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An Inverse BCC Model for Evaluating and Ordering Decision-Making Units under Fuzziness

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ABSTRACT
One of the most prevalent problems in linear programming as one of the convenient models in the field of operation research environment is Data Envelopment Analysis (DEA), which supports the efficiency of Decision-Making Units (DMUs). Usually, accurate data are common; however, in the real world, we are facing an inaccurate situation. In this paper, a new model for assessing DMUs in a fuzzy environment is presented; we consider the inverse DEA model with the variable return to scale with fuzzy numbers for fluctuating data. A case study is given to illustrate its performance.

KEYWORDS
fuzzy mathematical programming; fuzzy ordering; data envelopment analysis

1 Introduction

Data Envelopment Analysis (DEA) is a technique for deliberating the performance of Decision-Making Units (DMUs) with numerous inputs and outputs. Each DMU uses several inputs to generate several outputs. Moreover, the performance of DMUs is evaluated based on obtained inputs and outputs. DEA has been widely used in many fields of science. Pascoe et al.[1] used DEA to assess management alternatives in the presence of multiple objectives. Examples of the use of DEA to assess the financial effectiveness of insurance companies are presented in Ref. [2]. Nasseri and Khatir[3] organized a two-stage DEA model by taking into account undesirable output with fuzzy stochastic data. As one step forward, the Inverse of the DEA (IDEA) model has been recently introduced by Wei et al.[4] so that it tries to answer the question: If input (output) in a DMU is agitated, then how output (input) should be changed to keep the relative performance of the DMU? IDEA models have different applications in practical cases[5–7]. Furthermore, the IDEA model with a variable return to scale (Inverse Banker, Charnes, and Cooper model (IBCC)) is suggested by Lertworasirikul et al.[8] Next, Ghiyasi[9] proposed several problematic issues, and he challenged that the suggested Multi-Objective Linear Programming (MOLP) model by Lertworasirikul et al.[8] is not as effective as it was claimed.

Fuzzy uncertainty, grey uncertainty, and rough uncertainty are intertwined facets of imprecision in

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decision-making. Fuzzy uncertainty deals with vague boundaries and ambiguity, while grey uncertainty arises from incomplete or vague data. Rough uncertainty reflects imperfect or incomplete data, characterized by lower granularity. These uncertainties often coexist, making decision-making a complex challenge, as strategies need to navigate the nuances and complexities within each of these uncertainty types. The challenge in decision-making involves navigating these intertwined uncertainties, which require strategies to take into account the nuances and complexity within each of these uncertainty types. Grey uncertainty and rough uncertainty are used in many research such as Refs. [10, 11]. Bellman and Zadeh[12] proposed the concept of decision-making in a fuzzy environment. The DEA models with fuzzy data can more realistically represent real-world problems than the conventional DEA models. Fuzzy sets theory also allows linguistic data to be used directly within the DEA models. Fuzzy DEA models take the form of fuzzy linear programming models. Sheth and Triantis[13] suggested a fuzzy goal DEA frame in a fuzzy condition. A tunable approach of fuzzy data envelopment analysis was investigated by Peykani et al.[14], the model used for measuring the efficiency of hospitals in the USA. Two-stage network DEA in fully fuzzy surroundings with the existence of undesirable outputs was proposed by Mozaffari et al.[15] for greener petrochemical production. Chen et al.[16] considered a fuzzy DEA approach for the choice of design options for the smart product-service system. Zavieh et al.[17] applied fuzzy DEA as a solution method for Linear Semi-Infinite Programming (LSIP) problems. Their work aimed to address the challenges posed by the imprecision and uncertainty in the input data of LSIP models. By incorporating fuzzy numbers and DEA, they developed a new approach that could provide more robust and accurate solutions for LSIP problems.

In the real world, we often encounter problems that have different objectives[18], especially when it comes to ambiguous data. Many researchers extend this field of operations research[19–23]. Since IDEA is a kind of MOLP problem, therefore the necessity of the IDEA model is felt in fuzzy conditions. However, there are a lot of useful studies which are related to this area, and readers who are interested to have more information way find them in Refs. [24, 25]. Zavieh et al.[26] proposed an innovative technique called Fuzzy Inverse Markov Data Envelopment Analysis Process (FIMDEAP). Their method combines the strengths of the Fuzzy Inverse Data Envelopment Analysis (FIDEA) and Fuzzy Markov Decision Process (FMDP) techniques to select efficient physical and virtual machines in the fog and cloud environment. In this study, for the first time, we intend to present the multi-objective model with fuzzy fluctuation data with the IDEA model with variable returns to scale (Fuzzy Inverse BCC (FIBCC)) by using the modified IDEA in Ref. [6] and consider the solving approach for the FIDEA model in the real world. For this intent, Section 2 outlines the basic concepts required for the next sections. Section 3 presents a new model in inverse DEA with the fuzzy condition. In this way, we propose a novel MOLP model in a fuzzy environment. Then, in Section 4, a numerical example is given to illustrate its performance. In Section 5, the method is compared with some other existing methods, and Section 6 contains a conclusion and future works.

2 Preliminary

This section contains the basic definitions and concepts that we need in the other sections[27–29].

Definition 2.1 A fuzzy set $\tilde{A}$ on $\mathbb{R}$ is said to be a trapezoidal fuzzy number, if there exist real numbers a_1 and a_2, where $a_1 \leqslant a_2$ and $h_1, h_2 > 0$ such that

$$\tilde{A}(x) = \begin{cases} \frac{x}{h_1} + \frac{h_1 - a_1}{h_1}, & x \in (a_1 - h_1, a_1); \\ 1, & x \in [a_1, a_2]; \\ -\frac{x}{h_2} + \frac{a_2 + h_2}{h_2}, & x \in (a_2, a_2 + h_2); \\ 0, & \text{otherwise}. \end{cases}$$

where $\tilde{A}(x)$ is the membership function of a fuzzy number $\tilde{A}$.

We denote it by $\tilde{A} = [a_1, a_2, h_1, h_2]$. In the above definition, when $h_1 = h_2$, we called it a "symmetric trapezoidal fuzzy number"; if $h_1 \neq h_2$, we called it a "non-symmetric trapezoidal fuzzy number"; and also if $h_1 = h_2 = h$, then we call $\tilde{A} = [a_1, a_2, h]$ a "triangular fuzzy number". Moreover, when $h_1 = h_2 = 0$, $\tilde{A} = [a_1, a_2]$, as well as an interval form of data.

Definition 2.2 Assume that $\tilde{A}$ is the fuzzy number and $\alpha \in [0, 1]$, then an α-cut of $\tilde{A}$ is defined as $\{x | x \in \mathbb{R}, \mu_{\tilde{A}}(x) \geqslant \alpha\}$, and we briefly show it by $\tilde{A}_\alpha$.

A triangular fuzzy number is shown by a triple form as $\tilde{A} = (a^l, a^m, a^u)$, where a^m, a^l, and a^u are the core, the lower, and the upper limits of support of the triangular fuzzy number $\tilde{A}$, respectively. Also, the other type of the fuzzy number can be written $\tilde{A} = (a^m, a^\alpha, a^\beta)$, where a^m, a^α, and a^β are the core, the left, and the right spreads, respectively.

Assume that $F(\mathbb{R})$ gives the meaning of the set of all triangular fuzzy numbers on $\mathbb{R}$.

2.1 Ranking function

The ranking of fuzzy numbers is an important issue in the study of fuzzy sets theory. Ranking procedures are also useful in various applications, and one of them will be in the study of fuzzy mathematical programming in the later sections. There are numerous methods which are proposed in the literature for the ranking of fuzzy numbers, some of them seem that to be good in a particular context but not in general. Here, for introducing this concept, we describe only three simple methods for the ordering of fuzzy numbers.

The first approach is called the *k*-preference index approach. This approach has been suggested by Adamo[27]. Let $\tilde{a}$ be the given fuzzy number and $k \in [0, 1]$. The *k*-preference index of $\tilde{a}$ is defined as $F_k(\tilde{a}) = \max\{x : \mu_{\tilde{a}}(x) \geqslant k\}$. Now, using this *k*-preference index, for two fuzzy numbers $\tilde{a}$ and $\tilde{b}$, $\tilde{a} \preccurlyeq \tilde{b}$ with degree $k \in [0, 1]$, if and only if $F_k(\tilde{a}) \leqslant F_k(\tilde{b})$. And also $\preccurlyeq$ and $\succcurlyeq$ are fuzzy inequalities relations.

The second approach for ranking of fuzzy numbers is based on possibility theory. Dubois and Prade[30] studied the ranking of fuzzy numbers in the setting of possibility theory. To develop this suppose, we have two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. Then, following the extension principle of Zadeh[12], the crisp inequality $x \leqslant y$ can be extended to obtain the true value of the assertion that $\tilde{a}$ is less than or equal to $\tilde{b}$, as follows:

$$T(\tilde{a} \preccurlyeq \tilde{b}) = \sup_{x \leqslant y} (\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y))).$$

This true value $T(\tilde{a} \preccurlyeq \tilde{b})$ is also called the degree of possibility of the dominance of $\tilde{b}$ on $\tilde{a}$ and is denoted by $\mathrm{Poss}(\tilde{a} \preccurlyeq \tilde{b})$. Now define $\tilde{a} \preccurlyeq \tilde{b}$, if and only if

$$\mathrm{Poss}(\tilde{b} \preccurlyeq \tilde{a}) \leqslant \mathrm{Poss}(\tilde{a} \preccurlyeq \tilde{b}).$$

The third approach to the order of fuzzy numbers is based on the concept of comparison of fuzzy numbers by the use of ranking functions, in which a ranking function $R : F(\mathbb{R}) \to \mathbb{R}$ that maps each fuzzy number into the real line is defined for ranking the elements of $F(\mathbb{R})$. Thus, using the natural order of the real numbers, we can compare fuzzy numbers easily as follows[9]:

$$\tilde{a} \succcurlyeq \tilde{b}, \text{ if and only if } R(\tilde{a}) \geqslant R(\tilde{b});$$
$$\tilde{a} \succ \tilde{b}, \text{ if and only if } R(\tilde{a}) > R(\tilde{b});$$
$$\tilde{a} \approx \tilde{b}, \text{ if and only if } R(\tilde{a}) = R(\tilde{b});$$

where $\tilde{a}$ and $\tilde{b}$ are in $F(\mathbb{R})$. Also, we write $\tilde{a} \preccurlyeq \tilde{b}$ if and only if $\tilde{b} \succcurlyeq \tilde{a}$.

Lemma 2.1 For trapezoidal fuzzy numbers $\tilde{a}$ and $\tilde{b}$, if $\tilde{a} \preccurlyeq \tilde{b}$, then $-\tilde{a} \succcurlyeq -\tilde{b}$.

Several ranking functions have been proposed by researchers to suit the requirements of the problems under consideration. We restrict our attention to linear ranking functions that is a ranking function R such that $R(k\tilde{a} + \tilde{b}) = kR(\tilde{a}) + R(\tilde{b})$ for any $\tilde{a}$ and $\tilde{b}$ belonging to $F(\mathbb{R})$ and any $k \in \mathbb{R}$. We consider the linear ranking functions on $F(\mathbb{R})$ as $R(\tilde{a}) = c_L a^L + c_U a^U + c_\alpha \alpha + c_\beta \beta$, where $\tilde{a} = (a^L, a^U, \alpha, \beta)$, and c_L, c_U, c_α, and c_β are constants, at least one of which is nonzero. For a trapezoidal fuzzy number $\tilde{a} = (a^L, a^U, \alpha, \beta)$, some of these ranking functions are presented here:

(1) Yager's[29] ranking function:

$$Y_2(\tilde{a}) = \frac{1}{2}\int_0^1 (\mathrm{int}[\tilde{a}]_\alpha + \sup[\tilde{a}]_\alpha)\,\mathrm{d}\alpha = \frac{1}{2}\left[a^L + a^U + \frac{\beta - \alpha}{2}\right].$$

(2) de Campos Ibáñez and González Muñoz's[28] ranking function:

$$\mathrm{CM}_1^\lambda(\tilde{a}) = \int_0^1 (\lambda \mathrm{int}[\tilde{a}]_\alpha + (1-\lambda)\sup[\tilde{a}]_\alpha)\mathrm{d}\alpha = a^L + \lambda\left[(a^U - a^L) + \frac{\alpha + \beta}{2}\right] - \frac{\alpha}{2},$$

$$\mathrm{CM}_2^\lambda(\tilde{a}) = \int_0^1 \alpha(\lambda \mathrm{int}[\tilde{a}]_\alpha + (1-\lambda)\sup[\tilde{a}]_\alpha)\mathrm{d}\alpha = a^L + \lambda\left[(a^U - a^L) + \frac{\alpha + \beta}{3}\right] - \frac{\alpha}{3}.$$

In the later sections, the linear ranking functions will play a crucial role in ordering the trapezoidal fuzzy numbers being used for testing optimality, conditions, and deciding for pivoting, too. We especially use the linear ranking function $Y_2(\tilde{a})$ to illustrate our approaches. Then, for the trapezoidal fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$, we have $\tilde{a} \succcurlyeq \tilde{b}$, if and only if

$$R(\tilde{a}) = \frac{1}{2}\left(a^L + a^U + \frac{\beta - \alpha}{2}\right) \geqslant R(\tilde{b}) = \frac{1}{2}\left(b^L + b^U + \frac{\gamma - \theta}{2}\right).$$

Remark 2.1 For any trapezoidal fuzzy number the relation, $\tilde{a} \succcurlyeq \tilde{0}$ holds, if there exists $\varepsilon \geqslant 0$ and $\alpha \geqslant 0$ such that $\tilde{a} \succcurlyeq (-\varepsilon, \varepsilon, \alpha, \alpha)$. We realize that $R(-\varepsilon, \varepsilon, \alpha, \alpha) = 0$ (we also consider $\tilde{a} \approx \tilde{0}$, if and only if, $R(\tilde{\alpha}) = 0$. Thus, without loss of generality, throughout the paper, we let $\tilde{0} = (0, 0, 0, 0)$ be the zero trapezoidal fuzzy number.

Remark 2.2 We realize that the results obtained throughout the paper are independent of the choice of the linear ranking function. We can use any other linear ranking function based on the point of view of some decision-makers, and although the solution obtained may be different, the results are still valid for the new solution. As for the types of fuzzy data in the model and the assumption of fuzziness in the variables, the choice and compatibility of the ranking function for fuzzy linear programs should be the decision-maker's main concerns. For trapezoidal fuzzy numbers and variables, the linear ranking function used here is deemed to be appropriate.

The ranking function method is an appropriate approach to comparing fuzzy numbers and solving the Fuzzy Linear Programming (FLP) problems. Which schemes a fuzzy number into the real line[31–33].

2.2 Data envelopment analysis

DEA is a non-parametric approach to assess DMUs[34]. It is known as the Charnes, Cooper, and Rhodes (CCR) model, and the multiplier version of this model is presented as follows:

$$\max \sum_{j=1}^{n} u_r y_{rj} \tag{1}$$

$$\text{s.t.,} \quad \sum_{j=1}^{n} v_i x_{ij} = 1, i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} u_r y_{rj} - \sum_{j=1}^{n} v_i x_{ij} \leqslant 0, r = 1, 2, \ldots, s;$$

$$u_r, v_i \geqslant 0; i = 1, 2, \ldots, m; r = 1, 2, \ldots, s.$$

In the model, x_{ij} and y_{rj} are inputs and outputs of DMU_j, respectivily, and u_r and v_i are the weights of inputs and the weights of outputs, respectivily. And then, Banker et al.[35] presented the Banker, Charnes, and Cooper (BCC) model:

$$\max \sum_{j=1}^{n} u_r y_{rj} - u_o \tag{2}$$

$$\text{s.t.,} \quad \sum_{j=1}^{n} v_i x_{ij} = 1, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} u_r y_{rj} - \sum_{j=1}^{n} v_i x_{ij} - u_o \leqslant 0, \quad r = 1, 2, \ldots, s;$$

$$u_r, v_i \geqslant 0, i = 1, 2, \ldots, m, r = 1, 2, \ldots, s, \quad -\infty \leqslant u_o \leqslant +\infty.$$

In this investigation, our focus is directed towards the BCC model. Within the framework of the BCC model, especially in its input-oriented configuration, we witness the aggregation of multiple inputs into a virtual input entity and the amalgamation of multiple outputs into a virtual output entity. The formulation of the envelopment version of the BCC model is articulated as the Linear Programming (LP) model as follows:

$$\min \theta_o \tag{3}$$

$$\text{s.t.,} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta_o x_{io}, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geqslant y_{ro}, \quad r = 1, 2, \ldots, s;$$

$$\sum_{j=1}^{n} \lambda_j = 1;$$

$$\lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n;$$

where x_{ij} and y_{rj} are inputs and outputs of DMU_j and $\lambda_j \geqslant 0, j = 1, 2, \ldots, n$.

2.3 DEA with fuzzy parameters

Infact Fuzzy DEA (FDEA) models are a type of FLP model. One of the convenient models of FDEA is the BCC model with fuzzy coefficients which is defined as follows:

$$\min \tilde{\theta}_o \tag{4}$$

$$\text{s.t.,} \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leqslant \theta_o \tilde{x}_{io}, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \geqslant \tilde{y}_{ro}, \quad r = 1, 2, \ldots, s;$$

$$\sum_{j=1}^{n} \lambda_j = 1;$$

$$\lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n;$$

where $\tilde{x}_{ij}$ are the fuzzy inputs of DMUs for $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, $\tilde{x}_{io}$ is the vector of fuzzy inputs consumed by the target DMU (DMU_o), $\tilde{y}_{ro}$ is the vector of fuzzy outputs produced by the target DMU (DMU_o), and $\tilde{y}_{rj}$ are the fuzzy outputs of DMUs ($r = 1, 2, \ldots, s$, and $j = 1, 2, \ldots, n$).

The above model is not well-known and we cannot solve the fuzzy BCC model as well as the crisp form of the common BCC model. Since, the fuzzy BCC model is essentially a kind of FLP model, hence we can use the same approaches for solving the fuzzy BCC model that is usable for solving FLP problems. Usually, there are four approaches in the literature have been proposed for solving fuzzy DEA models: the tolerance method, defuzzification method, level-based method, and fuzzy ranking approach. Thus, a lot of valuable studies have been developed based on the above approaches[36–38]. In this study, we are going to use a combination of α-cut and ranking methods to solve the proposed model. However, we may use and extend the other approach in future study.

3 Proposed Model: IBCC Model with Fuzzy Data (FIBCC)

Lertworasirikul et al.[8] introduced an approach called the IBCC model for addressing the challenges posed by the inverse model. They considered that the output of DMU_o, y_o, changes to $\beta_o = y_o + \Delta y_o$, $\Delta y_o \in \mathbb{R}^s$, and presents the following model that gives us the amount of input that is necessary to maintain the relative efficiency of DMU_o.

Given that we are working with m inputs, we are confronted with a problem that encompasses multiple objectives. We dealt with the Dual form of the Inverse BCC model (DIBCC).

$$\min\left(\delta_1, \delta_2, \ldots, \delta_m\right) \tag{5}$$

$$\text{s.t.,} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta_o^* \delta_i, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geqslant \beta_{ro}, \quad r = 1, 2, \ldots, s;$$

$$\sum_{j=1}^{n} \lambda_j = 1;$$

$$\lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n;$$

where θ_o^* is the amount of optimal solution for the BCC model, δ_i is the fluctuate inputs, and $\delta_o = x_o + \Delta x_o$, $\Delta x_o \in \mathbb{R}^m$ is the necessary input that causes the relative performance for preserved DMU_o.

Within this segment, fuzzy number for inputs, outputs, fluctuated inputs, and fluctuated outputs are considered. In the fuzzy IBCC model, we are looking for fluctuated inputs in fuzzy environment that preserve efficiency of decision making units when the fluctuate outputs are changing in fuzzy mode. To address this query, the following model can be considered:

$$\min\left(\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_m\right) \tag{6}$$

$$\text{s.t.,} \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leqslant \theta_o^* \tilde{\delta}_i, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \geqslant \tilde{\beta}_{ro}, \quad r = 1, 2, \ldots, s;$$

$$\sum_{j=1}^{n} \lambda_j = 1;$$

$$\lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n;$$

where $\tilde{x}_{ij}$ is fuzzy input, $\tilde{\delta}_i$ is fuzzy fluctuate input, $\tilde{y}_{rj}$ is fuzzy output, and $\tilde{\beta}_{ro}$ is fuzzy fluctuate output for $i = 1, 2, \ldots, m; r = 1, 2, \ldots, s;$ and $j = 1, 2, \ldots, n$.

An interval solution for fuzzy inverse DEA model in cloud network is presented by Zavieh et al.[26] Here we intend to employ the ranking function methodology to handle the fuzzy objective function, while simultaneously exploring the optimal scenario through the utilization of an α-cut coefficient for the constraint. We assume that Formula (6) involves inputs and outputs in the format of fuzzy triangular numbers.

$$\tilde{x}_{ij} = (x_{ij}^{l}, x_{ij}^{m}, x_{ij}^{u}),$$

$$\tilde{y}_{rj} = (y_{rj}^{l}, y_{rj}^{m}, y_{rj}^{u}),$$

$$\tilde{\delta}_{i} = (\delta_{i}^{l}, \delta_{i}^{m}, \delta_{i}^{u}),$$

$$\tilde{\beta}_{ro} = (\beta_{ro}^{l}, \beta_{ro}^{m}, \beta_{ro}^{u}).$$

Now, we direct our attention toward the extraction of α-cut from fuzzy triangular numbers for each constraint and replace that into Formula (6):

$$\min \left(R\left(\tilde{\delta}_1\right), R\left(\tilde{\delta}_2\right), \ldots, R\left(\tilde{\delta}_m\right)\right) \tag{7}$$

$$\text{s.t.,}\quad \sum_{j=1}^{n} \lambda_j \left[\alpha x_{ij}^{m} + (1-\alpha)\, x_{ij}^{l}, \alpha x_{ij}^{m} + (1-\alpha)\, x_{ij}^{u}\right] \leqslant \theta_o^{*} \left[\alpha \delta_i^{m} + (1-\alpha)\, \delta_i^{l}, \alpha \delta_i^{m} + (1-\alpha)\, \delta_i^{u}\right],$$

$$\sum_{j=1}^{n} \lambda_j \left[\alpha y_{rj}^{m} + (1-\alpha)\, y_{rj}^{l}, \alpha y_{rj}^{m} + (1-\alpha)\, y_{rj}^{u}\right] \geqslant \left[\alpha \beta_{ro}^{m} + (1-\alpha)\, \beta_{ro}^{l}, \alpha \beta_{ro}^{m} + (1-\alpha)\, \beta_{ro}^{u}\right],$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n,$$

where we considered the ranking function as $R\left(\tilde{\delta}_i\right) = \delta_i^{m} + \dfrac{\delta_i^{u} - \delta_i^{l}}{2}$. We see that Formula (7) is a parametric linear problem based on the parameter α. The choice of α varies depending on our manager's choice between zero and one. For calculation θ_o^{*}, we will use the following model:

$$\min \theta_o \tag{8}$$

$$\text{s.t.,}\quad \sum_{j=1}^{n} \lambda_j x_{ij}^{U} \leqslant \theta_o x_{io}^{L}, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{L} \geqslant y_{ro}^{U}, \quad r = 1, 2, \ldots, s;$$

$$\sum_{j=1}^{n} \lambda_j = 1;$$

$$\lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n.$$

The current model considers the best situation for the problem.

We know that Formula (7) is the minimizing form, so, we consider a status for the constraints which is the largest area, it means that in the first constraint, the lower bound of α-cut for the left-hand side is considered, and also the upper bound of the α-cut for right-hand side is considered. Similar to the first one, in the second upper bound of α-cut for the left-hand side is considered, and the lower bound of α-cut for the right-hand side is considered.

Now, we can write the concluded model as follows:

$$\min\left(R\left(\tilde{\delta}_1\right), R\left(\tilde{\delta}_2\right), \ldots, R\left(\tilde{\delta}_m\right)\right) \tag{9}$$

$$\text{s.t.,} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^L \leqslant \theta_o^* \delta_i^U, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^U \geqslant \beta_{ro}^L, \quad r = 1, 2, \ldots, s;$$

$$\sum_{j=1}^{n} \lambda_j = 1;$$

$$\lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n.$$

The current linear programming problem is an MOLP problem that by solving this problem, the inputs are obtained in the best condition to maintain the relative efficiency of the units under evaluation.

Theorem 3.1 Suppose that the amount of the relative performance for DMU_o considering another DMU_s in a company of comparable DMU_s is θ_o^*. Also, suppose that the output with fuzzy values of DMU_o has fluctuated $\tilde{y}_o$ to $\tilde{\beta}_o \neq \tilde{0}$. At least one optimal solution to solve the FIBCC problem exists, if and only if $\beta_{ro}^L \in \mathrm{PPS}_{\mathrm{out}}^*$, where

$$\mathrm{PPS}_{\mathrm{out}} = \left\{ y | y \leqslant \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geqslant o, j = 1, 2, \ldots, n \right\},$$

$$\mathrm{PPS}_{\mathrm{out}}^* = \left\{ \beta_{ro}^L | \sum_{j=1}^{n} \lambda_j y_{rj}^U \geqslant \beta_{ro}^L, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geqslant 0, j = 1, 2, \ldots, n, r = 1, 2, \ldots, s \right\}.$$

Proof If $\beta_{ro}^L \in \mathrm{PPS}_{\mathrm{out}}^*$, then the constraints $\sum_{j=1}^{n} \lambda_j y_{rj}^U \geqslant \beta_{ro}^L, r = 1, 2, \ldots, s$ and $\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geqslant 0,$ $j = 1, 2, \ldots, n$ are satisfied in Formula (7). The constraints $\sum_{j=1}^{n} \lambda_j x_{ij}^L \leqslant \theta_o^* \delta_i^U, i = 1, 2, \ldots, m$ can be satisfied by finding a suitable value of $(\Delta x_{io})^L$. Also, from the $\sum_{j=1}^{n} \lambda_j x_{ij}^L \leqslant \theta_o^* \delta_i^U, i = 1, 2, \ldots, m$, we know that $\delta_i^U = (x_{io} + \Delta x_{io})^U \geqslant 0$. The objective function of the Formula (9) is to minimizing δ_i^L, so, at least one optimal solution to solve the problem has existed.

If at least one optimal solution to solve the FIBCC problem (Formula 9) exists, from the constraints $\sum_{j=1}^{n}\lambda_j y_{rj}^U \geqslant \beta_{ro}^L$ for $r=1,2,\ldots,s$ in the model, then $\mathrm{PPS}_{\mathrm{out}}^*$. ■

Now, we show the superior performance of our solution. We have performed a numerical example.

4 Case Study: Selecting an Efficient Car in Manufacturing and Preserving Efficiency in Uncertainty Condition

A car manufacturing plant is considering selecting the best car from its cars based on fuzzy inputs and outputs. In the decision-making process, management consideres several factors such as fuel consumption $(\tilde{x}_1)$ and operational costs $(\tilde{x}_2)$ for inputs and sales amount $(\tilde{y})$ for output. However, the available information usually is imprecise and uncertain. The company wants to use triangular fuzzy numbers to select the best car based on the criteria. Also, the company wants to know if the sales amount has some tolerance, then the fuel consumption and operational costs how much should be changed to preserve the efficiency of cars. The fuzzy inputs and output of each production unit comes in Table 1.

We would like to achieve the number of input changes by changing the output so that the efficiency of the decision units is maintained. In the case study, the condition of Theorem 3.1 is satisfied, it means that $\beta_{1o}^L \in \mathrm{PPS}_{\mathrm{out}}^*$.

Firstly, we conclude θ_o^* for $o=1,2,3,4$, the results of the α-cut method (the result is considered for different α) are as follows:

The results show that $\mathrm{Car}_1, \mathrm{Car}_2$, and Car_4 are efficiency units and Car_3 is non-efficiency with all of the parameter α, as shown in Table 2. Below, we give Formula (6) for $\alpha=0.5$:

$$\min\left(R\left(\tilde{\delta}_1\right), R\left(\tilde{\delta}_2\right)\right)=\left(\delta_1^m+\frac{\delta_1^U-\delta_1^L}{2}, \delta_2^m+\frac{\delta_2^U-\delta_2^L}{2}\right),$$

$$\text{s.t.,}\quad 1.5\lambda_1+9\lambda_2+6.5\lambda_3+16\lambda_4 \leqslant \theta_o^*\delta_1^U,$$

$$12.5\lambda_1+21.5\lambda_2+18.5\lambda_3+33.5\lambda_4 \leqslant \theta_o^*\delta_2^U,$$

$$72.5\lambda_1+85.5\lambda_2+74.5\lambda_3+99.5\lambda_4 \geqslant \beta_{1o}^L,$$

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=1,$$

$$\delta_1^u, \delta_2^u \geqslant 0, \lambda_j \geqslant 0, j=1,2,3,4.$$

Table 1 Cars with fuzzy inputs and output.

Car	Fuel consumption $(\tilde{x}_1)$	Operational cost $(\tilde{x}_2)$	Sales amount $(\tilde{y})$
Car_1	(1, 2, 3)	(12, 13, 15)	(69, 72, 73)
Car_2	(8, 10, 13)	(21, 22, 24)	(81, 85, 88)
Car_3	(6, 7, 9)	(18, 19, 20)	(72, 74, 75)
Car_4	(15, 17, 18)	(33, 34, 36)	(97, 99, 100)

Table 2 Efficiency of units (cars).

Car	Efficiency		
	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
Car_1	$\theta_1^* = 1$	$\theta_1^* = 1$	$\theta_1^* = 1$
Car_2	$\theta_2^* = 1$	$\theta_2^* = 1$	$\theta_2^* = 1$
Car_3	$\theta_3^* = 0.995$	$\theta_3^* = 0.751$	$\theta_3^* = 0.750$
Car_4	$\theta_4^* = 1$	$\theta_4^* = 1$	$\theta_4^* = 1$

We will use the weighted method to solve a multi-objective programming problem, and so we consider $(W_1, W_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Suppose that the output of units Car_1, Car_2, Car_3, and Car_4 are changed to (68, 73, 75), (82, 85, 89), (72, 75, 77), and (98, 100, 101), the result of the proposed method are as shown in Table 3.

For example, for an efficiency unit Car_1 with $\alpha = 0.5$, the change rate of inputs must be (−1.041, −1.845) until the efficiency of Car_1 is maintained, and for the non-efficiency unit Car_3, the change rate of inputs must be changed to (−5.234, −1.934) till the efficiency of Car_3 is preserved.

5 Comparison Analysis

By comparing our model with Lertworasirikul et al.[8] which is modified by Ghiyasi[9], we noted that:

(1) Our proposed model considers the IBCC model with fuzzy data in the objective function and constraints, and a solution is proposed for solving. In contrast, the model presented by Lertworasirikul et al.[8] has not proposed a solution to the problem with fuzzy fluctuation data.

(2) Our suggested method used a mixed ranking method with an α-cut approach to solving the model, and the model is considered for different manners that are given more results to compare DMUs.

Also by comparing our suggested model with all of the IDEA models, we found that:

All of the other presented models and applications in IDEA are used accurate data, but the suggested method can solve that kind of problem for ambiguous data that is more realistic in the real world instead of classical ones.

In most cases, when using the α-cut method to deal with fuzzy issues, only one α value is used, while in this paper, we used three different α values to examine and analyze more thoroughly and establish the efficiency of decision-making units in different situations.

6 Conclusion and Future Work

In practice, we face uncertain or fuzzy data for measuring the efficiency of units, so it plays a critical

Table 3 Amount of input changes to preserve the relative efficiency of DMUs (cars).

Car	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	Δx_1	Δx_2	Δx_1	Δx_2	Δx_1	Δx_2
Car_1	2.331	2.500	−1.041	−1.845	−0.619	−1.041
Car_2	−4.352	−3.35	−3.654	−2.885	−2.519	−2.091
Car_3	−2.651	−1.861	−5.234	−1.934	−3.975	−0.778
Car_4	−4.537	−3.148	−1.75	−1.929	−0.848	−0.81

role in unit performance evaluations. To address this important challenge, in this paper, a novel solution for achieving relative efficiency for DMUs with the leverage of a BCC reverse model with uncertain data has been proposed. Furthermore, to achieve a more efficient objective function, we used the ranking function. Moreover, we selected the best options for the IBCC model by calculating the α-cut of fuzzy numbers in constraints. Finally, by solving the inverse model with the fluctuating fuzzy outputs, the inputs have been changed in a way that we can preserve the relative performance of decision-making units. Based on the evaluation results, 75% of DMUs were efficient. For future work, we aim to leverage the method on top of various DEA models in fuzzy environments.

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