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Research article

Investigation the generalized extreme value under liner distribution parameters for progressive type-II censoring by using optimization algorithms

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Abstract: Several random phenomena have been modeled by using extreme value distributions. Based on progressive type-II censored data with three different distributions (i.e., fixed, discrete uniform, and binomial random removal), the statistical inference of the generalized extreme value distribution under liner normalization (GEVL distribution) parameters is investigated in this study. Since there is no analytical solution, determining the maximum likelihood parameters for the GEVL distribution is considered to be a problem. Standard numerical methods are frequently insufficient for this dilemma, requiring the use of artificial intelligence algorithms to address this difficulty. Here, nonlinear minimization and a genetic algorithm have been used to tackle that problem. In addition, Lindley approximation and Monte Carlo estimation were implemented via Metropolis-Hastings algorithms to carry out the Bayesian point estimation based on both the squared error loss function and LINEX loss functions. Moreover, the highest posterior density intervals were applied. The proposed theoretical inference techniques have been applied in a numerical simulation and a real-life example.

Keywords: GEVL; type-II progressive censoring; binomial random removal; discrete uniform random removal; genetic algorithm; Bayesian estimation
Mathematics Subject Classification: 60G70, 62G32

1. Introduction

Extreme value theory (EVT) models hold significant relevance in a wide array of disciplines, including environmental sciences, engineering, finance, insurance, and various other fields. In finance, particularly, extreme price fluctuations of financial assets or market indices are defined as the highest and lowest prices within a specific time frame. For a more comprehensive exploration of this concept, refer to the work in [1].

EVT studies the asymptotic behavior of extreme values that follow a well-defined pattern, regardless of the underlying return mechanism. It is primarily concerned with providing a probabilistic description of extreme occurrences within a sequence of random events. The foundational principles of EVT are elucidated in [2]. According to the fundamental theorem of EVT, the maxima of independent and identically distributed random variables can be categorized into one of three extreme value distributions. These include the Frechet distribution, characterized by an unbounded upper heavy tail, the Gumbel distribution, distinguished by an infinite upper tail that is lighter than that of the Frechet distribution, and the inverse Weibull distribution, which has a finite upper tail. These three distribution forms can be interconnected within the cumulative distribution function (CDF) family models known as generalized extreme value distributions under linear normalization (GEVL distributions). Numerous authors have introduced studies on GEVL distribution, detailed [3–8], and several others. A random variable denoted as X is said to follow a GEVL distribution if its probability density function (PDF) and CDF assume the following forms:

$$f(x;\mu,\sigma,\xi) = \begin{cases} \sigma^{-1}[\frac{1+\xi(x-\mu)}{\sigma}]^{-\frac{1}{\xi}-1} \exp\left[-[\frac{1+\xi(x-\mu)}{\sigma}]^{-\frac{1}{\xi}}\right] & \text{if} \quad \xi \neq 0, \\ \sigma^{-1} \exp(-\frac{(x-\mu)}{\sigma}) \exp[-\exp(-\frac{(x-\mu)}{\sigma})] & \text{if} \quad \xi \to 0, \end{cases} \tag{1.1}$$

$$F(x;\mu,\sigma,\xi) = \begin{cases} \exp\left[-[\frac{1+\xi(x-\mu)}{\sigma}]^{-\frac{1}{\xi}}\right] & \text{if} \quad \xi \neq 0, \\ \exp[-\exp(-\frac{(x-\mu)}{\sigma})] & \text{if} \quad \xi \to 0. \end{cases} \tag{1.2}$$

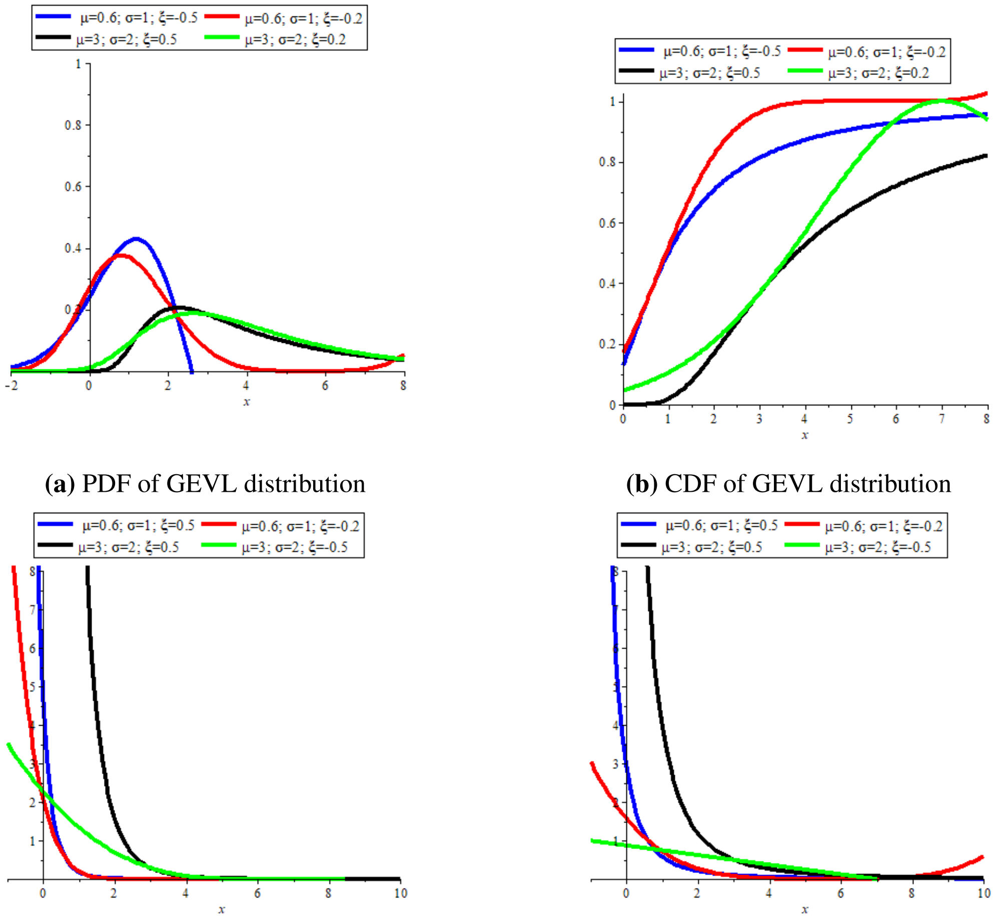
where $\sigma > 0$ and $-\infty < \mu < \infty$ are the scale and location parameters respectively, and ξ is the shape parameter that represents the behavior of the tail. Sub-models can be defined by $\xi \to 0, \xi > 0$, and $\xi < 0$, corresponding respectively to the Gumbel, Frechét, and Weibull distributions, which are mentioned above.

The hazard rate (HR) and reversed hazard rate (RHR) for GEVL distribution respectively obtained as follows:

$$h(x;\mu,\sigma,\xi) = \frac{f(x;\mu,\sigma,\xi)}{1 - F(x;\mu,\sigma,\xi)} = \begin{cases} \dfrac{[\frac{1+\xi(x-\mu)}{\sigma}]^{\frac{-1}{\xi}-1}}{\sigma[\exp[\frac{1+\xi(x-\mu)}{\sigma}]^{\frac{1}{\xi}}-1]} & \text{if} \quad \xi \neq 0, \\ \dfrac{\exp(-\frac{(x-\mu)}{\sigma})}{\sigma[\exp(\exp(-\frac{(x-\mu)}{\sigma}))-1]} & \text{if} \quad \xi \to 0, \end{cases} \tag{1.3}$$

$$rh(x;\mu,\sigma,\xi) = \frac{f(x;\mu,\sigma,\xi)}{F(x;\mu,\sigma,\xi)} = \begin{cases} \sigma^{-1}[\frac{1+\xi(x-\mu)}{\sigma}]^{\frac{-1}{\xi}-1} & \text{if} \quad \xi \neq 0, \\ \sigma^{-1} \exp(-\frac{(x-\mu)}{\sigma}) & \text{if} \quad \xi \to 0. \end{cases} \tag{1.4}$$

In this study, we only considered the distribution form at $\xi \neq 0$. Notice that, the GEVL distribution can be readily adapted to various models, as shown in plots of the PDF, CDF, HR, and RHR for some parameter values in Figure 1. These plots illustrate how the PDFs and CDFs respond to alterations in parameters, revealing their sensitivity and the resultant changes in the behavior of the parameter ξ configuration.



(a) PDF of GEVL distribution **(b)** CDF of GEVL distribution

(c) HR of GEVL distribution **(d)** RHR of GEVL distribution

Figure 1. Plots of the PDF, CDF, HR, and RHR of the GEVL distribution for several values of (μ, σ, ξ).

Over the years, both maximum likelihood estimation (MLE) and Bayesian methods have become widely recognized as crucial statistical techniques for parameter estimation. Moreover, the asymptotic properties associated with these statistical approaches have significantly contributed to their popularity.

In MLE, the parameters are treated as fixed but unknown values, often representing a measure such as the mean. On the other hand, Bayesian methods adopt a different perspective, treating the parameters as random variables with known prior distributions. This distinction allows Bayesian analysis to incorporate prior knowledge or beliefs about the parameters, resulting in a flexible and powerful approach to statistical inference.

For a comprehensive exploration of these methodologies and their applications, we recommend referring to the work of [9, 10]. In the present study, we study both MLE and Bayesian methods to investigate parameter estimation within the context of the GEVL distribution under a type-II progressive censored sample.

According to [11, 12], there is no analytical solution for the maximum likelihood equations (ML.eqs) by the classical root-finding algorithms, such as the Newton-Raphson, Broyden's methods, and thus, some artificial intelligence algorithms were used. Therefore the maximization of the log-likelihood function can be obtained by using a standard numerical optimization algorithm like Non-linear minimization (NLM). Moreover, we compare the result obtained via the NLM method with one of the most popular algorithms in artificial intelligence genetic algorithm (GA).

The NLM function is a valuable tool for numerical optimization, particularly for solving nonlinear optimization problems. It provides flexibility, stability, and efficiency, making it an important function for researchers, analysts, and practitioners working with complex optimization tasks. Moreover, the NLM function can handle optimization problems with multiple variables, allowing one to optimize functions of several parameters simultaneously. This makes it applicable in a wide range of fields, including statistics, econometrics, machine learning, and scientific research more details see [13]. The GA belongs to the field of artificial intelligence and specifically to the sub-field of Evolutionary computation. Evolutionary Computation encompasses computational methods inspired by biological evolution, and the GA is one of the prominent techniques within this domain. For more details in GA see [14, 15].

In certain experiments, it can be challenging to observe the failures of all of the units under investigation. This difficulty may arise from constraints such as limited time, budget constraints, or other practical considerations. Consequently, censored schemes are frequently employed in such situations. Censoring allows researchers to make inferences about the data, even when some of the information about failure times is incomplete or unavailable.

One of the commonly used censoring schemes in the investigations of parameter estimation for any distribution is the type-II progressive censored scheme. Type-II progressive censored data can be described as follows: Let N units being subjected to an experiment, and only a predetermined number of units, denoted as n (where $n < N$), be observed until their failure. After each failure represented by $x_{1:n:N}$, a specific quantity of units, R_1, is chosen at random from the remaining units $(N - 1)$. Following this, with the second failure, $x_{2:n:N}$, another batch of units, R_2, is randomly taken from the remaining $(N - R_1 - 2)$, and this sequence continues until the n failure, $x_{n:n:N}$ which is the endpoint of the experiment. The type-II progressive censored scheme allows researchers to collect and analyze data even when they are unable to observe the failure times of all units in the study. For more comprehensive information on this scheme, you can refer to the work of [16].

This paper is organized as follows: In Section 2, we consider an explanation of all three cases under investigation (i.e., fixed, discrete uniform, and binomial random removal). In Section 3, MLE of GEVL distribution parameters based on type-II progressively censored samples, for the three cases considered, are discussed for both point and approximate Confidence intervals. Section 4, Bayesian estimation of GEVL distribution parameters based on type-II progressively censored samples, using both Lindley's approximation and Metropolis-Hastings (MH) algorithms for both informative and non-informative priors that involve the use of squared error loss function (SELF) and LINEX loss functions are presented. Moreover, the Bayesian estimation of GEVL distribution parameters is carried out by adopting the Gibbs sampling method for both the point estimation and the highest posterior density (HPD) confidence intervals estimation. The paper includes a simulation analysis in Section 5 to demonstrate the practical application of the theoretical findings. Real data examples are discussed in Section 6 to provide an application of the methodology in real-life situations. Finally, Section 7 presents the study conclusions.

2. Cases under investigation

The associated sub-distribution hazards model, an established method for the regression analysis of time-to-event data in the presence of conflicting risks, depends heavily on the censoring distribution. Also, the features of the estimates from such a model when the censoring distribution is misspecified are examined by generating competing risk data under a proportional sub-distribution hazards model with various patterns of censoring. Models that correctly described the censoring distribution outperformed those that did not, providing estimates of the sub-distribution hazard ratio with less bias and risk. Estimates from the model based on these weights may not reflect the correct likelihood structure and, as a result, may have poor accuracy, particularly when the covariate of interest does not influence the censoring distribution but serves to calculate the risk set weights. In this section, we will indicate the three cases that will be utilized in this paper for the progressive type II censoring schemes given by $R = (R_1, R_2, \ldots, R_n)$. Suppose that, $X_{1:n:N}, X_{2:n:N}, \ldots, X_{n:n:N}$ be the progressively type-II censored ordering data from a GEVL distribution with the censoring scheme $R = (R_1, R_2, \ldots, R_n)$ and the observed data $x = (x_1, x_2, \ldots, x_n)$.

Case 1. Fixed removal censoring scheme

For this case, the censoring schemes donated by $R = (R_1, R_2, \ldots, R_n)$ are assumed to be predetermined fixed numbers.

Case 2. Removals with discrete uniform distribution

For this case, we assumed that R is an independent random variable following a discrete uniform distribution. The joint likelihood function of R is given by:

$$P(R = r) = P(R_n = r_n | R_{n-1} = r_{n-1}, R_{n-2} = r_{n-2} \\ , \ldots, R_1 = r_1) \ldots P(R_2 = r_2 | R_1 = r_1) P(R_1 = r_1), \tag{2.1}$$

where

$$P(R_1 = r_1) = \frac{1}{N - n + 1}, \tag{2.2}$$

where $0 < r_1 < N - n$ and for $i = 2, 3, \ldots, n - 1$,

$$P(R_i = r_i | R_{i-1} = r_{i-1}, R_{i-2} = r_{i-2} \ldots R_1 = r_1) = \frac{1}{N - n - \sum_{j=1}^{i-1} r_j + 1}, \tag{2.3}$$

where $0 < r_i < N - n - \sum_{j=1}^{i-1} r_j$ and $R_n = N - n - \sum_{j=1}^{n-1} R_j$.

Apply, Eqs (2.2) and (2.3) in Eq (2.1). Then the joint distribution of $R = (R_1, R_2, \ldots, R_n)$ can be easily obtained as follows:

$$P(R = r) = \frac{1}{N - n - 1} \prod_{i=1}^{n-1} \frac{1}{N - n - (\sum_{j=1}^{i-1} r_j) + 1}. \tag{2.4}$$

It is noticed that the joint PDF of R is parameter-free.

Case 3. Removals with binomial distribution

In this scenario, if the censoring scheme $R = (R_1, R_2, \ldots, R_n)$ is assumed to consist of independent random variables following binomial distributions with a probability of P, then the joint distribution of

R can be expressed as follows:

$$\begin{aligned} P(R=r|P) = P(R_n = r_n|R_{n-1} = r_{n-1}, R_{n-2} = r_{n-2}, \ldots, R_1 = r_1, P) \\ \ldots P(R_2 = r_2|R_1 = r_1, P)P(R_1 = r_1, P), \end{aligned} \tag{2.5}$$

where

$$P(R_1 = r_1|P) = \binom{N-n}{r_1} P^{r_1}(1-P)^{(N-n-r_1)}, \tag{2.6}$$

where $0 < r_1 < N-n$ and for $i = 2, 3, \ldots, n-1$,

$$P(R_i = r_i|P) = \binom{N-n-\sum_{j=1}^{i-1} r_j}{r_i} P^{r_i}(1-P)^{(N-n-\sum_{j=1}^{i} r_j)}, \tag{2.7}$$

where $0 < r_i < N-n-\sum_{j=1}^{i-1} r_j$ and $R_n = N-n-\sum_{j=1}^{n-1} R_j$. Then the joint distribution of $R = (R_1, R_2, \ldots, R_n)$ can be obtained easily by applying with Eqs (2.6) and (2.7) in Eq (2.5) as follows:

$$P(R=r|P) = \frac{(N-n)!}{(N-n-\sum_{j=1}^{n-1} r_j)!(\prod_{j=1}^{n-1} r_j!)} P^{\sum_{i=1}^{n-1} r_i}(1-P)^{[(N-n)(n-1)-\sum_{i=1}^{n-1}(n-i)r_j]}. \tag{2.8}$$

3. MLE of GEVL distribution parameters

MLE method, which is a classical point estimation technique, combines both the overall distribution information and the observed sample data to facilitate comprehensive inference analysis. In the context of the GEVL distribution and type-II progressive censoring, we have discussed the MLE approach for parameter estimation in three specific cases.

First, under the assumption that the sample size approaches infinity, we derived parameter estimates by using the MLE method. Subsequently, we recognized that traditional numerical optimization techniques such as the Newton-Raphson method, and Broyden's method, among others, are commonly employed. However, in our study, we employed both artificial intelligence algorithms and optimization techniques to numerically compute the corresponding maximum likelihood estimates.

Finally, we computed asymptotic confidence intervals to provide a comprehensive assessment of our findings and their statistical significance.

3.1. Point estimation of GEVL distribution's parameters from the perspective of the distribution of the censoring scheme

Case 1. Fixed removals

For this case, the joint likelihood function could be given by

$$L(\mu, \sigma, \xi) = C \prod_{i=1}^{n} f(x_i; \mu, \sigma, \xi)\left[1 - F(x_i; \mu, \sigma, \xi)\right]^{R_i}.$$

Then

$$L(\mu, \sigma, \xi) \propto \prod_{i=1}^{n} f(x_i; \mu, \sigma, \xi)\left[1 - F(x_i; \mu, \sigma, \xi)\right]^{R_i}, \tag{3.1}$$

where $x_1 < x_2 < \cdots < x_n$ and C is defined as follows:

$$C = N\,(N - R_1 - 1)\,(N - R_1 - R_2 - 2)\cdots\left(N - \sum_{i=1}^{n-1} R_i - n + 1\right).$$

Case 2. Removals with discrete uniforms

For this case, the joint likelihood function can be obtained as follows:

$$L^{*}(\mu, \sigma, \xi) = P(R = r)L(\mu, \sigma, \xi).$$

Then

$$L^{*}(\mu, \sigma, \xi) \propto \prod_{i=1}^{n} f\,(x_i; \mu, \sigma, \xi)\,[1 - F\,(x_i; \mu, \sigma, \xi)]^{R_i}\,, \tag{3.2}$$

where $P(R = r)$ and $L(\mu, \sigma, \xi)$ are as given in Eqs (2.4) and (3.1) respectively.

Case 3. Removals with binomial distribution

For this case, the joint likelihood function can be obtained as follows:

$$L^{**}(\mu, \sigma, \xi, P) = P(R = r|P)L(\mu, \sigma, \xi). \tag{3.3}$$

Then

$$L^{**}(\mu, \sigma, \xi, P) \propto \prod_{i=1}^{n} f\,(x_i; \mu, \sigma, \xi)\,[1 - F\,(x_i; \mu, \sigma, \xi)]^{R_i}\, P^{\sum_{i=1}^{n-1} r_i}(1 - p)^{(n-1)(N-n) - \sum_{i=1}^{n-1} r_i}. \tag{3.4}$$

where, $P(R = r|P)$ and $L(\mu, \sigma, \xi)$ are given as in Eqs (2.8) and (3.1) respectively.

Clearly, the ML.eqs for parameters μ, σ, and ξ for the three cases correspond to the same system of equations and the difference between the three cases is the distribution of the censored scheme $R = (R_1, R_2, \ldots, R_n)$. In addition, for Case 3, the MLE of P could be easily obtained as follows:,

$$\hat{P} = \frac{\sum_{i=1}^{n-1} r_i}{(N - n)(n - 1) - \sum_{i=1}^{n-1}(n - i - 1)r_i}. \tag{3.5}$$

When obtaining the ML.eqs or parameters μ, σ, and ξ by substituting this Eqs (1.1) and (1.2) in Eq (3.1), the likelihood function can be obtained as follows:

$$L(\mu, \sigma, \xi) \propto \prod_{i=1}^{n} \sigma^{-1} Z_i^{-\frac{1}{\xi}-1} U_i(1 - U_i)^{R_i}, \tag{3.6}$$

where, $Z_i = [1 + (\frac{\xi(x_i-\mu)}{\sigma})]$ and $U_i = \exp(-Z_i^{\frac{-1}{\xi}})$.

Then the log-likelihood function is given by

$$\log(L) \propto -n\log(\sigma) - \sum_{i=1}^{n}(\frac{1}{\xi} + 1)\log Z_i - \sum_{i=1}^{n} Z_i^{-\frac{1}{\xi}} + \sum_{i=1}^{n} R_i \log(1 - U_i). \tag{3.7}$$

The following three equations could be easily derived by taking the derivative of Eq (3.7) with respect to μ, σ, and ξ respectively, and making them equal to zero:

$$\frac{\delta \log L}{\delta \mu} = \sum_{i=1}^{n} (\frac{\xi + 1}{\sigma}) Z_i^{-1} + \sum_{i=1}^{n} \sigma^{-1} Z_i^{(-\frac{1}{\xi}-1)} + \sum_{i=1}^{n} \frac{R_i Z_i^{[\frac{-1}{\xi}-1]} U_i}{\sigma[1 - U_i]}, \tag{3.8}$$

$$\frac{\delta \log L}{\delta \sigma} = \frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(\xi + 1)(x_i - \mu)}{Z_i \sigma^2} - \sum_{i=1}^{n} \frac{(x_i - \mu)}{Z_i^{\frac{1}{\xi}+1} \sigma^2} - \sum_{i=1}^{n} \frac{R_i U_i (x_i - \mu)}{Z_i^{\frac{1}{\xi}+1} \sigma^2 [1 - U_i]}, \tag{3.9}$$

and

$$\frac{\delta \log L}{\delta \xi} = \sum_{i=1}^{n} \frac{\log(Z_i)[1 - Z_i^{\frac{-1}{\xi}} + \frac{R_i Z_i^{\frac{-1}{\xi}} U_i}{1-u_i}]}{\xi^{-2}} - \sum_{i=1}^{n} \frac{(x_i - \mu)[1 - Z_i^{\frac{-1}{\xi}} + \frac{R_i U_i Z_i^{\frac{-1}{\xi}}}{[1-U_i]}]}{\sigma Z_i \xi}. \tag{3.10}$$

The maximum likelihood estimators $\hat{\mu}, \hat{\sigma}$ and $\hat{\xi}$ based on type-II progressively censored data can be obtained by setting Eqs (3.8)–(3.10) equal to zero and solveing them. The nonlinear characteristics of those likelihood equations are considered to be a challenge due to their complexity and the classical techniques like the Newton-Raphson method, failed to solve them. In such cases, numerical optimization methods, and artificial intelligence algorithms are typically employed to find the maximum likelihood estimators that best fit the observed data and likelihood equations. So, NLM and one of the most popular algorithms in artificial intelligence, i.e., the GA, are employed as indicated in Tables 1–4. A quick tour of GA is provided in detail the following website: https://cran.r-project.org/web/packages/GA/vignettes/GA.html. NLM is explained on [17].

Table 1. Thebias (mean square error; MSE) for MLE and Bayesian estimation, results, obtained by using both Lindley and MH for informative (INF) and non-informative priors (NON) under SELF (sq) LINEX loss function (lx) at ($\beta = (-5, 5, 0.5)$) for $N = 1000$ under Case 1.

		Method	GA						NLM					
			At Beginning			At End			At Beginning			At End		
			$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$
			Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)
		MLE	0.0438	-0.0443	-0.0003	0.0455	-0.0438	-0.0042	0.0116	0.0075	-0.2005	-0.0243	-0.0695	-0.3364
			(0.0027)	(0.0024)	(0.0015)	(0.00300)	(0.0025)	(0.0015)	(0.0003)	(0.0002)	(0.0411)	(0.0007)	(0.0049)	(0.1141)
NON	Lindley	Sq	0.0438	-0.0443	-0.0003	0.0455	-0.0438	-0.0042	0.0123	0.0094	-0.2003	-0.0241	-0.0688	-0.3367
			(0.0027)	(0.0025)	(0.0015)	(0.0030)	(0.0025)	(0.0015)	(0.0003)	(0.0002)	(0.0410)	(0.0007)	(0.0048)	(0.1143)
		$lx_{(\beta=0.5)}$	0.04382	-0.0443	-0.0003	0.0455	-0.0438	-0.0042	0.0122	0.0084	-0.2006	-0.0241	-0.0692	-0.3368
			(0.0027)	(0.0025)	(0.0015)	(0.0030)	(0.0025)	(0.0015)	(0.0003)	(0.0002)	(0.0411)	(0.0007)	(0.0049)	(0.1144)
		$lx_{(\beta=-5)}$	0.04382	-0.0443	-0.0003	0.0455	-0.0437	-0.0042	0.0126	0.0098	-0.1979	-0.0239	-0.0687	-0.3352
			(0.0027)	(0.0025)	(0.0015)	(0.0029)	(0.0025)	(0.0015)	(0.0003)	(0.0003)	(0.0400)	(0.0007)	(0.0048)	(0.1133)
		$lx_{(\beta=5)}$	0.04382	-0.0443	-0.0003	0.0455	-0.0438	-0.0042	0.0119	0.009	-0.2027	-0.0243	-0.0689	-0.3381
			(0.0027)	(0.0025)	(0.0015)	(0.0030)	(0.0025)	(0.0015)	(0.0003)	(0.0002)	(0.0420)	(0.0007)	(0.0048)	(0.1153)
	MH	Sq	-0.0375	-0.1311	-0.3019	-0.0362	-0.1298	-0.3075	-0.0531	-0.105	-0.4056	-0.0708	-0.1442	-0.4746
			(0.0026)	(0.0182)	(0.1057)	(0.0025)	(0.0182)	(0.1098)	(0.0036)	(0.0128)	(0.1697)	(0.0057)	(0.0216)	(0.2280)
		$lx_{(\beta=0.5)}$	-0.0395	-0.1332	-0.3109	-0.038	-0.1319	-0.3166	-0.0548	-0.1078	-0.4115	-0.0721	-0.1461	-0.4782
			(0.0027)	(0.0188)	(0.1113)	(0.0026)	(0.0187)	(0.1154)	(0.0038)	(0.0134)	(0.1744)	(0.0059)	(0.0222)	(0.2314)
		$lx_{(\beta=-5)}$	-0.0183	-0.1103	-0.2178	-0.0168	-0.1091	-0.2246	-0.0375	-0.0784	-0.3508	-0.0586	-0.1264	-0.4401
			(0.0015)	(0.0133)	(0.0603)	(0.0016)	(0.0133)	(0.0646)	(0.0022)	(0.0078)	(0.1278)	(0.0041)	(0.0168)	(0.1964)
		$lx_{(\beta=5)}$	-0.0571	-0.1522	-0.3868	-0.0561	-0.1511	-0.3928	-0.0692	-0.1323	-0.4622	-0.0833	-0.1624	-0.5098
			(0.0043)	(0.0242)	(0.1639)	(0.0042)	(0.0240)	(0.1684)	(0.0055)	(0.0191)	(0.2183)	(0.0076)	(0.0272)	(0.2625)
INF	Lindley	Sq	0.04382	-0.0443	-0.0003	0.0455	-0.0437	-0.0042	0.0114	0.0084	-0.2018	-0.0244	-0.0691	-0.3375
			(0.0027)	(0.0025)	(0.0015)	(0.0030)	(0.0024)	(0.0015)	(0.0003)	(0.0002)	(0.0416)	(0.0007)	(0.0049)	(0.1148)
		$lx_{(\beta=0.5)}$	0.04382	-0.0443	-0.0003	0.0455	-0.0437	-0.0042	0.0114	0.0084	-0.202	-0.0245	-0.0691	-0.3376
			(0.0027)	(0.0025)	(0.0015)	(0.0030)	(0.0025)	(0.0015)	(0.0003)	(0.0002)	(0.0417)	(0.0007)	(0.0049)	(0.1149)
		$lx_{(\beta=-5)}$	0.04382	-0.0443	-0.0003	0.0452	-0.0438	-0.0042	0.0118	0.0088	-0.1994	-0.0242	-0.0690	-0.3360
			(0.0027)	(0.0025)	(0.0015)	(0.0030)	(0.0025)	(0.0015)	(0.0003)	(0.0002)	(0.0406)	(0.0007)	(0.0048)	(0.1138)
		$lx_{(\beta=5)}$	0.04382	-0.0443	-0.0002	0.0455	-0.0438	-0.0042	0.0111	0.008	-0.2042	-0.0246	-0.0693	-0.339
			(0.0027)	(0.0025)	(0.0015)	(0.0030)	(0.0025)	(0.0015)	(0.0003)	(0.0002)	(0.0425)	(0.0007)	(0.0049)	(0.1158)
	MH	Sq	-0.0384	-0.1296	-0.3031	-0.0354	-0.1306	-0.2925	-0.0542	-0.1054	-0.411	-0.0717	-0.1437	-0.4785
			(0.0025)	(0.0180)	(0.1069)	(0.0025)	(0.0183)	(0.1012)	(0.0037)	(0.0127)	(0.1746)	(0.0057)	(0.0215)	(0.2316)
		$lx_{(\beta=0.5)}$	-0.0404	-0.1318	-0.3121	-0.037	-0.1327	-0.3016	-0.0558	-0.1082	-0.4168	-0.0730	-0.1455	-0.4821
			(0.0026)	(0.0185)	(0.1125)	(0.0026)	(0.0188)	(0.1067)	(0.0038)	(0.0133)	(0.1793)	(0.0058)	(0.0221)	(0.2350)
		$lx_{(\beta=-5)}$	-0.0191	-0.1094	-0.2196	-0.0159	-0.1099	-0.2098	-0.0385	-0.0785	-0.3567	-0.0595	-0.1261	-0.4442
			(0.0015)	(0.0131)	(0.0617)	(0.0015)	(0.0134)	(0.0573)	(0.0022)	(0.0077)	(0.1323)	(0.0041)	(0.0168)	(0.1999)
		$lx_{(\beta=5)}$	-0.058	-0.1507	-0.3885	-0.0553	-0.1517	-0.3784	-0.0701	-0.1328	-0.466	-0.0842	-0.1617	-0.5134
			(0.0042)	(0.02380)	(0.1650)	(0.0041)	(0.0241)	(0.1591)	(0.0056)	(0.0192)	(0.2224)	(0.0076)	(0.0270)	(0.2660)

Table 2. The bais and (MSE) for MLE and Bayesian estimation results obtained by (using both Lindley and MH for informative (INF) and non-informative priors (NON) under SELF (sq) LINEX loss function (lx) at ($\beta = (-5, 5, 0.5)$) for $N = 50$ under Case 1.

Method			GA						NLM					
			At Beginning			At End			At Beginning			At End		
			$\mu = 0.2$ Bais (MSE)	$\sigma = 0.3$ Bais (MSE)	$\xi = 0.7$ Bais (MSE)	$\mu = 0.2$ Bais (MSE)	$\sigma = 0.3$ Bais (MSE)	$\xi = 0.7$ Bais (MSE)	$\mu = 0.2$ Bais (MSE)	$\sigma = 0.3$ Bais (MSE)	$\xi = 0.7$ Bais (MSE)	$\mu = 0.2$ Bais (MSE)	$\sigma = 0.3$ Bais (MSE)	$\xi = 0.7$ Bais (MSE)
NON		MLE	0.0586 (0.0040))	-0.0649 (0.0046)	0.014 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0102 (0.0016)	0.0178 (0.0032)	-0.0214 (0.0030)	-0.3621 (0.1514)	-0.0244 (0.0025)	-0.0848 (0.0085)	-0.4005 (0.1763)
	Lindley	Sq	0.0586 (0.0040)	-0.0648 (0.0046)	0.0139 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0104 (0.0017)	0.0271 (0.0037)	0.0072 (0.0032)	-0.3579 (0.1467)	-0.0196 (0.0023)	-0.0716 (0.0067)	-0.408 (0.1814)
		$lx_{(\beta=0.5)}$	0.0586 (0.0040)	-0.0648 (0.0046)	0.0139 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0104 (0.0017)	0.0266 (0.0036)	-0.0076 (0.0029)	-0.3637 (0.1508)	-0.02 (0.0023)	-0.0784 (0.0076)	-0.4119 (0.1845
		$lx_{(\beta=-5)}$	0.0586 (0.0040)	-0.0648 (0.0046)	0.0138 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0104 (0.0017)	0.0325 (0.0042)	0.0102 (0.0034)	-0.3079 (0.1150)	-0.016 (0.0023)	-0.0699 (0.0065)	-0.371 (0.1538)
		$lx_{(\beta=5)}$	0.0586 (0.0040))	-0.0648 (0.0046)	0.0139 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0103 (0.0016)	0.0212 (0.0032)	0.0028 (0.0030)	-0.4096 (0.1858)	-0.0235 (0.0024)	-0.0737 (0.0069)	-0.4419 (0.2098)
	MH	Sq	-0.0299 (0.0023)	-0.141 (0.0210)	-0.2873 (0.0985)	-0.0298 (0.0022)	-0.1409 (0.0209)	-0.2844 (0.0973)	-0.0505 (0.0041)	-0.1192 (0.0162)	-0.4901 (0.2478)	-0.0717 (0.0061)	-0.1499 (0.0236)	-0.5085 (0.2648)
		$lx_{(\beta=0.5)}$	-0.032 (0.0024)	-0.1429 (0.0215)	-0.2969 (0.1041)	-0.0319 (0.0023)	-0.1428 (0.0214)	-0.2937 (0.1028)	-0.0522 (0.0043)	-0.1217 (0.0168)	-0.4936 (0.2509)	-0.073 (0.0063)	-0.1516 (0.0241)	-0.5114 (0.2675)
		$lx_{(\beta=-5)}$	-0.0089 (0.0014)	-0.1228 (0.0162)	-0.2006 (0.0540)	-0.0089 (0.0014)	-0.1226 (0.0161)	-0.2002 (0.0544)	-0.0338 (0.0030)	-0.0956 (0.0114)	-0.4576 (0.2196)	-0.0592 (0.0047)	-0.1336 (0.0191)	-0.4809 (0.2396)
		$lx_{(\beta=5)}$	-0.0511 (0.0039)	-0.1596 (0.0265)	-0.3778 (0.1582)	-0.0511 (0.0038)	-0.1596 (0.0264)	-0.3728 (0.1551)	-0.0674 (0.0058)	-0.1433 (0.0222)	-0.5234 (0.2790)	-0.0845 (0.0079)	-0.1667 (0.0287)	-0.5366 (0.2922)
INF	Lindley	Sq	0.0586 (0.0040)	-0.0648 (0.0046)	0.0139 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0103 (0.0016)	0.0177 (0.0029)	-0.0031 (0.0026)	-0.4095 (0.1843)	-0.0232 (0.0023)	-0.075 (0.0070)	-0.4418 (0.2089)
		$lx_{(\beta=0.5)}$	0.0586 (0.0040)	-0.0648 (0.0046)	0.0139 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0103 (0.0016)	0.0171 (0.0028)	-0.0036 (0.0026)	-0.4145 (0.1885)	-0.0235 (0.0023)	-0.0752 (0.007)	-0.4451 (0.2118)
		$lx_{(\beta=-5)}$	0.0586 (0.0040)	-0.0648 (0.0046)	0.0139 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0104 (0.0016)	0.0235 (0.0033)	0.0013 (0.0028)	-0.3512 (0.1411)	-0.0194 (0.0022)	-0.073 (0.0068)	-0.4026 (0.1766)
		$lx_{(\beta=5)}$	0.0586 (0.0040)	-0.0648 (0.0046)	0.0139 (0.0016)	0.0592 (0.0041)	-0.0647 (0.0046)	0.0101 (0.0016)	0.0119 (0.0026)	-0.0085 (0.0025)	-0.4455 (0.2157)	-0.027 (0.0024)	-0.0772 (0.0073)	-0.4669 (0.2318)
	MH	Sq	-0.0302 (0.0022)	-0.1399 (0.0207)	-0.2877 (0.1001)	-0.0293 (0.0019)	-0.1425 (0.0213)	-0.2891 (0.0992)	-0.0492 (0.0042)	-0.1195 (0.0163)	-0.4891 (0.2474)	-0.071 (0.0060)	-0.1529 (0.0244)	-0.507 (0.2638)
		$lx_{(\beta=0.5)}$	-0.0323 (0.0023)	-0.1418 (0.0213)	-0.2969 (0.1057)	-0.0315 (0.0021)	-0.1443 (0.0218)	-0.2986 (0.1049)	-0.0509 (0.0043)	-0.1219 (0.0168)	-0.4926 (0.2506)	-0.0723 (0.0062)	-0.1545 (0.0249)	-0.5099 (0.2665)
		$lx_{(\beta=-5)}$	-0.0094 (0.0014)	-0.1219 (0.0160)	-0.203 (0.0562)	-0.0082 (0.0012)	-0.1242 (0.0165)	-0.2026 (0.0544)	-0.0326 (0.0031)	-0.0962 (0.0115)	-0.4563 (0.2191)	-0.0586 (0.0046)	-0.1367 (0.0199)	-0.4798 (0.2390)
		$lx_{(\beta=5)}$	-0.0513 (0.0038)	-0.1585 (0.0262)	-0.375 (0.1580)	-0.0509 (0.0035)	-0.1611 (0.0269)	-0.3774 (0.1584)	-0.0663 (0.0058)	-0.1433 (0.0222)	-0.5228 (0.2789)	-0.0837 (0.0078)	-0.1693 (0.0295)	-0.535 (0.2910)

Table 3. The Bais and (MSE) for MLE and Bayesian estimation results obtained by using both Lindley and MH for informative (INF) and non-informative priors (NON) under SELF (sq) LINEX loss function (lx) at ($\beta = (-5, 5, 0.5)$) for $N = 1000$ under Cases 2 and 3.

Method			GA						NLM					
			Case2			Case3			Case2			Case3		
			$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$
			Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)
		MLE	0.0326 (0.0038)	-0.0311 (0.0032)	0.0104 (0.0027)	0.0274 (0.0040)	-0.0273 (0.0034)	0.01229 (0.0030)	-0.0009 (0.0001)	-0.0061 (0.0002)	-0.0308 (0.0023)	0.0005 (0.0001)	-0.0028 (0.0002)	-0.0188 (0.0016)
NON	Lindley	Sq	0.0327 (0.0038)	-0.0308 (0.0033)	0.0112 (0.0029)	0.0276 (0.0039)	-0.0269 (0.0035)	0.01347 (0.0016)	-0.0001 (0.0001)	-0.0044 (0.0002)	-0.0304 (0.0023)	0.0013 (0.0001)	-0.0012 (0.0002)	-0.0183 (0.0016)
		$lx_{(\beta=0.5)}$	0.0326 (0.0038)	-0.0311 (0.0032)	0.0107 (0.0028)	0.0275 (0.0039)	-0.0271 (0.0035)	0.0134 (0.0015)	-0.0002 (0.0001)	-0.0053 (0.0002)	-0.0307 (0.0023)	0.0013 (0.0001)	-0.0020 (0.0002)	-0.0186 (0.0016)
		$lx_{(\beta=-5)}$	0.0327 (0.0038)	-0.031 (0.0033)	0.0112 (0.0029)	0.0276 (0.0039)	-0.0216 (0.0035)	-0.2964 (0.0017)	0.0002 (0.0001)	-0.0041 (0.0002)	-0.0272 (0.0021)	0.0013 (0.0001)	-0.002 (0.0002)	-0.0151 (0.0015)
		$lx_{(\beta=5)}$	0.0326 (0.0038)	-0.0311 (0.0032)	0.0106 (0.0028)	0.0274 (0.0039)	-0.0271 (0.0035)	-0.2967 (0.0886)	-0.0004 (0.0001)	-0.0048 (0.0002)	-0.0336 (0.0025)	0.0017 (0.0001)	-0.0008 (0.0002)	-0.0216 (0.0017)
	MH	Sq	0.0327 (0.0038)	-0.0307 (0.0033)	0.0116 (0.0030)	0.0276 (0.0039)	-0.0268 (0.0035)	-0.2881 (0.0844)	-0.0585 (0.0042)	-0.1128 (0.0140)	-0.3109 (0.1102)	0.001 (0.0001)	-0.0015 (0.0002)	-0.3080 (0.1089)
		$lx_{(\beta=0.5)}$	0.0326 (0.0038)	-0.031 (0.0033)	0.0111 (0.0029)	0.0275 (0.0039)	-0.027 (0.0035)	-0.2898 (0.0851)	-0.06 (0.0044)	-0.1154 (0.0146)	-0.3196 (0.1158)	-0.0594 (0.0043)	-0.1126 (0.0141)	-0.3125 (0.1114)
		$lx_{(\beta=-5)}$	0.0327 (0.0038)	-0.0308 (0.0033)	0.0107 (0.0028)	0.0276 (0.0039)	-0.0269 (0.0035)	-0.0156 (0.0889)	-0.0442 (0.0027)	-0.0877 (0.0089)	-0.231 (0.0651)	-0.0609 (0.0045)	-0.1153 (0.0147)	-0.3214 (0.1170)
		$lx_{(\beta=5)}$	0.0326 (0.0038)	-0.0311 (0.0032)	0.0102 (0.0027)	0.0274 (0.0040)	-0.0271 (0.0035)	-0.2972 (0.0889)	-0.0732 (0.0061)	-0.1384 (0.0203)	-0.3934 (0.1681)	-0.0448 (0.0028)	-0.0871 (0.0089)	-0.2312 (0.0660)
INF	Lindley	Sq	-0.0439 (0.0034)	-0.1242 (0.0172)	-0.2941 (0.1031)	-0.0444 (0.0038)	-0.123 (0.0171)	-0.2913 (0.1021)	-0.001 (0.0001)	-0.0056 (0.0002)	-0.0322 (0.0024)	-0.0743 (0.0062)	-0.1385 (0.0205)	-0.0183 (0.0016)
		$lx_{(\beta=0.5)}$	-0.0418 (0.0035)	-0.1234 (0.0171)	-0.2875 (0.0988)	-0.046 (0.0039)	-0.1227 (0.0171)	-0.2883 (0.0992)	-0.001 (0.0001)	-0.0057 (0.0002)	-0.0325 (0.0024)	0.0003 (0.0001)	-0.0022 (0.0002)	-0.0186 (0.0016)
		$lx_{(\beta=-5)}$	-0.0458 (0.0036)	-0.1265 (0.0178)	-0.3033 (0.1087)	-0.0462 (0.0040)	-0.1253 (0.0176)	0.01282 (0.1077)	-0.0007 (0.0001)	-0.0053 (0.0002)	-0.029 (0.0022)	0.0003 (0.0001)	-0.0023 (0.0002)	-0.0150 (0.0015)
		$lx_{(\beta=5)}$	-0.0437 (0.0036)	-0.1257 (0.0177)	-0.297 (0.1044)	-0.0478 (0.0040)	-0.1251 (0.0177)	-0.2979 (0.1049)	-0.0013 (0.0001)	-0.006 (0.0002)	-0.0353 (0.0026)	0.0006 (0.0001)	-0.0019 (0.0002)	-0.0215 (0.0017)
	MH	Sq	-0.0257 (0.0024)	-0.102 (0.0125)	-0.2092 (0.0585)	-0.0269 (0.0028)	-0.1002 (0.0123)	-0.2049 (0.0574)	-0.0585 (0.0043)	-0.1124 (0.0142)	-0.3176 (0.1141)	-0.2135 (0.0001)	-0.0026 (0.0002)	-0.2991 (0.0901)
		$lx_{(\beta=0.5)}$	-0.0235 (0.0025)	-0.1009 (0.0124)	-0.2015 (0.0548)	-0.0283 (0.0029)	-0.1003 (0.0124)	-0.2009 (0.0542)	-0.06 (0.0045)	-0.115 (0.0148)	-0.3263 (0.1198)	-0.0597 (0.0042)	-0.1124 (0.0139)	-0.3104 (0.1105)
		$lx_{(\beta=-5)}$	-0.0625 (0.0051)	-0.1469 (0.0231)	-0.3808 (0.1613)	-0.0623 (0.0054)	-0.1462 (0.0230)	-0.3797 (0.1611)	-0.0441 (0.0028)	-0.0876 (0.0091)	-0.2372 (0.0677)	-0.0612 (0.0044)	-0.1151 (0.0145)	-0.3194 (0.1162)
		$lx_{(\beta=5)}$	-0.0605 (0.0052)	-0.1464 (0.0230)	-0.3766 (0.1576)	-0.0641 (0.0055)	-0.146 (0.0230)	-0.3781 (0.1592)	-0.0732 (0.0062)	-0.1378 (0.0205)	-0.3993 (0.1722)	-0.0451 (0.0027)	-0.087 (0.0088)	-0.2275 (0.0643)

Table 4. The Bais and (MSE) for MLE and Bayesian estimation results obtained by using both Lindley and MH for informative (INF) and non-informative priors (NON) under SELF (sq) LINEX loss function (lx) at ($\beta = (-5, 5, 0.5)$) for $N = 50$ under Cases 2 and 3.

		Method	GA						NLM					
			Case2			Case3			Case2			Case3		
		Parameters	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$	$\mu = 0.2$	$\sigma = 0.3$	$\xi = 0.7$
			Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)	Bais (MSE)
		MLE	0.0561	0.0561	0.0262	0.0514	-0.0571	0.0241	0.0037	-0.0351	-0.2288	0.0056	-0.0325	-0.2179
			(0.0043)	(0.0041)	(0.0029)	(0.0041)	(0.0045)	(0.0030)	(0.0025)	(0.0035)	(0.0836)	(0.0026)	(0.0033)	(0.0825)
NON	Lindley	Sq	0.0564	-0.0553	0.0296	0.0521	-0.0531	-0.2584	0.0148	-0.009	-0.2175	0.0174	-0.0064	-0.2551
			(0.0043)	(0.0043)	(0.0043)	(0.0040)	(0.0060)	(0.0865)	(0.0027)	(0.0028)	(0.0770)	(0.0029)	(0.0028)	(0.0768)
		$lx_{(\beta=0.5)}$	0.0564	-0.0557	0.0296	0.0521	-0.0551	0.2594	0.0143	-0.0225	-0.225	0.0169	-0.0199	-0.2624
			(0.0043)	(0.0042)	(0.0043)	(0.0040)	(0.0050)	(0.0866)	(0.0026)	(0.0030)	(0.0799)	(0.0028)	(0.0029)	(0.0828)
		$lx_{(\beta=-5)}$	0.0564	-0.0553	0.0293	0.0521	-0.054	-0.2555	0.0197	-0.0061	-0.1575	0.0224	-0.0034	-0.1475
			(0.0043)	(0.0043)	(0.0041)	(0.0040)	(0.0055)	(0.0866)	(0.0030)	(0.0029)	(0.0582)	(0.0032)	(0.0029)	(0.0363)
		$lx_{(\beta=5)}$	0.0564	-0.0553	0.0278	0.0521	-0.0502	-0.2696	0.0094	-0.0129	-0.2826	0.0118	-0.0105	-0.3213
			(0.0043)	(0.0043)	(0.0034)	(0.0040)	(0.0087)	(0.0854)	(0.0024)	(0.0028)	(0.1070)	(0.0025)	(0.0027)	(0.1122)
	MH	Sq	-0.0296	-0.1381	-0.2805	-0.033	-0.1376	-0.2822	-0.0571	-0.1274	-0.4224	-0.0554	-0.1259	-0.4085
			(0.0024)	(0.0203)	(0.0951)	(0.0024)	(0.0204)	(0.0970)	(0.0046)	(0.0180)	(0.1926)	(0.0045)	(0.0174)	(0.1834)
		$lx_{(\beta=0.5)}$	-0.0317	-0.1401	-0.2901	-0.0351	-0.1395	-0.2917	-0.0587	-0.1297	-0.4279	-0.057	-0.1282	-0.4142
			(0.0025)	(0.0208)	(0.1006)	(0.0026)	(0.0209)	(0.1027)	(0.0047)	(0.0186)	(0.1968)	(0.0047)	(0.0179)	(0.1867)
		$lx_{(\beta=-5)}$	-0.0091	-0.1189	-0.1931	-0.0127	-0.1186	-0.1948	-0.042	-0.1055	-0.3706	-0.0401	-0.1036	-0.3549
			(0.0016)	(0.0154)	(0.0518)	(0.0017)	(0.0156)	(0.0529)	(0.0033)	(0.0131)	(0.1564)	(0.0033)	(0.0126)	(0.1471)
		$lx_{(\beta=5)}$	-0.0507	-0.1577	-0.3708	-0.0537	-0.1571	-0.3717	-0.0726	-0.1497	-0.4749	-0.0712	-0.1486	-0.4631
			(0.0039)	(0.0259)	(0.1535)	(0.0041)	(0.0259)	(0.1557)	(0.0063)	(0.0239)	(0.2356)	(0.0062)	(0.0233)	(0.2265)
INF	Lindley	Sq	0.0562	-0.0561	0.0277	0.0516	-0.0542	-0.2603	0.0046	-0.0234	-0.2698	0.0001	-0.0224	-0.2408
			(0.0043)	(0.0041)	(0.0033)	(0.0041)	(0.0055)	(0.0858)	(0.0021)	(0.0027)	(0.0980)	(0.0021)	(0.0027)	(0.0720)
		$lx_{(\beta=0.5)}$	0.0562	-0.0562	0.0275	0.0516	-0.0541	-0.2613	0.0041	-0.0238	-0.2765	-0.0005	-0.0229	-0.248
			(0.0043)	(0.0041)	(0.0032)	(0.0041)	(0.0056)	(0.0859)	(0.0021)	(0.0028)	(0.1014)	(0.0021)	(0.0027)	(0.0758)
		$lx_{(\beta=-5)}$	0.0563	-0.056	0.0286	0.0516	-0.0547	-0.2558	0.0101	-0.0189	-0.1978	0.0059	-0.0176	-0.1376
			(0.0043)	(0.0041)	(0.0037)	(0.0041)	(0.0052)	(0.0862)	(0.0024)	(0.0027)	(0.0687)	(0.0023)	(0.0027)	(0.0334)
		$lx_{(\beta=5)}$	0.0562	-0.0563	0.0255	0.0515	-0.0538	-0.2702	-0.0008	-0.0284	-0.3181	-0.0053	-0.0276	-0.3066
			(0.0043)	(0.0041)	(0.0028)	(0.0041)	(0.0060)	(0.0854)	(0.0020)	(0.0028)	(0.1262)	(0.0021)	(0.0027)	(0.1035)
	MH	Sq	-0.0309	-0.1372	-0.2846	-0.0325	-0.1378	-0.2820	-0.0570	-0.1265	-0.4215	-0.0564	-0.1269	-0.4111
			(0.0025)	(0.0200)	(0.0980)	(0.0026)	(0.0204)	(0.0963)	(0.0048)	(-0.0178)	(0.1931)	(0.0046)	(0.0177)	(0.1845)
		$lx_{(\beta=0.5)}$	-0.033	-0.1392	-0.2941	-0.0346	-0.1398	-0.2917	-0.0586	-0.1288	-0.4268	-0.058	-0.1292	-0.4169
			(0.0026)	(0.0205)	(0.1035)	(0.0027)	(0.0209)	(0.1019)	(0.0049)	(0.0183)	(0.1973)	(0.0048)	(0.0183)	(0.1888)
		$lx_{(\beta=-5)}$	-0.0104	-0.1178	-0.1978	-0.0121	-0.1187	-0.1942	-0.0419	-0.1044	-0.371	-0.0411	-0.1045	-0.3568
			(0.0017)	(0.0151)	(0.0544)	(0.0018)	(0.0156)	(0.0526)	(0.0035)	(0.0129)	(0.1571)	(0.0034)	(0.0128)	(0.1480)
		$lx_{(\beta=5)}$	-0.0519	-0.157	-0.3744	-0.0533	-0.1573	-0.3733	-0.0725	-0.1491	-0.4726	-0.0719	-0.1497	-0.4672
			(0.0040)	(0.0257)	(0.1565)	(0.0042)	(0.0259)	(0.1555)	(0.0065)	(0.0237)	(0.2351)	(0.0064)	(0.0236)	(0.2287)

3.2. Observed Fisher information and approximate confidence interval for the distribution of the censoring scheme

The observed Fisher information can be computed based on both the full likelihood and the approximate likelihood equations. These formulas serve as the foundation for the creation of pivotal quantities, which are used to explore the coverage probabilities in the context of the limiting normal distribution. To rigorously assess the performance of these pivotal quantities, Monte Carlo simulations were conducted. In situations in which the sample size is sufficiently large (i.e., N and n both exceed zero), we also consider the construction of approximate confidence intervals for the three specific cases that have been previously discussed. These confidence intervals provide valuable insights into the precision of parameter estimates and their statistical significance in various scenarios.

3.2.1. Observed Fisher information corresponding to with censored scheme cases

Under the assumption of censoring scheme distribution, the Fisher information matrix, based on log-likelihood functions given in Eqs (3.1) and (3.2) for Cases 1 and 2 can be obtained as follows:

$$I^{*}=\begin{bmatrix}-\log(L)_{\mu^2} & -\log(L)_{\mu\sigma} & -\log(L)_{\mu\xi}\\ -\log(L)_{\sigma\mu} & -\log(L)_{\sigma^2} & -\log(L)_{\sigma\xi}\\ -\log(L)_{\xi\mu} & -\log(L)_{\xi\sigma} & -\log(L)_{\xi^2}\end{bmatrix}, \tag{3.11}$$

while, for Case 3 the fisher information matrix $I^{**}(\theta)$ is obtained as follows:

$$I^{**}=\begin{bmatrix}-\log(L)_{\mu^2} & -\log(L)_{\mu\sigma} & -\log(L)_{\mu\xi} & 0\\ -\log(L)_{\sigma\mu} & -\log(L)_{\sigma^2} & -\log(L)_{\sigma\xi} & 0\\ -\log(L)_{\xi\mu} & -\log(L)_{\xi\sigma} & -\log(L)_{\xi^2} & 0\\ 0 & 0 & 0 & -\delta_{(P)}\end{bmatrix}, \tag{3.12}$$

where,

$$\log(L)_{\mu^2}=\sum_{i=1}^{n}\frac{(1+\frac{1}{\xi})\xi^2}{Z_i^2}-\sum_{i=1}^{n}\frac{(1+\xi)}{Z_i^{2+\frac{1}{\xi}}}+\sum_{i=1}^{n}\frac{R_i[1+\xi-Z_i^{\frac{-1}{\xi}}-\frac{(Z_i^{\frac{-1}{\xi}}U_i)}{(1-U_i)}]U_i}{Z_i^{2+\frac{1}{\xi}}[1-U_i]}, \tag{3.13}$$

$$\begin{aligned}\log(L)_{\mu\sigma}=\log(L)_{\sigma\mu}=&-\sum_{i=1}^{n}\frac{(1+\xi)[1-\frac{(x_i-\mu)\xi}{\sigma Z_i}]}{\sigma^2 Z_i}-\sum_{i=1}^{n}\frac{[\frac{(x_i-\mu)(1+\xi)}{\sigma}-Z_i]}{\sigma^2 Z_i^{2+\frac{1}{\xi}}}\\ &+\sum_{i=1}^{n}\frac{R_iU_i[\frac{(x_i-\mu)}{\sigma}(1+\xi-Z_i^{\frac{-1}{\xi}}-\frac{Z_i^{\frac{-1}{\xi}}U_i}{[1-U_i]})-Z_i]}{\sigma^2 Z_i^{2+\frac{1}{\xi}}[1-U_i]},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\log(L)_{\mu\xi}=\log(L)_{\xi\mu}=&\sum_{i=1}^{n}\frac{[1-\frac{(1+\xi)(x_i-\mu)}{\sigma Z_i}]}{\sigma Z_i}-\sum_{i=1}^{n}\frac{[\frac{\log(Z_i)}{\xi^2}-\frac{(x_i-\mu)[1+\frac{1}{\xi}]}{\sigma Z_i}]}{\sigma Z_i^{1+\frac{1}{\xi}}}\\ &+\sum_{i=1}^{n}\frac{R_iU_i[\frac{\log(Z_i)}{\xi^2}(1-Z_i^{\frac{-1}{\xi}}[1+\frac{U_i}{[1-U_i]}])-\frac{(x_i-\mu)}{\sigma Z_i}(1+\frac{1}{\xi}-\frac{Z_i^{\frac{-1}{\xi}}}{\xi}[1+\frac{U_i}{[1-U_i]}])]}{\sigma Z_i^{1+\frac{1}{\xi}}[1-U_i]},\end{aligned} \tag{3.15}$$

$$\log(L)_{\sigma^2} = \frac{n}{\sigma^2} - \sum_{i=1}^{n} \frac{(1+\xi)(x_i-\mu)[2-\frac{\xi^2}{\sigma Z_i}]}{\sigma^3 Z_i} - \sum_{i=1}^{n} \frac{(x_i-\mu)[\frac{(x_i-\mu)(1+\xi)}{\sigma Z_i}-2]}{\sigma^3 Z_i^{1+\frac{1}{\xi}}} + \sum_{i=1}^{n} \frac{R_i(x_i-\mu)^2 U_i[2-\frac{2\sigma Z_i}{(x_i-\mu)} - Z_i^{\frac{-1}{\xi}}[1+\frac{U_i}{[1-U_i]}]]}{\sigma^4 Z_i^{2+\frac{1}{\xi}}[1-U_i]}, \tag{3.16}$$

$$\log(L)_{\sigma\xi} = \log(L)_{\xi\sigma} = \sum_{i=1}^{n} \frac{(x_i-\mu)[\frac{1}{\xi} - \frac{(1+\frac{1}{\xi})[1+\xi(x_i-\mu)]}{\sigma Z_i}]}{\sigma^2 Z_i} - \sum_{i=1}^{n} \frac{(x_i-\mu)[\frac{\log(Z_i)}{\xi^2} - \frac{(x_i-\mu)(1+\frac{1}{\xi})}{\sigma Z_i}]}{\sigma^2 Z_i^{1+\frac{1}{\xi}}} + \sum_{i=1}^{n} \frac{R_i(x_i-\mu)U_i[\frac{\log(Z_i)}{\xi^2}(1-Z_i^{\frac{-1}{\xi}} - \frac{Z_i^{\frac{-1}{\xi}} U_i}{[1-U_i]}) + \frac{(x_i-\mu)}{\sigma Z_i}(-(\frac{1}{\xi}+1) + \frac{Z_i^{\frac{-1}{\xi}}}{\xi} + \frac{Z_i^{\frac{-1}{\xi}} U_i}{\xi[1-U_i]})]}{\sigma^2 Z_i^{1+\frac{1}{\xi}}[1-U_i]}, \tag{3.17}$$

$$\begin{aligned}\log(L)_{\xi^2} = &\sum_{i=1}^{n}[\frac{(x_i-\mu)}{\sigma Z_i}(\frac{2}{\xi^2} + \frac{(x_i-\mu)(1+\frac{1}{\xi})}{\sigma Z_i}) - \frac{2\log(Z_i)}{\xi^3}] \\ &- \sum_{i=1}^{n} Z_i^{\frac{-1}{\xi}}[\frac{\log(Z_i)}{\xi^2}(\frac{\log(Z_i)}{\xi^2} - \frac{2(x_i-\mu)}{\sigma Z_i \xi} - 2) + \frac{(x_i-\mu)}{\sigma\xi Z_i}(\frac{2}{\xi} + \frac{(x_i-\mu)(1+\frac{1}{\xi})}{\sigma Z_i})] \\ &+ \sum_{i=1}^{n} \frac{R_i Z_i^{\frac{-1}{\xi}} U_i[(\frac{\log(Z_i)}{\xi^2})^2(1-Z_i^{\frac{-1}{\xi}}) + \frac{(x_i-\mu)^2}{\sigma^2 Z_i^2 \xi^2}(1+\xi-Z_i^{\frac{-1}{\xi}})]}{[1-U_i]} \\ &- \sum_{i=1}^{n} \frac{R_i Z_i^{\frac{-1}{\xi}} U_i[\frac{2\log(Z_i)}{\xi^2}(\frac{(x_i-\mu)}{\sigma Z_i \xi} + 1 - \frac{(x_i-\mu)}{\xi\sigma Z_i^{1+\frac{1}{\xi}}})]}{[1-U_i]},\end{aligned} \tag{3.18}$$

and

$$\delta_{(P)} = \log(L_3)_{P^2} = \frac{-\sum_{i=1}^{n} r_i}{P^2} - \frac{(n-1)(N-n) - \sum_{i=1}^{n-1}(n-i)r_i}{(1-P)^2}, \tag{3.19}$$

where, $Z_i = [1+\xi\frac{(x_i-\mu)}{\sigma}]$ and $U_i = \exp(-Z_i^{\frac{-1}{\xi}})$.

Similarly, the observed Fisher information could be obtained by supplying the maximum likelihood values ofthe parameters for Eqs (3.8) and (3.10).

3.2.2. The asymptotic confidence interval for GEVL distribution parameters

The variance-covariance matrix could be obtained by applying the inverse of matrices I^* or I^{**} given by Eqs (3.11) and (3.12) which depend on the case under study. According to [18], the asymptotic distribution of parameter ν for the proposed cases (Cases 1–3) follows a normal distribution $\hat{\nu} \sim \mathcal{N}(\hat{\nu}, I^{-1}$ where I^{-1} is the variance-covariance matrix. Therefore, the asymptotic $100(1-\zeta)\%$ confidence interval for parameters ν with a significance level is given by

$$[\hat{\nu} - z_{\frac{\nu}{2}}\sqrt{I^{-1}} \quad , \quad \hat{\nu} + z_{\frac{\nu}{2}}\sqrt{I^{-1}}]. \tag{3.20}$$

Then the confidence intervals of distribution parameters are given by

$$\begin{aligned} &[\hat{\mu} - z_{\frac{\zeta}{2}} \sqrt{I^{-1}}, \qquad \hat{\mu} + z_{\frac{\zeta}{2}} \sqrt{I^{-1}}],\\ &[\hat{\sigma} - z_{\frac{\zeta}{2}} \sqrt{I^{-1}}, \qquad \hat{\sigma} + z_{\frac{\zeta}{2}} \sqrt{I^{-1}}],\\ &[\hat{\xi} - z_{\frac{\zeta}{2}} \sqrt{I^{-1}}, \qquad \hat{\xi} + z_{\frac{\zeta}{2}} \sqrt{I^{-1}}], \end{aligned} \tag{3.21}$$

where

$$I^{-1} = \begin{cases} I^{*-1}, & \text{For cases 1 and 2,} \\ I^{**-1}, & \text{For case 3.} \end{cases} \tag{3.22}$$

4. Bayesian estimation

The Bayesian estimation method is distinct in that it does not solely rely on the observed sample data ,as it also incorporates prior information about the distribution of samples. Consequently, Bayesian estimation leverages both the available population distribution information and prior probabilities. This approach allows for a more objective and rational description of unknown parameters. In the following section, we consider the estimations of parameters by using Bayesian methods and, two different loss functions: the square loss function and the LINEX loss function. These estimations are obtained under two scenarios: one utilizing informative priors and the other employing non-informative priors. The incorporation of these loss functions and prior information enables a comprehensive analysis of the parameter estimation in a Bayesian framework.

Informative priors

Let the unknown parameters for all cases considered to be independent of each other. The method of choosing parameter priors for the informative case depends on the parameter validation region, as introduced by [19, 20]. Suppose that the parameters μ, σ, ξ follow an exponential distribution with the hyperparameters a_1, a_2, and a_3, while the random variable P follows a beta distribution with the parameters $[\alpha, \gamma]$, Then the prior PDFs of parameters μ, σ, ξ, and P are given respectively by

$$\begin{aligned} \pi_1(\mu) &= \tfrac{1}{a_1} \exp(-a_1\mu), \qquad && a_1, \mu > 0,\\ \pi_2(\sigma) &= \tfrac{1}{a_2} \exp(-a_2\sigma), && a_2, \sigma > 0,\\ \pi_3(\sigma) &= \tfrac{1}{a_3} \exp(-a_3\xi), && a_3, \xi > 0, \end{aligned} \tag{4.1}$$

and,

$$\pi_4(P) = \frac{1}{B(\alpha, \gamma)} P^{\alpha-1}(1-p)^{1-\gamma}, \qquad \alpha, \gamma, P > 0, \tag{4.2}$$

where $B(\alpha, \gamma)$ is the beta function and all of the hyper-parameters $(a_1, a_2, a_3, \alpha, \gamma)$ are estimated by using the same method given in [21].

Non-informative priors

For this case, we assume that all prior PDFs of the parameters μ, σ, ξ, and P are equal to 1.

Based on type-II progressively censored data, for all distributions of the censored scheme discussed above, Bayesian estimation is discussed in this section for both point and interval estimation.

4.1. Bayesian point estimation of GEVL distribution's parameters according to censoring scheme and prior distribution

According to the informative prior functions of parameters μ, σ, ξ and P. The posterior PDFs are given by

$$\begin{aligned} \pi_j^*(\mu,\sigma,\xi) &= \frac{L(\mu,\sigma,\xi)\exp(-(a_1\mu+a_2\sigma+a_3\xi))}{\int_0^\infty\int_0^\infty\int_0^\infty L(\mu,\sigma,\xi)\exp(-(a_1\mu+a_2\sigma+a_3\xi))d\mu d\sigma d\xi}, \qquad j=1,2 \\ &= \frac{\prod_{i=1}^n \sigma^{-1} Z_i^{-\frac{1}{\xi}-1} U_i[1-U_i]^{R_i} e^{-(a_1\mu+a_2\sigma+a_3\xi)}}{\int_0^\infty\int_0^\infty\int_0^\infty \prod_{i=1}^n \sigma^{-1} Z_i^{-\frac{1}{\xi}-1} U_i[1-U_i]^{R_i} e^{-(a_1\mu+a_2\sigma+a_3\xi)} d\mu d\sigma d\xi} \\ &\propto \prod_{i=1}^n \sigma^{-1} Z_i^{-\frac{1}{\xi}-1} U_i[1-U_i]^{R_i} e^{-(a_1\mu+a_2\sigma+a_3\xi)} \end{aligned} \tag{4.3}$$

$$\pi_3^*(\mu,\sigma,\xi,P) = \frac{P^{\sum_{i=1}^{n-1} r_i+\alpha-1}(1-P)^{(n-1)(N-n)+\sum_{i=1}^{n-1}(n-i)r_i+\gamma-1}}{B(\sum_{i=1}^{n-1} r_i+\alpha,(n-1)(N-n)+\sum_{i=1}^{n-1}(n-i)r_i+\gamma)}\pi_j^*(\mu,\sigma,\xi), \qquad j=1,2, \tag{4.4}$$

where π_j^*, $j = 1,2,3$ depends on the case of the censoring scheme under study which was mentioned previously, $Z_i = [1+\xi\frac{(x_i-\mu)}{\sigma}]$ and $U_i = \exp(-Z_i^{\frac{-1}{\xi}})$.

Then the Bayesian estimation of parameters of the GEVL distribution based on progressive type-II censoring $\theta = (\mu,\sigma,\xi)$ for Case (1,2), and $\theta = (\mu,\sigma,\xi,P)$ for case 3 for the SELF loss function for the informative prior is given by

$$\hat{\theta}_S = \int \theta\pi_j^*(\theta)d\theta, \qquad j=1,2,3. \tag{4.5}$$

Also, the Bayesian estimation of GEVL distribution parameters is considered for the LINEX loss function $L_{lx}(\theta) = (\exp(\beta\theta) - (\beta\theta) - 1)$ for the informative prior which is given by, (see [22])

$$\hat{\theta}_{lx} = \frac{-1}{\beta}\log(\int \exp(-\beta\theta)\pi_j^*(\theta)d\theta), \qquad j=1,2,3 \tag{4.6}$$

where β is a shape parameter in which the negative value of β provides more weight to underestimation than the overestimation while for very (small or large) values of β the LINEX loss function is almost symmetric (see [22]).

For non-informative priors, the posterior PDFs are given by

$$\begin{aligned} \pi_j^{**}(\mu,\sigma,\xi) &= \frac{L(\mu,\sigma,\xi)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\mu,\sigma,\xi)d\mu d\sigma d\xi}, \qquad j=1,2 \\ &= \frac{\prod_{i=1}^n \sigma^{-1} Z_i^{-\frac{1}{\xi}-1} U_i[1-U_i]^{R_i}}{\int_0^\infty\int_0^\infty\int_0^\infty \prod_{i=1}^n \sigma^{-1} Z_i^{-\frac{1}{\xi}-1} U_i[1-U_i]^{R_i} d\mu d\sigma d\xi} \\ &\propto \prod_{i=1}^n \sigma^{-1} Z_i^{-\frac{1}{\xi}-1} U_i[1-U_i]^{R_i} \end{aligned} \tag{4.7}$$

$$\pi_3^{**}(\mu,\sigma,\xi,P) = \frac{P^{\sum_{i=1}^{n-1} r_i}(1-P)^{(n-1)(N-n)+\sum_{i=1}^{n-1}(n-i)r_i}}{B(\sum_{i=1}^{n-1} r_i+1,(n-1)(N-n)+\sum_{i=1}^{n-1}(n-i)r_i+1)}\pi_j^{**}(\mu,\sigma,\xi), \qquad j=1,2, \tag{4.8}$$

where π_j^{**}, $j = 1,2,3$ depends on the case of the censoring scheme under study, as mentioned above, $Z_i = [1+\xi\frac{(x_i-\mu)}{\sigma}]$ and $U_i = \exp(-Z_i^{\frac{-1}{\xi}})$.

Then the Bayesian estimation of parameters of the GEVL distribution based on progressive type-II censoring under the SELF and LINEX loss functions for the informative prior are given respectively by,

$$\hat{\theta}_S = \int \theta\pi_j^{**}(\theta)d\theta, \qquad j=1,2,3, \tag{4.9}$$

$$\hat{\theta}_{lx} \quad = \frac{-1}{\beta}\log\left(\int \exp(-\beta\theta)\pi_j^{**}(\theta)d\theta\right), \qquad j = 1, 2, 3, \tag{4.10}$$

Since the systems of equations given by Eqs (4.5), (4.6), (4.9) and (4.10) cannot be reduced analytically, we solve them numerically. The most popular numerical techniques for Bayesian estimation are Lindley's approximation method and Gibbs sampling with MH algorithms which are discussed in detail in the next subsection.

4.2. Numerical methods of Bayesian estimation

1) Lindley's approximation method

Among the various methods suggested to approximate the ratio of integrals given in the systems of equations given by Eqs (4.5), (4.6), (4.9) and (4.10) , e.g., the Markov chain Monte Carlo method, Gibbs sampler, and Lindley's approximation, perhaps the simplest one is Lindley's approximation method. The Lindley approximation method can be described as, finding the following expectation:

$$E(U(\vartheta)) = \frac{\int U(\vartheta)\exp(L(\vartheta) + \rho(\vartheta))d(\vartheta)}{\int \exp(L(\vartheta) + \rho(\vartheta))d(\vartheta)}, \tag{4.11}$$

where $\vartheta = (\vartheta_1, \vartheta_2, \ldots \vartheta_r)$, $U(\vartheta)$ is any function of ϑ, $L(\vartheta)$ is the log likelihood function of ϑ and $\rho(\vartheta)$ is the log of the joint prior of ϑ. Then Lindley's approximation of this integral is given by

$$\begin{aligned} E(U(\vartheta)) &= U(\hat{\vartheta}) + \tfrac{1}{2}\textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}[U_{i,j}(\hat{\vartheta}) + 2U_i(\hat{\vartheta})\rho_j(\hat{\vartheta})]I^{-1}(\hat{\vartheta})_{i,j} \\ &+\tfrac{1}{2}\textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} L_{i,j,k}(\hat{\vartheta})U_l(\hat{\vartheta})I^{-1}(\hat{\vartheta})_{i,j}I^{-1}(\hat{\vartheta})_{k,l}, \end{aligned} \tag{4.12}$$

where $U_i = \frac{\partial U}{\partial\vartheta_i}$, $\rho_j = \frac{\partial\rho}{\partial\vartheta_j}$, $U_{i,j} = \frac{\partial^2 U}{\partial\vartheta_i\partial\vartheta_j}$, $L_{i,j,k} = \frac{\partial^3 L}{\partial\vartheta_i\partial\vartheta_j\partial\vartheta_k}$, and $I^{-1}(\hat{\vartheta})_{i,j}$ is equal to the variance covariance matrix. All of the partial derivatives are evaluated by using the MLEs of ϑ (see [23]). For our case study, we considered $U(\vartheta)$ to be any parameter from θ for the proposed cases (Cases 1–3) and $L(\vartheta)$ is given by at Eq (3.6). Additionally,

$$\rho(\vartheta) = \begin{cases} \exp(-(a_1\mu + a_2\sigma + a_3\xi)), & for\ Cases\ 1\ and\ 2, \\ P^{\sum_{i=1}^{n-1} r_i+\alpha-1}(1-P)^{(n-1)(N-n)+\sum_{i=1}^{n-1}(n-i)r_i+\gamma-1}\exp(-(a_1\mu + a_2\sigma + a_3\xi)), & for\ Case\ 3, \end{cases} \tag{4.13}$$

2) MH algorithms

Gibbs sampling is a popular method and one of the commonly used Bayesian estimation methods for estimating specific distribution attributes. If it is not easy to generate samples directly from the posterior distribution, it is convenient to apply the Gibbs sampling method with MH algorithms. For more details on MH algorithms see [24].

Let $\vartheta = (\vartheta_1, \vartheta_2, \ldots \vartheta_r)$ be a vector of the parameter we want to estimate. The MH algorithm could be described as follows: suppose that our goal is to draw samples from some distribution $f(\vartheta|x) = \nu g(\vartheta)$, where ν is the normalizing constant which may not be known or very difficult to compute. The MH algorithm provides a way of sampling from $f(\vartheta|x)$ without requiring any knowledge of ν.

For any $\vartheta_j \in \vartheta$, let $q(\vartheta^b|\vartheta^a_j)$ be the transition kernel (proposal distribution), that is the probability of jumping from the current state ϑ^a_j to ϑ^b_j. The following steps will generate a sequence of values $(\vartheta^1_j, \vartheta^2_j, \ldots)$ which form a Markov chain with a stationary distribution given by $f(\vartheta_j|x)$.

(1) Choose an arbitrary starting point ϑ^0_j for which $f(\vartheta^0_j|x) > 0$.

(2) For $i = 1$, sample a candidate point or proposal ϑ_j^* from the proposal distribution $q(\vartheta_j^*|\vartheta_j^{i-1})$.

(3) Calculate the acceptance probability $\rho(\vartheta_j^{i-1}, \vartheta_j^*) = \min[1, \frac{f(\vartheta_j^*|x)q(\vartheta_j^{i-1}|\vartheta_j^i)}{f(\vartheta_j^*|x)q(\vartheta_j^i|\vartheta_j^{i-1})}]$.

(4) Generate $U \sim U(0, 1)$.

(5) If $U < \rho(\vartheta_j^{i-1}, \delta^*)$ accept the proposal and set $\vartheta_j^i = \vartheta_j^*$. Otherwise, reject the proposal and set $\delta^i = \delta^{i-1}$.

(6) Repeat Steps 2–5. If the proposal distribution is symmetric then the acceptance condition becomes $\rho(\delta^{i-1}, \delta^*) = \min[1, \frac{f(\vartheta_j^*|x)}{f(\vartheta_j^*|x)}]$.

(7) We could get a set of Gibbs (MH) samplings of parameters $\vartheta_j^1, \vartheta_j^2, \ldots, \vartheta_j^{10000}$. Now, the approximate expectation of reliability ϑ_j could be calculated as follows:

$$E(\vartheta_j) = \frac{1}{t - bb} \sum_{k=bb+1}^{t} \vartheta_j^k.$$

For our case of study, from Eq (4.3) the proper density function of parameters μ, σ and ξ can obtained as follows:

$$\begin{aligned}
f(\mu|\sigma, \xi) &\propto \prod_{i=1}^n [1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{-\frac{1}{\xi}-1} \exp(-(a_1\mu) - [1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{\frac{-1}{\xi}})[1 - \exp(-[1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{\frac{-1}{\xi}})]^{R_i},\\
f(\sigma|\mu, \xi) &\propto \prod_{i=1}^n \sigma^{-1}[1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{-\frac{1}{\xi}-1} \exp(-(a_2\sigma) - [1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{\frac{-1}{\xi}})[1 - \exp(-[1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{\frac{-1}{\xi}})]^{R_i},\\
f(\xi|\mu, \sigma) &\propto \prod_{i=1}^n \sigma^{-1}[1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{-\frac{1}{\xi}-1} \exp(-(a_3\xi) - [1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{\frac{-1}{\xi}})[1 - \exp(-[1 + \xi\tfrac{(x_i-\mu)}{\sigma}]^{\frac{-1}{\xi}})]^{R_i}.
\end{aligned} \tag{4.14}$$

Moreover, these proper densities cannot as be reduced analytically to a well-known distribution but its plots show that as they are similar to a normal distribution. So, the Mh algorithm with normal proposal distribution is used for Bayesian estimation of every parameter as by following the Steps 1–7 mentioned above.

4.3. HPD interval algorithm

To compute HPD credible interval of the parameters denoted by $\theta = (\mu, \sigma, \xi)$, first order the importance of MH samplings

$$(\hat{\mu}_{bb+1}, \hat{\sigma}_{bb+1}, \xi_{bb+1}), (\hat{\mu}_{bb+2}, \hat{\sigma}_{bb+2}, \hat{\xi}_{bb+2}), \ldots, (\hat{\mu}_t, \hat{\sigma}_t, \hat{\xi}_t),$$

Then we could obtain three sets of ascending order samplings:

$$\begin{aligned}
&\hat{\mu}_{aa+1} < \hat{\mu}_{aa+2} < \cdots < \hat{\mu}_t,\\
&\hat{\sigma}_{aa+1} < \hat{\sigma}_{aa+2} < \cdots < \hat{\sigma}_t,\\
&\hat{\xi}_{aa+1} < \hat{\xi}_{aa+2} < \cdots < \hat{\xi}_t,
\end{aligned} \tag{4.15}$$

Therefore, the $100(1 - \alpha)\%$ HPD credible interval of $\theta = (\mu, \sigma, \xi)$ can be calculated as follows:

$$\begin{aligned}
&(\hat{\mu}_{(\frac{\alpha}{2})(t-aa)}, \hat{\mu}_{(1-\frac{\alpha}{2})(t-aa)}),\\
&(\hat{\sigma}_{(\frac{\alpha}{2})(t-aa)}, \hat{\sigma}_{(1-\frac{\alpha}{2})(t-aa)}),\\
&(\hat{\xi}_{(\frac{\alpha}{2})(t-aa)}, \hat{\xi}_{(1-\frac{\alpha}{2})(t-aa)}).
\end{aligned} \tag{4.16}$$

5. Simulation

In this section, a Monte Carlo simulation presented to evaluate as well as compare the efficacy of different estimators in determining the parameters (μ, σ, ξ) of GEVL distribution, as introduced in Sections 3 and 4. This simulation was conducted to rigorously assess various estimation techniques and their performance in terms of accurately estimating these crucial parameters. To study the behavior of the GA and NLM methods under small and large samples, 1000 samples each of size 50 and 1000 were generated for X. The MLE of (μ, σ, ξ) was contrasted with Bayesian estimators utilizing both the SELF and the LINEX loss function. This comparison aims to assess the performance and robustness of these estimation techniques in terms of the accuracy of parameter estimation.

we performed a comprehensive comparison between various estimation techniques by using Bayesian approaches and MLE, considering different loss functions like LINEX and numerical techniques such as The Lindley and MH. The method of selection of hyperparameters was explored and derived by using a method previously employed in [21]. The comparison is based on estimated values , the MSE, and 95% HPD for both informative and non-informative Bayesian estimation. The LINEX loss function was examined across three parameter values: a very large value $\beta = -5$, a very small value $\beta = 0.5$, and a positive value $\beta = 5$, detailed in Tables 1–4. All computationsextcolorredwere executed by using the R program, assessing estimators with fixed random removals. The censoring schemes involve a 10 percent elimination either at the beginning or end of the sample. The algorithm for generating progressive censoring of type II can be seen in [16].

Clearly, from Tables 1–4, the Lindley method gives a better estimate than the MH method. The informative Bayesian estimation gives results that are very close to the non-informative Bayesian estimation in terms of Bais and (MSE). Bayesian estimation under the LINEX loss function gives results that are very close result to that under the SELF in term of Bias and MSE; the negative value of β provides more weight to underestimation than the overestimation while, for very (small or large) values of β the LINEX loss function is almost symmetric (see [22]). In most cases, the GA gives better results than the NLM method. Moreover, for Case 1, the elimination at the beginning gives a better estimate in terms of bias and MSE than the end. Concerning the sensitivity of the estimates for sample size, we find that the behavior of both methods (i.e., the GA and NLM)gives a better result for large sample sizes. In addition, MLE gives a better bais than Bayesian estimation. In Table 5, it's obvious that the 95% HPD credible interval of the Gibbs (MH) sampling method gives a better estimate for informative beginning elimination for Case 1. In general, the GA gives a better estimate than NLM in terms of the lower-length intervals containing the real value.

Table 5. 95% HPD credible interval of Gibbs (MH) sampling method for Bayes estimation of GEVL distribution parameters in terms of lower (LB) and upper bounds (UB), and the length interval (IL).

method			parameters	N=1000						N=50					
				NON			INF			NON			INF		
				LB	UB	IL	LB	UB	IL	LB	UB	IL	LB	UB	IL
Case1	GA	Beginning	$\mu = 0.2$	0.0124	0.3024	0.29	0.0111	0.302	0.2909	0.0056	0.3216	0.3161	0.0049	0.3211	0.3163
			$\sigma = 0.3$	0.0119	0.314	0.3021	0.0129	0.3139	0.301	0.005	0.2986	0.2936	0.0046	0.2988	0.2942
			$\xi = 0.7$	0.0744	0.7118	0.6375	0.0702	0.711	0.6408	0.0464	0.7504	0.704	0.0504	0.7462	0.6958
		End	$\mu = 0.2$	0.0038	0.3087	0.3049	0.0047	0.3088	0.3042	0.0045	0.3217	0.3171	0.0035	0.3224	0.3189
			$\sigma = 0.3$	0.0045	0.3195	0.315	0.0038	0.3194	0.3156	0.0043	0.299	0.2947	0.0033	0.2985	0.2951
			$\xi = 0.7$	0.0402	0.728	0.6878	0.0496	0.7344	0.6848	0.0479	0.7449	0.697	0.0489	0.7469	0.698
	NLM	Beginning	$\mu = 0.2$	0.0113	0.2716	0.2602	0.0101	0.2716	0.2615	0.0112	0.2771	0.2659	0.0121	0.2778	0.2657
			$\sigma = 0.3$	0.016	0.3623	0.3463	0.0157	0.3627	0.347	0.0155	0.3354	0.3199	0.0149	0.3351	0.3202
			$\xi = 0.7$	0.0321	0.5411	0.509	0.0325	0.5386	0.5062	0.0193	0.3886	0.3693	0.0199	0.3895	0.3696
		End	$\mu = 0.2$	0.0089	0.2377	0.2287	0.0086	0.2374	0.2288	0.0087	0.237	0.2283	0.0096	0.2373	0.2277
			$\sigma = 0.3$	0.0109	0.2898	0.2789	0.0115	0.2897	0.2783	0.0108	0.2754	0.2646	0.0101	0.2743	0.2643
			$\xi = 0.7$	0.0216	0.4162	0.3946	0.0189	0.4147	0.3958	0.0176	0.3544	0.3368	0.0189	0.3546	0.3358
Case2	GA		$\mu = 0.2$	0.0107	0.2914	0.2807	0.0129	0.2918	0.2789	0.014	0.3142	0.3002	0.0129	0.3137	0.3008
			$\sigma = 0.3$	0.0134	0.3257	0.3123	0.0141	0.3266	0.3126	0.0119	0.3023	0.2904	0.0122	0.3024	0.2902
			$\xi = 0.7$	0.0777	0.72	0.6423	0.0788	0.7251	0.6463	0.0825	0.7369	0.6544	0.0814	0.7321	0.6507
	NLM		$\mu = 0.2$	0.0107	0.2597	0.2490	0.0112	0.2599	0.2487	0.0107	0.2639	0.2531	0.011	0.2639	0.2529
			$\sigma = 0.3$	0.0144	0.3498	0.3354	0.0156	0.349	0.3334	0.0142	0.3223	0.3081	0.0131	0.3228	0.3097
			$\xi = 0.7$	0.0687	0.6901	0.6215	0.0649	0.6876	0.6227	0.036	0.5100	0.474	0.0386	0.5089	0.4702
Case3	GA		$\mu = 0.2$	0.0053	0.2914	0.2861	0.0034	0.2909	0.2875	0.0043	0.3149	0.3106	0.0051	0.3148	0.3097
			$\sigma = 0.3$	0.0042	0.3352	0.331	0.0051	0.3353	0.3302	0.0038	0.3061	0.3023	0.004	0.3062	0.3022
			$\xi = 0.7$	0.0488	0.7469	0.6981	0.0485	0.7506	0.7021	0.0536	0.7586	0.7050	0.0466	0.7577	0.7111
	NLM		$\mu = 0.2$	0.0027	0.2647	0.262	0.0025	0.2647	0.2623	0.0038	0.2701	0.2663	0.0037	0.2697	0.266
			$\sigma = 0.3$	0.0047	0.3587	0.354	0.0039	0.3588	0.3548	0.0029	0.3302	0.3274	0.0027	0.33	0.3272
			$\xi = 0.7$	0.0383	0.7178	0.6795	0.0383	0.7201	0.6818	0.0203	0.5417	0.5214	0.0168	0.5407	0.5239

6. Real data example

In this section, we discuss how real lifetime data were used to illustrate the above algorithms. Extreme weather events like high temperatures are often associated with extreme weather events such as heatwaves, droughts, and wildfires. Moreover, extremely high temperatures can have serious health implications, leading to heat-related illnesses. After the recent global crisis due to the unexpected rise in temperature degree, it became necessary to understand the historical highs for each country to avoid some of the catastrophic events that occurred as a result of this increase , and to help in the assessment of the frequency, intensity, and impact of these events, enabling better preparedness and response strategies.

So, we applied a real data set that included the highest ten daily temperatures for Queensland Australia from 1957–2022. It was obtained from the Australian government from the following website: http://www.bom.gov.au/cgi-bin/climate/extremes/annual_extremes.cgi?period=%2Fcgi-bin%2Fclimate%2Fextremes%2Fannual_extremes.cgi&climtab=tmax_high&area=qld&year=1957. This data set was fitted to the GEVL distribution by using the Kolmogorov-Smirnov (K-S) goodness of fit test at the level of significance of $\alpha = 0.01$. The K-S value was 0.0510 which is less than the tabulated value of 0.0681. Also, we constructed the Q-Q plot and returned the level for this data set, as in Figures 2 and 3. Figures 2, 3 and the K-S value verify that this data set isfit the GEVL distribution. In Table 6 some basic statistics are presented for this data set including the, mean, variance, median, and others.

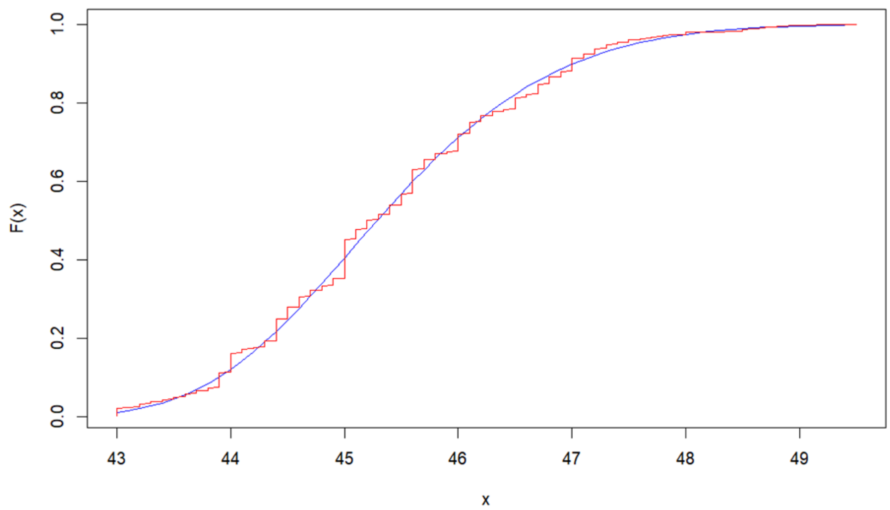


Figure 2. Q-Q plots regarding the fitting of the GEVL distribution for Queensland Australia's extreme temperature data.

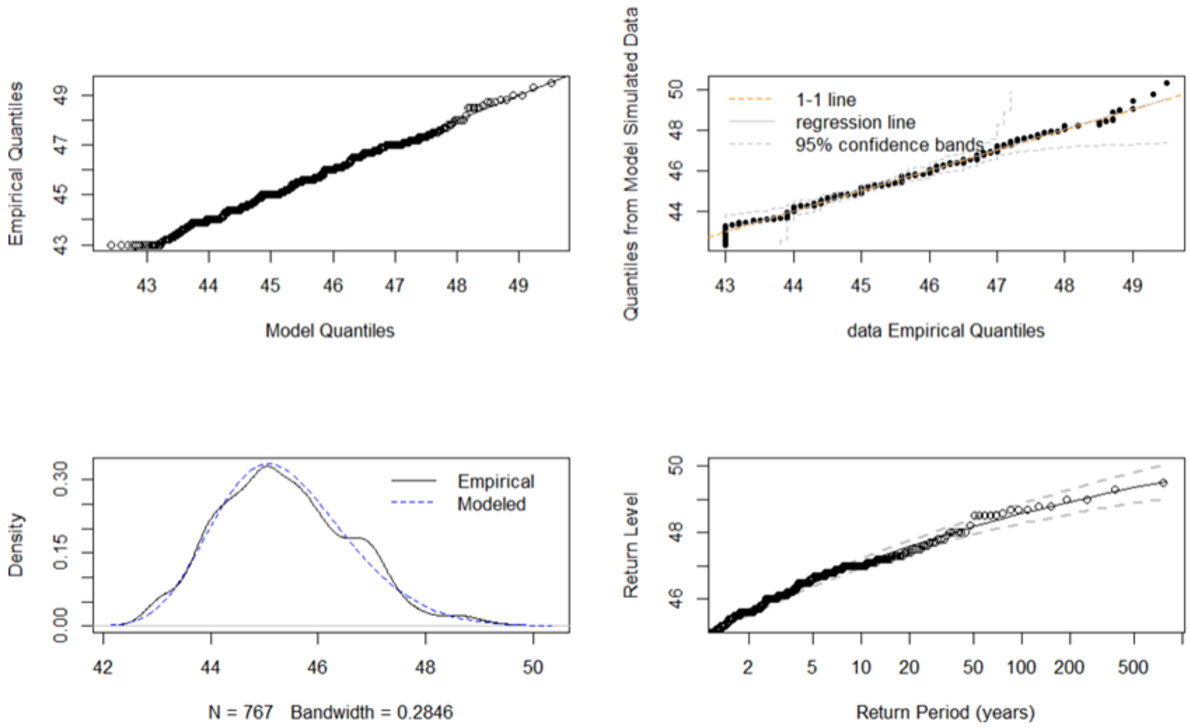


Figure 3. Returned level and empirical quantiles of the GEVL distribution for the Queensland Australia extreme temperaturedata.

Table 6. The basic statistics of the extreme temperature for Queensland (Australia) data sets.

Country	Mean	Median	VAR	Standard deviation	Minimum	Maximum	Range	Quartiles (25%, 50%, 75%)
Queensland(Australia)	45.3877	45.2	1.5019	1.2256	43	49.5	6.5	(44.5, 45.2, 46.1)

The real data serve as a practical example of the simulation technique, and we have used the methodology previously described in the simulation section in the process of estimation of the parameters and selection of the hyperparameters. The maximum likelihood estimate and Bayesian estimates considered in Sections 3 and 4 for the GEVL distribution parameters were obtained as in Tables 7 and 8 for elimination for the three cases considered above by using the two optimization methods considered i.e., (NLM and GA) for MLE and Bayesian estimation. Bayesian estimation is considered for informative and non-informative prior for both the Lindley and MH methods under the SELF (sq) and LINEX loss function ($lx_{(\beta=A)}$ and $A \in 0.5, -5, 5$).

Table 7. The MLE, Bayesian estimation, and 95% HPD credible interval of Gibbs (MH) sampling method for Bayesian estimation of GEVL distribution parameters for informative (INF) and non-informative priors (NON) under SELF (sq) LINEX loss function (lx) at ($\beta = (0.5, -5, 5)$) for Queensland (Australia) data set For Case 1.

			GA						NLM					
			Beginning			End			Beginning			Bnd		
			μ	σ	ξ	μ	σ	ξ	μ	σ	ξ	μ	σ	ξ
		MLE	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.949	1.1855	0.1186	44.6992	1.0072	0.1973
NON	Lindley	Sq	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.9512	1.1927	0.1206	44.7003	1.0094	0.1975
		$lx_{(\beta=0.5)}$	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.9506	1.1886	0.1203	44.6998	1.008	0.1973
		$lx_{(\beta=-5)}$	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.9571	1.1974	0.1234	44.7051	1.0119	0.1989
		$lx_{(\beta=5)}$	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.9451	1.1877	0.1178	44.6955	1.0067	0.196
	MH	Sq	44.6311	1.0327	0.1249	44.5174	0.4654	0.1404	44.6094	0.7503	0.077	44.3944	0.5559	0.1084
		$lx_{(\beta=0.5)}$	44.6136	1.0259	0.1236	44.4957	0.4455	0.1391	44.5921	0.7149	0.0764	44.3887	0.5456	0.1071
		$lx_{(\beta=-5)}$	44.7616	1.0916	0.1375	44.6843	0.6845	0.1535	44.7345	0.9839	0.0829	44.4549	0.6784	0.122
		$lx_{(\beta=5)}$	44.4584	0.964	0.1126	44.3081	0.3014	0.127	44.4039	0.4573	0.0714	44.3437	0.4741	0.0959
		UB	44.0901	0.6414	0.0015	43.8473	0.0035	0.0017	43.8255	0.0511	0.001	44.1044	0.1881	0.0011
		LB	44.9997	1.267	0.2582	44.9734	1.1611	0.2612	44.9956	1.2414	0.1836	44.7574	1.0552	0.2557
		LI	0.9095	0.6256	0.2567	1.1261	1.1576	0.2595	1.1701	1.1902	0.1826	0.653	0.8671	0.2546
INF	Lindley	Sq	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.9526	1.1823	0.1094	44.7016	1.0052	0.1938
		$lx_{(\beta=0.5)}$	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.952	1.1818	0.1091	44.7011	1.0049	0.1937
		$lx_{(\beta=-5)}$	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.9585	1.1873	0.1121	44.7063	1.0078	0.1952
		$lx_{(\beta=5)}$	44.9414	1.2063	0.2059	44.9207	1.2081	0.2047	44.9465	1.1774	0.1069	44.6967	1.0026	0.1924
	MH	Sq	44.3906	0.906	0.1641	44.0854	0.5644	0.1363	44.4551	0.898	0.0918	44.4891	0.6405	0.2666
		$lx_{(\beta=0.5)}$	44.3717	0.8866	0.1625	44.0418	0.5541	0.1348	44.4406	0.8877	0.0911	44.4845	0.6078	0.2664
		$lx_{(\beta=-5)}$	44.5951	1.0206	0.1789	44.5823	0.6466	0.1506	44.6093	0.9814	0.0983	44.5306	0.8291	0.2689
		$lx_{(\beta=5)}$	44.3717	0.8866	0.1625	44.0418	0.5541	0.1348	44.4406	0.8877	0.0911	44.4845	0.6078	0.2664
		UB	43.9698	0.2064	0.0021	43.4954	0.0544	0.0014	44.0197	0.4063	0.0014	44.1315	0.004	0.0652
		LB	45.0057	1.2466	0.2751	44.9853	0.9846	0.2686	44.9746	1.2358	0.1795	44.7534	1.0583	0.2752
		LI	1.0358	1.0402	0.273	1.4899	0.9302	0.2673	0.9549	0.8295	0.1782	0.6219	1.0543	0.21

Table 8. The MLE, Bayesian estimation and 95% HPD credible interval of Gibbs (MH) sampling method of Bayes estimation of GEVL distribution parameters for informative (INF) and non-informative priors (NON) under SELF (sq) LINEX loss function (lx) at ($\beta = (0.5, -5, 5)$) for Queensland (Australia) data sets for Cases 2 and 3.

			GA						NLM					
			Case2			Case3			Case2			Case3		
			μ	σ	ξ	μ	σ	ξ	μ	σ	ξ	μ	σ	ξ
		MLE	44.8468	1.09	0.2343	44.8081	1.1847	0.2417	44.9558	1.2257	0.1833	44.7978	1.2539	0.3541
NON	Lindley	Sq	44.8488	1.0925	0.2342	44.8096	1.1882	0.2421	44.9588	1.2319	0.1845	44.7758	1.2564	0.3593
		$lx_{(\beta=0.5)}$	44.8483	1.0909	0.2341	44.8089	1.1861	0.2419	44.9581	1.2282	0.1842	44.7743	1.255	0.3592
		$lx_{(\beta=-5)}$	44.8542	1.0951	0.2353	44.8162	1.1923	0.2443	44.9652	1.2374	0.1866	44.7925	1.258	0.36
		$lx_{(\beta=5)}$	44.8434	1.0897	0.2332	44.8029	1.184	0.2399	44.9522	1.2261	0.1823	44.7624	1.2548	0.3585
	MH	Sq	43.9561	0.7164	0.1517	44.2639	0.6625	0.1502	43.7289	1.2975	0.1221	44.7086	0.9467	0.1817
		$lx_{(\beta=0.5)}$	43.933	0.7043	0.1497	44.2314	0.6441	0.1484	43.7015	1.2971	0.121	44.7057	0.9182	0.178
		$lx_{(\beta=-5)}$	44.184	0.8143	0.1721	44.4741	0.8319	0.1676	43.9399	1.3008	0.1325	44.7344	1.1113	0.2194
		$lx_{(\beta=5)}$	43.7592	0.5964	0.1319	43.8342	0.4986	0.1331	43.4916	1.2905	0.1117	44.6783	0.6673	0.1477
		UB	43.2867	0.2697	0.0016	43.1805	0.0929	0.0016	44.8227	1.1008	0.1069	44.4412	0.2674	0.0021
		LB	44.5614	1.1159	0.3157	44.8466	1.2382	0.3089	43.1062	1.0118	0.002	44.8626	1.3128	0.4061
		LI	1.2747	0.8462	0.3142	1.6661	1.1454	0.3073	44.3877	1.3087	0.2471	0.4214	1.0454	0.404
INF	Lindley	Sq	44.8679	1.0524	0.1981	44.8086	1.1858	0.2409	1.2815	0.297	0.2451	44.7756	1.2559	0.3591
		$lx_{(\beta=0.5)}$	44.8674	1.0524	0.1983	44.808	1.1854	0.2406	44.9584	1.2245	0.1789	44.774	1.2557	0.3591
		$lx_{(\beta=-5)}$	44.8717	1.0517	0.1956	44.8153	1.1899	0.2431	44.9577	1.2239	0.1787	44.7923	1.2575	0.3599
		$lx_{(\beta=5)}$	44.863	1.0532	0.2001	44.802	1.1817	0.2387	44.9648	1.2302	0.1811	44.7622	1.2543	0.3583
	MH	Sq	44.7346	0.8012	0.1245	44.3627	0.8073	0.1525	44.9518	1.2188	0.1769	44.4404	1.0759	0.2183
		$lx_{(\beta=0.5)}$	44.7315	0.7929	0.1226	44.3425	0.7947	0.1507	44.2519	0.4094	0.1331	44.4297	1.0679	0.2143
		$lx_{(\beta=-5)}$	44.7618	0.8802	0.1435	44.5391	0.9331	0.1699	44.1865	0.3988	0.1316	44.5452	1.1419	0.2545
		$lx_{(\beta=5)}$	44.7315	0.7929	0.1226	44.3425	0.7947	0.1507	44.6945	0.5326	0.1477	44.4297	1.0679	0.2143
		UB	44.4411	0.3583	0.001	43.8676	0.3526	0.002	44.1865	0.3988	0.1316	43.999	0.6778	0.0016
		LB	44.9045	1.1519	0.2882	44.8653	1.2453	0.305	43.4544	0.0192	0.0013	44.8383	1.3196	0.4134
		LB	44.9045	1.1519	0.2882	44.8653	1.2453	0.305	43.4544	0.0192	0.0013	44.8383	1.3196	0.4134
		LI	0.4634	0.7936	0.2871	0.9978	0.8927	0.303	45.014	0.9599	0.2505	0.8394	0.6417	0.4118

7. Conclusions

In this paper, parameter estimation for GEVL distributions was derived under a type-II progressive censored scheme by using optimization methods (NLM) ; also, artificial intelligence (GA) for MLE parameters, and the Bayesian technique was studied. The Bayesian estimation of parameters has been considered for two loss functions (symmetric and asymmetric loss functions) for informative and non-informative priors. Furthermore, progressive censoring is considered for three cases of elimination (i.e., fixed, discrete uniform, and binomial), with random removal under 10% elimination from the sample size. For a fixed random removal censoring scheme two cases were considered i.e., beginning and end. A Monte Carlo simulation was performed to compare the performance of the different estimators of GEVL distribution parameters; the results revealed that the Lindley method outperforms the MH method in terms of estimation quality. Informative Bayesian estimation yielded results that are very close to those of non-informative Bayesian estimation in terms of the bias and MSE. When it comes to Bayesian estimation under the LINEX loss function, the results closely resembled those under the SELF, with the negative value of β favoring underestimation over overestimation. However, for very small or large values of β, the LINEX loss function exhibits nearly symmetrical behavior (refer to [22] for details). Moreover, in most cases, the GA method outperformed the NLM method. Furthermore, for Case 1, the initial elimination step provided more accurate estimates in terms of bias and MSE than the final elimination step. Regarding the sensitivity of the estimates to sample size, both the GA and NLM methods performed better with larger sample sizes. Additionally, MLE demonstrated superior bias compared to Bayesian estimation. The 95% HPD credible interval obtained via the Gibbs (MH) sampling method provided more accurate estimates with the informative initial elimination for Case 1. In general, the GA method yields more precise estimates than the NLM method, as indicated by narrower intervals containing the true values.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state no conflict of interest.

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