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Research article

Existence results for IBVP of (p,q)-fractional difference equations in Banach space

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Abstract: This article focuses on the problem of integral boundary value for Riemann-Liouville derivatives equipped with (p,q)-difference calculus in Banach space. To provide further clarification, our focus lies in establishing the existence of a solution to our problem using the measure of noncompactness (m.n.) and the Mönch's fixed point theorem. Our investigation in the Banach space encompasses two nonlinear terms with two distinct orders of derivatives. Our paper concludes with an illustrative example and conclusion.

Keywords: boundary value problem; existence of solutions; Banach space; Riemann-Liouville fractional derivatives; Mönch fixed point theorem; (p,q)-difference calculus
Mathematics Subject Classification: 26A33, 33D05, 34A05, 34A08, 34A12, 34B18, 39A13, 47H10

1. Introduction

In the last two decades, q-calculus has become an established field of study that has demonstrated its usefulness in various disciplines, such as hypergeometric series, quantum mechanics, complex analysis, and particle physics, as evidenced by the papers referenced in [1–4]. Subsequently, the concept of (p,q)-calculus underwent generalization and advancement of q-calculus theory, see here [5–10], some works concerning q-calculus. This particular mathematical framework has proven to be highly effective across various disciplines. Further information and findings are, [11, 12] for (p,q)-gamma and the (p,q)-beta functions, [13] for certain identities associated with (p,q)-binomial coefficients and (p,q)-Stirling polynomials, [14,15] for (p,q)-integral inequalities and teir applications, and others investigations of (p,q)-calculus can be explored in [16–18].

Researchers have extensively explored the subject of fractional order derivatives and (p,q)-calculus, recognizing its significance. They investigated the qualitative properties of solutions such as the existence [19, 20], positivity [21–23], and [24, 25] for M-truncated derivative, as these references will undoubtedly enrich the study of this subject.

Kuratowski [26] introduced the concept of m.n. in 1930. This measure finds numerous applications in mathematical research [27, 28]. Darbo [29] was the pioneering researcher who utilized the m.n. to investigate the correlation between contraction and compact mappings. However, the fixed-point theorems of Darbo, Sadovski [30], and Mönch [31, 32] have been widely recognized as valuable tools in the analysis of various classes of differential equations. These theorems, particularly in the context of fractional differential equations, have proven to be effective in studying the existence of solutions (see [33–39]).

Recently, Qin and Sun [23] conducted a study on positive solutions for BVPs of fractional (p,q)-difference

$$\begin{cases} \boldsymbol{D}_{p,q}^{\sigma} z(t) + \zeta\left(p^{\sigma} t, z\left(p^{\sigma} t\right)\right) = 0, & t \in (0,1), \\ z(0) = z(1) = 0, \end{cases} \tag{1.1}$$

where $0 < q < p \leq 1$, $1 < \sigma \leq 2$, $\boldsymbol{D}_{p,q}^{\sigma}$ is a (p,q)- fractional difference of Riemann-Liouville operator, and $\zeta : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function.

This paper aims to analyze an integral boundary value problem of (p,q)-fractional difference equations which encompasses two nonlinear terms defined as follows:

$$\begin{cases} \boldsymbol{D}_{p,q}^{\sigma} z(t) + \zeta\left(p^{\sigma} t, z\left(p^{\sigma} t\right)\right) = \boldsymbol{D}_{p,q}^{\sigma-1} h\left(pt, z(pt)\right), & t \in (0,1), \\ z(0) = 0, \quad z(1) = \int_0^1 h\left(pv, z(pv)\right) \mathrm{d}_{p,q} v, \end{cases} \tag{1.2}$$

where $\sigma \in (1,2]$, the functions $\zeta, h : [0,1] \times \mathcal{Z} \to \mathcal{Z}$ will be disclosed at a later stage, and $\mathcal{Z}$ is a Banach space supplied with the norm $\|\cdot\|$.

In the literature, Xu and Han in [21] investigated and proved the existence and uniqueness of a positive solution of (1.2) without (p,q)-calculus. Lachouri, Ardjouni, and Djoudi in [34] also considered problem (1.2) without (p,q)-calculus, but in Banach space and found interesting results for the existence of solutions. The authors in [40] studied and analyzed the existence of solutions for Caputo fractional q-difference equations in a Banach space with different nonlinear integral boundary conditions than in our problem and without the function h, their work is based on Mönch's fixed point theorem and m.n. tool. So, all the previous information invites us to study problem (1.2) in order to generalize and improve previous studies.

This document is created in the following manner. Section 2 will introduce key concepts, tools, and discoveries that are employed in the analysis. The third section is dedicated to examining the existence of solutions. An illustration is provided in Section 4. Section 5 encompasses the derivation of conclusions and generalizations.

2. Essential materials

This section provides a selection of necessary materials that are essential for our study. We initiate our discussion by presenting the essential tools of definitions and results of the q-calculus and (p,q)-calculus. Therefore, we direct the reader to the following references for more information [6, 12, 17, 18]. Given the numbers p, q such that $0 < q < p \leq 1$,

$$[S]_{p,q} := \begin{cases} \frac{p^{S} - q^{S}}{p-q} = p^{S-1} [S]_{\frac{q}{p}}, & S \in \mathbb{N}^{+}, \\ 1, & S = 0, \end{cases}$$

$$[S]_{p,q}! := \begin{cases} [S]_{p,q}[S-1]_{p,q}\dots[1]_{p,q} = \prod_{i=1}^{S}\frac{p^i-q^i}{p-q}, & S\in\mathbb{N}^+,\\ 1, & S=0.\end{cases}$$

Expressing the power function $(u-w)_q^{(n)}$ under the q-analogue is given by

$$(u-w)_q^{(n)} := \begin{cases} \prod_{i=0}^{n-1}\left(u-wq^i\right), & n\in\mathbb{N}^+,\ u,w\in\mathbb{R},\\ 1, & n=0.\end{cases}$$

Expressing the power function $(u-w)_{p,q}^{(n)}$ under the (p,q)-analogue is written as

$$(u-w)_{p,q}^{(n)} := \begin{cases} \prod_{i=0}^{n-1}\left(up^i-wq^i\right), & n\in\mathbb{N}^+,\ u,w\in\mathbb{R},\\ 1, & n=0,\end{cases}$$

and for $\sigma\in\mathbb{R}$, the general form of the above is given by

$$(u-w)_{p,q}^{(\sigma)} := p^{\binom{\sigma}{2}}(u-w)_{\frac{q}{p}}^{(\sigma)} = u^{\sigma}p^{\binom{\sigma}{2}}\prod_{i=0}^{\infty}\frac{u-w\left(\frac{q}{p}\right)^i}{u-w\left(\frac{q}{p}\right)^{\sigma+i}},\quad 0<w<u,$$

where $p^{\binom{\sigma}{2}} := \frac{\sigma(\sigma-1)}{2}$.

Let $C(A,\mathcal{Z})$ be a Banach space which contains the continuous functions defined from $A=[0,1]$ into $\mathcal{Z}$ equipped with the norm

$$\|z\|_\infty = \sup_{t\in A}\|z(t)\|.$$

Let $L^1(A,\mathcal{Z})$ be the Banach space which contains the measurable functions defined from A into $\mathcal{Z}$ that are Lebesgue integrable with norm

$$\|z\|_{L^1} = \int_0^1\|z(t)\|\,dt.$$

Definition 2.1. *[18] Given the numbers p,q such that $0<q<p\le 1$, the (p,q)-derivative of the function ζ is defined as*

$$\boldsymbol{D}_{p,q}\zeta(t) := \frac{\zeta(pt)-\zeta(qt)}{(p-q)t},\quad t\neq 0,$$

and $\left(\boldsymbol{D}_{p,q}\zeta\right)(0) = \lim_{t\to 0}\left(\boldsymbol{D}_{p,q}\zeta\right)(t)$ such that ζ is differentiable at 0. Moreover, the high order (p,q)-derivative $\boldsymbol{D}_{p,q}^n\zeta(t)$ is defined by

$$\boldsymbol{D}_{p,q}^n\zeta(t) = \begin{cases} \boldsymbol{D}_{p,q}\boldsymbol{D}_{p,q}^{n-1}\zeta(t), & n\in\mathbb{N}^+,\\ \zeta(t), & n=0.\end{cases}$$

Definition 2.2. *[18] Given the numbers p,q such that $0<q<p\le 1$, and ζ is an arbitrary function of a real number, the (p,q)-integral of the function ζ is defined as*

$$\boldsymbol{I}_{p,q}\zeta(t) := \int_0^t\zeta(v)\,\mathrm{d}_{p,q}v = (p-q)t\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}\zeta\left(\frac{q^i}{p^{i+1}}t\right).$$

Moreover, ζ is called (p,q)-integrable on $[0,t]$ if the series on the right side converges.

Definition 2.3. *[17] Let $0 < q < p \leq 1$, $\sigma > 0$, and $\zeta : A \to \mathbb{R}$ be an arbitrary function. The (p, q)-fractional integral of order σ is defined by*

$$\boldsymbol{I}^{\sigma}_{p,q}\zeta(t) = \frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^t (t - qv)^{(\sigma-1)}_{p,q} \zeta\left(\frac{v}{p^{\sigma-1}}\right) \mathrm{d}_{p,q}v,$$

and $\boldsymbol{I}^0_{p,q}\zeta(t) = \zeta(t)$.

Definition 2.4. *[17] Let $0 < q < p \leq 1$, $\sigma > 0$, and ζ be an arbitrary function on A. The (p, q)-fractional difference operator of Riemann-Liouville type of order σ is defined by*

$$\boldsymbol{D}^{\sigma}_{p,q}\zeta(t) = \boldsymbol{D}^{S}_{p,q}\boldsymbol{I}^{S-\sigma}_{p,q}\zeta(t),$$

and $\boldsymbol{D}^0_{p,q}\zeta(t) = \zeta(t)$, where S represents the smallest integer that is greater than or equal to σ. In addition,

$$\boldsymbol{D}^{\sigma}_{p,q}\boldsymbol{I}^{\sigma}_{p,q}\zeta(t) = \zeta(t).$$

Lemma 2.1. *[17] For $0 < q < p \leq 1$, $\sigma \in (S-1, S]$, $S \in \mathbb{N}$, and $\zeta : A \to \mathbb{R}$, obtain*

$$\boldsymbol{I}^{\sigma}_{p,q}\boldsymbol{D}^{\sigma}_{p,q}\zeta(t) = \zeta(t) + c_1 t^{\sigma-1} + c_2 t^{\sigma-2} + \ldots + c_S t^{\sigma-S},$$

with $c_j \in \mathbb{R}$, $j = 1, 2, \ldots, S$.

Lemma 2.2. *Let $1 < \sigma \leq 2$ and $0 < q < p \leq 1$. Then $z : A \to \mathcal{Z}$ is a solution of the IBVP (1.2) if and only if $z(t)$ satisfies*

$$z(t) = \int_0^1 \boldsymbol{\phi}(t, qv)\zeta(pv, z(pv))\,\mathrm{d}_{p,q}v + \int_0^t h(pv, z(pv))\,\mathrm{d}_{p,q}v,$$

where

$$\boldsymbol{\phi}(t, qv) = \frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \begin{cases} t^{\sigma-1}(1 - qv)^{(\sigma-1)}_{p,q} - (t - qv)^{(\sigma-1)}_{p,q}, & 0 \leq qv \leq t \leq 1, \\ t^{\sigma-1}(1 - qv)^{(\sigma-1)}_{p,q}, & 0 \leq t \leq qv \leq 1. \end{cases} \tag{2.1}$$

Proof. According to Definition 2.3 and Lemma 2.1, by employing the integral operator $\boldsymbol{I}^{\sigma}_{p,q}$ on the two sides of Eq (1.2), it can be deduced that

$$\begin{aligned} & z(t) + c_1 t^{\sigma-1} + c_2 t^{\sigma-2} + \frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)} \int_0^t (t - qv)^{(\sigma-1)}_{p,q} \zeta(pv, z(pv))\,\mathrm{d}_{p,q}v \\ &= \boldsymbol{I}_{p,q}\left(\boldsymbol{I}^{\sigma-1}_{p,q}\boldsymbol{D}^{\sigma-1}_{p,q}h(pt, z(pt))\right) \\ &= \boldsymbol{I}_{p,q}\left(h(pt, z(pt)) + c_3 t^{\sigma-2}\right) \\ &= \int_0^t h(pv, z(pv))\,\mathrm{d}_{p,q}v + \frac{c_3}{\sigma - 1} t^{\sigma-1}, \end{aligned}$$

for some constants $c_1, c_2, c_3 \in \mathbb{R}$. As a result of the initial condition $z(0) = 0$ and the last equation, we obtain $c_2 = 0$ and, based on the boundary condition

$$z(1) = \int_0^1 h(pv, z(pv))\,\mathrm{d}_{p,q}v,$$

we obtain

$$c_1 = -\frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\int_0^1 (1-qv)_{p,q}^{(\sigma-1)}\zeta(pv,z(pv))\,\mathrm{d}_{p,q}v + \frac{c_3}{\sigma-1}.$$

Therefore,

$$\begin{aligned}
z(t) =& \frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\int_0^1 t^{\sigma-1}(1-qv)_{p,q}^{(\sigma-1)}\zeta(pv,z(pv))\,\mathrm{d}_{p,q}v \\
&- \frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\int_0^t (t-qv)_{p,q}^{(\sigma-1)}\zeta(pv,z(pv))\,\mathrm{d}_{p,q}v + \int_0^t h(pv,z(pv))\,\mathrm{d}_{p,q}v \\
=& \frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\int_0^t \left[t^{\sigma-1}(1-qv)_{p,q}^{(\sigma-1)} - (t-qv)_{p,q}^{(\sigma-1)}\right]\zeta(pv,z(pv))\,\mathrm{d}_{p,q}v \\
&+ \frac{1}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\int_t^1 t^{\sigma-1}(1-qv)_{p,q}^{(\sigma-1)}\zeta(pv,z(pv))\,\mathrm{d}_{p,q}v + \int_0^t h(pv,z(pv))\,\mathrm{d}_{p,q}v \\
=& \int_0^1 \boldsymbol{\phi}(t,qv)\zeta(pv,z(pv))\,\mathrm{d}_{p,q}v + \int_0^t h(pv,z(pv))\,\mathrm{d}_{p,q}v.
\end{aligned}$$

This process is reversible, and The proof is finished. □

Lemma 2.3. *[23] The function $\boldsymbol{\phi}$ given by (2.1) satisfies the mentioned properties:*
(i) $\boldsymbol{\phi}(t,qv) \le t^{\sigma-1}(1-qv)_{p,q}^{(\sigma-1)} \le (1-qv)_{p,q}^{(\sigma-1)}$,
(ii) $0 \le \boldsymbol{\phi}(t,qv) \le 1$, *for all* $0 \le t, v \le 1$.

Next, let us revisit the concept of the Kuratowski m.n. and provide a concise overview of its key characteristics.

Definition 2.5. *[41] Let $\mathcal{Z}$ be a Banach space and let $Q_{\mathcal{Z}}$ be the family of bounded subsets of $\mathcal{Z}$. The Kuratowski m.n. is the map $\mu : Q_{\mathcal{Z}} \to [0,\infty)$ defined by*

$$\mu(Q) = \inf\left\{\varepsilon > 0 : Q \subset \cup_{i=0}^{S} Q_i \text{ and } \dim(Q_i) \le \varepsilon\right\}, \text{ where } Q \in Q_{\mathcal{Z}}.$$

Below we provide some information about m.n. μ (See [41, 42]). Let $\overline{Q}$ and $convQ$ be the closure and the convex hull of the bounded set Q, respectively.

1) $\mu(Q) = \mu\left(\overline{Q}\right)$,
2) $\mu(Q) = 0 \Leftrightarrow \overline{Q}$ is compact (Q is relatively compact),
3) $\mu(Q) = \mu(convQ)$,
4) $\mu(Q_1 + Q_2) \le \mu(Q_1) + \mu(Q_2)$,
5) $Q_1 \subset Q_2 \Rightarrow \mu(Q_1) \le \mu(Q_2)$,
6) $\mu(cQ) = |c|\mu(Q)$, for $c \in \mathbb{R}$.

Definition 2.6. *A mapping $\zeta : A \times \mathcal{Z} \to \mathcal{Z}$ is said to be Caratheodory if*
(i) $t \to \zeta(t,z)$ is a measurable function for each $z \in \mathcal{Z}$.
(ii) $t \to \zeta(t,z)$ is a continuous function for almost all $t \in A$.

Now, let the set K of functions $k : A \to \mathcal{Z}$, denoted by

$$K(t) = \{k(t) : k \in K\}, t \in A,$$
$$K(A) = \{k(t) : k \in K, \ \ t \in A\}.$$

Lemma 2.4. *[39] If $K \subset C(A, \mathcal{Z})$ is an equicontinuous bounded set, then*
i) The function $t \to \mu(K(t))$ is a continuous function on A.
ii) $\mu\left(\left\{\int_A z(t)\,dt, z \in K\right\}\right) \leq \int_A \mu(K(t))\,dt$.

Lemma 2.5. *[33] Let Ω be a bounded, closed, and convex subset of the Banach space $C(A, \mathcal{Z})$. Let Φ be a continuous function on $A \times A$ and ζ a function from $A \times \mathcal{Z} \to \mathcal{Z}$ which satisfies the conditions of Carathéodory, and suppose there exists $\theta \in L^1(A, \mathbb{R}^+)$ such that, for all $t \in A$ and any bounded set $Q \subset \mathcal{Z}$, we have*

$$\lim_{\epsilon \to 0^+} \mu(\zeta(A_\epsilon \times Q)) \leq \theta(t)\mu(Q),$$

with $A_\epsilon = [t - \epsilon, t] \cap A$. If K is an equicontinuous subset of Ω, then

$$\mu\left(\left\{\int_A \Phi(t, r)\zeta(r, z(r))\,dr, z \in K\right\}\right) \leq \int_A |\Phi(t, r)|\theta(r)\mu(K(r))\,dr.$$

Theorem 2.1. *(Mönch [32]) Let Ω be a bounded, closed, and convex subset of a Banach space such that $0 \in \Omega$, and let $\mathcal{H}$ be a continuous mapping of Ω into itself. In addition, if*

$$K = \overline{conv}((\mathcal{H}K)) \ \text{ or } K = \overline{conv}((\mathcal{H}K)) \cup \{0\} \Rightarrow \mu(K) = 0, \tag{2.2}$$

holds for every K of Ω, then $\mathcal{H}$ has a fixed point.

Theorem 2.2. *(Arzela-Ascoli's Theorem [43]) A bounded, uniformly Cauchy subset Q of $\mathcal{Z}$ is relatively compact.*

3. Main results

In this section, we establish the existence of solutions for our problem (1.2) through the utilization of Mönch's fixed point Theorem 2.1. To accomplish this, we ensure that all the required conditions are prepared.

(C1) The functions $\zeta; h : A \times \mathcal{Z} \to \ \mathcal{Z}$ satisfy the conditions of Caratheodory.
(C2) There exist $\theta, \vartheta \in L^1(A, \mathbb{R}^+) \cap (A, \mathbb{R}^+)$ satisfy

$$\|\zeta(t, z(t))\| \leq \theta(t)\|z\|, \text{ for a.e.} t \in A \text{ and each } z \in \mathcal{Z},$$

$$\|h(t, z(t))\| \leq \vartheta(t)\|z\|, \text{ for a.e.} t \in A \text{ and each } z \in \mathcal{Z},$$

and

$$\|\theta\|_\infty + \|\vartheta\|_\infty < 1.$$

(C3) Let $A_\epsilon = [t - \epsilon, t] \cap A$. Then, for all $t \in A$ and for any bounded set $Q \subset \mathcal{Z}$, we have

$$\lim_{\epsilon \to 0^+} \mu(\zeta(A_\epsilon \times Q)) \leq \theta(t)\mu(Q),$$

$$\lim_{\epsilon \to 0^+} \mu(h(A_\epsilon \times Q)) \leq \vartheta(t)\mu(Q).$$

As per Lemma 2.2, the operator $\mathcal{H}$ is defined from $C(A, \mathcal{Z})$ into itself as

$$(\mathcal{H}z)(t) = \int_0^1 \boldsymbol{\phi}(t, qv)\, \zeta(pv, z(pv))\, \mathrm{d}_{p,q}v + \int_0^t h(pv, z(pv))\, \mathrm{d}_{p,q}v, \tag{3.1}$$

where the fixed points of the operator $\mathcal{H}$ are the solution of problem (1.2).

To achieve this objective, we establish Ω as the closed, bounded, and convex subset of $C(A, \mathcal{Z})$ for $L > 0$,

$$\Omega = \{z \in C(A, \mathcal{Z}) : \|z\|_\infty \leq L\}.$$

To demonstrate the fundamental outcomes, we partition the verification into the subsequent lemmas.

Lemma 3.1. *Suppose that the condition (C2) holds. Then,*

(i) $\mathcal{H}$ maps Ω into itself.

(ii) $\mathcal{H}(\Omega)$ is bounded and equicontinuous.

Proof. Let t be an element of A and z be an element of Ω. By utilizing condition (C2) and referring to Lemma 2.3 (ii), we get

$$\begin{aligned}
\|(\mathcal{H}z)(t)\| &\leq \left\| \int_0^1 \boldsymbol{\phi}(t, qv)\, \zeta(pv, z(pv))\, \mathrm{d}_{p,q}v \right\| + \left\| \int_0^t h(pv, z(pv))\, \mathrm{d}_{p,q}v \right\| \\
&\leq \int_0^1 \boldsymbol{\phi}(t, qv)\, \|\zeta(pv, z(pv))\|\, \mathrm{d}_{p,q}v + \int_0^t \|h(pv, z(pv))\|\, \mathrm{d}_{p,q}v \\
&\leq \int_0^1 \boldsymbol{\phi}(t, qv)\, \theta(t)\, \|z\|\, \mathrm{d}_{p,q}v + \int_0^t \vartheta(t)\, \|z\|\, \mathrm{d}_{p,q}v \\
&\leq \left(\|\theta\|_\infty \int_0^1 \boldsymbol{\phi}(t, qv)\, \mathrm{d}_{p,q}v + \|\vartheta\|_\infty \int_0^t \mathrm{d}_{p,q}v \right) L \\
&\leq L,
\end{aligned}$$

this gives that $\mathcal{H}$ maps Ω into Ω and proves that $\mathcal{H}(\Omega)$ is bounded.

Next, let $t_1, t_2 \in A,\ t_1 < t_2$, and $z \in \Omega$. Then

$$\begin{aligned}
\|(\mathcal{H}z)(t_2) - (\mathcal{H}z)(t_1)\| = &\left\| \int_0^1 \boldsymbol{\phi}(t_2, qv)\, \zeta(pv, z(pv))\, \mathrm{d}_{p,q}v + \int_0^{t_2} h(pv, z(pv))\, \mathrm{d}_{p,q}v \right. \\
&\left. - \int_0^1 \boldsymbol{\phi}(t_1, qv)\, \zeta(pv, z(pv))\, \mathrm{d}_{p,q}v - \int_0^{t_1} h(pv, z(pv))\, \mathrm{d}_{p,q}v \right\| \\
\leq &\|\theta\|_\infty L \int_0^1 |\boldsymbol{\phi}(t_2, qv) - \boldsymbol{\phi}(t_1, qv)|\, \mathrm{d}_{p,q}v \\
&+ \left\| \int_0^{t_2} h(pv, z(pv))\, \mathrm{d}_{p,q}v - \int_0^{t_1} h(pv, z(pv))\, \mathrm{d}_{p,q}v \right\| \\
\leq &\frac{\|\theta\|_\infty L}{p^{\binom{\sigma}{2}} \Gamma_{p,q}(\sigma)} \int_0^{t_1} \left| (1 - qv)_{p,q}^{(\sigma-1)} \left(t_2^{\sigma-1} - t_1^{\sigma-1} \right) - (t_2 - qv)_{p,q}^{(\sigma-1)} + (t_1 - qv)_{p,q}^{(\sigma-1)} \right| \mathrm{d}_{p,q}v \\
&+ \frac{\|\theta\|_\infty L}{p^{\binom{\sigma}{2}} \Gamma_{p,q}(\sigma)} \int_{t_1}^{t_2} \left| (1 - qv)_{p,q}^{(\sigma-1)} \left(t_2^{\sigma-1} - t_1^{\sigma-1} \right) - (t_2 - qv)_{p,q}^{(\sigma-1)} \right| \mathrm{d}_{p,q}v
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\|\theta\|_{\infty} L}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\int_{t_2}^{1}\left|(1-qv)_{p,q}^{(\sigma-1)}\left(t_2^{\sigma-1}-t_1^{\sigma-1}\right)\right| \mathrm{d}_{p,q}v+(t_2-t_1)\|\vartheta\|_{\infty} L \\
=&\frac{\|\theta\|_{\infty} L}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\left\{\int_0^{t_1}\left(t_2^{\sigma-1}-t_1^{\sigma-1}\right)\mathrm{d}_{p,q}v+\int_0^{t_1}\left[(t_2-qv)_{p,q}^{(\sigma-1)}-(t_1-qv)_{p,q}^{(\sigma-1)}\right]\mathrm{d}_{p,q}v\right. \\
&\left.+\int_{t_1}^{t_2}\left(t_2^{\sigma-1}-t_1^{\sigma-1}+1\right)\mathrm{d}_{p,q}v+\int_{t_2}^{1}\left(t_2^{\sigma-1}-t_1^{\sigma-1}\right)\mathrm{d}_{p,q}v\right\}+(t_2-t_1)\|\vartheta\|_{\infty} L \\
\leq&\frac{\|\theta\|_{\infty} L}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\left\{\int_0^{1}\left(\left[(t_2-qv)_{p,q}^{(\sigma-1)}-(t_1-qv)_{p,q}^{(\sigma-1)}\right]+2(t_2-t_1)^{\sigma-1}\right)\mathrm{d}_{p,q}v\right. \\
&\left.+\int_{t_1}^{t_2}(t_2-t_1)^{\sigma-1}\mathrm{d}_{p,q}v+t_2-t_1\right\}+(t_2-t_1)\|\vartheta\|_{\infty} L,
\end{aligned}
\tag{3.2}
$$

Since $(\mathrm{t}-qv)_{p,q}^{(\sigma-1)}$ is continuous function with respect to t and v on $A\times A$, it can be inferred that the function is continuously uniform on $A\times A$. Consequently, for any $v\in A$, we can deduce the following:

$$
(t_2-qv)_{p,q}^{(\sigma-1)}-(t_1-qv)_{p,q}^{(\sigma-1)} \rightrightarrows 0 \text{ as} t_1\to t_2.
$$

As t_1 approaches t_2, it can be concluded that the right-hand side of the aforementioned inequality (3.2) tends to zero. This implies that $\mathcal{H}(\Omega)$ exhibits equicontinuity. □

Lemma 3.2. *Suppose that (C1) and (C2) hold. Then, $\mathcal{H}$ is continuous mapping on Ω.*

Proof. Let the sequence $\{z_n\}$ satisfy $z_n\to z\in C(A,\mathcal{Z})$. So, for each $t\in A$ and by Lemma 2.3 (ii), we have

$$
\begin{aligned}
\|(\mathcal{H}z_n)(t)-(\mathcal{H}z)(t)\|=&\left\|\int_0^1\boldsymbol{\phi}(t,qv)\zeta(pv,z_n(pv))\mathrm{d}_{p,q}v+\int_0^t h(pv,z_n(pv))\mathrm{d}_{p,q}v\right. \\
&\left.-\int_0^1\boldsymbol{\phi}(t,qv)\zeta(pv,z(pv))\mathrm{d}_{p,q}v-\int_0^t h(pv,z(pv))\mathrm{d}_{p,q}v\right\| \\
\leq&\int_0^1\|\zeta(pv,z_n(pv))-\zeta(pv,z(pv))\|\mathrm{d}_{p,q}v \\
&+\int_0^t\|h(pv,z_n(pv))-h(pv,z(pv))\|\mathrm{d}_{p,q}v.
\end{aligned}
$$

Given that the functions h and ζ adhere to the Carathéodory conditions, it can be deduced that

$$
\|(\mathcal{H}z_n)(t)-(\mathcal{H}z)(t)\|\to 0 \text{ as } n\to\infty,
$$

which gives us that $(\mathcal{H}z_n)$ converges to $\mathcal{H}z$ on A. However, as demonstrated in Lemma 3.1, the sequence $(\mathcal{H}z_n)$ exhibits equicontinuity. Consequently, $\mathcal{H}$ is continuous because $(\mathcal{H}z_n)$ converges uniformly to $\mathcal{H}z$. □

Theorem 3.1. *Suppose conditions (C1)–(C3) hold. Then, at least one solution exists for the boundary value problem (1.2).*

Proof. First, Lemmas 3.1 and 3.2 confirm the first part of the proof. In order to establish the validity of this theorem, it is only necessary to demonstrate (2.2).

Let K be a subset of the set Ω satisfies the relation $K \subset \overline{conv}\left(\left(\mathcal{H}K\right) \cup \{0\}\right)$. Given that K is both bounded and equicontinuous, it can be deduced the continuity on A of the function $k \to k(t) = \mu(K(t))$. According to Lemma 2.5, Lemma 2.3 (ii), and condition (C3), it can be deduced that for every $t \in A$, we have the following:

$$\begin{aligned} k(t) &\leq \mu(\mathcal{H}(K)(t) \cup \{0\}) \leq \mu(\mathcal{H}(K)(t)) \\ &\leq \int_0^1 \boldsymbol{\phi}(t, qv)\,\theta(s)\,\mu(K(s))\,ds + \int_0^t \vartheta(s)\,\mu(K(s))\,ds \\ &\leq \|k\|_\infty \left(\|\theta\|_\infty + \|\vartheta\|_\infty\right). \end{aligned}$$

Next, we get

$$\|k\|_\infty \left(1 - \|\theta\|_\infty - \|\vartheta\|_\infty\right) \leq 0.$$

Since $\|\theta\|_\infty + \|\vartheta\|_\infty < 1$, then $\|k\|_\infty = 0$, which gives that $k(t) = 0$ for all $t \in A$, hence $K(t)$ is relatively compact in $\mathcal{Z}$. Furthermore, due to the Ascoli-Arzela theorem, K is relatively compact within Ω. In conclusion, based on Theorem 2.1, we can deduce that $\mathcal{H}$ possesses a fixed point, serving as a solution to problem (1.2). □

Now, we give some restrictions of our study.

Theorem 3.2. *Let ζ be a function that fulfills the conditions (C1)–(C3). Then, at least one solution exists for the boundary value problem (1.1).*

Proof. The steps of the proof are similar to the proof of Theorem 3.1 such that $h \equiv 0$. □

On the other hand, problem (1.2) when $p = 1$ has not been studied in the literature. Consequently, we present the ensuing theorem.

Theorem 3.3. *Assume the conditions (C1)–(C3) hold. If $p = 1$. Then, At least one solution exists for the boundary value problem (1.2).*

Proof. The steps of the proof are similar to the proof of Theorem 3.1 when $p = 1$. □

4. An example

In this section, we aim to enhance the reader's understanding of Theorem 3.1 by providing an illustrative example, drawing inspiration from the examples presented in the papers by [21,34,40]. So, we propose the following IBVP of the (p,q)-fractional difference equation

$$\begin{cases} \boldsymbol{D}_{p,q}^{\frac{7}{4}} z(t) + \frac{z\left(p^{\frac{7}{4}} t\right)}{2+\exp\left(p^{\frac{7}{4}} t\right)} = \boldsymbol{D}_{p,q}^{\frac{1}{4}} \left(\frac{z(pt)}{6+\exp(pt)^2}\right), \text{for all } t \in (0,1), \\ z(0) = 0, \quad z(1) = \int_0^1 \frac{z(pv)}{6+\exp(pv)^2} \mathrm{d}_{p,q} v, \end{cases} \tag{4.1}$$

where $\sigma = \frac{7}{4}$, $p = 0.5$, $q = 0.4$, $z = (z_1, z_2, ..., z_i, ...)$, $\zeta = (\zeta_1, \zeta_2, ..., \zeta_i, ...)$, and $h = (h_1, h_2, ..., h_i, ...)$ such that

$$\zeta_i(t, z_i(t)) = \frac{z_i(t)}{2 + e^t} \text{ and } h_i(t, z_i(t)) = \frac{z_i(t)}{6 + e^{t^2}} \text{ for} t \in A,$$

and let the space

$$\mathcal{Z} = l^1 = \left\{ z = (z_1, z_2, ..., z_i, ...) \text{ such that } \sum_{i=1}^{\infty} |z_i| < \infty \right\},$$

equipped with the norm

$$\|z\|_{l^1} = \sum_{i=1}^{\infty} |z_i| .$$

Then, we get

$$|\zeta_i (t, z_i (t))| \leq \frac{1}{2 + e^t} |z_i (t)| \text{, for all } t \in A, \tag{4.2}$$

$$|h_i (t, z_i (t))| \leq \frac{1}{6 + e^{t^2}} |z_i (t)| \text{, for all } t \in A. \tag{4.3}$$

Therefore, it can be deduced that conditions (C1) and (C2) are met by $\theta(\mathrm{t}) = \frac{1}{4+e^{\mathrm{t}}}, \left(\|\theta\|_\infty = \frac{1}{3}\right)$ and $\vartheta(\mathrm{t}) = \frac{1}{6+e^{t^2}}, \left(\|\theta\|_\infty = \frac{1}{7}\right)$.

Furthermore, condition (C3) holds true as

$$\mu (\zeta (t, Q)) \leq \frac{1}{2 + e^t} \mu (Q),$$

$$\mu (h (t, Q)) \leq \frac{1}{6 + e^{t^2}} \mu (Q).$$

are satisfied for every $t \in A$ and any bounded set $Q \subset \mathcal{Z}$. As a result, it can be concluded from Theorem 3.1 that a solution to problem (4.1) does exist and is defined on A.

5. Conclusions

The existence of solutions in Banach spaces for the fractional (p, q)-difference equation with boundary conditions of nonlinear integral type has been proven in this study. Our approach involves the utilization of the Kuratowski m.n. and Mönch's fixed point theorem. The efficacy of our findings have been shown by providing an illustrative example.

The findings derived from this paper are commendable and significant becuase the results of existence for [23] can be obtained by the method used in this research when the function $h \equiv 0$. On the other hand, without (p, q)-calculus, our results are identical to those in paper [34].

One intriguing avenue for future investigation, undoubtedly, would involve contemplating the fractional (p, q)-difference equations of two orders derivatives different σ, β where $\sigma \in (1, 2]$ and $\beta \in (0, \sigma]$. Also, study of problems with Riemann-Stieltjes integral-type boundary conditions and including impulsive effects in the equation would broaden the scope of potential applications too.

Author contributions

Writing–original draft, M.B.M; Conceptualization M.B.M; Funding acquisition M.B.M, N.M.D; Writing–review and editing M.B.M, W.W.M; Project administration M.B.M, N.M.D, W.W.M; Supervision W.W.M.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

There are no conflicts of interest.

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