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*Research article*

# Almost sure convergence for a class of dependent random variables under sub-linear expectations

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**Abstract:** This article aimed to investigate the almost sure convergence theorem of widely negative orthant dependent (WNOD) random variables under sub-linear expectation space. The conclusions in this essay are an extension of the corresponding conclusions in the classical probability space.

**Keywords:** widely negative orthant dependent; almost sure convergence; sub-linear expectation
**Mathematics Subject Classification:** 60F15

## 1. Introduction

The limit theorem plays a pivotal role in the study of probability theory. Furthermore, the almost sure convergence is integral to the development of the limit theorem, a subject many scholars have studied. So far, a lot of excellent results have been obtained under the condition that the model holds with certainty. However, many uncertain phenomena of quantum mechanics and risk management cannot be explained by additive probability or expectation. To deal with this issue, many scholars have made great attempts and efforts. In particular, Peng [1,2] proposed the theory frame of the sub-linear expectations under a generic function space to solve this distributional uncertainty. In recent years, based on Peng, more and more scholars in the industry have done extensive research and obtained many related results; the study of the almost sure convergence has remained a hot-button issue. For example, Chen [3], Cheng [4], and Feng and Lan [5] obtained the SLLN (strong law of large numbers) of i.i.d.r.v. (independent identically distributed random variables), and Cheng [6] studied the SLLN of independent r.v. with $\sup_{i\geq 1} \hat{\mathbb{E}}\left[|X_i|\, l(|X_i|)\right] < \infty$. Through further research, Wu and Jiang [7] obtained the SLLN of the extended independent and identically distributed r.v.; Chen and Liu [8], Gao et al. [9], and Liang and Wu [10] proved the SLLN of ND (negatively dependent) r.v.; Zhang [11] built the exponential inequality and the law of logarithm of independent and ND r.v.; Wang and Wu [12] and Feng [13] offered the almost sure covergence for weighted sums of ND r.v.; Zhang [14] derived the

SLLN of the extended independent and END (extended negatively dependent) r.v.; Wang and Wu [15] obtained the almost sure convergence of END r.v.; Lin [16] achieved the SLLN of WND (widely negative dependence) r.v.; and Hwang [17] investigated the almost sure convergence of WND r.v..

Anna [18] proposed the definition of WNOD r.v. for the first time and obtained the limiting conclusions for WNOD r.v. in Peng's theory frame. Based on Yan's results [19], this paper promotes them to the sub-linear expectation space. Compared to the previously mentioned ND and END r.v., dominating coefficients $g(n)$ have been added to the definition of WNOD r.v., leading to a broader range. Besides, the sub-additivity property of the sub-linear expectation and capacity is added, making the research more meaningful and complex. Finally, the conclusions of almost sure convergence for WNOD r.v. are achieved. This paper contributes to the relevant research results of limiting behavior of WNOD r.v. in Peng's theory frame.

Our essay is arranged as follows: Section 2 recommends interrelated definitions and properties as well as some important lemmas in the frame. Section 3 gives the conclusions including two theorems and two corollaries. Section 4 shows that the process of proving the conclusions is given in detail.

Running through this essay, we point out that $c$ will be a positive constant, its value is not important, and it may take different values according to the situation. $a_x \sim b_x$ means $\lim_{x\to\infty} \frac{a_x}{b_x} = 1$. $a_n \ll b_n$ means there must be a positive number $c$, satifying $a_n \le cb_n$ when $n$ is large enough. Denote $\log(y) = \ln(\max\{\mathrm{e}, y\})$.

## 2. Preliminaries

This article uses the theory frame and concepts proposed by Peng [1, 2]. Suppose $(\Omega, \mathcal{F})$ is a given measurable space and $\mathcal{H}$ is a linear space of real functions defined on $(\Omega, \mathcal{F})$ so that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for every $\varphi \in C_{l,\,\mathrm{Lip}}(\mathbb{R}_n)$, where $\varphi \in C_{l,\,\mathrm{Lip}}(\mathbb{R}_n)$ shows the linear space of (local Lipschitz) functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \le c\,(1 + |x|^m + |y|^m)\,|x - y|, \quad \forall x, y \in \mathbb{R}_n,$$

for some $c > 0, m \in \mathbb{N}$ depending on $\varphi$. $\mathcal{H}$ is considered as a space of random variable. In this circumstance, we denote $X \in \mathcal{H}$.

**Definition 2.1.** (Peng [1]). A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: if $X \ge Y$, then $\hat{\mathbb{E}}(X) \ge \hat{\mathbb{E}}(Y)$;

(b) Constant preserving: $\hat{\mathbb{E}}(c) = c$;

(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \le \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$;

(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda\hat{\mathbb{E}}(X), \lambda \ge 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is known as a sub-linear expectation space.

Next, give the definition of the conjugate expectation $\hat{\mathcal{E}}$ of $\hat{\mathbb{E}}$ by

$$\hat{\mathcal{E}}(X) := -\hat{\mathbb{E}}(-X), \forall X \in \mathcal{H}.$$

By the above definitions of $\hat{\mathbb{E}}$ and $\hat{\mathcal{E}}$, the following inequality is feasible for all $X, Y \in \mathcal{H}$,

$$\hat{\mathcal{E}}(X) \le \hat{\mathbb{E}}(X),$$

$$\hat{\mathbb{E}}(X-Y) \geq \hat{\mathbb{E}}(X)-\hat{\mathbb{E}}(Y),$$
$$\hat{\mathbb{E}}(X+c) = \hat{\mathbb{E}}(X)+c,$$
$$|\hat{\mathbb{E}}(X-Y)| \leq \hat{\mathbb{E}}|X-Y|.$$

When we are talking about $\hat{\mathbb{E}}$ and $\hat{\mathcal{E}}$ in the course of the proof, we often use the above formula.

**Definition 2.2.** (Peng [1]). Make $\mathcal{G} \subset \mathcal{F}$, a function $V : \mathcal{G} \to [0,1]$ is described to be a capacity, when

$$V(\emptyset)=0, V(\Omega)=1 \text{ and } V(A) \leq V(B) \text{ for } A \subset B, A, B \in \mathcal{G}.$$

Similar to sub-linear expectations, it is known as sub-additive when $V(A \cup B) \leq V(A)+V(B)$ for every $A, B \in \mathcal{G}$. Now, represent $\mathbb{V}$ and $\mathcal{V}$, respectively corresponing to $\hat{\mathbb{E}}$ and $\hat{\mathcal{E}}$, using

$$\mathbb{V}(A):=\inf \left\{\hat{\mathbb{E}}[\xi], I_A \leq \xi, \xi \in \mathcal{H}\right\}, \mathcal{V}(A):=1-\mathbb{V}\left(A^c\right), A \in \mathcal{F},$$

where $A^c$ denotes the complement set of $A$.

From the definion and sub-additivity property of ($\mathbb{V}$,$\mathcal{V}$), the following formulas are true

$$\hat{\mathbb{E}}\zeta \leq \mathbb{V}(C) \leq \hat{\mathbb{E}}\eta, \hat{\mathcal{E}}\zeta \leq \mathcal{V}(C) \leq \hat{\mathcal{E}}\eta, \quad \text{if} \quad \zeta \leq I(C) \leq \eta, \quad \zeta, \eta \in \mathcal{H}.$$

And now we have Markov inequality:

$$\mathbb{V}(|Y| \geq y) \leq \hat{\mathbb{E}}|Y|^P / y^p, \forall y>0, p>0.$$

**Definition 2.3.** (Peng [1]). The Choquet integrals ($C_V$) is defined as follows

$$C_V(X)=\int_0^{\infty} V(X \geq x) \mathrm{d}x+\int_{-\infty}^0[V(X \geq x)-1] \mathrm{d}x,$$

where $\mathbb{V}$ and $\mathcal{V}$ can replace $V$ when required.

**Definition 2.4.** (Zhang [11]). (i) $\hat{\mathbb{E}}$ is referred to be countably sub-additive, when

$$\hat{\mathbb{E}}(X) \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(X_n\right), \quad \text{whenever} \quad X \leq \sum_{n=1}^{\infty} X_n, \quad X, X_n \in \mathcal{H}, \quad X \geq 0, X_n \geq 0, \quad n \geq 1.$$

(ii)$V$ is referred to be countably sub-additive when

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V\left(A_n\right), \forall A_n \in \mathcal{F}.$$

**Definition 2.5.** $\{X_n, n \geq 1\}$ is a sequence of r.v. and it is known to be stochastically dominated by a random variable $X$ if for a positive number $c$, there has

$$\hat{\mathbb{E}}\left[f\left(\left|X_n\right|\right)\right] \leq c \hat{\mathbb{E}}[f(|X|)], \text{for } n \geq 1, 0 \leq f \in C_{l,Lip}(\mathbb{R}).$$

**Definition 2.6.** (Anna [18]. Widely negative orthant dependent (WNOD)) $\{X_n, n \geq 1\}$ is called to be widely negative orthant dependent if there is a finite positive array $\{g(n), n \geq 1\}$ satisfying for every $n \geq 1$,

$$\hat{\mathbb{E}}\left(\prod_{i=1}^{n} \varphi_i(X_i)\right) \leq g(n) \prod_{i=1}^{n} \hat{\mathbb{E}}\ (\varphi_i(X_i)),$$

where $\varphi_i \in C_{b,Lip}(\mathbb{R}), \varphi_i \geq 0, 1 \leq i \leq n$ and all functions $\varphi_i$ are uniformly monotonous. Where the coefficients $g(n)$ $(n \geq 1)$ are known as dominating coefficients.

It is visible that, when $\{X_n, n \geq 1\}$ is widely negative orthant dependent and all functions $f_k(x) \in C_{l,Lip}(\mathbb{R})$ (where $k = 1, 2, \cdots, n$) are uniformly monotonous, then $\{f_n(X_n), n \geq 1\}$ is also widely negative orthant dependent .

**Definition 2.7.** (Seneta [20]). (i) A positive function $l(x)$ defined on $[a, \infty), a > 0$ is known to be a slowly varying function, satisfying

$$\lim_{x\to\infty} \frac{l(tx)}{l(x)} = 1, \text{for each}\, t \geq 0.$$

(ii) Each slowly varying function $l(x)$ can be expressed as

$$l(x) = C(x) \exp\left\{\int_1^x \frac{f(y)}{y} \mathrm{d}y\right\},$$

whenever $\lim_{x\to\infty} C(x) = c > 0$, as well as $\lim_{y\to\infty} f(y) = 0$.

In this article, we want to research the almost sure convergence of WNOD sequence under sub-linear expectations. Since $\mathbb{V}$ is only sub-addictive, the definition of almost sure convergence is a little different and is described in detail in Wu and Jiang [7].

Next, we give some lemmas before reaching our conclusions.

**Lemma 2.1.** *(Seneta [20]). For $\forall\alpha > 0$, there is a non-decreasing function $\varphi(x)$ and a non-increasing function $\xi(x)$ such that*

$$x^{\alpha} l(x) \sim \varphi(x), x^{-\alpha} l(x) \sim \xi(x), x \to \infty,$$

*where $l(x)$ is a slowly varying function.*

In the following section, we assume $l(x), x > 0$ is a non-decreasing slowly varying function that can be expressed as $l(x) = c\exp\left\{\int_1^x \frac{f(y)}{y} \mathrm{d}y\right\}$, where $c > 0$, $\lim_{x\to\infty} f(x) = 0$.

Let

$$\tau_n = nl(n)^{-1}, n \geq 1. \tag{2.1}$$

$$X_n' = -\tau_n^{1/p} I\left(X_n < -\tau_n^{1/p}\right) + X_n I\left(|X_n| \leq \tau_n^{1/p}\right) + \tau_n^{1/p} I\left(X_n > \tau_n^{1/p}\right). \tag{2.2}$$

$$X_n'' = X_n - X_n' = \left(X_n + \tau_n^{1/p}\right) I\left(X_n < -\tau_n^{1/p}\right) + \left(X_n - \tau_n^{1/p}\right) I\left(X_n > \tau_n^{1/p}\right). \tag{2.3}$$

**Lemma 2.2.** *Assume $X \in \mathcal{H}, 0 < p < 2, \tau_n$ defined by Eq (2.1).*

(i) *For every $c > 0$,*

$$C_{\mathbb{V}}(|X|^p) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} l^{-1}(n) \mathbb{V}(|X|^p > c\tau_n) < \infty. \tag{2.4}$$

(ii) *When* $C_{\mathbb{V}}(|X|^p) < \infty$, *and now for every* $c > 0$,

$$\sum_{k=1}^{\infty} \frac{2^k}{l(2^k)} \mathbb{V}(|X|^p > c\tau_{2^k}) < \infty. \tag{2.5}$$

*Proof.* (i) Obviously,

$$C_{\mathbb{V}}(|X|^p) < \infty \Longleftrightarrow C_{\mathbb{V}}(|X|^p/c) < \infty.$$

$$\begin{aligned} C_{\mathbb{V}}(|X|^p/c) &\sim \int_1^{\infty} \mathbb{V}(|X|^p > cx)\,\mathrm{d}x \\ &\sim \int_1^{\infty} \frac{l(y) - yl(y) \cdot \frac{f(y)}{y}}{l^2(y)} \mathbb{V}\left(|X|^p > c \cdot \frac{y}{l(y)}\right)\mathrm{d}y \quad \left(\text{make} \quad x = \frac{y}{l(y)}\right) \\ &\sim \int_1^{\infty} \frac{1}{l(y)} \mathbb{V}\left(|X|^p > c\tau_y\right)\mathrm{d}y. \end{aligned}$$

So

$$C_{\mathbb{V}}(|X|^p) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} l^{-1}(n)\,\mathbb{V}(|X|^p > c\tau_n) < \infty.$$

(ii) For every positive $c$, using the conclusion of (i), because of the monotonically increasing property of $l(x)$,

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} l^{-1}(n)\,\mathbb{V}(|X|^p > c\tau_n) \\ &= \sum_{k=1}^{\infty} \sum_{2^{k-1} \le n < 2^k} l^{-1}(n)\,\mathbb{V}(|X|^p > c\tau_n) \\ &\ge \sum_{k=1}^{\infty} \sum_{2^{k-1} \le n < 2^k} l^{-1}\left(2^k\right)\mathbb{V}(|X|^p > c\tau_{2^k}) \\ &= 2^{-1} \sum_{k=1}^{\infty} 2^k l^{-1}\left(2^k\right)\mathbb{V}(|X|^p > c\tau_{2^k}). \end{aligned}$$

As such, we have completed the proof of (ii). □

**Lemma 2.3.** $\{X_n, n \ge 1\}$ *is a sequence of random variables, as well as stochastically dominated by a r.v.* $X$ *and* $C_{\mathbb{V}}(|X|^p) < \infty, 1 \le p < 2$, $\hat{\mathbb{E}}$ *has countable sub-additivity, then*

$$\sum_{n=1}^{\infty} \tau_n^{-2/p} l^{-1}(n)\,\hat{\mathbb{E}}(X_n')^2 < \infty, \tag{2.6}$$

*moreover, when* $1 < p < 2$,

$$\sum_{n=1}^{\infty} \tau_n^{-1/p} l^{-1}(n)\,\hat{\mathbb{E}}\left|X_n''\right| < \infty. \tag{2.7}$$

*Where* $X_n', X_n''$ *are respectively defined by Eqs (2.2) and (2.3).*

*Proof.* For $0 < \mu < 1$, assume an even function $h(x) \in C_{l,Lip}(\mathbb{R})$ and $h(x) \downarrow$ when $x > 0$, so that the value of $h(x)$ is $[0,1]$, for $\forall x \in \mathbb{R}$ and $h(x) \equiv 1$ when $|x| \le \mu$, $h(x) \equiv 0$ when $|x| > 1$. We have

$$I(|x| \le \mu) \le h(|x|) \le I(|x| \le 1), I(|x| > 1) \le 1 - h(x) \le I(|x| > \mu). \tag{2.8}$$

For $\alpha = 1, 2$,

$$\begin{aligned} \left|X_k'\right|^\alpha &= |X_k|^\alpha I\left(|X_k| \le \tau_k^{1/p}\right) + \tau_k^{\alpha/p} I\left(|X_k| > \tau_k^{1/p}\right) \\ &\le \left|X_k'\right|^\alpha h\left(\frac{\mu |X_k|}{\tau_k^{1/p}}\right) + \tau_k^{\alpha/p}\left(1 - h\left(\frac{|X_k|}{\tau_k^{1/p}}\right)\right). \end{aligned} \tag{2.9}$$

$$\begin{aligned} \left|X_k''\right|^\alpha &= \left|X_k + \tau_k^{1/p}\right|^\alpha I\left(X_k < -\tau_k^{1/p}\right) + \left|X_k - \tau_k^{1/p}\right|^\alpha I\left(X_k > \tau_k^{1/p}\right) \\ &= \left|-|X_k| + \tau_k^{1/p}\right|^\alpha I\left(X_k < -\tau_k^{1/p}\right) + \left||X_k| - \tau_k^{1/p}\right|^\alpha I\left(X_k > \tau_k^{1/p}\right) \\ &= \left||X_k| - \tau_k^{1/p}\right|^\alpha I\left(|X_k| > \tau_k^{1/p}\right) \\ &\le |X_k|^\alpha I\left(|X_k| > \tau_k^{1/p}\right) \\ &\le |X_k|^\alpha \left(1 - h\left(\frac{|X_k|}{\tau_k^{1/p}}\right)\right). \end{aligned} \tag{2.10}$$

So, by (2.8) and Definition 2.7,

$$\begin{aligned} \hat{\mathbb{E}}\left|X_k'\right|^\alpha &\le \hat{\mathbb{E}}|X_k|^\alpha h\left(\frac{\mu |X_k|}{\tau_k^{1/p}}\right) + \tau_k^{\alpha/p}\hat{\mathbb{E}}\left(1 - h\left(\frac{|X_k|}{\tau_k^{1/p}}\right)\right) \\ &\le \hat{\mathbb{E}}|X|^\alpha h\left(\frac{\mu |X|}{\tau_k^{1/p}}\right) + \tau_k^{\alpha/p}\hat{\mathbb{E}}\left(1 - h\left(\frac{|X|}{\tau_k^{1/p}}\right)\right) \\ &\le \hat{\mathbb{E}}|X|^\alpha h\left(\frac{\mu |X|}{\tau_k^{1/p}}\right) + \tau_k^{\alpha/p}\mathbb{V}\left(|X| > \mu\tau_k^{1/p}\right). \end{aligned} \tag{2.11}$$

$$\hat{\mathbb{E}}\left|X_k''\right|^\alpha \le \hat{\mathbb{E}}|X|^\alpha \left(1 - h\left(\frac{|X|}{\tau_k^{1/p}}\right)\right). \tag{2.12}$$

Assume that $h_j(x) \in C_{l,Lip}(\mathbb{R}), j \ge 1$, consider that the value of $h_j(x)$ is $[0,1]$ for $\forall x \in \mathbb{R}$ . $h_j(x) \equiv 1$ when $\tau_{2^{j-1}}^{1/p} < |x| \le \tau_{2^j}^{1/p}$; $h_j(x) \equiv 0$ when $|x| \le \mu\tau_{2^{j-1}}^{1/p}$ or $|x| > (1+\mu)\tau_{2^j}^{1/p}$. The following formulas can be derived,

$$I\left(\tau_{2^{j-1}}^{1/p} < |x| \le \tau_{2^j}^{1/p}\right) \le h_j(|x|) \le I\left(\mu\tau_{2^{j-1}}^{1/p} < |x| \le (1+\mu)\tau_{2^j}^{1/p}\right). \tag{2.13}$$

$$|X|^r h\left(\frac{|X|}{\tau_{2^k}^{1/p}}\right) \le 1 + \sum_{j=1}^{k} |X|^r h_j(|X|), \quad r > 0. \tag{2.14}$$

$$|X|^r \left(1 - h\left(\frac{|X|}{\tau_{2^k}^{1/p}}\right)\right) \le \sum_{j=k}^{\infty} |X|^r h_j\left(\frac{|X|}{\mu}\right), \quad r > 0. \tag{2.15}$$

First, prove (2.6). For $1 \le p < 2$ , by (2.11) and (2.4),

$$
\begin{aligned}
H_1 := &\sum_{n=1}^{\infty} \tau_n^{-2/p} l^{-1}(n)\,\hat{\mathbb{E}}\,(X_n')^2 \\
\le &\sum_{n=1}^{\infty} \tau_n^{-2/p} l^{-1}(n)\left[\hat{\mathbb{E}}\left(X^2 h\left(\frac{\mu|X|}{\tau_n^{1/p}}\right)\right) + \tau_n^{2/p}\mathbb{V}\left(|X| > \mu\tau_n^{1/p}\right)\right] \\
= &\sum_{n=1}^{\infty} \tau_n^{-2/p} l^{-1}(n)\,\hat{\mathbb{E}}\left[X^2 h\left(\frac{\mu|X|}{\tau_n^{1/p}}\right)\right] + \sum_{n=1}^{\infty} l^{-1}(n)\,\mathbb{V}\left(|X| > \mu\tau_n^{1/p}\right) \\
\ll &\sum_{n=1}^{\infty} \tau_n^{-2/p} l^{-1}(n)\,\hat{\mathbb{E}}\left[X^2 h\left(\frac{\mu|X|}{\tau_n^{1/p}}\right)\right].
\end{aligned}
$$

Then, because $h(x)$ is decreasing in $(0,\infty)$, according to Lemma 2.1, $\tau_n^{-2/p} l^{-1}(n)$ is decreasing in $(0,\infty)$. So,

$$
\begin{aligned}
H_1 \ll &\sum_{k=1}^{\infty} \sum_{2^{k-1}\le n<2^k} \tau_n^{-2/p} l^{-1}(n)\,\hat{\mathbb{E}}\left[X^2 h\left(\frac{\mu|X|}{\tau_n^{1/p}}\right)\right] \\
\le &\sum_{k=1}^{\infty} \sum_{2^{k-1}\le n<2^k} \tau_{2^{k-1}}^{-2/p} l^{-1}\left(2^{k-1}\right)\hat{\mathbb{E}}\left[X^2 h\left(\frac{\mu|X|}{\tau_{2^k}^{1/p}}\right)\right] \\
\ll &\sum_{k=1}^{\infty} \sum_{2^{k-1}\le n<2^k} \tau_{2^k}^{-2/p} l^{-1}\left(2^k\right)\hat{\mathbb{E}}\left[X^2 h\left(\frac{\mu|X|}{\tau_{2^k}^{1/p}}\right)\right] \\
\ll &\sum_{k=1}^{\infty} 2^k \tau_{2^k}^{-2/p} l^{-1}\left(2^k\right)\hat{\mathbb{E}}\left[X^2 h\left(\frac{\mu|X|}{\tau_{2^k}^{1/p}}\right)\right].
\end{aligned}
$$

Last by (2.14), (2.13), and (2.5),

$$
\begin{aligned}
H_1 \ll &\sum_{k=1}^{\infty} 2^k \tau_{2^k}^{-2/p} l^{-1}\left(2^k\right) + \sum_{k=1}^{\infty} 2^k \tau_{2^k}^{-2/p} l^{-1}\left(2^k\right)\sum_{j=1}^{k}\hat{\mathbb{E}}\left(X^2 h_j(\mu|X|)\right) \\
\ll &\sum_{j=1}^{\infty}\sum_{k=j}^{\infty} 2^k \tau_{2^k}^{-2/p} l^{-1}\left(2^k\right)\hat{\mathbb{E}}\left(X^2 h_j(\mu|X|)\right) \\
\ll &\sum_{j=1}^{\infty} 2^j \tau_{2^j}^{-2/p} l^{-1}\left(2^j\right)\tau_{2^j}^{2/p}\mathbb{V}\left(|X| > \tau_{2^{j-1}}^{1/p}\right) \\
\ll &\sum_{j=1}^{\infty} \frac{2^j}{l(2^j)}\mathbb{V}\left(|X| > \tau_{2^j}^{1/p}\right) \\
< &\infty.
\end{aligned}
$$

Therefore, (2.6) holds.

Next, our proof of (2.7) is similar to (2.6). For $1 < p < 2$, by (2.12) and the monotonically decreasing propety of $h(x)$ in $(0,\infty)$, according to Lemma 2.1, $\tau_n^{-1/p} l^{-1}(n)$ is decreasing in $(0,\infty)$, we have,

$$
\begin{aligned}
H_2 := & \sum_{n=1}^{\infty} \tau_n^{-1/p} l^{-1}(n)\, \hat{\mathbb{E}}\left|X_n''\right| \\
\leq & \sum_{n=1}^{\infty} \tau_n^{-1/p} l^{-1}(n)\, \hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_n^{1/p}}\right)\right)\right] \\
= & \sum_{k=1}^{\infty} \sum_{2^{k-1}\leq n<2^k} \tau_n^{-1/p} l^{-1}(n)\, \hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_n^{1/p}}\right)\right)\right] \\
\leq & \sum_{k=1}^{\infty} 2^k \tau_{2^{k-1}}^{-1/p} l^{-1}\left(2^{k-1}\right) \hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_{2^{k-1}}^{1/p}}\right)\right)\right] \\
\ll & \sum_{k=1}^{\infty} 2^k \tau_{2^k}^{-1/p} l^{-1}\left(2^k\right) \hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_{2^k}^{1/p}}\right)\right)\right].
\end{aligned}
$$

Then, from (2.15), (2.13), and (2.5), countable sub-additivity of $\hat{\mathbb{E}}$,

$$
\begin{aligned}
H_2 \ll & \sum_{k=1}^{\infty} 2^k \tau_{2^k}^{-1/p} l^{-1}\left(2^k\right) \sum_{j=k}^{\infty} \hat{\mathbb{E}}\left(|X|\, h_j\left(\frac{|X|}{\mu}\right)\right) \\
= & \sum_{j=1}^{\infty} \sum_{k=1}^{j} 2^k \tau_{2^k}^{-1/p} l^{-1}\left(2^k\right) \hat{\mathbb{E}}\left(|X|\, h_j\left(\frac{|X|}{\mu}\right)\right) \\
\ll & \sum_{j=1}^{\infty} 2^j \tau_{2^j}^{-1/p} l^{-1}\left(2^j\right) \tau_{2^j}^{1/p} \mathbb{V}\left(|X| > \mu^2 \tau_{2^{j-1}}^{1/p}\right) \\
\ll & \sum_{j=1}^{\infty} \frac{2^j}{l\left(2^j\right)} \mathbb{V}\left(|X| > \mu^2 \tau_{2^j}^{1/p}\right) \\
< & \infty.
\end{aligned}
$$

Therefore, (2.7) holds. □

**Lemma 2.4.** *(Zhang [11] Borel-Cantelli Lemma ) Suppose $\{B_n; n \geq 1\}$ is an array of matters in $\mathcal{F}$. Suppose $\mathbb{V}$ has countable sub-additivity. We can obtain $\mathbb{V}(B_n; \text{i.o.}) = 0$ provided that $\sum_{n=1}^{\infty} \mathbb{V}(B_n) < \infty$, where $(B_n; \text{i.o.}) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B_m$.*

## 3. Main conclusions

**Theorem 3.1.** *Suppose $\{X_n, n \geq 1\}$ is a sequence of WNOD r.v., and its dominating coefficients are $g(n)$. The sequence is stochastically dominated by a r.v. $X$. $\hat{\mathbb{E}}$ and $\mathbb{V}$ both have countable sub-additivity, and satisfying*

$$C_{\mathbb{V}}\left(|X|^p\right) < \infty, 1 < p < 2. \tag{3.1}$$

*Make $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a positive sequence according to*

$$\max_{1\leq k\leq n} a_{nk} = O\left(\tau_n^{-1/p} l^{-1}(n)\right), n \to \infty, \tag{3.2}$$

*where $\tau_n$ is defined by (2.1) .*

*If for some* $0 < \delta < 1$,

$$\sum_{n=1}^{\infty} \mathrm{e}^{(\delta-2)l(n)} g(n) < \infty, \tag{3.3}$$

*then,*

$$\limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk} \left(X_k - \hat{\mathbb{E}}X_k\right) \le 0 \quad \text{a.s.}\mathbb{V}, \tag{3.4}$$

$$\liminf_{n\to\infty} \sum_{k=1}^{n} a_{nk} \left(X_k - \hat{\mathcal{E}}X_k\right) \ge 0 \quad \text{a.s.}\mathbb{V}, \tag{3.5}$$

*in particular, when* $\hat{\mathbb{E}}X_k = \hat{\mathcal{E}}X_k$, *then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} \left(X_k - \hat{\mathbb{E}}X_k\right) = 0 \quad \text{a.s.}\mathbb{V}. \tag{3.6}$$

**Remark 3.1.** *Theorem 3.1 under sub-linear expectations space is an extension of Theorem 2.1 of Yan [19] of the classical probability space.*

**Remark 3.2.** *If* $g(n) = M$, *for each* $n \ge 1$, *then the sequence is simplified to END. When let* $l(n) = \log n, n \ge 1$, *for* $0 < \delta < 1$,

$$\sum_{n=1}^{\infty} \mathrm{e}^{(\delta-2)l(n)} g(n) = M \sum_{n=1}^{\infty} n^{-(2-\delta)} < \infty,$$

*condition (3.3) is satisfied. By Theorem 3.1, Eqs (3.4)–(3.6) hold.*

**Remark 3.3.** *We can obtain different conclusions by taking different forms of slowly varying function* $l(x)$. *By taking* $l(n) = \log n$ *and* $l(n) = \exp\{(\log n)^{\nu}\}\,(0 < \nu < 1)$, *we will get the following two corollaries.*

**Corollary 3.1.** *Suppose* $\{X_n, n \ge 1\}$ *is a sequence of WNOD r.v., and its dominating coefficients are* $g(n)$. *The sequence is stochastically dominated by a r.v. X. Besides, the sequence is satisfied (3.1).* $\hat{\mathbb{E}}$ *and* $\mathbb{V}$ *both have countable sub-additivity. Make sure* $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ *is a positive sequence according to*

$$\max_{1\le k\le n} a_{nk} = O\left(\frac{1}{n^{1/p}\log^{1-1/p} n}\right), n \to \infty. \tag{3.7}$$

*For some* $0 < b < 1 - \delta$,

$$g(n)\, n^{-b} \le c, \tag{3.8}$$

*then (3.4)–(3.6) hold.*

**Corollary 3.2.** *Suppose* $\{X_n, n \ge 1\}$ *is a sequence of WNOD random variables, and its dominating coefficients are* $g(n)$. *The sequence is stochastically dominated by a r.v. X. Besides, the sequence is satisfied (3.1).* $\hat{\mathbb{E}}$ *and* $\mathbb{V}$ *both have countable sub-additivity. Make sure* $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ *is a positive sequence according to*

$$\max_{1\le k\le n} a_{nk} = O\left(n^{-1/p} \mathrm{e}^{(-1+1/p)\left(\log n\right)^{\nu}}\right), n \to \infty, \tag{3.9}$$

*where* $0 < \nu < 1$.

*For some* $m > 0$,

$$g(n)\, n^{-m} \le c, \tag{3.10}$$

*then (3.4)–(3.6) hold.*

Then, we will think about the situation of $p = 1$.

**Theorem 3.2.** *Suppose* $\{X_n, n \ge 1\}$ *is a sequence of WNOD r.v., and its dominating coefficients are* $g(n)$ *and are satisfied (3.3). The sequence is stochastically dominated by a r.v.* $X$. $\hat{\mathbb{E}}$ *and* $\mathbb{V}$ *both have countable sub-additivity, and satisfying*

$$C_{\mathbb{V}}(|X| \log |X|) < \infty. \tag{3.11}$$

*Suppose* $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ *is a positive sequence according to*

$$\max_{1\le k\le n} a_{nk} = O\left(n^{-1}\right), n \to \infty, \tag{3.12}$$

*then (3.4)–(3.6) hold.*

## 4. Proofs of the main conclusions

### 4.1. Proof of Theorem 3.1.

Because the sequence $\{-X_k, k \ge 1\}$ fulfills the criterion of Theorem 3.1, making $\{-X_k, k \ge 1\}$ as a substitute for $\{X_k, k \ge 1\}$ in formula (3.4), by $\hat{\mathcal{E}}X = -\hat{\mathbb{E}}(-X)$, there is

$$0 \ge \limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk}\left((-X_k) - \hat{\mathbb{E}}(-X_k)\right) = \limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk}\left((-X_k) + \hat{\mathcal{E}}X_k\right) = \limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk}\left(-\left(X_k - \hat{\mathcal{E}}X_k\right)\right).$$

$$\Rightarrow \liminf_{n\to\infty} \sum_{k=1}^{n} a_{nk}\left(X_k - \hat{\mathcal{E}}X_k\right) \ge 0.$$

Therefore, (3.5) holds. Then, by $\hat{\mathbb{E}}X_k = \hat{\mathcal{E}}X_k$, (3.4) and (3.5), we can get (3.6). So we just need to prove (3.4).

We denote $X'_n$, $X''_n$ respectively by equations (2.2) and (2.3). By Definition 2.6, $\left\{X'_k - \hat{\mathbb{E}}X'_k, k \ge 1\right\}$ is also WNOD. Denote $\widetilde{X}'_k := X'_k - \hat{\mathbb{E}}X'_k$.

Therefore,

$$\begin{aligned}\sum_{k=1}^{n} a_{nk}\left(X_k - \hat{\mathbb{E}}X_k\right) &= \sum_{k=1}^{n} a_{nk}\widetilde{X}'_k + \sum_{k=1}^{n} a_{nk}X''_k + \sum_{k=1}^{n} a_{nk}\left(\hat{\mathbb{E}}X'_k - \hat{\mathbb{E}}X_k\right) \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

So, if we want to prove (3.4), just prove

$$\limsup_{n\to\infty} I_i \le 0 \quad \text{a.s.}\mathbb{V}, i = 1, 2, \quad \textit{and} \quad \lim_{n\to\infty} I_3 = 0. \tag{4.1}$$

By (3.2) and the formula $\mathrm{e}^x \leq 1 + x + \frac{x^2}{2}\mathrm{e}^{|x|}, x \in [-\infty, \infty]$, for all $t > 0, 1 \leq k \leq n$ as well as large enough $n$,

$$\begin{aligned} \exp\left\{ta_{nk}\widetilde{X}'_k\right\} &\leq 1 + ta_{nk}\widetilde{X}'_k + \frac{t^2a_{nk}^2\left(\widetilde{X}'_k\right)^2}{2}\exp\left\{ta_{nk}\left|\widetilde{X}'_k\right|\right\} \\ &\leq 1 + ta_{nk}\widetilde{X}'_k + c\tau_n^{-2/p}l^{-2}(n)\,t^2\left(\widetilde{X}'_k\right)^2\exp\left\{ctl^{-1}(n)\right\}. \end{aligned} \tag{4.2}$$

By Definition 2.6, let $\varphi_i(x) = \mathrm{e}^{tX_i}, i \geq 1$, we can get for WNOD r.v.,

$$\hat{\mathbb{E}}\exp\left\{t\sum_{i=1}^{n}X_i\right\} \leq g(n)\prod_{i=1}^{n}\hat{\mathbb{E}}\exp\{tX_i\}. \tag{4.3}$$

By (4.2), (4.3), and the inequality $1 + x \leq \mathrm{e}^x, \forall x \in \mathbb{R}$, for all $t > 0$ as well as large enough $n$,

$$\begin{aligned} \hat{\mathbb{E}}\exp\left\{t\sum_{k=1}^{n}a_{nk}\widetilde{X}'_k\right\} &\leq g(n)\prod_{k=1}^{n}\hat{\mathbb{E}}\exp\left\{ta_{nk}\widetilde{X}'_k\right\} \\ &\leq g(n)\prod_{k=1}^{n}\hat{\mathbb{E}}\left[1 + ta_{nk}\widetilde{X}'_k + c\tau_n^{-2/p}l^{-2}(n)\,t^2\left(\widetilde{X}'_k\right)^2\exp\left\{ctl^{-1}(n)\right\}\right] \\ &\leq g(n)\prod_{k=1}^{n}\left[1 + c\tau_n^{-2/p}l^{-2}(n)\,t^2\exp\left\{ctl^{-1}(n)\right\}\hat{\mathbb{E}}\left(\widetilde{X}'_k\right)^2\right] \\ &\leq g(n)\exp\left\{c\tau_n^{-2/p}l^{-2}(n)\,t^2\exp\left\{ctl^{-1}(n)\right\}\sum_{k=1}^{n}\hat{\mathbb{E}}\left(\widetilde{X}'_k\right)^2\right]. \end{aligned}$$

For $\varepsilon > 0$, let $t = 2\varepsilon^{-1}l(n)$. According to Markov inequality, we can get

$$\begin{aligned} \mathbb{V}\left\{\sum_{k=1}^{n}a_{nk}\widetilde{X}'_k > \varepsilon\right\} &\leq \mathrm{e}^{-\varepsilon t}\hat{\mathbb{E}}\exp\left\{t\sum_{k=1}^{n}a_{nk}\widetilde{X}'_k\right\} \\ &\leq \mathrm{e}^{-\varepsilon t}g(n)\exp\left\{c\tau_n^{-2/p}l^{-2}(n)\,t^2\exp\left\{ctl^{-1}(n)\right\}\sum_{k=1}^{n}\hat{\mathbb{E}}\left(\widetilde{X}'_k\right)^2\right] \\ &\leq \mathrm{e}^{-2l(n)}g(n)\exp\left\{c\varepsilon^{-2}\exp\left\{c\varepsilon^{-1}\right\}l(n)\,\tau_n^{-2/p}l^{-1}(n)\sum_{k=1}^{n}\hat{\mathbb{E}}\left(\widetilde{X}'_k\right)^2\right\}. \end{aligned}$$

Combining $\hat{\mathbb{E}}\left(\widetilde{X}'_k\right)^2 \leq 4\hat{\mathbb{E}}\left(X'_k\right)^2$, (2.6), and Kronecker's Lemma,

$$\tau_n^{-2/p}l^{-1}(n)\sum_{k=1}^{n}\hat{\mathbb{E}}\left(\widetilde{X}'_k\right)^2 \to 0, n \to \infty.$$

So, for $\forall 0 < \delta < 1$, and large enough $n$, $l(n)$ is non-decreasing in $(0, \infty)$, we can get

$$c\varepsilon^{-2}\exp\left\{c\varepsilon^{-1}\right\}\tau_n^{-2/p}l^{-1}(n)\sum_{k=1}^{n}\hat{\mathbb{E}}\left(\widetilde{X}'_k\right)^2 l(n) \leq \delta l(1) \leq \delta l(n).$$

Therefore, by (3.3),

$$\sum_{n=1}^{\infty}\mathbb{V}\left\{\sum_{k=1}^{n}a_{nk}\widetilde{X}_k' > \varepsilon\right\} \le c\sum_{n=1}^{\infty}\mathrm{e}^{-2l(n)}g(n)\,\mathrm{e}^{\delta l(n)} = c\sum_{n=1}^{\infty}\mathrm{e}^{(\delta-2)l(n)}g(n) < \infty.$$

Because $\mathbb{V}$ has countable sub-additivity, and for every $\varepsilon > 0$, we obtain from Lemma 2.4,

$$\limsup_{n\to\infty} I_1 \le 0, \text{a.s.}\mathbb{V}. \tag{4.4}$$

For each $n$, there must be a $m$ such that $2^{m-1} \le n < 2^m$, by (2.12) and (3.2), $h(x)$ is decreasing in $(0,\infty)$, according to Lemma 2.1, $\tau_n^{-1/p}l^{-1}(n)$ is decreasing in $(0,\infty)$,

$$\begin{aligned}
H_3 := &\sum_{k=1}^{n}a_{nk}\left|\hat{\mathbb{E}}X_k - \hat{\mathbb{E}}X_k'\right| \\
\le &\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left|X_k''\right| \\
\le &\sum_{k=1}^{n}a_{nk}\hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_k^{1/p}}\right)\right)\right] \\
\ll &\,\tau_n^{-1/p}l^{-1}(n)\,n\hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_n^{1/p}}\right)\right)\right] \\
\le &\frac{2^m}{\tau_{2^{m-1}}^{1/p}l(2^{m-1})}\hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_{2^{m-1}}^{1/p}}\right)\right)\right] \\
\ll &\frac{2^m}{\tau_{2^m}^{1/p}l(2^m)}\hat{\mathbb{E}}\left[|X|\left(1-h\left(\frac{|X|}{\tau_{2^m}^{1/p}}\right)\right)\right].
\end{aligned}$$

Then, by (2.15) and (2.13), $\hat{\mathbb{E}}$ is countably sub-additive,

$$\begin{aligned}
H_3 \ll &\frac{2^m}{\tau_{2^m}^{1/p}l(2^m)}\sum_{j=m}^{\infty}\hat{\mathbb{E}}\left[|X|\,h_j\left(\frac{|X|}{\mu}\right)\right] \\
\le &\frac{2^m}{\tau_{2^m}^{1/p}l(2^m)}\sum_{j=m}^{\infty}\tau_{2^j}^{1/p}\mathbb{V}\left(|X| > \mu^2\tau_{2^{j-1}}^{1/p}\right) \\
\le &\sum_{j=m}^{\infty}\frac{2^j}{\tau_{2^j}^{1/p}l(2^j)}\tau_{2^j}^{1/p}\mathbb{V}\left(|X| > \mu^2\tau_{2^j}^{1/p}\right) \\
= &\sum_{j=m}^{\infty}\frac{2^j}{l(2^j)}\mathbb{V}\left(|X| > \mu^2\tau_{2^j}^{1/p}\right).
\end{aligned}$$

Combining (2.5), we get

$$\lim_{n\to\infty} I_3 = 0. \tag{4.5}$$

If we want to prove (3.4), just prove

$$\limsup_{n\to\infty} I_2 \le 0, \text{a.s.}\mathbb{V}. \tag{4.6}$$

Using (3.2) as well as the Lemma 2.1,

$$\begin{aligned}\max_{2^m\le n<2^{m+1}}\left|\sum_{k=1}^{n}a_{nk}X_k''\right| &\le c\max_{2^m\le n<2^{m+1}}\tau_n^{-1/p}l^{-1}(n)\sum_{k=1}^{n}\left|X_k''\right|\\ &\le c\tau_{2^m}^{-1/p}l^{-1}(2^m)\sum_{k=1}^{2^{m+1}}\left|X_k''\right|,\end{aligned}$$

for $\forall\varepsilon>0$, by (2.7) and Markov inequality,

$$\begin{aligned}\sum_{m=1}^{\infty}\mathbb{V}\left(\max_{2^m\le n<2^{m+1}}\left|\sum_{k=1}^{n}a_{nk}X_k''\right|>\varepsilon\right) &\le \sum_{m=1}^{\infty}\mathbb{V}\left(c\tau_{2^m}^{-1/p}l^{-1}(2^m)\sum_{k=1}^{2^{m+1}}\left|X_k''\right|>\varepsilon\right)\\ &\le c\sum_{m=1}^{\infty}\tau_{2^m}^{-1/p}l^{-1}(2^m)\sum_{k=1}^{2^{m+1}}\hat{\mathbb{E}}\left|X_k''\right|\\ &= c\sum_{k=1}^{\infty}\hat{\mathbb{E}}\left|X_k''\right|\sum_{m:2^{m+1}\ge k}\tau_{2^m}^{-1/p}l^{-1}(2^m)\\ &\ll \sum_{k=1}^{\infty}\tau_k^{-1/p}l^{-1}(k)\hat{\mathbb{E}}\left|X_k''\right|\\ &<\infty.\end{aligned}$$

By Lemma 2.4, for $\forall\varepsilon>0$,

$$\limsup_{m\to\infty}\max_{2^m\le n<2^{m+1}}\left|\sum_{k=1}^{n}a_{nk}X_k''\right|\le\varepsilon,\ \text{a.s.}\mathbb{V}.$$

Combining $\left|\sum_{k=1}^{n}a_{nk}X''\right|\le\max_{2^m\le n<2^{m+1}}\left|\sum_{k=1}^{n}a_{nk}X_k''\right|$ and the arbitrariness of $\varepsilon$, (4.6) holds. So far, Theorem 3.1 has been proved.

*4.2. Proof of Corollary 3.1.*

Let $l(n)=\log(n)$, for $0<b<1-\delta$, by (3.8), we have

$$\sum_{n=1}^{\infty}\mathrm{e}^{(\delta-2)l(n)}g(n)=\sum_{n=1}^{\infty}n^{\delta-2}g(n)=\sum_{n=1}^{\infty}n^{\delta-2+b}g(n)n^{-b}\le c\sum_{n=1}^{\infty}n^{\delta-2+b}<\infty.$$

Then, (3.4) holds. From Theorem 3.1, Eqs (3.4)–(3.6) hold.

*4.3. Proof of Corollary 3.2.*

Let $l(n)=\exp\{(\log n)^{\nu}\}, 0<\nu<1$. For $\forall q>0$, we have

$$(\log n)^{\nu}\ge q\log\log n,$$

so,

$$\exp\{(\log n)^{\nu}\}\ge\mathrm{e}^{q\log\log n}=\log^{q}n\ge q\log n.$$

By (3.10), $0<\delta<1$, when $q>\frac{m+1}{2-\delta}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\mathrm{e}^{(\delta-2)l(n)}g(n)&=\sum_{n=1}^{\infty}\exp\{(\delta-2)\exp\{\log^{\gamma}n\}\}g(n)\\&\le\sum_{n=1}^{\infty}\exp\{(\delta-2)q\log n\}g(n)\\&=\sum_{n=1}^{\infty}n^{(\delta-2)q+m}g(n)n^{-m}\\&\le c\sum_{n=1}^{\infty}n^{(\delta-2)q+m}\\&<\infty.\end{aligned}$$

Then, (3.4) holds. From Theorem 3.1, Eqs (3.4)–(3.6) hold.

*4.4. Proof of Theorem 3.2.*

When $p=1$, $C_{\mathbb{V}}(|X|)\le C_{\mathbb{V}}(|X|\log|X|)<\infty$, thus (4.4) and (4.5) are still valid, we just need to prove (4.6). Imitating the proof of Lemma 2.2, from $C_{\mathbb{V}}(|X|\log|X|)<\infty$, we can obtain

$$\sum_{k=1}^{\infty}\frac{2^k k}{l(2^k)}\mathbb{V}(|X|>c\tau_{2^k})<\infty. \tag{4.7}$$

Combining (2.12) and the monotonically decreasing property of $h(x)$ in $(0,\infty)$,

$$\begin{aligned}H_4:&=\sum_{n=1}^{\infty}\frac{1}{n}\hat{\mathbb{E}}\left|X_n''\right|\le\sum_{n=1}^{\infty}\frac{1}{n}\hat{\mathbb{E}}|X|\left(1-h\left(\frac{|X|}{\tau_n}\right)\right)\\&=\sum_{k=1}^{\infty}\sum_{2^{k-1}\le n<2^k}\frac{1}{n}\hat{\mathbb{E}}|X|\left(1-h\left(\frac{|X|}{\tau_n}\right)\right)\\&\le\sum_{k=1}^{\infty}2^{k-1}\frac{1}{2^{k-1}}\hat{\mathbb{E}}|X|\left(1-h\left(\frac{|X|}{\tau_{2^{k-1}}}\right)\right)\\&\ll\sum_{k=1}^{\infty}\hat{\mathbb{E}}|X|\left(1-h\left(\frac{|X|}{\tau_{2^k}}\right)\right).\end{aligned}$$

Then, by (2.15) and (4.7),

$$\begin{aligned}H_4&\ll\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\hat{\mathbb{E}}|X|h_j\left(\frac{|X|}{\mu}\right)\\&\le\sum_{j=1}^{\infty}j\tau_{2^j}\mathbb{V}\left(|X|>\mu^2\tau_{2^{j-1}}\right)\\&\ll\sum_{j=1}^{\infty}\frac{2^j j}{l(2^j)}\mathbb{V}\left(|X|>\mu^2\tau_{2^j}\right)\\&<\infty.\end{aligned}$$

For $\forall\varepsilon > 0$, by (3.12) and Markov inequality,

$$\begin{aligned}\sum_{m=1}^{\infty}\mathbb{V}\left(\max_{2^m\le n<2^{m+1}}\left|\sum_{k=1}^{n}a_{nk}X_k''>\varepsilon\right|\right) &\le c\sum_{m=1}^{\infty}\max_{2^m\le n<2^{m+1}}\frac{1}{n}\sum_{k=1}^{n}\hat{\mathbb{E}}\left|X_k''\right|\\ &\le c\sum_{m=1}^{\infty}\frac{1}{2^m}\sum_{k=1}^{2^{m+1}}\hat{\mathbb{E}}\left|X_k''\right|\\ &= c\sum_{k=1}^{\infty}\hat{\mathbb{E}}\left|X_k''\right|\sum_{m:2^{m+1}>k}\frac{1}{2^m}\\ &\ll c\sum_{k=1}^{\infty}\frac{1}{k}\hat{\mathbb{E}}\left|X_k''\right|\\ &<\infty.\end{aligned}$$

By Lemma 2.4, for $\forall\varepsilon > 0$,

$$\limsup_{m\to\infty}\max_{2^m\le n<2^{m+1}}\left|\sum_{k=1}^{n}a_{nk}X_k''\right|\le\varepsilon,\ \text{a.s.}\mathbb{V}.$$

Combining $\left|\sum_{k=1}^{n}a_{nk}X''\right| \le \max_{2^m\le n<2^{m+1}}\left|\sum_{k=1}^{n}a_{nk}X_k''\right|$ and the arbitrariness of $\varepsilon$, (4.6) holds. So far, Theorem 3.2 has been proved.

## 5. Conclusions

Almost sure convergence of WNOD r.v. in Peng's theory frame is built through this essay. It is based on the corresponding definition of stochastic domination in the sub-linear expectation space, as well as the properties of WNOD r.v. and the related proving methods. Compared with the previous research of ND, END, and so on, the research in this paper is suitable for a wider range of r.v.. So, broader conclusions are reached. In future research work, we will further consider investigating more meaningful conclusions.

## Author contributions

Baozhen Wang: Conceptualization, Formal analysis, Investigation, Methodology, Writing-original draft, Writing-review & editing; Qunying Wu: Funding acquisition, Formal analysis, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

In this article, all authors disclaim any conflict of interest.

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