}, \{\pi'_2\})$.

For $\rho'_3 : \mu^{-1}(\rho'_3) \cap \sigma = \{\rho_3\} \cap \{\rho_1, \rho_2, \rho_3\} = \{\rho_3\}$. Thus,

$$\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta)(\rho'_3) = \vartheta\Big(\bigcup_{\rho \in \mu^{-1}(\rho'_3)\cap\sigma} \Theta(\rho)\Big) = \vartheta\big(\Theta(\rho_3)\big) = \vartheta\big(\{\pi_3\}\big) = \{\pi'_3\}.$$

For $\neg\rho'_3 : \rho^{-1}(\neg\rho'_3) \cap \neg\sigma = \{\neg\rho_3\}$. Therefore,

$$\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Lambda)(\neg\rho'_3) = \vartheta\Big(\bigcap_{\neg\rho \in \rho^{-1}(\neg\rho'_3)\cap\neg\sigma} \Lambda(\neg\rho)\Big) = \vartheta\big(\Lambda(\neg\rho_3)\big) = \vartheta\big(\{\pi_1\}\big) = \{\pi'_1\}.$$

Hence, $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}\big((\Theta, \Lambda, \sigma)\big) = \{(\rho'_1, \{\pi'_1, \pi'_3\}, \{\pi'_2\}), (\rho'_2, \varnothing, \Omega'), (\rho'_3, \{\pi'_3\}, \{\pi'_1\}), (\rho'_4, \varnothing, \Omega')\}$.

Remark 4.1 In Ref. [16], Fadel and Dzul-Kifli defined the BS mapping, BS image, and the BS inverse image, and proved some of the properties, as a result, we found the following:

(1) In Proposition 2.1 (5), $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}\big((\Theta_1, \Lambda_1, \sigma_1) \ \tilde{\tilde{\cap}} \ (\Theta_2, \Lambda_2, \sigma_2) = (\Theta, \Lambda, \sigma_1 \cup \sigma_2)\big) \ \tilde{\tilde{\not\subseteq}} \ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1, \Lambda_1, \sigma_1) \ \tilde{\tilde{\cap}}$

$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2, \Lambda_2, \sigma_2)$.

(2) In Proposition 2.2 (5), $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\chi_1, \psi_1, \sigma'_1) \;\widetilde{\widetilde{\cap}}\; (\chi_2, \psi_2, \sigma'_2) = (\chi, \psi, \sigma'_1 \cup \sigma'_2)) \neq \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1, \psi_1, \sigma'_1) \;\widetilde{\widetilde{\cap}}\; \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2, \psi_2, \sigma'_2)$.

Example 4.2 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ be defined as in Example 4.1.

(1) Let $(\Theta_1, \Lambda_1, \sigma_1)$ and $(\Theta_2, \Lambda_2, \sigma_2)$ be two BSSs defined as

$$(\Theta_1, \Lambda_1, \sigma_1) = \{(\rho_1, \{\pi_1\}, \{\pi_2\}), (\rho_2, \{\pi_3\}, \{\pi_1, \pi_2\}), (\rho_3, \{\pi_3\}, \{\pi_1\})\},$$
$$(\Theta_2, \Lambda_2, \sigma_2) = \{(\rho_1, \{\pi_1, \pi_2\}, \varnothing), (\rho_2, \{\pi_3\}, \{\pi_1\}), (\rho_4, \Omega, \varnothing)\}.$$

Then,

$$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1, \Lambda_1, \sigma_1) = \{(\rho'_1, \{\pi'_1, \pi'_3\}, \{\pi'_2\}), (\rho'_2, \varnothing, \Omega'), (\rho'_3, \{\pi'_3\}, \{\pi'_1\}), (\rho'_4, \varnothing, \Omega')\},$$
$$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2, \Lambda_2, \sigma_2) = \{(\rho'_1, \{\pi'_1, \pi'_2\}, \varnothing), (\rho'_2, \varnothing, \Omega'), (\rho'_3, \{\pi'_3\}, \{\pi'_1\}), (\rho'_4, \{\pi'_1, \pi'_2, \pi'_3\}, \varnothing)\}.$$

Now,
$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1, \Lambda_1, \sigma_1) \;\widetilde{\widetilde{\cap}}\; \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2, \Lambda_2, \sigma_2) = \{(\rho'_1, \{\pi'_1\}, \{\pi'_2\}), (\rho'_2, \varnothing, \Omega'), (\rho'_3, \{\pi'_3\}, \{\pi'_1\}), (\rho'_4, \varnothing, \Omega')\}$.

On the other hand,
$(\Theta_1, \Lambda_1, \sigma_1) \;\widetilde{\widetilde{\cap}}\; (\Theta_2, \Lambda_2, \sigma_2) = \{(\rho_1, \{\pi_1\}, \{\pi_2\}), (\rho_2, \{\pi_3\}, \{\pi_1, \pi_2\}), (\rho_3, \{\pi_3\}, \{\pi_1\}), (\rho_4, \Omega', \varnothing)\}$.

Thus,
$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}((\Theta_1, \Lambda_1, \sigma_1) \;\widetilde{\widetilde{\cap}}\; (\Theta_2, \Lambda_2, \sigma_2)) = \{(\rho'_1, \{\pi'_1, \pi'_3\}, \{\pi'_2\}), (\rho'_2, \varnothing, \Omega'), (\rho'_3, \{\pi'_3\}, \{\pi'_1\}), (\rho'_4, \{\pi'_1, \pi'_2, \pi'_3\}, \varnothing)\}$.

Therefore,
$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1, \Lambda_1, \sigma_1) \;\widetilde{\widetilde{\cap}}\; \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2, \Lambda_2, \sigma_2) \;\widetilde{\widetilde{\not\supseteq}}\; \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}((\Theta_1, \Lambda_1, \sigma_1) \;\widetilde{\widetilde{\cap}}\; (\Theta_2, \Lambda_2, \sigma_2))$.

(2) Let $(\chi_1, \psi_1, \sigma'_1)$, $(\chi_2, \psi_2, \sigma'_2) \;\widetilde{\widetilde{\in}}\; \mathrm{BS}(\Omega'_{\varsigma'})$,

$$(\chi_1, \psi_1, \sigma'_1) = \{(\rho'_3, \Omega', \varnothing), (\rho'_4, \{\pi'_1, \pi'_3\}, \{\pi'_2\})\},$$
$$(\chi_2, \psi_2, \sigma'_2) = \{(\rho'_1, \{\pi'_4\}, \{\pi'_3\}), (\rho'_2, \{\pi'_1\}, \{\pi'_2\}), (\rho'_3, \{\pi'_1\}, \{\pi'_3, \pi'_4\}), (\rho'_4, \varnothing, \Omega')\}.$$

Then,

$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1, \psi_1, \sigma'_1) = \{(\rho_1, \varnothing, \Omega), (\rho_2, \varnothing, \Omega), (\rho_3, \Omega, \varnothing), (\rho_4, \{\pi_1, \pi_3\}, \{\pi_2\})\},$$
$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2, \psi_2, \sigma'_2) = \{(\rho_1, \varnothing, \{\pi_3\}), (\rho_2, \varnothing, \{\pi_3\}), (\rho_3, \{\pi_1\}, \{\pi_3\}), (\rho_4, \varnothing, \Omega)\}.$$

Now,
$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1, \psi_1, \sigma'_1) \;\widetilde{\widetilde{\cap}}\; \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2, \psi_2, \sigma'_2) = \{(\rho_1, \varnothing, \Omega), (\rho_2, \varnothing, \Omega), (\rho_3, \{\pi_1\}, \{\pi_3\}), (\rho_4, \varnothing, \Omega)\}$.

On the other hand, $(\chi_1, \psi_1, \sigma'_1) \;\widetilde{\widetilde{\cap}}\; (\chi_2, \psi_2, \sigma'_2) = (\chi_2, \psi_2, \sigma'_2)$, then

$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\chi_1, \psi_1, \sigma'_1)\widetilde{\widetilde{\cap}}(\chi_2, \psi_2, \sigma'_2)) = \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2, \psi_2, \sigma'_2).$$

$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}((\chi_1, \psi_1, \sigma'_1)\widetilde{\widetilde{\cap}}(\chi_2, \psi_2, \sigma'_2)) \neq \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1, \psi_1, \sigma'_1)\widetilde{\widetilde{\cap}}\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2, \psi_2, \sigma'_2).$$

Proposition 4.1 If $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_\varsigma) \longrightarrow \mathrm{BS}(\Omega'_{\varsigma'})$ is a BS bijective mapping, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega'_{\varsigma'}) \longrightarrow$

$\mathrm{BS}(\Omega_{\varsigma})$ is also a BS bijective mapping.

Proof Let $(\chi_1,\psi_1,\sigma_1') \neq (\chi_2,\psi_2,\sigma_2') \tilde{\tilde{\in}} \mathrm{BS}(\Omega'_{\varsigma'})$. As $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS bijective mapping, then there exists $(\Theta_1,\Lambda_1,\sigma_1) \neq (\Theta_2,\Lambda_2,\sigma_2) \tilde{\tilde{\in}} \mathrm{BS}(\Omega_{\varsigma})$ such that

$$\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1) = (\chi_1,\psi_1,\sigma_1'),$$
$$\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2) = (\chi_2,\psi_2,\sigma_2').$$

So, we have

$$(\Theta_1,\Lambda_1,\sigma_1) = \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\sigma_1'),$$
$$(\Theta_2,\Lambda_2,\sigma_2) = \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2,\psi_2,\sigma_2').$$

Hence,

$$\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_1,\psi_1,\sigma_1') \neq \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_2,\psi_2,\sigma_2') \text{ and } \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho} \text{ is a BS injective mapping.}$$

Now, let $(\Theta,\Lambda,\sigma) \tilde{\tilde{\in}} \mathrm{BS}(\Omega_{\varsigma})$. Then, there is $(\chi,\psi,\sigma') \tilde{\tilde{\in}} \mathrm{BS}(\Omega'_{\varsigma'})$ such that $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\sigma) = (\chi,\psi,\sigma')$. $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective mapping. Thus $(\Theta,\Lambda,\sigma) = \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\sigma')$. Therefore, $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}$ is a BS surjective mapping. Hence, $\tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho}$ is a BS bijective mapping.

Definition 4.1 Let $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_{1_{\varsigma}}) \longrightarrow \mathrm{BS}(\Omega_{2_{\varsigma'}})$ and $\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_{2_{\varsigma'}}) \longrightarrow \mathrm{BS}(\Omega_{3_{\varsigma''}})$ be two BS mappings. Then, the BS composite mapping $\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} \circ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_{1_{\varsigma}}) \longrightarrow \mathrm{BS}(\Omega_{3_{\varsigma''}})$ is defined by $(\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} \circ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho})(\Theta,\Lambda,\sigma) = \tilde{\tilde{\Delta}}_{\vartheta\mu\rho}(\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\sigma))$ for $(\Theta,\Lambda,\sigma) \tilde{\tilde{\in}} \mathrm{BS}(\Omega_{1_{\varsigma}})$.

Proposition 4.2 Let $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_{1_{\varsigma}}) \longrightarrow \mathrm{BS}(\Omega_{2_{\varsigma'}})$ and $\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_{2_{\varsigma'}}) \longrightarrow \mathrm{BS}(\Omega_{3_{\varsigma''}})$ be two BS bijective mappings. Then, the BS composite mapping $\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} \circ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho} : \mathrm{BS}(\Omega_{1_{\varsigma}}) \longrightarrow \mathrm{BS}(\Omega_{3_{\varsigma''}})$ is also a BS bijective mapping and $(\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} \circ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho})^{-1} = \tilde{\tilde{\varphi}}^{-1}_{\vartheta\mu\rho} \circ \tilde{\tilde{\Delta}}^{-1}_{\vartheta\mu\rho}$.

Proof Let $(\Theta_1,\Lambda_1,\sigma_1) \neq (\Theta_2,\Lambda_2,\sigma_2) \tilde{\tilde{\in}} \mathrm{BS}(\Omega_{1_{\varsigma}})$. Since $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS injective mapping, thus $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1) \neq \tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2)$. Again, since $\tilde{\tilde{\Delta}}_{\vartheta\mu\rho}$ is a BS injective mapping, we get

$$\tilde{\tilde{\Delta}}_{\vartheta\mu\rho}(\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_1,\Lambda_1,\sigma_1)) \neq \tilde{\tilde{\Delta}}_{\vartheta\mu\rho}(\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}(\Theta_2,\Lambda_2,\sigma_2)).$$

Therefore,

$$(\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} \circ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho})(\Theta_1,\Lambda_1,\sigma_1) \neq (\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} \circ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho})(\Theta_2,\Lambda_2,\sigma_2).$$

Hence, $\tilde{\tilde{\Delta}}_{\vartheta\mu\rho} \circ \tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS injective mapping.

Now, let $(\mathcal{H},\mathcal{P},\sigma'') \tilde{\tilde{\in}} \mathrm{BS}(\Omega_{3_{\varsigma''}})$. Then, there exists $(\chi,\psi,\sigma') \tilde{\tilde{\in}} \mathrm{BS}(\Omega_{2_{\varsigma'}})$ such that

$$\tilde{\tilde{\Delta}}_{\vartheta\mu\rho}(\chi,\psi,\sigma') = (\mathcal{H},\mathcal{P},\sigma''),$$

as $\tilde{\tilde{\Delta}}_{\vartheta\mu\rho}$ is a BS surjective mapping. Again, since $\tilde{\tilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective mapping, then there exists $(\Theta,\Lambda,\sigma) \tilde{\tilde{\in}} \mathrm{BS}(\Omega_{1_{\varsigma}})$ such that

$$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\sigma)=(\chi,\psi,\sigma').$$

Thus,

$$\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\big(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\sigma)\big)=(\mathcal{H},\mathcal{P},\sigma'').$$

Therefore, $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective mapping. Hence, $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS bijective mapping. So, let $(\Theta,\Lambda,\sigma)\ \widetilde{\widetilde{\in}}\ BS(\Omega_{1\varsigma})$, $(\chi,\psi,\sigma')\ \widetilde{\widetilde{\in}}\ BS(\Omega_{2\varsigma'})$, and $(\mathcal{H},\mathcal{P},\sigma'')\ \widetilde{\widetilde{\in}}\ BS(\Omega_{3\varsigma''})$, such that

$$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\sigma)=(\chi,\psi,\sigma')\text{ and }\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}(\chi,\psi,\sigma')=(\mathcal{H},\mathcal{P},\sigma'').$$

Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ are BS injective mappings, then

$$(\Theta,\Lambda,\sigma)=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\sigma')\text{ and }(\chi,\psi,\sigma')=\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\mathcal{H},\mathcal{P},\sigma'').$$

Thus, $\big(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\big)(\Theta,\Lambda,\sigma)=\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\big(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\sigma)\big)=\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}(\chi,\psi,\sigma')=(\mathcal{H},\mathcal{P},\sigma'')$. Since $\big(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\big)$ is a BS injective mapping, then

$$\big(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\big)^{-1}(\mathcal{H},\mathcal{P},\sigma'')=(\Theta,\Lambda,\sigma).$$

Also,

$$\big(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}\big)(\mathcal{H},\mathcal{P},\sigma'')=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\big(\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\mathcal{H},\mathcal{P},\sigma'')\big)=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\sigma')=(\Theta,\Lambda,\sigma).$$

Therefore, $\big(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\big)^{-1}=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}$.

5 Bipolar Soft Generalized Homeomorphism Mapping

This section, we present the novel types of BS $\widetilde{\widetilde{g}}$-continuous, BS $\widetilde{\widetilde{g}}$-open, BS $\widetilde{\widetilde{g}}$-closed, and BS $\widetilde{\widetilde{g}}$-homeomorphisms. In addition, we support the research with examples to make the main results understandable. Some results and counterexamples are given to explain this work.

Definition 5.1 A BS mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is said to be bipolar soft $\widetilde{\widetilde{g}}$-continuous, denoted by BS $\widetilde{\widetilde{g}}$-continuous, if $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\sigma')\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{g}}_{\Omega}$ for each $(\chi,\psi,\sigma')\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{g}}_{\Omega'}$.

Example 5.1 Let $\Omega=\{\pi_1,\pi_2,\pi_3\}$ and $\Omega'=\{\pi'_1,\pi'_2,\pi'_3,\pi'_4\}$ be two sets, $\sigma=\{\rho_1,\rho_2,\rho_3,\rho_4\}$ and $\sigma'=\{\rho'_1,\rho'_2,\rho'_3,\rho'_4\}$ be sets of parameters, $\vartheta:\Omega\longrightarrow\Omega'$ be a mapping defined as $\vartheta(\pi_i)=\pi'_i$ for $i=1,2,3$, the mapping $\mu:\varsigma\longrightarrow\varsigma'$ be defined as $\mu(\rho_1)=\mu(\rho_2)=\rho'_1$, $\mu(\rho_3)=\rho'_3$, and $\mu(\rho_4)=\rho'_4$, so the mapping $\rho:\neg\varsigma\longrightarrow\neg\varsigma'$ is defined as $\rho(\neg\rho_i)=\neg\mu(\rho_i)$ for $i=1,2,3,4$. Now, let $\widetilde{\widetilde{g}}_{\Omega}=\{(\Phi,\widetilde{\widetilde{\Omega}},\varsigma),(\Theta,\Lambda,\varsigma)\}$ and $\widetilde{\widetilde{g}}_{\Omega'}=\{(\Phi,\widetilde{\widetilde{\Omega}}',\varsigma),(\chi,\psi,\varsigma')\}$ be two BSGTSs defined on Ω and Ω', respectively, where $(\Theta,\Lambda,\varsigma)$ and (χ,ψ,ς') are BSSs defined as follows:

$$(\Theta,\Lambda,\varsigma)=\{(\rho_1,\varnothing,\{\pi_3\}),(\rho_2,\varnothing,\{\pi_3\}),(\rho_3,\{\pi_1\},\{\pi_3\}),(\rho_4,\varnothing,\Omega)\},$$
$$(\chi,\psi,\varsigma')=\{(\rho'_1,\{\pi'_4\},\{\pi'_3\}),(\rho'_2,\{\pi'_1\},\{\pi'_2\}),(\rho'_3,\{\pi'_1\},\{\pi'_3,\pi'_4\}),(\rho'_4,\varnothing,\Omega')\}.$$

Then, it is clear that $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.1 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ be a BS mapping, then the following statements are equivalent:

(1) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

(2) The BS inverse image of each BS $\widetilde{\widetilde{g}}$-closed set is BS $\widetilde{\widetilde{g}}$-closed.

(3) $c_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))$ for each $(\chi,\psi,\varsigma')\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}}',\Phi,\varsigma')$.

(4) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta,\Lambda,\varsigma))\ \widetilde{\widetilde{\subseteq}}\ c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))$ for each $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$.

(5) $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))\ \widetilde{\widetilde{\subseteq}}\ i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))$ for each $(\chi,\psi,\varsigma')\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}}',\Phi,\varsigma')$.

Proof

(1) $\Rightarrow$ (2): Let (χ,ψ,ς') be a BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$, thus $(\chi,\psi,\varsigma')^c$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. As $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')^c$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$. Since $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')^c=(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))^c$, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$.

(2) $\Rightarrow$ (3): Obviously, $c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-closed subset of $(\widetilde{\widetilde{\Omega}}',\Phi,\varsigma')$, by Statement (2), $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))$ is a BS $\widetilde{\widetilde{g}}$-closed subset of $(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$. Therefore, $c_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma')))=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))$. Since, $(\chi,\psi,\varsigma')\ \widetilde{\widetilde{\subseteq}}\ c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma')$ implies $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))$, therefore $c_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))\ \widetilde{\widetilde{\subseteq}}\ c_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma')))=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))$.

(3) $\Rightarrow$ (4): Let $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}}',\Phi,\varsigma')$. Since Statement (3), $c_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))$. Thus, $c_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ c_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta,\Lambda,\varsigma))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega'}}\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))\ \widetilde{\widetilde{\subseteq}}\ c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))$.

(4) $\Rightarrow$ (5): Let $(\chi,\psi,\varsigma')\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}}',\Phi,\varsigma')$. Then apply Statement (4) to $(\chi,\psi,\varsigma')^c$, we get $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')^c))\ \widetilde{\widetilde{\subseteq}}\ c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')^c))$.

Hence,

$\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')))^c\ \widetilde{\widetilde{\subseteq}}\ c_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma')^c=(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))^c$.

Thus, $(i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')))^c\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))^c$. Therefore,

$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))\ \widetilde{\widetilde{\subseteq}}\ i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))$.

(5) $\Rightarrow$ (1): Let (χ,ψ,ς') be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. Since Statement (5), $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma'))=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')\ \widetilde{\widetilde{\subseteq}}\ i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))$. But, $i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$. Therefore, $i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ and $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.2 A BS bijective mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping if and only if $i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta,\Lambda,\varsigma))$ for all $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}$

$(\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$.

Proof Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ be a BS $\widetilde{\widetilde{g}}$-continuous mapping and $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$ in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. Obviously, $i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. As, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ and

$$i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))) = \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))).$$

Clearly, $i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$. As, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS injective mapping, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))\ \widetilde{\widetilde{\subseteq}}\ (\Theta,\Lambda,\varsigma)$ and

$$i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))) = \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))\widetilde{\widetilde{\subseteq}} i_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta,\Lambda,\varsigma).$$

Again, as $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective mapping, then $i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta,\ \Lambda,\varsigma))$.

Conversely, let (χ,ψ,ς') be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$. By hypothesis,

$$i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')))\widetilde{\widetilde{\subseteq}}\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))).$$

Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective mapping, thus $i_{\widetilde{\widetilde{g}}_{\Omega'}}(\chi,\psi,\varsigma') = (\chi,\psi,\varsigma')\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_{\Omega}}\ (\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\ (\chi,\psi,\varsigma')))$. Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS injective mapping, then

$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')\widetilde{\widetilde{\subseteq}} i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')).$$

But $i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$. Therefore, $i_{\widetilde{\widetilde{g}}_{\Omega}}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\ \psi,\varsigma')) = \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\ \psi,\varsigma')$. Hence, $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ and consequently $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.3 A BS mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma) \longrightarrow (\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is BS $\widetilde{\widetilde{g}}$-continuous, if

(1) $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is an indiscrete BSGTS.

(2) $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ is a discrete BSGTS.

Proof

(1) Straightforward.

(2) Suppose that $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ is a discrete BSGTS. Then every BS $\widetilde{\widetilde{g}}$-open set (χ,ψ,ς') in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is the BS inverse image $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ which is also BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$. Therefore, BS mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.4 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma) \longrightarrow (\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ be a BS $\widetilde{\widetilde{g}}$-continuous mapping, then:

(1) If $\widetilde{\widetilde{g}}^{*}_{\Omega}\ \widetilde{\widetilde{\supseteq}}\ \widetilde{\widetilde{g}}_{\Omega}$, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega,\widetilde{\widetilde{g}}^{*}_{\Omega},\varsigma,\neg\varsigma) \longrightarrow (\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

(2) If $\widetilde{\widetilde{g}}^{*}_{\Omega'}\ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{g}}_{\Omega'}$, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma) \longrightarrow (\Omega',\widetilde{\widetilde{g}}^{*}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proof The proof is straightforward.

Proposition 5.5 A BS mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma) \longrightarrow (\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-continuous if and only if the BS inverse image of every member of a BS base $\widetilde{\widetilde{\mathfrak{B}}}$ for $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open in

$(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$.

Proof Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ be a BS $\widetilde{\widetilde{g}}$-continuous mapping and $(\Theta,\Lambda,\varsigma')$ be any BS basis element for $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. Since $(\Theta,\Lambda,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ and $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$.

Conversely, let $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma')$ be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ for every $(\Theta,\Lambda,\varsigma')\ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}$ and let (χ,ψ,ς') be any BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. Then,

$$(\chi,\psi,\varsigma')=\widetilde{\widetilde{\cup}}\{(\Theta,\Lambda,\varsigma'):(\Theta,\Lambda,\varsigma')\widetilde{\widetilde{\in}}\widetilde{\widetilde{\mathfrak{B}}}\},$$

$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')=\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\big(\widetilde{\widetilde{\cup}}\{(\Theta,\Lambda,\varsigma'):(\Theta,\Lambda,\varsigma')\widetilde{\widetilde{\in}}\widetilde{\widetilde{\mathfrak{B}}}\}\big)=\widetilde{\widetilde{\cup}}\{\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\{(\Theta,\Lambda,\varsigma'):(\Theta,\Lambda,\varsigma')\widetilde{\widetilde{\in}}\widetilde{\widetilde{\mathfrak{B}}}\})\}.$$

Hence, $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ since each $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$ by hypothesis. Therefore, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.6 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$ and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}:(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$ be two BS $\widetilde{\widetilde{g}}$-continuous mappings, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$ is also a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proof Let $(\mathcal{H},\mathcal{P},\varsigma'')$ be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$. Since $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\mathcal{H},\mathcal{P},\varsigma'')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$. Again, since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is BS $\widetilde{\widetilde{g}}$-continuous, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\big(\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\mathcal{H},\mathcal{P},\varsigma'')\big)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)$. But

$$\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\big(\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\mathcal{H},\mathcal{P},\varsigma'')\big)=\big(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}\big)(\mathcal{H},\mathcal{P},\varsigma'')=\big(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\big)^{-1}(\mathcal{H},\mathcal{P},\varsigma'').$$

Therefore, the BS inverse image under $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ of every BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)$. Hence, $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.7 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ be a BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\varphi}_{\vartheta\mu}:(\Omega,\widetilde{g}_{\Omega},\varsigma)\longrightarrow(\Omega',\widetilde{g}_{\Omega'},\varsigma')$ is $\mathcal{S}$ $\widetilde{g}$-continuous.

Proof Straightforward.

Proposition 5.8 Let the condition of constructing a BSGT since SGT as in Theorem 2.2 hold. If $\widetilde{\phi}_{\vartheta\mu}:(\Omega,\widetilde{g}_{\Omega},\varsigma)\longrightarrow(\Omega',\widetilde{g}_{\Omega'},\varsigma')$ is a $\mathcal{S}$ $\widetilde{g}$-continuous mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is BS $\widetilde{\widetilde{g}}$-continuous.

Proof Let $\widetilde{\phi}_{\vartheta\mu}$ be a $\mathcal{S}$ $\widetilde{g}$-continuous mapping and (χ,ψ,ς') be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{g}_{\Omega'},\varsigma',\neg\varsigma')$. Then, (χ,ς') is a $\mathcal{S}$ $\widetilde{g}$-open set in $(\Omega',g_{\Omega'},\varsigma')$. Since $\widetilde{\phi}_{\vartheta\mu}$ is a $\mathcal{S}$ $\widetilde{g}$-continuous mapping, thus $\widetilde{\phi}^{-1}_{\vartheta\mu}(\chi,\varsigma')$ is a $\mathcal{S}$ $\widetilde{g}$-open set in $(\Omega,g_{\Omega},\varsigma)$. Therefore, $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu,\rho}(\chi,\psi,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{g}_{\Omega},\varsigma,\neg\varsigma)$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu,\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.9 If $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping and $(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}},\Phi,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact set, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)\ \widetilde{\widetilde{\subseteq}}\ (\widetilde{\widetilde{\Omega}}',\phi,\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-compact set.

Proof Let $\{(\chi_\gamma,\psi_\gamma,\varsigma')\}_{\gamma\in\Gamma}$ be a BS $\widetilde{\widetilde{g}}$-open cover of $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$. Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping, then for each $\gamma \in \Gamma$, $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_\gamma, \psi_\gamma, \varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma)$. Then, the collection $\{\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_\gamma, \psi_\gamma, \varsigma')\}_{\gamma\in\Gamma}$ forms a BS $\widetilde{\widetilde{g}}$-open cover of $(\Theta, \Lambda, \varsigma)$. Since $(\Theta, \Lambda, \varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact set, we get

$$(\Theta, \Lambda, \varsigma) = \bigcup \{\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi_{\gamma_i}, \psi_{\gamma_i}, \varsigma')\}_{i=1}^{n} \widetilde{\widetilde{\subseteq}} \widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}\{\bigcup_{i=1}^{n}(\chi_{\gamma_i}, \psi_{\gamma_i}, \varsigma')\},$$

so that, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ \{\bigcup_{i=1}^{n}(\chi_{\gamma_i}, \psi_{\gamma_i}, \varsigma')\}$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)$ is a BS $\widetilde{\widetilde{g}}$-compact set.

Definition 5.2 A BS mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma) \longrightarrow (\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$ is said to be:

(1) BS $\widetilde{\widetilde{g}}$-open if the image of every BS $\widetilde{\widetilde{g}}$-open set in Ω is BS $\widetilde{\widetilde{g}}$-open in Ω';

(2) BS $\widetilde{\widetilde{g}}$-closed if the image of every BS $\widetilde{\widetilde{g}}$-closed set in Ω is BS $\widetilde{\widetilde{g}}$-closed in Ω'.

Proposition 5.10 A BS mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma) \longrightarrow (\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open if and only if $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)) \ \widetilde{\widetilde{\subseteq}} \ i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma))$ for each $(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$.

Proof Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ be a BS $\widetilde{\widetilde{g}}$-open mapping and $(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$. Since $i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma)$ and $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$ and $i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))) = \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))$. Obviously, $i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ (\Theta, \Lambda, \varsigma)$ and $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)) \ \widetilde{\widetilde{\subseteq}} \ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)$. Hence,

$$i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))) = \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)) \widetilde{\widetilde{\subseteq}} i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)).$$

Conversely, let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)) \ \widetilde{\widetilde{\subseteq}} \ i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma))$ for every $(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$ and let $(\Theta, \Lambda, \varsigma)$ be any BS $\widetilde{\widetilde{g}}$-open set in $(\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma)$ so that $i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma) = (\Theta, \Lambda, \varsigma)$. This implies $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(i_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)) = \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma))$. But $i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)) \ \widetilde{\widetilde{\subseteq}} \ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)$. Therefore, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma) = i_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma))$. Thus, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)$ is BS $\widetilde{\widetilde{g}}$-open set in $(\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping.

Proposition 5.11 A BS mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma) \longrightarrow (\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-closed mapping if and only if $c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)) \ \widetilde{\widetilde{\subseteq}} \ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))$ for each $(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$.

Proof Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ be a BS $\widetilde{\widetilde{g}}$-closed mapping and $(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$. Since $c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)$ is a BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma)$ and $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))$ is a BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$ and $c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))) = \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))$. Obviously, $(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)$ and $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))$. Hence,

$$c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)) \widetilde{\widetilde{\subseteq}} c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))) = \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)).$$

Conversely, let $c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)) \ \widetilde{\widetilde{\subseteq}} \ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma))$ for every $(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$ and let $(\Theta, \Lambda, \varsigma)$ be any BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega, \widetilde{\widetilde{g}}_\Omega, \varsigma, \neg\varsigma)$ so that $c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma) = (\Theta, \Lambda, \varsigma)$. This implies $c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)) \ \widetilde{\widetilde{\subseteq}} \ \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(c_{\widetilde{\widetilde{g}}_\Omega}(\Theta, \Lambda, \varsigma)) = \widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)$. But $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma) \ \widetilde{\widetilde{\subseteq}} \ c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma))$. Therefore, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma) = c_{\widetilde{\widetilde{g}}_{\Omega'}}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma))$. Thus, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)$ is BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping.

Proposition 5.12 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$, $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}:(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$, and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$ be three BS mappings, then:

(1) If $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ are BS $\widetilde{\widetilde{g}}$-open mappings, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is also a BS $\widetilde{\widetilde{g}}$-open mapping.

(2) If $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping and $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping.

(3) If $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS injective BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping.

Proof

(1) Let $(\Theta,\Lambda,\varsigma)$ be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)$. Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$. Again, since $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))=(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho})(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$. Hence, $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping.

(2) Let (χ,ψ,ς') be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$. Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')$ is BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)$. Again, since $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping, thus $(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho})(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$. As, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective, then

$$(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho})(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'))=\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}(\chi,\psi,\varsigma')))=\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}(\chi,\psi,\varsigma'),$$

is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$. Hence, $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping.

(3) Let $(\Theta,\Lambda,\varsigma)$ be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)$. Since $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open, then $(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho})(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$. Again, since $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}((\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho})(\Theta,\Lambda,\varsigma))$ is BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$. Also, since $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS injective mapping, we obtain that

$$\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma))=\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)))=\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma),$$

is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-open mapping.

Proposition 5.13 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$, $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}:(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$, and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$ be three BS mappings, then

(1) If $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ are BS $\widetilde{\widetilde{g}}$-closed mappings, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is also a BS $\widetilde{\widetilde{g}}$-closed mapping.

(2) If $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping and $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS surjective BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping.

(3) If $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS injective BS $\widetilde{\widetilde{g}}$-continuous mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping.

Proof The proof is similar to Proposition 5.12.

Proposition 5.14 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-open (BS $\widetilde{\widetilde{g}}$-closed) mapping, then $\widetilde{\phi}_{\vartheta_\mu}:(\Omega,\widetilde{g}_{\Omega},\varsigma)\longrightarrow(\Omega',\widetilde{g}_{\Omega'},\varsigma')$ is also $\mathcal{S}$ $\widetilde{g}$-open ($\mathcal{S}$ $\widetilde{g}$-closed).

Proof Straightforward.

Proposition 5.15 Let the condition of constructing a BSGT since SGT as in Theorem 2.2 hold. If $\widetilde{\phi}_{\vartheta_\mu}:(\Omega,\widetilde{g}_{\Omega},\varsigma)\longrightarrow(\Omega',\widetilde{g}_{\Omega'},\varsigma')$ is a $\mathcal{S}$ $\widetilde{g}$-open ($\mathcal{S}$ $\widetilde{g}$-closed) mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is BS $\widetilde{\widetilde{g}}$-open (BS $\widetilde{\widetilde{g}}$-closed).

Proof Similar to the proof of Proposition 5.8.

Definition 5.3 A BS bijective mapping $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is said to be a BS $\widetilde{\widetilde{g}}$-homeomorphism, if $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ and $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}$ are BS $\widetilde{\widetilde{g}}$-continuous mappings.

Proposition 5.16 If $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)\longrightarrow(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ is a BS bijective mapping, then the following statements are equivalent:

(1) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is BS $\widetilde{\widetilde{g}}$-homeomorphism.

(2) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is BS $\widetilde{\widetilde{g}}$-continuous and BS $\widetilde{\widetilde{g}}$-open.

(3) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is BS $\widetilde{\widetilde{g}}$-continuous and BS $\widetilde{\widetilde{g}}$-closed.

Proof

(1) $\Rightarrow$ (2): Let $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ be a BS inverse mapping of $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$. That is $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}=\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ and $\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}=\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$. Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS bijective mapping, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is also BS bijective. Let $(\Theta,\Lambda,\varsigma)$ be a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$. Since $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is BS $\widetilde{\widetilde{g}}$-continuous, then $\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. But $\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}=\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ so that $\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)=\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. Therefore, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is BS $\widetilde{\widetilde{g}}$-open. Also, by hypothesis, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

(2) $\Rightarrow$ (3): Let $(\Theta,\Lambda,\varsigma)$ be BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$, then $(\Theta,\Lambda,\varsigma)^c$ be BS $\widetilde{\widetilde{g}}$-open set, thus $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)^c$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. As $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS bijective mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)^c=\left(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)\right)^c$. Hence, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$ and consequently, $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping.

(3) $\Rightarrow$ (1): Let $(\Theta,\Lambda,\varsigma)$ be BS $\widetilde{\widetilde{g}}$-open set in $(\Omega,\widetilde{\widetilde{g}}_{\Omega},\varsigma,\neg\varsigma)$, then $(\Theta,\Lambda,\varsigma)^c$ be BS $\widetilde{\widetilde{g}}$-closed set. Since $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-closed mapping, then $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)^c=\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)^c=\left(\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)\right)^c$ is a BS $\widetilde{\widetilde{g}}$-closed set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$, therefore $\widetilde{\widetilde{\Delta}}^{-1}_{\vartheta\mu\rho}(\Theta,\Lambda,\varsigma)$ is a BS $\widetilde{\widetilde{g}}$-open set in $(\Omega',\widetilde{\widetilde{g}}_{\Omega'},\varsigma',\neg\varsigma')$. Hence, $\widetilde{\widetilde{\varphi}}^{-1}_{\vartheta\mu\rho}=\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}$ is a BS $\widetilde{\widetilde{g}}$-continuous mapping.

Proposition 5.17 Let $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')$ and $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}:(\Omega_2,\widetilde{\widetilde{g}}_{\Omega_2},\varsigma',\neg\varsigma')\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$ be two BS $\widetilde{\widetilde{g}}$-homeomorphism, then $\widetilde{\widetilde{\Delta}}_{\vartheta\mu\rho}\circ\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}:(\Omega_1,\widetilde{\widetilde{g}}_{\Omega_1},\varsigma,\neg\varsigma)\longrightarrow(\Omega_3,\widetilde{\widetilde{g}}_{\Omega_3},\varsigma'',\neg\varsigma'')$ is also a BS $\widetilde{\widetilde{g}}$-homeomorphism.

Proof Follows since Proposition 5.6, Proposition 5.12 (1), and Proposition 5.16.

Proposition 5.18 If $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho} : (\Omega, \widetilde{\widetilde{g}}_{\Omega}, \varsigma, \neg\varsigma) \longrightarrow (\Omega', \widetilde{\widetilde{g}}_{\Omega'}, \varsigma', \neg\varsigma')$ is a BS $\widetilde{\widetilde{g}}$-homeomorphisms, then the following statements hold for all $(\Theta, \Lambda, \varsigma) \mathrel{\widetilde{\widetilde{\subseteq}}} (\widetilde{\widetilde{\Omega}}, \Phi, \varsigma)$.

(1) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\left(c_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta, \Lambda, \varsigma)\right) = c_{\widetilde{\widetilde{g}}_{\Omega'}}\left(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)\right)$.

(2) $\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}\left(i_{\widetilde{\widetilde{g}}_{\Omega}}(\Theta, \Lambda, \varsigma)\right) = i_{\widetilde{\widetilde{g}}_{\Omega'}}\left(\widetilde{\widetilde{\varphi}}_{\vartheta\mu\rho}(\Theta, \Lambda, \varsigma)\right)$.

Proof

(1) Follows since Proposition 5.1, Proposition 5.11, and Proposition 5.16.

(2) Follows since Proposition 5.2, Proposition 5.10, and Proposition 5.16.

6 Conclusion and Future Research

The bipolar soft generalized compact sets and bipolar soft generalized compact spaces are introduced. The structures of $\widetilde{\widetilde{g}}$-centralized bipolar soft generalized closed sets collection is a bipolar soft generalized compact space have been discussed. In special case, this paper assumed BSGTSs is strong BSGTSs and it discussed their main features; especially, the properties and related to finite bipolar soft union of bipolar soft generalized compact sets and bipolar soft generalized closed sets in bipolar soft generalized compact sets are also bipolar soft generalized compact. Moreover, main properties and the relationships between them have been presented. The concept of a bipolar soft generalized compactness has been defined. Moreover, some further properties of bipolar soft mappings, such as bipolar soft composite mappings, have been investigated, and some of its characteristics have been explained. In addition, novel classes of bipolar soft mapping such as bipolar soft generalized continuous, bipolar soft generalized open, and bipolar soft generalized closed mappings have been introduced. In the forthcoming works, we will add some other concepts of BSGTS such as bipolar soft separation axioms interms of bipolar soft generalized open sets and bipolar soft points.

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