A classical approach to TQFT's*

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Abstract

We present a general framework for TQFT and related constructions using the language of monoidal categories. We construct a topological category $\mathcal{C}$ and an algebraic category $\mathcal{D}$, both monoidal, and a TQFT functor is

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then defined as a certain type of monoidal functor from $\mathcal{C}$ to $\mathcal{D}$. In contrast with the cobordism approach, this formulation of TQFT is closer in spirit to the classical functors of algebraic topology, like homology. The fundamental operation of gluing is incorporated at the level of the morphisms in the topological category through the notion of a gluing morphism, which we define. It allows not only the gluing together of two separate objects, but also the self-gluing of a single object to be treated in the same fashion. As an example of our framework we describe TQFT's for oriented 2D-manifolds, and classify a family of them in terms of a pair of tensors satisfying some relations.

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1. Introduction

The term Topological Quantum Field Theory, or TQFT for short, was introduced by Witten in [1] to describe a class of quantum field theories whose action is diffeomorphism invariant. To capture the common features of a number of examples that appeared, Atiyah, in a seminal article [2], formulated a set of axioms for TQFT that were modelled on similar axioms for conformal field theory, due to Segal [3]. A $(d+1)$-dimensional TQFT assigns to every closed oriented d-dimensional manifold A a finite-dimensional vector space V_A, and to every $(d+1)$-dimensional oriented manifold with boundary X an element Z_X of the vector space assigned to its boundary, subject to certain rules. The notion of TQFT has had a pervasive influence in several areas of mathematics, in particular in low-dimensional topology. Amongst the many constructions coming out of TQFT, we would single out the 3-manifold invariants due to Witten [4] and Reshetikhin-Turaev [5], and Turaev-Viro [6], the combinatorial formula for the signature of 4-manifolds due to Crane-Yetter [7], and the Dijkgraaf-Witten invariants for manifolds of any dimension, coming from a discretized path integral for gauge theories with finite group [8]. For a review of TQFT aimed at mathematicians, see [9].

As recognized in Atiyah's article, there are many possible modifications of the basic axioms. Indeed the Witten invariant from [4] does not fit into the axioms, since it is a combination of a 3-manifold invariant and an invariant for an embedded 1-manifold. Representations of braids [10] and tangles [11] also fit loosely into the TQFT framework, with the difference that they are trivial as 1-dimensional manifolds, but non-trivial due to their embedding in 3D space. Generalizations arise when one allows the manifolds to have extra geometrical structures. An example is Homotopy Quantum Field Theory (HQFT) [12, 13] (see also [14] and [15]), where the manifolds come equipped with a map to a fixed target space.

Turning to the algebraic side, many interesting ideas have appeared about the nature of the vector spaces and elements to be assigned to the manifolds. These ideas constitute, in a sense, extra structure, similar to the possible extra structure on the topological side considered above. Whilst not wishing to summarize all constructions, we would like to mention as examples, representations of the quantum group $SL(2)_q$ at roots of unity for 3-dimensional TQFT's [5], spherical categories for 3-dimensional TQFT's [16], Hopf categories [17, 18] and spherical 2-categories [19, 20] for 4-dimensional TQFT's, constructions coming from operator algebras [21], and the very interesting higher-dimensional algebra programme [22, 23] in which notions of higher algebra (n-categories) enter simultaneously on the topological and algebraic side.

To complete this brief survey of variations of TQFT, there are cases where both the topological and algebraic side come from geometry, namely parallel transport for vector bundles with connection. Here the underlying structure is a vector bundle (E, π, M) with connection, and the 1-dimensional TQFT assigns to each point a of the base space M its fibre $\pi^{-1}(a)$, and to each path from a to b the parallel transport operator from the fibre over a to the fibre over b, regarded as an element of $\pi^{-1}(a)^* \otimes \pi^{-1}(b)$. This is essentially the functorial viewpoint of connections given in [3]. An interpretation in terms of HQFT was recently given for parallel transport for gerbes with connection [24].

As stated earlier, in his article on the axioms of TQFT Atiyah already envisaged various modifications of the axioms to enlarge the scope, so it was natural that mathematicians would attempt to formulate the theory in a more general fashion. There are two such general definitions of TQFT that we would like to mention in this context, due to Quinn [25, 26] and Turaev [27]. Quinn achieved generality by giving a very general definition of a "domain category", endowed with a collection of structures which are abstractions of topological notions, such as boundary, cylinder or gluing. Turaev achieved generality by defining a "space structure" on a topological space, which includes as special cases a choice of orientation, a differentiable structure or a structure of a CW-complex. Both definitions are rather abstract, and both are in the so-called cobordism framework, which has some limitations as we now explain.

The cobordism approach to TQFT's arises by taking an algebraic idea and transporting it to the topological side. Suppose the manifold X has an in-boundary A_1^- and an out-boundary A_2^+, like the starting and end point of the curve on M in the parallel transport example mentioned above. To this boundary one assigns the tensor product $V_{A_1^-} \otimes V_{A_2^+}$, which is assumed to be isomorphic to $V^*_{A_1^+} \otimes V_{A_2^+}$. Thus Z_X can be regarded as a linear map from $V_{A_1^+}$ to $V_{A_2^+}$, i.e. a morphism in the algebraic category. In the cobordism approach, X is then also taken to be a morphism from A_1 to A_2, in a category on the topological side called the cobordism category. More precisely, appropriate equivalence class of manifolds X constitute the morphisms in the cobordism category, so as to ensure the associativity of composition and the existence of identity morphisms. An advantage of this shift of viewpoint is that the TQFT assignments become functorial both at the level of X's and A's, and thus a TQFT becomes a functor from the cobordism category to the category of vector spaces. The cobordism viewpoint is very natural for 1-dimensional TQFT's describing braids and tangles (these have been termed "embedded TQFT's" in [28]), since morphisms are intuitively associated to 1-dimensional topology, but is less natural, or rather, has to be amended to higher cobordism categories in the higher-dimensional algebra setting, when one is dealing with dimensions greater than one. Despite the attractive functorial feature of the cobordism approach, a disadvantage is that making the X's into morphisms on the topological side leaves no room for other morphisms between the A's, in particular isomorphisms between them. (In the context of HQFT, Rodrigues in [15] showed a nice way to get round this problem, by means of some quotient constructions, allowing both cobordisms and isomorphisms to appear as topological morphisms on the same footing.) A second disadvantage of the cobordism approach is that, although the gluing together of two manifolds X_1 and X_2 along a shared boundary comes in naturally as the composition of topological morphisms, the notion of self-gluing of a single manifold X by identifying two of its boundary components has no such natural interpretation, and has to be dealt with in theorems. A third disadvantage of the cobordism approach is that the algebraic notion of duality, which was necessary to pass from elements of a linear space to linear maps, requires a rather subtle treatment when it is transported to the topological side - see [15] for a definition in the context of HQFT and further references on the subject of duality in categories. Finally, a disadvantage, to our mind, of the cobordism approach is that the same topological object acquires a plethora of algebraic guises. Thus, for example, disregarding orientations, the cylinder can be viewed in three different ways as a morphism from the circle to the circle, or a morphism from two circles to the empty set, or a morphism from the empty set to two circles.

Thus we were motivated to return to Atiyah's original formulation where Z_X is simply an element of V_A. Now Atiyah's axioms in this form do not describe a functor, although they do contain functorial ingredients. One of our main goals was to define TQFT's as functors, analogously to the classical functors of algebraic topology such as homology and homotopy (hence our slightly tongue-in-cheek title). The approach we found worked best was to take objects on the topological side to be essentially pairs (X, A) (together with an "inclusion" morphism from A to X), and to allow only a very restricted class of morphisms between these objects, namely isomorphisms and gluing morphisms, where the latter are essentially morphisms from an object to a copy of the object that results after gluing some components of A together. A TQFT is then a functor from this category to an algebraic category whose objects are also pairs, essentially a vector space and an element of that space. Furthermore, the TQFT functors preserve structures, namely a monoidal structure (which is essentially the disjoint union on the topological side) as well as an endofunctor (which is an abstraction of the operation of changing orientation on the topological side). Thus we use the language of monoidal categories and monoidal functors. Generality in describing a wide range of topological situations is achieved by building up the topological category, in a series of steps, out of a fairly arbitrary topological starting category.

Our approach is rather detailed, since we wanted to arrive at a very concrete and explicit formulation, allowing calculations to be performed efficiently. The insight that is gained, after working through the formalism, is that TQFT is in essence a "topological tensor calculus", where manifolds get tensors assigned to them, with the number of indices equal to the number of connected components of A, and where gluing manifolds together corresponds to contracting the indices. This picture is made explicit in our illustrative example of 2-dimensional TQFT's, where we obtain a result precisely of this nature: all 2-manifolds with boundary are represented by a tensor built up from just two basic tensors, one with one index and one with three indices, representing the disk and the pair-of-pants, respectively. This result should be contrasted with Abrams' [29] characterization of 2-dimensional TQFT's in terms of Frobenius algebras, using the cobordism approach - see also earlier work by Sawin [30]. There the topological category (2-Cobord) is generated by five surfaces, two different disks, two different pairs-of-pants and one cylinder. The first four surfaces correspond to the (co)unit and (co)multiplication of the Frobenius algebra structure on the vector space assigned to the circle, which is again an example of topological objects acquiring multiple algebraic guises. Our result for 2-dimensional TQFT's is much closer in spirit to that of R. Lawrence [31], who gets a classification in terms of three tensors, corresponding to the disk, pair-of-pants and cylinder, respectively.

Rather than give an outline of the whole construction here, we refer to the introduction to each section for a description of the various stages involved. Since the whole setting is that of monoidal categories, we have included some relevant definitions in the introduction - see below. The general theory is interspersed with a detailed presentation of an example, namely 2-dimensional TQFT's, which are characterized in Theorems 4.6 and 5.6. A preliminary version of this work appeared in [32] and it was the subject of the Ph.D. thesis [33]. We are very grateful to Louis Crane for his suggestions and comments on an earlier version of this article.

To complete this introduction we recall some definitions concerning monoidal categories. For the background material on monoidal categories we refer the reader to the textbooks [34, pp. 161-170] and [35, pp. 281-288]:

Definition 1.1. *A* monoidal category *is a sextuple* $(\mathbf{C}, \otimes, I, a, l, r)$ *consisting of a category* $\mathbf{C}$, *a bifunctor* $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, *called the monoidal product, an object* $I \in \mathrm{Ob}(\mathbf{C})$, *called the unit object, and natural isomorphisms* $a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ *(associativity),* $l : I \otimes - \to \mathrm{Id}_{\mathbf{C}}$ *(left unit) and* $r : - \otimes I \to \mathrm{Id}_{\mathbf{C}}$ *(right unit), such that for all objects* A, B, C *and* D *the following diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C} \otimes id_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{A,B,C\otimes D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A \otimes a_{B,C,D}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "a_{A,I,B}"] \arrow[dr, "r_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes l_B"] \\
& A \otimes B &
\end{tikzcd}$$

commute. A monoidal category is often denoted simply by $\mathbf{C}$.
A monoidal category is said to be strict, *if the associativity, left and right unit are all identities of the category. A strict monoidal category is denoted by* $(\mathbf{C}, \otimes, I)$ *or* $\mathbf{C}$.

Remark 1.2. *According to a result by Mac Lane, any monoidal category is monoidal equivalent to a strict monoidal category. (See [34, pp. 256-257] for the definition of monoidal equivalence and the proof. See also [35, p. 291].)*
In the constructions of Sections 2-5 below, only strict monoidal categories will appear.

Definition 1.3. *Let* $(\mathbf{C}, \otimes, I, a, l, r)$ *and* $(\mathbf{D}, \otimes', I', a', l', r')$ *be monoidal categories. A* monoidal functor *from* $\mathbf{C}$ *to* $\mathbf{D}$ *is a triple* $(F, \varphi_2, \varphi_0)$*, which consists of a functor* $F : \mathbf{C} \to \mathbf{D}$*, a natural transformation* $\varphi_2 : F(-) \otimes' F(-) \to F(- \otimes -)$ *and a morphism* $\varphi_0 : I' \to F(I)$*, such that the following diagrams*

$$\begin{tikzcd}
(F(A) \otimes' F(B)) \otimes' F(C) \arrow[r, "a'_{F(A),F(B),F(C)}"] \arrow[d, "\varphi_{2(A,B)} \otimes' id_{F(C)}"'] & F(A) \otimes' (F(B) \otimes' F(C)) \arrow[d, "id_{F(A)} \otimes' \varphi_{2(B,C)}"] \\
F(A \otimes B) \otimes' F(C) \arrow[d, "\varphi_{2(A \otimes B,C)}"'] & F(A) \otimes' F(B \otimes C) \arrow[d, "\varphi_{2(A,B \otimes C)}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F(a_{A,B,C})"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

and

$$\begin{tikzcd}
I' \otimes' F(A) \arrow[r, "l'_{F(A)}"] \arrow[d, "\varphi_0 \otimes' id_{F(A)}"'] & F(A) \arrow[d, "F(l_A)"] \\
F(I) \otimes' F(A) \arrow[r, "\varphi_{2(I,A)}"'] & F(I \otimes A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(A) \otimes' I' \arrow[r, "r'_{F(A)}"] \arrow[d, "id_{F(A)} \otimes' \varphi_0"'] & F(A) \arrow[d, "F(r_A)"] \\
F(A) \otimes' F(I) \arrow[r, "\varphi_{2(A,I)}"'] & F(A \otimes I)
\end{tikzcd}$$

commute. We will frequently write a monoidal functor $(F, \varphi_2, \varphi_0)$ *simply as* F.

The monoidal functor is said to be strong, *if the natural transformation* φ_2 *is a natural isomorphism, and the morphism* φ_0 *is an isomorphism, and is said to be* strict *if they are identities.*

Definition 1.4. *A* monoidal natural transformation $\eta : (F, \varphi_2, \varphi_0) \to (G, \psi_2, \psi_0)$ *is a natural transformation* $\eta : F \to G$ *such that the following diagrams, for all objects* A, B

$$\begin{tikzcd}
F(A) \otimes' F(B) \arrow[r, "\eta_A \otimes' \eta_B"] \arrow[d, "\varphi_{2(A,B)}"'] & G(A) \otimes' G(B) \arrow[d, "\psi_{2(A,B)}"] \\
F(A \otimes B) \arrow[r, "\eta_{A \otimes B}"'] & G(A \otimes B)
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & F(I) \arrow[dd, "\eta_I"] \\
I' \arrow[ur, "\varphi_0"] \arrow[dr, "\psi_0"'] & \\
 & G(I)
\end{tikzcd}$$

commute.

Definition 1.5. *Let* $(\mathbf{C}, \otimes, I, a, l, r)$ *be a monoidal category and* $\tau : \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ *the flip functor, defined for all pairs of objects and morphisms by:*

$$\tau((A, B) \overset{(f,g)}{\to} (A', B')) = (B, A) \overset{(g,f)}{\to} (B', A').$$

A braiding *is a natural isomorphism*

$$c : - \otimes - \to (- \otimes -) \circ \tau,$$

such that, for all triples (A, B, C) *of objects of the category, the following two hexagonal diagrams*

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "c_{(A,B\otimes C)}"] & (B \otimes C) \otimes A \arrow[d, "a_{B,C,A}"] \\
(A \otimes B) \otimes C \arrow[u, "a_{A,B,C}"] \arrow[d, "c_{(A,B)} \otimes id_C"'] & B \otimes (C \otimes A) \\
(B \otimes A) \otimes C \arrow[r, "a_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[u, "id_B \otimes c_{(A,C)}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[rr, "c_{(A\otimes B,C)}"] & & C \otimes (A \otimes B) \arrow[d, "a^{-1}_{C,A,B}"] \\
A \otimes (B \otimes C) \arrow[u, "a^{-1}_{A,B,C}"] \arrow[d, "id_A \otimes c_{(B,C)}"'] & & (C \otimes A) \otimes B \\
A \otimes (C \otimes B) \arrow[rr, "a^{-1}_{A,B,C}"'] & & (A \otimes C) \otimes B \arrow[u, "c_{(A,C)} \otimes id_B"']
\end{tikzcd}$$

commute.

A braided monoidal category $\mathbf{C}$ *is a monoidal category endowed with a braiding, also denoted* $(\mathbf{C}, \otimes, I, a, l, r, c)$, *or* $(\mathbf{C}, \otimes, I, c)$ *if* $\mathbf{C}$ *is strict monoidal.*
A braided monoidal category $(\mathbf{C}, \otimes, I, a, l, r, c)$ *is called a* symmetric monoidal category *if the braiding* c *satisfies:*

$$c_{(B,A)} \circ c_{(A,B)} = \mathrm{id}_{A\otimes B}\,,$$

for all pairs (A, B) *of objects.*

Remark 1.6. *We note that the condition* $c_{(B,A)} \circ c_{(A,B)} = \mathrm{id}_{A\otimes B}$ *is equivalent to* $c_{(A,B)} = c^{-1}_{(B,A)}$, *which implies that the second hexagonal diagram is equal to the first. Therefore a symmetric monoidal category is a monoidal category in which the first hexagonal diagram commutes and the symmetry condition holds.*

Finally, we present the definition of a braided monoidal functor.

Definition 1.7. *Let* $(\mathbf{C}, \otimes, I, a, l, r, c)$ *and* $(\mathbf{D}, \otimes', I', a', l', r', c')$ *be braided monoidal categories. A monoidal functor* $(F, \varphi_2, \varphi_0)$ *from* $\mathbf{C}$ *to* $\mathbf{D}$ *is said to be a* braided monoidal functor *if, for all pairs of objects* (A, B), *the following diagram*

$$\begin{tikzcd}
F(A) \otimes' F(B) \arrow[r, "\varphi_{2(A,B)}"] \arrow[d, "c'_{(F(A),F(B))}"'] & F(A \otimes B) \arrow[d, "F(c_{(A,B)})"] \\
F(B) \otimes' F(A) \arrow[r, "\varphi_{2(B,A)}"'] & F(B \otimes A)
\end{tikzcd}$$

commutes.

A symmetric monoidal functor *is a braided monoidal functor between symmetric monoidal categories.*

2. The topological category

The construction of the topological category will proceed in stages, which we will first describe informally. The starting point is a category which is large enough to contain all the objects one wishes to describe, called the topological starting category. This might be, for instance, the category of all smooth oriented manifolds with boundary. Within this category we then focus attention on a class of "larger objects" and a class of "subobjects", by choosing a class of "inclusion" morphisms whose domains are the subobjects and whose codomains are the larger objects. For instance, the larger objects could be d-dimensional manifolds with boundary, and the subobjects $(d-1)$-dimensional manifolds with empty boundary, with the inclusion morphisms mapping the subobjects into boundary components of the larger objects. The subobjects form a category, called the category of subobjects, with isomorphisms between subobjects as its morphisms. The objects of the topological category itself are triples consisting of a larger object, a subobject and an inclusion map between them. The morphisms between these objects are restricted to be of two types only: isomorphisms and gluing morphisms. The isomorphisms correspond to the respective larger objects and subobjects being isomorphic in a compatible way. The gluing morphisms correspond to gluing larger objects together along one or more pairs of subobjects, and are best described as morphisms from the objects before gluing to a copy of the objects after gluing. The main result that has to be established is that these morphisms close under suitably defined composition. There is an extra piece of structure which plays a crucial role throughout the construction, namely an endofunctor on the topological starting category, which generalizes the operation of changing orientation in the example of oriented manifolds.

We now proceed with the details of the construction.

Definition 2.1. *A* topological starting category *is a triple* $(\mathbf{C}, F, P)$*, where* $\mathbf{C}$ *is a symmetric, strict monoidal category* $(\mathbf{C}, \sqcup, E, c)$*,* F *is a symmetric, strict monoidal forgetful functor from* $\mathbf{C}$ *to* Top *(the category of topological spaces and continuous maps with its standard monoidal structure and braiding) and* P *is a symmetric, strong monoidal endofunctor* (P, π_2, π_0) *on* $\mathbf{C}$*, such that*

$$F \circ P = F \quad \text{and} \quad F(\pi_2) = \mathrm{id}\,.$$

Instead of $(\mathbf{C}, F, P)$ *we will frequently write simply* $\mathbf{C}$*.*

Remark 2.2. *The forgetful functor F justifies the adjective topological. It sends each object X of $\mathbf{C}$ to its underlying topological space $F(X)$, sends each morphism of $\mathbf{C}$ to the underlying continuous map, sends $\sqcup$ to the disjoint union (or topological sum) of topological spaces, and sends E to the empty space $\emptyset$.*
One may think of the objects of $\mathbf{C}$ as topological spaces with some extra structure, e.g. oriented topological manifolds. The endofunctor P acts on the extra structure, e.g. by changing the orientation, but leaves the underlying topological space unchanged. We do not require $P^2 = \mathrm{Id}$, although the only examples we have in mind do have this property.

We will illustrate the constructions in Sections 2-5 with a simple example, which will keep returning.

Example: We take $\mathbf{C}$ to be the category whose objects are 1-dimensional, compact, oriented, topological manifolds without boundary or 2-dimensional, compact, oriented, topological manifolds with boundary[1], together with finite disjoint unions of these objects. We introduce the following symbols and a preferred presentation for some familiar objects:

- C_+ denotes the circle, identified with $\{z \in \mathbb{C} : |z| = 1\}$ with anticlockwise orientation.

- D_+ denotes the disk, identified with $\{z \in \mathbb{C} : |z| \leq 1\}$ with the orientation induced by the standard orientation of the complex plane (given by the choice of coordinates (x, y), where $z = x + iy$).

- A_+ denotes the annulus, identified with $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$ with the orientation induced by the standard orientation of the complex plane.

- P_+ denotes the pair-of-pants (i.e. the three holed sphere or trinion) identified with a disk in the complex plane with two holes removed, with the orientation induced by the standard orientation of the complex plane.

- S_+ denotes the 2-sphere, S^2, identified with the unit sphere in $\mathbb{R}^3$ and with the orientation given by the choice of coordinates (x, y) at $(0, 0, 1)$.

[1] The category of manifolds with boundary is more accurately described in terms of a category whose objects are pairs $(X, \partial X)$, where ∂X is the boundary of the manifold X. In our example, S^2, for instance, is regarded as a manifold with boundary corresponding to the pair $(S^2, \emptyset)$.

- Finally, T_+ denotes the torus $S^1 \times S^1$, identified with $\left\{(e^{i\theta}, e^{i\varphi}) \in \mathbb{C}^2 : 0 \leq \theta, \varphi < 2\pi\right\}$ and with the orientation given by the choice of coordinates (θ, φ).

For each of these objects replacing the "+" suffix by a "−" denotes the same object with the opposite orientation. We will abbreviate a disjoint union of the form $X_+ \sqcup X_- \sqcup \cdots \sqcup X_+$ by $X_{+-\cdots+}$. We will equate $X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$ with $X_1 \times \{1\} \cup X_2 \times \{2\} \cup \cdots \cup X_n \times \{n\}$. The objects:

$$\emptyset,\ C_\pm, D_\pm, P_\pm,$$

play a fundamental role, since, up to isomorphism, all other objects can be obtained from these by disjoint union and gluing, as we shall see later.

The morphisms of $\mathbf{C}$ are orientation-preserving maps between the above objects, for instance $f : C_+ \to C_+$, given by $f(z) = z$, or $f : C_- \to C_+$, given by $f(z) = \overline{z}$. Since the orientation of a manifold with boundary induces an orientation on its boundary, the morphisms of $\mathbf{C}$ include orientation-preserving maps from the oriented circle to oriented 2-dimensional manifolds with boundary whose image is contained in the boundary of the 2-dimensional manifold, such as $f : C_+ \to D_+$, given by $f(z) = z$. The monoidal product $\sqcup$ is the disjoint union, taken to be strictly associative, the unit object E is $\emptyset$, also regarded as strict, i.e. $X \sqcup \emptyset = \emptyset \sqcup X = X$, and the braiding $c_{(X,Y)} : X \sqcup Y \to Y \sqcup X$ is the usual flip map which sends $(x, 1)$ to $(x, 2)$ and $(y, 2)$ to $(y, 1)$. The forgetful functor F maps each oriented manifold to its underlying topological space and each orientation-preserving map to the underlying continuous map. Finally, the endofunctor P reverses the orientation of the objects and leaves the morphisms unchanged in the sense that $P(f)$ and f have the same transformation law, for any morphism f. Thus the endofunctor P is involutive, i.e. $P^2 = \mathrm{Id}$. The endofunctor P is strict in this example, meaning that π_2 and π_0 are identities. Thus, in particular, P leaves the unit object $\emptyset$ unchanged.▲

Having chosen a topological starting category $\mathbf{C}$, our next goal is to introduce a subcategory of $\mathbf{C}$, which we call the category of subobjects of $\mathbf{C}$. The idea is to separate the objects into "larger" objects and "smaller" objects, the latter behaving like the boundary components of the larger objects. For this purpose we first formalize the notion of a subobject following [36].

Definition 2.3. *Let $\mathcal{M}$ be a non-empty class of monomorphisms of $\mathbf{C}$. An $\mathcal{M}$-*subobject *of an object X is a pair (A, m), where $m : A \to X$ belongs to $\mathcal{M}$.*

The class $\mathcal{M}$ plays the role of a class of "inclusion" maps from "smaller", or "boundary", objects into "larger" objects.

In Definition 2.5 we will be formulating properties for an appropriate class $\mathcal{M}$, but first we need the following concepts:

Definition 2.4. *Let $(\mathbf{C}, F, P)$ be a topological starting category.*

(i) *An object A of $\mathbf{C}$ is said to be* irreducible, *if $F(A)$ is non-empty and connected.*

(ii) *An object of $\mathbf{C}$ is said to be* factorized, *if it is of the form $P^n(E)$, with $n \in \mathbb{N}$ (where $P^0 := \mathrm{Id}$) or a finite monoidal product of irreducible objects.*

Definition 2.5. *We say that $\mathcal{M}$ of Definition 2.3 is an* appropriate *class if it satisfies the following properties:*

1. For any $m \in \mathcal{M}$, $F(m)$ is an embedding in Top, i.e. a homeomorphism onto its image.

2. $\mathcal{M}$ is closed under the monoidal product.

3. $\mathcal{M}$ is closed under the endofunctor P.

4. The domain of any $m \in \mathcal{M}$ is either $P^n(E)$ with $n \in \mathbb{N}$ or any object that can be generated from irreducible objects by taking finite monoidal products and applying P.

5. (isomorphism-closure property)
   If $m \in \mathcal{M}$ and $f \in \mathrm{Iso}(\mathbf{C})$[2] such that $m \circ f$ exists, then $m \circ f \in \mathcal{M}$.

6. (subdivision property)
   Let $(A_i)_{i \in I}$ be a family of irreducible objects, where I is a finite ordered index set. We denote by $A_I := \sqcup_{i \in I} A_i$ the corresponding factorized object and set $A_\emptyset := E$.

   a) For any $\mathcal{M}$-subobject (A_I, m) of X and any proper ordered subset J of I there exists a morphism $s_{J,I} : A_J \to A_I$ (independent of m) such that $m_J := m \circ s_{J,I}$ belongs to $\mathcal{M}$, i.e. the following diagram

[2]The class $\mathrm{Iso}(\mathbf{C})$ is the class of all isomorphisms of $\mathbf{C}$.

$$
\begin{tikzcd}
A_J \arrow[rr, "m_J"] \arrow[dr, "s_{J,I}"'] & & X \\
& A_I \arrow[ur, "m"'] &
\end{tikzcd}
$$

commutes.

Under the forgetful functor the morphisms $F(s_{J,I}) : F(A)_J \to F(A)_I$ are the canonical monomorphisms.

b) For K a proper ordered subset of J one has:

$$s_{J,I} \circ s_{K,J} = s_{K,I}.$$

c) For any finite ordered sets $J \subsetneq I$, $L \subsetneq K$, with I and K disjoint, one has:

$$s_{J,I} \sqcup s_{L,K} = s_{JL,IK},$$

where IK denotes the finite ordered set consisting of the elements of I followed by the elements of K.

d) For any $\mathcal{M}$-subobjects (A_I, m) and $(A'_{I'}, m')$ with $\mathrm{card}(I') = \mathrm{card}(I)$, isomorphism $\alpha : A_I \to A'_{I'}$, and a proper ordered subset J' of I', there exists a proper ordered subset J of I with $\mathrm{card}(J) = \mathrm{card}(J')$ and an isomorphism $\alpha_{J,J'} := \sqcup_{i \in J} \alpha_i : A_J \to A'_{J'}$ (independent of m and m'), such that the following diagram

$$
\begin{tikzcd}
A_I \arrow[r, "\alpha"] & A'_{I'} \\
A_J \arrow[u, "s_{J,I}"] \arrow[r, "\alpha_{J,J'}"'] & A'_{J'} \arrow[u, "s_{J',I'}"']
\end{tikzcd}
$$

commutes.

Remark 2.6. *The intuitive content of the subdivision property is that when a collection of objects form a subobject of X any subcollection of them also do,*

with the respective elements of $\mathcal{M}$ related in the appropriate way.
The condition d) says that an isomorphism α induces a one-to-one correspondence between connected components of the domain and codomain subobjects, and isomorphisms α_i between corresponding components. Since $\alpha_{J,J'}$ is unique by the commutativity of the diagram in $\mathcal{M}$-6d) and the fact that $s_{J',I'}$ is a monomorphism (which in its turn follows from $\mathcal{M}$-6a)), for $K \subsetneq J$ and $K' \subsetneq J'$ we have, from $\mathcal{M}$-6b), $(\alpha_{J,J'})_{K,K'} = \alpha_{K,K'}$ and we use the simpler expression on the right hand side when it occurs.

Example: We choose $\mathcal{M}$ to be the class of monomorphisms whose domain is either $\emptyset$ or a finite disjoint union of oriented circles and whose codomain is either $\emptyset$ or an oriented surface with boundary, such that each circle in the domain is mapped isomorphically to a boundary component of the codomain surface. Examples of morphisms in $\mathcal{M}$ are the empty maps from $\emptyset$ to $\emptyset$, to $T_\pm$ or to $S_\pm$; the maps $C_\pm \to D_\pm$ sending z to z, and the map $C_{-+} \to A_+$, sending $(z,1)$ to z and $(z,2)$ to $2z$. The class $\mathcal{M}$ clearly satisfies the properties 1)-6) above. The morphisms $s_{J,I}$ in the subdivision property are the canonical monomorphisms, e.g. for any m with domain C_{+-+-} and $I = \{1,2,3,4\}$, $J = \{2,3\}$ one has $s_{J,I} : C_{-+} \to C_{+-+-}$ given by $(z,1) \mapsto (z,2)$ and $(z,2) \mapsto (z,3)$.▲

Definition 2.7. *Given a topological starting category ($\mathbf{C}$,F, P) and an appropriate class of monomorphisms $\mathcal{M}$, we define a symmetric, strict monoidal category $(\mathcal{S}(\mathbf{C}), \sqcup, E, c)$, also written $\mathcal{S}(\mathbf{C})$ for short, as follows:*

a) *the class of objects of $\mathcal{S}(\mathbf{C})$ is the subclass of objects of $\mathbf{C}$ which are the domain of some $m \in \mathcal{M}$,*

b) *the morphisms of $\mathcal{S}(\mathbf{C})$ are the isomorphisms of $\mathbf{C}$ restricted to the objects of $\mathcal{S}(\mathbf{C})$,*

c) *the composition, identity morphisms, monoidal product $\sqcup$, unit object E and braiding c are inherited from $\mathbf{C}$.*

We define a symmetric, strong monoidal endofunctor P on $\mathcal{S}(\mathbf{C})$ as the restriction to $\mathcal{S}(\mathbf{C})$ of P defined on $\mathbf{C}$. The category of subobjects, *is the pair $(\mathcal{S}(\mathbf{C}), P)$, where $\mathcal{S}(\mathbf{C})$ denotes the category with its symmetric, monoidal structure and P denotes the above monoidal endofunctor on $\mathcal{S}(\mathbf{C})$.*

Theorem 2.8. $(\mathcal{S}(\mathbf{C}), \sqcup, E, c)$ *is indeed a symmetric, strict monoidal category, and* P *restricts to a symmetric, strong monoidal endofunctor on* $\mathcal{S}(\mathbf{C})$*.*

Proof. One just has to check that the composition, identity morphisms, monoidal structure and braiding indeed restrict to $\mathcal{S}(\mathbf{C})$. For the composition and identity morphisms this is clear. The monoidal product is defined in $\mathcal{S}(\mathbf{C})$ because of property 2 of $\mathcal{M}$. The unit object E of $\mathbf{C}$ is an object of $\mathcal{S}(\mathbf{C})$, due to property $\mathcal{M}$-6a), since $\mathcal{M}$ is non-empty. The braiding isomorphisms restricted to the objects of $\mathcal{S}(\mathbf{C})$ are morphisms of $\mathcal{S}(\mathbf{C})$ because of property 5 (isomorphism-closure property) of $\mathcal{M}$. The functor P closes on the objects of $\mathcal{S}(\mathbf{C})$ because of $\mathcal{M}$-3), and on the morphisms of $\mathcal{S}(\mathbf{C})$ because, being a functor, P maps isomorphisms to isomorphisms. Thus P defines an endofunctor on $\mathcal{S}(\mathbf{C})$. For any $A, B \in \mathrm{Ob}(\mathcal{S}(\mathbf{C}))$, $P(A) \sqcup P(B)$ and $P(A \sqcup B)$ are objects of $\mathcal{S}(\mathbf{C})$ by $\mathcal{M}$-2) and $\mathcal{M}$-3), and thus $\pi_{2(A,B)}$, which is an isomorphism since P is strong monoidal, is a morphism of $\mathcal{S}(\mathbf{C})$. Likewise π_0 is a morphism of $\mathcal{S}(\mathbf{C})$ and, looking at Definition (1.3), it is clear that (P, π_2, π_0) is strong monoidal on $\mathcal{S}(\mathbf{C})$. It is also clearly symmetric. ■

Remark 2.9. *Using the isomorphism-closure property of* $\mathcal{M}$*, any object of* $\mathcal{S}(\mathbf{C})$ *is isomorphic to a factorized object of* $\mathcal{S}(\mathbf{C})$*, via a string of morphisms which are monoidal products of identity morphisms and morphisms of the form* $P^k(\pi^{-1}_{2(X,X')})$*, with* $k \in \mathbb{N}$*. We will proceed from now on to write all objects of* $\mathcal{S}(\mathbf{C})$ *as if they were factorized and let the necessary isomorphisms to achieve this be understood.*

We will distinguish two special cases of morphisms of $\mathcal{S}(\mathbf{C})$.

Definition 2.10. *A morphism of* $\mathcal{S}(\mathbf{C})$ *is called:*

1. a permuting isomorphism, *if it is of the form*

$$p_\sigma : A_1 \sqcup \cdots \sqcup A_n \to A_{\sigma(1)} \sqcup \cdots \sqcup A_{\sigma(n)},$$

   where A_i *are irreducible subobjects and* σ *is a permutation of* $\{1, 2, \ldots, n\} \subseteq \mathbb{N}$*, and it is generated, via the monoidal product and composition, by identity morphisms and braidings only. Because of naturality, a permuting isomorphism may always be written as a composition of elementary permuting isomorphisms of the form:*

$$(\bigsqcup_{j=1}^{i-1} \mathrm{id}_{A_j}) \sqcup c_{(A_i, A_{i+1})} \sqcup (\bigsqcup_{j=i+2}^{n} \mathrm{id}_{A_j}).$$

2. an order-preserving isomorphism*, if it is of the form*

$$\alpha_1 \sqcup \cdots \sqcup \alpha_n : A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n \to A_1' \sqcup A_2' \sqcup \cdots \sqcup A_n',$$

where A_i, A_i' are irreducible subobjects and each $\alpha_i : A_i \to A_i'$ is an isomorphism of $\mathbf{C}$.

Example: The category $\mathcal{S}(\mathbf{C})$ has as objects the empty space $\emptyset$, the circles $C_\pm$ and finite disjoint unions of these circles. The morphisms of $\mathcal{S}(\mathbf{C})$, apart from the identity morphism for $\emptyset$, are orientation-preserving automorphisms of circles and finite disjoint unions of these morphisms, as well as isomorphisms such as $\alpha : C_{-+} \to C_{++}$, given by $(z,1) \mapsto (\overline{z},2)$, $(z,2) \mapsto (z,1)$, which is the composition of a braiding (i.e. a permuting isomorphism) and an order-preserving isomorphism.▲

We are now ready to construct the topological category $\mathcal{C}$, which will be the domain category for the TQFT functor.

Definition 2.11. *Let $\mathcal{O}$ be a class of triples (X, A, m), where (A, m) is an $\mathcal{M}$-subobject of X, satisfying:*

1) $(E, E, \mathrm{id}_E) \in \mathcal{O}$*.*

2) (X, A, m), $(X', A', m') \in \mathcal{O}$ *implies* $(X \sqcup X', A \sqcup A', m \sqcup m') \in \mathcal{O}$*.*

3) $(X, A, m) \in \mathcal{O}$ *implies* $(P(X), P(A), P(m)) \in \mathcal{O}$*.*

4) $(X \sqcup X', A \sqcup A', m \sqcup m') \in \mathcal{O}$ *implies* $(X' \sqcup X, A' \sqcup A, m' \sqcup m) \in \mathcal{O}$*.*

We define $\mathcal{C}$ to be the category with $\mathrm{Ob}(\mathcal{C}) = \mathcal{O}$, morphisms consisting of two classes, isomorphisms and gluing morphisms, defined below (Definition 2.12 and Definition 2.14), and composition and identity morphisms defined below (Definition 2.16 and 2.19).

Example: For our example we choose the class of objects of $\mathcal{C}$ to be all triples (X, A, m), where A is $\emptyset$ or a finite disjoint union of oriented circles isomorphic to ∂X and the image of $m \in \mathcal{M}$ is ∂X. Thus for example we have the following objects:

- $X = A = \emptyset$ and $m = \mathrm{id}_\emptyset$,

- $X = S_\pm$ or $T_\pm$ (the sphere or the torus, with either orientation), $A = \emptyset$, and m is the corresponding empty map,

- $X = A_+$ (the annulus with positive orientation), $A = C_{-+}$ and m is given by $m(z,1) = z$, $m(z,2) = 2z$.

Note that, in this example, we do not admit objects having, e.g. $X = A_+$ and $A = C_+$, where A only corresponds to part of the boundary of X, although the general theory does not forbid this.▲

The morphisms of $\mathcal{C}$ are restricted to be of two types, isomorphisms and gluing morphisms. We start by defining the isomorphisms:

Definition 2.12. *Let (X, A, m) and (X', A', m') be two objects of $\mathcal{C}$. An isomorphism between them is a pair (f, α), where $f : X \to X'$ is an isomorphism in $\mathbf{C}$ and $\alpha : A \to A'$ is a morphism of $\mathcal{S}(\mathbf{C})$, such that the following diagram*

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ A \arrow[u, "m"] \arrow[r, "\alpha"'] & A' \arrow[u, "m'"'] \end{tikzcd}$$

is commutative.

Remark 2.13. *Two general types of isomorphisms are:*

- $(\mathrm{id}_X, \mathrm{id}_A)$ for any $(X, A, m) \in \mathrm{Ob}(\mathcal{C})$.

- $(c_{(X,X')}, c_{(A,A')}) : (X \sqcup X', A \sqcup A', m \sqcup m') \to (X' \sqcup X, A' \sqcup A, m' \sqcup m)$ for any $(X, A, m), (X', A', m') \in \mathrm{Ob}(\mathcal{C})$.

Example: Some examples of isomorphisms of $\mathcal{C}$, apart from the ones in the above remark, are:

- the reverse map

$$(r, \alpha) : (D_+, C_+, m_+) \to (D_-, C_-, m_-),$$

  where $m_\pm : C_\pm \to D_\pm$ are given by $m_\pm(z) = z$, $r : D_+ \to D_-$ is given by $r(z) = \overline{z}$ and $\alpha : C_+ \to C_-$ is given by $\alpha(z) = \overline{z}$.

- the following morphism, which can be interpreted as a reordering of the two boundary components of the annulus:

$$(\mathrm{id}_{A_+}, c_{(C_-,C_+)}) : (A_+, C_{-+}, m) \to (A_+, C_{+-}, m'),$$

  where $m(z,1) = z = m'(z,2)$ and $m(z,2) = 2z = m'(z,1)$.▲

Next we define the other class of morphisms of $\mathcal{C}$, which we call gluing morphisms. We recall the notation $A_I := \sqcup_{i\in I} A_i$ for a finite ordered index set, from property 6 of $\mathcal{M}$. Also we set $P(A)_I := \sqcup_{i\in I} P(A_i)$.

Definition 2.14. *Let (X, A, m), (X', A', m') be objects of $\mathcal{C}$, where $A := A_N$ for some finite ordered index set N. Let I, J and R be disjoint ordered subsets of N, with I, J non-empty and of the same cardinality, and such that, as sets, $N = I \cup J \cup R$. A gluing morphism from (X, A, m) to (X', A', m') is a triple (f, φ, α), where $X \xrightarrow{f} X'$ is a morphism of $\mathbf{C}$, and $\varphi : A_I \to P(A)_J$ and $\alpha : A_R \longrightarrow A'$ are morphisms of $\mathcal{S}(\mathbf{C})$, such that:*

i) $f \circ m_R = m' \circ \alpha$,

ii) *$F(f\circ m_I)$ is a (topological) embedding, the images of $F(f\circ m_I)$ and $F(f\circ m_R)$ are separated in $F(X')$, and the image under $F(f)$ of $\mathrm{Cls}(\mathrm{Im}(F(m_I)))$ is closed in $F(X')$, where* Cls *is the closure.*

iii) $F(f \circ m_J) \circ F(\varphi) = F(f \circ m_I)$,

iv) *$F(X) \setminus \mathrm{Im}(F(m_{IJ}))$ is isomorphic to $F(X') \setminus \mathrm{Im}(F(f \circ m_I))$, via $F(f)$.*

Remark 2.15. *Let us refer to the connected components of A, A', etc. as boundaries, although they are not necessarily boundary components of topological manifolds. Intuitively we are gluing the boundaries A_I to the boundaries A_J and the remaining boundaries A_R are not glued. The gluing morphism itself is really the morphism f from X "before gluing" to X', which is a copy of the quotient space "after gluing" (see figure below). Note that the gluing can involve gluing together disconnected spaces as well as self-gluing of a connected space. The morphisms φ and α give supplementary information about how the boundaries are glued together, and how the boundaries of X which are not glued map to the boundary of X'. We note that different choices of I, J and φ may be associated with the* *same morphisms f and α, since we may reorder A_I and A_J using permuting isomorphisms and compose φ with the respective inverse isomorphisms to make a different φ. Condition i) of the definition says that the boundaries of X which are not glued are isomorphic to the boundaries of X', via α. Conditions ii)-iv) are purely topological and capture the intuitive idea of gluing. Conditions ii) and iii) say that the glued boundaries map $2:1$ onto their image in X', which is isomorphic to each half of the glued boundaries. This image has to be separated from the boundaries of X' to avoid problems when composing two gluing morphisms. Note that $F(\varphi)$ in iii) is defined in the obvious way by regarding φ as a morphism of $\mathbf{C}$. (Indeed the forgetful functor F on $\mathbf{C}$ restricts to the subcategory $\mathcal{S}(\mathbf{C})$.) Finally, condition iv) says that, topologically, everything that is not glued in X is isomorphic to everything in X' that is not the image of the glued boundaries.*

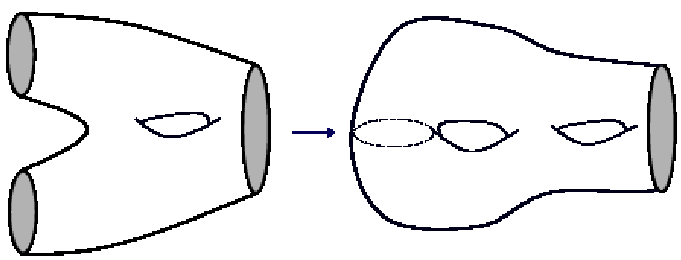

Figure 2.1: Self-gluing of an object

Example: Gluing morphisms can arise from the gluing together of separate objects, or the self-gluing of a single object, as well as a combination of both types of gluing.

a) The gluing together of two discs to give a sphere S may be described by the gluing morphism:

$$(f, \varphi, \alpha) : (D_{-+}, C_{-+}, m) \to (S_+, \emptyset, m'),$$

where $m(z,1) = (z,1)$, $m(z,2) = (z,2)$, m' is the empty map, and

$$I = \{1\}\,,\; J = \{2\}\,,\; R = \emptyset.$$

The morphisms f, φ and α are given by (writing $z = x + iy \in D$):

$$\begin{aligned} f(z,1) &= (x,y,-\sqrt{1-|z|^2}) \quad , \quad f(z,2) = (x,y,\sqrt{1-|z|^2}) \\ \varphi &= \mathrm{id}_{C_-} \quad , \quad \alpha = \mathrm{id}_\emptyset \,. \end{aligned}$$

b) A similar example is the gluing of a disc into one of the holes of a pair-of-pants P to give an annulus A. For convenience we take P to be

$$P = A \setminus \left\{ z : \left| z + \tfrac{3}{2} \right| < \tfrac{1}{4} \right\} .$$

The gluing morphism is then:

$$(f,\varphi,\alpha) : (D_+ \sqcup P_+, C_{+--+}, m) \to (A_+, C_{-+}, m'),$$

where $m(z,1) = (z,1)$, $m(z,2) = (-\frac{3}{2} + \frac{z}{4}, 2)$, $m(z,3) = (z,2)$, $m(z,4) = (2z,2)$, $m'(z,1) = z$, $m'(z,2) = 2z$ and

$$I = \{1\} \quad , \quad J = \{2\} \quad , \quad R = \{3,4\} \,.$$

The morphisms f, φ and α are given by:

$$f(z,1) = -\frac{3}{2} + \frac{z}{4} \quad , \quad f(z,2) = z \quad , \quad \varphi = \mathrm{id}_{C_+} \quad , \quad \alpha = \alpha_1 \sqcup \alpha_2 = \mathrm{id}_{C_{-+}} \,.$$

c) An example of a self-gluing is the gluing morphism from the annulus to the torus:

$$(g,\psi,\beta) : (A_+, C_{-+}, m') \to (T_+, \emptyset, m''),$$

where m' is as in b) and m'' is the empty map, and

$$I = \{1\} \quad , \quad J = \{2\} \quad , \quad R = \emptyset.$$

The morphisms g, ψ and β are given by:

$$g(z) = (\frac{z}{|z|}, e^{2\pi i(2-|z|)}) \quad , \quad \psi = \mathrm{id}_{C_-} \quad , \quad \beta = \mathrm{id}_\emptyset \,.$$

d) Finally we can glue a disc to a pair-of-pants simultaneously with the self-gluing of its remaining boundary components via the following gluing morphism:

$$(h,\chi,\gamma) : (D_+ \sqcup P_+, C_{+--+}, m) \to (T_+, \emptyset, m''),$$

where m, m'' are as in b) and c), and

$$I = \{1, 3\} \quad , \quad J = \{2, 4\} \quad , \quad R = \emptyset.$$

The morphisms h, χ and γ are given by:

$$h(z,1) \;=\; (\frac{-\frac{3}{2}+\frac{z}{4}}{\left|-\frac{3}{2}+\frac{z}{4}\right|}, e^{2\pi i(2-\left|-\frac{3}{2}+\frac{z}{4}\right|)}) \quad , \quad h(z,2) = (\frac{z}{|z|}, e^{2\pi i(2-|z|)}),$$
$$\chi := \chi_1 \sqcup \chi_2 = \mathrm{id}_{C_{+-}} \quad , \quad \gamma = \mathrm{id}_{\emptyset} \,.\blacktriangle$$

Next we define the composition of morphisms in $\mathcal{C}$:

Definition 2.16. *Let (X, A, m), (X', A', m') and (X'', A'', m'') be objects of $\mathcal{C}$.*

- *(isomorphism-isomorphism)*

  Let $(f, \alpha) : (X, A, m) \to (X', A', m')$ and $(g, \beta) : (X', A', m') \to (X'', A'', m'')$ be isomorphisms. Then we define:

  $$(g, \beta) \circ (f, \alpha) = (g \circ f, \beta \circ \alpha).$$

- *(gluing - isomorphism)*

  Let $(f, \varphi, \alpha) : (X, A, m) \to (X', A', m')$ be a gluing morphism with $\varphi : A_I \to P(A)_J$, $\alpha : A_R \to A'$ and let $(g, \beta) : (X', A', m') \to (X'', A'', m'')$ be an isomorphism. Then we define:

  $$(g, \beta) \circ (f, \varphi, \alpha) = (g \circ f, \varphi, \beta \circ \alpha).$$

- *(isomorphism - gluing)*

  Let $(f, \alpha) : (X, A, m) \to (X', A', m')$ be an isomorphism and let $(g, \psi, \beta) : (X', A', m') \to (X'', A'', m'')$ be a gluing morphism with $\psi : A'_{I'} \to P(A')_{J'}$ and $\beta : A'_{R'} \to A''$. By property $\mathcal{M}$-6d), corresponding to I', J' and R', there exist ordered sets I, J, R and isomorphisms

  $$\alpha_{I,I'} : A_I \to A'_{I'},$$
  $$\alpha_{J,J'} : A_J \to A'_{J'},$$
  $$\alpha_{R,R'} : A_R \to A'_{R'}.$$

Then we define:

$$(g,\psi,\beta)\circ(f,\alpha) = (g\circ f, P(\alpha^{-1})_{J,J'}\circ\psi\circ\alpha_{I,I'}, \beta\circ\alpha_{R,R'}),$$

where $P(\alpha^{-1})_{J,J'}$ *is defined to be the isomorphism* $\sqcup_{i\in J}P(\alpha_i^{-1})$*, and the* α_i*'s are the factor isomorphisms of property* $\mathcal{M}$*-6d).*

- *(gluing - gluing)*

  Let $(f,\varphi,\alpha) : (X,A,m) \to (X',A',m')$ *and* $(g,\psi,\beta) : (X',A',m') \to (X'',A'',m'')$ *be gluing morphisms with*

$$\begin{aligned}
&\varphi : A_I \to P(A)_J,\\
&\alpha : A_R \to A',\\
&\psi : A'_{I'} \to P(A')_{J'},\\
&\beta : A'_{R'} \to A'',
\end{aligned}$$

  By property $\mathcal{M}$*-6d), corresponding to* I'*,* J' *and* R'*, there exist* $\widetilde{I}$*,* $\widetilde{J}$ *and* $\widetilde{R}$*, ordered subsets of* R*, and isomorphisms:*

$$\alpha_{\widetilde{I},I'} : A_{\widetilde{I}} \to A'_{I'} \quad , \quad \alpha_{\widetilde{J},J'} : A_{\widetilde{J}} \to A'_{J'} \quad \text{and} \quad \alpha_{\widetilde{R},R'} : A_{\widetilde{R}} \to A'_{R'}.$$

  Then we define:

$$(g,\psi,\beta)\circ(f,\varphi,\alpha) = (g\circ f, \varphi\sqcup(P(\alpha^{-1})_{\widetilde{J},J'}\circ\psi\circ\alpha_{\widetilde{I},I'}), \beta\circ\alpha_{\widetilde{R},R'}).$$

Example: The composition

$$(g,\psi,\beta)\circ(f,\varphi,\alpha) = (g\circ f, \varphi\sqcup P(\alpha_2^{-1})\circ\psi\circ\alpha_1, \mathrm{id}_\emptyset)$$

from $(D_+\sqcup P_+, C_{+--+}, m)$ to $(T_+,\emptyset,m'')$ from our previous example b) and c) is equal to (h,χ,γ) from the same example d), as is easily checked.▲

The main result concerning the morphisms of $\mathcal{C}$ is:

Theorem 2.17. *The class of morphisms of* $\mathcal{C}$ *is closed under the above composition.*

Proof. We will just show this for the gluing-gluing case of Definition 2.16. The other combinations are proved analogously. Let $(f, \varphi, \alpha) : (X, A, m) \to (X', A', m')$ and $(g, \psi, \beta) : (X', A', m') \to (X'', A'', m'')$ be gluing morphisms, where $\varphi : A_I \to P(A)_J$, $\alpha : A_R \to A'$, $\psi : A'_{I'} \to P(A')_{J'}$ and $\beta : A'_{R'} \to A''$.

i) Set $A = A_N$ and $A' = A'_{N'}$, where N, N' are ordered index sets. By property $\mathcal{M}$-6d) there exist an ordered set $\widetilde{R} \subsetneq R$ and an isomorphism $\alpha_{\widetilde{R},R'}$ such that $\alpha \circ s_{\widetilde{R},R} = s_{R',N'} \circ \alpha_{\widetilde{R},R'}$. Thus

$$\begin{aligned} f \circ m_{\widetilde{R}} &= f \circ m_R \circ s_{\widetilde{R},R} \\ &= m' \circ \alpha \circ s_{\widetilde{R},R} \\ &= m' \circ s_{R',N'} \circ \alpha_{\widetilde{R},R'} \\ &= m'_{R'} \circ \alpha_{\widetilde{R},R'}, \end{aligned}$$

where we use i) for the gluing morphism (f, φ, α) in the second equality. Here and below we use $\mathcal{M}$-6a) and the simplified notation in Remark 2.6 without comment. Composing this equation with g and using i) for the gluing morphism (g, ψ, β) we have:

$$(g \circ f) \circ m_{\widetilde{R}} = g \circ m'_{R'} \circ \alpha_{\widetilde{R},R'} = m'' \circ (\beta \circ \alpha_{\widetilde{R},R'}).$$

To simplify the notation for the proof of the topological conditions, we will omit writing the forgetful functor F and consider all objects and morphisms to be in Top.

ii) First we show that $(g \circ f) \circ m_I : A_I \to X''$ and $(g \circ f) \circ m_{\widetilde{I}} : A_{\widetilde{I}} \to X''$ are topological embeddings and then we show that $\mathrm{Im}((g \circ f) \circ m_I)$ and $\mathrm{Im}((g \circ f) \circ m_{\widetilde{I}})$ are separated in X'', thus concluding that $(g \circ f) \circ m_{I\widetilde{I}}$ is a topological embedding. It is straightforward to show $\mathrm{Im}(m_{I\widetilde{I}}) = \mathrm{Im}(m_I) \cup \mathrm{Im}(m_{\widetilde{I}})$, using the commutativities $m_{I\widetilde{I}} \circ s_{I,I\widetilde{I}} = m_I$ and $m_{I\widetilde{I}} \circ s_{\widetilde{I},I\widetilde{I}} = m_{\widetilde{I}}$ and bearing in mind that $s_{I,I\widetilde{I}}$ and $s_{\widetilde{I},I\widetilde{I}}$ are the canonical monomorphisms in Top. Thus we have:

$$\mathrm{Im}((g \circ f) \circ m_{I\widetilde{I}}) = \mathrm{Im}((g \circ f) \circ m_I) \cup \mathrm{Im}((g \circ f) \circ m_{\widetilde{I}}).$$

Now $(g \circ f) \circ m_I$ is an embedding since $f \circ m_I$ is an embedding by ii) for (f, φ, α) and g restricted to $\mathrm{Im}(f \circ m_I) \subseteq X' \setminus \mathrm{Im}(m'_{I'J'})$ is an isomorphism by iv) for (g, ψ, β). Furthermore $(g \circ f) \circ m_{\widetilde{I}}$ is an embedding, since by the property $\mathcal{M}$-6d) there exist an ordered set $\widetilde{I} \subsetneq R$ and an isomorphism $\alpha_{\widetilde{I},I'}$ such that $\alpha \circ s_{\widetilde{I},R} = s_{I',N'} \circ \alpha_{\widetilde{I},I'}$. Thus we derive

$$f \circ m_{\widetilde{I}} = f \circ m_R \circ s_{\widetilde{I},R}$$

$$\begin{aligned} &= m' \circ \alpha \circ s_{\tilde{I},R} \\ &= m' \circ s_{I',N'} \circ \alpha_{\tilde{I},I'} \\ &= m'_{I'} \circ \alpha_{\tilde{I},I'}, \end{aligned}$$

which on composing with g, gives $g \circ f \circ m_{\tilde{I}} = g \circ m'_{I'} \circ \alpha_{\tilde{I},I'}$. Now since $g \circ m'_{I'}$ is an embedding from ii) for (g, ψ, β), and $\alpha_{\tilde{I},I'}$ is an isomorphism, we conclude that $g \circ f \circ m_{\tilde{I}}$ is an embedding.

Next we show that $\mathrm{Im}((g \circ f) \circ m_I)$ and $\mathrm{Im}((g \circ f) \circ m_{\tilde{I}})$ are separated in X'', which, together with the previous results proves that $(g \circ f) \circ m_{I\tilde{I}}$ is an embedding. For this we set $C := \mathrm{Im}(f \circ m_I)$ and $D := \mathrm{Im}(f \circ m_{\tilde{I}}) = \mathrm{Im}(m'_{I'} \circ \alpha_{\tilde{I},I'}) = \mathrm{Im}(m'_{I'})$. We need to show that $\mathrm{Cls}(g(C)) \cap g(D) = \emptyset$ and $g(C) \cap \mathrm{Cls}(g(D)) = \emptyset$. We have, by iv) for (g, ψ, β), $\mathrm{Cls}(g(C)) = g(\mathrm{Cls}(C)) \subseteq X'' \setminus g(D)$, since $\mathrm{Cls}(C) \subseteq X' \setminus \mathrm{Im}(m'_{I'J'})$, by ii) for (f, φ, α). Thus $\mathrm{Cls}(g(C)) \cap g(D) = \emptyset$. To show $g(C) \cap \mathrm{Cls}(g(D)) = \emptyset$, we have:

$$\begin{aligned} g(C) \cap \mathrm{Cls}(g(D)) &= g(C) \cap g(\mathrm{Cls}(D)) \\ &= (g(C) \cap g(D)) \cup (g(C) \cap g(\mathrm{Cls}(D) \setminus D)) \\ &= g(C) \cap g(\mathrm{Cls}(D) \setminus D) \\ &= g(C \cap (\mathrm{Cls}(D) \setminus D)) \\ &= \emptyset. \end{aligned}$$

The first equality follows from ii) for (g, ψ, β), which says that $g(\mathrm{Cls}(D))$ is closed in X'', since $g(D) \subseteq g(\mathrm{Cls}(D))$, hence $\mathrm{Cls}(g(D)) \subseteq g(\mathrm{Cls}(D))$ and hence, because g is continuous, $\mathrm{Cls}(g(D)) = g(\mathrm{Cls}(D))$. The second follows from properties of operations on sets. The third is clear since we have $g(C) \cap g(D) = \emptyset$. The fourth follows from the fact that g restricted to $X' \setminus \mathrm{Im}(m'_{I'J'})$ is an isomorphism, by iv) for (g, ψ, β) and by the fact that $\mathrm{Cls}(D) \setminus D \subseteq X' \setminus \mathrm{Im}(m'_{I'J'})$. This inclusion comes from the fact that $(\mathrm{Cls}(D) \setminus D) \cap \mathrm{Im}(m'_{I'J'})$ is indeed empty, since $A'_{I'}$ and $A'_{J'}$ are separated in $A'_{I'} \sqcup A'_{J'}$ and $m'_{I'J'}$ is an embedding by $\mathcal{M}$-1). The final equality follows from ii) for (f, φ, α), since $C = \mathrm{Im}(f \circ m_I)$ and $D = \mathrm{Im}(f \circ m_{\tilde{I}}) \subsetneq \mathrm{Im}(f \circ m_R)$ are separated in X'.

Now we show that $\mathrm{Im}(g \circ f \circ m_{I\tilde{I}})$ and $\mathrm{Im}(m'')$ are separated in X''. This holds since C and $\mathrm{Im}(m'_{R'}) \subseteq \mathrm{Im}(m')$ are separated in X', by ii) for (f, φ, α), and g is an isomorphism restricted to $X' \setminus \mathrm{Im}(m'_{I'J'})$, so that $g(C)$ and $\mathrm{Im}(g \circ m'_{R'}) = \mathrm{Im}(m'' \circ \beta) = \mathrm{Im}(m'')$ are separated, by iv) for (g, ψ, β). Furthermore $\mathrm{Im}(g \circ f \circ m_{\tilde{I}}) = g(D)$ and $\mathrm{Im}(m'')$ are separated in X'', by ii) for (g, ψ, β).

It remains to show that $(g \circ f)(\mathrm{Cls}(\mathrm{Im}(m_I)))$ and $(g \circ f)(\mathrm{Cls}(\mathrm{Im}(m_{\tilde{I}})))$ are closed. For this we have:

$$(g \circ f)(\mathrm{Cls}(\mathrm{Im}(m_I))) = g(\mathrm{Cls}(C)) = \mathrm{Cls}(g(C)),$$

which is closed. The first equality follows from the fact that $f(\mathrm{Cls}(\mathrm{Im}(m_I)))$ is closed, by ii) for (f, φ, α) and the fact that $C = \mathrm{Im}(f \circ m_I) \subseteq f(\mathrm{Cls}(\mathrm{Im}(m_I)))$, hence, because f is continuous, $\mathrm{Cls}(C) = f(\mathrm{Cls}(\mathrm{Im}(m_I)))$. The second follows from the fact that $\mathrm{Cls}(C) \subseteq X' \setminus \mathrm{Im}(m'_{I'J'})$ and from iv) for (g, ψ, β). For $(g \circ f)(\mathrm{Cls}(\mathrm{Im}(m_{\tilde{I}})))$ we have:

$$(g \circ f)(\mathrm{Cls}(\mathrm{Im}(m_{\tilde{I}}))) = g(\mathrm{Cls}(f(\mathrm{Im}(m_{\tilde{I}})))) = g(\mathrm{Cls}(D)),$$

which is closed by ii) for (g, ψ, β). The first equality follows from the fact that $\mathrm{Cls}(\mathrm{Im}(m_{\tilde{I}})) \subseteq X \setminus \mathrm{Im}(m_{IJ})$ and from iv) for (f, φ, α).
iii) We must show $(g \circ f \circ m_{J\tilde{J}}) \circ (\varphi \sqcup (\alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'})) = g \circ f \circ m_{I\tilde{I}}$. Let $x \in A_{I\tilde{I}}$. Then either there exists $y \in A_I$ such that $x = s_{I,I\tilde{I}}(y)$ or there exists $z \in A_{\tilde{I}}$ such that $x = s_{\tilde{I},I\tilde{I}}(z)$, since the morphisms $s_{I,I\tilde{I}}$ and $s_{\tilde{I},I\tilde{I}}$ are the canonical monomorphisms in Top. In the first case we have:

$$\begin{aligned} m_{J\tilde{J}} \circ (\varphi \sqcup (\alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'}))(x) &= m_{J\tilde{J}} \circ (\varphi \sqcup (\alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'})) \circ s_{I,I\tilde{I}}(y) \\ &= m_{J\tilde{J}} \circ s_{J,J\tilde{J}} \circ \varphi(y) \\ &= m_J \circ \varphi(y) \end{aligned}$$

using $\mathcal{M}$-6d) in the second equality. Thus

$$\begin{aligned} (g \circ f \circ m_{J\tilde{J}}) \circ (\varphi \sqcup (\alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'}))(x) &= (g \circ f \circ m_J \circ \varphi)(y) \\ &= (g \circ f \circ m_I)(y) \\ &= (g \circ f \circ m_{I\tilde{I}} \circ s_{I,I\tilde{I}})(y) \\ &= (g \circ f \circ m_{I\tilde{I}})(x), \end{aligned}$$

using (iii) for (f, φ, α) in the second equality. In the second case we have:

$$\begin{aligned} m_{J\tilde{J}} \circ (\varphi \sqcup (\alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'}))(x) &= (m_{J\tilde{J}} \circ (\varphi \sqcup (\alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'})) \circ s_{\tilde{I},I\tilde{I}})(z) \\ &= (m_{J\tilde{J}} \circ s_{\tilde{J},J\tilde{J}} \circ \alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'})(z) \\ &= (m_{\tilde{J}} \circ \alpha^{-1}_{\tilde{J}J'} \circ \psi \circ \alpha_{\tilde{I}I'})(z), \end{aligned}$$

using property $\mathcal{M}$-6d) in the second equality. Thus

$$\begin{aligned}(g\circ f\circ m_{J\tilde{J}})\circ(\varphi\sqcup(\alpha^{-1}_{\tilde{J}J'}\circ\psi\circ\alpha_{\tilde{I}I'}))(x) &= (g\circ f\circ m_{\tilde{J}}\circ\alpha^{-1}_{\tilde{J}J'}\circ\psi\circ\alpha_{\tilde{I}I'})(z)\\ &= (g\circ m'_{J'}\circ\psi\circ\alpha_{\tilde{I}I'})(z)\\ &= (g\circ m'_{I'}\circ\alpha_{\tilde{I}I'})(z)\\ &= (g\circ f\circ m_{\tilde{I}})(z)\\ &= (g\circ f\circ m_{I\tilde{I}}\circ s_{\tilde{I},I\tilde{I}})(z)\\ &= (g\circ f\circ m_{I\tilde{I}})(x),\end{aligned}$$

using (i) for (f,φ,α) and $\mathcal{M}$-6d) in the second and fourth equalities and (iii) for (g,ψ,β) in the third equality.
iv) Finally for this condition we have:

$$\begin{aligned}X\setminus\mathrm{Im}(m_{IJ\tilde{I}\tilde{J}}) &\cong (X'\setminus\mathrm{Im}(f\circ m_{IJ}))\setminus\mathrm{Im}(f\circ m_{\tilde{I}\tilde{J}})\\ &= (X'\setminus\mathrm{Im}(f\circ m_{I}))\setminus\mathrm{Im}(m'_{I'J'})\\ &= (X'\setminus\mathrm{Im}(m'_{I'J'}))\setminus\mathrm{Im}(f\circ m_{I})\\ &\cong (X''\setminus\mathrm{Im}(g\circ m'_{I'}))\setminus\mathrm{Im}(g\circ f\circ m_{I})\\ &= (X''\setminus\mathrm{Im}(g\circ f\circ m_{I}))\setminus\mathrm{Im}(g\circ f\circ m_{\tilde{I}})\\ &= (X''\setminus\mathrm{Im}(g\circ f\circ m_{I}))\setminus\mathrm{Im}(g\circ f\circ m_{\tilde{I}})\\ &= X''\setminus(\mathrm{Im}(g\circ f\circ m_{I})\cup\mathrm{Im}(g\circ f\circ m_{\tilde{I}}))\\ &= X''\setminus\mathrm{Im}(g\circ f\circ m_{I\tilde{I}}).\end{aligned}$$

These equalities and isomorphisms are proved by using iv) for the gluing morphisms (f,φ,α) and (g,ψ,β) and general properties of the operation of difference of sets. ■

Furthermore we have:

Theorem 2.18. *The composition in $\mathcal{C}$ is associative.*

Proof. We will show this for the case of three gluing morphisms. The other combinations are easily checked. We will consider the gluing morphisms

$$(f,\varphi,\alpha):(X,A,m)\to(X',A',m'),$$

where $\varphi:A_I\to P(A)_J$, $\alpha:A_R\to A'$,

$$(g,\psi,\beta):(X',A',m')\to(X'',A'',m''),$$

where $\psi : A'_{I'} \to P(A')_{J'}$, $\beta : A'_{R'} \to A''$ and

$$(h, \rho, \gamma) : (X'', A'', m'') \to (X''', A''', m'''),$$

where $\rho : A''_{I''} \to P(A'')_{J''}$ and $\gamma : A''_{R''} \to A'''$.
By the property $\mathcal{M}$-6d) applied to α, corresponding to the ordered subsets I', J', R', there exist ordered subsets $\widetilde{I}, \widetilde{J}, \widetilde{R} \subsetneq R$, and isomorphisms $\alpha_{\widetilde{I},I'}$, $\alpha_{\widetilde{J},J'}$, $\alpha_{\widetilde{R},R'}$. By the same property applied to β, corresponding to I'', J'', R'' there exist ordered subsets $\widetilde{I'}, \widetilde{J'}, \widetilde{R'} \subsetneq R'$ and isomorphisms $\beta_{\widetilde{I'},I''}$, $\beta_{\widetilde{J'},J''}$, $\beta_{\widetilde{R'},R''}$. Moreover, applying property $\mathcal{M}$-6d) to $\alpha_{\widetilde{R},R'}$ corresponding to the subsets $\widetilde{I'}, \widetilde{J'}, \widetilde{R'} \subsetneq R'$ there exist ordered subsets $\widehat{I}, \widehat{J}, \widehat{R} \subsetneq R$ and isomorphisms $\alpha_{\widehat{I},\widetilde{I'}}$, $\alpha_{\widehat{J},\widetilde{J'}}$ and $\alpha_{\widehat{R},\widetilde{R'}}$. For one side of the associativity equation, we have (denoting composition by juxtaposition for simplicity of notation):

$$\begin{aligned}
(h,\rho,\gamma)\left((g,\psi,\beta)(f,\varphi,\alpha)\right) &= (h,\rho,\gamma)\left(gf, \varphi \sqcup (P(\alpha^{-1})_{\widetilde{J},J'}\psi\alpha_{\widetilde{I},I'}), \beta\alpha_{\widetilde{R},R'}\right) \\
&= (h(gf), \varphi \sqcup (P(\alpha^{-1})_{\widetilde{J},J'}\psi\alpha_{\widetilde{I},I'}) \sqcup (P(\beta\alpha_{\widetilde{R},R'})^{-1}_{\widehat{J},J''} \\
&\quad \rho(\beta\alpha_{\widetilde{R},R'})_{\widehat{I},I''}), \gamma(\beta\alpha_{\widetilde{R},R'})_{\widehat{I},I''}) \\
&= (h(gf), \varphi \sqcup (P(\alpha^{-1})_{\widetilde{J},J'}\psi\alpha_{\widetilde{I},I'}) \sqcup (P(\alpha^{-1})_{\widehat{J},\widetilde{J'}} \\
&\quad P(\beta^{-1})_{\widetilde{J'},J''}\rho\beta_{\widetilde{I'},I''}\alpha_{\widehat{I},\widetilde{I'}}), \gamma(\beta_{\widetilde{R'},R''}\alpha_{\widehat{R},\widetilde{R'}})).
\end{aligned}$$

For the other side we have:

$$\begin{aligned}
\left((h,\rho,\gamma)(g,\psi,\beta)\right)(f,\varphi,\alpha) &= (hg, \psi \sqcup (P(\beta^{-1})_{\widetilde{J'},J''}\rho\beta_{\widetilde{I'},I''}), \gamma\beta_{\widetilde{R'},R''})(f,\varphi,\alpha) \\
&= ((hg)f, \varphi \sqcup ((P(\alpha^{-1})_{\widetilde{J}\widehat{J},J'\widetilde{J'}}(\psi \sqcup P(\beta^{-1})_{\widetilde{J'},J''}\rho\beta_{\widetilde{I'},I''}) \\
&\quad \alpha_{\widetilde{I}\widehat{I},I'\widetilde{I'}}, (\gamma\beta_{\widetilde{R'},R''})\alpha_{\widehat{R},\widetilde{R'}}) \\
&= ((hg)f, \varphi \sqcup (P(\alpha^{-1})_{\widetilde{J},J'}\psi\alpha_{\widetilde{I},I'}) \sqcup (P(\alpha^{-1})_{\widehat{J},\widetilde{J'}} \\
&\quad P(\beta^{-1})_{\widetilde{J'},J''}\rho\beta_{\widetilde{I'},I''}\alpha_{\widehat{I},\widetilde{I'}}), (\gamma\beta_{\widetilde{R'},R''})\alpha_{\widehat{R},\widetilde{R'}}),
\end{aligned}$$

using the interchange law in the third equality. The two expressions are equal because of the associativity of composition in **C**. ■

Definition 2.19. *Let $(X, A, m) \in \mathrm{Ob}(\mathcal{C})$. The identity morphism on (X, A, m) is defined as follows:*

$$\mathrm{id}_{(X,A,m)} = (\mathrm{id}_X, \mathrm{id}_A) : (X, A, m) \to (X, A, m).$$

Theorem 2.20. *$\mathcal{C}$ of Definition 2.11 is a category.*

Proof. This follows from the results already shown above and the fact that the morphisms of Definition 2.19 obviously satisfy the requirements to be identity morphisms. ■

Definition 2.21. *A monoidal product, unit object and braiding on $\mathcal{C}$ are defined as follows:*
monoidal product on objects:

$$(X, A, m) \sqcup (X', A', m') = (X \sqcup X', A \sqcup A', m \sqcup m'),$$

monoidal product on morphisms:

$$\begin{aligned} (f, \alpha) \sqcup (g, \beta) &= (f \sqcup g, \alpha \sqcup \beta), \\ (f, \varphi, \alpha) \sqcup (g, \beta) &= (f \sqcup g, \varphi, \alpha \sqcup \beta), \\ (f, \alpha) \sqcup (g, \psi, \beta) &= (f \sqcup g, \psi, \alpha \sqcup \beta), \\ (f, \varphi, \alpha) \sqcup (g, \psi, \beta) &= (f \sqcup g, \varphi \sqcup \psi, \alpha \sqcup \beta), \end{aligned}$$

unit object:

$$\widetilde{E} = (E, E, \mathrm{id}_E)$$

braiding:

$$c_{((X,A,m),(X',A',m'))} = \big(c_{(X,X')}, c_{(A,A')}\big).$$

Theorem 2.22. *$(\mathcal{C}, \sqcup, \widetilde{E}, c)$ is a symmetric, strict monoidal category.*

Proof. The monoidal product on $\mathcal{C}$ closes on objects by 2) of Definition 2.11, and clearly the four morphisms in Definition 2.21 are isomorphisms or gluing morphisms, so the monoidal product closes on morphisms too. The monoidal product is strictly associative in $\mathcal{C}$, since it is strictly associative in $\mathbf{C}$ and $\mathcal{S}(\mathbf{C})$. The unit object $\widetilde{E}$ is an object of $\mathcal{C}$ because of 1) of Definition 2.11, and is obviously a strict unit, since E is a strict unit in $\mathbf{C}$ and $\mathcal{S}(\mathbf{C})$. The braiding defined in Definition 2.21 is well-defined because of 4) of Definition 2.11 and is a braiding, because c is a braiding in $\mathbf{C}$ and $\mathcal{S}(\mathbf{C})$. It also clearly satisfies the symmetry condition. ■

For some TQFT functors only the endofunctor P on $\mathcal{S}(\mathbf{C})$ will play a role. For others we need a corresponding endofunctor on $\mathcal{C}$.

Definition 2.23. *The endofunctor* (P, π_2, π_0) *on* $\mathbf{C}$ *extends to an endofunctor* $(\mathbf{P}, \boldsymbol{\pi}_2, \boldsymbol{\pi}_0)$ *on* $\mathcal{C}$ *as follows:*
on objects:

$$\mathbf{P}(X, A, m) = (P(X), P(A), P(m)),$$

on morphisms

$$\begin{aligned} \mathbf{P}(f, \alpha) &= (P(f), P(\alpha)), \\ \mathbf{P}(f, \varphi, \alpha) &= (P(f), P(\varphi), P(\alpha)), \end{aligned}$$

and the natural isomorphism $\boldsymbol{\pi}_2$ *and the isomorphism* $\boldsymbol{\pi}_0$ *are given by:*

$$\begin{aligned} \boldsymbol{\pi}_{2((X,A,m),(X',A',m'))} &= (\pi_{2(X,X')}, \pi_{2(A,A')}), \\ \boldsymbol{\pi}_0 &= (\pi_0, \pi_0). \end{aligned}$$

Theorem 2.24. $(\mathbf{P}, \boldsymbol{\pi}_2, \boldsymbol{\pi}_0)$ *is a symmetric, strong monoidal endofunctor on* $\mathcal{C}$*.*

Proof. The natural isomorphism $\boldsymbol{\pi}_{2((X,A,m),(X',A',m'))}$ and the isomorphism $\boldsymbol{\pi}_0$ have domain and codomain in $\mathrm{Ob}(\mathcal{C})$, because of 1), 2) and 3) of Definition 2.11, and are isomorphisms of $\mathcal{C}$ since P is strong monoidal and because of the naturality of π_2. The naturality of $\boldsymbol{\pi}_2$ also follows directly from the naturality of π_2, by combining two commuting diagrams. The conditions on $\boldsymbol{\pi}_2$ and $\boldsymbol{\pi}_0$ follow directly from combining two corresponding diagrams for π_2 and π_0. ∎

We can now conclude this section by defining two types of topological categories, corresponding to two types of algebraic categories and two types of TQFT functors, as we shall see later.

Definition 2.25. *Let* $(\mathbf{C}, F, P)$ *be a topological starting category,* $\mathcal{M}$ *be an appropriate class of monomorphisms and* $\mathcal{O}$ *be a class of objects of* $\mathcal{C}$ *satisfying Definition 2.11.*

The topological category *for these data is the pair* $(\mathcal{C}, P)$*, where* $\mathcal{C}$ *denotes the symmetric, strict monoidal category* $(\mathcal{C}, \sqcup, \widetilde{E}, c)$ *and* P *is the monoidal endofunctor on* $\mathcal{S}(\mathbf{C})$ *defined in Definition 2.7.*
The full topological category *for these data is the pair* $(\mathcal{C}, \mathbf{P})$*, where* $\mathcal{C}$ *is as before and* $\mathbf{P}$ *is the monoidal endofunctor on* $\mathcal{C}$ *defined in Definition 2.23.*

Thus we have achieved our objective for this section, namely to define a general-purpose topological category capable of describing gluing and change of orientation in a generalized sense. Although the notation for the objects and morphisms of the topological category is a little cumbersome, it allows all features of a gluing operation to be specified clearly. The topological gluing operation lies at the heart of TQFT, and the topological category we have constructed distills out precisely that feature, suppressing all morphisms which do not fit into the scheme of gluing. Indeed the gluing is described at the lowest categorical level, namely at the level of morphisms, and the only extra structures on the category are the symmetric monoidal structure and, of course, the endofunctor representing change of orientation. This structure is about as simple as it can be, given the broad scope of the framework.

3. The algebraic category

The construction of the algebraic category in this section again proceeds in several stages, like in the previous section. The starting point is a category, the algebraic starting category, whose objects are (finite-dimensional) K-modules, where K is a ring, possibly endowed with further structures such as an inner product. Then we introduce a so-called evaluation on each object, which behaves very much like a hermitian structure on a K-module. This gives rise to a category called the evaluation-preserving category, which has the same objects as the algebraic starting category, and whose morphisms are isomorphisms preserving the evaluations. The evaluation-preserving category is the algebraic counterpart to the category of subobjects which appears in the construction of the topological category. The algebraic category itself has as its objects pairs consisting of an object of the algebraic starting category and an element of its underlying K-module. The morphisms of the algebraic category are morphisms of the algebraic starting category preserving the elements. (Thus a morphism implies an equation, and these equations will be crucial later on in determining TQFT functors.) Again there is an extra piece of structure which plays a role throughout the construction, coming from an endofunctor on the algebraic starting category. In the example, this is given by the operation of replacing a K-module by the same module with the same addition, but with a different scalar multiplication which uses the complex conjugate.

We now proceed with the details of the construction.

Definition 3.1. *An* algebraic starting category *is a triple* $(\mathbf{D}, G, Q)$*, where* $\mathbf{D}$ *is a symmetric, strict monoidal category* $(\mathbf{D}, \otimes, I, c)$*,* G *is a symmetric, strict* *monoidal forgetful functor from* $\mathbf{D}$ *to* K-Mod *(the category whose objects are finitely generated* K*-modules, where* K *is a fixed commutative ring with unit, and whose morphisms are* K*-homomorphisms) with its standard monoidal structure (tensor product) and braiding, and* Q *is a symmetric, strong monoidal endofunctor* (Q, θ_2, θ_0) *on* $\mathbf{D}$.

Remark 3.2. *The forgetful functor* G *justifies the adjective algebraic. It sends each object* V *of* $\mathbf{D}$ *to an underlying finitely generated* K*-module* $G(V)$ *and each morphism* f *of* $\mathbf{D}$ *to an underlying* K*-homomorphism* $G(f)$*. We may think of* Q *as assigning to each module* V *its "conjugate" module, a notion which will become clear in the example.*

Example: Let V be a finite-dimensional vector space over $\mathbb{C}$. We denote by $\overline{V}$ the vector space which has the same elements and addition operation as V, but a different operation of scalar multiplication denoted $\bar{\cdot}$, defined by $c \,\bar{\cdot}\, x = \overline{c}x$, where $c \in \mathbb{C}$, $x \in V$, $\overline{c}$ denotes the complex conjugate of c and scalar multiplication in V is denoted by juxtaposition. (We remark that the example could equally well be carried through using any commutative ring with an involution, instead of $\mathbb{C}$ and the operation of conjugation.) The objects of $\mathbf{D}$ are finite-dimensional complex vector spaces generated by $\mathbb{C}$ and V via the operation of tensor products, and the unary operation '$^-$' just introduced. Thus an example of an object of $\mathbf{D}$ is $\overline{V \otimes \overline{\overline{V}}}$. The motivation for taking this class of objects for $\mathbf{D}$, instead of simply all finite-dimensional vector spaces over $\mathbb{C}$, is that it is of minimal size for providing a good target category for the TQFT's in our example[3]. The morphisms of $\mathbf{D}$ are $\mathbb{C}$-linear maps between the objects of $\mathbf{D}$. The monoidal structure is given by the tensor product $\otimes$, taken to be strict (thus, e.g. $V \otimes \mathbb{C} = V$), and the unit object I, also strict, is $\mathbb{C}$. The braiding is given by the usual flip map

$$c_{(W,Y)} : W \otimes Y \to Y \otimes W, \quad w \otimes y \mapsto y \otimes w,$$

extended by linearity, which clearly satisfies the condition to be a symmetry. The forgetful functor G formally maps each object of $\mathbf{D}$ to the corresponding module over $\mathbb{C}$, regarded as a commutative ring, instead of a field, but G will be treated as the identity functor from now on.

[3]Roughly speaking, for the description of the TQFT functor we only need to give one vector space for each type of irreducible subobject on the topological side, and in this example there are just two types: positively oriented circles and negatively oriented circles.

The endofunctor Q acts on objects by

$$Q(W) = \overline{W}.$$

Before describing the action of Q on morphisms it is useful to introduce the following "identity" map,

$$k_W : W \to \overline{W},$$

defined by $k_W(x) = x$ for all x in W. Note that, although it preserves addition, k_W is not a morphism of $\mathbf{D}$, since $k_W(cx) = cx = \overline{c} \mathbin{\bar{\cdot}} x = \overline{c} \mathbin{\bar{\cdot}} k_W(x)$. We now define

$$Q(Y \xrightarrow{f} W) = \overline{Y} \xrightarrow{Q(f)} \overline{W},$$

where

$$Q(f) = k_W \circ f \circ k_Y^{-1}$$

is a linear map, i.e. a morphism of $\mathbf{D}$, as is easily checked. Finally the corresponding isomorphism

$$\theta_0 : \mathbb{C} \to \overline{\mathbb{C}}$$

is given by

$$\theta_0(z) = k_{\mathbb{C}}(\overline{z})$$

(note the conjugation in the argument which ensures that θ_0 is linear), and the natural isomorphism θ_2 is given by isomorphisms

$$\theta_{2(Y,W)} : \overline{Y} \otimes \overline{W} \to \overline{Y \otimes W},$$

where

$$\theta_{2(Y,W)} = k_{Y \otimes W} \circ (k_Y^{-1} \otimes k_W^{-1}),$$

is again a morphism of $\mathbf{D}$. Here the tensor product of k_Y^{-1} and k_W^{-1} is defined by

$$y \otimes w \mapsto k_Y^{-1}(y) \otimes k_W^{-1}(w)$$

which is well-defined and preserves addition, but not scalar multiplication. We omit the details of the simple check that (Q, θ_2, θ_0) satisfies the definition of a monoidal endofunctor.▲

We next introduce another category, which we denote by $\mathcal{S}(\mathbf{D})$, and which will correspond to the category $\mathcal{S}(\mathbf{C})$ of the previous section, when we start to talk about TQFT functors in the next section. $\mathcal{S}(\mathbf{D})$ has the same class of objects as $\mathbf{D}$, but its morphisms are restricted to a certain subclass of morphisms, namely isomorphisms preserving the so-called evaluations, which we now introduce.

Definition 3.3. *A* choice of evaluations *on an algebraic starting category* $(\mathbf{D}, G, Q)$ *is an assignment to each* $V \in \mathrm{Ob}(\mathbf{D})$ *of a morphism* $e_V : Q(V) \otimes V \to I$ *such that:*

- *(multiplicativity axiom)*

  for each pair of objects (V, W) *of* $\mathcal{S}(\mathbf{D})$

  $$e_{V\otimes W} \circ (\theta_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \circ (\mathrm{id}_{Q(V)} \otimes c_{(V,Q(W))} \otimes \mathrm{id}_W) = e_V \otimes e_W,$$

  i.e. the following diagram

  $$\begin{tikzcd} Q(V) \otimes V \otimes Q(W) \otimes W \arrow[r, "e_V \otimes e_W"] \arrow[d, "t"'] & I \otimes I = I \arrow[d, "\mathrm{id}_I"] \\ Q(V \otimes W) \otimes V \otimes W \arrow[r, "e_{V\otimes W}"'] & I \end{tikzcd}$$

  (where $t = (\theta_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \circ (\mathrm{id}_{Q(V)} \otimes c_{(V,Q(W))} \otimes \mathrm{id}_W)$*)*

  is commutative.

- *(conjugation axioms)*

  a) for each object V *of* $\mathcal{S}(\mathbf{D})$*,*

  $$Q(e_V) \circ \theta_{2(Q(V),V)} = \theta_0 \circ e_{Q(V)},$$

  i.e. the following diagram

  $$\begin{tikzcd} Q(Q(V)) \otimes Q(V) \arrow[r, "\theta_{2(Q(V),V)}"] \arrow[d, "e_{Q(V)}"'] & Q(Q(V) \otimes V) \arrow[d, "Q(e_V)"] \\ I \arrow[r, "\theta_0"'] & Q(I) \end{tikzcd}$$

is commutative.

b) for each pair (V, W) *of objects of* $\mathcal{S}(\mathbf{D})$,

$$e_{Q(V\otimes W)} \circ (Q(\theta_{2(V,W)}) \otimes \theta_{2(V,W)}) = e_{Q(V)\otimes Q(W)}.$$

c) for the unit object I,

$$e_{Q(I)} \circ (Q(\theta_0) \otimes \theta_0) = e_I.$$

Example: Because the objects of $\mathbf{D}$ in the example are generated from V and $\mathbb{C}$ by taking tensor products and applying Q, it is enough to specify e_V, since the other evaluations follow from the axioms. Let $(e_i)_i$ be a basis of V, and denote by $(\overline{e}_i)_i$ the same basis regarded as a basis of $\overline{V}$. Then we set

$$e_V(\overline{e}_i \otimes e_j) = a_{ij},$$

where $a_{ij} \in \mathbb{C}$. Now, by the conjugation axiom a), $e_{\overline{V}}$ is given by the composition $\theta_0^{-1} \circ \overline{e_V} \circ \theta_{2(\overline{V},V)}$. A short calculation using the previous definitions gives:

$$e_{\overline{V}}(e_i \otimes \overline{e}_j) = \overline{a}_{ij}.$$

Similar calculations using the multiplicativity axiom with $W = \mathbb{C}$ give

$$e_{\mathbb{C}}(k_{\mathbb{C}}(r) \otimes s) = \overline{r}s,$$

and for $e_{\overline{\mathbb{C}}}$ we find

$$e_{\overline{\mathbb{C}}}(r \otimes k_{\mathbb{C}}(s)) = r\overline{s}.$$

Finally to obtain, e.g. $e_{V\otimes\overline{V}}$, we apply the multiplicativity axiom to get

$$e_{V\otimes\overline{V}}(\overline{e_i \otimes \overline{e}_j} \otimes e_k \otimes \overline{e}_l) = a_{ik}\overline{a}_{jl}.$$

We still need to check that the conjugation axioms b) and c) are satisfied by these choices. For c) we have:

$$\begin{aligned}
e_{\overline{\mathbb{C}}} \circ (\overline{\theta}_0 \otimes \theta_0)(k_{\mathbb{C}}(r) \otimes s) &= e_{\overline{\mathbb{C}}}((k_{\overline{\mathbb{C}}} \circ \theta_0 \circ k_{\mathbb{C}}^{-1}) \otimes \theta_0)(k_{\mathbb{C}}(r) \otimes s) \\
&= e_{\overline{\mathbb{C}}}(k_{\overline{\mathbb{C}}}k_{\mathbb{C}}(\overline{r}) \otimes k_{\mathbb{C}}(\overline{s})) \\
&= e_{\overline{\mathbb{C}}}(\overline{r} \otimes k_{\mathbb{C}}(\overline{s})) \\
&= \overline{r}s \\
&= e_{\mathbb{C}}(k_{\mathbb{C}}(r) \otimes s),
\end{aligned}$$

and we leave the check of b) to the reader.▲

Definition 3.4. *Given a choice of evaluations* $(e_V)_{V\in \mathrm{Ob}(\mathbf{D})}$ *on an algebraic starting category* $(\mathbf{D}, G, Q)$*, we define a symmetric, strict monoidal category* $(\mathcal{S}(\mathbf{D}), \otimes, I, c)$*, also written* $\mathcal{S}(\mathbf{D})$ *for short, as follows:*

a) $\mathrm{Ob}(\mathcal{S}(\mathbf{D})) = \mathrm{Ob}(\mathbf{D})$,

b) *the morphisms of* $\mathcal{S}(\mathbf{D})$ *are the isomorphisms of* $\mathbf{D}$ *which preserve the evaluations, in the following sense:*

$$e_W \circ (Q(f) \otimes f) = e_V,$$

for $f : V \to W$ *belonging to* $\mathrm{Iso}(\mathbf{D})$,

c) *the composition, identity morphisms, monoidal product* $\otimes$*, unit object* I *and the braiding* c *are inherited from* $\mathbf{D}$.

Theorem 3.5. $(\mathcal{S}(\mathbf{D}), \otimes, I, c)$ *is a symmetric, strict monoidal category.*

Proof. $\mathcal{S}(\mathbf{D})$ is a category, since clearly all identity morphisms are morphisms of $\mathcal{S}(\mathbf{D})$, and since $\mathrm{Mor}(\mathcal{S}(\mathbf{D}))$ is closed under composition: for $f : V \to W$ and $g : W \to U$ morphisms of $\mathcal{S}(\mathbf{D})$ we have:

$$\begin{aligned} e_U \circ (Q(g \circ f) \otimes (g \circ f)) &= e_U \circ ((Q(g) \circ Q(f)) \otimes (g \circ f)) \\ &= e_U \circ ((Q(g) \otimes g) \circ (Q(f) \otimes f)) \\ &= e_W \circ (Q(f) \otimes f) \\ &= e_V \end{aligned}$$

using the interchange law in the second equality. $\mathrm{Mor}(\mathcal{S}(\mathbf{D}))$ is closed under the monoidal product since, for two morphisms of $\mathcal{S}(\mathbf{D})$ $f : V \to W$ and $g : U \to Y$, we have:

$$\begin{aligned} e_{W\otimes Y} \circ (Q(f \otimes g) \otimes f \otimes g) &= (e_W \otimes e_Y) \circ (\mathrm{id}_{Q(W)} \otimes c_{(Q(Y),W)} \otimes \mathrm{id}_Y) \circ \\ &\quad\ (\theta^{-1}_{2(W,Y)} \otimes \mathrm{id}_{W\otimes Y}) \circ (Q(f \otimes g) \otimes f \otimes g) \\ &= (e_W \otimes e_Y) \circ (\mathrm{id}_{Q(W)} \otimes c_{(Q(Y),W)} \otimes \mathrm{id}_Y) \circ \\ &\quad\ (Q(f) \otimes Q(g) \otimes f \otimes g) \circ (\theta^{-1}_{2(V,U)} \otimes \mathrm{id}_{V\otimes U}) \\ &= (e_W \otimes e_Y) \circ (Q(f) \otimes f \otimes Q(g) \otimes g) \circ \\ &\quad\ (\mathrm{id}_{Q(V)} \otimes c_{(Q(U),V)} \otimes \mathrm{id}_U) \circ (\theta^{-1}_{2(V,U)} \otimes \mathrm{id}_{V\otimes U}) \\ &= ((e_W \circ (Q(f) \otimes f)) \otimes (e_Y \circ (Q(g) \otimes g))) \circ \end{aligned}$$

$$
\begin{aligned}
& (\mathrm{id}_{Q(V)} \otimes c_{(Q(U),V)} \otimes \mathrm{id}_U) \circ (\theta^{-1}_{2(V,U)} \otimes \mathrm{id}_{V\otimes U}) \\
= \;& (e_V \otimes e_U) \circ (\mathrm{id}_{Q(V)} \otimes c_{(Q(U),V)} \otimes \mathrm{id}_U) \circ \\
& (\theta^{-1}_{2(V,U)} \otimes \mathrm{id}_{V\otimes U}) \\
= \;& e_{V\otimes U}.
\end{aligned}
$$

The first equality is the multiplicativity axiom, the second is the interchange law and the naturality of θ_2, the third is the interchange law and the naturality of the braiding, the fourth is the interchange law, the fifth is the preservation of the evaluations by f and g, and the last is again the multiplicativity axiom. Since we are assuming strictness the structural isomorphisms are identities and thus $\mathcal{S}(\mathbf{D})$ is monoidal.

Finally $\mathcal{S}(\mathbf{D})$ is symmetric since for any pair of objects (V, W) of $\mathcal{S}(\mathbf{D})$, $c_{(V,W)}$ belongs to $\mathrm{Mor}(\mathcal{S}(\mathbf{D}))$. To show this we need the following equation (writing composition as juxtaposition):

$$
\begin{aligned}
& c_{(Q(V)\otimes V, Q(W)\otimes W)}(\mathrm{id}_{Q(V)} \otimes c_{(Q(W),V)} \otimes \mathrm{id}_W) = \\
= \;& (((\mathrm{id}_{Q(W)} \otimes c_{(Q(V),W)})(c_{(Q(V),Q(W))} \otimes \mathrm{id}_W)) \otimes \mathrm{id}_V)(\mathrm{id}_{Q(V)\otimes Q(W)} \otimes c_{(V,W)}).
\end{aligned} \tag{3.1}
$$

which may be easily derived, using the commutativity of the second hexagonal diagram of the braiding and then the commutativity of the first hexagonal diagram, composing with the expression $\mathrm{id}_{Q(V)} \otimes c_{(Q(W),V)} \otimes \mathrm{id}_W$, and applying the interchange law twice.
Then we have:

$$
\begin{aligned}
e_{V\otimes W} &= (e_V \otimes e_W)(\mathrm{id}_{Q(V)} \otimes c_{(Q(W),V)} \otimes \mathrm{id}_W)(\theta^{-1}_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \\
&= c_{(I,I)}(e_V \otimes e_W)(\mathrm{id}_{Q(V)} \otimes c_{(Q(W),V)} \otimes \mathrm{id}_W)(\theta^{-1}_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \\
&= (e_W \otimes e_V) c_{(Q(V)\otimes V, Q(W)\otimes W)}(\mathrm{id}_{Q(V)} \otimes c_{(Q(W),V)} \otimes \mathrm{id}_W)(\theta^{-1}_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \\
&= (e_W \otimes e_V)(((\mathrm{id}_{Q(W)} \otimes c_{(Q(V),W)})(c_{(Q(V),Q(W))} \otimes \mathrm{id}_W)) \otimes \mathrm{id}_V) \\
&\quad (\mathrm{id}_{Q(V)\otimes Q(W)} \otimes c_{(V,W)})(\theta^{-1}_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \\
&= (e_W \otimes e_V)(\mathrm{id}_{Q(W)} \otimes c_{(Q(V),W)} \otimes \mathrm{id}_V)(c_{(Q(V),Q(W))} \otimes \mathrm{id}_{W\otimes V}) \\
&\quad (\mathrm{id}_{Q(V)\otimes Q(W)} \otimes c_{(V,W)})(\theta^{-1}_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \\
&= (e_W \otimes e_V)(\mathrm{id}_{Q(W)} \otimes c_{(Q(V),W)} \otimes \mathrm{id}_V)(c_{(Q(V),Q(W))} \otimes c_{(V,W)}) \\
&\quad (\theta^{-1}_{2(V,W)} \otimes \mathrm{id}_{V\otimes W}) \\
&= (e_W \otimes e_V)(\mathrm{id}_{Q(W)} \otimes c_{(Q(V),W)} \otimes \mathrm{id}_V)(c_{(Q(V),Q(W))}\theta^{-1}_{2(V,W)}) \otimes c_{(V,W)}) \\
&= (e_W \otimes e_V)(\mathrm{id}_{Q(W)} \otimes c_{(Q(V),W)} \otimes \mathrm{id}_V)((\theta^{-1}_{2(W,V)} Q(c_{(V,W)})) \otimes c_{(V,W)})
\end{aligned}
$$

$$
\begin{aligned}
&= (e_W \otimes e_V)(\mathrm{id}_{Q(W)} \otimes c_{(Q(V),W)} \otimes \mathrm{id}_V)(\theta^{-1}_{2(W,V)} \otimes \mathrm{id}_{W\otimes V}) \\
&\quad (Q(c_{(V,W)}) \otimes c_{(V,W)}) \\
&= e_{W\otimes V}(Q(c_{(V,W)}) \otimes c_{(V,W)}).
\end{aligned}
$$

The first equality is the multiplicativity condition of the evaluation, the second is the insertion of the identity $c_{(I,I)}$, the third is the naturality of the braiding, the fourth is Equation (3.1), the fifth is the insertion of id_V and the interchange law, the sixth and seventh is the interchange law, the eighth is the naturality of θ_2, the ninth is the insertion of $\mathrm{id}_{V\otimes W}$ and the interchange law, and the last is the multiplicativity condition of the evaluation. ■

Example: Regarding the morphisms of $\mathcal{S}(\mathbf{D})$, we will consider in detail the morphisms from V to V. Let $f : V \to V$ be given by

$$f(e_j) = b_{ij} e_i,$$

(using the summation convention over repeated indices). The condition for f to be a morphism of $\mathcal{S}(\mathbf{D})$

$$e_V \circ (\overline{f} \otimes f) = e_V$$

corresponds to the matrix equation:

$$\overline{B}^T AB = A,$$

where $A = [a_{ij}]$ is the matrix corresponding to the evaluation and $B = [b_{ij}]$ is the matrix corresponding to f. Here we use

$$\overline{f}(\overline{e}_i) = \overline{b}_{ki} \,\bar{\cdot}\, \overline{e}_k$$

and

$$e_V(\overline{b}_{ki} \,\bar{\cdot}\, \overline{e}_k \otimes b_{lj} e_l) = \overline{b}_{ki} b_{lj} a_{kl}.$$

In particular, if A is the identity matrix then B is a unitary matrix. ▲

Definition 3.6. *We define a symmetric, strong monoidal endofunctor Q on $\mathcal{S}(\mathbf{D})$ as the restriction to $\mathcal{S}(\mathbf{D})$ of Q defined on $\mathbf{D}$. The* evaluation-preserving category *is the pair $(\mathcal{S}(\mathbf{D}), Q)$, where $\mathcal{S}(\mathbf{D})$ denotes the category with its symmetric, monoidal structure and Q denotes the above monoidal endofunctor on $\mathcal{S}(\mathbf{D})$.*

Theorem 3.7. *Q indeed restricts to a symmetric, strong monoidal endofunctor on $\mathcal{S}(\mathbf{D})$.*

Proof. The conjugation axioms b) and c) imply that $\theta_{2(V,W)}$ for any V, W and θ_0 are morphisms of $\mathcal{S}(\mathbf{D})$. Therefore it is enough to show that $\mathrm{Mor}(\mathcal{S}(\mathbf{D}))$ is closed under Q. Let $f : V \to W$ be a morphism of $\mathcal{S}(\mathbf{D})$. Since f is an isomorphism, $Q(f)$ is an isomorphism too.

Then we have:

$$\begin{aligned}
e_{Q(W)}(Q(Q(f)) \otimes Q(f)) &= \theta_0^{-1} Q(e_W)\theta_{2(Q(W),W)}(Q(Q(f)) \otimes Q(f)) \\
&= \theta_0^{-1} Q(e_W) Q(Q(f) \otimes f)\theta_{2(Q(V),V)} \\
&= \theta_0^{-1} Q(e_V)\theta_{2(Q(V),V)} \\
&= e_{Q(V)},
\end{aligned}$$

using the conjugation axiom a) in the first and last equalities and the naturality of θ_2 in the second equality. ■

The evaluation-preserving category will play an important role in representing topological isomorphisms when we come to describing the TQFT functor. However the actual codomain category $\mathcal{D}$ of the TQFT is constructed directly from the algebraic starting category $\mathbf{D}$.

Definition 3.8. *The category $\mathcal{D}$ is defined as follows:*

- *the objects of $\mathcal{D}$ are all pairs of the form (V, x), where $V \in \mathrm{Ob}(\mathbf{D})$ and $x \in G(V)$, the underlying K-module of V.*

- *the morphisms of $\mathcal{D}$ from (V, x) to (W, y) are all triples (f, x, y), where $f \in \mathrm{Mor}_{\mathbf{D}}(V, W)$, $x \in G(V)$ and $y \in G(W)$, such that $G(f)(x) = y$.*
  (We will frequently write simply $f : (V, x) \to (W, y)$ for morphisms of $\mathcal{D}$, and say that the morphisms of $\mathcal{D}$ preserve elements.)

- *composition and identity morphisms are given by:*

$$\begin{aligned}
(g, y, z) \circ (f, x, y) &= (g \circ f, x, z), \\
\mathrm{id}_{(V,x)} &= (\mathrm{id}_V, x, x).
\end{aligned}$$

A monoidal product, unit object and braiding on $\mathcal{D}$ are defined as follows: monoidal product on objects:

$$(V, x) \otimes (W, y) = (V \otimes W, x \otimes y),$$

monoidal product on morphisms:

$$(f, x, y) \otimes (g, z, w) = (f \otimes g, x \otimes z, y \otimes w),$$

unit object:

$$\widetilde{I} := (I, i), \text{ where } i \text{ is the unit element of } G(I),$$

braiding:

$$c_{((V,x),(W,y))} = (c_{(V,W)}, x \otimes y, y \otimes x).$$

It is straightforward to show:

Theorem 3.9. *$(\mathcal{D}, \otimes, \widetilde{I}, c)$ is a symmetric, strict monoidal category.*

For some TQFT functors only the endofunctor Q on $\mathcal{S}(\mathbf{D})$ will play a role. For others we need a corresponding endofunctor on $\mathcal{D}$.

Definition 3.10. *Given an assignment:*
for any $V \in \mathrm{Ob}(\mathbf{D})$ and any $x \in G(V)$, an element $x_{Q(V)}$ of $G(Q(V))$, such that:

a) *for any $(f, x, y) \in \mathrm{Mor}_{\mathcal{D}}((V, x), (W, y))$*

$$G(Q(f))(x_{Q(V)}) = y_{Q(W)},$$

b) *for any $V, W \in \mathrm{Ob}(\mathbf{D})$, $x \in G(V)$, $y \in G(W)$*

$$G(\theta_{2(V,W)})(x_{Q(V)} \otimes y_{Q(W)}) = (x \otimes y)_{Q(V \otimes W)},$$

c) $G(\theta_0)(i) = i_{Q(I)}$,

we define a strong monoidal endofunctor $(\mathbf{Q}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_0)$, called an extension *of Q on $\mathcal{D}$, as follows:*
on objects:

$$\mathbf{Q}(V, x) = (Q(V), x_{Q(V)}),$$

on morphisms:

$$\mathbf{Q}(f, x, y) = (Q(f), x_{Q(V)}, y_{Q(W)}),$$

and the natural isomorphism $\boldsymbol{\theta}_2$ and the isomorphism $\boldsymbol{\theta}_0$ are given by:

$$\boldsymbol{\theta}_{2((V,x),(W,y))} = (\theta_{2(V,W)}, x_{Q(V)} \otimes y_{Q(W)}, (x \otimes y)_{Q(V \otimes W)}),$$

$$\boldsymbol{\theta}_0 = (\theta_0, i, i_{Q(I)}).$$

Theorem 3.11. $(\mathbf{Q}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_0)$ *defines a symmetric, strong monoidal endofunctor on* $\mathcal{D}$.

Proof. Conditions a), b) and c) guarantee that the relevant morphisms preserve elements, i.e. belong to $\mathrm{Mor}(\mathcal{D})$. ■

Example: For our example the objects, morphisms and monoidal structure of $\mathcal{D}$ are clear from the general theory. An endofunctor $\mathbf{Q}$ on $\mathcal{D}$ is given by the assignment:

$$x_{Q(V)} := k_V(x).$$

It is easy to check that the conditions for the extension apply:
for $f : (V, x) \to (W, y)$,

$$Q(f)(x_{Q(V)}) = (k_W \circ f \circ k_V^{-1})(k_V(x)) = k_W(y) = y_{Q(W)}$$

and, for the structural morphisms of $\mathbf{Q}$,

$$\begin{aligned} \theta_{2(V,W)}(x_{\overline{V}} \otimes y_{\overline{W}}) &= k_{V\otimes W}(k_V^{-1} \otimes k_W^{-1})(k_V(x) \otimes k_W(y)) \\ &= k_{V\otimes W}(x \otimes y) \\ &= (x \otimes y)_{\overline{V\otimes W}} \end{aligned}$$

and

$$\theta_0(1) = k_{\mathbb{C}}(\overline{1}) = k_{\mathbb{C}}(1) = 1_{\overline{\mathbb{C}}}. \blacktriangle$$

We are now able to give the formal definition of the two types of algebraic category which will appear as target categories for the TQFT functors.

Definition 3.12. *Let* $(\mathbf{D}, G, Q)$ *be an algebraic starting category and* $(e_V)_{V\in\mathrm{Ob}(\mathbf{D})}$ *be a choice of evaluations on* $(\mathbf{D}, G, Q)$. *The* algebraic category *for these data is the pair* $(\mathcal{D}, Q)$, *where* $\mathcal{D}$ *denotes the symmetric, strict monoidal category* $(\mathcal{D}, \otimes, \widetilde{I}, c)$ *and* Q *is the monoidal endofunctor on* $\mathcal{S}(\mathbf{D})$ *defined in Definition 3.6.*

Given, in addition, an assignment $(V, x) \mapsto x_{Q(V)}$ *satisfying conditions a)-c) of Definition 3.10, the* full algebraic category *for these data is the pair* $(\mathcal{D}, \mathbf{Q})$, *where* $\mathcal{D}$ *is as before and* $\mathbf{Q}$ *is the monoidal endofunctor on* $\mathcal{D}$ *defined in Definition 3.10.*

Having constructed these two types of algebraic category, corresponding to the two types of topological category defined at the end of Section 2, we are now ready to proceed to the definition of TQFT functors in the next section. However, since the example we have been using in this section is not the most generic, we will conclude with another, more generic, example of an algebraic category based on the category of hermitian spaces.

Example 3.13. *Let* $\mathbf{D}$ *be the category whose objects are hermitian linear spaces, i.e. pairs* (V, h)*, where* V *is a finite dimensional vector space over the field* $\mathbb{K}$ *with an involution* j *and* $h : V \times V \to \mathbb{K}$ *is a non-degenerate hermitian form on* V *(linear in the first argument and* j*-semilinear in the second). The morphisms are just linear maps. The monoidal product of two objects* (V, h) *and* (W, h') *is defined to be the pair* $(V \otimes W, h \otimes h')$*, where* $h \otimes h'$ *is the hermitian form given by:*

$$(h \otimes h')(x_1 \otimes y_1, x_2 \otimes y_2) = h(x_1, x_2)h'(y_1, y_2).$$

The unit object is the pair $(\mathbb{K}, h_1)$*, where* $h_1 : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ *is defined by* $(x, y) \mapsto xj(y)$*. Clearly* $\mathbf{D}$ *is a symmetric monoidal category with the obvious braiding* c*. The monoidal endofunctor* (Q, θ_2, θ_0) *is defined as follows. On objects we have* $Q(V, h) = (V^j, h^j)$*, where* V^j *is the involution vector space (i.e. the same additive group as* V *and with scalar multiplication* $\cdot_j$ *given by* $\alpha \cdot_j x = j(\alpha)x$*, where scalar multiplication in* V *is denoted by juxtaposition), and* h^j *is related to* h *by*

$$h(x, y) = h^j(k_V(y), k_V(x)),$$

where, for each V*,* $k_V : V \to V^j$ *is the* j*-semilinear 'identity' map, satisfying for each* $\alpha \in \mathbb{K}$ *and* $x \in V$*,* $k_V(\alpha x) = j(\alpha) \cdot_j x$ *(see [21, p. 63] for an equivalent condition for Hilbert spaces). On morphisms* Q *acts as follows: for* $f : (V, h) \to (W, h')$ *we have* $Q(f) : (V^j, h^j) \to (W^j, h'^j)$*, with*

$$Q(f) = k_W \circ f \circ k_V^{-1}.$$

The isomorphisms θ_2 *and* θ_0 *are given by:*

$$\theta_{2((V,h),(W,h'))} = k_{V \otimes W} \circ (k_V^{-1} \otimes k_W^{-1})$$

and

$$\theta_0(\alpha) = k_{\mathbb{K}}(j(\alpha)), \text{ for all } \alpha \in \mathbb{K}.$$

Then (Q, θ_2, θ_0) *defines a monoidal endofunctor on* $\mathbf{D}$ *in the same way as in the previous example, where* $\mathbb{K}$ *was* $\mathbb{C}$ *and* j *was complex conjugation.*

To define the category $\mathcal{S}(\mathbf{D})$ we must first specify evaluations $e_{(V,h)}$, for each object (V,h). These may be given in terms of the hermitian structures by:

$$e_{(V,h)}(x \otimes y) = h(y, k_V^{-1}(x)).$$

This choice of evaluations indeed satisfies the multiplication and conjugation axioms. The multiplication axiom follows directly from the definition of $(V,h) \otimes (W,h')$. The conjugation axiom a) is derived by acting on $x \otimes k_V(y) \in V \otimes V^j$:

$$\begin{aligned} Q(e_{(V,h)}) \circ \theta_{2((V^j,h^j),(V,h))}(x \otimes k_V(y)) &= Q(e_{(V,h)})(k_{V^j \otimes V}(k_V(x) \otimes y)) \\ &= k_{\mathbb{K}}(e_{(V,h)}(k_V(x) \otimes y)) \\ &= k_{\mathbb{K}}(h(y,x)) \\ &= \theta_0(h(x,y)) \\ &= \theta_0(h^j(k_V(y), k_V(x))) \\ &= \theta_0 \circ e_{(V^j,h^j)}(x \otimes k_V(y)). \end{aligned}$$

The conjugation axiom b) reads:

$$e_{Q((V,h)\otimes(W,h'))} \circ (Q(\theta_{2((V,h),(W,h'))}) \otimes \theta_{2((V,h),(W,h'))}) = e_{(V^j,h^j)\otimes(W^j,h'^j)}.$$

This equality is proved by applying both sides to $k_{V^j \otimes W^j}(k_V(x) \otimes k_W(y)) \otimes k_V(z) \otimes k_W(w)$ (using the multiplicativity axiom on the right hand side) to obtain $h(x,z)h'(y,w)$ in both cases. Details are left to the reader.
The conjugation axiom c)

$$e_{Q(\mathbb{K},h_1)} \circ (Q(\theta_0) \otimes \theta_0) = e_{(\mathbb{K},h_1)}$$

is derived by acting on $k_{\mathbb{K}}(\alpha) \otimes \beta \in \mathbb{K}^j \otimes \mathbb{K}$:

$$\begin{aligned} e_{(\mathbb{K}^j,h_1^j)} \circ (Q(\theta_0) \otimes \theta_0)(k_{\mathbb{K}}(\alpha) \otimes \beta) &= e_{(\mathbb{K}^j,h_1^j)}(j(\alpha) \otimes k_{\mathbb{K}}(j(\beta))) \\ &= h_1^j(k_{\mathbb{K}}(j(\beta)), k_{\mathbb{K}}(j(\alpha))) \\ &= h_1(j(\alpha) \otimes j(\beta)) \\ &= h_1(\beta, \alpha) \\ &= e_{(\mathbb{K},h_1)}(k_{\mathbb{K}}(\alpha) \otimes \beta). \end{aligned}$$

The morphisms of $\mathcal{S}(\mathbf{D})$ are the isomorphisms of $\mathbf{D}$ which preserve the evaluations: for $f : (V,h) \to (W,h')$ we have $e_{(W,h')} \circ (Q(f) \otimes f) = e_{(V,h)}$. This condition translates to the condition that f preserves the hermitian structures:

$$h'(f(x), f(y)) = h(x,y),$$

for all $(x, y) \in V \times V$. The construction of $\mathcal{D}$ and an endofunctor $\mathbf{Q}$ on $\mathcal{D}$ proceed in an analogous fashion to the previous example. As a final remark we note that this example could easily be adapted to categories of real or complex vector spaces endowed with inner products, giving rise to a consistent choice of evaluations in the same way as above.▲

4. TQFT functors

In this section we will be constructing a class of functors from the topological category to the algebraic category (not yet the respective full categories, which will be done in the next section). The first stage is to define a so-called pre-TQFT functor from the category of subobjects to the evaluation-preserving category, which can be thought of as an algebraic representation of topological isomorphisms amongst subobjects. A TQFT functor from the topological category to the algebraic category is a specific type of extension of the pre-TQFT functor. Its action on isomorphisms is determined by the pre-TQFT functor, and its action on gluing morphisms is given essentially by evaluations on the algebraic side. To take account of the extra structure of the endofunctors in both categories, the definition also involves a natural isomorphism which interpolates between the endofunctors and the pre-TQFT functor. This natural isomorphism makes it possible for some features on the algebraic side to be richer than on the topological side (in contrast with the usual purpose of a functor to simplify things). Thus, for instance, in the example of 2-dimensional TQFT's, P acts trivially on the empty set, whereas Q acts non-trivially on $\mathbb{C}$, and the natural isomorphism provides the additional flexibility for this to be so. In our analysis of the example of 2-dimensional TQFT's at the end of the section we prove a theorem which characterizes a class of TQFT functors for this case.

We now proceed with the details of the construction.

Definition 4.1. *Given a category of subobjects $(\mathcal{S}(\mathbf{C}), P)$ and an evaluation–preserving category $(\mathcal{S}(\mathbf{D}), Q)$, as introduced in Definitions 2.7 and 3.6 respectively, a* pre-TQFT functor *from $(\mathcal{S}(\mathbf{C}), P)$ to $(\mathcal{S}(\mathbf{D}), Q)$ is a pair (Z', η), where $Z' : \mathcal{S}(\mathbf{C}) \to \mathcal{S}(\mathbf{D})$ is a strict symmetric, monoidal functor, denoted on objects by $Z'(A) = V_A$ and on morphisms by $Z'(\alpha) = Z'_\alpha$, and η is a monoidal natural isomorphism*

$$\eta : (Z'P, Z'_{\pi_2}, Z'_{\pi_0}) \to (QZ', \theta_2, \theta_0)$$

from the monoidal functor $Z'P$ to the monoidal functor QZ', i.e. η is a natural isomorphism between the functors $Z'P$ and QZ' such that the following two diagrams, for each pair (A, B) of objects in $\mathcal{S}(\mathbf{C})$,

$$\begin{tikzcd} Z'P(A) \otimes Z'P(B) \arrow[r, "\eta_A \otimes \eta_B"] \arrow[d, "Z'_{\pi_2}"'] & QZ'(A) \otimes QZ'(B) \arrow[d, "\theta_2"] \\ Z'P(A \sqcup B) \arrow[r, "\eta_{A \sqcup B}"'] & QZ'(A \sqcup B) \end{tikzcd}$$

and

$$\begin{tikzcd} I \arrow[r, "Z'_{\pi_0}"] \arrow[dr, "\theta_0"'] & Z'P(E) \arrow[d, "\eta_E"] \\ & QZ'(E) \end{tikzcd}$$

commute in $\mathcal{S}(\mathbf{D})$.

Remark 4.2. *Thus the pre-TQFT functor Z' respects the endofunctors P and Q only up to monoidal natural isomorphism. The need for this weakening will be clear from the following example.*

Example: In our example a pre-TQFT functor Z' is given by the following assignments on objects:

$$V_{C_+} = V, \quad V_{C_-} = \overline{V}, \quad V_\emptyset = \mathbb{C} \quad \text{and} \quad V_{A \sqcup B} = V_A \otimes V_B.$$

The last two assignments express the strictness of Z'. On (iso)morphisms $\alpha : C_+ \to C_+$, $Z'_\alpha : V \to V$ is a morphism of $\mathcal{S}(\mathbf{D})$, i.e. given by a matrix B_α with respect to the basis $(e_i)_i$ of V, satisfying $\overline{B}_\alpha^T A B_\alpha = A$, where A corresponds to the evaluation on V (see the example on page 39). Similar statements hold for $\alpha : C_+ \to C_-$, $C_- \to C_+$ and $C_- \to C_-$, e.g., in the first case, Z'_α is given by B_α satisfying $\overline{B}_\alpha^T \overline{A} B_\alpha = A$.

There are some further restrictions on the matrices B_α, e.g. from functoriality we have

$$B_{\alpha\circ\beta} = B_\alpha B_\beta \quad \text{and} \quad B_{\mathrm{id}_{C_\pm}} = I.$$

Also, for $\alpha : C_+ \to C_-$ given by $\alpha(z) = \overline{z}$, we have $P(\alpha) = \alpha^{-1}$ and hence the corresponding matrix satisfies $\overline{B}_\alpha B_\alpha = I$. However, we will not study the many possibilities in depth, since the extension to a TQFT functor will impose strong extra conditions on the choices for Z'_α. As regards monoidal products and braidings, since Z' is a strict symmetric, monoidal functor we have:

$$Z'_{\alpha\sqcup\beta} = Z'_\alpha \otimes Z'_\beta \quad \text{and} \quad Z'_{c_{(A,B)}} = c_{(V_A,V_B)}.$$

The monoidal natural isomorphism η is given as follows. For $\emptyset$

$$\eta_\emptyset : Z'P(\emptyset) = \mathbb{C} \to QZ'(\emptyset) = \overline{\mathbb{C}}, \quad \eta_\emptyset = \theta_0,$$

for irreducible subobjects

$$\eta_{C_\pm} : Z'P(C_\pm) \to QZ'(C_\pm), \quad \eta_{C_+} = \mathrm{id}_{\overline{V}} \quad \text{and} \quad \eta_{C_-} = \mathrm{id}_V\,,$$

and for monoidal products $A \sqcup B$ with A and B irreducible

$$\eta_{A\sqcup B} : Z'P(A \sqcup B) = V_{P(A)} \otimes V_{P(B)} \to QZ'(A \sqcup B) = \overline{V}_{A\sqcup B}$$

is given by:

$$\eta_{A\sqcup B} = \theta_{2(V_A,V_B)} \circ (\eta_A \otimes \eta_B).$$

Thus η plays the role of interpolating between the monoidal features of "change of orientation" on the topological side (where π_2 and π_0 are trivial), and "passing to the conjugate module" on the algebraic side (where θ_2 and θ_0 are non-trivial).▲

We now come to our main definition, namely the definition of a TQFT functor, which is a suitable extension of a pre-TQFT functor to a functor from $\mathcal{C}$ to $\mathcal{D}$.

Definition 4.3. *Given a topological category $(\mathcal{C}, P)$, an algebraic category $(\mathcal{D}, Q)$ and a pre-TQFT functor (Z', η) from $(\mathcal{S}(\mathbf{C}), P)$ to $(\mathcal{S}(\mathbf{D}), Q)$, a* TQFT functor *from $(\mathcal{C}, P)$ to $(\mathcal{D}, Q)$ extending (Z', η) is a pair (Z, η), where η is as before and Z is a symmetric, strict monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ determined by assignments:*

Z1) to each object (X, A, m) of $\mathcal{C}$ an object $Z(X, A, m) = (V_A, Z_{(X,A,m)})$ of $\mathcal{D}$, where $V_A = Z'(A)$, and such that, for any pair of objects (X, A, m), (X', A', m') of $\mathcal{C}$

$$Z_{(X\sqcup X',A\sqcup A',m\sqcup m')} = Z_{(X,A,m)} \otimes Z_{(X',A',m')},$$

*Z*2) *to each isomorphism* $(f,\alpha):(X,A,m)\to(X',A',m')$ *of* $\mathcal{C}$ *the isomorphism* $Z_{(f,\alpha)}:(V_A,Z_{(X,A,m)})\to(V_{A'},Z_{(X',A',m')})$, *given by:*

$$Z_{(f,\alpha)}=Z'_\alpha$$

and to each gluing morphism of $\mathcal{C}$, $(f,\varphi,\alpha):(X,A,m)\to(X',A',m')$, *where* $\varphi:A_I\to P(A)_J$ *and* $\alpha:A_R\to A'$, *the morphism* $Z_{(f,\varphi,\alpha)}:(V_A,Z_{(X,A,m)})\to(V_{A'},Z_{(X',A',m')})$, *given by:*

$$Z_{(f,\varphi,\alpha)}=(e_{V_{A_J}}\otimes Z'_\alpha)\circ p'\circ((\eta_{A_J}\circ Z'_\varphi)\otimes \mathrm{id}_{V_{A_J}\otimes V_{A_R}})\circ p,$$

where $p:V_A\to V_{A_I}\otimes V_{A_J}\otimes V_{A_R}$ *and* $p':Q(V_A)_J\otimes V_{A_J}\otimes V_{A_R}\to(\otimes_{j\in J}(Q(V_{A_j})\otimes V_{A_j}))\otimes V_{A_R}$ *are the appropriate permuting isomorphisms, and denoting* $Q(V_A)_J:=\otimes_{j\in J}Q(V_{A_j})$, $e_{V_{A_J}}:=\otimes_{j\in J}e_{V_{A_j}}$ *and* $\eta_{A_J}:=\otimes_{j\in J}\eta_{A_j}$.

*Z*3) *to the unit object* (E,E,id_E), *the object*

$$Z(E,E,\mathrm{id}_E)=(I,i).$$

Remark 4.4. *A TQFT functor is then given by choosing the elements* $Z_{(X,A,m)}$ *consistently. Apart from the monoidal restrictions in* $Z1$) *and* $Z3$), *every morphism of* $\mathcal{C}$ *gives rise to an equation for the elements* $Z_{(X,A,m)}$ *through the condition* $Z2$). *Neither* $Z_{(f,\alpha)}$ *nor* $Z_{(f,\varphi,\alpha)}$ *depend explicitly on* f, *apart from the fact that* f *makes up a valid morphism of* $\mathcal{C}$, *together with* α, *or* φ *and* α. *The formula for* $Z_{(f,\varphi,\alpha)}$ *may be loosely described by saying that gluing on the topological side corresponds to evaluation on the algebraic side.*

Theorem 4.5. *A choice of assignments satisfying* $Z1$)-$Z3$) *determines a symmetric, strict monoidal functor* Z *from* $\mathcal{C}$ *to* $\mathcal{D}$.

Proof. We show that $Z2$) is functorial for the composition of two gluing morphisms. The other cases of composition can easily be checked. To simplify the proof we consider the special case of the following gluing morphisms $(X,A,m)\overset{(f,\varphi,\alpha)}{\to}(X',A',m')\overset{(g,\psi,\beta)}{\to}(X'',A'',m'')$, where $A=\sqcup_{i=1,\dots,5}A_i$, $A'=\sqcup_{i=3,4,5}A'_i$, $\varphi:A_1\to P(A_2)$, $\psi:A'_3\to P(A'_4)$, $\alpha=\sqcup_{i=3,4,5}\alpha_i$, with $\alpha_i:A_i\to A'_i$ and $\beta:A'_5\to A''$. We redefine the respective objects by $V_{A_1}=V$, $V_{A_2}=W$, $V_{A_3}=X$, $V_{A_4}=Y$, $V_{A_5}=T$, $V_{A'_3}=X'$, $V_{A'_4}=Y'$, $V_{A'_5}=T'$ and $V_{A''_5}=T''$, and write θ instead of θ_2. In the following calculations composition is denoted by juxtaposition, and compositions are performed before the monoidal product inside any bracketed expression:

$$
\begin{aligned}
Z_{(g,\psi,\beta)} Z_{(f,\varphi,\alpha)} &= (e_{Y'} \otimes Z'_\beta)(\eta_{A'_4} Z'_\psi \otimes \mathrm{id}_{Y' \otimes T'})(e_W \otimes Z'_\alpha)(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_{W \otimes X \otimes Y \otimes T}) \\
&= (e_{Y'} \otimes Z'_\beta)(\eta_{A'_4} Z'_\psi \otimes \mathrm{id}_{Y' \otimes T'})(e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W) \otimes Z'_\alpha) \\
&= e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W) \otimes e_{Y'}(\eta_{A'_4} Z'_\psi \otimes \mathrm{id}_{Y'})(Z'_{\alpha_3} \otimes Z'_{\alpha_4}) \otimes Z'_{\beta\alpha_5} \\
&= e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W) \otimes e_{Y'}(\eta_{A'_4} Z'_{\psi\alpha_3} \otimes Z'_{\alpha_4}) \otimes Z'_{\beta\alpha_5} \\
&= e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W) \otimes e_{Y'}(\eta_{A'_4} Z'_{P(\alpha_4)} Z'_{\widetilde{\psi}} \otimes Z'_{\alpha_4}) \otimes Z'_{\beta\alpha_5} \\
&= e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W) \otimes e_{Y'}(Q(Z'_{\alpha_4})\eta_{A_4} Z'_{\widetilde{\psi}} \otimes Z'_{\alpha_4}) \otimes Z'_{\beta\alpha_5} \\
&= e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W) \otimes e_{Y'}(Q(Z'_{\alpha_4}) \otimes Z'_{\alpha_4})(\eta_{A_4} Z'_{\widetilde{\psi}} \otimes \mathrm{id}_Y) \otimes Z'_{\beta\alpha_5} \\
&= e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W) \otimes e_Y(\eta_{A_4} Z'_{\widetilde{\psi}} \otimes \mathrm{id}_Y) \otimes Z'_{\beta\alpha_5} \\
&= (e_W \otimes e_Y \otimes Z'_{\beta\alpha_5})(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W \otimes \eta_{A_4} Z'_{\widetilde{\psi}} \otimes \mathrm{id}_{Y \otimes T}).
\end{aligned}
$$

The first equality is the definition of the gluing morphism, the second, third and fourth are the interchange law, the fifth is the equation $P(\alpha_4) \circ \widetilde{\psi} = \psi \circ \alpha_3$, the sixth is the naturality of η, and the seventh, eight and ninth are the interchange law.

For the other side, we have:

$$
\begin{aligned}
Z_{(gf,\varphi \sqcup \widetilde{\psi},\beta\alpha_5)} &= (e_W \otimes e_Y \otimes Z'_{\beta\alpha_5})(\mathrm{id}_W \otimes c_{(Q(Y),W)} \otimes \mathrm{id}_{Y \otimes T}) \\
&\quad ((\eta_{A_2} \otimes \eta_{A_4}) Z'_{\varphi \sqcup \widetilde{\psi}} \otimes \mathrm{id}_{W \otimes Y \otimes T})(\mathrm{id}_V \otimes c_{(W,X)} \otimes \mathrm{id}_{Y \otimes T}) \\
&= (e_W \otimes e_Y \otimes Z'_{\beta\alpha_5})(\mathrm{id}_W \otimes c_{(Q(Y),W)} \otimes \mathrm{id}_{Y \otimes T}) \\
&\quad (\eta_{A_2} Z'_\varphi \otimes \eta_{A_4} Z'_{\widetilde{\psi}} \otimes \mathrm{id}_{W \otimes Y \otimes T})(\mathrm{id}_V \otimes c_{(W,X)} \otimes \mathrm{id}_{Y \otimes T}) \\
&= (e_W \otimes e_Y \otimes Z'_{\beta\alpha_5})(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W \otimes \eta_{A_4} Z'_{\widetilde{\psi}} \otimes \mathrm{id}_{Y \otimes T}).
\end{aligned}
$$

The first equality is the definition of the gluing morphism, the second is the interchange law, and the third is the naturality and the symmetry of c. The functor Z preserves identity morphisms, since by $Z2)$

$$Z(\mathrm{id}_X, \mathrm{id}_A) = Z'(\mathrm{id}_A) = \mathrm{id}_{V_A}\,.$$

Therefore we conclude that Z is a functor.

The functor Z is strict monoidal on objects from $Z1)$ and $Z3)$. To simplify the proof that Z is monoidal on morphisms, we just consider the monoidal product of the following two gluing morphisms $(X, \sqcup_{i=1,2,3} A_i, m) \overset{(f,\varphi,\alpha)}{\longrightarrow} (X', A'_3, m')$ and

$(Y, \sqcup_{i=1,2,3} B_i, n) \stackrel{(g,\psi,\beta)}{\to} (Y', B_3', n')$, where $\varphi : A_1 \to P(A_2)$, $\psi : B_1 \to P(B_2)$, $\alpha : A_3 \to A_3'$ and $\beta : B_3 \to B_3'$, and we set $V_{A_1} = V$, $V_{A_2} = W$, $V_{A_3} = S$, $V_{B_1} = X$, $V_{B_2} = Y$ and $V_{B_3} = T$.

We have:

$$
\begin{aligned}
Z_{(f,\varphi,\alpha)\sqcup(g,\psi,\beta)} &= Z_{(f\sqcup g,\varphi\sqcup\psi,\alpha\sqcup\beta)} \\
&= (e_W \otimes e_Y \otimes Z'_{\alpha\sqcup\beta})p'((\eta_{A_2} \otimes \eta_{B_2})(Z'_\varphi \otimes Z'_\psi) \otimes \mathrm{id}_{W\otimes Y\otimes S\otimes T})p \\
&= (e_W \otimes e_Y \otimes Z'_\alpha \otimes Z'_\beta)p'(\eta_{A_2} Z'_\varphi \otimes \eta_{B_2} Z'_\psi \otimes \mathrm{id}_{W\otimes Y\otimes S\otimes T})p \\
&= (e_W \otimes e_Y \otimes Z'_\alpha \otimes Z'_\beta)\widetilde{p}(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_{W\otimes S} \otimes \eta_{B_2} Z'_\psi \otimes \mathrm{id}_{Y\otimes T}) \\
&= (e_W \otimes Z'_\alpha \otimes e_Y \otimes Z'_\beta)(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_{W\otimes S} \otimes \eta_{B_2} Z'_\psi \otimes \mathrm{id}_{Y\otimes T}) \\
&= (e_W \otimes Z'_\alpha)(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_{W\otimes S}) \otimes (e_Y \otimes Z'_\beta)(\eta_{B_2} Z'_\psi \otimes \mathrm{id}_{Y\otimes T}) \\
&= Z_{(f,\varphi,\alpha)} \otimes Z_{(g,\psi,\beta)}
\end{aligned}
$$

where, in the second, third and fourth equalities p, p' and $\widetilde{p}$ are the permuting isomorphisms

$$
\begin{aligned}
p &= \mathrm{id}_V \otimes (c_{(W,X)} \otimes \mathrm{id}_{Y\otimes S})(\mathrm{id}_W \otimes c_{(S,X\otimes Y)}) \otimes \mathrm{id}_T\,, \\
p' &= \mathrm{id}_{Q(W)} \otimes c_{(Q(Y),W)} \otimes \mathrm{id}_{Y\otimes S\otimes T}\,, \\
\widetilde{p} &= \mathrm{id}_{Q(W)\otimes W} \otimes c_{(S,Q(Y)\otimes Y)} \otimes \mathrm{id}_T\,,
\end{aligned}
$$

and we use naturality of the braiding in the third and fourth equalities, and the interchange law in the second and fifth equalities. Finally, since Z' is symmetric Z is also symmetric:

$$
\begin{aligned}
Z(c_{((X,A,m),(X',A',m'))}) &= Z((c_{(X,X')}, c_{(A,A')})) \\
&= Z'(c_{(A,A')}) \\
&= c_{(Z'(A),Z'(A'))} \\
&= c_{((Z'(A),Z_{(X,A,m)}),(Z'(A'),Z_{(X',A',m')}))} \\
&= c_{(Z_{(X,A,m)},Z_{(X',A',m')})}.
\end{aligned}
$$

This completes the proof. ■

Example: The possibility of extending a pre-TQFT functor Z' to a TQFT functor Z imposes strong restrictions on Z' as we will now discuss.

We consider first the gluing morphism

$$(f, \varphi, \alpha) : (D_{-+}, C_{-+}, m) \to (S_+, \emptyset, m')$$

corresponding to the gluing together of two disks to make a sphere, with $\varphi = \mathrm{id}_{C_-}$, described on page 21. Set $Z_{(D_+,C_+,m_+)} = d_i e_i \in V$ (again, and also in what follows, using the summation convention over repeated indices) and $Z_{(D_-,C_-,m_-)} = \delta_j \,\bar{\cdot}\, \overline{e}_j \in \overline{V}$, where $m = m_+ \sqcup m_-$. Then by $Z1$) and $Z2$)

$$\begin{aligned} Z_{(S_+,\emptyset,m')} &= Z_{(f,\varphi,\alpha)}(\delta_j \,\bar{\cdot}\, \overline{e}_j \otimes d_i e_i) \\ &= e_V(Z'_{\mathrm{id}_{C_-}} \otimes \mathrm{id}_V)(\delta_j \,\bar{\cdot}\, \overline{e}_j \otimes d_i e_i) \\ &= e_V(\delta_j \,\bar{\cdot}\, \overline{e}_j \otimes d_i e_i) = \delta_j d_i a_{ji}, \end{aligned}$$

where a_{ij} are the entries of the matrix corresponding to e_V - see page 36. Now suppose we glue the two disks together "with a twist", i.e. with a different identification along their boundaries. The result is still a sphere and the process is described by a gluing morphism of the form:

$$(g, \psi, \beta) : (D_{-+}, C_{-+}, m) \to (S_+, \emptyset, m'),$$

where $\psi : C_- \to C_-$ is no longer id_{C_-} and g is chosen to be compatible with ψ. This gives rise to the equation:

$$\begin{aligned} Z_{(S_+,\emptyset,m')} &= Z_{(g,\psi,\beta)}(\delta_j \,\bar{\cdot}\, \overline{e}_j \otimes d_i e_i) \\ &= e_V(Z'_\psi(\delta_j \,\bar{\cdot}\, \overline{e}_j) \otimes d_i e_i) \\ &= b_{kj}\delta_j d_i a_{ki}, \end{aligned}$$

where $B_\psi = [b_{ij}]$ is the matrix representing Z'_ψ with respect to the bases $(\overline{e}_i)_i$. More generally, for the gluing together of any two objects of the form (X, C_-, m_1) and (Y, C_+, m_2), we have, setting

$$Z_{(X,C_-,m_1)} = x_j \,\bar{\cdot}\, \overline{e}_j \quad \text{and} \quad Z_{(Y,C_+,m_2)} = y_i e_i,$$

the equation

$$b_{kj} x_j y_i a_{ki} = x_k y_i a_{ki},$$

for any $B_\psi = [b_{ij}]$. Therefore we will impose the following requirement on Z':

$$\text{for all} \quad \psi \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_-, C_-), \quad Z'_\psi = \mathrm{id}_{\overline{V}}\,,$$

which in turn implies, setting $\varphi = P(\psi) : C_+ \to C_+$ and using $Z'_\varphi = \overline{Z'_\psi}$,

$$\text{for all} \quad \varphi \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_+, C_+), \quad Z'_\varphi = \mathrm{id}_V\,.$$

Before considering constraints on Z'_φ for the remaining morphisms of $\mathcal{S}(\mathbf{C})$, we will derive a constraint on e_V, by considering gluing two disks together with their order swapped, described by the gluing morphism:

$$(h, \chi, \alpha) : (D_{+-}, C_{+-}, m_+ \sqcup m_-) \to (S_+, \emptyset, m'),$$

where $h(z, 1) = f(z, 2)$ and $h(z, 2) = f(z, 1)$ (see above and page 21) and $\chi = \mathrm{id}_{C_+}$. Applying the TQFT functor we get the equation:

$$\begin{aligned} Z_{(S_+,\emptyset,m')} &= Z_{(h,\chi,\alpha)}(d_i e_i \otimes \delta_j \,\bar{\cdot}\, \overline{e}_j) \\ &= e_{\overline{V}}(d_i e_i \otimes \delta_j \,\bar{\cdot}\, \overline{e}_j) \\ &= d_i \delta_j \overline{a}_{ij}, \end{aligned}$$

using the relation between e_V and $e_{\overline{V}}$ described on page 36 in the last equality. Comparing with the previous equation for $Z_{(S_+,\emptyset,m')}$, it is natural to impose

$$A = \overline{A}^T,$$

for the matrix A corresponding to the evaluation.

Now, given any $\alpha \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_+, C_-)$ and $\beta \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_-, C_+)$, we have from the requirements above:

$$Z'_\alpha \circ Z'_\beta = \mathrm{id}_{\overline{V}} \quad \text{and} \quad Z'_\beta \circ Z'_\alpha = \mathrm{id}_V \,.$$

Thus Z'_α and Z'_β are independent of α and β, respectively, and represented by the matrices B and B^{-1}, respectively, with respect to the bases $(e_i)_i$ and $(\overline{e}_i)_i$ of V and $\overline{V}$. Furthermore, from the example starting on page 46, we have the condition $\overline{B}B = I$, and so a natural choice is

$$B = I,$$

assuming that this choice satisfies the condition for Z'_α to belong to $\mathrm{Mor}(\mathcal{S}(\mathbf{D}))$, namely $\overline{B}^T \overline{A} B = A$. Thus we must have:

$$\overline{A} = A,$$

whilst we already had $\overline{A}^T = A$, so that A has to be symmetric and real. A natural choice, which also ensures non-degeneracy of the evaluation on V, is $A = I$.

Summarizing the above considerations, we will from now on restrict our attention to TQFT functors (Z, η) which extend the pre-TQFT functor (Z', η), where Z' satisfies:

$$\begin{array}{lll} \text{for all} & \alpha \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_+, C_+), & Z'_\alpha = \mathrm{id}_V \\ \text{for all} & \alpha \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_-, C_-), & Z'_\alpha = \mathrm{id}_{\overline{V}} \\ \text{for all} & \alpha \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_+, C_-), & Z'_\alpha(e_i) = \overline{e}_i \\ \text{for all} & \alpha \in \mathrm{Mor}_{\mathcal{S}(\mathbf{C})}(C_-, C_+), & Z'_\alpha(\overline{e}_i) = e_i, \end{array} \tag{4.1}$$

where η (introduced in the example starting on page 46) satisfies:

$$\begin{aligned} \eta_\emptyset &= \theta_0, \\ \eta_{C_+} &= \mathrm{id}_{\overline{V}} \quad \text{and} \quad \eta_{C_-} = \mathrm{id}_V\,, \\ \eta_{A \sqcup B} &= \theta_{2(V_A, V_B)} \circ (\eta_A \otimes \eta_B), \end{aligned} \tag{4.2}$$

and for which the evaluation in the algebraic category is given by:

$$e_V(\overline{e}_i \otimes e_j) = \delta_{ij}, \tag{4.3}$$

where δ_{ij} is the Kronecker symbol.

Having made these choices, we investigate some relations arising from topological isomorphisms. Consider first the isomorphism

$$(f, \alpha) : (S_+, \emptyset, m) \to (S_-, \emptyset, m'),$$

where m and m' are the respective empty maps, $\alpha = \mathrm{id}_\emptyset$, and f is given by $(x, y, z) \mapsto (-x, y, z)$. Under the TQFT functor (f, α) goes to the morphism of $\mathcal{D}$:

$$Z_{(f,\alpha)} : (\mathbb{C}, Z_{(S_+,\emptyset,m)}) \to (\mathbb{C}, Z_{(S_-,\emptyset,m')}).$$

Since $Z_{(f,\alpha)} = Z'_\alpha = \mathrm{id}_\mathbb{C}$, this implies the equation:

$$Z_{(S_+,\emptyset,m)} = Z_{(S_-,\emptyset,m')}.$$

A similar argument applied to the torus (and indeed to any manifold with empty boundary, as we will see shortly) gives:

$$Z_{(T_+,\emptyset,m)} = Z_{(T_-,\emptyset,m')}.$$

Next we will consider four objects in $\mathcal{C}$ associated with the disk D:

$$(D_+, C_+, m_1), \quad (D_-, C_-, m_2), \quad (D_+, C_-, m_3), \quad (D_-, C_+, m_4),$$

where $m_1(z) = m_2(z) = z$ and $m_3(z) = m_4(z) = \overline{z}$. Let $Z_{(D_+,C_+,m_1)} = d_i e_i \in V$. The isomorphism

$$(\mathrm{id}_D, \alpha) : (D_+, C_+, m_1) \to (D_+, C_-, m_3),$$

where $\alpha : C_+ \to C_-$ is given by $\alpha(z) = \overline{z}$, yields the equation:

$$Z_{(D_+,C_-,m_3)} = Z'_\alpha(d_i e_i) = d_i \mathbin{\bar{\cdot}} \overline{e}_i \in \overline{V}$$

(using the third equation of (4.1) above for Z'_α). The reverse map described in the example on page 19:

$$(r, \alpha) : (D_+, C_+, m_1) \to (D_-, C_-, m_2),$$

with $r(z) = \alpha(z) = \overline{z}$, gives rise to the equation:

$$Z_{(D_-,C_-,m_2)} = d_i \mathbin{\bar{\cdot}} \overline{e}_i \in \overline{V}.$$

Similarly

$$Z_{(D_-,C_+,m_4)} = d_i e_i \in V.$$

Thus it is enough to fix Z of one of the objects in order to determine Z of the other three.

Also, taking different monomorphisms m does not introduce anything new. For instance, consider the object $(D_+, C_+, \widetilde{m})$, where $\widetilde{m} \neq m_1$. Then there exists an automorphism $\alpha : C_+ \to C_+$ such that $\widetilde{m} = m_1 \circ \alpha$, i.e. we have an isomorphism

$$(\mathrm{id}_D, \alpha) : (D_+, C_+, \widetilde{m}) \to (D_+, C_+, m_1).$$

Since $Z_{(\mathrm{id}_D,\alpha)} = Z'_\alpha = \mathrm{id}_V$ by the first equation of (4.1) above, we have

$$Z_{(D_+,C_+,\widetilde{m})} = Z_{(D_+,C_+,m_1)} = d_i e_i \in V,$$

for any $\widetilde{m}$.

The annulus can appear as an object of $\mathcal{C}$ in sixteen different versions, by varying the orientation of the annulus, the orientations of the two subobject circles and the two ways of associating the subobject circles to the boundary circles of the annulus. These are all related by isomorphisms, which means that after choosing Z of one of them, Z of all the other combinations is determined. For instance, the isomorphism:

$$(\mathrm{id}_{A_+}, c_{(C_-,C_+)}) : (A_+, C_{-+}, m) \to (A_+, C_{+-}, m'),$$

where $m(z,1) = z = m'(z,2)$ and $m(z,2) = 2z = m'(z,1)$, implies the equation:

$$Z_{(A_+,C_{+-},m')} = c_{(\overline{V},V)}(Z_{(A_+,C_{-+},m)})$$

(since Z' and Z are symmetric monoidal). To pass to the other orientation of the annulus, we have the isomorphism:

$$(f, c_{(C_-,C_+)}) : (A_+, C_{-+}, m) \to (A_-, C_{+-}, m''),$$

where $f(re^{i\theta}) = (3-r)e^{i\theta}$, and m is given by $m(z,1) = z = m''(z,1)$ and $m(z,2) = 2z = m''(z,2)$, which implies the equation:

$$Z_{(A_-,C_{+-},m'')} = c_{(\overline{V},V)}(Z_{(A_+,C_{-+},m)}).$$

Finally we give a couple more examples of equations arising from gluing morphisms before giving a general result. The gluing morphism from the annulus to the torus

$$(g, \psi, \beta) : (A_+, C_{-+}, m) \to (T_+, \emptyset, m'),$$

described in example c) on page 22, leads to the relation:

$$Z_{(T_+,\emptyset,m')} = Z_{(g,\psi,\beta)}(Z_{(A_+,C_{-+},m)}).$$

Writing

$$Z_{(A_+,C_{-+},m)} = c_{ij}\overline{e}_i \otimes e_j \in \overline{V} \otimes V$$

and using Condition $Z2$) and Equation (4.3) this determines $Z_{(T_+,\emptyset,m')}$ in terms of $Z_{(A_+,C_{-+},m)}$:

$$Z_{(T_+,\emptyset,m')} = c_{ii} \in \mathbb{C}.$$

The gluing together of two annuli to make an annulus, may be described by the gluing morphism:

$$(f, \varphi, \alpha) : (A_{++}, C_{-+-+}, m \sqcup m) \to (A_+, C_{-+}, m),$$

with m as above, f given by

$$f(re^{i\theta}, 1) = \frac{r+1}{2} e^{i\theta} \quad \text{and} \quad f(re^{i\theta}, 2) = \frac{r+2}{2} e^{i\theta},$$

$I = \{2\}$, $J = \{3\}$, $\varphi = \mathrm{id}_{C_+}$ and $\alpha = \mathrm{id}_{C_{-+}}$. The corresponding morphism in $\mathcal{D}$ implies the condition:

$$Z_{(f,\varphi,\alpha)}(c_{ik} c_{lj} \overline{e}_i \otimes e_k \otimes \overline{e}_l \otimes e_j) = c_{ij} \overline{e}_i \otimes e_j,$$

which, using Condition $Z2$) and the equation $e_{\overline{V}}(e_k \otimes \overline{e}_l) = \overline{\delta}_{kl} = \delta_{kl}$ (coming from Equation (4.2) and the example on page 36) leads to the well-known constraint on the components c_{ij}:

$$c_{ik} c_{kj} = c_{ij}.$$

To conclude the example in this section we will now proceed to characterize the TQFT functors Z under the Conditions (4.1)-(4.3) above. First of all we show how various different topological gluing scenarios give the same algebraic result. Suppose we have a gluing morphism

$$(f, \varphi, \alpha) : (X, C_{-+}, m) \to (X', \emptyset, m'),$$

where X is a 2-manifold with two circles as its boundary, and $I = \{1\}$, $J = \{2\}$, $\varphi = \mathrm{id}_{C_-}$ and $\alpha = \mathrm{id}_{\emptyset}$. Fixing f, we can change the orientation of the boundary circles of the domain object:

$$(f, \psi, \alpha) : (X, C_{+-}, \overline{m}) \to (X', \emptyset, m'),$$

with $I = \{1\}$, $J = \{2\}$, $\psi = P(\varphi) = \mathrm{id}_{C_+}$, $\overline{m}(z,1) = m(\overline{z},1)$ and $\overline{m}(z,2) = m(\overline{z},2)$, or permute the boundary components in either orientation

$$(f, \chi, \alpha) : (X, C_{+-}, m_c) \to (X', \emptyset, m'),$$

where $m_c = m \circ c_{(C_+, C_-)}$, $I = \{2\}$, $J = \{1\}$, $\chi = \mathrm{id}_{C_-}$ and

$$(f, \rho, \alpha) : (X, C_{-+}, \overline{m}_c) \to (X', \emptyset, m'),$$

where $\overline{m}_c = \overline{m} \circ c_{(C_-, C_+)}$, $I = \{2\}$, $J = \{1\}$, $\rho = \mathrm{id}_{C_+}$. As we have seen above, isomorphisms between the different objects built from X imply relations. If $Z_{(X, C_{-+}, m)} = x_{ij} \overline{e}_i \otimes e_j$ then

$$\begin{aligned} Z_{(X, C_{+-}, \overline{m})} &= x_{ij} e_i \otimes \overline{e}_j, \\ Z_{(X, C_{+-}, m_c)} &= x_{ij} e_j \otimes \overline{e}_i, \\ Z_{(X, C_{-+}, \overline{m}_c)} &= x_{ij} \overline{e}_j \otimes e_i. \end{aligned}$$

However, applying the TQFT functor to the previous gluing morphisms, these elements are all mapped to $Z_{(X',\emptyset,m')} = x_{ii}$, since $e_V(\overline{e}_i \otimes e_j) = \delta_{ij} = e_{\overline{V}}(e_i \otimes \overline{e}_j)$. The same independence of the description holds for objects X with more than two boundary circles when we glue two of the boundary components together.

We will shortly be needing the following result on decomposing surfaces. Let X be a surface of genus g with n boundary circles. A marking on X is a set of $3g-3+n$ non-contractible pairwise non-isotopic circles on X, considered up to isotopy. Cutting along these circles decomposes X into $2g-2+n$ pairs-of-pants. The only surfaces for which there is no such pants decomposition are the sphere S, the disk D, the annulus A and the torus T. In [37, Lemma 1.2] Kohno proved that any two markings of X can be obtained from each other by a finite sequence of moves of the following two types:

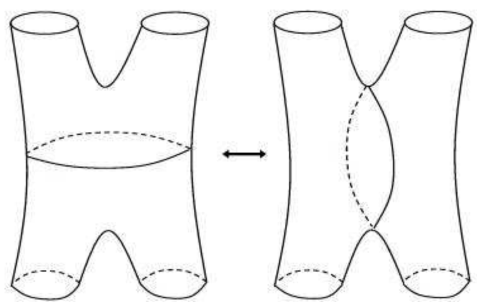

Figure 4.1: Type I move on markings

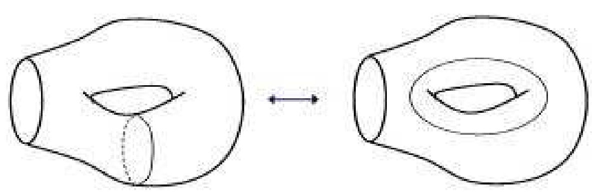

Figure 4.2: Type II move on markings

Now we can state our general result:

Theorem 4.6. *TQFT functors (Z, η), which extend (Z', η) for which the Conditions (4.1)-(4.3) hold, are in 1-1 correspondence with pairs of complex-valued tensors d_i, p_{ijk} satisfying the relations:*

1. $p_{ijk}p_{klm} = p_{ilk}p_{kjm}$,

2. $p_{ijk} = p_{jik} = p_{ikj}$,

3. $d_k p_{kij} d_j = d_i$,

4. $d_k p_{kij} p_{jlm} = p_{ilm}$.

Proof. Given such a TQFT functor (Z, η), we define d_i, p_{ijk} by

$$\begin{aligned} Z_{(D_+, C_+, m)} &= d_i e_i, \\ Z_{(P_+, C_{+++}, m')} &= p_{ijk} e_i \otimes e_j \otimes e_k. \end{aligned}$$

Relation 1 comes from looking at the two different ways of obtaining the "quaternion" surface Q in the type I move above by gluing two pairs-of-pants. Let us label the boundary circles of Q by i, j, m and l, starting from the top left hand component and going round anticlockwise. Consider the object $(Q_+, C_{++++}, m_{ijlm})$, where the notation m_{ijlm} means that the first circle get mapped to the boundary component labelled i, and so on. This object is obtained from a gluing morphism, say (f, φ, α), with domain $(P_{++}, C_{++-+++}, m_{ijknlm})$ and via a different gluing morphism , say (g, ψ, β), from $(P_{++}, C_{-+++++}, m_{knijlm})$.
Now, applying the TQFT functor, we get the equations

$$\begin{aligned} Z_{(Q_+, C_{++++}, m_{ijlm})} &= Z_{(f,\varphi,\alpha)}(p_{ijk}p_{nlm} e_i \otimes e_j \otimes \overline{e}_k \otimes e_n \otimes e_l \otimes e_m) \\ &= p_{ijk}p_{klm} e_i \otimes e_j \otimes e_l \otimes e_m \end{aligned}$$

and

$$\begin{aligned} Z_{(Q_+, C_{++++}, m_{ijlm})} &= Z_{(g,\psi,\beta)}(p_{ilk}p_{njm} \overline{e}_k \otimes e_n \otimes e_i \otimes e_j \otimes e_l \otimes e_m) \\ &= p_{ilk}p_{kjm} e_i \otimes e_j \otimes e_l \otimes e_m, \end{aligned}$$

and thus we have Relation 1.

The first equality in Relation 2 follows by applying Z to the isomorphism

$$(\mathrm{id}_{P_+}, c_{(C_+, C_+)} \sqcup \mathrm{id}_{C_+}) : (P_+, C_{+++}, m_{ijk}) \to (P_+, C_{+++}, m_{jik})$$

giving

$$(c_{(V,V)} \otimes \mathrm{id}_V)(p_{ijk} e_i \otimes e_j \otimes e_k) = p_{jik} e_j \otimes e_i \otimes e_k$$

i.e.

$$p_{ijk} = p_{jik}.$$

The second equality in Relation 2 is proved in identical fashion.

The third relation is shown by considering a gluing morphism

$$(f, \varphi, \alpha) : (P_+ \sqcup D_{++}, C_{-+-++}, m_{lkmji}) \to (D_+, C_+, m_i),$$

with the boundary circles labelled as in Fig. 4.3 and with $I = \{1, 3\}$, $J = \{2, 4\}$.

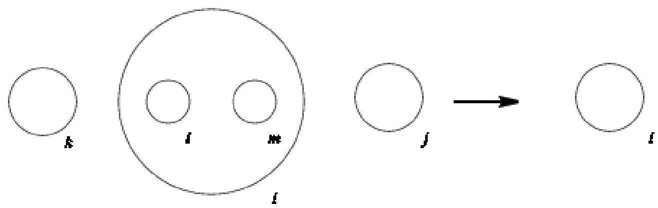

Figure 4.3: Relation 3

Applying Z gives the equation:

$$Z_{(f,\varphi,\alpha)}(d_k p_{lim} d_j \overline{e}_l \otimes e_k \otimes \overline{e}_m \otimes e_j \otimes e_i) = d_i e_i$$

i.e.

$$d_k p_{kij} d_j = d_i.$$

Finally the fourth relation is shown by considering a gluing morphism

$$(f, \varphi, \alpha) : (D_+ \sqcup P_{++}, C_{-+-++++}, m_{krjsilm}) \to (P_+, C_{+++}, m),$$

with the boundary circles labelled as in Fig. 4.4 and with $I = \{1, 3\}$, $J = \{2, 4\}$.

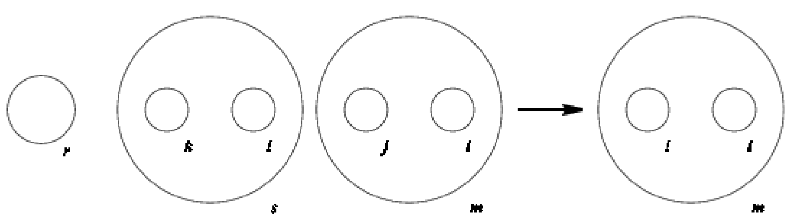

Figure 4.4: Relation 4

Applying Z gives:

$$Z_{(f,\varphi,\alpha)}(d_r p_{kis} p_{jlm} \overline{e}_k \otimes e_r \otimes \overline{e}_j \otimes e_s \otimes e_i \otimes e_l \otimes e_m) = p_{ilm} e_i \otimes e_l \otimes e_m$$

i.e.

$$d_k p_{kij} p_{jlm} = p_{ilm}.$$

Conversely, given tensors d_i, p_{ijk}, satisfying Relations 1-4, we define a TQFT functor Z as follows. Any object of $\mathcal{C}$ is of the form

$$(X_\pm, C_{\pm\pm\pm\cdots}, m_{ijk\cdots}),$$

where the boundary circles have been marked i, j, k, ... etc.. Set

$$Z_{(X_\pm, C_{\pm\pm\pm\cdots}, m_{ijk\cdots})} = x_{ijk\cdots} \overset{(-)}{e}_i \otimes \overset{(-)}{e}_j \otimes \overset{(-)}{e}_k \cdots,$$

where $\overset{(-)}{e}_i$ denotes e_i or $\overline{e}_i$ depending on the orientation of the corresponding subobject circle $C_\pm$. The tensors $x_{ijk\cdots}$ are given as follows:

empty set $\emptyset$	1
sphere S	$d_i d_i$
disk D	d_i
annulus A	$d_k p_{kij}$
torus T	$d_k p_{kii}$

and for any other connected object X, by taking a pants decomposition of X, labelling all circles in the decomposition, assigning a tensor to each labelled pair-of-pants and summing over all repeated indices (circles which are not in the boundary of X). This is well-defined because of the Kohno result above. For objects with more than one connected component we multiply the tensors associated to each connected component.

It remains to show that these assignments are consistent with the $Z2$) axiom for Z. For isomorphisms which change a subobject component from C_+ to C_-, or vice-versa, such as $(f, \alpha) : (D_+, C_+, m) \to (D_+, C_-, \overline{m})$, where $m(z) = z$, $\overline{m}(z) = \overline{z}$, $f = \mathrm{id}_{D_+}$ and $\alpha(z) = \overline{z}$, the components of the tensor are unchanged (since $Z'_\alpha(e_i) = \overline{e}_i$). For isomorphisms which are the identity on X and permute the boundary components, the consistency condition is the complete symmetry of the tensor $x_{ijk\cdots}$ under interchanges of indices. This is clear for the annulus and the pair-of-pants because of Relation 2, and for the general case one can show symmetry under the interchange of any pair of indices corresponding to boundary circles in the same connected component of X by repeated application of Relations 1 and 2, until the indices both belong to the same pants tensor in the expression. For any X one can construct an isomorphism $(X, A, m) \to (P(X), A', m')$ by

Figure 4.5: Reflection of X in the horizontal plane

reflecting in a plane of symmetry (see Fig. 4.5) so that the tensors for X_+ and X_- are the same.

To show consistency with Condition $Z2$) for gluing morphisms, it is enough to consider gluing morphisms which glue two boundary circles together, since any gluing morphism involving more than two circles can be written as a composition of such gluing morphisms. We will consider the six combinations of gluing together two objects, both of which are the disk, the annulus or X, where X denotes a surface with non-empty boundary admitting a pants decomposition.

1. $disk + disk \to sphere$
   The consistency condition is an identity:
   $$d_i d_i = d_i d_i.$$

2. $disk + annulus \to disk$
   The consistency condition is
   $$d_j d_k p_{kij} = d_i,$$
   which holds because of Relation 3.

3. $annulus + annulus \to annulus$
   The consistency condition is
   $$d_k p_{kij} d_l p_{ljm} = d_k p_{kim},$$
   which holds because of Relation 4.

4. $disk + X_{g,n} \to X_{g,n-1}$, where the indices g and n denote the genus and the number of boundary components, respectively.
   If $X = X_{0,3} = P$, then $X_{0,2} = A$ and the consistency condition is an identity:
   $$d_k p_{kij} = d_k p_{kij}.$$

Likewise, if X is $X_{1,1}$, then $X_{1,0} = T$ and we have an identity:

$$d_k p_{kii} = d_k p_{kii}.$$

Otherwise X has a pants decomposition into two or more pairs-of-pants and consistency follows from Relation 4 (see Fig. 4.6).

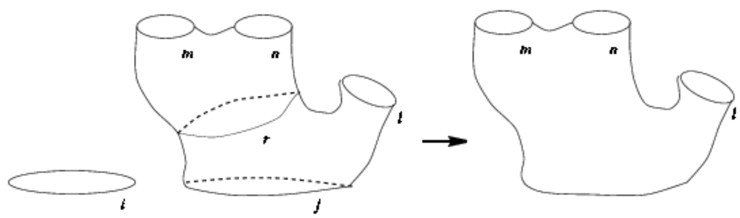

Figure 4.6: Gluing a disk and X

5. *annulus* $+ X_{g,n} \to X_{g,n}$.
   Consistency again follows from Relation 4 since suppose the annulus is attached to a boundary circle labelled j of $X_{g,n}$, belonging to a pair-of-pants labelled with j, l and m in some chosen pants decomposition. Then

$$d_k p_{kij} p_{jlm} = p_{ilm}.$$

6. $X_{g,n} + X'_{g',n'} \to X''_{g+g',n+n'-2}$.
   Here consistency is immediate since pants decompositions of X and X' induce a pants decomposition of X'' and the formulae for the corresponding tensors are clearly compatible. ■

5. Hermitian TQFT's

The TQFT functors of the previous section did not involve the full topological and algebraic categories, i.e. in terms of the internal structure on the categories, they only took account of the endofunctors P and Q defined on $\mathcal{S}(\mathbf{C})$ and $\mathcal{S}(\mathbf{D})$, respectively. In this final section we will define TQFT functors from a full topological category to a full algebraic category, which involve extended endofunctors $\mathbf{P}$ and $\mathbf{Q}$ defined on $\mathcal{C}$ and $\mathcal{D}$, respectively. This is achieved by extending the natural isomorphism $\eta : Z'P \to QZ'$ to a natural isomorphism $\boldsymbol{\eta} : Z\mathbf{P} \to \mathbf{Q}Z$.

Indeed we have seen in Section 2 that the monoidal endofunctor P defined on $\mathbf{C}$ and $\mathcal{S}(\mathbf{C})$ extends to a monoidal endofunctor $\mathbf{P}$ on $\mathcal{C}$ (Definition 2.23), and in Section 3 that, under certain conditions, the monoidal endofunctor Q defined on $\mathbf{D}$ and $\mathcal{S}(\mathbf{D})$ extends to a monoidal endofunctor $\mathbf{Q}$ on $\mathcal{D}$ (Definition 3.10). When Q extends in this way, we can consider extending the natural isomorphism $\eta : Z'P \to QZ'$ to a natural isomorphism $\boldsymbol{\eta} : Z\mathbf{P} \to \mathbf{Q}Z$.

Definition 5.1. *Given an extension* $\mathbf{Q}$ *of* Q*, the* extension *of* η *corresponding to* $\mathbf{Q}$ *(if it exists) is the monoidal natural isomorphism* $\boldsymbol{\eta} : Z\mathbf{P} \to \mathbf{Q}Z$ *satisfying, for every* $(X, A, m) \in \mathrm{Ob}(\mathcal{C})$,

$$\boldsymbol{\eta}_{(X,A,m)} := \eta_A.$$

For $\boldsymbol{\eta}_{(X,A,m)}$ to belong to $\mathrm{Mor}(\mathcal{D})$, we have the necessary condition,

$$\eta_A(Z_{\mathbf{P}(X,A,m)}) = (Z_{(X,A,m)})_{Q(V_A)}. \tag{5.1}$$

This condition is also sufficient:

Theorem 5.2. *If* $\eta_A(Z_{\mathbf{P}(X,A,m)}) = (Z_{(X,A,m)})_{Q(V_A)}$ *holds for every* $(X, A, m) \in \mathrm{Ob}(\mathcal{C})$*, then* $\boldsymbol{\eta}$ *as defined above is a monoidal natural isomorphism from* $Z\mathbf{P}$ *to* $\mathbf{Q}Z$*.*

Proof. For isomorphisms $(f, \alpha) : (X, A, m) \to (X', A', m')$ the naturality of $\boldsymbol{\eta}$ follows from the naturality of $\eta : Z'P \to QZ'$ for α, since $Z_{\mathbf{P}(f,\alpha)}$ and $\mathbf{Q}(Z_{(f,\alpha)})$ are morphisms of $\mathcal{D}$, i.e. preserve elements, and $\boldsymbol{\eta}_{(X,A,m)}$, $\boldsymbol{\eta}_{(X',A',m')}$ preserve elements by Equation (5.1).

For gluing morphisms $(f, \varphi, \alpha) : (X, A, m) \to (X', A', m')$ the naturality of $\boldsymbol{\eta}$ is given by

$$\mathbf{Q}(Z_{(f,\varphi,\alpha)}) \circ \boldsymbol{\eta}_{(X,A,m)} = \boldsymbol{\eta}_{(X',A',m')} \circ Z_{P(f,\varphi,\alpha)},$$

where again all morphisms preserve elements. For simplicity we will take $A = \sqcup_{i=1,2,3} A_i$, $\varphi : A_1 \to P(A_2)$ and $\alpha : A_3 \to A'$, set $V_{A_1} = V$, $V_{A_2} = W$, $V_{A_3} = Y$, $V_{A'} = Y'$, $V_{P(A_2)} = W_P$ and $V_{P(A_3)} = Y_P$, and denote π_2, θ_2 by π and θ, respectively. We prove the equivalent equation:

$$\mathbf{Q}(Z_{(f,\varphi,\alpha)}) \circ \boldsymbol{\eta}_{(X,A,m)} \circ Z'_{\pi_{(A_1,A_2,A_3)}} = \boldsymbol{\eta}_{(X',A',m')} \circ Z_{P(f,\varphi,\alpha)} \circ Z'_{\pi_{(A_1,A_2,A_3)}},$$

where

$$Z'_{\pi_{(A_1,A_2,A_3)}} = Z'_{\pi_{(A_1 \sqcup A_2, A_3)}} \circ (Z'_{\pi_{(A_1,A_2)}} \otimes \mathrm{id}_{Y_P}).$$

We have (writing composition as juxtaposition and denoting $Z'_{\pi_{(A_1,A_2,A_3)}}$ by Z'_π):

$$\begin{aligned}
\mathbf{Q}(Z_{(f,\varphi,\alpha)})\boldsymbol{\eta}_{(X,A,m)}Z'_\pi &= Q((e_W \otimes Z'_\alpha)(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_{W\otimes Y}))\eta_{A_1\sqcup A_2\sqcup A_3}Z'_\pi \\
&= Q((e_W \otimes Z'_\alpha)(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_{W\otimes Y}))\theta_{(V\otimes W,Y)}(\theta_{(V,W)} \otimes \mathrm{id}_{Q(Y)}) \\
&\quad (\eta_{A_1} \otimes \eta_{A_2} \otimes \eta_{A_3}) \\
&= Q(e_W \otimes Z'_\alpha)Q(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_{W\otimes Y}))\theta_{(V\otimes W,Y)} \\
&\quad (\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) \otimes \eta_{A_3}) \\
&= Q(e_W \otimes Z'_\alpha)\theta_{(Q(W)\otimes W,Y)}(Q(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_W \otimes \mathrm{id}_{Q(Y)}) \\
&\quad (\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) \otimes \eta_{A_3}) \\
&= \theta_{(I,Y')}(Q(e_W) \otimes Q(Z'_\alpha))(Q(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_W \otimes \mathrm{id}_{Q(Y)}) \\
&\quad (\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) \otimes \eta_{A_3}) \\
&= \theta_{(I,Y')}(Q(e_W(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_W)) \otimes Q(Z'_\alpha)) \\
&\quad (\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) \otimes \eta_{A_3}) \\
&= \theta_{(I,Y')}(Q(e_W(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_W))\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) \otimes Q(Z'_\alpha)\eta_{A_3} \\
&= \theta_{(I,Y')}(Q(e_W(\eta_{A_2}Z'_\varphi \otimes \mathrm{id}_W))\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) \otimes \eta_{A'}Z'_{P(\alpha)} \\
&= \theta_{(I,Y')}(\theta_0 e_{W_P}(\eta_{P(A_2)}Z'_{P(\varphi)} \otimes \mathrm{id}_{W_P}) \otimes \eta_{A'}Z'_{P(\alpha)} \\
&= \theta_{(I,Y')}(\theta_0 \otimes \eta_{A'})(e_{W_P}(\eta_{P(A_2)}Z'_{P(\varphi)} \otimes \mathrm{id}_{W_P}) \otimes Z'_{P(\alpha)}) \\
&= \theta_{(I,Y')}(\theta_0 \otimes \eta_{A'})(e_{W_P} \otimes Z'_{P(\alpha)})(\eta_{P(A_2)}Z'_{P(\varphi)} \otimes \mathrm{id}_{W_P\otimes Y_P}) \\
&= \eta_{A'}(e_{W_P} \otimes Z'_{P(\alpha)})(\eta_{P(A_2)}Z'_{P(\varphi)} \otimes \mathrm{id}_{W_P\otimes Y_P}) \\
&= \boldsymbol{\eta}_{(X',A',m')}Z_{P(f,\varphi,\alpha)}.
\end{aligned}$$

The first and the last equalities are definitions, the second and ninth are shown in Lemma 5.3, the third, sixth, seventh, tenth and eleventh are the interchange law, the fourth and fifth are the naturality of θ, the eighth is the naturality of η, and the twelfth is part of the definition of Q being a monoidal functor. Finally, since $\eta : Z'P \to QZ'$ is monoidal we also have the corresponding equations for $\boldsymbol{\eta} : Z\mathbf{P} \to \mathbf{Q}Z$:

$$\boldsymbol{\theta}_{(Z(X,A,m),Z(X',A',m'))} \circ (\boldsymbol{\eta}_{(X,A,m)} \otimes \boldsymbol{\eta}_{(X',A',m')}) = \boldsymbol{\eta}_{(X,A,m)\sqcup(X',A',m')} \circ Z_{\pi_{((X,A,m),(X',A',m'))}}$$

and the equation

$$\boldsymbol{\eta}_{(E,E,\mathrm{id}_E)}Z_{\boldsymbol{\pi}_0} = \boldsymbol{\theta}_0$$

in which all the morphisms preserve elements. ■

We now prove the required lemma:

Lemma 5.3. *Under the previous hypotheses, we have:*

$$Q(e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W))\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) = \theta_0 e_{W_P}(\eta_{P(A_2)} Z'_{P(\varphi)} \otimes \mathrm{id}_{W_P})$$

and

$$\eta_{A_1 \sqcup A_2 \sqcup A_3} Z'_{\pi_{(A_1,A_2,A_3)}} = \theta_{(V \otimes W,Y)}(\theta_{(V,W)} \otimes \mathrm{id}_{Q(Y)})(\eta_{A_1} \otimes \eta_{A_2} \otimes \eta_{A_3}).$$

Proof. To show the first equation we compose both sides with $(\eta_{P(A_2)} Z'_{P(\varphi)})^{-1} \otimes \mathrm{id}_{W_P}$. Then we have:

$$\begin{aligned}
\theta_0 e_{W_P} &= \theta_0 e_{Q(W)}(Q(\eta_{A_2}) \otimes \eta_{A_2}) \\
&= Q(e_W)\theta_{(Q(W),W)}(Q(\eta_{A_2}) \otimes \eta_{A_2}) \\
&= Q(e_W)\theta_{(Q(W),W)}(Q(\eta_{A_2} Z'_\varphi) \otimes \mathrm{id}_W)(Q(Z'_{\varphi^{-1}}) \otimes \eta_{A_2}) \\
&= Q(e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W))\theta_{(V,W)}(Q(Z'_{\varphi^{-1}}) \otimes \eta_{A_2}) \\
&= Q(e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W))\theta_{(V,W)}(\eta_{A_1}(\eta_{P(A_2)} Z'_{P(\varphi)})^{-1} \otimes \eta_{A_2}) \\
&= Q(e_W(\eta_{A_2} Z'_\varphi \otimes \mathrm{id}_W))\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2})(\eta_{P(A_2)} Z'_{P(\varphi)})^{-1} \otimes \mathrm{id}_{W_P}).
\end{aligned}$$

The first is the fact that η_{A_2} is a morphism of $\mathcal{S}(\mathbf{D})$, the second is the conjugation axiom of the evaluation map, the third is the interchange law, the fourth is the naturality of θ, the fifth is the naturality of η and the sixth is the interchange law. Now we prove the second equation:

$$\begin{aligned}
\eta_{A_1 \sqcup A_2 \sqcup A_3} Z'_{\pi_{(A_1,A_2,A_3)}} &= \theta_{(V \otimes W,Y)}(\eta_{A_1 \sqcup A_2} \otimes \eta_{A_3})(Z'_{\pi_{(A_1,A_2)}} \otimes \mathrm{id}_{Y_P}) \\
&= \theta_{(V \otimes W,Y)}(\eta_{A_1 \sqcup A_2} Z'_{\pi_{(A_1,A_2)}} \otimes \eta_{A_3}) \\
&= \theta_{(V \otimes W,Y)}(\theta_{(V,W)}(\eta_{A_1} \otimes \eta_{A_2}) \otimes \eta_{A_3}) \\
&= \theta_{(V \otimes W,Y)}(\theta_{(V,W)} \otimes \mathrm{id}_{Q(Y)})(\eta_{A_1} \otimes \eta_{A_2} \otimes \eta_{A_3})
\end{aligned}$$

using the monoidal property of η in the first and third equalities and the interchange law in the second and fourth equalities. ■

Example: Recall from Section 3 that in our example an extension of Q to $\mathcal{D}$ is given by:

$$\mathbf{Q}(V, x) = (\overline{V}, k_V(x)),$$

and a corresponding condition on morphisms. Consider an object (X, C_+, m) of $\mathcal{C}$, with a single boundary component. Then the Condition (5.1) for this object is:

$$\eta_{C_+}(Z_{(P(X),C_-,P(m))}) = k_V(Z_{(X,C_+,m)}).$$

Now $\eta_{C_+} = \mathrm{id}_{\overline{V}}$, so imposing the Conditions (4.1)-(4.3) as in Section 4, and bearing in mind the relation between $Z_{(X,C_+,m)}$ and $Z_{(P(X),C_-,P(m))}$ (see the discussion relating to Fig. 4.5), we have, setting $Z_{(X,C_+,m)} = x_i e_i$,

$$x_i \bar{\cdot}\, e_i = k_V(x_i e_i) = \overline{x}_i \bar{\cdot}\, e_i,$$

i.e. the tensor x_i is real.

For an object with empty boundary $(X, \emptyset, m)$ the Condition (5.1) is:

$$\eta_\emptyset(Z_{(P(X),\emptyset,P(m))}) = k_{\mathbb{C}}(Z_{(X,\emptyset,m)}).$$

Setting $Z_{(X,\emptyset,m)} = x$ we have $Z_{(P(X),\emptyset,P(m))} = x$ also, and using $\eta_\emptyset = \theta_0$ we get:

$$\theta_0(x) = k_{\mathbb{C}}(x)$$

i.e.

$$k_{\mathbb{C}}(\overline{x}) = k_{\mathbb{C}}(x)$$

so that x has to be real.

Finally, we will consider an object (X, C_{-+}, m) with two boundary components (but the reasoning can be extended to any number of components). The Condition (5.1) for this object reads:

$$\eta_{C_{-+}}(Z_{(P(X),C_{+-},P(m))}) = (Z_{(X,C_{-+},m)})_{Q(\overline{V}\otimes V)}.$$

Setting $Z_{(X,C_{-+},m)} = x_{ij}\overline{e}_i \otimes e_j$ we have $Z_{(P(X),C_{+-},P(m))} = x_{ij} e_i \otimes \overline{e}_j$. On the left hand side we use:

$$\eta_{C_{-+}} = \theta_{2(\overline{V},V)}(\eta_{C_-} \otimes \eta_{C_+}) = \theta_{2(\overline{V},V)}$$

and on the right hand side:

$$\begin{aligned}
(x_{ij}\overline{e}_i \otimes e_j)_{Q(\overline{V}\otimes V)} &= ((x_{ij} \bar{\cdot}\, \overline{e}_i) \otimes e_j)_{Q(\overline{V}\otimes V)} \\
&= \theta_{2(\overline{V},V)}((x_{ij} \bar{\cdot}\, \overline{e}_i)_{Q(\overline{V})} \otimes (e_j)_{Q(V)}) \\
&= \theta_{2(\overline{V},V)}((\overline{x}_{ij} e_i) \otimes \overline{e}_j) \\
&= \theta_{2(\overline{V},V)}(\overline{x}_{ij} e_i \otimes \overline{e}_j).
\end{aligned}$$

Thus we get

$$\theta_{2(\overline{V},V)}(x_{ij} e_i \otimes \overline{e}_j) = \theta_{2(\overline{V},V)}(\overline{x}_{ij} e_i \otimes \overline{e}_j),$$

i.e. the components x_{ij} are again real.▲

We now introduce some terminology to describe TQFT's for which η extends to $\boldsymbol{\eta}$. First we define the notion of an algebraic category with hermitian structure, based on the example in Section 3.

Definition 5.4. *We say that the full algebraic category* $(\mathcal{D}, \mathbf{Q})$ *has a hermitian structure if:*

1. *K has an involution, $j : K \to K$,*
2. *The endofunctor Q on $\mathbf{D}$ acts on the underlying K-modules and their morphisms as follows: for objects V of $\mathbf{D}$*

$$G(Q(V)) = G(V)^j,$$

   where $G(V)^j$ denotes the K-module with the same underlying set and addition as $G(V)$ and scalar multiplication $\cdot_j$ given by $\alpha \cdot_j x = j(\alpha)x$, where scalar multiplication in $G(V)$ is denoted by juxtaposition. For morphisms $f : V \to W$ of $\mathbf{D}$:

$$G(Q(f)) = k_{G(W)} \circ G(f) \circ k_{G(V)}^{-1},$$

   where $k_{G(V)} : G(V) \to G(V)^j$ is the j-semilinear map given by $k_{G(V)}(x) = x$, for all $x \in G(V)$,

3. *for every object V of $\mathcal{S}(\mathbf{D})$ the evaluation is non-degenerate, in the sense that the associated adjoint homomorphisms*

$$G(Q(V)) \to G(V)^* \text{ and } G(V) \to G(Q(V))^*,$$

   are isomorphisms, where $G(V)^* = \mathrm{Hom}(G(V), K)$,

4. *$\mathbf{Q} : \mathcal{D} \to \mathcal{D}$ is given, in terms of Definition 3.10, by*

$$x_{Q(V)} = k_{G(V)}(x).$$

Definition 5.5. *Let $(Z, \eta) : (\mathcal{C}, P) \to (\mathcal{D}, Q)$ be a TQFT functor and $\mathbf{Q}$ be an extension of Q. If η extends to $\boldsymbol{\eta} : Z\mathbf{P} \to \mathbf{Q}Z$, we say that the pair $(Z, \boldsymbol{\eta})$ is a* full TQFT functor *from $(\mathcal{C}, \mathbf{P})$ to $(\mathcal{D}, \mathbf{Q})$.*
If in addition, $\mathcal{D}$ has a hermitian structure, $(Z, \boldsymbol{\eta})$ is said to be a hermitian TQFT*. A hermitian TQFT for which $K = \mathbb{C}$, j is complex-conjugation and the evaluation is positive definite for every object $\mathcal{S}(\mathbf{D})$ (i.e. the bilinear map $Q(V) \times V \to K$, corresponding to the evaluation, is positive definite) is called a* unitary *TQFT.*

Example: The category $\mathcal{D}$ has hermitian structure both in our main example (see page 42), if we ensure that e_V is non-degenerate, and in the Example 3.13 of hermitian linear spaces.

For the 2-dimensional TQFT's considered in the previous section, our discussion above about the conditions for η to extend to $\boldsymbol{\eta}$ gives rise to the following characterization of unitary TQFT's:

Theorem 5.6. *Unitary TQFT's of the type considered in Theorem 4.6 are in 1-1 correspondence with pairs of real-valued tensors d_i, p_{ijk} satisfying the Relations 1-4 there.*▲

Remark 5.7. *In [2] Atiyah defined $(d+1)$-dimensional TQFT's over $\mathbb{C}$, for which X is a $(d+1)$-dimensional oriented differentiable manifold with boundary A, and V_A is a finite-dimensional complex vector space, with $Z_X \in V_A$. Our Condition (5.1) corresponds to Atiyah's hermitian axiom*

$$Z_{-X} = \overline{Z}_X,$$

where $-X$ denotes X with opposite orientation, and $\overline{Z}_X$ denotes the complex conjugate of Z_X when $A = \emptyset$, i.e. $V_A = \mathbb{C}$, and $k_{V_A}(Z_X)$ otherwise.
For a hermitian TQFT the evaluation on V_A gives rise to a hermitian form on V_A and induces an isomorphism

$$V_{-A} \cong V_A^*,$$

where $-A$ denotes A with opposite orientation, which is the involutory axiom of Atiyah's paper.

6. Final Comments

The example of 2-dimensional TQFT's was useful for illustrating the formalism, but describes a rather straightforward topological setup. In future work we hope to use our framework to gain new insight into more substantial examples, in particular Stallings manifolds [38] in 3-dimensional topology, which are obtained by self-gluing from manifolds of the form $\Sigma \otimes I$, where Σ is 2-dimensional, and parallel transport for gerbes [39, 24]. An interesting avenue on the algebraic side would be to try and incorporate infinite-dimensional vector spaces into the approach.

References

[1] E. Witten. Topological Quantum Field Theory. *Commun. Math. Phys.*, 117:353–386, 1988.

[2] M. Atiyah. Topological Quantum Field Theories. *Publ. Math. Inst. Hautes Etudes Sci.*, 68:175–186, 1989.

[3] G. Segal. Two-Dimensional Conformal Field Theories and Modular Functors. In *Proceedings of the XIth International Conference on Mathematical Physics*, pages 22–37, Swansea, 1988. Adam Hilger, Bristol, 1989.

[4] E. Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.

[5] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.

[6] V. Turaev and O. Viro. State sum invariants of three-manifolds and quantum $6j$-symbols. *Topology*, 31:865–902, 1992.

[7] L. Crane and D. Yetter. A categorical construction of 4D topological quantum field theories. In *Quantum Topology*, number 3 in Ser. Knots Everything, pages 120–130. World Sci. Publishing, River Edge, NJ, 1993.

[8] R. Dijkgraaf and E. Witten. Topological Gauge Theories and Group Cohomology. *Commun. Math. Phys.*, 129:393–429, 1990.

[9] S. Sawin. Links, Quantum Groups and TQFT's. *Bull. Amer. Math. Soc.*, 33(4):413–445, 1996.

[10] J. Birman. *Braids, Links and Mapping Class Groups*, volume 82 of *Annals Math. Studies*. Princeton Univ. Press, Princeton, 1975.

[11] V. Turaev. Operator invariants of tangles and R-matrices. *Math. USSR-Izv.*, 35(2):411–444, 1990.

[12] V. Turaev. Homotopy field theory in dimension 2 and group-algebras. 1999. math.QA/9910010.

[13] V. Turaev. Homotopy field theory in dimension 3 and crossed group-categories. 2000. math.GT/0005291.

[14] M. Brightwell and P. Turner. Representations of the homotopy surface category of a simply connected space. *J. Knot Theory Ramifications*, 9(7):855–864, 2000.

[15] G. Rodrigues. Homotopy quantum field theories and the homotopy cobordism category in dimension 1+1. 2001. to appear in J. Knot Theory Ramifications, math.QA/0105018.

[16] J. Barrett and B. Westbury. Spherical categories. *Adv. Math.*, 143(2):357–375, 1999.

[17] L. Crane and I. Frenkel. Four-dimensional Topological Quantum Field Theory, Hopf Categories, and the Canonical Bases. *J. Math. Phys.*, 35(10):5136–5154, 1994.

[18] L. Crane and D. Yetter. On algebraic structures implicit in topological quantum field theories. *J. Knot Theory Ramifications*, 8(2):125–163, 1999.

[19] M. Mackaay. Spherical 2-Categories and 4-Manifold Invariants. *Adv. Math.*, 143:288–348, 1999.

[20] M. Mackaay. Finite Groups, Spherical 2-Categories, and 4-Manifold Invariants. *Adv. Math.*, 153:353–390, 2000.

[21] D. Evans and Y. Kawahigashi. *Quantum Symmetries on Operator Algebras.* Oxford Mathematical Monographs. Oxford Science Publications, 1998.

[22] J. C. Baez and J. Dolan. Higher-dimensional Algebra and Topological Quantum Field Theory. *J. Math. Phys.*, 36:6073–6105, 1995.

[23] J. Baez and L. Langford. Higher-dimensional Algebra IV: 2-Tangles. to appear in Adv. Math.

[24] U. Bunke, P. Turner and S. Willerton. Gerbes and homotopy quantum field theories. 2002. math.AT/0201116.

[25] F. Quinn. *Lectures on Axiomatic Topological Quantum Field Theory*, volume I of *Math. Series*, pages 323–459. Amer. Math. Soc., IAS/Park City, 1995.

[26] F. Quinn. Group categories and their field theories. In *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, number 2 in Geom. Topol. Monogr., pages 407–453 (electronic). Geom. Topol. Publ., Coventry, 1999.

[27] V. Turaev. *Quantum Invariants of Knots and 3-Manifolds.* Walter de Gruyter, 1994.

[28] R. F. Picken. Reflections on Topological Quantum Field Theory. *Rep. Math. Phys.*, 40(2):295–303, 1997.

[29] L. Abrams. Two-dimensional Topological Quantum Field Theories and Frobenius Algebras. *J. Knot Theory Ramifications*, 5(5):569–587, 1996.

[30] S. Sawin. Direct Sum Decompositions and Indecomposable TQFT's. *J. Math. Phys.*, 36(12):6673–6680, 1995.

[31] R. J. Lawrence. An Introduction to Topological Field Theory. In *Proceedings of Symposia in Applied Mathematics*, volume 51, pages 89–128, 1996.

[32] R. Picken and P. Semião. TQFT - a new direction in algebraic topology. In *Proceedings of the Meeting "New Developments in Algebraic Topology"*, pages 85–101, Faro, Portugal, July 13-14 1998. Yu. Kubyshin et al. eds., Univ. do Algarve, 2000. math.QA/9912085.

[33] P. Semião. *Teorias Topológicas Quânticas do Campo no Contexto da Topologia Algébrica.* Ph.D. Dissertation, Univ. do Algarve, Área Departamental de Matemática, November 2001.

[34] S. Mac Lane. *Categories for the Working Mathematician.* Springer Verlag, 2nd edition, 1998.

[35] C. Kassel. *Quantum Groups.* Springer Verlag, 1995.

[36] J. Adámek, H. Herrlich and G. Strecker. *Abstract and Concrete Categories.* John Wiley, 1990.

[37] T. Kohno. Topological invariants for 3-manifolds using representations of mapping class groups I. *Topology*, 31(2):203–230, 1992.

[38] J. Stallings. On Fibering Certain 3-Manifolds. In *Topology of 3-Manifolds and related topics, Proceedings of the University of Georgia Institute, 1961*, pages 95–100. M. K. Fort, Jr. Ed., Prentice-Hall, New Jersey, 1962.

[39] M. Mackaay and R. Picken. Holonomy and parallel transport for abelian gerbes. *Adv. Math.*, 170(7):287–339, 2002.