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RANGE-DEPENDENT REGULARIZATION OF TRAVEL-TIME TOMOGRAPHY BASED ON THEORETICAL MODES

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Abstract: *Travel time inversion is a fundamental method of Ocean Acoustic Tomography, for the estimation of perturbations in sound speed. By discretizing the watercolumn into a system of layers, the method allows to introduce a system of linear equations, relating a known vector of perturbations in travel time, to an unknown vector of perturbations in sound speed, through the so-called "observation matrix". Inverting the system allows to determine a solution, which estimates the perturbation in sound speed in each layer of the watercolumn. However, in most problems of practical interest, the number of unknowns (i.e. the perturbations in sound speed) is larger that the number of equations (which correspond to the number of delays in travel time), which implies that inverting the system of linear equations can be viewed as an ill-posed problem. The discussion presented in this paper illustrates an approach to the problem of inversion, which is based on the usage of theoretical modes. Further, it is shown that for a range-dependent perturbation in sound speed, which corresponds to a superposition of plane waves, the inversion problem can be regularized (i.e. the system of linear equations can be rewritten in order to deal with more equations than unknowns) by estimating only the amplitudes and phases of the linear waves. Particular examples are given for simulated and real data.*

1 INTRODUCTION

Travel time inversion is a fundamental method of Ocean Acoustic Tomography, that allows to introduce a system of linear equations, relating a known vector of perturbations in travel time (called "travel time delays"), to an unknown vector of perturbations in sound speed, through the so-called "observation matrix", which can be calculated from a system of stable eigenrays. Inverting the system allows to determine the pertubations in sound speed, by estimating it in each layer of the watercolumn. In most cases inverting the system of linear equations can be viewed as an ill-posed problem since the number of unknowns (i.e. the perturbations in sound speed) uses to be larger that the number of equations, which correspond to the number of travel time delays. It can be shown that theoretical modes can be used efficiently to regularize the problem of inversion, by allowing to rewrite the system of linear equations obtaining more equations than unknowns, and that regularization proved to be robust when applied to real acoustic data [1]. The discussion presented in this paper explores further the regularization based on theoretical modes by developing a range-dependent inversion of the sound speed field, for the case of internal plane-wave propagation. In this case one estimates the amplitudes and phases of the plane waves. The proposed method is tested on both simulated and real acoustic data.

2 THEORETICAL BACKGROUND

2.1 Travel time inversion

A detailed description of travel time inversion can be found in [2]. Briefly, the method is based on the fundamental assumption that, for a small change of sound speed, $\delta c(z) = c(z) - c_0(z) \ll c_0(z)$, the perturbation in travel time of an acoustic pulse can be written as

$$\Delta\tau_j = \tau_j - \tau_j^0 = \int_{\Gamma_j} \frac{ds}{c(z)} - \int_{\Gamma_j} \frac{ds}{c_0(z)} \approx -\int_{\Gamma_j} \frac{\delta c(z)}{c_0^2(z)} ds \ , \tag{1}$$

where the integral Eq.(1) is taken along the unperturbed eigenray Γ_j. For a set of T perturbations in travel time, and discretizing the watercolumn into a system composed by L layers, one can relate a vector of delays, $\boldsymbol{\Delta\tau}$, to a vector of perturbations in sound speed, $\boldsymbol{\delta}\mathbf{c}$, through a linear system of equations:

$$\boldsymbol{\Delta\tau} = \mathbf{E}\boldsymbol{\delta}\mathbf{c} + \mathbf{n} \ , \tag{2}$$

where $\boldsymbol{\Delta\tau} = [\ \Delta\tau_1\ ,\ \Delta\tau_2\ ,\ \ldots\ ,\ \Delta\tau_{\mathsf{T}}\ ]^{\mathsf{t}}$, $\boldsymbol{\delta}\mathbf{c} = [\ \delta c_1\ ,\ \delta c_2\ ,\ \ldots\ ,\ \delta c_{\mathsf{L}}\ ]^{\mathsf{t}}$, and each δc_l corresponds to an average of $\delta c(z)$, in the lth layer; $[\ldots]^{\mathsf{t}}$ represents the transpose of vector $[\ldots]$. In Eq.(2) $\mathbf{n}$ accounts for rounding errors, and also for statistic contributions from noise sources. Matrix $\mathbf{E}$, dimension $\mathsf{T} \times \mathsf{L}$, is known as the *Observation Matrix* and can be calculated from unperturbed eigenrays. In most cases of practical interest $\mathsf{L} \gg \mathsf{T}$, so Eq.(2) corresponds to an undetermined system of equations and therefore has an infinite number of solutions. Providing that rank$\mathbf{E} = \mathsf{T}$ one can select the solution that has a minimum norm, which is given by the expression [3]:

$$\boldsymbol{\delta}\mathbf{c}^{\#} = \mathbf{E}^{\mathsf{t}} \left(\mathbf{E}\mathbf{E}^{\mathsf{t}}\right)^{-1} \boldsymbol{\Delta\tau} \ . \tag{3}$$

2.2 Multiple hydrophones

The treatment of the system of independent equations for a set of N hydrophones $\boldsymbol{\Delta\tau}_1 = \mathbf{E}_1\boldsymbol{\delta}\mathbf{c} + \mathbf{n}_1$, $\boldsymbol{\Delta\tau}_2 = \mathbf{E}_2\boldsymbol{\delta}\mathbf{c} + \mathbf{n}_2$, ..., $\boldsymbol{\Delta\tau}_{\mathsf{N}} = \mathbf{E}_{\mathsf{N}}\boldsymbol{\delta}\mathbf{c} + \mathbf{n}_{\mathsf{N}}$, sharing a common vector of perturbations in sound speed, $\boldsymbol{\delta}\mathbf{c}$, can be handled by introducing the following concatenated vectors and matrices:

$$\boldsymbol{\Delta\tau} = \begin{bmatrix} \boldsymbol{\Delta\tau}_1 \\ \boldsymbol{\Delta\tau}_2 \\ \vdots \\ \boldsymbol{\Delta\tau}_{\mathsf{N}} \end{bmatrix} \ , \ \mathbf{E} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_{\mathsf{N}} \end{bmatrix} \ \text{and } \mathbf{n} = \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \vdots \\ \mathbf{n}_{\mathsf{N}} \end{bmatrix} \ , \tag{4}$$

and further applying the solution Eq.(3).

2.3 Theoretical modes

For the hydrostatic linear rotationless case the set of theoretical modes, Ψ_m, can be calculated by solving a Sturm-Liouville problem of the form [4]:

$$\frac{d^2\Psi_m}{dz^2} + \frac{N^2}{C_m^2}\Psi_m = 0 \; + \; \text{Boundary Conditions} \; , \tag{5}$$

where $N^2(z)$ represents the buoyancy frequency. $N^2(z)$ can be calculated from temperature T and salinity S [4], although a satisfactory approximation can be obtained using only temperature [5]. In Eq.(5) C_m represents the propagation velocity of the linear wave associated to each Ψ_m; for a fixed frequency of internal wave propagation, ω, the wavenumber will be given by $k_m = \omega / C_m$. Under homogeneous top and bottom boundary conditions the $\Psi_m(z)$ form an orthogonal basis of functions, i.e. $\langle \Psi_m \, | N^2 | \, \Psi_n \rangle = 0$ for $m \neq n$, where the "inner product" is defined as $\langle f_1 \, | f_2 | \, f_3 \rangle = \int_0^D f_1 f_2 f_3 \; dz$.

2.4 Plave wave propagation

For internal plane wave propagation the range-dependent field of perturbation on sound speed can be represented as

$$\delta c\,(z,r) = \frac{dc_0}{dz} \sum_{m=1}^{M} \Psi_m(z) \left[\alpha_m \sin\left(k_m r \cos\theta\right) + \beta_m \cos\left(k_m r \cos\theta\right) \right] \tag{6}$$

where θ represents the direction of propagation related to the acoustic path (see Fig.1(a)). In Eq.(6) M represents the number of relevant theoretical modes.

2.5 Regularization using theoretical modes

Substituting Eq.(6) into Eq.(1) one can obtain a system of linear equations of the form

$$\boldsymbol{\Delta\tau} = \mathbf{Px} + \mathbf{n} \; , \tag{7}$$

where

$$\mathbf{P} = [\boldsymbol{\mathcal{S}} \;\; \boldsymbol{\mathcal{R}}] \quad , \quad \boldsymbol{\mathcal{S}} = \begin{bmatrix} \boldsymbol{\mathcal{S}}_1^{\mathsf{t}} \\ \boldsymbol{\mathcal{S}}_2^{\mathsf{t}} \\ \vdots \\ \boldsymbol{\mathcal{S}}_{\mathsf{T}}^{\mathsf{t}} \end{bmatrix} \quad , \quad \boldsymbol{\mathcal{R}} = \begin{bmatrix} \boldsymbol{\mathcal{R}}_1^{\mathsf{t}} \\ \boldsymbol{\mathcal{R}}_2^{\mathsf{t}} \\ \vdots \\ \boldsymbol{\mathcal{R}}_{\mathsf{T}}^{\mathsf{t}} \end{bmatrix} \quad ,$$
$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} \quad , \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{bmatrix} \quad , \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix} \; , \tag{8}$$

$$\begin{aligned} \boldsymbol{\mathcal{S}}_j^{\mathsf{t}} &= [\; \mathcal{S}_{j1} \; , \; \mathcal{S}_{j2} \; , \; \ldots \; , \; \mathcal{S}_{jM} \;] \; , \\ \boldsymbol{\mathcal{R}}_j^{\mathsf{t}} &= [\; \mathcal{R}_{j1} \; , \; \mathcal{R}_{j2} \; , \; \ldots \; , \; \mathcal{R}_{jM} \;] \; , \end{aligned} \tag{9}$$

and

$$\begin{bmatrix} \mathcal{S}_{jm} \\ \mathcal{R}_{jm} \end{bmatrix} = -\int_{\Gamma_j} \frac{dc_0}{dz} \frac{\Psi_m(z)}{c_0^2} ds \begin{bmatrix} \sin(k_m r \cos\theta) \\ \cos(k_m r \cos\theta) \end{bmatrix} . \tag{10}$$

Providing that $2M < \mathsf{T}$ one can regularize the original system of equations, Eq.(2), and write the solution of the system Eq.(7) as [3]

$$\mathbf{x}^{\#} = \left(\mathbf{P}^{\mathrm{t}}\mathbf{P}\right)^{-1} \mathbf{P}^{\mathrm{t}} \boldsymbol{\Delta\tau} . \tag{11}$$

The solution given by Eq.(11) allows to determine the modal amplitudes $\mathbf{x}$, made of two components $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which determine uniquely the range-dependent field of sound speed $c(z,r) = c_0(z) + \delta c(z,r)$ through Eq.(6). A multiple hydrophone system can be efficiently handled concatenating once more the corresponding systems of equations.

3 SIMULATIONS

The range-dependent regularization of travel time inversion was tested on simulated data, following the experimental scenarion of the INTIMATE'96 experiment, described in detail in [1]. Theoretical modes were calculated from temperature records acquired during the experiment (see Fig.1(b)). Further, as discussed in that reference analysis of hydrographic data acquired near the Vertical Line Array (hereafter VLA) allow to consider that *in average* $M = 3$. The propagation geometry for simulations correspondend to a source at depth of 90 m, a VLA with three hydrophones (at depths of 35, 105 and 115 m), a transmission distance $R = 5{,}6$ km and a bottom depth $D = 135$ m. Stable eigenrays were calculated using the background sound speed profile $c_0(z)$, derived from CTD records. The range-dependent field $c(z,r) = c_0(z) + \delta c(z,r)$ was calculated for an arbitrary set of realistic modal amplitudes α_m and β_m, and with $\theta = 75°$, which is an estimate of the direction of propapagation of the internal tide in the INTIMATE'96 scenario [5]. Perturbed travel times were calculated using a range-dependent ray-tracing procedure, which used a discretized distribution of $c(z,r)$ along range, with a reduced number of intervals. The first test of inversion using Eq.(11) failed providing unrealistic values of $\mathbf{x}$, which in part can be related to a mismatch between "truth" perturbed arrivals, and the arrivals obtained with the mentioned procedure. Additional tests optimized the solution Eq.(11), using different range discretizations, and calculating the matrix $\mathbf{P}$ over a commom range interval for each discretization. This optimization provided different solutions $\mathbf{x}^{\#}$, most of which were unrealistic, except the one that corresponded to the reliable estimation of $\mathbf{x}$.

4 APPLICATION TO REAL DATA

Tests with real data were based on a particular set of simultaneous data on both acoustic records and hydrographic measurements taken near the acoustic source and the VLA. Due to synchronization problems during the experiment absolute arrivals were not available directly. The lack of synchronization could be compensated through an accurate estimation of waveguide geometry, and by developing an accurate match of the inverse solution with direct observations taken near the VLA, i.e. by minimizing $\left|\left|\boldsymbol{\delta}\mathbf{c}^{\#} - \boldsymbol{\delta}\mathbf{c}\right|\right|$, where $\boldsymbol{\delta}\mathbf{c}^{\#}$ represented a range-independent estimate of the truth solution, $\boldsymbol{\delta}\mathbf{c}$ (see [1]).

This matching was repeated once more for the set of simultaneous data, but based on range dependent regularization and taking $\theta = 75°$. As indicated by simulations different range discretizations allowed to optimize the matching. However, it was also noticed that the accuracy could be significantly improved by taking $M = 4$. The optimized match can be seen on Fig.2(a). In fact it corresponds to an accurate estimation of $\boldsymbol{\beta}$, but not of $\boldsymbol{\alpha}$. Additional tests of optimization for slight variations of θ around 75° provided unrealistic estimates of both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Additionally, a realistic estimate of $\boldsymbol{\alpha}$ was found by optimizing once more the estimate

$$\boldsymbol{\alpha}^{\#} = \left(\boldsymbol{\mathcal{S}}^{\mathrm{t}}\boldsymbol{\mathcal{S}}\right)^{-1}\boldsymbol{\mathcal{S}}^{\mathrm{t}}\left(\boldsymbol{\Delta\tau} - \boldsymbol{\mathcal{R}}\boldsymbol{\beta}^{\#}\right) \ , \qquad (12)$$

for different range discretizations, and choosing the estimate with a minimum norm. The expected profile $\delta c(z, R)$, calculated using $\boldsymbol{\alpha}^{\#}$ and $\boldsymbol{\beta}^{\#}$, can be seen on Fig.2(b), and reveals a good agreement with the direct estimation of the sound speed perturbation, calculated from a temperature record acquired near the position of the acoustic source.

5 CONCLUSIONS

The feasibility of range dependent regularization based on theoretical modes was tested on simulated and real data for the case of internal plane-wave propagation in a shallow water environment. Inversion results proved to be accurate, and the inversion procedure was found to be robust to the presence of noise in real data, and able to resolve high order theoretical modes present in the sound speed field. However, the accurate numerical evaluation of the range-dependent matrix $\mathbf{P}$ still remains an open question, and the inversion procedure can be only applied to the propagation of internal plane-waves, at a fixed direction, across the scenario of acoustic propagation.

References

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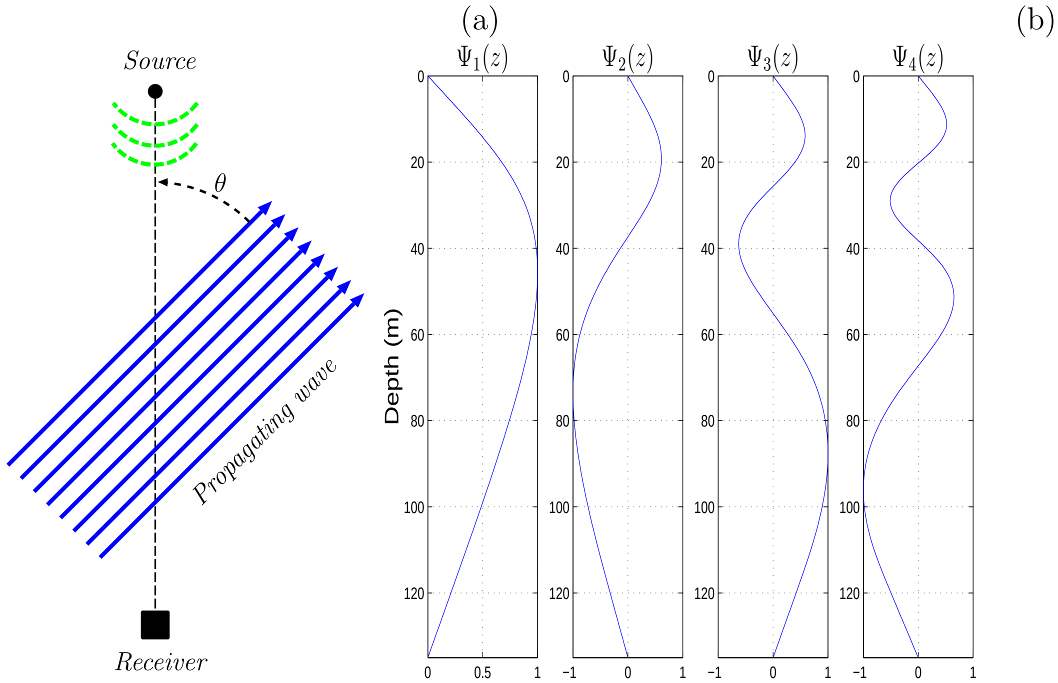

Figure 1: (a) Propagating direction of the linear internal wave related to the acoustic path; (b) theoretical modes.

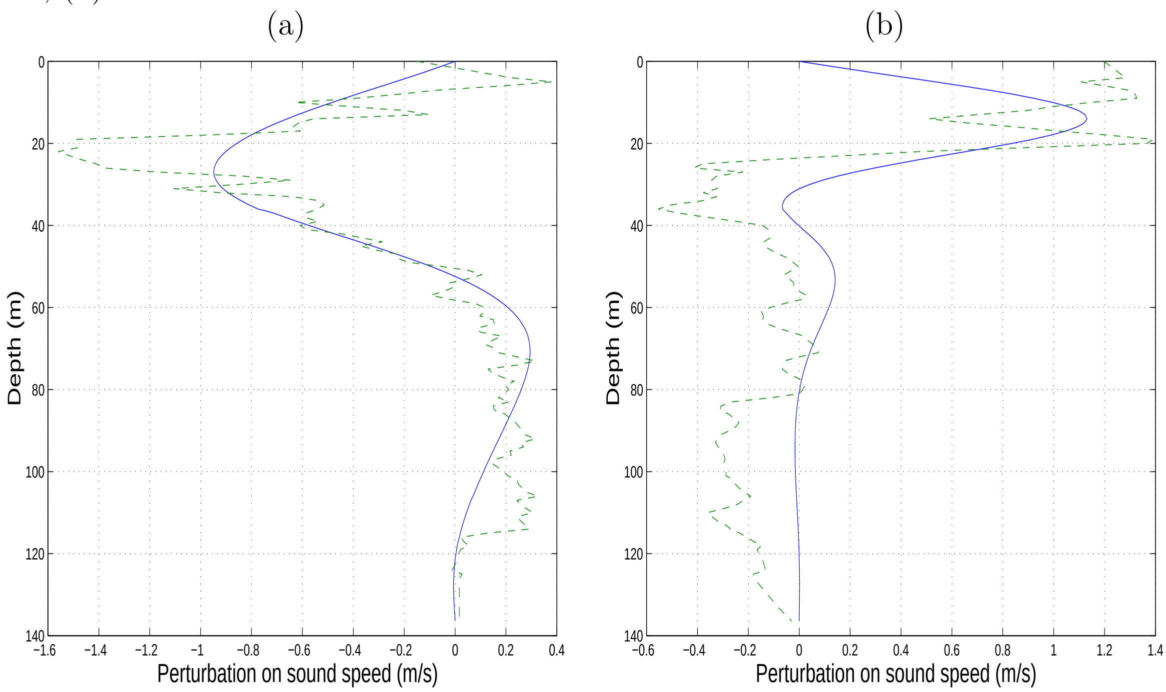

Figure 2: *(a)* Best match of sound speed perturbations near the Vertical Line Arrray (VLA); *(b)* extrapolated profile of sound speed perturbations near the position of the acoustic source. In both cases the continues line corresponds to the estimated value, while the dashed line represents direct measurements.