n	Number of triangles	Is the number of triangles a multiple of *n*?
3	1	NO
4	8	yes
5	35	yes
6	110	NO
7	287	yes
8	632	yes
9	1302	NO
10	2400	yes
11	4257	yes
12	6956	NO
13	11 297	yes
14	17 234	yes
15	25 935	yes
16	37 424	yes
17	53 516	yes
18	73 404	yes
19	101 745	yes
20	136 200	yes
21	181 279	NO

Table 1. The number of triangles in a regular n-gon in which all the diagonals are drawn; situations in which the number of triangles is not a multiple of the number of sides are shaded [1].

enough. There is no reason to believe that the results referred to would be of more use to a mathematics teacher than to any other user of mathematics. By contrast, the peripheral mathematical knowledge that I am seeking to describe lies on the verges of this main highway, yet is no less important for that.

Shulman (1987) argued that pedagogical content knowledge:

> represents a blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (p. 8)

It is perfectly possible to agree with Shulman's final sentence while also asserting that differing subject-matter knowledge may also be an important distinguishing feature.

Note

[1] See the On-Line Encyclopedia of Integer Sequences (http://oeis.org), sequence A006600.

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Teachers' advanced mathematical knowledge for solving mathematics teaching challenges: a response to Zazkis and Mamolo

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Like Zazkis and Mamolo (2011), we uphold the premise that a solid knowledge of advanced mathematics is needed for effective teaching of mathematics. With that in mind, we are interested in discussing the nature of horizon content knowledge, as used within the mathematical knowledge for teaching framework (Ball, Thames & Phelps, 2008).

Mathematical knowledge for teaching (MKT) continues to generate a huge number of papers and all kinds of praise and criticism in scientific settings. Far from adding to this work, our aim is to rescue the concept of horizon content knowledge and re-conceptualise it. We wish to highlight a fundamental premise underlying the MKT framework: teachers' mathematical knowledge belongs to their professional knowledge, and thus has to do with, and cannot be separated from, the teaching challenges that they approach in their practice (Stylanides & Stylanides, 2010). Our critique of Zazkis and Mamolo's paper is much more in terms of their assumptions about the nature of the mathematical knowledge that elementary and secondary teachers need, rather than in terms of their conceptualization of knowledge at the mathematical horizon.

Zazkis and Mamolo use several examples to illustrate how certain knowledge of what they consider to be advanced mathematics is used by teachers to deal with classroom situations. In the first one, the teacher asks the students to calculate the number of triangles in a regular convex pentagon with all diagonals drawn in. Some knowledge of symmetry helps her to see that the number of triangles has to be a multiple of five and to solve the mathematical problem. With this solution in mind, she directs her pupils to solve the problem using symmetries. It is clear that her advanced knowledge on this topic permits her to deal with a teaching situation in a very elegant way. This example reinforces the premise that advanced mathematics is the best - even the essential - background for teaching mathematics. However, after reading Zazkis and Mamolo's description of the situation, we doubt whether this way of using advanced knowledge, and probably the way in which that knowledge was acquired, allows teachers to build on students' knowledge or to interpret alternative solution paths implicit in students' answers. Mathematical problems like that of counting triangles behave very differently in a pure mathematical setting than in an educational context. They become much more complex in an educational context because, among other things, they necessarily involve the mathematical reasoning of the people we have the responsibility to teach.

There was probably no way for these 8-9 year old students to understand why the teacher directed them to identify different kinds of triangles, and then look for five triangles of each kind. Moreover, this approach may reinforce the preconception that problem solving necessarily calls for some brilliant idea, without which the solution remains utterly unattainable.

About the kind of mathematical knowledge that teachers need

Zazkis and Mamolo lead us to conclude that advanced mathematical knowledge is a necessary tool for the teachers to solve, in the classroom, the problems they pose. We certainly agree, but to us it feels like another case of those existence theorems that leave us longing for an explicit construction procedure. We maintain that another perspective is possible, one that also considers, and even advances, the knowledge of the mathematics education community. Let us go back again to one of the examples used in the paper: in Example 3 they describe how the teacher's knowledge about group theory sparks her insight to interpret several confusions and errors concerning the reciprocal and the inverse of a function in terms of a misgeneralization of previous work with negative exponents. Zazkis and Mamolo's description of the teaching situation speaks very well about this (fictional) teacher, and also about the advanced mathematical knowledge the teacher seems to have, but very badly about her mathematics education teachers. Students' confusion about $1/f(x)$ and $f^{-1}(x)$ is well known and the teacher should have heard about it in any course about teaching analysis. The interesting question for us, which is again a mathematics education problem, is what kind of solid education in analysis and group theory the teacher should have received in order to avoid the genesis of this misunderstanding. Perhaps she would not have been surprised about her students' confusion if she had been guided to reflect on the structure of the set of functions under composition immediately after the study of the multiplication of functions, trying to understand why the properties of multiplication in **Q** or **R** do not hold for general functions. In any case, giving advanced mathematical knowledge to teachers without taking into account its relevance for teaching practice is like providing a carpenter with a new, unknown tool without any information about how it can facilitate her work. Surely, with observation and reflection she will be able to elucidate some aspects of its uses and possible potentialities, but her professional problems are different.

Moreover, advanced mathematical knowledge is not meant to be directly applied in teaching situations, but instead is an essential ingredient for a deep understanding of basic mathematics, to an extent not usually covered in the syllabus of many mathematics faculties. To better explain what we intend to say, we use an example drawn from our own research, in which students had just started a unit on equivalent fractions:

> Mr. Paulino explains the idea of equivalent fractions. He takes a piece of paper, folds it twice in half and colors one of the rectangles obtained. He unfolds it and says: "We have colored one quarter of the paper". The students nod patiently. Afterwards, Mr Paulino folds the same piece of paper three times and, when unfolding it, he says: "The fraction colored is now two eighths. One quarter and two eighths are equivalent fractions because they represent the same quantity." He writes on the board:
>
> 
>
>
> and says: "Notice that 1 times 8 is 8, and 4 times 2 is 8. One quarter and two eighths are equivalent fractions because their crossed product is equal." He continues by saying: "The second fraction is obtained by multiplying both numerator and denominator of the first one by 2."

By folding a piece of paper, he implicitly defines a fraction as a part of a whole. Immediately afterwards, he refers to a fraction as the division between two numbers when he asserts that equivalent fractions represent the same quantity. The teacher knows different ways to define a fraction from his university studies and, in this point, we agree with Zazkis and Mamolo, but we consider that deeper reflection is needed.

There is an intraconceptual connection (inner horizon, using Zazkis and Mamolo's terminology) between these two meanings of a fraction which is not trivial for the students. This reflection is crucial for the mathematics teacher and, perhaps, not so much for others using mathematics professionally. The connection is not made explicit by the teacher, who freely moves between these two meanings and leaves the students to their own devices in the process of giving them coherence. Moreover, the definition of equivalent fractions is supposed to be generalised from one particular case to every pair of equivalent fractions. Immediately afterwards, the teacher writes down the same fraction and reduces to an observation what is usually taken as the definition of the equivalence relation in the field of fractions of **Z**: two ordered pairs of integers (a, b) and (c, d), with positive b and d, are equivalent if $a \cdot d = b \cdot c$. It is introduced as an almost mnemonic rule and it does not connect with the meaning of a fraction as a part of a whole. The "rule" is meaningful only once the students are familiar with the multiplication of fractions.

Moreover, the teacher's introduction to equivalent fractions shown in this episode ends by explaining a procedure to obtain equivalent fractions, namely, by multiplying both the numerator and denominator of the first one by 2. This procedure of generating equivalent fractions is normally presented using only integer multiples and produces a foundational misunderstanding for the students: they assume equivalent fractions to be characterized by one of them being the result of multiplying/dividing the numerator and denominator of the other by the same integer, which is not the most general operation possible (for instance, 4/6 and 6/9 are equivalent). If this is considered advanced mathematical knowledge, it is certainly not the focus of general university mathematics.

About the theoretical approach to horizon content knowledge

The notion of horizon content knowledge is given by Zazkis and Mamolo in terms of the application of the notion of "advanced mathematical knowledge", which corresponds to the "knowledge of the subject matter acquired during undergraduate studies at colleges or universities" (Zazkis & Leikin, 2010 p. 264). The notion is therefore grounded in the power of an institution and those who work there. This kind of approach leaves very little space for a deep intellectual debate about how we can understand the problems of mathematics education. We emphasize that our professional task of teaching mathematics to primary and secondary students, as well as to future elementary and secondary school teachers, requires a much broader perspective on the nature of knowledge.

We mentioned at the beginning of this communication that our purpose is to explore the conceptualization of horizon content knowledge. Zazkis and Mamolo's description in terms of inner and outer horizon is very stimulating and permits us to refine our own approach, which conceptualizes horizon content knowledge in terms of connections between mathematical concepts and ideas, grounded in the coherence of mathematics, in which all concepts and ideas are precisely defined and logically interwoven.

Mathematical content knowledge cannot be solid without connections, and this leads us to think about horizon content knowledge as a key necessary prerequisite of mathematical knowledge for teaching. However, after analyzing Zazkis and Mamolo's paper, we have the feeling that they articulate all their reflection around the premise that mathematical teaching problems, and thus theoretical outcomes in the field of mathematics education, should be subordinated to the problem of teachers' learning of advanced mathematics. We have focused our response on discussion of this aspect, emphasizing the need for teachers to construct deep knowledge of the connections within mathematical content as a basis to enhance students' learning of mathematical structure (Vale, McAndrew & Krishnan, 2011). We hope to have further opportunities to think together about the conceptualization of horizon content knowledge, and thus on its impact on practice and training.

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Apologies to Marc Schäfer, whose name was mis-spelt in several places in the *Communications* section of the last issue.

The selected quotations in this issue commemorate the life and work of Martin Hughes (1949-2011). *Children and Number,* originally published in 1986, was reprinted at least twelve times.
