AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 15325–15345.
DOI: 10.3934/math.2024744
Received: 08 January 2024
Revised: 13 April 2024
Accepted: 19 April 2024
Published: 28 April 2024

Research article

The equidistant dimension of graphs: NP-completeness and the case of lexicographic product graphs

Adrià Gispert-Fernández and Juan Alberto Rodríguez-Velázquez*

Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain

* **Correspondence:** Email: juanalberto.rodriguez@urv.cat.

Abstract: Let $V(G)$ be the vertex set of a simple and connected graph G. A subset $S \subseteq V(G)$ is a distance-equalizer set of G if, for every pair of vertices $u, v \in V(G) \setminus S$, there exists a vertex in S that is equidistant to u and v. The minimum cardinality among the distance-equalizer sets of G is the equidistant dimension of G, denoted by $\xi(G)$. In this paper, we studied the problem of finding $\xi(G \circ H)$, where $G \circ H$ denotes the lexicographic product of two graphs G and H. The aim was to express $\xi(G \circ H)$ in terms of parameters of G and H. In particular, we considered the cases in which G has a domination number equal to one, as well as the cases where G is a path or a cycle, among others. Furthermore, we showed that $\xi(G) \leq \xi(G \circ H) \leq \xi(G)|V(H)|$ for every connected graph G and every graph H and we discussed the extreme cases. We also showed that the general problem of finding the equidistant dimension of a graph is NP-hard.

Keywords: equidistant dimension; distance-equalizer; lexicographic product; NP-complete problem; distances in graphs
Mathematics Subject Classification: 05C12, 05C69, 05C76, 68Q25

1. Introduction

Let $V(G)$ be the vertex set of a simple and connected graph G. As usual, the distance between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is defined to be the length of a shortest path connecting u and v in G. A subset $S \subseteq V(G)$ is a distance-equalizer set of G if, for every pair of vertices $u, v \in V(G) \setminus S$, there exists a vertex $s \in S$ that is equidistant to u and v, i.e., $d_G(u, s) = d_G(v, s)$. The minimum cardinality among the distance-equalizers of G is the equidistant dimension of G, denoted by $\xi(G)$. This novel parameter was introduced in [9], where the authors explored its properties and proposed some applications to other problems not necessarily in the context of graph theory. They obtained general bounds on $\xi(G)$ and derived closed formulas for the case of several families of graphs. Furthermore, they have shown the usefulness of distance-equalizer sets for constructing doubly resolving sets.

In this paper, we deal with the problem of finding the equidistant dimension of the lexicographic product $G \circ H$ of two graphs G and H. The aim is to express $\xi(G \circ H)$ in terms of parameters of G and H. Since it is not usual to expect the existence of a unified formula of a parameter on $G \circ H$ for every graph G and every graph H, we will impose certain restrictions on the connected graph G, while H will be an arbitrary graph. In particular, we consider the cases in which G has a domination number equal to one, as well as the cases where G is a path or a cycle, among others. We show that if G has a domination number equal to one, then $\xi(G \circ H)$ can be expressed in terms of domination parameters of H, while if G is a path of a cycle of order n, then $\xi(G \circ H)$ can be expressed in terms of n and the order of H. As we can expect, $\xi(G) \leq \xi(G \circ H) \leq \xi(G)\,\mathrm{n}(H)$ for every connected graph G and every graph H. We discuss the extreme cases of these inequalities and, in particular, the analysis of the case $\xi(G \circ H) = \xi(G)$ leads to the challenge of investigating a new parameter that we call the total equidistant dimension of a graph. As a consequence of the study, we show that the general problem of finding the equidistant dimension of a graph is NP-hard.

The plan of the paper is described as follows. First, in Section 2, we declare some additional notations, define some concepts, and state some tools needed to develop the remaining sections. In Section 3, we discuss the class of lexicographic product graphs $G \circ H$ where the domination number of G is equal to one. Section 4 is devoted to considering the general bounds $\xi(G) \leq \xi(G \circ H) \leq \xi(G)\,\mathrm{n}(H)$ and discussing the extremal cases. The classes of lexicographic product graphs $G \circ H$ in which G is a path graph or a cycle graph are discussed in Sections 5 and 6, respectively. In Section 7, we show how the study of lexicographic product graphs provides us the tools to show that the general problem of finding the equidistant dimension of arbitrary graphs is NP-hard. Finally, Section 8 is devoted to describing some open problems and future works.

2. Notation, terminology and tools

To begin our study, we need to declare some notation, terminology and tools. The lexicographic product of two graphs G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and (g, h) is adjacent with (g', h') in $G \circ H$ if and only if either g is adjacent with g' in G or $g = g'$ and h is adjacent with h' in H. Notice that, for any vertex $g \in V(G)$, the subgraph of $G \circ H$ induced by $\{g\} \times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_g. Given a set $W \subseteq V(G) \times V(H)$, the restriction of W to $V(H_g)$ is $W_g = W \cap V(H_g)$, while the projection of W on $V(G)$ is given by

$$W_G = \{g \in V(G) : \ W_g \neq \emptyset\}.$$

For instance, Figure 1 shows the graph $P_4 \circ P_3$, where $V(P_4) = \{a, b, c, d\}$ and $V(P_3) = \{1, 2, 3\}$. In particular, $V(H_a) = \{(a, 1), (a, 2), (a, 3)\}$, and for $W = \{(a, 1), (a, 2), (b, 3), (c, 2)\}$ we have $W_a = \{(a, 1), (a, 2)\}$ and $W_{P_4} = \{a, b, c\}$.

The following claim, which states the distance formula in the lexicographic product of two graphs, is one of our main tools.

Remark 2.1. [10] *For any connected graph G of order $\mathrm{n}(G) \geq 2$ and any graph H, the following statements hold.*

(i) $d_{G\circ H}((g, h), (g', h')) = d_G(g, g')$ *for* $g \neq g'$.

(ii) $d_{G\circ H}((g,h),(g,h')) = \min\{2, d_H(h,h')\}$.

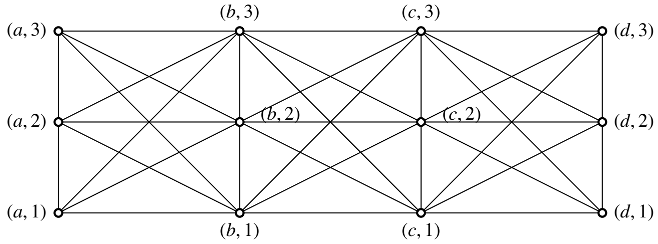

Figure 1. The graph $P_4 \circ P_3$.

For a basic introduction to the lexicographic product of two graphs, we suggest the books [10, 14]. One of the main problems in the study of $G \circ H$ consists of finding exact values or tight bounds for specific parameters of these graphs and expressing them in terms of known invariants of G and H. In particular, we cite the following works on metric parameters in lexicographic product graphs: metric dimension [15, 24, 26], fractional metric dimension [7], simultaneous metric dimension [23], simultaneous strong metric dimension [4], k-metric dimension [5, 6], strong metric dimension [21], local metric dimension [2], edge metric dimension [22], outer multiset dimension [18], general strong metric dimension [16], weak total resolvability [3], general position problem [20], general position achievement game [19], Steiner general position problem [17], universal lines and the Chen-Chvátal conjecture [25] and convex sets [1].

Next, we proceed to introduce additional notation and terminology. Complete graphs, empty graphs, cycle graphs, and path graphs of order n will be denoted by K_n, N_n, C_n, and P_n, respectively. As usual, the neighborhood of a vertex $g \in V(G)$ will be denoted by $N_G(g) = \{g' \in V(G) : \; g \text{ and } g' \text{ are adjacent}\}$, while the closed neighborhood will be denoted by $N_G[g] = \{g\} \cup N_G(g)$. Thus, the degree of a vertex $g \in V(G)$ is given by $|N_G(g)|$.

For short, if $(g,h) \in V(G \circ H)$, then we will use the notation $N_{G\circ H}(g,h)$, instead of $N_{G\circ H}((g,h))$.

Now, the open neighborhood of a set $S \subseteq V(G)$ is defined to be $N_G(S) = \cup_{g\in S} N_G(g)$, and the closed neighborhood as $N_G[S] = \cup_{g\in S} N_G[g] = S \cup N_G(S)$. The minimum degree of G will be denoted by $\delta(G)$, i.e.,

$$\delta(G) = \min\{|N_G(g)| : \; g \in V(G)\}.$$

Now, the maximum degree of G will be denoted by $\Delta(G)$, i.e.,

$$\Delta(G) = \max\{|N_G(g)| : \; g \in V(G)\}.$$

Given two different vertices $g, g' \in V(G)$, we define the bisector of g and g' as

$$B_{g|g'} = \{u \in V(G) : \; d_G(u,g) = d_G(u,g')\}.$$

Hence, we can express the concept of distance-equalizer set in term of bisectors, i.e., a set $X \subseteq V(G)$ is a distance-equalizer set of G if $X \cap B_{g|g'} \neq \emptyset$ for every pair of different vertices $g, g' \in V(G) \setminus X$.

Notice that, for every pair of different vertices $g, g' \in V(G)$ and every $h, h' \in V(H)$, the bisectors in G and the bisectors in $G \circ H$ are related as follows:

$$B_{g|g'} \times V(H) \subseteq B_{(g,h)|(g',h')}.$$

Furthermore,

$$(V(G) \setminus \{g\}) \times V(H) \subseteq B_{(g,h)|(g,h')}.$$

We define a $\xi(G)$-set as a distance-equalizer set X with $|X| = \xi(G)$. The same agreement will be assumed for optimal parameters associated to other characteristic sets used in the paper.

As described above, the aim of Section 3 is to provide a formula for the equidistant dimension of any lexicographic product graph $G \circ H$ whenever G has at least one universal vertex. To carry out the study, we can distinguish two cases depending on whether G has minimum degree one or not. In both cases, the result will depend on some domination parameters on H. Two of these parameters are the domination number and the total domination number, which are well known parameters. In contrast, the third parameter required for the study is completely new. We will call this new parameter the pairwise domination number.

A subset $S \subseteq V(H)$ is said to be a dominating set of H if $N_H[S] = V(H)$, while S is said to be a total dominating set if $N_H(S) = V(H)$. The minimum cardinality among all dominating sets of H is the domination number of H, denoted by $\gamma(H)$. The total domination number is defined by analogy, and is denoted by $\gamma_t(H)$. Since every total dominating set is dominating set, $\gamma_t(G) \geq \gamma(G)$. As mentioned above, the domination number and the total domination number of a graph have been extensively studied. For instance, we cite the books [11–13]. We recall that if u is a vertex of a graph G, then the vertex-deletion subgraph $G - u$ is the subgraph of G induced by $V(G) \setminus \{u\}$.

We shall show in Theorem 3.1 that if $\delta(G) = 1$, then either $\xi(G \circ H) = \gamma(H)$ or $\xi(G \circ H) = \gamma(H) + 1$, while if $\delta(G) \geq 2$, then either $\xi(G \circ H) = \min\{\gamma(H), \gamma_t(G-u)+1\}$ or $\xi(G \circ H) = 1 + \min\{\gamma(H), \gamma_t(G-u)\}$. In both cases, the problem of deciding the precise value of $\xi(G \circ H)$ will depend on the value of the pairwise domination number of H.

We say that a dominating set $X \subseteq V(H)$ is a pairwise dominating set of H if, for any pair of vertices $u, v \in V(H) \setminus X$, there exists a vertex $w \in X$ such that $\{u, v\} \subseteq N_H(w)$ or $\{u, v\} \cap N_H(w) = \emptyset$. The pairwise domination number of H, denoted by $\gamma_{\mathcal{P}}(H)$, is defined to be the minimum cardinality among all pairwise dominating sets of H. Since every pairwise dominating set is a dominating set, $\gamma_{\mathcal{P}}(H) \geq \gamma(H)$ for every graph H. Figure 2 shows three connected graphs with $\gamma_{\mathcal{P}}(H) = \gamma(H) = 3$. The sets of black coloured vertices correspond to $\gamma_{\mathcal{P}}(H)$-sets.

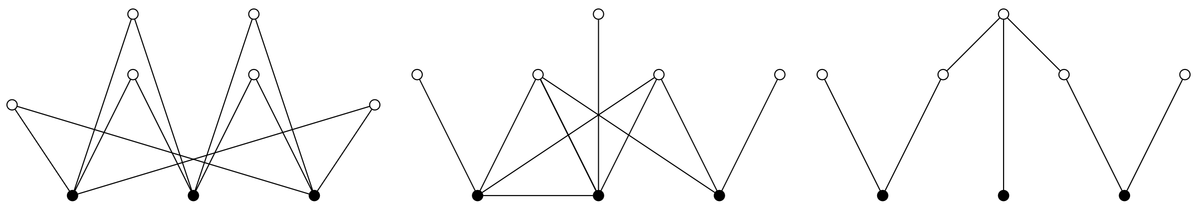

Figure 2. Three connected graphs with $\gamma_{\mathcal{P}}(H) = \gamma(H) = 3$.

We assume that the reader is familiar with the basic concepts, notation and terminology on graph theory. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

3. The case $\gamma(G) = 1$ & pairwise domination

We are now in a position to state the main result of this section.

Theorem 3.1. *Let G be a non-trivial graph with $\gamma(G) = 1$ and let $u \in V(G)$ be a vertex of degree $\Delta(G)$. The following statements hold for any graph H of order at least two.*

(i) *If $\delta(G) = 1$, then* $\xi(G \circ H) = \begin{cases} \gamma(H), & \textit{if } \gamma_{\wp}(H) = \gamma(H), \\ \gamma(H) + 1, & \textit{otherwise.} \end{cases}$

(ii) *If $\delta(G) \geq 2$, then* $\xi(G \circ H) = \begin{cases} \min\{\gamma(H), \gamma_t(G - u) + 1\}, & \textit{if } \gamma_{\wp}(H) = \gamma(H), \\ 1 + \min\{\gamma(H), \gamma_t(G - u)\}, & \textit{otherwise.} \end{cases}$

Proof. Let $u \in V(G)$ be a vertex of degree $\mathrm{n}(G) - 1$ and let $S \subseteq V(H)$ be a $\gamma(H)$-set. We proceed to show that $W = \{u\} \times S \cup \{(u', v)\}$ is a distance-equalizer set of $G \circ H$ for every $u' \in V(G) \setminus \{u\}$ and $v \in V(H)$. We differentiate the following cases for every pair of vertices $(g, h), (g', h') \in V(G \circ H) \setminus W$.

(a) If $g \neq u$ and $g' \neq u$, then (g, h) and (g', h') are adjacent to every vertex in $\{u\} \times S \subseteq W$.

(b) If $g = g' = u$, then (g, h) and (g', h') are adjacent to $(u', v) \in W$.

(c) If $g = u$ and $g' \neq u$, then (g, h) and (g', h') are adjacent to every vertex $(u, s) \in W$ such that $s \in S \cap N_H(h)$.

According to the three cases above, W is a distance-equalizer set of $G \circ H$, which implies that $\xi(G \circ H) \leq |W| = \gamma(H) + 1$.

In particular, if $\gamma_{\wp}(H) = \gamma(H)$, then for any $\gamma_{\wp}(H)$-set X we have that $\{u\} \times X$ is also a distance-equalizer set of $G \circ H$ (we leave the details to the reader). Therefore, in this case, $\xi(G \circ H) \leq |\{u\} \times X| = \gamma(H)$. In summary, for any graph H,

$$\xi(G \circ H) \leq \begin{cases} \gamma(H), & \text{if } \gamma_{\wp}(H) = \gamma(H), \\ \gamma(H) + 1, & \text{otherwise.} \end{cases} \tag{3.1}$$

Let Z be a $\xi(G \circ H)$-set where Z_G has maximum cardinality among all $\xi(G \circ H)$-sets. Without loss of generality, we can assume that $u \in Z_G$. We differentiate the following three cases for the set Z.

Case 1. $|Z_G| = 1$. Since $Z_u \subseteq N_{G \circ H}(u', v)$ for every $u' \in V(G) \setminus \{u\}$ and every $v \in V(H)$, we have that $N_{G \circ H}(u, h) \cap Z_u \neq \emptyset$ for every $(u, h) \in \{u\} \times V(H) \setminus Z_u$. Hence, $Z = Z_u$ is a dominating set of H_u. Moreover, if there exists a dominant set Y of H of cardinality less than $|Z_u|$, then for any vertex $(u', v) \in V(G \circ H)$, where $u' \neq u$, we have that $Z' = \{u\} \times Y \cup \{(u', v)\}$ is a $\xi(G \circ H)$-set with $|Z'_G| = 2 > 1 = |Z_G|$, which is a contradiction. Thus, Z_u is a $\gamma(H_u)$-set. Notice that Z_u is a pairwise dominating set of H_u, since for two vertices $(u, y), (u, y') \notin Z_u$ are required to have either a common neighbor in Z_u or a common non-neighbor in Z_u. In summary, $\xi(G \circ H) = |Z_u| = \gamma(H) = \gamma_{\wp}(H)$.

Case 2. $|Z_G| = 2$. Let $u' \in V(G) \setminus \{u\}$ such that $Z = Z_u \cup Z_{u'}$. Now, if $|Z_u| < \gamma(H)$ and $|Z_{u'}| < \gamma(H)$, then Z_u is not a dominating set of H_u and $Z_{u'}$ is not a dominating set of $H_{u'}$, which implies that there exist two vertices $(u, v) \in V(H_u) \setminus Z$ and $(u', v') \in V(H_{u'}) \setminus Z$ such that for every $(u, h) \in Z_u$,

$$d_{G \circ H}((u, v), (u, h)) = 2 \neq 1 = d_{G \circ H}((u', v'), (u, h)),$$

and for every $(u', h') \in Z_{u'}$,

$$d_{G \circ H}((u, v), (u', h')) = 1 \neq 2 = d_{G \circ H}((u', v'), (u', h')),$$

which is a contradiction. Therefore, either $|Z_u| \geq \gamma(H)$ or $|Z_{u'}| \geq \gamma(H)$, which implies that $\xi(G \circ H) = |Z| = |Z_u| + |Z_{u'}| \geq \gamma(H) + 1$, and by (3.1) we can conclude that $\xi(G \circ H) = \gamma(H) + 1$. Notice that, in this case, $\gamma(H) < \gamma_{\wp}(H)$.

Case 3. $|Z_G| \geq 3$. If $|Z_g| \geq \gamma(H)$ for some $g \in Z_G$, then $\xi(G \circ H) = |Z| \geq |Z_g| + 2 \geq \gamma(H) + 2$, which contradicts (3.1). Thus, $|Z_g| < \gamma(H)$ for every $g \in Z_G$. As a result, for any $g \in V(G)$, either $Z_g = \varnothing$ or there exists at least one vertex $(g, h_g) \in V(H_g) \setminus Z$, which is not dominated by any vertex in Z_g. Hence, for every pair of vertices of the form $(g, h_g) \in V(H_g) \setminus Z$ and $(u, h_u) \in V(H_u) \setminus Z$ with $g \neq u$, there exists $(g', h') \in Z \setminus (Z_g \cup Z_u)$, which is adjacent to (g, h_g) and (u, h_u). As a result, $Z_G \setminus \{u\}$ is a total dominating set of $G - u$, and so $\xi(G \circ H) = |Z| \geq |Z_G \setminus \{u\}| + 1 \geq \gamma_t(G - u) + 1$.

Notice that Case 3 does not occur when $\delta(G) = 1$. Therefore, we deduce (i) from Cases 1 and 2.

In order to conclude the proof of (ii), from now on we assume that $\delta(G) \geq 2$. Hence, $G - u$ does not have isolated vertices. We claim that, by taking any $\gamma_t(G - u)$-set X' and any vertex $y \in V(H)$, it follows that $W' = (\{u\} \cup X') \times \{y\}$ is a distance-equalizer set of $G \circ H$. To prove this claim, we consider the following possibilities for two different vertices $(g, h), (g', h') \in V(G \circ H) \setminus W'$.

(a') If $g \neq u$ and $g' \neq u$, then (g, h) and (g', h') are adjacent to $(u, y) \in W'$.

(b') If $g = g' = u$, then (g, h) and (g', h') are adjacent to every vertex $(x, y) \in X' \times \{y\} \subseteq W'$.

(c') If $g = u$ and $g' \neq u$, then (g, h) and (g', h') are adjacent to every vertex $(x, y) \in W'$ such that $x \in X' \cap N_{G-u}(g')$.

Thus, W' is a distance-equalizer set of $G \circ H$, which implies that

$$\xi(G \circ H) \leq \gamma_t(G - u) + 1. \tag{3.2}$$

Therefore, in Case 3 we have $\xi(G \circ H) = \gamma_t(G - u) + 1$. In summary, to conclude the proof of (ii) we only need to observe that if $\gamma_{\wp}(H) = \gamma(H)$, then the only possibilities are Cases 1 and 3, and so $\xi(G \circ H) = \min\{\gamma(H), \gamma_t(G - u) + 1\}$, while if $\gamma_{\wp}(H) > \gamma(H)$, then the possibilities are Cases 2 and 3, and then $\xi(G \circ H) = \min\{\gamma(H) + 1, \gamma_t(G - u) + 1\}$. □

According to the previous result, our challenge now is to study the pairwise domination number. In particular, notice that if $\gamma(H) = \mathrm{n}(H) - \Delta(H)$, then $\gamma_{\wp}(H) = \gamma(H)$. Therefore, the following corollary follows.

Corollary 3.1. *Let G be a graph with $\gamma(G) = 1$. If H is a graph with $\gamma(H) = \mathrm{n}(H) - \Delta(H)$, then the following statements hold.*

(i) *If $\delta(G) = 1$, then $\xi(G \circ H) = \mathrm{n}(H) - \Delta(H)$.*

(ii) *If $\delta(G) \geq 2$, then $\xi(G \circ H) = \min\{\mathrm{n}(H) - \Delta(H), \gamma_t(G - u) + 1\}$, where $u \in V(G)$ is a vertex of maximum degree.*

4. General bounds and graphs achieving the extreme cases

Lemma 4.1. *Let G be a connected graph and let H be a graph. If W is a $\xi(G \circ H)$-set, then W_G is a distance-equalizer set of G.*

Proof. Assume that there are two different vertices $g, g' \in V(G) \setminus W_G$. For every $h \in V(H)$ there exists $(u, v) \in W \cap B_{(g,h)|(g',h)}$, and so $u \in W_G \cap B_{g|g'}$, which implies that W_G is a distance-equalizer set of G. □

As we can expect, the following result holds.

Theorem 4.1. *For any connected graph G and any graph H,*

$$\xi(G) \leq \xi(G \circ H) \leq \xi(G)\,\mathrm{n}(H).$$

Proof. Let W be a $\xi(G \circ H)$-set. By Lemma 4.1, W_G is a distance-equalizer set of G. Therefore,

$$\xi(G \circ H) = |W| \geq |W_G| \geq \xi(G).$$

Now, let X be a $\xi(G)$-set. We proceed to show that $W' = X \times V(H)$ is a distance-equalizer set of $G \circ H$. To this end, let $(g, h), (g', h') \in V(G \circ H) \setminus W'$. Obviously, $g, g' \in V(G) \setminus X$. Hence, if $g = g'$, then $W' \subseteq (V(G) \setminus \{g\}) \times V(H) \subseteq B_{(g,h)|(g',h')}$. Now, if $g \neq g'$, then there exists $x \in X \cap B_{g|g'}$, which implies that $(x, y) \in W' \cap B_{(g,h)|(g',h')}$ for every $y \in V(H)$. Therefore, W' is a distance-equalizer set of $G \circ H$ and, as a result,

$$\xi(G \circ H) \leq |W'| = |X||V(H)| = \xi(G)\,\mathrm{n}(H).$$

Next, we discuss the tightness of these bounds. First, we characterize the graphs G such that $\xi(G \circ H) = \xi(G)$ for every graph H. If $B_{g|g'} \neq \emptyset$ for every pair of different vertices $g, g' \in V(G)$, then there exists a set $X \subseteq V(G)$ such that for every pair of different vertices $g, g' \in V(G)$, there exists a vertex $x \in X$ such that $d_G(g, x) = d_G(g', x)$. In such a case, we say that X is a total distance-equalizer set. Thus, we can define the total equidistant dimension of G, denoted by $\xi_t(G)$, as the minimum cardinality among all total distance-equalizer sets of G. If there exists $g, g' \in V(G)$ such that $B_{g|g'} = \emptyset$, then we can assume that $\xi_t(G) = +\infty$, and so we can agree that $\xi_t(G) \geq \xi(G)$ for every connected graph G.

For instance, for any complete graph of order $n \geq 3$, we have $\xi_t(K_n) = 3 > 1 = \xi(K_n)$. In the case of an odd order cycle, the bisector of every pair of vertices has cardinality one and every vertex forms a bisector, which implies that $\xi_t(C_{2k+1}) = 2k + 1$. In Figure 3, we show two examples of connected graphs with $B_{g|g'} \neq \emptyset$ for every pair of different vertices $g, g' \in V(G)$. In both cases, the set of black coloured vertices is a $\xi_t(G)$-set and $\xi_t(G) = \xi(G)$.

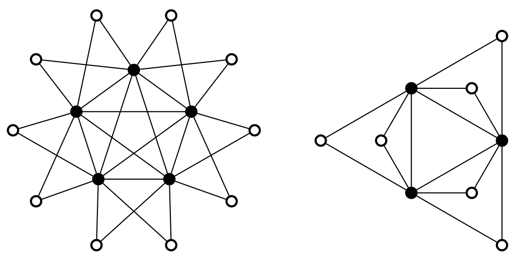

Figure 3. Two graphs with $B_{g|g'} \neq \emptyset$ for every pair of different vertices $g, g' \in V(G)$.

Proposition 4.1. *If G is a graph with $\xi_t(G) < +\infty$, then $\xi(G \circ H) \leq \xi_t(G)$ for every graph H.*

Proof. It is readily seen that for every $\xi_t(G)$-set X and every vertex h of a graph H, the set $X \times \{h\}$ is a distance-equalizer of $G \circ H$, which implies that $\xi(G \circ H) \leq \xi_t(G)$ for every graph H. □

Proposition 4.2. *Given a connected graph G of order at least two, the following statements are equivalent.*

(i) $\xi(G \circ H) = \xi(G)$ *for every graph H.*

(ii) $\xi_t(G) = \xi(G)$.

Proof. (ii)⇒(i). If $\xi_t(G) = \xi(G)$, then by the lower bound given in Theorem 4.1 and Proposition 4.1, we conclude that (i) follows.

(i)⇒(ii). Assume that $\xi(G \circ H) = \xi(G)$ for every graph H of order at least two. By Lemma 4.1, for any graph H and any $\xi(G \circ H)$-set W, we have that W_G is a $\xi(G)$-set, which implies that $|W_w| = 1$ for every $w \in W_G$. Now, suppose that there exist two different vertices $g, g' \in V(G)$ such that $B_{g|g'} \cap W_G = \emptyset$. We differentiate three cases.

Case 1. $d_G(g, g') \geq 3$. In this case, $W \cap B_{(g,h)|(g',h')} \subseteq (W_G \cap B_{g|g'}) \times V(H) = \emptyset$ for every $(g, h), (g', h') \notin W$, which is a contradiction.

Case 2. $d_G(g, g') = 2$. Let H be a graph with $\gamma(H) = 1$. Let us conveniently select two vertices $h, h' \in V(H)$. If $g \in W_G$, then we take $h \in V(H)$ such that $N_{G\circ H}(g, h) \cap W_g \neq \emptyset$, otherwise we take h as an arbitrary vertex of H. Analogously, if $g' \in W_G$, then we take $h' \in V(H)$ such that $N_{G\circ H}(g', h') \cap W_{g'} \neq \emptyset$, otherwise we take h' as an arbitrary vertex of H. Hence, $W \cap B_{(g,h)|(g',h')} \subseteq (W_G \cap B_{g|g'}) \times V(H) = \emptyset$, which is a contradiction.

Case 3. $d_G(g, g') = 1$. In this case, for any graph H with $\gamma(H) \geq 2$, there exist vertices $h, h' \in V(H)$ satisfying $W_g \cap (\{g\} \times N_H[h]) = \emptyset$ and $W_{g'} \cap (\{g'\} \times N_H[h']) = \emptyset$. Hence, $W \cap B_{(g,h)|(g',h')} \subseteq (W_G \cap B_{g|g'}) \times V(H) = \emptyset$, which is a contradiction.

According to the three cases above, W_G is a total distance-equalizer set of G with $|W_G| = \xi(G)$. Therefore, $\xi_t(G) = \xi(G)$. □

Next, we consider a class of graphs G where $\xi(G \circ H) = \xi(G)\,\mathrm{n}(H)$ for every graph H. To this end, we need to introduce the following result and some additional terminology.

Proposition 4.3. *[9] Let G be a bipartite graph with partite sets A and B. If X is a distance-equalizer set of G, then $A \subseteq X$ or $B \subseteq X$. Consequently, $\xi(G) \geq \min\{|A|, |B|\}$.*

We recall that a graph is 2-antipodal if every vertex has exactly one diametral vertex, i.e., G is 2-antipodal if for every vertex $v \in V(G)$ there exists exactly one vertex $v' \in V(G)$ such that $d_G(v, v') = \mathrm{diam}(G)$. For instance, every hypercube Q_k is a bipartite 2-antipodal graph and every cycle graph C_{2k} is a bipartite 2-antipodal graph. In particular, $\xi(Q_3) = 4$, and if k is even, then $\xi(C_{2k}) = k$. The equidistant dimension of cycle graphs was studied in [9].

Figure 4 shows two bipartite 2-antipodal graphs where $\xi(G) = \frac{\mathrm{n}(G)}{2}$. The graph on the left is 5-regular and has diameter 3, while the one on the right is 3-regular and has diameter 5.

Proposition 4.4. *Let G be a bipartite 2-antipodal graph G of diameter $2k + 1 \geq 3$. If $\xi(G) = \frac{\mathrm{n}(G)}{2}$, then for any graph H,*

$$\xi(G \circ H) = \frac{\mathrm{n}(G)\,\mathrm{n}(H)}{2}.$$

Proof. Let V_1 and V_2 be the partite sets of G. Since G is a 2-antipodal graph of odd diameter, every vertex in V_1 has its antipodal vertex in V_2, and so $|V_1| = |V_2|$.

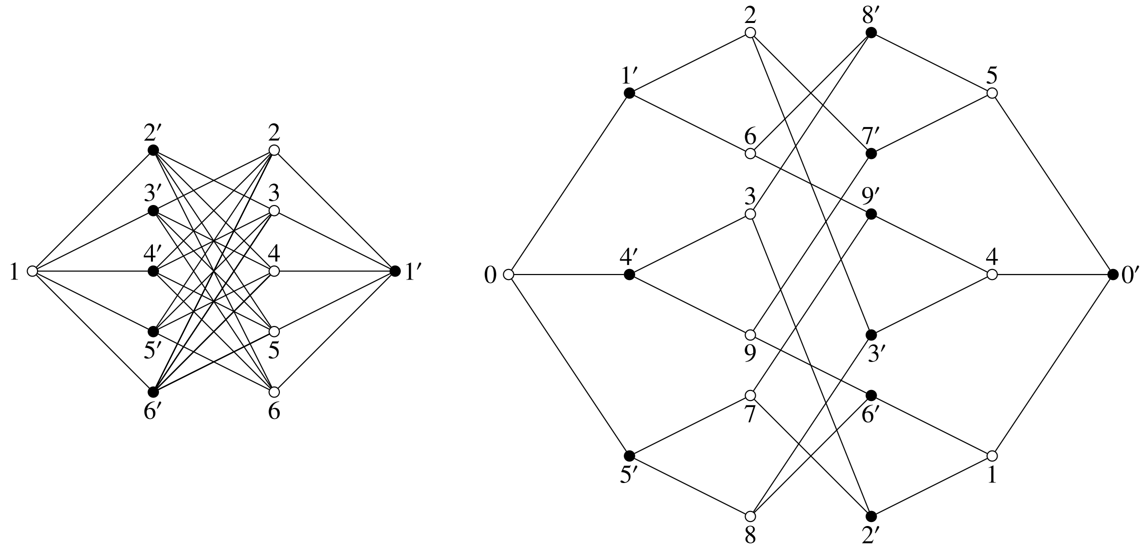

Figure 4. Two bipartite 2-antipodal graphs where $\xi(G) = \frac{\mathrm{n}(G)}{2}$.

Let W be a $\xi(G \circ H)$-set. By Lemma 4.1 and Proposition 4.3, we can assume that $V_1 \subseteq W_G$. Let $g \in V_1$ and let $g' \in V_2$ be the antipodal vertex of g. Since the distance between two vertices in the same partite set is even and the distance between vertices of different partite sets is odd, $B_{g|g'} = \emptyset$. Now, since $d_G(g, g') = 2k+1 \geq 3$, for every $h, h' \in V(H)$ we have that $B_{(g,h)|(g',h')} = B_{g|g'} \times V(H) = \emptyset$, which implies that either $V(H_g) \subseteq W$ or $V(H_{g'}) \subseteq W$. Therefore, $\xi(G \circ H) \geq |V_1|\,\mathrm{n}(H) = \dfrac{\mathrm{n}(G)}{2}\,\mathrm{n}(H) = \xi(G)\,\mathrm{n}(H)$, and by the upper bound given in Theorem 4.1 we complete the proof. □

5. The case where G is a path

In this section, we will use the following notation. Let $V(P_n) = \{u_1, \ldots, u_n\}$, where consecutive subscripts correspond to adjacent vertices of the path. Let H_i be the copy of H in $P_n \circ H$ associated to $u_i \in V(P_n)$. Given a set $W \subseteq V(P_n \circ H)$, we define the sets $W_i = W \cap V(H_i)$ for every $i \in \{1, \ldots, n\}$.

Lemma 5.1. *For any integer $n \geq 4$ and any graph H of order at least two, there exists a $\xi(P_n \circ H)$-set W satisfying the following properties.*

(i) $W_1 = \emptyset$.

(ii) $|W_2| \geq \gamma(H)$.

(iii) $|W_j| = \mathrm{n}(H)$ *for every even number $j \in \{4, \ldots, n\}$.*

Proof. Let W be a $\xi(P_n \circ H)$-set. First, we proceed to show that we can take W in such a way that $|W_1| \leq \mathrm{n}(H) - 1$. To this end, we define

$$l = \min_{j \in \{1,\ldots,n\}} \{j : \ |W_j| \leq \mathrm{n}(H) - 1\}.$$

Observe that l is well defined, as $\xi(P_n \circ H) < \mathrm{n}(P_n \circ H)$. Let $Y_i = \{h \in V(H) : \ (u_i, h) \in W\}$, i.e., $W_i = \{u_i\} \times Y_i$. If $|Y_1| \leq \mathrm{n}(H) - 1$, then we are done. Now, if $Y_1 = V(H)$, then from W we can construct a set $S = \cup_{i=1}^{n} S_i$ where $S_i = \{u_i\} \times Y_{i+l-1}$ for every $i \in \{1, \ldots, n-l+1\}$ while $S_i = \{u_i\} \times V(H)$ for every $i \in \{n-l+2, \ldots, n\}$. Since S is a $\xi(P_n \circ H)$-set with $|S_1| \leq \mathrm{n}(H) - 1$, we are done. With this fact in mind, from now on we can assume that W is a $\xi(P_n \circ H)$-set such that $|W_1|$ is minimum among all $\xi(P_n \circ H)$-sets. We claim that $W_1 = \emptyset$. Suppose, to the contrary, that there exists a vertex $(u_1, h) \in W_1$. Since $|W_1| \leq \mathrm{n}(H) - 1$, it is easy to check that $W' = (W \setminus \{(u_1, h)\}) \cup \{(u_2, h)\}$ is a $\xi(P_n \circ H)$-set with $|W_1'| < |W_1|$, which is a contradiction. Therefore, (i) follows.

In order to prove (ii) and (iii), we assume that $W_1 = \emptyset$. Thus, if W_2 is not a dominating set of H_2, then there exists $(u_2, h) \in V(H_2) \setminus N_{H_2}(W_2)$ and no vertex in W is equidistant to (u_2, h) and any vertex in $V(H_1)$, which is a contradiction. Therefore, W_2 is a dominating set of H_2, and so $|W_2| \geq \gamma(H_2) = \gamma(H)$.

Finally, to prove (iii), we only need to observe that for any even number $j \in \{4, \ldots, n\}$, the distance from any vertex in $V(H_1)$ to any vertex in $V(H_j)$ is odd, and since $W_1 = \emptyset$, we conclude that $|W_j| = \mathrm{n}(H)$. □

Since the cases $n = 2$ and $n = 3$ were previously considered in Theorem 3.1, we restrict ourselves to the case $n \geq 4$.

Proposition 5.1. *For any graph H of order at least two,* $\xi(P_4 \circ H) = \gamma(H) + \mathrm{n}(H)$.

Proof. For any $\xi(P_n \circ H)$-set X satisfying Lemma 5.1, we have $\xi(P_4 \circ H) = |X| \geq |X_2| + |X_4| = \gamma(H) + \mathrm{n}(H)$.

To conclude the proof, we only need to observe that for any $\gamma(H)$-set S, we have that $X' = \{u_2\} \times S \cup V(H_4)$ is a distance-equalizer set of $P_4 \circ H$, which implies that $\xi(P_4 \circ H) \leq |X'| = \gamma(H) + \mathrm{n}(H)$. □

Proposition 5.2. *For any integer* $n \geq 5$ *and any graph H of order at least three,*

$$\xi(P_n \circ H) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil.$$

Proof. For any $v \in V(H)$, we define the set

$$W = \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} V(H_{2i}) \right) \bigcup \left(\bigcup_{i=1}^{\lceil \frac{n-4}{2} \rceil} \{(u_{2i+1}, v)\} \right).$$

Let $(u_i, w), (u_j, w') \in V(P_n \circ H) \setminus W$ be two different vertices. Notice that i and j are odd. If $i = j$, then any vertex in $W_2 = V(H_2) \subseteq W$ is equidistant to (u_i, w) and (u_j, w'). Now, if $i \neq j$, then $W_{(i+j)/2} \neq \emptyset$, and so vertex $(u_{(i+j)/2}, v) \in W_{(i+j)/2} \subseteq W$ is equidistant to (u_i, w) and (u_j, w'). Therefore, W is a distance-equalizer set of $P_n \circ H$, which implies that

$$\xi(P_n \circ H) \leq |W| = \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil. \tag{5.1}$$

We proceed to prove the lower bound $\xi(P_n \circ H) \geq \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil$. To this end, we take a $\xi(P_n \circ H)$-set X satisfying Lemma 5.1 and assume first that n is odd. We consider two cases.

Case 1. $n = 5$. If $|X_2| \leq \mathrm{n}(H) - 1$, then $X_5 = V(H_5)$, as the distance from vertices in $V(H_2)$ to vertices in $V(H_5)$ is odd. Hence, $\xi(P_5 \circ H) = |X| \geq |X_2| + |X_4| + |X_5| \geq 1 + 2\,\mathrm{n}(H) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil$, as required. Now, assume $|X_2| = \mathrm{n}(H)$. If $X_3 = \emptyset$, then $|X_5| = \mathrm{n}(H)$, as no vertex in X is equidistant to vertices in

$V(H_1)$ and vertices in $V(H_5)$. Thus, $\xi(P_5 \circ H) = |X| \geq |X_2| + |X_4| + |X_5| = 3\,\mathrm{n}(H) > \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil$, which contradicts (5.1). Hence, $X_3 \neq \emptyset$, and so $\xi(P_5 \circ H) = |X| \geq |X_2| + |X_3| + |X_4| \geq 1 + 2\,\mathrm{n}(H) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil$, as required.

Case 2. $n \geq 7$. Suppose that $|X_2| \leq \mathrm{n}(H) - 1$. In such a case, $X_j = V(H_j)$ for every odd number $j \geq 5$, as the distance from vertices in $V(H_2)$ to vertices in $V(H_j)$ is odd. Since $n \geq 7$,

$$\xi(P_n \circ H) = |X| \geq |X_2| + \sum_{i=4}^{n} |X_i| \geq 1 + (n-3)\,\mathrm{n}(H) > \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil,$$

which contradicts (5.1). Hence, $X_i = V(H_i)$ for every even number $i \leq n$. Thus, by taking I as the set of odd numbers belonging to $\{1, \ldots, n\}$, we obtain that

$$\xi(P_n \circ H) = |X| = \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \sum_{j \in I} |X_j|. \tag{5.2}$$

We proceed now to show that $\sum_{j \in I} |X_j| \geq \left\lceil \frac{n-4}{2} \right\rceil$. Let $I^- = I \setminus \{1, n\}$. We already know that the upper bound is reached when $|X_j| = 1$ for every $j \in I^-$ and $|X_1| = |X_n| = 0$. Suppose to the contrary that there exists an odd number $l \in I^-$ such that $X_l = \emptyset$. Since in P_n we have that $B_{u_{l-2}|u_{l+2}} = \{u_l\}$, we conclude that $|X_{l-2}| = \mathrm{n}(H)$ or $|X_{l+2}| = \mathrm{n}(H)$. In general, if $3 \leq l \leq \frac{n+1}{2}$, then $|X_{l-2j}| = \mathrm{n}(H)$ or $|X_{l+2j}| = \mathrm{n}(H)$ for every $j \in \{1, \ldots, \frac{l-1}{2}\}$, while if $\frac{n+1}{2} \leq l \leq n-2$, then $|X_{l-2j}| = \mathrm{n}(H)$ or $|X_{l+2j}| = \mathrm{n}(H)$ for every $j \in \{1, \ldots, \frac{n-l}{2}\}$. With these facts in mind, we take

$$l = \max_{j \in I} \left\{ j : \ X_j = \emptyset \text{ and } j \leq \frac{n+1}{2} \right\}.$$

Hence, if $\frac{n}{3} \leq l \leq \frac{n+1}{2}$, then

$$\sum_{j \in I} |X_j| \geq \sum_{j=1}^{\frac{l-1}{2}} \left(|X_{l-2j}| + |X_{l+2j}| \right) \geq \frac{l-1}{2}\,\mathrm{n}(H) \geq \frac{n-3}{6}\,\mathrm{n}(H) \geq \frac{n-3}{2} = \left\lceil \frac{n-4}{2} \right\rceil. \tag{5.3}$$

Therefore, by combining (5.1) with (5.2) and (5.3), we obtain the result. The same reasoning works if we omit the restriction $l \geq \frac{n}{3}$ and consider that $X_j \neq \emptyset$ for every $j \in I$ with $j > \frac{n+1}{2}$. In such a case, we have

$$\sum_{j \in I} |X_j| \geq \frac{l-1}{2}\,\mathrm{n}(H) + ((|I| - 1) - (l-1)) \geq \frac{3}{2}(l-1) + |I^-| - (l-1)) \geq |I^-| = \left\lceil \frac{n-4}{2} \right\rceil. \tag{5.4}$$

Therefore, by combining (5.1) with (5.2) and (5.4), we obtain the result.

Assume that there exists $j \in I$ with $X_j = \emptyset$ and $j > \frac{n+1}{2}$. We define

$$l' = \min_{j \in I} \left\{ j : \ X_j = \emptyset \text{ and } j \geq \frac{n+1}{2} \right\}.$$

Now, if $\frac{n+1}{2} \leq l' \leq \frac{2n}{3} + 1$, then

$$\sum_{j \in I} |X_j| \geq \sum_{j=1}^{\frac{n-l'}{2}} \left(|X_{l'-2j}| + |X_{l'+2j}| \right) \geq \frac{n-l'}{2}\,\mathrm{n}(H) \geq \frac{n-3}{6}\,\mathrm{n}(H) \geq \frac{n-3}{2} = \left\lceil \frac{n-4}{2} \right\rceil. \tag{5.5}$$

Therefore, by combining (5.1) with (5.2) and (5.5), we obtain the result.

Now, if $1 \leq l \leq \frac{n}{3} - 1$ and $\frac{2n}{3} + 2 \leq l' \leq n$, then we have that $|X_j| \geq 1$ for every $j \in I$ such that $2l + 1 \leq j \leq 2l' - n - 2$. Thus, there are $l' - l - \frac{n+1}{2}$ different values for j with these requirements. In this case,

$$\begin{aligned}\sum_{j\in I} |X_j| &\geq \frac{l-1}{2}\,\mathrm{n}(H) + \frac{n-l'}{2}\,\mathrm{n}(H) + \left(l' - l - \frac{n+1}{2}\right)\\ &\geq \frac{3}{2}((n-1) - (l'-l)) + \frac{1}{2}(2(l'-l) - (n+1))\\ &= \frac{1}{2}(2n - 4 - (l'-l))\\ &\geq \frac{1}{2}(2n - 4 - (n-1))\\ &= \frac{n-3}{2} = \left\lceil \frac{n-4}{2} \right\rceil.\end{aligned}$$

Therefore, by combining this inequality with (5.1) and (5.2), we obtain the result.

Finally, if n is even, then we already know that $X_n = V(H_n)$, which implies that $X \setminus X_n$ is a distance-equalizer set of $P_{n-1} \circ H$. Hence,

$$\begin{aligned}\xi(P_n \circ H) &= |X|\\ &\geq \xi(P_{n-1} \circ H) + \mathrm{n}(H)\\ &= \left(\left\lfloor \frac{n-1}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-5}{2} \right\rceil\right) + \mathrm{n}(H)\\ &= \left(\frac{n-2}{2} \cdot \mathrm{n}(H) + \frac{n-4}{2}\right) + \mathrm{n}(H)\\ &= \left\lfloor \frac{n}{2} \right\rfloor \cdot \mathrm{n}(H) + \left\lceil \frac{n-4}{2} \right\rceil.\end{aligned}$$

Therefore, from this bound and (5.1), we conclude the proof. □

6. The case where G is a cycle

In this section, we will use the following notation. Let $V(C_n) = \{u_0, \ldots, u_{n-1}\}$, where consecutive subscripts correspond to adjacent vertices of the cycle, and the addition of subscripts will be modulo n. As above, given a set $W \subseteq V(C_n \circ H)$, we define the sets $W_i = W \cap V(H_i)$ for every $i \in \{0, \ldots, n-1\}$.

Since the case $n = 3$ was previously considered in Theorem 3.1, we restrict ourselves to the case $n \geq 4$.

Proposition 6.1. *Let H be a graph of order $\mathrm{n}(H) \geq 2$. If, for any $\gamma(H)$-set S, there exists a vertex $h \in V(H) \setminus S$ such that $S \subseteq N_H(h)$, then $\xi(C_4 \circ H) = 2\gamma(H) + 1$, otherwise $\xi(C_4 \circ H) = 2\gamma(H)$.*

Proof. It is readily seen that if S is a $\gamma(H)$-set, then for any $v \in V(H)$ we have that $S' = \{u_0, u_2\} \times S \cup \{(u_1, v)\}$ is a distance-equalizer set of $C_4 \circ H$. Thus,

$$\xi(C_4 \circ H) \leq |S'| = 2|S| + 1 = 2\gamma(H) + 1. \tag{6.1}$$

Now, if $S \setminus N_H(h) \neq \emptyset$ for every vertex $h \in V(H) \setminus S$, then $S'' = \{u_0, u_2\} \times S$ is a distance-equalizer set of $C_4 \circ H$, which implies that

$$\xi(C_4 \circ H) \leq |S''| = 2|S| = 2\gamma(H). \tag{6.2}$$

Notice that if H is an empty graph, then $S = V(H)$, which implies that S'' is a distance-equalizer set of $C_4 \circ H$, and so (6.2) follows.

From now on, let X be a $\xi(C_4 \circ H)$-set and let $i \in \{0, \ldots, 3\}$. If neither X_i nor X_{i+1} are dominating sets of H_i and H_{i+1}, respectively, then there exist two vertices $(u_i, h) \in V(H_i) \setminus X_i$ and $(u_{i+1}, h') \in V(H_{i+1}) \setminus X_{i+1}$ such that $B_{(u_i,h)|(u_{i+1},h')} \cap X = \emptyset$, which is a contradiction. Thus, there exists $j \in \{0, \ldots, 3\}$ such that X_j and X_{j+2} are dominating sets of H_i and H_{i+2}, respectively. Hence, $|X_j| \geq \gamma(H)$ and $|X_{j+2}| \geq \gamma(H)$, which implies that

$$\xi(C_4 \circ H) \geq |X_j| + |X_{j+2}| \geq 2\gamma(H). \tag{6.3}$$

From (6.2) and (6.3), we can conclude that if H is an empty graph or there exists a $\gamma(H)$-set S such that $S \setminus N_H(h) \neq \emptyset$ for every vertex $h \in V(H) \setminus S$, then $\xi(C_4 \circ H) = 2\gamma(H)$.

Finally, assume that for any $\gamma(H)$-set A, there exists a vertex $h \in V(H) \setminus A$ such that $A \subseteq N_H(h)$. Obviously, if $|X_j| + |X_{j+2}| > 2\gamma(H)$, then we are done by (6.1). Now, consider the case $|X_j| = |X_{j+2}| = \gamma(H) < \mathrm{n}(H)$. If $X_{j+1} = X_{j-1} = \emptyset$, then $B_{(u_j,h)|(u_{j+2},h')} \cap X = \emptyset$ for every pair of vertices $(u_j, h) \in V(H_j) \setminus X_j$ and $(u_{j+2}, h') \in V(H_{j+2}) \setminus X_{j+2}$ such that $X_j \subseteq N_{H_j}(u_j, h)$ and $X_{j+2} \subseteq N_{H_{j+2}}(u_{j+2}, h')$, which is a contradiction. Thus, $|X_{j+1}| + |X_{j-1}| \geq 1$, and so

$$\xi(C_4 \circ H) = |X| \geq |X_j| + |X_{j+2}| + 1 = 2\gamma(H) + 1. \tag{6.4}$$

Therefore, by (6.1) and (6.4), we conclude the proof. □

Proposition 6.2. *Let H be a graph of order at least two. If $\gamma(H) = \delta(H) = 1$, then $\xi(C_5 \circ H) = 4$, otherwise, $\xi(C_5 \circ H) = 5$.*

Proof. The proof is simple, we will describe the idea, leaving the details to the reader.

Assume $\gamma(H) = \delta(H) = 1$. It is readily seen that if $h \in V(H)$ is a vertex of degree one and $h' \in V(H) \setminus \{h\}$ is a universal vertex, then $X = \{u_0, u_2, u_3\} \times \{h'\} \cup \{(u_0, h)\}$ is a distance-equalizer set, and so $\xi(C_5 \circ H) \leq |X| = 4$.

It is easy to check that no set of cardinality three is a distance-equalizer set of $C_5 \circ H$. Therefore, $\xi(C_5 \circ H) = 4$.

From now on, assume that $\gamma(H) \geq 2$ or $\delta(H) \neq 1$. We already know that $\xi(C_5 \circ H) \leq 5$, by Proposition 4.1. Finally, through a case analysis, we can check that no set of cardinality four is a distance-equalizer set of $C_5 \circ H$. Therefore, $\xi(C_5 \circ H) \geq 5$. □

It was shown in [9] that $\xi(C_n) \geq \frac{n-1}{2}$ for every odd integer $n \geq 3$. We proceed too show that the bound can be improved for $n \geq 5$.

Lemma 6.1. *If $n \geq 5$ is an odd integer, then $\xi(C_n) \geq \frac{n+1}{2}$.*

Proof. Let W be a $\xi(C_n)$-set and let $V(C_n) = \{0, \ldots, n-1\}$, where consecutive vertices are adjacent. If for every pair of adjacent vertices of C_n, at least one of them belongs to W, then $\xi(C_n) \geq \frac{n+1}{2}$ and we are done. Thus, we can assume, without loss of generality, that $W \cap \{0, n-1\} = \emptyset$. Since n is an odd integer, for any $i \in \left\{1, \dots, \frac{n-1}{2}\right\}$, we have that $B_{i|n-i} = \{0\}$. Thus, from $0 \notin W$, we deduce that $i \in W$ or $n-i \in W$, which implies that $\xi(C_n) \geq \frac{n-1}{2}$.

Suppose that $\xi(C_n) = \frac{n-1}{2}$. Notice that, in this case, $|W \cap \{i, n-i\}| = 1$ for every $i \in \left\{1, \dots, \frac{n-1}{2}\right\}$. Now, since $W \cap \{0, n-1\} = \emptyset$, we have that $\frac{n-1}{2} \in W$, and since $\left|W \cap \left\{\frac{n-1}{2}, \frac{n+1}{2}\right\}\right| = 1$, we have that $\frac{n+1}{2} \notin W$. If $n \equiv 1 \pmod 4$, then $B_{n-1|\frac{n+1}{2}} = \left\{\frac{n-1}{4}\right\}$ and $B_{0|\frac{n+1}{2}} = \left\{n - \frac{n-1}{4}\right\}$, which is a contradiction, as $\left|W \cap \left\{\frac{n-1}{4}, n - \frac{n-1}{4}\right\}\right| = 1$. Analogously, if $n \equiv 3 \pmod 4$, then $B_{0|\frac{n+1}{2}} = \left\{\frac{n+1}{4}\right\}$ and $B_{n-1|\frac{n+1}{2}} = \left\{n - \frac{n+1}{4}\right\}$, which is a contradiction, as $\left|W \cap \left\{\frac{n+1}{4}, n - \frac{n+1}{4}\right\}\right| = 1$. Therefore, $\xi(C_n) \geq \frac{n-1}{2} + 1 = \frac{n+1}{2}$. □

Proposition 6.3. *Let H be a graph of order at least two. If $n \geq 7$ is an odd integer, then $\xi(C_n \circ H) = n$.*

Proof. By Proposition 4.1, we conclude that $\xi(C_n \circ H) \leq n$. Hence, we proceed to show that $\xi(C_n \circ H) \geq n$.

Let W be a $\xi(C_n \circ H)$-set and let $k = \frac{n-1}{2}$. If $W_i \neq \emptyset$ for every $i \in \{0, \dots, n-1\}$, then $\xi(C_n \circ H) = \sum_{i=0}^{n} |W_i| \geq n$ and we are done. From now on, we suppose, without loss of generality, that $W_0 = \emptyset$. Since two vertices $(u_i, y) \in V(H_i) \setminus W_i$ and $(u_{n-i}, y') \in V(H_{n-i}) \setminus W_{n-i}$ have to be equalized by W, we deduce the following constraints, where $\lambda = 1$ if $\delta(H) = 0$ while $\lambda \geq 2$ otherwise,

$$\begin{aligned}
|W_1| \geq \lambda \quad &\text{or} \quad |W_{n-1}| \geq \lambda, \\
|W_2| = \mathrm{n}(H) \quad &\text{or} \quad |W_{n-2}| = \mathrm{n}(H), \\
&\vdots \\
|W_{k-1}| = \mathrm{n}(H) \quad &\text{or} \quad |W_{k+2}| = \mathrm{n}(H), \\
|W_k| \geq \gamma(H) \quad &\text{or} \quad |W_{k+1}| \geq \gamma(H).
\end{aligned}$$

Hence,

$$\xi(C_n \circ H) = \sum_{i=0}^{n-1} |W_i| \geq \lambda + (k-2)\,\mathrm{n}(H) + \gamma(H). \tag{6.5}$$

Suppose that $\sum_{i=0}^{n-1} |W_i| = \lambda + (k-2)\,\mathrm{n}(H) + \gamma(H)$. In such a case, $|W_{C_n}| = \frac{n-1}{2}$, and by combining Lemmas 4.1 and 6.1, we obtain $\frac{n-1}{2} = |W_{C_n}| \geq \xi(C_n) \geq \frac{n+1}{2}$, which is a contradiction. Thus,

$$\xi(C_n \circ H) = \sum_{i=0}^{n-1} |W_i| \geq 1 + \lambda + (k-2)\,\mathrm{n}(H) + \gamma(H). \tag{6.6}$$

Furthermore, if $\delta(H) = 0$, then $\gamma(H) \geq 2$ and $\lambda = 1$, while if $\delta(H) \geq 1$, then $\gamma(H) \geq 1$ and $\lambda \geq 2$. This implies that in any case we have $\lambda + \gamma(H) \geq 3$. Therefore, if $\mathrm{n}(H) \geq 3$, then (6.6) leads to

$$\xi(C_n \circ H) = \sum_{i=0}^{n-1} |W_i| \geq 4 + (k-2)\,\mathrm{n}(H) \geq 4 + 3(k-2) = (2k+1) + (k-3) \geq 2k+1 = n,$$

as required.

From now on, assume $\mathrm{n}(H) = 2$. We consider first $H \cong K_2$. In this case, $\lambda = 2$ and $\gamma(H) = 1$. Without loss of generality, we can assume that $|W_1| = 2$. Thus, if $|W_{n-1}| = 2$, then $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 4 + 2(k-2) + 1 = 2k+1 = n$ and we are done. Suppose that $|W_{n-1}| \leq 1$.

Case 1. $|W_{n-1}| = 1$. If $|W_{k+1}| = 2$, then we are done, as $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 3 + 2(k-2) + 2 = 2k + 1 = n$. Now, we consider the case $|W_{k+1}| \leq 1$. For the sake of clarity, we will simplify the notation by taking three vertices $z_0 \in V(H_0)$, $z_{n-1} \in V(H_{n-1}) \setminus W_{n-1}$ and $z_{k+1} \in V(H_{k+1}) \setminus W_{k+1}$. Notice that if $n \equiv 1 \pmod 4$, then $B_{z_{n-1}|z_{k+1}} = V(H_{\frac{k}{2}})$ and $B_{z_0|z_{k+1}} = V(H_{n-\frac{k}{2}})$, which implies that $W_{\frac{k}{2}} \neq \emptyset$ and $W_{n-\frac{k}{2}} \neq \emptyset$, and so $|W_{\frac{k}{2}}| + |W_{n-\frac{k}{2}}| \geq 3$. Hence, $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 3 + (2(k-2)+1) + 1 = 2k+1 = n$, as required. Analogously, if $n \equiv 3 \pmod 4$, then $B_{z_0|z_{k+1}} = V(H_{\frac{k+1}{2}})$ and $B_{z_{n-1}|z_{k+1}} = V(H_{\frac{k+1}{2}})$, which implies that $W_{\frac{k+1}{2}} \neq \emptyset$ and $W_{n-\frac{k+1}{2}} \neq \emptyset$, and so $|W_{\frac{k+1}{2}}| + |W_{n-\frac{k+1}{2}}| \geq 3$. Hence, $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 3 + (2(k-2) + 1) + 1 = 2k + 1 = n$, as required.

Case 2. $W_{n-1} = \emptyset$. Observe that $|W_k| \geq 1$, as $B_{z_0|z_{n-1}} = V(H_k)$. Now, if $|W_{k+1}| = 2$, then we are done, as $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 2 + 2(k-2) + 3 = 2k + 1 = n$. Assume $|W_{k+1}| \leq 1$. Notice also that $|W_{n-2}| = 2$. If $|W_2| = 2$, then we are done, as $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 2 + (2(k-2)+2) + 1 = 2k+1 = n$. Assume $|W_2| \leq 1$. Thus, $|W_{k+1}| = 1$, as for every $z_2 \in V(H_2) \setminus W_2$ we have that $B_{z_2|z_{n-1}} = V(H_{k+1})$. As in Case 1, if $n \equiv 1 \pmod 4$, then we reach to $|W_{\frac{k}{2}}| + |W_{n-\frac{k}{2}}| \geq 3$, and so $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 2 + (2(k-2) + 1) + 2 = 2k + 1 = n$, as required. Analogously, if $n \equiv 3 \pmod 4$, then we reach to $|W_{\frac{k+1}{2}}| + |W_{n-\frac{k+1}{2}}| \geq 3$. Hence, $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 2 + (2(k-2) + 1) + 2 = 2k + 1 = n$, as required.

According to the two cases above, the proof is complete for $H \cong K_2$. From now on we assume that H is the empty graph, i.e., $H \cong N_2$. In this case, $\lambda = 1$ and $\gamma(H) = 2$. Without loss of generality, we can assume that $|W_k| = 2$. Thus, if $|W_{k+1}| = 2$, then $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 1 + 2(k-2) + 4 = 2k+1 = n$ and we are done. From now on, we assume that $|W_{k+1}| \leq 1$. Now, if $|W_1| + |W_{n-1}| \geq 3$, then we are done, as $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 3 + 2(k-2) + 2 = 2k+1 = n$. Thus, we assume $1 \leq |W_1| + |W_{n-1}| \leq 2$ and we differentiate the following cases.

Case 1'. $|W_1| = 0$ and $|W_{n-1}| = 2$. In this case, $|W_{k+1}| = 1$ and so $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 2+2(k-2)+3 = 2k + 1 = n$.

Case 2'. $|W_1| = 2$ and $|W_{n-1}| = 0$. By analogy to Case 1, we deduce that either $|W_{\frac{k}{2}}| + |W_{n-\frac{k}{2}}| \geq 3$ or $|W_{\frac{k+1}{2}}| + |W_{n-\frac{k+1}{2}}| \geq 3$. Hence, we reach to $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 2 + (2(k-2) + 1) + 2 = 2k + 1 = n$.

Case 3'. $|W_1| = 1$ and $|W_{n-1}| \leq 1$. In this case, we have that $|W_{k+1}| = 1$ and, as above, we deduce that either $|W_{\frac{k}{2}}| + |W_{n-\frac{k}{2}}| \geq 3$ or $|W_{\frac{k+1}{2}}| + |W_{n-\frac{k+1}{2}}| \geq 3$. Hence, we reach to $\xi(C_n \circ K_2) = \sum_{i=0}^{n-1} |W_i| \geq 1 + (2(k-2) + 1) + 3 = 2k + 1 = n$.

Case 4'. $|W_1| \leq 1$ and $|W_{n-1}| = 1$. This case is analogous to the previous one.

According to the four cases above, the proof is complete. □

Proposition 6.4. *Let $n \geq 6$ be an integer. The following statements hold for any graph H of order at least two.*

(a) *If $n \equiv 2 \pmod 4$, then $\xi(C_n \circ H) = \frac{n}{2} \cdot \mathrm{n}(H)$.*

(b) *If $n \equiv 0 \pmod 4$, then $\xi(C_n \circ H) = \frac{n}{2} \cdot \mathrm{n}(H) + \frac{n}{4}$.*

Proof. If $n \equiv 2 \pmod 4$, then Proposition 4.4 leads to $\xi(C_n \circ H) = \frac{n}{2} \cdot \mathrm{n}(H)$. From now on, we assume that $n \equiv 0 \pmod 4$ and $n \geq 8$. Let $U \subseteq V(C_n \circ H)$ be a set satisfying the following properties:

- $|U_i| = \mathrm{n}(H)$ whenever i is odd.
- $|U_i \cup U_{i+n/2}| = 1$ whenever $i \equiv 0 \pmod 2$.

A simple case analysis shows that U is a distance-equalizer set of $C_n \circ H$. Hence,

$$\xi(C_n \circ H) \leq |U| = \frac{n}{2} \cdot \mathrm{n}(H) + \frac{n}{4}. \tag{6.7}$$

It remains necessary to prove that $\xi(C_n \circ H) \geq \frac{n}{2} \cdot \mathrm{n}(H) + \frac{n}{4}$. Let W be a $\xi(C_n \circ H)$-set. Suppose that there are two consecutive copies of H, say H_l and H_{l+1}, such that $|W_l| < \mathrm{n}(H)$ and $|W_{l+1}| < \mathrm{n}(H)$. Without loss of generality, we can consider that $l = 0$. Notice that for any $h, h' \in V(H)$ and $k \in \{1, \ldots, \frac{n-4}{2}\}$, we have that $B_{(u_0,h)|(u_{2k+1},h')} = \varnothing$. Thus, $|W_{2k+1}| = \mathrm{n}(H)$ for every $k \in \{1, \ldots, \frac{n-4}{2}\}$. By analogy, we deduce that $|W_{2k+2}| = \mathrm{n}(H)$ for every $k \in \{1, \ldots, \frac{n-4}{2}\}$. Furthermore, $|W_{n-1}| = \mathrm{n}(H)$ or $|W_2| = \mathrm{n}(H)$, as $B_{(u_{n-1},h)|(u_2,h')} = \varnothing$ for every $h, h' \in V(H)$. In addition, if a vertex $(u_i, v) \in W$ is equidistant to a vertex in $V(H_0) \setminus W_0$ and a vertex in $V(H_1) \setminus W_1$, then $i = 0$ or $i = 1$, which implies that $|W_0| + |W_1| \geq 1$. In summary,

$$\xi(C_n \circ H) \geq 1 + \mathrm{n}(H) + (n-4)\,\mathrm{n}(H) > \frac{n}{2} \cdot \mathrm{n}(H) + \frac{n}{4},$$

which contradicts (6.7). Hence, we conclude that for every pair of consecutive copies of H, say H_l and H_{l+1}, we have that $|W_l| = \mathrm{n}(H)$ or $|W_{l+1}| = \mathrm{n}(H)$. Without loss of generality, we assume that $|W_i| = \mathrm{n}(H)$ whenever i is odd. Now, if $W_0 = W_{n/2} = \varnothing$, then $|W_{2j}| = \mathrm{n}(H)$ or $|W_{n-2j}| = \mathrm{n}(H)$ for every $j \in \{1, \ldots, \frac{n-4}{4}\}$. Thus, by the minimality of $|W|$, we have that $|W_0| + |W_{n/2}| \geq 1$. The same reasoning applies to conclude that $|W_i| + |W_{i+n/2}| \geq 1$ whenever i is even. Therefore,

$$\xi(C_n \circ H) = |W| = \sum_{i=0}^{\frac{n-2}{2}} |W_{2l+1}| + \sum_{i=0}^{\frac{n-2}{2}} |W_{2l}| \geq \frac{n}{2} \cdot \mathrm{n}(H) + \frac{n}{4}. \tag{6.8}$$

From (6.7) and (6.8), we conclude the proof. □

7. NP-completeness

In this section, we discuss the following basic question: How difficult is it to compute the equidistant dimension of a graph? The decision problem associated with the problem of finding the equidistant dimension of a graph can be expressed in the following form.

Distance-equalizer set problem (DESP)
Instance: A connected graph G and a positive integer $k \leq \mathrm{n}(G)$.
Question: Does G have a distance-equalizer set of cardinality at most k?

We shall now formally define another known decision problem that we need to continue our study.

Dominating set problem (DSP)
Instance: A graph H and a positive integer $k \leq \mathrm{n}(H)$.
Question: Does H have a dominating set of cardinality at most k?

The following known result will be one of our main tools.

Theorem 7.1. [8] *DSP is NP-complete.*

Our second tool will be the graph shown in the following figure.
We are now in a position to present the following result.

Theorem 7.2. *DESP is NP-complete, even for the class of lexicographic product graphs.*

Proof. We must do two things. First, we must show that DESP belongs to the class NP, which is easy to do, since it is easy to verify a "yes" instance of DESP in polynomial time. That is, for a graph G, a positive integer k, and an arbitrary set of vertices $X \subseteq V(G)$ with $|X| \leq k$, it is easy to verify in polynomial time (by computing first the distance matrix) whether X is a distance-equalizer set of G.

Second, we must construct a reduction from a known NP-complete problem to DESP. We can use DSP, by Theorem 7.1. Given an instance $I = (H, k)$ of DSP, we construct an instance $I' = (G^* \circ H, k')$ of DESP, where G^* is the graph shown in Figure 5. We must show that I is a "yes" instance of DSP if and only if I' is a "yes" instance of DESP, in this case for $k' = 3k$. To do so, let $X = \{a_1, a_2, a_3\}$ be the set of vertices of degree 4 in G^* and let $b_i \in V(G^*)$ be the vertex of degree 1 that is adjacent to a_i for every $i \in \{1, 2, 3\}$, and let z be the vertex of degree 3 in G^*.

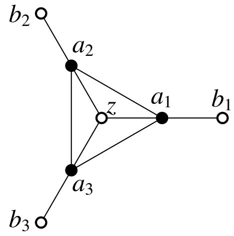

Figure 5. The graph G^* used in the proof of Theorem 7.2. The set $X = \{a_1, a_2, a_3\}$ forms a $\xi(G^*)$-set.

Now, we proceed to show that if $S \subseteq V(H)$ is a dominating set of H with $|S| \leq k$, then $X \times S$ is a distance-equalizer of $G^* \circ H$ with $|X \times S| \leq k'$. To this end, we differentiate the following cases for two different vertices $(g, h), (g', h') \in V(G \circ H) \setminus X \times S$ and, for each of these cases, we will show that there exists at least one vertex $(x, s) \in X \times S$ such that $d_{G\circ H}((x, s), (g, h)) = d_{G\circ H}((x, s), (g', h'))$.

Case 1. $g = g'$. For every $a_i \in X \setminus \{g\}$ and every $s \in S$ we have that $(a_i, s) \in X \times S$ and by Remark 2.1 we have

$$d_{G\circ H}((a_i, s), (g, h)) = d_G(a_i, g) = d_G(a_i, g') = d_{G\circ H}((a_i, s), (g', h')).$$

Case 2. $\{g, g'\} = \{a_i, b_i\}$ for some $i \in \{1, 2, 3\}$. Without loss of generality, we can take $g = a_i$ and $g' = b_i$. In this case, since $h \in V(H) \setminus S$ and S is a dominating set of H, there exists at least one vertex $s \in S \cap N_H(h)$. Hence, $(a_i, s) \in X \times S$ and by Remark 2.1 we have

$$\begin{aligned} d_{G\circ H}((a_i, s), (g, h)) &= d_{G\circ H}((a_i, s), (a_i, h)) = d_H(s, h) = 1 \\ &= d_G(a_i, b_i) = d_{G\circ H}((a_i, s), (b_i, h)) = d_{G\circ H}((a_i, s), (g', h')). \end{aligned}$$

Case 3. $g \neq g'$ and $\{g, g'\} \neq \{a_i, b_i\}$ for every $i \in \{1, 2, 3\}$. If $\{g, g'\} = \{b_l, b_m\}$, then for $j \in \{1, 2, 3\} \setminus \{l, m\}$ we have $d_G(g, a_j) = 2 = d_G(g', a_j)$. Thus, for every $s \in S$ we have that $(a_j, s) \in X \times S$ and by Remark 2.1 we have

$$d_{G\circ H}((a_j, s), (g, h)) = d_G(a_j, g) = d_G(a_j, g') = d_{G\circ H}((a_j, s), (g', h')). \tag{7.1}$$

Now, if $\{g, g'\} = \{a_l, a_m\}$, then for $j \in \{1, 2, 3\} \setminus \{l, m\}$ we have $d_G(g, a_j) = 1 = d_G(g', a_j)$, and so we arrive to (7.1). If $(g, g') = (a_l, b_j)$ for $j \neq l$, then $d_G(g, a_j) = 1 = d_G(g', a_j)$, and again we arrive to (7.1). Assume, without loss of generality, that $g = z$ is the central vertex of G^*. In such a case, we have $g' = b_j$ or $g' = a_i$ for $i, j \in \{1, 2, 3\}$. In the first case, $d_G(g, a_j) = 1 = d_G(g', a_j)$, and so we arrive to (7.1). In the second case, since S is a dominating set, there exists $s \in S \cap N_H(h')$ and so $d_{G\circ H}((g', h'), (a_i, s)) = d_H(h', s) = 1 = d_G(g, a_i) = d_{G\circ H}((g, h), (a_i, s))$. According to the three cases above, $X \times S$ is a distance-equalizer of $G^* \circ H$.

Therefore, we conclude that if I is a "yes" instance of DSP, then I' is a "yes" instance of DESP.

Conversely, let W be a distance-equalizer set of $G^* \circ H$ with $|W| \leq k'$. Suppose that there exists $i \in \{1, 2, 3\}$ such that neither W_{a_i} nor W_{b_i} are dominating sets of H_{a_i} and H_{b_i}, respectively. Thus, there exists at least one vertex $(a_i, h) \in V(H_{a_i}) \setminus W_{a_i}$ such that $W_{a_i} \cap N_{G\circ H}(a_i, h) = \emptyset$ and there exists at least one vertex $(b_i, h') \in V(H_{b_i}) \setminus W_{b_i}$ such that $W_{b_i} \cap N_{G\circ H}(b_i, h') = \emptyset$. We proceed to show that, with the assumptions above, $d_{G\circ H}((x, y), (a_i, h)) \neq d_{G\circ H}((x, y), (b_i, h'))$ for every $(x, y) \in W$. To this end, we differentiate the following three cases.

Case 1'. $(x, y) \in W \setminus (W_{a_i} \cup W_{b_i})$. In this case, $x \neq a_i$ and $x \neq b_i$, which implies that

$$d_{G\circ H}((x, y), (a_i, h)) = d_G(x, a_i) \neq d_G(x, a_i) + d_G(a_i, b_i) = d_G(x, b_i) = d_{G\circ H}((x, y), (b_i, h')).$$

Case 2'. $(x, y) \in W_{a_i}$. In this case, $x = a_i$. Since $(a_i, h) \in V(H_{a_i}) \setminus W_{a_i}$ and $W_{a_i} \cap N_{G\circ H}(a_i, h) = \emptyset$, we have that $d_{G\circ H}((x, y), (a_i, h)) = 2$. On the other side, $d_{G\circ H}((x, y), (b_i, h')) = d_{G\circ H}((a_i, y), (b_i, h')) = d_G(a_i, b_i) = 1$. Therefore, $d_{G\circ H}((x, y), (a_i, h)) \neq d_{G\circ H}((x, y), (b_i, h'))$.

Case 3'. $(x, y) \in W_{b_i}$. In this case, $x = b_i$. Now, since $(b_i, h') \in V(H_{b_i}) \setminus W_{b_i}$ and $W_{b_i} \cap N_{G\circ H}(b_i, h') = \emptyset$, we have that $d_{G\circ H}((x, y), (b_i, h')) = 2$. On the other side, $d_{G\circ H}((x, y), (a_i, h)) = d_{G\circ H}((b_i, y), (a_i, h)) = d_G(b_i, a_i) = 1$. Therefore, $d_{G\circ H}((x, y), (a_i, h)) \neq d_{G\circ H}((x, y), (b_i, h'))$.

According to the three cases above, $d_{G\circ H}((x, y), (a_i, h)) \neq d_{G\circ H}((x, y), (b_i, h'))$ for every $(x, y) \in W$, which contradicts the fact that W is a distance-equalizer set of $G^* \circ H$. Hence, either W_{a_i} is a dominating set of H_{a_i} or W_{b_i} is a dominating set of H_{b_i} for every $i \in \{1, 2, 3\}$, which implies that H has a dominating set of cardinality at most $\frac{1}{3}|W| \leq k$. Therefore, we conclude that if I' is a "yes" instance of DESP, then I is a "yes" instance of DSP. □

If S is the set of vertices of degree two of P_4, then the set $X \times S$ described in the proof of Theorem 7.2 corresponds to the set of black coloured vertices shown in Figure 6, which forms a $\xi(G^* \circ P_4)$-set.

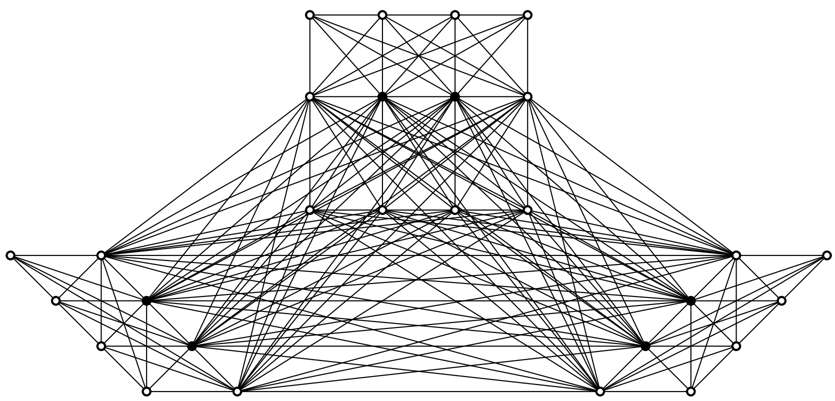

Figure 6. The graph $G^* \circ P_4$.

8. Concluding remarks and open problems

In this paper, we deal with the problem of finding the equidistant dimension of the lexicographic product $G \circ H$ of two graphs G and H. To carry out the study when the domination number of G is equal to one, we have distinguished two cases depending on whether G has minimum degree one or not. In both cases, the result depends on some domination parameters on H. Two of these parameters are the domination number and the total domination number, which are well-known parameters. In contrast, the third parameter required for the study is completely new (the pairwise domination number).

As we can expect, $\xi(G) \leq \xi(G \circ H) \leq \xi(G)\,\mathrm{n}(H)$ for every connected graph G and every graph H. The discussion of the case $\xi(G \circ H) = \xi(G)$ leads to the challenge of investigating a new parameter that we call the total equidistant dimension of a graph.

We also obtain the value of $\xi(G \circ H)$ for some particular classes of graphs, including the cases where G is a path or a cycle, and we show that the general problem of finding the equidistant dimension of a graph is NP-hard.

Some specific open problems derived from the discussion are listed below.

- Theorem 3.1 shows the need for an extensive study on pairwise domination. In particular, it would be convenient to characterize the graphs with $\gamma_{\wp}(H) = \gamma(H)$.
- We learned from Proposition 4.2 that $\xi(G \circ H) = \xi(G)$ for every graph H if and only if $\xi_t(G) = \xi(G)$. This statement shows the need to develop a detailed study on the total equidistant dimension of a graph.
- In Proposition 4.4, the assumptions are that G is a bipartite 2-antipodal graph G of diameter $2k+1 \geq 3$ with $\xi(G) = \frac{\mathrm{n}(G)}{2}$. However, a problem to be clarified is the existence (or non-existence) of bipartite 2-antipodal graphs of diameter $2k+1 \geq 3$ with $\xi(G) > \frac{\mathrm{n}(G)}{2}$.
- To complete the study proposed in Section 5, it remains to investigate the following cases. If $\mathrm{n}(H) = 2$, then for $n \geq 5$ and $n \notin \{9, 10\}$ we conjecture that $\xi(P_n \circ H) = \frac{3n-5}{2}$ whenever n is odd, and $\xi(P_n \circ H) = \frac{3n-4}{2}$ whenever n is even, while $\xi(P_9 \circ H) = 10$ and $\xi(P_{10} \circ H) = 12$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors were supported by the SGR Grant 2021 00115.

Conflict of interest

Juan Alberto Rodríguez-Velázquez is an editorial board member and was not involved in the editorial review or the decision to publish this article. The authors declare that they have no conflicts of interest.

References

1. B. S. Anand, M. Changat, S. Klavžar, I. Peterin, Convex sets in lexicographic products of graphs, *Graphs Combin.*, **28** (2012), 77–84. https://doi.org/10.1007/s00373-011-1031-4
2. G. A. Barragán-Ramírez, A. Estrada-Moreno, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez, The local metric dimension of the lexicographic product of graphs, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 2481–2496. https://doi.org/10.1007/s40840-018-0611-3
3. K. Casel, A. Estrada-Moreno, H. Fernau, J. A. Rodríguez-Velázquez, Weak total resolvability in graphs, *Discuss. Math. Graph Theory*, **36** (2016), 185–210.
4. A. Estrada-Moreno, C. García-Gómez, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez, The simultaneous strong metric dimension of graph families, *Bull. Malays. Math. Sci. Soc.*, **39** (2016), 175–192. https://doi.org/10.1007/s40840-015-0268-0
5. A. Estrada-Moreno, I. G. Yero, J. A. Rodríguez-Velázquez, Relationships between the 2-metric dimension and the 2-adjacency dimension in the lexicographic product of graphs, *Graphs Combin.*, **32** (2016), 2367–2392. https://doi.org/10.1007/s00373-016-1736-5
6. A. Estrada-Moreno, I. G. Yero, J. A. Rodríguez-Velázquez, The *k*-metric dimension of the lexicographic product of graphs, *Discrete Math.*, **339** (2016), 1924–1934. https://doi.org/10.1016/j.disc.2015.12.024
7. M. Feng, K. S. Wang, On the fractional metric dimension of corona product graphs and lexicographic product graphs, 2012, arXiv: 1206.1906.
8. M. R. Garey, D. S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness*, New York: W. H. Freeman and Company, 1979.
9. A. González, C. Hernando, M. Mora, The equidistant dimension of graphs, *Bull. Malays. Math. Sci. Soc.*, **45** (2022), 1757–1775. https://doi.org/10.1007/s40840-022-01295-z
10. R. Hammack, W. Imrich, S. Klavžar, *Handbook of product graphs*, 2 Eds., Boca Raton: CRC Press, 2011. https://doi.org/10.1201/b10959
11. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of domination in graphs*, Boca Raton: CRC Press, 1998. https://doi.org/10.1201/9781482246582
12. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Domination in graphs: Volume 2: Advanced topics*, New York: Routledge, 1998. https://doi.org/10.1201/9781315141428
13. M. A. Henning, A. Yeo, *Total domination in graphs*, New York: Springer, 2013. https://doi.org/10.1007/978-1-4614-6525-6
14. W. Imrich, S. Klavžar, *Product graphs: structure and recognition*, New York: Wiley, 2000.
15. M. Jannesari, B. Omoomi, The metric dimension of the lexicographic product of graphs, *Discrete Math.*, **312** (2012), 3349–3356. https://doi.org/10.1016/j.disc.2012.07.025
16. C. X. Kang, I. G. Yero, E. Yi, The fractional strong metric dimension in three graph products, *Discrete Appl. Math.*, **251** (2018), 190–203. https://doi.org/10.1016/j.dam.2018.05.051
17. S. Klavžar, D. Kuziak, I. Peterin, I. G. Yero, A Steiner general position problem in graph theory, *Comput. Appl. Math.*, **40** (2021), 223. https://doi.org/10.1007/s40314-021-01619-y

18. S. Klavžar, D. Kuziak, I. G. Yero, Further contributions on the outer multiset dimension of graphs, *Results Math.*, **78** (2023), 50. https://doi.org/10.1007/s00025-022-01829-8

19. S. Klavžar, P. K. Neethu, S. V. Ullas Chandran, The general position achievement game played on graphs, *Discrete Appl. Math.*, **317** (2022), 109–116. https://doi.org/10.1016/j.dam.2022.04.019

20. S. Klavžar, I. G. Yero, The general position problem and strong resolving graphs, *Open Math.*, **17** (2019), 1126–1135. https://doi.org/10.1515/math-2019-0088

21. D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Closed formulae for the strong metric dimension of lexicographic product graphs, *Discuss. Math. Graph Theory*, **36** (2016), 1051–1064. https://doi.org/10.7151/dmgt.1911

22. I. Peterin, I. G. Yero, Edge metric dimension of some graph operations, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 2465–2477. https://doi.org/10.1007/s40840-019-00816-7

23. Y. Ramírez-Cruz, A. Estrada-Moreno, J. A. Rodríguez-Velázquez, The simultaneous metric dimension of families composed by lexicographic product graphs, *Graphs Combin.*, **32** (2016), 2093–2120. https://doi.org/10.1007/s00373-016-1675-1

24. J. A. Rodríguez-Velázquez, Lexicographic metric spaces: basic properties and the metric dimension, *Appl. Anal. Discrete Math.*, **14** (2020), 20–32.

25. J. A. Rodríguez-Velázquez, Solution of the Chen-Chvátal conjecture for specific classes of metric spaces, *AIMS Math.*, **6** (2021), 7766–7781. https://doi.org/10.3934/math.2021452

26. S. W. Saputro, R. Simanjuntak, S. Uttunggadewa, H. Assiyatun, E. T. Baskoro, A. N. M. Salman, et al., The metric dimension of the lexicographic product of graphs, *Discrete Math.*, **313** (2013), 1045–1051. https://doi.org/10.1016/j.disc.2013.01.021

AIMS Press