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Exploiting Waveguide Networks to Calculate Solutions of Partial Differential Equations

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***Abstract* – In this contribution, we present a method to calculate the solution of partial differential equations (PDE) by exploiting metatronic elements within arrays of parallel plate waveguides connected at series junctions. A full physical and mathematical description will be discussed with several examples such as the solution of the equation $\nabla^2 f + 7.2 f = 0$ using an array of 3×3 waveguide junctions arranged in a square lattice.**

I. INTRODUCTION

Analogue computing with waves is an exciting field of research which in recent years has seen a resurgence in scientific interest, motivated by the requirement for faster and more energy efficient computing systems[1]–[3]. In this context metamaterials and metasurfaces[4] have seen significant application due to their ability to arbitrarily control the propagation of electromagnetic (EM) waves in both space and time [5]–[11]. In this regard metamaterial processors have been reported performing mathematical operations such as differentiation, integration, and convolution[12]–[14] directly onto the envelope of the incident signals in both spatial and temporal domains. Other operations such as computing the solutions to ordinary differential equations or integral equations have also been reported[15], [16].

Classical electronics concepts for analogue computing can be traced back from the last century where, for instance, it has been shown that networks of resistors can be exploited to calculate the solutions of partial differential equations (PDE's) such as Laplace's and Poisson's equations[17]. Recently, it has been demonstrated how this technique can be exploited within the EM domain by using a grid-based EM analogue computing system[18] and also by exploiting epsilon-near-zero media and metatronic elements[19] (effectively emulating the performance of traditional circuit elements [19], [20]). Inspired by the interesting features of metatronic elements and the exciting opportunities of analogue computing as new paradigms for future computing systems, in this work we investigate the usage of waveguide networks[21] to solve PDEs via the engineering of metatronic filters [22]. We make use of a network of parallel plate waveguides connected at series junctions and engineer the metatronic elements within them such that the distribution of the magnetic field calculated at the center of the waveguide junctions will corresponds to the solution of a chosen PDE[23]. These results may open new avenues in the field of analogue computing using EM waves, leading to the development of networks capable of solving higher order PDE solutions.

II. RESULTS AND DISCUSSION

Our proposed method to solve PDEs exploits a network of series waveguide junctions arranged in a square lattice, as can be seen in Fig. 1b. Each junction (formed by 4 waveguides each) is connected to its nearest neighbor junctions (one junction left, right, up, and down) by a metatronic filter made from three thin dielectric slabs separated by a distance $\lambda_0/4$ (λ_0 as the wavelength in free space). This is done to mimic metatronic elements such as series inductors or parallel capacitors. Complete details will be provided during the conference showing how these thin dielectrics are carefully designed by choosing their values of relative permittivity and thickness. The filling material of the waveguides between the metatronic elements is vacuum ($\varepsilon_r = 1$, $\mu_r = 1$). An example of the equivalent circuit of a single waveguide junction having four interconnected waveguides and three metatronic elements within each waveguide is shown in Fig. 1a along with the schematic representation of the designed structure in Fig. 1b. Each connected metatronic filter consists of two series impedances Z_{se} and one parallel impedance, Z_{pa}. In this transmission line (TL) representation the magnetic *H*-field at the center of the junction is represented by the current I_0. In this realm, by using Kirchhoff's laws at the junction between waveguides (voltage law) and taking into account the division of the incident currents at the T circuits (formed by the impedances Z_{se}, Z_{pa}, Z_{se}, see Fig. 1a) within each waveguide, one can arrive to the following expression $Z_{pa}(I_1+I_2+I_3+I_4-4I_0)-4Z_{se}I_0 = 0$ with I_1, I_2, I_3 and I_4 referring to the currents circulating in the structure as shown in Fig. 1a. By comparison to the well-known second order finite difference equation in two dimensions[24], it is possible to represent the finite difference step size h as $h^2 = 1/Z_{pa}$, arriving to a PDE of the form $0 = \nabla^2 I - 4Z_{se}I$, where $\nabla^2 I$ represents the Laplace operator applied to the distribution of the currents around the junctions of the metatronic network. As observed, this expression represents the Helmholtz equation for a wave, assuming solutions with a harmonic time dependance only [$I(\mathbf{r},t) = I(\mathbf{r})e^{-i\omega t}$]. In this generalized representation, the parameter h is a complex number as it depends on Z_{pa}. However, this can be solved by applying a normalization such that $h = 1/|Z_{pa}|$ instead, as it will be further explained in the conference.

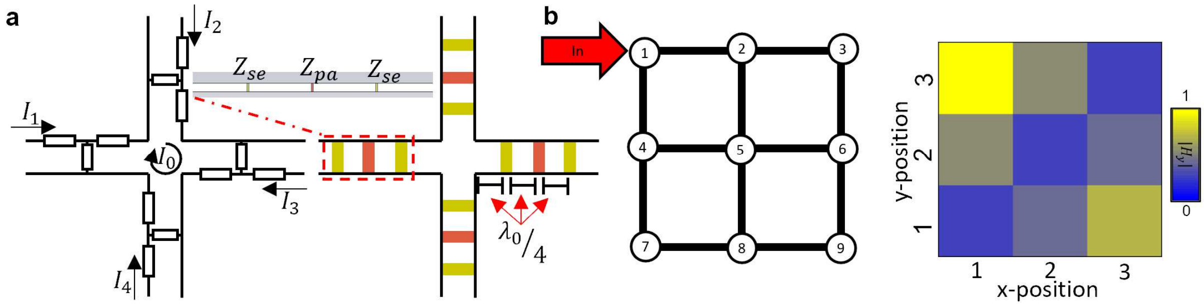

Fig. 1. (a) Schematic representation of a single 4-port waveguide junction with metatronic elements. The left panel shows the circuit representation of the network and the right panel shows the metatronic elements which emulate this circuit. (b) Transfer matrix method results for a 3×3 metatronic network designed to solve a PDE with $h = 0.5$ and $k^2 = 7.2$. The left panel shows a nodal representation of the metatronic network, with junctions represented by circles and the connections between them by a solid black line. The right panel shows the calculated absolute value of the out-of-plane *H*-field at each junction of the 3×3 network when a monochromatic (10 GHz) wave is excited at junction 1 (top-left junction, as labeled in the left panel). These results are normalized with respect to the *H*-field value calculated at junction 1.

To corroborate that the structure performs the solution of PDEs, we present transfer matrix method calculations using TL theory of the magnitude of the *H*-field distribution in a 3×3 waveguide metatronic network with $Z_{pa} = -2iZ_0$ and $Z_{se} = 0.9iZ_0$, where Z_0 is the characteristic impedance of the connecting waveguides. The parallel and series impedances presented here are used to model dielectric slabs with a width w (in the direction of propagation) $w_{pa} = 0.0058\lambda_0$ and $w_{se} = 0.0064\lambda_0$, respectively and relative permittivity values of $\varepsilon_{pa} = 11.9292$ and, $\varepsilon_{se} = 21.4361$, respectively. These values correspond to a PDE equation with $h = 0.5$ and $k^2 = 7.2$ (with $k^2 = -4Z_{se}\ Z_{pa}^*$ and "*" representing the complex conjugate). The 3×3 network is excited with an incident monochromatic wave (10 GHz) from the top left (junction 1 in Fig. 1b). The calculated magnitude of the *H*-field at each of the junctions (1 to 9) are presented in Fig. 1b. As expected, the solution resembles that of an emitting dipole centered at junction one.

III. Conclusion

In this work, a method for the computation of the solution to PDE wave equations by exploiting a square network of waveguides connected at junctions in series has been presented. It has been shown how by tailoring the values of parallel and series impedances modelled via metatronic elements, it is possible to compute the solution to PDE boundary value problems such as the Helmholtz or the Schrodinger equation. This is done by measuring the value of the magnetic field at the center of the waveguide junction. We envision that this work may enable the design of additional processors capable of solving higher order PDE equations based on the finite difference method.

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